1 The Nakaoka setting

Given a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{C}$, it was proved in [1] that it is possible to use it to construct a subcategory $\mathcal{W} \subseteq \mathcal{C}$ such that in the quotient category $\dfrac{\mathcal{C}}{\mathcal{W}} = \underline{\mathcal{C}}$ a heart $\mathcal{H}$ is defined in such a way that it is an abelian category and there is a homological functor $\mathcal{C} \to \mathcal{H}$.

Our goal is to provide a set of axioms for a (nice) additive category $\mathcal{A}$ and a couple of torsion pairs in it, in such a way that they will guarantee the existence of an abelian heart in $\mathcal{A}$. In a sense, we want to axiomatize $\underline{\mathcal{C}}$ and the pairs which are referred in Nakaoka's work as $(\underline{\mathcal{C}}^-, \underline{\mathcal{V}})$ and $(\underline{\mathcal{U}}, \underline{\mathcal{C}}^+)$.

Now we will briefly recall Nakaoka's setting. Assume that $\mathcal{C}$ is a triangulated category, and $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in $\mathcal{C}$, i.e.

1. $\mathcal{C}(\mathcal{U}, \mathcal{V}[1]) = 0$
2. $\mathcal{C} = \mathcal{U} * \mathcal{V}[1]$, where $X \in \mathcal{M} * \mathcal{N}$ if and only if there is a distinguished triangle
$$M \to X \to N \to M[1]$$
with $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

Then, we put $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ and define $\underline{\mathcal{C}} = \mathcal{C}/\mathcal{W}$, and similarly for $\underline{\mathcal{U}}$, $\underline{\mathcal{V}}$, etc.

We define the following full subcategories of $\mathcal{C}$:

- $\mathcal{C}^+ = \mathcal{W} * \mathcal{V}[1]$
- $\mathcal{C}^- = \mathcal{U}[-1] * \mathcal{W}$

together with their respective quotients $\underline{\mathcal{C}}^+$ and $\underline{\mathcal{C}}^-$. Then, we have the following lemma.

Lemma 1. *Let $X \in \mathcal{C}$, TFAE:*

1. $\underline{X} \in \underline{\mathcal{C}}^+$,
2. *There is a monomorphism $\underline{X} \to \underline{V[1]}$ in $\underline{\mathcal{C}}$, for some $V \in \mathcal{V}$.*

The dual also holds:

Lemma 2. *Let $X \in \mathcal{C}$, TFAE:*

1. $\underline{X} \in \underline{\mathcal{C}}^-$,
2. *There is an epimorphism $\underline{U[-1]} \to \underline{X}$ in $\underline{\mathcal{C}}$, for some $U \in \mathcal{U}$.*

Corollary 3. *If $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in a triangulated category $\mathcal{C}$, then:*

1. ${}^{\perp}\underline{\mathcal{V}} = \underline{\mathcal{C}}^-$
2. $\underline{\mathcal{U}}^{\perp} = \underline{\mathcal{C}}^+$

Lemma 4. *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{C}}^+$ be the left adjoint of the inclusion functor $j : \underline{\mathcal{C}}^+ \hookrightarrow \underline{\mathcal{C}}$. If $\lambda : 1_{\underline{\mathcal{C}}} \to j \circ F$ is the unit of the adjunction, then there is a pseudokernel-pseudocokernel sequence*
$$U_C \xrightarrow{u} C \xrightarrow{\lambda_C} (j \circ F)(C) \xrightarrow{+}$$
in $\underline{\mathcal{C}}$ such that $U_C \in \mathcal{U}$.

With dual:

Lemma 5. *Let $G : \underline{\mathcal{C}} \to \underline{\mathcal{C}}^-$ be the left adjoint of the inclusion functor $i : \underline{\mathcal{C}}^- \hookrightarrow \underline{\mathcal{C}}$. If $\varepsilon : i \circ G \to 1_{\underline{\mathcal{C}}}$ is the co-unit of the adjunction, then there is a pseudokernel-pseudocokernel sequence*
$$(i \circ G)(C) \xrightarrow{\varepsilon_C} C \xrightarrow{V}{}_C \xrightarrow{+}$$
in $\underline{\mathcal{C}}$ such that $V_C \in \mathcal{V}$.

Corollary 6. *$(\underline{\mathcal{C}}^-, \underline{\mathcal{V}})$ and $(\underline{\mathcal{U}}, \underline{\mathcal{C}}^+)$ are orthogonal pairs in $\underline{\mathcal{C}}$ provided $\mathcal{C}$ has split idempotents.*

Remark 1.

1. By prop 5.3 Nakaoka we have that $\underline{\mathcal{C}}^+$ has cokernels and, dually, $\underline{\mathcal{C}}^-$ has kernels. add better reference
2. We have inclusions $\mathcal{V} \subseteq \mathcal{C}^+$ and $\mathcal{U} \subseteq \mathcal{C}^-$ see page 5.5 of the notes

2 Torsion pairs

We fix an additive category $\mathcal{X}$ with pseudokernels and pseudocokernels on which idempotents split.

Definition 1. A pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{X}$ is a torsion pair in $\mathcal{X}$ if:

1.
$$\mathcal{F} = \mathcal{T}^{\perp} = \{X \in \mathcal{X} \mid \mathcal{X}(\mathcal{T}, X) = 0\},$$
$$\mathcal{T} = {}^{\perp}\mathcal{F} = \{X \in \mathcal{X} \mid \mathcal{X}(X, \mathcal{F}) = 0\};$$

2. FOr each $M \in \mathcal{X}$ there is a pseudokernel-pseudocokernel sequence
$$T_M \xrightarrow{\varepsilon_M} M \xrightarrow{\lambda_M} F^M$$
where $T_M \in \mathcal{T}$ and $F^M \in \mathcal{F}$.

If in addition the assignment $M \mapsto t(M) := T_M$ (resp. $M \mapsto f(M) := F^M$) is functorial and defines an adjoint pair (i, t) (resp. (f, j)), where $i : \mathcal{T} \hookrightarrow \mathcal{X}$ (resp. $j : \mathcal{F} \hookrightarrow \mathcal{X}$) is the inclusion functor, then we say that $\mathfrak{t}$ is left (resp. right) functorial. In such a case, ε (resp. λ) is the counit (resp. unit) of the given adjoint pair. We say that $\mathfrak{t}$ is functorial if it is right and left functorial.

Remark 2. Let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a left functorial torsion pair in $\mathcal{X}$. Then

(a) For any $M \in \mathcal{X}, T' \in \mathcal{T}$ and $\alpha \in \mathcal{X}(T', M)$ there is a unique $\alpha' \in \mathcal{X}(T', t(M))$ such that $\varepsilon_M \circ \alpha' = \alpha$, i.e.
$$\begin{tikzcd} & T' \arrow[dl, dashed, "\exists!\, \alpha'"'] \arrow[d, "\alpha"] \\ t(M) \arrow[r, "\varepsilon_M"] & M \end{tikzcd}$$

(b) Let $g : T_1 \to T_2$ be a morphism in $\mathcal{T}$, which admits a pseudocokernel $g^C : T_2 \to \operatorname{PCok}_{\mathcal{T}}(g)$ in $\mathcal{T}$. Then g^C is a pseudocokernel of g in $\mathcal{X}$.

Proof. (a) Since (i, t) is an adjoint pair, we have a functorial isomorphism
$$\Theta : \operatorname{Hom}_{\mathcal{T}}(T', t(M)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{X}}(i(T'), M) = \operatorname{Hom}_{\mathcal{X}}(T', M).$$
Let $\alpha' := \Theta^{-1}(\alpha)$. Then, $\varepsilon_M \circ \alpha' = \Theta(\alpha') = \alpha$.

(b) Let $X \in \mathcal{X}$ and $h : T_2 \to X$ such that $hg = 0$. Consider the commutative diagram
$$\begin{tikzcd} T_1 \arrow[r, "g"] & T_2 \arrow[r, "g^C"] \arrow[d, "h"] \arrow[dr, dashed, "\exists h'"] & \operatorname{PCok}_{\mathcal{T}}(g) \arrow[d, dashed, "\exists h''"] \\ & X & t(X). \arrow[l, "\varepsilon_X"] \end{tikzcd}$$
Then, $h' \circ g = 0$. In fact, $0 = h \circ g = \varepsilon_X \circ h' \circ g$ and $\varepsilon_X \circ 0 = 0$, so, by (a), $h' \circ g = 0$. Since $h' \circ g = 0$, it follows that there is a map $h'' : \operatorname{PCok}_{\mathcal{T}}(g) \to t(X)$ such that $h'' \circ g^C = h'$. Finally, $h = (\varepsilon_X \circ h'') \circ g^C$.

