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**University of Toronto**  
**Faculty of Arts and Sciences**

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## **MAT244H1F Homework 2**

### *Special Instructions:*

- See the **submission instructions** on Quercus for details about submitting this assignment.
- Include **all pages** when submitting your assignment.
- Write legibly and darkly (if scanned) or digitally on a tabled.
- For questions with a boxed area, ensure your answer is completely within the box.
- Fill in your bubbles completely.

Good: ☒ A ☐ B

Bad: ☐ A ☒ B ☒ C

1. Certain non-separable equations can be solved using the method of *integrating factors*. It works by exploiting the product rule as follows. Since  $[r(x)y]' = r(x)y' + r'(x)y$ , if you are given an equation of the form  $A(x)y' + B(x)y = C(x)$ , you may be able to multiply both sides by a non-zero function  $p(x)$  so that the left hand side of the equation looks like a product rule.

(a) Consider  $2y' + 4y$ . Find non-zero functions  $p(x)$  and  $r(x)$  so that  $p(x) \cdot (2y' + 4y) = [r(x)y]'$ .

$p(x) =$

$r(x) =$

(b) Consider  $y' + \cos(x)y$ . Find non-zero functions  $p(x)$  and  $r(x)$  so that  $p(x) \cdot (y' + \cos(x)y) = [r(x)y]'$ .

$p(x) =$

$r(x) =$

(c) Solve the differential equation  $[e^{2x}y]' = \frac{x^2}{2}$  for  $y$ . (Hint: most of the work has been done for you already.)

Solution:

(d) Use the method of integrating factors to solve the differential equation  $2y' + 4y = \frac{x^2 e^{-2x}}{2}$ . Explain your steps.

2. To answer this question, you may need to review linear combinations and change of basis from MAT223.

Consider the differential equation  $y'' = 0$  which has solutions  $s_1(x) = x + 1$  and  $s_2(x) = x - 1$  (among many others).

(a) Express the solution  $s_3(x) = 2x + 4$  as a linear combination of  $s_1$  and  $s_2$ .

$$s_3(x) = \boxed{\phantom{000000}} s_1(x) + \boxed{\phantom{000000}} s_2(x)$$

(b) The complete solution to  $y'' = 0$  is  $r(x) = Ax + B$  where  $A, B \in \mathbb{R}$ . Express  $r$  as a linear combination of  $s_1$  and  $s_2$ . (*Hint: your solution will involve the constants  $A$  and  $B$ .*)

$$r(x) = \boxed{\phantom{000000}} s_1(x) + \boxed{\phantom{000000}} s_2(x)$$

(c) Give two different bases for the solution set to  $y'' = 0$ .

Basis 1:  $\left\{ \begin{array}{c} \vdots \\ \vdots \end{array} \right\}$

Basis 2:  $\left\{ \begin{array}{c} \vdots \\ \vdots \end{array} \right\}$

(d) The differential equation  $y''(x) = J_1(x)$  has a solution  $F_{\text{ugly}}(x) = (\frac{2}{x} - 1)J_1(x) - \frac{1}{x}J_0(x)$  where  $J_1$  and  $J_0$  are Bessel Functions of the First Kind<sup>1</sup>. Use your knowledge from the previous parts to find the complete solution to  $y'' = J_1(x)$ . Explain your reasoning.

*Hint: You do not need to know what Bessel functions are to answer this question.*

<sup>1</sup>[https://en.wikipedia.org/wiki/Bessel\\_function](https://en.wikipedia.org/wiki/Bessel_function)

3. Read the appendix on *complex numbers* in your textbook (you may seek additional resources if needed).

- (a) Let  $x = 3 + 2i$  and  $y = -2 + 5i$ . Compute  $xy$  and  $\frac{x}{y}$ . Express your answer in  $a + bi$  form.

$xy =$

$\frac{x}{y} =$

- (b) The *absolute value* or *modulus* of a complex number  $a + bi$  is  $|a + bi| = \sqrt{a^2 + b^2}$ , and when a vector  $\vec{v} = (z_1, z_2, \dots, z_n)$  has complex entries, its norm is computed by the formula  $\|\vec{v}\| = \sqrt{|z_1|^2 + |z_2|^2 + \dots + |z_n|^2}$

- i. Let  $x = a + bi$ . Find a complex number  $y$  so that  $|x|^2 = xy$ . (Such a number is called the *complex conjugate* of  $x$ .)

