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University of Toronto

Faculty of Arts and Sciences

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MAT244H1F Homework 2

Special Instructions:

- See the **submission instructions** on Quercus for details about submitting this assignment.
- Include **all pages** when submitting your assignment.
- Write legibly and darkly (if scanned) or digitally on a tabled.
- For questions with a boxed area, ensure your answer is completely within the box.
- Fill in your bubbles completely.

Good: ● A ○ B   Bad: ◐ A ⊙ B ⊗ C

1. Certain non-separable equations can be solved using the method of *integrating factors*. It works by exploiting the product rule as follows. Since $[r(x)y]' = r(x)y' + r'(x)y$, if you are given an equation of the form $A(x)y' + B(x)y = C(x)$, you may be able to multiply both sides by a non-zero function $p(x)$ so that the left hand side of the equation looks like a product rule.

   (a) Consider $2y' + 4y$. Find non-zero functions $p(x)$ and $r(x)$ so that $p(x) \cdot (2y' + 4y) = [r(x)y]'$.

   $p(x) =$ $\quad$ $r(x) =$

   (b) Consider $y' + \cos(x)y$. Find non-zero functions $p(x)$ and $r(x)$ so that $p(x) \cdot (y' + \cos(x)y) = [r(x)y]'$.

   $p(x) =$ $\quad$ $r(x) =$

   (c) Solve the differential equation $[e^{2x}y]' = \frac{x^2}{2}$ for y. (Hint: most of the work has been done for you already.)

   Solution:

   (d) Use the method of integrating factors to solve the differential equation $2y' + 4y = \frac{x^2e^{-2x}}{2}$. Explain your steps.

2. *To answer this question, you may need to review linear combinations and change of basis from MAT223.*

   Consider the differential equation $y'' = 0$ which has solutions $s_1(x) = x + 1$ and $s_2(x) = x - 1$ (among many others).

   (a) Express the solution $s_3(x) = 2x + 4$ as a linear combination of s_1 and s_2.

   $s_3(x) = \boxed{}\, s_1(x) + \boxed{}\, s_2(x)$

   (b) The complete solution to $y'' = 0$ is $r(x) = Ax + B$ where $A, B \in \mathbb{R}$. Express r as a linear combination of s_1 and s_2. (*Hint: your solution will involve the constants A and B.*)

   $r(x) = \boxed{}\, s_1(x) + \boxed{}\, s_2(x)$

   (c) Give two different bases for the solution set to $y'' = 0$.

   Basis 1: $\{ \qquad \}$

   Basis 2: $\{ \qquad \}$

   (d) The differential equation $y''(x) = J_1(x)$ has a solution $F_{\text{ugly}}(x) = (\frac{2}{x} - 1)J_1(x) - \frac{1}{x}J_0(x)$ where J_1 and J_0 are Bessel Functions of the First Kind[1]. Use your knowledge from the previous parts to find the complete solution to $y'' = J_1(x)$. Explain your reasoning.

   Hint: You do not need to know what Bessel functions are to answer this question.

[1] https://en.wikipedia.org/wiki/Bessel_function

3. Read the appendix on *complex numbers* in your textbook (you may seek additional resources if needed).

(a) Let $x = 3 + 2i$ and $y = -2 + 5i$. Compute xy and $\frac{x}{y}$. Express your answer in $a + bi$ form.

$xy =$ $\qquad$ $\frac{x}{y} =$

(b) The *absolute value* or *modulus* of a complex number $a + bi$ is $|a + bi| = \sqrt{a^2 + b^2}$, and when a vector $\vec{v} = (z_1, z_2, \ldots, z_n)$ has complex entries, its norm is computed by the formula $\|\vec{v}\| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$

i. Let $x = a + bi$. Find a complex number y so that $|x|^2 = xy$. (Such a number is called the *complex conjugate* of x.)

$y =$

ii. Let $\vec{v} = (1 + 2i, c)$. Find a *real number* c so that $\|\vec{v}\| = 4$.

$c =$

(c) A complex number $a+bi$ is equal to zero if and only if $a = 0$ and $b = 0$. Let $x = a+bi$ and $y = c+di$ and suppose $a, b, c, d \neq 0$. Show that $xy \neq 0$.

4. For each of the following, sketch a phase portrait satisfying the criteria or explain why no such phase portrait exists. (Assume all differential equations/systems are continuous.)

   (a) A phase portrait for a **system** of two differential equations with exactly one attracting and one repelling equilibrium solution and no other equilibrium solutions.

   (b) A phase portrait for a **single** differential equation with exactly one attracting and one repelling equilibrium solution and no other equilibrium solutions.

   (c) A phase portrait for a **system** of two differential equations with exactly two attracting equilibrium solutions and no other equilibrium solutions.

(d) A phase portrait for a **single** differential equations with exactly two attracting equilibrium solutions and no other equilibrium solutions.

(e) A phase portrait for a **system** of two differential equations with infinitely many repelling equilibrium solutions and no other equilibrium solutions.

5. **Report Preparation.** *At the end of the term, you will hand in a final report. This problem is intended to help you develop the skills needed for your final report.*

Consider the model of how a bee travels in a particular field of flowers

$$S'(t) = \sin(D(t))$$
$$D'(t) = S(t) - D(t)$$

where

$S(t)$ = intensity of the smell of flowers at time t (on a logarithmic scale from $-\infty$ to $+\infty$)
$D(t)$ = (positive) distance between the bee and its hive at time t

Additionally, near a flower and the hive, the smell intensity is measured to be constant.

In this problem, you will be analyzing what this model claims about a bee's behaviour. Your analyses should be a mix of analytic arguments (those coming from mathematical formulas), numeric arguments (computer-based simulations), and qualitative arguments (e.g., analyzing pictures and phase portraits).

(a) At what distance(s) is smell intensity constant? Justify your answer.

(b) In the field being modeled, a bee will only stay still in its hive or on a flower. At what distance(s) from the hive are the flowers? Justify your answer.

(c) If a bee starts at distance 5 from its hive and detects a neutral smell (i.e., smell intensity 0), what will the bee do and where will it end up? Justify with a simulation. Include details about your simulation in your answer. (For example, screenshots of a spreadsheet, details about what methods you used, graphs of results with labeled axes, etc.)

(d) Which flowers can attract bees? Justify with simulations and with qualitative techniques.

(e) If a bee starts in its hive, what is the minimum smell intensity that would cause it to venture to the **third** flower (i.e., the third-closest flower to the hive)? Justify your answer.

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