Tutorial 4

Let H and K be groups and consider the set of pairs

$$H \times K = \{(h, k) : h \in H, k \in K\}$$

a.k.a. the Cartesian product. Define a binary operation on $H \times K$ by

$$(h, k)(h', k') = (hh', kk')$$

i.e. coordinatewise multiplication.

Problem 1. Show that $H \times K$ is a group. What is its order?

Solution

That the operation is associative (and hence a composition law) is straightforward. The identity element of $H \times K$ is the pair (e_H, e_K) because

$$(h, k)(e_H, e_K) = (he_H, ke_K) = (h, k)$$

for all $(h, k) \in H \times K$. And the inverse of (h, k) is (h^{-1}, k^{-1}) because

$$(h, k)(h^{-1}, k^{-1}) = (hh^{-1}, kk^{-1}) = (e_H, e_K).$$

The claim about $o(H \times K)$ follows from the fact that $|H \times K| = |H| \cdot |K|$ (as sets).

Definition — The group $H \times K$ is called the *direct product* of H and K. For ease of notation, we'll refer to (e_H, e_K) simply as (e, e).

Problem 2.

a) What is $C_2 \times C_2$?

b) What is $C_2 \times C_3$?

c) What is $\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$ (with the operation on $\mathbb{R}$ being $+$)?

Solution

a) Let $C_2 = \langle a \rangle$. Then

$$C_2 \times C_2 = \{(e, e), (a, e), (e, a), (a, a)\}$$

is a group of order 4. From Tutorial 2 we know this is either C_4 or V. Now

$$(a, e)^2 = (e, a)^2 = (a, a)^2 = (e, e).$$

Since there are no elements of order 4, we must have $C_2 \times C_2 \cong V$.

b) Let $C_2 = \langle a \rangle$ and let $C_3 = \langle b \rangle$. Then

$$C_2 \times C_3 = \{(e,e),(e,b),(e,b^2),(a,e),(a,b),(a,b^2)\}$$

is a group of order 6. What group is it?

Well, the element (a,b) has order 6 because neither $(a,b)^2 = (e,b^2)$ nor $(a,b)^3 = (a,e)$ is trivial, so $C_2 \times C_3 \cong C_6$.

c) The direct product of n copies of $\mathbb{R}$ has

$$\mathbb{R} \times \ldots \times \mathbb{R} = \{(x_1,\ldots,x_n) : x_1,\ldots,x_n \in \mathbb{R}\}$$

as its underlying set. The operation is coordinatewise addition and so $\mathbb{R} \times \ldots \times \mathbb{R} = \mathbb{R}^n$.

Problem 3. Let H and K be groups and consider the direct product $H \times K$. Show that, if h and k have finite order, then $o((h,k)) = \operatorname{lcm}\{o(h), o(k)\}$.

Solution

Let $n = o\big((h,k)\big)$ and let $m = \operatorname{lcm}\{o(h), o(k)\}$. We want to show that $n = m$.

Since n is the order of (h,k), we have $(h,k)^n = (e,e)$, which means $h^n = e$ and $k^n = e$. By the Division Lemma, n is a common multiple of $o(h)$ and $o(k)$. In particular, $n \geq \operatorname{lcm}\{o(h), o(k)\} = m$. But since m is a common multiple of $o(h)$ and $o(k)$, we have $(h,k)^m = (h^m, k^m) = (e,e)$, so $n = o\big((h,k)\big) \leq m$.

Problem 4. Show that $C_n \times C_m$ is cyclic iff $\gcd(n,m) = 1$.

Solution

For any $a \in C_n, b \in C_m$, we have $o((a,b)) = \operatorname{lcm}(o(a), o(b))$. Using the formula

$$\operatorname{lcm}(n,m) = \frac{nm}{\gcd(n,m)},$$

we know that $o((a,b)) = nm$ iff a, b are generators of C_n, C_m respectively and $\gcd(n,m) = 1$.