WORKSHEET WEEK 2 MONDAY
CSC165 — 2025 WINTER

Universally Quantified Implication

A **universally quantified implication** is a common form of universal quantification, used to express a **restriction** of the domain using a special form of body:

$$\forall\, v \in D, (r_v \Rightarrow e_v) \text{ means } \forall\, v \in \left\{v' \in D : r_{v'}\right\}, e_v$$

Active

Do you recall the terminology for the parts of a (symbolic) quantification?
Identify those parts in this instance of the new notation:

$$\forall\, j \in \mathbb{N}, (j \leq 2 \Rightarrow \text{row } j \text{ has a zero})$$

Then, match its body against the body form in the new notation, and identify r_v and e_v.
Write out the meaning of this instance of the notation, using its particular v, D, e_v, and r_v.
Expand that, and summarize the meaning in simple natural prose.

Solution

Quantified variable j, domain $\mathbb{N}$, body $(j \leq 2 \Rightarrow \text{row } j \text{ has a zero})$.
Lining up the body notation form and this instance of it:

$$\begin{array}{ccccc} (& r_v & \Rightarrow & e_v &) \\ (& j \leq 2 & \Rightarrow & \text{row } j \text{ has a zero} &) \end{array}$$

So r_v (which is r_j for this instance) is "$j \leq 2$" and e_v (i.e., e_j) is "row j has a zero".
Using j as v, $\mathbb{N}$ as D, e_j as "row j has a zero", and $r_{v'}$ as "$j' \leq 2$", it means

$$\forall\, j \in \left\{j' \in \mathbb{N} : j' \leq 2\right\}, \text{row } j \text{ has a zero}$$

which expands to

$$\forall\, j \in \{0, 1, 2\}, \text{row } j \text{ has a zero} \equiv \text{row } 0 \text{ has a zero} \wedge \text{row } 1 \text{ has a zero} \wedge \text{row } 2 \text{ has a zero}.$$

(note: to symbolically express that two boolean expressions produce equal values it's common to use "$\equiv$" instead of "$=$", and the expressions are then called "equivalent" instead of "equal").
The effect of "$j \leq 2 \Rightarrow$" was to restrict the domain $\mathbb{N}$ to its elements less than or equal to two.
Prose summary: rows 0, 1, and 2 each have a zero.

Here are some other (mostly) prose phrasings, using a variable:

- For each (every) $j = 0$, 1, and 2, row j has a zero.
- For every (each) natural j less than or equal to two, row j has a zero.
- Row j has a zero for all naturals j less than or equal to two.
- If natural j is at most two then row j has a zero.

The last phrase doesn't explicitly quantify j with terms such as each / every / all / natural*s*. An unquantified variable is usually implicitly universally quantified, especially if it can be treated directly as a parameter. For example, if we treat the entire phrase as parameterized by j and instantiate it with 1 we get the meaningful phrase:

If natural 1 is at most two then row 1 has a zero.

We can also make the quantification explicit:

For each natural j, if j is at most two then row j has a zero.

English allows a lot of variety of phrasing, and mathematical English still does as well.
If we consider those phrasings as just rephrasings of one concept, then we might consider

$$\forall\, j \in \{0, 1, 2\}, \text{row } j \text{ has a zero}, \text{ and}$$

$$\forall\, j \in \mathbb{N}, (j \leq 2 \Rightarrow \text{row } j \text{ has a zero})$$

as just rephrasings, one in terms of a set of elements, the other in terms of a type and a condition. We want to understand them as equivalent, in the same way that we would understand any of the prose phrasings as expressing the "same" concept.

Define sets

$$A = \{n \in \mathbb{N} : n < 10 \wedge n \text{ is prime}\}$$
$$B = \{n \in \mathbb{N} : n < 10\}$$
$$C = \{n \in \mathbb{N} : n \text{ is prime}\}$$

Active

Which of these sets is a (**proper**) **subset** of another?
Are any of these sets a combination of the other two with a common set operation?
For each set, if it's a subset of another one of the sets, express it as a restriction of the other set: a set comprehension with an appropriate condition further restricting it.

