Faculty of Information Technology, Monash University

FIT2004: Algorithms and Data Structures

Week 6: Retrieval Data Structures for Strings

These slides include materials prepared by M. A. Cheema, Arun Koagurthu and Lloyd Allison.

Recommended readings

- Unit notes (Chapters 9&10)
- Cormen et al. “Introduction to Algorithms" (Chapter 18)
- For a more advanced treatment of Trie and suffix trees: Dan Gusfield, Algorithms on Strings, Trees and Sequences, Cambridge University Press. (Chapter 5) - Book available in the library!

Outline

1. Introduction
2. Trie
   A. Construction
   B. Query Processing
3. Suffix Trie
   A. Construction
   B. Query Processing
   C. Suffix Tree
4. Suffix Array
   A. Introduction
   B. Query Processing
   C. Reducing Construction Cost

Introduction

Suppose you have a large text containing N strings. You want to pre-process it such that searching on this text is efficient.

Sorting based approach:

- Pre-processing: Sort the strings
- Searching: Binary search to find

Let M be the average length of strings (M can be quite large, e.g., for DNA sequences). Comparison between two strings takes O(M).

Time complexity:

Pre-processing → O(MN log N) using merge sort or O(MN) using radix sort

Searching → O(M log N)

Can we do better?

Yes! ReTrieval data structures allow answering different string queries efficiently

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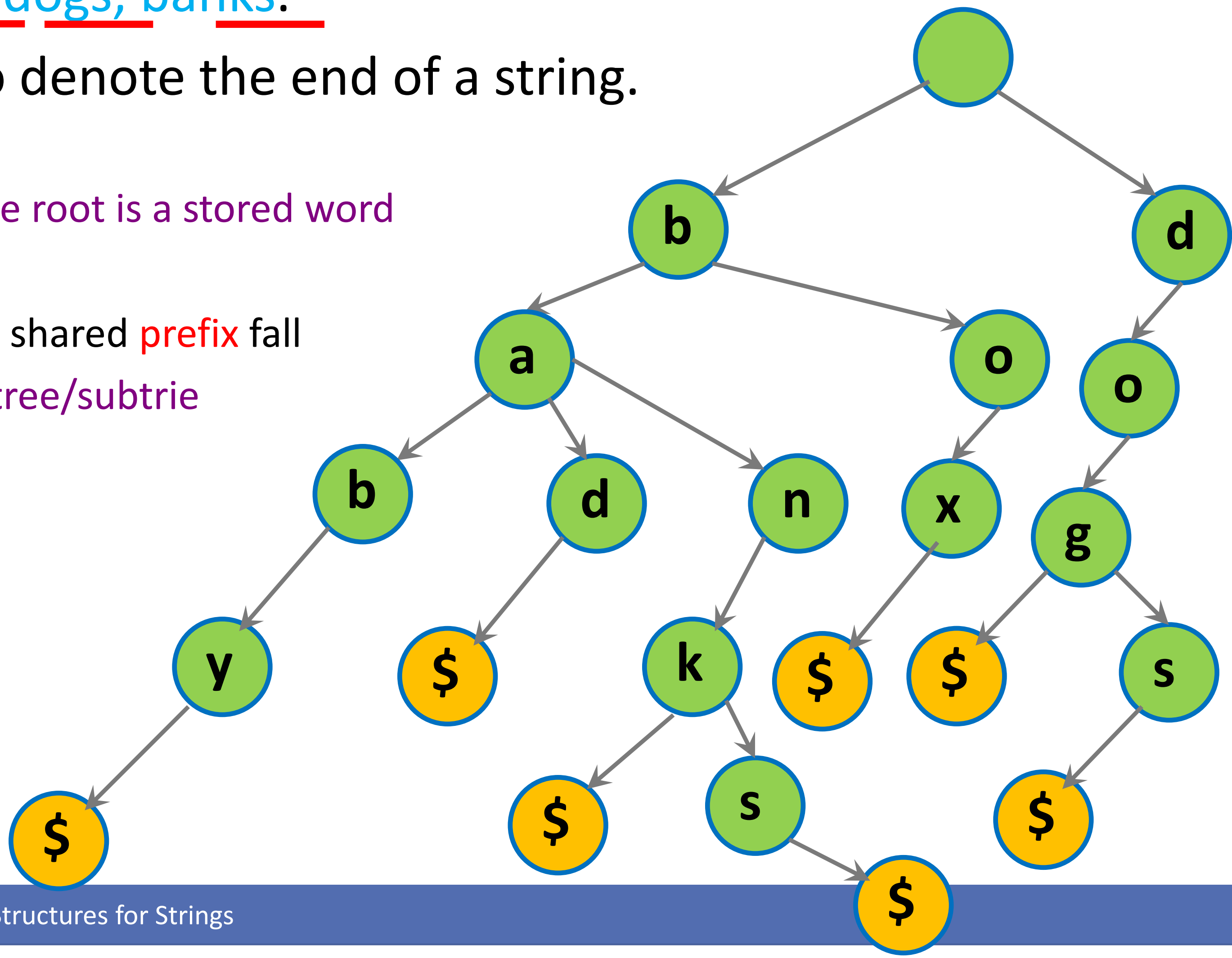
Tries: Prefix-based Structures

Let's look at an example : a Trie that stores baby, bad, bank, box, dog, dogs, banks.

We append $ to denote the end of a string.

- Each path from the root is a stored word
- all words with the shared prefix fall within the same subtree/subtrie



Trie

- ReTRIEval tree = Trie
- Often pronounced as ‘Try'.
- Trie is an N-way (or multi-way) tree, where N is the size of the alphabet
  - E.g., N=2 for binary
  - N = 26 for English letters
  - N = 4 for DNA
- In a standard Trie, all words with the shared prefix fall within the same subtree/subtrie
- In fact, it is the shortest possible tree that can be constructed such that all prefixes fall within the same subtree.

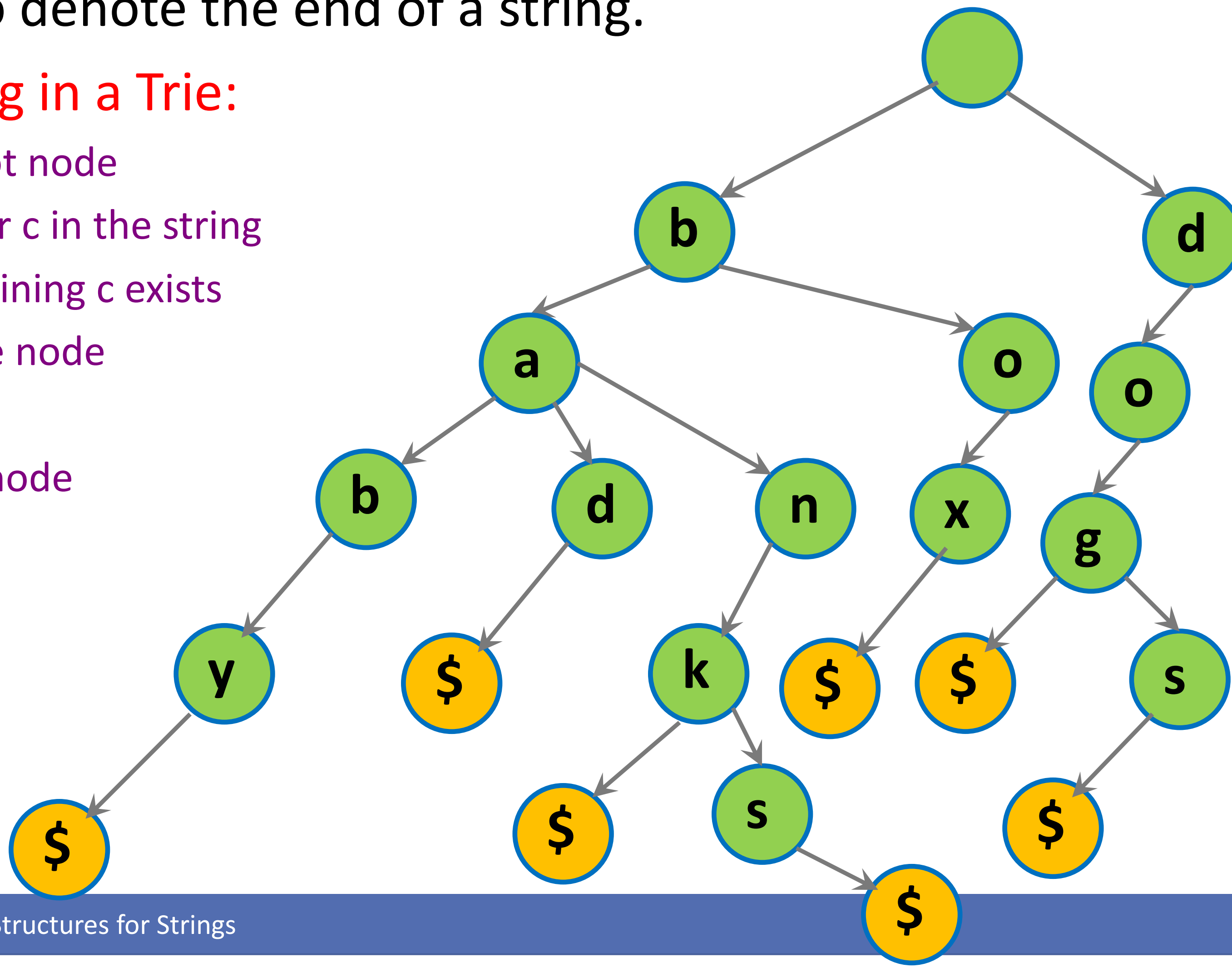
Trie Example: Insertion

baby, bad, bank, box, dog, dogs, banks.

We append $ to denote the end of a string.

Inserting a string in a Trie:

- Start from the root node
- For each character c in the string
  - If a node containing c exists
    - Move to the node
  - Else
    - Create the node
    - Move to it



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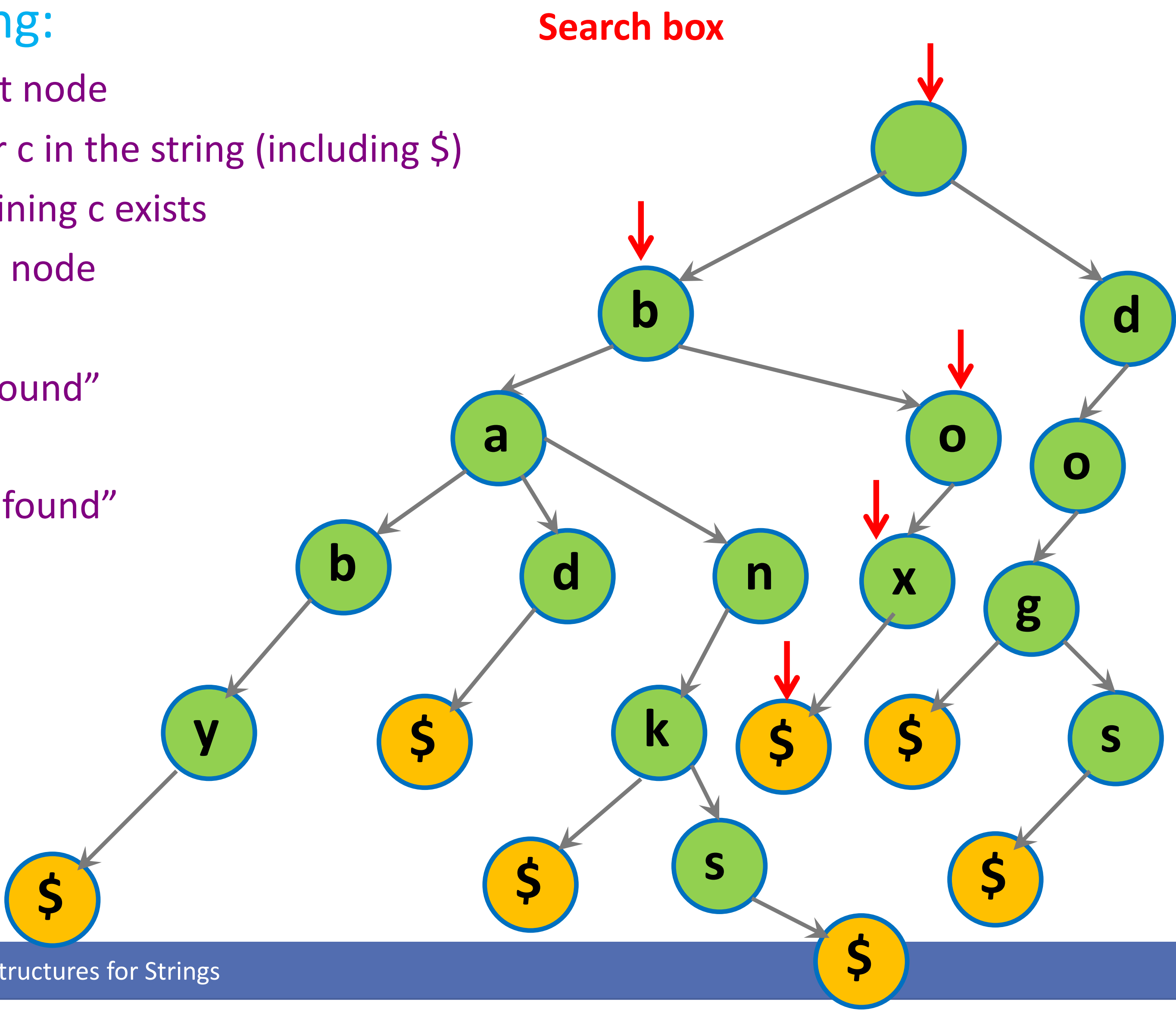
Trie Example: Search

Searching a string:

- Start from the root node
- For each character c in the string (including $)
  - If a node containing c exists
    - Move to the node
    - If c == $
      - Return “found”
  - Else
    - Return “not found”



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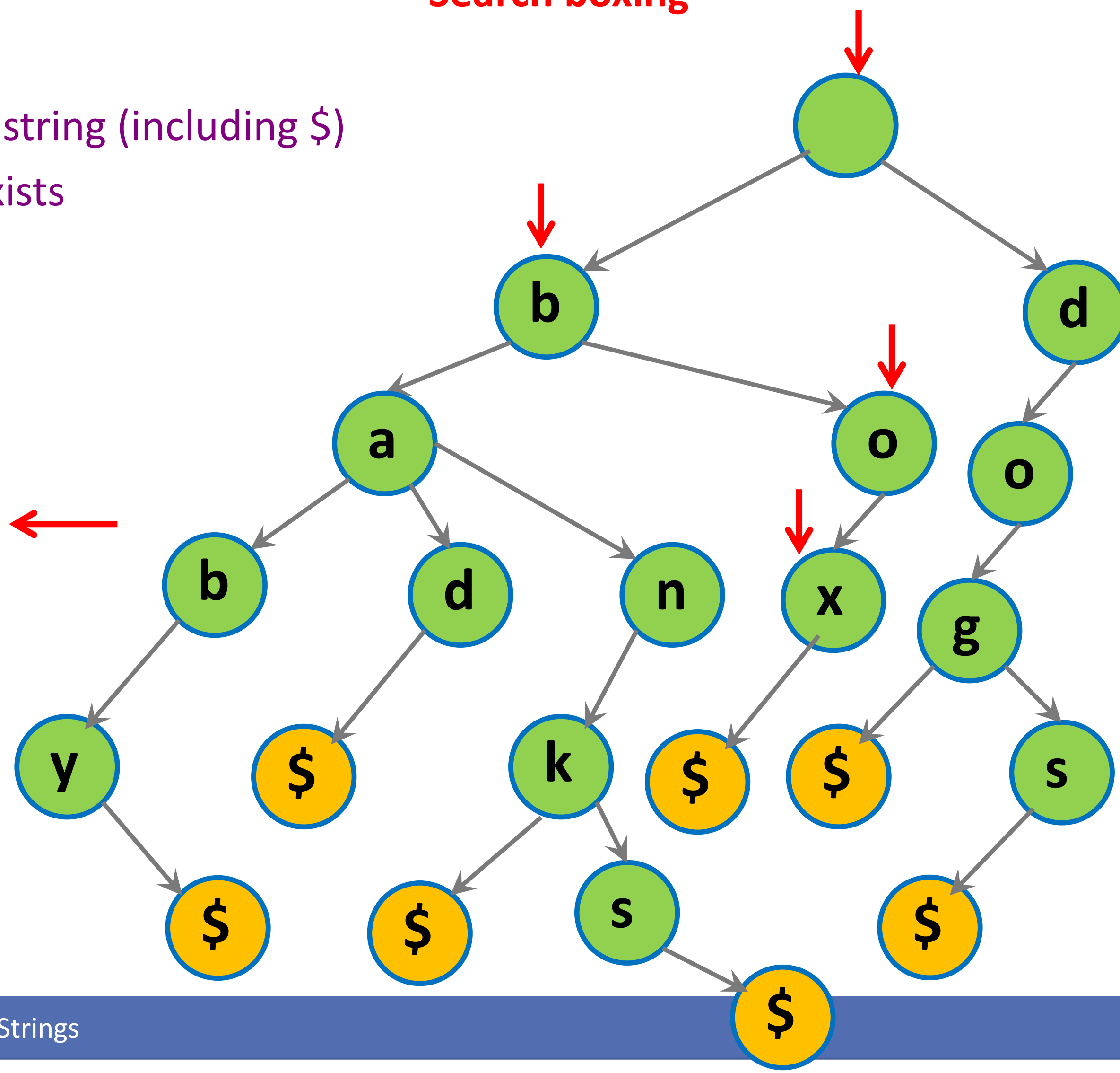
Search boxing



Trie Example: Search

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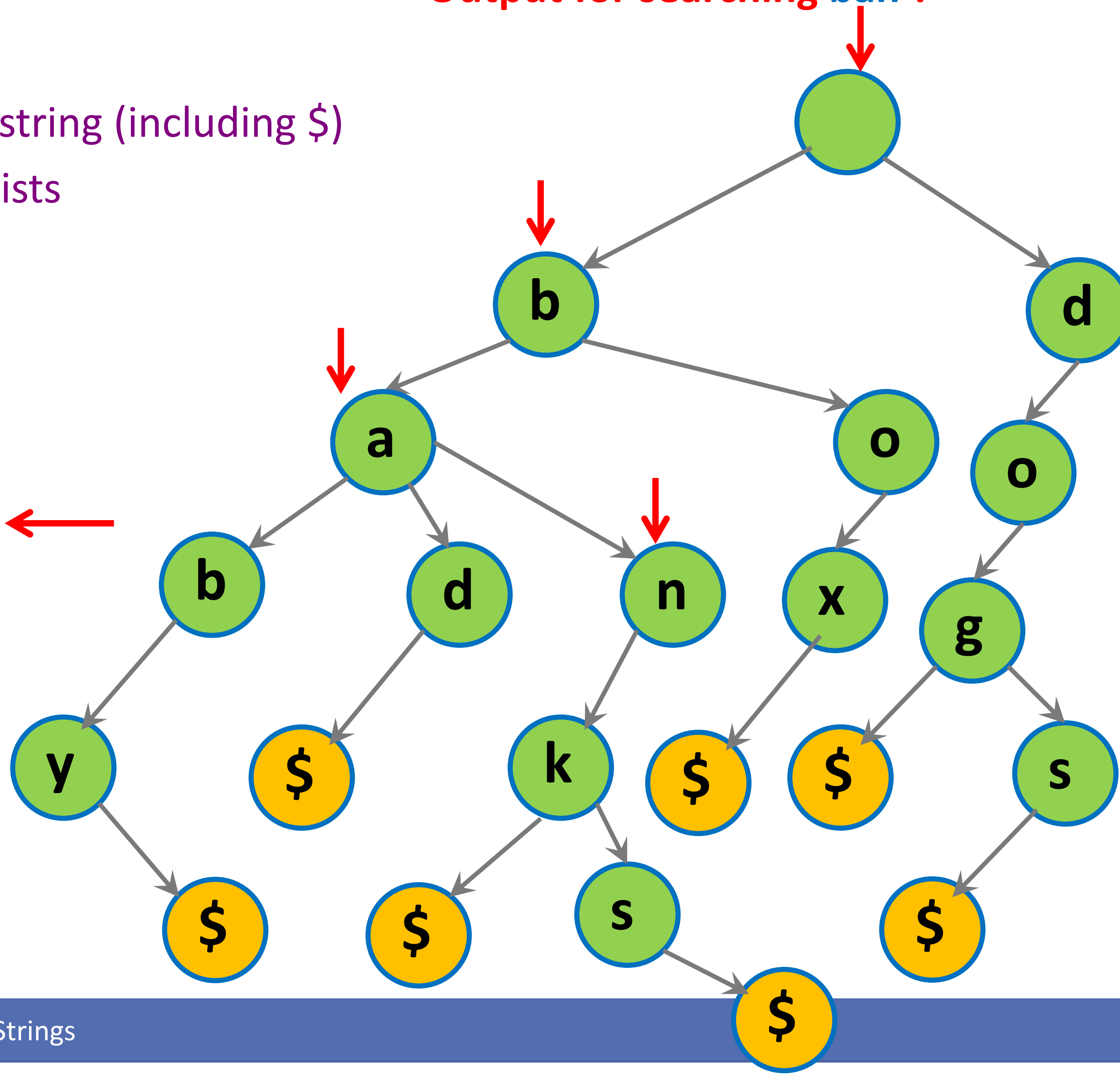
Time Complexity?:

- For loop runs O(M) times.
- Time to check if a node containing c exists?
  - Depends on implementation, and on whether alphabet size is constant

