WORKSHEET WEEK 1 WEDNESDAY
CSC165 — 2025 WINTER

Predicate Logic: Introducing Universal and Existential Quantification

Repeated addition and multiplication of terms of "the same form" can be expressed with summation notation and product notation. Similarly, conjunction and disjunction of terms of the same form are expressed by the notations

$$\forall\, v \in D, e_v \text{ and } \exists\, v \in D, e_v$$

Conjunction and disjunction in this form are called **universal quantification** and **existential quantification**. The notation has parameter v, which stands for a variable name (the **quantified variable**), parameter D, which is a set (the **domain**), and parameter e_v (the **body**), which is a propositional expression that may (and usually does) involve the variable name. The expansions of them are

$$\begin{aligned} \forall\, v \in \{x_0, x_1, x_2, \ldots\}, e_v &\text{ means } e_{x_0} \wedge e_{x_1} \wedge e_{x_2} \wedge \cdots \\ \text{and } \exists\, v \in \{x_0, x_1, x_2, \ldots\}, e_v &\text{ means } e_{x_0} \vee e_{x_1} \vee e_{x_2} \vee \cdots \end{aligned}$$

where e_{x_0}, e_{x_1}, e_{x_2}, ... are the instantiations of the quantified variable in e_v with the values x_0, x_1, x_2, ...
Identify the kind of quantification, quantified variable, domain, and body of

$$\exists\, c \in \{\mathbf{csc108}, \mathbf{csc165}, \mathbf{csc148}\}, c \text{ is fun}$$

and determine whether it's true or false.

Line up the parameterized notation with the instance of it

$$\begin{array}{ccccc} \exists & v & \in & D & , & e_v \\ \exists & c & \in & \{\mathbf{csc108}, \mathbf{csc165}, \mathbf{csc148}\} & , & c \text{ is fun} \end{array}$$

We see that the quantification is existential, quantified variable is c, domain is $\{\mathbf{csc108}, \mathbf{csc165}, \mathbf{csc148}\}$, and body is "$c$ is fun". Line it up with the meaning of the notation:

$$\begin{array}{cccccc} \exists\, v \in & \{x_0, x_1, x_2, \ldots\} & , & e_v & \text{means} & e_{x_0} \vee e_{x_1} \vee e_{x_2} \vee \cdots \\ \exists\, c \in & \{\mathbf{csc108}, \mathbf{csc165}, \mathbf{csc148}\} & , & c \text{ is fun} & \text{means} & \mathbf{csc108} \text{ is fun} \vee \mathbf{csc165} \text{ is fun} \vee \mathbf{csc148} \text{ is fun} \end{array}$$

which we hope is true (you find at least one of these courses fun)!

Identify the kind of quantification, quantified variable, domain, and body of each of the following, and determine which are true and which are false:

$$\forall\, n \in \left\{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\right\}, n \text{ is odd}$$

$$\exists\, z \in \mathbb{Z}, z^2 < z$$

$$\forall\, x \in \{1, 6, 5\}, (\exists\, y \in \{1, 0, 8\}, y - x \text{ is even})$$

Determine which ones are true and which are false.

Universal, quantified variable n, domain $\{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\}$, body "$n$ is odd":

$$\begin{aligned} \forall\, n \in \left\{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\right\}, n \text{ is odd} \quad &\text{means} \quad \forall\, n \in \{2, 3, 5, 7\}, n \text{ is odd} \\ &\text{means} \quad 2 \text{ is odd} \wedge 3 \text{ is odd} \wedge 5 \text{ is odd} \wedge 7 \text{ is odd} \end{aligned}$$

which is false since 2 isn't odd.
Existential, quantified variable z, domain $\mathbb{Z}$, body "$z^2 < z$", expansion (after substituting the value of $\mathbb{Z}$):

$$\left[0^2 < 0\right] \vee \left[(-1)^2 < -1\right] \vee \left[1^2 < 1\right] \vee \left[(-2)^2 < 2\right] \vee \left[2^2 < 2\right] \vee \cdots$$

which is false since none of the disjuncts are ever true.
Universal, quantified variable x, domain $\{1, 6, 5\}$, body "$\exists\, y \in \{1, 0, 8\}, y - x$ is even", expansion:

$$\begin{array}{llcll} & (\exists\, y \in \{1, 0, 8\}, y - 1 \text{ is even}) & & & (1 - 1 \text{ is even } \vee 0 - 1 \text{ is even} \vee 8 - 1 \text{ is even}) \\ \wedge & (\exists\, y \in \{1, 0, 8\}, y - 6 \text{ is even}) & \text{which means} & \wedge & (1 - 6 \text{ is even } \vee 0 - 6 \text{ is even} \vee 8 - 6 \text{ is even}) \\ \wedge & (\exists\, y \in \{1, 0, 8\}, y - 5 \text{ is even}) & & \wedge & (1 - 5 \text{ is even } \vee 0 - 5 \text{ is even} \vee 8 - 5 \text{ is even}) \end{array}$$

which is true, becuase each conjunct is true, because at least one disjunct in each conjunct is true (the first, second (and third), and first disjunct, respectively).

