Homework for Week 10

Question 1

Recall that a linear functional on V is a linear map from V to $\mathbb{F}$ where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

Let $f \in V'$ be a linear functional. It follows that $\operatorname{ran}(f)$ is a subspace of $\mathbb{F}$ which can have dimension 0 or 1. If $\operatorname{ran}(f) = \{0\}$, then $f = 0$. Otherwise, $\operatorname{ran}(f) = \mathbb{F}$ and f is surjective.

Question 3

Let $v \in V$ be such that $v \neq 0$. We want to show that there exists a linear functional $f \in V'$ such that $f(v) = 1$.

To that end, let $b = \{b_k\}$ be a basis for V. Recall that $v = \sum_{k=1}^{n} b_k'(v) b_k$ where $b' = \{b_k'\}$ is the dual basis of b. Since $v \neq 0$, there exists a b_k such that $b_k'(v) \neq 0$ and therefore

$$f = \frac{1}{b_k'(v)} b_k'$$

is a linear functional such that $f(v) = 1$.

Question 5

Let T be a linear map from V to W and let $w = \{w_k\}$ be a basis for $\operatorname{ran}(T)$. It follows that for each $v \in V$ we have

$$T(v) = \sum_{k=1}^{n} w_k'(T(v)) w_k$$

where $w' = \{w_k'\}$ is the dual basis of w.

We find that each map $\varphi_k \in W'$ defined in the question is actually the linear functional $T'(w_k')$ where T' is the dual map of T and is therefore a linear functional on V.

Question 8

Writing these maps out explicitly, we find that

$$\Lambda \circ \Gamma(v) = \sum_{k=1}^{n} \varphi_k(v) v_k$$

where $\{v_k\}$ is a basis for V and $\{\varphi_k\}$ is its dual basis (we called these dual basis elements v_k' in class).

Using the resolution of the identity in terms of the dual basis, we find that $\Lambda \circ \Gamma = \mathrm{id}_V$ as required.

These maps are therefore inverses of each other due to the resolution of the identity.

Question 9

Let

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}$$

where $p \in \mathbb{P}_m$.

Writting $p(x) = \sum_{i=0}^{m} a_i x^i$, we find that

$$p^{(k)}(x) = \sum_{i=k}^{m} a_i \frac{i!}{(i-k)!} x^{i-k} \,.$$

Evaluating at $x = 0$, most terms vanish expect for when $i = k$ and we find that

$$p^{(k)}(0) = k! a_k \,.$$

Rearranging, we find that

$$a_k = \frac{p^{(k)}(0)}{k!} = \varphi_k(p)$$

and thus φ_k is the dual basis of the standard basis of $\mathbb{P}_m$.

Question 11

Let $v = \{v_k\}$ be a basis for V and let $\varphi = \{\varphi_k\}$ be its dual basis.

To show that $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$ it suffices to show that they agree on each basis element v_k.

To that end, we find that

$$\sum_{k=1}^{n} \psi(v_k)\varphi_k(v_j) = \psi(v_j)$$

since $\varphi_k(v_j) = \delta_{kj}$ (by the definition of the dual basis).

Question 12

Let S and T be linear maps from V to W. Then

$$(S+T)'(\varphi)(v) = \varphi((S+T)(v)) = \varphi(S(v) + T(v)) = \varphi(S(v)) + \varphi(T(v)) = S'(\varphi)(v) + T'(\varphi)(v) \,.$$

It follows that for each $\varphi \in W'$, $(S+T)'(\varphi) = S'(\varphi) + T'(\varphi)$ and thus $(S+T)' = S' + T'$.

$(\lambda S)' = \lambda S'$ follows similarly.

Question 13

Let us denote the identity map on V as $\mathbb{1}$. Then by definition, for each $\varphi \in V'$ we have

$$\mathbb{1}'(\varphi)(v) = \varphi(\mathbb{1}(v)) = \varphi(v)$$

and thus $\mathbb{1}'$ is the identity map on V'.

Question 16

Suppose that $T = 0$ is the zero map from V to W. Then for each $\varphi \in W'$ and $v \in V$ we have

$$T'(\varphi)(v) = \varphi(T(v)) = \varphi(0) = 0$$

and therefore $T' = 0$.

