WORKSHEET WEEK 2 MONDAY
CSC165 — 2025 WINTER

Universally Quantified Implication

A **universally quantified implication** is a common form of universal quantification, used to express a **restriction** of the domain using a special form of body:

$$\forall\, v \in D, (r_v \Rightarrow e_v) \text{ means } \forall\, v \in \left\{v' \in D : r_{v'}\right\}, e_v$$

Active

Do you recall the terminology for the parts of a (symbolic) quantification?
Identify those parts in this instance of the new notation:

$$\forall\, j \in \mathbb{N}, (j \leq 2 \Rightarrow \text{row } j \text{ has a zero})$$

Then, match its body against the body form in the new notation, and identify r_v and e_v.
Write out the meaning of this instance of the notation, using its particular v, D, e_v, and r_v.
Expand that, and summarize the meaning in simple natural prose.

Here are some other (mostly) prose phrasings, using a variable:

· For each (every) $j = 0$, 1, and 2, row j has a zero.
· For every (each) natural j less than or equal to two, row j has a zero.
· Row j has a zero for all naturals j less than or equal to two.
· If natural j is at most two then row j has a zero.

The last phrase doesn't explicitly quantify j with terms such as each / every / all / natural*s*. An unquantified variable is usually implicitly universally quantified, especially if it can be treated directly as a parameter. For example, if we treat the entire phrase as parameterized by j and instantiate it with 1 we get the meaningful phrase:

If natural 1 is at most two then row 1 has a zero.

We can also make the quantification explicit:

For each natural j, if j is at most two then row j has a zero.

English allows a lot of variety of phrasing, and mathematical English still does as well.
If we consider those phrasings as just rephrasings of one concept, then we might consider

$$\forall\, j \in \{0, 1, 2\}, \text{row } j \text{ has a zero}, \text{ and}$$

$$\forall\, j \in \mathbb{N}, (j \leq 2 \Rightarrow \text{row } j \text{ has a zero})$$

as just rephrasings, one in terms of a set of elements, the other in terms of a type and a condition. We want to understand them as equivalent, in the same way that we would understand any of the prose phrasings as expressing the "same" concept.

Define sets

$$A = \{n \in \mathbb{N} : n < 10 \wedge n \text{ is prime}\}$$
$$B = \{n \in \mathbb{N} : n < 10\}$$
$$C = \{n \in \mathbb{N} : n \text{ is prime}\}$$

Active

Which of these sets is a (**proper**) **subset** of another?
Are any of these sets a combination of the other two with a common set operation?
For each set, if it's a subset of another one of the sets, express it as a restriction of the other set: a set comprehension with an appropriate condition further restricting it.

Active

Express
$$\forall\, n \in \left\{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\right\}, n \text{ is odd}$$
as a universally quantified implication by moving the entire restriction into the body.
Then, based on the previous question, express it as two other universally quantified implications by keeping part of the restriction in the domain and moving part of it into the body.
Expand your answers enough to check.

Mathematical English also has various phrasings for this, for example:

- Every natural less than ten and prime is odd.
- If a natural is less than ten and prime then it's odd.
- If a natural less than ten is prime then it's odd.
- If a prime natural is less than ten then it's odd.

Notice that these make the same kinds of choices of whether to use a set/type/category versus a condition as the four symbolic quantifications do. Presumably you understand the prose phrasings as all "saying the same thing" as each other. A goal is to understand the symbolic phrasings as ultimately "the same".
The latter three prose phrases aren't explicitly quantified nor contain an explicit variable that might be implicitly universally quantified, but an indefinite object ("a natural", as opposed to a definite "the natural") often functions as an implicit universally quantified parameter.

Active

Consider the phrasing “if a natural is less than ten, then if it’s prime, then it’s odd”. This restricts the naturals to the ones less than ten, then restricts to the primes ones, then makes the claim that they’re odd. Expand

$$\forall\, n \in \mathbb{N},\, (n < 10 \Rightarrow (n \text{ is prime} \Rightarrow n \text{ is odd}))\ ,$$

which is inspired by this phrasing, to see whether it makes the same claim as the other universals. How about if we swap the conditions “$n < 10$” and “n is prime”?

