WORKSHEET WEEK 1 WEDNESDAY
CSC165 — 2025 WINTER

Predicate Logic: Introducing Universal and Existential Quantification

Repeated addition and multiplication of terms of "the same form" can be expressed with summation notation and product notation. Similarly, conjunction and disjunction of terms of the same form are expressed by the notations

$$\forall\, v \in D,\, e_v \text{ and } \exists\, v \in D,\, e_v$$

Conjunction and disjunction in this form are called **universal quantification** and **existential quantification**. The notation has parameter v, which stands for a variable name (the **quantified variable**), parameter D, which is a set (the **domain**), and parameter e_v (the **body**), which is a propositional expression that may (and usually does) involve the variable name. The expansions of them are

$$\forall\, v \in \{x_0, x_1, x_2, \ldots\},\, e_v \text{ means } e_{x_0} \wedge e_{x_1} \wedge e_{x_2} \wedge \cdots$$
$$\text{and } \exists\, v \in \{x_0, x_1, x_2, \ldots\},\, e_v \text{ means } e_{x_0} \vee e_{x_1} \vee e_{x_2} \vee \cdots$$

where e_{x_0}, e_{x_1}, e_{x_2}, ... are the instantiations of the quantified variable in e_v with the values x_0, x_1, x_2, ...
Identify the kind of quantification, quantified variable, domain, and body of

$$\exists\, c \in \{\textbf{csc108}, \textbf{csc165}, \textbf{csc148}\},\, c \text{ is fun}$$

and determine whether it's true or false.

Identify the kind of quantification, quantified variable, domain, and body of each of the following, and determine which are true and which are false:

$$\forall\, n \in \left\{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\right\},\, n \text{ is odd}$$
$$\exists\, z \in \mathbb{Z}, z^2 < z$$
$$\forall\, x \in \{1, 6, 5\},\, (\exists\, y \in \{1, 0, 8\},\, y - x \text{ is even})$$

Determine which ones are true and which are false.

Recall equation $E_{\Sigma\Pi}$ from Worksheet Week 1 Monday. Consider the instance with $n = 3$. Express the summary of when $E_{\Sigma\Pi}$ is true using quantification, using a body "row j has a zero".

Use quantification to express "row j has a zero" in that, using a body "$a_{j,k} = 0$".

Repeat that to express when $E_{\Sigma\Pi}$ is false. Then express the condition for $E_{\Sigma\Pi}$ to be true, and the condition for it to be false, in general (with n uninstantiated), and give the corresponding expansions.

It's convenient to be able refer to the result of an instantiation by just mentioning the value of the parameter, so we might refer to the instance "165 is fun" as "instance $c = 165$" or even just "instance 165". We already did this in the first worksheet, but it's worth being aware of this explicitly for our next definition. Also, a universal quantification is often referred to as just "a universal", and an existential quantification as just "an existential".

A conjunction is true if every conjunct is true, and is false if at least one conjunct is false. So $\forall\, v \in D,\, e_v$ is true if every instance (with values from D) of e_v is true, and is false if at least one of the instances is false. A false instance is called a **counter-example**, and (if one exists) *disproves* (**prove is false**) a *universal*.
A disjunction is true if at least one disjunct is true, and is false if every disjunct is false. So $\exists\, v \in D,\, e_v$ is true if at least one instance of e_v is true, and is false if every instance is false. A true instance is called a **witness**, and (if one exists) *proves* an *existential*.

Use the terminology of counter-examples and witnesses to phrase why the first four examples on the first page are true or false, when applicable.

Homework.
Our "Numeric Types" reference has four quantifications of the form we just discussed:

$$\forall x \in \mathbb{Q}^*, \frac{1}{x} \in \mathbb{Q}^*$$

$$\forall z \in \mathbb{Z}, |z| \in \mathbb{N}$$

$$\forall x \in \mathbb{R}, \lfloor x \rfloor \in \mathbb{Z} \land \lceil x \rceil \in \mathbb{Z}$$

$$\forall x \in \mathbb{R}^{\geq 0}, \lfloor x \rfloor \in \mathbb{N} \land \lceil x \rceil \in \mathbb{N}$$

Expand the universal in each one, and replace the instances of $\frac{1}{x}$, $|z|$, $\lfloor x \rfloor$, and $\lceil x \rceil$ that appear with specific values.

