Faculty of Information Technology, Monash University

FIT2004: Algorithms and Data Structures

Week 4:
Dynamic Programming

These slides include materials prepared by M. A. Cheema, Arun Koagurthu and Lloyd Allison.

Recommended Reading

- Unit Notes (Chapter 5)
- Weiss “Data Structures and Algorithm Analysis"
  (10.3, Pages 462-466.)

Things to remember/note

- If you don’t understand some lecture content, come to consultations! Times are available on Moodle
- Assignment 1 is due at the end of week 4
- Assignment 2 will be released at the end of week 4

An interesting problem...

Problem: We are programming a vending machine and need to determine the minimum number of coins and notes to use to pay a certain amount of change.

More precisely:

We have a number of denominations D= {d1, d2, …, dN}.

Given a value V, find the minimum number of d_j that sum up to V

Example: Change $123

A naïve solution…

You have just applied an algorithm in your head to solve this.

What type of algorithm was this?

Quiz time!
https://flux.qa/YTJMAZ

The naive algorithm

```
Coin_change_naive (C)
    let res := []  (* empty list *)
    repeat until (c=0)
        let tmp=0
        foreach d in D=[d1, …, dn]
            if d<=C and d>tmp then let tmp := d
        let C := C-tmp
        append tmp to C
    return res
```

Aside: as an exercise you could try to prove this with using invariants and pre and post conditions

Let’s code this

```
D=[1,2,5,10]

def coinsT(C):
  res=[]
  while C>0:
    tmp=0
    for d in D:
      if (d<=C and d>tmp):
        tmp = d
    C = C-tmp
    res.append(tmp)
  return res
```

What is wrong with the naïve solution?

Are you happy with the naïve solution?

What would you criticise (if anything)

Quiz time!
https://flux.qa/YTJMAZ

Think recursively!

$$\text{change}(c) = \begin{cases} 0 & \text{if} \quad c = 0 \\ min_i(1 + \text{change}(c - d_i)) & \text{otherwise} \end{cases}$$

Let’s code this

```
D=[1,2,5,10]

def coinsR(C):
    if C==0:
        return 0
    else:
        min = math.inf
        for d in D:
            if  d <= C:
                tmp = coinsR(C-d)
                if tmp+1 < min:
                    min  = tmp+1
        return min
```

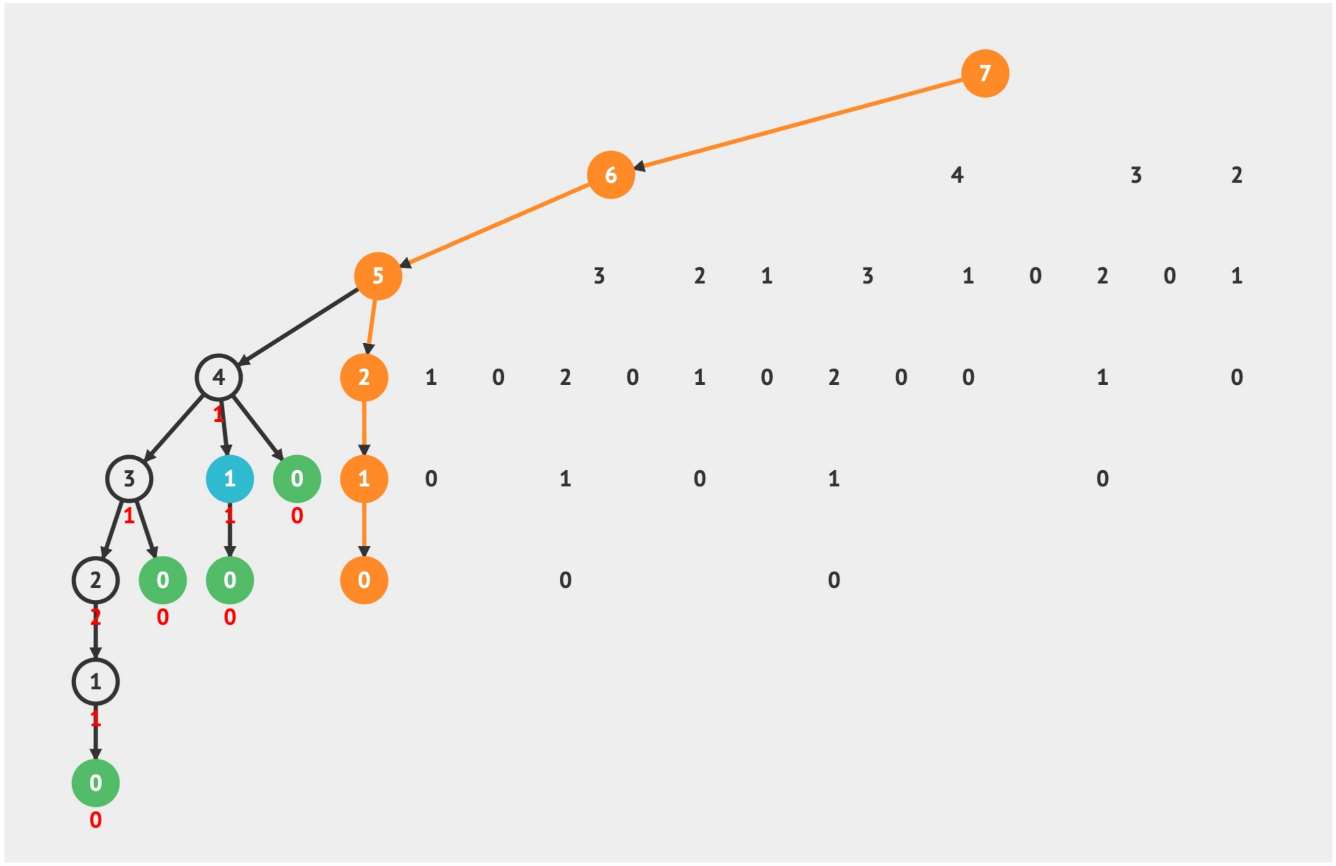
Anything wrong with the recursive solution?

Are you happy with the recursive solution?

What would you criticise (if anything)

Quiz time!
https://flux.qa/YTJMAZ

Calltree



see https://visualgo.net/en/recursion

Let’s code this

```
D=[1,2,5,10]

def coinsR(C):
    if C==0:
        return 0
    else:
        min = math.inf
        for d in D:
            if  d <= C:
                tmp = coinsR(C-d)
                if tmp+1 < min:
                    min  = tmp+1
        return min
```

