Course Review

April 16, 2023 4:24 PM

Important Definitions

1. Even/Odd numbers

   x is even $\Leftrightarrow \exists n \in \mathbb{N}$ s.t. $x = 2n$

   x is odd $\Leftrightarrow \exists n \in \mathbb{N}$ s.t. $x = 2n+1$

2. One-to-one Function

   $\forall x_1, x_2 \in D, f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

3. Epsilon-delta Definitions

   Limit exists

   Let $a \in \mathbb{R}$

   $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

   Limit does not exist

   Let $a \in \mathbb{R}$

   $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $|x-a| < \delta \wedge |f(x) - f(a)| > \varepsilon$

   Limit is infinity

   $\forall M > 0, \exists \delta > 0$ s.t. $|x| > \delta \Rightarrow |f(x)| > \varepsilon$

4 Epsilon-delta definitions of "f is continuous at a"

Let $a \in \mathbb{R}$

Let f be a function defined, at least, on an interval centered at a

f is continuous at $a \Leftrightarrow \lim_{x \to a} f(x) = f(a) \Rightarrow$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

f is continuous on an interval $(a,b) \Leftrightarrow \forall c \in (a,b)$, f is continuous at c

f is continuous on a closed interval $[a,b]$:

1. $\lim_{x \to a^+} f(x) = f(a)$
2. f is continuous on (a,b)
3. $\lim_{x \to b^-} f(x) = f(b)$

5. Definition of derivative of a function at a as its limit

$$\frac{dy}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

6. Inverse Function

   A function f if it is injective or one-to-one

7. Inverse Trig functions

   $\arcsin x = \sin^{-1}(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$

   $-1 < x < 1$

   $\arctan x = \tan^{-1}(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$

   $-\infty < x < \infty$

   $\arccos x =$

8. Global max/min and Local max/min

   Let f be a function with domain I

   Let $c \in I$

   f has a global maximum at c when:

   $\forall x \in I, f(x) \le c$

   f has a local maximum at c when:

   $\exists \delta > 0$ s.t. $|x-c| < \delta \Rightarrow f(x) \le f(c)$

   IF
   - f has a local extremum at c
   - c is an interior point to I (not endpoint)

   THEN

   $f'(c) = 0$ or DNE

9. Monotonicity of a Function

   Let f be defined on I
   - f is increasing on I when

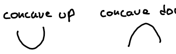
     $\forall x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

   IF $\forall x \in I, f'(x) > 0$

   THEN f is increasing on I

10. Definition of concavity

    concave up concave down



Let f be a differentiable function defined on interval I

f is concave-up when f' is increasing on I $f''(x) > 0$

f is concave-down when f' is decreasing on I $f''(x) < 0$

Let $c \in I$

f has an inflection point when "f changes concavity at c" $f''(c) = 0$ or DNE

11. Asymtotes

    Vertical

    Let f be a function. Let $a \in \mathbb{R}$

    The vertical line $x = a$ is an asymtote of f when $\lim_{x \to a^\pm} f(x) = \pm\infty$

    Horizontal

    Let f be a function. Let $L \in \mathbb{R}$

    The horizontal line $y = L$ is an asymtote of f when $\lim_{x \to \pm\infty} f(x) = L$

Important Theorems

1. Limit Laws + Squeeze Theorem

   Laws:

   $\forall a \in \mathbb{R}$,

   $\lim_{x \to a} f(x) = f(a)$ $\lim_{x \to a} g(x) = g(a)$

   $\lim_{x \to a} [f(x) + g(x)] = f(a) + g(a)$

   $\lim_{x \to a} [f(x)g(x)] = f(a)g(a)$

   $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{f(a)}{g(a)}$ (assuming $g(a) \neq 0$)

   Only Apply if Limits exist

   Squeeze Theorem

   Let $a, L \in \mathbb{R}$

   Let f, g, and h be functions defined near a, except possibly at a

   If
   - For x close to but not a, $h(x) \le g(x) \le f(x)$
   - $\lim_{x \to a} f(x) = L$
   - $\lim_{x \to a} h(x) = L$

   THEN
   - $\lim_{x \to a} g(x) = L$

2. Continuity theorems

   The main continuity theorem:

   Any function we can construct with sum, product, quotient, and composition of polynomials, roots, trigonometric fuctions, exponentials, logarithms, and absolute values is continuous (on its domain)

