1 Introduction

Let $\mathcal{A}$ be a *good* category (abelian/exact/triangulated). The precise meaning of this will have to be clarified later (probably, the recent Nakaoka-Palu's paper is the right setting). But, whatever the choice, two things should happen. First, idempotents should split in $\mathcal{A}$. Secondly, each torsion pair considered in $\mathcal{A}$ should be functorial on both sides. If $(\mathcal{T},\mathcal{F})$ is such a torsion pair, we will denote by $t:\mathcal{A}\longrightarrow\mathcal{T}$ (resp. $f:\mathcal{A}\longrightarrow\mathcal{F}$) the right (resp. left) adjoint of the inclusion functor and, also, the composition $\mathcal{A}\xrightarrow{t}\mathcal{T}\overset{i}{\hookrightarrow}\mathcal{A}$ (resp. $\mathcal{A}\xrightarrow{f}\mathcal{F}\overset{j}{\hookrightarrow}\mathcal{A}$), where $\mathcal{T}\overset{i}{\hookrightarrow}\mathcal{A}$ (resp. $\mathcal{F}\overset{j}{\hookrightarrow}\mathcal{A}$) is the inclusion functor. The functoriality should then give rise to an admissible sequence $t(M)\longrightarrow M\longrightarrow f(M)$, for each object $M\in\mathcal{A}$ (e.g. if $\mathcal{A}$ is abelian, that sequence should be short exact, if $\mathcal{A}$ is exact it should be a conflation, if A is triangulated it should be a triangle).

$\mathcal{W}$ a full subcategory of $\mathcal{A}$ closed by direct summands and extensions, and consider the category $\underline{\mathcal{A}}=\dfrac{\mathcal{A}}{\mathcal{W}}$.

Let $(\mathcal{X},\mathcal{Y})$ be a orthogonal pair in $\underline{\mathcal{A}}$ and consider the following classes in $\mathcal{A}$:

$$\mathcal{T}=\{T\in\mathcal{A}|\underline{T}\in\mathcal{X}\}$$
$$\mathcal{F}=\{F\in\mathcal{A}|\underline{F}\in\mathcal{Y}\}.$$

Lemma 1. *In the previous notation, $(\mathcal{T},\mathcal{T}^{\perp})$ is a orthogonal pair.*

Proof. In order to prove it we need to show that ${}^{\perp}(\mathcal{T}^{\perp})=\mathcal{T}$.

Let $M\in{}^{\perp}(\mathcal{T}^{\perp})$, this means that

$$\mathcal{A}(M,Y)=0 \tag{1}$$

whenever

$$\mathcal{A}(T,Y)=0\,\forall T\in\mathcal{T}. \tag{2}$$

However, if $\mathcal{A}(T,Y)=0\ \forall T\in\mathcal{T}$, then $\underline{\mathcal{A}}(\underline{X},\underline{Y})=0\ \forall\underline{X}\in\mathcal{X}$. Hence, $\underline{Y}\in\mathcal{Y}$. So $\underline{\mathcal{A}}(\underline{M},\underline{Y})=0\ \forall\underline{Y}\in\mathcal{Y}$. Hence, $\underline{M}\in\mathcal{X}$ and so $M\in\mathcal{T}$.

We have proved that ${}^{\perp}(\mathcal{T}^{\perp})\subseteq\mathcal{T}$, the converse inclusion is trivial. □

Remark. The dual statement holds for $\mathcal{F}$. Notice that have we also proved that if $\mathcal{A}(T,Y)=0\ \forall T\in\mathcal{T}$, then $\underline{Y}\in\mathcal{Y}$ and hence $Y\in\mathcal{F}$. That is, $\mathcal{T}^{\perp}\subseteq\mathcal{F}$ and dually ${}^{\perp}\mathcal{F}\subseteq\mathcal{T}$.

Properties of $(\mathcal{T},\mathcal{T}^{\perp})$ and $({}^{\perp}\mathcal{F},\mathcal{F})$:

1. ${}^{\perp}\mathcal{F}\subseteq\mathcal{T}$ and $\mathcal{T}^{\perp}\subseteq\mathcal{F}$.
2. $\mathcal{T}\cap\mathcal{F}=\mathcal{W}$. In fact, $M\in\mathcal{T}\cap\mathcal{F}$ iff $\underline{M}\in\mathcal{X}\cap\mathcal{Y}=0$, which happens iff $M<_{\oplus}W$ for some $W\in\mathcal{W}$, but $\mathcal{W}$ is closed by direct summands, hence $M\in\mathcal{W}$.
3. If $N\in\mathcal{T}^{\perp}\cap{}^{\perp}\mathcal{F}$, then $N=0$. It follows from $N\in\mathcal{T}^{\perp}\cap{}^{\perp}\mathcal{F}\subseteq\mathcal{F}\cap\mathcal{T}=\mathcal{W}$. But $\mathcal{W}\subseteq\mathcal{T}$, hence $\mathcal{A}(W',N)=0\ \forall W'\in\mathcal{W}$, in particular $\mathcal{A}(N,N)=0$, i.e. $N=0$.

