1 Homework for Week 2

1.1 Calculational Question 1

To show that $\{(1,1,1),(1,2,3),(2,-1,1)\}$ is linearly independent, we solve the system of equations $a(1,1,1)+b(1,2,3)+c(2,-1,1)=(0,0,0)$.

This gives us the following system of equations:

$$\begin{aligned} a+b+2c&=0\\ a+2b-c&=0\\ a+3b+c&=0 \end{aligned}$$

By subtracting the first equation from the second and third equations, we are left with the equivalent system of equations:

$$\begin{aligned} a+b+2c&=0\\ b-3c&=0\\ 2b-c&=0 \end{aligned}$$

The last two equations are equivalent to $b=3c$ and $c=2b$ so $b=6b \implies b=0$ and $c=0$. We conclude that $a=0$ as well and thus the set is linearly independent.

To write the vector $(1,-2,5)$ as a linear combination of the vectors above, we solve the system of equations

$$\begin{aligned} a+b+2c&=1\\ a+2b-c&=-2\\ a+3b-c&=5 \end{aligned}$$

which we similarly reduce to the system

$$\begin{aligned} a+b+2c&=1\\ b-3c&=-3\\ 2b-c&=4 \end{aligned}$$

This gives $c=2b-4$ which we substitute into the second equation to get $b-3(2b-4)=-3 \implies -5b+12=-3 \implies b=3$. We then have $c=2(3)-4=2$ and $a=1-3-4=-6$ and so

$$(1,-2,5)=-6(1,1,1)+3(1,2,3)+2(2,-1,1)$$

1.2 Calculational Question 4

1.2.1 Part (a)

By inspection, we see $(\lambda,-1,-1),(-1,\lambda,-1),(-1,-1,\lambda)$ is linearly dependent if $\lambda=-1$ since then the set of vectors is three copies of $(-1,-1,-1)$ which is clearly linearly dependent.

Is this the only solution? We can recall from an earlier linear algebra course that a set of n vectors in $\mathbb{R}^n$ is linearly independent if and only if the determinant of the matrix formed by the vectors is non-zero. In this case, we have the matrix

$$\begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix}$$

whose determinant is $\lambda(\lambda^2 - 1) - (-1)(-\lambda - 1) + (-1)(1 + \lambda) = (\lambda + 1)(\lambda(\lambda - 1) - 2)$ which is zero when either $\lambda = -1$ or $\lambda(\lambda - 1) - 2 = 0 \implies \lambda^2 - \lambda - 2 = 0 \implies \lambda = -1, 2$ so have another solution when $\lambda = 2$.

In this case $(2, -1, -1) + (-1, 2, -1) + (-1, -1, 2) = (0, 0, 0)$.

1.2.2 Part (b)

The key observation here is that $\cos 2x = 2\cos^2 x - 1$ and thus $\{\cos 2x, \sin^2 x, \lambda\}$ are linearly dependent if and only if $\{2\cos^2 x - 1, \sin^2 x, \lambda\}$ are linearly dependent.

Since $2\cos^2 x - 1 + 2(\sin^2 x) = 1$, we have that $\{2\cos^2 x - 1, \sin^2 x, \lambda\}$ are linearly dependent when $\lambda = 1$. In fact, we can see that $\{\cos 2x, \sin^2 x, \lambda\}$ are linearly dependent for any λ since scaling a collection of linearly dependent vectors does not change their linear dependence.

1.3 Calculational Question 5

1.3.1 Part (b)

It is not so, take $v = (1, 0, 1, 1)$. If we try to write $v = a(1, 1, 0, 0) + b(0, 1, 1, 0) + c(0, 0, 1, 1)$ we have the system of equations

$$\begin{aligned} a &= 1 \\ a + b &= 0 \\ b + c &= 1 \\ c &= 1 \end{aligned}$$

which can only be satisifed if $b = -1$ *and* $b = 0$ which clearly cannot be the case.

1.3.2 Part (c)

It is not so as $(1, -1, 0, 0) + (0, 1, -1, 0) + (0, 0, 1, -1) + (-1, 0, 0, 1) = (0, 0, 0, 0)$.

1.3.3 Part (d)

This is true. We show by contradiction. Suppose that $\operatorname{span}\{v_1, v_2, v_3, v_4\} = V$ and $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. Since $\{v_1, v_2, v_3, v_4\}$ is linearly dependent, it follows V is equal to the span of some subcollection of the vectors $\{v_1, v_2, v_3, v_4\}$ of length three (Linear Dependence Lemma) and therefore $\dim V \leq 3$ which is a contradiction since $\dim V = 4$.

1.3.4 Part (e)

This is a direct consequence of the Basis Extension Theorem (Theorem 5.3.7).

1.4 Proof Question 1

To show that $U = \text{span}\{v, u_1, \ldots, u_n\}$ it is necessary and sufficient to show that $U \leq \text{span}\{v, u_1, \ldots, u_n\}$ and $\text{span}\{v, u_1, \ldots, u_n\} \leq U$.

We first show that $U \leq \text{span}\{v, u_1, \ldots, u_n\}$. Let $u \in U$. Since $u \in U$ we can write $u = a_1u_1 + \ldots + a_nu_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$. Since 0 is the additive identity, we have $u = 0v + a_1u_1 + \ldots + a_nu_n$ and so $u \in \text{span}\{v, u_1, \ldots, u_n\}$.

