1 The Nakaoka setting

Given a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{C}$, it was proved in [1] that it is possible to use it to construct a subcategory $\mathcal{W} \subseteq \mathcal{C}$ such that in the quotient category $\frac{\mathcal{C}}{\mathcal{W}} = \underline{\mathcal{C}}$ a heart $\mathcal{H}$ is defined in such a way that it is an abelian category and there is a homological functor $\mathcal{C} \to \mathcal{H}$.

Our goal is to provide a set of axioms for a (nice) additive category $\mathcal{A}$ and a couple of torsion pairs in it, in such a way that they will guarantee the existence of an abelian heart in $\mathcal{A}$. In a sense, we want to axiomatize $\underline{\mathcal{C}}$ and the pairs which are referred in Nakaoka's work as $(\underline{\mathcal{C}}^-, \underline{\mathcal{V}})$ and $(\underline{\mathcal{U}}, \underline{\mathcal{C}}^+)$.

Now we will briefly recall Nakaoka's setting. Assume that $\mathcal{C}$ is a triangulated category, and $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in $\mathcal{C}$, i.e.

1. $\mathcal{C}(\mathcal{U}, \mathcal{V}[1]) = 0$
2. $\mathcal{C} = \mathcal{U} * \mathcal{V}[1]$, where $X \in \mathcal{M} * \mathcal{N}$ if and only if there is a distinguished triangle
$$M \to X \to N[1] \to M[1]$$
with $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

Then, we put $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ and define $\underline{\mathcal{C}} = \mathcal{C}/\mathcal{W}$, and similarly for $\underline{\mathcal{U}}$, $\underline{\mathcal{V}}$, etc.

We define the following full subcategories of $\mathcal{C}$:

- $\mathcal{C}^+ = \mathcal{W} * \mathcal{V}[1]$
- $\mathcal{C}^- = \mathcal{U}[-1] * \mathcal{W}$

together with their respective quotients $\underline{\mathcal{C}}^+$ and $\underline{\mathcal{C}}^-$. Then, we have the following lemma.

Lemma 1. *Let $X \in \mathcal{C}$, TFAE:*

1. *$\underline{X} \in \underline{\mathcal{C}}^+$,*
2. *There is a monomorphism $\underline{X} \to \underline{V[1]}$ in $\underline{\mathcal{C}}$, for some $V \in \mathcal{V}$.*

The dual also holds:

Lemma 2. *Let $X \in \mathcal{C}$, TFAE:*

1. *$\underline{X} \in \underline{\mathcal{C}}^-$,*
2. *There is an epimorphism $\underline{U[-1]} \to \underline{X}$ in $\underline{\mathcal{C}}$, for some $U \in \mathcal{U}$.*

Corollary 3. *If $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in a triangulated category $\mathcal{C}$, then:*

1. *${}^{\perp}\underline{\mathcal{V}} = \underline{\mathcal{C}}^-$*
2. *$\underline{\mathcal{U}}^{\perp} = \underline{\mathcal{C}}^+$*

Lemma 4. *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{C}}^+$ be the left adjoint of the inclusion functor $j : \underline{\mathcal{C}}^+ \hookrightarrow \underline{\mathcal{C}}$. If $\lambda : 1_{\underline{\mathcal{C}}} \to j \circ F$ is the unit of the adjunction, then there is a pseudokernel-pseudocokernel sequence*
$$U_C \xrightarrow{u} C \xrightarrow{\lambda_C} (j \circ F)(C) \xrightarrow{+}$$
in $\underline{\mathcal{C}}$ such that $U_C \in \mathcal{U}$.

With dual:

Lemma 5. *Let $G : \underline{\mathcal{C}} \to \underline{\mathcal{C}}^-$ be the left adjoint of the inclusion functor $i : \underline{\mathcal{C}}^- \hookrightarrow \underline{\mathcal{C}}$. If $\varepsilon : i \circ G \to 1_{\underline{\mathcal{C}}}$ is the co-unit of the adjunction, then there is a pseudokernel-pseudocokernel sequence*
$$(i \circ G)(C) \xrightarrow{\varepsilon_C} C \xrightarrow{V}{}_C \xrightarrow{+}$$
in $\underline{\mathcal{C}}$ such that $V_C \in \mathcal{V}$.

Corollary 6. *$(\underline{\mathcal{C}}^-, \underline{\mathcal{V}})$ and $(\underline{\mathcal{U}}, \underline{\mathcal{C}}^+)$ are orthogonal pairs in $\underline{\mathcal{C}}$ provided $\mathcal{C}$ has split idempotents.*

Remark. 1. By prop 5.3 Nakaoka we have that $\underline{\mathcal{C}}^+$ has cokernels and, dually, $\underline{\mathcal{C}}^-$ has kernels. add better reference

2. We have inclusions $\mathcal{V} \subseteq \mathcal{C}^+$ and $\mathcal{U} \subseteq \mathcal{C}^-$ see page 5.5 of the notes

2 Torsion pairs

We fix an additive category $\mathcal{X}$ with pseudokernels and pseudocokernels on which idempotents split.

Definition 1. A pair $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{X}$ is a torsion pair in $\mathcal{X}$ if:

1. $$\begin{aligned} \mathcal{F} &= \mathcal{T}^{\perp} = \{X \in \mathcal{X} \mid \mathcal{X}(\mathcal{T}, X) = 0\}, \\ \mathcal{T} &= {}^{\perp}\mathcal{F} = \{X \in \mathcal{X} \mid \mathcal{X}(X, \mathcal{F}) = 0\}; \end{aligned}$$

2. FOr each $M \in \mathcal{X}$ there is a pseudokernel-pseudocokernel sequence
$$T_M \xrightarrow{\varepsilon_M} M \xrightarrow{\lambda_M} F^M$$
where $T_M \in \mathcal{T}$ and $F^M \in \mathcal{F}$.

If in addition the assignment $M \mapsto t(M) := T_M$ (resp. $M \mapsto f(M) := F^M$) is functorial and defines an adjoint pair (i, t) (resp. (f, j)), where $i : \mathcal{T} \hookrightarrow \mathcal{X}$ (resp. $j : \mathcal{F} \hookrightarrow \mathcal{X}$) is the inclusion functor, then we say that $\mathbb{t}$ is left (resp. right) functorial. In such a case, ε (resp. λ) is the counit (resp. unit) of the given adjoint pair. We say that $\mathbb{t}$ is functorial if it is right and left functorial.

