Supplemental Material: Unified topological characterization of electronic states in spin textures from noncommutative K-theory

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The Supplemental Material provides details on the derivations from the manuscript. In order to make the discussions in the Supplemental Material more or less self-contained, table I summarizes the central definitions from the manuscript. Additionally, a brief introduction to K-theory is given in the context of this work.

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TABLE I. **Summary of the notation in the manuscript.** The following table lists important symbols and notation which is used throughout the manuscript and gives a brief explanation.

Symbol	Explanation
t	Hopping parameter
Δ_{xc}	Exchange-correlation energy
d	Dimensionality of the lattice
N	Number of sites in the lattice; Thermodynamic limit: $N \to \infty$
$\mathbf{a}_i$	Bravais lattice vectors
$\mathbf{b}_j$	Reciprocal lattice vectors; $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}$
$\mathbf{x}_{\mathbf{k}}$	Real-space coordinates; $\mathbf{x}_{\mathbf{k}} = \sum_{i=1}^{3} k_i \mathbf{a}_i$
$\mathbf{q}_i$	The i-th $\mathbf{q}$-vector of the multi-$\mathbf{q}$ texture; $\mathbf{q}_i = \sum_{j=1}^{d} \theta_{ij} \mathbf{b}_j$
$\boldsymbol{\theta}_i$	i-th column vector in θ_{ij}; $(\boldsymbol{\theta}_i)_j = \theta_{ij}$
r	Number of distinct $\mathbf{q}$-vectors
T^r	The r-dimensional torus
$\omega_i(\mathbf{x}_{\mathbf{k}})$	Phase with values in T^1, associated with $\mathbf{q}_i$; $\omega_i(\mathbf{x}_{\mathbf{k}}) = (\mathbf{x}_{\mathbf{k}} \cdot \mathbf{q}_i/(2\pi) + \varphi_i) \mod 1$
$\boldsymbol{\omega}$	The collection of all $\omega_i \in T^1$ into a vector $\boldsymbol{\omega} \in T^r$
$\lvert\mathbf{k}\rangle$	Position ket corresponding to the atomic site at location $\mathbf{x}_{\mathbf{k}}$
$\lvert\sigma\rangle$	Spin ket labelled by the eigenstates of Pauli matrix σ_z
$\lvert\mathbf{k},\sigma\rangle$	Tensor product state $\lvert\mathbf{k},\sigma\rangle = \lvert\mathbf{k}\rangle \otimes \lvert\sigma\rangle$
$\hat{T}_{\mathbf{m}}$	Lattice translation operator acting on kets with $\mathbf{m} \in \mathbb{Z}^d$; $\hat{T}_{\mathbf{m}} \lvert\mathbf{k},\sigma\rangle = \lvert\mathbf{k}+\mathbf{m},\sigma\rangle$
$\hat{T}_i$	Unit lattice translation in the i-th direction
$\tau_{\mathbf{m}}$	Action of the translation group $\mathbb{Z}^d \ni \mathbf{m}$ on the phases; $\tau_{\mathbf{m}}\omega_i(\mathbf{x}_{\mathbf{k}}) = (\omega_i(\mathbf{x}_{\mathbf{k}}) - (\mathbf{m}\cdot\boldsymbol{\theta}_i \mod 1)) \mod 1$
τ_i	Unit lattice translation of the phases in the i-th direction
$\boldsymbol{\phi}$	The phase vector at one arbitrary, but fixed reference point $\mathbf{x}_0 \in \mathbb{R}^d$; $\boldsymbol{\phi} \equiv \boldsymbol{\omega}(\mathbf{x}_0) \in T^r$
Ω	Hull of the magnetic pattern; $\Omega = \{\tau_{\mathbf{m}}\boldsymbol{\phi} \mid \mathbf{m} \in \mathbb{Z}^d\} \subset T^r$
u_k	Fourier amplitude $u_k = e^{2\pi i \phi_k}$; generator of periodic functions (functions on the T^r)
Θ	Generalized flux matrix; $\Theta = ((0, -\theta^T), (\theta, 0))$
$\boldsymbol{\alpha}$	Vector of generators; $\boldsymbol{\alpha} = (\tau_1, \ldots, \tau_d, u_1, \ldots u_r)$
$\mathcal{A}_\Theta$	The universal C^*-algebra of the noncommutative torus; $\mathcal{A}_\Theta = \left\langle \alpha_1, \cdots, \alpha_{r+d} \,\middle\vert\, \alpha_l\alpha_k = e^{2\pi i\Theta_{lk}}\alpha_k\alpha_l \right\rangle$
d_{eff}	Effective dimension; $d_{\mathrm{eff}} = r + d$
P	Projection operator in $\mathcal{A}_\Theta$; $P^2 = P$
$[P]$	Equivalence class of unitarily equivalent projection operators in $\mathcal{A}_\Theta$;
$K_0(\mathcal{A}_\Theta)$	The (Grothendieck) group of all $[P]$; $K_0(\mathcal{A}_\Theta) = \mathbb{Z}^{2^{d_{\mathrm{eff}}-1}}$
$\mathcal{I}$	Index set which labels the generators; $\mathcal{I} = \{\tau_1 \cdots, \tau_d, u_1 \cdots, u_r\}$
J	An even cardinality subset of $\mathcal{I}$; $J \subseteq \mathcal{I}$: $\lvert J\rvert$ even
E_J	Generators of $K_0(\mathcal{A}_\Theta)$
$\Theta_{J\setminus J'}$	Restriction of the flux matrix to the submatrix for the restricted index set $J \setminus J'$
$\mathrm{Ch}_J(g)$	$\lvert J\rvert/2$-th Chern number of the gap with label g and w.r.t. to the indices J
$\mathrm{Pf}(A)$	The Pfaffian of a matrix A; For $A^T = -A$: $(\mathrm{Pf}(A))^2 = \det A$
$\mathrm{IDS}(g)$	Integrated density of states within the gap g; $\mathrm{IDS}(g) = \lim_{N\to\infty} \frac{1}{N} \mathrm{tr}\, P_{E<E_g}$, where E_g is an energy in g and $P_{E<E_g}$ projects onto states below E_g

A. Bringing the Hamiltonian into its covariant form

We begin by demonstrating how the Hamiltonian can indeed be written in the canonical form presented in the manuscript. For the hopping term, one finds

$$\begin{aligned} H_t &= t \sum_{\langle \mathbf{k},\mathbf{l}\rangle \in \mathbb{Z}^{2d}} |\mathbf{k}\rangle \langle \mathbf{l}| \\ &= t \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{l=1}^{d} \left(|\mathbf{k}\rangle \langle \mathbf{k}+\mathbf{e}_l| + |\mathbf{k}+\mathbf{e}_l\rangle \langle \mathbf{k}| \right) \\ &= t \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{l=1}^{d} (\hat{T}_l + \hat{T}_l^\dagger) |\mathbf{k}\rangle \langle \mathbf{k}| \\ &= \sum_{l=1}^{d} (\hat{T}_l + \hat{T}_l^\dagger) \sum_{\mathbf{k}\in\mathbb{Z}^d} t |\mathbf{k}\rangle \langle \mathbf{k}| \\ &= t \sum_{l=1}^{d} (\hat{T}_l + \hat{T}_l^\dagger), \end{aligned} \tag{1}$$

where $\hat{T}_l$ is a unit-translation in the direction $\mathbf{e}_l \in \mathbb{Z}^d$. It is therefore invariant under translations: $\hat{T}_\mathbf{m} H_t \hat{T}_\mathbf{m}^\dagger = H_t$. The exchange term is given by

$$H_{\mathrm{xc}} = \Delta_{\mathrm{xc}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\hat{\mathbf{n}}(\boldsymbol{\omega}(\mathbf{x}_\mathbf{k})) \cdot \boldsymbol{\sigma} \right) \; |\mathbf{k}\rangle \langle \mathbf{k}| \, . \tag{2}$$

It is not invariant under lattice translations, but transforms as

$$\begin{aligned} \hat{T}_\mathbf{m} H_{\mathrm{xc}} \hat{T}_\mathbf{m}^\dagger &= \Delta_{\mathrm{xc}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\hat{\mathbf{n}}(\boldsymbol{\omega}(\mathbf{x}_\mathbf{k})) \cdot \boldsymbol{\sigma} \right) \; |\mathbf{k}+\mathbf{m}\rangle \langle \mathbf{k}+\mathbf{m}| \\ &= \Delta_{\mathrm{xc}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\hat{\mathbf{n}}(\boldsymbol{\omega}(\mathbf{x}_{\mathbf{k}-\mathbf{m}})) \cdot \boldsymbol{\sigma} \right) \; |\mathbf{k}\rangle \langle \mathbf{k}| \\ &= \Delta_{\mathrm{xc}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\hat{\mathbf{n}}(\tau_\mathbf{m} \boldsymbol{\omega}(\mathbf{x}_\mathbf{k})) \cdot \boldsymbol{\sigma} \right) \; |\mathbf{k}\rangle \langle \mathbf{k}| \, . \end{aligned} \tag{3}$$