$y =$

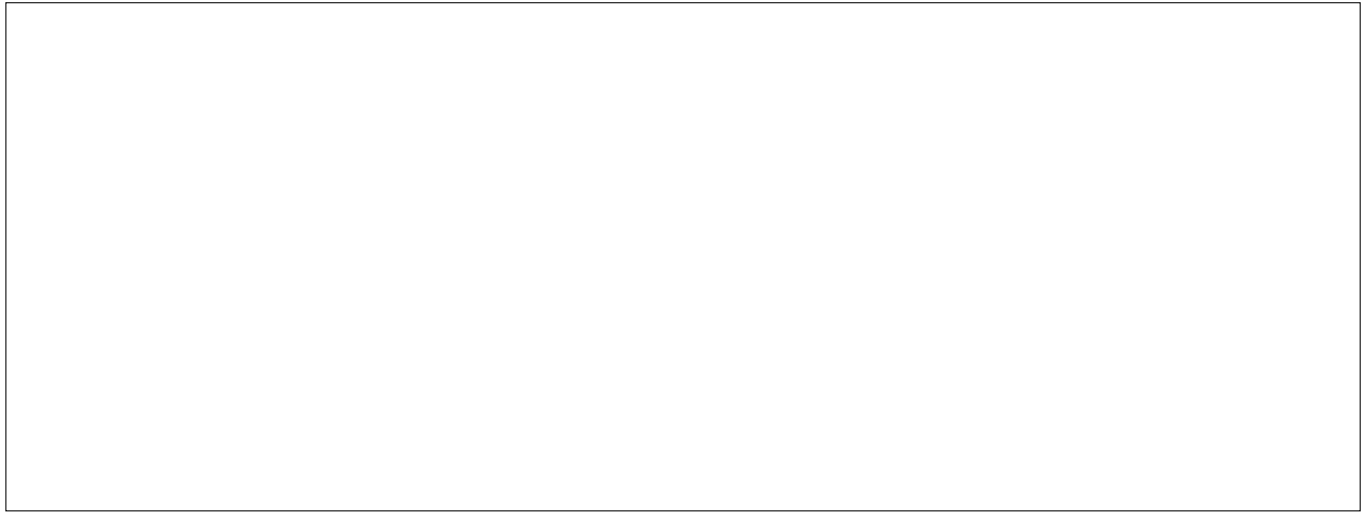
- ii. Let  $\vec{v} = (1 + 2i, c)$ . Find a *real number*  $c$  so that  $\|\vec{v}\| = 4$ .

$c =$

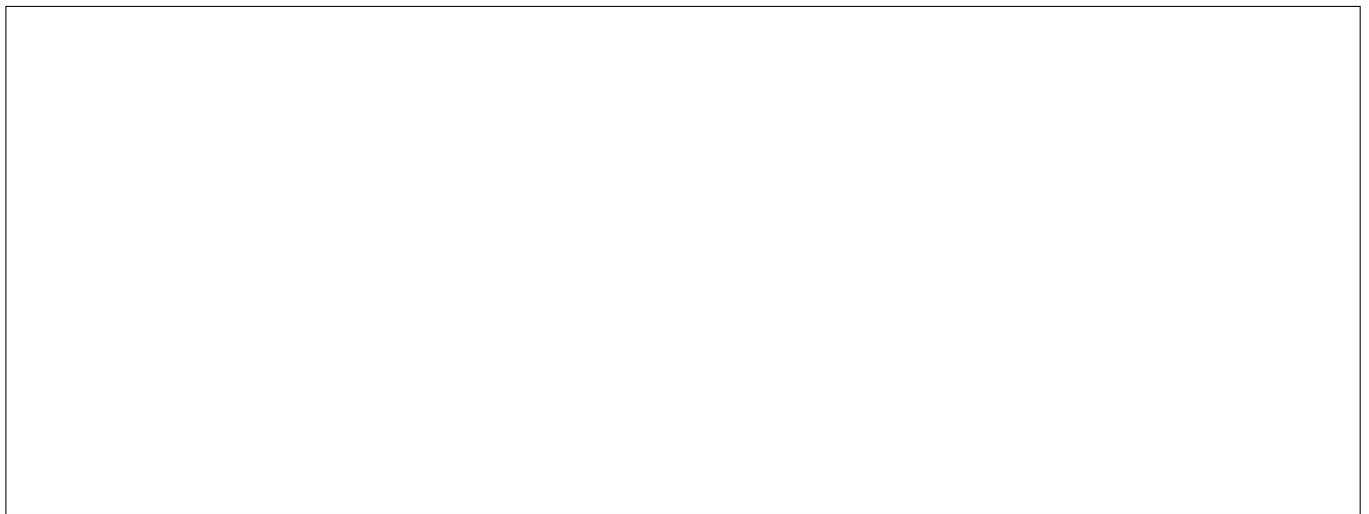
- (c) A complex number  $a + bi$  is equal to zero if and only if  $a = 0$  and  $b = 0$ . Let  $x = a + bi$  and  $y = c + di$  and suppose  $a, b, c, d \neq 0$ . Show that  $xy \neq 0$ .

