

# AN EXPRESSION FOR PROBABILITY AND FIRST PASSAGE TIMES OF BD-PROCESSES

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## 1 Introduction and definitions

Let us consider a finite continuous/discrete-time birth-death process  $\{X_t, t \geq 0\}$  with  $X_t \in \{0, \dots, N\}$ . Every position  $x$  has a probability/rate to go left and right of  $q_x$  and  $p_x$  respectively. The space includes two absorbers being  $\{0, N\}$  for which  $q_N = p_0 = 0$ : the process stops when  $X_t \in \{0, N\}$ . Physically, these two positions can be seen as the PAM and the cleavage position respectively.

We are interested in the length of time of a process. This can be defined as the time that the process hits either position 0 or  $N$ :

$$T_{0,N} = \min\{t \geq 0 : X_t \in \{0, N\}\} \quad (1)$$

For this two more definitions are needed:

$$T_0 = \min\{t \geq 0 : X_t = 0\} \quad (2)$$

$$T_N = \min\{t \geq 0 : X_t = N\} \quad (3)$$

Two more things need to be defined which will be used often:

$$\varphi(x) = \begin{cases} \prod_{j=1}^x \frac{q_j}{p_j} & \text{if } x \geq 1 \\ 1 & \text{if } x = 0 \end{cases} \quad (4)$$

$$\mathbb{P}_x(A) = \mathbb{P}(A|X_0 = x) \quad (5)$$

### Linear operator

The following linear operator plays a central role in this document:

$$Lf(x) = p(x)[f(x+1) - f(x)] - q(x)[f(x) - f(x-1)] \quad (6)$$

## 2 Probability

First we can derive a closed expression for the probability that  $X_t$  hits  $N$  before 0. Let us define the following for  $0 \leq x \leq N$ :

$$\Lambda(x) = \mathbb{P}_x(T_N < T_0) \quad (7)$$

It is clear that  $\Lambda(0) = 0$  and  $\Lambda(N) = 1$ , these are the boundary conditions. Observe the following:

$$\mathbb{P}_x(T_N < T_0) = \mathbb{P}_{x+1}(T_N < T_0)p_x + \mathbb{P}_{x-1}(T_N < T_0)q_x \quad (8)$$

As  $q_x + p_x = 1$ :

$$(p_x + q_x)\Lambda(x) = p_x\Lambda(x+1) + q_x\Lambda(x-1) \quad (9)$$

$$p_x[\Lambda(x+1) - \Lambda(x)] - q_x[\Lambda(x) - \Lambda(x-1)] = 0 \quad (10)$$

From this we see that  $L\Lambda(x) = 0$ . Therefore  $\Lambda$  is a homogeneous solution of the linear operator mentioned in equation (6). We try to find  $\Lambda$ :

$$\Lambda(x+1) - \Lambda(x) = \frac{q(x)}{p(x)}[\Lambda(x) - \Lambda(x-1)] \quad (11)$$

Plugging this equation in itself we find:

$$\Lambda(x+1) - \Lambda(x) = \frac{q(x)}{p(x)} \frac{q(x-1)}{p(x-1)} [\Lambda(x-1) - \Lambda(x-2)]$$

which eventually gives

$$\begin{aligned} \Lambda(x+1) - \Lambda(x) &= \frac{q(x)}{p(x)} \frac{q(x-1)}{p(x-1)} \dots \frac{q(1)}{p(1)} [\Lambda(1) - \Lambda(0)] \\ &= \prod_{\eta=1}^x \left( \frac{q_{\eta}}{p_{\eta}} \right) \cdot [\Lambda(1) - \Lambda(0)] \end{aligned} \quad (12)$$

Summing (12) over  $x$  from 0 to  $y-1$  results in a telescope series on the left hand side.

$$\Lambda(y) - \Lambda(0) = \sum_{x=0}^{y-1} \left( \prod_{\eta=1}^x \frac{q_{\eta}}{p_{\eta}} \right) \cdot [\Lambda(1) - \Lambda(0)] \quad (13)$$

