Subspaces Handout

1. Use the definition of a subspace to prove that for any nonzero vector $\vec{v} \in \mathbb{R}^n$, the set $S = \{\vec{u} \in \mathbb{R}^n \mid \vec{u} \cdot \vec{v} = 0\}$ of all vectors orthogonal to $\vec{v}$ is a subspace of $\mathbb{R}^n$.

2. Let $S = \left\{ \begin{bmatrix} x \\ 3x \end{bmatrix} \mid x \in \mathbb{R} \right\}$.

   (a) Show that S is a subspace of $\mathbb{R}^2$ using the definition of a subspace.

   (b) Express S as a span and prove that it is a subspace using the Subspace-Span Theorem.

3. For the following subsets, determine whether they are subspaces or not. If a subset is a subspace, find a basis for it and determine its dimension.

   (a) $A = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x + 2y = 3z \right\}$.

   (b) $B = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid 3x = 2y \right\}$.

   (c) $C = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x, y, z \geq 0 \right\}$.

   (d) $D = \left\{ \begin{bmatrix} 0 \\ a \\ a+1 \end{bmatrix} \mid a \in \mathbb{R} \right\}$.

   (e) $E = \left\{ a \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + b \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix} \mid a, b \in \mathbb{R} | \right\}$.

   (f) $F = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x^2 = z^2 \right\}$.

   (g) G is the set of all vectors in $\mathbb{R}^3$ with third component equal to -1.

   (h) H is the set of all vectors in $\mathbb{R}^2$ with rational coefficients.

   (i) $J = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

   (j) $K = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \\ 2 \end{bmatrix} \right\}$.

Solutions

1. First, we check that S is non-empty. Since the zero vector is orthogonal to any vector, we must have $\vec{0}\cdot\vec{v}=0$, and so $\vec{0}\in S$ and S is non-empty.

   Next, we check if S is closed under addition. Let $\vec{u},\vec{w}\in S$, then we know that $\vec{u}\cdot\vec{v}=0$ and $\vec{w}\cdot\vec{v}=0$. But then

$$\begin{aligned}(\vec{u}+\vec{v})\cdot\vec{v}&=\vec{u}\cdot\vec{v}+\vec{w}\cdot\vec{v}\\&=0+0\\&=0.\end{aligned}$$

   And so, $\vec{u}+\vec{w}\in S$ and S is closed under addition.

   Lastly, we check that S is closed under scalar multiplication. Let $\vec{u}\in S$, and let $k\in\mathbb{R}$ be a scalar. Then we know that $\vec{u}\cdot\vec{v}=0$, and so $(k\vec{u})\cdot\vec{v}=k(\vec{u}\cdot\vec{v})=k\cdot 0=0$. But this means that $k\vec{u}\in S$ and S is closed under scalar multiplication. This completes the proof that S is a subspace of $\mathbb{R}^n$.

2. (a) Note that for $x=1$, we have that $\begin{bmatrix}1\\3\end{bmatrix}\in S$, so S is non-empty.

   Next, we check if S is closed under addition. Let $\vec{u}=\begin{bmatrix}u\\3u\end{bmatrix}, \vec{w}=\begin{bmatrix}w\\3w\end{bmatrix}\in S$, where $u,w\in\mathbb{R}$. Then

$$\begin{aligned}\vec{u}+\vec{w}&=\begin{bmatrix}u\\3u\end{bmatrix}+\begin{bmatrix}w\\3w\end{bmatrix}\\&=\begin{bmatrix}u+w\\3u+3w\end{bmatrix}\\&=\begin{bmatrix}u+w\\3(u+w)\end{bmatrix}.\end{aligned}$$

   And so, $\vec{u}+\vec{w}\in S$ and S is closed under addition.

   Lastly, we check that S is closed under scalar multiplication. Let $\vec{u}=\begin{bmatrix}u\\3u\end{bmatrix}\in S$, and let $k\in\mathbb{R}$ be a scalar. Then $(k\vec{u})=k\begin{bmatrix}u\\3u\end{bmatrix}=\begin{bmatrix}ku\\3ku\end{bmatrix}$. But this means that $k\vec{u}\in S$ and S is closed under scalar multiplication. This completes the proof that S is a subspace of $\mathbb{R}^2$.

   (b) Notice that $S=\left\{x\begin{bmatrix}1\\3\end{bmatrix} \mid x\in\mathbb{R}\right\}=\operatorname{span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$. Since S can be expressed as a span, then by the Subspace-Span Theorem, S must be a subspace of $\mathbb{R}^2$.

3. Note that there are many correct counterexamples that show that a subset is not a subspace, the ones shown here are just a sample. You may be able to come up with a counterexample for a different property of subspaces than the one that was demonstrated here, and that is okay. Also note that you may prove that a subset is a subspace directly using the definition of a subspace as was done in the first problem, or you could use the Subspace-Span Theorem as done in the solutions below.