4. For each of the following, sketch a phase portrait satisfying the criteria or explain why no such phase portrait exists. (Assume all differential equations/systems are continuous.)

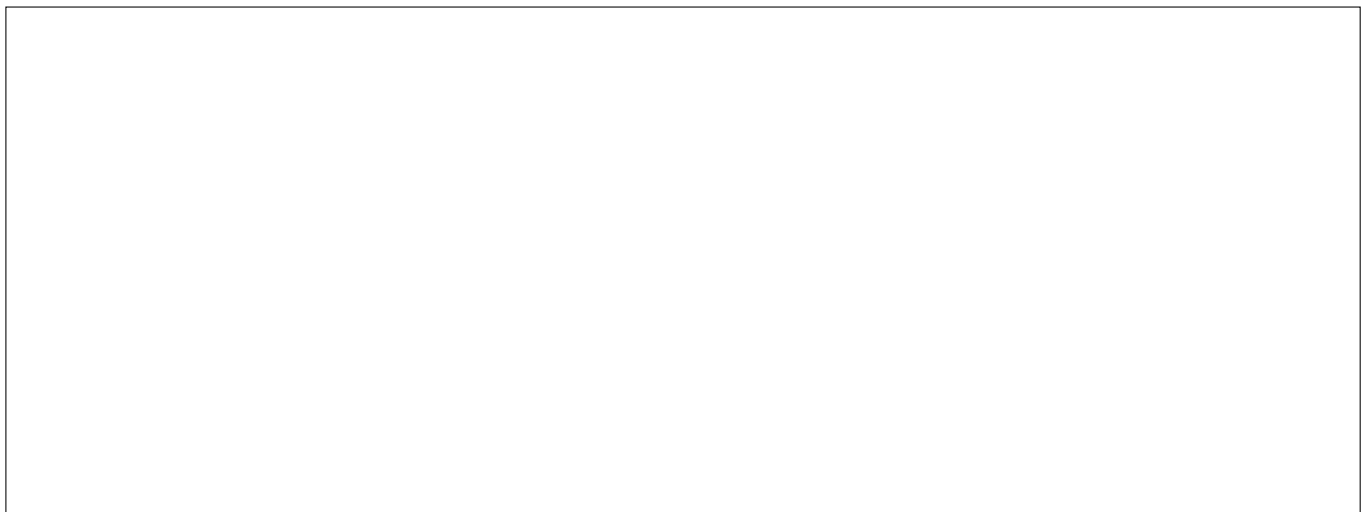
- (a) A phase portrait for a **system** of two differential equations with exactly one attracting and one repelling equilibrium solution and no other equilibrium solutions.



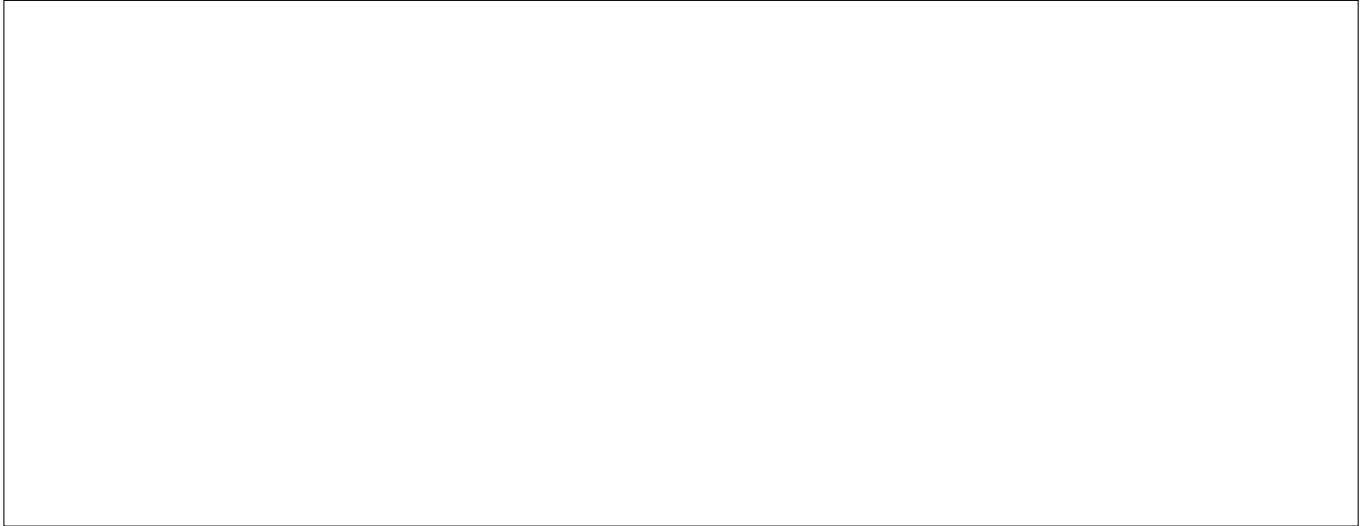
- (b) A phase portrait for a **single** differential equation with exactly one attracting and one repelling equilibrium solution and no other equilibrium solutions.