A homogeneous solution can be multiplied by a constant. Therefore we can fix the expression for  $\Lambda(x)$  such that it adheres to the boundary conditions:  $\Lambda(0) = 0$  and  $\Lambda(N) = 1$ . From this it follows that

$$\Lambda(y) = \mathbb{P}_y(T_N < T_0) = \frac{\sum_{x=0}^{y-1} \left( \prod_{\eta=1}^x \frac{q_{\eta}}{p_{\eta}} \right)}{\sum_{x=0}^{N-1} \left( \prod_{\eta=1}^x \frac{q_{\eta}}{p_{\eta}} \right)} = \frac{\sum_{x=0}^{y-1} \varphi(x)}{\sum_{x=0}^{N-1} \varphi(x)} \quad (14)$$

### 3 Expectation of $T_{0,N}$

We are interested in the time that it takes the Birth-Death process to end; the process reaches state 0 or  $N$ . From this we can easily derive the time either to end in state 0 or to end in state  $N$  which will be done in the next section.

First define the following:

$$\psi(x) = \mathbb{E}_x(T_{0,N}) \quad (15)$$

Clearly, the boundary conditions are  $\psi(0) = \psi(N) = 0$ . When one assumes discrete time, equation (16) holds:

$$\psi(x) = \mathbb{E}_x(T_{0,N} | X_1 = x+1) p_x + \mathbb{E}_x(T_{0,N} | X_1 = x-1) q_x \quad (16)$$

$$= \mathbb{E}_{x+1}(T_{0,N} + 1) p_x + \mathbb{E}_{x-1}(T_{0,N} + 1) q_x \quad (17)$$

$$= [\psi(x+1) + 1] p_x + [\psi(x-1) + 1] q_x \quad (18)$$

Assuming  $p(x)$  and  $q(x)$  add up to 1:

$$p_x[\psi(x+1) - \psi(x)] - q_x[\psi(x) - \psi(x-1)] = -1 \quad (19)$$

Now one can see that  $\psi(x)$  adheres to  $L\psi = -1$ . A homogeneous solution to this operator was found in the previous section, let us find a particular solution now.

$$\psi(x+1) - \psi(x) = \frac{q_x}{p_x} [\psi(x) - \psi(x-1)] - \frac{1}{p_x} \quad (20)$$

Notice that we can plug in this equation in itself as was done in the previous section. Let us write out two iterations:

$$\begin{aligned}
\psi(x+1) - \psi(x) &= \frac{q_x}{p_x} \left( \frac{q_{x-1}}{p_{x-1}} [\psi(x-1) - \psi(x-2)] - \frac{1}{p_{x-1}} \right) - \frac{1}{p_x} \\
&= \frac{q_x q_{x-1}}{p_x p_{x-1}} [\psi(x-1) - \psi(x-2)] - \frac{q_x}{p_x p_{x-1}} - \frac{1}{p_x} \\
&\vdots \\
&= \frac{q_x q_{x-1} q_{x-2}}{p_x p_{x-1} p_{x-2}} [\psi(x-2) - \psi(x-3)] - \frac{q_x q_{x-1}}{p_x p_{x-1} p_{x-2}} - \frac{q_x}{p_x p_{x-1}} - \frac{1}{p_x}
\end{aligned}$$

Generalizing our result:

$$\psi(x+1) - \psi(x) = \varphi(x)[\psi(1) - \psi(0)] - \sum_{i=0}^{x-1} \left\{ \frac{1}{p_{x-i}} \frac{\varphi(x)}{\varphi(x-i)} \right\} \quad (21)$$

Summing over  $x$  from 0 to  $y-1$  gives a telescope series, which gives the particular solution:

$$\psi(y) - \psi(0) = \sum_{x=0}^{y-1} \left\{ \varphi(x) [\psi(1) - \psi(0)] - \sum_{i=0}^{x-1} \left( \frac{1}{p_{x-i}} \frac{\varphi(x)}{\varphi(x-i)} \right) \right\} \quad (22)$$

For the general solution, the homogeneous solution can be added to the particular solution as often as necessary to meet the boundary conditions  $\psi(0) = \psi(N) = 0$ . From this the following expression can be derived:

$$\psi(y) = \sum_{x=0}^{y-1} \left\{ \frac{\sum_{\xi=0}^{N-1} \sum_{j=0}^{\xi-1} \frac{1}{p_{\xi-j}} \frac{\varphi(\xi)}{\varphi(\xi-j)}}{\sum_{\xi=0}^{N-1} \varphi(\xi)} \varphi(x) - \sum_{i=0}^{x-1} \frac{1}{p_{x-i}} \frac{\varphi(x)}{\varphi(x-i)} \right\} \quad (23)$$

which is the expression for  $\mathbb{E}_y(T_{0,N})$ .

