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# Do We Actually Need Dense Over-Parameterization? In-Time Over-Parameterization in Sparse Training

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## Abstract

In this paper, we introduce a new perspective on training deep neural networks capable of state-of-the-art performance without the need for the expensive over-parameterization by proposing the concept of In-Time Over-Parameterization (ITOP) in sparse training. By starting from a random sparse network and continuously exploring sparse connectivities during training, we can perform an Over-Parameterization over the course of training, closing the gap in the expressibility between sparse training and dense training. We further use ITOP to understand the underlying mechanism of Dynamic Sparse Training (DST) and discover that the benefits of DST come from its ability to consider across time all possible parameters when searching for the optimal sparse connectivity. As long as sufficient parameters have been reliably explored, DST can outperform the dense neural network by a large margin. We present a series of experiments to support our conjecture and achieve the state-of-the-art sparse training performance with ResNet-50 on ImageNet. More impressively, ITOP achieves dominant performance over the overparameterization-based sparse methods at extreme sparsities. When trained with ResNet-34 on CIFAR-100, ITOP can match the performance of the dense model at an extreme sparsity of 98%.

## 1. Introduction

Over-Parameterization has been shown to be crucial to the dominating performance of deep neural networks in practice, despite the fact that the training objective function

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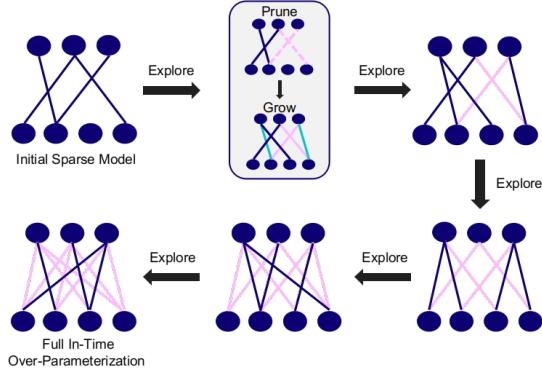


Figure 1. As the figure proceeds, we perform an Over-Parameterization in time. Blue lines refer to the currently activated connections. Pink lines are the connections that have been activated previously. While exploring In-Time Over-Parameterization, the parameter count (blue lines) of the sparse model is fixed throughout training.

is usually non-convex and non-smooth (Goodfellow et al., 2015; Brutzkus et al., 2017; Li & Liang, 2018; Safran & Shamir, 2018; Soudry & Carmon, 2016; Allen-Zhu et al., 2019; Du et al., 2019; Zou et al., 2020; Zou & Gu, 2019). Meanwhile, advanced deep models (Simonyan & Zisserman, 2014; He et al., 2016; Devlin et al., 2018; Brown et al., 2020; Dosovitskiy et al., 2021) are continuously achieving state-of-the-art results in numerous machine-learning tasks. While achieving impressive performance, the size of the state-of-the-art models is also exploding. The resources required to train and deploy those highly over-parameterized models are prohibitive.

Motivated for inference, a large body of research (Mozer & Smolensky, 1989; Han et al., 2015) attempts to discover a sparse model that can sufficiently match the performance of the corresponding dense model while substantially reduce the number of parameters. While effective, these techniques involve pre-training a highly over-parameterized model for either at least a full converged training time (full dense over-parameterization) (Janowsky, 1989; LeCun et al., 1990; Hassibi & Stork, 1993; Molchanov et al., 2017; Han et al., 2016; Gomez et al., 2019; Dai et al., 2018a) or a partial converged training time (partial dense over-parameterization) (Louizos et al., 2017; Zhu & Gupta, 2017; Gale et al., 2019; Savarese et al., 2019; Kusupati et al., 2020; You et al., 2019).

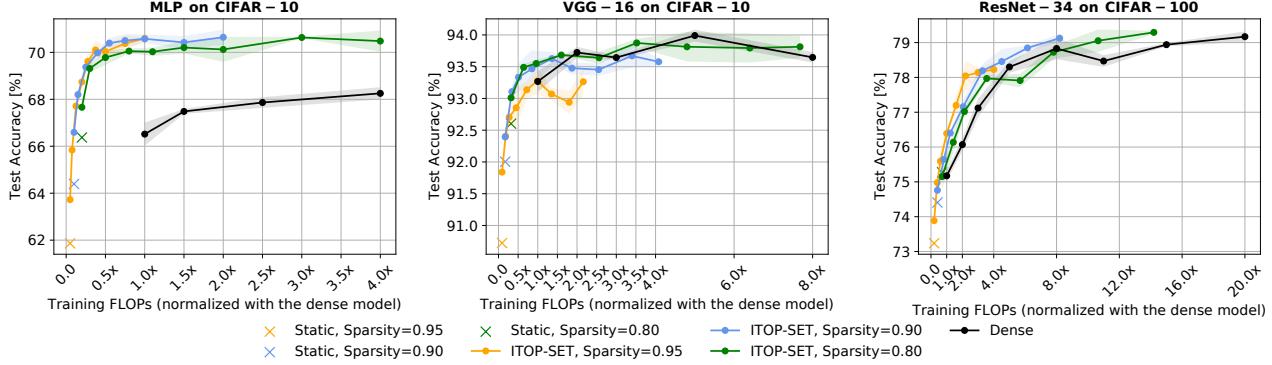


Figure 2. Training FLOPs of sparse models trained with In-Time Over-Parameterization.  $\Delta T = 1500$  and batch size is 128.

Given the fact that the training costs of state-of-the-art models e.g., GPT-3 (Brown et al., 2020) and Vision Transformer (Dosovitskiy et al., 2020), have increasingly exploded, this heavily over-parameterized dependency leads to a situation where state-of-the-art models are beyond the reach of the majority of the machine learning community.

