

1 The Nakaoka setting

Given a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category \mathcal{C} , it was proved in [1] that it is possible to use it to construct a subcategory $\mathcal{W} \subseteq \mathcal{C}$ such that in the quotient category $\frac{\mathcal{C}}{\mathcal{W}} = \underline{\mathcal{C}}$ a heart \mathcal{H} is defined in such a way that it is an abelian category and there is a homological functor $\mathcal{C} \rightarrow \mathcal{H}$.

Our goal is to provide a set of axioms for a (nice) additive category \mathcal{A} and a couple of torsion pairs in it, in such a way that they will guarantee the existence of an abelian heart in \mathcal{A} . In a sense, we want to axiomatize $\underline{\mathcal{C}}$ and the pairs which are referred in Nakaoka's work as $(\underline{\mathcal{C}}^-, \underline{\mathcal{V}})$ and $(\underline{\mathcal{U}}, \underline{\mathcal{C}}^+)$.

Now we will briefly recall Nakaoka's setting. Assume that \mathcal{C} is a triangulated category, and $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in \mathcal{C} , i.e.

1. $\mathcal{C}(\mathcal{U}, \mathcal{V}[1]) = 0$
2. $\mathcal{C} = \mathcal{U} * \mathcal{V}[1]$, where $X \in \mathcal{M} * \mathcal{N}$ if and only if there is a distinguished triangle

$$M \rightarrow X \rightarrow N[1] \rightarrow M[1]$$

with $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

Then, we put $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ and define $\underline{\mathcal{C}} = \mathcal{C}/\mathcal{W}$, and similarly for $\underline{\mathcal{U}}, \underline{\mathcal{V}}$, etc.

We define the following full subcategories of $\underline{\mathcal{C}}$:

- $\underline{\mathcal{C}}^+ = \mathcal{W} * \mathcal{V}[1]$
- $\underline{\mathcal{C}}^- = \mathcal{U}[-1] * \mathcal{W}$

together with their respective quotients $\underline{\mathcal{C}}^+$ and $\underline{\mathcal{C}}^-$. Then, we have the following lemma.

Lemma 1. *Let $X \in \underline{\mathcal{C}}$, TFAE:*

1. $\underline{X} \in \underline{\mathcal{C}}^+$,
2. *There is a monomorphism $\underline{X} \rightarrow \underline{V}[1]$ in $\underline{\mathcal{C}}$, for some $V \in \mathcal{V}$.*

The dual also holds:

Lemma 2. *Let $X \in \underline{\mathcal{C}}$, TFAE:*

1. $\underline{X} \in \underline{\mathcal{C}}^-$,
2. *There is an epimorphism $\underline{U}[-1] \rightarrow \underline{X}$ in $\underline{\mathcal{C}}$, for some $U \in \mathcal{U}$.*

Corollary 3. *If $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in a triangulated category \mathcal{C} , then:*

1. ${}^\perp \underline{\mathcal{V}} = \underline{\mathcal{C}}^-$
2. $\underline{\mathcal{U}}^\perp = \underline{\mathcal{C}}^+$

Lemma 4. *Let $F : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}^+$ be the left adjoint of the inclusion functor $j : \underline{\mathcal{C}}^+ \hookrightarrow \underline{\mathcal{C}}$. If $\lambda : 1_{\underline{\mathcal{C}}} \rightarrow j \circ F$ is the unit of the adjunction, then there is a pseudokernel-pseudocokernel sequence*

$$U_C \xrightarrow{u} C \xrightarrow{\lambda_C} (j \circ F)(C) \xrightarrow{\perp}$$

in $\underline{\mathcal{C}}$ such that $U_C \in \mathcal{U}$.

With dual:

Lemma 5. *Let $G : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}^-$ be the left adjoint of the inclusion functor $i : \underline{\mathcal{C}}^- \hookrightarrow \underline{\mathcal{C}}$. If $\varepsilon : i \circ G \rightarrow 1_{\underline{\mathcal{C}}}$ is the co-unit of the adjunction, then there is a pseudokernel-pseudocokernel sequence*

$$(i \circ G)(C) \xrightarrow{\varepsilon_C} C \xrightarrow{V_C} \xrightarrow{\perp}$$

in $\underline{\mathcal{C}}$ such that $V_C \in \mathcal{V}$.

Corollary 6. *$(\underline{\mathcal{C}}^-, \underline{\mathcal{V}})$ and $(\underline{\mathcal{U}}, \underline{\mathcal{C}}^+)$ are orthogonal pairs in $\underline{\mathcal{C}}$ provided \mathcal{C} has split idempotents.*

Remark. 1. By prop 5.3 Nakaoka we have that $\underline{\mathcal{C}}^+$ has cokernels and, dually, $\underline{\mathcal{C}}^-$ has kernels.

add better reference

2. We have inclusions $\mathcal{V} \subseteq \underline{\mathcal{C}}^+$ and $\mathcal{U} \subseteq \underline{\mathcal{C}}^-$

see page 5.5 of the notes

2 Torsion pairs

We fix an additive category \mathcal{X} with pseudokernels and pseudocokernels on which idempotents split.

Definition 1. A pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ of full subcategories of \mathcal{X} is a torsion pair in \mathcal{X} if:

1.

$$\begin{aligned}\mathcal{F} &= \mathcal{T}^\perp = \{X \in \mathcal{X} \mid \mathcal{X}(\mathcal{T}, X) = 0\}, \\ \mathcal{T} &= {}^\perp \mathcal{F} = \{X \in \mathcal{X} \mid \mathcal{X}(X, \mathcal{F}) = 0\};\end{aligned}$$

2. For each $M \in \mathcal{X}$ there is a pseudokernel-pseudocokernel sequence

$$T_M \xrightarrow{\varepsilon_M} M \xrightarrow{\lambda_M} F^M$$

where $T_M \in \mathcal{T}$ and $F^M \in \mathcal{F}$.

If in addition the assignment $M \mapsto t(M) := T_M$ (resp. $M \mapsto f(M) := F^M$) is functorial and defines an adjoint pair (i, t) (resp. (f, j)), where $i : \mathcal{T} \hookrightarrow \mathcal{X}$ (resp. $j : \mathcal{F} \hookrightarrow \mathcal{X}$) is the inclusion functor, then we say that \mathfrak{t} is left (resp. right) functorial. In such a case, ε (resp. λ) is the counit (resp. unit) of the given adjoint pair. We say that \mathfrak{t} is functorial if it is right and left functorial.

Remark. Let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a left functorial torsion pair in \mathcal{X} . Then

- (a) For any $M \in \mathcal{X}, T' \in \mathcal{T}$ and $\alpha \in \mathcal{X}(T', M)$ there is a unique $\alpha' \in \mathcal{X}(T', t(M))$ such that $\varepsilon_M \circ \alpha' = \alpha$, i.e.

$$\begin{array}{ccc} & & T' \\ & \swarrow \exists! \alpha' & \downarrow \alpha \\ t(M) & \xrightarrow{\varepsilon_M} & M \end{array}$$

- (b) Let $g : T_1 \rightarrow T_2$ be a morphism in \mathcal{T} , which admits a pseudocokernel $g^C : T_2 \rightarrow \text{PCok}_{\mathcal{T}}(g)$ in \mathcal{T} . Then g^c is a pseudocokernel of g in \mathcal{X} .

Proof. see Octavio's notes

□

The dual also holds.

3 Axiomatization

Definition 2. Let \mathcal{X} be an additive category with pseudokernels and pseudocokernels, a *compatible* torsion pair $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ in \mathcal{X} consists of the two pairs $\mathbb{k}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{k}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ of full subcategories of \mathcal{X} satisfying the following axioms:

- (CT1) $\mathbb{k}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{k}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are respectively a left functorial and a right functorial torsion pair
- (CT2) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equiv. $\mathcal{F}_1 \subseteq \mathcal{F}_2$)
- (CT3) Any morphism $g : T_1 \rightarrow T'_1$ in \mathcal{T}_1 admits a pseudocokernel $g^C : T'_1 \rightarrow \text{PCok}_{\mathcal{X}}(g)$, with $T''_1 := \text{PCok}_{\mathcal{X}}(g) \in \mathcal{T}_1$, such that

$$0 \longrightarrow (T''_1, -)_{|\mathcal{F}_2} \xrightarrow{(g^C, -)} (T'_1, -)_{|\mathcal{F}_2} \xrightarrow{(g, -)} (T_1, -)_{|\mathcal{F}_2}$$

is an exact sequence of functors.