### Alternative expression for (23)

Our starting point is:

$$\begin{aligned}
L\psi &= -(p_x + q_x) \quad (24) \\
\psi(x+1) - \psi(x) &= \frac{q_x}{p_x} \left[ \frac{q_{x-1}}{p_{x-1}} [\psi(x-1) - \psi(x-2)] - 1 - \frac{q_{x-1}}{p_{x-1}} \right] - 1 - \frac{q_x}{p_x} \\
&= \frac{q_x q_{x-1}}{p_x p_{x-1}} [\psi(x-1) - \psi(x-2)] - \frac{q_x q_{x-1}}{p_x p_{x-1}} - 2 \frac{q_x}{p_x} - 1 \\
&\vdots \\
&= \frac{q_x q_{x-1} q_{x-2}}{p_x p_{x-1} p_{x-2}} [\psi(x-2) - \psi(x-3)] - \frac{q_x q_{x-1} q_{x-2}}{p_x p_{x-1} p_{x-2}} - 2 \frac{q_x q_{x-1}}{p_x p_{x-1}} - 2 \frac{q_x}{p_x} - 1
\end{aligned}$$

Generalizing these findings:

$$\psi(x+1) - \psi(x) = \varphi(x)[\psi(1) - \psi(0)] - \sum_{i=0}^{x-1} \left\{ \frac{\varphi(x)}{\varphi(x-i-1)} + \frac{\varphi(x)}{\varphi(x-i)} \right\} \quad (25)$$

Now we can sum over  $x$  from 0 to  $y - 1$  again to find a telescope series:

$$\psi(y) - \psi(0) = \sum_{x=0}^{y-1} \left\{ \varphi(x)[\psi(1) - \psi(0)] - \sum_{i=0}^{x-1} \left( \frac{\varphi(x)}{\varphi(x-i-1)} + \frac{\varphi(x)}{\varphi(x-i)} \right) \right\} \quad (26)$$

Normalising this expression:

$$\psi(y) = \sum_{x=0}^{y-1} \left\{ \frac{\sum_{\xi=0}^{N-1} \sum_{j=0}^{\xi-1} \left( \frac{\varphi(\xi)}{\varphi(\xi-j-1)} + \frac{\varphi(\xi)}{\varphi(\xi-j)} \right)}{\sum_{\xi=0}^{N-1} \varphi(\xi)} \varphi(x) - \sum_{i=0}^{x-1} \left( \frac{\varphi(x)}{\varphi(x-i-1)} + \frac{\varphi(x)}{\varphi(x-i)} \right) \right\} \quad (27)$$

## 4 The conditional expectation

Finally we would like to derive an expression for  $\tau_x := \mathbb{E}_x(T_{0,N} | T_0 < T_N)$ . For this we can use the following identity:

$$\tau_x := \mathbb{E}_x(T_{0,N} | T_0 < T_N) = \frac{\mathbb{E}_x(T_{0,N} I(T_0 < T_N))}{\mathbb{P}_x(T_0 < T_N)}. \quad (28)$$

In this expression  $I(T_0 < T_N)$  represents the indicator function. We define the following:

$$\mathbb{P}_x(T_0 < T_N) = 1 - \Lambda(x) := \Lambda^*(x)$$

We would like to find an expression for  $\kappa_x = \mathbb{E}_x(T_{0,N} I(T_0 < T_N))$ . Again our boundary conditions are  $\kappa(0) = \kappa(N) = 0$ . Assuming discrete time we can see the following:

$$\kappa_x = \mathbb{E}_{x+1}[(T_{0,N} + 1)I(T_0 < T_N)]p_x + \mathbb{E}_{x-1}[(T_{0,N} + 1)I(T_0 < T_N)]q_x \quad (29)$$