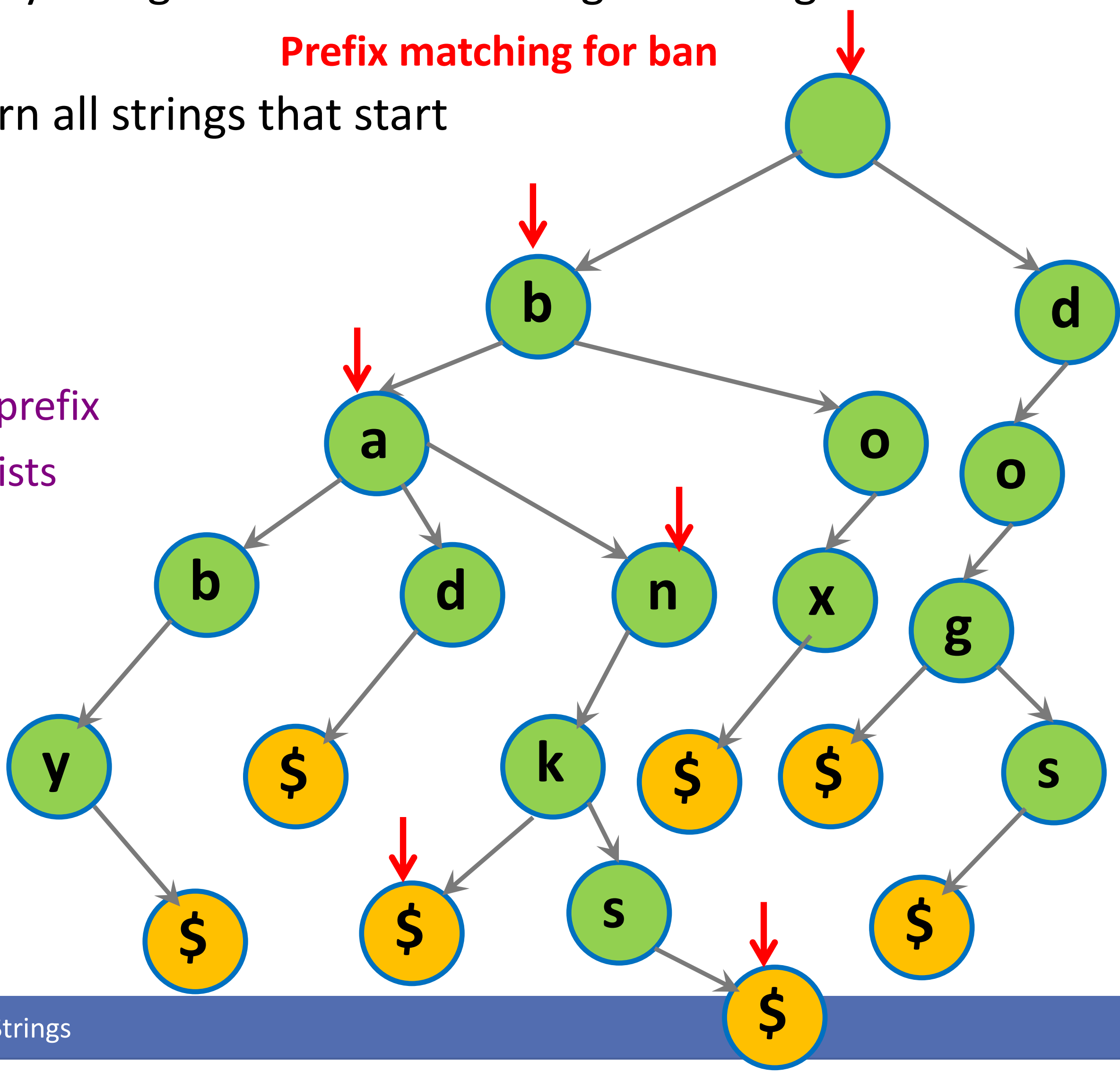
Output for searching ban ?



Trie Example: Prefix Matching

Prefix matching returns every string in text that has the given string as its **prefix**.

E.g., **Autocompletion**. Return all strings that start with "ban"



Prefix matching:

- Start from the root node
- For each character c in the prefix
  - If a node containing c exists
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  - Else
    - Return "not found"
- Return all strings in the subtree rooted at the last node

Trie Example: Prefix Matching

Prefix matching returns every string in text that has the given string as its **<u>prefix</u>**.

E.g., Autocompletion. Return all strings that start with "b"

Prefix matching:

- Start from the root node
- For each character c in the prefix
  - If a node containing c exists
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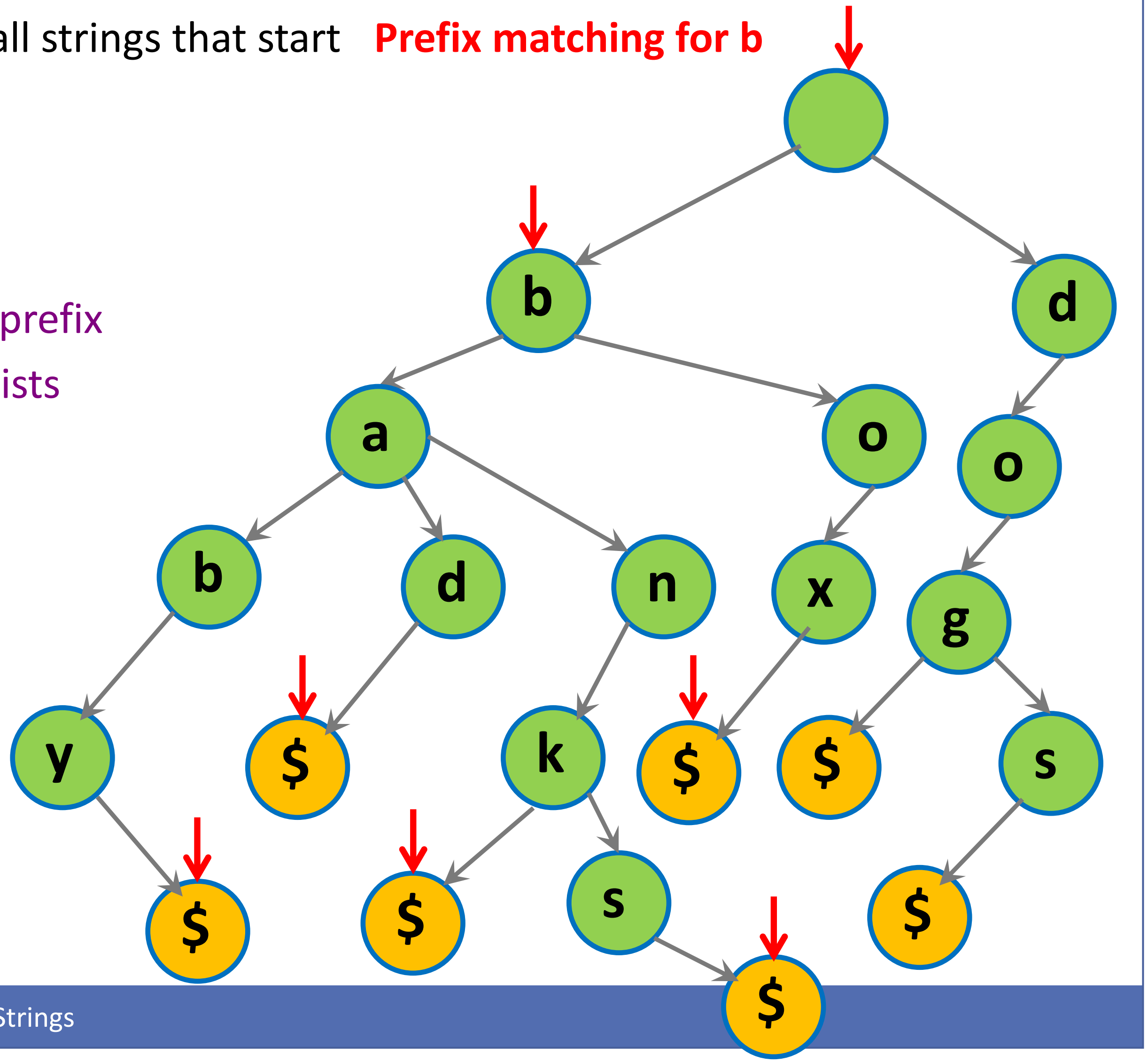
- Return all strings in the subtree rooted at the last node



Trie Node Implementation

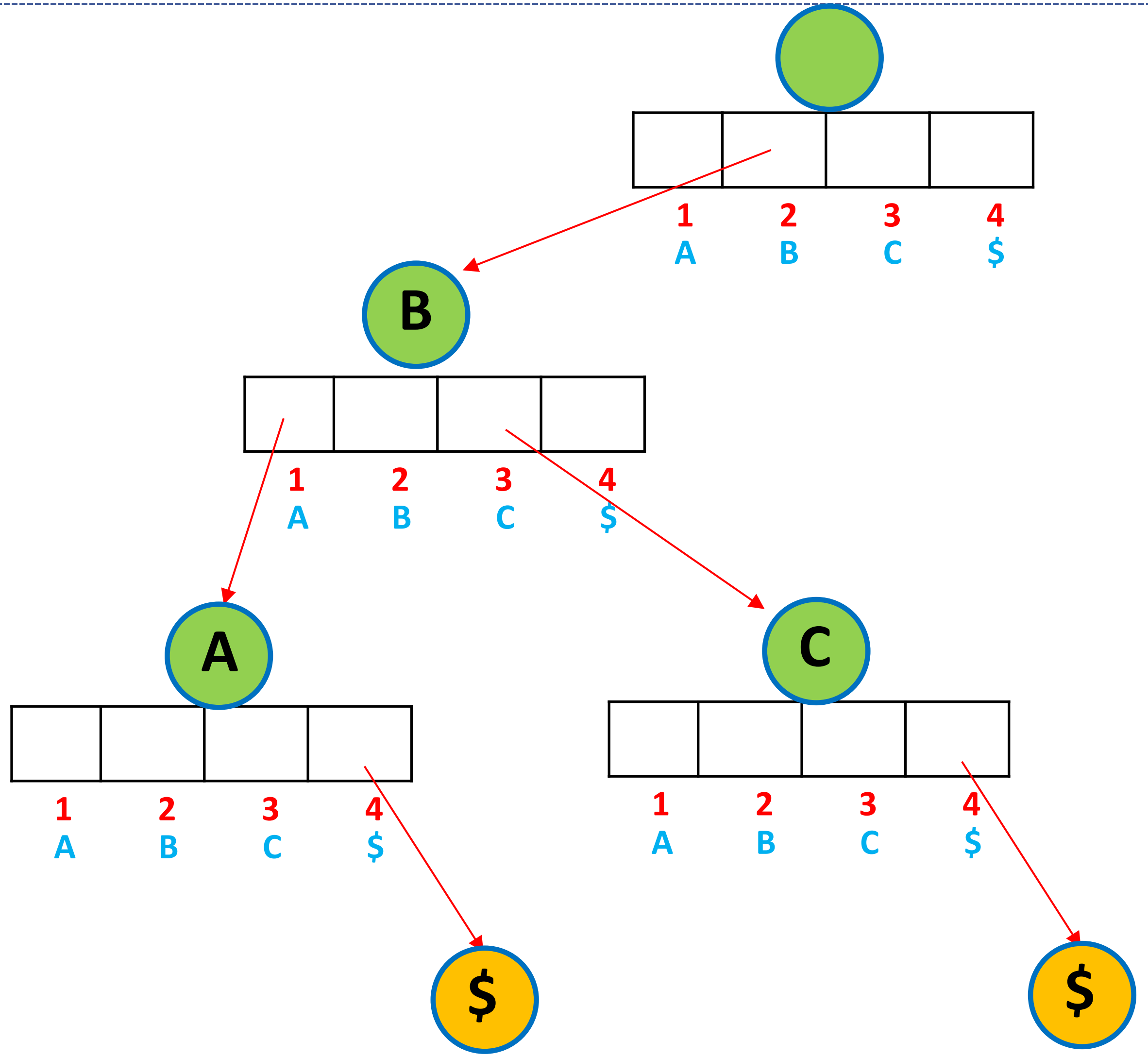
Implementation using an array:

- At each node, create an array of alphabets size (e.g., 26 for English letters, 4 for DNA strings)
- If i-th node exists, store it in array[i] (pointer to it, resp.)
- Otherwise, array[i] = Nil.

The above implementation allows checking whether a node exists or not in O(1).

Other implementations are possible (e.g., using linked lists or hash tables).

Example: Implementing a Trie using arrays (assuming only three letters A,B,C)



Insert BA$

Insert BC$

Some properties of Trie

- The maximum depth is the length of longest string in the collection.
- Insertion, Deletion, Lookup operations take time proportional to the length of the string/pattern being inserted, deleted, or searched.
- But we waste a lot of space if
  - each node has 1 pointer per symbol in the alphabet.
  - deeper nodes typically have mostly null pointers.
- Can reduce total space usage by turning each node into a linked list or binary search tree etc, trading off time for space.

Advantages and Disadvantages of Trie

Advantages

- A faster search structure than a binary search tree with string keys.
- A more versatile search structure than hash table
  - Allows lookup on prefix matching in O(M)-time where M is the length of prefix.
  - Allows sorting collection of strings in O(MN) time where MN is the total number of characters in all strings

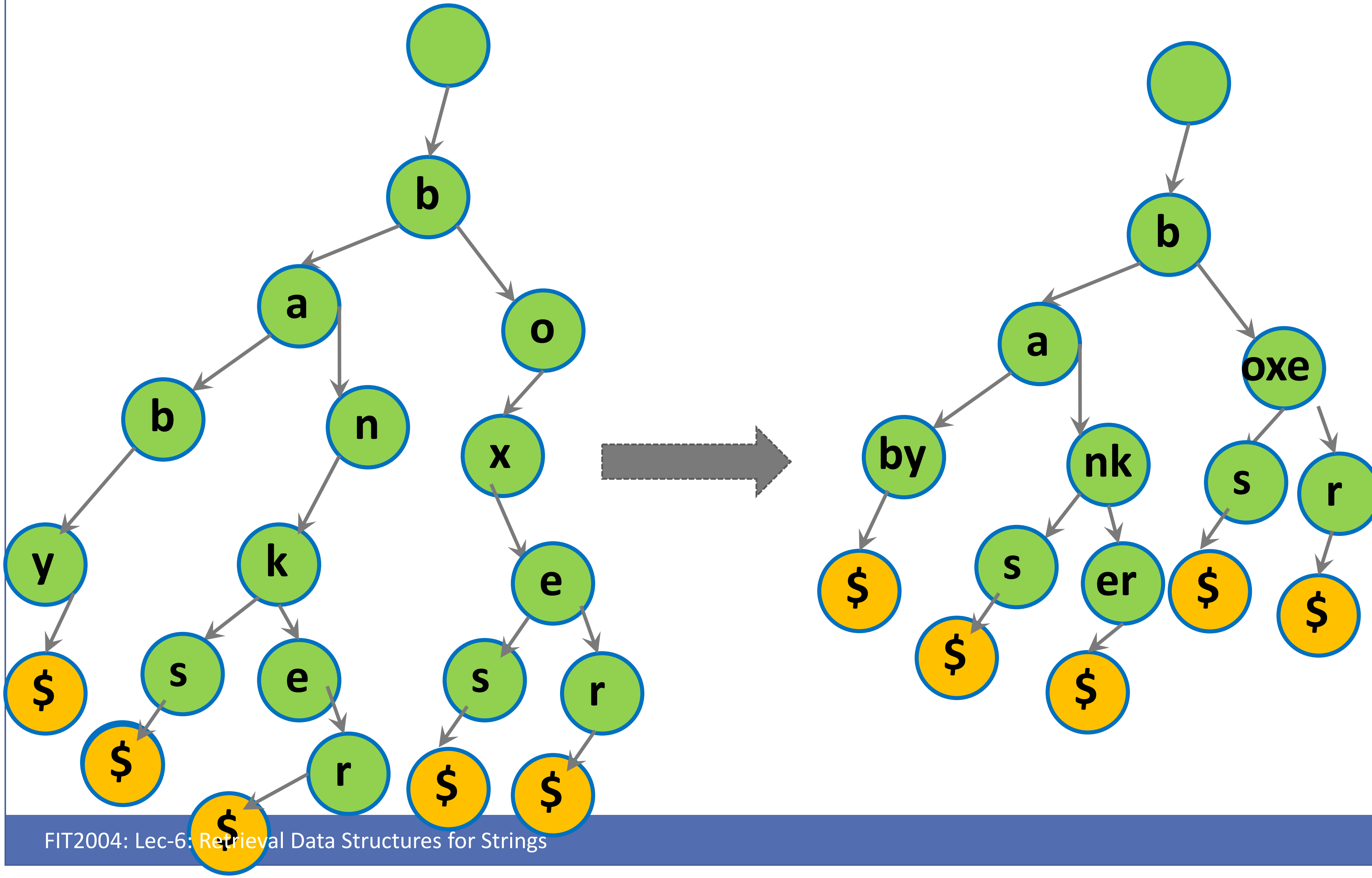
Disadvantages

- On average Tries can be slower (in some cases) than hash tables for looking up patterns/queries.
- Wastes space, since even when a node has few children, you need to create an array of size alphabet

Radix/PATRICIA Tree (<u>NOT EXAMINABLE</u> BUT WORTH MENTIONING)

- Radix/PATRICIA tree is a space-optimized/compact Trie data structure
- Unlike regular tries, edges can be **labeled with substrings** of characters.
- The nodes along a path having exactly one child are merged
- This makes them much more efficient for sets of strings that share long prefixes or substrings.

Radix Tree – path compression (NOT EXAMINABLE BUT WORTH MENTIONING)



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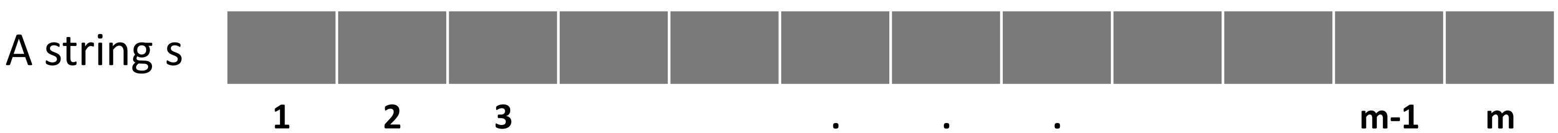
Substring search

- Standard tries are very useful for quickly looking up
  - whole words
  - **prefixes** of words
- But what if we want to look up **substrings** anywhere in a word?