With the definition $\boldsymbol{\phi} = \boldsymbol{\omega}(\mathbf{x}_0)$, the exchange term can therefore also be written as

$$\begin{aligned} H_{\mathrm{xc}}(\boldsymbol{\phi}) &= \Delta_{\mathrm{xc}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\hat{\mathbf{n}}(\tau_{-\mathbf{k}} \boldsymbol{\phi}) \cdot \boldsymbol{\sigma} \right) \; |\mathbf{k}\rangle \langle \mathbf{k}| \, , \\ &= \Delta_{\mathrm{xc}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\hat{\mathbf{n}}(\boldsymbol{\phi} + \theta \mathbf{k}) \cdot \boldsymbol{\sigma} \right) \; |\mathbf{k}\rangle \langle \mathbf{k}| \end{aligned} \tag{4}$$

and the translation of the Hamiltonian $H = H_t + H_{\mathrm{xc}}(\boldsymbol{\phi})$ can be expressed in the compact, covariant form

$$\hat{T}_\mathbf{m} H(\boldsymbol{\phi}) \hat{T}_\mathbf{m}^\dagger = H(\tau_\mathbf{m} \boldsymbol{\phi}), \tag{5}$$

or alternatively

$$\hat{T}_\mathbf{m}^\dagger H(\boldsymbol{\phi}) \hat{T}_\mathbf{m} = H(\boldsymbol{\phi} + \theta \mathbf{m}). \tag{6}$$

Combining the results above, the Hamiltonian can finally be cast into the form

$$H = \sum_{\mathbf{n}\in\mathbb{Z}^d} \hat{T}_\mathbf{n} \sum_{\mathbf{m}\in\mathbb{Z}} h_\mathbf{n}(\boldsymbol{\phi} + \theta \mathbf{m}) \, |\mathbf{m}\rangle \langle \mathbf{m}| \, , \tag{7}$$

with the definition

$$h_\mathbf{n}(\boldsymbol{\phi}) \equiv \begin{cases} \Delta_{\mathrm{xc}} (\hat{\mathbf{n}}(\boldsymbol{\phi}) \cdot \boldsymbol{\sigma}), & \mathbf{n} = 0 \\ t \, \mathrm{id}_2, & \exists l \in \{1, \cdots, d\} \colon \mathbf{n} = \mathbf{e}_l \text{ or } \mathbf{n} = -\mathbf{e}_l \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

B. Derivation of the torus commutation relation

The covariant form of the Hamiltonian demonstrates that it fits into a generic form which combines the action of the translation operator with matrix-valued functions on the r-torus T^r. A continuous function $f\colon T^r \to \mathbb{C}$ can be decomposed into a Fourier series as

$$\begin{aligned} f(\boldsymbol{\phi}) &= \sum_{\mathbf{n}} f_{\mathbf{n}}\, e^{2\pi i \boldsymbol{\phi}\cdot\mathbf{n}} \\ &= \sum_{\mathbf{n}} f_{\mathbf{n}}\, e^{2\pi i \phi_1 n_1} \cdots e^{2\pi i \phi_r n_r} \\ &= \sum_{\mathbf{n}} f_{\mathbf{n}}\, (e^{2\pi i \phi_1})^{n_1} \cdots (e^{2\pi i \phi_r})^{n_r} \\ &\equiv \sum_{\mathbf{n}} f_{\mathbf{n}}\, u_1^{n_1} \cdots u_r^{n_r}. \end{aligned} \tag{9}$$

In other words, the algebra of continuous functions on the torus is generated by $u_k = e^{2\pi i \phi_k}$. One can condense this result into the presentation

$$C(T^r) = \langle u_1, \ldots, u_r \mid [u_i, u_j] = 0 \rangle . \tag{10}$$

The commutation relation between the unit lattice translation τ_l and the Fourier factor u_k can be derived as

$$\begin{aligned} \tau_l u_k &= \exp\{2\pi i(\tau_l \phi_k)\}\tau_l \\ &= \exp\{2\pi i((\phi_k - (\mathbf{e}_l \cdot \boldsymbol{\theta}_k \mod 1)) \mod 1)\}\tau_l \\ &= \exp\{2\pi i(\phi_k - (\mathbf{e}_l \cdot \boldsymbol{\theta}_k \mod 1))\}\tau_l \\ &= \exp\{-2\pi i(\mathbf{e}_l \cdot \boldsymbol{\theta}_k \mod 1)\}u_k\tau_l \\ &= \exp\{-2\pi i(\mathbf{e}_l \cdot \boldsymbol{\theta}_k)\}u_k\tau_l \\ &= \exp\{-2\pi i\theta_{kl}\}u_k\tau_l \end{aligned} \tag{11}$$

which gives the relation presented in the manuscript.

C. A more precise description of the noncommutative torus

The following section is adapted from [1]. By defining $\boldsymbol{\alpha} = (\tau_1, \ldots, \tau_d, u_1, \ldots u_r)$, the commutation relations can be summarized to $\alpha_l\alpha_k = e^{2\pi i \Theta_{lk}}\alpha_k\alpha_l$, where

$$\Theta = \begin{pmatrix} 0 & -\theta^T \\ \theta & 0 \end{pmatrix}. \tag{12}$$

The manuscript summarizes the observable algebra of a multi-$\mathbf{q}$ texture as the universal C^*-algebra given by the presentation

$$\mathcal{A}_\Theta = \left\langle \alpha_1, \cdots, \alpha_{d_{\text{eff}}} \,\middle|\, \alpha_l\alpha_k = e^{2\pi i\Theta_{lk}}\alpha_k\alpha_l \right\rangle, \tag{13}$$

with $d_{\text{eff}} = r + d$. Θ_{lk} is considered as an antisymmetric $d_{\text{eff}} \times d_{\text{eff}}$ matrix with entries from $\mathbb{R}/\mathbb{Z}$. A generic element of the algebra can be presented in the form

$$\begin{aligned} a &= \sum_{\mathbf{q}\in\mathbb{Z}^{d_{\text{eff}}}} a_{\mathbf{q}}\,\alpha_{\mathbf{q}}, \quad \alpha_{\mathbf{q}} = \alpha_1^{q_1}\ldots\alpha_{d_{\text{eff}}}^{q_{d_{\text{eff}}}}, \quad a_{\mathbf{q}} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\ &= \sum_{\mathbf{q}\in\mathbb{Z}^d} a(\boldsymbol{\phi}, \mathbf{q})\,\alpha_1^{q_1}\ldots\alpha_d^{q_d}, \end{aligned} \tag{14}$$

where $a(\boldsymbol{\phi}, \mathbf{q})$ is a continuous function $T^r \times \mathbb{Z}^d \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$ with compact support. The noncommutative torus accepts the trace

$$\mathcal{T}\Big(\sum_{\mathbf{q}\in\mathbb{Z}^{d_{\text{eff}}}} a_{\mathbf{q}}\,\alpha_{\mathbf{q}}\Big) = \mathrm{tr}\; a_{\mathbf{0}}. \tag{15}$$

We define a representation of the noncommutative torus $\pi_{\boldsymbol{\phi}}\colon \mathcal{A}_\Theta \to \mathcal{B}(\ell^2(\mathbb{Z}^d \otimes \mathbb{C}^2))$ via the matrix elements

$$\langle \mathbf{q}, \alpha | \pi_{\boldsymbol{\phi}}(a) | \mathbf{q}', \beta \rangle = a_{\alpha\beta}(\tau_{-\mathbf{q}}\boldsymbol{\phi}, \mathbf{q}' - \mathbf{q}). \tag{16}$$

Constructed in this way, the representation fulfills the covariance condition

$$\hat{T}_{\mathbf{m}} \pi_{\boldsymbol{\phi}}(a) \hat{T}_{\mathbf{m}}^\dagger = \pi_{\tau_{\mathbf{m}}\boldsymbol{\phi}}(a), \tag{17}$$

which we previously confirmed to hold for the Hamiltonian. Additionally, an involution is defined by

$$a^*(\boldsymbol{\phi}, \mathbf{q}) = a(\tau_{-\mathbf{q}}\boldsymbol{\phi}, -\mathbf{q})^\dagger. \tag{18}$$

The C^*-algebra associated to $\mathcal{A}_\Theta$ is then given by the completion with respect to the norm

$$\|a\| = \sup_{\boldsymbol{\phi} \in T^r} \|\pi_{\boldsymbol{\phi}} a\|. \tag{19}$$

D. Some general elements of K-theory

The following section is adapted from [1]. The general goal of the K-theory of operator algebras is to supply all independent topological invariants that can be associated to projections and unitary elements of an algebra. The complex K-theory of the algebra $\mathcal{A}_\Theta$ contains two K-groups, which can be described as follows. The first one is the $K_0(\mathcal{A}_\Theta)$ group, which classifies the projections

$$p \in \mathcal{M}_\infty \otimes \mathcal{A}_\Theta, \quad p^2 = p^* = p, \tag{20}$$

with respect to the von Neumann equivalence relation

$$p \sim p' \quad \text{iff} \quad p = vv' \text{ and } p' = v'v, \tag{21}$$

for some partial isometries v and v' with $vv', v'v \in \mathcal{M}_\infty \otimes \mathcal{A}_\Theta$. Above, $\mathcal{M}_N$ is the algebra of $N \times N$ matrices with complex entries and M_∞ is the direct limit of these algebras. For any projection p from $\mathcal{M}_\infty \otimes \mathcal{A}_\Theta$, there exists $N \in \mathbb{N}$ such that $p \in \mathcal{M}_N \otimes \mathcal{A}_\Theta$, hence we do not really need to work with infinite matrices. However, $\mathcal{M}_N$ can be canonically embedded into $\mathcal{M}_\infty$ and this convenient, because it enables N to take flexible values.

