1 Homework for Week 4

1.1 Calculational Question 1

We apply the Gram-Schmidt process to the basis

$$\begin{aligned} f_1 &= e_1 + e_2 + e_3 \\ f_2 &= e_2 + e_3 \\ f_3 &= e_3 \end{aligned}$$

To begin, we have

$$v_1 = \frac{1}{\|f_1\|} f_1 \text{ where } \|f_1\|^2 = \langle f_1, f_1 \rangle = 3 \text{ and thus } v_1 = \frac{1}{\sqrt{3}} f_1 = \frac{1}{\sqrt{3}} (e_1 + e_2 + e_3)$$

Next, we have $w_2 = f_2 - \langle f_2, v_1 \rangle v_1$ and therefore we compute

$$\langle f_2, v_1 \rangle = \frac{1}{\sqrt{3}} \langle e_2 + e_3, e_1 + e_2 + e_3 \rangle = \frac{2}{\sqrt{3}}$$

which gives

$$w_2 = f_2 - \frac{2}{\sqrt{3}} v_1 = e_2 + e_3 - \frac{2}{3}(e_1 + e_2 + e_3) = -\frac{2}{3} e_1 + \frac{1}{3} e_2 + \frac{1}{3} e_3 \,.$$

We have

$$v_2 = \frac{1}{\|w_2\|} w_2 \text{ where } \|w_2\|^2 = \langle w_2, w_2 \rangle = \frac{2}{3} \text{ and thus } v_2 = \frac{1}{\sqrt{6}} (e_2 + e_3 - 2e_1) \,.$$

Lastly, we have $w_3 = f_3 - \langle f_3, v_1 \rangle v_1 - \langle f_3, v_2 \rangle v_2$ and therefore we compute the inner products,

$$\langle f_3, v_1 \rangle = \frac{1}{\sqrt{3}} \text{ and } \langle f_3, v_2 \rangle = \frac{1}{\sqrt{6}}$$

from which we conclude that

$$w_3 = e_3 - \frac{1}{3}(e_1 + e_2 + e_3) - \frac{1}{6}(e_2 + e_3 - 2e_1) = -\frac{1}{2} e_2 + \frac{1}{2} e_3$$

which after normalisation gives

$$v_3 = \frac{1}{\sqrt{2}} (e_3 - e_2) \,.$$

Verify that if we apply Gram-Schmidt in the opposite order we get

$$v_1 = e_3, \; v_2 = e_2, \quad \text{and } v_3 = e_1$$

1.2 Calculational Question 2

We have three types of functions in the collection. The constant function

$$c_0(x) = \frac{1}{\sqrt{2\pi}} \,,$$

the sine functions

$$s_k(x) = \frac{\sin(kx)}{\sqrt{\pi}} \text{ for } k = 1, 2, 3, \ldots$$

and the cosine functions

$$c_k(x) = \frac{\cos(kx)}{\sqrt{\pi}} \text{ for } k = 1, 2, 3, \ldots$$

To show that the collection is orthogonal, we need to show that

$$\langle c_0, c_k \rangle = \langle c_0, s_k \rangle = \langle c_k, s_k \rangle = 0 \text{ for } k = 1, 2, 3, \ldots$$

while

$$\langle c_i, c_j \rangle = \langle s_i, s_j \rangle = 0 \text{ for all } i, j = 1, 2, 3, \ldots \text{ such that } i \neq j$$

To show that the collection is orthonormal, we additionally need to show that

$$\langle c_0, c_0 \rangle = \langle c_k, c_k \rangle = \langle s_k, s_k \rangle = 1 \text{ for } k = 1, 2, 3, \ldots$$

There are two key facts we will use, firstly that if k is a (non-zero in the case of cosine) integer then

$$\int_{-\pi}^{\pi} \sin(kx)\, \mathrm{d}x = \int_{-\pi}^{\pi} \cos(kx)\, \mathrm{d}x = 0$$

and secondly, the product-to-sum identities for sine and cosine functions

$$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right), \sin(a)\sin(b) = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

$$\cos(a)\sin(b) = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right), \sin(a)\cos(b) = \frac{1}{2}\left(\sin(a+b) - \sin(a-b)\right).$$

We calculate,

$$\langle c_0, c_k \rangle = \frac{1}{\pi\sqrt{2}} \int_{-\pi}^{\pi} \cos(kx)\, \mathrm{d}x = 0 \text{ as } k \text{ is a non-zero integer}$$

$$\langle c_0, s_k \rangle = \frac{1}{\pi\sqrt{2}} \int_{-\pi}^{\pi} \sin(kx)\, \mathrm{d}x = 0 \text{ as } k \text{ is an integer}$$

$$\langle c_k, s_k \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx)\sin(kx)\, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}\sin(2kx)\, \mathrm{d}x = 0 \text{ as } 2k \text{ is an integer}.$$

$$\langle c_i, c_j \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ix)\cos(jx)\, \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((i-j)x) + \cos((i+j)x)\, \mathrm{d}x$$

by the product-to-sum identity for cosine functions. $i + j$ is certainly a non-zero integer while $i - j$ is a non-zero integer if $i \neq j$ and therefore we conclude that $\langle c_i, c_j \rangle = 0$ unless $i = j$ in which case (letting $i = j = k$)

$$\langle c_k, c_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}x = 1$$

The sine case follows similarly, and lastly

$$\langle c_0, c_0 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}x = 1$$

1.3 Calculational Question 3

We apply the Gram-Schmidt process to the standard basis $\{1, x, x^2\}$ of $\mathbb{R}_2[x]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x\,.$$

Let us define

$$u_1(x) = 1\,, u_2(x) = x\,, u_3(x) = x^2$$

and we will call our new orthonormal basis $\{v_1, v_2, v_3\}$. We will use the notation $\{w_1, w_2, w_3\}$ for the intermediate, unnormalised vectors.