- (CT3)* Dual of **(CT3)**. **(CT3)***

Notation. In the case of a compatible torsion pair $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ in \mathcal{X} , we have the adjoint pairs

$$(i_1, t_1) : \mathcal{T}_1 \xrightleftharpoons[t_1]{i_1} \mathcal{X} \quad \text{and} \quad (t_2, j_2) : \mathcal{F}_2 \xrightleftharpoons[j_2]{t_2} \mathcal{X}$$

In this case there are also the counit $\varepsilon_{1,M} : t_1(M) \rightarrow M$ and the unit $\lambda_{2,M} : M \rightarrow f_2(M)$. The heart of \mathbb{k} is defined as $\mathcal{H} = \mathcal{H}_{\mathbb{k}} := \mathcal{T}_1 \cap \mathcal{F}_2$.

Lemma 7. Let $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ be a compatible torsion pair in \mathcal{X} . Then the following statements hold true.

- (a) $\mathcal{F}_1 \cap \mathcal{T}_2 = 0$,
- (b) $f_2(\mathcal{T}_1) \subseteq \mathcal{H}$ and $t_1(\mathcal{F}_2) \subseteq \mathcal{H}$.

Proposition 8. Let $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ be a compatible torsion pair in \mathcal{X} and $f : H_1 \rightarrow H_2$ be a morphism in \mathcal{H} . Then

- (a) if $f^C : H_2 \rightarrow T_1$ is the pseudocokernel of f , given by **(CT3)** and $\lambda_{2,T_1} : T_1 \rightarrow f_2(T_1)$, then

$$\text{Coker}(H_1 \xrightarrow{f} H_2) = (H_2 \xrightarrow{\tilde{f}^C} f_1(T_1))$$

in \mathcal{H} , where $\tilde{f}^C := \lambda_{2,T_1} \circ f^C$;

- (b) if $f^K : F_2 \rightarrow H_1$ is the pseudokernel of f , given by **(CT3)*** and $\varepsilon_{1,F_2} : t_1(F_2) \rightarrow F_2$, then

$$\text{Ker}(H_1 \xrightarrow{f} H_2) = (t_1(F_2) \xrightarrow{\tilde{f}^K} H_1)$$

in \mathcal{H} , where $\tilde{f}^K = f^K \circ \varepsilon_{1,F_2}$.

Proposition 9. Let $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ be a compatible torsion pair in \mathcal{X} . Then, for any $f : H \rightarrow H'$ in \mathcal{H} we have that:

- (a) f is a monomorphism in \mathcal{H} if and only if there is a pseudokernel $\text{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$;
- (b) f is an epimorphism in \mathcal{H} if and only if there is a pseudocokernel $\text{PCok}_{\mathcal{T}_1}(f) \in \mathcal{T}_2$.

Definition 3. A compatible torsion pair $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ in \mathcal{X} is strong if the following axioms hold:

define torsion pairs in additive categories

(CT4) Let $f : H_1 \rightarrow H_2$ in \mathcal{H} be such that there is a pseudokernel $\text{PKer}_{\mathcal{F}_2}(f) \in \mathcal{F}_1$. Then, for the commutative diagram

$$\begin{array}{ccccccc} H_1 & \xrightarrow{f} & H_2 & \xrightarrow{f^C} & T_1 := \text{PCok}_{\mathcal{X}}(f) \\ \downarrow a & & \parallel & & \downarrow \lambda_{2,T_1} \\ t_1(F)_2 & \xrightarrow{\varepsilon_{1,F_2}} & F_2 := \text{PKer}_{\mathcal{X}}(g) & \xrightarrow{g^K} & H_2 & \xrightarrow{g} & f_2(T_1) \end{array}$$

there exists a morphism $b : t_1(F_2) \rightarrow H_1$ such that $ab = \varepsilon_{1,F_2}$.

(CT4)*Dual

With these axioms we can prove that the heart has kernels and cokernels.

Theorem 10. Let $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ be a compatible torsion pair in \mathcal{X} . Then, the following are equivalent:

1. \mathbb{k} is strong,
2. \mathcal{H} is an abelian category.

3.1 The case of an abelian category

Let's consider the case $\mathcal{X} = \mathcal{A}$ of an Abelian category with two torsion pairs $\mathbb{k}_i = (\mathcal{T}_i, \mathcal{F}_i)$ for $i = 1, 2$. Consider $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$.

Remark. In the case of an Abelian category $\mathcal{X} = \mathcal{A}$, we have that \mathbb{k} is compatible if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proof. Let $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we need to show that item (CT1), item (CT3) and item (CT3) hold.

(CT1) It is well known that any torsion pair in an abelian category is functorial.

(CT3) Let $g : T_1 \rightarrow T'_1$ be a morphism in \mathcal{T}_1 . Consider the cokernel morphism of g in \mathcal{A}

$$\text{Coker}_{\mathcal{A}}(T_1 \xrightarrow{g} T'_1) = (T'_1 \xrightarrow{c_g} \text{Coker}(g)).$$

Since \mathcal{T}_1 is closed under quotient objects, we get that $\text{Coker}(g) \in \mathcal{T}_1$. Therefore, we can choose $c_g : T'_1 \rightarrow \text{Coker}(g)$ as $g^C : T'_1 \rightarrow \text{PCok}_{\mathcal{A}}(g)$.

(CT3)*Analogous to the previous.

□

Corollary 11. Let $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ be a torsion pair in \mathcal{A} with $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Then, for $f : H_1 \rightarrow H_2$ in \mathcal{H} , the following statements hold:

(a) the cokernel of f in \mathcal{H} is the composition of the morphisms

$$H_2 \xrightarrow{c_f} \text{Coker}(f) \xrightarrow{\lambda_{2, \text{Coker}(f)}} f_2(\text{Coker}(f));$$

(b) the kernel of f in \mathcal{H} is the composition of the morphisms

$$t_1(\text{Ker}(f)) \xrightarrow{\varepsilon_{1, \text{Ker}(f)}} \text{Ker}(f) \xrightarrow{k_f} H_1;$$

(c) f is an epimorphism in \mathcal{H} if and only if $\text{Coker}(f) \in \mathcal{T}_2$;

(d) f is a monomorphism in \mathcal{H} if and only if $\text{Ker}(f) \in \mathcal{F}_1$.

Proof. (a) and (b) follow from the proof of section 3.1 and proposition 8.

(c)

Is there an explicit choice that we made somewhere before when we talk about $\text{PCok}_{\mathcal{A}}$?

this should be a remark, fix it

\Leftarrow is trivial.

\Rightarrow By proposition 9 (b) there exists $f^C : H_2 \rightarrow T_2$, where $T_2 = \text{Coker } T_1(f) \in \mathcal{T}_2$. Then, we have

$$\begin{array}{ccc} H_2 & \xrightarrow{f^C} & T_2 \\ & \searrow c_f & \nearrow u \\ & \text{Coker}(f) & \nearrow v \end{array} \quad \text{such that } \begin{cases} uf^C = c_f, \\ vc_f = f^C. \end{cases}$$

Hence, $uvc_f = c_f$, but c_f is epi, therefore $uv = 1$. Hence, $\text{Coker}(f)$ is a direct summand of $T_2 \in \mathcal{T}_2$ so $\text{Coker}(f) \in \mathcal{T}_2$.