We need to answer two questions: 1) How does the equivalence relation (21) supply topological information? 2) Why do we need the tensoring by $\mathcal{M}_\infty$? Both questions find their answers in the following remark. There are two additional equivalence relations for projections [2, p. 18]:

- Similarity equivalence:

$$p \sim_u p' \quad \text{iff} \quad p' = upu^* \tag{22}$$

  for some unitary element u from $\mathcal{M}_\infty \otimes \mathcal{A}_\Theta$;

- Homotopy equivalence:

$$p \sim_h p' \quad \text{iff} \quad \boldsymbol{p}(0) = p \text{ and } \boldsymbol{p}(1) = p' \tag{23}$$

  for some continuous function $\boldsymbol{p} : [0,1] \to \mathcal{M}_\infty \otimes \mathcal{A}_\Theta$, which always returns a projection.

The homotopy equivalence is certainly the topological equivalence as understood by condensed matter physicists. Now, in general, the three equivalence relations are different, but tensoring $\mathcal{A}_\Theta$ by $\mathcal{M}_\infty$ makes them entirely equivalent. For topological classification, $\sim_h$ is the most interesting relation, but, as we shall see, the relation $\sim$ is essential for understanding the spectral properties of Hamiltonians.

The equivalence class of a projection p will be denoted by $[p]_0$, hence, $[p]_0$ is the set

$$[p]_0 = \{p' \in \mathcal{M}_\infty \otimes \mathcal{A}_\Theta\,,\ p' \sim p\}. \tag{24}$$

If $p \in \mathcal{M}_N \otimes \mathcal{A}_\Theta$ and $p' \in \mathcal{M}_M \otimes \mathcal{A}_\Theta$ are two projections, then $\begin{pmatrix} p & 0 \\ 0 & p' \end{pmatrix}$ is a projection from $\mathcal{M}_{N+M} \otimes \mathcal{A}_\Theta$ and one can define the addition

$$[p]_0 \oplus [p']_0 = \begin{bmatrix} p & 0 \\ 0 & p' \end{bmatrix}_0, \tag{25}$$

which provides a semigroup structure on the set of equivalence classes. Then $K_0(\mathcal{A}_\Theta)$ is its enveloping group [3] and, for the non-commutative $d_{\rm eff}$-torus,

$$K_0(\mathcal{A}_\Theta) = \mathbb{Z}^{2^{d_{\rm eff}-1}}, \tag{26}$$

regardless of Θ and where $d_{\rm eff} = r + d$. As such, there are $2^{d_{\rm eff}-1}$ generators $[e_J]_0$, which can be uniquely labeled by the subsets of indices $J \subseteq \{1, \ldots, d\}$ of even cardinality [4]. Throughout, the cardinality of a set will be indicated by $|\cdot|$. Eq. (26) assures us that, for any projection p from $\mathcal{M}_\infty \otimes \mathcal{A}_\Theta$, one has

$$[p]_0 = \sum_{J\subseteq\{1,\ldots,d_{\rm eff}\}}^{|J|={\rm even}} n_J\, [e_J]_0, \tag{27}$$

where the coefficients n_J are integer numbers that do not change as long as p is deformed inside its K_0-class. Specifically, two homotopically equivalent projections will display the same coefficients, hence $\{n_J\}_{|J|={\rm even}}$ represent the *complete* set of topological invariants associated to the projection p. Furthermore, two projections that display the same set of coefficients are necessarily in the same K_0-class. Let us point out that the coefficient n_J corresponding to $J = \{1, 2, \ldots, d_{\rm eff}\}$ is called the top coefficient and is equal to the strong Chern number associated to the projection p [4, Sec. 5.7].

E. Differential calculus on the noncommutative torus

As a preliminary step to the calculation of Chern numbers on the noncommutative torus, a differential calculus needs to be established. Let $\lambda_i \in \mathbb{C}$, $|\lambda_i| = 1$ and observe that commutation relations of $\mathcal{A}_\Theta$ are invariant with respect to:

$$\alpha_j \mapsto \lambda_j \alpha_j. \tag{28}$$

As such, we can define a $d_{\rm eff}$-torus action:

$$\mathbb{T}^{d_{\rm eff}} \ni \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{d_{\rm eff}}) \mapsto \rho_{\boldsymbol{\lambda}} : \mathcal{A}_\Theta \to \mathcal{A}_\Theta \tag{29}$$

where the latter is the algebra automorphism:

$$A = \sum_{\mathbf{q}\in\mathbb{Z}^{d_{\rm eff}}} a_{\mathbf{q}}\, \alpha_1^{q_1} \ldots \alpha_{d_{\rm eff}}^{q_{d_{\rm eff}}} \mapsto \sum_{\mathbf{q}\in\mathbb{Z}^{d_{\rm eff}}} a_{\mathbf{q}}\, \lambda_1^{q_1} \ldots \lambda_{d_{\rm eff}}^{q_{d_{\rm eff}}}\, \alpha_1^{q_1} \ldots \alpha_{d_{\rm eff}}^{q_{d_{\rm eff}}}. \tag{30}$$

Then the generators of the torus action:

$$\partial_i(A) = i\partial_{\lambda_i}\rho_{\boldsymbol{\lambda}}(A)|_{\boldsymbol{\lambda}\to 1} = \sum_{\mathbf{q}\in\mathbb{Z}^{d_{\rm eff}}} iq_i a_{\mathbf{q}}\, \alpha_1^{q_1} \ldots \alpha_{d_{\rm eff}}^{q_{d_{\rm eff}}} \tag{31}$$

provide derivations on the noncommutative $d_{\rm eff}$-torus. We again define our indices with respect to the index set $\mathcal{I} = \{\tau_1 \cdots, \tau_d, u_1 \cdots, u_r\}$. Since

$$\partial_{\phi_k} e^{2\pi i \boldsymbol{\phi}\cdot\mathbf{n}} = 2\pi i n_k e^{2\pi i \boldsymbol{\phi}\cdot\mathbf{n}}, \tag{32}$$

one finds that the u-derivations are just given by the partial derivatives

$$\partial_{u_k} A = (2\pi)^{-1}\partial_{\phi_k} A. \tag{33}$$

For the τ-derivations, the representation on the Hilbert space evaluates to

$$\pi_{\boldsymbol{\phi}}(\partial_{\tau_k} A) = i[\hat{X}_k, \pi_{\boldsymbol{\phi}}(A)], \tag{34}$$

where $\hat{\mathbf{X}} = \sum_{\mathbf{q}\in\mathbb{Z}^d} |\mathbf{q}\rangle\langle\mathbf{q}|$ is the position operator on the Hilbert space.

If the multi-$\mathbf{q}$ texture is commensurate with the lattice, a Bloch basis can be chosen. We introduce the new basis notation for the orbital wave functions

$$|\mathbf{R}, \mathbf{q}, \alpha\rangle = |\mathbf{R} + \mathbf{x}_{\mathbf{q}}, \alpha\rangle\,, \tag{35}$$

here $\mathbf{R}$ describes the lattice of the superstructure. The lattice Fourier transform (Wannier basis) is given by:

$$|\mathbf{R},\mathbf{q},\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}\in 1.\mathrm{BZ}} e^{-i\mathbf{k}\cdot\mathbf{R}} \, |\psi_{\mathbf{k}\mathbf{q}\alpha}\rangle \,, \tag{36}$$

where N is now the number of primitive cells in the system. Let $\hat{A}$ now represent a translationally invariant operator (w.r.t. the superstructure), i.e.,

$$\begin{aligned}
\hat{A} &= \sum_{\mathbf{R},\mathbf{R}'} \sum_{\mathbf{q},\mathbf{q}',\alpha,\beta} A^{\alpha,\beta,\mathbf{q},\mathbf{q}'}_{\mathbf{R}-\mathbf{R}'} \, |\mathbf{R},\mathbf{q},\alpha\rangle \, \langle \mathbf{R}',\mathbf{q}',\beta| \\
&= \frac{1}{N} \sum_{\mathbf{k},\mathbf{k}'\in 1.\mathrm{BZ}} \sum_{\mathbf{R},\mathbf{R}'} \sum_{\mathbf{q},\mathbf{q}',\alpha,\beta} A^{\alpha,\beta,\mathbf{q},\mathbf{q}'}_{\mathbf{R}-\mathbf{R}'} e^{-i\mathbf{k}\cdot\mathbf{R}} e^{+i\mathbf{k}'\cdot\mathbf{R}'} \, |\psi_{\mathbf{k}\mathbf{q}\alpha}\rangle \, \langle \psi_{\mathbf{k}'\mathbf{q}'\beta}| \\
&= \sum_{\mathbf{k},\mathbf{k}'\in 1.\mathrm{BZ}} \sum_{\mathbf{q},\mathbf{q}',\alpha,\beta} A^{\alpha,\beta,\mathbf{q},\mathbf{q}'}_{\mathbf{k},\mathbf{k}'} \, |\psi_{\mathbf{k}\mathbf{q}\alpha}\rangle \, \langle \psi_{\mathbf{k}'\mathbf{q}'\beta}| \,,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
A^{\alpha,\beta,\mathbf{q},\mathbf{q}'}_{\mathbf{k},\mathbf{k}'} &= \frac{1}{N} \sum_{\mathbf{R},\mathbf{R}'} A^{\alpha,\beta,\mathbf{q},\mathbf{q}'}_{\mathbf{R}-\mathbf{R}'} e^{-i\mathbf{k}\cdot\mathbf{R}} e^{+i\mathbf{k}'\cdot\mathbf{R}'} \\
&= \frac{1}{N} \sum_{\mathbf{R},\mathbf{R}'} A^{\alpha,\beta,\mathbf{q},\mathbf{q}'}_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}'} e^{+i\mathbf{k}'\cdot\mathbf{R}'} \\
&= \delta_{\mathbf{k},\mathbf{k}'} \sum_{\mathbf{R}} A^{\alpha,\beta,\mathbf{q},\mathbf{q}'}_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}} \\
&\equiv \delta_{\mathbf{k},\mathbf{k}'} (A_{\mathbf{k}})_{\alpha,\beta,\mathbf{q},\mathbf{q}'}.
\end{aligned} \tag{38}$$

