Problem 1

Let $f : [0,1] \to \mathbb{C}$ such that $f(x) = \frac{1}{i-x}$. Find the real and imaginary parts of f. Compute $f'(x)$ and $\int_0^1 f(x)\,\mathrm{d}x$.

Solution

We find the real and imaginary parts of f by multiplying the numerator and denominator by the conjugate of the denominator.

$$f(x) = \frac{1}{i-x} = \frac{1}{i-x} \cdot \frac{-i-x}{-i-x} = -\frac{x+i}{x^2+1} = -\frac{x}{x^2+1} - \frac{i}{x^2+1}.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{x}{1+x^2}\right) = \frac{x^2-1}{\left(x^2+1\right)^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{x^2+1}\right) = \frac{2x}{\left(x^2+1\right)^2}$$

we have

$$f'(x) = \frac{x^2+2ix-1}{(x^2+1)^2} = \left(\frac{x+i}{x^2+1}\right)^2 = \left(\frac{1}{i-x}\right)^2.$$

Similarly, we can notice that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{2}\log\left(x^2+1\right)\right) = -\frac{x}{x^2+1} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(-\arctan(x)\right) = -\frac{1}{x^2+1}$$

and therefore

$$\int_0^1 f(x)\,\mathrm{d}x = \left[-\frac{1}{2}\log\left(x^2+1\right) - \arctan(x)\right]_0^1 = -\frac{1}{2}\log(2) - \arctan(1).$$

Problem 2

Show that the maps $Tf : f \mapsto f'$ and $Sf : f \mapsto \int_0^x f(t)\,\mathrm{d}t$ are $\mathbb{C}$-linear maps from $C^1([0,1],\mathbb{C})$ to $C([0,1],\mathbb{C})$ and $C([0,1],\mathbb{C})$ to $\mathbb{C}$, respectively.

Solution

The problem has a typo. We should have $Sf : f \mapsto \int_0^1 f(t)\,\mathrm{d}t$ although as written you can indeed show that S is linear from $C([0,1],\mathbb{C})$ to itself.

Let $f(x) = f_r(x) + if_i(x)$, $g(x) = g_r(x) + ig_i(x)$ and $z = a + ib$. Then

$$T(zf+g) = ((a+ib)(f_r+if_i) + g_r + ig_i)' = ((af_r - bf_i + g_r) + i(af_i + bf_r + g_i))'$$

which by definition is equal to

$$(af_r - bf_i + g_r)' + i(af_i + bf_r + g_i)'.$$

Using linearity of the *real-valued* derivative we conclude that

$$T(zf+g) = af_r' - bf_i' + g_r' + iaf_i' + ibf_r' + ig_i' = (a+bi)(f_r' + if_i') + (g_r' + ig_i)' = zT(f) + T(g)\,.$$

The case of S follows similarly.

Problem 3

Using the fundamental theorem of calculus for real functions, prove the fundamental theorem of calculus for complex functions, i.e.

$$\int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0)$$

for $f \in C^1([0,1],\mathbb{C})$.

Solution

Let $f = f_r + if_i$ with f_r and f_i real-valued. Then by the definition of the complex-valued derivative and integral we have

$$\int_0^1 f'(x)\,\mathrm{d}x = \int_0^1 f_r'(x)\,\mathrm{d}x + i\int_0^1 f_i'(x)\,\mathrm{d}x\,.$$

Now using the fundamental theorem of calculus for real-valued functions, this is equal to

$$f_r(1) - f_r(0) + if_i(1) - if_i(0) = f(1) - f(0)$$

as required.

Problem 4

Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{zt} = ze^{zt}$$

for $z \in \mathbb{C}$.

Solution

See class notes.

Problem 5

Consider the integral $I = \int_0^\infty e^{-at}\cos{(bt)}\,\mathrm{d}t$ where $a > 0$ and $b \in \mathbb{R}$ are real numbers.

1. Calculate I using integration by parts.
2. Show that $I = \mathrm{Re}\left[\int_0^\infty e^{-(a-ib)t}\,\mathrm{d}t\right]$ where Re denotes the real part of a complex number.
3. Calculate I using the formula above. Which do you prefer?

Solution

Part (1) is a standard example from first-year calculus.

By the definition of the complex-valued integral, we have

$$\mathrm{Re}\left[\int_0^\infty f(t)\,\mathrm{d}t\right] = \int_0^\infty f_r(t)\,\mathrm{d}t$$

where $f(x) = f_r(x) + if_i(x)$ for f_r and f_i real-valued.

Using exponential rules and Euler's formula, we have

$$e^{-(a-ib)t} = e^{-at}\cos(bt) + ie^{-at}\sin(bt)$$

which is of the form $f(x) = f_r(x) + if_i(x)$ with $f_r(x) = e^{-at}\cos(bt)$ as desired.

For part (3), instead of using integration by parts, we first calculate $\int_0^\infty e^{-(a-ib)t}\,\mathrm{d}t$ and then take the real part.

To that end, we have

$$\int_0^\infty e^{-(a-ib)t}\,\mathrm{d}t = -\frac{1}{a-ib}\int_0^\infty (ib-a)e^{-(a-ib)t}\,\mathrm{d}t = -\frac{1}{a-ib}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-(a-ib)t}\right)\,\mathrm{d}t = \frac{1}{a-ib}\,.$$

Taking the real part, we find that $I = \frac{a}{a^2+b^2}$.

Problem 6

Let V be a finite-dimensional inner product space over $\mathbb{C}$ with inner product $\langle\cdot,\cdot\rangle$ and $b = \{b_1,\ldots,b_n\}$ an orthonormal basis of V.

Using the *resolution of the identity* formula, i.e.

$$v = \sum_{i=1}^{n}\langle v, b_i\rangle b_i$$

for $v \in V$, show that the matrix elements of a linear operator $A : V \to V$ with respect to the basis b are given by

$$A_{ij} = \langle A(b_j), b_i\rangle\,.$$

Solution

Recall that $M_b(T)$ is the matrix of T with respect to the basis b if and only if

$$T(b_j) = \sum_{i=1}^{n} M_b(T)_{ij}\, b_i$$

holds for all $j \in \{1, \ldots, n\}$.

Using the resolution of the identity formula, applied to the vector $A(b_j)$,

$$A(b_j) = \sum_{i=1}^{n} \langle A(b_j), b_i \rangle\, b_i\,.$$

This holds for all $j \in \{1, \ldots, n\}$ and therefore the elements of $M_b(A)$ (denoted here A_{ij}) must be given by

$$A_{ij} = \langle A(b_j), b_i \rangle\,.$$