(d) Similar to the previous proof. \square

Theorem 12. Let $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ be a torsion pair in an abelian category \mathcal{A} , for $i = 1, 2$, such that $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Then, for $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ the following statements are equivalent:

- (a) \mathcal{H} is an abelian category.
- (b) The following conditions hold:
 - (b1) For any $f : H \rightarrow H'$ in \mathcal{H} , with $\text{Ker}(f) \in \mathcal{F}_1$, we have that $\text{Ker}(f) = 0$.
 - (b2) For any $f : H \rightarrow H'$ in \mathcal{H} , with $\text{Coker}(f) \in \mathcal{T}_2$, we have that $\text{Coker}(f) = 0$.
 - (b3) \mathcal{H} is closed under kernels (resp. cokernels) of epimorphisms (resp. monomorphisms) in \mathcal{A} .
- (c) \mathcal{H} is closed under kernels and cokernels in \mathcal{A} .

3.2 Related torsion pairs in triangulated categories

Let $\mathcal{X} = \mathcal{T}$ be a triangulated category on which idempotents split. We start by recalling the definition of a t-structure in \mathcal{T} .

Definition 4. A pair $(\mathcal{A}, \mathcal{B})$ of full subcategories of \mathcal{T} is a t-structure in \mathcal{T} if

- (a) $\mathbb{t} = (\mathcal{A}, \mathcal{B}[-1])$ is a torsion pair in \mathcal{T} , and
- (b) $\mathcal{A}[1] \subseteq \mathcal{A}$.

Remark. It is well known that any t-structure $(\mathcal{A}, \mathcal{B})$ in \mathcal{T} gives a functional torsion pair $\mathbb{t} = (\mathcal{A}, \mathcal{B}[-1])$ and $\mathcal{B}[-1] \subseteq \mathcal{B}$. Furthermore, \mathcal{A} and \mathcal{B} are closed under extensions and direct summands. Note that the t-structure $(\mathcal{A}, \mathcal{B})$ depends only on \mathcal{A} , since $\mathcal{B} = \mathcal{A}^\perp[1]$.

Definition 5. A *related* torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in triangulated category \mathcal{T} consists of the torsion pairs $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t} = (\mathcal{T}_2, \mathcal{F}_2)$ in \mathcal{T} such that $\mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proposition 13. Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related torsion pair in \mathcal{T} . Then

- (a) $(\mathcal{T}_1, \mathcal{F}_1[1])$ and $(\mathcal{T}_2, \mathcal{F}_2[1])$ are t-structures in \mathcal{T} ;
- (b) \mathbb{t} is a compatible torsion pair in \mathcal{T} ;
- (c) the heart $\mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2[1]$ is a preabelian category.

Definition 6. A related torsion pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in the triangulated category \mathcal{T} is *strong* if for any morphism $f : H_1 \rightarrow H_2$, in $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$, and a distinguished triangle $Z \rightarrow H_1 \xrightarrow{f} H_2 \rightarrow Z[1]$, the following conditions hold true

- (RST1) $Z \in \mathcal{F}_1$ if and only if $Z \in \mathcal{F}_2[-1]$;
- (RST2) $Z[1] \in \mathcal{T}_2$ if and only if $Z \in \mathcal{T}_1$.

Theorem 14. Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a strongly related torsion pair in the triangulated category \mathcal{T} . Then, the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ is an abelian category.

Example 1. Let $(\mathcal{A}, \mathcal{B})$ be a t-structure in \mathcal{T} . Consider $\mathbb{k}_1 := (\mathcal{A}, \mathcal{B}[-1])$ and $\mathbb{k}_2 := (\mathcal{A}[1], \mathcal{B})$. It is not hard to see that $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ is a strongly related torsion pair in \mathcal{T} . In this case, by theorem 14, we get that $\mathcal{H} = \mathcal{A} \cap \mathcal{B}$ is an abelian category (BBD theorem).

Example 2. Let R be any (associative with 1) ring. Consider the triangulated category $\mathcal{T} := \mathcal{D}(R)$. The derived category $\mathcal{D}(R)$ has the so called natural t-structure $(\mathcal{D}^{\leq 0}(R), \mathcal{D}^{\geq 0}(R))$ where

$$\begin{aligned}\mathcal{D}^{\leq 0}(R) &:= \{X \in \mathcal{D}(R) \mid H^i(X) = 0 \text{ for } i > 0\}, \\ \mathcal{D}^{\geq 0}(R) &:= \{X \in \mathcal{D}(R) \mid H^i(x) = 0 \text{ for } i < 0\}.\end{aligned}$$

Theorem 15. *Let $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ be a related torsion pair in a triangulated category \mathcal{T} . Then, the following statements are equivalent.*

- (a) *For any distinguished triangle $V \rightarrow H_1 \xrightarrow{f} H_2 \xrightarrow{+}$, with f a morphism in $\mathcal{H} = \mathcal{H}_{\mathbb{k}} := \mathcal{T}_1 \cap \mathcal{F}_2$, we have that*

$$V \in \mathcal{F}_1 \Rightarrow V[1] \in \mathcal{F}_2.$$
- (b) *For any monomorphism $\alpha : H_1 \hookrightarrow H_2$, in the abelian category $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$, with $H_1, H_2 \in \mathcal{H}$, we have that $\text{Coker}_{\mathcal{H}_1}(\alpha) \in \mathcal{H}$.*
- (c) *\mathcal{H} is closed under kernels and cokernels in the abelian category \mathcal{H}_1*
- (d) *\mathcal{H} is an abelian category.*
- (e) *For any epimorphism $H \twoheadrightarrow X$ in \mathcal{H}_1 , with $H \in \mathcal{H}$, we have that $X \in \mathcal{H}$ (i.e. \mathcal{H} is closed under quotients in \mathcal{H}_1).*

Let $t = (\mathcal{A}, \mathcal{B})$ be a pair of full subcategories of the triangulated category \mathcal{T} . We will use the following notation:

$$\begin{aligned}t[1] &:= (\mathcal{A}[1], \mathcal{B}[1]), \\ \bar{t} &:= (\mathcal{A}, \mathcal{B}[1]).\end{aligned}$$

Note that \bar{t} is a t-structure in \mathcal{T} if and only if t is a torsion pair \mathcal{T} such that $\mathcal{A}[1] \subseteq \mathcal{A}$.

Remark. Consider $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$, where $\mathbb{k}_i := (\mathcal{U}_i, \mathcal{U}_i^\perp)$ for $i = 1, 2$. We have

1. $\mathcal{H}_{\mathbb{k}} := \mathcal{U}_1 \cap \mathcal{U}_2^\perp$, $\mathcal{H}_i := \mathcal{U}_i \cap \mathcal{U}_i^\perp[1]$,
2. $\mathbb{k}' := (\mathbb{k}_2, \mathbb{k}_1[1])$

Note that

3. $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ is a related torsion pair in \mathcal{T}

$$\Leftrightarrow \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1 \Leftrightarrow \mathcal{U}_2[1] \subseteq \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \Leftrightarrow \mathbb{k}' = (\mathbb{k}_2, \mathbb{k}_1[1]) \text{ is a related torsion pair}$$

4. Let $\mathbb{k} = (\mathbb{k}_1, \mathbb{k}_2)$ is a related torsion pair in \mathcal{T} . In this case, we have

$$\begin{aligned}\mathcal{H}_{\mathbb{k}} &= \mathcal{U}_1 \cap \mathcal{U}_2^\perp, \quad \mathcal{H}_{\mathbb{k}'} = \mathcal{U}_2 \cap \mathcal{U}_1^\perp[1], \\ \text{Pol}_{\mathcal{H}_1}^{-1}(\bar{\mathbb{k}}_2) &= \text{Pol}_{\mathcal{H}_1}^{-1}(\mathcal{U}_2, \mathcal{U}_2^\perp[1]) = (\mathcal{H}_{\mathbb{k}'}, \mathcal{H}_{\mathbb{k}}), \\ \text{Pol}_{\mathcal{H}_2}^{-1}(\bar{\mathbb{k}}_1[1]) &= \text{Pol}_{\mathcal{H}_1}^{-1}(\mathcal{U}_1[1], \mathcal{U}_1^\perp[2]) = (\mathcal{H}_{\mathbb{k}}[1], \mathcal{H}_{\mathbb{k}'}).\end{aligned}$$

Thus $(\mathcal{H}_{\mathbb{k}'}, \mathcal{H}_{\mathbb{k}})$ is a torsion pair in the abelian category \mathcal{H}_1 , $(\mathcal{H}_{\mathbb{k}}[1], \mathcal{H}_{\mathbb{k}'})$ is a torsion pair in the abelian category \mathcal{H}_2 .