Recently, the lottery ticket hypothesis (LTH) (Frankle & Carbin, 2019) shows the possibility to train a sub-network from scratch (sparse training) to match the performance of the dense network. However, these “winning tickets” are found under the guidance of a fully dense over-parameterized process (iterative pruning a fully converged network), and solutions that are discovered through either a partial dense over-parameterization (pruning at initialization (Lee et al., 2019; Wang et al., 2020; de Jorge et al., 2020)) or no over-parameterization (randomly-initialized static sparse training (Mocanu et al., 2016; Evci et al., 2019)) typically are not able to match the accuracy achieved by their dense counterpart. A common-sense explanation would be that in comparison with dense training, sparse training, especially at extreme high sparsities, does not have the over-parameterization property, and hence suffers from a poor expressibility. One approach to address this problem is to leverage the knowledge learned from dense training, e.g., LTH (Frankle & Carbin, 2019). While effective, the computational costs and memory requirements attached to the over-parameterized dense training are prohibitive.

### 1.1. Our Contribution

In this paper, we propose a concept that we call *In-Time Over-Parameterization*<sup>1</sup> to close the gap in over-parameterization along with expressibility between sparse training and dense training, illustrated in Figure 1. Instead of inheriting weights from a dense and pre-trained model, allowing a continuous parameter exploration across the training time performs an over-parameterization in the

<sup>1</sup><https://github.com/Shiweiliuiiiiiii/In-Time-Over-Parameterization>

space-time manifold, which can significantly improve the expressibility of sparse training.

We find the concept of In-Time Over-Parameterization useful (1) in exploring the expressibility of sparse training, especially for extreme sparsities, (2) in reducing training and inference costs (3) in understanding the underlying mechanism of dynamic sparse training (DST) (Mocanu et al., 2018; Evci et al., 2020a), (4) in preventing overfitting and improving generalization.

Based on In-Time Over-Parameterization, we improve the state-of-the-art sparse training performance with ResNet-50 on ImageNet. We further assess the ITOP concept by applying it to the main class of sparse training methods, DST, in comparison with the overparameterization-based sparse methods including LTH, gradual magnitude pruning (GMP), and pruning at initialization (PI). Our results show that, when a sufficient and reliable parameter exploration is reached (as required by ITOP), DST consistently outperforms those overparameterization-based methods. Since ITOP eliminates the dense Over-Parameterization throughout the whole course of training, it can match the performance of the corresponding dense networks with much fewer training FLOPs, as shown in Figure 2.

## 2. Related Work

### 2.1. Dense Over-Parameterization

Sparsity-inducing techniques that depend on dense over-parameterization (dense-to-sparse training) have been extensively studied. We divide them into three categories according to their degrees of dependence on the dense over-parameterization.

**Full dense over-parameterization.** Techniques sought to inherit weights from a fully pre-trained dense model have a long history and were first introduced by Janowsky (1989) and Mozer & Smolensky (1989), autonomously evolving as the iterative pruning and retaining method. The basic idea of iterative pruning and retaining involves a three-step pro-

cess: (1) fully pre-training a dense model until converged, (2) pruning the weights or the neurons that have the lowest influence on the performance, and (3) re-training the pruned model to further improve the performance. The pruning and retraining cycle is required at least once (Liu et al., 2019), and usually many times (Han et al., 2016; Guo et al., 2016; Frankle & Carbin, 2019). The criteria used for pruning includes but are not limited to magnitude (Mozer & Smolensky, 1989; Han et al., 2016; Guo et al., 2016), Hessian (LeCun et al., 1990; Hassibi & Stork, 1993), mutual information (Dai et al., 2018a), Taylor expansion (Molchanov et al., 2016; 2019). Except for pruning, other techniques including variational dropout (Molchanov et al., 2017), targeted dropout (Gomez et al., 2019), reinforcement learning (Lin et al., 2017) also yield a sparse model from a pre-trained dense model.

**Partial dense over-parameterization.** Another class of methods start from a dense network and continuously sparsify the model during training. Gradual magnitude pruning (GMP) (Narang et al., 2017; Zhu & Gupta, 2017; Gale et al., 2019) was proposed to reduce the number of pruning-and-retaining rounds by pruning the dense network to the target sparsity gradually over the course of training. There are some examples Louizos et al. (2017) and Wen et al. (2016) that utilize  $L_0$  and  $L_1$  regularization to gradually learn the sparsity by explicitly penalizing parameters for being different from zero, respectively. Recently, Srinivas et al. (2017); Liu et al. (2020a); Savarese et al. (2019); Xiao et al. (2019); Kusupati et al. (2020); Zhou et al. (2021) moved further by introducing trainable masks to learn the desirable sparse connectivity during training. Since these techniques start from a dense model, the training cost is smaller than training a dense network, depending on the stage at which the final sparse models are learned.

**One-Shot dense over-parameterization.** Very recently, works on pruning at initialization (PI) (Lee et al., 2019; 2020; Wang et al., 2020; Tanaka et al., 2020; de Jorge et al., 2021) have emerged to obtain trainable sparse neural networks before the main training process based on some salience criteria. These methods fall into the category of dense over-parameterization mainly because the dense model is required to train for at least one iteration to obtain those trainable sparse networks.

## 2.2. In-Time Over-Parameterization

**Dynamic Sparse Training.** Evolving in parallel with LTH, DST is a growing class of methods to train sparse networks from scratch with a fixed parameter count throughout training (sparse-to-sparse training). This paradigm starts from a (random) sparse neural network and allows the sparse connectivity to evolve dynamically during training. It has been first introduced in Mocanu (2017) and became well-

established in Mocanu et al. (2018) by proposing the Sparse Evolutionary Training (SET) algorithm which achieves better performance than static sparse neural networks. In addition to the proper classification performance, it also helps to detect important input features (Atashgahi et al., 2020). Bellec et al. (2018) proposed Deep Rewiring to train sparse neural networks with a strict connectivity constraint by sampling sparse configurations and weights from a posterior distribution. Follow-up works further introduced weight redistribution (Mostafa & Wang, 2019; Dettmers & Zettlemoyer, 2019; Liu et al., 2021), gradient-based weight growth (Dettmers & Zettlemoyer, 2019; Evcı et al., 2020a), and extra weights update in the backward pass (Raihan & Aamodt, 2020; Jayakumar et al., 2020) to improve the sparse training performance. By relaxing the constraint of the fixed parameter count, Dai et al. (2019; 2018b) proposed a grow-and-prune strategy based on gradient-based growth and magnitude-based pruning to yield an accurate, yet very compact sparse network. More recently, Liu et al. (2020b) illustrated for the first time the true potential of using dynamic sparse training. By developing an independent framework, they can train truly sparse neural networks without masks with over one million neurons on a typical laptop.

