Projections and Vector Components Handout

1. Let C be the unit circle in $\mathbb{R}^2$. Find the following projections:

   (a) $\mathrm{proj}_C \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

   (b) $\mathrm{proj}_C \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

   (c) $\mathrm{proj}_C \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (d) $\mathrm{proj}_C \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

   (e) $\mathrm{proj}_C \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2. Let S be the set of convex linear combinations of the vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find the following projections:

   (a) $\mathrm{proj}_S \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

   (b) $\mathrm{proj}_S \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

   (c) $\mathrm{proj}_S \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

   (d) $\mathrm{proj}_S \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

   (e) $\mathrm{proj}_S \begin{bmatrix} 0 \\ 10 \end{bmatrix}$

   (f) $\mathrm{proj}_S \begin{bmatrix} 0 \\ -10 \end{bmatrix}$

3. Let $\ell = \mathrm{span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$, $L = \mathrm{span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and let S be the set of convex linear combinations of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$. Find the following projections for $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$:

   (a) $\mathrm{proj}_\ell \vec{v}$
   (b) $\mathrm{proj}_L \vec{v}$
   (c) $\mathrm{proj}_S \vec{v}$

4. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\ell = \mathrm{span}\{\vec{u}\}$. Find the following:

   (a) $\mathrm{proj}_\ell \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

   (b) $\mathrm{vcomp}_{\vec{u}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

   (c) The angle between $\mathrm{proj}_\ell \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix} - \mathrm{vcomp}_{\vec{u}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Solutions

Note that there are multiple ways to find the projection of a point onto a set. The solutions here are just sample answers. You may use a different technique than the ones used here and reach the same solution.

1. (a) The projection of a vector on C is the closest point in C to that vector. Notice that $\begin{bmatrix}-1\\0\end{bmatrix}$ is a point in C, therefore, it is the closest point in C to itself! So we have $\mathrm{proj}_C \begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$.

   (b) The projection of a vector on C is the closest point in C to that vector. Suppose $\vec{v} = \begin{bmatrix}x\\y\end{bmatrix} \in C$ is the projection of $\begin{bmatrix}2\\0\end{bmatrix}$ on C. Then the distance $\left|\left|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix}\right|\right|$, or equivalently, $\left|\left|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix}\right|\right|^2$ is minimized. We can rewrite this expression as

$$\begin{aligned}
\left|\left|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix}\right|\right|^2 &= (x-2)^2 + (y-0)^2 \\
&= (x-2)^2 + y^2 \\
&= x^2 - 4x + 4 + y^2 \\
&= (x^2 + y^2) - 4x + 4 \\
&= ||\vec{v}||^2 - 4x + 4 \\
&= 1 - 4x + 4 \\
&= 5 - 4x.
\end{aligned}$$

   Since $\vec{v} \in C$, we know that $|x| \leq 1$. To minimize the above expression, we choose the maximal value for x, so we should have $x = 1$. In this case, $y = 0$ (since $\vec{v} \in C$, so we must have $x^2 + y^2 = 1$). Therefore, $\vec{v} = \mathrm{proj}_C \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$.

   (c) The projection of a vector on C is the closest point in C to that vector. We can solve this the same way we solved part(b), or we can solve it by drawing a picture. First, let's solve it using the same technique we used in part (b). Suppose $\vec{v} = \begin{bmatrix}x\\y\end{bmatrix} \in C$ is the projection of $\begin{bmatrix}1\\1\end{bmatrix}$ on C. Then the distance $\left|\left|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right|\right|$, or equivalently, $\left|\left|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right|\right|^2$ is minimized. We can rewrite this expression as

$$\begin{aligned}
\left|\left|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right|\right|^2 &= (x-1)^2 + (y-1)^2 \\
&= x^2 - 2x + 1 + y^2 - 2y + 1 \\
&= (x^2 + y^2) - 2x - 2y + 2 \\
&= ||\vec{v}||^2 - 2(x+y) + 2 \\
&= 1 - 2(x+y) + 2 \\
&= 3 - 2(x+y).
\end{aligned}$$

   Since $\vec{v} \in C$, we know that $|x|, |y| \leq 1$. To minimize the above expression, we choose the maximal value for $x+y$. Note that $x^2 + y^2 = 1$, so we have $y = \pm\sqrt{1-x^2}$. Maximizing $x+y$ is equivalent to maximizing $x \pm \sqrt{1-x^2}$. We know that maximizing $x+y$ would give us positive x and y. So we can drop the $\pm$ sign and try to maximize $x+\sqrt{1-x^2}$. To find x that maximizes this expression, we need to differentiate and equate to zero. Now

$$\begin{aligned}\frac{d}{dx}(x+\sqrt{1-x^2}) &= 1+\tfrac{1}{2}(1-x^2)^{\frac{-1}{2}}(-2x)\\ &= 1+\tfrac{-x}{\sqrt{1-x^2}}.\end{aligned}$$

