Solutions to Selected Exercises - Week 3

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MAT246H1F: CONCEPTS IN ABSTRACT MATHEMATICS

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The following exercises are retrieved from Chapter 5 of the textbook [LNS16].

Calculational Exercises

Exercise 2. Consider the complex vector space $V = \mathbb{C}^3$ and the list (v_1, v_2, v_3) of vectors in V where:

$$v_1 = (i, 0, 0),\ v_2 = (i, 1, 0),\ v_3 = (i, i, -1).$$

(a) Prove that $\mathrm{span}(v_1, v_2, v_3) = V$.

(b) Prove or disprove: (v_1, v_2, v_3) is a basis for V.

Solution. (a) Denote $U = \mathrm{span}(v_1, v_2, v_3)$. Since $v_1, v_2, v_3 \in V$, by Lemma 5.1.2 U is a subspace of V and hence $U \subset V$. To complete the proof, one needs to show the reverse inclusion i.e. $V \subset U$, that is to say, any vector $w \in V$ can be expressed as a linear combination of v_1, v_2 and v_3. To do so, let $w = (\alpha, \beta, \gamma) \in V$ for $\alpha, \beta, \gamma \in \mathbb{C}$ and then impose the condition:

$$av_1 + bv_2 + cv_3 = w,$$

for some $a, b, c, \in \mathbb{C}$ to be determined. This can be expressed as:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = a \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} i \\ i \\ -1 \end{pmatrix} = \begin{pmatrix} ai + bi + ci \\ b + ci \\ -c \end{pmatrix}.$$

This can be interpreted as a linear system of equations:

$$\begin{cases} ai + bi + ci & = \alpha \\ b + ci & = \beta \\ -c & = \gamma \end{cases}$$

This system admits one and only one solution: starting from the third line and substituting back one can solve for a, b and c in terms of α, β and γ as desired:

$$a = (\gamma - \beta) - i(\alpha + \gamma),\ b = \beta + i\gamma,\ c = -\gamma.$$

This proves that $w = (\alpha, \beta, \gamma) = av_1 + bv_2 + bv_3 \in \mathrm{span}(v_1, v_2 v_3) = U$. Recalling that w was chosen arbitrarily, the above implies that $U = V$ and the proof is complete.

(b) In order to be a basis for V, vectors (v_1, v_2, v_3) need to:

1. generate V, i.e. $\mathrm{span}(v_1, v_2, v_3) = V$
2. be linearly independent.

Item (i) was proven in part (a) of the exercise. To prove item (ii) let $a_1, a_2, a_3 \in \mathbb{C}$. Assume that

$$0 = a_1 v_1 + a_2 v_2 + a_3 v_3.$$

The above implies:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} i \\ i \\ -1 \end{pmatrix} = \begin{pmatrix} a_1 i + a_2 i + a_3 i \\ a_2 + a_3 i \\ -a_3 \end{pmatrix}.$$

This can be interpreted as a linear system of equations:

$$\begin{cases} ia_1 + ia_2 + ia_3 & = 0 \\ a_2 + a_3 & = 0 \\ -a_3 & = 0 \end{cases}$$

The third row implies $a_3 = 0$. Substituting this in the second row, one gets $a_2 = 0$, finally, substituting $a_1 = a_2 = 0$ in the first row, one finds $a_1 = 0$. All in all, one has $a_1 = a_2 = a_3 = 0$ and recalling the definitions, this shows that v_1, v_2, v_3 are linearly independent. Recalling (i) and (ii), (v_1, v_2, v_3) is a basis for V.

□

Remark. Parts (a) and (b) of the above exercise could be solved simultaneously remarking that $\dim V = 3$ and that (v_1, v_2, v_3) are linearly independent. By Theorem 5.4.4, this implies at once that (v_1, v_2, v_3) is a basis for V and as a consequence $\operatorname{span}(v_1, v_2, v_3) = V$.

Exercise 3(d). Determine the dimension of the following subspaces of $\mathbb{F}^4$.

(d) $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 | x_4 = x_1 + x_2, x_3 = x_1 - x_2, x_3 + x_4 = 2x_1\}$.

