NUMERIC TYPES
CSC165 — 2025 WINTER

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of Natural numbers, aka the Naturals.
Whether zero is a natural number depends on context, but our course always considers zero to be a natural number.

$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \ldots\}$ is the set of Integral numbers, aka the Integers.

$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \wedge q \neq 0 \right\}$ is the set of Rational numbers, aka the Rationals.

$\mathbb{R}$ is the set of Real numbers, aka the Reals.

A superscript *, $^{\geq 0}$, or $^+$ on these sets means the non-zero, non-negative, or (strictly) positive elements, respectively. In particular, $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$, $\mathbb{Z}^* = \{-1, 1, -2, 2, -3, \ldots\}$, $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, $\mathbb{R}^* = (-\infty, 0) \cup (0, \infty)$, $\mathbb{R}^{\geq 0} = [0, \infty)$, and $\mathbb{R}^+ = (0, \infty)$.

Closure Properties

$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are "closed" under addition, multiplication, and raising to a natural number power:

$$\forall S \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}\}, x, y \in S, n \in \mathbb{N}, x + y \in S \wedge x \cdot y \in S \wedge x^n \in S$$

For 0^0 that assumes a context that defines 0^0 to be 1.

$\mathbb{Z}$ and $\mathbb{Q}$ are closed under taking the negative and subtraction:

$$\forall S \in \{\mathbb{Z}, \mathbb{Q}\}, x, y \in S, -x \in S \wedge x - y \in S$$

Subtracting an integer from an integer at least as large produces a natural:

$$\forall x, y \in \mathbb{Z}, x \geq y \Rightarrow x - y \in \mathbb{N}$$

$\mathbb{Q}$ is closed under taking the reciprocal on the subset $\mathbb{Q}^*$ where reciprocal is defined, and it stays in that subset, i.e.,

$$\forall x \in \mathbb{Q}^*, \frac{1}{x} \in \mathbb{Q}^*,$$

and it's closed under division when division is defined:

$$\forall y \in \mathbb{Q}, x \in \mathbb{Q}^*, \frac{y}{x} \in \mathbb{Q}$$

$\mathbb{Q}$ is closed under raising to an integer power when defined, so in addition to closure under natural number power:

$$\forall x \in \mathbb{Q}^*, z \in \mathbb{Z}, x^z \in \mathbb{Q}$$

$\mathbb{Z}$ is closed under taking the absolute value, and furthermore produces a natural, i.e.,

$$\forall z \in \mathbb{Z}, |z| \in \mathbb{N},$$

so $\mathbb{N}$ is also closed under it.

$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are closed under taking the minimum or maximum:

$$\forall S \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}\}, x, y \in S, \min(x, y) \in S \wedge \max(x, y) \in S$$

Floor and ceiling produce an integer, and for non-negative numbers produce specifically a natural, i.e.,

$$\forall x \in \mathbb{R}, \quad \lfloor x \rfloor \in \mathbb{Z} \wedge \lceil x \rceil \in \mathbb{Z},$$
$$\text{and } \forall x \in \mathbb{R}^{\geq 0}, \lfloor x \rfloor \in \mathbb{N} \wedge \lceil x \rceil \in \mathbb{N}.$$

so $\mathbb{Z}$ and $\mathbb{N}$ are also closed under them.

To use an instance of a closure property one can refer to the closed set, e.g., "since $\mathbb{Z}$ is closed under addition / +", or to the elements of the set, e.g., "since the difference of (any) two integers is an integer".

These properties are all universally quantified implications, so when it's clear that the conclusion of an instance of one of these properties is being used the instance can be referred to by just its hypotheses / pre-conditions. For example, if x and y are clearly integers and it's clear that one wants to conclude that $x - y$ is an integer one often just says "since x and y are integers".