A quantification whose body is also quantified is called a **nested quantification** and the body is referred to as the **inner** quantification. For the nested quantifications earlier in this worksheet, expand them again by expanding only the **inner** quantifications.

False Universal and Existential Quantifications

Notice that we described when a universal is false as there being at least one counter-example (false instance), which suggests that a universal being false can be expressed as an existential; express

$$\forall\, n \in \left\{n' \in \mathbb{N} : n' < 10 \wedge n' \text{ is prime}\right\},\, n \text{ is odd}$$

in natural prose, what it being false means as an existential, and express that in natural prose.

Similarly, an existential being false was described as every instance being false: express what

$$\exists\, c \in \{\mathbf{csc108}, \mathbf{csc165}, \mathbf{csc148}\}\,,\, c \text{ is fun}$$

being false means, as a universal.

Express what $\forall\, x \in \{1, 6, 5\}\,,\, (\exists\, y \in \{1, 0, 8\}\,,\, y - x \text{ is even})$ being false means, as a nested quantification. We know it's true, which means the quantification you produced is false: explain directly why it's false.

Keep in mind that you did this kind of thinking naturally when expressing what it means for $E_{\Sigma\Pi}$ to be true versus false in the previous worksheet; every time you do these transformations *do them by thinking about what it means for the specific universal or existential to be false.*

Predicates

A function that produces booleans is called a **predicate**. We write

$$P : D \to \mathbb{B}$$

to indicate that P is a **unary** predicate (has one parameter) with **domain** D (the set of values that P can be instantiated with). If P has domain D we also say that P is a predicate **on** D, or **over** D.
E.g., for each $n \in \mathbb{N}$, define $E\,(n)$ to be: n is even. Then $E : \mathbb{N} \to \mathbb{B}$, i.e., E is a predicate on / over the natural numbers. Instantiate E with 108 and 165 and verify that the instances produce a specific boolean. Use E to (slightly) shorten $\forall\, x \in \{1, 6, 5\}\,,\, (\exists\, y \in \{1, 0, 8\}\,,\, y - x \text{ is even})$.

Suppose someone tries to define $Q : \mathbb{N} \to \mathbb{B}$ by $Q\,(n) = \forall\, n \in \mathbb{N}, n$ is even. What happens when you instantiate that?

A predicate on the natural numbers can be thought of as a table. For example, for E:

n	0	1	2	3	4	5	6	7	$\cdots$
$E(n)$	T	F	T	F	T	F	T	F	$\cdots$

In the rest of this section let $P : \mathbb{N} \to \mathbb{B}$, and do everything asked and answer every question.

Expand the statement $\exists\, n \in \mathbb{N},\, P(n)$. Make at least three edge case predicates P, with illustrative tables, to explore that existential; make at least one that makes the existential true and one that makes it false. Express $\exists\, n \in \mathbb{N},\, P(n)$ in simple natural prose (try to avoid just writing out the symbols in words, and try to avoid mentioning variable n).

Repeat the previous question for $\forall\, n,\, P(n)$. Does $\exists\, n \in \mathbb{N},\, P(n)$ entail $\forall\, n,\, P(n)$ (does each setting of parameter P that makes $\exists\, n \in \mathbb{N},\, P(n)$ true make $\forall\, n,\, P(n)$ true)? If not, give an example P where $\exists\, n \in \mathbb{N},\, P(n)$ is true but $\forall\, n,\, P(n)$ is false (and do that whenever we ask whether a statement entails another). Repeat this for whether $\forall\, n,\, P(n)$ entails $\exists\, n \in \mathbb{N},\, P(n)$.

Expand $\exists\, n \in \mathbb{N},\, P\,(n+1)$ and simplify the additions. Re-express the result in simple natural prose. Then re-express $\exists\, n \in \mathbb{N},\, P\,(n+1)$ as a quantification with body $P\,(n)$, by changing the domain. Hint: look at the prose.

Determine whether $\exists\, n \in \mathbb{N},\, P\,(n+1)$ entails $\exists\, n \in \mathbb{N},\, P\,(n)$, and vice-versa (if you use any example Ps that weren't defined before include tables for them).

Repeat the two previous questions for $\forall\, n \in \mathbb{N},\, P\,(n+1)$ versus $\forall\, n \in \mathbb{N},\, P\,(n)$.