Heart construction for twin torsion pairs in additive categories

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Contents

1 Introduction

Given a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{C}$, it was proved in [**nakaoka2011general**] [redacted] that it is possible to use it to construct a subcategory $\mathcal{W} \subseteq \mathcal{C}$ such that in the quotient category $\dfrac{\mathcal{C}}{\mathcal{W}} = \underline{\mathcal{C}}$ a heart $\mathcal{H}$ is defined in such a way that it is an abelian category and there is a homological functor $\mathcal{C} \to \mathcal{H}$.

Our goal is to provide a set of axioms for a (nice) additive category $\mathcal{A}$ and a couple of torsion pairs in it, in such a way that they will guarantee the existence of an abelian heart in $\mathcal{A}$. In a sense, we want to axiomatize $\underline{\mathcal{C}}$ and the pairs which are referred in Nakaoka's work as $(\underline{\mathcal{C}}^-, \underline{\mathcal{V}})$ and $(\underline{\mathcal{U}}, \underline{\mathcal{C}}^+)$.

The Nakaoka setting

Let us briefly recall Nakaoka's setting. Assume that $\mathcal{C}$ is a triangulated category, and $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in $\mathcal{C}$, i.e.

1. $\mathcal{C}(\mathcal{U}, \mathcal{V}[1]) = 0$
2. $\mathcal{C} = \mathcal{U} * \mathcal{V}[1]$, where $X \in \mathcal{M} * \mathcal{N}$ if and only if there is a distinguished triangle

$$M \to X \to N \to M[1]$$

   with $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

Then, we put $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ and define $\underline{\mathcal{C}} = \mathcal{C}/\mathcal{W}$, and similarly for $\underline{\mathcal{U}}$, $\underline{\mathcal{V}}$, etc.

We define the following full subcategories of $\mathcal{C}$:

- $\mathcal{C}^+ = \mathcal{W} * \mathcal{V}[1]$
- $\mathcal{C}^- = \mathcal{U}[-1] * \mathcal{W}$

together with their respective quotients $\underline{\mathcal{C}}^+$ and $\underline{\mathcal{C}}^-$. Then, we have the following lemma.

Lemma 1.1. *Let $X \in \mathcal{C}$, TFAE:*

1. $\underline{X} \in \underline{\mathcal{C}}^+$,
2. *There is a monomorphism $\underline{X} \to \underline{V[1]}$ in $\underline{\mathcal{C}}$, for some $V \in \mathcal{V}$.*

The dual also holds:

Lemma 1.2. *Let $X \in \mathcal{C}$, TFAE:*

1. $\underline{X} \in \underline{\mathcal{C}}^-$,
2. *There is an epimorphism $\underline{U[-1]} \to \underline{X}$ in $\underline{\mathcal{C}}$, for some $U \in \mathcal{U}$.*

Corollary 1.3. *If $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in a triangulated category $\mathcal{C}$, then:*

1. ${}^{\perp}\underline{\mathcal{V}} = \underline{\mathcal{C}}^-$
2. $\underline{\mathcal{U}}^{\perp} = \underline{\mathcal{C}}^+$

Lemma 1.4. *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{C}}^+$ be the left adjoint of the inclusion functor $j : \underline{\mathcal{C}}^+ \hookrightarrow \underline{\mathcal{C}}$. If $\lambda : 1_{\underline{\mathcal{C}}} \to j \circ F$ is the unit of the adjunction, then there is a pseudokernel-pseudocokernel sequence*

$$U_C \xrightarrow{u} C \xrightarrow{\lambda_C} (j \circ F)(C) \xrightarrow{+}$$

in $\underline{\mathcal{C}}$ such that $U_C \in \mathcal{U}$.

With dual:

Lemma 1.5. *Let $G : \underline{\mathcal{C}} \to \underline{\mathcal{C}}^-$ be the left adjoint of the inclusion functor $i : \underline{\mathcal{C}}^- \hookrightarrow \underline{\mathcal{C}}$. If $\varepsilon : i \circ G \to 1_{\underline{\mathcal{C}}}$ is the co-unit of the adjunction, then there is a pseudokernel-pseudocokernel sequence*

$$(i \circ G)(C) \xrightarrow{\varepsilon_C} C \xrightarrow{V}{}_C \xrightarrow{+}$$

in $\underline{\mathcal{C}}$ such that $V_C \in \mathcal{V}$.

Corollary 1.6. *$(\underline{\mathcal{C}}^-, \underline{\mathcal{V}})$ and $(\underline{\mathcal{U}}, \underline{\mathcal{C}}^+)$ are orthogonal pairs in $\underline{\mathcal{C}}$ provided $\mathcal{C}$ has split idempotents.*

Remark 1. 1. By prop 5.3 Nakaoka we have that $\underline{\mathcal{C}}^+$ has cokernels and, dually, $\underline{\mathcal{C}}^-$ has kernels. add better reference

2. We have inclusions $\mathcal{V} \subseteq \mathcal{C}^+$ and $\mathcal{U} \subseteq \mathcal{C}^-$ see page 5.5 of the notes

2 Nakaoka contexts in additive categories

Let $\mathcal{X}$ be a pointed category. Recall that, given a map $\phi\colon X \to Y$, a map $\psi\colon Y \to Z$ such that $\psi\phi = 0$ is said to be a *pseudocokernel* for ϕ if any other morphism $\psi'\colon Y \to Z'$ such that $\psi'\phi = 0$ factors (not necessarily uniquely) through ψ. Pseudokernels can be defined dually.

Convention. We fix trough this section an additive category $\mathcal{X}$ where idempotents split, and in which any morphism has at least a pseudokernel and a pseudocokernel.

In this section, after some generalities about torsion pairs, we introduce the notion of a "Nakaoka context" in $\mathcal{X}$ and we associate to it a full subcategory $\mathcal{H}$ of $\mathcal{X}$, called the heart. We then give conditions on a given Nakaoka context for the heart to be pre-Abelian, semi-Abelian, integral and, lastly, Abelian.

2.1 Generalities on torsion pairs

For a class of objects $\mathcal{A} \subseteq \mathcal{X}$ we introduce the following notations

$$\mathcal{A}^{\perp} = \{X \in \mathcal{X} : \mathcal{X}(\mathcal{A}, X) = 0\} \quad \text{and} \quad {}^{\perp}\mathcal{A} = \{X \in \mathcal{X} : \mathcal{X}(X, \mathcal{A}) = 0\}.$$

In the following definition we recall the concept of torsion pair in $\mathcal{X}$, which is just an adaptation of the notion of torsion pair in Abelian categories to our broader context:

Definition 2.1. A pair $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{X}$ is a *torsion pair* if:

(TP.1) $\mathcal{F} = \mathcal{T}^{\perp}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$;

(TP.2) given $X \in \mathcal{X}$ there is a pseudokernel-pseudocokernel sequence

$$T_X \xrightarrow{\varepsilon_X} X \xrightarrow{\lambda_X} F^X,$$

where $T_X \in \mathcal{T}$ and $F^X \in \mathcal{F}$.

Definition 2.2. A torsion pair $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ is said to be *left* (resp. *right*) *functorial* if $\mathcal{T}$ (resp. $\mathcal{F}$) is a coreflective (resp. reflective) subcategory of $\mathcal{X}$. Furthermore, we say that $\mathbb{t}$ is *functorial* if it is both left and right functorial.

It is a classical result that torsion pairs in Abelian categories are automatically functorial. Similarly, t-structures in triangulated categories are examples of functorial torsion pairs.

In the following lemma we collect two basic observations about left functorial torsion pairs. The proof is omitted as it is essentially an exercise on the definitions.

Lemma 2.3. *Let $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ be a left functorial torsion pair in $\mathcal{X}$. Then*

(a) for any $M \in \mathcal{X}$, $T' \in \mathcal{T}$ and $\alpha \in \mathcal{X}(T', M)$ there is a unique $\alpha' \in \mathcal{X}(T', t(M))$ such that $\varepsilon_M \circ \alpha' = \alpha$;

(b) for a morphis $g : T_1 \to T_2$ in $\mathcal{T}$, any pseudocokernel $g^C : T_2 \to C$ in $\mathcal{T}$ is also a pseudocokernel of g in $\mathcal{X}$.

