1 Introduction

Let $\mathcal{A}$ be a *good* category (abelian/exact/triangulated). The precise meaning of this will have to be clarified later (probably, the recent Nakaoka-Palu's paper is the right setting). But, whatever the choice, two things should happen. First, idempotents should split in $\mathcal{A}$. Secondly, each torsion pair considered in $\mathcal{A}$ should be functorial on both sides. If $(\mathcal{T}, \mathcal{F})$ is such a torsion pair, we will denote by $t : \mathcal{A} \longrightarrow \mathcal{T}$ (resp. $f : \mathcal{A} \longrightarrow \mathcal{F}$) the right (resp. left) adjoint of the inclusion functor and, also, the composition $\mathcal{A} \xrightarrow{t} \mathcal{T} \overset{i}{\hookrightarrow} \mathcal{A}$ (resp. $\mathcal{A} \xrightarrow{f} \mathcal{F} \overset{j}{\hookrightarrow} \mathcal{A}$), where $\mathcal{T} \overset{i}{\hookrightarrow} \mathcal{A}$ (resp. $\mathcal{F} \overset{j}{\hookrightarrow} \mathcal{A}$) is the inclusion functor. The functoriality should then give rise to an admissible sequence $t(M) \longrightarrow M \longrightarrow f(M)$, for each object $M \in \mathcal{A}$ (e.g. if $\mathcal{A}$ is abelian, that sequence should be short exact, if $\mathcal{A}$ is exact it should be a conflation, if A is triangulated it should be a triangle).

$\mathcal{W}$ a full subcategory of $\mathcal{A}$ closed by direct summands and extensions, and consider the category $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$.

Let $(\mathcal{X}, \mathcal{Y})$ be a orthogonal pair in $\underline{\mathcal{A}}$ and consider the following classes in $\mathcal{A}$:

$$\mathcal{T} = \{T \in \mathcal{A} | \underline{T} \in \mathcal{X}\}$$
$$\mathcal{F} = \{F \in \mathcal{A} | \underline{F} \in \mathcal{Y}\}.$$

Lemma 1. *In the previous notation, $(\mathcal{T}, \mathcal{T}^{\perp})$ is a orthogonal pair.*

Proof. In order to prove it we need to show that ${}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$.

Let $M \in {}^{\perp}(\mathcal{T}^{\perp})$, this means that

$$\mathcal{A}(M, Y) = 0 \tag{1}$$

whenever

$$\mathcal{A}(T, Y) = 0 \, \forall T \in \mathcal{T}. \tag{2}$$

However, if $\mathcal{A}(T, Y) = 0 \ \forall T \in \mathcal{T}$, then $\underline{\mathcal{A}}(\underline{X}, \underline{Y}) = 0 \ \forall \underline{X} \in \mathcal{X}$. Hence, $\underline{Y} \in \mathcal{Y}$. So $\underline{\mathcal{A}}(\underline{M}, \underline{Y}) = 0 \ \forall \underline{Y} \in \mathcal{Y}$. Hence, $\underline{M} \in \mathcal{X}$ and so $M \in \mathcal{T}$.

We have proved that ${}^{\perp}(\mathcal{T}^{\perp}) \subseteq \mathcal{T}$, the converse inclusion is trivial. □

Remark. The dual statement holds for $\mathcal{F}$. Notice that have we also proved that if $\mathcal{A}(T, Y) = 0 \ \forall T \in \mathcal{T}$, then $\underline{Y} \in \mathcal{Y}$ and hence $Y \in \mathcal{F}$. That is, $\mathcal{T}^{\perp} \subseteq \mathcal{F}$ and dually ${}^{\perp}\mathcal{F} \subseteq \mathcal{T}$.

Properties of $(\mathcal{T}, \mathcal{T}^{\perp})$ and $({}^{\perp}\mathcal{F}, \mathcal{F})$:

1. ${}^{\perp}\mathcal{F} \subseteq \mathcal{T}$ and $\mathcal{T}^{\perp} \subseteq \mathcal{F}$.
2. $\mathcal{T} \cap \mathcal{F} = \mathcal{W}$. In fact, $M \in \mathcal{T} \cap \mathcal{F}$ iff $\underline{M} \in \mathcal{X} \cap \mathcal{Y} = 0$, which happens iff $M <_{\oplus} W$ for some $W \in \mathcal{W}$, but $\mathcal{W}$ is closed by direct summands, hence $M \in \mathcal{W}$.
3. If $N \in \mathcal{T}^{\perp} \cap {}^{\perp}\mathcal{F}$, then $N = 0$. It follows from $N \in \mathcal{T}^{\perp} \cap {}^{\perp}\mathcal{F} \subseteq \mathcal{F} \cap \mathcal{T} = \mathcal{W}$. But $\mathcal{W} \subseteq \mathcal{T}$, hence $\mathcal{A}(W', N) = 0 \ \forall W' \in \mathcal{W}$, in particular $\mathcal{A}(N, N) = 0$, i.e. $N = 0$.

