

UNIT-5

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Joint Density Function

“The probability that the point (x, y) will lie in the infinitesimal rectangular region, of area $dxdy$ is given by

$$P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}, y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\right) = f_{XY}(x, y)dxdy$$

where the function $f_{XY}(x, y)$ is called the joint probability density function of X and Y . This function is defined as:

$$f_{XY}(x, y) = \lim_{\delta x \rightarrow 0, \delta y \rightarrow 0} \frac{P(x \leq X \leq x + \delta x, y \leq Y \leq y + \delta y)}{\delta x \delta y}$$

Marginal Density Function

The marginal density function is defined as:

Definition 1

Of X:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

Of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$$

Conditional Probability Density Function

Definition 2

The conditional probability density function of Y given X for two random variables X and Y which are jointly continuously distributed is defined as follows, for two real numbers x and y :

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Similarly, you can define the conditional probability density function of X given Y .

Marginal Distribution Functions

Recall

Definition 3

The distribution function of the two-dimensional r.v. (X, Y) is a real valued function F defined for all real x and y by the relation:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

- We can obtain *individual distribution functions*, $F_X(x)$ and $F_Y(y)$ from Joint Distribution Function $F_{XY}(x, y)$.
- $F_X(x)$ and $F_Y(y)$ are termed as marginal distribution functions of X and Y respectively w.r.t. the joint distribution function $F_{XY}(x, y)$.

Marginal Distribution Function

Marginal distribution function is defined as:

Definition 4

$$F_X(x) = P(X \leq x) = P(X \leq, y < \infty) = \lim_{y \rightarrow \infty} F_{XY}(x, y) = F_{XY}(x, \infty)$$

Similarly,

$$F_Y(y) = P(Y \leq y) = P(X < \infty, Y \leq y) = \lim_{x \rightarrow \infty} F_{XY}(x, y) = F_{XY}(\infty, y)$$

Marginal Distribution Function

- In case of jointly discrete random variables:

Definition 5

$$F_X(x) = \sum_y P(X \leq x, Y = y), \text{ and}$$

$$F_Y(y) = \sum_x P(X = x, Y \leq y)$$

- In case of jointly continuous random variables:

Definition 6

$$F_X(x) = \int_{-\infty}^x \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y) dy \right\} dx,$$

$$F_Y(y) = \int_{-\infty}^y \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y) dx \right\} dy,$$

Conditional Distribution Function

Recall, the joint distribution function is given by:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

- A: Event $X \leq x$. Using conditional probabilities we can write:

$$\begin{aligned} F_{XY}(x, y) &= \int_{-\infty}^y P(A|Y = y) dF_Y(y) \\ &= \int_{-\infty}^y F_{X|Y}(x|y) dF_Y(y) \end{aligned}$$

UNIT-4: Self Study Topics

Self study topics:

- Moments
- Moments Generating Functions
- Characteristics Function

QUIZ-2 will contain at most 50% questions from these topics.

Assignment-2

- Will be shared soon.
- Will contain 2 questions.
- Last Date: 30-November-2025

2D Random Variable

- So far, one random variable on a sample space.

Example 7

Interest in the height and weight of every person in a certain educational institution.

- ⇒ **More than one** random variable on the **same** sample space.

2D Random Variable

Definition 8

Let X and Y be two random variables defined on the same sample space S , then the function (X, Y) that assigns a point in $\mathbb{R}^2 (= R \times R)$, is called a two-dimensional random variable.

- The value of (X, Y) at $\omega \in S$ is defined as:

$$\{X(\omega), Y(\omega)\}$$

- Notation $\{X(\omega), Y(\omega)\}$: the set of all events $\omega \in S$ such that $X(\omega) \leq a$ and $Y(\omega) \leq b$.
- $\{a < X \leq b, c < Y \leq d\} = \{a < X \leq b\} \cap \{c < Y \leq d\} = A \cap B$
 - Probability can be defined in same way.

text

2D Random Variable

Remarks

- Two dimensional discrete variables takes at most a countable number of points in \mathbb{R}^2 .
- Two random variables are said to be **jointly distributed** if they are defined on the same probability space.
- In joint distribution, a sample point is represented by a 2-tuple (e.g. (x, y)).

Joint Probability Mass Function

Definition 9

Let (X, Y) is a two-dimensional discrete random variable, then the joint discrete function of X, Y , also called the joint probability mass function of X, Y , denoted by $p_{X,Y}$ is defined as:

$$p_{XY}(x_i, y_j) = P(X = x_i, Y = y_j) \text{ for a value } (x_i, y_j) \text{ of } (X, Y),$$

and

$$p_{XY}(x_i, y_j) = 0, \text{ otherwise}$$

Remark

The sum of the probabilities of all possible values of (X, Y) denoted as $\sum \sum p_{XY}(x_i, y_j)$ is 1.

Marginal Probability Function of JPMF

Definition 10

Let (X, Y) be a discrete two-dimensional r.v. which takes up countable number of values (x_i, y_i) . Then the probability distribution of X is determined as follows:

$$\begin{aligned} p_X(x_i) &= P(X = x_i) = P(X = x_i \cap Y = y_1) + \dots + P(X = x_i \cap Y = y_m) \\ &= p_{i1} + p_{i2} + \dots + p_{im} = \sum_{j=1}^m p_{ij} = \sum_{j=1}^m p(x_i, y_j) \end{aligned}$$

It is also known as the *marginal probability mass function* of X .

– Similarly, it can be proved for Y .

Conditional Probability Function

Definition 11

Let (X, Y) be a discrete two-dimensional random variable. Then the conditional probability mass function of X , given $Y = y$, denoted by $p_{X|Y}(x|y)$, is defined as:

$$p_{X|Y}(x|y) = \frac{P(X = x | Y = y)}{P(Y = y)} = \frac{P(x, y)}{P(y)}, \text{ provided } P(Y = y) \neq 0$$

Two-Dimensional Distribution Function

Definition 12

The distribution function of the two-dimensional r.v. (X, Y) is a real valued function F defined for all real x and y by the relation:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

Properties:

- ① $0 \leq F(x, y) \leq 1, -\infty < x < \infty, -\infty < y < \infty$
- ② $F(x, y)$ is monotone increasing in both the variables; that is if $x_1 \leq x_2$ and $y_1 \leq y_2$, then $F(x_1, y_1) \leq F(x_2, y_2)$.
- ③ If either x or y approaches $-\infty$, then $F(x, y)$ approaches 0, and if both x and y approach ∞ , then $F(x, y)$ approaches 1.
- ④ $P(a < X \leq b \text{ and } c < Y \leq d) = F(b, d) - F(a, d) - F(b, c) + F(a, c).$
- ⑤ If X and Y are continuous r.v. then $F(x, y)$ is continuous.

References I