Prefixes and suffixes

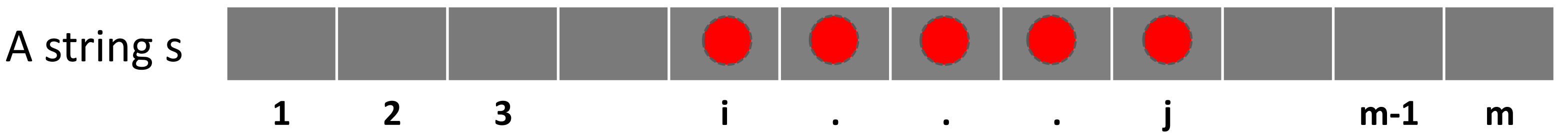
- Standard tries are very useful for quickly looking up
  - whole words
  - **prefixes** of words
- But what if we want to look up substrings anywhere in a word?
- A prefix of a word s[1..m] is some string s[1..i] where 1<=i<=m

Prefixes and suffixes

- (Prefix) Tries are very useful for quickly looking up whole words, but more generally, **prefixes** of words
- A prefix of a word s[1..m] is s[1..i] where 1<=i<=m
- A suffix of a word s[1..m] is s[i..m] where 1<=i<=m

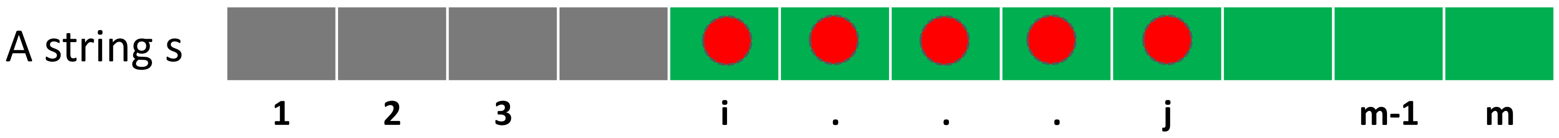
Prefixes and suffixes

- Any **substring** of a word is a **prefix** of some **suffix**
- In other words, a substring of s[1..m] is s[i..j]



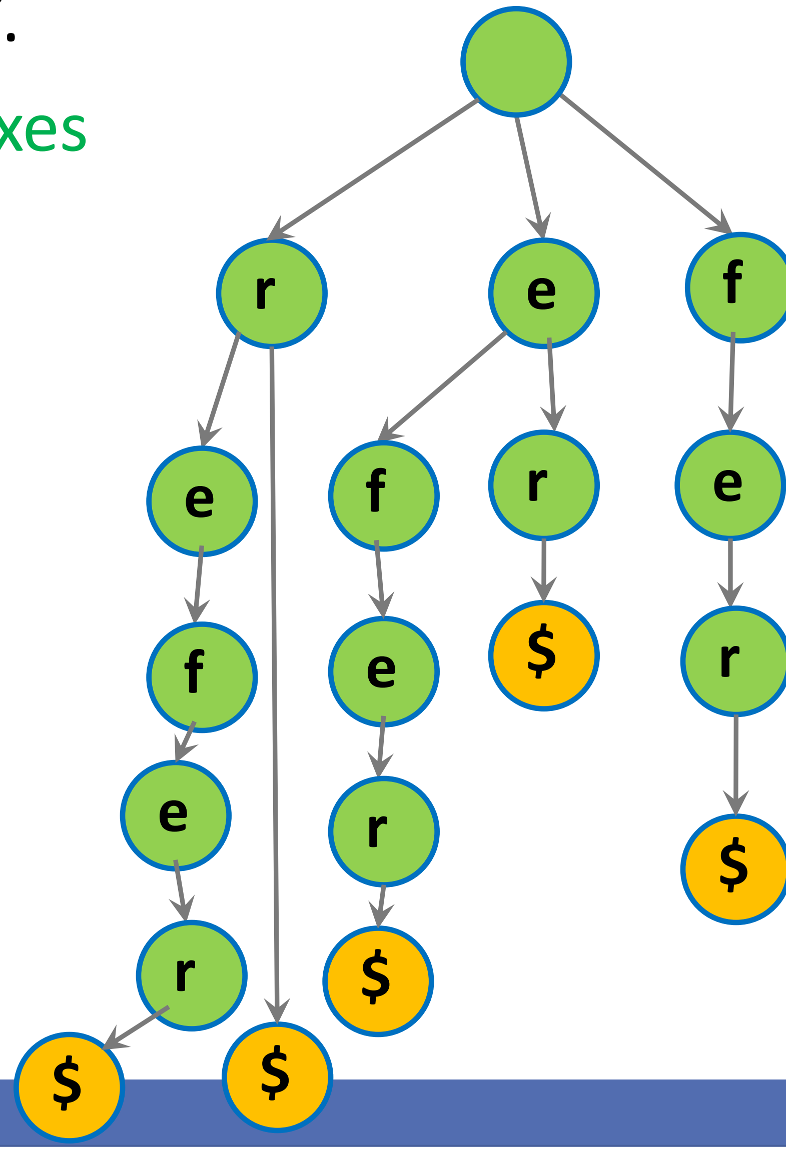
Prefixes and suffixes

- Any substring of a word is a **prefix** of some **suffix**
- In other words, a substring of s[1..m] is s[i..j]
- s[i..j] is a prefix of s[i..m] (which is a suffix of s[1..m])
- To be able to efficiently search **substrings**…
- Just make a prefix trie of suffixes



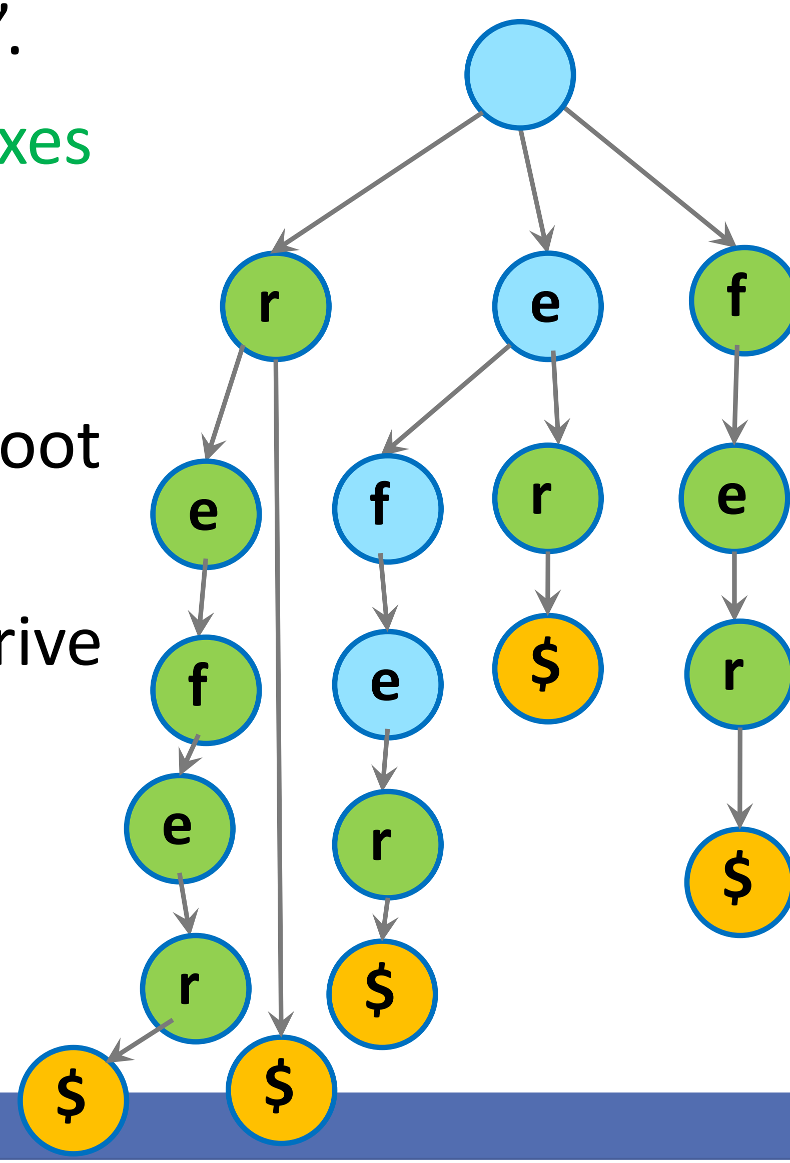
Suffix Trie

- Consider some text, e.g., “refer”.
- A Trie constructed using all suffixes of the text is called a Suffix Trie
- Pick any substring, eg “efe”



Suffix Trie

- Consider some text, e.g., “refer”.
- A Trie constructed using all suffixes of the text is called a Suffix Trie
- Pick any substring, eg “efe”
- Note that it traces a path from root to some node
- Note that you do not have to arrive at $



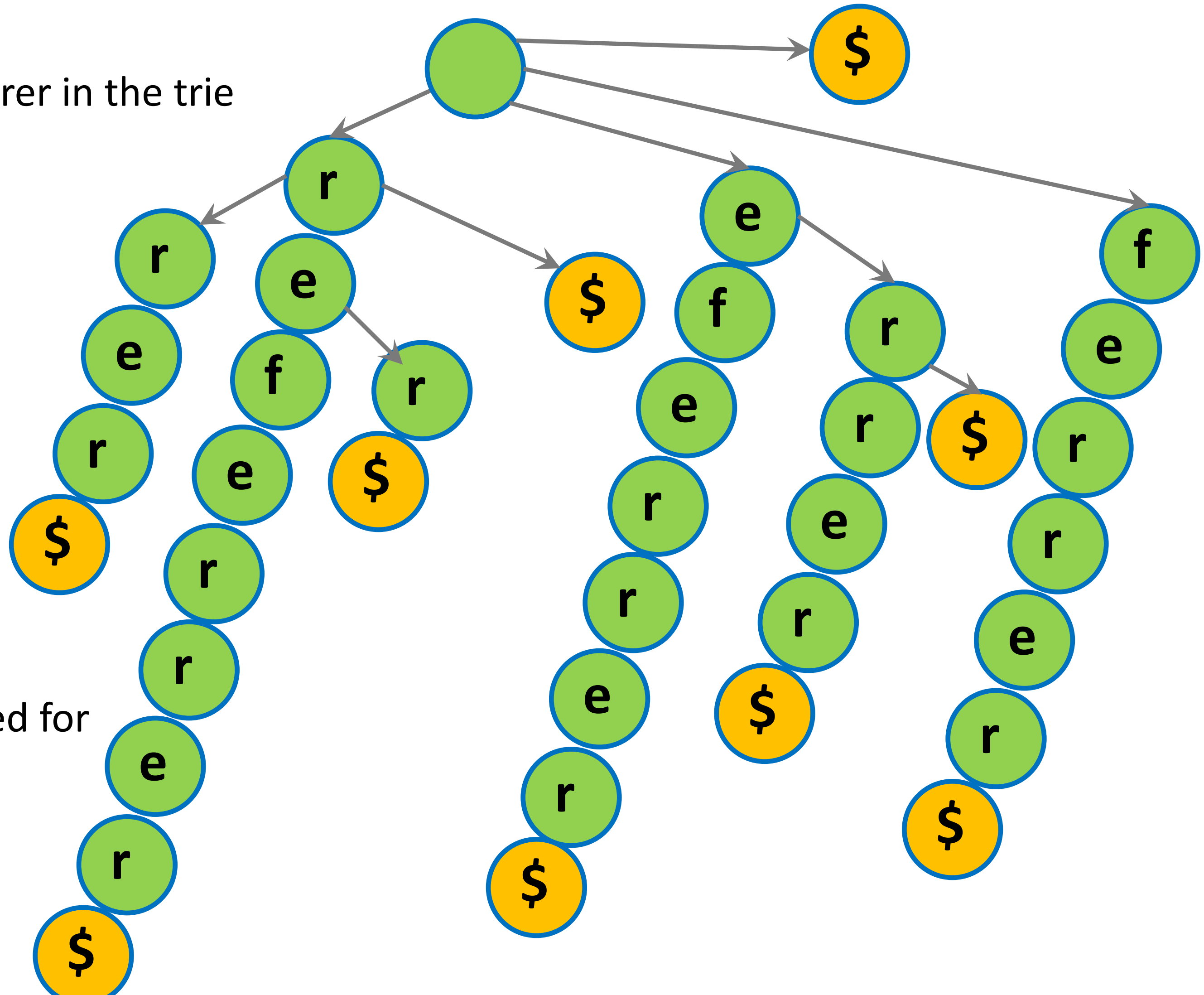
Constructing Suffix Trie

Suffix trie of referrer$

Insert all suffixes of referrer in the trie

1. referrer$
2. eferrer$
3. ferrer$
4. errer$
5. rrer$
6. rer$
7. er$
8. r$
9. $

Many arrows are removed for better visualization



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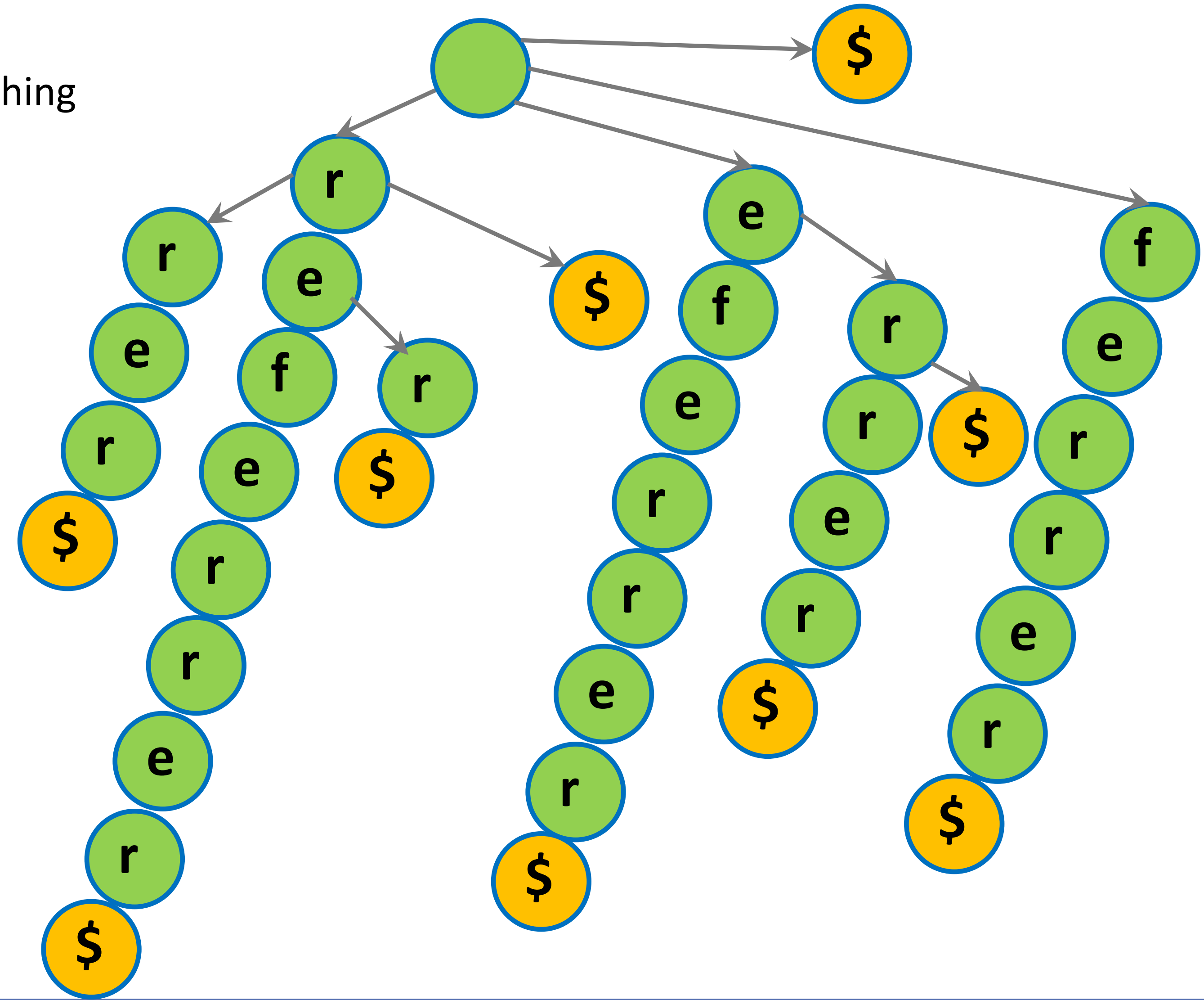
Substring search on Suffix Trie

Substring search for str

- Similar to prefix matching

search “err”



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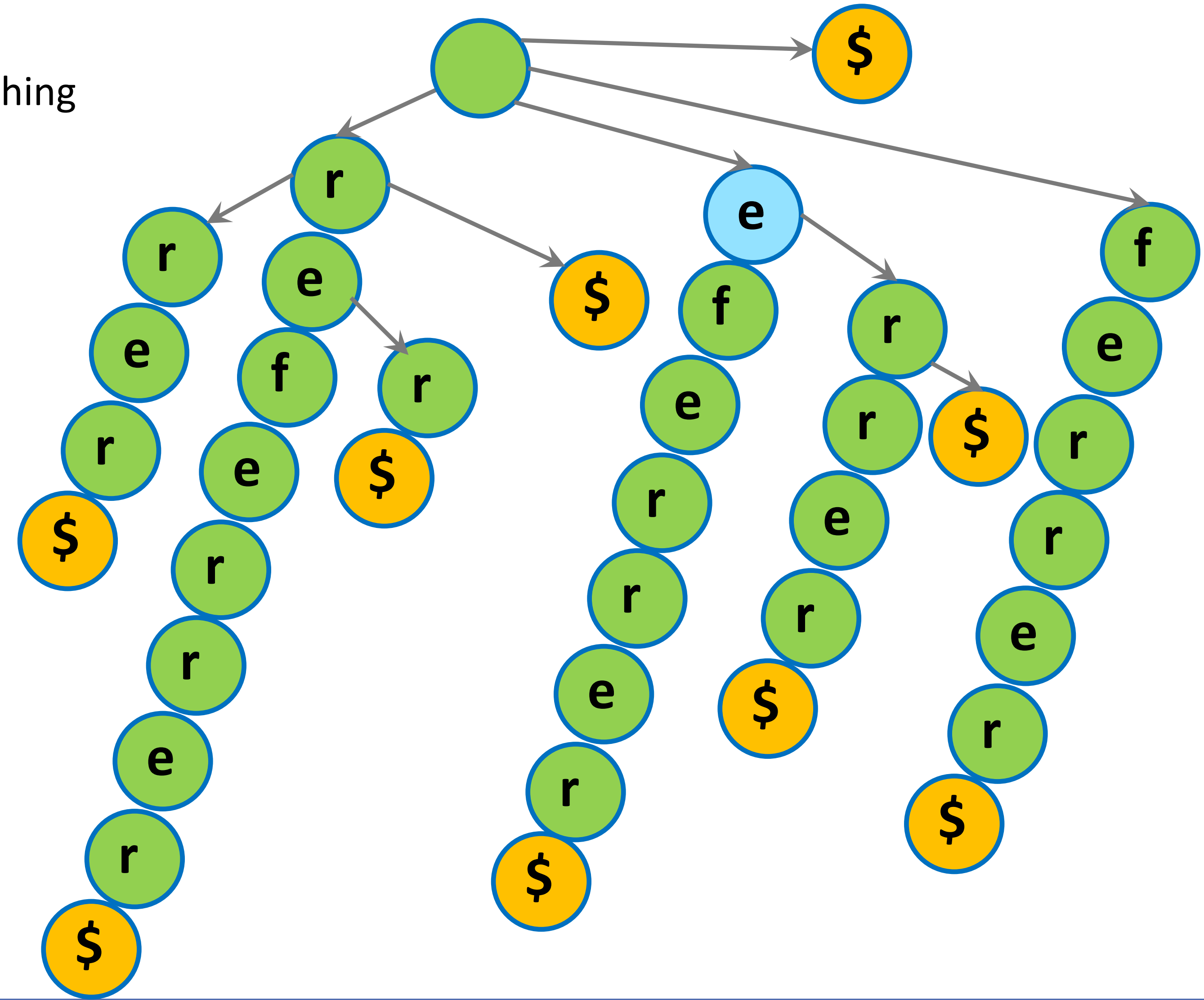

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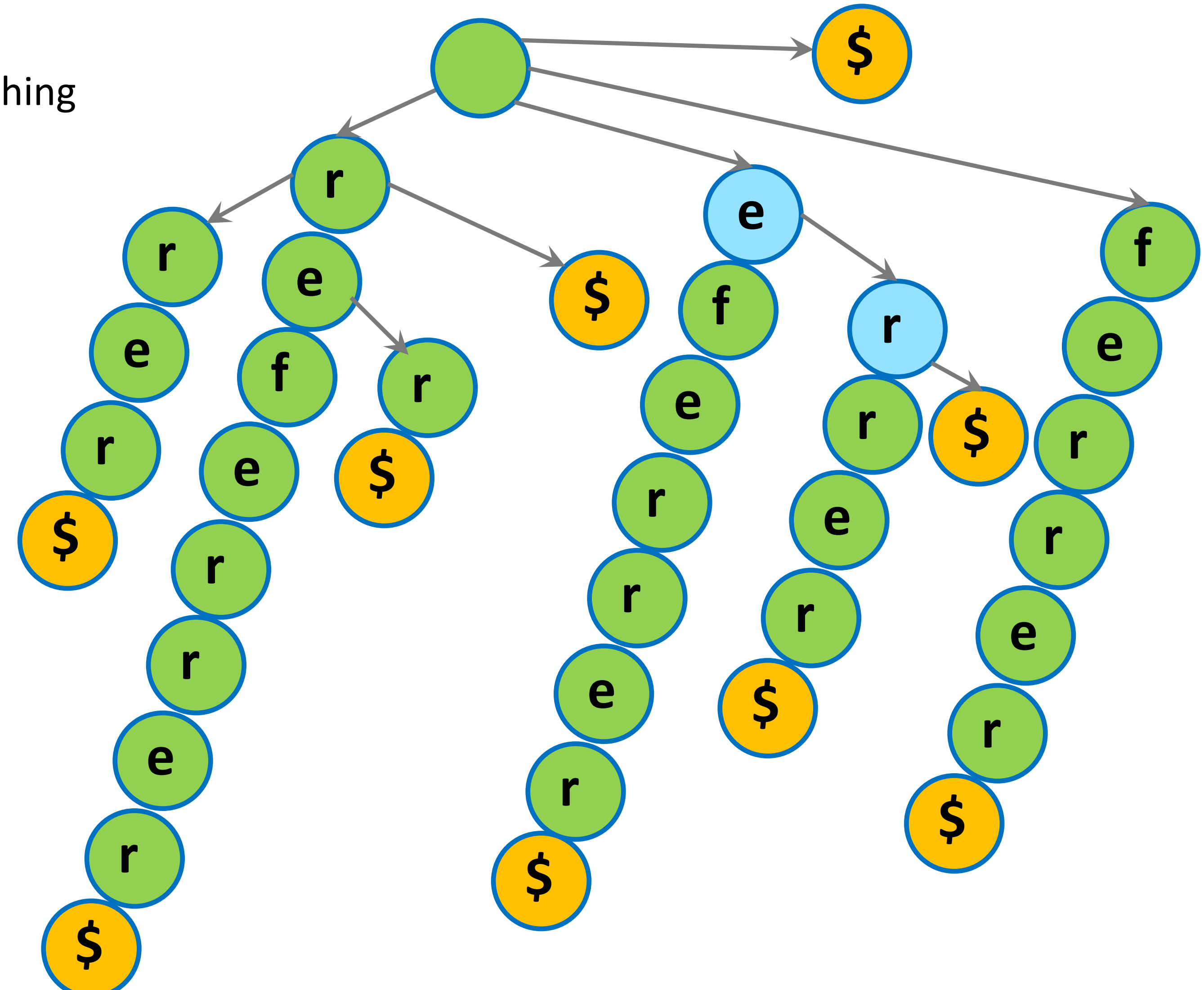