If $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ is a orthogonal pair in a cocomplete and locally small abelian category $\mathcal{A}$, then $\mathbb{t}$ is a torsion pair. Indeed, if M is any object and we consider the set $\mathcal{T}_M$ of subobjects of M which are in $\mathcal{T}$, then $t(M) := \sum_{T \in \mathcal{T}_M} T$ is subobject of M which is an epimorphic image of $\coprod_{T \in \mathcal{T}_M}$ and, hence, we have that $t(M) \in \mathcal{T}_M$. If we had a nonzero morphism $f : T' \longrightarrow M/t(M)$, where $T' \in \mathcal{T}$, then we would have that $\mathrm{Im}(f) = \tilde{T}/t(M)$ is a nonzero submodule of $M/t(M)$ which is in $\mathcal{T}$. Since $\mathcal{T}$ is closed under extensions and we have an exact sequence $0 \to t(M) \longrightarrow \tilde{T} \longrightarrow \tilde{T}/t(M) \to 0$, we conclude that $\tilde{T} \in \mathcal{T}$. But then we have that $\tilde{T} \in \mathcal{T}_M$, which is a contradiction since $t(M)$ contains all subobjects in $\mathcal{T}_M$.

Lemma 2. *Let $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in $\mathcal{A}$, with associated radical functors t_i and coradical functors f_i $(i = 1, w)$, respectively. Suppose that they satisfy the following conditions:*

a) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ *(equivalently,* $\mathcal{F}_1 \subseteq \mathcal{F}_2$*)*

b) $\mathcal{T}_2 \cap \mathcal{F}_1 = 0$.

c) $(\underline{\mathcal{T}}_1, \underline{\mathcal{F}}_2)$ *is an orthogonal pair in* $\underline{\mathcal{A}} := \mathcal{A}/\mathcal{W}$*, where* $\mathcal{W} = \mathcal{T}_1 \cap \mathcal{F}_2$.

Then the following assertions hold:

1. $\mathcal{T}_1$ *consists of those objects* $X \in \mathcal{A}$ *such that* $f_2(X) \in \mathcal{W}$*. We will write* $\mathcal{T}_1 = \mathcal{T}_2 \star \mathcal{W}$.

2. $\mathcal{F}_2$ *consists of those objects* $Y \in \mathcal{A}$ *such that* $t_1(Y) \in \mathcal{W}$*. We will write* $\mathcal{F}_2 = \mathcal{W} \star \mathcal{F}_1$.

Proof. We just prove assertion 1, and assertion 2 will follow by duality. Let us take $X \in \mathcal{T}_2 \star \mathcal{W}$. Since we have an admissible sequence $t_2(X) \longrightarrow X \longrightarrow f_2(X)$ whose outer terms are in $\mathcal{T}_2$ and $\mathcal{W}$, respectively, and these two classes are contained in $\mathcal{T}_1$ we conclude that $\mathcal{T}_1' \subseteq \mathcal{T}_1$, because $\mathcal{T}_1$ is closed under taking extensions in $\mathcal{A}$.

Let T_1 be in $\mathcal{T}_1$ and consider its canonical admissible sequence

$$t_2(T_1) \to T_1 \xrightarrow{f} f_2(T_1). \tag{3}$$

Note that $\underline{f} = 0$ because of condition c) in the statement. It follows that f decomposes in the form $f : T_1 \xrightarrow{\gamma} W \xrightarrow{\phi} f_2(T_1)$, where $W \in \mathcal{W}$. We then consider the following admissible pullback diagram

$$\begin{tikzcd}
t_2(T_1) \arrow[r] \arrow[d, equal] & \widehat{T_1} \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\
t_2(T_1) \arrow[r] & T_1 \arrow[r, "f"] & f_2(T_1)
\end{tikzcd} \tag{4}$$

Then, there exist a (non necessarely unique) $\eta : T_1 \to \widehat{T_1}$ making the following diagram commute.

$$\begin{tikzcd}
T_1 \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "1"'] & & \\
& \widehat{T_1} \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\
& T_1 \arrow[r, "f"] & f_2(T_1)
\end{tikzcd} \tag{5}$$