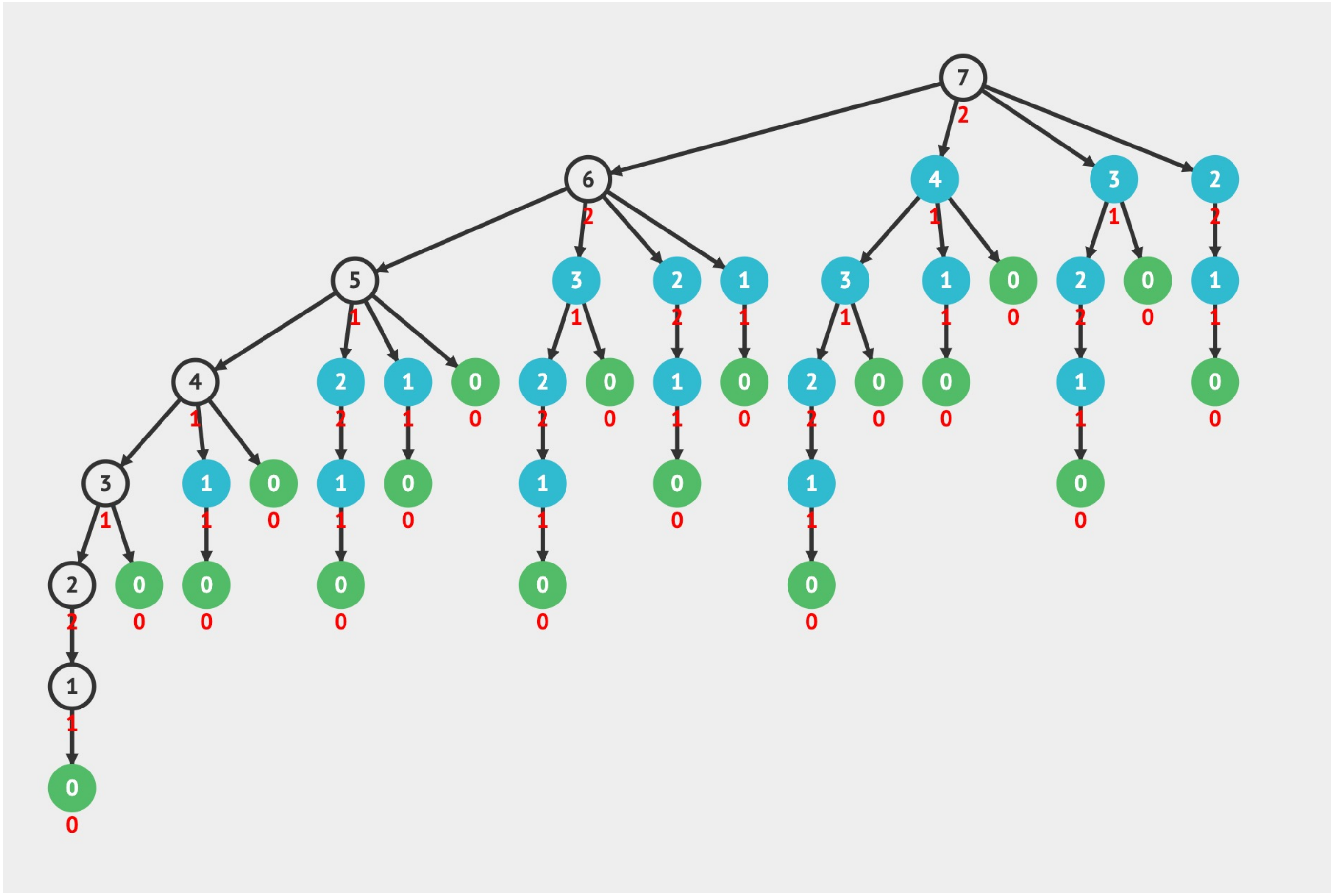
Memoization

```
Let M be an Array to store all solutions one computed
Initialise M[1..C] = None

def coinsMemo(C):

    if M[C] is None
       if C==0:
           return 0
       else:
           min = math.inf
           for d in D:
               if  d <= C:
                   tmp = coinsR(C-d)
                   if tmp+1 < min:
                       min  = tmp+1
           return min
   else (* if M[C] is a number already *)
      return M[C]
```

Execution Memoized



Mark the subtrees that are pruned

Dynamic Programming

- Core Idea: Computing the recursion tree bottom-up

- Required problem properties:
  - **Overlapping subproblems**: the same subproblem needs to (potentially) be solved multiple times
  - **Subproblem optimality** or **Optimal substructure**: optimal solutions to subproblems are components of the optimal solution of the larger problem, ie. the solution is not context dependent.

Dynamic Programming

- Core Idea: Computing the recursion tree bottom-up
- Steps:
  - Keep a solution array M (just as in memorization)
  - Fill M in all spots where the solution is trivial. These are the base cases of the recursion.
  - Compute the remaining fields in the order of “coming up the recursion tree”
- The tricky bits:
  - What? Finding the right memo structure
  - How? Finding the right computation order

The dynamic programming algorithm

```
Coin_change_DP(C)

 let M be an array of length C+1 with all elements set to +infinity
 let M[0]=0 (* from base case *)
  for i from 1 to C (* computation order bottom-up *)
    foreach d in D=[d1, …, dn]
      if (d<i) (* can fit this coin in *)
        if  (M[i] > M[i-d]+1) (* new solution is better then best solution so far *)
          M[i] := 1+M[i-d]
 return M[C]
```

Let’s code this

```
import math

D=[1,2,5,10]

def coinsDP(C):
    A = [math.inf]*(C+1)
    A[0] = 0
    for i in range(1, C+1):
        for d in D:
            if d <= i:
                if A[i]>1+A[i-d]:
                    A[i] = 1+A[i-d]
    return A[C]
```

Dynamic Programming Paradigm

- A powerful optimization technique in computer science
- Practice is the key to be good at dynamic programming
- It frequently allows us to reduce exponential (unacceptable) complexity to polynomial complexity (acceptable)
- Ideally, you start to think about the problem recursively
- The core is to get the recursion right, ie. to identify what the "reusable" sub problems are.

Origin of Dynamic Programming

- Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

- The name “dynamic programming” was given to the technique to market it…
- According to Bellman’s auto-biography the Secretary of Defense (funding!) at that time was hostile to mathematical research and the name was chosen because
  - "it's impossible to use *dynamic* in a pejorative sense"
  - "something not even a Congressman could object to"

DP vs D&C

- Divide a complicated problem by breaking it down into simpler subproblems in a recursive manner and solve these.
- Question: But how does this differ from `Divide and Conquer' approach?
- **Overlapping subproblems**: the same subproblem needs to be (potentially) be used multiple times
- We also need **Optimal substructure**: optimal solutions to subproblems help us find optimal solutions to larger problems.

If you have a recursion already…

```
fib(N)
    if N == 0 or N == 1
        return N
    else
        return fib(N - 1) + fib(N - 2)
```

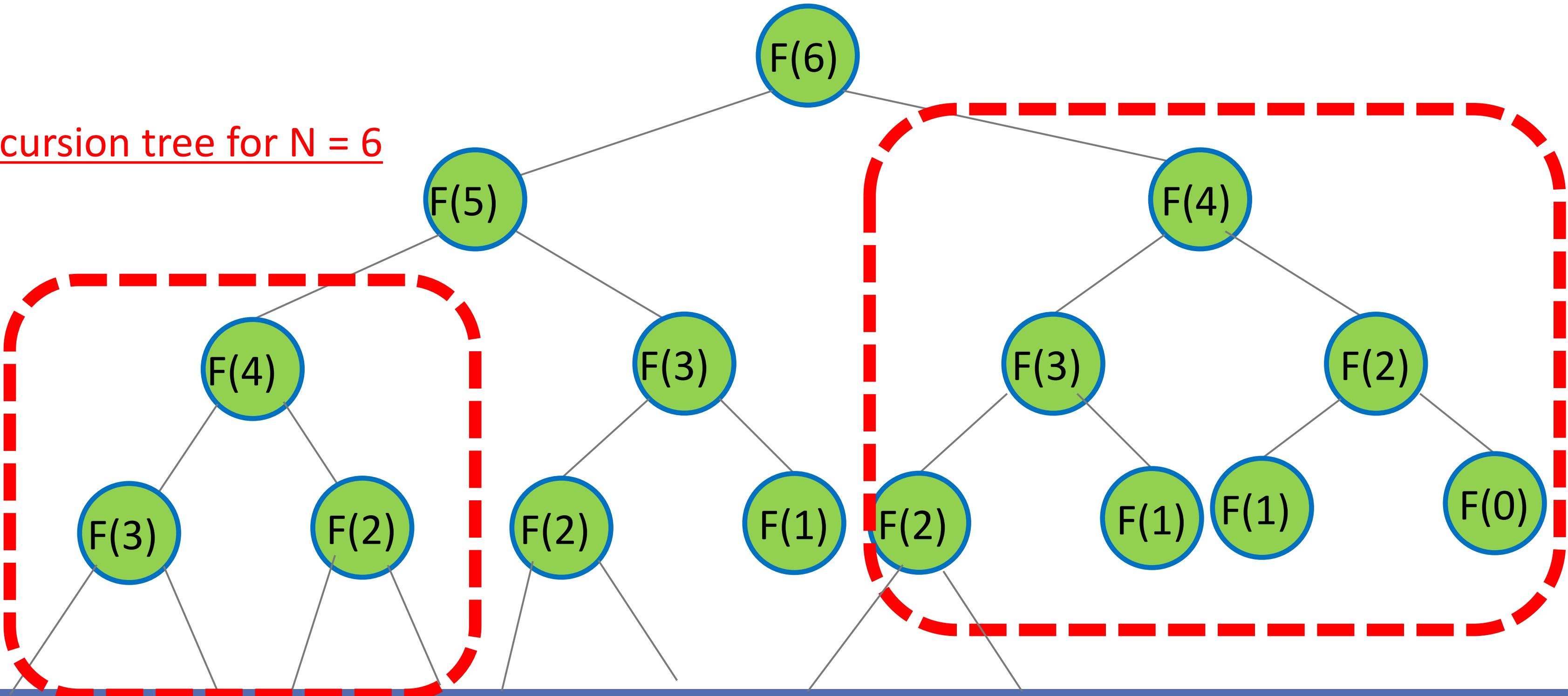
Time Complexity

$T(1) = b$ // b and c are constants

$T(N) = T(N-1) + T(N-2) + c$

$= O(2^N)$

Recursion tree for N = 6



N-th Fibonacci Number

```
fib(N)
    if N == 0 or N == 1
        return N
    else
        return fib(N - 1) + fib(N - 2)
```