Solution

Note: a previous worksheet demonstrated systematically expanding such set expressions. Make sure you can do that, and do it (at least on your mental whiteboard) whenever you're unsure about the meaning of such an expression. In particular, that makes the next two claims about the relationships between A, B, and C clear, and explains why the set comprehensions all produce A.

$A \subseteq B$ and $A \subseteq C$.
$A = B \cap C$.
$A = \{n \in B : n \text{ is prime}\} = \{n \in B : n \in C\}$ (either answers the question, but be aware of both).
$A = \{n \in C : n < 10\} = \{n \in C : n \in B\}$.
Note that $\{n \in \mathbb{N} : n < 10 \wedge n \text{ is prime}\} = \{n \in \mathbb{N} : n \text{ is prime} \wedge n < 10\}$, so it's unsurprising that we can treat B and C symmetrically.

Active

Express
$$\forall n \in \left\{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\right\}, n \text{ is odd}$$
as a universally quantified implication by moving the entire restriction into the body.
Then, based on the previous question, express it as two other universally quantified implications by keeping part of the restriction in the domain and moving part of it into the body.
Expand your answers enough to check.

Solution

$\forall n \in \mathbb{N}, ((n < 10 \wedge n \text{ is prime}) \Rightarrow n \text{ is odd})$ immediately expands to the original.
$\forall n \in \{n' \in \mathbb{N} : n' < 10\}, (n \text{ is prime} \Rightarrow n \text{ is odd}) \equiv \forall n \in \{n'' \in \{n' \in \mathbb{N} : n' < 10\} : n'' \text{ is prime}\}, n \text{ is odd}$, which has the same domain by the previous question.
$\forall n \in \{n' \in \mathbb{N} : n' \text{ is prime}\}, (n < 10 \Rightarrow n \text{ is odd}) \equiv \forall n \in \{n'' \in \{n' \in \mathbb{N} : n' \text{ is prime}\} : n < 10\}, n \text{ is odd}$, which also has the same domain.
If any of those are too abstract/symbolic, expand them completely to an enumerated domain.
For example, and expanding the domain first:

$$\begin{aligned}
\forall n \in \left\{n' \in \mathbb{N} : n' \text{ is prime}\right\}, (n < 10 \Rightarrow n \text{ is odd}) &\equiv \forall n \in \{2, 3, 5, 7, 11, 13, \ldots\}, (n < 10 \Rightarrow n \text{ is odd}) \\
&\equiv \forall n \in \left\{n' \in \{2, 3, 5, 7, 11, 13, \ldots\} : n' < 10\right\}, n \text{ is odd} \\
&\equiv \forall n \in \{2, 3, 5, 7\}, n \text{ is odd}
\end{aligned}$$

Mathematical English also has various phrasings for this, for example:

- Every natural less than ten and prime is odd.
- If a natural is less than ten and prime then it's odd.
- If a natural less than ten is prime then it's odd.
- If a prime natural is less than ten then it's odd.

Notice that these make the same kinds of choices of whether to use a set/type/category versus a condition as the four symbolic quantifications do. Presumably you understand the prose phrasings as all "saying the same thing" as each other. A goal is to understand the symbolic phrasings as ultimately "the same".
The latter three prose phrases aren't explicitly quantified nor contain an explicit variable that might be implicitly universally quantified, but an indefinite object ("a natural", as opposed to a definite "the natural") often functions as an implicit universally quantified parameter.

Active

Consider the phrasing "if a natural is less than ten, then if it's prime, then it's odd". This restricts the naturals to the ones less than ten, then restricts to the primes ones, then makes the claim that they're odd. Expand

$$\forall n \in \mathbb{N}, (n < 10 \Rightarrow (n \text{ is prime} \Rightarrow n \text{ is odd})) ,$$

which is inspired by this phrasing, to see whether it makes the same claim as the other universals. How about if we swap the conditions "$n < 10$" and "n is prime"?

