Cartesian, Vector, and Normal Forms Handout

Let $\mathcal{P}_1$ be the plane in $\mathbb{R}^3$ with vector form

$$\mathcal{P}_1 : \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} s + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

Let $\mathcal{P}_2$ be the plane in $\mathbb{R}^3$ with equation $x + y + z = 1$. And let ℓ be the line given in vector form by

$$\ell : \vec{x} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}.$$

(a) Find $\mathcal{P}_1 \cap \ell$.

(b) Express the plane $\mathcal{P}_2$ in vector form.

(c) Find $\mathcal{P}_2 \cap \ell$.

(d) Using the vector forms of the planes, find $\mathcal{P}_1 \cap \mathcal{P}_2$.

(e) Using the vector form of $\mathcal{P}_1$, find a plane $\mathcal{P}_3$ such that $\mathcal{P}_1 \cap \mathcal{P}_3 = \emptyset$.

(f) Express the plane $\mathcal{P}_1$ in normal form and in Cartesian form.

(g) Using the Cartesian forms of the planes, find $\mathcal{P}_1 \cap \mathcal{P}_2$.

(h) Using the Cartesian form of $\mathcal{P}_1$, find a plane $\mathcal{P}_4$ such that $\mathcal{P}_1 \cap \mathcal{P}_4 = \emptyset$.

Solutions

(a) First, let's rewrite the vector form of the line ℓ with a different parameter to avoid confusion. So we'll have

$$\ell : \vec{x} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} r + \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}.$$

Next, we need to find points that are both on the line and the plane, so we are looking for solutions to the following,

$$\begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} s + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} r + \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}.$$

We can rearrange and write this as

$$\begin{aligned} \begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} s + \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix} r &= \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}. \end{aligned}$$

Now, we convert the vector equation into an augmented matrix,

$$\begin{bmatrix} -1 & 2 & -4 & : & 0 \\ 1 & 1 & 3 & : & -1 \\ 7 & -5 & -1 & : & -3 \end{bmatrix}.$$

Row reduction then gives us the solution $r = 0, s = \frac{-1}{3}, t = \frac{-2}{3}$. Substituting $r = 0$ in the equation for the line, we get the point $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$. To check our answer, we can substitute $t = \frac{-2}{3}$ and $s = \frac{-1}{3}$ in the plane's equation, and we get the same point. Therefore, $\mathcal{P}_1 \cap \ell = \left\{ \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right\}$.

(b) Note that the vector form of a plane is not unique, you may choose different points, and you may calculate the direction vectors differently. To express a plane in vector form, we need three non-colinear points on the plane. To make calculations simple, we may choose the following three points which satisfy the Cartesian equation of the plane:

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then, we find two direction vectors for the plane,

$$\vec{d}_1 = B - A = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{d}_2 = C - A = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We can express $\mathcal{P}_2$ in vector form as

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Following the same steps as in part (a) we reach the following solution,

$$\mathcal{P}_2 \cap \ell = \left\{ \begin{bmatrix} 7 \\ -5 \\ -1 \end{bmatrix} \right\}.$$

(d) Similar to what was done in part (a), we rewrite the vector form of $\mathcal{P}_2$ using different dummy variables to avoid confusion, so we have,

$$\mathcal{P}_2 : \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} b + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We need to find the points that lie on both planes, so we have to find the solutions to

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} b + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} s + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

If we rearrange, we get the following vector equation,

$$\begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} s + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} a + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} b = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}.$$

Converting this vector equation into an augmented matrix gives

$$\begin{bmatrix} -1 & 2 & 1 & 1 & : & 2 \\ 1 & 1 & -1 & 0 & : & -2 \\ 7 & -5 & 0 & -1 & : & 0 \end{bmatrix}.$$

Row reduction then gives us the solution set $a = 2 - 3r$ and $b = 7r$, where $r \in \mathbb{R}$. Substituting a and b in the equation for $\mathcal{P}_2$, we get the solution set

$$\vec{x} = \begin{bmatrix} -4 \\ -3 \\ 7 \end{bmatrix} r + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

This is the vector form of a line and so the two planes intersect in a line.