Time Complexity

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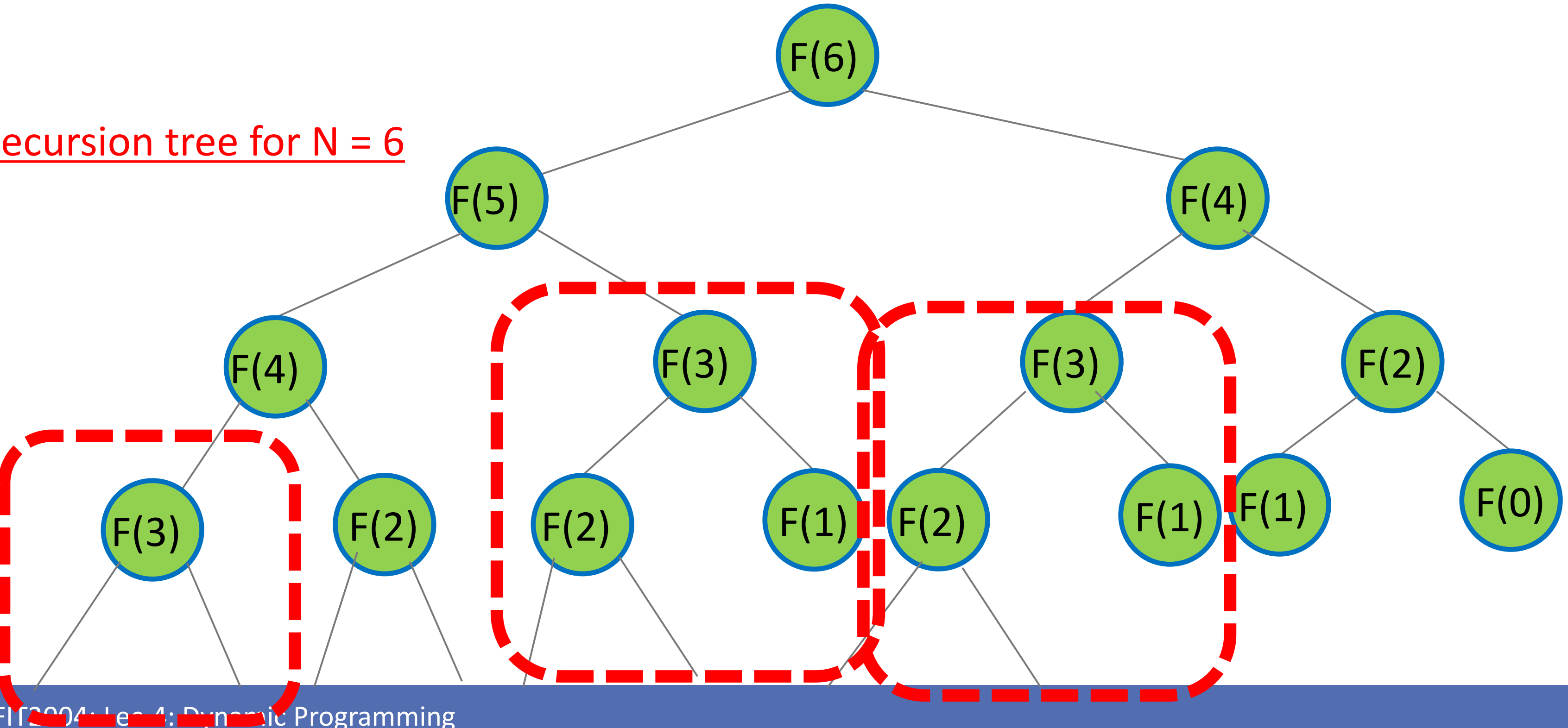

N-th Fibonacci Number

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        return fib(N - 1) + fib(N - 2)
```

Time Complexity

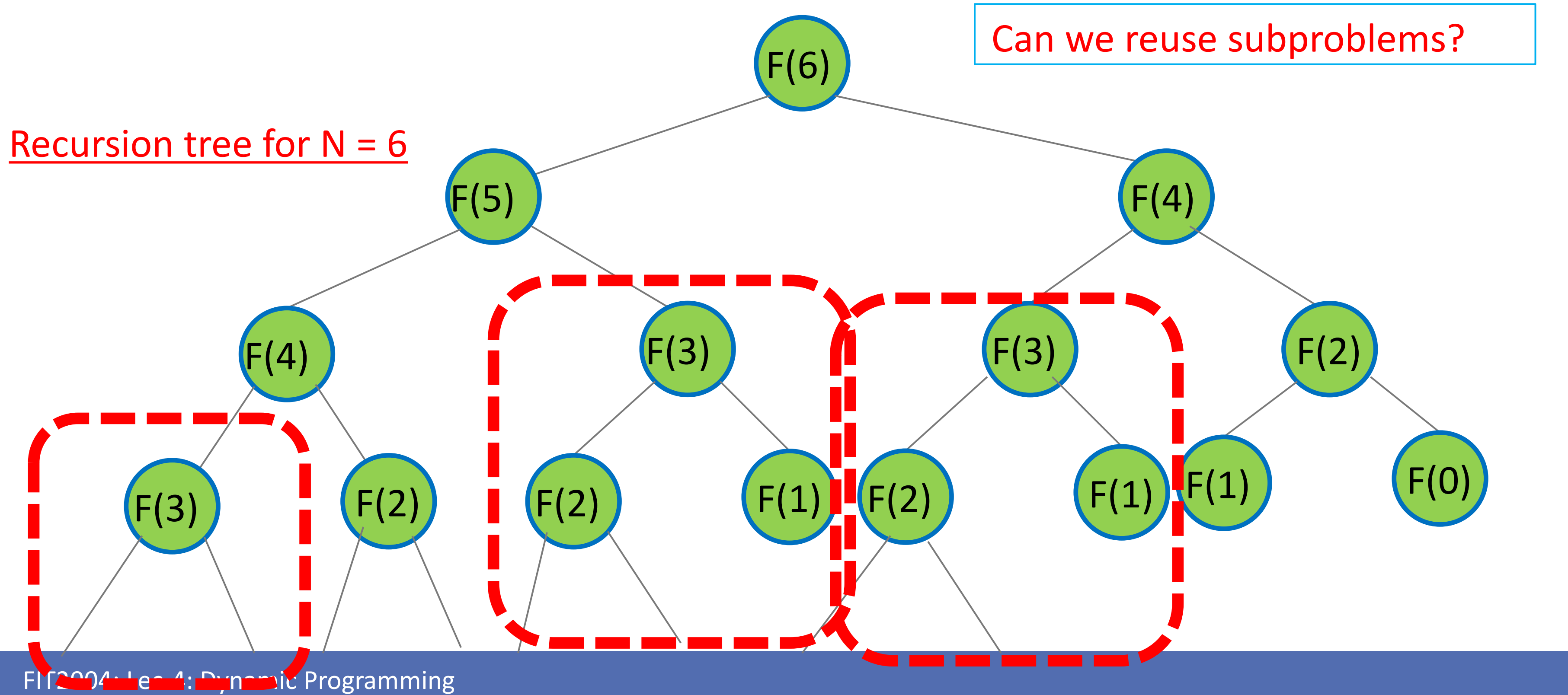
T(1) = b // b and c are constants

T(N) = T(N-1) + T(N-2) + c

= $O(2^N)$

Can we reuse subproblems?

Recursion tree for N = 6



Fibonacci with DP