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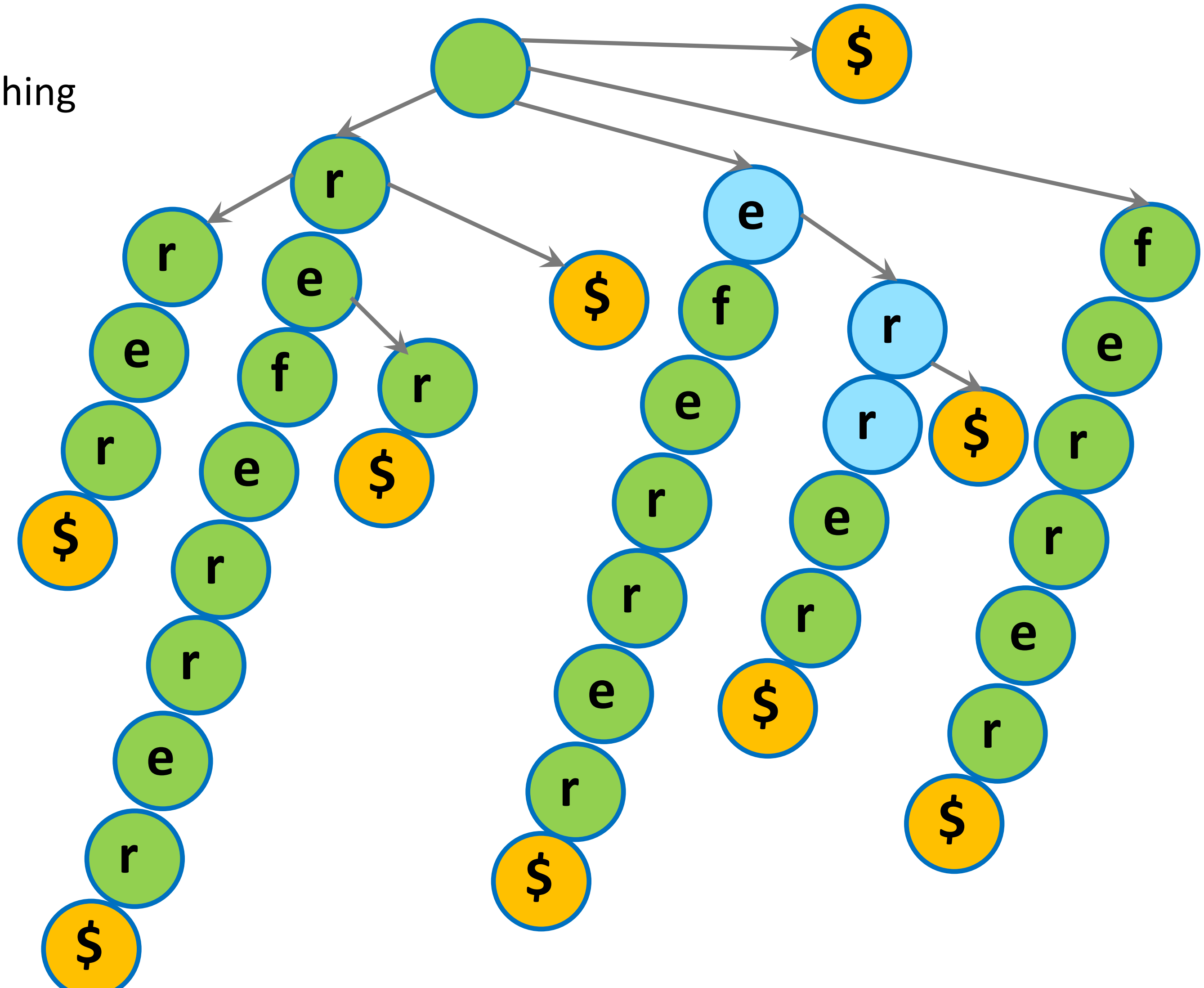


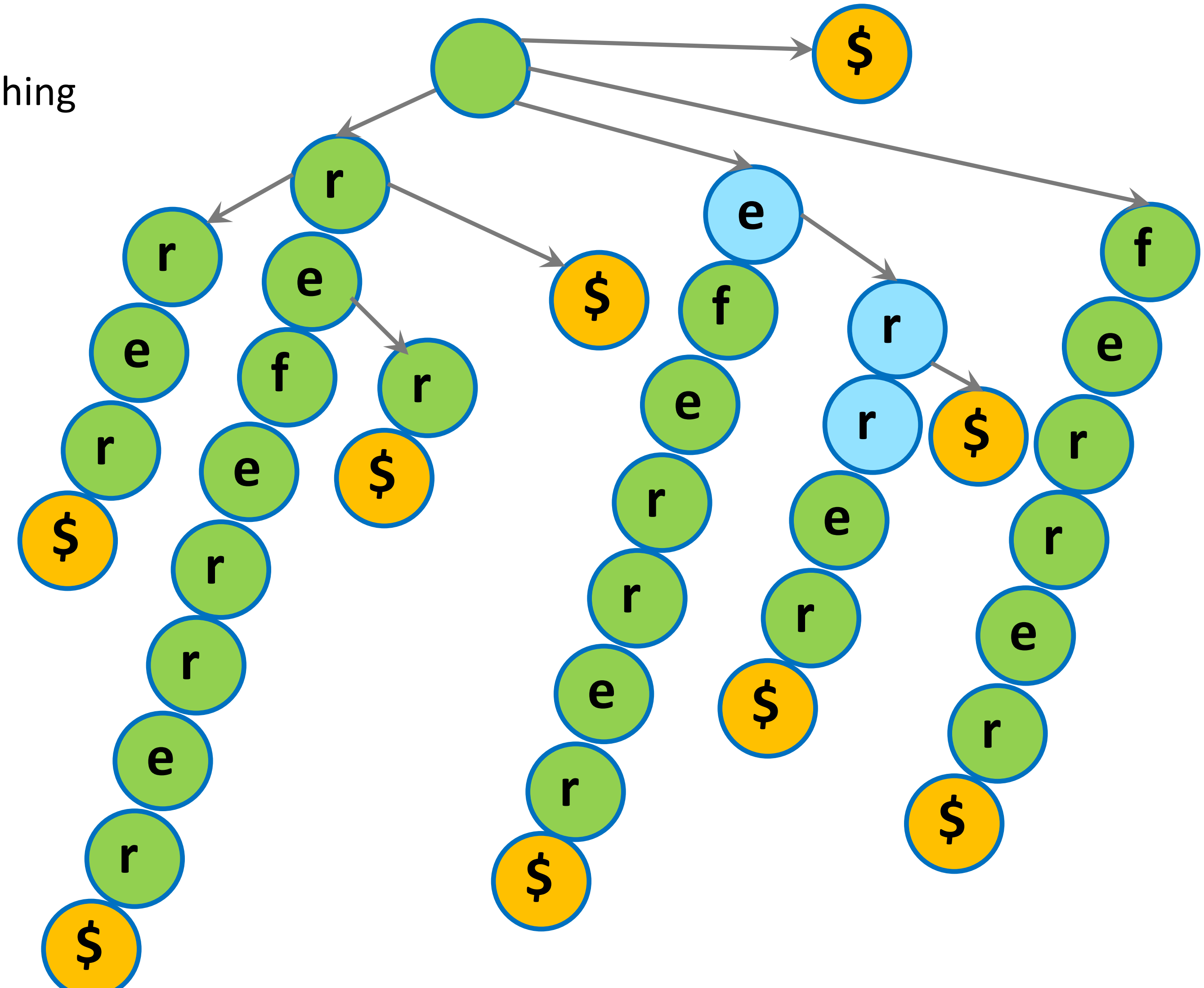
Substring search on Suffix Trie

Substring search for str

- Similar to prefix matching

search “err”

Found!



Substring search on Suffix Trie

Substring search for str

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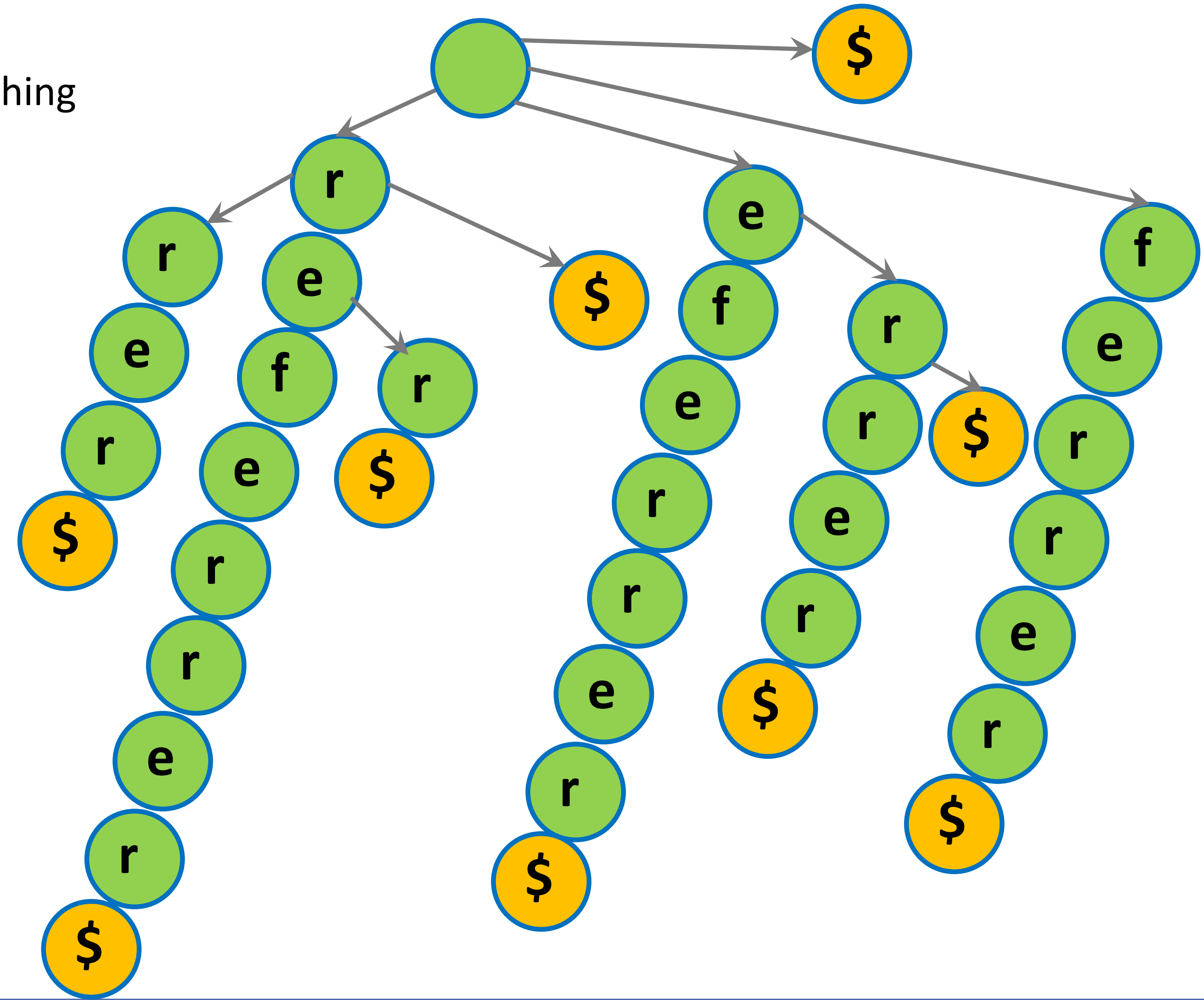
search "err"

search "fers"



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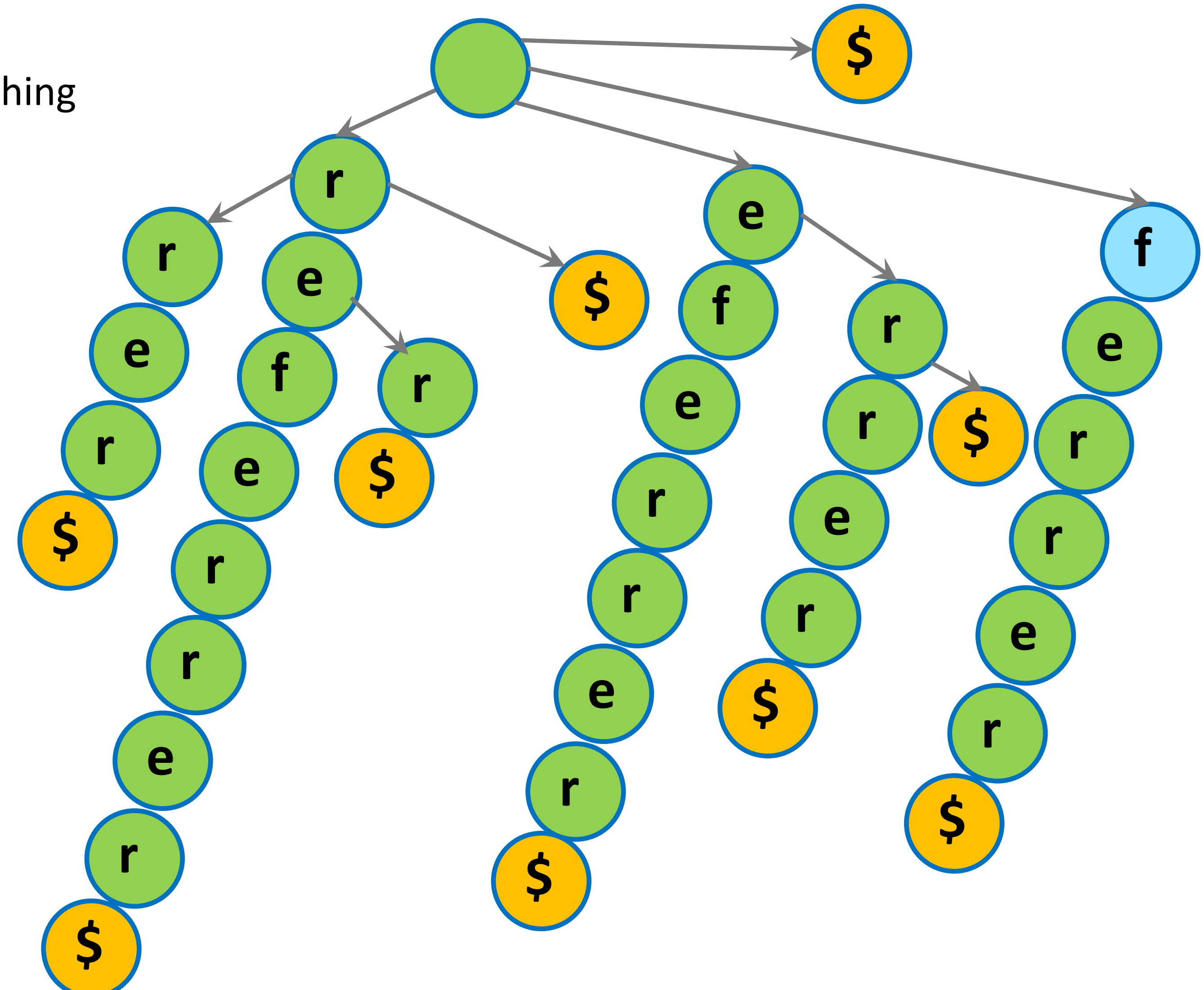

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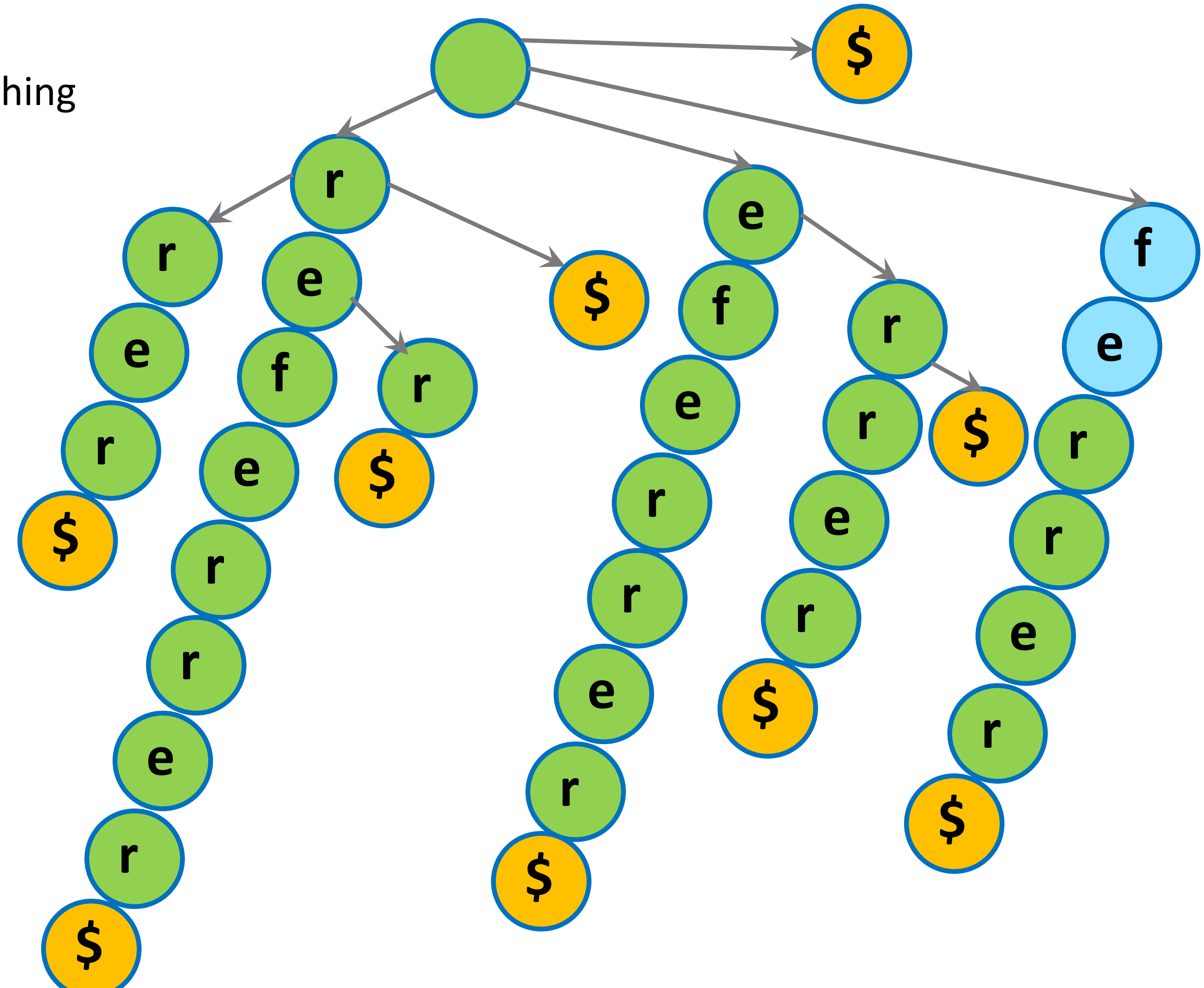

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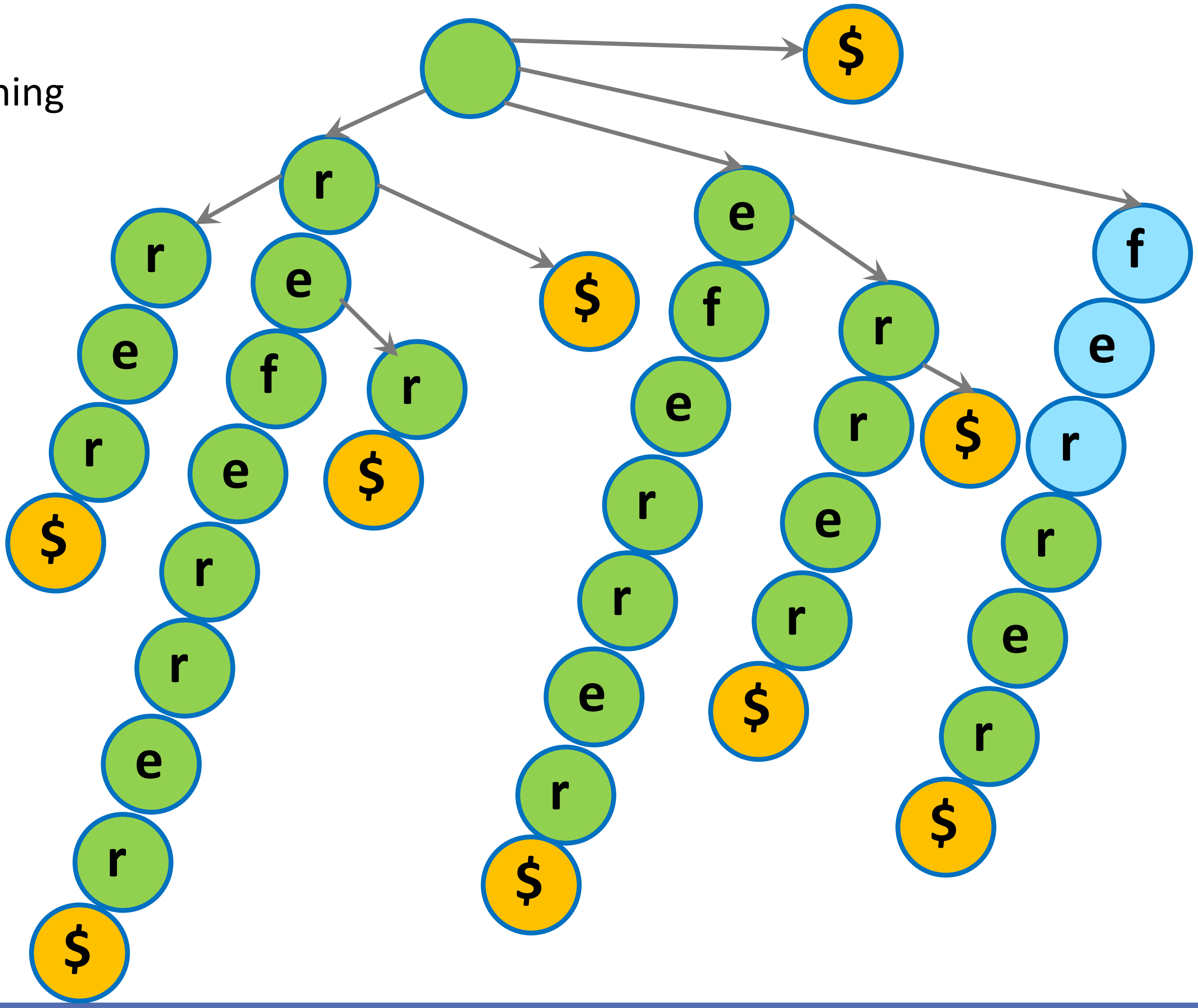