**Understanding Dynamic Sparse Training.** Concurrently, some works attempt to understand Dynamic Sparse Training. Liu et al. (2020c) found that DST gradually optimizes the initial sparse topology towards a completely different one. Although there exist many low-loss sparse solutions that can achieve similar loss, they are very different in the topological space. Evcı et al. (2020b) found that sparse neural networks that are initialized by a dense initialization e.g., He et al. (2015), suffer from a poor gradient flow, whereas DST can improve the gradient flow during training significantly. Although promising, the capability of sparse training has not been fully explored and the mechanism underlying DST is not clear yet. Questions like: *Why Dynamic Sparse Training can improve the performance of sparse training? How Dynamic Sparse Training can enable sparse neural network models to match - and even to outperform - their dense counterparts?* are required to be answered.

## 3. In-Time Over-Parameterization

In this section, we describe in detail In-Time Over-Parameterization, a concept that we proposed to be an alternative way to train deep neural networks without the expensive over-parameterization. We refer In-Time Over-Parameterization as a variant of dense over-parameterization, which can be achieved by encouraging a continuous parameter exploration across the training time. Note that different from the over-parameterization of dense models which refers to the spatial dimensionality of the parameter space, In-Time Over-Parameterization refers to the overall dimen-

sionality explored in the space-time manifold.

### 3.1. In-Time Over-Parameterization Hypothesis

Based on In-Time Over-Parameterization, we propose the following hypothesis to understand Dynamic Sparse Training:

**Hypothesis.** *The benefits of Dynamic Sparse Training come from its ability to consider across time all possible parameters when searching for the optimal sparse neural network connectivity.* Concretely, this hypothesis can be divided into three main pillars which can explain the performance of DST:

1. Dynamic Sparse Training can significantly improve the performance of sparse training mainly due to the parameter exploration across the training time.
2. The performance of Dynamic Sparse Training is highly related to the total number of the reliably explored parameters throughout training. The reliably explored parameters refer to those newly-explored (newly-grown) weights that have been updated for long enough to exceed the pruning threshold.
3. As long as there are sufficient parameters that have been reliably explored, sparse neural network models trained by Dynamic Sparse Training can match or even outperform their dense counterparts by a large margin, even at extremely high sparsity levels.

We name our hypothesis as In-Time Over-Parameterization hypothesis for convenience.

Formally, given a dataset containing  $N$  samples  $\mathbf{D} = \{(x_i, y_i)\}_{i=1}^N$  and a dense network  $f(x; \theta)$  parameterized by  $\theta$ . We train the dense network to minimize the loss function  $\sum_{i=1}^N L(f(x_i; \theta), y_i)$ . When optimizing with a certain optimizer,  $f(x; \theta)$  reaches a minimum validation loss function  $l$  with a test accuracy  $a$ . Differently, sparse training starts with a sparse neural network  $f(x; \theta_s)$  parameterized by a fraction of parameters  $\theta_s$ . The basic mechanism of Dynamic Sparse Training is to train the sparse neural network  $f(x; \theta_s)$  to minimize the loss  $\sum_{i=1}^N L(f(x_i; \theta_s), y_i)$  while periodically update the sparse connectivity  $\theta_s$  every  $\Delta T$  iterations based on some criteria.  $f(x; \theta_s^u)$  reaches a minimum validation loss  $l'$  at sparse connectivity update  $u$  with a test accuracy  $a'$ , where  $\theta_s^u$  is the sparse connectivity parameters obtained at the iteration  $u$ . Let us denote  $R_s$  as the ratio of the total number of reliably explored parameters during training to the total number of parameters, or simply In-Time Over-Parameterization rate, computed as  $R_s = \frac{\|\theta_s^1 \cup \theta_s^2 \cup \dots \cup \theta_s^u\|_0}{\|\theta\|_0}$ , where  $\|\cdot\|_0$  is the  $\ell_0$ -norm.

Our hypothesis states that when  $\Delta T \geq T_0$ ,  $\exists R_0$  as long as  $R_s \geq R_0$ , for which  $a' \geq a$  (commensurate accuracy) and

$\|\theta_s^u\|_0 \ll \|\theta\|_0$  (fewer parameters in the final sparse model), where  $T_0$  is the minimum threshold of update interval to guarantee the reliable parameter exploration, and  $R_0$  is the threshold of In-Time Over-Parameterization rate where DST can match the performance of the dense model.

Similar to dense training, there are many factors affecting the performance of dynamic sparse training, learning rate, batch size, regularization, optimizers, sparsity distribution, etc. In this paper, we limit our study to parameter exploration, since it is the fundamental difference between dynamic sparse training and static sparse training. By “reliable”, we mainly focus on the newly-activated weights that are updated for a long time (guaranteed by  $\Delta T \geq T_0$ ) so that they are not pruned in the next update iteration. We believe this is a good starting point, since even in this simple setting, our community does not have a satisfactory answer.

