# Solutions to Selected Exercises - Week 5

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MAT246H1F: CONCEPTS IN ABSTRACT MATHEMATICS

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## Exercises from [LNS16]

The following exercises are retrieved from Chapter 10 of the textbook [LNS16].

### Calculational Exercises

**Exercise 1.** Consider $\mathbb{R}^3$ with two orthonormal bases: the canonical basis $e = (e_1, e_2, e_3)$ and the basis $f = (f_1, f_2, f_3)$ where:

$$f_1 = \frac{1}{\sqrt{3}}(1,1,1), f_2 = \frac{1}{\sqrt{6}}(1,-2,1), f_3 = \frac{1}{\sqrt{2}}(1,0,-1).$$

Find the matrix, $S$, of the change of basis transformation such that:

$$[v]_f = S[v]_e, \quad \text{for all } v \in \mathbb{R}^3,$$

where $[v]_b$ denotes the column vector of $v$ with respect to the basis $b$.

*Solution.* Since $e$ and $f$ are orthonormal bases, we know that:

$$S = (s_{ij})_{i,j=1}^3 \text{ with } s_{ij} = \langle e_j, f_i \rangle.$$

Hence:

$$s_{11} = \langle e_1, f_1 \rangle = \langle (1,0,0), \frac{1}{\sqrt{3}}(1,1,1) \rangle = \frac{1}{\sqrt{3}},$$

$$s_{12} = \langle e_2, f_1 \rangle = \langle (0,1,0), \frac{1}{\sqrt{3}}(1,1,1) \rangle = \frac{1}{\sqrt{3}},$$

$$s_{13} = \langle e_3, f_1 \rangle = \langle (0,0,1), \frac{1}{\sqrt{3}}(1,1,1) \rangle = \frac{1}{\sqrt{3}},$$

$$s_{21} = \langle e_1, f_2 \rangle = \langle (1,0,0), \frac{1}{\sqrt{6}}(1,-2,1) \rangle = \frac{1}{\sqrt{6}},$$

$$s_{22} = \langle e_2, f_2 \rangle = \langle (0,1,0), \frac{1}{\sqrt{6}}(1,-2,1) \rangle = -\frac{2}{\sqrt{6}},$$

$$s_{23} = \langle e_3, f_2 \rangle = \langle (0,0,1), \frac{1}{\sqrt{6}}(1,-2,1) \rangle = \frac{1}{\sqrt{6}},$$

$$s_{31} = \langle e_1, f_3 \rangle = \langle (1,0,0), \frac{1}{\sqrt{2}}(1,0,-1) \rangle = \frac{1}{\sqrt{2}},$$

$$s_{32} = \langle e_2, f_3 \rangle = \langle (0,1,0), \frac{1}{\sqrt{2}}(1,0,-1) \rangle = 0,$$

$$s_{33} = \langle e_3, f_3 \rangle = \langle (0,0,1), \frac{1}{\sqrt{2}}(1,0,-1) \rangle = -\frac{1}{\sqrt{2}}.$$

Thus:

$$S = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

□

## Proof-Writing Exercises

**Exercise 1.** Let $V$ be a finite-dimensional vector space over $\mathbb{F}$ with dimension $n \in \mathbb{Z}_+$, and suppose that $b = (v_1, v_2, \ldots, v_n)$ is a basis for $V$. Prove that the coordinate vectors $[v_1]_b, [v_2]_b, \ldots, [v_n]_b$ with respect to $b$ form a basis for $\mathbb{F}^n$.

*Solution.* Since $b$ is a basis, for every $w \in V$ there exist unique coefficients[1] $a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that:

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

Hence:

$$[w]_b = [a_1, a_2, \ldots, a_n]^\top.$$

In the light of this:

$$\begin{aligned} [v_1]_b &= (1, 0, \ldots, 0)^\top; \\ [v_2]_b &= (0, 1, \ldots, 0)^\top; \\ [v_n]_b &= (0, 0, \ldots, 1)^\top; \end{aligned}$$

that is, for all $i = 1, \ldots, n$ $[v_i]_b = e_i$ where $e_i$ is the $i$-th vector of the canonical basis $e$ of $\mathbb{F}^n$. All in all, $[v_1]_b, [v_2]_b, \ldots, [v_n]_b$ are a basis for $\mathbb{F}^n$, as desired. □

**Exercise 2.** Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V)$ is a linear operator having the following property: given any two bases $b$ and $c$ for $V$, the matrix $M(T, b)$ for $T$ with respect to $b$ is the same as the matrix $M(T, c)$ for $T$ with respect to $c$. Prove that there exists a scalar $\alpha \in \mathbb{F}$ such that $T = \alpha I_V$ where $I_V$ denotes the identity map on $V$.

*Solution.* If $\dim V = 1$, there is nothing to prove. Hence, in the following, $n = \dim V \geq 2$ will be assumed.

Let $v \in V \setminus \{0\}$. Define $w = Tv$. Assume that $v$ and $w$ are linearly independent. Then $w \neq 0$ and the lists $(v, w)$ and $(w, v)$ can be extended to two bases $b$ and $c$ for $V$ as follows:

$$b = (v, w, u_3, \ldots, u_n);$$

$$c = (w, v, u_3, \ldots, u_n).$$

By assumption $M(T, b) = M(T, c)$. As a consequence:

$$\begin{aligned} [Tw]_c &= \text{ first column of } M(T, c) \\ &= \text{ first column of } M(T, b) \\ &= [0, 1, \ldots, 0]^\top \\ &= [v]_c; \end{aligned}$$

Hence: $Tw = v$. Since $v$ and $w$ are independent, also:

$$d = (v + w, v, u_3, \ldots, u_n);$$

is a basis for $V$ and by assumption $M(T, d) = M(T, b) = M(T, c)$. But:

$$T(v + w) = Tv + Tw = w + v = v + w$$

[1]See Lemma 5.2.6 p. 51.

