Solutions to Selected Exercises - Week 5

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MAT246H1F: CONCEPTS IN ABSTRACT MATHEMATICS

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The following exercises are retrieved from Chapter 6 of the textbook [LNS16].

Calculational Exercises

Exercise 4. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ having the property that

$$\forall\, a \in \mathbb{R}, \forall\, v \in \mathbb{R}^2, f(av) = af(v) \tag{1}$$

but such that f is not a linear map.

Solution. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} 0, & \text{if } y = 0 \\ \frac{x^2}{y}, & \text{if } y \neq 0. \end{cases}$$

This function has the required property, indeed, let $a \in \mathbb{R}$, then:

1. for every $(x,0) \in \mathbb{R}^2$, $f(x,0) = 0$ hence:

$$f(a(x,0)) = f(ax,0) = 0 = a \cdot 0 = af(x,0).$$

2. for every $(x,y) \in \mathbb{R}^2$ such that $y \neq 0$, one has:

$$f(a(x,y)) = f(ax,ay) = \begin{cases} f(0,0) = 0, & \text{if } a = 0 \\ \frac{(ax)^2}{ay}, & \text{if } a \neq 0 \end{cases} = \begin{cases} 0 \cdot f(x,y), & \text{if } a = 0 \\ \frac{ax^2}{y}, & \text{if } a \neq 0. \end{cases} = af(x,y).$$

Since a was chosen arbitrarily, one has that, for all $a \in \mathbb{R}$ and for all $(x,y) \in \mathbb{R}^2$, $f(a(x,y)) = af(x,y)$, i.e. property (1) holds true. Finally, one can remark that: $f(1,1) = 1$ and $f(1,0) = 0$ but

$$f((1,1) + (1,0)) = f(2,1) = 4 \neq 1 + 0 = f(1,1) + f(1,0).$$

This shows that f is not linear. $\square$

Proof-Writing Exercises

Exercise 2. Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V,W)$ is injective. Given any linearly independent list $(v_1, \ldots, v_n)$ of vectors in V, prove that the list $(T(v_1), \ldots, T(v_n))$ is linearly independent in W.

Solution. Let $a_i \in \mathbb{F}$ for $i = 1 \dots n$ be such that:

$$\sum_{i=1}^{n} a_i T(v_i) = 0. \tag{2}$$

Since T is linear, the above reads:

$$0 = \sum_{i=1}^{n} T(a_i v_i) = T\left(\sum_{i=1}^{n} a_i v_i\right).$$

Thus:

$$\sum_{i=1}^{n} a_i v_i \in \text{null}(T).$$

By assumption T is injective, hence $\text{null}(T) = \{0\}$. This implies:

$$\sum_{i=1}^{n} a_i v_i = 0.$$

But $(v_1, \dots, v_n)$ is a linearly independent list of vectors in V, hence:

$$\forall\ i = 1, \dots, n\ a_i = 0.$$

Recalling that this argument begun with equation (2), this implies that $(T(v_1), \dots, T(v_n))$ is a linearly independent list of vectors in W. □

Exercise 3. Let U, V and W be vector spaces over $\mathbb{F}$, and suppose that the linear maps $S \in \mathcal{L}(U, V)$and $T \in \mathcal{L}(V, W)$ are both injective. Prove that the composition $T \circ S$ is injective.

Solution. Recall that a linear function between vector spaces is injective if and only if its null space is $\{0\}$.

Assume $u \in \text{null}(T \circ S)$, that is to say, $(T \circ S)(u) = 0$. This means that $T(Su) = 0$, i.e. $Su \in \text{null}(T)$. But T is injective, hence $\text{null}(T) = \{0\}$. This implies $Su = 0$. As a consequence, $u \in \text{null}(S)$. But S is injective as well, so $\text{null}(S) = \{0\}$. This implies $u = 0$.

Summarizing, if $u \in \text{null}(T \circ S)$, then $u = 0$. This shows that $\text{null}(S \circ T) = \{0\}$ and so $T \circ S$ is injective. □

Exercise 4. Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is surjective. Given a spanning list $(v_1, \dots, v_n)$ for V, prove that:

$$\text{span}(T(v_1), \dots, T(v_n)) = W.$$

Solution. By definition of T, for all $i = 1, \dots, n$ $T(v_i) \in W$. Hence:

$$\text{span}(T(v_1), \dots, T(v_n)) \subset W,$$

To complete the proof one has to show the reverse inclusion. To this aim let $w \in W$. Since T is surjective, there exists $v \in V$ such that $w = Tv$. Since $(v_1, \dots, v_n)$ is a spanning list for V, there exist $a_i \in \mathbb{F}$, with $i = 1, \dots, n$ such that $v = \sum_{i=1}^{n} a_i v_i$. Since T is linear:

$$w = Tv = T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} T(a_i v_i) = \sum_{i=1}^{n} a_i T(v_i).$$

All in all, w is a linear combination of $T(v_1), \dots, T(v_n)$.

The choice of $w \in W$ was arbitrary, hence for every $w \in W$ there exist $a_i \in \mathbb{F}$ with $i = 1, \dots, n$ such that $w = \sum_{i=1}^{n} a_i T(v_i)$. This amounts to say that

$$\text{span}(T(v_1), \dots, T(v_n)) \supset W.$$

This completes the proof. □

References

[LNS16] Isaia Lankham, Bruno Nachtergaele, and Anne Schilling. *Linear Algebra As an Introduction to Abstract Mathematics*. Nov. 15, 2016. URL: https://www.math.ucdavis.edu/~anne/linear_algebra/.