```
memo[0] = 0  // 0th Fibonacci number
memo[1] = 1  // 1st Fibonacci number
for i=2 to i=N:
        memo[i] = memo[i-1] + memo[i-2]
```

Time Complexity

O(N)

0	1	1	2	3	5	8	13	21	34

Version 2 is called **Bottom-up** because it starts from the bottom – solving the smallest problem first, e.g., F(0), F(1), and so on

Fibonacci with Memoization

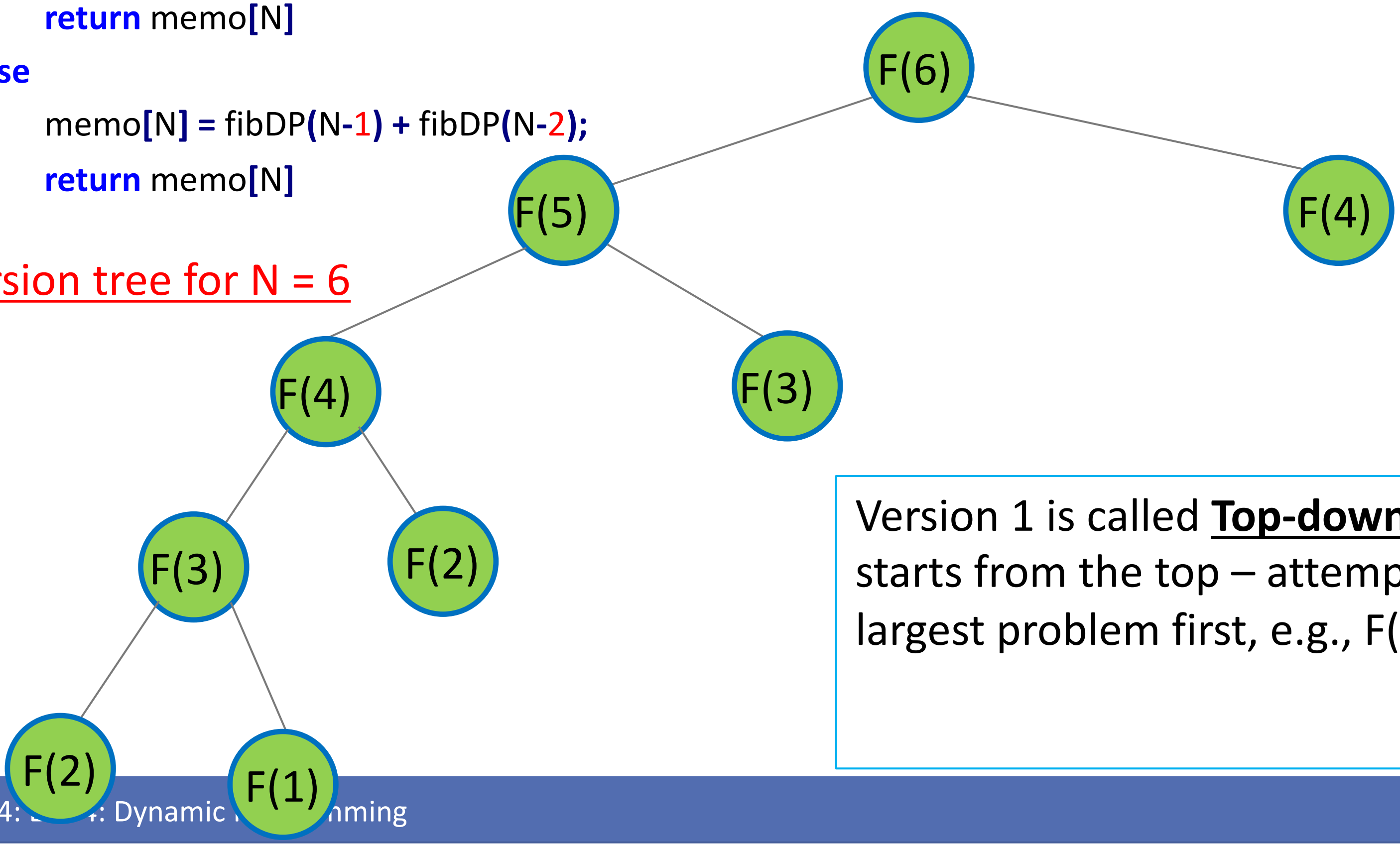
```
memo[0] = 0  // 0th Fibonacci number
memo[1] = 1  // 1st Fibonacci number
for i=2 to i=N:
    memo[i] = -1
fibDP(N)
    if memo[N] != -1
        return memo[N]
    else
        memo[N] = fibDP(N-1) + fibDP(N-2);
        return memo[N]
```

Time Complexity

calls fibDP() roughly 2*N times

So the complexity is O(N)

Recursion tree for N = 6



Version 1 is called **Top-down** because it starts from the top – attempting the largest problem first, e.g., F(6)

Outline

Unbounded Knapsack Problem

Problem: Given a capacity C and a set of items with their weights and values, you need to pick items such that their total weight is at most C and their total value is maximized. What is the maximum value you can take? In unbounded knapsack, you can pick an item as many times as you want.

Example: What is the maximum value for the example given below given capacity is 12 kg?

Answer: $780 (take two Bs and two Ds)
Greedy solution does not always work.

Item	**A**	**B**	**C**	**D**
Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

DP Solution for Unbounded Knapsack

Problem: Given a capacity C and a set of items with their weights and values, you need to pick items such that their total weight is at most C and their total value is maximized. What is the maximum value you can take? In unbounded knapsack, you can pick an item as many times as you want.

- Overlapping subproblems: Memo[i] = Most value with capacity at most i
- If we know optimal solutions to all subproblems, how can we build an optimal solution to a larger problem?

- Similar logic to coin change: We need to choose an item
- For each possible item choice, find out how much value we could get (using subproblems) and then take the best one

Memo structure

Item	1	2	3	4
Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

What would you use as the index on the memo array?

(i.e. how can you reduce the problem size in a useful way)

Quiz time!
https://flux.qa/YTJMAZ

DP Solution for Unbounded Knapsack

- Similar logic to coin change: We need to choose an item
- For each possible item choice, obtain the solution of the subproblem left (memo or recursively) and then take the best one
- Example: Capacity 12. If we take item 1, then we have 3kg left
- The best we can do with 3kg is memo[3] = $120
- So one option for memo[12] would be value[1] + memo[12-weight[1]]

Item	**1**	**2**	**3**	**4**
Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

Memo

1	2	3	4	5	6	7	8	9	10	11	12
40	80	120	160	350	390	430	470	550	700	740	

DP Solution for Unbounded Knapsack

Memo[12] could be:

- value[1] + memo[12-weight[1]] = 550 + 120 = 670
- Value[2] + memo[12-weight[2]] = 350 + 430 = 780
- Value[3] + memo[12-weight[3]] = 390 + 180 = 570
- Value[4] + memo[12-weight[4]] = 740 + 40   = 780
- Choose the best!

Item	**1**	**2**	**3**	**4**
Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

Memo	40	80	120	160	350	390	430	470	550	700	740	
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DP Solution for Unbounded Knapsack

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Memo	40	80	120	160	350	390	430	470	550	700	740	780
	1	2	3	4	5	6	7	8	9	10	11	12

DP Solution for Unbounded Knapsack

Lets write our recurrence

- What is our base case?
- With no capacity, we cannot take any items
- Also note, as before, that if an item is heavier than the capacity we have left, we cannot take it
- Otherwise, we want the maximum over all values (1 <= i <= n, v_i) of items that we could take (w_i <= c)
- But also taking into account the optimal value we could fit into the rest of our knapsack, one we took that item (MaxValue[c-w_i])

Quiz time!
https://flux.qa/YTJMAZ

DP Solution for Unbounded Knapsack

Lets write our recurrence

- What is our base case?
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- But also taking into account the optimal value we could fit into the rest of our knapsack, one we took that item (MaxValue[c-w_i])

$$\text{MaxValue}[c] = \begin{cases} 0 & \text{if } c < w_i \text{ for all } i, \\ \max\limits_{\substack{1 \le i \le n \\ w_i \le c}} (v_i + \text{MaxValue}[c - w_i]) & \text{otherwise.} \end{cases}$$

DP Solution for Unbounded Knapsack

Overlapping subproblems: Memo[i] = Most value with capacity at most i

Optimal substructure:

$$\text{MaxValue}[c] = \begin{cases} 0 & \text{if } c < w_i \text{ for all } i, \\ \max\limits_{\substack{1 \le i \le n \\ w_i \le c}} (v_i + \text{MaxValue}[c - w_i]) & \text{otherwise.} \end{cases}$$

Top-down Recursive Solution

