STA237H1F (LEC0201)

Week 3: Discrete random variables

(Dekking et al., 2005; Wagaman & Dobrow, 2021)

Introduction

- *Definition*. A random variable X is a measurable function with domain Ω and codomain $\mathbb{R}$, i.e.

$$X\colon \Omega \longrightarrow \mathbb{R}$$

  - A random variable simplifies the calculations of probabilities
- A random experiment is characterised by a probability space (sample space, events, probability)
- $\mathcal{E} \Leftrightarrow (\Omega, \mathcal{B}, P) \overset{X}{\Rightarrow} (\mathbb{R}, \mathcal{B}_X, P_X)$ new probability space
- e.g. $\mathcal{E} =$ toss one coin, $\Omega = \{H, T\}$,
  - $X = \#$ of heads $\Rightarrow x \in \mathcal{X} = \{1,0\}$ discrete r.v. (countable)
- e.g. $\mathcal{E}$=place of an accident in road of length L, $\Omega = [0, L]$,
  - $X =$ shortest distance to help, $x \in [0, L/2]$ continuous r.v. (uncountale)

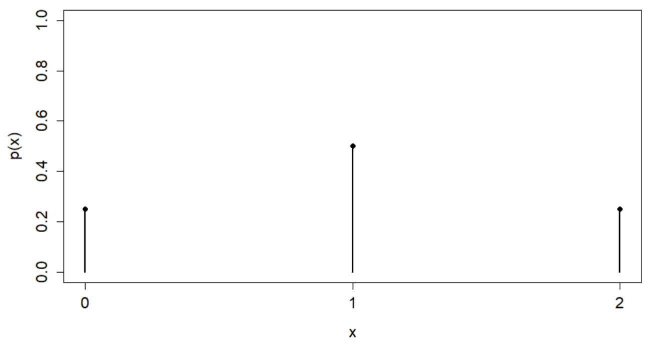
Characterisation

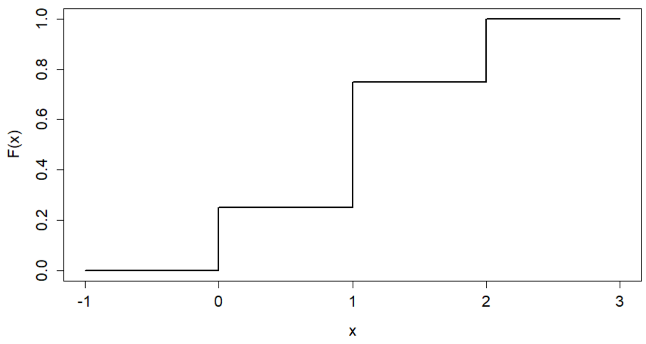
- Let X be a discrete r.v. with values x in a discrete set $\mathcal{X}$
- We charcaterise the probabilities in X in 3 ways

1. *Probability mass function (density function):*
$$p(x) = P(X = x)$$
2. *Cumulative distribution function (cdf):*
$$F(x) = P(X \leq x)$$
3. *Moment generating function:*
$$M_X(t) = E(e^{tX})$$
$$= \sum_{x \in \mathcal{X}} e^{tx} p(x)$$
    - "E" is a linear operator: $E\{ag(X) + b\} = aE\{g(X)\} + b$

Cont…

- *e.g.* Ɛ= toss two coins, $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$
  - X = number of heads $\mathrm{x} \in \mathcal{X} = \{0,1,2\}$
  - Density of X:
    - $P(X = 0) = P\{(T,T)\} = 1/4$
    - $P(X = 1) = P\{(H,T),(T;H)\} = 2/4$
    - $P(X = 2) = P\{(H,H)\} = 1/4$
  - cdf of X:
  - $F(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \le x < 1 \\ 3/4, & 1 \le x < 2 \\ 1, & 2 \le x \end{cases}$





Mean and Variance

- *Mean of X*: measure of location

$$\mu = E(X) = \sum_{x \in \mathcal{X}} x p(x)$$

  - weighted average
- *Variance of X*: measure of dispersion

$$\sigma^2 = Var(X) = E\{(X - \mu)^2\} = \sum_{x \in \mathcal{X}} (x - \mu)^2 p(x)$$