□

The dual also holds.

3 Axiomatization

Definition 2. Let $\mathcal{X}$ be an additive category with pseudokernels and pseudocokernels, a *compatible* torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{X}$ consists of the two pairs $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ of full subcategories of $\mathcal{X}$ satisfying the following axioms:

(CT1) $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are respectively a left functorial and a right functorial torsion pair

(CT2) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equiv. $\mathcal{F}_1 \subseteq \mathcal{F}_2$)

(CT3) Any morphism $g : T_1 \to T_1'$ in $\mathcal{T}_1$ admits a pseudocokernel $g^C : T_1' \to \mathrm{PCok}_{\mathcal{X}}(g)$, with $T_1'' := \mathrm{PCok}_{\mathcal{X}}(g) \in \mathcal{T}_1$, such that

$$0 \longrightarrow (T_1'', -)_{|\mathcal{F}_2} \xrightarrow{(g^C,-)} (T_1', -)_{|\mathcal{F}_2} \xrightarrow{(g,-)} (T_1, -)_{|\mathcal{F}_2}$$

is an exact sequence of functors.

(CT3)* Dual of **(CT3)**.

Notation. In the case of a compatible torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{X}$, we have the adjoint pairs

$$(i_1, t_1) : \mathcal{T}_1 \underset{t_1}{\overset{i_1}{\rightleftarrows}} \mathcal{X} \qquad \text{and} \qquad (t_2, j_2) : \mathcal{F}_2 \underset{j_2}{\overset{t_2}{\leftrightarrows}} \mathcal{X}$$

In this case there are also the counit $\varepsilon_{1,M} : t_1(M) \to M$ and the unit $\lambda_{2,M} : M \to f_2(M)$. The heart of $\mathbb{t}$ is defined as $\mathcal{H} = \mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$.

Lemma 7. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$. Then the following statements hold true.*

(a) $\mathcal{F}_1 \cap \mathcal{T}_2 = 0$*,*

(b) $f_2(\mathcal{T}_1) \subseteq \mathcal{H}$ *and* $t_1(\mathcal{F}_2) \subseteq \mathcal{H}$*.*

Proof. (a) Since $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we have $\mathcal{F}_1 \cap \mathcal{T}_2 \subseteq \mathcal{F}_1 \cap \mathcal{T}_1 = 0$.

(b) Let $T_2 \in \mathcal{T}_2$ and $F_2 \in \mathcal{F}_2$. Then,

$$\mathrm{Hom}_{\mathcal{X}}(T_2, t_1(F_2)) \cong \mathrm{Hom}_{\mathcal{X}}(T_2, F_2) = 0.$$

Hence, $t_1(F_2) \in \mathcal{T}_2^{\perp} = \mathcal{F}_2$. An analogous proof shows that $f_2(\mathcal{T}_1) \subseteq \mathcal{T}_1$.

□

Proposition 8. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$ and $f : H_1 \to H_2$ be a morphism in $\mathcal{H}$. Then*

*(a) if $f^C : H_2 \to T_1$ is the pseudocokernel of f, given by **(CT3)** and $\lambda_{2,T_1} : T_1 \to f_2(T_1)$, then*

$$\mathrm{Coker}\,(H_1 \xrightarrow{f} H_2) = (H_2 \xrightarrow{\tilde{f}^C} f_1(T_1))$$

in $\mathcal{H}$, where $\tilde{f}^C := \lambda_{2,T_1} \circ f^C$;

*(b) if $f^K : F_2 \to H_1$ is the pseudokernel of f, given by **(CT3)*** and $\varepsilon_{1,F_2} : t_1(F_2) \to F_2$, then*

$$\mathrm{Ker}\,(H_1 \xrightarrow{f} H_2) = (t_1(F_2) \xrightarrow{\tilde{f}^K} H_1)$$

in $\mathcal{H}$, where $\tilde{f}^K = f^K \circ \varepsilon_{1,F_2}$.

Proof. (a) Let $g : H_2 \to H$ in $\mathcal{H}$ such that $gf = 0$. And consider the solid part of the following diagram

$$\begin{tikzcd}
H_1 \arrow[r, "f"] & H_2 \arrow[r, "f^C"] \arrow[d, "g"] & T_1 \arrow[r, "\lambda_{2,T_1}"] \arrow[dl, dashed, "f'"'] & f_2(T_1) \arrow[dll, dashed, "f''"] \\
& H & &
\end{tikzcd}$$

with $T_1 \in \mathcal{T}_1$ and $f_2(T_1) \in \mathcal{H}$ (by lemma 7). Since $H \in \mathcal{H} \subseteq \mathcal{F}_2$, by **(CT3)** there is a unique $f' : T_1 \to H$ such that $f' f^C = g$. By Remark 2(a), there is a $f'' : f_2(T_1) \to H$ making the diagram commute. Hence, $g = f'' \circ \tilde{f}^C$.

As for unicity, let $r : f_2(T_1) \to H$, such that $g = r \circ \tilde{f}^C$. Then, $(f'' \circ \lambda_{2,T_1}) \circ f^C = (r \circ \lambda_{2,T_1}) \circ f^C$, so $f'' \circ \lambda_{2,T_1} = r \circ \lambda_{2,T_1}$ by **(CT3)**, and $f'' = r$ by Remark 2(a)

(b) Dual.

□

Proposition 9. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$. Then, for any $f : H \to H'$ in $\mathcal{H}$ we have that:*

(a) f is a monomorphism in $\mathcal{H}$ if and only if there is a pseudokernel $\mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$*;*

(b) f is an epimorphism in $\mathcal{H}$ if and only if there is a pseudocokernel $\mathrm{PCok}_{\mathcal{T}_1}(f) \in \mathcal{T}_2$.

Proof. (a) Let $f : H \to H'$ be a morphism in $\mathcal{H}$. By item (b) there is a diagram

$$\begin{tikzcd}
F_2 \arrow[r, "f^K"] & H \arrow[r, "f"] & H' \\
t_1 F_2 \arrow[u, "\varepsilon_{1,F_2}"] \arrow[ur, "\tilde{f}^K"'] & &
\end{tikzcd}$$

with $\mathrm{Ker}\,(H \xrightarrow{f} H') = (t_1(F_2) \xrightarrow{\tilde{f}^K} H)$.

Assume that f is a monomorphism, then $t_1(F_2) = 0$, so $0 = \mathrm{Hom}_{\mathcal{X}}(Z, t_1(F_2)) \cong \mathrm{Hom}_{\mathcal{X}}(Z, F_2)$ for all $Z \in \mathcal{T}_1$. Hence, $F_2 \in \mathcal{F}_1$.

Conversely, let $f' : F_1 \to H$ be a pseudokernel of f in $\mathcal{F}_2$, with $F_1 \in \mathcal{F}_1$. Consider the solid part of the diagram

$$\begin{tikzcd}
t_1(F_2) \arrow[d, "\varepsilon_{1,F_2}"] \arrow[dr, "\tilde{f}^K"] & & \\
F_2 \arrow[r, "f^K"] \arrow[d, dashed, shift right, "u"'] & H \arrow[r, "f"] & H' \\
F_1 \arrow[u, dashed, shift right, "v"'] \arrow[ur, "f'"] & &
\end{tikzcd}$$

Since f^K and f' are pseudokernel of f in $\mathcal{F}_2$, there exist u and v such that $f' = f^K \circ v$ and $f^K = f' \circ u$. Therefore, $f^K = f^K \circ v \circ u$, and so $\tilde{f}^K = f^K \circ \varepsilon_{1,F_2} = f^K \circ v \circ u \circ \varepsilon_{1,F_2}$. Notice that $u \circ \varepsilon_{1,F_2} = 0$ since $\mathrm{Hom}_{\mathcal{X}}(\mathcal{T}_1, \mathcal{F}_1) = 0$. Thus, f has the zero morphism as its kernel in $\mathcal{H}$, i.e. f is a monomorphism.