```
Initialize Memo[ ] to contain None for all indices
// -1 indicates solution for this index has not yet been computed
Memo[0] = 0
function knapsack(Capacity)
    maxValue = 0
    for i=1 to N
      if Weight[ i ] <= Capacity
        thisValue = Value[i]  + knapsack(Capacity - Weight[ i ] )
        if thisValue > maxValue
          maxValue = thisValue
     return maxValue
```

Top-down Memo Solution

```
Initialize Memo[ ] to contain None for all indices
// -1 indicates solution for this index has not yet been computed
Memo[0] = 0
function knapsack(Capacity)
    if Memo[ Capacity ] is not None:
        return Memo[Capacity]
    else:
        maxValue = 0
        for i=1 to N
            if Weight[ i ] <= Capacity
                thisValue = Value[i]  + knapsack(Capacity - Weight[ i ] )
                if thisValue > maxValue
                    maxValue = thisValue
        Memo[Capacity] = maxValue
        return Memo[Capacity]
```

Bottom up solution:

Values[i] + Memo[Capacity – Weights[i]]

Bottom-up Solution

```
// Construct Memo[ ] starting from 1 until C in a way similar to previous slide .
Initialize Memo[ ] to contain 0 for all indices
for c = 1 to C
    maxValue = 0
    for i=1 to N
        if Weight[ i ] <= c
            thisValue = Value[i]  + Memo[c - Weight[ i ] ]
            if thisValue > maxValue
                maxValue = thisValue
    Memo[c] = maxValue
```

Time Complexity:

O(NC)

Space Complexity:

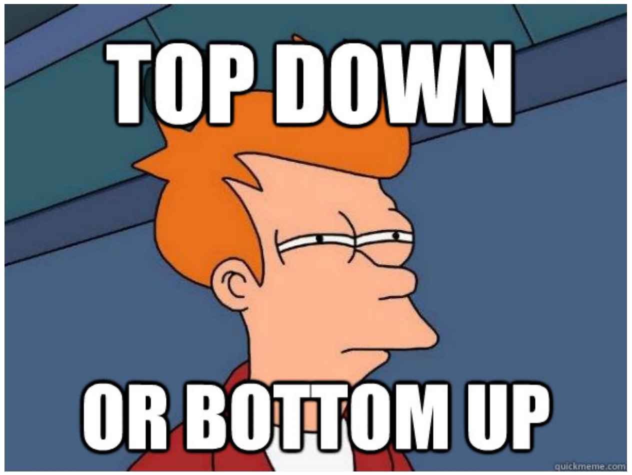
O(C + N)

E.g., Fill Memo[13]

Item	1	2	3	4
Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

Memo	40	80	120	160	350	390	430	470	550	700	740	780	
	1	2	3	4	5	6	7	8	9	10	11	12	13

Top Down vs Bottom Up



- Top-down **may** save some computations (E.g., some subproblems may not needed to be solved)
- Usually easier to think about top-down (recursively) but implement bottom-up
- In some cases, the solution cannot be written bottom-up without some silly contortions

Outline

1. Introduction to Dynamic Programming
2. Coins Change
3. Unbounded Knapsack
4. 0/1 Knapsack
5. Edit Distance
6. Constructing Optimal Solution

0/1 Knapsack Problem

Same as unbounded knapsack except that each item can only be picked at most once.

Example: What is the maximum value for the example given below given capacity is 11 kg?

Answer: $590 (B and D)

Greedy solution may not always work.

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

0/1 Knapsack Problem

Same as unbounded knapsack except that each item can only be picked at most once.

*** Why can't we just solve this like unbounded knapsack?**

Quiz time!
https://flux.qa/YTJMAZ

0/1 Knapsack Problem

Same as unbounded knapsack except that each item can only be picked at most once.

Different from unbounded: If we pick an item X, giving us a remaining capacity R, we have to somehow make sure that X is not part of the optimal solution to our new subproblem of size R

Idea: Lets have two axes on which we think about subproblems.

- Capacity
- Which items are part of the subproblem

0/1 Knapsack Problem

Problem: What is the solution for 0/1 knapsack for items {A,B,C,D} where capacity = 11.

Assume that we have computed solutions for every capacity<=11 considering only items {A,B,C} (see table below).

What is the solution for capacity=11 and set {A,B,C,D} ?

- Case 1: the knapsack must **NOT** contain D
    - Solution for 0/1 knapsack with set {A,B,C} and capacity 11.

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

	1	2	3	4	5	6	7	8	9	10	11
{A,B,C}	40	40	40	40	350	390	390	390	390	390	580

0/1 Knapsack Problem

Problem: What is the solution for 0/1 knapsack for items {A,B,C,D} where capacity = 11.

Assume that we have computed solutions for every capacity<=11 considering the items {A,B,C} (see table below).

What is the solution for capacity=11 and set {A,B,C,D} ?

- Case 1: the knapsack must **NOT** contain D to achieve optimality
  - Solution for 0/1 knapsack with set {A,B,C} and capacity 11 = 580
- Case 2: the knapsack **must** contain D
  - The value of item D + solution for 0/1 knapsack with set {A,B,C} and capacity 11-9=2
  - This gives a value of 550+40
- **Solution = max(Case1, Case2)**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

	1	2	3	4	5	6	7	8	9	10	11
{A,B,C}	40	40	40	40	350	390	390	390	390	390	580

0/1 Knapsack Problem

Does this mean we need a <u>set</u> as one dimension of the memo structure to describe which items are already used?

No!

- We only need to keep a memory of which items have been *considered*
- we are allowed to do this in an arbitrary order

Thus, we can fix an arbitrary order and only keep track of "items up to number *i* have been considered". This is just a single number.

0/1 Knapsack Solution

Recursive Solution

"No space" base case

"No items" base case

$$knap10(i,c) = \begin{cases} 0 & \text{if} \quad c = 0 \\ 0 & \text{if} \quad i = 0 \\ knap10(i-1,c) & \text{if} \quad w_i > c \\ max(knap10(i-1,c), v_i + knap10(i-1,c-w_i)) & \end{cases}$$

Try to use item

Can't use item

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

Memo[i][c] contains the solution of knapsack for **Set[1 … i]** and capacity **c**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590		

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Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
3	C	40	40	40	40	350	390	390	390	390	390	580	620
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max(580

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
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Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(580, 550 + 40)

			1	2	3	4	5	6	7	8	9	10	11	12
	0	**Φ**	0	0	0	0	0	0	0	0	0	0	0	0
	1	**A**	0	0	0	0	0	230	230	230	230	230	230	230
	2	**B**	40	40	40	40	40	230	270	270	270	270	270	270
	3	**C**	40	40	40	40	350	390	390	390	390	390	580	620
i	**4**	**D**	40	40	40	40	350	390	390	390	550	590		

0/1 Knapsack Problem

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Memo[i][c] contains the solution of knapsack for **Set[1 … i]** and capacity **c**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(580, 550 + 40)

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

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max(

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	

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Value	$230	$40	$350	$550

max(620

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
3	C	40	40	40	40	350	390	390	390	390	390	580	620
i 4	D	40	40	40	40	350	390	390	390	550	590	590	