2.2 Nakaoka contexts and the heart construction

We start this subsection giving the main definition of the paper:

Definition 2.4. A *Nakaoka context* is a pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ of torsion pairs in $\mathcal{X}$, satisfying the following axioms:

(CT.1) $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are respectively a left functorial and a right functorial torsion pair;

(CT.2) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equiv. $\mathcal{F}_1 \subseteq \mathcal{F}_2$).

Notation. For a Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{X}$, we will always take $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$. Furthermore, we will use the following notation for the corresponding coreflection and reflection:

$$(i_1, t_1)\colon \mathcal{T}_1 \underset{t_1}{\overset{i_1}{\rightleftarrows}} \mathcal{X} \qquad \text{and} \qquad (t_2, j_2)\colon \mathcal{F}_2 \underset{j_2}{\overset{f_2}{\leftrightarrows}} \mathcal{X}.$$

The counit of the first adjunction will be denoted by $\varepsilon_1\colon t_1 i_1 \to \mathrm{id}_{\mathcal{X}}$, while the unit of the second will be denoted by $\lambda_2\colon \mathrm{id}_{\mathcal{X}} \to f_2 i_2$.

Definition 2.5. The *heart* of a Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is $\mathcal{H} = \mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$.

In the following lemma, we collect two general observations about Nakaoka contexts.

Lemma 2.6. *The following statements hold true for a Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$:*

(a) $\mathcal{F}_1 \cap \mathcal{T}_2 = 0$;

(b) $f_2(\mathcal{T}_1) \subseteq \mathcal{H}$ *and* $t_1(\mathcal{F}_2) \subseteq \mathcal{H}$.

Proof. (a) Since $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we have $\mathcal{F}_1 \cap \mathcal{T}_2 \subseteq \mathcal{F}_1 \cap \mathcal{T}_1 = 0$.
(b) Let $T_2 \in \mathcal{T}_2$ and $F_2 \in \mathcal{F}_2$. Then, $\mathcal{X}(T_2, t_1(F_2)) \cong \mathcal{X}(T_2, F_2) = 0$. Hence, $t_1(F_2) \in \mathcal{T}_2^{\perp} = \mathcal{F}_2$. An analogous proof shows that $f_2(\mathcal{T}_1) \subseteq \mathcal{T}_1$. $\square$

In the following lemma we introduce a technical condition under which we can easily construct kernels of morphisms in the heart of a given Nakaoka context:

Lemma 2.7. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a Nakaoka context and let $f\colon H \to H'$ be a morphism in the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$. If there is a morphism $f^K\colon K \to H$, with $K \in \mathcal{F}_2$, such that the following sequence is exact in* $\mathrm{Func}(\mathcal{T}_1, \mathrm{Ab})$

$$(*) \qquad 0 \to (-, K)\restriction_{\mathcal{T}_1} \to (-, H)\restriction_{\mathcal{T}_1} \to (-, H')\restriction_{\mathcal{T}_1},$$

then the composition $f^K \circ \varepsilon_{1,K}\colon t_1 K \to K \to H$ is a kernel for f in $\mathcal{H}$.

Proof. Consider the exact sequence in $(*)$ and notice that it gives, by restriction of the functors, an exact sequence of the form:

$$0 \to (-, K)\restriction_{\mathcal{H}} \to (-, H)\restriction_{\mathcal{H}} \to (-, H')\restriction_{\mathcal{H}}.$$

The map $\varepsilon_{1,K}\colon t_1 K \to K$, induces a natural isomorphism $(-, t_1 K)\restriction_{\mathcal{H}} \to (-, K)\restriction_{\mathcal{H}}$, so we get an exact sequence

$$0 \longrightarrow (-, t_1 K)\restriction_{\mathcal{H}} \xrightarrow{k \circ \varepsilon_{1,K} \circ -} (-, H)\restriction_{\mathcal{H}} \longrightarrow (-, H')\restriction_{\mathcal{H}}.$$

To conclude one notices that, since $t_1 K \in \mathcal{H}$ by Lemma 2.6, the above exact sequence means exactly that $f^K \circ \varepsilon_{1,K}$ is a kernel of f. $\square$

2.3 Pre-abelian Nakaoka contexts

Let us start recalling that an additive category is *pre-Abelian* if any of its morphisms has a kernel and a cokernel. In view of Lemma 2.7, it is natural to introduce the following definition:

Definition 2.8. A Nakaoka context is said to be *pre-Abelian* if it satisfies the following axioms:

(CT.3) any $g\colon H \to H'$ in $\mathcal{H}(\subseteq \mathcal{T}_1)$ admits a pseudocokernel $g^C\colon H' \to C$ in $\mathcal{T}_1$, such that

$$0 \longrightarrow (C, -)_{|\mathcal{F}_2} \xrightarrow{(g^C, -)} (H', -)_{|\mathcal{F}_2} \xrightarrow{(g, -)} (H, -)_{|\mathcal{F}_2}$$

is an exact sequence in $\mathrm{Func}(\mathcal{F}_2, \mathrm{Ab})$.

(CT.3*) Dual of **(CT.3)**.

Theorem 2.9. *For a pre-Abelian Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$, the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ is a pre-Abelian category.*

Proof. This is a consequence of the axioms, Lemma 2.7 and its dual. ☐

In the following proposition we give a characterization of those morphisms that are monomorphisms in the heart. For this, remember that, in a pre-Abelian category, a morphism is mono if and only if its kernel is trivial.

Proposition 2.10. *The following are equivalent for a morphism $f\colon H \to H'$ in the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ of a pre-Abelian Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$:*

(a) f is a monomorphism (in $\mathcal{H}$);

(b) there is a pseudokernel $f^K\colon K \to H$ of f in $\mathcal{F}_2$ such that $K \in \mathcal{F}_1$.

Proof. For any morphism $f\colon H \to H'$ in $\mathcal{H}$, by the axiom (CT.3*), we can consider a diagram as follows

$$\begin{tikzcd} F_2 \arrow[r, "f^K"] & H \arrow[r, "f"] & H' \\ t_1F_2 \arrow[u, "\varepsilon_{1,F_2}"] \arrow[ur, "\tilde{f}^K"'] & & \end{tikzcd}$$

where $F_2 \in \mathcal{F}_2$ is a pseudo-kernel of f in $\mathcal{F}_2$ and, by Lemma ********, $t_1F_2 \to H$ is the kernel of f in $\mathcal{H}$.

(a)⇒(b). Since f is a monomorphism in $\mathcal{H}$, its kernel is trivial, that is, $t_1F_2 = 0$. Hence, $0 = \mathcal{X}(Z, t_1(F_2)) \cong \mathcal{X}(Z, F_2)$ for all $Z \in \mathcal{T}_1$, that is, $F_2 \in \mathcal{T}_1^{\perp} = \mathcal{F}_1$.

(b)⇒(a). Conversely, let $f' : F_1 \to H$ be a pseudokernel of f in $\mathcal{F}_2$, with $F_1 \in \mathcal{F}_1$. Consider the solid part of the diagram

$$\begin{tikzcd} t_1(F_2) \arrow[d, "\varepsilon_{1,F_2}"] \arrow[dr, "\tilde{f}^K"] & & \\ F_2 \arrow[r, "f^K"] \arrow[d, dashed, shift right, "u"'] & H \arrow[r, "f"] & H' \\ F_1 \arrow[u, dashed, shift right, "v"'] \arrow[ur, "f'"] & & \end{tikzcd}$$

Since f^K and f' are pseudokernel of f in $\mathcal{F}_2$, there exist u and v such that $f' = f^K \circ v$ and $f^K = f' \circ u$. Therefore, $f^K = f^K \circ v \circ u$, and so $\tilde{f}^K = f^K \circ \varepsilon_{1,F_2} = f^K \circ v \circ u \circ \varepsilon_{1,F_2}$. Notice that $u \circ \varepsilon_{1,F_2} = 0$ since $\mathrm{Hom}_{\mathcal{X}}(\mathcal{T}_1, \mathcal{F}_1) = 0$. Thus, f has the zero morphism as its kernel in $\mathcal{H}$, i.e. f is a monomorphism. ☐

Proposition 2.11. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$ and $f : H_1 \to H_2$ be a morphism in $\mathcal{H}$. Then*

*(a) if $f^C : H_2 \to T_1$ is the pseudocokernel of f, given by **(CT.3)** and $\lambda_{2,T_1} : T_1 \to f_2(T_1)$, then*

$$\mathrm{Coker}\,(H_1 \xrightarrow{f} H_2) = (H_2 \xrightarrow{\tilde{f}^C} f_1(T_1))$$

in $\mathcal{H}$, where $\tilde{f}^C := \lambda_{2,T_1} \circ f^C$;

*(b) if $f^K : F_2 \to H_1$ is the pseudokernel of f, given by **(CT.3*)** and $\varepsilon_{1,F_2} : t_1(F_2) \to F_2$, then*

$$\mathrm{Ker}\,(H_1 \xrightarrow{f} H_2) = (t_1(F_2) \xrightarrow{\tilde{f}^K} H_1)$$

in $\mathcal{H}$, where $\tilde{f}^K = f^K \circ \varepsilon_{1,F_2}$.