### 3.2. Hypothesis Evaluation

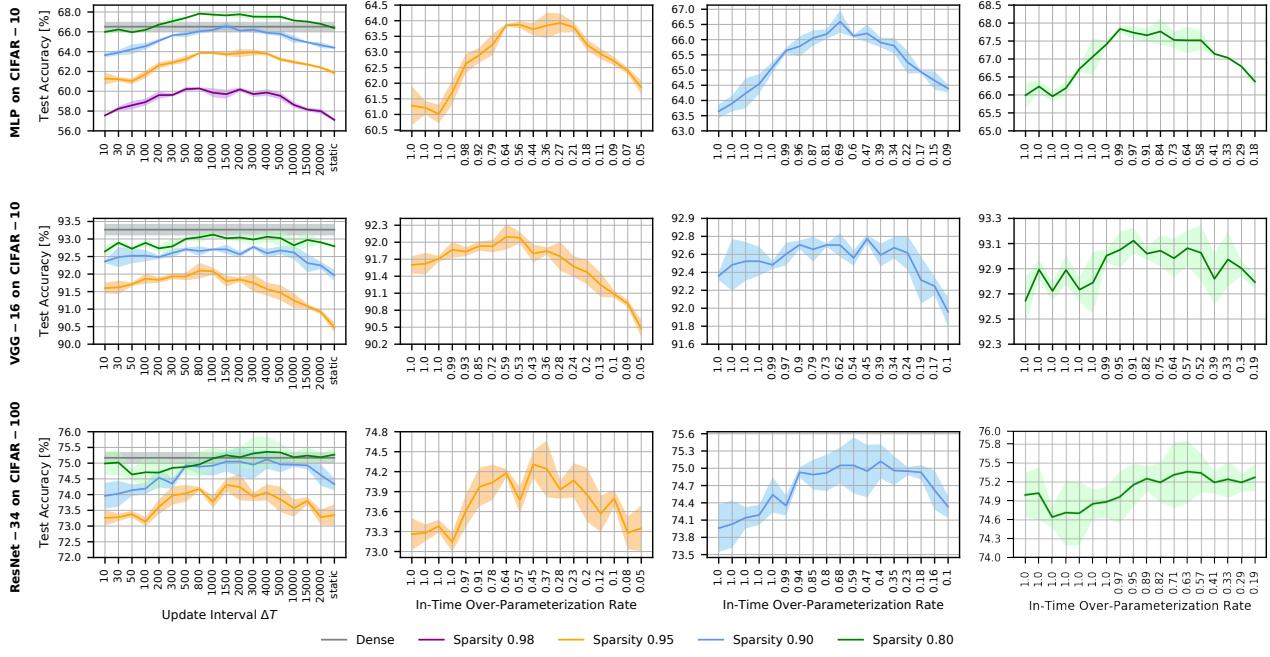
In this section, we work through the In-Time Over-Parameterization hypothesis and study the effect of In-Time Over-Parameterization on the performance of DST. We choose Sparse Evolutionary Training (SET) as our DST method as SET activates new weights in a random fashion which naturally considers all possible parameters to explore. It also helps to avoid the dense over-parameterization bias introduced by the gradient-based methods e.g., The Rigged Lottery (RigL) (Evci et al., 2020a) and Sparse Networks from Scratch (SNFS) (Dettmers & Zettlemoyer, 2019), as the latter utilize dense gradients in the backward pass to explore new weights. To work through the proposed hypothesis, we conduct a set of step-wise fashion experiments with image classification. We study Multi-layer Perceptron (MLP) on CIFAR-10, VGG-16 on CIFAR-10, ResNet-34 on CIFAR-100, and ResNet-50 on ImageNet. We use PyTorch as our library. All results are averaged from three different runs and reported with the mean and standard deviation. See Appendix A for the experimental details.

#### 3.2.1. TYPICAL TRAINING TIME

Our first evaluation of the In-Time Over-Parameterization hypothesis is to see what happens when different over-parameterization rates  $R_s$  are reached during training within a typical training time (200 or 250 epochs). A direct way to control  $R_s$  is to vary  $\Delta T$ , a hyperparameter that determines the update interval of sparse connectivities (the number of iterations between two sparse connectivity updates). We train MLP, VGG-16, and ResNet-34 with various  $\Delta T$  and report the test accuracy.

**Expected results.** Gradually decreasing  $\Delta T$  will explore more parameters, and thus lead to increasingly higher test accuracy. However, when  $\Delta T$  gets smaller than the reliable exploration threshold  $T_0$ , the test accuracy will start

## In-Time Over-Parameterization



**Figure 3.** Effect of In-Time Over-Parameterization on sparse training MLPs (**top**), VGG-16 (**middle**), and ResNet-34 (**bottom**) with a typical training time. All sparse models are trained with SET. Each line is averaged from three different runs. “Static” refers to the static sparse training without parameter exploration.

to decrease since the new weights can not receive enough updates to exceed the pruning threshold.

**Experimental results.** For a better overview, we plot the performance achieved at different sparsities together in the leftmost column of Figure 3. To understand better the relationship between  $R_s$  and test accuracy, we report the final  $R_s$  associated with  $\Delta T$  separately in the rest columns of Figure 3.

Overall, a similar pattern can be found existing in all lines. Starting from the static sparse training, sparse training consistently benefits from the increased  $R_s$  as  $\Delta T$  decreases. However, the test accuracy starts to drop rapidly after it reaches a peak value, especially at high sparsities (yellow and blue lines). For example, even if MLPs and ResNet-34 eventually reach a 100% exploration rate with extremely small  $\Delta T$  values (e.g., 10, 30), their performance is much worse than the static sparse training. This behavior is perfectly in line with our hypothesis. While small  $\Delta T$  encourages sparse models to maximally explore the search space spanned over the dense model, the benefits provided by In-Time Over-Parameterization is heavily limited by the unreliable parameter exploration. Interestingly, the negative effect of the unreliable exploration on lower sparsities (green lines) is less than the one on high sparsities (yellow lines). We regard this as trivial sparsities (Frankle et al., 2020a) as the remaining models are still over-parameterized to fit the data.