Remark. Let $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ be a left functorial torsion pair in $\mathcal{X}$. Then

(a) For any $M \in \mathcal{X}, T' \in \mathcal{T}$ and $\alpha \in \mathcal{X}(T', M)$ there is a unique $\alpha' \in \mathcal{X}(T', t(M))$ such that $\varepsilon_M \circ \alpha' = \alpha$, i.e.
$$\begin{tikzcd} & T' \arrow[dl, dashed, "\exists!\,\alpha'"'] \arrow[d, "\alpha"] \\ t(M) \arrow[r, "\varepsilon_M"'] & M \end{tikzcd}$$

(b) Let $g : T_1 \to T_2$ be a morphism in $\mathcal{T}$, which admits a pseudocokernel $g^C : T_2 \to \mathrm{PCok}_{\mathcal{T}}(g)$ in $\mathcal{T}$. Then g^c is a pseudocokernel of g in $\mathcal{X}$.

Proof. see Octavio's notes □

The dual also holds.

3 Axiomatization

Definition 2. Let $\mathcal{X}$ be an additive category with pseudokernels and pseudocokernels, a *compatible* torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{X}$ consists of the two pairs $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ of full subcategories of $\mathcal{X}$ satisfying the following axioms:

(CT1) $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are respectively a left functorial and a right functorial torsion pair

define torsion pairs in additive categories

(CT2) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equiv. $\mathcal{F}_1 \subseteq \mathcal{F}_2$)

(CT3) Any morphism $g : T_1 \to T_1'$ in $\mathcal{T}_1$ iadmits a pseudocokernel $g^C : T_1' \to \mathrm{PCok}_{\mathcal{X}}(g)$, with $T_1'' := \mathrm{PCok}_{\mathcal{X}}(g) \in \mathcal{T}_1$, such that

$$0 \longrightarrow (T_1'', -)_{|\mathcal{F}_2} \xrightarrow{(g^C,-)} (T_1', -)_{|\mathcal{F}_2} \xrightarrow{(g,-)} (T_1, -)_{|\mathcal{F}_2}$$

is an exact sequence of functors.

(CT3)*Dual of **(CT3)**. **(CT3)***

Notation. In the case of a compatible torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{X}$, we have the adjoint pairs

$$(i_1, t_1) : \mathcal{T}_1 \underset{t_1}{\overset{i_1}{\rightleftarrows}} \mathcal{X} \qquad \text{and} \qquad (t_2, j_2) : \mathcal{F}_2 \underset{j_2}{\overset{t_2}{\leftrightarrows}} \mathcal{X}$$

In this case there are also the counit $\varepsilon_{1,M} : t_1(M) \to M$ and the unit $\lambda_{2,M} : M \to f_2(M)$. The heart of $\mathbb{t}$ is defined as $\mathcal{H} = \mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$.

Lemma 7. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$. Then the following statements hold true.*

(a) $\mathcal{F}_1 \cap \mathcal{T}_2 = 0$,

(a) $f_2(\mathcal{T}_1) \subseteq \mathcal{H}$ *and* $t_1(\mathcal{F}_2) \subseteq \mathcal{H}$.

Proposition 8. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$ and $f : H_1 \to H_2$ be a morphism in $\mathcal{H}$. Then*

*(a) if $f^C : H_2 \to T_1$ is the pseudocokernel of f, given by **(CT3)** and $\lambda_{2,T_1} : T_1 \to f_2(T_1)$, then*

$$\mathrm{Coker}\,(H_1 \xrightarrow{f} H_2) = (H_2 \xrightarrow{\tilde{f}^C}) f_1(T_1))$$

in $\mathcal{H}$, where $\tilde{f}^C := \lambda_{2,T_1} \circ f^C$;

*(a) if $f^K : F_2 \to H_1$ is the pseudokernel of f, given by **(CT3)*** and $\varepsilon_{1,F_2} : t_1(F_2) \to F_2$, then*

$$\mathrm{Ker}\,(H_1 \xrightarrow{f} H_2) = (t_1(F_2) \xrightarrow{\tilde{f}^K} H_1)$$

in $\mathcal{H}$, where $\tilde{f}^K = f^K \circ \varepsilon_{1,F_2}$.

Proposition 9. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$. Then, for any $f : H \to H'$ in $\mathcal{H}$ we have that:*

(a) f is a monomorphism in $\mathcal{H}$ if and only if there is a pseudokernel $\mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$*;*

(a) f is an epimorphism in $\mathcal{H}$ if and only if there is a pseudocokernel $\mathrm{PCok}_{\mathcal{T}_1}(f) \in \mathcal{T}_2$.

Definition 3. A compatible torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{X}$ is strong if the following axioms hold:

(CT4) Let $f : H_1 \to H_2$ in $\mathcal{H}$ be such that there is a pseudokernel $\mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$. Then, for the commutative diagram

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 := \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F)_2 \arrow[r, "\varepsilon_{1,F_2}"] & F_2 := \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1)
\end{tikzcd}$$

there exists a morphism $b : t_1(F_2) \to H_1$ such that $ab = \varepsilon_{1,F_2}$.

(CT4)* Dual

With these axioms we can prove that the heart has kernels and cokernels.

Theorem 10. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$. Then, the following are equivalent:*

1. *$\mathbb{t}$ is strong,*
2. *$\mathcal{H}$ is an abelian category.*

3.1 The case of an abelian category

Let's consider the case $\mathcal{X} = \mathcal{A}$ of an Abelian category with two torsion pairs $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ for $i = 1, 2$. Consider $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$.

Remark. In the case of an Abelian category $\mathcal{X} = \mathcal{A}$, we have that $\mathbb{t}$ is compatible if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proof. Let $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we need to show that item **(CT1)**, item **(CT3)** and item **(CT3)** hold.

(CT1) It is well known that any torsion pair in an abelian category is functorial.

(CT3) Let $g : T_1 \to T_1'$ be a morphism in $\mathcal{T}_1$. Consider the cokernel morphism of g in $\mathcal{A}$

$$\mathrm{Coker}_{\mathcal{A}}(T_1 \xrightarrow{g} T_1') = (T_1' \xrightarrow{c_g} \mathrm{Coker}\,(g)).$$

Since $\mathcal{T}_1$ is closed under quotient objects, we get that $\mathrm{Coker}\,(g) \in \mathcal{T}_1$. Therefore, we can choose $c_g : T_1' \to \mathrm{Coker}\,(g)$ as $g^C : T_1' \to \mathrm{PCok}_{\mathcal{A}}(g)$.