This means that the trace of any operator product of translationally invariant operators is given by

$$\begin{aligned}
\mathcal{T}\left(\hat{A}^1 \cdots \hat{A}^j\right) &= \frac{1}{V} \lim_{N\to\infty} \frac{1}{N} \sum_{\mathbf{k}\in 1.\mathrm{BZ}} \mathrm{tr}\; A^1_{\mathbf{k}} \cdots A^j_{\mathbf{k}} \\
&= \int\limits_{1.\mathrm{BZ}} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \, \mathrm{tr}\; A^1_{\mathbf{k}} \cdots A^j_{\mathbf{k}},
\end{aligned} \tag{39}$$

where V is the volume of the primitive unit cell and the trace tr includes the internal lattice degrees of freedom within the unit cell (in the addition to the spin degree). Take now a covariant operator

$$\hat{A} = \sum_{\mathbf{R}} \sum_{\alpha\beta} \sum_{\mathbf{q}} A_{\alpha,\beta}(\tau_{-\mathbf{q}}\boldsymbol{\phi}) \; |\mathbf{R},\mathbf{q},\alpha\rangle \, \langle \mathbf{R},\mathbf{q},\beta| \,, \tag{40}$$

and therefore

$$(A_{\mathbf{k}})_{\alpha,\beta,\mathbf{q},\mathbf{q}'} = \delta_{\mathbf{q},\mathbf{q}'} (A_{\mathbf{k}}(\tau_{-\mathbf{q}}\boldsymbol{\phi}))_{\alpha,\beta}. \tag{41}$$

We split the trace in two parts $\mathrm{tr} = \mathrm{tr}_{\mathbf{q}}\mathrm{tr}_\sigma$ according to the atomistic degrees of freedom and the spin degree of freedom. By carrying out the operator product of covariant operators, one finds

$$\begin{aligned}
\mathcal{T}\left(\hat{A}^1 \cdots \hat{A}^j\right) &= \int\limits_{1.\mathrm{BZ}} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \, \mathrm{tr}_{\mathbf{q}}\mathrm{tr}_\sigma \; A^1_{\mathbf{k}} \cdots A^j_{\mathbf{k}} \\
&= \sum_{\mathbf{q}} \int\limits_{1.\mathrm{BZ}} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \, \mathrm{tr}_\sigma \; A^1_{\mathbf{k}}(\tau_{-\mathbf{q}}\boldsymbol{\phi}) \cdots A^j_{\mathbf{k}}(\tau_{-\mathbf{q}}\boldsymbol{\phi}) \\
&\to \int\limits_{1.\mathrm{BZ}} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \int\limits_{\Omega} \mathrm{d}^r\boldsymbol{\phi} \; \mathrm{tr}_\sigma \; A^1_{\mathbf{k}}(\boldsymbol{\phi}) \cdots A^j_{\mathbf{k}}(\boldsymbol{\phi}).
\end{aligned} \tag{42}$$

Here, the limit $\to$ indicates the transition to a smooth magnetic texture, which is supported by a larger and larger amount of atomic sites in the primitive cell of the superstructure. As a further ingredient, one needs that the action of the translation operator is ergodic on Ω in the smooth limit.

Assuming $\hat{A}$ is diagonal in $\mathbf{q}$ (as is the case for the covariant operators):

$$\begin{aligned}
i[\hat{X}_i, A] &= \sum_{\mathbf{R},\mathbf{R}'} \sum_{\mathbf{q},\alpha,\beta} i(\mathbf{R}-\mathbf{R}')_i A^{\alpha,\beta,\mathbf{q}}_{\mathbf{R}-\mathbf{R}'} \left|\mathbf{R},\mathbf{q},\alpha\right\rangle \left\langle \mathbf{R}',\mathbf{q},\beta\right| \\
&= \sum_{\mathbf{k}\in 1.\mathrm{BZ}} \sum_{\mathbf{q},\alpha,\beta} \sum_{\mathbf{R}} iR_i \, A^{\alpha,\beta,\mathbf{q}}_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}} \left|\psi_{\mathbf{kq}\alpha}\right\rangle \left\langle \psi_{\mathbf{kq}\beta}\right| \\
&= -\sum_{\mathbf{k}\in 1.\mathrm{BZ}} \sum_{\mathbf{q},\alpha,\beta} \partial_{k_i} \sum_{\mathbf{R}} A^{\alpha,\beta,\mathbf{q}}_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}} \left|\psi_{\mathbf{kq}\alpha}\right\rangle \left\langle \psi_{\mathbf{kq}\beta}\right| \\
&= \sum_{\mathbf{k}\in 1.\mathrm{BZ}} \sum_{\mathbf{q},\alpha,\beta} (-\partial_{k_i} A_{\mathbf{k}})_{\alpha,\beta,\mathbf{q}} \left|\psi_{\mathbf{kq}\alpha}\right\rangle \left\langle \psi_{\mathbf{kq}\beta}\right|
\end{aligned} \tag{43}$$

For covariant operators, we therefore have the correspondence dictionary for the covariant Bloch representation

$$\pi_{\boldsymbol{\phi}}(A) \to A_{\mathbf{k}}(\boldsymbol{\phi}) \tag{44}$$

$$\pi_{\boldsymbol{\phi}}(\partial_{u_j} A) \to \partial_{\phi_j} A_{\mathbf{k}}(\boldsymbol{\phi})/(2\pi) \tag{45}$$

$$\pi_{\boldsymbol{\phi}}(\partial_{\tau_j} A) \to -\partial_{k_j} A_{\mathbf{k}}(\boldsymbol{\phi}) \tag{46}$$

$$\mathcal{T} \to \int_{1.\mathrm{BZ}} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \, \mathrm{tr}_\sigma \, \sum_{\mathbf{q}} \tau_{\mathbf{q}} \triangleright, \tag{47}$$

where $\tau_{\mathbf{q}}\triangleright$ denotes the action:

$$\tau_{\mathbf{q}} \triangleright A^1(\boldsymbol{\phi}) \cdots A^j(\boldsymbol{\phi}) \equiv A^1(\tau_{-\mathbf{q}}\boldsymbol{\phi}) \cdots A^j(\tau_{-\mathbf{q}}\boldsymbol{\phi}) \tag{48}$$

And, in the limit of smooth textures and ergodic action,

$$\sum_{\mathbf{q}} \tau_{\mathbf{q}} \triangleright \to \int_{\Omega} \mathrm{d}^r \boldsymbol{\phi}. \tag{49}$$

F. Chern number

Now that the differential calculus on the torus is established, the Chern numbers can be defined. The Chern number of a projection P to gap g and associated to a subset of indices J of even cardinality is given by

$$\mathrm{Ch}_{J'}(g) = \frac{(2\pi i)^{|J'|/2}}{(|J'|/2)!} \sum_{\sigma\in\mathcal{S}_{|J'|}} (-1)^\sigma \mathcal{T}\Big(P \prod_{j\in J'} \partial_{\sigma_j} P\Big), \tag{50}$$

where for $J = \emptyset$, we define $\mathrm{Ch}_\emptyset(P) = \mathcal{T}(P)$. The structure of the noncommutative torus imposes relations on the Chern numbers. These can be found by studying the values of the Chern numbers on the K_0-generators of $\mathcal{A}_\Theta$, which can be found in [4][p. 141]:

$$\mathrm{Ch}_{J'}[E_J]_0 = \begin{cases} 0 \text{ if } J' \not\subseteq J, \\ 1 \text{ if } J' = J, \\ \mathrm{Pf}(\Theta_{J\setminus J'}) \text{ if } J' \subset J, \end{cases} \qquad J, J' \subset \{1,\ldots,d_{\mathrm{eff}}\}. \tag{51}$$

Since the Chern numbers are also linear maps, their values on the gap projection $[P_G]_0 = \sum_J n_J \, [e_J]_0$ can be straightforwardly computed from Eq. (51):

$$\mathrm{Ch}_{J'}(g) = n_{J'}(g) + \sum_{J'\subsetneq J} n_J(g) \, \mathrm{Pf}(\Theta_{J\setminus J'}). \tag{52}$$

The K-theory of the noncommutative torus therefore imposes relations among the various Chern numbers. Top Chern number corresponding to $J' = \{1,\ldots,d_{\mathrm{eff}}\}$ is always an integer, but the lower Chern numbers may not be.

