

Let \mathcal{A} be a *good* category (abelian/exact/triangulated). The precise meaning of this will have to be clarified later (probably, the recent Nakaoka-Palu's paper is the right setting). But, whatever the choice, two things should happen. First, idempotents should split in \mathcal{A} . Secondly, each torsion pair considered in \mathcal{A} should be functorial on both sides. If $(\mathcal{T}, \mathcal{F})$ is such a torsion pair, we will denote by $t : \mathcal{A} \rightarrow \mathcal{T}$ (resp. $f : \mathcal{A} \rightarrow \mathcal{F}$) the right (resp. left) adjoint of the inclusion functor and, also, the composition $\mathcal{A} \xrightarrow{t} \mathcal{T} \xrightarrow{i} \mathcal{A}$ (resp. $\mathcal{A} \xrightarrow{f} \mathcal{F} \xrightarrow{j} \mathcal{A}$), where $\mathcal{T} \xrightarrow{i} \mathcal{A}$ (resp. $\mathcal{F} \xrightarrow{j} \mathcal{A}$) is the inclusion functor. The functoriality should then give rise to an admissible sequence $t(M) \rightarrow M \rightarrow f(M)$, for each object $M \in \mathcal{A}$ (e.g. if \mathcal{A} is abelian, that sequence should be short exact, if \mathcal{A} is exact it should be a conflation, if \mathcal{A} is triangulated it should be a triangle).

Let \mathcal{W} a full subcategory of \mathcal{A} closed by direct summands and extensions, and consider the category $\underline{\mathcal{A}} = \frac{\mathcal{A}}{\mathcal{W}}$.

Let $(\mathcal{X}, \mathcal{Y})$ be a orthogonal pair in $\underline{\mathcal{A}}$ and consider the following classes in \mathcal{A} :

$$\begin{aligned}\mathcal{T} &= \{T \in \mathcal{A} | \underline{T} \in \mathcal{X}\} \\ \mathcal{F} &= \{F \in \mathcal{A} | \underline{F} \in \mathcal{Y}\}.\end{aligned}$$

Lemma 1. *In the previous notation, $(\mathcal{T}, \mathcal{T}^\perp)$ is a orthogonal pair.*

Proof. In order to prove it we need to show that ${}^\perp(\mathcal{T}^\perp) = \mathcal{T}$.

Let $M \in {}^\perp(\mathcal{T}^\perp)$, this means that

$$\mathcal{A}(M, Y) = 0 \tag{1}$$

whenever

$$\mathcal{A}(T, Y) = 0 \forall T \in \mathcal{T}. \tag{2}$$

However, if $\mathcal{A}(T, Y) = 0 \forall T \in \mathcal{T}$, then $\underline{\mathcal{A}}(\underline{X}, \underline{Y}) = 0 \forall \underline{X} \in \mathcal{X}$. Hence, $\underline{Y} \in \mathcal{Y}$. So $\underline{\mathcal{A}}(\underline{M}, \underline{Y}) = 0 \forall \underline{Y} \in \mathcal{Y}$. Hence, $\underline{M} \in \mathcal{X}$ and so $M \in \mathcal{T}$.

We have proved that ${}^\perp(\mathcal{T}^\perp) \subseteq \mathcal{T}$, the converse inclusion is trivial. \square

Remark. The dual statement holds for \mathcal{F} . Notice that have we also proved that if $\mathcal{A}(T, Y) = 0 \forall T \in \mathcal{T}$, then $\underline{Y} \in \mathcal{Y}$ and hence $Y \in \mathcal{F}$. That is, $\mathcal{T}^\perp \subseteq \mathcal{F}$ and dually ${}^\perp\mathcal{F} \subseteq \mathcal{T}$.

Properties of $(\mathcal{T}, \mathcal{T}^\perp)$ and $({}^\perp\mathcal{F}, \mathcal{F})$:

1. ${}^\perp\mathcal{F} \subseteq \mathcal{T}$ and $\mathcal{T}^\perp \subseteq \mathcal{F}$.
2. $\mathcal{T} \cap \mathcal{F} = \mathcal{W}$. In fact, $M \in \mathcal{T} \cap \mathcal{F}$ iff $\underline{M} \in \mathcal{X} \cap \mathcal{Y} = 0$, which happens iff $M <_\oplus W$ for some $W \in \mathcal{W}$, but \mathcal{W} is closed by direct summands, hence $M \in \mathcal{W}$.
3. If $N \in \mathcal{T}^\perp \cap {}^\perp\mathcal{F}$, then $N = 0$. It follows from $N \in \mathcal{T}^\perp \cap {}^\perp\mathcal{F} \subseteq \mathcal{F} \cap \mathcal{T} = \mathcal{W}$. But $\mathcal{W} \subseteq \mathcal{T}$, hence $\mathcal{A}(W', N) = 0 \forall W' \in \mathcal{W}$, in particular $\mathcal{A}(N, N) = 0$, i.e. $N = 0$.

