Solutions to Selected Exercises - Week 8

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MAT246H1F: CONCEPTS IN ABSTRACT MATHEMATICS

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Exercises from Chapter 2 of [LNS16]

Calculational Exercises

Exercise 2. Compute the real and imaginary part of e^{e^z} for $z \in \mathbb{C}$.

Solution. Since $z \in \mathbb{C}$, then $z = x + iy$ with $x = \Re z \in \mathbb{R}$, $y = \Im z \in \mathbb{R}$. Then:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

As a consequence:

$$e^{e^z} = e^{e^x(\cos y + i\sin y)} = e^{e^x \cos y + ie^x \sin y} = e^{e^x \cos y} e^{ie^x \sin y} = e^{e^x \cos y} \left(\cos(e^x \sin y) + i \sin(e^x \sin y) \right).$$

Thus:

$$e^{e^z} = e^{e^x \cos y} \cos(e^x \sin y) + ie^{e^x \cos y} \sin(e^x \sin y).$$

All in all:

$$\Re\left(e^{e^z}\right) = e^{e^x \cos y} \cos(e^x \sin y) = e^{e^{\Re z} \cos(\Im z)} \cos(e^{\Re z} \sin(\Im z));$$
$$\Im\left(e^{e^z}\right) = e^{e^x \cos y} \sin(e^x \sin y) = e^{e^{\Re z} \cos(\Im z)} \sin(e^{\Re z} \sin(\Im z)).$$

$\square$

Proof-Writing Exercises

The following exercises are retrieved from Chapter 2 of the textbook [LNS16].

Exercise 3. Let $z, w \in \mathbb{C}$. Prove the *parallelogram law* $|z-w|^2 + |z+w| = 2(|z|^2 = |w|^2)$.

Solution. Let $z, w \in \mathbb{C}$. Then:

$$|z-w|^2 = (z-w)\overline{(z-w)} = (z-w)(\overline{z} - \overline{w}) = z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w} = |z|^2 - z\overline{w} - w\overline{z} + |w|^2;$$
$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} = |z|^2 + z\overline{w} + w\overline{z} + |w|^2.$$

Thus, summing both sides of the equations above:

$$|z-w|^2 + |z+w|^2 = |z|^2 - z\overline{w} - w\overline{z} + |w|^2 + |z|^2 + z\overline{w} + w\overline{z} + |w|^2 = 2|z|^2 + 2|w|^2.$$

$\square$

Exercises from Chapter 9 of [LNS16]

The following exercises are retrieved from Chapter 9 of the textbook [LNS16].

Proof-Writing Exercises

Exercise 5. Let V be a finite-dimensional inner product space over $\mathbb{C}$. Given $u, v \in V$, prove that:

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + \frac{\|u+iv\|^2 - \|u-iv\|^2}{4} i$$

Solution. Let $u, v \in V$. Then:

$$\begin{aligned}
\|u+v\|^2 = \langle u+v, u+v\rangle &= \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle \\
&= \|u\|^2 + \overline{\langle u,v\rangle} + \langle u,v\rangle + \|v\|^2 \\
&= \|u\|^2 + 2\Re\langle u,v\rangle + \|v\|^2;
\end{aligned}$$

$$\begin{aligned}
\|u-v\|^2 = \langle u+v, u+v\rangle &= \langle u,u\rangle - \langle v,u\rangle - \langle u,v\rangle + \langle v,v\rangle \\
&= \|u\|^2 - \overline{\langle u,v\rangle} - \langle u,v\rangle + \|v\|^2 \\
&= \|u\|^2 - 2\Re\langle u,v\rangle + \|v\|^2.
\end{aligned}$$

Thus:

$$\frac{\|u+v\|^2 + \|u-v\|^2}{4} = \frac{\|u\|^2 + 2\Re\langle u,v\rangle + \|v\|^2 - \left(\|u\|^2 - 2\Re\langle u,v\rangle + \|v\|^2\right)}{4} = \frac{4\Re\langle u,v\rangle}{4} = \Re\langle u,v\rangle.$$

On the other hand:

$$\begin{aligned}
\|u+iv\|^2 = \langle u+iv, u+iv\rangle &= \langle u,u\rangle + \langle iv,u\rangle + \langle u,iv\rangle + \langle iv,iv\rangle \\
&= \|u\|^2 + \langle iv,u\rangle + \langle u,iv\rangle + \|iv\|^2 \\
&= \|u\|^2 + i\langle v,u\rangle - i\langle u,v\rangle + |i|^2\|v\|^2 \\
&= \|u\|^2 + i\overline{\langle u,v\rangle} - i\langle u,v\rangle + \|v\|^2 \\
&= \|u\|^2 - i\left(\langle u,v\rangle - \overline{\langle u,v\rangle}\right) + \|v\|^2 \\
&= \|u\|^2 - i\left(2i\Im\langle u,v\rangle\right) + \|v\|^2 \\
&= \|u\|^2 + 2\Im\langle u,v\rangle + \|v\|^2;
\end{aligned}$$

$$\begin{aligned}
\|u-iv\|^2 = \langle u-iv, u-iv\rangle &= \langle u,u\rangle + \langle -iv,u\rangle + \langle u,-iv\rangle + \langle -iv,-iv\rangle \\
&= \|u\|^2 + \langle -iv,u\rangle + \langle u,-iv\rangle + \|-iv\|^2 \\
&= \|u\|^2 - i\langle v,u\rangle + i\langle u,v\rangle + |-i|^2\|v\|^2 \\
&= \|u\|^2 - i\overline{\langle u,v\rangle} + i\langle u,v\rangle + \|v\|^2 \\
&= \|u\|^2 + i\left(\langle u,v\rangle - \overline{\langle u,v\rangle}\right) + \|v\|^2 \\
&= \|u\|^2 + i\left(2i\Im\langle u,v\rangle\right) + \|v\|^2 \\
&= \|u\|^2 - 2\Im\langle u,v\rangle + \|v\|^2.
\end{aligned}$$

$$\frac{\|u+iv\|^2 + \|u-iv\|^2}{4} = \frac{\|u\|^2 + 2\Im\langle u,v\rangle + \|v\|^2 - \left(\|u\|^2 - 2\Im\langle u,v\rangle + \|v\|^2\right)}{4} = \frac{4\Im\langle u,v\rangle}{4} = \Im\langle u,v\rangle.$$

All in all:

$$\langle u,v\rangle = \Re\langle u,v\rangle + i\Im\langle u,v\rangle = \frac{\|u+v\|^2 + \|u-v\|^2}{4} + i\frac{\|u+iv\|^2 + \|u-iv\|^2}{4};$$

as desired. □

Remark. The equality proved above is called *Polarization Identity*.

References

[LNS16] Isaia Lankham, Bruno Nachtergaele, and Anne Schilling. *Linear Algebra As an Introduction to Abstract Mathematics*. Nov. 15, 2016. URL: https://www.math.ucdavis.edu/~anne/linear_algebra/.