Hence, $T_1 <_\oplus \widehat{T_1} \in \mathcal{T}_2 * \mathcal{W}$. This implies that $\mathcal{T}_1 \subseteq \mathrm{add}(\mathcal{T}_2 * \mathcal{W})$. The proof will be finished once we check that $\mathcal{T}_2 * \mathcal{W}$ is closed under direct summands. But this is a direct consequence of the functoriality of the torsion pair. Indeed if we have admissible torsion sequences $t_2(M) \longrightarrow M \longrightarrow f_2(M)$ and $t_2(N) \longrightarrow N \longrightarrow f_2(N)$, then the coproduct sequence $t_2(M) \oplus t_2(N) \longrightarrow M \oplus N \longrightarrow f_2(M) \oplus f_2(N)$ is the admissible torsion sequence for $M \oplus N$. The fact that $M \oplus N \in \mathcal{T}_2 * \mathcal{W}$ is then equivalent to the fact that $f_2(M) \oplus f_2(N) \in \mathcal{W}$. Since $\mathcal{W}$ is closed under direct summands, we conclude that $f_2(M) \in \mathcal{W}$ and, hence, that $M \in \mathcal{T}_2 * \mathcal{W}$.

□

2 Induced torsion theories

Lemma 3. *Let $\mathcal{A}$ be a (nice) category with a torsion pair $({}^{\perp}\mathcal{F}, \mathcal{F})$ and a precovering class $\mathcal{W} \subseteq \mathcal{F}$ such that for any $F \in \mathcal{F}$ there is an admissible sequence*

$$F' \to W \to F$$

such that $F' \in \mathcal{F}$.

Then the torsion pair $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial.

Recall that the truncation $t : \underline{\mathcal{A}} \to {}^{\perp}(\underline{\mathcal{F}})$ is given by the following construction.

Let $M \in \mathcal{A}$ be any object, take an admissible sequence

$$T_M \to M \to F^M$$

with $T_M \in {}^{\perp}\mathcal{F}$ and $F^M \in \mathcal{F}$. Moreover, consider $W_F \to F^M$ with $W_M \in \mathcal{W}$ as before, and take the admissible pullback:

$$\begin{tikzcd}
t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\
M \arrow[r] & F
\end{tikzcd}$$

Then, t restricts to a functor $\underline{t} : \underline{\mathcal{A}} \to {}^{\perp}\underline{\mathcal{F}}$.

In order to prove that $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial we need to show that $\underline{t}$ admits a right adjoint.

Proof. Let $M \in \mathcal{A}$ and consider $M \to F^M$ and $W_F \to F^M$ as above. For any $T \in {}^{\perp}\mathcal{F} * \mathcal{W}$ consider any morphism $f : T \to M$. Since $T \to M \to F^M$ is 0 in $\underline{\mathcal{A}}$ we that the solid part of the following diagram commutes.

$$\begin{tikzcd}
T \arrow[rrr] \arrow[dr, dotted, "\psi"] \arrow[ddr, bend right, "f"'] & & & W \arrow[dl, dashed, "\phi"'] \arrow[ddl, bend left] \\
& t(M) \arrow[r] \arrow[d] & W_F \arrow[d] & \\
& M \arrow[r] & F^M &
\end{tikzcd}$$

□

Since $W_F \to F^M$ is a precover there is a morphism $\phi : W \to W_F$ making the diagram commute, and since the square is an admissible pullback there is a morphism $\psi : T \to t(M)$ making the diagram commutative.

Hence, $\mathcal{A}(T, t(M)) \to \mathcal{A}(T, M)$ is surjective. To conclude the proof we need to show that when restricted to $\underline{\mathcal{A}}$ it becomes an iso. Assume that there are two morphisms $\underline{\psi}$ and $\underline{\psi}'$ in $\underline{\mathcal{A}}$ such that the following commutes:

$$\begin{tikzcd}
& t(M) \arrow[d] \\
T \arrow[ur, shift left, "\underline{\psi}"] \arrow[ur, shift right, "\underline{\psi}'"'] \arrow[r, "\underline{f}"'] & M
\end{tikzcd}$$

So, if we call $h = \psi - \psi'$ in $\mathcal{A}$, we have that $T \xrightarrow{h} t(M) \to M$ factors through W so that we have that the solid part of the following diagram commutes:

$$\begin{tikzcd}
W \arrow[dddrr, bend right] \arrow[ddrr, dotted, bend left, "\gamma"] \arrow[ddrrr, dashed, bend left, "\eta"] & & & \\
& T \arrow[ul, "p"'] \arrow[dr, "h"] \arrow[ddr, "g"'] & & \\
& & t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\
& & M \arrow[r] & F^M
\end{tikzcd}$$

where $\eta : W \to W_F$ comes from the fact that $W_F \to F^M$ is a precover, and γ from the fact that the square is an admissible pullback, and they make the complete diagram commute.