Working out the brackets gives:

$$= \kappa_{x+1}p_x + \mathbb{E}_{x+1}[I(T_0 < T_N)]p_x + \kappa_{x-1}q_x + \mathbb{E}_{x-1}[I(T_0 < T_N)]q_x \quad (30)$$

$$= \kappa_{x+1}p_x + \kappa_{x-1}q_x + \Lambda^*(x+1)p_x + \Lambda^*(x-1)q_x \quad (31)$$

Assuming  $p_x$  and  $q_x$  add up to 1 and using identity 9, we find:

$$p_x[\kappa_{x+1} - \kappa_x] = q_x[\kappa_x - \kappa_{x-1}] - \Lambda^*(x) \quad (32)$$

Using the linear operator we see that the following needs to be solved:  $L\kappa = -\Lambda^*$ . Working out the recursion gives us the following:

$$\kappa_{x+1} - \kappa_x = \frac{q_x}{p_x} [\kappa_x - \kappa_{x-1}] - \frac{1}{p_x} \Lambda_x^* \quad (33)$$

$$= \frac{q_x}{p_x} \left[ \frac{q_{x-1}}{p_{x-1}} [\kappa_{x-1} - \kappa_{x-2}] - \frac{1}{p_{x-1}} \Lambda_{x-1}^* \right] - \frac{1}{p_x} \Lambda_x^* \quad (34)$$

$$= \frac{q_x q_{x-1}}{p_x p_{x-1}} [\kappa_{x-1} - \kappa_{x-2}] - \frac{q_x}{p_x p_{x-1}} \Lambda_{x-1}^* - \frac{1}{p_x} \Lambda_x^* \quad (35)$$

$$= \frac{q_x q_{x-1} q_{x-2}}{p_x p_{x-1} p_{x-2}} [\kappa_{x-2} - \kappa_{x-3}] - \frac{q_x q_{x-1}}{p_x p_{x-1} p_{x-2}} \Lambda_{x-2}^* - \frac{q_x}{p_x p_{x-1}} \Lambda_{x-1}^* - \frac{1}{p_x} \Lambda_x^* \quad (36)$$

$$\vdots \quad (37)$$

$$= \varphi(x) [\kappa_1 - \kappa_0] - \sum_{i=0}^{x-1} \frac{\varphi(x)}{\varphi(x-i)} \frac{1}{p_{x-i}} \Lambda_{x-i}^* \quad (38)$$

Using the telescope series the following expression can be found:

$$\kappa_y - \kappa_0 = \sum_{x=0}^{y-1} \left\{ \varphi(x) [\kappa_1 - \kappa_0] - \sum_{i=0}^{x-1} \frac{\varphi(x)}{\varphi(x-i)} \frac{1}{p_{x-i}} \Lambda_{x-i}^* \right\} \quad (39)$$

The normalization can be done by adding the homogeneous solution a number of times such that the equation adheres to the boundary conditions. Then the expression for  $\kappa_y$  equals:

$$\kappa_y = \sum_{x=0}^{y-1} \left\{ \frac{\sum_{\xi=0}^{N-1} \sum_{j=0}^{\xi-1} \frac{\varphi(\xi)}{\varphi(\xi-j)} \frac{1}{p_{\xi-j}} \Lambda_{\xi-j}^*}{\sum_{\xi=0}^{N-1} \varphi(\xi)} \varphi(x) - \sum_{i=0}^{x-1} \frac{\varphi(x)}{\varphi(x-i)} \frac{1}{p_{x-i}} \Lambda_{x-i}^* \right\} \quad (40)$$

Given this expression, the expression for  $\tau_y := \mathbb{E}_y(T_{0,N} | T_0 < T_N)$  can be found by:

$$\tau_y = \frac{\kappa_y}{\Lambda_y^*} \quad (41)$$

### Openstaande punten

1. Omschrijven naar rates.
2. x-afhankelijke p en q in Python
3. Wat gebeurt er bij grote  $N$  en  $p < q$ ?