(b) Dual.

□

Definition 3. A compatible torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{X}$ is strong if the following axioms hold:

(CT4) Let $f : H_1 \to H_2$ in $\mathcal{H}$ be such that there is a pseudokernel $\mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$. Then, for the commutative diagram

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 := \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F)_2 \arrow[r, "\varepsilon_{1,F_2}"] & F_2 := \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1)
\end{tikzcd}$$

there exists a morphism $b : t_1(F_2) \to H_1$ such that $ab = \varepsilon_{1,F_2}$.

(CT4)* Dual

With these axioms we can prove that the heart has kernels and cokernels.

Theorem 10. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a compatible torsion pair in $\mathcal{X}$. Then, the following are equivalent:*

(a) $\mathbb{t}$ is strong,

(b) $\mathcal{H}$ is an abelian category.

Proof.

$(a) \Rightarrow (b)$ By definition $\mathcal{H}$ is an additive subcategory of $\mathcal{X}$. In order to prove that (a) implies (b) observe that $\mathcal{H}$ is preabelian by proposition 8. To prove that $\mathcal{H}$ is abelian, we Check referencing in TeX just need to show that any monomorphism (resp. epimorphism) is the kernel (resp. cokernel) of some morphism in $\mathcal{H}$.

Let $f : H_1 \to H_2$ be a monomorphism in $\mathcal{H}$. By **(CT3)** and **(CT3)*** we can consider the solid part of the commutative diagram

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 = \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F_2) \arrow[r, "\varepsilon_{1,F_2}"] \arrow[ur, dashed, "b"] & F_2 = \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1)
\end{tikzcd}$$

where $gf = \lambda_{2,T_1}(f^C f) = 0$, so there exists $a : H_1 \to F_2$. Note that $\exists \mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$ since f is a monomorphism in $\mathcal{H}$ (by proposition 9(a)). So by **(CT4)** there is a Add proper reference map $b : t_1(F_2) \to H_1$ making the diagram commute.

We claim that $\mathrm{Ker}\, g = f$. Let $\alpha : H \to H_2$ be a morphism such that $g\alpha = 0$. Since $F_2 = \mathrm{PKer}_{\mathcal{X}}(g)$, there is a morphism $\alpha' : H \to F_2$ such that $g^K \alpha' = \alpha$. By remark 2, α' factors as $\varepsilon_{1,F_2}\alpha''$, as in the diagram:

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 = \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F_2) \arrow[r, "\varepsilon_{1,F_2}"] \arrow[ur, "b"] & F_2 = \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1) \\
 & H. \arrow[ul, dashed, "\alpha''"] \arrow[u, dashed, "\alpha'"] \arrow[ur, "\alpha"] & &
\end{tikzcd}$$

By setting $\alpha''' := b\alpha'' : H \to H_1$, we get $f\alpha''' = g^K ab\alpha'' = g^K \varepsilon_{1,F_2}\alpha'' = g^K \alpha' = \alpha$. Thus, any morphism $\alpha : H \to H_2$ such that $g\alpha = 0$ factors through f. To conclude that f is a kernel of g, it suffice to observe that f is a monomorphism, so α''' must be unique.

$(b) \Rightarrow (a)$ Let $\mathcal{H}$ be an abelian category. We only check **(CT4)** since the proof of **(CT4)*** is Fix reference analogous.

Consider the solid part of the commutative diagram

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] \arrow[dl, dashed, "\beta"'] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 = \mathrm{PCok}_{\mathcal{X}}(f) \arrow[d, "\lambda_{2,T_1}"] \\
t_1(F^2) \arrow[r, "\varepsilon_{1,F_2}"] & F_2 = \mathrm{PKer}_{\mathcal{X}}(g) \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2(T_1)
\end{tikzcd}$$

where $f : H_1 \to H_2$ is a morphism in $\mathcal{H}$ such that $\mathrm{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$. Since f is a monomorphism in $\mathcal{H}$ by 9(a) and $\mathcal{H}$ is abelian, we have that $f = \mathrm{Ker}\,\mathrm{Coker}\,(f)$. On the other hand, by 2 (a) there is $\beta : H_1 \to t_1(F_2)$ such that $\varepsilon_{1,F_2}\beta = a$ completing the diagram above. Since $f = \mathrm{Ker}\,\mathrm{Coker}\,(f)$, it follows that β is an isomorphism. Therefore, **(CT4)** follows by setting $b := \beta^{-1}$.

Fix reference

□

3.1 The case of an abelian category

Let's consider the case $\mathcal{X} = \mathcal{A}$ of an Abelian category with two torsion pairs $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ for $i = 1, 2$. Consider $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$.

Remark 3. In the case of an Abelian category $\mathcal{X} = \mathcal{A}$, we have that $\mathbb{t}$ is compatible if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proof. Let $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we need to show that item **(CT1)**, item **(CT3)** and item **(CT3)** hold.

(CT1) It is well known that any torsion pair in an abelian category is functorial.

(CT3) Let $g : T_1 \to T_1'$ be a morphism in $\mathcal{T}_1$. Consider the cokernel morphism of g in $\mathcal{A}$

$$\mathrm{Coker}_{\mathcal{A}}(T_1 \xrightarrow{g} T_1') = (T_1' \xrightarrow{c_g} \mathrm{Coker}\,(g)).$$

Since $\mathcal{T}_1$ is closed under quotient objects, we get that $\mathrm{Coker}\,(g) \in \mathcal{T}_1$. Therefore, we can choose $c_g : T_1' \to \mathrm{Coker}\,(g)$ as $g^C : T_1' \to \mathrm{PCok}_{\mathcal{A}}(g)$.

Is there an explicit choice that we made somewhere before when we talk about $\mathrm{PCok}_{\mathcal{A}}$?

(CT3)* Anologous to the previous.

□

Corollary 11. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a torsion pair in $\mathcal{A}$ with $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Then, for $f : H_1 \to H_2$ in $\mathcal{H}$, the following statements hold:*

(a) the cokernel of f in $\mathcal{H}$ is the composition of the morphisms

$$H_2 \xrightarrow{c_f} \mathrm{Coker}\,(f) \xrightarrow{\lambda_{2,\mathrm{Coker}\,(f)}} f_2(\mathrm{Coker}\,(f));$$

(b) the kernel of f in $\mathcal{H}$ is the composition of the morphisms

$$t_1(\mathrm{Ker}\,(f)) \xrightarrow{\varepsilon_{1,\mathrm{Ker}\,(f)}} \mathrm{Ker}\,(f) \xrightarrow{k_f} H_1;$$

(c) f is an epimorphism in $\mathcal{H}$ if and only if $\mathrm{Coker}\,(f) \in \mathcal{T}_2$;

(d) f is a monomorphism in $\mathcal{H}$ if and only if $\mathrm{Ker}\,(f) \in \mathcal{F}_1$.

Proof. (a) and (b) follow from the proof of Remark 3and proposition 8.

this should be a remark, fix it

(c)

$\Leftarrow$ is trivial.

$\Rightarrow$ By proposition 9 (b) there exists $f^C : H_2 \to T_2$, where $T_2 = \operatorname{Coker}_{T_1}(f) \in \mathcal{T}_2$. Then, we have

$$\begin{tikzcd} H_2 \arrow[rr, "f^C"] \arrow[dr, "c_f"'] & & T_2 \arrow[dl, dashed, "v"] \\ & \operatorname{Coker}(f) \arrow[ur, dashed, "u"] & \end{tikzcd} \qquad \text{such that } \begin{cases} uf^C = c_f, \\ vc_f = f^C. \end{cases}$$

Hence, $uvc_f = c_f$, but c_f is epi, therefore $uv = 1$. Hence, $\operatorname{Coker}(f)$ is a direct summand of $T_2 \in \mathcal{T}_2$ so $\operatorname{Coker}(f) \in \mathcal{T}_2$.

(d) Similar to the previous proof. □

Theorem 12. *Let $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ be a torsion pair in an abelian category $\mathcal{A}$, for $i = 1, 2$, such that $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Then, for $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ the following statements are equivalent:*

(a) $\mathcal{H}$ is an abelian category.