To illustrate the special case of a commensurate texture, consider the special case of $d = r = 2$ and $d_{\text{eff}} = d + r = 4$. Via the correspondence dictionary, we find the top Chern number provided by the expression (for $J = \{\tau_1, \tau_2, u_1, u_2\}$)

$$\mathrm{Ch}_J(g) = -\frac{1}{2} \int\limits_{1.\mathrm{BZ}} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \sum_{\mathbf{q}} \tau_{\mathbf{q}} \triangleright \sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma \mathrm{tr}_\sigma \; P_{\mathbf{k}}(\boldsymbol{\phi}) \prod_{j \in J} \partial_{\sigma_j} P_{\mathbf{k}}(\boldsymbol{\phi}), \tag{53}$$

where the representations of Eq. 44 - Eq. 45 have already been inserted. We identify the Berry curvature

$$F_{\sigma_1,\sigma_2}(\mathbf{k}, \boldsymbol{\phi}) = i P_{\mathbf{k}}(\boldsymbol{\phi})[\partial_{\sigma_1} P_{\mathbf{k}}(\boldsymbol{\phi}), \partial_{\sigma_2} P_{\mathbf{k}}(\boldsymbol{\phi})] \tag{54}$$

and write

$$\begin{aligned}
\sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma \mathrm{tr}_\sigma \; P_{\mathbf{k}}(\boldsymbol{\phi}) \prod_{j \in J} \partial_{\sigma_j} P_{\mathbf{k}}(\boldsymbol{\phi}) &= \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr}_\sigma \; P_{\mathbf{k}}(\boldsymbol{\phi}) \partial_{\sigma_\alpha} P_{\mathbf{k}}(\boldsymbol{\phi}) \partial_{\sigma_\beta} P_{\mathbf{k}}(\boldsymbol{\phi}) \partial_{\sigma_\gamma} P_{\mathbf{k}}(\boldsymbol{\phi}) \partial_{\sigma_\delta} P_{\mathbf{k}}(\boldsymbol{\phi}) \\
&= \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr}_\sigma \; P_{\mathbf{k}}(\boldsymbol{\phi}) \partial_{\sigma_\alpha} P_{\mathbf{k}}(\boldsymbol{\phi}) \partial_{\sigma_\beta} P_{\mathbf{k}}(\boldsymbol{\phi}) P_{\mathbf{k}}(\boldsymbol{\phi}) \partial_{\sigma_\gamma} P_{\mathbf{k}}(\boldsymbol{\phi}) \partial_{\sigma_\delta} P_{\mathbf{k}}(\boldsymbol{\phi}) \\
&= -\frac{1}{4} \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr}_\sigma \; F_{\alpha\beta}(\mathbf{k}, \boldsymbol{\phi}) F_{\gamma\delta}(\mathbf{k}, \boldsymbol{\phi}).
\end{aligned} \tag{55}$$

Inserting this result into the expression for the Chern number gives

$$\begin{aligned}
\mathrm{Ch}_J(g) &= \frac{1}{8} \sum_{\mathbf{q}} \int\limits_{1.\mathrm{BZ}} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \, \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr}_\sigma \; F_{\alpha\beta}(\mathbf{k}, \tau_{-\mathbf{q}}\boldsymbol{\phi}) F_{\gamma\delta}(\mathbf{k}, \tau_{-\mathbf{q}}\boldsymbol{\phi}) \\
&= \frac{1}{32\pi^2} \sum_{\mathbf{q}} \int\limits_{1.\mathrm{BZ}} \mathrm{d}^d\mathbf{k} \, \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr}_\sigma \; F_{\alpha\beta}(\mathbf{k}, \tau_{-\mathbf{q}}\boldsymbol{\phi}) F_{\gamma\delta}(\mathbf{k}, \tau_{-\mathbf{q}}\boldsymbol{\phi}) \\
&\to \frac{1}{32\pi^2} \int\limits_{\Omega} \mathrm{d}^r\boldsymbol{\phi} \int\limits_{1.\mathrm{BZ}} \mathrm{d}^d\mathbf{k} \, \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr}_\sigma \; F_{\alpha\beta}(\mathbf{k}, \boldsymbol{\phi}) F_{\gamma\delta}(\mathbf{k}, \boldsymbol{\phi}) \\
&= \frac{1}{32\pi^2} \int\limits_{T^{d_{\text{eff}}}} \mathrm{d}^{d_{\text{eff}}}\boldsymbol{\lambda} \, \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr}_\sigma \; F_{\alpha\beta}(\boldsymbol{\lambda}) F_{\gamma\delta}(\boldsymbol{\lambda}),
\end{aligned} \tag{56}$$

which is the familiar expression for the second Chern number in terms of the Berry curvature [5]. Repeating the same calculation for the case of $d = r = 1$ and $d_{\text{eff}} = d + r = 2$, with $J = \{\tau u\}$, one finds

$$\begin{aligned}
\mathrm{Ch}_J(g) &= -\frac{1}{2\pi} \sum_{\mathbf{q}} \int\limits_{1.\mathrm{BZ}} \mathrm{d}^d\mathbf{k} \; \mathrm{tr}_\sigma \; F_{\tau u}(\mathbf{k}, \tau_{-\mathbf{q}}\boldsymbol{\phi}) \\
&\to -\frac{1}{2\pi} \int\limits_{\Omega} \mathrm{d}^r\boldsymbol{\phi} \int\limits_{1.\mathrm{BZ}} \mathrm{d}^d\mathbf{k} \; \mathrm{tr}_\sigma \; F_{\tau u}(\mathbf{k}, \boldsymbol{\phi})) \\
&= -\frac{1}{2\pi} \int\limits_{T^{d_{\text{eff}}}} \mathrm{d}^{d_{\text{eff}}}\boldsymbol{\lambda} \; \mathrm{tr}_\sigma \; F_{\tau u}(\boldsymbol{\lambda}),
\end{aligned} \tag{57}$$

which, in this case, is representing the usual expression for the first Chern number in terms of the Berry curvature [5].

G. The Θ-matrix for 3q states on the triangular lattice

In this section, we discuss the construction of the skyrmion $3\mathbf{q}$-state on the triangular lattice as it appears in the manuscript. Real- and reciprocal space lattice vectors are introduced via

$$\mathbf{a}_1 = (1, 0)^T \tag{58}$$

$$\mathbf{a}_2 = (1/2, \sqrt{3}/2)^T \tag{59}$$

$$\mathbf{b}_1 = 2\pi(1, -1/\sqrt{3})^T \tag{60}$$

$$\mathbf{b}_2 = 2\pi(0, 2/\sqrt{3})^T. \tag{61}$$

With respect to these lattice vectors, the $\mathbf{q}$-vectors of the texture are given by

$$\mathbf{q_1} = \theta_1 \mathbf{b}_1 \tag{62}$$
$$\mathbf{q_2} = \theta_1 \mathbf{b}_2 \tag{63}$$
$$\mathbf{q_3} = \theta_1(-\mathbf{b}_1 - \mathbf{b_2}). \tag{64}$$

One can confirm that these vectors form the vertices of an equilateral triangle and that $\sum_i \mathbf{q}_i = 0$. From the definition follows that the θ-matrix is given by

$$\theta = \theta_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{65}$$

As initial phases we take $\boldsymbol{\phi} = (0, 0, \pi)$. The respective Chern number decomposition can be found in table IV at the end of this document (the analogous case for a 2$\mathbf{q}$-state in $d = 1$ and $d = 2$ is shown in II and III respectively). Let $R^z_{2\pi/3}$ represent a $2\pi/3$ rotation around the z-axis. Then we write

$$\hat{\mathbf{n}}_{\mathrm{SkX}}(\mathbf{x}) = \sum_{i=1}^{3} (R^z_{2\pi/3})^{i-1} \hat{\mathbf{n}}_{\mathrm{hx}}(((R^z_{2\pi/3})^{i-1}\mathbf{q}_1) \cdot \mathbf{x}/(2\pi) + \phi_i) \tag{66}$$
$$\hat{\mathbf{n}}_{\mathrm{XY-V}}(\mathbf{x}) = \sum_{i=1}^{3} (R^z_{2\pi/3})^{i-1} \hat{\mathbf{n}}_{\mathrm{sdw}}(((R^z_{2\pi/3})^{i-1}\mathbf{q}_1) \cdot \mathbf{x}/(2\pi) + \phi_i). \tag{67}$$

Here, the skyrmion lattice $\hat{\mathbf{n}}_{\mathrm{SkX}}$ is therefore constructed from a coherent superposition of three spin helices (hx), and the vortex lattice $\hat{\mathbf{n}}_{\mathrm{XY-V}}$ is constructed from a coherent superposition of spin density waves (sdw). Respectively, these are defined by

$$\hat{\mathbf{n}}_{\mathrm{hx}}(\psi) = (0, \sin(\psi), \cos(\psi))^T, \tag{68}$$
$$\hat{\mathbf{n}}_{\mathrm{sdw}}(\psi) = (\sin(\psi), 0, 0)^T. \tag{69}$$

For the SkX is state, the result of the formula is always normalized by $\hat{\mathbf{n}}_{\mathrm{SkX}}(\mathbf{x}) \to \hat{\mathbf{n}}_{\mathrm{SkX}}(\mathbf{x})/\|\hat{\mathbf{n}}_{\mathrm{SkX}}(\mathbf{x})\|$, while for the $XY - V$ state, one scales the result such that

$$\sup_{\mathbf{x}} \|\hat{\mathbf{n}}_{\mathrm{XY-V}}(\mathbf{x})\| = 1. \tag{70}$$

As the exact diagonalization of the Hamiltonian is computationally more demanding in $d = 2$ dimensions compared to the $d = 1$ case, we combine the spectra of different linear system sizes $N \in [19, 79]$ (i.e. there are N lattice unit cells per dimension). The θ_1 are sampled again at rational values $\theta_1 = m/N$ with $m \in \mathbb{Z}$ and $0 \leq m \leq N$. Since N is not necessarily prime, some θ_1-values would be sampled multiple times. When this occurs for two different values of N, we always choose the larger system size to obtain a better spectral resolution.

