Let $\mathcal{A}$ be a *good* category (abelian/exact/triangulated) and $\mathcal{W}$ a full subcategory of $\mathcal{A}$ closed by direct summands and extensions, and consider the category $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$.

Let $(\mathcal{X}, \mathcal{Y})$ be a orthogonal pair in $\underline{\mathcal{A}}$ and consider the following classes in $\mathcal{A}$:

$$\mathcal{T} = \{T \in \mathcal{A} | \underline{T} \in \mathcal{X}\}$$
$$\mathcal{F} = \{F \in \mathcal{A} | \underline{F} \in \mathcal{Y}\}.$$

Lemma 1. *In the previous notation, $(\mathcal{T}, \mathcal{T}^{\perp})$ is a orthogonal pair.*

Proof. In order to prove it we need to show that ${}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$.

Let $M \in {}^{\perp}(\mathcal{T}^{\perp})$, this means that

$$\mathcal{A}(M, Y) = 0 \tag{1}$$

whenever

$$\mathcal{A}(T, Y) = 0\, \forall T \in \mathcal{T}. \tag{2}$$

However, if $\mathcal{A}(T, Y) = 0\ \forall T \in \mathcal{T}$, then $\underline{\mathcal{A}}(\underline{X}, \underline{Y}) = 0\ \forall \underline{X} \in \mathcal{X}$. Hence, $\underline{Y} \in \mathcal{Y}$. So $\underline{\mathcal{A}}(\underline{M}, \underline{Y}) = 0\ \forall \underline{Y} \in \mathcal{Y}$. Hence, $\underline{M} \in \mathcal{X}$ and so $M \in \mathcal{T}$.

We have proved that ${}^{\perp}(\mathcal{T}^{\perp}) \subseteq \mathcal{T}$, the converse inclusion is trivial. □

Remark. The dual statement holds for $\mathcal{F}$. Notice that have we also proved that if $\mathcal{A}(T, Y) = 0\ \forall T \in \mathcal{T}$, then $\underline{Y} \in \mathcal{Y}$ and hence $Y \in \mathcal{F}$. That is, $\mathcal{T}^{\perp} \subseteq \mathcal{F}$ and dually ${}^{\perp}\mathcal{F} \subseteq \mathcal{T}$.

Properties of $(\mathcal{T}, \mathcal{T}^{\perp})$ and $({}^{\perp}\mathcal{F}, \mathcal{F})$:

1. ${}^{\perp}\mathcal{F} \subseteq \mathcal{T}$ and $\mathcal{T}^{\perp} \subseteq \mathcal{F}$.
2. $\mathcal{T} \cap \mathcal{F} = \mathcal{W}$. In fact, $M \in \mathcal{T} \cap \mathcal{F}$ iff $\underline{M} \in \mathcal{X} \cap \mathcal{Y} = 0$, which happens iff $M <_{\oplus} W$ for some $W \in \mathcal{W}$, but $\mathcal{W}$ is closed by direct summands, hence $M \in \mathcal{W}$.
3. If $N \in \mathcal{T}^{\perp} \cap {}^{\perp}\mathcal{F}$, then $N = 0$. It follows from $N \in \mathcal{T}^{\perp} \cap {}^{\perp}\mathcal{F} \subseteq \mathcal{F} \cap \mathcal{T} = \mathcal{W}$. But $\mathcal{W} \subseteq \mathcal{T}$, hence $\mathcal{A}(W', N) = 0\ \forall W' \in \mathcal{W}$, in particular $\mathcal{A}(N, N) = 0$, i.e. $N = 0$.

It follows the following

Lemma 2. *Let $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ be two orthogonal pairs in $\mathcal{A}$ such that $\mathcal{T}_2 \subseteq \mathcal{T}_1$ and $\mathcal{T}_2 \cap \mathcal{F}_1 = 0$. Then, $\mathcal{W} = \mathcal{T}_1 \cap \mathcal{F}_2$.*

add proof or at least explanation

If $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ is a orthogonal pair in an abelian and locally small category $\mathcal{A}$, then $\mathbb{t}$ is a torsion pair. add proof?

Lemma 3. *Let $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in $\mathcal{A}$ such that $(\underline{\mathcal{T}_1}, \underline{\mathcal{F}_2})$ is a orthogonal pair in $\underline{\mathcal{A}}$. Then $\operatorname{add}(\mathcal{T}_2 * \mathcal{W}) = \mathcal{T}_1$ and $\operatorname{add}(\mathcal{W} * \mathcal{F}_1) = \mathcal{F}_2$.*

Proof. Let T_1 be in $\mathcal{T}_1$ and consider its approximation sequence Explain this better

$$t_2(T_1) \to T_1 \xrightarrow{f} f_2(T_1) \xrightarrow{+} . \tag{3}$$

Consider the following diagram

$$\begin{tikzcd}
t_2(T_1) \arrow[r] \arrow[d, equal] & \widehat{T}_1 \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\
t_2(T_1) \arrow[r] & T_1 \arrow[r, "f"] & f_2(T_1)
\end{tikzcd} \tag{4}$$

where the square on the right is a homological pullback, and the arrow $\phi : W \to f_2(T_1)$ is such that $f = \phi \circ \gamma$ for some $\gamma : T_1 \to W$, since $\underline{\mathcal{A}}(T_1, f_2(T_1)) = 0$.

Then, there exist a (non necessarely unique) $\eta : T_1 \to \widehat{T_1}$ making the following diagram commute.

$$
\begin{tikzcd}
T_1 \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "1"'] & & \\
& \widehat{T_1} \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\
& T_1 \arrow[r, "f"] & f_2(T_1)
\end{tikzcd}
\tag{5}
$$

Hence, $T_1 <_\oplus \widehat{T_1} \in \mathcal{T}_2 * \mathcal{W}$, i.e. $\mathrm{add}(\mathcal{T}_2 * \mathcal{W}) = \mathcal{T}_1$.

Dually, $\mathrm{add}(\mathcal{W} * \mathcal{F}_1) = \mathcal{F}_2$. □