2.4 Abelian Nakaoka contexts

Definition 2.12. A compatible torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{X}$ is strong if the following axioms hold:

(CT.4) Let $f : H_1 \to H_2$ in $\mathcal{H}$ be such that there is a pseudokernel $\mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$. Then, for the commutative diagram

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 := \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F)_2 \arrow[r, "\varepsilon_{1,F_2}"] & F_2 := \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1)
\end{tikzcd}$$

there exists a morphism $b : t_1(F_2) \to H_1$ such that $ab = \varepsilon_{1,F_2}$.

(CT.4*) Dual

With these axioms we can prove that the heart has kernels and cokernels.

Theorem 2.13. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$. Then, the following are equivalent:*

(a) $\mathbb{t}$ is strong,

(b) $\mathcal{H}$ is an abelian category.

Proof.

$(a) \Rightarrow (b)$ By definition $\mathcal{H}$ is an additive subcategory of $\mathcal{X}$. In order to prove that (a) implies (b) observe that $\mathcal{H}$ is preabelian by proposition 2.11. To prove that $\mathcal{H}$ is abelian, we just need to show that any monomorphism (resp. epimorphism) is the kernel (resp. cokernel) of some morphism in $\mathcal{H}$.

Check referencing in TeX

Let $f : H_1 \to H_2$ be a monomorphism in $\mathcal{H}$. By **(CT.3)** and **(CT.3*)** we can consider the solid part of the commutative diagram

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 = \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F_2) \arrow[r, "\varepsilon_{1,F_2}"] \arrow[ur, dashed, "b"] & F_2 = \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1)
\end{tikzcd}$$

where $gf = \lambda_{2,T_1}(f^C f) = 0$, so there exists $a : H_1 \to F_2$. Note that $\exists \mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$ since f is a monomorphism in $\mathcal{H}$ (by proposition 2.10**??**). So by **(CT.4)** there is a map $b : t_1(F_2) \to H_1$ making the diagram commute.

Add proper reference

We claim that $\mathrm{Ker}\, g = f$. Let $\alpha : H \to H_2$ be a morphism such that $g\alpha = 0$. Since $F_2 = \mathrm{PKer}_{\mathcal{X}}(g)$, there is a morphism $\alpha' : H \to F_2$ such that $g^K \alpha' = \alpha$. By remark 2.3, α' factors as $\varepsilon_{1,F_2}\alpha''$, as in the diagram:

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 = \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F_2) \arrow[r, "\varepsilon_{1,F_2}"] \arrow[ur, "b"] & F_2 = \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1) \\
 & H. \arrow[ul, dashed, "\alpha''"] \arrow[u, dashed, "\alpha'"] \arrow[ur, "\alpha"'] & &
\end{tikzcd}$$

By setting $\alpha''' := b\alpha'' : H \to H_1$, we get $f\alpha''' = g^K ab\alpha'' = g^K \varepsilon_{1,F_2}\alpha'' = g^K \alpha' = \alpha$. Thus, any morphism $\alpha : H \to H_2$ such that $g\alpha = 0$ factors through f. To conclude that f is a kernel of g, it suffice to observe that f is a monomorphism, so α''' must be unique.

$(b) \Rightarrow (a)$ Let $\mathcal{H}$ be an abelian category. We only check **(CT.4)** since the proof of **(CT.4*)** is analogous.

Fix reference

Consider the solid part of the commutative diagram

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] \arrow[dl, dashed, "\beta"'] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 = \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F^2) \arrow[r, "\varepsilon_{1,F_2}"] & F_2 = \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1)
\end{tikzcd}$$

where $f : H_1 \to H_2$ is a morphism in $\mathcal{H}$ such that $\mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$. Since f is a monomorphism in $\mathcal{H}$ by 2.10(a) and $\mathcal{H}$ is abelian, we have that $f = \mathrm{Ker}\,\mathrm{Coker}\,(f)$. On the other hand, by 2.3 (a) there is $\beta : H_1 \to t_1(F_2)$ such that $\varepsilon_{1,F_2}\beta = a$ completing the diagram above. Since $f = \mathrm{Ker}\,\mathrm{Coker}\,(f)$, it follows that β is an isomorphism. Therefore, **(CT.4)** follows by setting $b := \beta^{-1}$.

Fix reference

□

3 Particular cases (título provisional)

3.1 The case of an abelian category

Let's consider the case $\mathcal{X} = \mathcal{A}$ of an Abelian category with two torsion pairs $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ for $i = 1, 2$. Consider $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$.

Remark 2. In the case of an Abelian category $\mathcal{X} = \mathcal{A}$, we have that $\mathbb{t}$ is compatible if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proof. Let $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we need to show that item **(CT.1)**, item **(CT.3)** and item **(CT.3)** hold.

(CT.1) It is well known that any torsion pair in an abelian category is functorial.

(CT.3) Let $g : T_1 \to T_1'$ be a morphism in $\mathcal{T}_1$. Consider the cokernel morphism of g in $\mathcal{A}$

$$\mathrm{Coker}_{\mathcal{A}}(T_1 \xrightarrow{g} T_1') = (T_1' \xrightarrow{c_g} \mathrm{Coker}\,(g)).$$

Since $\mathcal{T}_1$ is closed under quotient objects, we get that $\mathrm{Coker}\,(g) \in \mathcal{T}_1$. Therefore, we can choose $c_g : T_1' \to \mathrm{Coker}\,(g)$ as $g^C : T_1' \to \mathrm{PCok}_{\mathcal{A}}(g)$.

Is there an explicit choice that we made somewhere before when we talk about $\mathrm{PCok}_{\mathcal{A}}$?

(CT.3*) Anologous to the previous.

□

Corollary 3.1. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a torsion pair in $\mathcal{A}$ with $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Then, for $f : H_1 \to H_2$ in $\mathcal{H}$, the following statements hold:*

(a) the cokernel of f in $\mathcal{H}$ is the composition of the morphisms

$$H_2 \xrightarrow{c_f} \mathrm{Coker}\,(f) \xrightarrow{\lambda_{2,\mathrm{Coker}\,(f)}} f_2(\mathrm{Coker}\,(f));$$

(b) the kernel of f in $\mathcal{H}$ is the composition of the morphisms

$$t_1(\mathrm{Ker}\,(f)) \xrightarrow{\varepsilon_{1,\mathrm{Ker}\,(f)}} \mathrm{Ker}\,(f) \xrightarrow{k_f} H_1;$$

(c) f is an epimorphism in $\mathcal{H}$ if and only if $\mathrm{Coker}\,(f) \in \mathcal{T}_2$;

(d) f is a monomorphism in $\mathcal{H}$ if and only if $\mathrm{Ker}\,(f) \in \mathcal{F}_1$.

Proof. (a) and (b) follow from the proof of Remark 2and proposition 2.11.

this should be a remark, fix it

(c)

$\Leftarrow$ is trivial.

$\Rightarrow$ By proposition 2.10 **??** there exists $f^C : H_2 \to T_2$, where $T_2 = \operatorname{Coker}_{\mathcal{T}_1}(f) \in \mathcal{T}_2$. Then, we have

$$\begin{tikzcd} H_2 \arrow[rr, "f^C"] \arrow[dr, "c_f"'] & & T_2 \arrow[dl, dashed, bend left, "v"] \\ & \operatorname{Coker}(f) \arrow[ur, dashed, "u"] & \end{tikzcd} \qquad \text{such that } \begin{cases} uf^C = c_f, \\ vc_f = f^C. \end{cases}$$

Hence, $uvc_f = c_f$, but c_f is epi, therefore $uv = 1$. Hence, $\operatorname{Coker}(f)$ is a direct summand of $T_2 \in \mathcal{T}_2$ so $\operatorname{Coker}(f) \in \mathcal{T}_2$.

