An expression for probability and first passage times of BD-processes

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1 Introduction and definitions

Let us consider a finite continuous/discrete-time birth-death process $\{X_t, t \geq 0\}$ with $X_t \in \{0, \dots, N\}$. Every position x has a probability/rate to go left and right of q_x and p_x respectively . The space includes two absorbers being $\{0, N\}$ for which $q_N = p_0 = 0$: the process stops when $X_t \in \{0, N\}$. Physically, these two positions can be seen as the PAM and the cleavage position respectively.

We are interested in the length of time of a process. This can be defined as the time that the process hits either position 0 or N:

$$T_{0,N} = \min\{t \geq 0 : X_t \in \{0, N\}\} \tag{1}$$

For this two more definitions are needed:

$$T_0 = \min\{t \geq 0 : X_t = 0\} \tag{2}$$
$$T_N = \min\{t \geq 0 : X_t = N\} \tag{3}$$

Two more things need to be defined which will be used often:

$$\varphi(x) = \begin{cases} \prod_{j=1}^{x} \frac{q_j}{p_j} & \text{if } x \geq 1 \\ 1 & \text{if } x = 0 \end{cases} \tag{4}$$
$$\mathbb{P}_x(A) = \mathbb{P}(A|X_0 = x) \tag{5}$$

Linear operator

The following linear operator plays a central role in this document:

$$Lf(x) = p(x)\left[f(x+1) - f(x)\right] - q(x)\left[f(x) - f(x-1)\right] \tag{6}$$

2 Probability

First we can derive a closed expression for the probability that X_t hits N before 0. Let us define the following for $0 \leq x \leq N$:

$$\Lambda(x) = \mathbb{P}_x(T_N < T_0) \tag{7}$$

It is clear that $\Lambda(0) = 0$ and $\Lambda(N) = 1$, these are the boundary conditions. Observe the following:

$$\mathbb{P}_x(T_N < T_0) = \mathbb{P}_{x+1}(T_N < T_0)p_x + \mathbb{P}_{x-1}(T_N < T_0)q_x \tag{8}$$

As $q_x + p_x = 1$:

$$(p_x + q_x)\Lambda(x) = p_x\Lambda(x+1) + q_x\Lambda(x-1) \tag{9}$$
$$p_x\left[\Lambda(x+1) - \Lambda(x)\right] - q_x\left[\Lambda(x) - \Lambda(x-1)\right] = 0 \tag{10}$$

From this we see that $L\Lambda(x) = 0$. Therefore Λ is a homogeneous solution of the linear operator mentioned in equation (6). We try to find Λ:

$$\Lambda(x+1) - \Lambda(x) = \frac{q(x)}{p(x)}\left[\Lambda(x) - \Lambda(x-1)\right] \tag{11}$$

Plugging this equation in itself we find:

$$\Lambda(x+1) - \Lambda(x) = \frac{q(x)}{p(x)} \frac{q(x-1)}{p(x-1)} \left[\Lambda(x-1) - \Lambda(x-2)\right]$$

which eventually gives

$$\begin{aligned}\Lambda(x+1) - \Lambda(x) &= \frac{q(x)}{p(x)} \frac{q(x-1)}{p(x-1)} \cdots \frac{q(1)}{p(1)} \left[\Lambda(1) - \Lambda(0)\right] \\ &= \prod_{\eta=1}^{x} \left(\frac{q_\eta}{p_\eta}\right) \cdot \left[\Lambda(1) - \Lambda(0)\right] \qquad (12)\end{aligned}$$

Summing (12) over x from 0 to $y-1$ results in a telescope series on the left hand side.

$$\Lambda(y) - \Lambda(0) = \sum_{x=0}^{y-1} \left(\prod_{\eta=1}^{x} \frac{q_\eta}{p_\eta}\right) \cdot \left[\Lambda(1) - \Lambda(0)\right] \tag{13}$$

A homogeneous solution can be multiplied by a constant. Therefore we can fix the expression for $\Lambda(x)$ such that it adheres to the boundary conditions: $\Lambda(0) = 0$ and $\Lambda(N) = 1$. From this it follows that

$$\Lambda(y) = \mathbb{P}_y(T_N < T_0) = \frac{\sum_{x=0}^{y-1} \left(\prod_{\eta=1}^{x} \frac{q_\eta}{p_\eta}\right)}{\sum_{x=0}^{N-1} \left(\prod_{\eta=1}^{x} \frac{q_\eta}{p_\eta}\right)} = \frac{\sum_{x=0}^{y-1} \varphi(x)}{\sum_{x=0}^{N-1} \varphi(x)} \tag{14}$$

3 Expectation of $T_{0,N}$

We are interested in the time that it takes the Birth-Death process to end; the process reaches state 0 or N. From this we can easily derive the time either to end in state 0 or to end in state N which will be done in the next section.

First define the following:

$$\psi(x) = \mathbb{E}_x\left(T_{0,N}\right) \tag{15}$$

Clearly, the boundary conditions are $\psi(0) = \psi(N) = 0$. When one assumes discrete time, equation (16) holds:

$$\begin{aligned}\psi(x) &= \mathbb{E}_x\left(T_{0,N} | X_1 = x+1\right) p_x + \mathbb{E}_x(T_{0,N} | X_1 = x-1) q_x \qquad (16)\\ &= \mathbb{E}_{x+1}(T_{0,N}+1) p_x + \mathbb{E}_{x-1}(T_{0,N}+1) q_x \qquad (17)\\ &= \left[\psi(x+1)+1\right] p_x + \left[\psi(x-1)+1\right] q_x \qquad (18)\end{aligned}$$

