Norm and Dot Product Handout

1. Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.
   (a) Calculate $||\vec{u}||, ||\vec{v}||$, and $||\vec{w}||$.
   (b) What is the distance between $\vec{u}$ and $\vec{v}$?
   (c) What is the angle between $\vec{u}$ and $\vec{w}$?
   (d) Calculate $\vec{u} \cdot (\vec{v} + \vec{w})$.
   (e) Suppose $\vec{x} \in \mathbb{R}^3$ such that $\vec{x}$ is orthogonal to $\vec{u}$ and $\vec{w}$ and $\vec{x} \cdot \vec{v} = 3$. Calculate $(2\vec{u} - 4\vec{v} + \vec{w}) \cdot \vec{x}$.

2. Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ such that $\vec{u}$ and $\vec{v}$ are orthogonal to each other, $||\vec{v}|| = 4$ and $\vec{v} \cdot \vec{w} = 8$. Find a scalar t such that $\vec{u} = \vec{v} + t\vec{w}$.

3. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be unit vectors such that $\vec{u} \cdot \vec{v} = -1$. What is the distance between $\vec{u}$ and $\vec{v}$?

4. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be orthogonal to each other, with $||\vec{u}|| = 8$, and $||\vec{v}|| = 3$. What is the distance between $\vec{u}$ and $\vec{v}$?

5. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $||\vec{u}|| = 10$, $||\vec{v}|| = 5$, and $||\vec{u} + \vec{v}|| = 13$. What is the angle between $\vec{u}$ and $\vec{v}$?

6. Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ such that $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $||\vec{v}|| = 2$, and the distance between $\vec{u}$ and $\vec{v}$ is 3. What is the angle between $\vec{u}$ and $\vec{v}$?

Solutions

1. (a)

$$||\vec{u}|| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$
$$||\vec{v}|| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$
$$||\vec{w}|| = \sqrt{5^2 + (-3)^2 + 1^2} = \sqrt{35}.$$

(b) Note that $\vec{u} - \vec{v} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$, and so $||\vec{u} - \vec{v}|| = \sqrt{(-1)^2 + (-2)^2 + (-3)^2} = \sqrt{14}$.

(c) We will use both formulas for the dot product to find the angle between the two vectors. Now

$$\vec{u} \cdot \vec{w} = 1 \cdot 5 + (-1) \cdot (-3) + 0 \cdot 1 = 8.$$

And using the geometric formula, we also have that

$$\vec{u} \cdot \vec{w} = ||\vec{u}|| \cdot ||\vec{w}|| \cdot \cos\theta = \sqrt{2} \cdot \sqrt{35} \cdot \cos\theta,$$

where θ is the angle between the two vectors. Substituting the value we've got for the dot product gives us

$$\sqrt{2} \cdot \sqrt{35} \cos\theta = 8,$$

and so $\theta = \cos^{-1}(\frac{8}{\sqrt{70}}) = 17.0°$

(d) Note that $\vec{v} + \vec{w} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix}$, and so

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix} = 1 \cdot 7 + (-1) \cdot (-2) + 0 \cdot 4 = 9.$$

(e) Since $\vec{x}$ is orthogonal to $\vec{u}$ and $\vec{w}$, we have $\vec{u} \cdot \vec{x} = \vec{w} \cdot \vec{x} = 0$. Then

$$\begin{aligned}(2\vec{u} - 4\vec{v} + \vec{w}) \cdot \vec{x} &= 2\vec{u} \cdot \vec{x} - 4\vec{v} \cdot \vec{x} + \vec{w} \cdot \vec{x} \\ &= 2 \cdot 0 - 4 \cdot 3 + 0 \\ &= -12.\end{aligned}$$

2. Since $\vec{u}$ and $\vec{v}$ are orthogonal to each other, we have $\vec{u} \cdot \vec{v} = 0$. We also have that $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = 4$, so $\vec{v} \cdot \vec{v} = 4^2 = 16$, and we're given $\vec{w} \cdot \vec{v} = 8$. We can take the equation $\vec{u} = \vec{v} + t\vec{w}$ and take the dot product of both sides with $\vec{v}$. We get

$$\begin{aligned}\vec{u} \cdot \vec{v} &= (\vec{v} + t\vec{w}) \cdot \vec{v} \\ &= \vec{v} \cdot \vec{v} + t\vec{w} \cdot \vec{v}\end{aligned}$$

Substituting the values we have into this equation gives us

$$0 = 16 + 8t.$$

And so we must have $t = -2$.

