Tutorial 7

Given a group G and some of its elements, we will construct some normal subgroups that are related to these elements.

Problem 1. Let G be a group and let $H \leq G$. We define the *normalizer* of H in G to be the set of elements in G that commute with H. That is,

$$N_G(H) = \{g \in G : gHg^{-1} = H\},$$

where $gHg^{-1} = \{ghg^{-1} : h \in H\}$.

a) Show that $H \leq N_G(H) \leq G$.

b) Show that $H \trianglelefteq N_G(H)$.

c) Show that H is normal in G if and only if $N_G(H) = G$.

Solution

a) We have that $e \in N_G(H)$ because $eHe^{-1} = H$. We note that if $x, y \in N_G(H)$ then

$$xy^{-1}(H)yx^{-1} = xy^{-1}(yHy^{-1})yx^{-1} = xHx^{-1} = H.$$

Thus, $xy^{-1} \in N_G(H)$ so $N_G(H) \leq G$.

Now for $h \in H$, $hHh^{-1} = H$ because $hh'h^{-1} \in H$ for any $h' \in H$. That is, $h \in N_G(H)$ and hence $H \subset N_G(H)$.

$$H \leq N_G(H) \leq G.$$

b) Take $x \in N_G(H)$. By definition of $N_G(H)$, we know that $xHx^{-1} = H \subseteq H$.

Since this holds for all $x \in N_G(S)$, we have $H \trianglelefteq N_G(H)$.

c) H is normal in G if and only if $gHg^{-1} \subseteq H$ for all $g \in G$ if and only if $N_G(H) = G$.

Problem 2. Let G be a group. Recall that the *conjugacy class* of $x \in G$ is defined as its orbit under the conjugation action, that is,

$$\mathrm{cl}(x) := \{gxg^{-1} : g \in G\}.$$

If $H \leq G$, show that $H \trianglelefteq G$ if and only if

$$H = \bigcup_{x \in H} \mathrm{cl}(x).$$

Solution

If $H \trianglelefteq G$, then $gHg^{-1} \subseteq H$ for any $g \in G$. That is, if $h \in H$ then $ghg^{-1} \in H$ for all $g \in G$. in other words, $\mathrm{cl}(x) \in H$.

It follows that

$$H = \bigcup_{x \in H} \mathrm{cl}(x).$$

If

$$H = \bigcup_{x \in H} \mathrm{cl}(x),$$

then for all $x \in H$ and $g \in G$ we have $gxg^{-1} \in \mathrm{cl}(x) \subseteq H$. That is, $gHg^{-1} \subseteq H$ and hence $H \trianglelefteq G$.

Problem 3. Let G be a group and let $H \leq G$. Consider the intersection of conjugates of H, $K_H = \cap_{g \in G} gHg^{-1}$. [We don't have a name for it.]

a) Show that $K_H \leq H \leq G$.

b) Show that $K_H \trianglelefteq G$.

Solution

a) We present the following lemma.

Lemma

An arbitrary intersection of subgroups is a group.

Proof. If $\{A_i\}_{i \in I}$ is a set of subgroups of G, then $e \in A_i$ for all i and hence $e \in \bigcap_{i \in I} A_i$.

Now $x, y \in \bigcap_{i \in I} A_i$ implies that for every $i \in I$,

$$x \in A_i \text{ and } y \in A_i.$$

Since each A_i is a subgroup of G, we know $xy^{-1} \in A_i$ for all i. That is, $xy^{-1} \bigcap_{i \in I} A_i$.

Subgroup criterion gives

$$\bigcap_{i \in I} A_i \leq G. \qquad \square$$

$K_H \leq G$ follows.

b) Take any $g \in G$. For any $x \in K_H$ and every $g' \in G$, we know

$$x \in (g^{-1}g')H(g^{-1}g')^{-1} = g^{-1}g'Hg'^{-1}g.$$

That is, $gxg^{-1} = g'hg'^{-1}$ for some $h \in H$. Then

$$gxg^{-1} \in \bigcap_{g' \in G} g'Hg'^{-1} = K_H.$$

So we know $gK_Hg^{-1} \subseteq K_H$ for any g.