  - weighted average of squared deviations
  - $\sigma = \sqrt{\sigma^2}$ = standard deviation of X
- Moments:
  - 1st moment: $E(X)$
  - 2nd moment: $E(X^2)$

$$Var(X) = E(X^2) - E^2(X)$$

Cont…

- *e.g.* Ɛ= toss two coins, $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$
  - X = number of heads x ∈ $\mathcal{X} = \{0,1,2\}$
  - Moments of X
  - $\mu = E(X) = 0\left(\frac{1}{4}\right) + 1\left(\frac{2}{4}\right) + 2\left(\frac{1}{4}\right) = 1$
    - Probability mass is symmetric at one
  - $\sigma^2 = Var(X) = (0-1)^2\left(\frac{1}{4}\right) + (1-1)^2\left(\frac{2}{4}\right) + (2-1)^2\left(\frac{1}{4}\right) = \frac{1}{2}$
  - $E(X^2) = (0)^2\left(\frac{1}{4}\right) + (1)^2\left(\frac{2}{4}\right) + (2)^2\left(\frac{1}{4}\right) = \frac{3}{2}$
  - $\sigma^2 = Var(X) = \frac{3}{2} - (1)^2 = \frac{1}{2}$
  - $\sigma = 0.7071$

Moment generating function

- *We have defined*

$$M_X(t) = E(e^{tX}) = \sum_{x \in \mathcal{X}} e^{tx} p(x)$$

- But where are the moments?
- Using Taylor expansion we get

$$M_X(t) = E\left(\sum_{i=1}^{\infty} \frac{t^i X^i}{i!}\right) = \sum_{i=1}^{\infty} \frac{t^i}{i!} E\left(X^i\right)$$

- We have to derivate

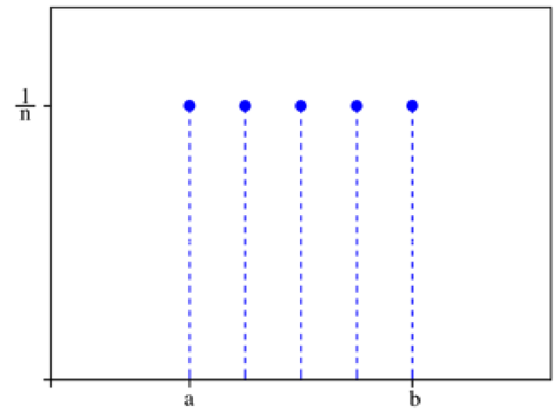
$$E\left(X^i\right) = \frac{\partial^i}{\partial t^i} M_X(t)\Big|_{t=0}$$

Uniform distribution

- A r.v. X is uniformly distributed on $S = \{s_1, s_2, \dots, s_k\}$ if

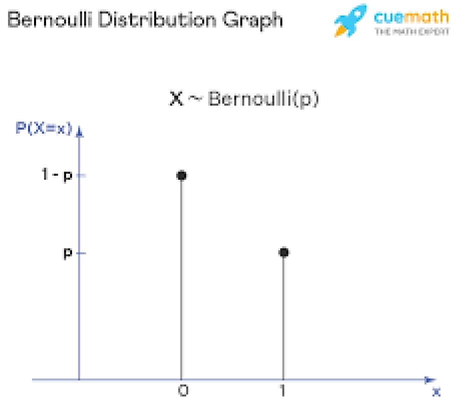
$$p(s_i) = P(X = s_i) = \frac{1}{k}, \qquad i = 1, \dots, k$$

- Notation: $X \sim U\{s_1, s_2, \dots, s_k\}$
- If $X \sim U\{1,2, \dots, k\}$ then $E(X) = \frac{k+1}{2}$ and $Var(X) = \frac{k^2-1}{12}$
- e.g. Ɛ=select a number between 1 and 10⇒ $X \sim U\{1, \dots, 10\}$
  - $P(select\ 5) = \frac{1}{10}$
  - $P(3 \leq X \leq 6) = \frac{4}{10} = \frac{2}{5}$
  - $P(X\ is\ prime) = \frac{4}{10}$



Bernoulli distribution

- A r.v. X has a Bernoulli distribution with parameter θ if
$$p(x) = P(X = x) = \theta^x (1-\theta)^{1-x}, \qquad x \in \{0,1\}$$
- Notation: $X \sim Ber(\theta)$, for $\theta \epsilon (0,1)$
- Moments: $E(X) = \theta, Var(X) = \theta(1-\theta), M_X(t) = 1 - \theta + \theta e^t$
- e.g. Ɛ= A manufacturer produces electronic components and one of every 100 is defective
  - $X = \begin{cases} 1, if\ defective \\ 0, if\ not \end{cases}$ then $X \sim Ber(0.01)$
  - In a batch of 500 how many will be defective (in average)? $500E(X) = 500(0.01) = 5$

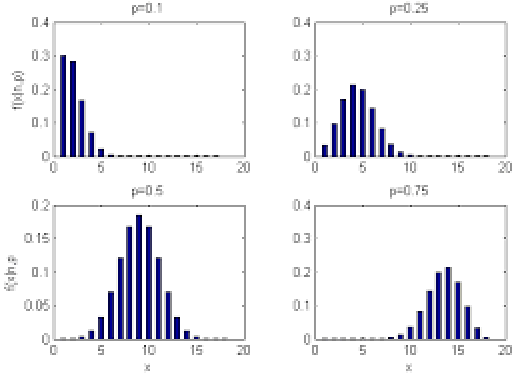


Binomial distribution

- A r.v. X has a Binomial dist. with parameters (n, θ) if

$$p(x) = P(X = x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \qquad x \in \{0,1,\dots,n\}$$