Solution

$$\forall n \in \mathbb{N}, (n < 10 \Rightarrow (n \text{ is prime} \Rightarrow n \text{ is odd})) \equiv \forall n \in \{0, 1, 2, \ldots, 10\}, (n \text{ is prime} \Rightarrow n \text{ is odd})$$

(that skipped a step, which you can try putting in if you like), which is one of the earlier phrasings (or an expansion of an earlier phrasing).

$$\forall n \in \mathbb{N}, (n \text{ is prime} \Rightarrow (n < 10 \Rightarrow n \text{ is odd})) \equiv \forall n \in \{2, 3, 5, 7, 11, 13, \ldots\}, (n < 10 \Rightarrow n \text{ is odd})$$

which is also one of the earlier ones.

In general, the following are equivalent:

$$\begin{aligned}
&\forall v \in D, ((r_v \wedge s_v) \Rightarrow e_v)\\
&\forall v \in D, ((s_v \wedge r_v) \Rightarrow e_v)\\
&\forall v \in D, (r_v \Rightarrow (s_v \Rightarrow e_v))\\
&\forall v \in D, (s_v \Rightarrow (r_v \Rightarrow e_v))
\end{aligned}$$

Active

Let $P : \mathbb{N} \to \mathbb{B}$. Expand $\exists n \in \mathbb{N}, (\forall k \in \mathbb{N}, (k > n \Rightarrow P(k)))$.

Solution

$$\begin{aligned}
& \quad\ (\forall k \in \mathbb{N}, (k > 0 \Rightarrow P(k)))\\
& \vee (\forall k \in \mathbb{N}, (k > 1 \Rightarrow P(k)))\\
& \vee (\forall k \in \mathbb{N}, (k > 2 \Rightarrow P(k)))\\
& \vee \cdots\\
\equiv & \quad\ \big(\forall k \in \big\{k' \in \mathbb{N} : k' > 0\big\}, P(k)\big)\\
& \vee \big(\forall k \in \big\{k' \in \mathbb{N} : k' > 1\big\}, P(k)\big)\\
& \vee \big(\forall k \in \big\{k' \in \mathbb{N} : k' > 2\big\}, P(k)\big)\\
& \vee \cdots
\end{aligned}$$

$$\equiv (\forall k \in \{1, 2, 3, \ldots\}, P(k)) \vee (\forall k \in \{2, 3, 4, \ldots\}, P(k)) \vee (\forall k \in \{3, 4, 5, \ldots\}, P(k)) \vee \cdots$$

$$\equiv (P(1) \wedge P(2) \wedge P(3) \wedge \cdots) \vee (P(2) \wedge P(3) \wedge P(4) \wedge \cdots) \vee (P(3) \wedge P(4) \wedge P(5) \wedge \cdots) \vee \cdots$$

Active

Consider again the statement

$$\exists\, n \in \mathbb{N}, (\forall\, k \in \mathbb{N}, (k > n \Rightarrow P(k)))\ .$$

Generate a good set of example predicates, exploring ways it can be true and ways it can be false. During that exploration, develop a simple prose summary of the predicates which make it true, and the ones which make it false.

Solution

Prose summary, based on the earlier expansion, of Ps for which the statement is true:
P is true from some positive natural onwards.

Thought: if P is true for all natural numbers that's still sufficient.
Alternative condition: P is true from some natural onwards. (Not necessarily positive.)
People also say this as: P is eventually always true.

Some true instances P:

For each $n \in \mathbb{N}$ define $P(n)$ to be: n is prime $\vee\ n \geq 6$.
(typical: intermittently false, then always true)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	F	T	T	F	T	T	T	T

For each $n \in \mathbb{N}$ define $P(n)$ to be: $n \geq 4$.
(but doesn't have to be intermittent)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	F	F	F	T	T	T	T	T

For each $n \in \mathbb{N}$ define $P(n)$ to be true.
(and could never be false)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	T	T	T	T	T	T	T	T	T

No set of examples (unless there are only a finite number and we list them all) can determine the exact statement, but we're trying to give enough to clear up any ambiguity in the prose phrasings, and possibly enough that if someone looks for a relatively simple statement to match the examples they would likely produce our statement. The false examples then continue to clarify the statement. These are also considerations when producing examples in software documentation.