(d) Similar to the previous proof. $\square$

Theorem 3.2. *Let $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ be a torsion pair in an abelian category $\mathcal{A}$, for $i = 1, 2$, such that $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Then, for $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ the following statements are equivalent:*

(a) $\mathcal{H}$ is an abelian category.

(b) The following conditions hold:

(b1) For any $f : H \to H'$ in $\mathcal{H}$, with $\operatorname{Ker}(f) \in \mathcal{F}_1$, we have that $\operatorname{Ker}(f) = 0$.

(b2) For any $f : H \to H'$ in $\mathcal{H}$, with $\operatorname{Coker}(f) \in \mathcal{T}_2$, we have that $\operatorname{Coker}(f) = 0$.

(b3) $\mathcal{H}$ is closed under kernels (resp. cokernels) of epimorphisms (resp. monomorphisms) in $\mathcal{A}$.

(c) $\mathcal{H}$ is closed under kernels and cokernels in $\mathcal{A}$.

3.2 Twin torsion pairs in triangulated categories

Let $\mathcal{X} = \mathcal{T}$ be a triangulated category on which idempotents split. We start by recalling the definition of a t-structure in $\mathcal{T}$.

Definition 3.3. A pair $(\mathcal{A}, \mathcal{B})$ of full subcategories of $\mathcal{T}$ is a t-structure in $\mathcal{T}$ if

(a) $\mathbb{t} = (\mathcal{A}, \mathcal{B}[-1])$ is a torsion pair in $\mathcal{T}$, and

(b) $\mathcal{A}[1] \subseteq \mathcal{A}$.

Remark 3. It is well known that any t-structure $(\mathcal{A}, \mathcal{B})$ in $\mathcal{T}$ gives a functional torsion pair $\mathbb{t} = (\mathcal{A}, [-1])$ and $\mathbb{B}[-1] \subseteq \mathcal{B}$. Furthermore, $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions and direct summands. Note that the t-structure $(\mathcal{A}, \mathcal{B})$ depends only on $\mathcal{A}$, since $\mathcal{B} = \mathcal{A}^{\perp}[1]$.

Definition 3.4. A *related* torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in triangulated category $\mathcal{T}$ consists of the torsion pairs $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t} = (\mathcal{T}_2, \mathcal{F}_2)$ in $\mathcal{T}$ such that $\mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proposition 3.5. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related torsion pair in $\mathcal{T}$. Then*

(a) $(\mathcal{T}_1, \mathcal{F}_1[1])$ and $(\mathcal{T}_2, \mathcal{F}_2[1])$ are t-structures in $\mathcal{T}$;

(b) $\mathbb{t}$ is a compatible torsion pair in $\mathcal{T}$;

(c) the heart $\mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2[1]$ is a preabelian category.

Definition 3.6. A related torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in the triangulated category $\mathcal{T}$ is *strong* if for any morphism $f : H_1 \to H_2$, in $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$, and a distinguished triangle $Z \to H_1 \xrightarrow{f} H_2 \to Z[1]$, the following conditions hold true

(RST.1) $Z \in \mathcal{F}_1$ if and only if $Z \in \mathcal{F}_2[-1]$;

(RST.2) $Z[1] \in \mathcal{T}_2$ if and only if $Z \in \mathcal{T}_1$.

Theorem 3.7. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a strongly related torsion pair in the triangulated category $\mathcal{T}$. Then, the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ is an abelian category.*

Example 1. Let $(\mathcal{A},\mathcal{B})$ be a t-structure in $\mathcal{T}$. Consider $\mathbb{t}_1 := (\mathcal{A},\mathcal{B}[-1])$ and $\mathbb{t}_2 := (\mathcal{A}[1],\mathcal{B})$. It is not hard to see that $\mathbb{t} = (\mathbb{t}_1,\mathbb{t}_2)$ is a strongly related torsion pair in $\mathcal{T}$. In this case, by theorem 3.7, we get that $\mathcal{H} = \mathcal{A}\cap\mathcal{B}$ is an abelian category (BBD theorem).

Example 2. Let $\mathcal{D}$ be a triangulated category with two t-structures $(\mathcal{U}_1,\mathcal{U}_1^{\perp})$ and $(\mathcal{U}_2,\mathcal{U}_2^{\perp})$ such that $\mathcal{U}_1[1]\subseteq\mathcal{U}_2\subseteq\mathcal{U}_1$. Then, these satisfy axioms **(CT.1)-(CT.3)**,**(CT.3*)**, hence $\mathcal{H} = \mathcal{U}_1\cap\mathcal{U}_2^{\perp}$ has kernels and cokernels. Moreover, TFAE:

1.a **(CT.4)** holds.

1.b If $V_1 \to H_1 \xrightarrow{f} H_2 \xrightarrow{+}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $V_1 \in \mathcal{U}_1^{\perp}$, then $V_1 \in \mathcal{U}_2^{\perp}[-1]$.

And, dually, there is an equivalence of the following:

2.a **(CT.4*)** holds.

2.b If $H_1 \xrightarrow{f} H_2 \to U_2 \xrightarrow{+}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $U_2 \in \mathcal{U}_2$, then $U_2 \in \mathcal{U}_1[1]$.

Proof of the equivalences in example 2. Let's $\mathcal{D}$ be a triangulated category with two t-structures as in example 3. The pseudocokernel of a morphism in $\mathcal{U}_1$ can be computed by taking the cone in $\mathcal{D}$, i.e. given a morphism $f : U_1 \to U_1'$ in $\mathcal{U}_1$ we can compute a pseudocokernel in $\mathcal{U}_1$ by completing f to a triangle

$$U_1 \xrightarrow{f} U_1' \to \mathrm{Cone}(f) \xrightarrow{+} .$$

Moreover, this pseudocokernel satisfies **(CT.3)**.

Now, assume that **(CT.1)-(CT.3)**,**(CT.3*)** are satisfied together with axiom **1.b**, and consider the solid part of the diagram as in **(CT.4)**:

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_1 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
\tau_{\mathcal{U}_1}(V_2) \arrow[r, "\varepsilon"] \arrow[ur, dotted, "\beta"] & V_2 \arrow[r] & H_2 \arrow[r] & \tau^{\mathcal{U}_2^{\perp}}\mathrm{Cone}(f)
\end{tikzcd}$$

with $\mathrm{Cone}(f)[-1]\in\mathcal{U}_1^{\perp}$ and where the upper row is a distinguished triangle. By **1.b** then it belongs to $\mathcal{U}_2^{\perp}[-1]$, i.e. $\mathrm{Cone}(f)\in\mathcal{U}_2^{\perp}$, so λ is an iso, conseguently α is an iso and so is ε, so there exist a map $\beta = \alpha^{-1}\circ\varepsilon$ making the diagram commute, that is **(CT.4)** holds.

Conversely, assume that **(CT.1)-(CT.3)**,**(CT.3*)** are satisfied together with **(CT.4)**. Consider again the solid part of the diagram

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "{\lambda[-1]}"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_1 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
\tau_{\mathcal{U}_2^{\perp}}(\mathrm{Cone}(f))[-1] \arrow[r, dashed] & V_2 \arrow[r] & H_2 \arrow[r] & \tau^{\mathcal{U}_2^{\perp}}\mathrm{Cone}(f) \\
 & \tau_{\mathcal{U}_1}(V_2) \arrow[u, "\varepsilon"] \arrow[uu, dotted, bend left, "\beta"] & &
\end{tikzcd}$$

with $\mathrm{Cone}(f)[-1]\in\mathcal{U}_1^{\perp}$. Neeman guarantees that α can be taken so that the square on the left is a pullback. Axiom **(CT.4)** gives the existence of $\beta : \tau_{\mathcal{U}_1}(V_2)\to H_1$ such that $\alpha\circ\beta = \varepsilon$.