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Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(620, 550 + 40)

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
3	C	40	40	40	40	350	390	390	390	390	390	580	620
4	D	40	40	40	40	350	390	390	390	550	590	590	

i

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

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max(620, 550 + 40)

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	620

0/1 Knapsack Problem

Complexity:

- We need to fill the grid
- Filling each cell is O(1) since it is the max of 2 numbers, each of which can be computer in a constant number of lookups
- Therefore, the time and space complexity are both O(NC) where N is the number of items and C is the capacity of the knapsack

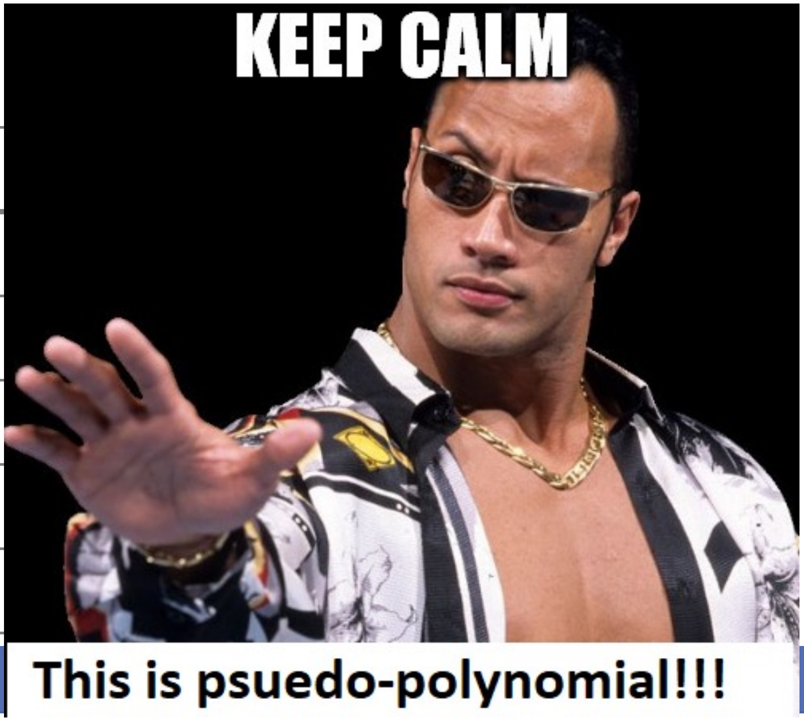
			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	620

0/1 Knapsack Problem

Complexity:

, each of which can be computer in a

NC) where N is the number of items and C is



1	A	0	0	0	0	0	230
2	B	40	40	40	40	40	230
3	C	40	40	40	40	350	390
4	D	40	40	40	40	350	390

i

0/1 Knapsack Problem

Complexity:

- We do not just need a larger table, each cell also requires more memory!
- Think about how many bits it takes to specify input
- N is the number of items. The N items require O(N) bits to describe
- C is the capacity. C can be described with log(C) bits
- Instead of C, lets talk about B, the number of bits to specify C
- $\log_2 C = B \Rightarrow C = 2^B$
- Now we can say our algorithm runs in O(CN) = O($2^B N$), which is not polynomial in the size of the input (as expected for an NP-complete problem)

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	620

Reducing Space Complexity

- Works if the neighbourhood for lockup is restricted
- While generating each row, we may only need to look at values from the previous row (e.g. in 0-1 knapsack)

- All values from the earlier rows may be discarded
- Reduces space complexity to O(C) (or O(2^B) as we saw)

Note: Space saving not possible for top-down dynamic programming (since we don't know the order we solve subproblems)

Note: Space saving **not possible when we want to reconstruct the solution**. We usually want to do this!!!

		1	2	3	4	5	6	7	8	9	10	11	12
3	C	40	40	40	40	350	390	390	390	390	390	580	620
4	D	40	40	40	40	350	390	390	390	550	590		

Outline

1. Introduction to Dynamic Programming
2. Coins Change
3. Unbounded Knapsack
4. 0/1 Knapsack
5. Edit Distance
6. Constructing Optimal Solution

Edit Distance

- The words computer and commuter are very similar, and a change of just one letter, p → m, will change the first word into the second.
- The word sport can be changed into sort by the deletion of p, or equivalently, sort can be changed into sport by the insertion of p'.
- Notion of editing provides a simple and handy formalisation to compare two strings.
- Problem: how ”far” is a word (from a dictionary) from misspelt word?
- The goal is to convert the first string (i.e., sequence) into the second through a minimal series of edit operations
- The permitted edit operations are:
    1. insertion of a symbol into a sequence.
    2. deletion of a symbol from a sequence.
    3. substitution or replacement of one symbol with another in a sequence.

Edit Distance

Edit distance between two sequences

- Edit distance is the minimum number of edit operations required to convert one sequence into another

For example:

- Edit distance between computer and commuter is 1
- Edit distance between sport and sort is 1.
- Edit distance between shine and sings is ?
- Edit distance between dnasgivethis and dentsgnawstrims is ?

Some Applications of Edit Distance

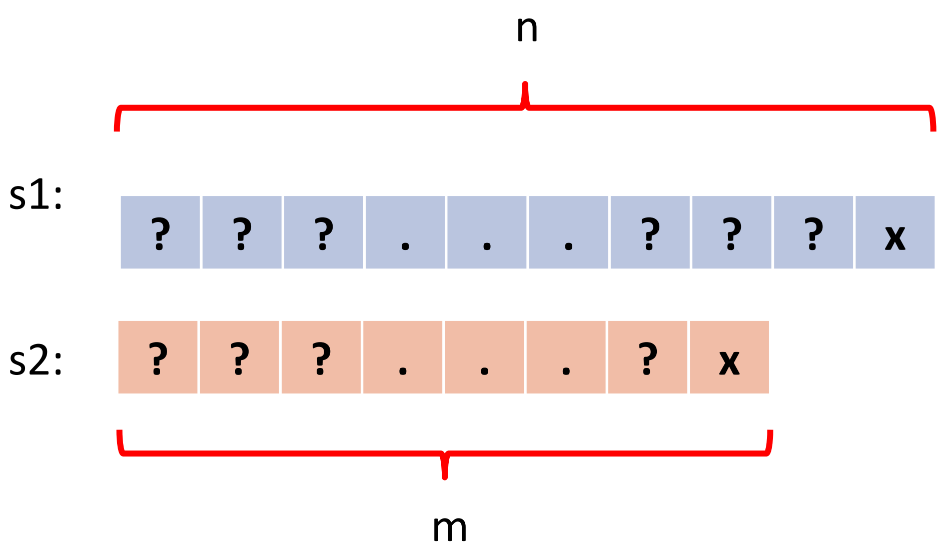
- Natural Language Processing
  - Auto-correction
  - Query suggestions
- BioInformatics
  - DNA/Protein sequence alignment

Computing Edit Distance

We want to convert s1 to s2 containing n and m letters, respectively

To gain an intuition for this problem, lets look at some situations we might run into

This is a good technique in general, try playing around with the problem and see what happens



How much does it cost to turn s1 into s2 if the last characters are the same?

We can leave the last character, and just convert the front part of one string into the front part of the other

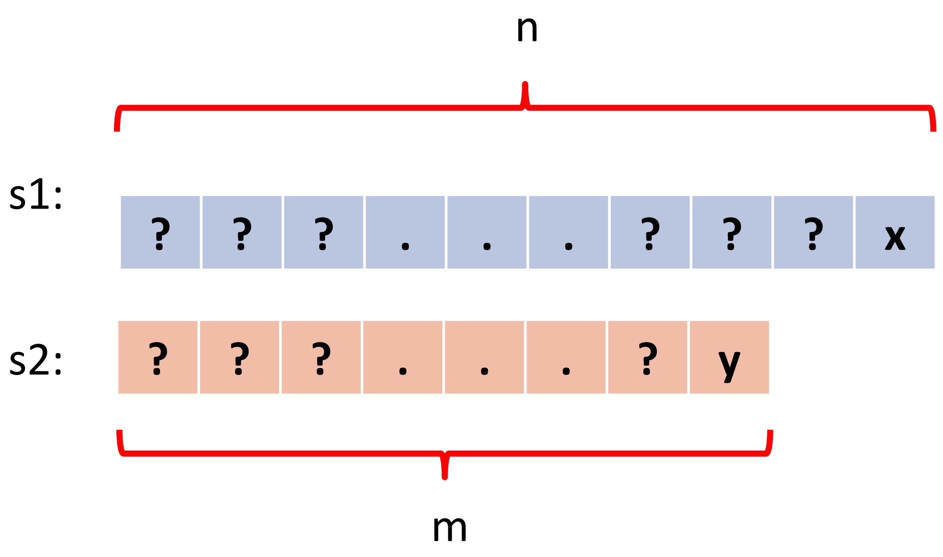
edit(s1[1..n], s2[1..m])
=edit(s1[1..n-1], s2[1..m-1])

Computing Edit Distance

We want to convert s1 to s2 containing n and m letters, respectively

To gain an intuition for this problem, lets look at some situations we might run into

This is a good technique in general, try playing around with the problem and see what happens



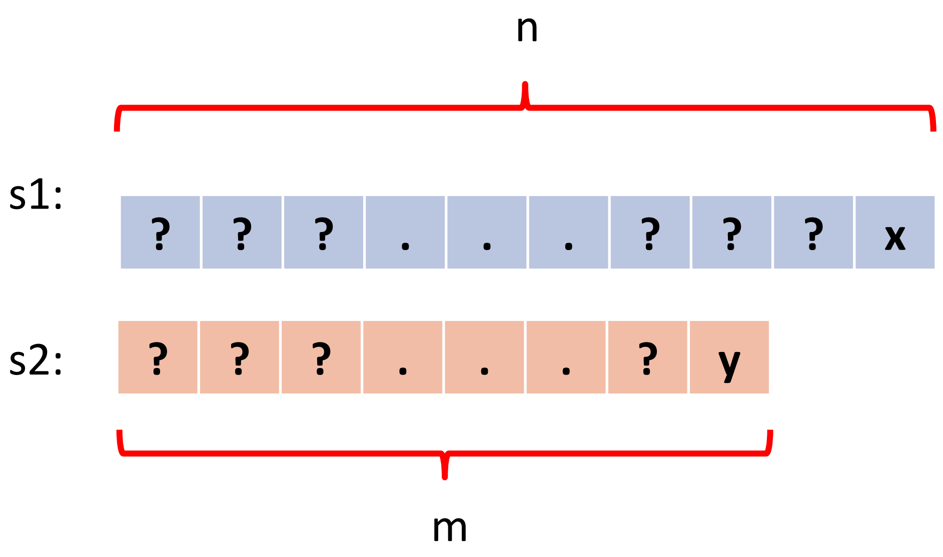
How much does it cost to turn s1 into s2 if the last characters are different?

We have some options

Computing Edit Distance

We only need to look at S1. This is the sequence we are trying to change.

What are we allowed to do with the last character in S1?



1. Replace x with y: easy, we can just make x,y the same and then drop them
2. Delete x: compare the rest of the S1 and hope for the best
3. Add y at the end of S1 and continue comparing the original S1 to S2 with y dropped

Computing Edit Distance

Base cases?

Quiz time!
https://flux.qa/YTJMAZ

Computing Edit Distance

Base cases?

- When one string is empty, the cost is just the length of the other string
- we would have to insert each character in the other string, starting from nothing
- So edit(s1[], s2[1..j]) = j
- edit(s1[1..i], s2[]) = i

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ & \\ & \end{cases}$$

Computing Edit Distance

If the last characters are the same:

- edit(s1[1..n-1], s2[1..m-1])

If the last characters are different, three options:

- cost(delete) + edit(s1[1..n-1], s2[1..m])
- edit(s1[1..n], s2[1..m-1]) + cost(insert)
- edit(s1[1..n-1], s2[1..m-1]) + cost(replace)

We want the minimum cost, so our optimal substructure will be (if all costs are 1)

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

Computing Edit Distance

Overlapping subproblems: Dist[I,j] = cost of operations to turn S[1...i] into S[1…j]

Optimal substructure:

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

Computing Edit Distance

- Remember! We can assume that we have solved ALL subproblems already. In other words, we know
- Edit(s1[1..i], s2[1..j]) for all i<=n, j<=m BUT NOT when i = n AND j = m (since this is the exact problem we are trying to solve)