Solution. For convenience, denote the assigned vector space $V_{(d)}$. First of all, remark that the conditions $x_4 = x_1 + x_2$ and $x_3 = x_1 - x_2$ do imply $x_3 + x_4 = 2x_1$, hence in the definition of $V_{(d)}$, the latter condition can be dropped. Hence:

$$V_{(d)} = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 | x_4 = x_1 + x_2, x_3 = x_1 - x_2\}.$$

Notice that:

$$\begin{aligned} V_{(d)} &= \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 | x_4 = x_1 + x_2, x_3 = x_1 - x_2\} \\ &= \{(x_1, x_2, x_1 - x_2, x_1 + x_2) | x_1, x_2 \in \mathbb{F}^4\} \\ &\overset{\star}{=} \{x_1(1, 0, 1, 1) + x_2(0, 1, -1, 1) | x_1, x_2 \in \mathbb{F}^4\} \\ &= \operatorname{span}((1, 0, 1, 1), (0, 1, -1, 1)) \end{aligned}$$

Denoting $u_1 = (1, 0, 1, 1)$ and $u_2 = (0, 1, -1, 1)$, the above shows that $V_{(d)} = \operatorname{span}(u_1, u_2)$. Vectors u_1, u_2 are also linearly independent. To see this, let $a_1, a_2 \in \mathbb{F}$ be such that $a_1 u_1 + a_2 u_2 = 0$. That is to say:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_1 - a_2 \\ a_1 + a_2 \end{pmatrix}.$$

This can be interpreted as a linear system of equations:

$$\begin{cases} a_1 & = 0 \\ a_2 & = 0 \\ a_1 - a_2 & = 0 \\ a_1 + a_2 & = 0 \end{cases}$$

Clearly the only possible solution is $a_1 = a_2 = 0$. All in all u_1, u_2 are linearly independent[1], recalling that these vectors also span $V_{(d)}$, one has that: (u_1, u_2) is a basis for $V_{(d)}$. This implies that $\dim V_{(d)} = 2$. □

[1] A much faster proof that u_1, u_2 are linearly independent can be given in the light of Chapter 9, showing that $\langle u, v \rangle = 0$, however, the proof above shows that the structure of inner-product space is not necessary.

Remark. The starred equality in the previous exercise can be deducted with the following procedure: in $(x_1, x_2, x_1 - x_2, x_1 + x_2)$ substitute $x_1 = 1$ and $x_2 = 0$ to obtain $(1, 0, 1, 1)$, substitute $x_1 = 0$ and $x_2 = 1$ to obtain $(0, 1, -1, 1)$. If there are more than two different variables, this procedure should be iterated more times obtaining more vectors.

Proof-Writing Exercises

Exercise 5. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose that U is a subspace of V for which $\dim(U) = \dim(V)$. Prove that $U = V$.

Remark. A solution for this exercise, different from the one proposed below, can be found in the notes from Federico's Office Hours on 2024-09-25, available on Quercus.

Proof. Let $n = \dim(U) = \dim(V)$. Then U admits a basis $(u_1, u_2, \dots, u_n)$, that is to say $(u_1, u_2, \dots, u_n)$ are linearly independent and $U = \operatorname{span}(u_1, u_2, \dots, u_n)$. Since U is a subspace of V, $U \subset V$, as a consequence, $u_1, u_2, \dots, u_n \in V$. Thus $(u_1, u_2, \dots, u_n)$ is a family of n linearly independent vectors in V. By part 3 of Theorem 5.4.4 from [LNS16], $(u_1, u_2, \dots, u_n)$ is a basis for V as $\dim(V) = n$. This implies $V = \operatorname{span}(u_1, u_2, \dots, u_n)$. Recalling that $U = \operatorname{span}(u_1, u_2, \dots, u_n)$ as well, one has:

$$U = \operatorname{span}(u_1, u_2, \dots, u_n) = V.$$

□

The following exercises are retrieved from Chapter 9 of the textbook [LNS16].

Proof-Writing Exercises

Exercise 1. Let V be a finite-dimensional inner product space over $\mathbb{F}$. Given any vectors $u, v \in V$, prove that the following statements are equivalent:

(a) $\langle u, v \rangle = 0$

(b) $\|u\| \leq \|u + \alpha v\|$ for every $\alpha \in \mathbb{F}$.