(b) The following conditions hold:

- *(b1) For any $f : H \to H'$ in $\mathcal{H}$, with $\operatorname{Ker}(f) \in \mathcal{F}_1$, we have that $\operatorname{Ker}(f) = 0$.*
- *(b2) For any $f : H \to H'$ in $\mathcal{H}$, with $\operatorname{Coker}(f) \in \mathcal{T}_2$, we have that $\operatorname{Coker}(f) = 0$.*
- *(b3) $\mathcal{H}$ is closed under kernels (resp. cokernels) of epimorphisms (resp. monomorphisms) in $\mathcal{A}$.*

(c) $\mathcal{H}$ is closed under kernels and cokernels in $\mathcal{A}$.

3.2 Related torsion pairs in triangulated categories

Let $\mathcal{X} = \mathcal{T}$ be a triangulated category on which idempotents split. We start by recalling the definition of a t-structure in $\mathcal{T}$.

Definition 4. A pair $(\mathcal{A}, \mathcal{B})$ of full subcategories of $\mathcal{T}$ is a t-structure in $\mathcal{T}$ if

(a) $\mathbb{t} = (\mathcal{A}, \mathcal{B}[-1])$ is a torsion pair in $\mathcal{T}$, and

(b) $\mathcal{A}[1] \subseteq \mathcal{A}$.

Remark 4. It is well known that any t-structure $(\mathcal{A}, \mathcal{B})$ in $\mathcal{T}$ gives a functional torsion pair $\mathbb{t} = (\mathcal{A}, [-1])$ and $\mathbb{B}[-1] \subseteq \mathcal{B}$. Furthermore, $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions and direct summands. Note that the t-structure $(\mathcal{A}, \mathcal{B})$ depends only on $\mathcal{A}$, since $\mathcal{B} = \mathcal{A}^{\perp}[1]$.

Definition 5. A *related* torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in triangulated category $\mathcal{T}$ consists of the torsion pairs $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t} = (\mathcal{T}_2, \mathcal{F}_2)$ in $\mathcal{T}$ such that $\mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proposition 13. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related torsion pair in $\mathcal{T}$. Then*

(a) $(\mathcal{T}_1, \mathcal{F}_1[1])$ and $(\mathcal{T}_2, \mathcal{F}_2[1])$ are t-structures in $\mathcal{T}$;

(b) $\mathbb{t}$ is a compatible torsion pair in $\mathcal{T}$;

(c) the heart $\mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2[1]$ is a preabelian category.

Definition 6. A related torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in the triangulated category $\mathcal{T}$ is *strong* if for any morphism $f : H_1 \to H_2$, in $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$, and a distinguished triangle $Z \to H_1 \xrightarrow{f} H_2 \to Z[1]$, the following conditions hold true

(RST1) $Z \in \mathcal{F}_1$ if and only if $Z \in \mathcal{F}_2[-1]$;

(RST2) $Z[1] \in \mathcal{T}_2$ if and only if $Z \in \mathcal{T}_1$.

Theorem 14. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a strongly related torsion pair in the triangulated category $\mathcal{T}$. Then, the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ is an abelian category.*

Example 1. Let $(\mathcal{A},\mathcal{B})$ be a t-structure in $\mathcal{T}$. Consider $\mathbb{t}_1 := (\mathcal{A},\mathcal{B}[-1])$ and $\mathbb{t}_2 := (\mathcal{A}[1],\mathcal{B})$. It is not hard to see that $\mathbb{t} = (\mathbb{t}_1,\mathbb{t}_2)$ is a strongly related torsion pair in $\mathcal{T}$. In this case, by theorem 14, we get that $\mathcal{H} = \mathcal{A}\cap\mathcal{B}$ is an abelian category (BBD theorem).

Example 2. Let $\mathcal{D}$ be a triangulated category with two t-structures $(\mathcal{U}_1,\mathcal{U}_1^{\perp})$ and $(\mathcal{U}_2,\mathcal{U}_2^{\perp})$ such that $\mathcal{U}_1[1]\subseteq\mathcal{U}_2\subseteq\mathcal{U}_1$. Then, these satisfy axioms **(CT1)-(CT3)**,**(CT3)***, hence $\mathcal{H}=\mathcal{U}_1\cap\mathcal{U}_2^{\perp}$ has kernels and cokernels. Moreover, TFAE:

1.a **(CT4)** holds.

1.b If $V_1 \to H_1 \xrightarrow{f} H_2 \xrightarrow{+}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $V_1 \in \mathcal{U}_1^{\perp}$, then $V_1 \in \mathcal{U}_2^{\perp}[-1]$.

And, dually, there is an equivalence of the following:

2.a **(CT4)*** holds.

2.b If $H_1 \xrightarrow{f} H_2 \to U_2 \xrightarrow{+}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $U_2 \in \mathcal{U}_2$, then $U_2 \in \mathcal{U}_1[1]$.

Proof of the equivalences in example 2. Let's $\mathcal{D}$ be a triangulated category with two t-structures as in example 3. The pseudocokernel of a morphism in $\mathcal{U}_1$ can be computed by taking the cone in $\mathcal{D}$, i.e. given a morphism $f : U_1 \to U_1'$ in $\mathcal{U}_1$ we can compute a pseudocokernel in $\mathcal{U}_1$ by completing f to a triangle

$$U_1 \xrightarrow{f} U_1' \to \mathrm{Cone}(f) \xrightarrow{+} .$$

Moreover, this pseudocokernel satisfies **(CT3)**.

Now, assume that **(CT1)-(CT3)**,**(CT3)*** are satisfied together with axiom **1.b**, and consider the solid part of the diagram as in **(CT4)**:

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_1 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
\tau_{\mathcal{U}_1}(V_2) \arrow[r, "\varepsilon"] \arrow[ur, dotted, "\beta"] & V_2 \arrow[r] & H_2 \arrow[r] & \tau^{\mathcal{U}_2^{\perp}}\mathrm{Cone}(f)
\end{tikzcd}$$

with $\mathrm{Cone}(f)[-1]\in\mathcal{U}_1^{\perp}$ and where the upper row is a distinguished triangle. By **1.b** then it belongs to $\mathcal{U}_2^{\perp}[-1]$, i.e. $\mathrm{Cone}(f)\in\mathcal{U}_2^{\perp}$, so λ is an iso, conseguently α is an iso and so is ε, so there exist a map $\beta=\alpha^{-1}\circ\varepsilon$ making the diagram commute, that is **(CT4)** holds.

Conversely, assume that **(CT1)-(CT3)**,**(CT3)*** are satisfied together with **(CT4)**. Consider again the solid part of the diagram

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "\lambda[-1]"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_1 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
\tau_{\mathcal{U}_2^{\perp}}(\mathrm{Cone}(f))[-1] \arrow[r, dashed] & V_2 \arrow[r] & H_2 \arrow[r] & \tau^{\mathcal{U}_2^{\perp}}\mathrm{Cone}(f) \\
& \tau_{\mathcal{U}_1}(V_2) \arrow[u, "\varepsilon"] \arrow[uu, dotted, bend left, "\beta"] & &
\end{tikzcd}$$

with $\mathrm{Cone}(f)[-1]\in\mathcal{U}_1^{\perp}$. Neeman guarantees that α can be taken so that the square on the left is a pullback. Axiom **(CT4)** gives the existence of $\beta : \tau_{\mathcal{U}_1}(V_2)\to H_1$ such that $\alpha\circ\beta=\varepsilon$. Add reference

Since $\tau_{\mathcal{U}_1}$ is a functor, there is also a morphism $\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(H_1) = H_1 \to \tau_{\mathcal{U}_1}(V_2)$ such that $\varepsilon\circ\tau_{\mathcal{U}_1}(\alpha)=\alpha$, hence $\varepsilon\circ\tau_{\mathcal{U}_1}(\alpha)\circ\beta=\varepsilon$. By the functoriality of the torsion pair $(\mathcal{U}_1,\mathcal{U}_1^{\perp})$, this means that $\tau_{\mathcal{U}_1}(\alpha)\circ\beta = 1_{\tau_{\mathcal{U}_1}(V_2)}$. Then, β is a section.

