Solutions to Selected Exercises - Week 11

Federico Manganello

MAT246H1F: CONCEPTS IN ABSTRACT MATHEMATICS

November 27, 2024

Exercises from Section 3E of [Axl24]

Exercise 3. Suppose T is a function from V to W. The *graph* of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Solution. Suppose T is linear, let $(v_1, Tv_1), (v_2, Tv_2) \in$ graph of T, $\alpha, \beta \in \mathbb{F}$. Then in $V \times W$:

$$\alpha(v_1, Tv_1) + \beta(v_2, Tv_2) = (\alpha v_1, \alpha Tv_1) + (\beta v_2, \beta Tv_2) = (\alpha v_1 + \beta v_2, \alpha Tv_1 + \beta Tv_2) = (\alpha v_1 + \beta v_2, T(\alpha v_1 + \beta v_2)).$$

Thus $\alpha(v_1, Tv_1) + \beta(v_2, Tv_2) \in$ graph of T and graph of T is a subspace of $V \times W$.

Conversely, assume graph of T is a subspace of $V \times W$. Let $v_1, v_2 \in V$, $\alpha, \beta \in \mathbb{F}$. Then:

$$\alpha(v_1, Tv_1) + \beta(v_2, Tv_2) = (\alpha v_1 + \beta v_2, \alpha Tv_1 + \beta Tv_2) \in \text{graph of } T.$$

By definition of graph of T, this implies that $T(\alpha v_1 + \beta v_2) = \alpha Tv_1 + \beta Tv_2$ and hence T is linear. $\square$

Exercise 5. For m a positive integer, define V^m by

$$V^m = \underbrace{V \times \cdots \times V}_{m \text{ times}}.$$

Prove that V^m and $\mathcal{L}(\mathbb{F}^m, V)$ are isomorphic vector spaces.

Hint. Consider the map Φ such that:

$$V^m \ni (v_1, \ldots, v_m) \mapsto \Phi(v) \in \mathcal{L}(\mathbb{F}^m, V),$$

where for every $v = (v_1, \ldots, v_m) \in V^m$, $\Phi(v)$ is the map:

$$\begin{aligned} \Phi(v): \quad \mathbb{F}^m &\longrightarrow V \\ f &\longmapsto \sum_{i=1}^{m} f_i v_i; \end{aligned}$$

where $f = (f_1, \ldots, f_m)$.

Then one can check that Φ is well-defined, i.e. for every $v \in V$, $\Phi(v) \in \mathcal{L}(\mathbb{F}^m, V)$. Then one verifies that Φ is linear. Finally, one needs to show that Φ is a bijection i.e. it is both injective and surjective. Thus, one can conclude that Φ is the sought isomorphism.

Note: if one manages to show that $\dim V^m = \dim \mathcal{L}(\mathbb{F}^m, V)$, it is enough to show that Φ is injective or to show that Φ is surjective, in order to show that Φ is a bijection. $\square$

Exercises from Section 9D of [Axl24]

Exercise 1. Suppose $v \in V$ and $w \in W$. Prove that $v \otimes w = 0$ if and only if $v = 0$ or $w = 0$.

Solution. Recall that by definition $V \otimes W = \mathcal{B}(V', W')$.

Let $v \otimes w = 0$. Assume by contradiction that both $v \neq 0$, $w \neq 0$; then there exist $\varphi \in V'$, $\psi \in W'$ such that $\varphi(v) = 1$ and $\psi(w) = 1$. This implies:

$$(v \otimes w)(\varphi, \psi) = \varphi(v)\psi(w) = 1 \cdot 1 = 1 \neq 0.$$

This contradicts the initial assumption $v \otimes w = 0$ and hence $v = 0$ or $w = 0$.

Conversely, assume that $v = 0$. Then for every $\varphi \in V'$, $\psi \in W'$ one has

$$(v \otimes w)(\varphi, \psi) = \varphi(v)\psi(w) = \varphi(0)\psi(w) = 0\psi(w) = 0,$$

Hence $v \otimes w = 0$. Analogously one shows that if $w = 0$ then $v \otimes w = 0$. □

Exercise 3. Suppose that $v_1, \ldots, v_m$ is a linearly independent list in V. Suppose also that $w_1, \ldots, w_m$ is a list in W such that:

$$v_1 \otimes w_1 + \cdots + v_m \otimes w_m = 0.$$

Prove that $w_1 = \cdots = w_m = 0$.