(e) A plane that does not intersect $\mathcal{P}_1$, must be a plane that is parallel to $\mathcal{P}_1$ and does not coincide with it (meaning it is not the same plane). If we choose the direction vectors of $\mathcal{P}_3$ to be the same as those for $\mathcal{P}_1$, we guarantee that the two planes are parallel. To make sure that they are not the same plane, we need to add a point that is not on the plane $\mathcal{P}_1$. Let's check if $\mathcal{P}_1$ passes through the origin. If it does, then there would be scalars t and s, such that

$$\begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} s + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can convert this to the augmented matrix

$$\begin{bmatrix} -1 & 2 & : & 1 \\ 1 & 1 & : & -2 \\ 7 & -5 & : & 0 \end{bmatrix}.$$

Row reduction then shows us that this system is inconsistent, which means that the origin (the point $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$) does not lie on $\mathcal{P}_1$. So, we may choose our plane $\mathcal{P}_3$ to be a plane through the origin, and it will have an empty intersection with $\mathcal{P}_1$. Then the vector form for the plane $\mathcal{P}_3$ will be

$$\mathcal{P}_3 : \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} s.$$

(f) Note that the normal form is not unique. You may use a different normal vector (but it must be a scalar multiple of the one calculated here), you may also choose a different point on the plane. To express the plane in normal form, we need a normal vector to the plane and a point on the plane. The normal vector must be a nonzero vector orthogonal to both direction vectors of the plane. Let $\vec{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a normal vector to $\mathcal{P}_1$, then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix} = 0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}.$$

Expanding this, we get the following system of equations

$$\begin{aligned} -x + y + 7z &= 0 \\ 2x + y - 5z &= 0. \end{aligned}$$

Converting this system into an augmented matrix gives

$$\begin{bmatrix} 1 & -1 & -7 & : & 0 \\ 2 & 1 & -5 & : & 0 \end{bmatrix}.$$

Row reduction gives the solution $x = 4t, y = -3t, z = t$, where $t \in \mathbb{R}$. Choosing $t = 1$, we get $\vec{n} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$. We can now give the normal form for $\mathcal{P}_1$

$$\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \cdot \left(\vec{x} - \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right) = 0.$$

We can use the normal form to find the Cartesian form of $\mathcal{P}_1$. All we have to do is expand the dot product, take $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then expanding the dot product, we get

$$\begin{aligned}
\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right) &= 0 \\
\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} &= 0 \\
4x - 3y + z - (-4 - 6 + 0) &= 0 \\
4x - 3y + z &= -10.
\end{aligned}$$

Therefore, the Cartesian form for $\mathcal{P}_1$ is given by the equation $4x - 3y + z = -10$. (Note that the coefficients of x, y and z are the entries of the normal vector.)

(g) We already found the intersection between the two planes in part (d) using the vector forms of the planes. In this part, we are asked to find it using the Cartesian forms of the planes. So, we need to find the solution to the following system of equations

$$\begin{aligned}
x + y + z &= 1 \\
4x - 3y + z &= -10.
\end{aligned}$$

Converting to the augmented matrix, we get

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 4 & -3 & 1 & : & -10 \end{bmatrix}.$$

Row reduction gives us the solution $x = -1 - 4t, y = 2 - 3t, z = 7t$, where $t \in \mathbb{R}$. This gives us the vector form of a line

$$\vec{x} = \begin{bmatrix} -4 \\ -3 \\ 7 \end{bmatrix} t + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

This solution agrees with that of part (d).

(h) We found a plane $\mathcal{P}_3$ in part (e) that satisfies these conditions. However, we now have the Cartesian form of the plane $\mathcal{P}_1$, and we can use another method to find a non-intersecting plane. Note that a non-intersecting plane must be parallel to $\mathcal{P}_1$ and not equal to $\mathcal{P}_1$. But if a plane is parallel to $\mathcal{P}_1$, then it must have the same normal vectors. So we know that the vector $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is a normal vector to the plane $\mathcal{P}_4$. And so $\mathcal{P}_4$ can be any plane with equation $4x - 3y + z = c$, where c is any real number other than -10. If we choose $c = 0$, then we get the same plane $\mathcal{P}_3$, that we saw in part (e).