Let's call $h' = \gamma \circ p$, then composing both h and h' with $\rho : t(M) \to M$ gives the same morphism g. Hence, $\rho \circ (h - h') = 0$. But since $F' \xrightarrow{i} t(M) \to M$ is an admissible sequence we have the following exact sequence of abelian groups:

$$\begin{array}{ccccc}
\mathcal{A}(T, F) & \longrightarrow & \mathcal{A}(T, t(M)) & \longrightarrow & \mathcal{A}(T, M) \\
 & & & & \\
 & & h - h' & \longmapsto & 0
\end{array}$$

So there is a map $k : T \to F'$ such that $i \circ k = h - h'$, but $\underline{k} = 0$ so $\underline{h} = \underline{h'} = 0$, hence $\underline{\psi} - \underline{\psi}' = 0$ which proves that $\underline{\mathcal{A}}(T, t(M)) \cong \underline{\mathcal{A}}(T, M)$.

The naturality of the isomorphism in T and M is clear.

3 Abelian categories

Now we work in an abelian category with two torsion pairs $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ such that $t_2(\mathcal{F}_1) \subseteq \mathcal{F}_1$ and $f_1(\mathcal{T}_2) \subseteq \mathcal{T}_2$ and let $\mathcal{W} = \mathcal{T}_2 \cap \mathcal{F}_1$.

Recall that $(\underline{\mathcal{T}_1 * \mathcal{W}}, \underline{\mathcal{F}_1})$ (resp. $(\underline{\mathcal{T}_2}, \underline{\mathcal{W} * \mathcal{F}_2})$) is a left (resp. right) functorial torsion pair in $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$. Moreover, they satisfy $TC1-3, 3^*$.

Lemma 4. *The inclusion $i : \mathcal{T}_1 * \mathcal{W} \hookrightarrow \mathcal{A}$ admits a right adjoint $\widehat{t}$.*

Proof. For $M \in \mathcal{A}$ consider the exact sequence

$$0 \to T_1 \to M \to f_1(M) \to 0$$

with $T_1 \in \mathcal{T}_1$ and $f_1(M) \in \mathcal{F}$. Take $t_2 f_1(M) \hookrightarrow f_1(M)$ and observe that $t_2 f_1(M) \in \mathcal{W}$. Call it W_M and take the pullback diagram

$$\begin{tikzcd}
\widehat{t}(M) \arrow[r] \arrow[d] & W_M \arrow[d, tail] \\
M \arrow[r, two heads] & f_1(M)
\end{tikzcd}$$

then $\widehat{t}(M) \in \mathcal{T}_1 * \mathcal{W}$.

Now for any morphism $\widehat{T} \to M$ with $\widehat{T} \in \widehat{\mathcal{T}}$ the solid part of the following diagram commutes

$\widehat{T} \to M$ is mono by Buhler prop. 2.14: pullback of monic along epic is monic

$$\begin{tikzcd}
T \arrow[r, tail] \arrow[d, equal] & \widehat{t}(M) \arrow[r, two heads] \arrow[d] & W \arrow[d] \\
T \arrow[r, tail] & M \arrow[r, two heads] & f_1(M) \\
T_1 \arrow[r, tail] & \widehat{T} \arrow[r, two heads] \arrow[u] \arrow[uu, dotted, bend left] & W_1 \arrow[u, dashed] \arrow[uu, dashed, bend right]
\end{tikzcd}$$

Since the composition $T_1 \to \widehat{T} \to M \to f_1(M)$ is zero, there exists the dashed morphism $W_1 \to f_1(M)$, which lifts to the morphism $W_1 \to W$ (since $W \to f_1(M)$ is a $\mathcal{W}$-precover). Hence, there is a morphism $\widehat{T} \to \widehat{t}(M)$ making the diagram commutative. This means that

$$\mathcal{A}(\widehat{T}, \widehat{t}(M)) \xrightarrow{\mathcal{A}(\widehat{T}, \widehat{t}(M) \to M)} \mathcal{A}(\widehat{T}, M)$$

is surjective. But it is also injective, since $\operatorname{Ker}(\widehat{t}(M), M) = 0$. Hence, it is an iso and $\widehat{t}$ is right adjoint to i. ☐

