Basics of Applied Stochastic Processes

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Introduction

A discrete-time *stochastic process* $\{X_n : n \geqslant 0\}$ on a countable set S is a collection of S-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The $\mathcal{P}$ is a probability measure on a family of events $\mathcal{F}$ (a σ-field) in an event-space Ω. The set S is the *state space* of the process, and the value $X_n \in S$ is the *state* of the process at *time* n. The n may represent a parameter other than time such as a length or a job number. The *finite-dimensional* distributions of the process are

$$P\{X_0 = i_0, \ldots, X_n = i_n\}, \qquad i_0, \ldots, i_n \in S, n \geqslant 0 \tag{1}$$

These probabilities uniquely determine the probabilities of all events of the process.

Markov Chains

A Markov chain is defined as follows.

Definition 1. A stochastic process $X = \{X_n : n \geqslant 0\}$ on a countable set S is a *Markov Chain* if, for any $i, j \in S$ and $n \geqslant 0$,

$$P\{X_n + 1 = j | X_0, \ldots, X_n\} = P\{X_n + 1 = j | X_n\} \tag{2}$$

$$P\{X_n + 1 = j | X_n\} = p_{ij} \tag{3}$$

The p_{ij} is the probability that the Markov chain jumps from state i to state j. These *transition probabilities* satisfy $\sum_{j \in S} p_{ij} = 1, i \in S$, and the matrix $\mathbf{P} = (pij)$ is the *transition matrix* of the chain.

Markov Chains

Condition (2), called the *Markov property*, says that, at any time n, the next state X_{n+1} is conditionally independent of the past $X_0, \dots, X_{n-1}$ given the present state X_n.

Condition (3) simply says the transition probabilities do not depend on the time parameter n; the Markov chain is therefore "time-homogeneous".