UNIT-5 Part-3 Expectation of two Random Variables

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Recap

Expectation

The average value of a random phenomenon is termed as its mathematical expectation.

Definition 1

Expected value of a discrete random variable is a weighted average of all possible values of the random variable, and is given below:

$$E(X) = \sum_{x} x \cdot p(x)$$

Definition 2

For continuous random variable it is:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Expectation Recall Example

Example 3

The number of failures of a computer system in a week of ooperation has the following pmf:

No. of failures	0	1	2	3	4	5	6
Probability	0.18	0.28	0.25	0.18	0.06	0.04	0.01

(a) Find the number of failures in a week.

(b) Find the variance of the number of failures in a week.

Expectaton Recall: Examples

Example 4

Let X be an exponentially distributed random variable with parameter λ. Find its expectation.
An exponentially distributed function with parameter λ is defined as: $\lambda e^{-\lambda x}$
The integration of $e^{-\lambda x}$ is $-\frac{1}{\lambda}e^{\lambda x} + C$ where C is the constant of integration. For this question you can ignore the constant C.

Expected Value of Function of a Random Variable

- Assume X a r.v. with distribution function $F(X)$.
- Let $g(\cdot)$ be a function s.t. $g(X)$ is a r.v., and
- $E[g(X)]$ exists, then

Definition 5

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx \qquad \text{for continuous r.v.}$$

$$E[g(X)] = \sum_{x} g(x)p(x) \qquad \text{for discrete r.v.}$$

Expectation of a Linear Combination of Random Variables

Let $X_1, X_2, \dots X_n$ be any n random variables, and if $a_1, a_2, \dots a_n$ are any n contants, then

$$E\Big(\sum_{i=1}^{n} a_i X_i\Big) = \sum_{i=1}^{n} a_i E(X_i)$$

provided all the expectations exists.

Expectation Based on Multiple Random Variables

Let $X_1, X_2, \ldots, X_n$ be n random variables defined on the same probability space, and let $Y = \phi(X_1, X_2, \ldots, X_n)$. Then,

$$E[Y] = E[\phi(X_1, X_2, \ldots, X_n)]$$
$$= \begin{cases} \sum_{x_1} \sum_{x_2} \ldots \sum_{x_n} \phi(x_1, x_2, \ldots, x_n) p(x_1, x_2, \ldots, x_n) & \text{discrete case} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \phi(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n \\ \text{continuous case} \end{cases}$$

Conditional Expectation

- Let X and Y are continuous r.v.
- $f_{Y|X} :=$ conditional density function
- **Conditional expectation** of Y given $[X = x]$ is denoted by $E[Y|X = x]$ or $E[Y|x]$, and is defined as:

$$
\begin{aligned}
E[Y|x] &= \int_{-\infty}^{\infty} y f(y|x) dy \\
&= \frac{\int_{-\infty}^{\infty} y f(x, y) dy}{f_X(x)}
\end{aligned}
$$

This quantity is also known as the regression function of Y on X.

Conditional Expectation

- Let X and Y are discrete r.v.
- $f_{Y|X} :=$ conditional mass function
- **Conditional expectation** of Y given $[X = x]$ is denoted by $E[Y|X = x]$ or $E[Y|x]$, and is defined as:

$$\begin{aligned} E[Y|x] &= \sum_y yP(Y = y|X = x) \\ &= \sum_y yp_{Y|X}(y|x) \end{aligned}$$

This quantity is also known as the regression function of Y on X.

Conditional Expectation

- Similar definitions can be given in mixed situations.
- Conditional expectation of a function $\phi(Y)$

$$E[\phi(Y)|X=x] = \begin{cases} \int_{-\infty}^{\infty} \phi(y) f_{Y|X}(y|x)dy, & \text{contunuous } Y \\ \sum_i \phi(y_i) p_{Y|X}(y_i|x) & \text{discrete } Y \end{cases}$$

Joint Density Function

"The probabiblity that the point (x, y) will lie in the infinitesimal rectangular region, of area $dxdy$ is given by

$$P(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}, y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}) = f_{XY}(x, y)dxdy$$

where the function $f_{XY}(x, y)$ is called the joint probability density function of X and Y. This function is defined as:

$$f_{XY}(x, y) = \lim_{\delta x \to 0, \delta y \to 0} \frac{P(x \leq X \leq x + \delta x, y \leq Y \leq y + \delta y)}{\delta x \delta y}$$

Marginal Density Function

The marginal density function is defined as:

Definition 6

Of X:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

Of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$$

Conditional Probability Density Function

Definition 7

The conditional probability density function of Y given X for two random variables X and Y which are jointly continuously distributed is defined as follows, for two real numbers x and y:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Similarly, you can define the conditional probability density function of X given Y.

Marginal Distribution Functions

Recall

Definition 8

The distribution function of the two-dimensional r.v. (X, Y) is a real valued functon F defined for all real x and y by the relation:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

– We can obtain *individual distribution functions*, $F_X(x)$ and $F_Y(y)$ from Joint Distribution Function $F_{XY}(x, y)$.

– $F_X(x)$ and $F_Y(y)$ are termed as marginal distribution functions of X and Y respectively w.r.t. the joint distribution function $F_{XY}(x, y)$.