Solution. Since $v_1, \ldots, v_m$ are linearly independent, for every $i = 1, \ldots, m$ there exists $\varphi_i \in V'$ such that

$$\varphi_i(v_j) = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Fix $i \in \{1, \ldots, n\}$. Let $\psi \in W'$. Then:

$$0 = (v_1 \otimes w_1 + \cdots + v_m \otimes w_m)(\varphi_i, \psi) = \sum_{j=1}^{m} (v_j \otimes w_j)(\varphi_i, \psi) = \sum_{j=1}^{m} \varphi_i(v_j)\psi(w_j) = \sum_{j=1}^{m} \delta_{i,j}\psi(w_j) = \psi(w_i).$$

All in all, by arbitrariness of $\psi \in W'$ one has that: for every $\psi \in W'$, $\psi(w_i) = 0$. This implies[1] that $w_i = 0$. Since the choice of i was arbitrary, too, this applies for every $i \in \{1, \ldots, n\}$. All in all, $w_i = 0$ for every $i = 1, \ldots, m$, $w_i = 0$. □

Exercise 5. Suppose that m and n are positive integers. For $v \in \mathbb{F}^m$ and $w \in \mathbb{F}^n$, identify $v \otimes w$ with an m-by-n matrix as in Example 9.76. With that identification show that the set

$$\{v \otimes w : v \in \mathbb{F}^m \text{ and } w \in \mathbb{F}^n\},$$

is the set of m-by-n matrices with entries in $\mathbb{F}$ that have rank at most one.

Hint. With the same notation as in Example 9.76, let $v \otimes w \simeq A$, where $A = (w_{jk})_{j=1,\ldots,m; k=1,\ldots,n}$. Denote $A_k \in \mathbb{F}^m$ for $k = 1, \ldots, n$ the column vector constituting the k-th column of A. Then for every $k = 1, \ldots, n$:

$$A_k = w_k \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}.$$

This implies that for every $l, s \in \{1, \ldots, n\}$:

$$w_s A_l = w_l A_s.$$

As a consequence, for every $k = 2, \ldots, n$ $A_k \in \operatorname{span}(A_1)$. This implies that A has at most one linearly independent column, i.e. it has at most rank 1.

Conversely, given a matrix $A = (w_{jk})_{j=1,\ldots,m; k=1,\ldots,n}$ whose rank is at most one, we know that there exists a matrix $B \in GL(n, \mathbb{R})$ such that $\tilde{A} = AB$ is a matrix with all the non-zero entries, if any, in the first column; let $v \in \mathbb{F}^m$ be the column vector corresponding to the first column of $\tilde{A}$. Take $\tilde{w} = (1, 0, \ldots, 0)^\top \in \mathbb{F}^n$. Take $w = (B^{-1})^\top \tilde{w}$. Then $v \otimes w \simeq A$. □

[1] Recall that in a finite dimensional vector space V, if $v \in V$ is non-zero, then there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Alternative approach to show that A has rank at most one. Remark that $A = vw^\top$. Let $x \in \mathbb{F}^n$. Then:

$$Ax = vw^\top x = v \underbrace{(w^\top x)}_{\in \mathbb{F}} \in \operatorname{span} v.$$

Hence $\operatorname{range} A \subset \operatorname{span} v$. This implies that $\dim \operatorname{range} A \leq 1$, and hence A has rank at most 1.

Exercise 9. Suppose $S \in \mathcal{L}(V)$, $T \in \mathcal{L}(W)$. Prove that there exists a unique operator on $V \otimes W$ that takes $v \otimes w$ to $Sv \otimes Tw$ for all $v \in V$ and $w \in W$.

Hint. Consider the operator:

$$\begin{aligned} \Gamma : V \times W &\longrightarrow V \otimes W \\ (v, w) &\longmapsto Sv \otimes Tw. \end{aligned}$$

By the bilinearity of tensor product, one can check that this map is bilinear. Then, by the universal property of tensor product, there exists a unique linear map $\hat{\Gamma} : V \otimes W \to V \otimes W$ such that $\hat{\Gamma}(v \otimes w) = \Gamma(v, w) = Sv \otimes Tw$ for every $v \in V$ and $w \in W$. □

Exercise 10. Suppose $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(W)$. Prove that $S \otimes T$ is an invertible operator on $V \otimes W$ if and only if both S and T are invertible operators. Also, prove that if both S and T are invertible operators then $(S \otimes T)^{-1} = S^{-1} \otimes T^{-}1$.