Add reference

Since $\tau_{\mathcal{U}_1}$ is a functor, there is also a morphism $\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(H_1) = H_1 \to \tau_{\mathcal{U}_1}(V_2)$ such that $\varepsilon\circ\tau_{\mathcal{U}_1}(\alpha) = \alpha$, hence $\varepsilon\circ\tau_{\mathcal{U}_1}(\alpha)\circ\beta = \varepsilon$. By the functoriality of the torsion pair $(\mathcal{U}_1,\mathcal{U}_1^{\perp})$, this means that $\tau_{\mathcal{U}_1}(\alpha)\circ\beta = 1_{\tau_{\mathcal{U}_1}(V_2)}$. Then, β is a section.

Hence, we can write $\tau_{\mathcal{U}_1}(\alpha) : H_1 \to \tau_{\mathcal{U}_1}(V_2)$ as

$$\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(V_2)\oplus H_1' \xrightarrow{(*\ 0)} \tau_{\mathcal{U}_1}(V_2)$$

for some $H_1' \underset{\oplus}{<} H_1$ such that α vanishes on H_1'. If we consider the solid part of the diagram

$$\begin{tikzcd} & & H_1' \arrow[dll, dashed] \arrow[d, "\binom{1}{0}"] \arrow[dr, "0"] & & \\ \mathrm{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1] \arrow[rr] & & H_1 \arrow[r, "\tau_{\mathcal{U}_1}(\alpha)"] & \tau_{\mathcal{U}_1}(V_2) \arrow[r, dashed, "+"] & {} \end{tikzcd}$$

we can construct the dashed arrow, and the fact that the triangle commutes means that $H_1' \underset{\oplus}{<} \mathrm{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1]$.

Observe that $\mathrm{Cone}(\alpha) = \mathrm{Cone}(\lambda)[-1]$, since the square

$$\begin{tikzcd} \mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "\lambda[-1]"] & H_1 \arrow[d, "\alpha"] \\ \tau_{\mathcal{U}_2^\perp}(\mathrm{Cone}(f))[-1] \arrow[r] & V_2 \end{tikzcd}$$

is a pullback. Moreover, $\mathrm{Cone}(\lambda)[-1] = (\tau_{\mathcal{U}_2}(\mathrm{Cone}(f))[1])[-1] = \tau_{\mathcal{U}_2}(\mathrm{Cone}(f))$. Hence, $\mathrm{Cone}(\alpha) \in \mathcal{U}_2$ and $\tau^{\mathcal{U}_1^\perp}(\mathrm{Cone}(\alpha)) = 0$, that is, $\mathrm{Cone}(\alpha) \in \mathcal{U}_1$, and since there is a distinguished triangle

$$H_1 \xrightarrow{\alpha} V_2 \to \mathrm{Cone}(\alpha) \xrightarrow{+}$$

with $H_1, \mathrm{Cone}(\alpha) \in \mathcal{U}_1$ it follows that $V_2 \in \mathcal{U}_1$. Hence, $\tau_{\mathcal{U}_1}(V_2) \cong V_2$.

We can then write $V_2 \underset{\oplus}{<} H_1$ and consider the commutative diagram

$$\begin{tikzcd} H_1 \cong H_1' \oplus V_2 \arrow[r, "(f' \ \tilde{f})"] \arrow[d, "(0\ 1)"] & H_2 \arrow[d, equal] \\ V_2 \arrow[r] & H_2 \end{tikzcd}$$

so $f' = 0$. Hence, the inclusion $\binom{1}{0} : H_1' \to H_1' \oplus V_2$ can be lifted to $\mathrm{Cone}(f)[-1]$ and $H_1' \underset{\oplus}{<} \mathrm{Cone}(f)[-1]$. Since $\mathrm{Cone}(f)[-1] \in \mathcal{U}_1^\perp$, so does H_1'. Similarly, $H_1' \in \mathcal{U}_1$ because $H_1 \in \mathcal{U}_1$. Hence, $H_1' = 0$ and $\alpha : H_1 \to V_2$ is an iso. The same follows for λ. Therefore, $\mathrm{Cone}(f) \in \mathcal{U}_2^\perp$ which proves **1.b**. □

Example 3. Let R be any (associative with 1) ring. Consider the triangulated category $\mathcal{T} := \mathcal{D}(R)$. The derived category $\mathcal{D}(R)$ has the so called natural t-structure $(\mathcal{D}^{\leq 0}(R), \mathcal{D}^{\geq}(R))$ where

$$\mathcal{D}^{\leq 0}(R) := \{X \in \mathcal{D}(R) \mid H^i(X) = 0 \text{ for } i > 0\},$$
$$\mathcal{D}^{\geq 0}(R) := \{X \in \mathcal{D}(R) \mid H^i(x) = 0 \text{ for } i < 0\}.$$

For any ideal $I \trianglelefteq R$, we have the TTF-triple $(\mathcal{C}_I, \mathcal{T}_I, \mathcal{F}_I)$ associated to I, where

$$\mathcal{C}_I := \{M \in \text{Mod-}R \mid IM = M\},$$
$$\mathcal{T}_I := \{M \in \text{Mod-}R \mid IM = 0\} \cong \text{Mod-}\frac{R}{I},$$
$$\mathcal{F}_I := \{M \in \text{Mod-}R \mid Ix = 0 \text{ and } x \in M \Rightarrow x = 0\}.$$

Consider the t-structure (Happel-Reiten-Smalo) $(\mathcal{D}^{\leq 0}_{t_I}(R), \mathcal{D}^{\geq 0}_{t_I}(R))$ associated to the torsion pair $t_I = (\mathcal{C}_I, \mathcal{T}_I)$, where

$$\mathcal{D}^{\leq 0}_{t_I}(R) := \{X \in \mathcal{D}^{\leq 0}(R) \mid H^0(X) \in \mathcal{C}_I\},$$
$$\mathcal{D}^{\geq 0}_{t_I}(R) := \{X \in \mathcal{D}^{\geq 0}(R) \mid H^0(X) \in \mathcal{T}_I\}.$$

It can be seen that $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ where $\mathbb{t}_1 := (\mathcal{D}^{\leq 0}(R), \mathcal{D}^{\geq 1}(R))$ and $\mathbb{T}_2 := (\mathcal{D}^{\leq 0}_{t_I}(R), \mathcal{D}^{\geq 1}_{t_I}(R))$ [redacted] is a strongly related torsion pair in $\mathcal{T} = \mathcal{D}(R)$.

3.3 Polishchuk correspondence

We recall the following bijection given by A. Polishchuk, and in order to do that, for a t-structure $(\mathcal{U}_1,\mathcal{U}_1^{\perp}[1])$ in $\mathcal{T}$, we have the cohomological functor $H_1^0 : \mathcal{T} \to \mathcal{H}_1 := \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1]$ ($\mathcal{H}_1$ is an abelian category).

Proposition 3.8 (Polishchuk)**.** *Let $(\mathcal{U}_1,\mathcal{U}_1^{\perp}[1])$ be a t-structure in a triangulated category. Then we have a bijection (Polishchuk's bijection)*

$$\left\{ \begin{array}{c} \textit{torsion pairs in} \\ \mathcal{H}_1 = \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1] \end{array} \right\} \overset{\mathrm{Pol}_{\mathcal{H}_1}}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{t-structures } (\mathcal{U}_2,\mathcal{U}_2^{\perp}) \\ \textit{in } \mathcal{D} \textit{ satisfying} \\ \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1 \end{array} \right\}$$

$$(\mathcal{X},\mathcal{Y}) \longmapsto (\mathcal{U}_2,\mathcal{U}_2^{\perp}[1])$$

$$(\mathcal{U}_2 \cap \mathcal{H}_1,\mathcal{U}_2^{\perp} \cap \mathcal{H}_1) \longleftarrow\!\shortmid (\mathcal{U}_2,\mathcal{U}_2^{\perp}[1])$$

where

$$\mathcal{U}_2 = \{X \in \mathcal{U}_1 \mid H_1^0(X) \in \mathcal{X}\}$$
$$\mathcal{U}_2^{\perp} = \{Y \in \mathcal{U}_1^{\perp} \mid H_1^0(Y) \in \mathcal{Y}\}.$$

Remark 4. (1) Note that $\mathrm{Pol}_{\mathcal{H}_1}^{-1}(\mathcal{U}_2,\mathcal{U}_2^{\perp}[1]) = (\mathcal{U}_2 \cap \mathcal{U}_1^{\perp}[1], \mathcal{H})$, where $\mathcal{H} := \mathcal{U}_1 \cap \mathcal{U}_2^{\perp}$.