functoriality should follow immediately

Lemma 5. *Let $\widehat{T}_1 \in \mathcal{T}_1 * \mathcal{W}$, i.e. there is an exact sequence*

$$0 \to t_1(\widehat{T}_1) \to \widehat{T}_1 \to W_1 \to 0.$$

If

$$\begin{tikzcd}
\widehat{T}_1 \arrow[r, two heads, "p"] \arrow[d, "g"] & W_1 \arrow[d, "g'"] \\
\widehat{T}'_1 \arrow[r, "q"] & N
\end{tikzcd}$$

is a pushout diagram, then $N \in \mathcal{T}_1 * \mathcal{W}$.

Proof. Since it is a pushout, $\widehat{T}'_1 \to N$ is epi, then consider $\widehat{t}(N)$ and the following commutative diagram

$$\begin{tikzcd}
\widehat{T}'_1 \arrow[rr, two heads, "q"] \arrow[dr, dashed, "\rho"'] & & N \\
& \widehat{t}(N) \arrow[ur, tail, "\varepsilon"'] &
\end{tikzcd}$$

where the map $\widehat{T}'_1 \to \widehat{t}(N)$ is given by the adjunction $(i, \widehat{t})$. Since $q = \varepsilon \circ \rho$ is epi, then ε is epi. But it is mono, so it is an isomorphism, hence $N \in \mathcal{T}_1 * \mathcal{W}$. ☐

Lemma 6. *In the same notation as the previous lemma, if $\varphi : N \to P$ is any map s.t. $\underline{\varphi} \circ \underline{q} = \underline{0}$ in $\underline{\mathcal{A}}$, then φ factors through $\mathcal{W}$.*

Proof. Since $\underline{\varphi} \circ \underline{q} = \underline{0}$ it means that $\varphi \circ q$ factors through $\mathcal{W}$, hence we have that the solid part of the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & t_1(T_1') \arrow[r] \arrow[d] & \widehat{T}_1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] \arrow[dd, dashed, bend left] & 0 \\
0 \arrow[r] & X \arrow[r] & \widehat{T}_1' \arrow[r] \arrow[d, equal] & N \arrow[dr, "\varphi"] \arrow[d, dotted] & \\
 & & \widehat{T}_1' \arrow[r] & W \arrow[r] & P
\end{tikzcd}$$

Since $t_1(T_1') \to \widehat{T}_1 \to \widehat{T}_1' \to W$ is zero, there is the dashed morphism $W_1 \to W$ making the diagram commute. Since the square on the right is a pushout there is a map $N \to W$, and again the diagram commutes. Hence φ factors through $\mathcal{W}$. □

Lemma 12. *If $\mathcal{H}$ is balanced (i.e. mono and epi implies iso), then whenever $f : H_1 \to H_2$ is mono and epi in $\mathcal{H}$, there are bicartesian squares in $\mathcal{A}$*

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & H_1 \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner", very near start] & W_1 \arrow[d] \\
W_2 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & H_2 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & T_2
\end{tikzcd}$$

where $W_1 = f_1(H_1)$ and $W_2 = t_2(H_2)$. In particular there is an exact sequence

$$0 \to F_1 \to W_1 \oplus W_2 \to T_2 \to 0.$$

Proof. We can build the pullback on the left and the pushout on the right as usual

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & H_1 \arrow[r, "r"] \arrow[d, "f"] & W_1 \arrow[d, "s"] \\
W_2 \arrow[r] & H_2 \arrow[r, "f^C"'] \arrow[ur, phantom, "\lrcorner", very near end] & T_2
\end{tikzcd} \tag{6}$$

We will only prove that the square on the right hand side is a pullback, since the proof that the left square is a pushout is dual. The statetment that the square on the right is a pushout is equivalent to saying that there is an exact sequence

$$H_1 \xrightarrow{\binom{f}{r}} H_1 \oplus W_1 \xrightarrow{(f^C\ s)} T_2 \longrightarrow 0 \tag{7}$$