H. Discussion of the relation to emergent magnetic fields

In the adiabatic limit of smooth textures and strong exchange coupling, our theory should reduce to the well-known language of emergent magnetic fields. To discuss the adiabatic limit, we introduce the unitary transformation

$$U^\dagger(\mathbf{x})(\hat{\mathbf{n}}(\mathbf{x}) \cdot \boldsymbol{\sigma})U(\mathbf{x}) = \sigma_z. \tag{71}$$

By parameterizing the magnetization vector in polar coordinates $\hat{\mathbf{n}} = \hat{\mathbf{n}}(\theta, \phi)$ in spherical coordinates, this transformation can be formulated explicitly as $\mathcal{U} = \hat{\mathbf{n}}(\theta/2, \phi) \cdot \boldsymbol{\sigma} \equiv \mathbf{m} \cdot \boldsymbol{\sigma}$. The discretization on the lattice is given by

$$U(\hat{\mathbf{x}}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} U(\mathbf{x_k}) \left|\mathbf{k}\right\rangle \left\langle\mathbf{k}\right|. \tag{72}$$

Applying the transformation to the Hamiltonian, one finds

$$U(\hat{\mathbf{x}})^\dagger H U(\hat{\mathbf{x}}) = \sum_{\langle \mathbf{k},\mathbf{l}\rangle \in \mathbb{Z}^{2d}} t_{\mathbf{kl}} \left|\mathbf{k}\right\rangle \left\langle\mathbf{l}\right| + \Delta_{\mathrm{xc}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sigma_z \; \left|\mathbf{k}\right\rangle \left\langle\mathbf{k}\right|, \tag{73}$$

where $t_{\mathbf{kl}} = tU^\dagger(\mathbf{x_k})U(\mathbf{x_l})$. In the limit $\Delta_{\mathrm{xc}}/t \to \infty$, one can project onto the eigenstates $\sigma = \pm 1$ of σ_z in order to arrive at the effective Hamiltonian

$$H^\sigma_{\mathrm{eff}} = \sum_{\langle \mathbf{k},\mathbf{l}\rangle \in \mathbb{Z}^{2d}} t^{\mathrm{eff}}_{\mathbf{kl},\sigma} \left|\mathbf{k}\right\rangle \left\langle \mathbf{l}\right|, \tag{74}$$

where $t^{\mathrm{eff}}_{\mathbf{kl},\sigma} = t \left\langle \sigma \middle| U^\dagger(\mathbf{x_k})U(\mathbf{x_l}) \middle| \sigma \right\rangle$. In the continuous case, a vector potential can be defined as $A_i = -i\hbar U^\dagger \partial_i U/e$, which has the adiabatic projection

$$A_i^\pm = \pm\frac{\hbar}{e}(\mathbf{m} \times \partial_i \mathbf{m})_z. \tag{75}$$

From this, one obtains the emergent magnetic field

$$B_z^\pm = (\nabla \times \mathbf{A}^\pm)_z = \pm\frac{\hbar}{2e}\hat{\mathbf{n}} \cdot (\partial_x \hat{\mathbf{n}} \times \partial_y \hat{\mathbf{n}}). \tag{76}$$

For an isolated skyrmion of topological charge

$$\mathcal{Q} = \frac{1}{4\pi}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, \hat{\mathbf{n}} \cdot (\partial_x \hat{\mathbf{n}} \times \partial_y \hat{\mathbf{n}}) \in \mathbb{Z} \tag{77}$$

is quantized. The emergent flux in this case is

$$\Phi^\pm = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, B_z^\pm = \pm\frac{\hbar}{2e}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, \hat{\mathbf{n}} \cdot (\partial_x \hat{\mathbf{n}} \times \partial_y \hat{\mathbf{n}}) = \pm 2\pi\frac{\hbar}{e}\mathcal{Q}. \tag{78}$$

Transformed translation operator in two dimensions

Applying the translation operator to the previously defined unitary operator, we find

$$\begin{aligned}\hat{T}_{\mathbf{m}} U(\hat{\mathbf{x}}) \hat{T}^\dagger_{\mathbf{m}} &= \sum_{\mathbf{k}} U(\mathbf{x_k}) \left|\mathbf{k}+\mathbf{m}\right\rangle \left\langle \mathbf{k}+\mathbf{m}\right| \\ &= \sum_{\mathbf{k}} U(\mathbf{x_{k-m}}) \left|\mathbf{k}\right\rangle \left\langle \mathbf{k}\right| \\ &= U(\hat{\mathbf{x}} - \mathbf{x_m}),\end{aligned} \tag{79}$$

from which one can obtain the relation $\hat{T}_{\mathbf{m}} U(\hat{\mathbf{x}}) = U(\hat{\mathbf{x}} - \mathbf{x_m})\hat{T}_{\mathbf{m}}$. Within the changed frame of reference, the new unit translation operator is given by

$$\begin{aligned}\hat{S}_i &\equiv U^\dagger(\hat{\mathbf{x}}) \hat{T}_i U(\hat{\mathbf{x}}) \\ &= U^\dagger(\hat{\mathbf{x}}) U(\hat{\mathbf{x}} - \mathbf{a}_i) \hat{T}_i.\end{aligned} \tag{80}$$

We now assume that $\hat{\mathbf{n}}$ is given by a multi-$\mathbf{q}$ state in $d = 2$ dimensions, characterized by a single pitch variable θ_1. For a smoothly varying texture (limit of small θ_1), the pre-factor can be expanded:

$$\begin{aligned}U^\dagger(\mathbf{x}) U(\mathbf{x} - \mathbf{a}_i) &= \mathrm{id}_2 - U^\dagger(\mathbf{x})(\mathbf{a}_i \cdot \nabla) U(\mathbf{x}) + \mathcal{O}(\theta_1^2) \\ &= \mathrm{id}_2 + ie\, \mathbf{a}_i \cdot \mathbf{A}/\hbar + \mathcal{O}(\theta_1^2) \\ &= \mathrm{id}_2 + \frac{ie}{\hbar}\int_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_i} \mathrm{d}\mathbf{r}\, \mathbf{A} + \mathcal{O}(\theta_1^2) \\ &= \exp\left(\frac{ie}{\hbar}\int_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_i} \mathrm{d}\mathbf{r}\, \mathbf{A}\right) + \mathcal{O}(\theta_1^2),\end{aligned} \tag{81}$$

were we have implicitly used the adiabatic projection into a spin-subspace. Introduce the shorthand notation

$$\uparrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}} \equiv \exp\left(\frac{ie}{\hbar}\int\limits_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_i} \mathrm{d}\mathbf{r}\,\mathbf{A}\right), \tag{82}$$

$$\downarrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}} \equiv \exp\left(-\frac{ie}{\hbar}\int\limits_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_i} \mathrm{d}\mathbf{r}\,\mathbf{A}\right). \tag{83}$$

Using this notation, one can derive the commutation relations

$$\begin{aligned}
S_1 S_2 =& \uparrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1} T_1 \uparrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_2} T_2 \\
=& \uparrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1} \uparrow_{\mathbf{x}-\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_2-\mathbf{a}_1} T_2 T_1 \\
=& \uparrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1} \uparrow_{\mathbf{x}-\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_2-\mathbf{a}_1} T_2 \downarrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1} S_1 \\
=& \uparrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1} \uparrow_{\mathbf{x}-\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_2-\mathbf{a}_1} \downarrow_{\mathbf{x}+\mathbf{a}_2}^{\mathbf{x}+\mathbf{a}_1+\mathbf{a}_2} T_2 S_1 \\
=& \uparrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1} \uparrow_{\mathbf{x}-\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_2-\mathbf{a}_1} \downarrow_{\mathbf{x}+\mathbf{a}_2}^{\mathbf{x}+\mathbf{a}_1+\mathbf{a}_2} \downarrow_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_2} S_2 S_1.
\end{aligned} \tag{84}$$