Hence, we can write $\tau_{\mathcal{U}_1}(\alpha) : H_1 \to \tau_{\mathcal{U}_1}(V_2)$ as

$$\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(V_2)\oplus H_1' \xrightarrow{(*\ 0)} \tau_{\mathcal{U}_1}(V_2)$$

for some $H_1' \underset{\oplus}{<} H_1$ such that α vanishes on H_1'. If we consider the solid part of the diagram

$$\begin{tikzcd}
 & H_1' \arrow[dl, dashed] \arrow[d, "\binom{1}{0}"] \arrow[dr, "0"] & & \\
\mathrm{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1] \arrow[r] & H_1 \arrow[r, "\tau_{\mathcal{U}_1}(\alpha)"] & \tau_{\mathcal{U}_1}(V_2) \arrow[r, dashed, "+"] & {}
\end{tikzcd}$$

we can construct the dashed arrow, and the fact that the triangle commutes means that $H_1' \underset{\oplus}{<} \mathrm{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1]$.

Observe that $\mathrm{Cone}(\alpha) = \mathrm{Cone}(\lambda)[-1]$, since the square

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "\lambda[-1]"] & H_1 \arrow[d, "\alpha"] \\
\tau_{\mathcal{U}_2^{\perp}}(\mathrm{Cone}(f))[-1] \arrow[r] & V_2
\end{tikzcd}$$

is a pullback. Moreover, $\mathrm{Cone}(\lambda)[-1] = (\tau_{\mathcal{U}_2}(\mathrm{Cone}(f))[1])[-1] = \tau_{\mathcal{U}_2}(\mathrm{Cone}(f))$. Hence, $\mathrm{Cone}(\alpha) \in \mathcal{U}_2$ and $\tau^{\mathcal{U}_1^{\perp}}(\mathrm{Cone}(\alpha)) = 0$, that is, $\mathrm{Cone}(\alpha) \in \mathcal{U}_1$, and since there is a distinguished triangle

$$H_1 \xrightarrow{\alpha} V_2 \to \mathrm{Cone}(\alpha) \xrightarrow{+}$$

with $H_1, \mathrm{Cone}(\alpha) \in \mathcal{U}_1$ it follows that $V_2 \in \mathcal{U}_1$. Hence, $\tau_{\mathcal{U}_1}(V_2) \cong V_2$.

We can then write $V_2 \underset{\oplus}{<} H_1$ and consider the commutative diagram

$$\begin{tikzcd}
H_1 \cong H_1' \oplus V_2 \arrow[r, "\begin{pmatrix} f' & \tilde{f} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] & H_2 \arrow[d, equal] \\
V_2 \arrow[r] & H_2
\end{tikzcd}$$

so $f' = 0$. Hence, the inclusion $\binom{1}{0} : H_1' \to H_1' \oplus V_2$ can be lifted to $\mathrm{Cone}(f)[-1]$ and $H_1' \underset{\oplus}{<} \mathrm{Cone}(f)[-1]$. Since $\mathrm{Cone}(f)[-1] \in \mathcal{U}_1^{\perp}$, so does H_1'. Similarly, $H_1' \in \mathcal{U}_1$ because $H_1 \in \mathcal{U}_1$. Hence, $H_1' = 0$ and $\alpha : H_1 \to V_2$ is an iso. The same follows for λ. Therefore, $\mathrm{Cone}(f) \in \mathcal{U}_2^{\perp}$ which proves **1.b**. $\square$

Example 3. Let R be any (associative with 1) ring. Consider the triangulated category $\mathcal{T} := \mathcal{D}(R)$. The derived category $\mathcal{D}(R)$ has the so called natural t-structure $(\mathcal{D}^{\leq 0}(R), \mathcal{D}^{\geq}(R))$ where

$$\mathcal{D}^{\leq 0}(R) := \{X \in \mathcal{D}(R) \mid H^i(X) = 0 \text{ for } i > 0\},$$
$$\mathcal{D}^{\geq 0}(R) := \{X \in \mathcal{D}(R) \mid H^i(x) = 0 \text{ for } i < 0\}.$$

For any ideal $I \trianglelefteq R$, we have the TTF-triple $(\mathcal{C}_I, \mathcal{T}_I, \mathcal{F}_I)$ associated to I, where

$$\mathcal{C}_I := \{M \in \text{Mod-}R \mid IM = M\},$$
$$\mathcal{T}_I := \{M \in \text{Mod-}R \mid IM = 0\} \cong \text{Mod-}\frac{R}{I},$$
$$\mathcal{F}_I := \{M \in \text{Mod-}R \mid Ix = 0 \text{ and } x \in M \Rightarrow x = 0\}.$$

Consider the t-structure (Happel-Reiten-Smalo) $(\mathcal{D}_{t_I}^{\leq 0}(R), \mathcal{D}_{t_I}^{\geq 0}(R))$ associated to the torsion pair $t_I = (\mathcal{C}_I, \mathcal{T}_I)$, where

$$\mathcal{D}_{t_I}^{\leq 0}(R) := \{X \in \mathcal{D}^{\leq 0}(R) \mid H^0(X) \in \mathcal{C}_I\},$$
$$\mathcal{D}_{t_I}^{\geq 0}(R) := \{X \in \mathcal{D}^{\geq 0}(R) \mid H^0(X) \in \mathcal{T}_I\}.$$

It can be seen that $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ where $\mathbb{t}_1 := (\mathcal{D}^{\leq 0}(R), \mathcal{D}^{\geq 1}(R))$ and $\mathbb{T}_2 := (\mathcal{D}_{t_I}^{\leq 0}(R), \mathcal{D}_{t_I}^{\geq 1}(R))$ [redacted] is a strongly related torsion pair in $\mathcal{T} = \mathcal{D}(R)$.

3.3 Polishchuk correspondence

We recall the following bijection given by A. Polishchuk, and in order to do that, for a t-structure $(\mathcal{U}_1, \mathcal{U}_1^{\perp}[1])$ in $\mathcal{T}$, we have the cohomological functor $H_1^0 : \mathcal{T} \to \mathcal{H}_1 := \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1]$ ($\mathcal{H}_1$ is an abelian category).

Proposition 15 (Polishchuk)**.** *Let $(\mathcal{U}_1, \mathcal{U}_1^{\perp}[1])$ be a t-structure in a triangulated category. Then we have a bijection (Polishchuk's bijection)*

$$\left\{ \begin{array}{c} \textit{torsion pairs in} \\ \mathcal{H}_1 = \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1] \end{array} \right\} \overset{\mathrm{Pol}_{\mathcal{H}_1}}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{t-structures } (\mathcal{U}_2, \mathcal{U}_2^{\perp}) \\ \textit{in } \mathcal{D} \textit{ satisfying} \\ \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1 \end{array} \right\}$$

$$(\mathcal{X}, \mathcal{Y}) \longmapsto (\mathcal{U}_2, \mathcal{U}_2^{\perp}[1])$$

$$(\mathcal{U}_2 \cap \mathcal{H}_1, \mathcal{U}_2^{\perp} \cap \mathcal{H}_1) \longleftarrow\!\shortmid (\mathcal{U}_2, \mathcal{U}_2^{\perp}[1])$$

where

$$\mathcal{U}_2 = \{X \in \mathcal{U}_1 \mid H_1^0(X) \in \mathcal{X}\}$$
$$\mathcal{U}_2^{\perp} = \{Y \in \mathcal{U}_1^{\perp} \mid H_1^0(Y) \in \mathcal{Y}\}.$$

Remark 5. (1) Note that $\mathrm{Pol}_{\mathcal{H}_1}^{-1}(\mathcal{U}_2, \mathcal{U}_2^{\perp}[1]) = (\mathcal{U}_2 \cap \mathcal{U}_1^{\perp}[1], \mathcal{H})$, where $\mathcal{H} := \mathcal{U}_1 \cap \mathcal{U}_2^{\perp}$.

(2) By (1), it follows that $\mathcal{H}$ is a torsion free class in the abelian category $\mathcal{H}_1 := \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1]$.