Substring search on Suffix Trie

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- Similar to prefix matching

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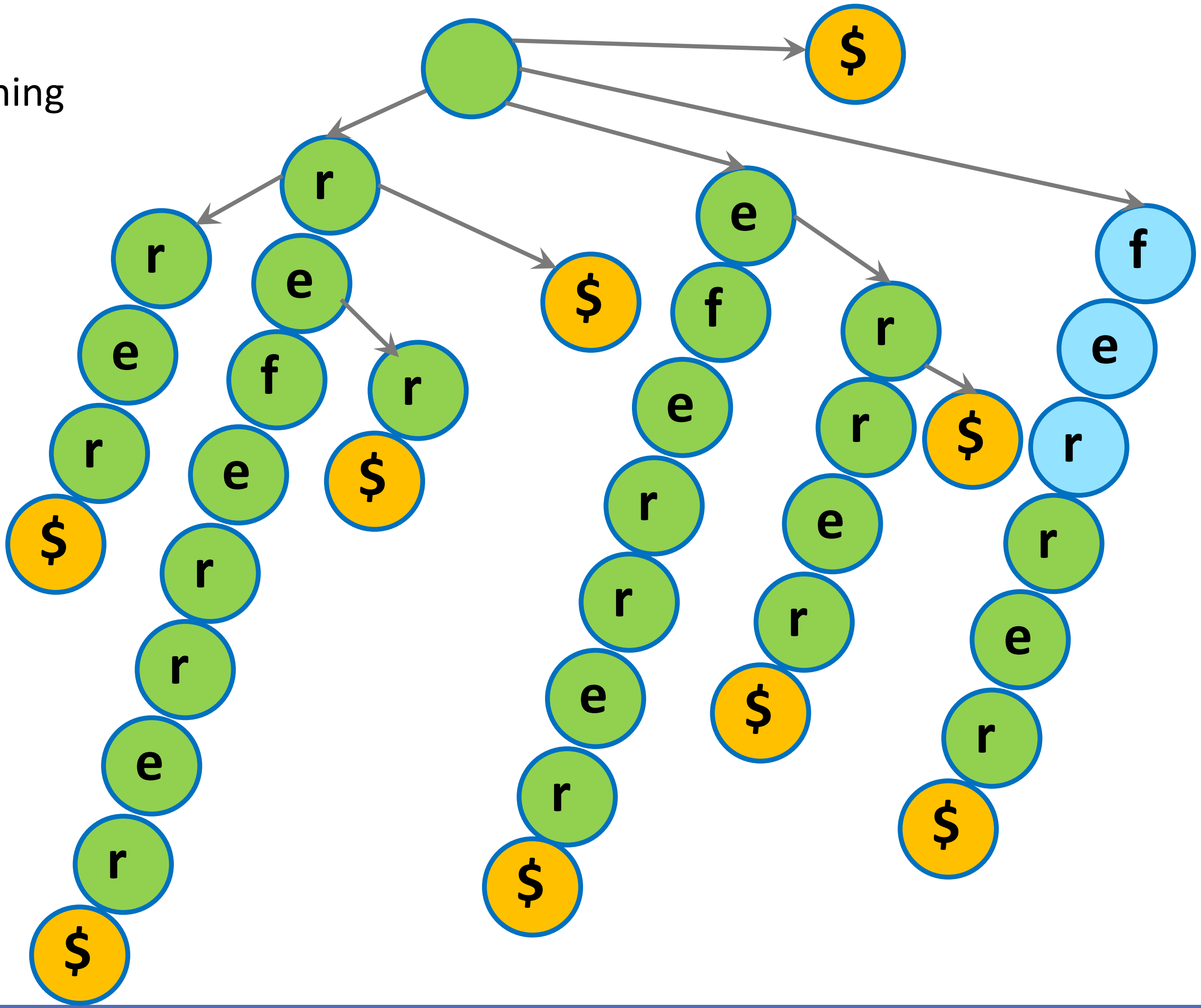
Not found :(



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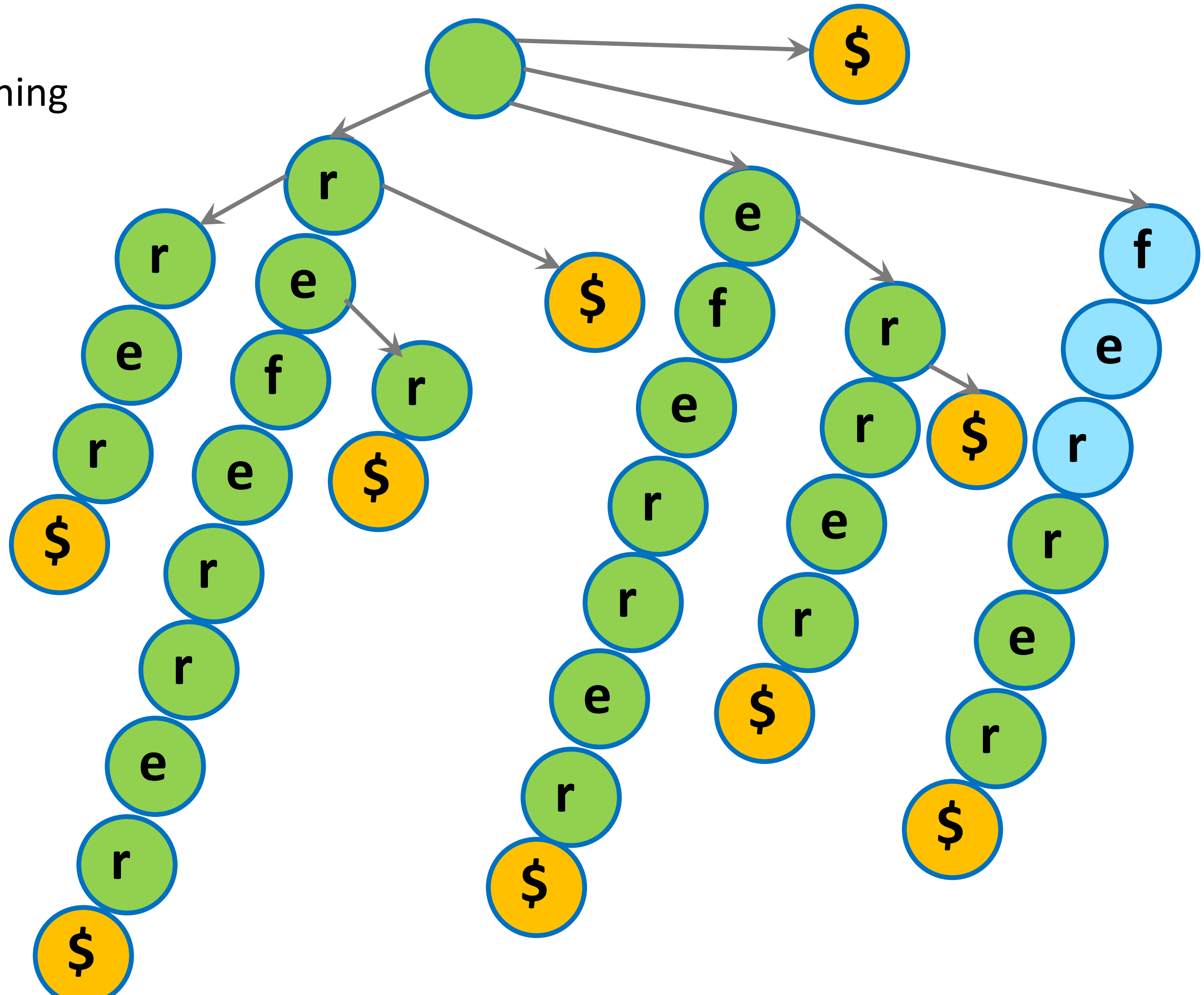

Time Complexity:

O(M) where M is the length of substring

Counting # of occurences of a substring

Quiz time!
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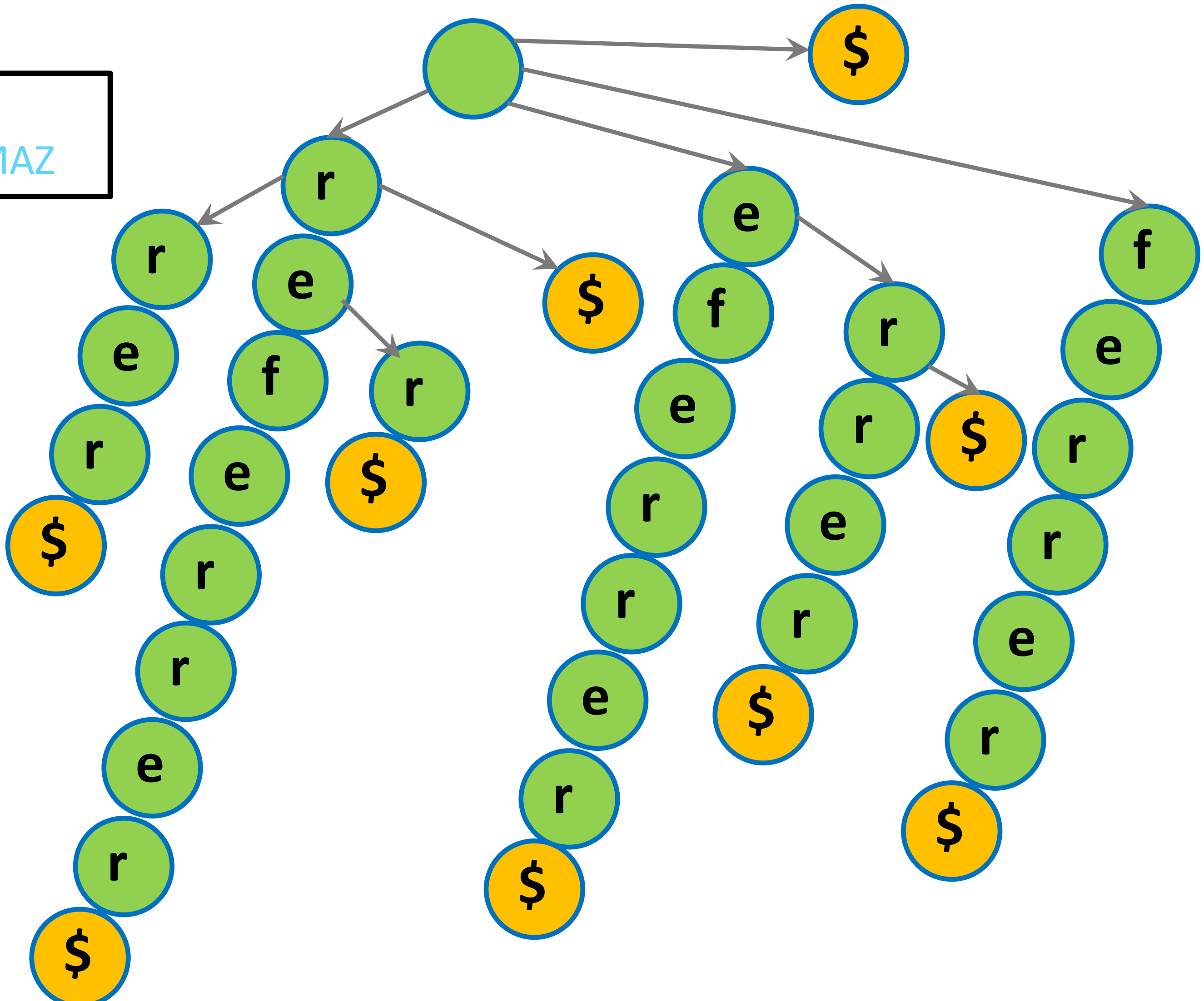
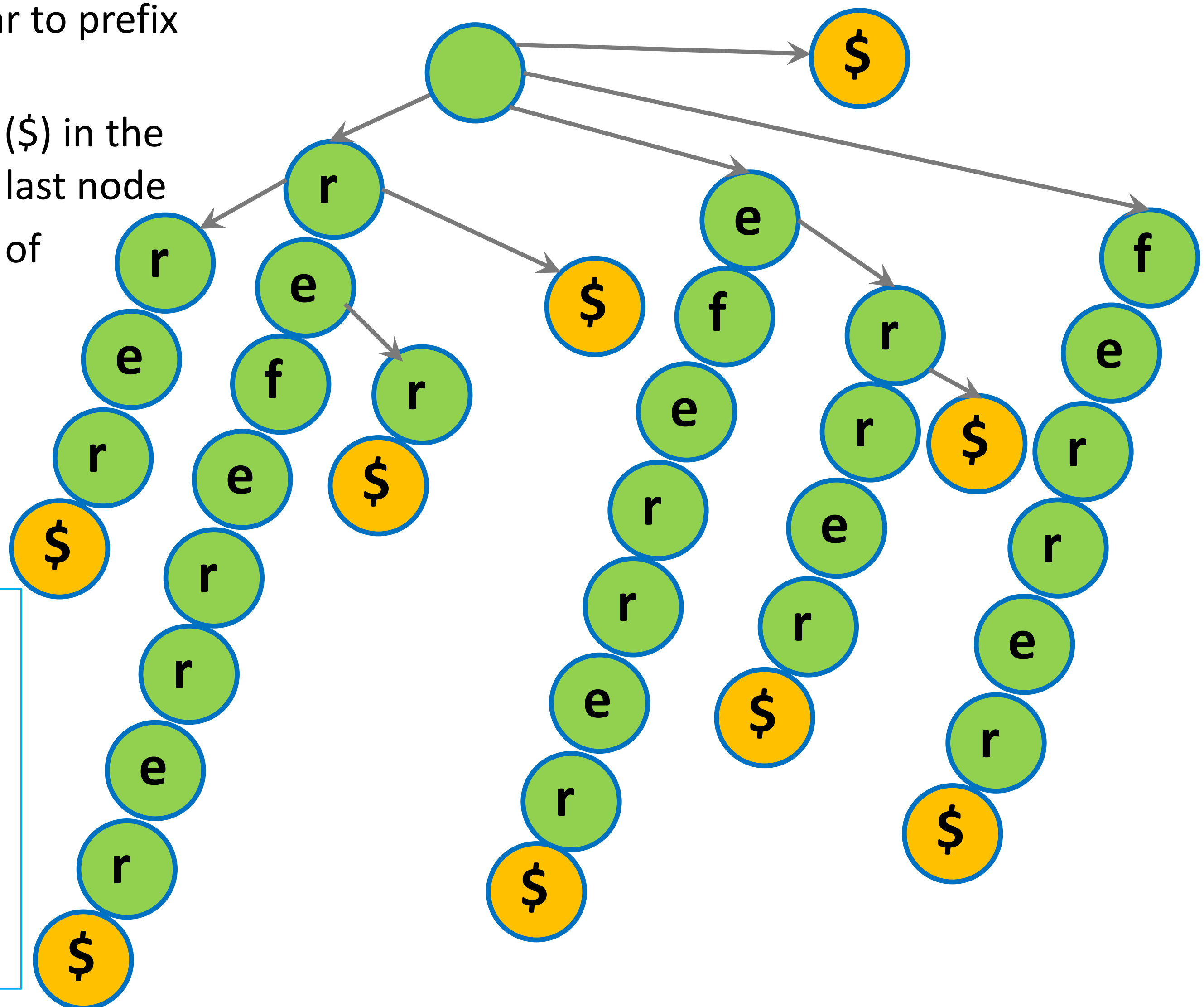


Counting # of occurences of a substring

- Follow the path similar to prefix matching
- Count # of leaf nodes ($) in the subtree rooted at the last node
- E.g Count occurences of "er"



Time Complexity:

Can be done in O(M) if number of leaf nodes is maintained during construction of suffix trie

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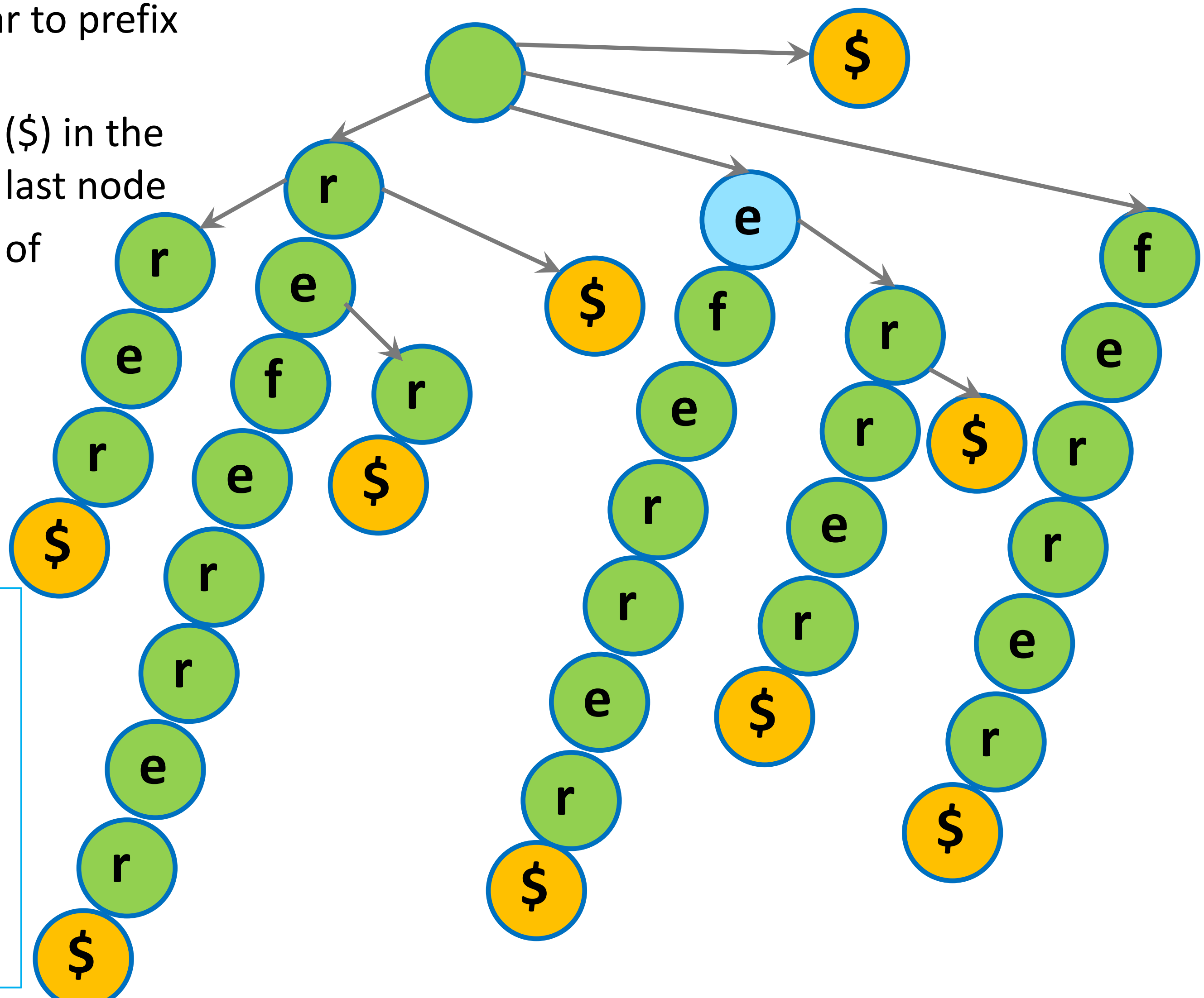