Recall equation $E_{\Sigma\Pi}$ from Worksheet Week 1 Monday. Consider the instance with $n = 3$. Express the summary of when $E_{\Sigma\Pi}$ is true using quantification, using a body "row j has a zero".

We want to express: row 0 has a zero, row 1 has a zero, and row 2 has a zero:

$$\forall\, j \in \{0,1,2\}\,,\ \text{row } j \text{ has a zero}$$

Use quantification to express "row j has a zero" in that, using a body "$a_{j,k} = 0$".

$\forall\, j \in \{0,1,2\}\,,\ (\exists\, k \in \{0,1,2\}\,,\ a_{j,k} = 0)$. Make sure you could expand that to see that it expands to the conjunction of disjunctions from the previous worksheet.

Repeat that to express when $E_{\Sigma\Pi}$ is false. Then express the condition for $E_{\Sigma\Pi}$ to be true, and the condition for it to be false, in general (with n uninstantiated), and give the corresponding expansions.

$$\exists\, j \in \{0,1,2\}\,,\ (\forall\, k \in \{0,1,2\}\,,\ a_{j,k} > 0)$$
$$\forall\, j \in \{0,1,\ldots,n-1\}\,,\ (\exists\, k \in \{0,1,\ldots,n-1\}\,,\ a_{j,k} = 0)$$

$$\begin{aligned} &(a_{0,,0} = 0 \ \vee\ a_{0,1} = 0 \ \vee\ a_{0,2} = 0 \ \vee\ \cdots) \\ \wedge\ &(a_{1,0} = 0 \ \vee\ a_{1,1} = 0 \ \vee\ a_{1,2} = 0 \ \vee\ \cdots) \\ \wedge\ &(a_{2,0} = 0 \ \vee\ a_{2,1} = 0 \ \vee\ a_{2,2} = 0 \ \vee\ \cdots) \\ &\vdots \end{aligned}$$

$$\exists\, j \in \{0,1,\ldots,n-1\}\,,\ (\forall\, k \in \{0,1,\ldots,n-1\}\,,\ a_{j,k} > 0)$$

$$\begin{aligned} &(a_{0,,0} > 0 \ \wedge\ a_{0,1} > 0 \ \wedge\ a_{0,2} > 0 \ \wedge\ \cdots) \\ \vee\ &(a_{1,0} > 0 \ \wedge\ a_{1,1} > 0 \ \wedge\ a_{1,2} > 0 \ \wedge\ \cdots) \\ \vee\ &(a_{2,0} > 0 \ \wedge\ a_{2,1} > 0 \ \wedge\ a_{2,2} > 0 \ \wedge\ \cdots) \\ &\vdots \end{aligned}$$

It's convenient to be able refer to the result of an instantiation by just mentioning the value of the parameter, so we might refer to the instance "165 is fun" as "instance $c = 165$" or even just "instance 165". We already did this in the first worksheet, but it's worth being aware of this explicitly for our next definition. Also, a universal quantification is often referred to as just "a universal", and an existential quantification as just "an existential".

A conjunction is true if every conjunct is true, and is false if at least one conjunct is false. So $\forall\, v \in D, e_v$ is true if every instance (with values from D) of e_v is true, and is false if at least one of the instances is false. A false instance is called a **counter-example**, and (if one exists) *disproves* (**prove is false**) a *universal*.
A disjunction is true if at least one disjunct is true, and is false if every disjunct is false. So $\exists\, v \in D, e_v$ is true if at least one instance of e_v is true, and is false if every instance is false. A true instance is called a **witness**, and (if one exists) *proves* an *existential*.

Use the terminology of counter-examples and witnesses to phrase why the first four examples on the first page are true or false, when applicable.

"$\exists\, c \in \{\textbf{csc108}, \textbf{csc165}, \textbf{csc148}\}\,,\ c$ is fun" is true, and **csc165** is a witness to that.
"$\forall\, n \in \{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\}\,,\ n$ is odd" is false, and 2 is a counter-example.
"$\forall\, x \in \{1,6,5\}\,,\ (\exists\, y \in \{1,0,8\}\,,\ y - x \text{ is even})$" is true: for $x = 1$ the body existential has the witness $y = 1$, for $x = 6$ the body existential has the witness $y = 0$ (and $y = 8$), and for $x = 5$ the body existential has the witness $y = 8$.