Is there an explicit choice that we made somewhere before when we talk about $\mathrm{PCok}_{\mathcal{A}}$?

(CT3)* Anologous to the previous.

□

4 Introduction

Let $\mathcal{A}$ be a *good* category (abelian/exact/triangulated). The precise meaning of this will have to be clarified later (probably, the recent Nakaoka-Palu's paper is the right setting). But, whatever the choice, two things should happen. First, idempotents should split in $\mathcal{A}$. Secondly, each torsion pair considered in $\mathcal{A}$ should be functorial on both sides. If $(\mathcal{T}, \mathcal{F})$ is such a torsion pair, we will denote by $t : \mathcal{A} \longrightarrow \mathcal{T}$ (resp. $f : \mathcal{A} \longrightarrow \mathcal{F}$) the right (resp. left) adjoint of the inclusion functor and, also, the composition $\mathcal{A} \xrightarrow{t} \mathcal{T} \overset{i}{\hookrightarrow} \mathcal{A}$ (resp. $\mathcal{A} \xrightarrow{f} \mathcal{F} \overset{j}{\hookrightarrow} \mathcal{A}$), where $\mathcal{T} \overset{i}{\hookrightarrow} \mathcal{A}$ (resp. $\mathcal{F} \overset{j}{\hookrightarrow} \mathcal{A}$) is the inclusion functor. The functoriality should then give rise to an admissible sequence $t(M) \longrightarrow M \longrightarrow f(M)$, for each object $M \in \mathcal{A}$ (e.g. if $\mathcal{A}$ is abelian, that sequence should be short exact, if $\mathcal{A}$ is exact it should be a conflation, if A is triangulated it should be a triangle).

$\mathcal{W}$ a full subcategory of $\mathcal{A}$ closed by direct summands and extensions, and consider the category $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$.

Let $(\mathcal{X}, \mathcal{Y})$ be a orthogonal pair in $\underline{\mathcal{A}}$ and consider the following classes in $\mathcal{A}$:

$$\mathcal{T} = \{T \in \mathcal{A} | \underline{T} \in \mathcal{X}\}$$
$$\mathcal{F} = \{F \in \mathcal{A} | \underline{F} \in \mathcal{Y}\}.$$

Lemma 11 (empty)**.**

If $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ is a orthogonal pair in a cocomplete and locally small abelian category $\mathcal{A}$, then $\mathbb{t}$ is a torsion pair. Indeed, if M is any object and we consider the set $\mathcal{T}_M$ of subobjects of M which are in $\mathcal{T}$, then $t(M) := \sum_{T \in \mathcal{T}_M} T$ is subobject of M which is an epimorphic image of $\coprod_{T \in \mathcal{T}_M}$ and, hence, we have that $t(M) \in \mathcal{T}_M$. If we had a nonzero morphism $f : T' \longrightarrow M/t(M)$, where $T' \in \mathcal{T}$, then we would have that $\mathrm{Im}(f) = \tilde{T}/t(M)$ is a nonzero submodule of $M/t(M)$ which is in $\mathcal{T}$. Since $\mathcal{T}$ is closed under extensions and we have an exact sequence $0 \to t(M) \longrightarrow \tilde{T} \longrightarrow \tilde{T}/t(M) \to 0$, we conclude that $\tilde{T} \in \mathcal{T}$. But then we have that $\tilde{T} \in \mathcal{T}_M$, which is a contradiction since $t(M)$ contains all subobjects in $\mathcal{T}_M$.

Lemma 12. *Let $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in $\mathcal{A}$, with associated radical functors t_i and coradical functors f_i $(i = 1, w)$, respectively. Suppose that they satisfy the following conditions:*

a) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equivalently, $\mathcal{F}_1 \subseteq \mathcal{F}_2$)

b) $\mathcal{T}_2 \cap \mathcal{F}_1 = 0$.

c) $(\underline{\mathcal{T}}_1, \underline{\mathcal{F}}_2)$ is an orthogonal pair in $\underline{\mathcal{A}} := \mathcal{A}/\mathcal{W}$, where $\mathcal{W} = \mathcal{T}_1 \cap \mathcal{F}_2$.

Then the following assertions hold:

1. $\mathcal{T}_1$ consists of those objects $X \in \mathcal{A}$ such that $f_2(X) \in \mathcal{W}$. We will write $\mathcal{T}_1 = \mathcal{T}_2 \star \mathcal{W}$.

2. $\mathcal{F}_2$ consists of those objects $Y \in \mathcal{A}$ such that $t_1(Y) \in \mathcal{W}$. We will write $\mathcal{F}_2 = \mathcal{W} \star \mathcal{F}_1$.

Proof. We just prove assertion 1, and assertion 2 will follow by duality. Let us take $X \in \mathcal{T}_2 \star \mathcal{W}$. Since we have an admissible sequence $t_2(X) \longrightarrow X \longrightarrow f_2(X)$ whose outer terms are in $\mathcal{T}_2$ and $\mathcal{W}$, respectively, and these two classes are contained in $\mathcal{T}_1$ we conclude that $\mathcal{T}_1' \subseteq \mathcal{T}_1$, because $\mathcal{T}_1$ is closed under taking extensions in $\mathcal{A}$.

Let T_1 be in $\mathcal{T}_1$ and consider its canonical admissible sequence

$$t_2(T_1) \to T_1 \xrightarrow{f} f_2(T_1). \tag{1}$$

Note that $\underline{f} = 0$ because of condition c) in the statement. It follows that f decomposes in the form $f : T_1 \xrightarrow{\gamma} W \xrightarrow{\phi} f_2(T_1)$, where $W \in \mathcal{W}$. We then consider the following admissible pullback diagram

$$\begin{tikzcd}
t_2(T_1) \arrow[r] \arrow[d, equal] & \widehat{T_1} \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\
t_2(T_1) \arrow[r] & T_1 \arrow[r, "f"] & f_2(T_1)
\end{tikzcd} \tag{2}$$

Then, there exist a (non necessarely unique) $\eta : T_1 \to \widehat{T_1}$ making the following diagram commute.

$$\begin{tikzcd}
T_1 \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "1"'] & & \\
& \widehat{T_1} \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\
& T_1 \arrow[r, "f"] & f_2(T_1)
\end{tikzcd} \tag{3}$$