4 Introduction

Let \mathcal{A} be a *good* category (abelian/exact/triangulated). The precise meaning of this will have to be clarified later (probably, the recent Nakaoka-Palu's paper is the right setting). But, whatever the choice, two things should happen. First, idempotents should split in \mathcal{A} . Secondly, each torsion pair considered in \mathcal{A} should be functorial on both sides. If $(\mathcal{T}, \mathcal{F})$ is such a torsion pair, we will denote by $t : \mathcal{A} \rightarrow \mathcal{T}$ (resp. $f : \mathcal{A} \rightarrow \mathcal{F}$) the right (resp. left) adjoint of the inclusion functor and, also, the composition $\mathcal{A} \xrightarrow{t} \mathcal{T} \xrightarrow{i} \mathcal{A}$ (resp. $\mathcal{A} \xrightarrow{f} \mathcal{F} \xrightarrow{j} \mathcal{A}$), where $\mathcal{T} \xrightarrow{i} \mathcal{A}$ (resp. $\mathcal{F} \xrightarrow{j} \mathcal{A}$) is the inclusion functor. The functoriality should then give rise to an admissible sequence $t(M) \rightarrow M \rightarrow f(M)$, for each object $M \in \mathcal{A}$ (e.g. if \mathcal{A} is abelian, that sequence should be short exact, if \mathcal{A} is exact it should be a conflation, if \mathcal{A} is triangulated it should be a triangle).

Let \mathcal{W} a full subcategory of \mathcal{A} closed by direct summands and extensions, and consider the category $\underline{\mathcal{A}} = \frac{\mathcal{A}}{\mathcal{W}}$.

Let $(\mathcal{X}, \mathcal{Y})$ be a orthogonal pair in $\underline{\mathcal{A}}$ and consider the following classes in \mathcal{A} :

$$\begin{aligned}\mathcal{T} &= \{T \in \mathcal{A} | \underline{T} \in \mathcal{X}\} \\ \mathcal{F} &= \{F \in \mathcal{A} | \underline{F} \in \mathcal{Y}\}.\end{aligned}$$

Lemma 16 (empty).

If $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ is a orthogonal pair in a cocomplete and locally small abelian category \mathcal{A} , then \mathfrak{t} is a torsion pair. Indeed, if M is any object and we consider the set \mathcal{T}_M of subobjects of M which are in \mathcal{T} , then $t(M) := \sum_{T \in \mathcal{T}_M} T$ is subobject of M which is an epimorphic image of $\coprod_{T \in \mathcal{T}_M} T$ and, hence, we have that $t(M) \in \mathcal{T}_M$. If we had a nonzero morphism $f : T' \rightarrow M/t(M)$, where $T' \in \mathcal{T}$, then we would have that $\text{Im}(f) = \tilde{T}/t(M)$ is a nonzero submodule of $M/t(M)$ which is in $\tilde{\mathcal{T}}$. Since \mathcal{T} is closed under extensions and we have an exact sequence $0 \rightarrow t(M) \rightarrow \tilde{T} \rightarrow \tilde{T}/t(M) \rightarrow 0$, we conclude that $\tilde{T} \in \mathcal{T}$. But then we have that $\tilde{T} \in \mathcal{T}_M$, which is a contradiction since $t(M)$ contains all subobjects in \mathcal{T}_M .

Lemma 17. *Let $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in \mathcal{A} , with associated radical functors t_i and coradical functors f_i ($i = 1, 2$), respectively. Suppose that they satisfy the following conditions:*

- a) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equivalently, $\mathcal{F}_1 \subseteq \mathcal{F}_2$)
- b) $\mathcal{T}_2 \cap \mathcal{F}_1 = 0$.
- c) $(\underline{\mathcal{T}}_1, \underline{\mathcal{F}}_2)$ is an orthogonal pair in $\underline{\mathcal{A}} := \mathcal{A}/\mathcal{W}$, where $\mathcal{W} = \mathcal{T}_1 \cap \mathcal{F}_2$.

Then the following assertions hold:

- 1. \mathcal{T}_1 consists of those objects $X \in \mathcal{A}$ such that $f_2(X) \in \mathcal{W}$. We will write $\mathcal{T}_1 = \mathcal{T}_2 \star \mathcal{W}$.
- 2. \mathcal{F}_2 consists of those objects $Y \in \mathcal{A}$ such that $t_1(Y) \in \mathcal{W}$. We will write $\mathcal{F}_2 = \mathcal{W} \star \mathcal{F}_1$.

Proof. We just prove assertion 1, and assertion 2 will follow by duality. Let us take $X \in \mathcal{T}_2 \star \mathcal{W}$. Since we have an admissible sequence $t_2(X) \rightarrow X \rightarrow f_2(X)$ whose outer terms are in \mathcal{T}_2 and \mathcal{W} , respectively, and these two classes are contained in \mathcal{T}_1 we conclude that $\mathcal{T}'_1 \subseteq \mathcal{T}_1$, because \mathcal{T}_1 is closed under taking extensions in \mathcal{A} .

Let T_1 be in \mathcal{T}_1 and consider its canonical admissible sequence

$$t_2(T_1) \rightarrow T_1 \xrightarrow{f} f_2(T_1). \quad (1)$$

Note that $\underline{f} = 0$ because of condition c) in the statement. It follows that f decomposes in the form $f : T_1 \xrightarrow{\gamma} W \xrightarrow{\phi} f_2(T_1)$, where $W \in \mathcal{W}$. We then consider the following admissible pullback diagram

$$\begin{array}{ccccc} t_2(T_1) & \longrightarrow & \widehat{T}_1 & \longrightarrow & W \\ \parallel & & \downarrow & & \downarrow \phi \\ t_2(T_1) & \longrightarrow & T_1 & \xrightarrow{f} & f_2(T_1) \end{array} \quad (2)$$

Then, there exist a (non necessarily unique) $\eta : T_1 \rightarrow \widehat{T}_1$ making the following diagram commute.

$$\begin{array}{ccccc} T_1 & & \xrightarrow{\gamma} & & W \\ & \searrow \eta & & \searrow & \downarrow \phi \\ & & \widehat{T}_1 & \longrightarrow & W \\ & \searrow 1 & \downarrow & & \downarrow \phi \\ & & T_1 & \xrightarrow{f} & f_2(T_1) \end{array} \quad (3)$$

Hence, $T_1 <_{\oplus} \widehat{T}_1 \in \mathcal{T}_2 * \mathcal{W}$. This implies that $\mathcal{T}_1 \subseteq \text{add}(\mathcal{T}_2 * \mathcal{W})$. The proof will be finished once we check that $\mathcal{T}_2 * \mathcal{W}$ is closed under direct summands. But this is a direct consequence of the functoriality of the torsion pair. Indeed if we have admissible torsion sequences $t_2(M) \rightarrow M \rightarrow f_2(M)$ and $t_2(N) \rightarrow N \rightarrow f_2(N)$, then the coproduct sequence $t_2(M) \oplus t_2(N) \rightarrow M \oplus N \rightarrow f_2(M) \oplus f_2(N)$ is the admissible torsion sequence for $M \oplus N$. The fact that $M \oplus N \in \mathcal{T}_2 * \mathcal{W}$ is then equivalent to the fact that $f_2(M) \oplus f_2(N) \in \mathcal{W}$. Since \mathcal{W} is closed under direct summands, we conclude that $f_2(M) \in \mathcal{W}$ and, hence, that $M \in \mathcal{T}_2 * \mathcal{W}$. \square