(2) By (1), it follows that $\mathcal{H}$ is a torsion free class in the abelian category $\mathcal{H}_1 := \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1]$.

Theorem 3.9. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related torsion pair in a triangulated category $\mathcal{T}$. Then, the following statements are equivalent.*

(a) For any distinguished triangle $V \to H_1 \xrightarrow{f} H_2 \xrightarrow{+}$, with f a morphism in $\mathcal{H} = \mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$, we have that
$$V \in \mathcal{F}_1 \Rightarrow V[1] \in \mathcal{F}_2.$$

(b) For any monomorphism $\alpha : H_1 \hookrightarrow H_2$, in the abelian category $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$, with $H_1, H_2 \in \mathcal{H}$, we have that $\mathrm{Coker}_{\mathcal{H}_1}(\alpha) \in \mathcal{H}$.

(c) $\mathcal{H}$ is closed under kernels and cokernels in the abelian category $\mathcal{H}_1$

(d) $\mathcal{H}$ is an abelian category.

(e) For any epimorphism $H \twoheadrightarrow X$ in $\mathcal{H}_1$, with $H \in \mathcal{H}$, we have that $X \in \mathcal{H}$ (i.e. $\mathcal{H}$ is closed under quotients in $\mathcal{H}_1$).

Let $t = (\mathcal{A}, \mathcal{B})$ be a pair of full subcategories of the triangulated category $\mathcal{T}$. We will use the following notation:

$$t[1] := (\mathcal{A}[1], \mathcal{B}[1]),$$
$$\bar{t} := (\mathcal{A}, \mathcal{B}[1]).$$

Note that $\bar{t}$ is a t-structure in $\mathcal{T}$ if and only if t is a torsion pair $\mathcal{T}$ such that $\mathcal{A}[1] \subseteq \mathcal{A}$.

Remark 5. Consider $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$, where $\mathbb{t}_i := (\mathcal{U}_i, \mathcal{U}_i^{\perp})$ for $i = 1, 2$. We have

1. $\mathcal{H}_{\mathbb{t}} := \mathcal{U}_1 \cap \mathcal{U}_2^{\perp}$, $\mathcal{H}_i := \mathcal{U}_i \cap \mathcal{U}_i^{\perp}[1]$,
2. $\mathbb{t}' := (\mathbb{t}_2, \mathbb{t}_1[1])$

Note that

3. $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is a related torsion pair in $\mathcal{T}$

$$\begin{aligned}
&\Leftrightarrow \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1 \\
&\Leftrightarrow \mathcal{U}_2[1] \subseteq \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \\
&\Leftrightarrow \mathbb{t}' = (\mathbb{t}_2, \mathbb{t}_1[1]) \text{ is a related torsion pair in } \mathcal{T}.
\end{aligned}$$

4. Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is a related torsion pair in $\mathcal{T}$. In this case, we have

$$\begin{aligned}
&\mathcal{H}_{\mathbb{t}} = \mathcal{U}_1 \cap \mathcal{U}_2^{\perp}, \ \mathcal{H}_{\mathbb{t}'} = \mathcal{U}_2 \cap \mathcal{U}_1^{\perp}[1], \\
&\mathrm{Pol}^{-1}_{\mathcal{H}_1}(\bar{\mathbb{t}}_2) = \mathrm{Pol}^{-1}_{\mathcal{H}_1}(\mathcal{U}_2, \mathcal{U}_2^{\perp}[1]) = (\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}}), \\
&\mathrm{Pol}^{-1}_{\mathcal{H}_2}(\bar{\mathbb{t}}_1[1]) = \mathrm{Pol}^{-1}_{\mathcal{H}_1}(\mathcal{U}_1[1], \mathcal{U}_1^{\perp}[2]) = (\mathcal{H}_{\mathbb{t}}[1], \mathcal{H}_{\mathbb{t}'}).
\end{aligned}$$

Thus, $(\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}})$ is a torsion pair in the abelian category $\mathcal{H}_1$, $(\mathcal{H}_{\mathbb{t}}[1], \mathcal{H}_{\mathbb{t}'})$ is a torsion pair in the abelian category $\mathcal{H}_2$.

Corollary 3.10. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related torsion pair in a triangulated category $\mathcal{T}$. Then, the following statements are equivalent:*

(a) For any distinguished triangle $V \to H_1 \xrightarrow{f} H_2 \xrightarrow{+}$, with f a morphism in $\mathcal{H}_{\mathbb{t}'} = \mathcal{T}_2 \cap \mathcal{F}_1[1]$, we have that $V \in \mathcal{F}_2$ implies $V \in \mathcal{F}_1$.

(b) For any monomorphism $\alpha : H_1 \hookrightarrow H_2$, in the abelian category $\mathcal{H}_2 := \mathcal{T}_2 \cap \mathcal{F}_2[1]$, with $H_1, H_2 \in \mathcal{H}_{\mathbb{t}'}$, we have that $\mathrm{Coker}_{\mathcal{H}_2}(\alpha) \in \mathcal{H}_{\mathbb{t}'}$.

(c) $\mathcal{H}_{\mathbb{t}'}$ is closed under kernels and cokernels in the abelian category $\mathcal{H}_2$.

(d) $\mathcal{H}_{\mathbb{t}'}$ is an abelian category.

(e) $\mathcal{H}_{\mathbb{t}'}$ is closed under quotients in $\mathcal{H}_2$.

We recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ is cohereditary if the class $\mathcal{F}$ is closed under quotients in $\mathcal{A}$.

Definition 3.11. For a triangulated category $\mathcal{T}$, we consider the following classes:

1. $\mathit{RtAb}(\mathcal{T}) := \{$related torsion pairs $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{T}$ s.t. $\mathcal{H}_{\mathbb{t}}$ is abelian$\}$;

2.

$$\mathrm{t}-\mathit{stCoh}(\mathcal{T}) := \left\{ \begin{array}{c} \text{pairs } (\bar{\mathbb{t}}_1, \tau) \text{ s.t. } \bar{\mathbb{t}}_1 \text{ is a t-structure in } \mathcal{T} \text{ and } \tau \text{ is a} \\ \text{cohereditary torsion pair in the abelian category} \\ \mathcal{H}_1 := \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1] \end{array} \right\};$$

1′. $\mathit{RtAb}'(\mathcal{T}) := \{$related torsion pairs $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{T}$ s.t. $\mathcal{H}_{\mathbb{t}'}$ is abelian$\}$;

2′.

$$\mathrm{t}-\mathit{stCoh}'(\mathcal{T}) := \left\{ \begin{array}{c} \text{pairs } (\bar{\mathbb{t}}_2, \tau) \text{ s.t. } \bar{\mathbb{t}}_2 \text{ is a t-structure in } \mathcal{T} \text{ and } \tau \text{ is a} \\ \text{cohereditary torsion pair in the abelian category} \\ \mathcal{H}_2 := \mathcal{U}_2 \cap \mathcal{U}_2^{\perp}[1] \end{array} \right\}.$$

Theorem 3.12. *For a triangulated category $\mathcal{T}$, the following statements hold true.*

(a) There is a bijective correspondence

$$\begin{array}{ccc}
\mathit{RtAb}(\mathcal{T}) & \xleftrightarrow{\alpha} & \mathrm{t} - \mathit{stCoh}(\mathcal{T}) \\
\mathbb{t} & \longmapsto & (\bar{\mathbb{t}}_1, \mathrm{Pol}^{-1}_{\mathcal{H}_1}(\bar{\mathbb{t}}_2)) \\
(\mathbb{t}_1, \mathbb{t}_2) & \longleftarrow\!\shortmid & (\bar{\mathbb{t}}_1, \tau)
\end{array}$$

where $\bar{\mathbb{t}}_2 = \mathrm{Pol}_{\mathcal{H}_1}(\tau)$.