Hence, $T_1 <_{\oplus} \widehat{T_1} \in \mathcal{T}_2 * \mathcal{W}$. This implies that $\mathcal{T}_1 \subseteq \mathrm{add}(\mathcal{T}_2 * \mathcal{W})$. The proof will be finished once we check that $\mathcal{T}_2 * \mathcal{W}$ is closed under direct summands. But this is a direct consequence of the functoriality of the torsion pair. Indeed if we have admissible torsion sequences $t_2(M) \longrightarrow M \longrightarrow f_2(M)$ and $t_2(N) \longrightarrow N \longrightarrow f_2(N)$, then the coproduct sequence $t_2(M) \oplus t_2(N) \longrightarrow M \oplus N \longrightarrow f_2(M) \oplus f_2(N)$ is the admissible torsion sequence for $M \oplus N$. The fact that $M \oplus N \in \mathcal{T}_2 * \mathcal{W}$ is then equivalent to the fact that $f_2(M) \oplus f_2(N) \in \mathcal{W}$. Since $\mathcal{W}$ is closed under direct summands, we conclude that $f_2(M) \in \mathcal{W}$ and, hence, that $M \in \mathcal{T}_2 * \mathcal{W}$.

□

5 Induced torsion theories

Lemma 13. *Let $\mathcal{A}$ be a (nice) category and $\mathcal{W} \subseteq \mathcal{A}$ a subcategory such that* $\operatorname{add}(\mathcal{W}) = \mathcal{W}$*. If $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a torsion pair such that $\mathcal{W} \subseteq \mathcal{F}$, then $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is an orthogonal pair in $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$.*

Lemma 14. *Let $\mathcal{A}$ and $\mathcal{W} \subseteq \mathcal{A}$ be defined as above and let $({}^{\perp}\mathcal{F}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$ such that $\mathcal{W} \subseteq \mathcal{F}$. Call $p : \mathcal{A} \to \underline{\mathcal{A}}$ the quotient functor. The following assertions hold:*

1. *$p^{-1}({}^{\perp}(\underline{\mathcal{F}})) = \operatorname{add}({}^{\perp}\mathcal{F} * \mathcal{W})$.*
2. *If $\mathcal{W}$ is precovering in $\mathcal{F}$, then $({}^{\perp}\underline{\mathcal{F}}, \underline{\mathcal{F}})$ is a torsion pair in $\underline{\mathcal{A}}$.*

Lemma 15. *Let $\mathcal{A}$ be a (nice) category with a torsion pair $({}^{\perp}\mathcal{F}, \mathcal{F})$ and a precovering class $\mathcal{W} \subseteq \mathcal{F}$ such that for any $F \in \mathcal{F}$ there is an admissible sequence*

$$F' \to W \to F$$

such that $F' \in \mathcal{F}$.

Then the torsion pair $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial.

Recall that the truncation $t : \underline{\mathcal{A}} \to {}^{\perp}(\underline{\mathcal{F}})$ is given by the following construction.

Let $M \in \mathcal{A}$ be any object, take an admissible sequence

$$T_M \to M \to F^M$$

with $T_M \in {}^{\perp}\mathcal{F}$ and $F^M \in \mathcal{F}$. Moreover, consider $W_F \to F^M$ with $W_M \in \mathcal{W}$ as before, and take the admissible pullback:

$$\begin{tikzcd} t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\ M \arrow[r] & F \end{tikzcd}$$

Then, t restricts to a functor $\underline{t} : \underline{\mathcal{A}} \to {}^{\perp}\underline{\mathcal{F}}$.

In order to prove that $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial we need to show that $\underline{t}$ admits a right adjoint.

Proof. Let $M \in \mathcal{A}$ and consider $M \to F^M$ and $W_F \to F^M$ as above. For any $T \in {}^{\perp}\mathcal{F} * \mathcal{W}$ consider any morphism $f : T \to M$. Since $T \to M \to F^M$ is 0 in $\underline{\mathcal{A}}$ we that the solid part of the following diagram commutes.

$$\begin{tikzcd} T \arrow[rrr] \arrow[dr, dotted, "\psi"] \arrow[ddr, bend right, "f"'] & & & W \arrow[dl, dashed, "\phi"'] \arrow[ddl, bend left] \\ & t(M) \arrow[r] \arrow[d] & W_F \arrow[d] & \\ & M \arrow[r] & F^M & \end{tikzcd}$$

□

Since $W_F \to F^M$ is a precover there is a morphism $\phi : W \to W_F$ making the diagram commute, and since the square is an admissible pullback there is a morphism $\psi : T \to t(M)$ making the diagram commutative.

Hence, $\mathcal{A}(T, t(M)) \to \mathcal{A}(T, M)$ is surjective. To conclude the proof we need to show that when restricted to $\underline{\mathcal{A}}$ it becomes an iso. Assume that there are two morphisms $\underline{\psi}$ and $\underline{\psi}'$ in $\underline{\mathcal{A}}$ such that the following commutes:

$$\begin{tikzcd} & t(M) \arrow[d] \\ T \arrow[ur, shift left, "\underline{\psi}"] \arrow[ur, shift right, "\underline{\psi}'"'] \arrow[r, "\underline{f}"'] & M \end{tikzcd}$$

So, if we call $h = \psi - \psi'$ in $\mathcal{A}$, we have that $T \xrightarrow{h} t(M) \to M$ factors through W so that we have that the solid part of the following diagram commutes:

$$\begin{tikzcd}
W \arrow[ddrr, bend right] \arrow[drr, dotted, bend left, "\gamma"] \arrow[drrr, dashed, bend left, "\eta"] & & & \\
& T \arrow[ul, "p"'] \arrow[dr, "h"] \arrow[ddr, "g"'] & & \\
& & t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\
& & M \arrow[r] & F^M
\end{tikzcd}$$

where $\eta : W \to W_F$ comes from the fact that $W_F \to F^M$ is a precover, and γ from the fact that the square is an admissible pullback, and they make the complete diagram commute.

Let's call $h' = \gamma \circ p$, then composing both h and h' with $\rho : t(M) \to M$ gives the same morphism g. Hence, $\rho \circ (h - h') = 0$. But since $F' \xrightarrow{i} t(M) \to M$ is an admissible sequence we have the following exact sequence of abelian groups:

$$\begin{array}{ccccc}
\mathcal{A}(T, F) & \longrightarrow & \mathcal{A}(T, t(M)) & \longrightarrow & \mathcal{A}(T, M) \\
 & & & & \\
 & & h - h' & \longmapsto & 0
\end{array}$$

So there is a map $k : T \to F'$ such that $i \circ k = h - h'$, but $\underline{k} = 0$ so $\underline{h} = \underline{h'} = 0$, hence $\underline{\psi} - \underline{\psi'} = 0$ which proves that $\underline{\mathcal{A}}(T, t(M)) \cong \underline{\mathcal{A}}(T, M)$.

