Sec 2.1

#3 Determine whether the following functions $T: V \to W$ defines linear transformation
(c)(f)(g)

strategy: If T is linear, $T(0_V) = 0_W$. Therefore, if $T(0_V) \neq 0_W$, T can't be linear. What if $T(0_V) = 0_W$? Then check whether T satisfies 2 conditions: $T(v+u) = T(v) + T(u)$ and $T(\alpha u) = \alpha T(u)$.

Caution

(g) Define $T: \mathbb{R} \to \mathbb{R}^+$ by $T(x) = e^x$.

Notice that the vector addition and scalar multiplication of $\mathbb{R}^+$ are different: $\forall x, y \in \mathbb{R}^+$ (a set of positive real numbers)

vector addition $x \underset{\smile}{+} y = xy$

Scalar multiplication $c \cdot x = x^c$

($+'$ = notation for vector addition in the book

It is okay to use just $+$

For example, if you want to check whether T satisfies the first condition, you should show that

$$T(x+y) \overset{?}{=} T(x) +' T(y)$$

the vector addition of $\mathbb{R}$

the vector addition of $\mathbb{R}^+$

Notice that the additive identity of $\mathbb{R}^+$ is 1

#5 Check the 2 conditions of linear mapping

(b) $\mathrm{Int}(f) = \int_a^b f(x)\,dx$

↳ the definite integral is linear?

a function

a real number

You can use T instead of D. $T(f) = \int_a^b f(x)\,dx$

So $T(f+g) = \int_a^b (f+g)\,dx$

and $T(cf) = \int_a^b (cf)\,dx$

#11 $T: \mathbb{R}^2 \to P_2(\mathbb{R})$ linear

with $T(1,1) = x + x^2$ and $T(3,0) = x - x^3$ what is $T(2,2)$?

Hint Write $(2,2)$ as a linear combination of $(1,1)$ and $(3,0)$ and use the linearity of T to find $T(2,2)$.

Sec 2.2

#3 (a)(b)(c) Let $T: V \to W$

You should find $[T]_\alpha^\beta$ where α is a standard basis of V and β is a standard basis of W.

Say $\alpha = \{v_1 \cdots v_n\}$ and $\beta = \{w_1 \cdots w_m\}$

$$[T]_\alpha^\beta = \left[\, [T(v_1)]_\beta \;\cdots\; [T(v_n)]_\beta \,\right]$$

$[T(v_n)]_\beta$: the coordinates of $T(w_n)$ w.r.t β

$[T(v_1)]_\beta$: the coordinates of $T(v_1)$ w.r.t β

For example, $T(v_1) = t_1 w_1 + t_2 w_2 + \cdots + t_m w_m$

Then $[T(v_1)]_\beta = (t_1, t_2 \ldots t_m)$

(a) $\alpha = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a standard basis of $\mathbb{R}^3$

$\beta = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ is a standard basis of $\mathbb{R}^4$. Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$ and $T(\vec{e}_3)$

Then find $[T(\vec{e}_1)]_\beta \cdots [T(\vec{e}_4)]_\beta$.

(b) $\alpha = \{\vec{e}_1 \ldots \vec{e}_n\}$ is a standard basis of $\mathbb{R}^n$.

$\beta = \{1\}$ is a standard basis of $\mathbb{R}$. Similar to (a)

(c) $D : \operatorname{span}\{\sin x, \cos x\} \longrightarrow \operatorname{span}\{\sin x, \cos x\}$

$D(f) = f'(x)$

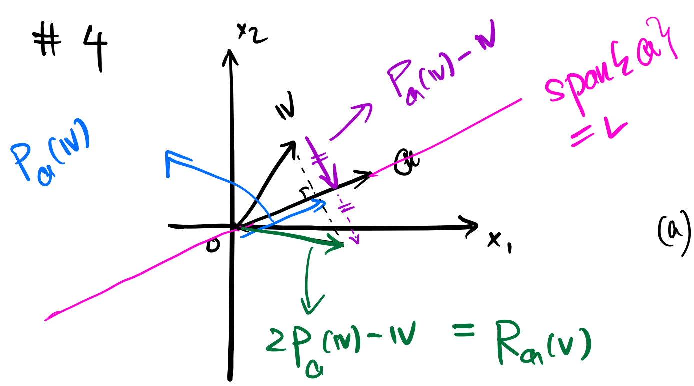
↑ linear function

$\alpha = \{\sin x, \cos x\}$ is a standard basis of $\operatorname{span}\{\sin x, \cos x\}$