   Proof

   1. Prove "basic" functions are continuous

      $f(x) = c$ $f(x) = e^x$ $f(x) = \sin x$

      $f(x) = x$ $f(x) = \ln(x)$ $f(x) = |x|$

   2. Prove sum, product, quotient, and composition of continuous functions is continuous

   Sum, product, quotient are Proved through limit laws

   Example

   Assume f and g are continuous at a

   $\lim_{x \to a} f(x) = f(a)$, $\lim_{x \to a} g(x) = g(a)$

   Limit Law

   $\lim_{x \to a} [f(x) + g(x)] = f(a) + g(a)$

   $\therefore f + g$ is continuous at a

3. Extreme Value Theorem (EVT)

   Let f be a function

   IF f is continuous on an interval $[a,b]$

   THEN f has a max and min on $[a,b]$

4. Intermediate Value Theorem [IVT

   Let f be a function defined on the interval $[a,b]$

   IF
   - $f(a) < M$
   - $f(b) > M$
   - f is continuous on $[a,b]$

   THEN
   - $\exists c \in (a,b)$ s.t. $f(c) = M$

5. Differentiability implies Continuity

   Let $c \in \mathbb{R}$

   Let f be a function defined at and near c

   IF f is differentiable at c

   THEN f is continuous at c

6. Differentiation rules

   $\frac{d}{dx}[c] = 0$ $\frac{d}{dx}\sin x = \cos x$ $\frac{d}{dx}\cos x = -\sin x$

   $\frac{d}{dx}[x^c] = cx^{c-1}$ $\frac{d}{dx}\tan x = \sec^2 x$ $\frac{d}{dx}\cot x = -\csc^2 x$

   $(f+g)' = f' + g'$ $\frac{d}{dx}\sec x = \sec x \tan x$ $\frac{d}{dx}\csc x = -\csc x \cot x$

   $(cf)' = cf'$

   $(f \cdot g)' = f'g + fg'$

   $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

   $(f(g))' = f'(g(x))g'(x)$

7. Rolle's Theorem

   Let $a < b$. Let f be a function defined on $[a,b]$

   IF
   - f is continuous on $[a,b]$
   - f is differentiable on $[a,b]$
   - $f(a) = f(b)$

   THEN

   $\exists c \in [a,b]$ s.t. $f'(c) = 0$

8. Mean Value Theorem (MVT)

   Let $a < b$. Let f be a function defined on $[a,b]$

   IF
   - f is continuous on $[a,b]$
   - f is differentiable on (a,b)

   THEN

   $\exists c \in (a,b)$ s.t.:

   $f'(c) = \frac{f(b) - f(a)}{b - a}$

9. Zero Derivative $\Rightarrow$ Constant

   Let $a < b$. Let f be a function defined on $[a,b]$

   IF
   - $\forall x \in (a,b), f'(x) = 0$
   - f is continuous on $[a,b]$

   THEN
   - f is constant on $[a,b]$

10. L'Hopital's Rule

    An Indeterminate form is

    $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$

    Let f, g be functions. Let $a \in \mathbb{R}$

Slant

Let f be a function Let $m, b \in \mathbb{R}$

The line $y = mx+b$ is an asymtote for x

when $\lim_{x \to \pm\infty} [f(x) - (mx+b)] = 0$

12. Supremum/Infimum

Let $A \subseteq \mathbb{R}$. Let $c \in \mathbb{R}$

- c is an upper bound of A means $\forall x \in A, x \leq c$
- c is a least upper bound, or supremum of A means
  - c is an upper bound of A
  - If b is an upper bound of A, $c \leq b$
- If the supremum of A is in A, it is a maximum
- A is bounded above means it has at least one upper bound
  - f is bounded above $\Rightarrow$ supremum

Infimum is the reverse

Define supremum as $\sup\{f(x) : x \in I\}$

13. Definition of Integrable

Let $f(x)$ be a bounded function on $[a,b]$

Define a partition P of an interval $[a,b]$ s.t.