Homework.
Our "Numeric Types" reference has four quantifications of the form we just discussed:

$$\forall\, x \in \mathbb{Q}^*,\, \frac{1}{x} \in \mathbb{Q}^*$$

$$\forall\, z \in \mathbb{Z},\, |z| \in \mathbb{N}$$

$$\forall\, x \in \mathbb{R},\, \lfloor x \rfloor \in \mathbb{Z} \,\wedge\, \lceil x \rceil \in \mathbb{Z}$$

$$\forall\, x \in \mathbb{R}^{\geq 0},\, \lfloor x \rfloor \in \mathbb{N} \,\wedge\, \lceil x \rceil \in \mathbb{N}$$

Expand the universal in each one, and replace the instances of $\frac{1}{x}$, $|z|$, $\lfloor x \rfloor$, and $\lceil x \rceil$ that appear with specific values.

A quantification whose body is also quantified is called a **nested quantification** and the body is referred to as the **inner** quantification. For the nested quantifications earlier in this worksheet, expand them again by expanding only the **inner** quantifications.

False Universal and Existential Quantifications

Notice that we described when a universal is false as there being at least one counter-example (false instance), which suggests that a universal being false can be expressed as an existential; express

$$\forall n \in \{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\}, n \text{ is odd}$$

in natural prose, what it being false means as an existential, and express that in natural prose.

> Every prime natural number less than ten is odd.
> $\exists n \in \{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\}, n$ is even: some prime natural number less than ten is even.

Similarly, an existential being false was described as every instance being false: express what

$$\exists c \in \{\mathbf{csc108}, \mathbf{csc165}, \mathbf{csc148}\}, c \text{ is fun}$$

being false means, as a universal.

> $\forall c \in \{\mathbf{csc108}, \mathbf{csc165}, \mathbf{csc148}\}, c$ isn't fun

Express what $\forall x \in \{1, 6, 5\}, (\exists y \in \{1, 0, 8\}, y - x \text{ is even})$ being false means, as a nested quantification. We know it's true, which means the quantification you produced is false: explain directly why it's false.

> $\exists x \in \{1, 6, 5\}, (\exists y \in \{1, 0, 8\}, y - x \text{ is even})$ is false
> $\exists x \in \{1, 6, 5\}, (\forall y \in \{1, 0, 8\}, y - x \text{ is odd})$
> In terms of the expansion: no disjunct is true, since for each one at least one of its conjuncts is false (the first, second or third, and first, respectively).
> Alternatively: for each x in $\{1, 6, 5\}$, $y - x$ isn't odd for at least one of the ys in $\{1, 0, 8\}$ ($y = 1$ for $x = 1$, $y = 0$ (or 8) for $x = 6$, and $y = 8$ for $x = 5$).

Keep in mind that you did this kind of thinking naturally when expressing what it means for $E_{\Sigma\Pi}$ to be true versus false in the previous worksheet; every time you do these transformations *do them by thinking about what it means for the specific universal or existential to be false.*

Predicates

A function that produces booleans is called a **predicate**. We write

$$P : D \to \mathbb{B}$$

to indicate that P is a **unary** predicate (has one parameter) with **domain** D (the set of values that P can be instantiated with). If P has domain D we also say that P is a predicate **on** D, or **over** D.
E.g., for each $n \in \mathbb{N}$, define $E(n)$ to be: n is even. Then $E : \mathbb{N} \to \mathbb{B}$, i.e., E is a predicate on / over the natural numbers. Instantiate E with 108 and 165 and verify that the instances produce a specific boolean. Use E to (slightly) shorten $\forall x \in \{1, 6, 5\}, (\exists y \in \{1, 0, 8\}, y - x \text{ is even})$.

$$\forall x \in \{1, 6, 5\}, (\exists y \in \{1, 0, 8\}, E(y - x))$$

Suppose someone tries to define $Q : \mathbb{N} \to \mathbb{B}$ by $Q(n) = \forall n \in \mathbb{N}, n$ is even. What happens when you instantiate that?

> E.g., instance $n = 165$, produces $\forall 165 \in \mathbb{N}, 165$ is even, which doesn't have the proper form for a quantification: 165 is not a proper domain *variable*. It's not a proper definition.

A predicate on the natural numbers can be thought of as a table. For example, for E:

n	0	1	2	3	4	5	6	7	$\cdots$
$E(n)$	T	F	T	F	T	F	T	F	$\cdots$

In the rest of this section let $P : \mathbb{N} \to \mathbb{B}$, and do everything asked and answer every question.

Expand the statement $\exists\, n \in \mathbb{N}, P(n)$. Make at least three edge case predicates P, with illustrative tables, to explore that existential; make at least one that makes the existential true and one that makes it false. Express $\exists\, n \in \mathbb{N}, P(n)$ in simple natural prose (try to avoid just writing out the symbols in words, and try to avoid mentioning variable n).

$P(0) \vee P(1) \vee P(2) \vee P(3) \vee \cdots$

For each $n \in \mathbb{N}$ define $P(n)$ to be false.