$\beta = \alpha$. Compute $D(\sin x)$ and $D(\cos x)$

Then find $[D(\sin x)]_\beta$ and $[D(\cos x)]_\beta$

#4



Show that for any v_1 and $v_2 \in \mathbb{R}^2$,

(a) $R_a(v_1+v_2) = 2P_a(v_1+v_2) - (v_1+v_2)$

$= R_a(v_1) + R_a(v_2)$

$R_a(cv) = 2P_a(cv) - (cv)$

$= c\,R_a(v)$

(b) See #3.

#8 (a) $T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(b) Find the matrix of $P_{(1,2)}$ (see #3)

Sec 2.3

#1 Strategy: ① Find $[T]_\alpha^\beta$ where α and β are bases of V and W respectively.

② Find a REF of $[T]_\alpha^\beta$.

③ Find a basis of $\ker(T)$ and a basis of $\mathrm{Im}(T)$

④ Say $\{T(v_1), \dots T(v_k)\}$ is a basis of $\mathrm{Im}(T)$, then a basis of V is the union of a basis of $\ker(T)$ and $\{v_1, \dots, v_k\}$.

#3 (b) follow the strategy of #1 Solve #3(b) and #1

#8 Hint: $\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = \dim(V)$

(b) Consider bases of $\ker(T)$ and $\mathrm{Im}(T)$.

For example, $V = \mathbb{R}^2$ and $[T]_\alpha^\alpha = [T(\vec{e}_1)\ T(\vec{e}_2)] \sim$ a REF

Say a REF of $[T]_\alpha^\alpha$ = $\begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \\ \uparrow \text{ basic variable} & \uparrow \text{ free variable} \end{array}$

Find bases of $\ker(T)$ and $\mathrm{Im}(T)$, and think about under what condition $\ker(T) = \mathrm{Im}(T)$.

(a) Apply the idea of (b) to (a) when the $\dim(V)$ is even.

Sec 2.4

#(1) (a)(c)(d) Use the definition of injectivity or a theorem such as

T is injective $\Leftrightarrow$ $\ker(T) = \{0\}$

or if $T: V \to W$ is injective, $\dim(V) \leq \dim(W)$

Likewise you can use the definition of surjectivity or theorems to check whether the given linear mapping is surjective.

#5 the same strategy as #(1)

#9 Hint: $T|_U : U \to W$ is defined by $T|_U(x) = T(x)$ for any $x \in U$.

and $\ker(T|_U) = \{x \in U \mid T|_U(x) = 0_W\}$

Show that $\ker(T|_U) = \{0_V\}$

#11 (a) Check the two conditions $T(\mathbb{v}+\mathbb{w}) = T(\mathbb{v}) + T(\mathbb{w})$ for any $\mathbb{v}, \mathbb{w} \in V$

$T(c\mathbb{v}) = c\,T(\mathbb{v})$ for any $\mathbb{v} \in V$ and $c \in \mathbb{R}$

using T_1 and T_2 are $\underbrace{\text{linear transformations}}$.

so, satisfying the 2 conditions

(b) and (c) Like #1, 5, use the definitions or theorems.

For example, Suppose $T(\mathbb{v}) = T(\mathbb{w})$ for any $\mathbb{v}, \mathbb{w} \in V$

Then $T_1(\mathbb{v}_1) + T_2(\mathbb{v}_2) = T_1(\mathbb{w}_1) + T_2(\mathbb{w}_2)$ where

$\mathbb{v} = \mathbb{v}_1 + \mathbb{v}_2$, $\mathbb{w} = \mathbb{w}_1 + \mathbb{w}_2$, and

$\mathbb{v}_1, \mathbb{w}_1 \in U_1$ and $\mathbb{v}_2, \mathbb{w}_2 \in U_2$

...