Theorem 16. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related torsion pair in a triangulated category $\mathcal{T}$. Then, the following statements are equivalent.*

(a) For any distinguished triangle $V \to H_1 \xrightarrow{f} H_2 \xrightarrow{+}$, with f a morphism in $\mathcal{H} = \mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$, we have that
$$V \in \mathcal{F}_1 \Rightarrow V[1] \in \mathcal{F}_2.$$

(b) For any monomorphism $\alpha : H_1 \hookrightarrow H_2$, in the abelian category $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$, with $H_1, H_2 \in \mathcal{H}$, we have that $\mathrm{Coker}_{\mathcal{H}_1}(\alpha) \in \mathcal{H}$.

(c) $\mathcal{H}$ is closed under kernels and cokernels in the abelian category $\mathcal{H}_1$

(d) $\mathcal{H}$ is an abelian category.

(e) For any epimorphism $H \twoheadrightarrow X$ in $\mathcal{H}_1$, with $H \in \mathcal{H}$, we have that $X \in \mathcal{H}$ (i.e. $\mathcal{H}$ is closed under quotients in $\mathcal{H}_1$).

Let $t = (\mathcal{A}, \mathcal{B})$ be a pair of full subcategories of the triangulated category $\mathcal{T}$. We will use the following notation:

$$t[1] := (\mathcal{A}[1], \mathcal{B}[1]),$$
$$\bar{t} := (\mathcal{A}, \mathcal{B}[1]).$$

Note that $\bar{t}$ is a t-structure in $\mathcal{T}$ if and only if t is a torsion pair $\mathcal{T}$ such that $\mathcal{A}[1] \subseteq \mathcal{A}$.

Remark 6. Consider $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$, where $\mathbb{t}_i := (\mathcal{U}_i, \mathcal{U}_i^{\perp})$ for $i = 1, 2$. We have

1. $\mathcal{H}_{\mathbb{t}} := \mathcal{U}_1 \cap \mathcal{U}_2^{\perp}$, $\mathcal{H}_i := \mathcal{U}_i \cap \mathcal{U}_i^{\perp}[1]$,
2. $\mathbb{t}' := (\mathbb{t}_2, \mathbb{t}_1[1])$

Note that

3. $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is a related torsion pair in $\mathcal{T}$

$$\begin{aligned}&\Leftrightarrow \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1\\ &\Leftrightarrow \mathcal{U}_2[1] \subseteq \mathcal{U}_1[1] \subseteq \mathcal{U}_2\\ &\Leftrightarrow \mathbb{t}' = (\mathbb{t}_2, \mathbb{t}_1[1]) \text{ is a related torsion pair in } \mathcal{T}.\end{aligned}$$

4. Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is a related torsion pair in $\mathcal{T}$. In this case, we have

$$\begin{aligned}&\mathcal{H}_{\mathbb{t}} = \mathcal{U}_1 \cap \mathcal{U}_2^{\perp},\ \mathcal{H}_{\mathbb{t}'} = \mathcal{U}_2 \cap \mathcal{U}_1^{\perp}[1],\\ &\mathrm{Pol}_{\mathcal{H}_1}^{-1}(\bar{\mathbb{t}}_2) = \mathrm{Pol}_{\mathcal{H}_1}^{-1}(\mathcal{U}_2, \mathcal{U}_2^{\perp}[1]) = (\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}}),\\ &\mathrm{Pol}_{\mathcal{H}_2}^{-1}(\bar{\mathbb{t}}_1[1]) = \mathrm{Pol}_{\mathcal{H}_1}^{-1}(\mathcal{U}_1[1], \mathcal{U}_1^{\perp}[2]) = (\mathcal{H}_{\mathbb{t}}[1], \mathcal{H}_{\mathbb{t}'}).\end{aligned}$$

Thus, $(\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}})$ is a torsion pair in the abelian category $\mathcal{H}_1$, $(\mathcal{H}_{\mathbb{t}}[1], \mathcal{H}_{\mathbb{t}'})$ is a torsion pair in the abelian category $\mathcal{H}_2$.

Corollary 17. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related torsion pair in a triangulated category $\mathcal{T}$. Then, the following statements are equivalent:*

(a) For any distinguished triangle $V \to H_1 \xrightarrow{f} H_2 \xrightarrow{+}$, with f a morphism in $\mathcal{H}_{\mathbb{t}'} = \mathcal{T}_2 \cap \mathcal{F}_1[1]$, we have that $V \in \mathcal{F}_2$ implies $V \in \mathcal{F}_1$.

(b) For any monomorphism $\alpha : H_1 \hookrightarrow H_2$, in the abelian category $\mathcal{H}_2 := \mathcal{T}_2 \cap \mathcal{F}_2[1]$, with $H_1, H_2 \in \mathcal{H}_{\mathbb{t}'}$, we have that $\mathrm{Coker}_{\mathcal{H}_2}(\alpha) \in \mathcal{H}_{\mathbb{t}'}$.

(c) $\mathcal{H}_{\mathbb{t}'}$ is closed under kernels and cokernels in the abelian category $\mathcal{H}_2$.

(d) $\mathcal{H}_{\mathbb{t}'}$ is an abelian category.

(e) $\mathcal{H}_{\mathbb{t}'}$ is closed under quotients in $\mathcal{H}_2$.

We recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ is cohereditary if the class $\mathcal{F}$ is closed under quotients in $\mathcal{A}$.

Definition 7. For a triangulated category $\mathcal{T}$, we consider the following classes:

1. $RtAb(\mathcal{T}) := \{$related torsion pairs $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{T}$ s.t. $\mathcal{H}_{\mathbb{t}}$ is abelian$\}$;

2.

$$\mathrm{t}{-}stCoh(\mathcal{T}) := \left\{ \begin{array}{c} \text{pairs } (\bar{\mathbb{t}}_1, \tau) \text{ s.t. } \bar{\mathbb{t}}_1 \text{ is a t-structure in } \mathcal{T} \text{ and } \tau \text{ is a} \\ \text{cohereditary torsion pair in the abelian category} \\ \mathcal{H}_1 := \mathcal{U}_1 \cap \mathcal{U}_1^{\perp}[1] \end{array} \right\};$$

1′. $RtAb'(\mathcal{T}) := \{$related torsion pairs $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathcal{T}$ s.t. $\mathcal{H}_{\mathbb{t}'}$ is abelian$\}$;

2′.

$$\mathrm{t}{-}stCoh'(\mathcal{T}) := \left\{ \begin{array}{c} \text{pairs } (\bar{\mathbb{t}}_2, \tau) \text{ s.t. } \bar{\mathbb{t}}_2 \text{ is a t-structure in } \mathcal{T} \text{ and } \tau \text{ is a} \\ \text{cohereditary torsion pair in the abelian category} \\ \mathcal{H}_2 := \mathcal{U}_2 \cap \mathcal{U}_2^{\perp}[1] \end{array} \right\}.$$

Theorem 18. *For a triangulated category $\mathcal{T}$, the following statements hold true.*

(a) There is a bijective correspondence

$$\begin{aligned} RtAb(\mathcal{T}) &\overset{\alpha}{\longleftrightarrow} \mathrm{t} - stCoh(\mathcal{T})\\ \mathbb{t} &\longmapsto (\bar{\mathbb{t}}_1, \mathrm{Pol}_{\mathcal{H}_1}^{-1}(\bar{\mathbb{t}}_2))\\ (\mathbb{t}_1, \mathbb{t}_2) &\longleftarrow\!\!\shortmid (\bar{\mathbb{t}}_1, \tau)\end{aligned}$$

where $\bar{\mathbb{t}}_2 = \mathrm{Pol}_{\mathcal{H}_1}(\tau)$.

(b) There is a bijective correspondence

$$
\begin{array}{ccc}
RtAb'(\mathcal{T}) & \xleftrightarrow{\alpha'} & \mathrm{t}-stCoh'(\mathcal{T}) \\
\\
\mathbb{t} & \longmapsto & (\bar{\mathbb{t}}_2, \mathrm{Pol}_{\mathcal{H}_2}^{-1}(\bar{\mathbb{t}}_1[1])) \\
\\
(\mathbb{t}_1, \mathbb{t}_2) & \longleftarrow\!\shortmid & (\bar{\mathbb{t}}_2, \tau)
\end{array}
$$

where $\bar{\mathbb{t}}_1 = \mathrm{Pol}_{\mathcal{H}_2}(\tau)[-1]$.