5 Induced torsion theories

Lemma 18. *Let \mathcal{A} be a (nice) category and $\mathcal{W} \subseteq \mathcal{A}$ a subcategory such that $\text{add}(\mathcal{W}) = \mathcal{W}$. If $({}^\perp \mathcal{F}, \mathcal{F})$ is a torsion pair such that $\mathcal{W} \subseteq \mathcal{F}$, then $({}^\perp(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is an orthogonal pair in $\underline{\mathcal{A}} = \frac{\mathcal{A}}{\mathcal{W}}$.*

Lemma 19. *Let \mathcal{A} and $\mathcal{W} \subseteq \mathcal{A}$ be defined as above and let $({}^\perp \mathcal{F}, \mathcal{F})$ be a torsion pair in \mathcal{A} such that $\mathcal{W} \subseteq \mathcal{F}$. Call $p : \mathcal{A} \rightarrow \underline{\mathcal{A}}$ the quotient functor. The following assertions hold:*

1. $p^{-1}({}^\perp(\underline{\mathcal{F}})) = \text{add}({}^\perp \mathcal{F} * \mathcal{W})$.
2. *If \mathcal{W} is precovering in \mathcal{F} , then $({}^\perp \underline{\mathcal{F}}, \underline{\mathcal{F}})$ is a torsion pair in $\underline{\mathcal{A}}$.*

Lemma 20. *Let \mathcal{A} be a (nice) category with a torsion pair $({}^\perp \mathcal{F}, \mathcal{F})$ and a precovering class $\mathcal{W} \subseteq \mathcal{F}$ such that for any $F \in \mathcal{F}$ there is an admissible sequence*

$$F' \rightarrow W \rightarrow F$$

such that $F' \in \mathcal{F}$.

Then the torsion pair $({}^\perp(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial.

Recall that the truncation $t : \mathcal{A} \rightarrow {}^\perp(\underline{\mathcal{F}})$ is given by the following construction.

Let $M \in \mathcal{A}$ be any object, take an admissible sequence

$$T_M \rightarrow M \rightarrow F^M$$

with $T_M \in {}^\perp \mathcal{F}$ and $F^M \in \mathcal{F}$. Moreover, consider $W_F \rightarrow F^M$ with $W_M \in \mathcal{W}$ as before, and take the admissible pullback:

$$\begin{array}{ccc} t(M) & \longrightarrow & W_F \\ \downarrow & & \downarrow \\ M & \longrightarrow & F \end{array}$$

Then, t restricts to a functor $\underline{t} : \underline{\mathcal{A}} \rightarrow {}^\perp \underline{\mathcal{F}}$.

In order to prove that $({}^\perp(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial we need to show that \underline{t} admits a right adjoint.

Proof. Let $M \in \mathcal{A}$ and consider $M \rightarrow F^M$ and $W_F \rightarrow F^M$ as above. For any $T \in {}^\perp \mathcal{F} * \mathcal{W}$ consider any morphism $f : T \rightarrow M$. Since $T \rightarrow M \rightarrow F^M$ is 0 in $\underline{\mathcal{A}}$ we have that the solid part of the following diagram commutes.

$$\begin{array}{ccc} T & \xrightarrow{\quad} & W \\ \downarrow \scriptstyle f & \searrow \scriptstyle \psi & \swarrow \scriptstyle \phi \\ & t(M) \longrightarrow W_F & \\ & \downarrow & \downarrow \\ & M \longrightarrow F^M & \end{array}$$

□

Since $W_F \rightarrow F^M$ is a precover there is a morphism $\phi : W \rightarrow W_F$ making the diagram commute, and since the square is an admissible pullback there is a morphism $\psi : T \rightarrow t(M)$ making the diagram commutative.

Hence, $\mathcal{A}(T, t(M)) \rightarrow \mathcal{A}(T, M)$ is surjective. To conclude the proof we need to show that when restricted to $\underline{\mathcal{A}}$ it becomes an iso. Assume that there are two morphisms $\underline{\psi}$ and $\underline{\psi}'$ in $\underline{\mathcal{A}}$ such that the following commutes:

$$\begin{array}{ccc} & & t(M) \\ & \nearrow \underline{\psi} & \downarrow \\ T & \xrightarrow{\quad} & M \\ & \nwarrow \underline{\psi}' & \end{array}$$

So, if we call $h = \psi - \psi'$ in \mathcal{A} , we have that $T \xrightarrow{h} t(M) \rightarrow M$ factors through W so that we have that the solid part of the following diagram commutes:

$$\begin{array}{ccccc}
 & & W & & \\
 & \swarrow p & \nearrow \gamma & \searrow \eta & \\
 T & \xrightarrow{h} & t(M) & \longrightarrow & W_F \\
 & \searrow g & \downarrow & & \downarrow \\
 & & M & \longrightarrow & F^M
 \end{array}$$

where $\eta : W \rightarrow W_F$ comes from the fact that $W_F \rightarrow F^M$ is a precover, and γ from the fact that the square is an admissible pullback, and they make the complete diagram commute.

Let's call $h' = \gamma \circ p$, then composing both h and h' with $\rho : t(M) \rightarrow M$ gives the same morphism g . Hence, $\rho \circ (h - h') = 0$. But since $F' \xrightarrow{i} t(M) \rightarrow M$ is an admissible sequence we have the following exact sequence of abelian groups:

$$\mathcal{A}(T, F) \longrightarrow \mathcal{A}(T, t(M)) \longrightarrow \mathcal{A}(T, M)$$

$$h - h' \longmapsto 0$$

So there is a map $k : T \rightarrow F'$ such that $i \circ k = h - h'$, but $\underline{k} = 0$ so $\underline{h} = \underline{h}' = 0$, hence $\underline{\psi} - \underline{\psi}' = 0$ which proves that $\underline{\mathcal{A}}(T, t(M)) \cong \underline{\mathcal{A}}(T, M)$.

The naturality of the isomorphism in T and M is clear.

6 Abelian categories

Now we work in an abelian category with two torsion pairs $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ such that $t_2(\mathcal{F}_1) \subseteq \mathcal{F}_1$ and $f_1(\mathcal{T}_2) \subseteq \mathcal{T}_2$ and let $\mathcal{W} = \mathcal{T}_2 \cap \mathcal{F}_1$.

Recall that $(\mathcal{T}_1 * \mathcal{W}, \mathcal{F}_1)$ (resp. $(\mathcal{T}_2, \mathcal{W} * \mathcal{F}_2)$) is a left (resp. right) functorial torsion pair in $\underline{\mathcal{A}} = \frac{\mathcal{A}}{\mathcal{W}}$. Moreover, they satisfy $TC1 - 3, 3^*$.

Lemma 21. *The inclusion $i : \mathcal{T}_1 * \mathcal{W} \hookrightarrow \mathcal{A}$ admits a right adjoint \hat{t} .*

Proof. For $M \in \mathcal{A}$ consider the exact sequence

$$0 \rightarrow T_1 \rightarrow M \rightarrow f_1(M) \rightarrow 0$$

with $T_1 \in \mathcal{T}_1$ and $f_1(M) \in \mathcal{F}$. Take $t_2 f_1(M) \hookrightarrow f_1(M)$ and observe that $t_2 f_1(M) \in \mathcal{W}$. Call it W_M and take the pullback diagram

$$\begin{array}{ccc} \hat{t}(M) & \longrightarrow & W_M \\ \downarrow & & \downarrow \\ M & \longrightarrow & f_1(M) \end{array}$$

then $\hat{t}(M) \in \mathcal{T}_1 * \mathcal{W}$.