- P is finite
- $P \subseteq [a,b]$
- $a \in P$ and $b \in P$

Define

$P_n = \{x_0, x_1, \ldots x_n\}$ as a partion of $[a,b]$

For $i \in [1, n]$

Let $m_i = \inf\{f(x) : x \in (x_{i-1}, x_i)\}$

$M_i = \sup\{f(x) : x \in (x_{i-1}, x_i)\}$

$\Delta x_i = x_i - x_{i-1}$

$L_p(f) = \sum_{i=1}^{n} m_i \Delta x_i$

$U_p(f) = \sum_{i=1}^{n} M_i \Delta x_i$

For all possible partitions

$\underline{I}_a^b(f) = \sup\{\text{All possible } L_p(f)\}$

If f is continuous

Then it is integrable

$\overline{I}_a^b(f) = \inf\{\text{All possible } U_p(f)\}$

If $\overline{I}_a^b(f) = \underline{I}_a^b(f)$, then we say f is integrable on $[a,b]$

14. Definition of Riemann Sum

Let f be a bounded function on $[a,b]$

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$

For each $i = 1, 2, \ldots n$

Let $\Delta x_i = x_i - x_{i-1}$

Choose $x_i^* \in [x_{i-1}, x_i]$

Then

$S_p^*(f) = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$

This a Riemann-sum for f and P

15. Definition of sequence is convergent

"Sequence is convergent to $L \in \mathbb{R}$" means

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$,

$n \geq n_0 \Rightarrow L - \varepsilon < a_n < L + \varepsilon$

$|a_n - L| < \varepsilon$

A sequence is convergent when it has a limit and divergent when it doesn't

16. Definition of sequence is increasing/decreasing

A sequence $\{a_n\}_{n=0}^{\infty}$ is:

increasing when:

$\forall n \in \mathbb{N}, a_n < a_{n+1}$ <u>or</u>

$\forall n, m \in \mathbb{N}, n < m \Rightarrow a_n < a_m$

decreasing when:

$\forall n \in \mathbb{N}, a_n > a_{n+1}$ <u>or</u>

$\forall n, m \in \mathbb{N}, n < m \Rightarrow a_n > a_m$

A sequence is <u>monotonic</u> if it is increasing, non decreasing, decreasing, non increasing

17. Definition of sequence is bounded

A sequence $\{a_n\}_{n=0}^{\infty}$ is:

Bounded below when:

$\exists A \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, A \leq a_n$

Bounded above when:

$\exists B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, B \geq a_n$

Bounded when

It is bounded above <u>and</u> below

Let f, g be functions. Let $a \in \mathbb{R}$

IF

$\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of $\begin{cases} \frac{0}{0} \\ \frac{\pm\infty}{\pm\infty} \end{cases}$

f and g are differentiable as $x \to a$

g and g' is never $= 0$ as $x \to a$

$\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, or is $\pm\infty$

THEN

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

11. Riemann Sum

Let f be bounded and integrable on $[a,b]$

Pick a sequence of partitions $P_1, P_2, P_3, \ldots$ of $[a,b]$ s.t.

$\lim_{n \to \infty} \|P_n\| = 0$

One example would be to break $[a,b)$ into partitions of equal length

Then,

$\int_a^b f(x)dx = \lim_{N \to \infty} S_{P_n}^*(f)$

12. Properties of definite integrals

1. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\int_a^b [C f(x)]dx = C\int_a^b f(x)dx$
3. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \quad a < b < c$
4. Integral is the area underneath the graph
5. $f(x) < g(x) \Rightarrow \int_a^b f(x)dx < \int_a^b g(x)dx$

13. Fundemental Theorem of Calculus

FTC part 1.

Let I be an interval. Let $a \in I$

Let f be a function on I

Define

$F(x) = \int_a^x f(x)dx$

IF f is continuous

THEN F is differentiable and $F' = f$

$\frac{d}{dx} \int_a^x f(t)dx = f(x)$

Part 2

Let $a < b$

Let f be a continuous function on $[a,b]$

Let G be any antiderivative G of f

Then $\int_a^b f(x)dx = G(b) - G(a)$

14. Convergent Sequences must be bounded

Convergent $\Rightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow |a_n - L| < \varepsilon$

Bounded $\Rightarrow \exists A, B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, A < a_n < B$

Assume $\{a_n\}_{n=0}^{\infty}$ is convergent. Let L be limit

$\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - 1 < a_n < L + 1$

Take $A = \min\{L-1, a_0, a_1, a_2, \ldots, a_{n_0-1}\}$

$B = \max\{L+1, a_0, a_1, a_2, \ldots, a_{n_0-1}\}$

* we are allowed to do this, because $\{a_0, a_1, \ldots, a_{n_0-1}\}$ is a finite set

$\forall n \in \mathbb{N}$

$n \geq n_0 \Rightarrow A \leq L - 1 < a_n < L + 1 \leq B$

$n < n_0$, by definition of A and B, $A \leq a_n \leq B$

15. Monotone convergence theorem

We know sequence is convergent $\Rightarrow$ bounded

Proving

IF

- sequence is eventually monotonic
- bounded

THEN

- it is convergent

Proof that if a sequence is increasing and bounded above, it is convergent

WTS $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t.