Marginal Distribution Function

Marginal distribution function is defined as:

Definition 9

$F_X(x) = P(X \leq x) = P(X \leq, y < \infty) = \lim_{y \to \infty} F_{XY}(x, y) = F_{XY}(x, \infty)$
Similarly,
$F_Y(y) = P(Y \leq y) = P(X < \infty, Y \leq y) = \lim_{x \to \infty} F_{XY}(x, y) =$
$F_{XY}(\infty, y)$

Marginal Distribution Function

– In case of jointly discrete random variables:

Definition 10

$F_X(x) = \sum_y P(X \leq x, Y = y)$, and

$F_Y(y) = \sum_x P(X = x, Y \leq y)$

– In case of jointly continuous random variables:

Definition 11

$F_X(x) = \int_{-\infty}^{x} \{\int_{-\infty}^{\infty} f_{XY}(x, y) dy\} dx$,

$F_Y(y) = \int_{-\infty}^{y} \{\int_{-\infty}^{\infty} f_{XY}(x, y) dx\} dy$,

Conditional Distribution Function

Recall, the joint distribution function is given by:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

- A: Event $X \leq x$. Using conditional probabilities we can write:

$$F_{XY}(x, y) = \int_{-\infty}^{x} P(A|Y = y) dF_Y(y)$$

Conditional distribution function $F_{X|Y}(x|y)$

$$F_{X|Y}(x|y) = P(X \leq x|Y = y) = P(A|Y = y)$$

$$= \int_{-\infty}^{x} F_{X|Y}(x|y) dF_Y(y)$$

Conditional Distribution Function

Definition 12

Conditional Distribution function is defined as:

$$F_{X|Y}(x|y) = P(X \le x|Y = y) = \frac{\int_{-\infty}^{x} f(x,y)dx}{f_Y(y)}$$
$$= \int_{-\infty}^{x} f_{X|Y}(x|y)dx$$

Definition 13

Conditional Distribution function is defined as:

$$F_{X|Y}(x|y) = P(X \le x|Y = y) = \frac{P(X \le x \text{ and } Y = y)}{P(Y = y)}$$
$$= \frac{\sum_{x} p(x,y)}{p_Y(y)} = \sum_{x} p_{X|Y}(x|y)$$

UNIT-4: Self Study Topics

Self study topics:

- Moments
- Moments Generating Functions
- Characteristics Function

QUIZ-2 will contain at most 50% questions from these topics.

Assignment-2

– Will be shared soon.
– Will contain 2 questions.
– Last Date: 30-November-2025

2D Random Variable

- So far, one random variable on a sample space.

Example 14

Interest in the height and weight of every person in a certain educational institution.

- $\Longrightarrow$ **More than one** random variable on the **same** sample space.

2D Random Variable

Definition 15

Let X and Y be two random variables defined on the same sample space S, then then function (X, Y) that assigns a point in $\mathbb{R}^2 (= R \times R)$, is called a two-dimensional random variable.

- The value of (X, Y) at $\omega \in S$ is defined as:

$$\{X(\omega), Y(\omega)\}$$

- Notation $\{X(\omega), Y(\omega)\}$: the set of all events $\omega \in S$ such that $X(\omega) \leq a$ and $Y(\omega) \leq b$.
- $\{a < X \leq b, c < Y \leq d\} = \{a < X \leq b\} \cap \{c < Y \leq d\} = A \cap B$

– Probability can be defined in same way.
text

Remarks

- Two dimentional discrete variables takes at most a countable number of points in $\mathbb{R}^2$.
- Two random variables are said to be **jointly distributed** if they are defined on the *same probability space*.
- In joint distribution, a sample point is represented by a $2-$tuple (e.g. (x, y)).

Joint Probability Mass Function

Definition 16

Let (X, Y) is a two-dimensional discrete random variable, then the joint discrete function of X, Y, also called the joint probability mass function of X, Y, denoted by $p_{X,Y}$ is defined as:

$$p_{XY}(x_i, y_j) = P(X = x_i, Y = y_j) \text{ for a value } (x_i, y_j) \text{ of } (X, Y),$$

and

$$p_{XY}(x_i, y_j) = 0, \text{ otherwise}$$

Remark

The sum of the probablities of all possible values of (X, Y) denoted as $\sum \sum p_{XY}(x_i, y_j)$ is 1.

Marginal Probability Function of JPMF

Definition 17

Let (X, Y) be a discrete two-dimensional r.v. which takes up countable number of values (x_i, y_i). Then the probability distribution of X is determined as follows:

$$
\begin{aligned}
p_X(x_i) = P(X = x_i) &= P(X = x_i \cap Y = y_1) + \ldots + P(X = x_i \cap Y = y_m) \\
&= p_{i1} + p_{i2} + \ldots + p_{im} = \sum_{j=1}^{m} p_{ij} = \sum_{j=1}^{m} p(x_i, y_j)
\end{aligned}
$$

It is also known as the *marginal probability mass function* of X.
– Similarly, it can be proved for Y.

Conditional Probability Function

Definition 18

Let (X, Y) be a discrete two-dimensional random variable. Then the conditional probability mass function of X, given $Y = y$, denoted by $p_{X|Y}(x|y)$, is defined as:

$$p_{X|Y}(x|y) = \frac{P(X = x|Y = y)}{P(Y = y)} = \frac{P(x, y)}{P(y)}, \text{ provided } P(Y = y) \neq 0$$

Two-Dimensional Distribution Function

Definition 19

The distribution function of the two-dimensional r.v. (X, Y) is a real valued functon F defined for all real x and y by the relation:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

Properties:

1. $0 \leq F(x, y) \leq 1, -\infty < x < \infty, -\infty < y < \infty$
2. $F(x, y)$ is monotone increasing in both the variables; that is if $x_1 \leq x_2$ and $y_1 \leq y_2$, then $F(x_1, y_1) \leq F(x_2, y_2)$.
3. If either x or y approaches $-\infty$, then $F(x, y)$ approaches 0, and if both x and y approach ∞, then $F(x, y)$ approaches 1.
4. $P(a < X \leq b \text{ and } c < Y \leq d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$.
5. If X and Y are continuous r.v. then $F(x, y)$ is continuous.

References I