Solutions to Selected Exercises - Week 1

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These exercises are retrieved from Chapter 4 of the textbook [LNS16].

Calculational Exercises

Exercise 3. For each of the following sets, either show that the set is a subspace of $\mathcal{C}(\mathbb{R})$ or explain why it is not a subspace.

(a) The set $\{f \in \mathcal{C}(\mathbb{R}) | f(x) \leq 0,\ \forall x \in \mathbb{R}\}$.

(e) The set $\{\alpha + \beta \sin(x) | \alpha, \beta \in \mathbb{R}\}$.

Solution. (a) The given set is not a subspace of $\mathcal{C}(\mathbb{R})$. Indeed, the function $f(x) = -1$ does belong to the set. But multiplying the function by the scalar -1 one obtains the function $g(x) := (-1)f(x) = (-1)(-1) = 1$. The function g does not belong to the given set, hence the given set is not closed under scalar multiplication.

(e) The given set is a subspace of $\mathcal{C}(\mathbb{R})$. To show that the given set is indeed a subspace one has to check the 3 conditions listed in Lemma 4.3.2. do apply.

Choosing $\alpha = \beta = 0$ one has the function:

$$h(x) = 0 + 0\sin(x) = 0.$$

This is the additive identity of $\mathcal{C}(\mathbb{R})$ and so the given set contains it.

If f, g belong to the proposed set, then:

$$f(x) = \alpha_1 + \beta_1 \sin(x) \text{ and } g(x) = \alpha_2 + \beta_2 \sin(x),$$

for some $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$. Then:

$$(f+g)(x) = f(x) + g(x) = (\alpha_1 + \beta_1 \sin(x)) + (\alpha_2 + \beta_2 \sin(x)) = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)\sin(x).$$

Since $\alpha_1 + \alpha_2 \in \mathbb{R}$ and $\beta_1 + \beta_2 \in \mathbb{R}$ then $f + g$ belongs to the given set. Hence the given set is closed under addition.

If f belongs to the proposed set, then $f(x) = \alpha + \beta \sin(x)$ for some $\alpha, \beta \in \mathbb{R}$. Let $a \in \mathbb{R}$. Then:

$$(af)(x) = a(\alpha + \beta \sin(x)) = (a\alpha) + (a\beta)\sin(x).$$

Since $a\alpha \in \mathbb{R}$ and $a\beta \in \mathbb{R}$ then af belongs to the given set. Hence the given set is closed under scalar multiplication.

All in all, the proposed set contains the additive identity of $\mathcal{C}(\mathbb{R})$, is closed under addition and is closed under scalar multiplication. This implies that the proposed set is a subspace of $\mathcal{C}(\mathbb{R})$.

□

Exercise 4. Give an example of a nonempty subset $U \subset \mathbb{R}^2$ such that U is closed under scalar multiplication but is not a subspace of $\mathbb{R}^2$.

Solution. Consider:

$$U := \{(x, y) \in \mathbb{R}^2 : |y| \geq |x|\}.$$

Clearly U is nonempty as $(0, 0) \in U$ (and U contains the additive identity).

If $(x, y) \in U$ and $a \in \mathbb{R}$ then $|y| \geq |x|$. Multiplying both sides by $|a|$ one obtains:

$$|a||y| \geq |a||x|.$$

This is equivalent to:

$$|ay| \geq |ax|.$$

This shows that $a(x, y) = (ax, ay) \in U$, i.e. U is closed under scalar multiplication. However, the vectors $v = (1, 1)$ and $w = (1, -1)$ do belong to U, but their sum $v + w = (2, 0)$ does not. Hence U is not closed under addition and so it is not a subspace of $\mathbb{R}^2$. $\square$

Proof-Writing Exercises

Exercise 1. Let v be a vector space over $\mathbb{F}$. Then, given $a \in \mathbb{F}$ and $v \in V$ such that $av = 0$, prove that either $a = 0$ or $v = 0$.

Solution. It is enough to show that, when $av = 0$ and $a \neq 0$, then $v = 0$.[1]
If $a \neq 0$, then a^{-1} exists. So we can multiply by a^{-1} both sides of the equation $av = 0$ obtaining:

$$a^{-1}(av) = a^{-1}0.$$

Recalling Associativity from Definition 4.1.1. the equation is equivalent to:

$$(a^{-1}a)v = a^{-1}0.$$

Since $a^{-1}a = 1$ and by Proposition 4.2.4. $a^{-1}0 = 0$, the above equation reads:

$$v = 0,$$

and this completes the proof. $\square$

Exercise 2. Let V be a vector space over $\mathbb{F}$, and suppose that W_1 and W_2 are subspaces of V. Prove that their intersection $W_1 \cap W_2$ is also a subspace of V.

Solution. Recall that by definition:

$$W_1 \cap W_2 = \{v \in V | v \in W_1 \text{ and } v \in W_2\}.$$

To show that $W_1 \cap W_2$ is indeed a subspace of V one has to check the 3 conditions listed in Lemma 4.3.2. In addition, remark that W_1 and W_2 are subspaces of W by assumption, so the 3 conditions listed in Lemma 4.3.2. do apply to W_1 and W_2.

1. Since both W_1 and W_2 are subspaces of V, then:

   $$0 \in W_1 \text{ and } 0 \in W_2.$$

   This implies that $0 \in W_1 \cap W_2$.

2. Let $u, v \in W_1 \cap W_2$. Then $u, v \in W_1$ and since W_1 is a subspace, then $u + v \in W_1$. But $u, v \in W_2$ as well, and since W_2 is a subspace, then $u + v \in W_2$. All in all, $u + v \in W_1$ and $u + v \in W_2$. Thus $u + v \in W_1 \cap W_2$.

3. Let $a \in \mathbb{F}$ and $u \in W_1 \cap W_2$. Then $u \in W_1$ and since W_1 is a subspace, then $au \in W_1$. But $u \in W_2$ as well, and since W_1 is a subspace, then $au \in W_2$. All in all, $au \in W_1$ and $au \in W_2$. Thus $au \in W_1 \cap W_2$.

All in all, $W_1 \cap W_2$ contains the additive identity, is closed under addition and is closed under scalar multiplication, thus it is a vector subspace of V. $\square$

[1] Indeed we have to show that: if $av = 0$ then at least one among a and v must be zero. Thus, when $a = 0$, we are done. Now we have to show that when $a \neq 0$ then v must necessarily be 0. All in all, at least one among a and v is 0.

References

[LNS16] Isaia Lankham, Bruno Nachtergaele, and Anne Schilling. *Linear Algebra As an Introduction to Abstract Mathematics*. Nov. 15, 2016. URL: https://www.math.ucdavis.edu/~anne/linear_algebra/.