Now for any morphism $\hat{T} \rightarrow M$ with $\hat{T} \in \hat{\mathcal{T}}$ the solid part of the following diagram commutes

$$\begin{array}{ccccc} T & \twoheadrightarrow & \hat{t}(M) & \longrightarrow & W \\ \parallel & & \downarrow & & \downarrow \\ T & \twoheadrightarrow & M & \longrightarrow & f_1(M) \\ & & \uparrow & & \uparrow \\ T_1 & \twoheadrightarrow & \hat{T} & \longrightarrow & W_1 \end{array}$$

$\hat{T} \rightarrow M$ is mono by Buhler prop. 2.14: pullback of monic along epic is monic

Since the composition $T_1 \rightarrow \hat{T} \rightarrow M \rightarrow f_1(M)$ is zero, there exists the dashed morphism $W_1 \rightarrow f_1(M)$, which lifts to the morphism $W_1 \rightarrow W$ (since $W \rightarrow f_1(M)$ is a \mathcal{W} -precover). Hence, there is a morphism $\hat{T} \rightarrow \hat{t}(M)$ making the diagram commutative. This means that

$$\mathcal{A}(\hat{T}, \hat{t}(M)) \xrightarrow{\mathcal{A}(\hat{T}, \hat{t}(M) \rightarrow M)} \mathcal{A}(\hat{T}, M)$$

is surjective. But it is also injective, since $\text{Ker}(\hat{t}(M), M) = 0$. Hence, it is an iso and \hat{t} is right adjoint to i . □

functoriality should follow immediately

Lemma 22. *Let $\hat{T}_1 \in \mathcal{T}_1 * \mathcal{W}$, i.e. there is an exact sequence*

$$0 \rightarrow t_1(\hat{T}_1) \rightarrow \hat{T}_1 \rightarrow W_1 \rightarrow 0.$$

If

$$\begin{array}{ccc} \hat{T}_1 & \xrightarrow{p} & W_1 \\ \downarrow g & & \downarrow g' \\ \hat{T}'_1 & \xrightarrow{q} & N \end{array}$$

*is a pushout diagram, then $N \in \mathcal{T}_1 * \mathcal{W}$.*

Proof. Since it is a pushout, $\hat{T}'_1 \rightarrow N$ is epi, then consider $\hat{t}(N)$ and the following commutative diagram

$$\begin{array}{ccc} \hat{T}'_1 & \xrightarrow{q} & N \\ \searrow \rho & & \nearrow \varepsilon \\ & \hat{t}(N) & \end{array}$$

where the map $\hat{T}'_1 \rightarrow \hat{t}(N)$ is given by the adjunction (i, \hat{t}) . Since $q = \varepsilon \circ \rho$ is epi, then ε is epi. But it is mono, so it is an isomorphism, hence $N \in \mathcal{T}_1 * \mathcal{W}$. □

Lemma 23. *In the same notation as the previous lemma, if $\varphi : N \rightarrow P$ is any map s.t. $\underline{\varphi} \circ \underline{q} = \underline{0}$ in $\underline{\mathcal{A}}$, then φ factors through \mathcal{W} .*

Proof. Since $\underline{\varphi} \circ \underline{q} = \underline{0}$ it means that $\varphi \circ q$ factors through \mathcal{W} , hence we have that the solid part of the following diagram is commutative.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & t_1(T'_1) & \longrightarrow & \widehat{T}_1 & \longrightarrow & W_1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X & \longrightarrow & \widehat{T}'_1 & \longrightarrow & N \\
 & & & & \parallel & & \downarrow \\
 & & & & \widehat{T}'_1 & \longrightarrow & W \longrightarrow P
 \end{array}$$

Since $t_1(T'_1) \rightarrow \widehat{T}_1 \rightarrow \widehat{T}'_1 \rightarrow W$ is zero, there is the dashed morphism $W_1 \rightarrow W$ making the diagram commute. Since the square on the right is a pushout there is a map $N \rightarrow W$, and again the diagram commutes. Hence φ factors through \mathcal{W} . \square

Lemma 29. *If \mathcal{H} is balanced (i.e. mono and epi implies iso), then whenever $f : H_1 \rightarrow H_2$ is mono and epi in \mathcal{H} , there are bicartesian squares in \mathcal{A}*

$$\begin{array}{ccccc} F_1 & \longrightarrow & H_1 & \longrightarrow & W_1 \\ \downarrow & \lrcorner & & \lrcorner & \downarrow \\ W_2 & \longrightarrow & H_2 & \longrightarrow & T_2 \end{array}$$

where $W_1 = f_1(H_1)$ and $W_2 = t_2(H_2)$. In particular there is an exact sequence

$$0 \rightarrow F_1 \rightarrow W_1 \oplus W_2 \rightarrow T_2 \rightarrow 0.$$

Proof. We can build the pullback on the left and the pushout on the right as usual

$$\begin{array}{ccccc} F_1 & \longrightarrow & H_1 & \xrightarrow{r} & W_1 \\ \downarrow & \lrcorner & \downarrow f & & \downarrow s \\ W_2 & \longrightarrow & H_2 & \xrightarrow{f^C} & T_2 \end{array} \quad (4)$$

We will only prove that the square on the right hand side is a pullback, since the proof that the left square is a pushout is dual. The statetment that the square on the right is a pushout is equivalent to saying that there is an exact sequence

$$H_1 \xrightarrow{\begin{pmatrix} f \\ r \end{pmatrix}} H_1 \oplus W_1 \xrightarrow{\begin{pmatrix} f^C & s \end{pmatrix}} T_2 \longrightarrow 0 \quad (5)$$

Since f is both a mono and an epi in \mathcal{H} , then it is an iso and hence both a section and a retraction. Consider $g : H_2 \rightarrow H_1$ such that $\underline{g} \circ \underline{f} = \underline{1}_{H_1}$, that is there are maps $\alpha : H_1 \rightarrow W$ and $\beta : W \rightarrow H_1$ such that

$$\begin{array}{ccc} H_1 & \xrightarrow{\begin{pmatrix} f \\ \alpha \end{pmatrix}} & H_2 \oplus W \xrightarrow{(g \ \beta)} H_1 \\ & \searrow \quad \quad \quad \nearrow & \\ & 1_{H_1} & \end{array}$$

is commutative in \mathcal{A} , and hence H_1 is a direct summand of $H_2 \oplus W$. We can actually choose $W = W_1$, in fact consider the commutative diagram

$$\begin{array}{ccccc} H_1 & \xrightarrow{\begin{pmatrix} f \\ r \end{pmatrix}} & H_1 \oplus W_1 & \xrightarrow{\begin{pmatrix} f^C & s \end{pmatrix}} & T_2 \\ \parallel & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} & & \\ H_1 & \xrightarrow{\begin{pmatrix} f \\ \alpha \end{pmatrix}} & H_2 \oplus W & \xrightarrow{(g \ \beta)} & H_1 \\ & \searrow \quad \quad \quad \nearrow & \\ & 1_{H_1} & \end{array}$$

where $\rho : W_1 \rightarrow W$ comes from the fact that $H_1 \rightarrow W_1$ is a \mathcal{W} -preenvelope. Hence, $(g \ \beta) \circ \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \circ \begin{pmatrix} f \\ r \end{pmatrix} = 1_{H_1}$, that is H_1 is a direct summand of $H_2 \oplus W_1$. Moreover, it means that $(g \ \beta)$ is a section, that is the sequence in (5) is also exact on the left and the corresponding square in (4) is a pullback diagram.

Since both squares in (4) are bicartesian, it follows that the square

$$\begin{array}{ccc} F_1 & \longrightarrow & W_1 \\ \downarrow & \lrcorner & \downarrow \\ W_2 & \longrightarrow & T_2 \end{array}$$

is bicartesian as well. □

7 Second approach to axiomatization

We give another set of axioms:

TC1 $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ are two respectively left functorial and right functorial torsion pairs in \mathcal{X} .

TC2 $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equivalently $\mathcal{F}_1 \subseteq \mathcal{F}_2$).

TC3 For any morphism $g : T_1 \rightarrow T'_1$ in \mathcal{T}_1 has a pseudocokernel in \mathcal{T}_1 which completes diagrams in a unique way wrt \mathcal{F}_2 .

TC3* Dual of **TC3**.