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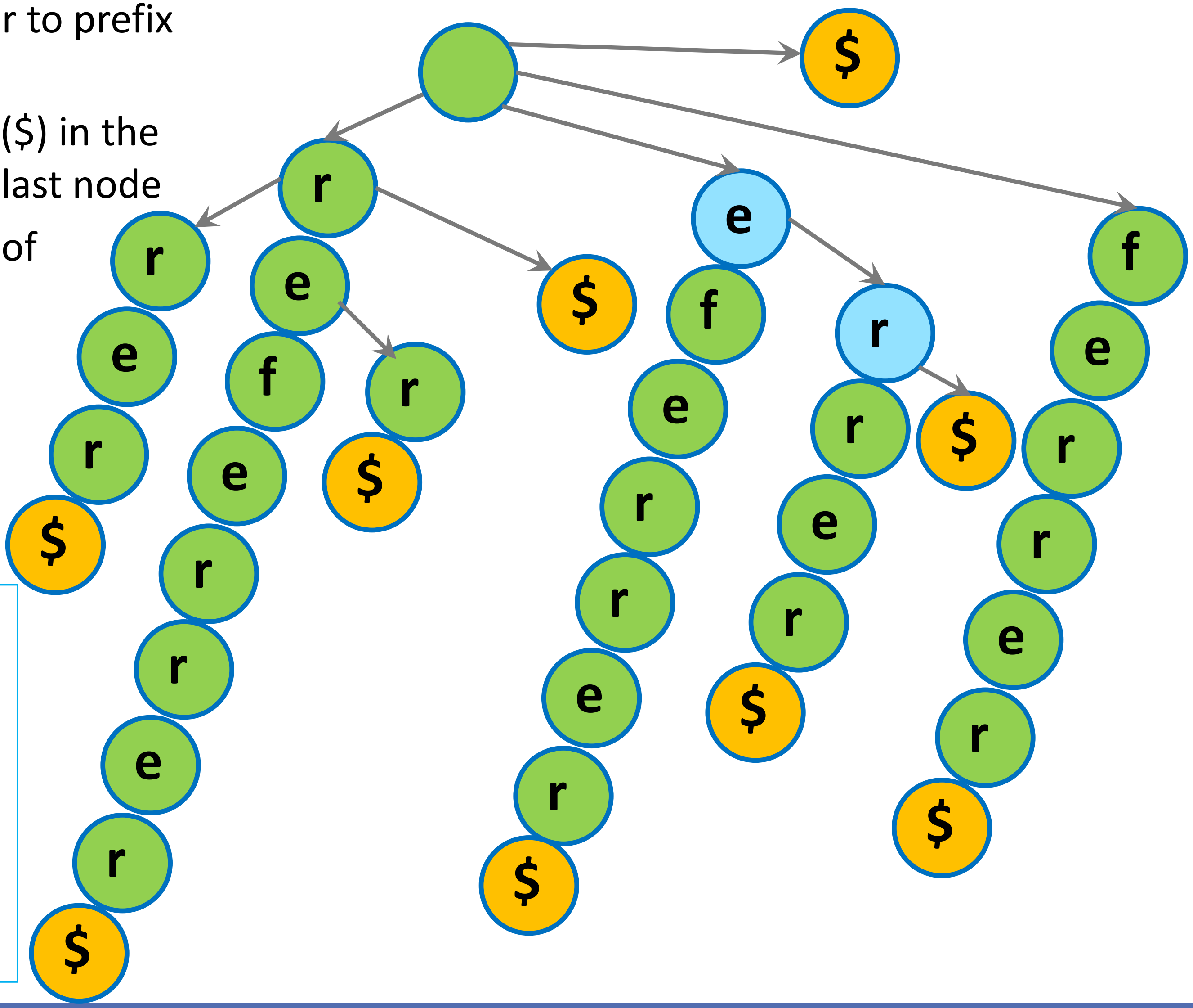

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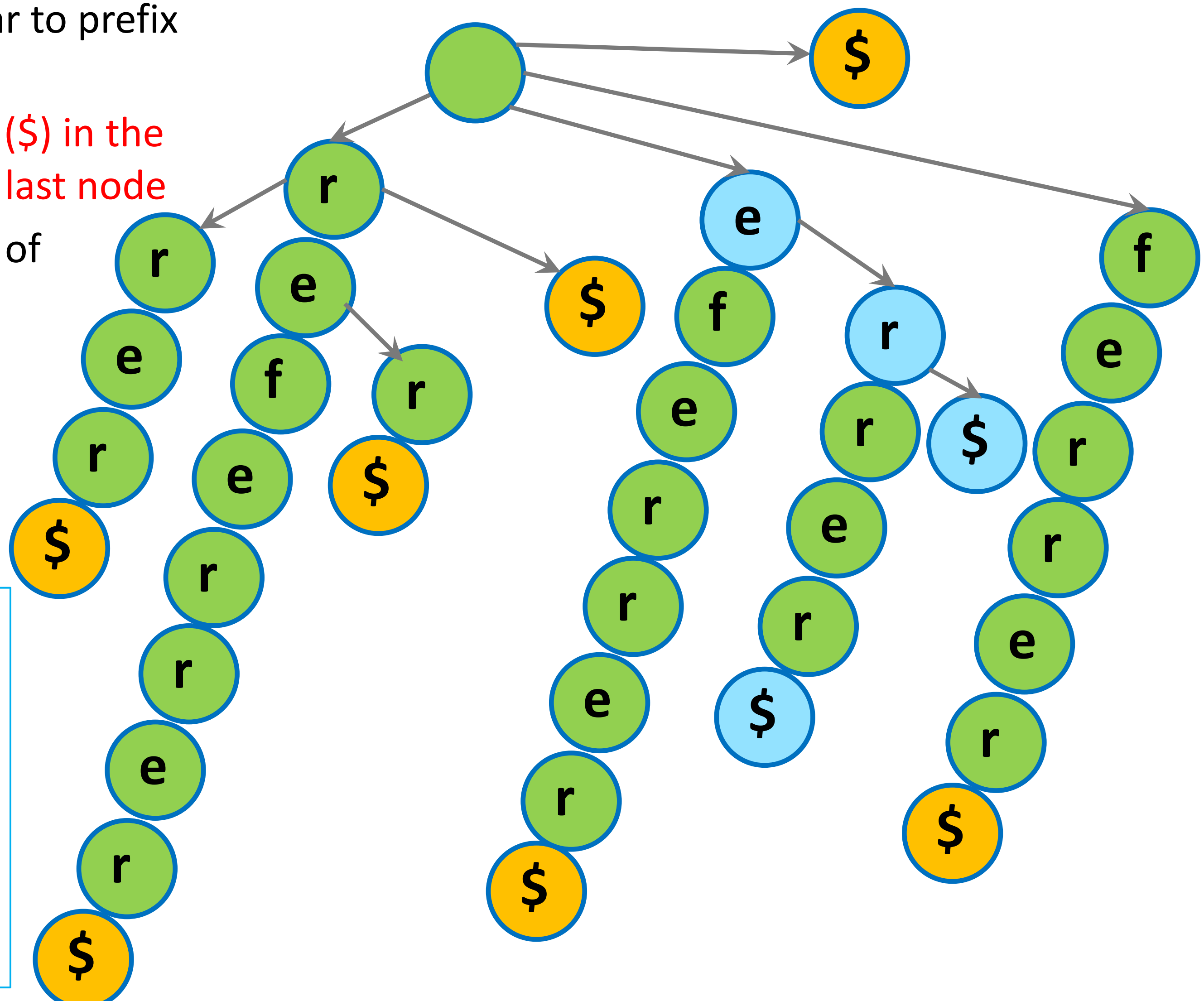

Time Complexity:

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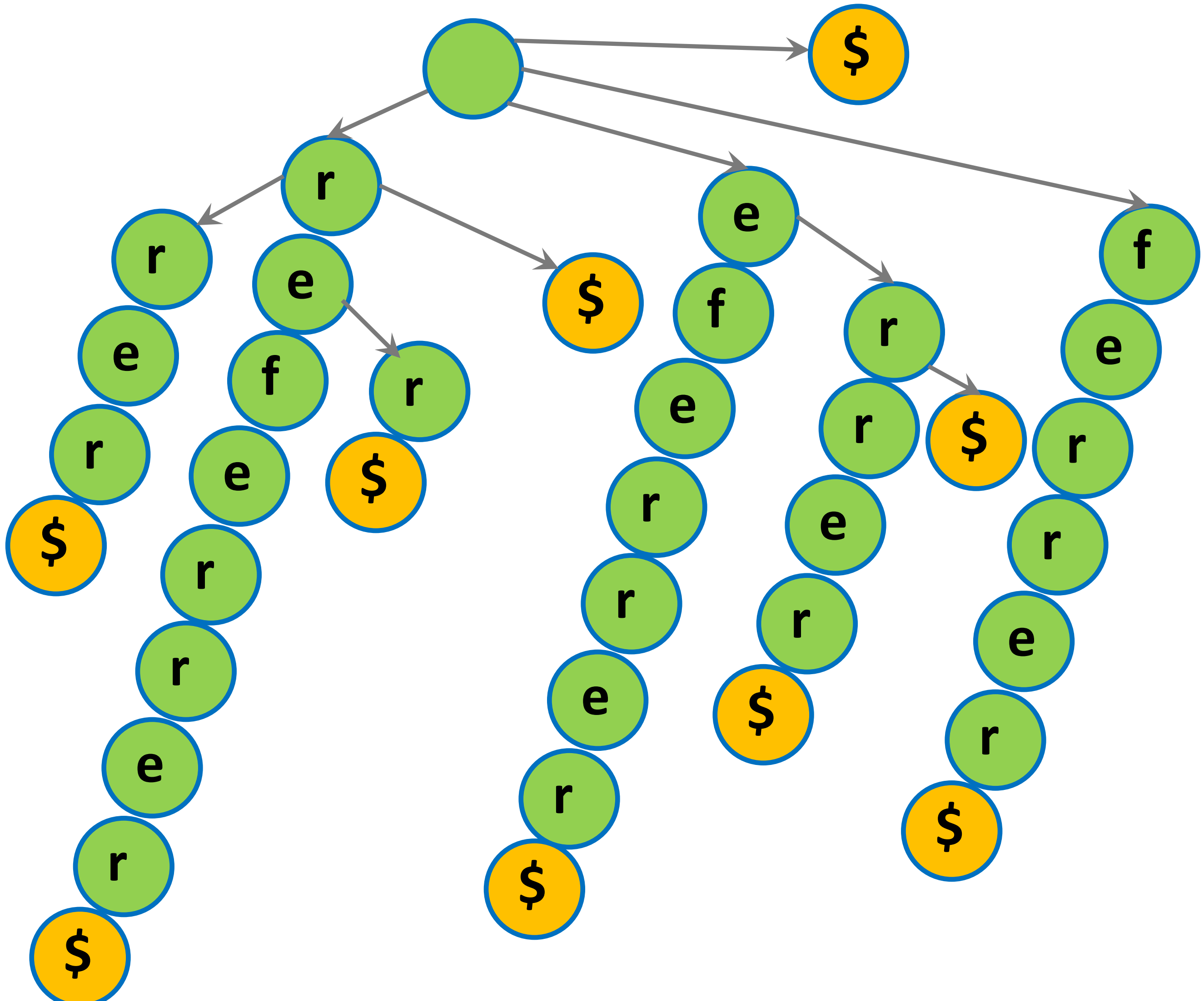
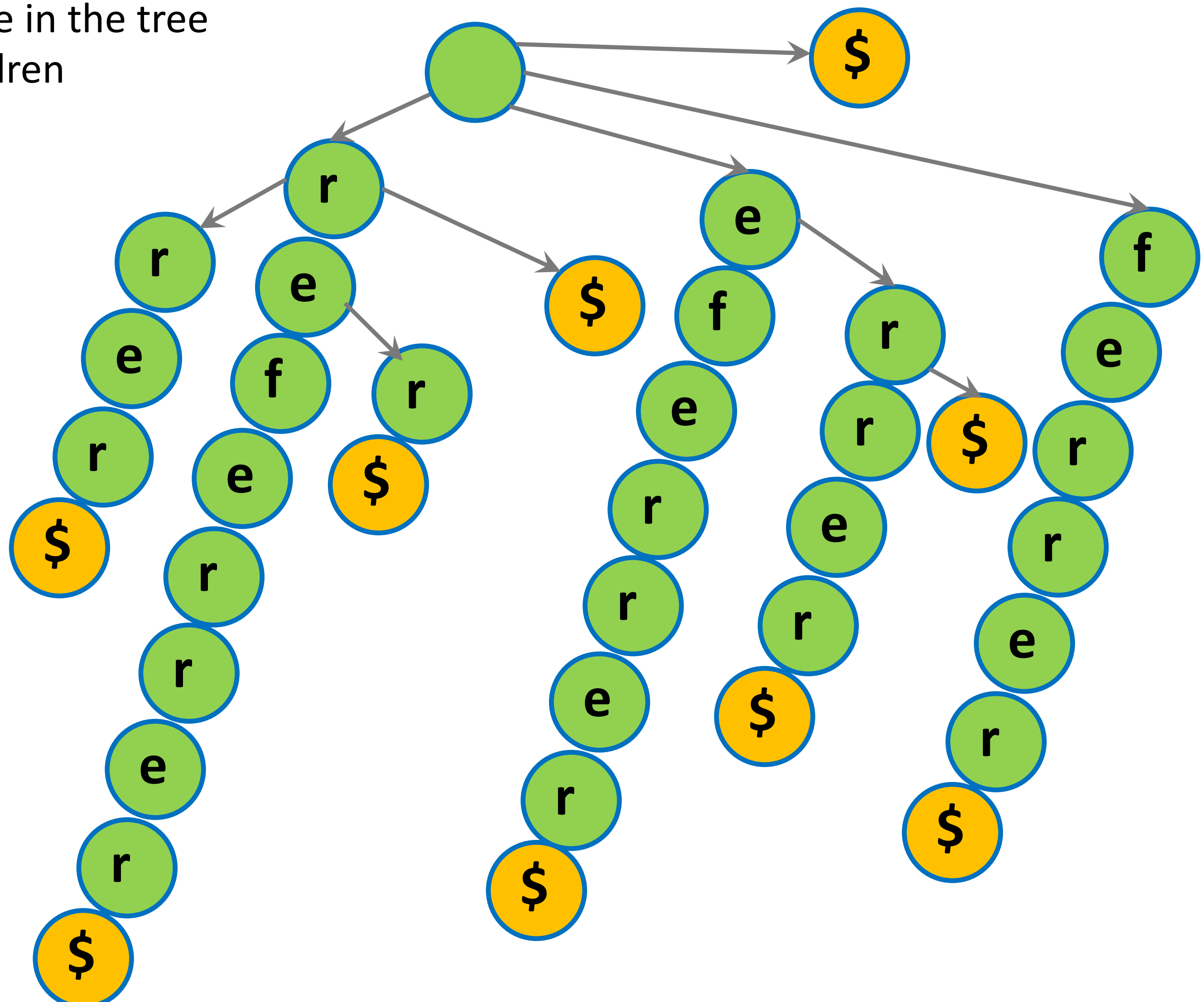
Finding longest repeated substring



Finding longest repeated substring

- Find the deepest node in the tree with at least two children



Space complexity of suffix trie

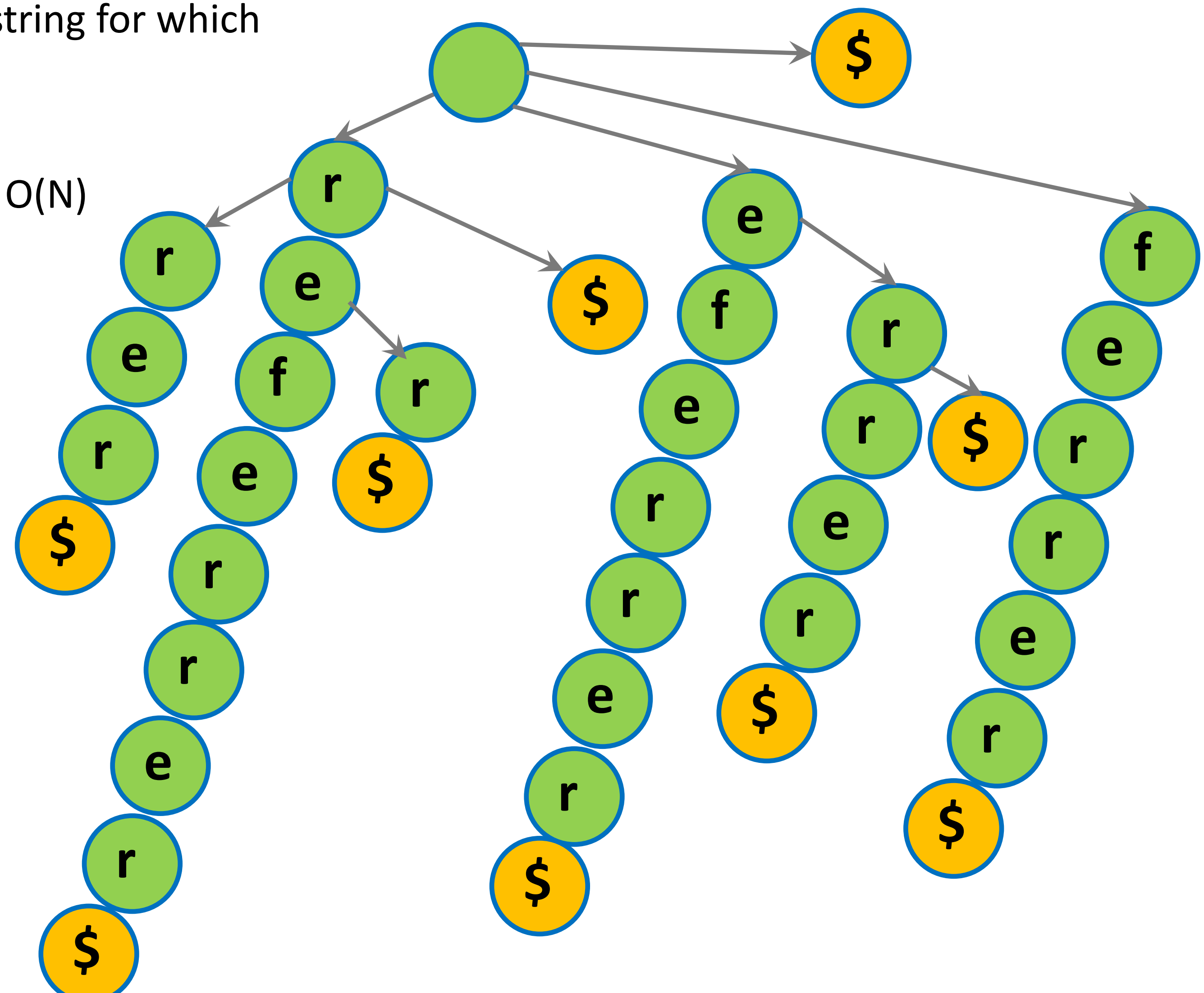
Let N be the size of the string for which suffix trie is constructed

Space complexity?:

- # number of suffixes: O(N)
- Cost for each suffix is

linear to its size

Total space cost: $O(N^2)$



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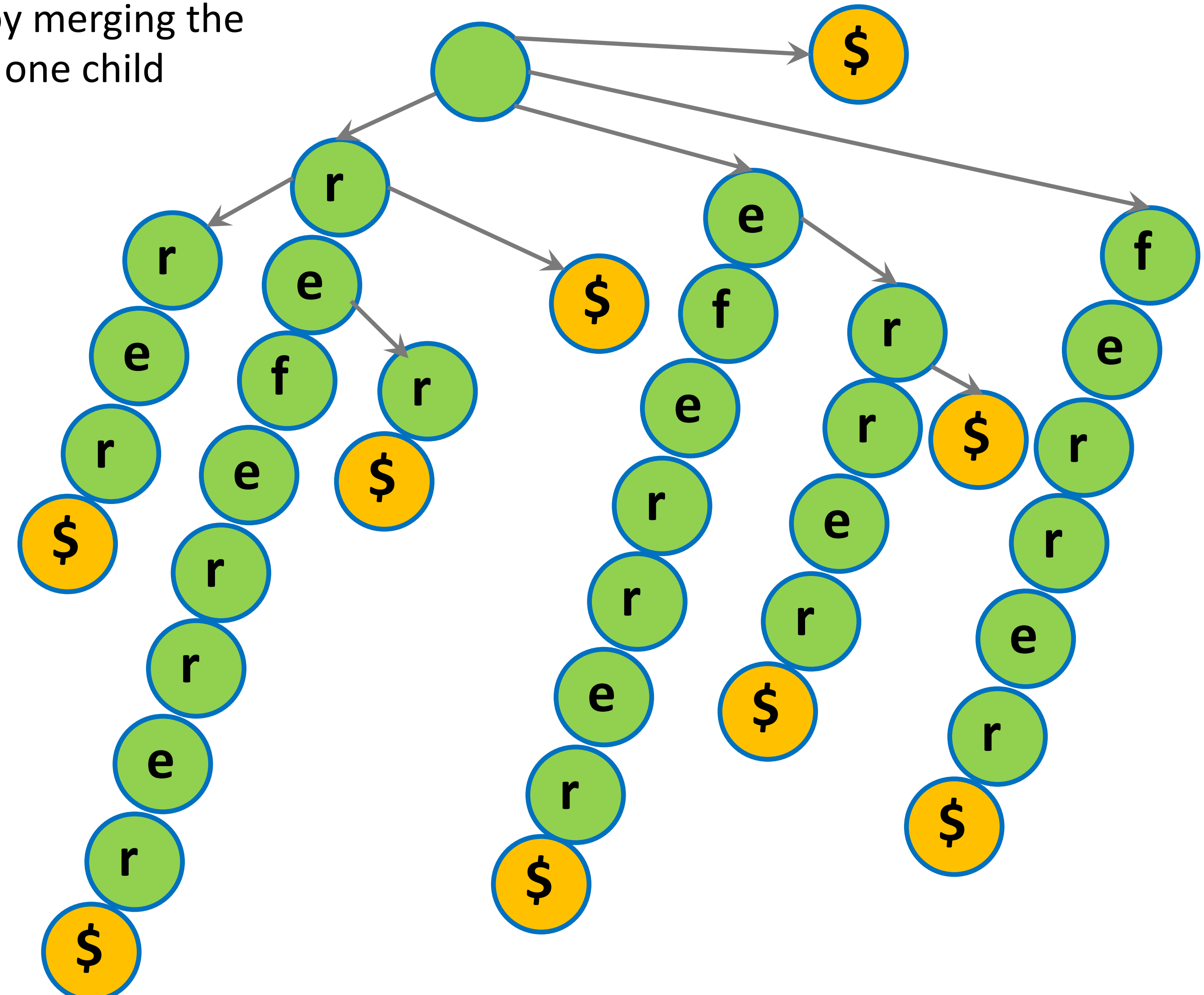
Suffix Tree is a compact Suffix Trie

- Compress branches by merging the nodes that have only one child



Suffix Tree

- Compress branches by merging the nodes that have only one child
- But the total complexity is still the same as the same number of letters are stored

- Can we better this?