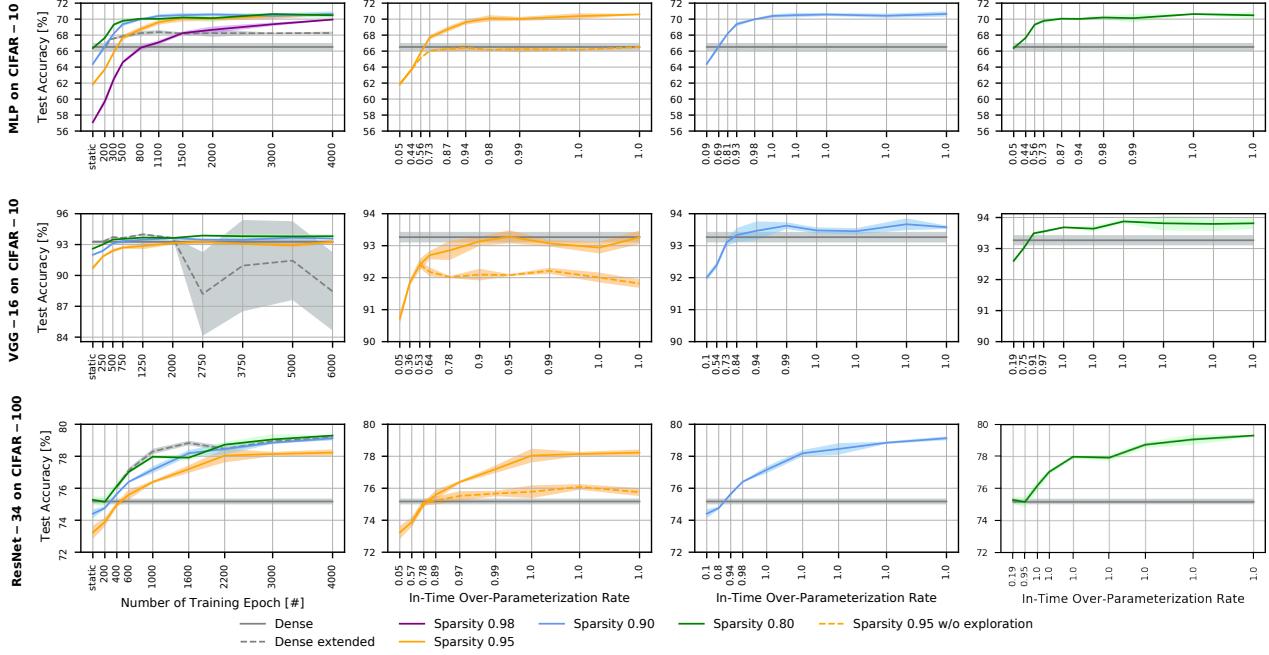
### 3.2.2. EXTENDED TRAINING TIME

Until now, we have already learned the trade-off between test accuracy and  $R_s$  for the typical training time. A direct approach to alleviating this trade-off is to extend the training time while using large  $\Delta T$ . We train MLP, VGG-16, and ResNet-34 for an extended training time with a large  $\Delta T$ . We safely choose  $\Delta T$  as 1500 for MLPs, 2000 for VGG-16, and 1000 for ResNet-34 according to the trade-off shown in Figure 3. In addition to the training time, the anchor points of the learning rate schedule are also scaled by the same factor.

**Expected results.** In this setting, we expect that, in addition to the benefits brought by the extended training time, sparse training would benefit significantly from the increased  $R_s$ .

**Experimental results.** The results are shown in Figure 4. Static sparse training without parameter exploration consistently achieves the lowest accuracy. However, all models at different sparsities substantially benefit from an extended training time accompanied by an increased  $R_s$ . In other words, reliably exploring the parameter space in time continuously improves the expressibility of sparse training. Importantly, after matching the performance of the dense baseline (black line), the performance of sparse training continues to improve, yielding a notable improvement over the dense baseline. Furthermore, the models with lower sparsities require less time to match their full accuracy plateau than

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**Figure 4.** Effect of In-Time Over-Parameterization on sparse training MLPs (**top**), VGG-16 (**middle**) and ResNet-34 (**bottom**) with an extended training time. All sparse models are trained with SET. Each line is averaged from three different runs. ‘‘Static’’ refers to the static sparse training without parameter exploration. ‘‘Dense extended’’ refers to training a dense model for an extended time. ‘‘Sparsity 0.95 w/o exploration’’ means we train the model for the same extended time but stop exploring parameters after a typical training time (200 or 250 epochs).

higher sparsities; the cause appears to be that models with lower sparsity can explore more parameters in the same training time.

To show that the performance gains are not only caused by the longer training time, we make a controlled experiment by stopping the parameter exploration immediately after the typical training time (the sparse connectivity remains fixed after a typical training time), shown as the orange dashed lines. As we can see, even though improved, the accuracy is much lower than the accuracy achieved by In-Time Over-Parameterization.

We also report the performance of dense models with an extended training time as the dashed black lines. Training a dense model with an extended time leads to either inferior (MLPs and VGG-16), or equal solutions (ResNet-34). Different from the dense over-parameterization where overfitting usually occurs when the model has been overtrained for long enough, the test accuracy of dynamic sparse training is continuously increasing as  $R_s$  increases until a plateau is reached with a full In-Time Over-Parameterization. This observation highlights the advantage of In-Time Over-Parameterization to prevent overfitting over the dense over-parameterization.

## 4. Effect of Hyperparameter Choices

### 4.1. Effect of Weight Growth methods on ITOP

We next investigate the effect of gradient-based weight growth (used in RigL and SNFS) and random-based weight growth (used in SET) on In-Time Over-Parameterization. Since gradient-based methods have access to the dense over-parameterization in the backward pass (occasionally using dense gradients to activate new weights), we hypothesize that they can reach a converged accuracy without a high  $R_s$ . We make a comparison between RigL and SET for both the typical training and the extended training in Figure 5. We study them on MLPs where the model size is relatively small so that we can easily achieve a full In-Time Over-Parameterization and have a better understanding of these two methods.