In general, the following are equivalent:

$$\begin{aligned}
&\forall\, v \in D,\, ((r_v \wedge s_v) \Rightarrow e_v)\\
&\forall\, v \in D,\, ((s_v \wedge r_v) \Rightarrow e_v)\\
&\forall\, v \in D,\, (r_v \Rightarrow (s_v \Rightarrow e_v))\\
&\forall\, v \in D,\, (s_v \Rightarrow (r_v \Rightarrow e_v))
\end{aligned}$$

Active

Let $P : \mathbb{N} \to \mathbb{B}$. Expand $\exists\, n \in \mathbb{N},\, (\forall\, k \in \mathbb{N},\, (k > n \Rightarrow P(k)))\ .$

Active

Consider again the statement

$$\exists\, n \in \mathbb{N},\, (\forall\, k \in \mathbb{N},\, (k > n \Rightarrow P\,(k)))\;.$$

Generate a good set of example predicates, exploring ways it can be true and ways it can be false. During that exploration, develop a simple prose summary of the predicates which make it true, and the ones which make it false.

Homework

Express the statement in terms of something being finite or infinite. How about when it's false? Compare it with $\exists\, n \in \mathbb{N},\, P\,(n)$, $\forall\, n \in \mathbb{N},\, P\,(n)$, $\exists\, n \in \mathbb{N},\, P\,(n+1)$, and $\forall\, n \in \mathbb{N},\, P\,(n+1)$: which of those statements are entailed by it, and which ones entail it?

Let P and Q be predicates on $\mathbb{N}$.
Recall from Worksheet Week 1 Wednesday: $\exists\, n \in \mathbb{N}, P\,(n+1)$ is equivalent to $\exists\, n \in \mathbb{N}^{+}, P\,(n)$.

Active

Expand $\exists\, n \in \mathbb{N}, (Q\,(n) \wedge P\,(n))$.
Comparing expansions, can you find a Q so that that statement is equivalent to $\exists\, n \in \mathbb{N}^{+}, P\,(n)$?
Notice: for some naturals n you want $Q\,(n)$ to have a value that makes the value of $P\,(n)$ irrelevant, and for others you want it to have a value that makes only $P\,(n)$ relevant.

Active

Expand $\exists\, n \in \mathbb{N}, (Q\,(n) \wedge P\,(n))$ with that particular Q, and convince yourself (if you weren't already convinced, or needed to peek at the solution to find it) that it's equivalent to $\exists\, n \in \mathbb{N}^{+}, P\,(n)$.

Active

Expand the statement

$$\forall\, n \in \mathbb{N}, (\exists\, k \in \mathbb{N}, (k \geq n \wedge P\,(k)))\,.$$

Simplify the expansion by removing disjuncts that can't be true and conjuncts that have no effect.

In general, if p is a proposition then $\mathbf{false} \wedge p \equiv \mathbf{false}$, $\mathbf{true} \wedge p \equiv p$, and $\mathbf{false} \vee p \equiv p$.
But always keep in mind that logic, even when symbolic, is mainly used as a precise (and concise, when symbolic) language, not a system of calculation. Those equivalences are summarizing what we all understand if we say "both of **false** and p are true", "at least one of **false** and p is true", and "both of **true** and p are true": the first is obviously impossible, the latter two are determined by whether p is true.

Active

Express the simplified expansion with quantification, by restricting one of the domains.

In general, when the body of an existential is a conjunction, the conjunction can be viewed as restricting the domain:

$$\exists\, v \in D, (r_v \wedge e_v) \;\equiv\; \exists\, v \in \left\{v' \in D : r_{v'}\right\}, e_v$$

Only the elements of the domain for which r_v is true matter, and for those then only e_v matters.

Active

Explore

$$\forall\, n \in \mathbb{N}, (\exists\, k \in \mathbb{N}, (k \geq n \wedge P\,(k)))\,,$$

producing a good set of examples and prose summaries for when it's true versus false.

Homework

Compare the statement with $\exists\, n \in \mathbb{N}$, $P\,(n)$, $\forall\, n \in \mathbb{N}$, $P\,(n)$, $\exists\, n \in \mathbb{N}$, $P\,(n+1)$, $\forall\, n \in \mathbb{N}$, $P\,(n+1)$, and $\exists\, n \in \mathbb{N}$, $(\forall\, k \in \mathbb{N}, (k > n \Rightarrow P\,(k)))$.