If $\mathbb{t}=(\mathcal{T},\mathcal{F})$ is a orthogonal pair in a cocomplete and locally small abelian category $\mathcal{A}$, then $\mathbb{t}$ is a torsion pair. Indeed, if M is any object and we consider the set $\mathcal{T}_M$ of subobjects of M which are in $\mathcal{T}$, then $t(M):=\sum_{T\in\mathcal{T}_M}T$ is subobject of M which is an epimorphic image of $\coprod_{T\in\mathcal{T}_M}$ and, hence, we have that $t(M)\in\mathcal{T}_M$. If we had a nonzero morphism $f:T'\longrightarrow M/t(M)$, where $T'\in\mathcal{T}$, then we would have that $\operatorname{Im}(f)=\tilde{T}/t(M)$ is a nonzero submodule of $M/t(M)$ which is in $\mathcal{T}$. Since $\mathcal{T}$ is closed under extensions and we have an exact sequence $0\to t(M)\longrightarrow\tilde{T}\longrightarrow\tilde{T}/t(M)\to 0$, we conclude that $\tilde{T}\in\mathcal{T}$. But then we have that $\tilde{T}\in\mathcal{T}_M$, which is a contradiction since $t(M)$ contains all subobjects in $\mathcal{T}_M$.

Lemma 2. *Let $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in $\mathcal{A}$, with associated radical functors t_i and coradical functors f_i $(i = 1, w)$, respectively. Suppose that they satisfy the following conditions:*

a) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ *(equivalently, $\mathcal{F}_1 \subseteq \mathcal{F}_2$)*

b) $\mathcal{T}_2 \cap \mathcal{F}_1 = 0$.

c) $(\underline{\mathcal{T}}_1, \underline{\mathcal{F}}_2)$ *is an orthogonal pair in* $\underline{\mathcal{A}} := \mathcal{A}/\mathcal{W}$, *where* $\mathcal{W} = \mathcal{T}_1 \cap \mathcal{F}_2$.

Then the following assertions hold:

1. *$\mathcal{T}_1$ consists of those objects $X \in \mathcal{A}$ such that $f_2(X) \in \mathcal{W}$. We will write $\mathcal{T}_1 = \mathcal{T}_2 \star \mathcal{W}$.*

2. *$\mathcal{F}_2$ consists of those objects $Y \in \mathcal{A}$ such that $t_1(Y) \in \mathcal{W}$. We will write $\mathcal{F}_2 = \mathcal{W} \star \mathcal{F}_1$.*

Proof. We just prove assertion 1, and assertion 2 will follow by duality. Let us take $X \in \mathcal{T}_2 \star \mathcal{W}$. Since we have an admissible sequence $t_2(X) \longrightarrow X \longrightarrow f_2(X)$ whose outer terms are in $\mathcal{T}_2$ and $\mathcal{W}$, respectively, and these two classes are contained in $\mathcal{T}_1$ we conclude that $\mathcal{T}_1' \subseteq \mathcal{T}_1$, because $\mathcal{T}_1$ is closed under taking extensions in $\mathcal{A}$.

Let T_1 be in $\mathcal{T}_1$ and consider its canonical admissible sequence

$$t_2(T_1) \to T_1 \xrightarrow{f} f_2(T_1). \tag{3}$$

Note that $\underline{f} = 0$ because of condition c) in the statement. It follows that f decomposes in the form $f : T_1 \xrightarrow{\gamma} W \xrightarrow{\phi} f_2(T_1)$, where $W \in \mathcal{W}$. We then consider the following admissible pullback diagram

$$\begin{tikzcd}
t_2(T_1) \arrow[r] \arrow[d, equal] & \widehat{T}_1 \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\
t_2(T_1) \arrow[r] & T_1 \arrow[r, "f"] & f_2(T_1)
\end{tikzcd} \tag{4}$$

Then, there exist a (non necessarely unique) $\eta : T_1 \to \widehat{T}_1$ making the following diagram commute.

$$\begin{tikzcd}
T_1 \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "1"'] & & \\
& \widehat{T}_1 \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\
& T_1 \arrow[r, "f"] & f_2(T_1)
\end{tikzcd} \tag{5}$$

Hence, $T_1 <_{\oplus} \widehat{T}_1 \in \mathcal{T}_2 * \mathcal{W}$. This implies that $\mathcal{T}_1 \subseteq \mathrm{add}(\mathcal{T}_2 * \mathcal{W})$. The proof will be finished once we check that $\mathcal{T}_2 * \mathcal{W}$ is closed under direct summands. But this is a direct consequence of the functoriality of the torsion pair. Indeed if we have admissible torsion sequences $t_2(M) \longrightarrow M \longrightarrow f_2(M)$ and $t_2(N) \longrightarrow N \longrightarrow f_2(N)$, then the coproduct sequence $t_2(M) \oplus t_2(N) \longrightarrow M \oplus N \longrightarrow f_2(M) \oplus f_2(N)$ is the admissible torsion sequence for $M \oplus N$. The fact that $M \oplus N \in \mathcal{T}_2 * \mathcal{W}$ is then equivalent to the fact that $f_2(M) \oplus f_2(N) \in \mathcal{W}$. Since $\mathcal{W}$ is closed under direct summands, we conclude that $f_2(M) \in \mathcal{W}$ and, hence, that $M \in \mathcal{T}_2 * \mathcal{W}$.