**Typical Training Time.** It is clear that RigL also heavily suffers from the unreliable exploration. As  $\Delta T$  decreases, the test accuracy of RigL presents a trend of rising, falling, and rising again. Compared with the random-based growth, RigL receives larger gains from the reliable parameter exploration and also a larger forfeit from the unreliable exploration. These differences are potentially due to that RigL grows new weights with high gradient magnitude, which leads to a faster loss decrease when the exploration is faithful, but also requires a higher  $\Delta T$  to guarantee a faithful

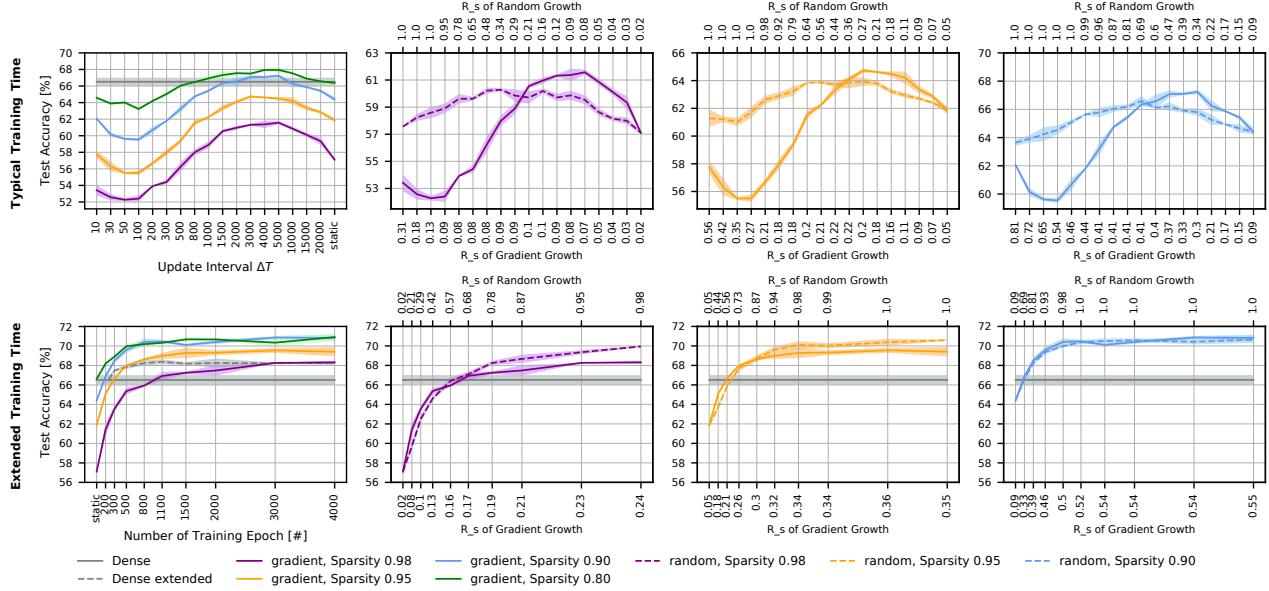


Figure 5. Comparisons between RigL and SET with MLP on CIFAR-10. We vary the update interval  $\Delta T$  for the typical training time setting, and keep it fixed for the extended training time setting (1500 for SET and 4000 for RigL).

exploration as the weight with large gradients is likely to end up with high magnitude, resulting in a large pruning threshold.

**Extended Training Time.** For RigL, we choose  $\Delta T = 4000$  to ensure the reliable exploration (the performance of RigL with a smaller  $\Delta T = 1500$  is much worse as shown in Appendix D). We can see that RigL also significantly benefits from an increased  $R_s$ . Surprisingly, although RigL achieves higher accuracy than the SET with a limited training time, it ends up with lower accuracy than SET with a sufficient training time. From the perspective of  $R_s$ , we can see that the  $R_s$  of RigL is much smaller than SET, indicating that gradient weight growth drives the sparse connectivity into some similar structures and in turn limits its expressibility. On the contrary, random growth naturally considers the whole search space to explore parameters and has a larger possibility of finding better local optima. Similar results are also reported for sparse Recurrent Neural Networks (RNNs) in Liu et al. (2021). However, similar results are not shared with large-scale architectures on large datasets. For instance, RigL achieves better performance than SET with ResNet-50 on ImageNet. This result is reasonable since the dense gradients help RigL easily find the most promising weights at each sparse connectivity update. In contrast, it would take a much longer time (high  $R_s$ ) for SET (random weight growth) to discover these promising weights within large-scale architectures, especially at high sparsities.

#### 4.2. Effect of Batch Size on ITOP

Intuitively, our hypothesis uncovers ways to improve the existing DST methods within a limited training time. A direct

way to reliably explore more parameters within a typical training time is to train with a small batch size. Using a smaller batch size equally means having more updates, and therefore leads to a higher  $R_s$ . We simply demonstrate the effectiveness of this conjecture on SET with  $\Delta T = 1500$  in Figure 6 (see Appendix E for RigL). With a large batch size, the parameter exploration is insufficient to achieve a high In-Time Over-Parameterization rate  $R_s$ , and the test accuracy is subsequently much lower than the dense model. As we expected, the reduction in batch size consistently increases  $R_s$  as well as the test accuracy, until the batch size gets smaller than 16. However, the performance of the dense model remarkably decreases as the batch size decreases. More interestingly, when the batch size is smaller than 16, the performance of sparse models flips and the sparsest model starts to achieve the highest accuracy. The performance drop is likely caused by the increased ‘noise scale’ of SGD where extremely small batch sizes lead to large noise scale and large accuracy drop (Smith et al. (2017)).

#### 4.3. Effect of Pruning Rate on ITOP

The initial pruning rate (denoted as  $P$ ) of parameter exploration also affects the overall number of parameters visited during training. Relatively large pruning rates encourage a large range of exploration, resulting in higher accuracy, whereas a too-large pruning rate hurts the model capacity as it prunes too many parameters. We confirm this with ResNet-18 on CIFAR-10 trained with various initial pruning rates  $P \in [0.1, 0.3, 0.5, 0.7, 0.9]$  as shown in Table 1. The similar pattern as we expected is shared across all update intervals.