Conversely, suppose that $T' = 0$. Unpacking the definition, this means that for each $\varphi \in W'$ and $v \in V$ we have

$$T'(\varphi)(v) = \varphi(T(v)) = 0\,.$$

Let $\{w_k\}$ be a basis for W and $\{\varphi_k\}$ be its dual basis. Then by the resolution of the identity, we find that

$$T(v) = \sum_{k=1}^{n} \varphi_k(T(v)) w_k = 0$$

and thus $T(v) = 0$ for each $v \in V$. We conclude that $T = 0$.

Question 25

Recall that Γ is injective if and only if Γ' is surjective. It therefore suffices to show that $\{\phi_1, \ldots, \phi_n\}$ spans V' if and only if Γ' is surjective.

It is not obvious why we would want to consider Γ' instead of Γ. One way to motivate it is that Γ does not actually involve the vector space V' at all which makes it difficult to relate to the dual space V' while Γ' does.

Γ' is a map from $(\mathbb{F}^n)' \to V'$ so let's evaluate it on the standard basis of $(\mathbb{F}^n)'$. We have

$$\Gamma'(e'_k)(v) = e'_k(\Gamma(v)) = e'_k\left([\phi_1(v), \ldots, \phi_n(v)]\right) = \phi_k(v)$$

so $\Gamma'(e'_k) = \phi_k$.

Suppose that Γ' is surjective and let $\varphi \in V'$. Then there exists $f \in (\mathbb{F}^n)'$ such that $\Gamma'(f) = \varphi$. Expressing $f = \sum_{k=1}^n f_k e'_k$, we find that

$$\Gamma'(f) = \sum_{k=1}^{n} f_k \Gamma'(e'_k) = \sum_{k=1}^{n} f_k e'_k(\Gamma) = \sum_{k=1}^{n} f_k \phi_k = \varphi$$

and therefore φ is in the span of $\{\phi_1, \ldots, \phi_n\}$.

Conversely, suppose that $\{\phi_1, \ldots, \phi_n\}$ spans V' and let $\varphi \in V'$. To show that Γ' is surjective, we need to find $f \in (\mathbb{F}^n)'$ such that $\Gamma'(f) = \varphi$. To that end

$$\varphi = \sum_{k=1}^{n} a_k \phi_k = \sum_{k=1}^{n} a_k \Gamma'(e'_k) = \Gamma'\left(\sum_{k=1}^{n} a_k e'_k\right)\,.$$

Similarly, recall that Γ is surjective if and only if Γ' is injective. For part (b), it then suffices to show that $\{\phi_1, \ldots, \phi_n\}$ is linearly independent if and only if Γ' is injective.

To that end, suppose that Γ' is injective and let $a_1, \ldots, a_n$ be such that

$$\sum_{k=1}^{n} a_k \phi_k = 0\,.$$

By the same calculation as above, it follows that

$$\Gamma'\left(\sum_{k=1}^{n} a_k e'_k\right) = 0$$

and therefore $\sum_{k=1}^{n} a_k e'_k = 0$. The standard basis of $(\mathbb{F}^n)'$ is linearly independent so $a_1 = \cdots = a_n = 0$.

Conversely, suppose $\{\phi_1, \ldots, \phi_n\}$ is linearly independent and let $f \in (\mathbb{F}^n)'$ be such that $\Gamma'(f) = 0$. Then

$$\Gamma'(f) = \sum_{k=1}^{n} f_k \Gamma'(e'_k) = \sum_{k=1}^{n} f_k \phi_k = 0$$

but since the ϕ_k are linearly independent, it follows that $f_k = 0$ for each k and therefore $f = 0$.

Question 32

For parts (a) and (c), see the class notes. The map they call Λ is equivalent to our ev.

For part (b), let $f \in V'$. Then

$$(T'' \circ \Lambda)(v)(f) = T''(\Lambda(v))(f) = \Lambda(v)(T'(f)) = T'(f)(v) = f(T(v)) = \Lambda(T(v))(f) = (\Lambda \circ T)(v)(f)\,.$$

We conclude that $T'' \circ \Lambda = \Lambda \circ T$.