Quiz time!
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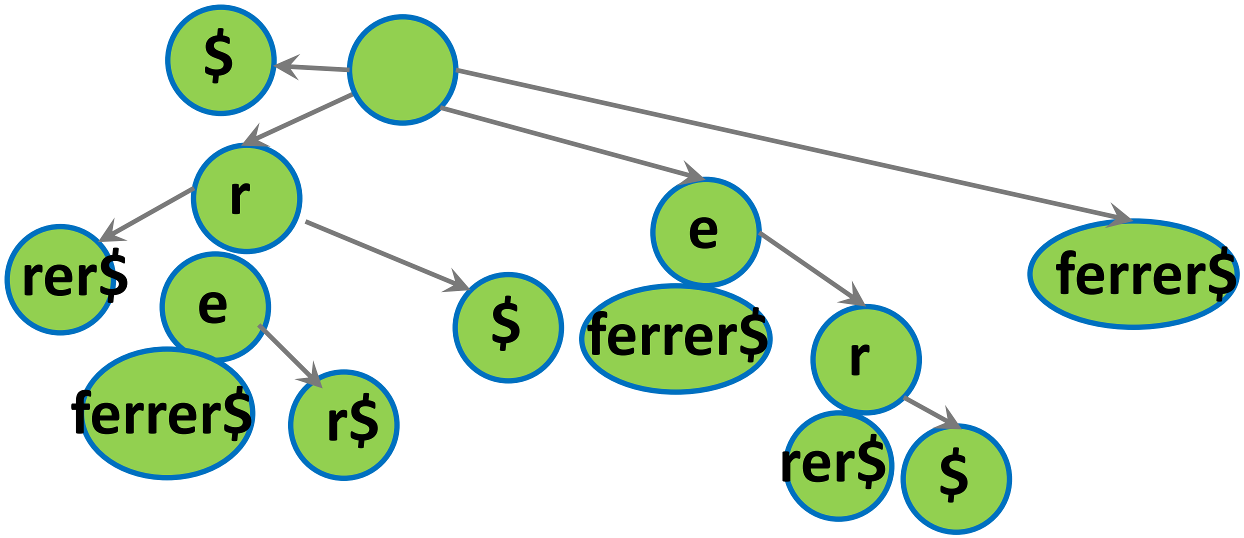


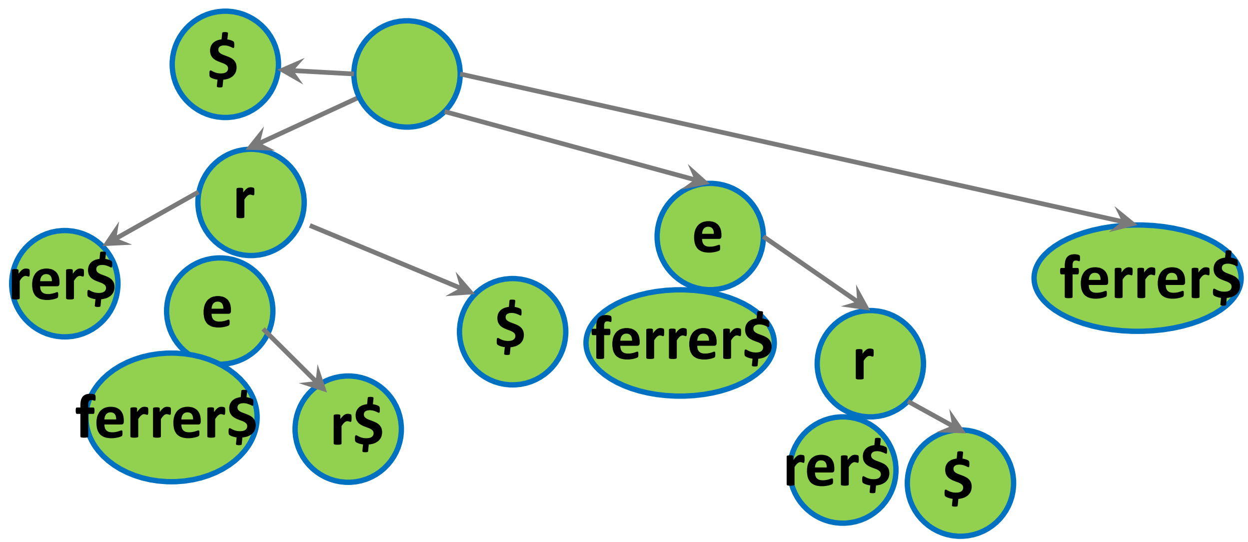
Space complexity of suffix tree

- Compress branches by merging the nodes that have only one child
- But the total complexity is still the same as the same number of letters are stored
- Replace every substring with numbers (x,y) where x is the starting index of the substring and y is its length

e.g., ferrer is represented as (3,6)

rer is represented as (6,3)

r	e	f	e	r	r	e	r	$
1	2	3	4	5	6	7	8	9



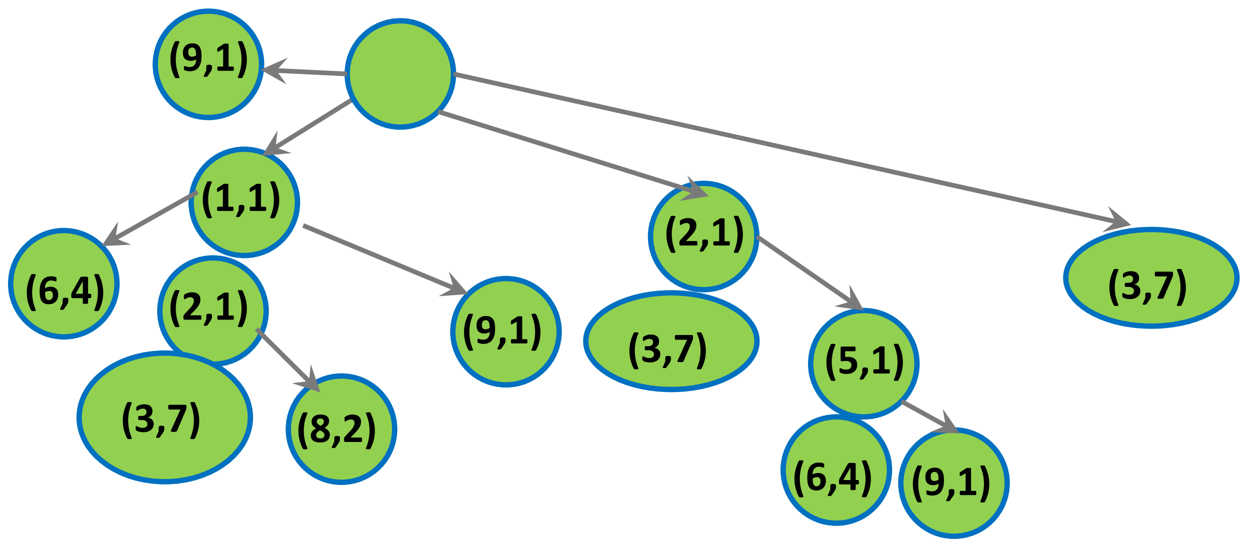
Space complexity of suffix tree

- Compress branches by merging the nodes that have only one child
- But the total complexity is still the same as the same number of letters are stored
- Replace every substring with numbers (x,y) where x is the starting index of the substring and y is its length

e.g., ferrer$ is represented as (3,7)

rer$ is represented as (6,4)

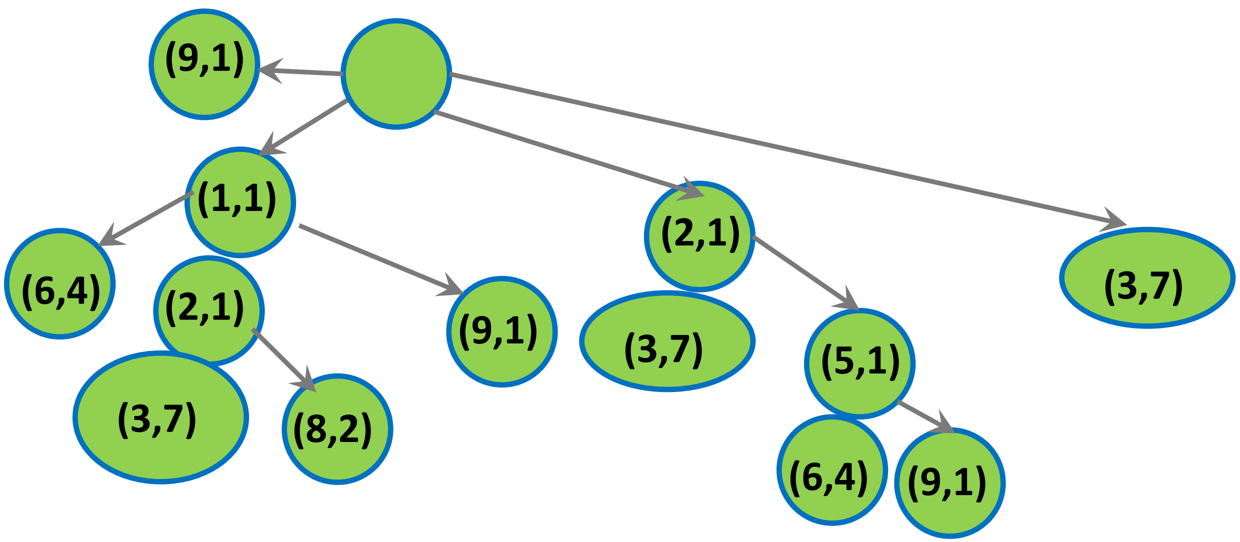
r	e	f	e	r	r	e	r	$
1	2	3	4	5	6	7	8	9



Space complexity of suffix tree

- There are exactly n+1 leaves
- Every (internal) node has at least 2 children
- There are at most n internal nodes
- Suffix trees are O(N) space
- Exercise: Prove this by induction

r	e	f	e	r	r	e	r	$
1	2	3	4	5	6	7	8	9



Time complexity of constructing suffix tree

- The algorithm described earlier inserts O(N) suffixes
- Insertion cost of each suffix is linear in the size of suffix
- Average suffix size is O(N)
- Compressing the trie requires traversing it, $O(N^2)$
- Thus, total time complexity to construct **tree** is $O(N^2)$

It is possible to construct suffix tree in O(N)

- Esko Ukkonen in 1995 gave a beautiful (but involved) algorithm to construct a Suffix Tree in linear time. If you ever get interested in doing this in linear time, consider reading the source:

Ukkonen, E. (1995). "On-line construction of suffix trees". Algorithmica 14 (3): 249260.

Outline

1. Introduction
2. Trie
   A. Construction
   B. Query Processing
3. Suffix Trie
   A. Construction
   B. Query Processing
   C. Suffix Tree
4. Suffix Array
   A. Introduction
   B. Reducing Construction Cost

Suffix Array

- A space-efficient alternative to Suffix Trees
  - Same complexity class O(N) but implementation allows smaller constants
- A sorted array of all suffixes of a string

Sorted Suffixes

String	M	I	S	S	I	S	S	I	P	P	I	$

	Suffix	Sorted
1	M I S S I S S I P P I $	$
2	I S S I S S I P P I $	I $
3	S S I S S I P P I $	I P P I $
4	S I S S I P P I $	I S S I P P I $
5	I S S I P P I $	I S S I S S I P P I $
6	S S I P P I $	M I S S I S S I P P I $
7	S I P P I $	P I $
8	I P P I $	P P I $
9	P P I $	S I P P I $
10	P I $	S I S S I P P I $
11	I $	S S I P P I $
12	$	S S I S S I P P I $

Sort →

Querying on Sorted Suffixes

String	M	I	S	S	I	S	S	I	P	P	I	$

Substring search:

- Is "IPP" in the String?
  - Binary search on sorted suffices
- Let M be the number of characters in substring and N be the size of string.
- Worst-case cost of substring search is?
  - O (M log N)

$

I $

I P P I $

I S S I P P I $

I S S I S S I P P I $

M I S S I S S I P P I $

P I $

P P I $

S I P P I $

S I S S I P P I $

S S I P P I $

S S I S S I P P I $

Querying on Sorted Suffixes

String												
	M	I	S	S	I	S	S	I	P	P	I	$

Longest repeated substring:

- For each consecutive pair in sorted suffices
  - Compute the size of longest common prefix (LCP) among the pair
  - Maintain the one with the maximum size
- Scan the LCP for the maximum
- Complexity:
  - Cost of building suffix array + cost of building LCP array + O(N)

$

0

I $

1

I P P I $

1

I S S I P P I $

4

I S S I S S I P P I $

0

M I S S I S S I P P I $

P I $

P P I $

S I P P I $

S I S S I P P I $

S S I P P I $

3

S S I S S I P P I $

Sorted Suffixes

String	M	I	S	S	I	S	S	I	P	P	I	$

Space complexity of Sorted Suffixes:

- $O(N^2)$

- Can we do better?

Yes! Suffix Array reduces it to O(N) without losing effectiveness

$
I $
I P P I $
I S S I P P I $
I S S I S S I P P I $
M I S S I S S I P P I $
P I $
P P I $
S I P P I $
S I S S I P P I $
S S I P P I $
S S I S S I P P I $

Suffix Array

Suffix ID

index	1	2	3	4	5	6	7	8	9	10	11	12
String	**M**	**I**	**S**	**S**	**I**	**S**	**S**	**I**	**P**	**P**	**I**	**$**

1	M I S S I S S I P P I $
2	I S S I S S I P P I $
3	S S I S S I P P I $
4	S I S S I P P I $
5	I S S I P P I $
6	S S I P P I $
7	S I P P I $
8	I P P I $
9	P P I $
10	P I $
11	I $
12	$

Sort

12	$
11	I $
8	I P P I $
5	I S S I P P I $
2	I S S I S S I P P I $
1	M I S S I S S I P P I $
10	P I $
9	P P I $
7	S I P P I $
4	S I S S I P P I $
6	S S I P P I $
3	S S I S S I P P I $

Suffix Array:
Only stores IDs of suffixes. The sorted suffices are shown just for illustration

Practice

What will be the suffix array of ABAB$?

Remember that $ < A, B, C, ..., Z

Quiz time!
https://flux.qa - YTJMAZ

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   B. Query Processing
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Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Ranks

- Tell us the relative order of strings
- Only with respect to the chars which have been considered so far

What's the naïve complexity of sorting this array?

Quiz time!
https://flux.qa - YTJMAZ

Rank	ID	
-	1	M I S S I S S I P P I $
-	2	I S S I S S I P P I $
-	3	S S I S S I P P I $
-	4	S I S S I P P I $
-	5	I S S I P P I $
-	6	S S I P P I $
-	7	S I P P I $
-	8	I P P I $
-	9	P P I $
-	10	P I $
-	11	I $
-	12	$

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Basic Idea:

- Generate suffixes

Rank	ID	
-	1	M I S S I S S I P P I $
-	2	I S S I S S I P P I $
-	3	S S I S S I P P I $
-	4	S I S S I P P I $
-	5	I S S I P P I $
-	6	S S I P P I $
-	7	S I P P I $
-	8	I P P I $
-	9	P P I $
-	10	P I $
-	11	I $
-	12	$

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Basic Idea:

- Generate suffixes
- Use ascii values of first chars to set up ranks (pertaining to the first character)

Rank	ID	Suffix
77	1	M I S S I S S I P P I $
73	2	I S S I S S I P P I $
83	3	S S I S S I P P I $
83	4	S I S S I P P I $
73	5	I S S I P P I $
83	6	S S I P P I $
83	7	S I P P I $
73	8	I P P I $
80	9	P P I $
80	10	P I $
73	11	I $
36	12	$

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	**M**	**I**	**S**	**S**	**I**	**S**	**S**	**I**	**P**	**P**	**I**	**$**

Basic Idea:

- Generate suffixes
- Use ascii values of first chars to set up ranks
- Sort the strings on first 2 chars, using ranks for first char

Rank	ID	Suffix
36	12	$
73	11	I $
73	8	I P P I $
73	2	I S S I S S I P P I $
73	5	I S S I P P I $
77	1	M I S S I S S I P P I $
80	10	P I $
80	9	P P I $
83	4	S I S S I P P I $
83	7	S I P P I $
83	3	S S I S S I P P I $
83	6	S S I P P I $

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Basic Idea:
- Generate suffixes
- Use ascii values of first chars to set up ranks
- Sort the strings on first 2 chars, using ranks for first char
    - Update ranks
    - Ranks now pertain to relative order of first 2 chars

Rank	ID		
1	12	$	
2	11	I $	
3	8	I P	P I $
4	2	I S	S I S S I P P I $
4	5	I S	S I P P I $
5	1	M I	S S I S S I P P I $
6	10	P I	$
7	9	P P	I $
8	4	S I	S S I P P I $
8	7	S I	P P I $
9	3	S S	I S S I P P I $
9	6	S S	I P P I $

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Basic Idea:

- Generate suffixes
- Use ascii values of first chars to set up ranks
- Sort the strings on first 2 chars, using ranks for first char
  - Update ranks
  - Ranks now pertain to relative order of first 2 chars
- Sort strings on first 4 chars, using ranks for first 2 chars

Rank	ID	Suffix
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	7	S I P P I $
8	4	S I S S I P P I $
9	6	S S I P P I $
9	3	S S I S S I P P I $

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Basic Idea:

- Generate suffixes
- Use ascii values of first chars to set up ranks
- Sort the strings on first 2 chars, using ranks for first char
  - Update ranks
  - Ranks now pertain to relative order of first 2 chars
- Sort strings on first 4 chars, using ranks for first 2 chars
  - Update ranks
  - Ranks now pertain to relative order of first 4 chars

Rank	ID												
1	12	$											
2	11	I	$										
3	8	I	P	P	I	$							
4	2	I	S	S	I	S	S	I	P	P	I	$	
4	5	I	S	S	I	P	P	I	$				
5	1	M	I	S	S	I	S	S	I	P	P	I	$
6	10	P	I	$									
7	9	P	P	I	$								
8	7	S	I	P	P	I	$						
9	4	S	I	S	S	I	P	P	I	$			
10	6	S	S	I	P	P	I	$					
11	3	S	S	I	S	S	I	P	P	I	$		

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	**M**	**I**	**S**	**S**	**I**	**S**	**S**	**I**	**P**	**P**	**I**	**$**

Basic Idea:

- Generate suffixes
- Use ascii values of first chars to set up ranks
- Sort the strings on first 2 chars, using ranks for first char
  - Update ranks
  - Ranks now pertain to relative order of first 2 chars
- Sort strings on first 4 chars, using ranks for first 2 chars
  - Update ranks
  - Ranks now pertain to relative order of first 4 chars

…

Rank	ID	
1	**12**	$
2	**11**	I $
3	**8**	I P P I $
4	**5**	I S S I P P I $
4	**2**	I S S I S S I P P I $
5	**1**	M I S S I S S I P P I $
6	**10**	P I $
7	**9**	P P I $
8	**7**	S I P P I $
9	**4**	S I S S I P P I $
10	**6**	S S I P P I $
11	**3**	S S I S S I P P I $

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Basic Idea:

- Generate suffixes
- Use ascii values of first chars to set up ranks
- Sort the strings on first 2 chars, using ranks for first char
    - Update ranks
    - Ranks now pertain to relative order of first 2 chars
- Sort strings on first 4 chars, using ranks for first 2 chars
    - Update ranks
    - Ranks now pertain to relative order of first 4 chars

…

Rank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	5	I S S I P P I $
5	2	I S S I S S I P P I $
6	1	M I S S I S S I P P I $
7	10	P I $
8	9	P P I $
9	7	S I P P I $
10	4	S I S S I P P I $
11	6	S S I P P I $
12	3	S S I S S I P P I $

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Basic Idea:

- Generate suffixes
- Use ascii values of first chars to set up ranks
- Sort the strings on first 2 chars, using ranks for first char
    - Update ranks
    - Ranks now pertain to relative order of first 2 chars
- Sort strings on first 4 chars, using ranks for first 2 chars
    - Update ranks
    - Ranks now pertain to relative order of first 4 chars

…

Rank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	5	I S S I P P I $
5	2	I S S I S S I P P I $
6	1	M I S S I S S I P P I $
7	10	P I $
8	9	P P I $
9	7	S I P P I $
10	4	S I S S I P P I $
11	6	S S I P P I $
12	3	S S I S S I P P I $

Constructing Suffix Array: Prefix Doubling

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

- What do we use the ranks for?
- Why do we do all these intermediate sorts?
- Naïve sorting would take N*N time just for the final sort (with radix sort)
- How many sorts do you need?

Quiz time!
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- To reach O(N log N) – or at least O(N log^2 N), the next better "normal" classes under O(N^2), we need O(1) comparisons.

Rank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	5	I S S I P P I $
5	2	I S S I S S I P P I $
6	1	M I S S I S S I P P I $
7	10	P I $
8	9	P P I $
9	7	S I P P I $
10	4	S I S S I P P I $
11	6	S S I P P I $
12	3	S S I S S I P P I $

- How do we compare suffixes in O(1) though?