The naturality of the isomorphism in T and M is clear.

6 Abelian categories

Now we work in an abelian category with two torsion pairs $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ such that $t_2(\mathcal{F}_1) \subseteq \mathcal{F}_1$ and $f_1(\mathcal{T}_2) \subseteq \mathcal{T}_2$ and let $\mathcal{W} = \mathcal{T}_2 \cap \mathcal{F}_1$.

Recall that $(\underline{\mathcal{T}_1 * \mathcal{W}}, \underline{\mathcal{F}_1})$ (resp. $(\underline{\mathcal{T}_2}, \underline{\mathcal{W} * \mathcal{F}_2})$) is a left (resp. right) functorial torsion pair in $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$. Moreover, they satisfy $TC1-3, 3^*$.

Lemma 16. *The inclusion $i : \mathcal{T}_1 * \mathcal{W} \hookrightarrow \mathcal{A}$ admits a right adjoint $\widehat{t}$.*

Proof. For $M \in \mathcal{A}$ consider the exact sequence

$$0 \to T_1 \to M \to f_1(M) \to 0$$

with $T_1 \in \mathcal{T}_1$ and $f_1(M) \in \mathcal{F}$. Take $t_2 f_1(M) \hookrightarrow f_1(M)$ and observe that $t_2 f_1(M) \in \mathcal{W}$. Call it W_M and take the pullback diagram

$$\begin{tikzcd}
\widehat{t}(M) \arrow[r] \arrow[d] & W_M \arrow[d, tail] \\
M \arrow[r, two heads] & f_1(M)
\end{tikzcd}$$

then $\widehat{t}(M) \in \mathcal{T}_1 * \mathcal{W}$.

Now for any morphism $\widehat{T} \to M$ with $\widehat{T} \in \widehat{\mathcal{T}}$ the solid part of the following diagram commutes

$\widehat{T} \to M$ is mono by Buhler prop. 2.14: pullback of monic along epic is monic

$$\begin{tikzcd}
T \arrow[r, tail] \arrow[d, equal] & \widehat{t}(M) \arrow[r, two heads] \arrow[d] & W \arrow[d] \\
T \arrow[r, tail] & M \arrow[r, two heads] & f_1(M) \\
T_1 \arrow[r, tail] & \widehat{T} \arrow[r, two heads] \arrow[u] \arrow[uu, dotted, bend left] & W_1 \arrow[u, dashed] \arrow[uu, dashed, bend right]
\end{tikzcd}$$

Since the composition $T_1 \to \widehat{T} \to M \to f_1(M)$ is zero, there exists the dashed morphism $W_1 \to f_1(M)$, which lifts to the morphism $W_1 \to W$ (since $W \to f_1(M)$ is a $\mathcal{W}$-precover). Hence, there is a morphism $\widehat{T} \to \widehat{t}(M)$ making the diagram commutative. This means that

$$\mathcal{A}(\widehat{T}, \widehat{t}(M)) \xrightarrow{\mathcal{A}(\widehat{T}, \widehat{t}(M) \to M)} \mathcal{A}(\widehat{T}, M)$$

is surjective. But it is also injective, since $\operatorname{Ker}(\widehat{t}(M), M) = 0$. Hence, it is an iso and $\widehat{t}$ is right adjoint to i. $\square$

functoriality should follow immediately

Lemma 17. *Let $\widehat{T}_1 \in \mathcal{T}_1 * \mathcal{W}$, i.e. there is an exact sequence*

$$0 \to t_1(\widehat{T}_1) \to \widehat{T}_1 \to W_1 \to 0.$$

If

$$\begin{tikzcd}
\widehat{T}_1 \arrow[r, two heads, "p"] \arrow[d, "g"] & W_1 \arrow[d, "g'"] \\
\widehat{T}_1' \arrow[r, "q"] & N
\end{tikzcd}$$

is a pushout diagram, then $N \in \mathcal{T}_1 * \mathcal{W}$.

Proof. Since it is a pushout, $\widehat{T}_1' \to N$ is epi, then consider $\widehat{t}(N)$ and the following commutative diagram

$$\begin{tikzcd}
\widehat{T}_1' \arrow[rr, two heads, "q"] \arrow[dr, dashed, "\rho"'] & & N \\
& \widehat{t}(N) \arrow[ur, tail, "\varepsilon"'] &
\end{tikzcd}$$

where the map $\widehat{T}_1' \to \widehat{t}(N)$ is given by the adjunction $(i, \widehat{t})$. Since $q = \varepsilon \circ \rho$ is epi, then ε is epi. But it is mono, so it is an isomorphism, hence $N \in \mathcal{T}_1 * \mathcal{W}$. $\square$

Lemma 18. *In the same notation as the previous lemma, if $\varphi : N \to P$ is any map s.t. $\underline{\varphi} \circ \underline{q} = \underline{0}$ in $\underline{\mathcal{A}}$, then φ factors through $\mathcal{W}$.*

Proof. Since $\underline{\varphi} \circ \underline{q} = \underline{0}$ it means that $\varphi \circ q$ factors through $\mathcal{W}$, hence we have that the solid part of the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & t_1(T_1') \arrow[r] \arrow[d] & \widehat{T}_1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] \arrow[dd, dashed, bend left] & 0 \\
0 \arrow[r] & X \arrow[r] & \widehat{T}_1' \arrow[r] \arrow[d, equal] & N \arrow[d, dotted] \arrow[dr, "\varphi"] & \\
& & \widehat{T}_1' \arrow[r] & W \arrow[r] & P
\end{tikzcd}$$

Since $t_1(T_1') \to \widehat{T}_1 \to \widehat{T}_1' \to W$ is zero, there is the dashed morphism $W_1 \to W$ making the diagram commute. Since the square on the right is a pushout there is a map $N \to W$, and again the diagram commutes. Hence φ factors through $\mathcal{W}$. $\square$

Lemma 24. *If $\mathcal{H}$ is balanced (i.e. mono and epi implies iso), then whenever $f : H_1 \to H_2$ is mono and epi in $\mathcal{H}$, there are bicartesian squares in $\mathcal{A}$*