- Alternatively, we could think about it visually
- In this table, cell[i][j] is the cost of turning s1[1..i] into s2[1..j]

	1	2	3	.	.	.	m
1	🟦	🟦	🟦	🟦	🟦	🟦	🟦
2	🟦	🟦	🟦	🟦	🟦	🟦	🟦
3	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
n	🟦	🟦	🟦	🟦	🟦	🟦	🟧

Known:	🟦
Unknown:	🟧

Computing Edit Distance

We know:

Edit(s1[1..i], s2[1..j]) for all i<=n, j<=m
BUT NOT i = n AND j = m

Equivalently, we know all the blue cells

	1	2	3	.	.	.	m
1	🟦	🟦	🟦	🟦	🟦	🟦	🟦
2	🟦	🟦	🟦	🟦	🟦	🟦	🟦
3	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
n	🟦	🟦	🟦	🟦	🟦	🟦	🟧

From the bottom right corner, there are three things we can do:

Match of last characters:

- Go left and up

Mismatch of last characters:

- Go left and up (replace)
- Go up (S1 shorter, delete)
- Go left (S2 shorter, insert)

Computing Edit Distance

We know:

Edit(s1[1..i], s2[1..j])for all i<=n, j<=m

BUT NOT i = n AND j = m

Equivalently, we know all the blue cells

	1	2	3	.	.	.	m
1	🟦	🟦	🟦	🟦	🟦	🟦	🟦
2	🟦	🟦	🟦	🟦	🟦	🟦	🟦
3	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟥
n	🟦	🟦	🟦	🟦	🟦	🟦	🟧

From the bottom right corner, there are three things we can do:

- Go up
- Go left
- Go left and up

Going up:

Subproblem: edit(s1[1..n-1], s2[1..m])

- First delete s1[n]
- Then turn s1[1..n-1] into s2[1..m]

Total cost:

cost(delete) + edit(s1[1..n-1], s2[1..m])

Computing Edit Distance

We know:

Edit(s1[1..i], s2[1..j]) for all i<=n, j<=m
BUT NOT i = n AND j = m

Equivalently, we know all the blue cells

	1	2	3	.	.	.	m
1	🟦	🟦	🟦	🟦	🟦	🟦	🟦
2	🟦	🟦	🟦	🟦	🟦	🟦	🟦
3	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
n	🟦	🟦	🟦	🟦	🟦	🟥	🟧

From the bottom right corner, there are three things we can do:

- Go up
- Go left
- Go left and up

Going left:
Subproblem: edit(s1[1..n], s2[1..m-1])

- First turn s1[1..n] into s2[1..m-1]
- First insert s2[m] at the end of s2

Total cost:
edit(s1[1..n], s2[1..m-1]) + cost(insert)

Computing Edit Distance

We know:

Edit(s1[1..i], s2[1..j]) for all i<=n, j<=m
BUT NOT i = n AND j = m

Equivalently, we know all the blue cells

	1	2	3	.	.	.	m
1	🟦	🟦	🟦	🟦	🟦	🟦	🟦
2	🟦	🟦	🟦	🟦	🟦	🟦	🟦
3	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟦	🟦
.	🟦	🟦	🟦	🟦	🟦	🟥	🟦
n	🟦	🟦	🟦	🟦	🟦	🟦	🟧

From the bottom right corner, there are three things we can do:

- Go up
- Go left
- Go left and up

Going left:
Subproblem: edit(s1[1..n-1], s2[1..m-1])

- Replace s1[n] with s2[m]
- Turn s1[1..n-1] into s2[1..m-1]

Total cost:
edit(s1[1..n-1], s2[1..m-1]) + cost(replace)

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ						
S						
I						
N						
G						
S						

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S						
I						
N						
G						
S						

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1					
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	**S**	**H**	**I**	**N**	**E**
Φ	0	1	2	3	4	5
S	1					
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0				
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0				
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1,j]+1 \\ \text{Dist}[i,j-1]+1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1			
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j]=\begin{cases} i & \text{if } j=0,\\ j & \text{if } i=0,\\ \min\begin{cases}\text{Dist}[i-1,j-1]+1_{S_1[i]\neq S_2[j]}\\ \text{Dist}[i-1,j]+1\\ \text{Dist}[i,j-1]+1\end{cases} & \text{otherwise}\end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1			
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1,j]+1 \\ \text{Dist}[i,j-1]+1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2		
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1,j] + 1 \\ \text{Dist}[i,j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2					
N	3					
G	4					
S	5					

Example

$$\mathrm{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min\begin{cases} \mathrm{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \mathrm{Dist}[i-1,j]+1 \\ \mathrm{Dist}[i,j-1]+1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1				
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1,j] + 1 \\ \text{Dist}[i,j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1			
N	3					
G	4					
S	5					

Example

$$\mathrm{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \mathrm{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \mathrm{Dist}[i-1,j] + 1 \\ \mathrm{Dist}[i,j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1			
N	3					
G	4		Now	you	Try!	
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1,j] + 1 \\ \text{Dist}[i,j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Outline

1. Introduction to Dynamic Programming
2. Coins Change
3. Unbounded Knapsack
4. 0/1 Knapsack
5. Edit Distance
6. Constructing Optimal Solution

Constructing optimal solutions

- The algorithms we have seen determine optimal values, e.g.,
  - Minimum number of coins
  - Maximum value of knapsack
  - Edit distance
- How do we construct optimal solution that gives the optimal value, e.g.,
  - The coins to give the change
  - The items to put in knapsack
  - Converting one string to the other
- There may be multiple optimal solutions and our goal is to return just one solution!
- Two strategies can be used.
  1. Backtracking
  2. Create an additional array recording decision at each step

Backtracking

- Start in bottom right
- Are the letters the same?

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Start in bottom right
- Are the letters the same?
- No

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Start in bottom right
- Are the letters the same?
- No
- From recurrence…
- We know that our current value (3) was obtained from any of the three previous cells by adding 1

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Start in bottom right
- Are the letters the same?
- No
- From recurrence…
- We know that our current value (3) was obtained from any of the three previous cells by adding 1
- So our options are up or up-and-left
- Choose one arbitrarily

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue from our new cell (but remember the path)

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue from our new cell (but remember the path)
- Letters are different

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue from our new cell (but remember the path)
- Letters are different
- Can have come from the cell above

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Letter are the same
- From our recurrence, we know
  - IF our value is the same as the up-and-left cell, then we could have came from there
  - IF our value is one more than the up cell or the left cell, then we could have come from there