Note for students: in what follows the same symbol 0 is used to denote the zero vector in many different vector spaces. Try to figure out when it represents 0_V, 0_W, $0_{V \otimes W}$.

Hint. Assume $S \otimes T$ is invertible. Assume by contradiction that at least one between S and T is not invertible. For simplicity, assume S is invertible and T is not invertible. Then, there exists $w \in W$ such that $w \neq 0$ and $Tw = 0$. Let $v \in V$ be such that $v \neq 0$. Then by Exercise 1, $v \otimes w \neq 0$. But:

$$(S \otimes T)(v \otimes w) = Sv \otimes Tw = Sv \otimes 0 = 0.$$

Then $0 \neq v \otimes w \in \ker S \otimes T$, contradicting the assumption that $S \otimes T$ is invertible.

Analogously, one can give the proof in the case S is not invertible and T is invertible and in the case both S and T are not invertible.

Conversely, assume S and T are invertible. Consider the map:

$$\begin{aligned} \Lambda : V \times W &\longrightarrow V \otimes W \\ (v, w) &\longmapsto S^{-1}v \otimes T^{-1}w. \end{aligned}$$

One can check that this map is bilinear and hence by the universal property, there exists a unique map $\hat{\Lambda} : V \otimes W \to V \otimes W$ such that for every $v \otimes w \in V \otimes W$:

$$\hat{\Lambda}(v \otimes w) = \Lambda(v, w) = S^{-1}v \otimes T^{-1}w.$$

With respect to Exercise 9 notation, notice that $\hat{\Lambda} = S^{-1} \otimes T^{-1}$. Now one can remark that:

$$\hat{\Gamma} \circ \hat{\Lambda}(v \otimes w) = \hat{\Gamma}(\hat{\Lambda}(v \otimes w)) = \hat{\Gamma}(\Lambda(v, w)) = \hat{\Gamma}(S^{-1}v \otimes T^{-1}w) = \Gamma(S^{-1}v, T^{-1}w) = (SS^{-1}v) \otimes (TT^{-1}w) = v \otimes w;$$

and:

$$\hat{\Lambda} \circ \hat{\Gamma}(v \otimes w) = \hat{\Lambda}(\hat{\Gamma}(v \otimes w)) = \hat{\Lambda}(\Gamma(v, w)) = \hat{\Lambda}(Sv \otimes Tw) = \Lambda(Sv, Tw) = (S^{-1}Sv) \otimes (T^{-1}Tw) = v \otimes w.$$

All in all, $\hat{\Lambda}$ is the inverse of $\hat{\Gamma}$ and using the notation from Exercise 9 this means:

$$(T \otimes S)^{-1} = \hat{\Gamma}^{-1} = \hat{\Lambda} = S^{-1} \otimes T^{-1}.$$

□

Exercise 12. Suppose that $V_1, \dots, V_m$ are finite-dimensional inner product spaces. Prove that there is a unique inner product on $V_1 \otimes \cdots \otimes V_m$ such that:

$$\langle v_1 \otimes \cdots v_m, u_1 \otimes \cdots \otimes u_m \rangle = \langle v_1, u_1 \rangle \cdots \langle v_m, u_m \rangle$$

for all $(v_1, \dots, v_m)$ and $(u_1, \dots, u_m)$ in $V_1 \times \cdots \times V_m$.

Hint. Generalize Proposition 9.80 from [Axl24], p. 376. □

Exercise 13. Suppose that $V_1, \dots, V_m$ are finite-dimensional inner product spaces and $V_1 \otimes \cdots \otimes V_m$ is made into an inner product space using the inner product from Exercise 12. Suppose $e_1^k, \dots, e_{n_k}^k$ is an orthonormal basis of V_k for each $k = 1, \dots, m$. Show that the list

$$\{e_{j_1}^1 \otimes \cdots \otimes e_{j_m}^m\}_{j_1=1,\dots,n_1;\cdots;j_m=1,\dots,n_m}$$

is an orthonormal basis of $V_1 \otimes \cdots \otimes V_m$.

Hint. Generalize Proposition 9.83 from [Axl24], p. 377. □

References

[Axl24] Sheldon Axler. *Linear algebra done right.* Fourth. Undergraduate Texts in Mathematics. Springer, Cham, 2024, pp. xvii+390. ISBN: 978-3-031-41025-3; 978-3-031-41026-0. DOI: `10.1007/978-3-031-41026-0`. URL: `https://doi.org/10.1007/978-3-031-41026-0`.