4 Induced torsion theories

In this section and the following we study the case of a torsion pair $(\mathcal{T}, \mathcal{F})$ in a (nice) category $\mathcal{A}$, such that there is a subcategory $\mathcal{W} \subseteq \mathcal{F}$ of $\mathcal{A}$ for which it makes sense to consider $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{W}$. The goal is to describe the cases where $(\mathcal{T}, \mathcal{F})$ induces a torsion pair in $\underline{\mathcal{A}}$.

add a better description of the setting

4.1 Torsion pairs in $\underline{\mathcal{A}}$

Lemma 19. *Let $\mathcal{A}$ be a (nice) category and $\mathcal{W} \subseteq \mathcal{A}$ a subcategory such that $\operatorname{add}(\mathcal{W}) = \mathcal{W}$. If $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a torsion pair such that $\mathcal{W} \subseteq \mathcal{F}$, then $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is an orthogonal pair in $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$.*

Lemma 20. *Let $\mathcal{A}$ and $\mathcal{W} \subseteq \mathcal{A}$ be defined as above and let $({}^{\perp}\mathcal{F}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$ such that $\mathcal{W} \subseteq \mathcal{F}$. Call $p : \mathcal{A} \to \underline{\mathcal{A}}$ the quotient functor. The following assertions hold:*

1. *$p^{-1}({}^{\perp}(\underline{\mathcal{F}})) = \operatorname{add}({}^{\perp}\mathcal{F} * \mathcal{W})$.*
2. *If $\mathcal{W}$ is precovering in $\mathcal{F}$, then $({}^{\perp}\underline{\mathcal{F}}, \underline{\mathcal{F}})$ is a torsion pair in $\underline{\mathcal{A}}$.*

Lemma 21. *Let $\mathcal{A}$ be a (nice) category with a torsion pair $({}^{\perp}\mathcal{F}, \mathcal{F})$ and a precovering class $\mathcal{W} \subseteq \mathcal{F}$ such that for any $F \in \mathcal{F}$ there is an admissible sequence*

$$F' \to W \to F$$

such that $F' \in \mathcal{F}$.

Then the torsion pair $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial.

Recall that the truncation $t : \underline{\mathcal{A}} \to {}^{\perp}(\underline{\mathcal{F}})$ is given by the following construction.

Let $M \in \mathcal{A}$ be any object, take an admissible sequence

$$T_M \to M \to F^M$$

with $T_M \in {}^{\perp}\mathcal{F}$ and $F^M \in \mathcal{F}$. Moreover, consider $W_F \to F^M$ with $W_M \in \mathcal{W}$ as before, and take the admissible pullback:

$$\begin{tikzcd}
t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\
M \arrow[r] & F
\end{tikzcd}$$

Then, t restricts to a functor $\underline{t} : \underline{\mathcal{A}} \to {}^{\perp}\underline{\mathcal{F}}$.

In order to prove that $({}^{\perp}(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial we need to show that $\underline{t}$ admits a right adjoint.

Proof. Let $M \in \mathcal{A}$ and consider $M \to F^M$ and $W_F \to F^M$ as above. For any $T \in {}^{\perp}\mathcal{F} * \mathcal{W}$ consider any morphism $f : T \to M$. Since $T \to M \to F^M$ is 0 in $\underline{\mathcal{A}}$ we that the solid part of the following diagram commutes.

$$\begin{tikzcd}
T \arrow[rrr] \arrow[dr, dotted, "\psi"] \arrow[ddr, bend right, "f"'] & & & W \arrow[dl, dashed, "\phi"'] \arrow[ddl, bend left] \\
& t(M) \arrow[r] \arrow[d] & W_F \arrow[d] & \\
& M \arrow[r] & F^M &
\end{tikzcd}$$

$\square$

Since $W_F \to F^M$ is a precover there is a morphism $\phi : W \to W_F$ making the diagram commute, and since the square is an admissible pullback there is a morphism $\psi : T \to t(M)$ making the diagram commutative.

Hence, $\mathcal{A}(T, t(M)) \to \mathcal{A}(T, M)$ is surjective. To conclude the proof we need to show that when restricted to $\underline{\mathcal{A}}$ it becomes an iso. Assume that there are two morphisms $\underline{\psi}$ and $\underline{\psi'}$ in $\underline{\mathcal{A}}$ such that the following commutes:

$$\begin{tikzcd} & t(M) \arrow[d] \\ T \arrow[ur, shift left, "\underline{\psi}"] \arrow[ur, shift right, "\underline{\psi'}"'] \arrow[r, "\underline{f}"'] & M \end{tikzcd}$$

So, if we call $h = \psi - \psi'$ in $\mathcal{A}$, we have that $T \xrightarrow{h} t(M) \to M$ factors through W so that we have that the solid part of the following diagram commutes:

$$\begin{tikzcd} W \arrow[ddrr, dotted, bend left, "\gamma"] \arrow[ddrrr, dashed, bend left, "\eta"] \arrow[dddrr, bend right] & & & \\ & T \arrow[ul, "p"'] \arrow[dr, "h"] \arrow[ddr, "g"'] & & \\ & & t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\ & & M \arrow[r] & F^M \end{tikzcd}$$

where $\eta : W \to W_F$ comes from the fact that $W_F \to F^M$ is a precover, and γ from the fact that the square is an admissible pullback, and they make the complete diagram commute.

Let's call $h' = \gamma \circ p$, then composing both h and h' with $\rho : t(M) \to M$ gives the same morphism g. Hence, $\rho \circ (h - h') = 0$. But since $F' \xrightarrow{i} t(M) \to M$ is an admissible sequence we have the following exact sequence of abelian groups:

$$\begin{tikzcd} \mathcal{A}(T, F) \arrow[r] & \mathcal{A}(T, t(M)) \arrow[r] & \mathcal{A}(T, M) \\ & h - h' \arrow[r, mapsto] & 0 \end{tikzcd}$$

So there is a map $k : T \to F'$ such that $i \circ k = h - h'$, but $\underline{k} = 0$ so $\underline{h} = \underline{h'} = 0$, hence $\underline{\psi} - \underline{\psi'} = 0$ which proves that $\underline{\mathcal{A}}(T, t(M)) \cong \underline{\mathcal{A}}(T, M)$.

The naturality of the isomorphism in T and M is clear.

4.2 Abelian categories

Now we work in an abelian category with two torsion pairs $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ such that $t_2(\mathcal{F}_1) \subseteq \mathcal{F}_1$ and $f_1(\mathcal{T}_2) \subseteq \mathcal{T}_2$ and let $\mathcal{W} = \mathcal{T}_2 \cap \mathcal{F}_1$.

Recall that $(\underline{\mathcal{T}_1 * \mathcal{W}}, \underline{\mathcal{F}_1})$ (resp. $(\underline{\mathcal{T}_2}, \underline{\mathcal{W} * \mathcal{F}_2})$) is a left (resp. right) functorial torsion pair in $\underline{\mathcal{A}} = \dfrac{\mathcal{A}}{\mathcal{W}}$. Moreover, they satisfy $TC1 - 3, 3^*$.

Lemma 22. *The inclusion $i : \mathcal{T}_1 * \mathcal{W} \hookrightarrow \mathcal{A}$ admits a right adjoint $\widehat{t}$.*

Proof. For $M \in \mathcal{A}$ consider the exact sequence

$$0 \to T_1 \to M \to f_1(M) \to 0$$

with $T_1 \in \mathcal{T}_1$ and $f_1(M) \in \mathcal{F}$. Take $t_2 f_1(M) \hookrightarrow f_1(M)$ and observe that $t_2 f_1(M) \in \mathcal{W}$. Call it W_M and take the pullback diagram

$$\begin{tikzcd} \widehat{t}(M) \arrow[r] \arrow[d] & W_M \arrow[d, tail] \\ M \arrow[r, two heads] & f_1(M) \end{tikzcd}$$

then $\widehat{t}(M) \in \mathcal{T}_1 * \mathcal{W}$.