3. Since $\vec{u}$ and $\vec{v}$ are unit vectors, then $||\vec{u}|| = \sqrt{\vec{u}\cdot\vec{u}} = 1$. Similarly, $||\vec{v}|| = \sqrt{\vec{v}\cdot\vec{v}} = 1$. Now the distance between $\vec{u}$ and $\vec{v}$ is given by $||\vec{u}-\vec{v}||$. Using the relation between the dot product and the length of a vector, and the properties of the dot product, we have

$$\begin{aligned}||\vec{u}-\vec{v}||^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})\\ &= \vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v}\\ &= ||\vec{u}||^2-2\vec{u}\cdot\vec{v}+||\vec{v}^2||\\ &= 1^2-2\cdot(-1)+1^2\\ &= 4.\end{aligned}$$

Then the distance between $\vec{u}$ and $\vec{v}$ is $||\vec{u}-\vec{v}|| = \sqrt{4} = 2$.

4. Following the same steps as we did in the previous problem, and noting that $\vec{u}\cdot\vec{v} = 0$, we get

$$\begin{aligned}||\vec{u}-\vec{v}||^2 &= \vec{u}\cdot\vec{u}-2\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v}\\ &= ||\vec{u}||^2-2\vec{u}\cdot\vec{v}+||\vec{v}||^2\\ &= 8^2-2\cdot0+3^2\\ &= 73.\end{aligned}$$

So the distance between $\vec{u}$ and $\vec{v}$ is $||\vec{u}-\vec{v}|| = \sqrt{73}$.

5. To find the angle between the two vectors, we need to use the geometric formula for the dot product $\vec{u}\cdot\vec{v} = ||\vec{u}||\cdot||\vec{v}||\cdot\cos\theta$. The unknowns here are the dot product $\vec{u}\cdot\vec{v}$ and the angle θ. We can find the dot product from the information given if we apply some algebra. Using the relation between the dot product and the length of a vector, and the properties of the dot product, we have

$$\begin{aligned}||\vec{u}+\vec{v}||^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})\\ &= \vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v}\\ &= ||\vec{u}||^2+2\vec{u}\cdot\vec{v}+||\vec{v}^2||.\end{aligned}$$

Substituting the values that we have gives us $13^2 = 10^2+2\vec{u}\cdot\vec{v}+5^2$. Solving for the dot product gives us $\vec{u}\cdot\vec{v} = 22$. We can now use the geometric formula for the dot product to find θ, the angle between the two vectors,

$$\begin{aligned}\theta &= \cos^{-1}\left(\tfrac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}\right)\\ &= \cos^{-1}\left(\tfrac{22}{10\cdot5}\right)\\ &= 63.9^\circ.\end{aligned}$$

6. As in the previous problem, we need to use the geometric formula for the dot product to find the angle between the two vectors. The unknowns in the formula $\vec{u}\cdot\vec{v} = ||\vec{u}||\cdot||\vec{v}||\cdot\cos\theta$ are the dot product $\vec{u}\cdot\vec{v}$, $||\vec{u}||$ and the angle θ. We can directly calculate $||\vec{u}|| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$. We are also given the distance between $\vec{u}$ and $\vec{v}$, which is $||\vec{u}-\vec{}v||$. We can use this information to find the dot product.

$$\begin{aligned}||\vec{u}-\vec{v}||^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})\\ &= \vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v}\\ &= ||\vec{u}||^2-2\vec{u}\cdot\vec{v}+||\vec{v}^2||.\end{aligned}$$

Substituting the values that we have gives us $3^2 = (\sqrt{14})^2 - 2\vec{u}\cdot\vec{v} + 2^2$. Solving for the dot product gives us $\vec{u}\cdot\vec{v} = 4.5$. We can now use the geometric formula for the dot product to find θ, the angle between the two vectors,

$$\begin{aligned}\theta &= \cos^{-1}\left(\tfrac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}\right)\\ &= \cos^{-1}\left(\tfrac{4.5}{(\sqrt{14})(2)}\right)\\ &= 53.0^\circ.\end{aligned}$$