Thought: for the statement to be false, we need each disjunct to be false, so after each natural we need P to be false somewhere.

Prose summary of Ps for which the statement is false:
Each natural has a point afterwards where P is false.

People also say this as: there are arbitrarily large naturals for which P is false (or just: P is false for arbitrarily large naturals). Note that the phrase "arbitrarily large naturals" can sound like "arbitrarily large" is a property of a number (as if we're saying $\{n \in \mathbb{N} : n \text{ is arbitrarily large}\}$, which is meaningless), the same way that just "large" could be, and, e.g., "prime" is, but the phrase is actually describing a property of a *set* (the set $\{n \in \mathbb{N} : P(n)\}$).

Some false instances P:

For each $n \in \mathbb{N}$ define $P(n)$ to be: $n < 5 \vee n$ is even. (typical)

n	0	1	2	3	4	5	6	7	8	9	$\cdots$	164	165	$\cdots$
$P(n)$	T	T	T	T	T	F	T	F	T	F	$\cdots$	T	F	$\cdots$

For each $n \in \mathbb{N}$ define $P(n)$ to be: $n < 4$. (could stop being true at some point)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	T	T	T	T	F	F	F	F	F

For each $n \in \mathbb{N}$ define $P(n)$ to be false. (could never be true)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	F	F	F	F	F	F	F	F

Homework

Express the statement in terms of something being finite or infinite. How about when it's false?
Compare it with $\exists\, n \in \mathbb{N}, P(n)$, $\forall\, n \in \mathbb{N}, P(n)$, $\exists\, n \in \mathbb{N}, P(n+1)$, and $\forall\, n \in \mathbb{N}, P(n+1)$: which of those statements are entailed by it, and which ones entail it?

Let P and Q be predicates on $\mathbb{N}$.
Recall from Worksheet Week 1 Wednesday: $\exists\, n \in \mathbb{N}, P\,(n+1)$ is equivalent to $\exists\, n \in \mathbb{N}^+, P\,(n)$.

Active

Expand $\exists\, n \in \mathbb{N}, (Q\,(n) \wedge P\,(n))$.
Comparing expansions, can you find a Q so that that statement is equivalent to $\exists\, n \in \mathbb{N}^+, P\,(n)$?
Notice: for some naturals n you want $Q\,(n)$ to have a value that makes the value of $P\,(n)$ irrelevant, and for others you want it to have a value that makes only $P\,(n)$ relevant.

Solution

$$\exists\, n \in \mathbb{N}, (Q\,(n) \wedge P\,(n)) \;\equiv\; (Q\,(0) \wedge P\,(0)) \,\vee\, (Q\,(1) \wedge P\,(1)) \,\vee\, (Q\,(2) \wedge P\,(2)) \,\vee\, (Q\,(3) \wedge P\,(3)) \,\vee\, \cdots$$

Each parenthesized term is a sufficient condition, any one being true would make it true.
We want the statement to be equivalent to $P\,(1) \,\vee\, P\,(2) \,\vee\, P\,(3) \,\vee\, \cdots$, so we need to make it impossible for zero to be a witness. We can do that by making $Q\,(0)$ false: $Q\,(0) \wedge P\,(0)$ will then be false, regardless of the value of $P\,(0)$, and so will be irrelevant for the existential.
In the remaining terms $(Q\,(1) \wedge P\,(1)) \vee (Q\,(2) \wedge P\,(2)) \vee (Q\,(3) \wedge P\,(3)) \vee \cdots$ we want $Q\,(1)$ to not be false, so that $P\,(1)$ remains relevant. If we make $Q\,(1)$ true, then it has no effect in $Q\,(1) \wedge P\,(1)$: that conjunction is then true exactly when $P\,(1)$ is true.
So for each $n \in \mathbb{N}$, define $Q\,(n)$ to be: $n \geq 1$.
Then $\exists\, n \in \mathbb{N}, (Q\,(n) \wedge P\,(n))$, i.e., $\exists\, n \in \mathbb{N}, (n \geq 1 \wedge P\,(n))$, is equivalent to $\exists\, n \in \mathbb{N}^+, P\,(n)$, i.e., $\exists\, n \in \{n' \in \mathbb{N} : n' \geq 1\}, P\,(n)$.

