$z = a+bi$, $\bar{z} = a-bi$ $\overline{(\bar{z})} = a+bi = z$

A5. $z, w \in \mathbb{C}$

$\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$, $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ . . .

If $z = \bar{z}$, z is a real number, ~~that is,~~ $Im(z) = 0$

In order to understand Sec 5.3

Sec 5.2

Let F be a fied

$F^n = F \times \cdots \times F = \{ (x_1 \ldots x_n) \mid x_i \in F, \ 1 \le i \le n \}$

F^n is a vector space over F with the operations

vector addition: $\forall\, x, y \in F^n$, $x = (x_1, \ldots x_n)$, $y = (y_1 \ldots y_n)$

$x + y = (x_1 + y_1, \ \ldots \ x_n + y_n)$

vector addition field addition field multiplication.

Scalar multiplication: $\forall\, c \in F$, $cx = (cx_1, cx_2, \ldots, cx_n)$

$x \in F^n$,

scalar multiplication

That is, F^n with the operations satisfies the 8 conditions.

Ex1 $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C} = \{ (x_1, x_2) \mid x_1, x_2 \in \mathbb{C} \}$

is a vector space over $\mathbb{C}$. with the operations above

scalar

For example, $x_1 = (1+i, i)$ and $x_2 = (i, -1)$ $c = i \in \mathbb{C}$

$x_1 + x_2 = ((1+i) + i, \ i + (-1)) = (1+2i, -1+i)$

vector addition complex addition

$c x_1 = i(1+i, i) = (i(1+i), i(i))$

scalar multiplication

$= (i + i^2, \ i^2)$

$= (-1+i, -1)$

$\forall A \in M_{n\times m}(\mathbb{C})$ $A : \mathbb{C}^m \to \mathbb{C}^n$

Ex2 $A = \begin{bmatrix} -i & 2+3i \\ 1 & 1-i \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$

Is it invertible? If it is invertible, find A^{-1}.

Sol $\det(A) = (-i)(1-i) - (1)(2+3i) = (-1-i) - (2+3i) = -3-4i$

$\ne 0$

A is invertible.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1-i & -(2+3i) \\ -1 & -i \end{bmatrix}$$

$\det(A) = -3-4i$

$$\frac{1}{-3-4i} = \frac{-3+4i}{(-3-4i)(-3+4i)} = \frac{-3+4i}{9+16} = \frac{1}{25}(-3+4i)$$

$$= \begin{bmatrix} \frac{1}{25}(1-i)(-3+4i) & -\frac{1}{25}(2+3i)(-3+4i) \\ \frac{-1}{25}(-3+4i) & -\frac{1}{25}i(-3+4i) \end{bmatrix}$$

$0+0i$

Ex3 Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$ Is A diagonalizable?

Sol $\det(A - \lambda I) = \lambda^2 + 1 \overset{\text{let}}{=} 0$

$\lambda = \pm i \to$ different eigenvalues of 2×2 matrix $\Rightarrow A$ is diagonalizable.

For $\lambda = i$, $A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow[R_2 \to (i)R_1 + R_2]{} \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \to$ REF

$ix_1 + x_2 = 0$

Let $x_2 = t$. $x_1 = -\frac{t}{i} = -\frac{it}{ii} = it$

$(x_1, x_2) = (it, t) = t(i, 1)$

$E_{\lambda=i} = \text{span}\{(i,1)\}$ $\dim(E_{\lambda=i}) = 1$

For $\lambda = -i$ HW : $E_{\lambda=-i} = \text{span}\{(-i, 1)\}$

Let $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$P^{-1}AP = D \leftarrow$ true ✓

Sec5.3 Geometry in a complex vector spaces — vector space over $\mathbb{C}$.

Def A Hermitian inner product on V: $\langle\ ,\ \rangle$ is a function from $V \times V$ to $\mathbb{C}$ satisfying the following properties

① $\forall\, u, v, w \in V$, $\forall\, a, b \in \mathbb{C}$,

$$\langle au + bv, w \rangle = a\langle u, w\rangle + b\langle v, w\rangle$$

② $\langle u, v \rangle = \overline{\langle v, u\rangle}$

③ $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0 \Leftrightarrow v = 0$

Remark1 $\langle au, v\rangle = a\langle u, v\rangle$ by ①

$$\langle u, av\rangle = \bar{a}\,\langle u, v\rangle$$

$$\overline{zw} = \bar{z}\,\bar{w} \qquad \overline{(\bar{z})} = z$$

Pf

$$\boxed{\langle u, av\rangle} \underset{\text{by } (2)}{=} \boxed{\overline{\langle av, u\rangle}} \underset{(1)}{=} \overline{a\langle v, u\rangle} = \bar{a}\,\overline{\langle v, u\rangle}$$

$$= \bar{a}\,\overline{\overline{\langle u, v\rangle}}$$

$$= \bar{a}\,\langle u, v\rangle$$