If $\mathfrak{k} = (\mathcal{T}, \mathcal{F})$ is a orthogonal pair in a cocomplete and locally small abelian category \mathcal{A} , then \mathfrak{k} is a torsion pair. Indeed, if M is any object and we consider the set \mathcal{T}_M of subobjects of M which are in \mathcal{T} , then $t(M) := \sum_{T \in \mathcal{T}_M} T$ is subobject of M which is an epimorphic image of $\coprod_{T \in \mathcal{T}_M} T$ and, hence, we have that $t(M) \in \mathcal{T}_M$. If we had a nonzero morphism $f : T' \rightarrow M/t(M)$, where $T' \in \mathcal{T}$, then we would have that $\text{Im}(f) = \tilde{T}/t(M)$ is a nonzero submodule of $M/t(M)$ which is in \mathcal{T} . Since \mathcal{T} is closed under extensions and we have an exact sequence $0 \rightarrow t(M) \rightarrow \tilde{T} \rightarrow \tilde{T}/t(M) \rightarrow 0$, we conclude that $\tilde{T} \in \mathcal{T}$. But then we have that $\tilde{T} \in \mathcal{T}_M$, which is a contradiction since $t(M)$ contains all subobjects in \mathcal{T}_M .

Lemma 2. Let $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in \mathcal{A} , with associated radical functors t_i and coradical functors f_i ($i = 1, 2$), respectively. Suppose that they satisfy the following conditions:

- a) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equivalently, $\mathcal{F}_1 \subseteq \mathcal{F}_2$)
- b) $\mathcal{T}_2 \cap \mathcal{F}_1 = 0$.
- c) $(\mathcal{T}_1, \mathcal{F}_2)$ is an orthogonal pair in $\underline{\mathcal{A}} := \mathcal{A}/\mathcal{W}$, where $\mathcal{W} = \mathcal{T}_1 \cap \mathcal{F}_2$.

Then the following assertions hold:

- 1. \mathcal{T}_1 consists of those objects $X \in \mathcal{A}$ such that $f_2(X) \in \mathcal{W}$. We will write $\mathcal{T}_1 = \mathcal{T}_2 \star \mathcal{W}$.
- 2. \mathcal{F}_2 consists of those objects $Y \in \mathcal{A}$ such that $t_1(Y) \in \mathcal{W}$. We will write $\mathcal{F}_2 = \mathcal{W} \star \mathcal{F}_1$.

Proof. We just prove assertion 1, and assertion 2 will follow by duality. Let us take $X \in \mathcal{T}_2 \star \mathcal{W}$. Since we have an admissible sequence $t_2(X) \rightarrow X \rightarrow f_2(X)$ whose outer terms are in \mathcal{T}_2 and \mathcal{W} , respectively, and these two classes are contained in \mathcal{T}_1 we conclude that $\mathcal{T}_1' \subseteq \mathcal{T}_1$, because \mathcal{T}_1 is closed under taking extensions in \mathcal{A} .

Let T_1 be in \mathcal{T}_1 and consider its canonical admissible sequence

$$t_2(T_1) \rightarrow T_1 \xrightarrow{f} f_2(T_1). \quad (3)$$

Note that $\underline{f} = 0$ because of condition c) in the statement. It follows that f decomposes in the form $f : T_1 \xrightarrow{\gamma} W \xrightarrow{\phi} f_2(T_1)$, where $W \in \mathcal{W}$. We then consider the following admissible pullback diagram

$$\begin{array}{ccccc} t_2(T_1) & \longrightarrow & \widehat{T}_1 & \longrightarrow & W \\ \parallel & & \downarrow & & \downarrow \phi \\ t_2(T_1) & \longrightarrow & T_1 & \xrightarrow{f} & f_2(T_1) \end{array} \quad (4)$$

Then, there exist a (non necessarily unique) $\eta : T_1 \rightarrow \widehat{T}_1$ making the following diagram commute.

$$\begin{array}{ccccc} T_1 & & \xrightarrow{\gamma} & & W \\ & \searrow \eta & & \searrow & \\ & \widehat{T}_1 & \longrightarrow & & W \\ & \downarrow & & \downarrow \phi & \\ & T_1 & \xrightarrow{f} & & f_2(T_1) \end{array} \quad (5)$$

Hence, $T_1 <_{\oplus} \widehat{T}_1 \in \mathcal{T}_2 \star \mathcal{W}$. This implies that $\mathcal{T}_1 \subseteq \text{add}(\mathcal{T}_2 \star \mathcal{W})$. The proof will be finished once we check that $\mathcal{T}_2 \star \mathcal{W}$ is closed under direct summands. But this is a direct consequence of the functoriality of the torsion pair. Indeed if we have admissible torsion sequences $t_2(M) \rightarrow M \rightarrow f_2(M)$ and $t_2(N) \rightarrow N \rightarrow f_2(N)$, then the coproduct sequence $t_2(M) \oplus t_2(N) \rightarrow M \oplus N \rightarrow f_2(M) \oplus f_2(N)$ is the admissible torsion sequence for $M \oplus N$. The fact that $M \oplus N \in \mathcal{T}_2 \star \mathcal{W}$ is then equivalent to the fact that $f_2(M) \oplus f_2(N) \in \mathcal{W}$. Since \mathcal{W} is closed under direct summands, we conclude that $f_2(M) \in \mathcal{W}$ and, hence, that $M \in \mathcal{T}_2 \star \mathcal{W}$. \square