Since

$$\uparrow_{\mathbf{x}-\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_2-\mathbf{a}_1} = \uparrow_{\mathbf{x}+\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_2+\mathbf{a}_1} + \mathcal{O}(\theta_1^2), \tag{85}$$

the combination of integrals amounts to clockwise line integral around the unit cell anchored at $\mathbf{x}$. We change this to a counter-clockwise orientation and apply the Stokes theorem to write the emergent flux as

$$\Phi(\mathbf{x}) = \oint\limits_{\partial \mathrm{uc}(\mathbf{x})} \mathrm{d}\mathbf{r}\cdot\mathbf{A} = \int\limits_{\mathrm{uc}(\mathbf{x})} \mathrm{d}^2\mathbf{r}\,(\nabla\times\mathbf{A})_z. \tag{86}$$

We therefore find the commutation relation

$$S_1 S_2 = e^{-i\hbar\Phi(\mathbf{x})/e} S_2 S_1 + \mathcal{O}(\theta_1^2). \tag{87}$$

For the lattice of skyrmions with charge $\mathcal{Q} = 1$, the emergent flux per magnetic unit cell is quantized, i.e., it is given by $|e\Phi_{\mathrm{sk}}/\hbar| = 2\pi$. On average, the flux per unit cell of the lattice is therefore given by

$$\langle\Phi(\mathbf{x})\rangle = \frac{2\pi}{\langle N_{\mathrm{uc}}\rangle}, \tag{88}$$

where $\langle N_{\mathrm{uc}}\rangle$ is the average number of lattice unit cells within a magnetic unit cell. In $d = 2$ dimensions, one has $\langle N_{\mathrm{uc}}\rangle = 1/\theta_1^2$ The algebra can then be approximated by replacing the exact flux $\Phi(\mathbf{x})$ per lattice unit cell by this average and one obtains the commutation relation

$$S_1 S_2 \approx e^{-i2\pi\theta_1^2} S_2 S_1, \tag{89}$$

while at same time, S_i commutes with the Fourier factors since the non-collinear magnetism has been transformed away. All possible Chern numbers are then summarized by the table

$$\begin{array}{l|l}
J' & \mathrm{Ch}_{J'} \\
\hline
\{\} & \theta_1^2 n_{\{s_1,s_2\}} + n_{\{\}} \\
\{s_1,s_2\} & n_{\{s_1,s_2\}}
\end{array} \tag{90}$$

Consequently, the IDS in the gap g for the effective system is given by the expansion

$$\mathrm{IDS}(g) = n_{\emptyset}(g) + n_{s_1,s_2}(g)\theta_1^2. \tag{91}$$

By matching the coefficients, of the two limits one therefore finds

$$n_{\{s_1,s_2\}}(g) \sim n_{t^2u^2}(g),\ \text{for } |\Delta_{\mathrm{xc}}/t| \to \infty. \tag{92}$$

Further, the left-hand side can also be calculated directly as Chern number, since

$$\mathrm{Ch}_{\{s_1,s_2\}}(g) = n_{\{s_1,s_2\}}(g). \tag{93}$$

Since the Chern number is invariant under unitary transformations of the Hamiltonian, this then leads to

$$\mathrm{Ch}_{\{t_1,t_2\}}(g) \sim \mathrm{Ch}_{\{s_1,s_2\}}(g) \sim n_{t^2u^2}(g), \text{ for } |\Delta_{\mathrm{xc}}/t| \to \infty, \tag{94}$$

which means that the presence of a quantum anomalous Hall effect can be deduced from the IDS (where $n_{t^2u^2}(g)$ can be extracted). To rephrase this result: the relation holds, because we have shown that the physics of the asymptotic limit is described by a 2-dimensional subalgebra of the full $(2+r)$-dimensional noncommutative torus generated by $\hat{S}_i = \langle\sigma|U^\dagger(\hat{\mathbf{x}})U(\hat{\mathbf{x}}-\mathbf{a}_i)|\sigma\rangle\,\hat{T}_i$. This subalgebra is completely characterized by two topological integers $n_{\{\}}$ and $n_{\{s_1,s_2\}}$ which can be directly extracted from the IDS.

I. The Θ-matrix on the cubic hedgehog lattice

In the case of the cubic hedgehog lattice, one deals with three linearly independent, mutually orthogonal $\mathbf{q}$-vectors: $\mathbf{q}_i = q\mathbf{e}_i$. This means that the θ-matrix is just the identity in $d = 3$ dimensions: $\theta = \theta_1 \mathrm{id}_3$. Table V summarizes the possible Chern numbers in this case. Noteworthy is the emergence of a third Chern numbers in the IDS:

$$\mathrm{IDS}(g) = n_\emptyset + n_{\tau u}\theta_1 + n_{\tau^2u^2}\theta_1^2 + n_{\tau^3u^3}\theta_1^3, \tag{95}$$

where

$$n_{\tau u} = n_{\{\tau_1 u_1\}} + n_{\{\tau_2 u_2\}} + n_{\{\tau_3 u_3\}} \tag{96}$$

$$n_{\tau^2u^2} = n_{\{\tau_1\tau_2 u_1 u_2\}} + n_{\{\tau_1\tau_3 u_1 u_3\}} + n_{\{\tau_2\tau_3 u_2 u_3\}} \tag{97}$$

$$n_{\tau^3u^3} = n_{\{\tau_1\tau_2\tau_3 u_1 u_2 u_3\}}. \tag{98}$$

The latter can be identified with the top-level Chern number

$$\mathrm{Ch}_{\{\tau_1\tau_2\tau_3 u_1 u_2 u_3\}} = n_{\{\tau_1\tau_2\tau_3 u_1 u_2 u_3\}}. \tag{99}$$

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J'	$\mathrm{Ch}_{J'}$
$\{\}$	$\theta_1 n_{\{\tau_1,u_1\}} + \theta_2 n_{\{\tau_1,u_2\}} + n_{\{\}}$
$\{\tau_1, u_1\}$	$n_{\{\tau_1,u_1\}}$
$\{\tau_1, u_2\}$	$n_{\{\tau_1,u_2\}}$
$\{u_1, u_2\}$	$n_{\{u_1,u_2\}}$

TABLE II. Chern number expansion for a 2$\mathbf{q}$-state in $d = 1$ dimensions with $\theta = (\theta_1, \theta_2)^T$ (e.g. the 2-$\mathbf{q}$ helicoids)

J'	$\mathrm{Ch}_{J'}$
$\{\}$	$\theta_1^2 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1 n_{\{\tau_1,u_2\}} + \theta_1 n_{\{\tau_2,u_1\}} + n_{\{\}}$
$\{\tau_1, \tau_2\}$	$n_{\{\tau_1,\tau_2\}}$
$\{\tau_1, u_1\}$	$n_{\{\tau_1,u_1\}}$
$\{\tau_1, u_2\}$	$\theta_1 n_{\{\tau_2,u_1\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_2\}} + n_{\{\tau_1,u_2\}}$
$\{\tau_2, u_1\}$	$\theta_1 n_{\{\tau_1,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_2\}} + n_{\{\tau_2,u_1\}}$
$\{\tau_2, u_2\}$	$n_{\{\tau_2,u_2\}}$
$\{u_1, u_2\}$	$n_{\{u_1,u_2\}}$
$\{\tau_1, \tau_2, u_1, u_2\}$	$n_{\{\tau_1,\tau_2,u_1,u_2\}}$

TABLE III. Chern number expansion for a 2$\mathbf{q}$-state in $d = 2$ dimensions with $\theta = \theta_1 \mathrm{id}_2$ (an example would be the 2-$\mathbf{q}$ skyrmion lattice).

J'	$\mathrm{Ch}_{J'}$
$\{\}$	$\theta_1^2 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1^2 n_{\{\tau_1,\tau_2,u_1,u_3\}} + \theta_1^2 n_{\{\tau_1,\tau_2,u_2,u_3\}} + \theta_1 n_{\{\tau_1,u_2\}} + \theta_1 n_{\{\tau_1,u_3\}} + \theta_1 n_{\{\tau_2,u_1\}} + \theta_1 n_{\{\tau_2,u_3\}} + n_{\{\}}$
$\{\tau_1,\tau_2\}$	$n_{\{\tau_1,\tau_2\}}$
$\{\tau_1,u_1\}$	$\theta_1 n_{\{\tau_2,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_3\}} + n_{\{\tau_1,u_1\}}$
$\{\tau_1,u_2\}$	$\theta_1 n_{\{\tau_2,u_1\}} + \theta_1 n_{\{\tau_2,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,u_2,u_3\}} + n_{\{\tau_1,u_2\}}$
$\{\tau_1,u_3\}$	$\theta_1 n_{\{\tau_2,u_1\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_3\}} + n_{\{\tau_1,u_3\}}$
$\{\tau_2,u_1\}$	$\theta_1 n_{\{\tau_1,u_2\}} + \theta_1 n_{\{\tau_1,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_3\}} + n_{\{\tau_2,u_1\}}$
$\{\tau_2,u_2\}$	$\theta_1 n_{\{\tau_1,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,u_2,u_3\}} + n_{\{\tau_2,u_2\}}$
$\{\tau_2,u_3\}$	$\theta_1 n_{\{\tau_1,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,u_2,u_3\}} + n_{\{\tau_2,u_3\}}$
$\{u_1,u_2\}$	$\theta_1 n_{\{\tau_1,u_3\}} + \theta_1 n_{\{\tau_2,u_3\}} + \theta_1 n_{\{\tau_1,u_1,u_2,u_3\}} + \theta_1 n_{\{\tau_2,u_1,u_2,u_3\}} + n_{\{u_1,u_2\}}$
$\{u_1,u_3\}$	$\theta_1 n_{\{\tau_1,u_2\}} + \theta_1 n_{\{\tau_1,u_1,u_2,u_3\}} + n_{\{u_1,u_3\}}$
$\{u_2,u_3\}$	$\theta_1 n_{\{\tau_2,u_1\}} + \theta_1 n_{\{\tau_2,u_1,u_2,u_3\}} + n_{\{u_2,u_3\}}$
$\{\tau_1,\tau_2,u_1,u_2\}$	$n_{\{\tau_1,\tau_2,u_1,u_2\}}$
$\{\tau_1,\tau_2,u_1,u_3\}$	$n_{\{\tau_1,\tau_2,u_1,u_3\}}$
$\{\tau_1,\tau_2,u_2,u_3\}$	$n_{\{\tau_1,\tau_2,u_2,u_3\}}$
$\{\tau_1,u_1,u_2,u_3\}$	$n_{\{\tau_1,u_1,u_2,u_3\}}$
$\{\tau_2,u_1,u_2,u_3\}$	$n_{\{\tau_2,u_1,u_2,u_3\}}$

