Let $\mathcal{A}$ be a *good* category (abelian/exact/triangulated). The precise meaning of this will have to be clarified later (probably, the recent Nakaoka-Palu's paper is the right setting). But, whatever the choice, two things should happen. First, idempotents should split in $\mathcal{A}$. Secondly, each torsion pair considered in $\mathcal{A}$ should be functorial on both sides. If $(\mathcal{T}, \mathcal{F})$ is such a torsion pair, we will denote by $t : \mathcal{A} \longrightarrow \mathcal{T}$ (resp. $f : \mathcal{A} \longrightarrow \mathcal{F}$) the right (resp. left) adjoint of the inclusion functor and, also, the composition $\mathcal{A} \xrightarrow{t} \mathcal{T} \overset{i}{\hookrightarrow} \mathcal{A}$ (resp. $\mathcal{A} \xrightarrow{f} \mathcal{F} \overset{j}{\hookrightarrow} \mathcal{A}$), where $\mathcal{T} \overset{i}{\hookrightarrow} \mathcal{A}$ (resp. $\mathcal{F} \overset{j}{\hookrightarrow} \mathcal{A}$) is the inclusion functor. The functoriality should then give rise to an admissible sequence $t(M) \longrightarrow M \longrightarrow f(M)$, for each object $M \in \mathcal{A}$ (e.g. if $\mathcal{A}$ is abelian, that sequence should be short exact, if $\mathcal{A}$ is exact it should be a conflation, if A is triangulated it should be a triangle).

$\mathcal{W}$ a full subcategory of $\mathcal{A}$ closed by direct summands and extensions, and consider the category $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$.

Let $(\mathcal{X}, \mathcal{Y})$ be a orthogonal pair in $\underline{\mathcal{A}}$ and consider the following classes in $\mathcal{A}$:

$$\mathcal{T} = \{T \in \mathcal{A} | \underline{T} \in \mathcal{X}\}$$
$$\mathcal{F} = \{F \in \mathcal{A} | \underline{F} \in \mathcal{Y}\}.$$

Lemma 1. *In the previous notation, $(\mathcal{T}, \mathcal{T}^{\perp})$ is a orthogonal pair.*

Proof. In order to prove it we need to show that ${}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$.

Let $M \in {}^{\perp}(\mathcal{T}^{\perp})$, this means that

$$\mathcal{A}(M, Y) = 0 \tag{1}$$

whenever

$$\mathcal{A}(T, Y) = 0 \, \forall T \in \mathcal{T}. \tag{2}$$

However, if $\mathcal{A}(T, Y) = 0$ $\forall T \in \mathcal{T}$, then $\underline{\mathcal{A}}(\underline{X}, \underline{Y}) = 0$ $\forall \underline{X} \in \mathcal{X}$. Hence, $\underline{Y} \in \mathcal{Y}$. So $\underline{\mathcal{A}}(\underline{M}, \underline{Y}) = 0$ $\forall \underline{Y} \in \mathcal{Y}$. Hence, $\underline{M} \in \mathcal{X}$ and so $M \in \mathcal{T}$.

We have proved that ${}^{\perp}(\mathcal{T}^{\perp}) \subseteq \mathcal{T}$, the converse inclusion is trivial. □

Remark. The dual statement holds for $\mathcal{F}$. Notice that have we also proved that if $\mathcal{A}(T, Y) = 0$ $\forall T \in \mathcal{T}$, then $\underline{Y} \in \mathcal{Y}$ and hence $Y \in \mathcal{F}$. That is, $\mathcal{T}^{\perp} \subseteq \mathcal{F}$ and dually ${}^{\perp}\mathcal{F} \subseteq \mathcal{T}$.

Properties of $(\mathcal{T}, \mathcal{T}^{\perp})$ and $({}^{\perp}\mathcal{F}, \mathcal{F})$:

1. ${}^{\perp}\mathcal{F} \subseteq \mathcal{T}$ and $\mathcal{T}^{\perp} \subseteq \mathcal{F}$.
2. $\mathcal{T} \cap \mathcal{F} = \mathcal{W}$. In fact, $M \in \mathcal{T} \cap \mathcal{F}$ iff $\underline{M} \in \mathcal{X} \cap \mathcal{Y} = 0$, which happens iff $M <_{\oplus} W$ for some $W \in \mathcal{W}$, but $\mathcal{W}$ is closed by direct summands, hence $M \in \mathcal{W}$.
3. If $N \in \mathcal{T}^{\perp} \cap {}^{\perp}\mathcal{F}$, then $N = 0$. It follows from $N \in \mathcal{T}^{\perp} \cap {}^{\perp}\mathcal{F} \subseteq \mathcal{F} \cap \mathcal{T} = \mathcal{W}$. But $\mathcal{W} \subseteq \mathcal{T}$, hence $\mathcal{A}(W', N) = 0$ $\forall W' \in \mathcal{W}$, in particular $\mathcal{A}(N, N) = 0$, i.e. $N = 0$.