Solution. $(a) \Rightarrow (b)$. Assuming that statement (a) holds, we want to prove statement (b). Let $\alpha \in \mathbb{F}$, notice that:

$$\begin{aligned} \|u + \alpha v\|^2 &= \langle u + \alpha v, u + \alpha v \rangle \\ &= \langle u, u \rangle + \alpha \langle v, u \rangle + \overline{\alpha} \langle u, v \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \|v\|^2 \\ &\geq \|u\|^2; \end{aligned}$$

where the third equality follows from the fact that $\alpha \langle v, u \rangle = \overline{\alpha} \langle u, v \rangle = 0$ as (a) is assumed to hold. Recalling that the function $t \mapsto \sqrt{t}$ is increasing on its domain $[0, +\infty)$, one can drop the power 2 in the above chain obtaining:

$$\|u + \alpha v\| \geq \|u\|.$$

By the arbitrariness of the choice of α, statement (b) is proven.

$(b) \Rightarrow (a)$. Assuming that statement (b) holds, we want to prove statement (a). If $v = 0$ statement (a) is true as for every $u \in V$, $\langle u, 0 \rangle = 0$. Hence, in what follows, it is not restrictive to assume that $v \neq 0$. In the light of this, $u = u_1 + u_2$, where[2] $u_1 = \beta v$ for some $\beta \in \mathbb{F}$ and $u_2 \perp v$ (this implies: $u_1 \perp u_2$). Thus, by Pythagorean Theorem:

$$\|u_1\|^2 + \|u_2\|^2 = \|u\|^2 \leq \|u + \alpha v\|^2 = \|u_1 + u_2 + \alpha v\|^2 = \|u_2 + (\alpha + \beta) v\|^2 = \|u_2\|^2 + \|(\alpha + \beta) v\|^2.$$

[2]For the explicit expression see Equation (9.3) from [LNS16].

Pythagorean Theorem was applied twice: in the first equality and in the last equality, thanks to the fact that $u_2 \perp v$. Notice that the inequality in the above chain follows from (b) and the fact that $t \mapsto t^2$ is increasing on $[0, +\infty)$. Considering the first term and last term of the above chain, the term $\|u_2\|^2$ can be cancelled and recalling the properties of norms:

$$\|u_1\|^2 \leq |\alpha + \beta|^2 \|v\|^2.$$

Thanks to the assumptions in (b), the above must hold true for every $\alpha \in \mathbb{F}$, in particular it must hold true for $\alpha = -\beta$. This implies $\|u_1\|^2 = 0$ and hence $u_1 = 0$. Finally, $u = u_2$ and since $u_2 \perp v$, $u \perp v$ as well, i.e. $\langle u, v \rangle = 0$. □

Exercise 4. Let V be a finite-dimensional inner product space over $\mathbb{R}$. Given $u, v \in V$, prove that:

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4}.$$

Remark. The above identity is called **polarization identity** for vector spaces over $\mathbb{R}$. There is also a polarization identity for vector spaces over $\mathbb{C}$, namely:

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2}{4}.$$

Solution. Remark that:

$$\begin{aligned}
\|u + v\|^2 =& \langle u + v, u + v \rangle \\
=& \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\
=& \|u\|^2 + 2\langle u, v \rangle + \|v\|^2.
\end{aligned}$$

Analogously:

$$\begin{aligned}
\|u - v\|^2 =& \langle u - v, u - v \rangle \\
=& \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\
=& \|u\|^2 - 2\langle u, v \rangle + \|v\|^2.
\end{aligned}$$

Subtracting the first and last terms of these chains of equalities one obtains:

$$\|u + v\|^2 - \|u - v\|^2 = 4\langle u, v \rangle.$$

Dividing both terms by 4 one obtains the thesis. □

References

[LNS16] Isaia Lankham, Bruno Nachtergaele, and Anne Schilling. *Linear Algebra As an Introduction to Abstract Mathematics*. Nov. 15, 2016. URL: `https://www.math.ucdavis.edu/~anne/linear_algebra/`.