Tutorial 6

Problem 1. Let p be an odd prime. Let G be a group of order $2p$.

a) What are the possible orders of non-identity elements in G?

b) (Exercise 29) Show that if $g^2 = e$ for all $g \in G$, then G is abelian.

c) If G has an element r of order p, and $s \notin \langle r \rangle$, show that $o(s) = 2$ or $2p$.

Solution

a) By Lagrange theorem, any $g \in G$ has $o(g) \mid o(G) = 2p$. The divisors of $2p$ are 1, 2, p, and $2p$. Now $o(g) = 1$ iff $g = e$, so non-identity elements have orders 2, p, or $2p$.

b) Any elements $a, b \in G$ satisfy

$$abab = (ab)^2 = e = a^2b^2 = aabb.$$

Cancelling out a on the left and b on the right gives $ba = ab$.

c) In this case we note that $[G : \langle r \rangle] = 2$. Thus, if $s \notin \langle r \rangle$ then $s^2\langle r \rangle = \langle r \rangle$. That is, $s^2 \in \langle r \rangle$.

Now if $o(s) \neq 2p$ or 2, then $o(s) = p$. We have

$$s^p = (s^2)^{\frac{p-1}{2}} s = e \in \langle r \rangle,$$

and hence

$$s = (s^2)^{-\frac{p-1}{2}} \in \langle r \rangle.$$

But this contradicts our assumption that $s \notin \langle r \rangle$. Thus, $o(s) = 2p$ or $o(s) = 2$.

Problem 2. Let p be an odd prime. Suppose that G is an abelian group of order $2p$.

a) Show that G must not have more than one element of order 2. [*Hint*: Take two distinct elements $a, b \in G$ of order 2. Show that the subgroup they generate is $\{e, a, b, ab\}$.]

b) Conclude that $G \cong C_{2p}$.

Solution

a) Consider $H = \langle a, b \rangle$. We claim this is $\{e, a, b, ab\}$. If G is abelian, we have

$$ab^{-1} = ab, ba^{-1} = ba = ab.$$

So $\{e, a, b, ab\}$ is the smallest subgroup containing a, b (because e and ab must be in the group) and thus is H.

Now $H \leq G$ so we have $o(H) = 4 \mid o(G) = 2p$, which is not true.

b) Suppose G has no element of order $2p$. Then b) tells us it has an element r of order p. Q1c) gives us an element s of order 2, and so $o(rs) = \text{lcm}(o(r), o(s)) = 2p$, which yields a contradiction. So we must have an element of order $2p$ in G which generates G.

Problem 3. Let p be an odd prime. Suppose that G is a nonabelian group of order $2p$.

a) Show that G must contain an element r of order p.

b) Show that G contains at least two involutions s and sr.

c) Conclude that $G \cong D_p$.

Solution

a) G contains no element of order $2p$ because that would make G cyclic (and abelian). If G contains only elements of order 2, Q1b) tells us G is abelian.

b) Q1c) tells us there is some element s of order 2 not in $\langle r \rangle$. Now consider $o(sr)$. We know $s \notin \langle r \rangle$ so $sr \notin \langle r \rangle$. Applying Q1c) again tells us $o(sr) = 2$.

c) $o(sr) = 2$ and s, sr together generate the group G (since $ssr = r$). So G is generated by two involutions. Thus it is the dihedral group of order $2p$.