Hence the first column of $M(T,d)$ will be $[1,0,\dots,0]^\top$. This is different from the first column of $M(T,b)$. This contradicts the hypotheses of the exercise. As a consequence, $w = Tv$ and $v$ are linearly dependent and there exists $\lambda_v \in \mathbb{F}$ such that $w = Tv = \lambda_v v$. Recalling that $v \neq 0$, there exists $n-1$ vectors $g_2, g_3, \dots, g_n$ such that $e = (v, g_2, g_3, \dots, g_n)$ is a basis for $V$. Clearly the first column of $M(T,e)$ is $[\lambda_v, 0, \dots, 0]^\top$. Let $e_2 = (g_2, v, g_3, \dots, g_n)$. This is another basis for $V$. Then $M(T,e) = M(T,e_2)$ by assumption, and hence the second column of $M(T,e)$ is equal to the second column of $M(T,e_2)$ that is $[0,\lambda_v,0,\dots,0]^\top$. Iterating this procedure one finds that $M(T,e) = \lambda_v I_n$, where $I_n$ is the identity $n \times n$ matrix. This implies that for every $u \in V$, $Tu = \lambda_v u$. Choosing $\alpha = \lambda_v$ the exercise is done. $\square$

# Exercises from [DD10]

The following exercises are retrieved from Section 7.4 of the textbook [DD10].

**Exercise J.** Let $\mathrm{Tr}(A) = \sum_{i=1}^n a_{ii}$ denote the trace on the space $\mathcal{M}_n$ of all $n \times n$ matrices. Show that there is an inner product on $\mathcal{M}_n$ given by $\langle A, B\rangle = \mathrm{Tr}(AB^t) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij}$. The norm $\|A\|_2 = \langle A, A\rangle^{1/2}$ is called the **Hilbert-Schmidt** norm.

*Solution.* To avoid confusion, denote $Z$ the zero matrix, i.e the matrix whose entries are all 0.

1. Let $A \in \mathcal{M}_n$ be arbitrarily chosen. Then:
$$\langle A, A\rangle = \mathrm{Tr}(AA^t) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} a_{ij} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \geq 0;$$
indeed in the last term, all summands are non-negative. Hence, for every $A \in \mathcal{M}_n$, $\langle A, A\rangle \geq 0$. If $A \in \mathcal{M}_n$ is such that $\langle A, A\rangle = 0$, then from the above chain of equalities:
$$0 = \langle A, A\rangle = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2$$
Hence $a_{ij}^2 = 0$ for all $i = 1,\dots,n$ and all $j = 1,\dots,n$; and hence $a_{ij} = 0$ for all $i = 1,\dots,n$ and all $j = 1,\dots,n$. Thus $A = Z$, i.e. $A$ is the zero-matrix. Conversely, $\langle Z, Z\rangle = 0$. All in all, $\langle A, A\rangle = 0$ if and only if $A = Z$ i.e. $A$ is the zero matrix.

2. Let $A, B \in \mathcal{M}_n$ be arbitrarily chosen, then:
$$\langle A, B\rangle = \mathrm{Tr}(AB^t) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij} = \sum_{i=1}^n \sum_{j=1}^n b_{ij} a_{ij} = \mathrm{Tr}(BA^t) = \langle B, A\rangle.$$
All in all, for every $A, B \in \mathcal{M}_n$, $\langle A, B\rangle = \langle B, A\rangle$.

3. Let $A, B, C \in \mathcal{M}_n$, $\lambda, \mu \in \mathbb{R}$ be arbitrarily chosen, then:
$$\begin{aligned}\langle \lambda A + \mu B, C\rangle = \mathrm{Tr}((\lambda A + \mu B)C^t) &= \sum_{i=1}^n \sum_{j=1}^n (\lambda A + \mu B)_{ij} c_{ij} \\ &= \sum_{i=1}^n \sum_{j=1}^n (\lambda a_{ij} + \mu b_{ij}) c_{ij} \\ &= \sum_{i=1}^n \sum_{j=1}^n \lambda a_{ij} c_{ij} + \sum_{i=1}^n \sum_{j=1}^n \mu b_{ij} c_{ij} \\ &= \lambda \sum_{i=1}^n \sum_{j=1}^n a_{ij} c_{ij} \mu \sum_{i=1}^n \sum_{j=1}^n b_{ij} c_{ij} \\ &= \lambda \,\mathrm{Tr}(AC^t) + \mu \,\mathrm{Tr}(BC^t) \\ &= \lambda \langle A, C\rangle + \mu \langle B, C\rangle.\end{aligned}$$
All in all, $\langle \lambda A + \mu B, C\rangle = \lambda\langle A, C\rangle + \mu\langle B, C\rangle$ for every $A, B, C \in \mathcal{M}_n$, $\lambda, \mu \in \mathbb{R}$.

This shows that all the axioms of inner product are satisfied by the proposed $\langle \cdot, \cdot \rangle$ and hence it is an inner product on $\mathcal{M}_n$. $\square$

# References


[DD10] Kenneth R. Davidson and Allan P. Donsig. *Real analysis and applications*. Undergraduate Texts in Mathematics. Theory in practice. Springer, New York, 2010, pp. xii+513. ISBN: 978-0-387-98097-3. DOI: 10.1007/978-0-387-98098-0. URL: https://doi.org/10.1007/978-0-387-98098-0.

[LNS16] Isaia Lankham, Bruno Nachtergaele, and Anne Schilling. *Linear Algebra As an Introduction to Abstract Mathematics*. Nov. 15, 2016. URL: https://www.math.ucdavis.edu/~anne/linear_algebra/.