Equating this to zero, we get

$$\begin{aligned}1+\tfrac{-x}{\sqrt{1-x^2}} &= 0\\ \tfrac{x}{\sqrt{1-x^2}} &= 1\\ x &= \sqrt{1-x^2}\\ x^2 &= 1-x^2\\ 2x^2 &= 1\\ x &= \pm\sqrt{\tfrac{1}{2}} = \pm\tfrac{\sqrt{2}}{2}.\end{aligned}$$

But we know that x should be positive, so we have $x = \frac{\sqrt{2}}{2}$, which means that $y = \frac{\sqrt{2}}{2}$ (since we know that y is also positive and $y = \sqrt{1-x^2}$). Therefore $\text{proj}_C \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}$.

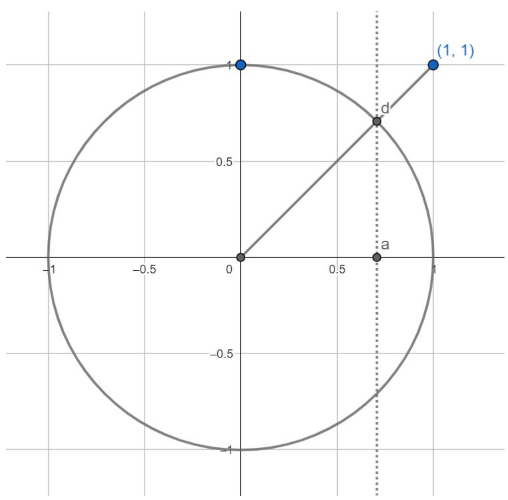

Figure 1: Finding the projection of the point $\begin{bmatrix}1\\1\end{bmatrix}$ onto the unit circle geometrically.

Another way to find the projection is geometrically (see Figure 1). If we draw C and the point $\begin{bmatrix}1\\1\end{bmatrix}$, we can see that the closest point on C to $\begin{bmatrix}1\\1\end{bmatrix}$, let's call it d, is the point of intersection of the circle C with the line segment connecting the origin with the point $\begin{bmatrix}1\\1\end{bmatrix}$. This line segment makes a 45° angle with the x-axis, and we know that the radius of the circle (the length of the segment from the origin to the point d) is 1. If we drop a perpendicular line from d to the $x-axis$, we can use geometry to find the coordinates of the point d. If $d = \begin{bmatrix}x\\y\end{bmatrix}$, then $x = \cos 45 = \frac{\sqrt{2}}{2}$ and $y = \sin 45 = \frac{\sqrt{2}}{2}$. So we get the same answer as before, $\text{proj}_C \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$.

(d) We can solve this geometrically (see Figure 2) or algebraically in the same way we did part (c). The solution will be $\text{proj}_C \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$.

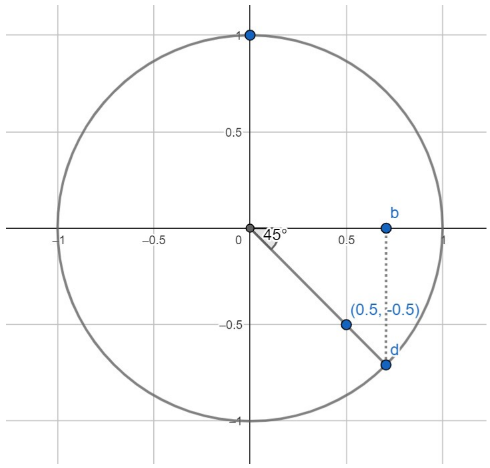

Figure 2: Finding the projection of the point $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ onto the unit circle geometrically.

(e) Note that the distance between any point in C and the origin is 1. So, there is no 'closest' point in C to the origin and this projection ($\text{proj}_C \begin{bmatrix} 0 \\ 0 \end{bmatrix}$) is undefined.