TC4 explain this axiom

$$\begin{array}{ccccccc}
 F_1 & \xrightarrow{f^K} & H_1 & \xrightarrow{\forall f} & H_2 & \xrightarrow{f^C} & T_1 \\
 & \nearrow \text{dashed} & \downarrow \text{dashed} & & \parallel & & \downarrow \\
 i_1 t_1(F_2) & \xrightarrow{\varepsilon} & F & \longrightarrow & H_2 & \longrightarrow & j_2 f_2(T_1)
 \end{array}$$

TC4* Dual of **TC4**.

EXAMPLES

add examples from page 2, 9/11/16

2 If $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in a triangulated category (as in Nakaoka's work) produces an example.

Add reference

3 Let \mathcal{D} be a triangulated category with two t -structures $(\mathcal{U}_1, \mathcal{U}_1^\perp)$ and $(\mathcal{U}_2, \mathcal{U}_2^\perp)$ such that $\mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1$. Then, these satisfy axioms **TC1-TC3, TC3***, hence $\mathcal{H} = \mathcal{U}_1 \cap \mathcal{U}_2^\perp$ has kernels and cokernels. Moreover, TFAE:

1.a **TC4** holds.

1.b If $V_1 \rightarrow H_1 \xrightarrow{f} H_2 \xrightarrow{+}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $V_1 \in \mathcal{U}_1^\perp$, then $V_1 \in \mathcal{U}_2^\perp[-1]$.

And, dually, there is an equivalence of the following:

2.a **TC4*** holds.

2.b If $H_1 \xrightarrow{f} H_2 \rightarrow U_2 \xrightarrow{+}$ is a distinguished triangle such that $H_1, H_2 \in \mathcal{H}$ and $U_2 \in \mathcal{U}_2$, then $U_2 \in \mathcal{U}_1[1]$.

Proof of the equivalences in example 3. Let's \mathcal{D} be a triangulated category with two t -structures as in example 3. The pseudocokernel of a morphism in \mathcal{U}_1 can be computed by taking the cone in \mathcal{D} , i.e. given a morphism $f : U_1 \rightarrow U'_1$ in \mathcal{U}_1 we can compute a pseudocokernel in \mathcal{U}_1 by completing f to a triangle

$$U_1 \xrightarrow{f} U'_1 \rightarrow \text{Cone}(f) \xrightarrow{+}.$$

Moreover, this pseudocokernel satisfies **TC3**.

Now, assume that **TC1-TC3, TC3*** are satisfied together with axiom **1.b**, and consider the solid part of the diagram as in **TC4**:

$$\begin{array}{ccccccc}
 \text{Cone}(f)[-1] & \xrightarrow{f^K} & H_1 & \xrightarrow{f} & H_1 & \xrightarrow{f^C} & \text{Cone}(f) \\
 & \nearrow \beta \text{ dashed} & \downarrow \alpha & & \parallel & & \downarrow \lambda \\
 \tau_{\mathcal{U}_1}(V_2) & \xrightarrow{\varepsilon} & V_2 & \longrightarrow & H_2 & \longrightarrow & \tau^{\mathcal{U}_2^\perp} \text{Cone}(f)
 \end{array}$$

with $\text{Cone}(f)[-1] \in \mathcal{U}_1^\perp$ and where the upper row is a distinguished triangle. By **1.b** then it belongs to $\mathcal{U}_2^\perp[-1]$, i.e. $\text{Cone}(f) \in \mathcal{U}_2^\perp$, so λ is an iso, consequently α is an iso and so is ε , so there exist a map $\beta = \alpha^{-1} \circ \varepsilon$ making the diagram commute, that is **TC4** holds.

Conversely, assume that **TC1-TC3, TC3*** are satisfied together with **TC4**. Consider again the solid part of the diagram

$$\begin{array}{ccccccc}
 \text{Cone}(f)[-1] & \xrightarrow{f^K} & H_1 & \xrightarrow{f} & H_1 & \xrightarrow{f^C} & \text{Cone}(f) \\
 \downarrow \lambda[-1] & & \downarrow \alpha & & \parallel & & \downarrow \lambda \\
 \tau_{\mathcal{U}_2^\perp}(\text{Cone}(f))[-1] & \xrightarrow{\quad} & V_2 & \xrightarrow{\quad} & H_2 & \xrightarrow{\quad} & \tau_{\mathcal{U}_2^\perp} \text{Cone}(f) \\
 & & \uparrow \varepsilon & & & & \\
 & & \tau_{\mathcal{U}_1}(V_2) & & & &
 \end{array}$$

(A dotted arrow labeled β connects $\tau_{\mathcal{U}_2^\perp}(\text{Cone}(f))[-1]$ to H_1 .)

with $\text{Cone}(f)[-1] \in \mathcal{U}_1^\perp$. Neeman guarantees that α can be taken so that the square on the left is a pullback. Axiom **TC4** gives the existence of $\beta : \tau_{\mathcal{U}_1}(V_2) \rightarrow H_1$ such that $\alpha \circ \beta = \varepsilon$.

Since $\tau_{\mathcal{U}_1}$ is a functor, there is also a morphism $\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(H_1) = H_1 \rightarrow \tau_{\mathcal{U}_1}(V_2)$ such that $\varepsilon \circ \tau_{\mathcal{U}_1}(\alpha) = \alpha$, hence $\varepsilon \circ \tau_{\mathcal{U}_1}(\alpha) \circ \beta = \varepsilon$. By the functoriality of the torsion pair $(\mathcal{U}_1, \mathcal{U}_1^\perp)$, this means that $\tau_{\mathcal{U}_1}(\alpha) \circ \beta = 1_{\tau_{\mathcal{U}_1}(V_2)}$. Then, β is a section.

Hence, we can write $\tau_{\mathcal{U}_1}(\alpha) : H_1 \rightarrow \tau_{\mathcal{U}_1}(V_2)$ as

$$\tau_{\mathcal{U}_1}(\alpha) : \tau_{\mathcal{U}_1}(V_2) \oplus H'_1 \xrightarrow{(* \ 0)} \tau_{\mathcal{U}_1}(V_2)$$

for some $H'_1 \leq_{\oplus} H_1$ such that α vanishes on H'_1 . If we consider the solid part of the diagram

$$\begin{array}{ccccc}
 & & H'_1 & & \\
 & \swarrow & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \searrow 0 & \\
 \text{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1] & \longrightarrow & H_1 & \xrightarrow{\tau_{\mathcal{U}_1}(\alpha)} & \tau_{\mathcal{U}_1}(V_2) \dashrightarrow
 \end{array}$$

we can construct the dashed arrow, and the fact that the triangle commutes means that $H'_1 \leq_{\oplus} \text{Cone}(\tau_{\mathcal{U}_1}(\alpha))[-1]$.

Observe that $\text{Cone}(\alpha) = \text{Cone}(\lambda)[-1]$, since the square

$$\begin{array}{ccc}
 \text{Cone}(f)[-1] & \xrightarrow{f^K} & H_1 \\
 \downarrow \lambda[-1] & & \downarrow \alpha \\
 \tau_{\mathcal{U}_2^\perp}(\text{Cone}(f))[-1] & \longrightarrow & V_2
 \end{array}$$

is a pullback. Moreover, $\text{Cone}(\lambda)[-1] = (\tau_{\mathcal{U}_2}(\text{Cone}(f)))[1][-1] = \tau_{\mathcal{U}_2}(\text{Cone}(f))$. Hence, $\text{Cone}(\alpha) \in \mathcal{U}_2$ and $\tau_{\mathcal{U}_1^\perp}(\text{Cone}(\alpha)) = 0$, that is, $\text{Cone}(\alpha) \in \mathcal{U}_1$, and since there is a distinguished triangle

$$H_1 \xrightarrow{\alpha} V_2 \rightarrow \text{Cone}(\alpha) \xrightarrow{+}$$

with $H_1, \text{Cone}(\alpha) \in \mathcal{U}_1$ it follows that $V_2 \in \mathcal{U}_1$. Hence, $\tau_{\mathcal{U}_1}(V_2) \cong V_2$.