O(1) Comparison

	1	2	3	4	5	6	7	8	9	10	11	12
String	**M**	**I**	**S**	**S**	**I**	**S**	**S**	**I**	**P**	**P**	**I**	**$**

Comparing suffixes in O(1):

- Suppose already sorted on first k characters (2 in this example)
- We have ranks for first 2 characters
- Now sorting on 2k characters (4 in this example)

Observation 1:

- If current ranks are different, suffix with smaller rank is smaller (because its first k characters are smaller)
  - E.g., PPI$ < SSIP
  - Note comparison cost is O(1)

Rank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	4	S I S S I P P I $
8	7	S I P P I $
9	3	S S I S S I P P I $
9	6	S S I P P I $

O(1) Comparison

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Observation 2:

If current ranks are the same

- First k characters must be the same
- The tie is to be broken on the next k characters, e.g.,

Rank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	4	S I S S I P P I $
8	7	S I P P I $
9	3	S S I S S I P P I $
9	6	S S I P P I $

O(1) Comparison

	1	2	3	4	5	6	7	8	9	10	11	12
String	**M**	**I**	**S**	**S**	**I**	**S**	**S**	**I**	**P**	**P**	**I**	**$**

Observation 2:

If current ranks are the same

- First k characters must be the same
- The tie is to be broken on the next k characters, e.g.,
  - We need to compare “SSIPPI$” and “PPI$” on the first 2 characters

Rank	ID		
1	**12**	$	
2	**11**	I $	
3	**8**	I P	P I $
4	**2**	I S	S I S S I P P I $
4	**5**	I S	S I P P I $
5	**1**	M I	S S I S S I P P I $
6	**10**	P I	$
7	**9**	P P	I $
8	**4**	S I	S S I P P I $
8	**7**	S I	P P I $
9	**3**	S S	I S S I P P I $
9	**6**	S S	I P P I $

O(1) Comparison

	1	2	3	4	5	6	7	8	9	10	11	12
String	**M**	**I**	**S**	**S**	**I**	**S**	**S**	**I**	**P**	**P**	**I**	**$**

Observation 2:

If current ranks are the same

- First k characters must be the same
- The tie is to be broken on the next k characters, e.g.,
  - We need to compare “SSIPPI$” and “PPI$” on the first 2 characters
  - **SSIPPI$ and PPI$ are suffixes and are already ranked on first 2 characters**
    - E.g., PPI$ < SSIPPI$ because its rank is smaller
    - Therefore, suffix id7< suffix id4

Rank	ID		
1	**12**	$	
2	**11**	I $	
3	**8**	I P	P I $
4	**2**	I S	S I S S I P P I $
4	**5**	I S	S I P P I $
5	**1**	M I	S S I S S I P P I $
6	**10**	P I	$
7	**9**	P P	I $
8	**4**	S I	S S I P P I $
8	**7**	S I	P P I $
9	**3**	S S	I S S I P P I $
9	**6**	S S	I P P I $

Practice

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

- **BUT WAIT!**
- **How did we do that quickly? Surely looking up the "second half" suffixes is O(N)?**

Rank	ID	Suffix
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	7	S I P P I $
9	4	S I S S I P P I $
10	6	S S I P P I $
11	3	S S I S S I P P I $

Practice

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Suppose we are comparing suffix with ID 2 and 5:

- We need to compare SSIPPI$ and PPI$

Rank	ID	Suffix
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	7	S I P P I $
9	4	S I S S I P P I $
10	6	S S I P P I $
11	3	S S I S S I P P I $

Practice

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Suppose we are comparing suffix with ID 2 and 5:

- We need to compare SSIPPI$ and PPI$
- How do we find their ranks quickly?

Rank	ID	Suffix
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	7	S I P P I $
9	4	S I S S I P P I $
10	6	S S I P P I $
11	3	S S I S S I P P I $

Practice

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Suppose we are comparing suffix with ID 2 and 5:

- We need to compare SSIPPI$ and PPI$
- How do we find their ranks quickly?
- We want the ranks of the suffixes with id 2+k and id 5+k
- I.e. suffixes 6 and 9
- This means we can calculate the IDs of the suffixes we want in O(1)
- Now we need to get from IDs to ranks in O(1)

Rank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	7	S I P P I $
9	4	S I S S I P P I $
10	6	S S I P P I $
11	3	S S I S S I P P I $

Practice

	1	2	3	4	5	6	7	8	9	10	11	12
String	M	I	S	S	I	S	S	I	P	P	I	$

Suppose we are comparing suffix with ID 2 and 5:

- We need to compare SSIPPI$ and PPI$
- How do we find their ranks quickly?

- We want the ranks of the suffixes with id 2+k and id 5+k
- I.e. suffixes 6 and 9

- To have O(1) access to their ranks, we need an array indexed by ID which contains the ranks!

- In other words, the way the ranks are arranged on this slide is useless

Rank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	7	S I P P I $
9	4	S I S S I P P I $
10	6	S S I P P I $
11	3	S S I S S I P P I $

O(1) Comparison

Index/ID	1	2	3	4	5	6	7	8	9	10	11	12
Rank	5	4	11	9	4	10	8	3	7	6	2	1
String	**M**	**I**	**S**	**S**	**I**	**S**	**S**	**I**	**P**	**P**	**I**	**$**

Note: The greyed out oldRank array has been left for reference, but does not exist in implementation

- If we want the rank of ID i, look at Rank[i]
- Going back to our example…

oldRank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	7	S I P P I $
9	4	S I S S I P P I $
10	6	S S I P P I $
11	3	S S I S S I P P I $

O(1) Comparison

Index/ID	1	2	3	4	5	6	7	8	9	10	11	12
Rank	5	4	11	9	4	10	8	3	7	6	2	1
String	**M**	**I**	**S**	**S**	**I**	**S**	**S**	**I**	**P**	**P**	**I**	**$**

Note: The greyed out oldRank array has been left for reference, but does not exist in implementation

- If we want the rank of ID i, look at Rank[i]
- Going back to our example…

- We wanted to find the second parts of suffixes 2 and 5

oldRank	**ID**	
1	**12**	$
2	**11**	I $
3	**8**	I P P I $
4	**2**	I S S I S S I P P I $
4	**5**	I S S I P P I $
5	**1**	M I S S I S S I P P I $
6	**10**	P I $
7	**9**	P P I $
8	**7**	S I P P I $
9	**4**	S I S S I P P I $
10	**6**	S S I P P I $
11	**3**	S S I S S I P P I $

O(1) Comparison

Index/ID	1	2	3	4	5	6	7	8	9	10	11	12
Rank	5	4	11	9	4	10	8	3	7	6	2	1
String	M	I	S	S	I	S	S	I	P	P	I	$

oldRank	ID	
1	12	$
2	11	I $
3	8	I P P I $
4	2	I S S I S S I P P I $
4	5	I S S I P P I $
5	1	M I S S I S S I P P I $
6	10	P I $
7	9	P P I $
8	7	S I P P I $
9	4	S I S S I P P I $
10	6	S S I P P I $
11	3	S S I S S I P P I $

Note: The greyed out oldRank array has been left for reference, but does not exist in implementation

- If we want the rank of ID i, look at Rank[i]
- Going back to our example…

- We wanted to find the second parts of suffixes 2 and 5
- I.e. ID 6 and 9

- Rank[9] < rank[6]
- So ID 5 should come before ID 2 in the suffix array

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	-
2	I	-
3	S	-
4	S	-
5	I	-
6	S	-
7	S	-
8	I	-
9	P	-
10	P	-
11	I	-
12	$	-

REMEMBER:
This rank array does not exist

It contains the same values as the other rank array, its just to make the algorithm easier to follow

Rank	SA	
-	1	M I S S I S S I P P I $
-	2	I S S I S S I P P I $
-	3	S S I S S I P P I $
-	4	S I S S I P P I $
-	5	I S S I P P I $
-	6	S S I P P I $
-	7	S I P P I $
-	8	I P P I $
-	9	P P I $
-	10	P I $
-	11	I $
-	12	$

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	-
2	I	-
3	S	-
4	S	-
5	I	-
6	S	-
7	S	-
8	I	-
9	P	-
10	P	-
11	I	-
12	$	-

Rank the first characters of each suffix

Rank	SA	
-	1	M I S S I S S I P P I $
-	2	I S S I S S I P P I $
-	3	S S I S S I P P I $
-	4	S I S S I P P I $
-	5	I S S I P P I $
-	6	S S I P P I $
-	7	S I P P I $
-	8	I P P I $
-	9	P P I $
-	10	P I $
-	11	I $
-	12	$

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Rank the first characters of each suffix using ord()

Rank	SA	
77	1	M I S S I S S I P P I $
73	2	I S S I S S I P P I $
83	3	S S I S S I P P I $
83	4	S I S S I P P I $
73	5	I S S I P P I $
83	6	S S I P P I $
83	7	S I P P I $
73	8	I P P I $
80	9	P P I $
80	10	P I $
73	11	I $
36	12	$

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Sort SA

Since we have ranks for the first character, our sort will sort the suffixes based on their first **two characters** (using the doubling trick)

Rank	SA	
77	1	M I S S I S S I P P I $
73	2	I S S I S S I P P I $
83	3	S S I S S I P P I $
83	4	S I S S I P P I $
73	5	I S S I P P I $
83	6	S S I P P I $
83	7	S I P P I $
73	8	I P P I $
80	9	P P I $
80	10	P I $
73	11	I $
36	12	$

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Sort SA by ranks

Note that this does not change the rank array, since IDs have kept the same ranks

We just rearranged the SA

Ranks still relate only to first char

Rank	SA	
36	12	$
73	11	I $
73	8	I P P I $
73	2	I S S I S S I P P I $
73	5	I S S I P P I $
77	1	M I S S I S S I P P I $
80	10	P I $
80	9	P P I $
83	4	S I S S I P P I $
83	7	S I P P I $
83	3	S S I S S I P P I $
83	6	S S I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Now we update the ranks to relate to first 2 chars

Rank	SA												
36	12	$											
73	11	I	$										
73	8	I	P	P	I	$							
73	2	I	S	S	I	S	S	I	P	P	I	$	
73	5	I	S	S	I	P	P	I	$				
77	1	M	I	S	S	I	S	S	I	P	P	I	$
80	10	P	I	$									
80	9	P	P	I	$								
83	4	S	I	S	S	I	P	P	I	$			
83	7	S	I	P	P	I	$						
83	3	S	S	I	S	S	I	P	P	I	$		
83	6	S	S	I	P	P	I	$					

Lets do an example (by hand in lecture)

Make an array, "Temp" to hold the new ranks

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	1
9	P	80	1
10	P	80	1
11	I	73	1
12	$	36	1

Rank	SA	
36	12	$
73	11	I $
73	8	I P P I $
73	2	I S S I S S I P P I $
73	5	I S S I P P I $
77	1	M I S S I S S I P P I $
80	10	P I $
80	9	P P I $
83	4	S I S S I P P I $
83	7	S I P P I $
83	3	S S I S S I P P I $
83	6	S S I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	1
9	P	80	1
10	P	80	1
11	I	73	1
12	$	36	1

For each pair of adjacent suffixes, compare them

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	1
9	P	80	1
10	P	80	1
11	I	73	1
12	$	36	1

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

If they have different ranks already, then the second suffix is certainly larger

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	1
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

Set Temp[11] to Temp[12]+1

Rank	SA	Suffix
36	12	$
73	11	I $
73	8	I P P I $
73	2	I S S I S S I P P I $
73	5	I S S I P P I $
77	1	M I S S I S S I P P I $
80	10	P I $
80	9	P P I $
83	4	S I S S I P P I $
83	7	S I P P I $
83	3	S S I S S I P P I $
83	6	S S I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	1
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

If they have the same rank

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	1
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

We need to use the O(1) trick

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Temp
1
1
1
1
1
1
1
1
1
1
2
1

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

First characters have the same rank, so compare the suffixes which start with next characters

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	1
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

$ vs PPI$ (ID 11+1 and 8+1)

Rank	SA												
36	12	$											
73	11	I	$										
73	8	I	P	P	I	$							
73	2	I	S	S	I	S	S	I	P	P	I	$	
73	5	I	S	S	I	P	P	I	$				
77	1	M	I	S	S	I	S	S	I	P	P	I	$
80	10	P	I	$									
80	9	P	P	I	$								
83	4	S	I	S	S	I	P	P	I	$			
83	7	S	I	P	P	I	$						
83	3	S	S	I	S	S	I	P	P	I	$		
83	6	S	S	I	P	P	I	$					

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	1
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

12 has lower rank than 9, so 11 should have lower rank than 8

Lets do an example (by hand in lecture)

Set Temp[8] = Temp[11] + 1

ID	String	Rank	Temp
1	M	77	1
2	I	73	1
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	3
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

Rank	SA												
36	12	$											
73	11	I	$										
73	8	I	P	P	I	$							
73	2	I	S	S	I	S	S	I	P	P	I	$	
73	5	I	S	S	I	P	P	I	$				
77	1	M	I	S	S	I	S	S	I	P	P	I	$
80	10	P	I	$									
80	9	P	P	I	$								
83	4	S	I	S	S	I	P	P	I	$			
83	7	S	I	P	P	I	$						
83	3	S	S	I	S	S	I	P	P	I	$		
83	6	S	S	I	P	P	I	$					

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	4
3	S	83	1
4	S	83	1
5	I	73	1
6	S	83	1
7	S	83	1
8	I	73	3
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

Continue in this way

Rank	SA												
36	12	$											
73	11	I	$										
73	8	I	P	P	I	$							
73	2	I	S	S	I	S	S	I	P	P	I	$	
73	5	I	S	S	I	P	P	I	$				
77	1	M	I	S	S	I	S	S	I	P	P	I	$
80	10	P	I	$									
80	9	P	P	I	$								
83	4	S	I	S	S	I	P	P	I	$			
83	7	S	I	P	P	I	$						
83	3	S	S	I	S	S	I	P	P	I	$		
83	6	S	S	I	P	P	I	$					

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	1
2	I	73	4
3	S	83	1
4	S	83	1
5	I	73	4
6	S	83	1
7	S	83	1
8	I	73	3
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

Continue in this way

Rank	SA												
36	12	$											
73	11	I	$										
73	8	I	P	P	I	$							
73	2	I	S	S	I	S	S	I	P	P	I	$	
73	5	I	S	S	I	P	P	I	$				
77	1	M	I	S	S	I	S	S	I	P	P	I	$
80	10	P	I	$									
80	9	P	P	I	$								
83	4	S	I	S	S	I	P	P	I	$			
83	7	S	I	P	P	I	$						
83	3	S	S	I	S	S	I	P	P	I	$		
83	6	S	S	I	P	P	I	$					

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	5
2	I	73	4
3	S	83	1
4	S	83	1
5	I	73	4
6	S	83	1
7	S	83	1
8	I	73	3
9	P	80	1
10	P	80	1
11	I	73	2
12	$	36	1

Continue in this way

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	5
2	I	73	4
3	S	83	1
4	S	83	1
5	I	73	4
6	S	83	1
7	S	83	1
8	I	73	3
9	P	80	1
10	P	80	6
11	I	73	2
12	$	36	1

Continue in this way

Rank	SA	Suffix
36	12	$
73	11	I $
73	8	I P P I $
73	2	I S S I S S I P P I $
73	5	I S S I P P I $
77	1	M I S S I S S I P P I $
80	10	P I $
80	9	P P I $
83	4	S I S S I P P I $
83	7	S I P P I $
83	3	S S I S S I P P I $
83	6	S S I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Temp
5
4
1
1
4
1
1
3
7
6
2
1

Continue in this way

Rank	SA												
36	12	$											
73	11	I	$										
73	8	I	P	P	I	$							
73	2	I	S	S	I	S	S	I	P	P	I	$	
73	5	I	S	S	I	P	P	I	$				
77	1	M	I	S	S	I	S	S	I	P	P	I	$
80	10	P	I	$									
80	9	P	P	I	$								
83	4	S	I	S	S	I	P	P	I	$			
83	7	S	I	P	P	I	$						
83	3	S	S	I	S	S	I	P	P	I	$		
83	6	S	S	I	P	P	I	$					

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Temp
5
4
1
8
4
1
1
3
7
6
2
1

Continue in this way

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Temp
5
4
1
8
4
1
8
3
7
6
2
1

Continue in this way

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	77
2	I	73
3	S	83
4	S	83
5	I	73
6	S	83
7	S	83
8	I	73
9	P	80
10	P	80
11	I	73
12	$	36

Temp
5
4
9
8
4
1
8
3
7
6
2
1

Continue in this way

Rank	SA	
36	12	$
73	11	I $
73	8	I P P I $
73	2	I S S I S S I P P I $
73	5	I S S I P P I $
77	1	M I S S I S S I P P I $
80	10	P I $
80	9	P P I $
83	4	S I S S I P P I $
83	7	S I P P I $
83	3	S S I S S I P P I $
83	6	S S I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	5
2	I	73	4
3	S	83	9
4	S	83	8
5	I	73	4
6	S	83	9
7	S	83	8
8	I	73	3
9	P	80	7
10	P	80	6
11	I	73	2
12	$	36	1

Continue in this way

Rank	SA		
36	12	$	
73	11	I $	
73	8	I P	P I $
73	2	I S	S I S S I P P I $
73	5	I S	S I P P I $
77	1	M I	S S I S S I P P I $
80	10	P I	$
80	9	P P	I $
83	4	S I	S S I P P I $
83	7	S I	P P I $
83	3	S S	I S S I P P I $
83	6	S S	I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	77	5
2	I	73	4
3	S	83	9
4	S	83	8
5	I	73	4
6	S	83	9
7	S	83	8
8	I	73	3
9	P	80	7
10	P	80	6
11	I	73	2
12	$	36	1

Rank	SA	
36	12	$
73	11	I $
73	8	I P P I $
73	2	I S S I S S I P P I $
73	5	I S S I P P I $
77	1	M I S S I S S I P P I $
80	10	P I $
80	9	P P I $
83	4	S I S S I P P I $
83	7	S I P P I $
83	3	S S I S S I P P I $
83	6	S S I P P I $

This is our new Rank array, so overwrite the old one

Lets do an example (by hand in lecture)

ID	String	Rank	Temp
1	M	5	5
2	I	4	4
3	S	9	9
4	S	8	8
5	I	4	4
6	S	9	9
7	S	8	8
8	I	3	3
9	P	7	7
10	P	6	6
11	I	2	2
12	$	1	1

Rank	SA		
1	12	$	
2	11	I $	
3	8	I P	P I $
4	2	I S	S I S S I P P I $
4	5	I S	S I P P I $
5	1	M I	S S I S S I P P I $
6	10	P I	$
7	9	P P	I $
8	4	S I	S S I P P I $
8	7	S I	P P I $
9	3	S S	I S S I P P I $
9	6	S S	I P P I $

This is our new Rank array, so overwrite the old one

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	5
2	I	4
3	S	9
4	S	8
5	I	4
6	S	9
7	S	8
8	I	3
9	P	7
10	P	6
11	I	2
12	$	1

Rank	SA		
1	12	$	
2	11	I $	
3	8	I P	P I $
4	2	I S	S I S S I P P I $
4	5	I S	S I P P I $
5	1	M I	S S I S S I P P I $
6	10	P I	$
7	9	P P	I $
8	4	S I	S S I P P I $
8	7	S I	P P I $
9	3	S S	I S S I P P I $
9	6	S S	I P P I $

Now we have ranks for 2 characters, we can sort on 4 characters

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	5
2	I	4
3	S	11
4	S	9
5	I	4
6	S	10
7	S	8
8	I	3
9	P	7
10	P	6
11	I	2
12	$	1

Rank	SA											
1	12	$										
2	11	I	$									
3	8	I	P	P	I	$						
4	2	I	S	S	I	S	S	I	P	P	I	$
4	5	I	S	S	I	P	P	I	$			
5	1	M	I	S	S	I	S	S	I	P	P	I	$
6	10	P	I	$								
7	9	P	P	I	$							
8	7	S	I	P	P	I	$					
9	4	S	I	S	S	I	P	P	I	$		
10	6	S	S	I	P	P	I	$				
11	3	S	S	I	S	S	I	P	P	I	$	

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	6
2	I	5
3	S	12
4	S	10
5	I	4
6	S	11
7	S	9
8	I	3
9	P	8
10	P	7
11	I	2
12	$	1

Rank	SA	
1	12	$
2	11	I $
3	8	I P P I $
4	5	I S S I P P I $
5	2	I S S I S S I P P I $
6	1	M I S S I S S I P P I $
7	10	P I $
8	9	P P I $
9	7	S I P P I $
10	4	S I S S I P P I $
11	6	S S I P P I $
12	3	S S I S S I P P I $

Lets do an example (by hand in lecture)

ID	String	Rank
1	M	6
2	I	5
3	S	12
4	S	10
5	I	4
6	S	11
7	S	9
8	I	3
9	P	8
10	P	7
11	I	2
12	$	1

Rank	SA	
1	12	$
2	11	I $
3	8	I P P I $
4	5	I S S I P P I $
5	2	I S S I S S I P P I $
6	1	M I S S I S S I P P I $
7	10	P I $
8	9	P P I $
9	7	S I P P I $
10	4	S I S S I P P I $
11	6	S S I P P I $
12	3	S S I S S I P P I $

Suffix arrays

- Prefix doubling allows construction in
  - O(N log^2(N)) time if we use a O(N log N) sorting algorithm
  - O(N log N) is we use a radix sort (possible since ranks are in [1…N]!)
- The O(1) comparison idea is very powerful

- Linear time construction algorithms exist for suffix array
- In order to match the speed of a suffix tree, LCP array is necessary

- Suffix arrays are more compact than suffix trees (no links)
- Suffix arrays are localised in memory
  - Suffix arrays allow better caching behaviour

- Final question: How do we know that construction of a suffix array in linear time must be possible?

Quiz time!
https://flux.qa – YTJMAZ

Summary

Take home message

- Tries, Suffix trees and Suffix array provide efficient text search and pattern matching (typically linear in number of characters in string)
- Linear time construction for both is possible, but beyond the scope of this unit

Things to do (this list is not exhaustive)

- Implement Trie, Suffix trees and Suffix array and run various pattern matching queries
- Study visualisations at https://visualgo.net/

Coming Up Next

- Burrows-Wheeler Transform - A beautiful space-time efficient pattern matching algorithm on text