Following the algorithm, we let $w_1 = u_1$ and calculate

$$\langle w_1, w_1 \rangle = \int_0^1 1 \cdot 1\,\mathrm{d}x = 1\,.$$

Since w_1 is already normalised, we set $v_1 = w_1 = 1$.

Next, we let $w_2 = u_2 - \langle u_2, v_1 \rangle\, v_1$. Calculating,

$$\langle u_2, v_1 \rangle = \int_0^1 x \cdot 1\,\mathrm{d}x = \frac{1}{2} \implies w_2(x) = x - \frac{1}{2}\,,$$

and to normalize, we compute

$$\langle w_2, w_2 \rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2 \mathrm{d}x = \frac{1}{3}\left(x - \frac{1}{2}\right)^3\Bigg|_0^1 = \frac{1}{12}$$

and thus

$$v_2(x) = \sqrt{12}\left(x - \frac{1}{2}\right)\,.$$

Finally, we let $w_3 = u_3 - \langle u_3, v_1 \rangle\, v_1 - \langle u_3, v_2 \rangle\, v_2$. Calculating,

$$\langle u_3, v_1 \rangle = \int_0^1 x^2 \cdot 1\,\mathrm{d}x = \frac{1}{3}\,, \langle u_3, v_2 \rangle = \sqrt{12}\int_0^1 x^3 - \frac{1}{2}x^2\,\mathrm{d}x$$

where

$$\int_0^1 x^3 - \frac{1}{2}x^2\,\mathrm{d}x = \frac{1}{4}x^4 - \frac{1}{6}x^3\Bigg|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \implies \langle u_3, v_2 \rangle = \frac{1}{\sqrt{12}}\,,$$

and thus

$$w_3(x) = x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}\,.$$

To conclude

$$\langle w_3, w_3 \rangle = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 \mathrm{d}x = \int_0^1 \left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{12}\right)^2 \mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(u^2 - \frac{1}{12}\right)^2 \mathrm{d}u = 2\int_0^{\frac{1}{2}} \left(u^2 - \frac{1}{12}\right)^2 \mathrm{d}u = \frac{1}{180}$$

and thus

$$v_3(x) = \sqrt{180}\left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{12}\right) = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right)\,.$$

1.4 Calculational Question 4

This problem is similar to Question 1 so we will be brief. We have

$$v_1 = (1,2,1)\ , v_2 = (1,-2,1)\ , v_3 = (1,2,-1)$$

and want to construct an orthonormal basis $\{u_1, u_2, u_3\}$.

$$\langle v_1, v_1\rangle = 6 \implies u_1 = \frac{1}{\sqrt{6}}(1,2,1)$$

$$\langle v_2, u_1\rangle = \frac{1}{\sqrt{6}}\cdot(1-4+1) = -\frac{2}{\sqrt{6}}$$

and so

$$w_2 = v_2 - \langle v_2, u_1\rangle u_1 = (1,-2,1) + \frac{1}{3}(1,2,1) = \left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right) = \frac{4}{3}(1,-1,1)\ .$$

Normalizing,

$$u_2 = \frac{1}{\sqrt{3}}(1,-1,1)$$

$$\langle v_3, u_1\rangle = \frac{1}{\sqrt{6}}\cdot(1+4-1) = \frac{4}{\sqrt{6}}\ , \langle v_3, u_2\rangle = \frac{1}{\sqrt{3}}\cdot(1-2-1) = -\frac{2}{\sqrt{3}}$$

so

$$w_3 = (1,2,-1) - \frac{2}{3}(1,2,1) + \frac{2}{3}(1,-1,1) = (1,0,-1)$$

and thus

$$u_3 = \frac{1}{\sqrt{2}}(1,0,-1)\ .$$

1.5 Proof Question 3

The claim is false. Suppose that $\langle -, -\rangle$ is an inner product such that

$$\langle (x_1,x_2),(x_1,x_2)\rangle = \|(x_1,x_2)\|^2 = (|x_1| + |x_2|)^2$$

We show that $\langle -, -\rangle$ cannot satisfy the parallelogram law, thereby contradicting the claim. To that end, let $u = (1,1)$ and $v = (1,-1)$. Then the parallelogram law demands that

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

but from the supposed inner product, we have

$$\|u\|^2 = \|v\|^2 = 4$$

while on the other hand,

$$u + v = (2,0) \implies \|u+v\|^2 = 4$$

$$u - v = (0,2) \implies \|u-v\|^2 = 4$$

We therefore find that

$$\|u+v\|^2 + \|u-v\|^2 = 8 \neq 16 = 2\left(\|u\|^2 + \|v\|^2\right)$$