$\forall n \in \mathbb{N}, n > n_0 \Rightarrow |a_n - L| < \varepsilon$

Take set $A = \{a_n | n \in \mathbb{N}\}$

a_n is non-empty and bounded above

so it has a supremum

Take $L = \sup(A)$

Prove $L = \lim_{n \to \infty} a_n$

$\varepsilon > 0$

By def of supremum

$\exists n_0 \in \mathbb{N}$ s.t. $L - \varepsilon < a_{n_0}$

We know $L - \varepsilon < a_n$

18. Definition of improper integrals as limits

Let $a \in \mathbb{R}$

Let f be continuous on $[a, \infty]$

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

The integral is convergent when this limit exists and divergent when it does not

19. Definition of series is convergent

Construct a sequence of partial sums $\{S_k\}_{k=1}^{\infty}$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_k = a_1 + a_2 + \ldots + a_k = \sum_{n=1}^{k} a_n$

Compute the limit $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k$

The series is CONVERGENT when the limit exists

20. Geometric Series

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $-1<x<1$

Otherwise, $\sum_{n=0}^{\infty} x^n$ is DIVERGENT

Computing

$$S = \sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} S_k$$

$$S_k = 1 + x + x^2 + \ldots + x^k$$

$$xS_k = x + x^2 + x^3 + \ldots + x^{k+1}$$

$$S_k - xS_k = 1 - x^{k+1}$$

$$S_k(1-x) = 1 - x^{k+1}$$

$$S_k = \frac{1-x^{k+1}}{1-x}$$

$$= \frac{1 - \lim_{k\to\infty} x^{k+1}}{1-x}$$

if $|x|>1$, $\lim_{k\to\infty} x^{k+1}$ DNE,

if $|x|<1$, $\lim_{k\to\infty} x^{k+1} = 0$

if $x=1$, $\frac{1}{1-x}$ DNE

if $x=-1$, $\lim_{k\to\infty} x^{k+1}$ DNE

Thus, if $-1<x<1$, $S = \frac{1}{1-x}$

21. Definition of series is conditionally, absolutely convergent, or divergent

Absolute convergence Test

If $\sum_n^\infty |a_n|$ is convergent

Then $\sum_n^\infty a_n$ is convergent

A convergent series $\sum_n^\infty a_n$ is:

- absolutely convergent when $\sum_n^\infty |a_n|$ is also convergent
- conditionally convergent when $\sum_n^\infty |a_n|$ is divergent

If $\sum_n^\infty a_n$ and $\sum_n^\infty |a_n|$ are both divergent, it is simply divergent

22. Definition of power series

Let $a \in \mathbb{R}$

A power series centered at a is one defined like

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

where $c_0, c_1, c_2, \ldots \in \mathbb{R}$

Domain $f = \{x \in \mathbb{R} : \text{When } f(x) \text{ is convergent}\}$

The domain of f is an interval centered at a:

$(a-R, a+R)$, $(a-R, a+R]$, $\mathbb{R}$

$[a-R, a+R]$, $[a-R, a+R)$, $\{a\}$

The domain is called the interval of convergence

The radius is called the radius of convergence

On $(a-R, a+R)$, or the interior, $f(x)$ is absolutely convergent

The endpoints may be divergent, or absolutely/conditionally convergent

Otherwise, $f(x)$ is divergent

23. Definition of Taylor Polynomials

Definition 1:

Let $a \in \mathbb{R}$

Let f be a continuous function defined at and near a

Let $n \in \mathbb{N}$

The n-th Taylor Polynomial at a is a polynomial P_n which is an approximation for f near a of order n:

meaning $\lim_{x\to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$

with a degree at most n

Because sequence is increasing, $a_{n_0} \le a_n$

By def of supremum, $a_n < L$

$\therefore L - \varepsilon < a_{n_0} \le a_n \le L < L + \varepsilon$

16. Big Theorem

$\ln(n) << n^a << c^n << n! << n^n$
$(a>0)$ $(c>1)$

17. Basic and Limit Comparison Test

Basic Comparison Test

Let $a \in \mathbb{R}$

Let f and g be continuous functions on $[a, \infty]$

IF $\forall x > a$:

$0 \le f(x) \le g(x)$

THEN

$\int_a^\infty f(x)dx = \infty \Rightarrow \int_a^\infty g(x)dx = \infty$

$\int_a^\infty g(x)dx < \infty \Rightarrow \int_a^\infty f(x)dx < \infty$

Limit Comparison Test

Let $a \in \mathbb{R}$

Let f and g be positive, continuous functions on $[a, \infty]$

IF

$L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists, and $L > 0$, not ∞

THEN

$\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)$

are both convergent or divergent

18. Series Tests

Necessary condition

IF $\sum_{n=0}^{\infty} a_n$ is convergent,

THEN $\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ Divergent

$\lim_{n\to\infty} a_n = 0 \Rightarrow$ nothing :(

BCT and LCT function the same, refer to those theorems from improper integrals

Integral Test

Let $a \in \mathbb{R}$

Let f be a continuous, positive, decreasing function on $[a, \infty]$

THEN

$\int_a^\infty f(x)dx$ is convergent $\Leftrightarrow \sum_n^\infty f(n)$ is convergent

Alternating Series Test

A series is alternating when $\forall n, a_n a_{n+1} < 0$

Consider a series of the form

$\sum_n^\infty (-1)^n b_n$ or $\sum_n^\infty (-1)^{n+1} b_n$

IF

- $\forall n, b_n > 0$
- $\{b_n\}_n^\infty$ is decreasing
- $\lim_{n\to\infty} b_n = 0$

THEN

- The series is convergent

Ratio Test

Let $\sum_n^\infty a_n$ be a series. Assume $\forall n, a_n \neq 0$

Assume $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞

$L < 1 \Rightarrow \sum_n^\infty a_n$ is absolutely convergent

$L = 1$ is inconclusive

$L > 1 \Rightarrow \sum_n^\infty a_n$ is divergent

19. 4 main Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad -\infty < x < \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad -\infty < x < \infty$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

20. Lagrange Remainder Theorem and Alternating Series Estimation

Lagrange's Remainder Theorem

Let I be an open interval, Let $a \in I$

Let $n \in \mathbb{N}$

Let f be a C^{n+1} function on I

Let $P_n(x) = \sum^n \frac{f^{(k)}(a)}{k!}(x-a)^k$ be its n-th Taylor Polynomial which is an approximation for f near a of order n:

meaning $\lim_{x\to a} \frac{f(x)-P_n(x)}{(x-a)^n} = 0$

with a degree at most n

Definition 2:

Let $a \in \mathbb{R}$
Let $n \in \mathbb{N}$
Let f be a C^n function at a

The n-th Taylor polynomial for f at a is a polynomial P_n s.t.

$P_n(a)=f(a),\ P_n'(a)=f'(a),\ \dots,\ P_n^{(n)}(a)=f^{(n)}(a)$

with a degree at most n

Definition 3

Let $a \in \mathbb{R}$
Let $n \in \mathbb{N}$
Let f be a C^n function at a

The n-th Taylor polynomial for f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(Swift) me/

24. Definition of Taylor Series

Let $a \in \mathbb{R}$
Let f be a C^∞ function at a

The Taylor Series for f at a is the power series

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$\forall k \in \mathbb{N},\ S^{(k)}(a) = f^{(k)}(a)$

25. Definition of an analytic function

Let f be a C^∞ function defined on an open interval I

Let $a \in I$ Let $S_a(x)$ be the Taylor series of f at a

f is analytic at a when:

$\exists$ an open interval J_a centered at a s.t.

$\forall x \in J_a,\ f(x) = S_a(x)$

f is analytic when:

$\forall a \in I$, f is analytic at a

1. Polynomials are analytic
2. Sums, products, quotients and composition of analytic functions are analytic
3. Derivatives and antiderivatives of analytic functions are analytic

Let I be an open interval, Let $a \in I$
Let $n \in \mathbb{N}$
Let f be a C^{n+1} function on I
Let $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ be its n-th Taylor Polynomial

Let $R_n(x) = f(x) - P_n(x)$ be the remainder

THEN

$\exists \xi$ between a and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Alternating Series Estimation

Take an alternating series of the form

$$\sum_{n}^{\infty} (-1)^n b_n \text{ or } \sum_{n}^{\infty} (-1)^{n+1} b_n$$

IF it is convergent from alternating series test

THEN $|S-S_k| < b_{k+1}$