Matrix Representations Handout

1. Find the intersection point of the following planes

$$\begin{aligned} 2x + y + z &= -4 \\ 3x - 2z &= 2 \\ 3x + y + 2z &= -7. \end{aligned}$$

2. Express the vector $\begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}_{\mathcal{E}}$ in the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\}$.

3. Find a vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that satisfies the following

$$\vec{v} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = -4, \vec{v} \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = 2, \vec{v} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = -7.$$

4. Find all vectors orthogonal to

   (a) $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

5. Let $\mathcal{P} = \{(x, y, z) : x - y + z = -2\}$, and let $\mathcal{Q}$ be the plane with vector form

$$\vec{x} = t \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix},$$

   and let $\mathcal{R}$ be the plane with normal form

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \left(\vec{x} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right) = 0.$$

   Find $\mathcal{P} \cap \mathcal{Q} \cap \mathcal{R}$.

Solutions

1. The point of intersection of the three planes is the solution to the system of linear equations. First, we express these three equations representing the planes using a matrix equation.

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & -2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ -7 \end{bmatrix}.$$

We can now row reduce the augmented matrix for this equation to find the solution of the system of linear equations.

$$\begin{bmatrix} 2 & 1 & 1 & : & -4 \\ 3 & 0 & -2 & : & 2 \\ 3 & 1 & 2 & : & -7 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & : & -\frac{4}{5} \\ 0 & 1 & 0 & : & -\frac{1}{5} \\ 0 & 0 & 1 & : & -\frac{11}{5} \end{bmatrix}.$$

And so, the point of intersection of the three planes is $\begin{bmatrix} -\frac{4}{5} \\ -\frac{1}{5} \\ -\frac{11}{5} \end{bmatrix}$.

2. We need to find the coefficients of the vector $\begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}_{\mathcal{E}}$ when it is expressed as a linear combination of the vectors in $\mathcal{B}$. But this is equivalent to finding the solution to the following matrix equation (the column picture).

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & -2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ -7 \end{bmatrix}.$$

As seen in the first problem, we get that $x = -\frac{4}{5}, y = -\frac{1}{5}, z = -\frac{11}{5}$. In other words, $\begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{1}{5} \\ -\frac{11}{5} \end{bmatrix}_{\mathcal{B}}$

3. This problem is equivalent to finding the solution to the following matrix equation (the row picture).

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & -2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ -7 \end{bmatrix}.$$

As seen in the first problem, we get that $x = -\frac{4}{5}, y = -\frac{1}{5}, z = -\frac{11}{5}$. In other words, $\vec{v} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{1}{5} \\ -\frac{11}{5} \end{bmatrix}$.

4. (a) We can find the orthogonal vectors by solving the following matrix equation (row picture)

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Row reducing the augmented matrix gives us

$$\begin{bmatrix} 3 & 1 & 2 & : & 0 \\ 2 & 1 & 3 & : & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 & : & 0 \\ 0 & 1 & 5 & : & 0 \end{bmatrix}.$$

So taking $z = t$, we get $y = -5t$, and $x = t$, meaning that the vectors orthogonal to the given vectors are those in the span of $\begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}$.

(b) We can find the orthogonal vectors by solving the following matrix equation (row picture)

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Row reducing the augmented matrix gives us

$$\begin{bmatrix} 1 & 1 & 2 & 2 & : & 0 \\ 2 & 2 & 1 & 1 & : & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & 1 & : & 0 \end{bmatrix}.$$

So we have two free variables, taking $y = t$, and $z = s$, we get $x = -t$, and $w = -s$, meaning that the vectors orthogonal to the given vectors are those in the span of $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

(c) We can find the orthogonal vectors by solving the following matrix equation (row picture)

$$\begin{bmatrix} 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$

We do not need to row-reduce, we know that there is only one pivot, and so we have two free variables, let $y = 3t$ and $z = 3s$, then $x = -2t - 5s$. And so the vectors orthogonal to $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ are those in the span of $\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$.

5. We first need to express all three planes in the same form (any form would work), in these solutions we choose the Cartesian form so we can then use the matrix equation to find the intersection. The plane $\mathcal{P}$ is already given in Cartesian form. To find the Cartesian form of $\mathcal{Q}$, we could find its normal form and convert that to the Cartesian form. We need to find a non-zero vector $\vec{n}$ that is orthogonal to both direction vectors of $\mathcal{Q}$. So we need

$$\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \cdot \vec{n} = 0, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \vec{n} = 0.$$

This is equivalent to solving the following matrix equation

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We can see that $c = 2b$ and $a = -c$, where we have one free variable. Choosing $c = 2$, we get $a = -2$ and $b = 1$. So we have the following normal form for $\mathcal{Q}$,

$$\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \cdot \left(\vec{x} - \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix} \right) = 0.$$

Expanding this dot product out, we get the Cartesian form for $\mathcal{Q}$, which is $-2x + y + 2z = 9$.

Finally, expanding the dot product in the normal form given for $\mathcal{R}$, we get the Cartesian form $3x + y - z = -2$. We can now find $\mathcal{P} \cap \mathcal{Q} \cap \mathcal{R}$ by finding the solution to the matrix equation

$$\begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 2 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ -2 \end{bmatrix}.$$

Row-reducing the augmented matrix gives us,

$$\begin{bmatrix} 1 & -1 & 1 & : & -2 \\ -2 & 1 & 2 & : & 9 \\ 3 & 1 & -1 & : & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 2 \end{bmatrix}.$$

So $\mathcal{P} \cap \mathcal{Q} \cap \mathcal{R} = \left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \right\}$.