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near end] & H_1 \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near end] & W_1 \arrow[d] \\
W_2 \arrow[r] & H_2 \arrow[r] & T_2
\end{tikzcd}$$

where $W_1 = f_1(H_1)$ and $W_2 = t_2(H_2)$. In particular there is an exact sequence

$$0 \to F_1 \to W_1 \oplus W_2 \to T_2 \to 0.$$

Proof. We can build the pullback on the left and the pushout on the right as usual

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & H_1 \arrow[r, "r"] \arrow[d, "f"] & W_1 \arrow[d, "s"] \\
W_2 \arrow[r] & H_2 \arrow[r, "f^C"] \arrow[ur, phantom, "\lrcorner", very near end] & T_2
\end{tikzcd} \tag{4}$$

We will only prove that the square on the right hand side is a pullback, since the proof that the left square is a pushout is dual. The statetment that the square on the right is a pushout is equivalent to saying that there is an exact sequence

$$H_1 \xrightarrow{\left(\begin{smallmatrix} f \\ r \end{smallmatrix}\right)} H_1 \oplus W_1 \xrightarrow{\left(\begin{smallmatrix} f^C & s \end{smallmatrix}\right)} T_2 \longrightarrow 0 \tag{5}$$

Since f is both a mono and an epi in $\mathcal{H}$, then it is an iso and hence both a section and a retraction. Consider $g : H_2 \to H_1$ such that $\underline{g} \circ \underline{f} = \underline{1_{H_1}}$, that is there are maps $\alpha : H_1 \to W$ and $\beta : W \to H_1$ such that

$$\begin{tikzcd}
H_1 \arrow[r, "\left(\begin{smallmatrix} f \\ \alpha \end{smallmatrix}\right)"] \arrow[rr, bend right, "1_{H_1}"'] & H_2 \oplus W \arrow[r, "\left(\begin{smallmatrix} g & \beta \end{smallmatrix}\right)"] & H_1
\end{tikzcd}$$

is commutative in $\mathcal{A}$, and hence H_1 is a direct summand of $H_2 \oplus W$. We can actually choose $W = W_1$, in fact consider the commutative diagram

$$\begin{tikzcd}
H_1 \arrow[r, "\left(\begin{smallmatrix} f \\ r \end{smallmatrix}\right)"] \arrow[d, equal] & H_1 \oplus W_1 \arrow[r, "\left(\begin{smallmatrix} f^C & s \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix}\right)"] & T_2 \\
H_1 \arrow[r, "\left(\begin{smallmatrix} f \\ \alpha \end{smallmatrix}\right)"] \arrow[rr, bend right, "1_{H_1}"'] & H_2 \oplus W \arrow[r, "\left(\begin{smallmatrix} g & \beta \end{smallmatrix}\right)"] & H_1
\end{tikzcd}$$

where $\rho : W_1 \to W$ comes from the fact that $H_1 \to W_1$ is a $\mathcal{W}$-preenvelope. Hence, $\left(\begin{smallmatrix} g & \beta \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} f \\ r \end{smallmatrix}\right) = 1_{H_1}$, that is H_1 is a direct summand of $H_2 \oplus W_1$. Moreover, it means that $\left(\begin{smallmatrix} g & \beta \end{smallmatrix}\right)$ is a section, that is the sequence in (5) is also exact on the left and the corresponding square in (4) is a pullback diagram.

Since both squares in (4) are bicartesian, it follows that the square

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near end] & W_1 \arrow[d] \\
W_2 \arrow[r] & T_2
\end{tikzcd}$$

is bicartesian as well. □

7 Second approach to axiomatization

We give another set of axioms:

TC1 $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ are two respectively left functorial and right functorial torsion pairs in $\mathcal{X}$.

TC2 $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equivalently $\mathcal{F}_1 \subseteq \mathcal{F}_2$).

TC3 For any morphism $g : T_1 \to T_1'$ in $\mathcal{T}_1$ has a pseudocokernel in $\mathcal{T}_1$ which completes diagrams in a unique way wrt $\mathcal{F}_2$.

TC3* Dual of **TC3**.

TC4

explain this axiom

$$\begin{tikzcd}
F_1 \arrow[r, "f^K"] & H_1 \arrow[r, "\forall f"] \arrow[d, dashed] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 \arrow[d] \\
i_1 t_1(F_2) \arrow[r, "\varepsilon"] \arrow[ur, dashed] & F \arrow[r] & H_2 \arrow[r] & j_2 f_2(T_1)
\end{tikzcd}$$

TC4* Dual of **TC4**.

EXAMPLES

add examples from page 2, 9/11/16

2 If $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in a triangulated category (as in Nakaoka's work) produces an example.

Add reference

3 Let $\mathcal{D}$ be a triangulated category with two t-structures $(\mathcal{U}_1, \mathcal{U}_1^\perp)$ and $(\mathcal{U}_2, \mathcal{U}_2^\perp)$ such that $\mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1$. Then, these satisfy axioms **TC1-TC3**,**TC3***, hence $\mathcal{H} = \mathcal{U}_1 \cap \mathcal{U}_2^\perp$ has kernels and cokernels. Moreover, TFAE:

1.a **TC4** holds.

1.b If $V_1 \to H_1 \xrightarrow{f} H_2 \xrightarrow{+}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $V_1 \in \mathcal{U}_1^\perp$, then $V_1 \in \mathcal{U}_2^\perp[-1]$.

And, dually, there is an equivalence of the following:

2.a **TC4*** holds.

2.b If $H_1 \xrightarrow{f} H_2 \to U_2 \xrightarrow{+}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $U_2 \in \mathcal{U}_2$, then $U_2 \in \mathcal{U}_1[1]$.

Proof of the equivalences in example 3. Let's $\mathcal{D}$ be a triangulated category with two t-structures as in example 3. The pseudocokernel of a morphism in $\mathcal{U}_1$ can be computed by taking the cone in $\mathcal{D}$, i.e. given a morphism $f : U_1 \to U_1'$ in $\mathcal{U}_1$ we can compute a pseudocokernel in $\mathcal{U}_1$ by completing f to a triangle

$$U_1 \xrightarrow{f} U_1' \to \mathrm{Cone}(f) \xrightarrow{+} .$$

Moreover, this pseudocokernel satisfies **TC3**.