To conclude, we show that $\text{span}\{v, u_1, \ldots, u_n\} \leq U$. Let $w \in \text{span}\{v, u_1, \ldots, u_n\}$. Then $w = b_1v + b_2u_1 + \ldots + b_nu_n$ for some $b_1, \ldots, b_n \in \mathbb{R}$. Since $v \in U$ and each $u_k \in U$ (as $U = \text{span}\{u_1, \ldots, u_n\}$), we have by the closure of U under addition that $w \in U$. ■

1.5 Proof Question 2

Recall that a set of vectors $\{u_1, \ldots, u_n\}$ **spans** a vector space V if $\text{span}\{u_1, \ldots, u_n\} = V$. We are given that $\{u_1, \ldots, u_n\}$ spans V and we are asked to show that $\{u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_n\}$ also spans V.

To simplify notation, let $v_k = u_k - u_{k+1}$ for $k \in \{1, \ldots, n-1\}$. Rephrasing the question, we are asked to show that $\{v_1, \ldots, v_{n-1}, u_n\}$ spans V when $\{u_1, \ldots, u_n\}$ spans V.

A direct approach is possible but we will show an inductive proof. By induction, it is sufficient to show that

1. $\{v_1, u_2, \ldots, u_n\}$ spans V (the base case).
2. $\{v_1, \ldots, v_k, u_{k+1}, \ldots, u_n\}$ spans V whenever $\{v_1, \ldots, v_{k-1}, u_k, \ldots, u_n\}$ spans V (the induction step).

For the base case, we will show that $\text{span}\{v_1, u_2, \ldots, u_n\} \leq V$ and $V \leq \text{span}\{v_1, u_2, \ldots, u_n\}$. To that end, let $w \in \text{span}\{v_1, u_2, \ldots, u_n\}$. Then $w = a_1u_1 - a_1u_2 + a_2u_2 + \ldots + a_nu_n = a_1u_1 + (a_2 - a_1)u_2 + \ldots + a_nu_n$ and thus $w \in V$ since V is spanned by $\{u_1, \ldots, u_n\}$. Conversely, let $w \in V$. Then $w = b_1u_1 + b_2u_2 + \ldots + b_nu_n$ for some $b_1, \ldots, b_n \in \mathbb{R}$. Adding and subtracting a term, we write $w = b_1u_1 - b_1u_2 + b_1u_2 + b_2u_2 + \ldots + b_nu_n = b_1v_1 + (b_2 - b_1)u_2 + \ldots + b_nu_n$ and thus $w \in \text{span}\{v_1, u_2, \ldots, u_n\}$.

The inductive step is similar so we will only show $V \leq \text{span}\{v_1, \ldots, v_k, u_{k+1}, \ldots, u_n\}$. To that end, let $w \in V$. Since this is an inductive step, we can assume that $w \in \text{span}\{v_1, \ldots, v_{k-1}, u_k, \ldots, u_n\}$. Then $w = c_1v_1 + \ldots + c_{k-1}v_{k-1} + c_ku_k + c_{k+1}u_{k+1} + \ldots + c_nu_n$ for some $c_1, \ldots, c_n \in \mathbb{R}$. Adding and subtracting a term, we write $w = c_1v_1 + \ldots + c_{k-1}v_{k-1} + c_ku_k - (c_ku_{k+1} - c_ku_{k+1}) + c_{k+1}u_{k+1} + \ldots + c_nu_n$ which is equal to $c_1v_1 + \ldots + c_{k-1}v_{k-1} + c_kv_k + (c_{k+1} - c_k)u_{k+1} + \ldots + c_nu_n$ and thus $w \in \text{span}\{v_1, \ldots, v_k, u_{k+1}, \ldots, u_n\}$.[1] ■

[1] While it may take some time to get used to the inductive approach, the key conceptual idea is that at each step we can write $u_k = v_k + u_{k+1}$ and so we can replace u_k with v_k in any linear combination.

1.6 Proof Question 3

We are asked to prove that if $\{v_1, v_2, \ldots, v_n\}$ are linearly independent vectors and $\{v_1 + w, v_2 + w, \ldots, v_n + w\}$ are linearly dependent vectors, then w is in the span of $\{v_1, v_2, \ldots, v_n\}$.

In this case, it is convenient to apply a logical transformation. You should convince yourself that proving the above is equivalent to proving that if $\{v_1 + w, v_2 + w, \ldots, v_n + w\}$ are linearly dependent vectors then *either* w is in the span of $\{v_1, v_2, \ldots, v_n\}$ or $\{v_1, v_2, \ldots, v_n\}$ are linearly dependent vectors as well.

To prove this, let $w, v_1, \ldots, v_n \in V$ such that $\{v_1 + w, v_2 + w, \ldots, v_n + w\}$ are linearly dependent vectors. Then there exist $a_1, \ldots, a_n \in \mathbb{R}$ not all zero such that $a_1(v_1 + w) + \ldots + a_n(v_n + w) = 0$. Rearranging, we find that $a_1v_1 + \ldots + a_nv_n + (a_1 + \ldots + a_n)w = 0$.

We now consider two cases, if $a_1 + \ldots + a_n = 0$ then $a_1v_1 + \ldots + a_nv_n = 0$ and thus $\{v_1, \ldots, v_n\}$ are linearly dependent otherwise

$$w = -\frac{a_1}{a_1 + \ldots + a_n}v_1 - \ldots - \frac{a_n}{a_1 + \ldots + a_n}v_n$$

and thus w is in the span of $\{v_1, \ldots, v_n\}$. ■