We can then write $V_2 \leq_{\oplus} H_1$ and consider the commutative diagram

$$\begin{array}{ccc}
 H_1 \cong H'_1 \oplus V_2 & \xrightarrow{(f' \ \bar{f})} & H_2 \\
 \downarrow \begin{pmatrix} 0 & 1 \end{pmatrix} & & \parallel \\
 V_2 & \longrightarrow & H_2
 \end{array}$$

so $f' = 0$. Hence, the inclusion $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : H'_1 \rightarrow H'_1 \oplus V_2$ can be lifted to $\text{Cone}(f)[-1]$ and $H'_1 \leq_{\oplus} \text{Cone}(f)[-1]$. Since $\text{Cone}(f)[-1] \in \mathcal{U}_1^\perp$, so does H'_1 . Similarly, $H'_1 \in \mathcal{U}_1$ because $H_1 \in \mathcal{U}_1$. Hence, $H'_1 = 0$ and $\alpha : H_1 \rightarrow V_2$ is an iso. The same follows for λ . Therefore, $\text{Cone}(f) \in \mathcal{U}_2^\perp$ which proves **1.b**. \square

Add reference

We can see a special case of example 3 in the case of the derived category of a ring. Let R be a commutative ring, consider the t-structure $(\mathcal{U}_1, \mathcal{U}_1^\perp) = (\mathcal{D}^{\leq 0}(R), \mathcal{D}^{> 0}(R))$ in $\mathcal{D}(R)$. Given an idempotent ideal $I = I^2 \triangleleft R$, it defines three classes of modules

$$\begin{aligned}\mathcal{C}_I &= \{C \in \text{Mod-}R \mid IC = C\} \\ \mathcal{T}_I &= \{T \in \text{Mod-}R \mid IT = 0\} \cong \text{Mod-}\frac{R}{I} \\ \mathcal{F}_I &= \{F \in \text{Mod-}R \mid Ix \neq 0 \forall x \in F \setminus \{0\}\}\end{aligned}$$

such that $(\mathcal{C}_I, \mathcal{T}_I)$ and $(\mathcal{T}_I, \mathcal{F}_I)$ make two torsion pairs. We call the triple $(\mathcal{C}_I, \mathcal{T}_I, \mathcal{F}_I)$ a TTP triple.

We define the t-structure $(\mathcal{U}_2, \mathcal{U}_2^\perp)$ as the Happel-Reiten-Smalø t-structure associated to the torsion pair $(\mathcal{C}_I, \mathcal{T}_I)$ in $\text{Mod-}R$:

$$\begin{aligned}\mathcal{U}_2 &= \{U_2 \in \mathcal{D}^{\leq 0}(R) \mid H^0(U_2) \in \mathcal{C}_I\} \\ \mathcal{U}_2^\perp &= \{V_2 \in \mathcal{D}^{\geq 0}(R) \mid H^0(V_2) \in \mathcal{T}_I\}.\end{aligned}$$

In this case we can check that condition **1.b** holds. In fact, let \mathcal{H} be the heart

$$\begin{aligned}\mathcal{U}_1 \cap \mathcal{U}_2^\perp &= \mathcal{D}^{\leq 0}(R) \cap \mathcal{U}_2^\perp \\ &= \{T[0] \mid T \in \mathcal{T}_I\} \cong \text{Mod-}\frac{R}{I}.\end{aligned}$$

Hence, \mathcal{H} is abelian.

Now, consider $V_1 \in \mathcal{U}_1^\perp$ such that there is an exact triangle

$$V_1 \rightarrow T_1[0] \xrightarrow{f[0]} T_2[0] \xrightarrow{+}$$

with $T_1, T_2 \in \mathcal{H}$. Of course, $V_1 = \text{Cone}(f)[-1]$, i.e.

$$V_1 = \cdots \rightarrow 0 \xrightarrow{0} T_1 \xrightarrow{1} T_2 \rightarrow 0 \rightarrow \cdots$$

where the numbers over T_1 and T_2 represent their cohomological degree.

The fact that $V_1 \in \mathcal{U}_1^\perp = \mathcal{D}^{> 0}(R)$ implies that $H^0(V_1) = 0$, i.e. f is mono. To prove that $V_1 \in \mathcal{U}_2^\perp[-1]$ we would need to show that $\text{Coker}(f) = H^1(V_1)$ belongs to \mathcal{T}_I , but this follows from the fact that f is a mono in \mathcal{T}_I which is a torsion class.

7.1 Polishchuk correspondence

Given a t-structure $(\mathcal{U}_1, \mathcal{U}_1^\perp)$ in a triangulated category \mathcal{D} there is a correspondence between t-structures $(\mathcal{U}_2, \mathcal{U}_2^\perp)$ satisfying $\mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1$ in \mathcal{D} and torsion pairs in $\mathcal{H}_1 = \mathcal{U}_1 \cap \mathcal{U}_1^\perp[1]$, namely

$$\left\{ \begin{array}{l} \text{torsion pairs in} \\ \mathcal{H}_1 = \mathcal{U}_1 \cap \mathcal{U}_1^\perp[1] \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{t-structures } (\mathcal{U}_2, \mathcal{U}_2^\perp) \\ \text{in } \mathcal{D} \text{ satisfying} \\ \mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1 \end{array} \right\}$$

$$(\mathcal{X}, \mathcal{Y}) \longrightarrow \left\{ \begin{array}{l} \mathcal{U}_2 = \{X \in \mathcal{U}_1 \mid H^0(X) \in \mathcal{X}\} \\ \mathcal{U}_2^\perp = \{Y \in \mathcal{U}_1^\perp \mid H^0(Y) \in \mathcal{Y}\} \end{array} \right\}$$

$$(\mathcal{U}_2 \cap \mathcal{H}_1, \mathcal{U}_2^\perp \cap \mathcal{H}_1) \longleftarrow (\mathcal{U}_2, \mathcal{U}_2^\perp)$$

Observe that $(\mathcal{U}_2 \cap \mathcal{H}_1, \mathcal{U}_2^\perp \cap \mathcal{H}_1)$ is actually $(\mathcal{U}_1^\perp \cap \mathcal{U}_2, \mathcal{U}_1 \cap \mathcal{U}_2^\perp)$, where the right hand side $\mathcal{U}_1 \cap \mathcal{U}_2^\perp$ is exactly our heart \mathcal{H} .

Hence, under this correspondence, given a t-structure $(\mathcal{U}_1, \mathcal{U}_1^\perp)$ in a triangulated, the t-structures $(\mathcal{U}_2, \mathcal{U}_2^\perp)$ satisfying $\mathcal{U}_1[1] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1$ and such that the heart $\mathcal{H} = \mathcal{U}_1 \cap \mathcal{U}_2^\perp$ is abelian and in bijection with the abelian torsion free classes of $\mathcal{H}_1 = \mathcal{U}_1 \cap \mathcal{U}_1^\perp[1]$.

The following lemma clarifies which properties characterize an abelian torsion free class in an abelian category.

Lemma 30. *Let \mathcal{A} be an abelian category and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in \mathcal{A} . The following assertions hold:*

1. *If $f : F \rightarrow F'$ is a morphism in \mathcal{F} , then the composition*

$$F' \xrightarrow{f^C} \text{Coker}_{\mathcal{A}}(f) \rightarrow \frac{\text{Coker}_{\mathcal{A}}(f)}{t(\text{Coker}_{\mathcal{A}}(f))}$$

is the cokernel map of f in \mathcal{F} .

2. *$f : F \rightarrow F'$ as above is an epimorphism in \mathcal{F} if and only if $\text{Coker}_{\mathcal{A}}(f) \in \mathcal{T}$.*
3. *Epimorphisms and cokernel maps coincide in \mathcal{F} if and only if each morphism f in \mathcal{F} with $\text{Coker}_{\mathcal{A}}(f) \in \mathcal{T}$ is an epimorphism in \mathcal{A} .*
4. *\mathcal{F} is an abelian category if and only if \mathcal{F} is closed under quotients in \mathcal{A} .*

References

- [1] Hiroyuki Nakaoka. “General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories”. In: *Applied Categorical Structures* 19.6 (2011), pp. 879–899.