2. (a) Note that $S = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 1 \\ 0 \end{bmatrix} | \alpha \in [0,1] \right\} = \left\{ \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} | \alpha \in [0,1] \right\}$. The projection of a vector on S is the closest point in S to that vector. Suppose $\vec{v} = \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} \in S$ is the projection of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ on S. Then the distance $\left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\|$, or equivalently, $\left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\|^2$ is minimized. We can rewrite this expression as

$$\begin{aligned} \left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\|^2 &= (1-\alpha)^2 + \alpha^2 \\ &= 1 - 2\alpha + \alpha^2 + \alpha^2 \\ &= 1 - 2\alpha + 2\alpha^2. \end{aligned}$$

To minimize this expression, we differentiate and equate it to zero. Now

$$\frac{d}{d\alpha}(1 - 2\alpha + 2\alpha^2) = -2 + 4\alpha.$$

Equating this to zero, gives us $4\alpha - 2 = 0$, and so $\alpha = \frac{1}{2}$, which is in $[0,1]$. Therefore, $\text{proj}_S \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

(b) Since $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \in S$, it is the closest point in S to itself. Therefore, $\text{proj}_S \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

(c) The projection of a vector on S is the closest point in S to that vector. Suppose $\vec{v} = \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} \in S$ is the projection of $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ on S. Then the distance $\left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\|$, or equivalently, $\left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\|^2$ is minimized. We can rewrite this expression as

$$\begin{aligned} \left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\|^2 &= (1-\alpha-2)^2 + \alpha^2 \\ &= (-1-\alpha)^2 + \alpha^2 \\ &= 1 + 2\alpha + \alpha^2 + \alpha^2 \\ &= 1 + 2\alpha + 2\alpha^2. \end{aligned}$$

To minimize this expression, we differentiate and equate it to zero. Now

$$\tfrac{d}{d\alpha}(1 + 2\alpha + 2\alpha^2) = 2 + 4\alpha.$$

Equating this to zero, gives us $4\alpha + 2 = 0$, and so $\alpha = -\frac{1}{2}$, which is not in $[0,1]$. Therefore, the projection has to be one of the endpoints. Checking the distance between $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and each endpoint of the line segment S, we find that it is closer to the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. And so, $\text{proj}_S \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(d) The projection of a vector on S is the closest point in S to that vector. Suppose $\vec{v} = \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} \in S$ is the projection of $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ on S. Then the distance $\left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\|$, or equivalently, $\left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\|^2$ is minimized. We can rewrite this expression as

$$\begin{aligned} \left\| \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\|^2 &= (1-\alpha+2)^2 + \alpha^2 \\ &= (3-\alpha)^2 + \alpha^2 \\ &= 9 - 6\alpha + \alpha^2 + \alpha^2 \\ &= 9 - 6\alpha + 2\alpha^2. \end{aligned}$$

To minimize this expression, we differentiate and equate it to zero. Now

$$\tfrac{d}{d\alpha}(9 - 6\alpha + 2\alpha^2) = -6 + 4\alpha.$$

Equating this to zero, gives us $4\alpha - 6 = 0$, and so $\alpha = \frac{3}{2}$, which is not in $[0,1]$. Therefore, the projection has to be one of the endpoints. Checking the distance between $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$, and each endpoint of the line segment S, we find that it is closer to the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\text{proj}_S \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(e) Following the same strategy used in previous parts, we find that $\text{proj}_S \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(f) Following the same strategy used in previous parts, we find that $\text{proj}_S \begin{bmatrix} 0 \\ -10 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3. (a) Since $\ell = \text{span}\{\vec{u}\}$ is the span of one vector, where $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, the following equality holds, $\text{proj}_\ell \vec{\text{v}} = \text{vcomp}_{\vec{\text{u}}} \vec{\text{v}}$. Then

$$\begin{aligned}
\text{proj}_\ell \vec{\text{v}} &= \text{vcomp}_{\vec{\text{u}}} \vec{\text{v}} \\
&= \tfrac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}} \vec{u} \\
&= \frac{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
&= \tfrac{6}{13} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{18}{13} \\ \frac{12}{13} \end{bmatrix}.
\end{aligned}$$

(b) Again, let $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $L = \text{span}\{\vec{u}\} + \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. So $\text{proj}_{\text{L}} \vec{\text{v}} = \text{t}\vec{\text{u}} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for some scalar t which minimizes the distance between L and $\vec{v}$. In particular, t minimizes $\left\| \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \left(t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right\|^2$. This expression can be written as

$$\begin{aligned}
\left\| \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \left(t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right\|^2 &= (2 - 3t - 1)^2 + (0 - 2t - 1)^2 \\
&= (1 - 3t)^2 + (-1 - 2t)^2 \\
&= (1 - 6t + 9t^2) + (1 + 4t + 4t^2) \\
&= 2 - 2t + 13t^2.
\end{aligned}$$

To minimize this quantity, we differentiate it and equate to 0, to get

$$\tfrac{d}{dt}(2 - 2t + 13t^2) = -2 + 26t.$$

Equating this to zero, we find that the quantity is minimized at $t = \frac{1}{13}$. Therefore, $\text{proj}_{\text{L}} \vec{\text{v}} = \frac{1}{13}\vec{\text{u}} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{13} \\ \frac{2}{13} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{16}{13} \\ \frac{15}{13} \end{bmatrix}$.