Since f is both a mono and an epi in $\mathcal{H}$, then it is an iso and hence both a section and a retraction. Consider $g : H_2 \to H_1$ such that $\underline{g} \circ \underline{f} = \underline{1_{H_1}}$, that is there are maps $\alpha : H_1 \to W$ and $\beta : W \to H_1$ such that

$$\begin{tikzcd}
H_1 \arrow[r, "\binom{f}{\alpha}"] \arrow[rr, bend right, "1_{H_1}"'] & H_2 \oplus W \arrow[r, "(g\ \beta)"] & H_1
\end{tikzcd}$$

is commutative in $\mathcal{A}$, and hence H_1 is a direct summand of $H_2 \oplus W$. We can actually choose $W = W_1$, in fact consider the commutative diagram

$$\begin{tikzcd}
H_1 \arrow[r, "\binom{f}{r}"] \arrow[d, equal] & H_1 \oplus W_1 \arrow[r, "(f^C\ s)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix}\right)"] & T_2 \\
H_1 \arrow[r, "\binom{f}{\alpha}"] \arrow[rr, bend right, "1_{H_1}"'] & H_2 \oplus W \arrow[r, "(g\ \beta)"] & H_1
\end{tikzcd}$$

where $\rho : W_1 \to W$ comes from the fact that $H_1 \to W_1$ is a $\mathcal{W}$-preenvelope. Hence, $(g\ \beta) \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix}\right) \circ \binom{f}{r} = 1_{H_1}$, that is H_1 is a direct summand of $H_2 \oplus W_1$. Moreover, it means that $(g\ \beta)$ is a section, that is the sequence in (7) is also exact on the left and the corresponding square in (6) is a pullback diagram.

Since both squares in (6) are bicartesian, it follows that the square

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & W_1 \arrow[d] \\
W_2 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & T_2
\end{tikzcd}$$

is bicartesian as well. □

4 Second approach to axiomatization

We give another set of axioms:

TC1 $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ are two respectively left functorial and right functorial torsion pairs in $\mathcal{X}$.

TC2 $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equivalently $\mathcal{F}_1 \subseteq \mathcal{F}_2$).

TC3 For any morphism $g : T_1 \to T_1'$ in $\mathcal{T}_1$ has a pseudocokernel in $\mathcal{T}_1$ which completes diagrams in a unique way wrt $\mathcal{F}_2$.

TC3* Dual of **TC3**.

TC4

explain this axiom

$$\begin{tikzcd}
F_1 \arrow[r, "f^K"] & H_1 \arrow[r, "\forall f"] \arrow[d, dashed] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 \arrow[d] \\
i_1t_1(F_2) \arrow[ur, dashed] \arrow[r, "\varepsilon"] & F \arrow[r] & H_2 \arrow[r] & j_2f_2(T_1)
\end{tikzcd}$$

TC4* Dual of **TC4**.

EXAMPLES

add examples from page 2, 9/11/16

2 If $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in a triangulated category (as in Nakaoka's work) produces an example.

Add reference

3 Let $\mathcal{D}$ be a triangulated category with two t-structures $(\mathcal{U}_1, \mathcal{U}_1^{\perp})$ and $(\mathcal{U}_2, \mathcal{U}_2^{\perp})$ such that $\mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1$. Then, these satisfy axioms **TC1-TC3**,**TC3***, hence $\mathcal{H} = \mathcal{U}_1 \cap \mathcal{U}_2^{\perp}$ has kernels and cokernels. Moreover, TFAE:

1.a **TC4** holds.

1.b If $V_1 \to H_1 \overset{f}{\twoheadrightarrow} H_2 \overset{+}{\to}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $V_1 \in \mathcal{U}_1^{\perp}$, then $V_1 \in \mathcal{U}_2^{\perp}[-1]$.

And, dually, there is an equivalence of the following:

2.a **TC4*** holds.

2.b If $H_1 \overset{f}{\to} H_2 \to U_2 \overset{+}{\to}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $U_2 \in \mathcal{U}_2$, then $U_2 \in \mathcal{U}_1[1]$.

Proof of the equivalences in example 3. Let's $\mathcal{D}$ be a triangulated category with two t-structures as in example 3. The pseudocokernel of a morphism in $\mathcal{U}_1$ can be computed by taking the cone in $\mathcal{D}$, i.e. given a morphism $f : U_1 \to U_1'$ in $\mathcal{U}_1$ we can compute a pseudocokernel in $\mathcal{U}_1$ by completing f to a triangle

$$U_1 \overset{f}{\to} U_1' \to \mathrm{Cone}(f) \overset{+}{\to} .$$

Moreover, this pseudocokernel satisfies **TC3**.

