Row Reduction and Systems of Linear Equations Handout

1. Determine which of the following matrices are in reduced row echelon form:

   (a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. Row reduce the augmented matrix

$$\begin{bmatrix} 2 & 5 & : & 1 \\ 1 & 3 & : & 0 \end{bmatrix}$$

3. Row reduce $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

4. For each of the following systems of linear equations

   (i) Convert the system of equations to an augmented matrix.

   (ii) Find the reduced row echelon form of the augmented matrix.

   (iii) Is the system consistent? If yes, what is the solution?

   (a)

$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$$

   (b)

$$\begin{aligned} 2x + 8y &= 10 \\ 4x + 16y &= 12 \end{aligned}$$

   (c)

$$\begin{aligned} 5x + 2y &= 12 \\ 10x + 4y &= 24 \end{aligned}$$

(d)

$$\begin{aligned} x - 3z &= -2 \\ 3x + y - 2z &= 5 \\ 2x + 2y + z &= 4 \end{aligned}$$

(e)

$$\begin{aligned} 2x + y - 3z &= 0 \\ 4x + 2y - 6z &= 0 \\ x - y + z &= 0 \end{aligned}$$

5. Determine if the vector $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}$ is in span$\{\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$. If it is, find the coefficients of the vectors.

Solutions

1. (a) The matrix is in RREF.
   (b) The matrix is in RREF.
   (c) The matrix is not in RREF, because there is a row of zeros at the top of the matrix (and the matrix is non-zero).
   (d) The matrix is in RREF.

2. There are many ways to go about this, but the end result is the same.

$$\begin{bmatrix} 2 & 5 & : & 1 \\ 1 & 3 & : & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & : & 0 \\ 2 & 5 & : & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & : & 0 \\ 0 & -1 & : & 1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 3 & : & 0 \\ 0 & 1 & : & -1 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & : & 3 \\ 0 & 1 & : & -1 \end{bmatrix}$$

3. There are many ways to go about this, but the end result is the same.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow[\frac{R_3}{-6}]{\frac{R_2}{-3}} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 - 4R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

4. (a) The augmented matrix is $\begin{bmatrix} 1 & 2 & : & 7 \\ 2 & 1 & : & 8 \end{bmatrix}$. Its RREF is $\begin{bmatrix} 1 & 0 & : & 3 \\ 0 & 1 & : & 2 \end{bmatrix}$. This means that the system is consistent with a unique solution, $x = 3$ and $y = 2$.
   (b) The augmented matrix is $\begin{bmatrix} 2 & 8 & : & 10 \\ 4 & 16 & : & 12 \end{bmatrix}$. Its RREF is $\begin{bmatrix} 1 & 4 & : & 5 \\ 0 & 0 & : & -8 \end{bmatrix}$. This means that the system is inconsistent, because if we rewrite the equation represented by the last row, we get $0x + 0y = -8$, which is impossible.
   (c) The augmented matrix is $\begin{bmatrix} 5 & 2 & : & 12 \\ 10 & 4 & : & 24 \end{bmatrix}$. Its RREF is $\begin{bmatrix} 1 & \frac{2}{5} & : & \frac{12}{5} \\ 0 & 0 & : & 0 \end{bmatrix}$. This means that the system is consistent with infinite solutions. The complete solution for this system is $x = \frac{12}{5} - \frac{2}{5}t$ and $y = t$ for $t \in \mathbb{R}$.
   (d) The augmented matrix is $\begin{bmatrix} 1 & 0 & -3 & : & -2 \\ 3 & 1 & -2 & : & 5 \\ 2 & 2 & 1 & : & 4 \end{bmatrix}$. Its RREF is $\begin{bmatrix} 1 & 0 & 0 & : & 4 \\ 0 & 1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 2 \end{bmatrix}$. This means that the system is consistent with a unique solution, $x = 3$, $y = -3$ and $z = 2$.
   (e) The augmented matrix is $\begin{bmatrix} 2 & 1 & -3 & : & 0 \\ 4 & 2 & -6 & : & 0 \\ 1 & -1 & 1 & : & 0 \end{bmatrix}$. Its RREF is $\begin{bmatrix} 1 & 0 & \frac{-2}{3} & : & 0 \\ 0 & 1 & \frac{-5}{3} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$. This means that the system is consistent with infinite solutions. The complete solution for this system is $x = \frac{2}{3}t$, $y = \frac{5}{3}t$ and $z = t$ for $t \in \mathbb{R}$.

5. We begin by writing a vector equation,

$$a\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

If this equation has a solution, then the vector $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and therefore is in their span. If there is no solution, then it is not in the span of these vectors. Next, we convert the vector equation to a system of linear equations and then to an augmented matrix (or directly to an augmented matrix). The corresponding system of linear equations is

$$\begin{aligned} 4a + b &= 5 \\ 3a + 2b &= 0 \\ a + 3b &= -7. \end{aligned}$$

And the corresponding augmented matrix is

$$\begin{bmatrix} 4 & 1 & : & 5 \\ 3 & 2 & : & 0 \\ 1 & 3 & : & -7 \end{bmatrix}.$$

The RREF of this matrix is

$$\begin{bmatrix} 1 & 0 & : & 2 \\ 0 & 1 & : & -3 \\ 0 & 0 & : & 0 \end{bmatrix}.$$

So, we have a unique solution and $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, with coefficients 2 and -3, respectively. Consequently, the vector is in the span of the other two vectors.