□

2 Induced torsion theories

Lemma 3. *Let $\mathcal{A}$ be a (nice) category with a torsion pair $({}^\perp\mathcal{F}, \mathcal{F})$ and a precovering class $\mathcal{W} \subseteq \mathcal{F}$ such that for any $F \in \mathcal{F}$ there is an admissible sequence*

$$F' \to W \to F$$

such that $F' \in \mathcal{F}$.

Then the torsion pair $({}^\perp(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial.

Recall that the truncation $t : \underline{\mathcal{A}} \to {}^\perp(\underline{\mathcal{F}})$ is given by the following construction.

Let $M \in \mathcal{A}$ be any object, take an admissible sequence

$$T_M \to M \to F^M$$

with $T_M \in {}^\perp\mathcal{F}$ and $F^M \in \mathcal{F}$. Moreover, consider $W_F \to F^M$ with $W_M \in \mathcal{W}$ as before, and take the admissible pullback:

$$\begin{tikzcd} t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\ M \arrow[r] & F \end{tikzcd}$$

Then, t restricts to a functor $\underline{t} : \underline{\mathcal{A}} \to {}^\perp\underline{\mathcal{F}}$.

In order to prove that $({}^\perp(\underline{\mathcal{F}}), \underline{\mathcal{F}})$ is left functorial we need to show that $\underline{t}$ admits a right adjoint.

Proof. Let $M \in \mathcal{A}$ and consider $M \to F^M$ and $W_F \to F^M$ as above. For any $T \in {}^\perp\mathcal{F} * \mathcal{W}$ consider any morphism $f : T \to M$. Since $T \to M \to F^M$ is 0 in $\underline{\mathcal{A}}$ we that the solid part of the following diagram commutes.

$$\begin{tikzcd} T \arrow[rrr] \arrow[dr, dotted, "\psi"] \arrow[ddr, bend right, "f"'] & & & W \arrow[dl, dashed, "\phi"'] \arrow[ddl, bend left] \\ & t(M) \arrow[r] \arrow[d] & W_F \arrow[d] & \\ & M \arrow[r] & F^M & \end{tikzcd}$$

∎

Since $W_F \to F^M$ is a precover there is a morphism $\phi : W \to W_F$ making the diagram commute, and since the square is an admissible pullback there is a morphism $\psi : T \to t(M)$ making the diagram commutative.

Hence, $\mathcal{A}(T, t(M)) \to \mathcal{A}(T, M)$ is surjective. To conclude the proof we need to show that when restricted to $\underline{\mathcal{A}}$ it becomes an iso. Assume that there are two morphisms $\underline{\psi}$ and $\underline{\psi}'$ in $\underline{\mathcal{A}}$ such that the following commutes:

$$\begin{tikzcd} & t(M) \arrow[d] \\ T \arrow[ur, shift left, "\underline{\psi}"] \arrow[ur, shift right, "\underline{\psi}'"'] \arrow[r, "\underline{f}"'] & M \end{tikzcd}$$

So, if we call $h = \psi - \psi'$ in $\mathcal{A}$, we have that $T \xrightarrow{h} t(M) \to M$ factors through W so that we have that the solid part of the following diagram commutes:

$$\begin{tikzcd}
W \arrow[ddrr, dotted, bend left, "\gamma"] \arrow[ddrrr, dashed, bend left, "\eta"] \arrow[dddrr, bend right] & & & \\
& T \arrow[ul, "p"'] \arrow[dr, "h"] \arrow[ddr, "g"'] & & \\
& & t(M) \arrow[r] \arrow[d] & W_F \arrow[d] \\
& & M \arrow[r] & F^M
\end{tikzcd}$$

where $\eta : W \to W_F$ comes from the fact that $W_F \to F^M$ is a precover, and γ from the fact that the square is an admissible pullback, and they make the complete diagram commute.

Let's call $h' = \gamma \circ p$, then composing both h and h' with $\rho : t(M) \to M$ gives the same morphism g. Hence, $\rho \circ (h - h') = 0$. But since $F' \xrightarrow{i} t(M) \to M$ is an admissible sequence we have the following exact sequence of abelian groups:

$$\begin{tikzcd}
\mathcal{A}(T, F) \arrow[r] & \mathcal{A}(T, t(M)) \arrow[r] & \mathcal{A}(T, M) \\
& h - h' \arrow[r, mapsto] & 0
\end{tikzcd}$$

So there is a map $k : T \to F'$ such that $i \circ k = h - h'$, but $\underline{k} = 0$ so $\underline{h} = \underline{h'} = 0$, hence $\underline{\psi} - \underline{\psi}' = 0$ which proves that $\underline{\mathcal{A}}(T, t(M)) \cong \underline{\mathcal{A}}(T, M)$.

The naturality of the isomorphism in T and M is clear.