Assuming $p(x)$ and $q(x)$ add up to 1:

$$p_x[\psi(x+1) - \psi(x)] - q_x[\psi(x) - \psi(x-1)] = -(p_x + q_x) \tag{19}$$

Now one can see that $\psi(x)$ adheres to $L\psi = -(p_x + q_x)$. A homogeneous solution to this operator was found in the previous section, let us find a particular solution now.

$$\psi(x+1) - \psi(x) = \frac{q_x}{p_x}[\psi(x) - \psi(x-1)] - 1 - \frac{q_x}{p_x} \tag{20}$$

Notice that we can plug in this equation in itself as was done in the previous section. Let us write out two iterations:

$$\begin{aligned}
\psi(x+1)-\psi(x) &= \frac{q_x}{p_x}\left[\frac{q_{x-1}}{p_{x-1}}[\psi(x-1)-\psi(x-2)]-1-\frac{q_{x-1}}{p_{x-1}}\right]-1-\frac{q_x}{p_x} \\
&= \frac{q_x q_{x-1}}{p_x p_{x-1}}[\psi(x-1)-\psi(x-2)]-\frac{q_x q_{x-1}}{p_x p_{x-1}}-2\frac{q_x}{p_x}-1 \\
&\vdots \\
&= \frac{q_x q_{x-1} q_{x-2}}{p_x p_{x-1} p_{x-2}}[\psi(x-2)-\psi(x-3)]-\frac{q_x q_{x-1} q_{x-2}}{p_x p_{x-1} p_{x-2}}-2\frac{q_x q_{x-1}}{p_x p_{x-1}}-2\frac{q_x}{p_x}-1
\end{aligned}$$

Generalizing our result:

$$\psi(x+1)-\psi(x)=\varphi(x)[\psi(1)-\psi(0)]-\sum_{i=0}^{x-1}\left\{\frac{\varphi(x)}{\varphi(x-i-1)}+\frac{\varphi(x)}{\varphi(x-i)}\right\} \tag{21}$$

Summing over x from 0 to $y-1$ gives a telescope series, which gives the particular solution:

$$\psi(y)-\psi(0)=\sum_{x=0}^{y-1}\left\{\varphi(x)[\psi(1)-\psi(0)]-\sum_{i=0}^{x-1}\left(\frac{\varphi(x)}{\varphi(x-i-1)}+\frac{\varphi(x)}{\varphi(x-i)}\right)\right\} \tag{22}$$

For the general solution, the homogeneous solution can be added to the particular solution as often as necessary to meet the boundary conditions $\psi(0)=\psi(N)=0$. From this the following expression can be derived:

$$\psi(y)=\sum_{x=0}^{y-1}\left\{\frac{\sum_{\xi=0}^{N-1}\sum_{j=0}^{\xi-1}\left(\frac{\varphi(\xi)}{\varphi(\xi-j-1)}+\frac{\varphi(\xi)}{\varphi(\xi-j)}\right)}{\sum_{\xi=0}^{N-1}\varphi(\xi)}\varphi(x)-\sum_{i=0}^{x-1}\left(\frac{\varphi(x)}{\varphi(x-i-1)}+\frac{\varphi(x)}{\varphi(x-i)}\right)\right\} \tag{23}$$

which is the expression for $\mathbb{E}_y(T_{0,N})$.

Alternative expression for (23)

Our starting point is:

$$L\psi=-(p_x+q_x)=-1 \tag{24}$$

$$\begin{aligned}
\psi(x+1)-\psi(x) &= \frac{q_x}{p_x}\left(\frac{q_{x-1}}{p_{x-1}}[\psi(x-1)-\psi(x-2)]-\frac{1}{p_{x-1}}\right)-\frac{1}{p_x} \\
&= \frac{q_x q_{x-1}}{p_x p_{x-1}}[\psi(x-1)-\psi(x-2)]-\frac{q_x}{p_x p_{x-1}}-\frac{1}{p_x} \\
&\vdots \\
&= \frac{q_x q_{x-1} q_{x-2}}{p_x p_{x-1} p_{x-2}}[\psi(x-2)-\psi(x-3)]-\frac{q_x q_{x-1}}{p_x p_{x-1} p_{x-2}}-\frac{q_x}{p_x p_{x-1}}-\frac{1}{p_x}
\end{aligned}$$

Generalizing these findings:

$$\psi(x+1)-\psi(x)=\varphi(x)[\psi(1)-\psi(0)]-\sum_{i=0}^{x-1}\left\{\frac{1}{p_{x-i}}\frac{\varphi(x)}{\varphi(x-i)}\right\} \tag{25}$$

Now we can sum over x from 0 to $y-1$ again to find a telescope series:

$$\psi(y) - \psi(0) = \sum_{x=0}^{y-1} \left\{ \varphi(x) \left[\psi(1) - \psi(0) \right] - \sum_{i=0}^{x-1} \left(\frac{1}{p_{x-i}} \frac{\varphi(x)}{\varphi(x-i)} \right) \right\} \tag{26}$$

Normalising this expression:

$$\psi(y) = \sum_{x=0}^{y-1} \left\{ \frac{\sum_{\xi=0}^{N-1} \sum_{j=0}^{\xi-1} \frac{1}{p_{\xi-j}} \frac{\varphi(\xi)}{\varphi(\xi-j)}}{\sum_{\xi=0}^{N-1} \varphi(\xi)} \varphi(x) - \sum_{i=0}^{x-1} \frac{1}{p_{x-i}} \frac{\varphi(x)}{\varphi(x-i)} \right\} \tag{27}$$

Openstaande punten

1. Omschrijven naar rates.
2. x-afhankelijke p en q in Python
3. Wat gebeurt er bij grote xi en $p < q$?
4. Uitdrukking voor $\mathbb{E}_y(T_{0,N} I(T_0 < T_N))$