   (a) A is a plane with Cartesian equation $x+2y-3z=0$. Since A is a plane which passes through the origin, we know we can express it as a span. Then by the Subspace-Span Theorem, A is a subspace of $\mathbb{R}^3$. The dimension of A is 2 since it is a plane. The direction vectors of A will give us a basis for the plane (think of the vector form, and how we can express A as a span of two vectors). We know that $\vec{0}$ is on the plane, and if we take $y = 1, z = 0$, then $x = -2$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is a point on the plane. If we take $y = 0, z = 1$, then $x = 3$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is another point on the plane. We get two direction vectors for the plane that are linearly independent (they are not scalar multiples of each other), and a basis for A is the set $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b) If we let $y = 3s$, where $s \in \mathbb{R}$, then $x = \frac{2}{3}y = 2s$. If we also take $z = t$ for $t \in \mathbb{R}$, we can write

$$\begin{aligned} B &= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid 3x = 2y \right\} \\ &= \left\{ s \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} | s, t \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

Since we can express B as a span of two linearly independent vectors, then by the Subspace-Span theorem, B is a subspace of $\mathbb{R}^3$ of dimension 2. The two vectors that we found to span the set are linearly independent and therefore form a basis for B. In particular, $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for B.

(c) C is not a subspace of $\mathbb{R}^3$. To prove this, we need to find one counter-example to show that C does not satisfy one of the conditions of being a subspace. Consider the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in C$, if we multiply this vector by the scalar -1, we get $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$, which is not a vector in C. This shows that C is not closed under scalar multiplication and is therefore not a subspace of $\mathbb{R}^3$.

(d) D is not a subspace of $\mathbb{R}^3$. Note that the vector $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in D$, but if we multiply it by the scalar 2, we get $\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$, which is not in D. This shows that D is not closed under scalar multiplication and is therefore not a subspace of $\mathbb{R}^3$.

(e) We can write $E = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix} \right\}$. Therefore, E is a subspace of $\mathbb{R}^3$ by the Subspace-Span Theorem. Since the two vectors spanning the set are linearly independent (they are not scalar multiples of each other), those vectors form a basis for E. In particular, $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix} \right\}$ is a basis for E and the dimension of E is 2.

(f) F is not a subspace of $\mathbb{R}^3$. Note that the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in F$, but $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is not in F. This shows that F is not closed under addition and is therefore not a subspace of $\mathbb{R}^3$.

(g) G is not a subspace of $\mathbb{R}^3$. Note that the vector $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \in G$, but if we multiply it by the scalar 2, we get $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$, which is not in G. This shows that G is not closed under scalar multiplication and is therefore not a subspace of $\mathbb{R}^3$.

(h) H is not a subspace of $\mathbb{R}^2$. Note that the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H$, but if we multiply it by the scalar π, we get $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$, which is not in H, since π is irrational. This shows that H is not closed under scalar multiplication and is therefore not a subspace of $\mathbb{R}^2$.

(i) Since J is expressed as a span, the Subspace-Span Theorem tells us that it must be a subspace. We need to find a basis for it. In order to do this, we need to find the maximum number of linearly independent vectors in the spanning set. Consider the equation

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We will use this equation to find the linearly independent vectors. Let's convert the vector equation into an augmented matrix. We get

$$\begin{bmatrix} 1 & -1 & 2 & 1 & : & 0 \\ 2 & -2 & 6 & 1 & : & 0 \\ 1 & -1 & -2 & 3 & : & 0 \end{bmatrix}.$$

This matrix has RREF

$$\begin{bmatrix} 1 & -1 & 0 & 2 & : & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}.$$

Note that we have pivots in the first and third columns, this means that the first and third columns are linearly independent. To see this, think about the equation

$$b_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If we convert it to an augmented matrix we get the same matrix as before but with only the first and third columns on the left. Namely, we get

$$\begin{bmatrix} 1 & 2 & : & 0 \\ 2 & 6 & : & 0 \\ 1 & -2 & : & 0 \end{bmatrix}.$$

Row reducing this smaller matrix gives us the RREF we reached before but with only the first and third columns on the left, namely we get the following matrix

$$\begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}.$$

This means that the equation $b_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ will have only the trivial solution, with $b_1 = b_2 = 0$, meaning that these two vectors are in fact linearly independent. This means that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} \right\}$ is a basis for J and so J has dimension 2.

(j) Since K is expressed as a span, the Subspace-Span Theorem tells us that it must be a subspace. We need to find a basis for it. In order to do this, we need to find the maximum number of linearly independent vectors in the spanning set. Consider the equation

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -1 \\ 5 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} + c_5 \begin{bmatrix} 2 \\ 7 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We will use this equation to find the linearly independent vectors. Let's convert the vector equation into an augmented matrix. We get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & : & 0 \\ 2 & 3 & 5 & 1 & 7 & : & 0 \\ 2 & 1 & -1 & 4 & 0 & : & 0 \\ -1 & 1 & 5 & -1 & 2 & : 0 & \end{bmatrix}.$$

This matrix has RREF

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 & : & 0 \\ 0 & 1 & 3 & 0 & 2 & : & 0 \\ 0 & 0 & 0 & 1 & -1 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : 0 & \end{bmatrix}.$$

Note that we have pivots in the first, second and fourth columns, this means that the first, second and fourth columns are linearly independent. Therefore, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\}$ is a basis for K and so K has dimension 3.