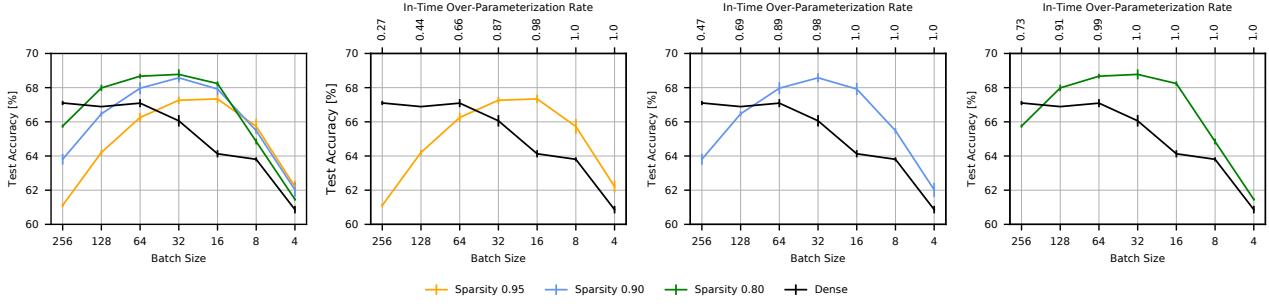


Figure 6. Test accuracy of SET with various batch sizes. The update interval  $\Delta T$  is set as 1500.

Table 1. Performance of sparse ResNet-18 on CIFAR-10 with various pruning rates. The results are run three times and reported with (mean  $\pm$  std,  $R_s$ ). The highest test accuracies are marked in bold.

| Sparsity | Method | P   | $\Delta T = 15000$                          | $\Delta T = 10000$                          | $\Delta T = 8000$                           | $\Delta T = 5000$                           | $\Delta T = 3000$                           |
|----------|--------|-----|---|---|---|---|---|
|          |        |     | (93.53 $\pm$ 0.02, 0.113)                   | (93.57 $\pm$ 0.05, 0.150)                   | (93.44 $\pm$ 0.12, 0.176)                   | (93.70 $\pm$ 0.19, 0.247)                   | (93.78 $\pm$ 0.09, 0.353)                   |
| 0.95     | SET    | 0.9 | <b>(93.54 <math>\pm</math> 0.09, 0.100)</b> | (93.54 $\pm$ 0.18, 0.130)                   | (93.76 $\pm$ 0.07, 0.151)                   | (93.91 $\pm$ 0.17, 0.210)                   | (93.63 $\pm$ 0.01, 0.300)                   |
|          | SET    | 0.7 | (93.51 $\pm$ 0.01, 0.086)                   | <b>(93.77 <math>\pm</math> 0.21, 0.109)</b> | <b>(93.84 <math>\pm</math> 0.10, 0.125)</b> | <b>(93.93 <math>\pm</math> 0.09, 0.170)</b> | <b>(93.94 <math>\pm</math> 0.08, 0.241)</b> |
|          | SET    | 0.5 | (93.28 $\pm$ 0.01, 0.071)                   | (93.66 $\pm$ 0.05, 0.086)                   | (93.80 $\pm$ 0.01, 0.096)                   | (93.75 $\pm$ 0.18, 0.126)                   | (93.86 $\pm$ 0.10, 0.174)                   |
|          | SET    | 0.3 | (93.24 $\pm$ 0.02, 0.056)                   | (93.35 $\pm$ 0.18, 0.061)                   | (93.29 $\pm$ 0.06, 0.065)                   | (93.50 $\pm$ 0.03, 0.076)                   | (93.34 $\pm$ 0.03, 0.096)                   |
|          | SET    | 0.1 | (93.24 $\pm$ 0.02, 0.056)                   | (93.35 $\pm$ 0.18, 0.061)                   | (93.29 $\pm$ 0.06, 0.065)                   | (93.50 $\pm$ 0.03, 0.076)                   | (93.34 $\pm$ 0.03, 0.096)                   |

#### 4.4. Boosting the Performance of DST

Based on the above-mentioned insights, we demonstrate the state-of-the-art sparse training performance with ResNet-50 on ImageNet. More precisely, we choose an update interval  $\Delta T$  of 4000, a batch size of 64, and an initial pruning rate of 0.5 so that we can achieve a high  $R_s$  within a typical training time. We briefly name the improved method as RigL-ITOP. Please see Appendix B for the implementation details. Table 2 shows that without any advanced techniques, our method boosts the accuracy of RigL over the overparameterization-based method (GMP and Lottery Ticket Rewinding (LTR)) (Frankle et al., 2020a)). More importantly, our method requires only 2× training time to match the performance of dense ResNet-50 at 80% sparsity, far less than RigL (5× training time) (Evci et al., 2020a).

Instead of using small batch size, another trick to encourage parameter exploration is sampling from the non-activated weights first when growing new weights. We demonstrate the effectiveness of this idea in Appendix F.

#### 5. The Versatility of ITOP

Although we mainly focus on understanding DST from the ITOP point of view, ITOP can be potentially generalized to other sparsity-inducing categories. Here, we demonstrate its versatility by applying ITOP to two recently popular methods, LTH and PI. We choose SNIP (Lee et al., 2019) as the PI method, as it consistently performs well among different methods for pruning at initialization as shown by Frankle et al. (2020b). Compared with ITOP, LTH and SNIP are two overparameterization-based methods designed for a better initial subnetwork but without consulting any

information yielded during training. We choose magnitude-based weight pruning and random-based weight growth for SNIP and LTH to achieve ITOP and name the corresponding methods as SNIP-SET-ITOP and LTH-SET-ITOP. To make a fair comparison between different pruning criteria, we use global and one-shot pruning for both SNIP and LTH. We train all models for 200 epochs and report the best test accuracy in Figure 7. See Appendix C for the experimental details.

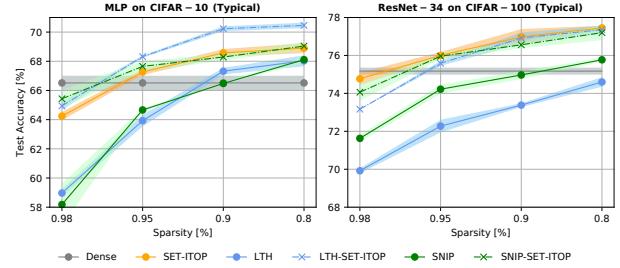


Figure 7. Effect of In-Time Over-Parameterization on SNIP and LTH. “Typical” means training a model for a typical time of 200 epochs.