- In this case, we came from up-and-left

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Letter are the same
- From our recurrence, we know
  - IF our value is the same as the up-and-left cell, then we could have came from there
  - IF our value is one more than the up cell or the left cell, then we could have come from there
- In this case, we came from up-and-left

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

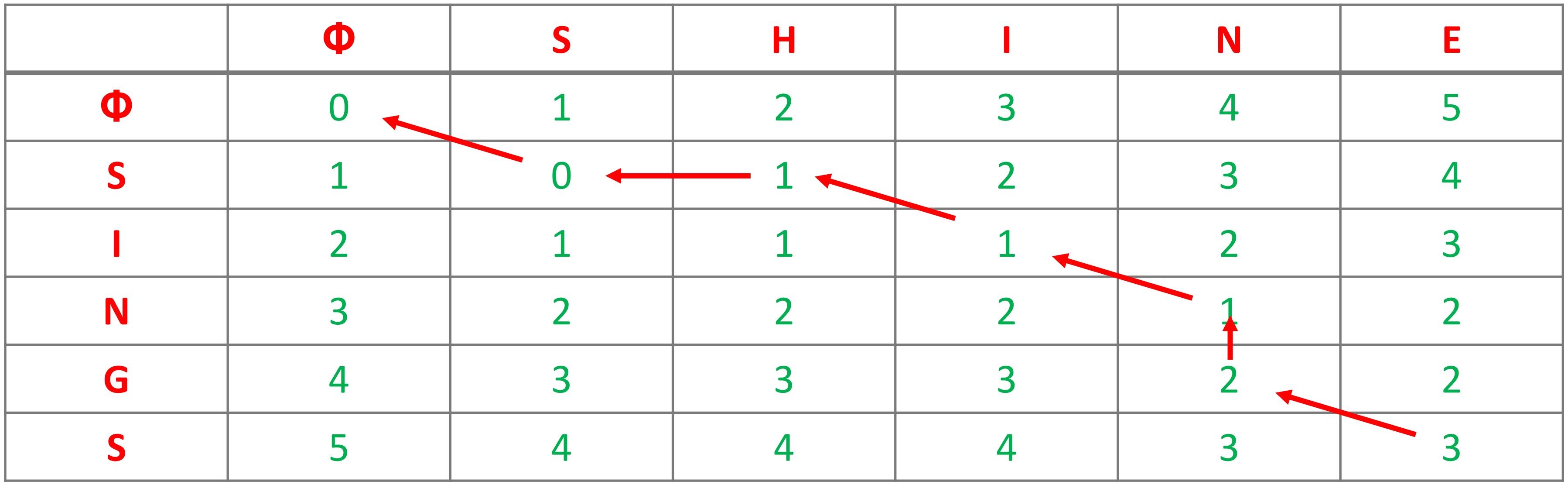
Backtracking

- Continue in this way

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	↖ 0	← 1	2	3	4
I	2	1	1	↖ 1	2	3
N	3	2	2	2	↖ 1	2
G	4	3	3	3	↑ 2	2
S	5	4	4	4	3	↖ 3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is



	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E)

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4]

	Φ	**S**	**H**	**I**	**N**	**E**
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing, nothing

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing, nothing, insert(2, H)

	Φ	**S**	**H**	**I**	**N**	**E**
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing, nothing, insert(2, H), nothing

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing, nothing, insert(2, H), nothing
- SINGS
- SINGE (replace position 5 with E)
- SINE (delete G in position 4)
- SHINE (insert H at position 2)

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

- Make a second array of the same size
- Each time you fill in a cell of the memo array, record your decision in the decision array
- Remember, going right (or coming from the left) is insert
- Going down (or coming from up) is delete
- Going down and right (or coming from up and left) is replace OR do nothing

	Φ	S	H	I	N	E
Φ	0					
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S				
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1				
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H			
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1	2			
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I		
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1	2	3		
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S					
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1					
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing				
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0				
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H			
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1			
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I		
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2		
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I				
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1				
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	Choice?	Choice?	Do nothing	Insert E
G	Delete G	Delete G	Choice?	Choice?	Delete G	replace G, E
S	Delete S	Delete S	Choice?	Choice?	Delete S	Choice?

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

Sequence of operations:
Delete S (from position 5)
replace G with E (at position 4)
insert H (at position 2)

- SINGS
- SING
- SINE
- SHINE

Backtracking Vs Decision array?

- Space usage
  - Backtracking requires less space as it does not require creating an additional array
  - However, space **complexity** is the same
- Efficiency
  - Backtracking requires to **determine** for a second time what decision was made which costs additional computation
  - However, time **complexity** is the same
- Note the space saving tricks discussed for 0/1 knapsack and edit distance can only be used when solution is not to be constructed
  - e.g., all rows are needed for backtracking, and all rows must be stored for 2D-decision array

Practice problems I

- Number of ways to give change: The setting is the same as in the original coin change problem discussed in the lecture. This time, tough, we want to compute the number of different ways in which change can given for a certain amount rather than the (number of) coins to use minimally.

Practice problems II

- Deploying charging stations: Your task is to decide how to deploy charging stations for electric cars. To make this task easier in the context of this problem, we are only looking at a single (very long) stretch of road, not at a network of roads. Let’s say, Highway 1 in Australia.
  - You are starting from a given placement of stations of which you know that they are placed more densely than needed.
  - Looking at the range of electric cars you decide that you can remove any (!) station but never two that are next two each other. The gap would become too large.
  - Each station has a given cost to build.
- Design a DP algorithm to find the set of stations that has to be removed to minimize the cost of building all required stations.

Practice problems III

- Rod Cutting: We want to solve a planning problem for a company that buys steel rods, cuts them into varying lengths, and resells them again. The question that arises is how to cut a rod of a given length n in order to achieve the maximum profit.

- The answer is not entirely straight forward, because the price for each rod length is set individually, i.e. it is not a simple "per meter" price. The company only sells rods in multiples of full meters, and the cost for cutting is negligible so that it does not need to be taken into account.
- You have to write an algorithm that determines the maximum total selling price that can be achieved for the pieces cut from a rod of length n.
- Your algorithm is given as input the total length n and the prices p_i that are charged for a single piece of length $i \in \{1 \dots n\}$.
- We are only interested in the profit we can make. Your algorithm does not need to output where to cut the rod!

Concluding Remarks

Dynamic Programming Strategy

- Work from a recursive definition
- Assume you already know the optimal solutions for all subproblems and have memoized these solutions
- Observe how you can solve the original problem using this memoization
- Iteratively solve the sub-problems and memoize

Things to do (this list is not exhaustive)

- Practice, practice, practice
  - http://www.geeksforgeeks.org/tag/dynamic-programming/
  - https://www.topcoder.com/community/data-science/data-science-tutorials/dynamic-programming-from-novice-to-advanced/
  - http://weaklearner.com/problems/search/dp
- Revise hash tables and binary search tree

Coming Up Next

- Hashing, Binary Search Tree, AVL Tree

as Søren Kierkegaard said…

Life can only be understood backwards,

but it must be lived forwards.