TABLE IV. Chern number expansion for $\theta = \theta_1((0,1),(1,0),(-1,-1))$ (the 3-$\mathbf{q}$, triangular skyrmion lattice).

J'	$\mathrm{Ch}_{J'}$
$\{\}$	$\theta_1^3 n_{\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}} + \theta_1^2 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1^2 n_{\{\tau_1,\tau_3,u_1,u_3\}} + \theta_1^2 n_{\{\tau_2,\tau_3,u_2,u_3\}} + \theta_1 n_{\{\tau_1,u_1\}} + \theta_1 n_{\{\tau_2,u_2\}} + \theta_1 n_{\{\tau_3,u_3\}} + n_{\{\}}$
$\{\tau_1,\tau_2\}$	$\theta_1 n_{\{\tau_3,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_3\}} + n_{\{\tau_1,\tau_2\}}$
$\{\tau_1,\tau_3\}$	$\theta_1 n_{\{\tau_2,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_2\}} + n_{\{\tau_1,\tau_3\}}$
$\{\tau_1,u_1\}$	$\theta_1^2 n_{\{\tau_2,\tau_3,u_2,u_3\}} + \theta_1^2 n_{\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}} + \theta_1 n_{\{\tau_2,u_2\}} + \theta_1 n_{\{\tau_3,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1 n_{\{\tau_1,\tau_3,u_1,u_3\}} + n_{\{\tau_1,u_1\}}$
$\{\tau_1,u_2\}$	$\theta_1 n_{\{\tau_3,u_3\}} + \theta_1 n_{\{\tau_1,\tau_3,u_2,u_3\}} + n_{\{\tau_1,u_2\}}$
$\{\tau_1,u_3\}$	$\theta_1 n_{\{\tau_2,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,u_2,u_3\}} + n_{\{\tau_1,u_3\}}$
$\{\tau_2,\tau_3\}$	$\theta_1 n_{\{\tau_1,u_1\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_1\}} + n_{\{\tau_2,\tau_3\}}$
$\{\tau_2,u_1\}$	$\theta_1 n_{\{\tau_3,u_3\}} + \theta_1 n_{\{\tau_2,\tau_3,u_1,u_3\}} + n_{\{\tau_2,u_1\}}$
$\{\tau_2,u_2\}$	$\theta_1^2 n_{\{\tau_1,\tau_3,u_1,u_3\}} + \theta_1^2 n_{\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}} + \theta_1 n_{\{\tau_1,u_1\}} + \theta_1 n_{\{\tau_3,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1 n_{\{\tau_2,\tau_3,u_2,u_3\}} + n_{\{\tau_2,u_2\}}$
$\{\tau_2,u_3\}$	$\theta_1 n_{\{\tau_1,u_1\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_3\}} + n_{\{\tau_2,u_3\}}$
$\{\tau_3,u_1\}$	$\theta_1 n_{\{\tau_2,u_2\}} + \theta_1 n_{\{\tau_2,\tau_3,u_1,u_2\}} + n_{\{\tau_3,u_1\}}$
$\{\tau_3,u_2\}$	$\theta_1 n_{\{\tau_1,u_1\}} + \theta_1 n_{\{\tau_1,\tau_3,u_1,u_2\}} + n_{\{\tau_3,u_2\}}$
$\{\tau_3,u_3\}$	$\theta_1^2 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1^2 n_{\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}} + \theta_1 n_{\{\tau_1,u_1\}} + \theta_1 n_{\{\tau_2,u_2\}} + \theta_1 n_{\{\tau_1,\tau_3,u_1,u_3\}} + \theta_1 n_{\{\tau_2,\tau_3,u_2,u_3\}} + n_{\{\tau_3,u_3\}}$
$\{u_1,u_2\}$	$\theta_1 n_{\{\tau_3,u_3\}} + \theta_1 n_{\{\tau_3,u_1,u_2,u_3\}} + n_{\{u_1,u_2\}}$
$\{u_1,u_3\}$	$\theta_1 n_{\{\tau_2,u_2\}} + \theta_1 n_{\{\tau_2,u_1,u_2,u_3\}} + n_{\{u_1,u_3\}}$
$\{u_2,u_3\}$	$\theta_1 n_{\{\tau_1,u_1\}} + \theta_1 n_{\{\tau_1,u_1,u_2,u_3\}} + n_{\{u_2,u_3\}}$
$\{\tau_1,\tau_2,\tau_3,u_1\}$	$n_{\{\tau_1,\tau_2,\tau_3,u_1\}}$
$\{\tau_1,\tau_2,\tau_3,u_2\}$	$n_{\{\tau_1,\tau_2,\tau_3,u_2\}}$
$\{\tau_1,\tau_2,\tau_3,u_3\}$	$n_{\{\tau_1,\tau_2,\tau_3,u_3\}}$
$\{\tau_1,\tau_2,u_1,u_2\}$	$\theta_1 n_{\{\tau_3,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_3\}} + \theta_1 n_{\{\tau_1,\tau_3,u_1,u_3\}} + \theta_1 n_{\{\tau_1,\tau_3,u_2,u_3\}} + \theta_1 n_{\{\tau_2,\tau_3,u_1,u_3\}} + \theta_1 n_{\{\tau_2,\tau_3,u_2,u_3\}} + \theta_1 n_{\{\tau_3,u_1,u_2,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}} + n_{\{\tau_1,\tau_2,u_1,u_2\}}$
$\{\tau_1,\tau_2,u_1,u_3\}$	$n_{\{\tau_1,\tau_2,u_1,u_3\}}$
$\{\tau_1,\tau_2,u_2,u_3\}$	$n_{\{\tau_1,\tau_2,u_2,u_3\}}$
$\{\tau_1,\tau_3,u_1,u_2\}$	$n_{\{\tau_1,\tau_3,u_1,u_2\}}$
$\{\tau_1,\tau_3,u_1,u_3\}$	$\theta_1 n_{\{\tau_2,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,u_2,u_3\}} + \theta_1 n_{\{\tau_2,\tau_3,u_1,u_2\}} + \theta_1 n_{\{\tau_2,\tau_3,u_2,u_3\}} + \theta_1 n_{\{\tau_2,u_1,u_2,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}} + n_{\{\tau_1,\tau_3,u_1,u_3\}}$
$\{\tau_1,\tau_3,u_2,u_3\}$	$n_{\{\tau_1,\tau_3,u_2,u_3\}}$
$\{\tau_1,u_1,u_2,u_3\}$	$n_{\{\tau_1,u_1,u_2,u_3\}}$
$\{\tau_2,\tau_3,u_1,u_2\}$	$n_{\{\tau_2,\tau_3,u_1,u_2\}}$
$\{\tau_2,\tau_3,u_1,u_3\}$	$n_{\{\tau_2,\tau_3,u_1,u_3\}}$
$\{\tau_2,\tau_3,u_2,u_3\}$	$\theta_1 n_{\{\tau_1,u_1\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_1\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_2\}} + \theta_1 n_{\{\tau_1,\tau_2,u_1,u_3\}} + \theta_1 n_{\{\tau_1,\tau_3,u_1,u_2\}} + \theta_1 n_{\{\tau_1,\tau_3,u_1,u_3\}} + \theta_1 n_{\{\tau_1,u_1,u_2,u_3\}} + \theta_1 n_{\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}} + n_{\{\tau_2,\tau_3,u_2,u_3\}}$
$\{\tau_2,u_1,u_2,u_3\}$	$n_{\{\tau_2,u_1,u_2,u_3\}}$
$\{\tau_3,u_1,u_2,u_3\}$	$n_{\{\tau_3,u_1,u_2,u_3\}}$
$\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}$	$n_{\{\tau_1,\tau_2,\tau_3,u_1,u_2,u_3\}}$

TABLE V. Chern number expansion for the 3-$\mathbf{q}$ cubic hedgehog lattice in three dimensions with $\theta = \theta_1 \mathrm{id}_3$.