1 Homework for Week 6

1.1 Calculational Question 1

1.1.1 Parts (d) and (e)

Recall that to find the matrix representation of a linear map $T : V \to W$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V and W respectively, we can apply T to each basis vector in $\mathcal{B}$ and express the result as a linear combination of the basis vectors in $\mathcal{C}$. The coefficients of these linear combinations form the columns of the matrix representation of T.

To that end, we have the following linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x)$.

Let $\mathcal{B} = \{(1, 0), (0, 1)\}$ and $\mathcal{C} = \{(1, 0), (0, 1)\}$ be the standard bases of $\mathbb{R}^2$. Then $T(1, 0) = (1, 1) = 1 \cdot (1, 0) + 1 \cdot (0, 1)$ and $T(0, 1) = (1, 0) = 1 \cdot (1, 0) + 0 \cdot (0, 1)$ and thus the matrix representation of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Similarly, let $\mathcal{B} = \{(1, 0), (0, 1)\}$ and $\mathcal{C} = \{(1, 1), (1, -1)\}$. Then $T(1, 0) = (1, 1) = 1 \cdot (1, 1) + 0 \cdot (1, -1)$ and $T(0, 1) = (1, 0) = \frac{1}{2} \cdot (1, 1) + \frac{1}{2} \cdot (1, -1)$ and thus the matrix representation of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

1.1.2 Part (f)

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $F(x, y) = (x + y, x + 1)$. To show that F is not linear, we need to find a counterexample to one of the properties of linearity.

In this case, we have $F(\vec{0}) = F(0, 0) = (0, 1) \neq 0 \cdot F(\vec{0}) = (0, 0)$ and thus F is not linear.

1.2 Proof Question 1

We construct the map $T : V \to W$ such that $T(u) = S(u)$ for all $u \in U$ explicitly. The following lemma is useful.

Lemma 1. *Let V, W be vector spaces and let $\{v_1, \ldots, v_n\}$ be a basis of V. Then for any vectors $w_1, \ldots, w_n \in W$, there exists a unique linear map $T : V \to W$ such that $T(v_i) = w_i$.*

Proof. Let $v \in V$ be arbitrary. Then $v = a_1v_1 + \cdots + a_nv_n$ for some scalars $a_1, \ldots, a_n$. Define $T(v) = a_1w_1 + \cdots + a_nw_n$. Then T is linear and $T(v_i) = w_i$ for all i.

To show uniqueness, suppose T' is another linear map such that $T'(v_i) = w_i$. Then $T'(v) = T'(a_1v_1 + \cdots + a_nv_n) = a_1T'(v_1) + \cdots + a_nT'(v_n) = a_1w_1 + \cdots + a_nw_n = T(v)$. Thus $T = T'$. ■

Since V is finite dimensional and U is a subspace of V, U is also finite dimensional. Let $\{u_1, \ldots, u_m\}$ be a basis of U. Since V is finite dimensional and the collection $\{u_1, \ldots, u_m\}$ is linearly independent (as a basis of U), we can extend this collection to a basis $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ of V.

Define $T : V \to W$ to be the unique linear map (whose existence is guaranteed by the lemma) such that $T(u_i) = S(u_i)$ for all $i = 1, \ldots, m$ and $T(v_j) = 0$ for all $j = 1, \ldots, n$.

We claim that T is the desired linear map. To see this, let $u \in U$ be arbitrary. Then $u = a_1u_1 + \cdots + a_mu_m$ for some scalars $a_1, \ldots, a_m$ and indeed by linearity of T we have

$$T(u) = T(a_1u_1 + \cdots + a_mu_m) = a_1T(u_1) + \cdots + a_mT(u_m) = a_1S(u_1) + \cdots + a_mS(u_m) = S(u)$$

as required. This concludes the proof.

1.3 Proof Question 5

Let $T : V \to W$ be a linear map. We wish to find a subspace whose intersection with $\text{null}\,(T)$ is trivial and image under T is equal to the image of T. i.e. $\text{null}\,(T) \cap U = \{\vec{0}\}$ and $T(U) = T(V)$.

We consider two cases. First, suppose $\text{null}\,(T) = \{\vec{0}\}$. Then we can simply take $U = V$ as $\text{null}\,(T) \cap V = \text{null}\,(T) = \{\vec{0}\}$ and $T(U) = T(V)$ by definition.

Otherwise, since V is finite dimensional, $\text{null}\,(T)$ is a subspace of V and thus is also finite dimensional. Let $\{n_1, \ldots, n_m\}$ be a basis of $\text{null}\,(T)$. Since $\text{null}\,(T) \neq \{\vec{0}\}$, we have $m > 0$.

As in the previous question, we can extend this basis to a basis $\{n_1, \ldots, n_m, u_1, \ldots, u_n\}$ of V. Define $U = \text{span}\,\{u_1, \ldots, u_n\}$. We claim that U is the desired subspace.

To begin, we note that $\dim V = m + n$ and $\dim U = n$ by counting the number of basis vectors. Next we show that $T(U) = T(V)$.

To see this, first note that $T(U) \subseteq T(V)$ by definition. To show the reverse inclusion, let $w \in T(V)$ be arbitrary. Then $w = T(v)$ for some $v \in V$. We can write $v = a_1n_1 + \cdots + a_mn_m + b_1u_1 + \cdots + b_nu_n$ for some scalars $a_1, \ldots, a_m, b_1, \ldots, b_n$ but then

$$w = T(v) = T(a_1n_1 + \cdots + a_mn_m + b_1u_1 + \cdots + b_nu_n) = b_1T(u_1) + \cdots + b_nT(u_n) \in T(U)$$

We conclude by dimension counting. Let $T|_U : U \to W$ be the restriction of T to U. Since $T_U(U) = T(U) = T(V)$, the range of T_U is the same as the range of T.

Applying the rank-nullity theorem to T, we have $\dim T(V) = \dim V - \dim \text{null}\,(T) = (n + m) - m = n$. Applying the rank-nullity theorem to T_U, we have $\dim \text{null}\,(T_U) = \dim U - \dim T(U) = n - n = 0$ and thus $\text{null}\,(T_U) = U \cap \text{null}\,(T) = \{\vec{0}\}$ as required.

1.4 Proof Question 8

We are asked to show that T and S are invertible if and only if the composition $T \circ S$ is invertible.

To that end, let T and S be invertible linear maps. It suffices to show that $T \circ S$ is injective. To see this, suppose that $(T \circ S)(v) = \vec{0}$ for some $v \in V$. Since T is invertible (and thus injective), we have $S(v) = \vec{0}$ and since S is invertible (and thus injective), we have $v = \vec{0}$. Thus $T \circ S$ is injective.

Next, suppose that $T \circ S$ is invertible. We wish to show that T and S are invertible. We first show S is invertible. To see this, suppose that $S(v) = \vec{0}$. Then $(T \circ S)(v) = T(S(v)) = T(\vec{0}) = \vec{0}$ and since $T \circ S$ is invertible, this implies $v = \vec{0}$. Thus S is injective and therefore invertible.

To conclude, we show that T is invertible. To see this, suppose that $T(v) = 0$. Since S is invertible, and thus surjective, there exists $w \in V$ such that $S(w) = v$. Then $T(v) = T(S(w)) = (T \circ S)(w) = 0$ and since $T \circ S$ is invertible, we have $w = \vec{0}$. Since S is linear, this implies $v = S(w) = S(\vec{0}) = \vec{0}$ and thus T is injective.