Now, assume that **TC1-TC3**,**TC3*** are satisfied together with axiom **1.b**, and consider the solid part of the diagram as in **TC4**:

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_1 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
\tau_{\mathcal{U}_1}(V_2) \arrow[r, "\varepsilon"] \arrow[ur, dotted, "\beta"] & V_2 \arrow[r] & H_2 \arrow[r] & \tau^{\mathcal{U}_2^\perp}\mathrm{Cone}(f)
\end{tikzcd}$$

with $\mathrm{Cone}(f)[-1] \in \mathcal{U}_1^\perp$ and where the upper row is a distinguished triangle. By **1.b** then it belongs to $\mathcal{U}_2^\perp[-1]$, i.e. $\mathrm{Cone}(f) \in \mathcal{U}_2^\perp$, so λ is an iso, conseguently α is an iso and so is ε, so there exist a map $\beta = \alpha^{-1} \circ \varepsilon$ making the diagram commute, that is **TC4** holds.

Conversely, assume that **TC1-TC3**,**TC3*** are satisfied together with **TC4**. Consider again the solid part of the diagram

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "\lambda[-1]"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_1 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
\tau_{\mathcal{U}_2^\perp}(\mathrm{Cone}(f))[-1] \arrow[r] & V_2 \arrow[r] & H_2 \arrow[r] & \tau^{\mathcal{U}_2^\perp}\mathrm{Cone}(f) \\
& \tau_{\mathcal{U}_1}(V_2) \arrow[u, "\varepsilon"] \arrow[uu, dotted, bend left, "\beta"] & &
\end{tikzcd}$$

with $\mathrm{Cone}(f)[-1] \in \mathcal{U}_1^\perp$. Neeman guarantees that α can be taken so that the square on the left is a pullback. Axiom **TC4** gives the existence of $\beta : \tau_{\mathcal{U}_1}(V_2) \to H_1$ such that $\alpha \circ \beta = \varepsilon$.

Add reference

Since $\tau_{\mathcal{U}_1}$ is a functor, there is also a morphism $\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(H_1) = H_1 \to \tau_{\mathcal{U}_1}(V_2)$ such that $\varepsilon \circ \tau_{\mathcal{U}_1}(\alpha) = \alpha$, hence $\varepsilon \circ \tau_{\mathcal{U}_1}(\alpha) \circ \beta = \varepsilon$. By the functoriality of the torsion pair $(\mathcal{U}_1, \mathcal{U}_1^\perp)$, this means that $\tau_{\mathcal{U}_1}(\alpha) \circ \beta = 1_{\tau_{\mathcal{U}_1}(V_2)}$. Then, β is a section.

Hence, we can write $\tau_{\mathcal{U}_1}(\alpha) : H_1 \to \tau_{\mathcal{U}_1}(V_2)$ as

$$\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(V_2) \oplus H_1' \xrightarrow{(\,*\ 0\,)} \tau_{\mathcal{U}_1}(V_2)$$

for some $H_1' \underset{\oplus}{<} H_1$ such that α vanishes on H_1'. If we consider the solid part of the diagram

$$\begin{tikzcd}
& H_1' \arrow[dl, dashed] \arrow[d, "\binom{1}{0}"] \arrow[dr, "0"] & & \\
\mathrm{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1] \arrow[r] & H_1 \arrow[r, "\tau_{\mathcal{U}_1}(\alpha)"] & \tau_{\mathcal{U}_1}(V_2) \arrow[r, dashed, "+"] & {}
\end{tikzcd}$$

we can construct the dashed arrow, and the fact that the triangle commutes means that $H_1' \underset{\oplus}{<} \mathrm{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1]$.

Observe that $\mathrm{Cone}(\alpha) = \mathrm{Cone}(\lambda)[-1]$, since the square

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "\lambda[-1]"] & H_1 \arrow[d, "\alpha"] \\
\tau_{\mathcal{U}_2^\perp}(\mathrm{Cone}(f))[-1] \arrow[r] & V_2
\end{tikzcd}$$

is a pullback. Moreover, $\mathrm{Cone}(\lambda)[-1] = (\tau_{\mathcal{U}_2}(\mathrm{Cone}(f))[1])[-1] = \tau_{\mathcal{U}_2}(\mathrm{Cone}(f))$. Hence, $\mathrm{Cone}(\alpha) \in \mathcal{U}_2$ and $\tau^{\mathcal{U}_1^\perp}(\mathrm{Cone}(\alpha)) = 0$, that is, $\mathrm{Cone}(\alpha) \in \mathcal{U}_1$, and since there is a distinguished triangle

$$H_1 \xrightarrow{\alpha} V_2 \to \mathrm{Cone}(\alpha) \xrightarrow{+}$$

with $H_1, \mathrm{Cone}(\alpha) \in \mathcal{U}_1$ it follows that $V_2 \in \mathcal{U}_1$. Hence, $\tau_{\mathcal{U}_1}(V_2) \cong V_2$.

We can then write $V_2 \underset{\oplus}{<} H_1$ and consider the commutative diagram

$$\begin{tikzcd}
H_1 \cong H_1' \oplus V_2 \arrow[r, "(\,f'\ \tilde{f}\,)"] \arrow[d, "(\,0\ 1\,)"] & H_2 \arrow[d, equal] \\
V_2 \arrow[r] & H_2
\end{tikzcd}$$

so $f' = 0$. Hence, the inclusion $\binom{1}{0} : H_1' \to H_1' \oplus V_2$ can be lifted to $\mathrm{Cone}(f)[-1]$ and $H_1' \underset{\oplus}{<} \mathrm{Cone}(f)[-1]$. Since $\mathrm{Cone}(f)[-1] \in \mathcal{U}_1^\perp$, so does H_1'. Similarly, $H_1' \in \mathcal{U}_1$ because $H_1 \in \mathcal{U}_1$. Hence, $H_1' = 0$ and $\alpha : H_1 \to V_2$ is an iso. The same follows for λ. Therefore, $\mathrm{Cone}(f) \in \mathcal{U}_2^\perp$ which proves **1.b**. $\square$

We can see a special case of example 3 in the case of the derived category of a ring. Let R be a commutative ring, consider the t-structure $(\mathcal{U}_1, \mathcal{U}_1^{\perp}) = (\mathcal{D}^{\leq 0}(R), \mathcal{D}^{>0}(R))$ in $\mathcal{D}(R)$. Given an idempotent ideal $I = I^2 \lhd R$, it defines three classes of modules

$$\begin{aligned}
\mathcal{C}_I &= \{C \in \text{Mod-}R | IC = C\} \\
\mathcal{T}_I &= \{T \in \text{Mod-}R | IT = 0\} \cong \text{Mod-}\frac{R}{I} \\
\mathcal{F}_I &= \{F \in \text{Mod-}R | Ix \neq 0 \forall x \in F \setminus \{0\}\}
\end{aligned}$$

such that $(\mathcal{C}_I, \mathcal{T}_I)$ and $(\mathcal{T}_I, \mathcal{F}_I)$ make two torsion pairs. We call the triple $(\mathcal{C}_I, \mathcal{T}_I, \mathcal{F}_I)$ a TTP triple.