Now, assume that **TC1-TC3**,**TC3*** are satisfied together with axiom **1.b**, and consider the solid part of the diagram as in **TC4**:

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_1 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
\tau_{\mathcal{U}_1}(V_2) \arrow[ur, dotted, "\beta"] \arrow[r, "\varepsilon"] & V_2 \arrow[r] & H_2 \arrow[r] & \tau^{\mathcal{U}_2^{\perp}}\mathrm{Cone}(f)
\end{tikzcd}$$

with $\mathrm{Cone}(f)[-1] \in \mathcal{U}_1^{\perp}$ and where the upper row is a distinguished triangle. By **1.b** then it belongs to $\mathcal{U}_2^{\perp}[-1]$, i.e. $\mathrm{Cone}(f) \in \mathcal{U}_2^{\perp}$, so λ is an iso, conseguently α is an iso and so is ε, so there exist a map $\beta = \alpha^{-1} \circ \varepsilon$ making the diagram commute, that is **TC4** holds.

Conversely, assume that **TC1-TC3**,**TC3*** are satisfied together with **TC4**. Consider again the solid part of the diagram

$$
\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "\lambda[-1]"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_1 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
\tau_{\mathcal{U}_2^\perp}(\mathrm{Cone}(f))[-1] \arrow[r, dashed] & V_2 \arrow[r] & H_2 \arrow[r] & \tau^{\mathcal{U}_2^\perp}\mathrm{Cone}(f) \\
& \tau_{\mathcal{U}_1}(V_2) \arrow[u, "\varepsilon"] \arrow[uu, dotted, bend left, "\beta"] & &
\end{tikzcd}
$$

with $\mathrm{Cone}(f)[-1] \in \mathcal{U}_1^\perp$. Neeman guarantees that α can be taken so that the square on the left is a pullback. Axiom **TC4** gives the existence of $\beta : \tau_{\mathcal{U}_1}(V_2) \to H_1$ such that $\alpha \circ \beta = \varepsilon$.

Add reference

Since $\tau_{\mathcal{U}_1}$ is a functor, there is also a morphism $\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(H_1) = H_1 \to \tau_{\mathcal{U}_1}(V_2)$ such that $\varepsilon \circ \tau_{\mathcal{U}_1}(\alpha) = \alpha$, hence $\varepsilon \circ \tau_{\mathcal{U}_1}(\alpha) \circ \beta = \varepsilon$. By the functoriality of the torsion pair $(\mathcal{U}_1, \mathcal{U}_1^\perp)$, this means that $\tau_{\mathcal{U}_1}(\alpha) \circ \beta = 1_{\tau_{\mathcal{U}_1}(V_2)}$. Then, β is a section.

Hence, we can write $\tau_{\mathcal{U}_1}(\alpha) : H_1 \to \tau_{\mathcal{U}_1}(V_2)$ as

$$\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(V_2) \oplus H_1' \xrightarrow{(\,*\ 0\,)} \tau_{\mathcal{U}_1}(V_2)$$

for some $H_1' \underset{\oplus}{<} H_1$ such that α vanishes on H_1'. If we consider the solid part of the diagram

$$
\begin{tikzcd}
& H_1' \arrow[dl, dashed] \arrow[d, "\binom{1}{0}"] \arrow[dr, "0"] & & \\
\mathrm{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1] \arrow[r] & H_1 \arrow[r, "\tau_{\mathcal{U}_1}(\alpha)"] & \tau_{\mathcal{U}_1}(V_2) \arrow[r, dashed, "+"] & {}
\end{tikzcd}
$$

we can construct the dashed arrow, and the fact that the triangle commutes means that $H_1' \underset{\oplus}{<} \mathrm{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1]$.

Observe that $\mathrm{Cone}(\alpha) = \mathrm{Cone}(\lambda)[-1]$, since the square

$$
\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "\lambda[-1]"] & H_1 \arrow[d, "\alpha"] \\
\tau_{\mathcal{U}_2^\perp}(\mathrm{Cone}(f))[-1] \arrow[r] & V_2
\end{tikzcd}
$$

is a pullback. Moreover, $\mathrm{Cone}(\lambda)[-1] = (\tau_{\mathcal{U}_2}(\mathrm{Cone}(f))[1])[-1] = \tau_{\mathcal{U}_2}(\mathrm{Cone}(f))$. Hence, $\mathrm{Cone}(\alpha) \in \mathcal{U}_2$ and $\tau^{\mathcal{U}_1^\perp}(\mathrm{Cone}(\alpha)) = 0$, that is, $\mathrm{Cone}(\alpha) \in \mathcal{U}_1$, and since there is a distinguished triangle

$$H_1 \xrightarrow{\alpha} V_2 \to \mathrm{Cone}(\alpha) \xrightarrow{+}$$

with $H_1, \mathrm{Cone}(\alpha) \in \mathcal{U}_1$ it follows that $V_2 \in \mathcal{U}_1$. Hence, $\tau_{\mathcal{U}_1}(V_2) \cong V_2$.