If $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ is a orthogonal pair in a cocomplete and locally small abelian category $\mathcal{A}$, then $\mathbb{t}$ is a torsion pair. Indeed, if M is any object and we consider the set $\mathcal{T}_M$ of subobjects of M which are in $\mathcal{T}$, then $t(M) := \sum_{T \in \mathcal{T}_M} T$ is subobject of M which is an epimorphic image of $\coprod_{T \in \mathcal{T}_M}$ and, hence, we have that $t(M) \in \mathcal{T}_M$. If we had a nonzero morphism $f : T' \longrightarrow M/t(M)$, where $T' \in \mathcal{T}$, then we would have that $\mathrm{Im}(f) = \tilde{T}/t(M)$ is a nonzero submodule of $M/t(M)$ which is in $\mathcal{T}$. Since $\mathcal{T}$ is closed under extensions and we have an exact sequence $0 \to t(M) \longrightarrow \tilde{T} \longrightarrow \tilde{T}/t(M) \to 0$, we conclude that $\tilde{T} \in \mathcal{T}$. But then we have that $\tilde{T} \in \mathcal{T}_M$, which is a contradiction since $t(M)$ contains all subobjects in $\mathcal{T}_M$.

Lemma 2. *Let $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in $\mathcal{A}$, with associated radical functors t_i and coradical functors f_i $(i = 1, w)$, respectively. Suppose that they satisfy the following conditions:*

a) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ *(equivalently, $\mathcal{F}_1 \subseteq \mathcal{F}_2$)*

b) $\mathcal{T}_2 \cap \mathcal{F}_1 = 0$.

c)

If we put $\mathcal{W} = \mathcal{T}_1 \cap \mathcal{F}_2$. Then the following assertions hold:

1. *$\mathcal{T}_1$ consists of those objects $X \in \mathcal{A}$ such that $f_2(X) \in \mathcal{W}$. We will write $\mathcal{T}_1 = \mathcal{T}_2 \star \mathcal{W}$.*
2. *$\mathcal{F}_2$ consists of those objects $Y \in \mathcal{A}$ such that $t_1(Y) \in \mathcal{W}$. We will write $\mathcal{F}_2 = \mathcal{W} \star \mathcal{F}_1$.*

Proof. We just prove assertion 1, and assertion 2 will follow by duality. Let us take $X \in \mathcal{T}_2 \star \mathcal{W}$. Since we have an admissible sequence $t_2(X) \longrightarrow X \longrightarrow f_2(X)$ whose outer terms are in $\mathcal{T}_2$ and $\mathcal{W}$, respectively, and these two classes are contained in $\mathcal{T}_1$ we conclude that $\mathcal{T}_1' \subseteq \mathcal{T}_1$, because $\mathcal{T}_1$ is closed under taking extensions in $\mathcal{A}$.

Let T_1 be in $\mathcal{T}_1$ and consider its canonical admissible sequence

$$t_2(T_1) \to T_1 \xrightarrow{f} f_2(T_1). \tag{3}$$

Note that $\underline{f} = 0$ because of condition c) in the statement. It follows that f decomposes in the form $f : T_1 \xrightarrow{\gamma} W \xrightarrow{\phi} f_2(T_1)$, where $W \in \mathcal{W}$. We then consider the following admissible pullback diagram

$$\begin{tikzcd} t_2(T_1) \arrow[r] \arrow[d, equal] & \widehat{T_1} \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\ t_2(T_1) \arrow[r] & T_1 \arrow[r, "f"] & f_2(T_1) \end{tikzcd} \tag{4}$$

Then, there exist a (non necessarely unique) $\eta : T_1 \to \widehat{T_1}$ making the following diagram commute.

$$\begin{tikzcd} T_1 \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "1"'] & & \\ & \widehat{T_1} \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\ & T_1 \arrow[r, "f"] & f_2(T_1) \end{tikzcd} \tag{5}$$

Hence, $T_1 <_{\oplus} \widehat{T_1} \in \mathcal{T}_2 * \mathcal{W}$. This implies that $\mathcal{T}_1 \subseteq \mathrm{add}(\mathcal{T}_2 * \mathcal{W})$. The proof will be finished once we check that $\mathcal{T}_2 * \mathcal{W}$ is closed under direct summands. But this is a direct consequence of the functoriality of the torsion pair. Indeed if we have admissible torsion sequences $t_2(M) \longrightarrow M \longrightarrow f_2(M)$ and $t_2(N) \longrightarrow N \longrightarrow f_2(N)$, then the coproduct sequence $t_2(M) \oplus t_2(N) \longrightarrow M \oplus N \longrightarrow f_2(M) \oplus f_2(N)$ is the admissible torsion sequence for $M \oplus N$. The fact that $M \oplus N \in \mathcal{T}_2 * \mathcal{W}$ is then equivalent to the fact that $f_2(M) \oplus f_2(N) \in \mathcal{W}$. Since $\mathcal{W}$ is closed under direct summands, we conclude that $f_2(M) \in \mathcal{W}$ and, hence, that $M \in \mathcal{T}_2 * \mathcal{W}$.

$\square$