We define the t-structure $(\mathcal{U}_2, \mathcal{U}_2^{\perp})$ as the Happel-Reiten-Smalo t-structure associated to the torsion pair $(\mathcal{C}_I, \mathcal{T}_I)$ in Mod-R:

$$\begin{aligned}
\mathcal{U}_2 &= \{U_2 \in \mathcal{D}^{\leq 0}(R) | H^0(U_2) \in \mathcal{C}_I\} \\
\mathcal{U}_2^{\perp} &= \{V_2 \in \mathcal{D}^{\geq 0}(R) | H^0(V_2) \in \mathcal{T}_I\}.
\end{aligned}$$

In this case we can check that condition **1.b** holds. In fact, let $\mathcal{H}$ be the heart

$$\begin{aligned}
\mathcal{U}_1 \cap \mathcal{U}_2^{\perp} &= \mathcal{D}^{\leq 0}(R) \cap \mathcal{U}_2^{\perp} \\
&= \{T[0] | T \in \mathcal{T}_I\} \cong \text{Mod-}\frac{R}{I}.
\end{aligned}$$

Hence, $\mathcal{H}$ is abelian.

Now, consider $V_1 \in \mathcal{U}_1^{\perp}$ such that there is an exact triangle

$$V_1 \to T_1[0] \xrightarrow{f[0]} T_2[0] \xrightarrow{+}$$

with $T_1, T_2 \in \mathcal{H}$. Of course, $V_1 = \text{Cone}(f)[-1]$, i.e.

$$V_1 = \cdots \to 0 \to \overset{0}{T_1} \xrightarrow{f} \overset{1}{T_2} \to 0 \to \cdots$$

where the numbers over T_1 and T_2 represent their cohomological degree.

The fact that $V_1 \in \mathcal{U}_1^{\perp} = \mathcal{D}^{>0}(R)$ implies that $H^0(V_1) = 0$, i.e. f is mono. To prove that $V_1 \in \mathcal{U}_2^{\perp}[-1]$ we would need to show that $\text{Coker}(f) = H^1(V_1)$ belongs to $\mathcal{T}_I$, but this follows from the fact that f is a mono in $\mathcal{T}_I$ which is a torsion class.

7.1 Polishchuk correspondence

Given a t-structure $(\mathcal{U}_1, \mathcal{U}_1^{\perp})$ in a triangulated category $\mathcal{D}$ there is a correspondence between t-structures $(\mathcal{U}_2, \mathcal{U}_2^{\perp})$ satisfying $\mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1$ in $\mathcal{D}$ and torsion pairs in $\mathcal{H}_1 = \mathcal{U}_1 \cap \mathcal{U}^{\perp}[1]$, namely

$$\left\{ \begin{array}{c} \text{torsion pairs in} \\ \mathcal{H}_1 = \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1] \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{t-structures } (\mathcal{U}_2, \mathcal{U}_2^{\perp}) \\ \text{in } \mathcal{D} \text{ satisfying} \\ \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1 \end{array} \right\}$$

$$(\mathcal{X}, \mathcal{Y}) \longrightarrow \left\{ \begin{array}{l} \mathcal{U}_2 = \{X \in \mathcal{U}_1 \mid H^0(X) \in \mathcal{X}\} \\ \mathcal{U}_2^{\perp} = \{Y \in \mathcal{U}_1^{\perp} \mid H^0(Y) \in \mathcal{Y}\} \end{array} \right.$$

$$(\mathcal{U}_2 \cap \mathcal{H}_1, \mathcal{U}_2^{\perp} \cap \mathcal{H}_1) \longleftarrow (\mathcal{U}_2, \mathcal{U}_2^{\perp})$$

Observe that $(\mathcal{U}_2 \cap \mathcal{H}_1, \mathcal{U}_2^{\perp} \cap \mathcal{H}_1)$ is actually $(\mathcal{U}_1^{\perp} \cap \mathcal{U}_2, \mathcal{U}_1 \cap \mathcal{U}_2^{\perp})$, where the right hand side $\mathcal{U}_1 \cap \mathcal{U}_2^{\perp}$ is exactly our heart $\mathcal{H}$.

Hence, under this correspondence, given a t-structure $(\mathcal{U}_1, \mathcal{U}_1^{\perp})$ in a triangulated, the t-structures $(\mathcal{U}_2, \mathcal{U}_2^{\perp})$ satisfying $\mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1$ and such that the heart $\mathcal{H} = \mathcal{U}_1 \cap \mathcal{U}_2^{\perp}$ is abelian and in bijection with the abelian torsion free classes of $\mathcal{H}_1 = \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1]$.

The following lemma clarifies which properties characterize an abelian torsion free class in an abelian category.

Lemma 25. *Let $\mathcal{A}$ be an abelian category and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. The following assertions hold:*

1. *If $f : F \to F'$ is a morphism in F, then the composition*

$$F' \xrightarrow{f^C} \operatorname{Coker}_{\mathcal{A}}(f) \to \frac{\operatorname{Coker}_{\mathcal{A}}(f)}{t(\operatorname{Coker}_{\mathcal{A}}(f))}$$

   is the cokernel map of f in $\mathcal{F}$.

2. *$f : F \to F'$ as above is an epimorphism in $\mathcal{F}$ if and only if $\operatorname{Coker}_{\mathcal{A}}(f) \in \mathcal{T}$.*

3. *Epimorphisms and cokernel maps coincide in $\mathcal{F}$ if and only if each morphism f in $\mathcal{F}$ with $\operatorname{Coker}_{\mathcal{A}}(f) \in \mathcal{T}$ is an epimorphism in $\mathcal{A}$.*

4. *$\mathcal{F}$ is an abelian category if and only if $\mathcal{F}$ is closed under quotients in $\mathcal{A}.m$*

References

[1] Hiroyuki Nakaoka. "General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories". In: *Applied Categorical Structures* 19.6 (2011), pp. 879–899.