We can then write $V_2 \underset{\oplus}{<} H_1$ and consider the commutative diagram

$$
\begin{tikzcd}
H_1 \cong H_1' \oplus V_2 \arrow[r, "(\,f'\ \tilde{f}\,)"] \arrow[d, "(\,0\ 1\,)"] & H_2 \arrow[d, equal] \\
V_2 \arrow[r] & H_2
\end{tikzcd}
$$

so $f' = 0$. Hence, the inclusion $\binom{1}{0} : H_1' \to H_1' \oplus V_2$ can be lifted to $\mathrm{Cone}(f)[-1]$ and $H_1' \underset{\oplus}{<} \mathrm{Cone}(f)[-1]$. Since $\mathrm{Cone}(f)[-1] \in \mathcal{U}_1^\perp$, so does H_1'. Similarly, $H_1' \in \mathcal{U}_1$ because $H_1 \in \mathcal{U}_1$. Hence, $H_1' = 0$ and $\alpha : H_1 \to V_2$ is an iso. The same follows for λ. Therefore, $\mathrm{Cone}(f) \in \mathcal{U}_2^\perp$ which proves **1.b**. $\square$

We can see a special case of example 3 in the case of the derived category of a ring. Let R be a commutative ring, consider the t-structure $(\mathcal{U}_1, \mathcal{U}_1^{\perp}) = (\mathcal{D}^{\leq 0}(R), \mathcal{D}^{>0}(R))$ in $\mathcal{D}(R)$. Given an idempotent ideal $I = I^2 \lhd R$, it defines three classes of modules

$$\begin{aligned}
\mathcal{C}_I &= \{C \in \text{Mod-}R | IC = C\} \\
\mathcal{T}_I &= \{T \in \text{Mod-}R | IT = 0\} \cong \text{Mod-}\frac{R}{I} \\
\mathcal{F}_I &= \{F \in \text{Mod-}R | Ix \neq 0 \forall x \in F \setminus \{0\}\}
\end{aligned}$$

such that $(\mathcal{C}_I, \mathcal{T}_I)$ and $(\mathcal{T}_I, \mathcal{F}_I)$ make two torsion pairs. We call the triple $(\mathcal{C}_I, \mathcal{T}_I, \mathcal{F}_I)$ a TTP triple.

We define the t-structure $(\mathcal{U}_2, \mathcal{U}_2^{\perp})$ as the Happel-Reiten-Smalo t-structure associated to the torsion pair $(\mathcal{C}_I, \mathcal{T}_I)$ in Mod-R:

$$\begin{aligned}
\mathcal{U}_2 &= \{U_2 \in \mathcal{D}^{\leq 0}(R) | H^0(U_2) \in \mathcal{C}_I\} \\
\mathcal{U}_2^{\perp} &= \{V_2 \in \mathcal{D}^{\geq 0}(R) | H^0(V_2) \in \mathcal{T}_I\}.
\end{aligned}$$

In this case we can check that condition **1.b** holds. In fact, let $\mathcal{H}$ be the heart

$$\begin{aligned}
\mathcal{U}_1 \cap \mathcal{U}_2^{\perp} &= \mathcal{D}^{\leq 0}(R) \cap \mathcal{U}_2^{\perp} \\
&= \{T[0] | T \in \mathcal{T}_I\} \cong \text{Mod-}\frac{R}{I}.
\end{aligned}$$

Hence, $\mathcal{H}$ is abelian.

Now, consider $V_1 \in \mathcal{U}_1^{\perp}$ such that there is an exact triangle

$$V_1 \to T_1[0] \xrightarrow{f[0]} T_2[0] \xrightarrow{+}$$

with $T_1, T_2 \in \mathcal{H}$. Of course, $V_1 = \text{Cone}(f)[-1]$, i.e.

$$V_1 = \cdots \to 0 \to \overset{0}{T_1} \xrightarrow{f} \overset{1}{T_2} \to 0 \to \cdots$$

where the numbers over T_1 and T_2 represent their cohomological degree.

The fact that $V_1 \in \mathcal{U}_1^{\perp} = \mathcal{D}^{>0}(R)$ implies that $H^0(V_1) = 0$, i.e. f is mono. To prove that $V_1 \in \mathcal{U}_2^{\perp}[-1]$ we would need to show that $\text{Coker}(f) = H^1(V_1)$ belongs to $\mathcal{T}_I$, but this follows from the fact that f is a mono in $\mathcal{T}_I$ which is a torsion class.