With a high In-Time Over-Parameterization rate, SET-ITOP consistently outperforms the overparameterization-based methods as well as the dense training, by a large margin. For instance, SET-ITOP can easily match the performance of the corresponding dense models with at most 5% parameters. More importantly, SET-ITOP has dominant performance at the extreme sparsity (98%) over LTH and SNIP, indicating the potential of In-Time Over-Parameterization to address the poor expressibility problem of the extremely sparse neural networks.

Table 2. Performance of sparse ResNet-50 on ImageNet dataset with a typical training time. All results of other methods are obtained from Evcı et al. (2020a) except LTR which is the late-rewinding LTH version obtained from Evcı et al. (2020b). RigL-ITOP<sub>2×</sub> is obtained by extending the training time by 2 times.

| Methods                 | Top-1 Acc                          | $R_s$ | Training FLOPs      | Test FLOPs         | Top-1 Acc                          | $R_s$ | Training FLOPs      | Test FLOPs         |
|-------------------------|------------------------------------|-------|---------------------|--------------------|------------------------------------|-------|---------------------|--------------------|
| Dense                   | $76.8 \pm 0.09$                    | 1.00  | $1 \times (3.2e18)$ | $1 \times (8.2e9)$ | $76.8 \pm 0.09$                    | 1.00  | $1 \times (3.2e18)$ | $1 \times (8.2e9)$ |
| sparsity=0.9            |                                    |       |                     |                    | sparsity=0.8                       |       |                     |                    |
| Static                  | $67.7 \pm 0.12$                    | 0.10  | $0.24 \times$       | $0.24 \times$      | $72.1 \pm 0.04$                    | 0.20  | $0.42 \times$       | $0.42 \times$      |
| SET                     | $69.6 \pm 0.23$                    | -     | $0.10 \times$       | $0.10 \times$      | $72.9 \pm 0.39$                    | -     | $0.23 \times$       | $0.23 \times$      |
| SNFS                    | $72.9 \pm 0.06$                    | -     | $0.50 \times$       | $0.24 \times$      | $75.2 \pm 0.11$                    | -     | $0.61 \times$       | $0.42 \times$      |
| RigL                    | $73.0 \pm 0.04$                    | -     | $0.25 \times$       | $0.24 \times$      | $75.1 \pm 0.05$                    | -     | $0.42 \times$       | $0.42 \times$      |
| GMP                     | 73.9                               | -     | $0.56 \times$       | $0.23 \times$      | 75.6                               | -     | $0.51 \times$       | $0.10 \times$      |
| LTR                     | -                                  | -     | -                   | -                  | $75.75 \pm 0.12$                   | -     | -                   | -                  |
| RigL-ITOP               | $73.82 \pm 0.08$                   | 0.83  | $0.25 \times$       | $0.24 \times$      | $75.84 \pm 0.05$                   | 0.93  | $0.42 \times$       | $0.42 \times$      |
| RigL-ITOP <sub>2×</sub> | <b><math>75.50 \pm 0.09</math></b> | 0.89  | $0.50 \times$       | $0.24 \times$      | <b><math>76.91 \pm 0.07</math></b> | 0.97  | $0.84 \times$       | $0.42 \times$      |

It is maybe more interesting that In-Time Over-Parameterization brings large benefits to SNIP and LTH as well. While LTH and SNIP fall short of SET-ITOP, SNIP-SET-ITOP and LTH-SET-ITOP can match or even exceed the performance of SET-ITOP with both MLP and ResNet-34. This observation confirms that ITOP is a foundational concept and can potentially improve any existing sparse training methods.

## 6. Generalization Improvement of ITOP

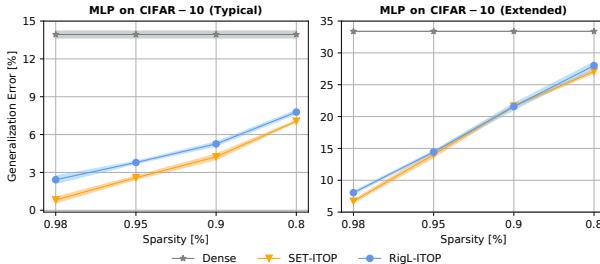


Figure 8. Generalization errors of SET-ITOP, RigL-ITOP, and the dense models.

We further observe the ability of In-Time Over-Parameterization to improve generalization. Figure 8 shows that the generalization error (the difference between the training accuracy and the test accuracy) of In-Time Over-Parameterization (SET-ITOP and RigL-ITOP) and the dense over-parameterization with MLPs on CIFAR-10. It is clear to see that models with the In-Time Over-Parameterization property generalize much better than the corresponding dense models. The generalization error gradually increases as the model gets denser. Together with the results in Figure 7, we can see that the reductions in sparsity lead to better classification performance but worse generalization.

## 7. Conclusion and Future Work

In this paper, we propose In-Time Over-Parameterization, a variant of dense over-parameterization in the space-time manifold, to be an alternative way to train deep neural networks without the prohibitive dense over-parameterized dependency. We demonstrate the ability of In-Time Over-Parameterization (1) to improve the expressibility of sparse training, (2) to accelerate both training and inference, (3) to understand the underlying mechanism of DST, (4) to prevent overfitting and improve generalization. In addition, we empirically found that, with a sufficient and reliable parameter exploration, randomly-initialized sparse models consistently achieve better performance over those specially-initialized static sparse models. Our paper suggests that it is more effective and efficient to allocate the limited resources to explore more the sparse connectivity space, rather than allocating all resources to find a good sparse initialization.

Our paper discovers the importance of parameter exploration for sparse training. Even though we adjust the hyperparameters of RigL and reach the state of art sparse training performance, the usage of small batch size slows down the training speed of modern architectures. It is interesting to pursue a high In-Time Over-Parameterization rate with large batch size under a typical training time. Moreover, we believe that ITOP has potentials to help people to interpret the networks' decisions (Wong et al., 2021), to improve the robustness out of distribution and uncertainty performance (Zhang et al., 2021), to detect non-spurious correlation (Sagawa et al., 2020), etc.

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