We may also find the solution using a normal line. The closest point on L to $\vec{v}$ lies on the line through $\vec{v}$ that is normal to the line L. Denote this line through $\vec{v}$ by L_2, then L_2 has a direction vector that is orthogonal to $\vec{u}$, the direction vector of L. We may take the direction vector of L_2 to be $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. We know that L_2 passes through $\vec{v}$, and so we can now express it in vector form,

$$L_2 : \vec{x} = s \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The projection of $\vec{v}$ onto L is the point of intersection between L and L_2. To find this point, we equate the equations of the two lines making sure we use different symbols for the parameters. So we get

$$t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = s \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Rearranging this, we have

$$t\begin{bmatrix}3\\2\end{bmatrix}+s\begin{bmatrix}2\\-3\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}.$$

We can then express this vector equation as an augmented matrix,

$$\begin{bmatrix}3 & 2 & : & 1\\ 2 & -3 & : & -1\end{bmatrix}.$$

We can row reduce the matrix above to reach the solution $t=\frac{1}{13}, s=\frac{5}{13}$. Substituting in the equations of L or L_2 gives the point of intersection as $\begin{bmatrix}\frac{16}{13}\\ \frac{15}{13}\end{bmatrix}$, which is the same point we got before using the minimum distance method. In conclusion, we have $\text{proj}_\text{L}\vec{\text{v}}=\begin{bmatrix}\frac{16}{13}\\ \frac{15}{13}\end{bmatrix}$.

(c) Notice that in part(a), we found that the projection of $\vec{v}$ onto $\ell=\text{span}\left\{\begin{bmatrix}3\\2\end{bmatrix}\right\}=\left\{t\begin{bmatrix}3\\2\end{bmatrix}\mid t\in\mathbb{R}\right\}$ is $\frac{6}{13}\begin{bmatrix}3\\2\end{bmatrix}$. But $S=\left\{t\begin{bmatrix}3\\2\end{bmatrix}\mid t\in[1,2]\right\}$ and so $S\subseteq\ell$. Since $\text{proj}_\ell\vec{\text{v}}$ is not in S, it follows that $\text{proj}_\text{S}\vec{\text{v}}$ is one of the endpoints of the line segment S. Now calculating the distances between $\vec{v}$ and each endpoint of S, we find that $\left\|\begin{bmatrix}3\\2\end{bmatrix}-\begin{bmatrix}2\\0\end{bmatrix}\right\|^2=1^2+2^2=5$, while $\left\|\begin{bmatrix}6\\4\end{bmatrix}-\begin{bmatrix}2\\0\end{bmatrix}\right\|^2=4^2+4^2=32$. Therefore, $\text{proj}_\text{S}\vec{\text{v}}=\begin{bmatrix}3\\2\end{bmatrix}$.

4. (a) Since ℓ is the span of one vector $\vec{u}$, the equality $\text{proj}_\ell\vec{\text{v}}=\text{vcomp}_{\vec{\text{u}}}\vec{\text{v}}$ holds for any vector $\vec{v}$. Let $\vec{v}=\begin{bmatrix}-3\\1\end{bmatrix}$, then

$$\begin{aligned}\text{proj}_\ell\vec{\text{v}}&=\text{vcomp}_{\vec{\text{u}}}\vec{\text{v}}\\&=\tfrac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}\\&=\frac{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}-3\\1\end{bmatrix}}{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}}\begin{bmatrix}1\\2\end{bmatrix}\\&=\tfrac{-1}{5}\begin{bmatrix}1\\2\end{bmatrix}\\&=\begin{bmatrix}\frac{-1}{5}\\ \frac{-2}{5}\end{bmatrix}.\end{aligned}$$

(b) As explained in part(a), the equality $\text{proj}_\ell\vec{\text{v}}=\text{vcomp}_{\vec{\text{u}}}\vec{\text{v}}$ holds for any vector $\vec{v}$. And so, $\text{vcomp}_{\vec{\text{u}}}\vec{\text{v}}=\begin{bmatrix}\frac{-1}{5}\\ \frac{-2}{5}\end{bmatrix}$, as in part(a).

(c) The angle is $90°$, because $\text{proj}_\ell\begin{bmatrix}-3\\1\end{bmatrix}=\text{vcomp}_{\vec{\text{u}}}\begin{bmatrix}-3\\1\end{bmatrix}$, and $\text{vcomp}_{\vec{\text{u}}}\vec{\text{v}}$ is always orthogonal to $\vec{v}-\text{vcomp}_{\vec{\text{u}}}\vec{\text{v}}$.