Active

Expand $\exists\, n \in \mathbb{N}, (Q\,(n) \wedge P\,(n))$ with that particular Q, and convince yourself (if you weren't already convinced, or needed to peek at the solution to find it) that it's equivalent to $\exists\, n \in \mathbb{N}^+, P\,(n)$.

Solution

$$\begin{aligned}\exists\, n \in \mathbb{N}, (n \geq 1 \wedge P\,(n)) &\equiv (0 \geq 1 \,\wedge\, P\,(0)) \,\vee\, (1 \geq 1 \,\wedge\, P\,(1)) \,\vee\, (2 \geq 1 \,\wedge\, P\,(2)) \,\vee\, (3 \geq 1 \,\wedge\, P\,(3)) \,\vee\, \cdots \\ &\equiv P\,(1) \,\vee\, P\,(2) \,\vee\, P\,(3) \,\vee\, \cdots\end{aligned}$$

Since $0 \geq 1$ is false the first conjunction can't be true, so it doesn't affect the disjunction, and $1 \geq 1$, $2 \geq 1$, $3 \geq 1$, ..., are all true, so they don't affect their conjunctions and can be removed.

Active

Expand the statement

$$\forall\, n \in \mathbb{N}, (\exists\, k \in \mathbb{N}, (k \geq n \wedge P\,(k)))\,.$$

Simplify the expansion by removing disjuncts that can't be true and conjuncts that have no effect.

Solution

$$\begin{aligned}\forall\, n \in \mathbb{N}, (\exists\, k \in \mathbb{N}, (k \geq n \wedge P\,(k))) \equiv&\quad (\exists\, k \in \mathbb{N}, (k \geq 0 \wedge P\,(k))) \\ &\wedge (\exists\, k \in \mathbb{N}, (k \geq 1 \wedge P\,(k))) \\ &\wedge (\exists\, k \in \mathbb{N}, (k \geq 2 \wedge P\,(k))) \\ &\wedge \cdots \\ \equiv&\quad [(0 \geq 0 \wedge P\,(0)) \vee (1 \geq 0 \wedge P\,(1)) \vee (2 \geq 0 \wedge P\,(2)) \vee \cdots] \\ &\wedge [(0 \geq 1 \wedge P\,(0)) \vee (1 \geq 1 \wedge P\,(1)) \vee (2 \geq 1 \wedge P\,(2)) \vee \cdots] \\ &\wedge [(0 \geq 2 \wedge P\,(0)) \vee (1 \geq 2 \wedge P\,(1)) \vee (2 \geq 2 \wedge P\,(2)) \vee \cdots] \\ &\wedge \cdots \\ \equiv&\quad [P\,(0) \vee P\,(1) \vee P\,(2) \vee \cdots] \\ &\wedge [P\,(1) \vee P\,(2) \vee P\,(3) \vee \cdots] \\ &\wedge [P\,(2) \vee P\,(3) \vee P\,(4) \vee \cdots] \\ &\wedge \cdots\end{aligned}$$

In general, if p is a proposition then $\textbf{false} \wedge p \;\equiv\; \textbf{false}$, $\textbf{true} \wedge p \;\equiv\; p$, and $\textbf{false} \vee p \;\equiv\; p$.
But always keep in mind that logic, even when symbolic, is mainly used as a precise (and concise, when symbolic) language, not a system of calculation. Those equivalences are summarizing what we all understand if we say "both of **false** and p are true", "at least one of **false** and p is true", and "both of **true** and p are true": the first is obviously impossible, the latter two are determined by whether p is true.