Now for any morphism $\widehat{T} \to M$ with $\widehat{T} \in \widehat{\mathcal{T}}$ the solid part of the following diagram commutes

$\widehat{T} \to M$ is mono by Buhler prop. 2.14: pullback of monic along epic is monic

$$\begin{tikzcd}
T \arrow[r, tail] \arrow[d, equal] & \widehat{t}(M) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & W \arrow[d, tail] \\
T \arrow[r, tail] & M \arrow[r, two heads] & f_1(M) \\
T_1 \arrow[r, tail] & \widehat{T} \arrow[r, two heads] \arrow[u] \arrow[uu, dotted, bend left] & W_1 \arrow[u, dashed] \arrow[uu, dashed, bend right]
\end{tikzcd}$$

Since the composition $T_1 \to \widehat{T} \to M \to f_1(M)$ is zero, there exists the dashed morphism $W_1 \to f_1(M)$, which lifts to the morphism $W_1 \to W$ (since $W \to f_1(M)$ is a $\mathcal{W}$-precover). Hence, there is a morphism $\widehat{T} \to \widehat{t}(M)$ making the diagram commutative. This means that

$$\mathcal{A}(\widehat{T}, \widehat{t}(M)) \xrightarrow{\mathcal{A}(\widehat{T}, \widehat{t}(M) \to M)} \mathcal{A}(\widehat{T}, M)$$

is surjective. But it is also injective, since $\operatorname{Ker}\left(\widehat{t}(M) \to M\right) = 0$. Hence, it is an iso and $\widehat{t}$ is right adjoint to i. $\square$

functoriality should follow immediately

Lemma 23. *Let $\widehat{T}_1 \in \mathcal{T}_1 * \mathcal{W}$, i.e. there is an exact sequence*

$$0 \to t_1(\widehat{T}_1) \to \widehat{T}_1 \to W_1 \to 0.$$

If

$$\begin{tikzcd}
\widehat{T}_1 \arrow[r, two heads, "p"] \arrow[d, "g"] & W_1 \arrow[d, "g'"] \\
\widehat{T}_1' \arrow[r, "q"] & N
\end{tikzcd}$$

is a pushout diagram, then $N \in \mathcal{T}_1 * \mathcal{W}$.

Proof. Since it is a pushout, $\widehat{T}_1' \to N$ is epi, then consider $\widehat{t}(N)$ and the following commutative diagram

$$\begin{tikzcd}
\widehat{T}_1' \arrow[rr, two heads, "q"] \arrow[dr, dashed, "\rho"'] & & N \\
& \widehat{t}(N) \arrow[ur, tail, "\varepsilon"'] &
\end{tikzcd}$$

where the map $\widehat{T}_1' \to \widehat{t}(N)$ is given by the adjunction $(i, \widehat{t})$. Since $q = \varepsilon \circ \rho$ is epi, then ε is epi. But it is mono, so it is an isomorphism, hence $N \in \mathcal{T}_1 * \mathcal{W}$. $\square$

Lemma 24. *In the same notation as the previous lemma, if $\varphi : N \to P$ is any map s.t. $\underline{\varphi} \circ \underline{q} = \underline{0}$ in $\underline{\mathcal{A}}$, then φ factors through $\mathcal{W}$.*

Proof. Since $\underline{\varphi} \circ \underline{q} = \underline{0}$ it means that $\varphi \circ q$ factors through $\mathcal{W}$, hence we have that the solid part of the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & t_1(T_1') \arrow[r] \arrow[d] & \widehat{T}_1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] \arrow[dd, dashed, bend left] & 0 \\
0 \arrow[r] & X \arrow[r] & \widehat{T}_1' \arrow[r] \arrow[d, equal] & N \arrow[d, dotted] \arrow[dr, "\varphi"] & \\
& & \widehat{T}_1' \arrow[r] & W \arrow[r] & P
\end{tikzcd}$$

Since $t_1(T_1') \to \widehat{T}_1 \to \widehat{T}_1' \to W$ is zero, there is the dashed morphism $W_1 \to W$ making the diagram commute. Since the square on the right is a pushout there is a map $N \to W$, and again the diagram commutes. Hence φ factors through $\mathcal{W}$. $\square$

Lemma 25. *If $\mathcal{H}$ is balanced (i.e. mono and epi implies iso), then whenever $f : H_1 \to H_2$ is mono and epi in $\mathcal{H}$, there are bicartesian squares in $\mathcal{A}$*

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near end] & H_1 \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near end] & W_1 \arrow[d] \\
W_2 \arrow[r] & H_2 \arrow[r] & T_2
\end{tikzcd}$$

where $W_1 = f_1(H_1)$ and $W_2 = t_2(H_2)$. In particular there is an exact sequence

$$0 \to F_1 \to W_1 \oplus W_2 \to T_2 \to 0.$$

Proof. We can build the pullback on the left and the pushout on the right as usual

$$\begin{tikzcd}
F_1 \arrow[r, "f^K"] \arrow[d, "u"] \arrow[dr, phantom, "\ulcorner", very near start] & H_1 \arrow[r, "r"] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near end] & W_1 \arrow[d, "s"] \\
W_2 \arrow[r, "v"] & H_2 \arrow[r, "f^C"] & T_2
\end{tikzcd} \tag{1}$$

We will only prove that the square on the right hand side is a pullback, since the proof that the left square is a pushout is dual. The statetment that the square on the right is a pushout is equivalent to saying that there is an exact sequence

$$H_1 \xrightarrow{\left(\begin{smallmatrix} f \\ r \end{smallmatrix}\right)} H_2 \oplus W_1 \xrightarrow{\left(\begin{smallmatrix} f^C & s \end{smallmatrix}\right)} T_2 \longrightarrow 0 \tag{2}$$

$H_1 \oplus T_2 \cong H_2 \cong W_1 \Rightarrow T_2 \in \mathcal{W} * \mathcal{F}$ $W' \hookrightarrow T_2 \overset{0}{\twoheadrightarrow} F'$, with $W' \in \mathcal{W}, T_2 \in \mathcal{T}_2, F' \in \mathcal{F}_2$, hence $T_2 \in \mathcal{W}$.

Since f is both a mono and an epi in $\mathcal{H}$, then it is an iso and hence both a section and a retraction. Consider $g : H_2 \to H_1$ such that $\underline{g} \circ \underline{f} = \underline{1_{H_1}}$, that is there are maps $\alpha : H_1 \to W$ and $\beta : W \to H_1$ such that

$$\begin{tikzcd}
H_1 \arrow[r, "\left(\begin{smallmatrix} f \\ \alpha \end{smallmatrix}\right)"] \arrow[rr, bend right, "1_{H_1}"'] & H_2 \oplus W \arrow[r, "\left(\begin{smallmatrix} g & \beta \end{smallmatrix}\right)"] & H_1
\end{tikzcd}$$

is commutative in $\mathcal{A}$, and hence H_1 is a direct summand of $H_2 \oplus W$. We can actually choose $W = W_1$, in fact consider the commutative diagram

$$\begin{tikzcd}
H_1 \arrow[r, "\left(\begin{smallmatrix} f \\ r \end{smallmatrix}\right)"] \arrow[d, equal] & H_2 \oplus W_1 \arrow[r, "\left(\begin{smallmatrix} f^C & -s \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix}\right)"] & T_2 \\
H_1 \arrow[r, "\left(\begin{smallmatrix} f \\ \alpha \end{smallmatrix}\right)"] \arrow[rr, bend right, "1_{H_1}"'] & H_2 \oplus W \arrow[r, "\left(\begin{smallmatrix} g & \beta \end{smallmatrix}\right)"] & H_1
\end{tikzcd}$$

where $\rho : W_1 \to W$ comes from the fact that $H_1 \to W_1$ is a $\mathcal{W}$-preenvelope. Hence, $\left(\begin{smallmatrix} g & \beta \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} f \\ r \end{smallmatrix}\right) = 1_{H_1}$, that is H_1 is a direct summand of $H_2 \oplus W_1$. Moreover, it means that $\left(\begin{smallmatrix} f \\ r \end{smallmatrix}\right)$ is a section, that is the sequence in (2) is also exact on the left and the corresponding square in (1) is a pullback diagram.

Since both squares in (1) are bicartesian, it follows that the square

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near end] & W_1 \arrow[d] \\
W_2 \arrow[r] & T_2
\end{tikzcd}$$

is bicartesian as well. ◻

References

[1] Hiroyuki Nakaoka. “General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories”. In: *Applied Categorical Structures* 19.6 (2011), pp. 879–899.