(b) There is a bijective correspondence

$$\begin{aligned} RtAb'(\mathcal{T}) &\xleftrightarrow{\alpha'} \mathrm{t}-stCoh'(\mathcal{T}) \\ \mathbb{t} &\longmapsto (\bar{\mathbb{t}}_2, \mathrm{Pol}^{-1}_{\mathcal{H}_2}(\bar{\mathbb{t}}_1[1])) \\ (\mathbb{t}_1, \mathbb{t}_2) &\longleftarrow\!\shortmid (\bar{\mathbb{t}}_2, \tau) \end{aligned}$$

where $\bar{\mathbb{t}}_1 = \mathrm{Pol}_{\mathcal{H}_2}(\tau)[-1]$.

4 Induced torsion theories

In this section and the following we study the case of a torsion pair $(\mathcal{T}, \mathcal{F})$ in a (nice) category $\mathcal{A}$, such that there is a subcategory $\mathcal{W} \subseteq \mathcal{F}$ of $\mathcal{A}$ for which it makes sense to consider $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{W}$. The goal is to describe the cases where $(\mathcal{T}, \mathcal{F})$ induces a torsion pair in $\underline{\mathcal{A}}$.

add a better description of the setting

4.1 Torsion pairs in the stable category

Lemma 4.1. *Let $\mathcal{A}$ be a (nice) category and $\mathcal{W} \subseteq \mathcal{A}$ a subcategory such that $\mathrm{add}(\mathcal{W}) = \mathcal{W}$. If $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a torsion pair such that $\mathcal{W} \subseteq \mathcal{F}$, then $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is an orthogonal pair in $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$.*

Lemma 4.2. *Let $\mathcal{A}$ and $\mathcal{W} \subseteq \mathcal{A}$ be defined as above and let $({}^{\perp}\mathcal{F}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$ such that $\mathcal{W} \subseteq \mathcal{F}$. Call $p : \mathcal{A} \to \underline{\mathcal{A}}$ the quotient functor. The following assertions hold:*

1. *$p^{-1}({}^{\perp}(\underline{\mathcal{F}})) = \mathrm{add}({}^{\perp}\mathcal{F} * \mathcal{W})$.*
2. *If $\mathcal{W}$ is precovering in $\mathcal{F}$, then $({}^{\perp}\underline{\mathcal{F}}, \underline{\mathcal{F}})$ is a torsion pair in $\underline{\mathcal{A}}$.*

Lemma 4.3. *Let $\mathcal{A}$ be a (nice) category with a torsion pair $({}^{\perp}\mathcal{F}, \mathcal{F})$ and a precovering class $\mathcal{W} \subseteq \mathcal{F}$ such that for any $F \in \mathcal{F}$ there is an admissible sequence*

$$F' \to W \to F$$

such that $F' \in \mathcal{F}$.

Then the torsion pair $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial.

Recall that the truncation $t : \underline{\mathcal{A}} \to {}^{\perp}(\underline{\mathcal{F}})$ is given by the following construction.

Let $M \in \mathcal{A}$ be any object, take an admissible sequence

$$T_M \to M \to F^M$$

with $T_M \in {}^{\perp}\mathcal{F}$ and $F^M \in \mathcal{F}$. Moreover, consider $W_F \to F^M$ with $W_M \in \mathcal{W}$ as before, and take the admissible pullback:

$$\begin{tikzcd}
t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\
M \arrow[r] & F
\end{tikzcd}$$

Then, t restricts to a functor $\underline{t} : \underline{\mathcal{A}} \to {}^{\perp}\underline{\mathcal{F}}$.

In order to prove that $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial we need to show that $\underline{t}$ admits a right adjoint.

Proof. Let $M \in \mathcal{A}$ and consider $M \to F^M$ and $W_F \to F^M$ as above. For any $T \in {}^{\perp}\mathcal{F} * \mathcal{W}$ consider any morphism $f : T \to M$. Since $T \to M \to F^M$ is 0 in $\underline{\mathcal{A}}$ we that the solid part of the following diagram commutes.

$$\begin{tikzcd}
T \arrow[rrr] \arrow[dr, dotted, "\psi"] \arrow[ddr, bend right, "f"'] & & & W \arrow[dl, dashed, "\phi"'] \arrow[ddl, bend left] \\
& t(M) \arrow[r] \arrow[d] & W_F \arrow[d] & \\
& M \arrow[r] & F^M &
\end{tikzcd}$$

$\square$

Since $W_F \to F^M$ is a precover there is a morphism $\phi : W \to W_F$ making the diagram commute, and since the square is an admissible pullback there is a morphism $\psi : T \to t(M)$ making the diagram commutative.

Hence, $\mathcal{A}(T, t(M)) \to \mathcal{A}(T, M)$ is surjective. To conclude the proof we need to show that when restricted to $\underline{\mathcal{A}}$ it becomes an iso. Assume that there are two morphisms $\underline{\psi}$ and $\underline{\psi}'$ in $\underline{\mathcal{A}}$ such that the following commutes:

$$\begin{tikzcd}
 & t(M) \arrow[d] \\
T \arrow[ur, shift left, "\underline{\psi}"] \arrow[ur, shift right, "\underline{\psi}'"'] \arrow[r, "\underline{f}"'] & M
\end{tikzcd}$$

So, if we call $h = \psi - \psi'$ in $\mathcal{A}$, we have that $T \xrightarrow{h} t(M) \to M$ factors through W so that we have that the solid part of the following diagram commutes:

$$\begin{tikzcd}
W \arrow[ddrr, dotted, "\gamma"] \arrow[ddrrr, dashed, bend left, "\eta"] \arrow[dddrr, bend right] & & & \\
 & T \arrow[ul, "p"'] \arrow[dr, "h"] \arrow[ddr, "g"'] & & \\
 & & t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\
 & & M \arrow[r] & F^M
\end{tikzcd}$$

where $\eta : W \to W_F$ comes from the fact that $W_F \to F^M$ is a precover, and γ from the fact that the square is an admissible pullback, and they make the complete diagram commute.

Let's call $h' = \gamma \circ p$, then composing both h and h' with $\rho : t(M) \to M$ gives the same morphism g. Hence, $\rho \circ (h - h') = 0$. But since $F' \xrightarrow{i} t(M) \to M$ is an admissible sequence we have the following exact sequence of abelian groups:

$$\begin{tikzcd}
\mathcal{A}(T, F) \arrow[r] & \mathcal{A}(T, t(M)) \arrow[r] & \mathcal{A}(T, M) \\
 & h - h' \arrow[r, mapsto] & 0
\end{tikzcd}$$

So there is a map $k : T \to F'$ such that $i \circ k = h - h'$, but $\underline{k} = 0$ so $\underline{h} = \underline{h'} = 0$, hence $\underline{\psi} - \underline{\psi}' = 0$ which proves that $\underline{\mathcal{A}}(T, t(M)) \cong \underline{\mathcal{A}}(T, M)$.

The naturality of the isomorphism in T and M is clear.

4.2 Abelian categories

Now we work in an abelian category with two torsion pairs $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ such that $t_2(\mathcal{F}_1) \subseteq \mathcal{F}_1$ and $f_1(\mathcal{T}_2) \subseteq \mathcal{T}_2$ and let $\mathcal{W} = \mathcal{T}_2 \cap \mathcal{F}_1$.

Recall that $(\underline{\mathcal{T}_1 * \mathcal{W}}, \underline{\mathcal{F}_1})$ (resp. $(\underline{\mathcal{T}_2}, \underline{\mathcal{W} * \mathcal{F}_2})$) is a left (resp. right) functorial torsion pair in $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$. Moreover, they satisfy $TC1-3, 3^*$.

Lemma 4.4. *The inclusion $i : \mathcal{T}_1 * \mathcal{W} \hookrightarrow \mathcal{A}$ admits a right adjoint $\widehat{t}$.*

Proof. For $M \in \mathcal{A}$ consider the exact sequence

$$0 \to T_1 \to M \to f_1(M) \to 0$$

with $T_1 \in \mathcal{T}_1$ and $f_1(M) \in \mathcal{F}$. Take $t_2 f_1(M) \hookrightarrow f_1(M)$ and observe that $t_2 f_1(M) \in \mathcal{W}$. Call it W_M and take the pullback diagram

$$\begin{tikzcd}
\widehat{t}(M) \arrow[r] \arrow[d] & W_M \arrow[d, tail] \\
M \arrow[r, two heads] & f_1(M)
\end{tikzcd}$$

then $\widehat{t}(M) \in \mathcal{T}_1 * \mathcal{W}$.