Active

Express the simplified expansion with quantification, by restricting one of the domains.

Solution

$$\begin{aligned} & (P(0) \vee P(1) \vee P(2) \vee \cdots) \\ & \wedge (P(1) \vee P(2) \vee P(3) \vee \cdots) \\ & \wedge (P(2) \vee P(3) \vee P(4) \vee \cdots) \\ & \wedge \cdots \\ \equiv\ & (\exists k \in \{0,1,2,\ldots\}, P(k)) \\ & \wedge (\exists k \in \{1,2,3,\ldots\}, P(k)) \\ & \wedge (\exists k \in \{2,3,4,\ldots\}, P(k)) \\ & \wedge \cdots \\ \equiv\ & \forall n \in \mathbb{N}, \exists k \in \{n, n+1, n+2, \ldots\}, P(k) \\ \equiv\ & \forall n \in \mathbb{N}, \exists k \in \left\{k' \in \mathbb{N} : k' \geq n\right\}, P(k) \end{aligned}$$

In general, when the body of an existential is a conjunction, the conjunction can be viewed as restricting the domain:

$$\exists v \in D, (r_v \wedge e_v) \equiv \exists v \in \left\{v' \in D : r_{v'}\right\}, e_v$$

Only the elements of the domain for which r_v is true matter, and for those then only e_v matters.

Active

Explore

$$\forall n \in \mathbb{N}, (\exists k \in \mathbb{N}, (k \geq n \wedge P(k))),$$

producing a good set of examples and prose summaries for when it's true versus false.

Solution

$$(\exists k \in \{0,1,2,\ldots\}, P(k)) \wedge (\exists k \in \{1,2,3,\ldots\}, P(k)) \wedge (\exists k \in \{2,3,4,\ldots\}, P(k)) \wedge \cdots$$

says there's a witness in $\{0,1,2,\ldots\}$, a witness in $\{1,2,3,\ldots\}$, a witness in $\{2,3,4,\ldots\}$, and so on.
Thought: a witness for one existential would witness all earlier existentials, but not most later existentials (as usual, since this thought is implicitly a universal claim about "a witness", instantiate it if you're not sure what it means or why it's true).
Prose summary of Ps for which the statement is true: there are arbitrarily large naturals for which P is true (or just: P is true for arbitrarily large numbers). Alternative: P is true for an infinite number of naturals (or just: P is true an infinite number of times).
Thought: A counter-example would be when one of the existentials has no witness.
Prose summary of Ps for which the statement is false: there's a natural after which P is never true, i.e., always false (or just: P is false from some point onward). Alternative: P is true for only a finite number of numbers (or just: P is true a finite number of times).
Some true instances P:

For each $n \in \mathbb{N}$ define $P(n)$ to be: $n \geq 5 \wedge n$ is odd. (typical)

n	0	1	2	3	4	5	6	7	8	9	$\cdots$	164	165	$\cdots$
$P(n)$	F	F	F	F	F	T	F	T	F	T	$\cdots$	F	T	$\cdots$

For each $n \in \mathbb{N}$ define $P(n)$ to be: $n \geq 5$. (could stop being false at some point)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	F	F	F	F	T	T	T	T

For each $n \in \mathbb{N}$ define $P(n)$ to be true. (and doesn't have to ever be false)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	T	T	T	T	T	T	T	T	T

Some false instances P:

For each $n \in \mathbb{N}$ define $P(n)$ to be: n is odd $\wedge\ n \leq 5$. (typical)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	T	F	T	F	T	F	F	F

For each $n \in \mathbb{N}$ define $P(n)$ to be false. (doesn't have to ever be true)

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	F	F	F	F	F	F	F	F

Homework

Compare the statement with $\exists n \in \mathbb{N}, P(n)$, $\forall n \in \mathbb{N}, P(n)$, $\exists n \in \mathbb{N}, P(n+1)$, $\forall n \in \mathbb{N}, P(n+1)$, and $\exists n \in \mathbb{N}, (\forall k \in \mathbb{N}, (k > n \Rightarrow P(k)))$.