Problem 1

Let $f : [0, 1] \to \mathbb{C}$ such that $f(x) = \frac{1}{i-x}$. Find the real and imaginary parts of f. Compute $f'(x)$ and $\int_0^1 f(x)\,\mathrm{d}x$.

Problem 2

Show that the maps $Tf : f \mapsto f'$ and $Sf : f \mapsto \int_0^x f(t)\,\mathrm{d}t$ are $\mathbb{C}$-linear maps from $C^1([0, 1], \mathbb{C})$ to $C([0, 1], \mathbb{C})$ and $C([0, 1], \mathbb{C})$ to $\mathbb{C}$, respectively.

Problem 3

Using the fundamental theorem of calculus for real functions, prove the fundamental theorem of calculus for complex functions, i.e.

$$\int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0)$$

for $f \in C^1([0, 1], \mathbb{C})$.

Problem 4

Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{zt} = ze^{zt}$$

for $z \in \mathbb{C}$.

Problem 5

Consider the integral $I = \int_0^\infty e^{-at} \cos(bt)\,\mathrm{d}t$ where $a > 0$ and $b \in \mathbb{R}$ are real numbers.

1. Calculate I using integration by parts.
2. Show that $I = \mathrm{Re}\left[\int_0^\infty e^{-(a-ib)t}\,\mathrm{d}t\right]$ where Re denotes the real part of a complex number.
3. Calculate I using the formula above. Which do you prefer?

Problem 6

Let V be a finite-dimensional inner product space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and $b = \{b_1, \ldots, b_n\}$ an orthonormal basis of V.

Using the *resolution of the identity* formula, i.e.

$$v = \sum_{i=1}^{n} \langle v, b_i \rangle b_i$$

for $v \in V$, show that the matrix elements of a linear operator $A : V \to V$ with respect to the basis b are given by

$$A_{ij} = \langle A(b_j), b_i \rangle \,.$$