Now for any morphism $\widehat{T} \to M$ with $\widehat{T} \in \widehat{\mathcal{T}}$ the solid part of the following diagram commutes

$\widehat{T} \to M$ is mono by Buhler prop. 2.14: pullback of monic along epic is monic

$$\begin{tikzcd}
T \arrow[r, tail] \arrow[d, equal] & \widehat{t}(M) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & W \arrow[d, tail] \\
T \arrow[r, tail] & M \arrow[r, two heads] & f_1(M) \\
T_1 \arrow[r, tail] & \widehat{T} \arrow[r, two heads] \arrow[u] \arrow[uu, dotted, bend left] & W_1 \arrow[u, dashed] \arrow[uu, dashed, bend right]
\end{tikzcd}$$

Since the composition $T_1 \to \widehat{T} \to M \to f_1(M)$ is zero, there exists the dashed morphism $W_1 \to f_1(M)$, which lifts to the morphism $W_1 \to W$ (since $W \to f_1(M)$ is a $\mathcal{W}$-precover). Hence, there is a morphism $\widehat{T} \to \widehat{t}(M)$ making the diagram commutative. This means that

$$\mathcal{A}(\widehat{T}, \widehat{t}(M)) \xrightarrow{\mathcal{A}(\widehat{T}, \widehat{t}(M) \to M)} \mathcal{A}(\widehat{T}, M)$$

is surjective. But it is also injective, since $\operatorname{Ker}(\widehat{t}(M) \to M) = 0$. Hence, it is an iso and $\widehat{t}$ is right adjoint to i. ☐

functoriality should follow immediately

Lemma 4.5. *Let $\widehat{T}_1 \in \mathcal{T}_1 * \mathcal{W}$, i.e. there is an exact sequence*

$$0 \to t_1(\widehat{T}_1) \to \widehat{T}_1 \to W_1 \to 0.$$

If

$$\begin{tikzcd}
\widehat{T}_1 \arrow[r, two heads, "p"] \arrow[d, "g"] & W_1 \arrow[d, "g'"] \\
\widehat{T}_1' \arrow[r, "q"] & N
\end{tikzcd}$$

is a pushout diagram, then $N \in \mathcal{T}_1 * \mathcal{W}$.

Proof. Since it is a pushout, $\widehat{T}_1' \to N$ is epi, then consider $\widehat{t}(N)$ and the following commutative diagram

$$\begin{tikzcd}
\widehat{T}_1' \arrow[rr, two heads, "q"] \arrow[dr, dashed, "\rho"'] & & N \\
& \widehat{t}(N) \arrow[ur, tail, "\varepsilon"'] &
\end{tikzcd}$$

where the map $\widehat{T}_1' \to \widehat{t}(N)$ is given by the adjunction $(i, \widehat{t})$. Since $q = \varepsilon \circ \rho$ is epi, then ε is epi. But it is mono, so it is an isomorphism, hence $N \in \mathcal{T}_1 * \mathcal{W}$. ☐

Lemma 4.6. *In the same notation as the previous lemma, if $\varphi : N \to P$ is any map s.t. $\underline{\varphi} \circ \underline{q} = \underline{0}$ in $\underline{\mathcal{A}}$, then φ factors through $\mathcal{W}$.*

Proof. Since $\underline{\varphi} \circ \underline{q} = \underline{0}$ it means that $\varphi \circ q$ factors through $\mathcal{W}$, hence we have that the solid part of the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & t_1(T_1') \arrow[r] \arrow[d] & \widehat{T}_1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] \arrow[dd, dashed, bend left] & 0 \\
0 \arrow[r] & X \arrow[r] & \widehat{T}_1' \arrow[r] \arrow[d, equal] & N \arrow[d, dotted] \arrow[dr, "\varphi"] & \\
& & \widehat{T}_1' \arrow[r] & W \arrow[r] & P
\end{tikzcd}$$

Since $t_1(T_1') \to \widehat{T}_1 \to \widehat{T}_1' \to W$ is zero, there is the dashed morphism $W_1 \to W$ making the diagram commute. Since the square on the right is a pushout there is a map $N \to W$, and again the diagram commutes. Hence φ factors through $\mathcal{W}$. ☐

Lemma 4.7. *If $\mathcal{H}$ is balanced (i.e. mono and epi implies iso), then whenever $f : H_1 \to H_2$ is mono and epi in $\mathcal{H}$, there are bicartesian squares in $\mathcal{A}$*

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\quad\lrcorner"] & H_1 \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner\quad\lrcorner"] & W_1 \arrow[d] \\
W_2 \arrow[r] & H_2 \arrow[r] & T_2
\end{tikzcd}$$

where $W_1 = f_1(H_1)$ and $W_2 = t_2(H_2)$. In particular there is an exact sequence

$$0 \to F_1 \to W_1 \oplus W_2 \to T_2 \to 0.$$

Proof. We can build the pullback on the left and the pushout on the right as usual

$$\begin{tikzcd}
F_1 \arrow[r, "f^K"] \arrow[d, "u"] \arrow[dr, phantom, "\ulcorner"] & H_1 \arrow[r, "r"] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner"] & W_1 \arrow[d, "s"] \\
W_2 \arrow[r, "v"] & H_2 \arrow[r, "f^C"] & T_2
\end{tikzcd} \tag{1}$$

We will only prove that the square on the right hand side is a pullback, since the proof that the left square is a pushout is dual. The statetment that the square on the right is a pushout is equivalent to saying that there is an exact sequence

$$H_1 \xrightarrow{\begin{pmatrix} f \\ r \end{pmatrix}} H_2 \oplus W_1 \xrightarrow{\begin{pmatrix} f^C & s \end{pmatrix}} T_2 \longrightarrow 0 \tag{2}$$

Since f is both a mono and an epi in $\mathcal{H}$, then it is an iso and hence both a section and a retraction. Consider $g : H_2 \to H_1$ such that $\underline{g} \circ \underline{f} = \underline{1_{H_1}}$, that is there are maps $\alpha : H_1 \to W$ and $\beta : W \to H_1$ such that

$H_1 \oplus T_2 \cong H_2 \cong W_1 \Rightarrow T_2 \in \mathcal{W} * \mathcal{F}$ $W' \hookrightarrow T_2 \overset{0}{\twoheadrightarrow} F'$, with $W' \in \mathcal{W}, T_2 \in \mathcal{T}_2, F' \in \mathcal{F}_2$, hence $T_2 \in \mathcal{W}$.

$$\begin{tikzcd}
H_1 \arrow[r, "\begin{pmatrix} f \\ \alpha \end{pmatrix}"] \arrow[rr, bend right, "1_{H_1}"'] & H_2 \oplus W \arrow[r, "\begin{pmatrix} g & \beta \end{pmatrix}"] & H_1
\end{tikzcd}$$

is commutative in $\mathcal{A}$, and hence H_1 is a direct summand of $H_2 \oplus W$. We can actually choose $W = W_1$, in fact consider the commutative diagram

$$\begin{tikzcd}
H_1 \arrow[r, "\begin{pmatrix} f \\ r \end{pmatrix}"] \arrow[d, equal] & H_2 \oplus W_1 \arrow[r, "\begin{pmatrix} f^C & -s \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}"] & T_2 \\
H_1 \arrow[r, "\begin{pmatrix} f \\ \alpha \end{pmatrix}"] \arrow[rr, bend right, "1_{H_1}"'] & H_2 \oplus W \arrow[r, "\begin{pmatrix} g & \beta \end{pmatrix}"] & H_1
\end{tikzcd}$$

where $\rho : W_1 \to W$ comes from the fact that $H_1 \to W_1$ is a $\mathcal{W}$-preenvelope. Hence, $\begin{pmatrix} g & \beta \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \circ \begin{pmatrix} f \\ r \end{pmatrix} = 1_{H_1}$, that is H_1 is a direct summand of $H_2 \oplus W_1$. Moreover, it means that $\begin{pmatrix} f \\ r \end{pmatrix}$ is a section, that is the sequence in (2) is also exact on the left and the corresponding square in (1) is a pullback diagram.

Since both squares in (1) are bicartesian, it follows that the square

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\quad\lrcorner"] & W_1 \arrow[d] \\
W_2 \arrow[r] & T_2
\end{tikzcd}$$

is bicartesian as well. $\square$