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	F	F	F	F	F	F	F	$\cdots$

$\exists\, n \in \mathbb{N}, P(n)$ is false.

For each $n \in \mathbb{N}$ define $P(n)$ to be: $n = 6$.

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	F	F	F	F	F	T	F	$\cdots$

$\exists\, n \in \mathbb{N}, P(n)$ is true, 6 is a witness.

For each $n \in \mathbb{N}$ define $P(n)$ to be true.

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	T	T	T	T	T	T	T	T	$\cdots$

$\exists\, n \in \mathbb{N}, P(n)$ is true, 165 is a witness.

$\exists\, n \in \mathbb{N}, P(n)$ means: P is true for at least one natural number.

Repeat the previous question for $\forall\, n, P(n)$. Does $\exists\, n \in \mathbb{N}, P(n)$ entail $\forall\, n, P(n)$ (does each setting of parameter P that makes $\exists\, n \in \mathbb{N}, P(n)$ true make $\forall\, n, P(n)$ true)? If not, give an example P where $\exists\, n \in \mathbb{N}, P(n)$ is true but $\forall\, n, P(n)$ is false (and do that whenever we ask whether a statement entails another). Repeat this for whether $\forall\, n, P(n)$ entails $\exists\, n \in \mathbb{N}, P(n)$.

$P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \cdots$

For each $n \in \mathbb{N}$ define $P(n)$ to be false.

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	F	F	F	F	F	F	F	$\cdots$

$\forall\, n, P(n)$ is false, 165 is a counter-example.

For each $n \in \mathbb{N}$ define $P(n)$ to be: $n \neq 6$.

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	T	T	T	T	T	T	F	T	$\cdots$

$\forall\, n, P(n)$ is false, 6 is a counter-example.

For each $n \in \mathbb{N}$ define $P(n)$ to be true.

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	T	T	T	T	T	T	T	T	$\cdots$

$\exists\, n \in \mathbb{N}, P(n)$ is true.

The 2nd example from the previous solution shows $\exists\, n \in \mathbb{N}, P(n)$ can be true but $\forall\, n, P(n)$ false.

$\exists\, n \in \mathbb{N}, P(n)$ means: P is true for every natural number (or even: P is always true).

$\forall\, n, P(n)$ entails $\exists\, n \in \mathbb{N}, P(n)$, since if P is always true it's true for at least one natural number.

Expand $\exists\, n \in \mathbb{N},\, P(n+1)$ and simplify the additions. Re-express the result in simple natural prose. Then re-express $\exists\, n \in \mathbb{N},\, P(n+1)$ as a quantification with body $P(n)$, by changing the domain. Hint: look at the prose.

> $P(0+1) \vee P(1+1) \vee P(2+1) \vee P(3+1) \vee \cdots$, i.e., $P(1) \vee P(2) \vee P(3) \vee P(4) \vee \cdots$
> P is true for at least one positive natural number: $\exists\, n \in \mathbb{N}^{+},\, P(n)$.

Determine whether $\exists\, n \in \mathbb{N},\, P(n+1)$ entails $\exists\, n \in \mathbb{N},\, P(n)$, and vice-versa (if you use any example Ps that weren't defined before include tables for them).

> If P is true for at least one positive natural number then it's true for at least one natural number, so $\exists\, n \in \mathbb{N},\, P(n+1)$ entails $\exists\, n \in \mathbb{N},\, P(n)$.
> For each $n \in \mathbb{N}$, define $P(n)$ to be: $n = 0$.

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	T	F	F	F	F	F	F	F	$\cdots$

> Then P is true for zero, so true for at least one natural number, but not true for any other number, so not true for any positive natural number. That demonstrates that $\exists\, n \in \mathbb{N},\, P(n)$ doesn't entail $\exists\, n \in \mathbb{N},\, P(n+1)$.

Repeat the two previous questions for $\forall\, n \in \mathbb{N},\, P(n+1)$ versus $\forall\, n \in \mathbb{N},\, P(n)$.

> $\forall\, n \in \mathbb{N},\, P(n+1)$: $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge \cdots$
> P is true for every positive natural number.
> If P is true for every natural number then it's true for every positive natural number, so $\forall\, n \in \mathbb{N},\, P(n)$ entails $\forall\, n \in \mathbb{N},\, P(n+1)$.
> For each $n \in \mathbb{N}$, define $P(n)$ to be: $n \geq 1$.

n	0	1	2	3	4	5	6	7	$\cdots$
$P(n)$	F	T	T	T	T	T	T	T	$\cdots$

> Then P is true for every positive number, but not for zero so not for every natural number. That demonstrates that $\forall\, n \in \mathbb{N},\, P(n+1)$ doesn't entail $\forall\, n \in \mathbb{N},\, P(n)$.