- Notation: $X \sim Bin(n, \theta)$, for $n \in \mathbb{N}$ and $\theta \epsilon (0,1)$
- Moments:
$E(X) = n\theta, Var(X) = n\theta(1-\theta), M_X(t) = (1-\theta+\theta e^t)^n$
- e.g. Ɛ= A manufacturer produces electronic components and one of every 100 is defective
  - $X =$ number of defective in a batch of 500 $\Rightarrow X \sim Bin(500, 0.01)$
  - $P(at\ least\ one\ defective) = P(X \geq 1) = 1 - P(X = 0) = 0.99343$
  - $P(X = 0) = (1-\theta)^n = (0.99)^{500} = 0.00657$
  - $E(X) = 500(0.01) = 5$
  - $Var(X) = 500(0.01)(0.99) = 4.95$
  - $\sigma = 2.22$

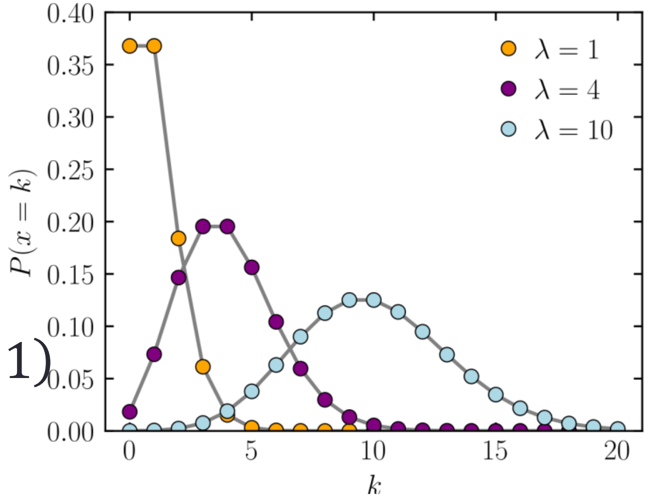


Poisson distribution

- A r.v. X has a Poisson dist. with parameter λ if

$$p(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x \in \{0,1,\ldots\}$$

- Notation: $X \sim Po(\lambda)$, for $\lambda > 0$
- Moments: $E(X) = \lambda = Var(X), M_X(t) = e^{\lambda(e^t - 1)}$
- e.g. X= # scam phone calls in a day
  - In average I receive 3 ⇒ $X \sim Po(3)$
  - $P(more\ than\ 5) = P(X > 5) = 1 - P(X \leq 5)$
  - $= 1 - \sum_{x=1}^{5} e^{-\lambda}\lambda^x / x! = 1 - 0.9160 = 0.084$
  - Y = # scan phone calls in a week ⇒ $Y \sim Po(21)$
  - $P(Y = y) = \text{dpois(y,}\lambda\text{)}$
  - $P(Y \leq y) = \text{ppois(y,}\lambda\text{)}$



Poisson and Binomial

- Let $X \sim Bin(n, \theta)$ and let $Y \sim Po(\lambda)$ with
- $E(X) = n\theta = \lambda = E(Y)$
- If $n \to \infty$ then

$$P(X = x) = P(Y = x)$$

- $P(X = x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$
- $$= \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = e^{-\lambda}\frac{\lambda^x}{x!}$$
  - $\frac{n!}{(n-x)!n^x} \xrightarrow{n \to \infty} 1$
  - $\left(1-\frac{\lambda}{n}\right)^n \xrightarrow{n \to \infty} e^{-\lambda}$
  - $\left(1-\frac{\lambda}{n}\right)^{-x} \xrightarrow{n \to \infty} 1$

Geometric distribution

- A r.v. X has a geometric dist. with parametersθ if
$$p(x) = P(X = x) = \theta(1 - \theta)^{x-1}, \qquad x \in \{1,2, \dots\}$$
- Notation: $X \sim Geo(\theta)$, for $\theta \epsilon (0,1)$
- cdf: $F(x) = 1 - (1 - \theta)^x$
- Moments: $E(X) = 1/\theta, Var(X) = 1/\theta^2, M_X(t) = \frac{\theta e^t}{1-(1-\theta)e^t}$
- Memory less: if $s, t > 0$
$$P(X > t + s | X > s) = P(X > t)$$
- e.g. Ɛ= Suppose a player has 30% chance of getting a hit
  - $X =$ # times at bat until getting a hit $\Rightarrow X \sim Geo(0.3)$
  - $E(X) = \frac{1}{0.3} = 3.33$
  - $P(X > 7 | X > 3) = P(X > 4) = (0.7)^4 = 0.24$