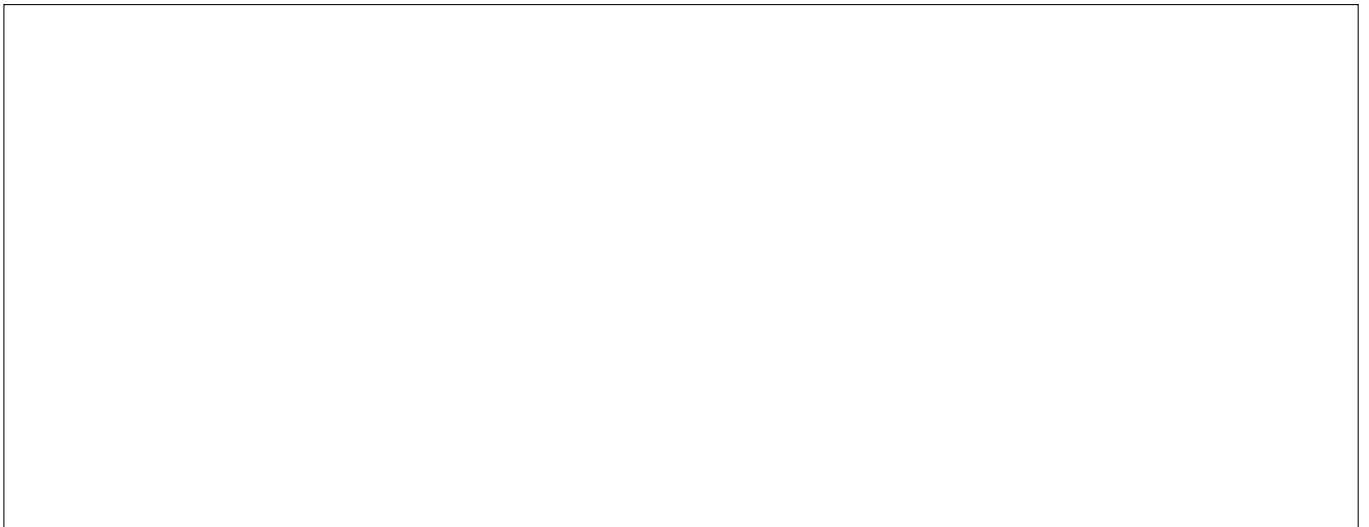
- (c) A phase portrait for a **system** of two differential equations with exactly two attracting equilibrium solutions and no other equilibrium solutions.



- (d) A phase portrait for a **single** differential equations with exactly two attracting equilibrium solutions and no other equilibrium solutions.



- (e) A phase portrait for a **system** of two differential equations with infinitely many repelling equilibrium solutions and no other equilibrium solutions.



5. **Report Preparation.** *At the end of the term, you will hand in a final report. This problem is intended to help you develop the skills needed for your final report.*

Consider the model of how a bee travels in a particular field of flowers

$$\begin{aligned}S'(t) &= \sin(D(t)) \\ D'(t) &= S(t) - D(t)\end{aligned}$$

where

$S(t)$  = intensity of the smell of flowers at time  $t$  (on a logarithmic scale from  $-\infty$  to  $+\infty$ )

$D(t)$  = (positive) distance between the bee and its hive at time  $t$

Additionally, near a flower and the hive, the smell intensity is measured to be constant.

In this problem, you will be analyzing what this model claims about a bee's behaviour. Your analyses should be a mix of analytic arguments (those coming from mathematical formulas), numeric arguments (computer-based simulations), and qualitative arguments (e.g., analyzing pictures and phase portraits).

- (a) At what distance(s) is smell intensity constant? Justify your answer.

- (b) In the field being modeled, a bee will only stay still in its hive or on a flower. At what distance(s) from the hive are the flowers? Justify your answer.

- (c) If a bee starts at distance 5 from its hive and detects a neutral smell (i.e., smell intensity 0), what will the bee do and where will it end up? Justify with a simulation. Include details about your simulation in your answer. (For example, screenshots of a spreadsheet, details about what methods you used, graphs of results with labeled axes, etc.)



(d) Which flowers can attract bees? Justify with simulations and with qualitative techniques.

- (e) If a bee starts in its hive, what is the minimum smell intensity that would cause it to venture to the **third** flower (i.e., the third-closest flower to the hive)? Justify your answer.

**YOU MUST SUBMIT THIS PAGE.**

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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