DIFFERENTIATION

Derivative

Definition

The derivative quantifies the sensitivity to change of a function's output with respect to its input.

Notations

- For a given function we denote the derivative of a function $f(x)$ to be $f'(x)$, which is referred to as prime notation.
- For a given function f, the derivative f' is often denoted by:

$$\frac{d}{dx}f(x) \quad \text{or} \quad \frac{df(x)}{dx}$$

  This notation is called Leibniz notation.

Differentiability

Definition

We say that $f : D \to \mathbb{R}$ is differentiable at x if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. We call $f'(x)$ the derivative of $f(x)$. If $f'(x)$ exists for all $x \in D$ then we say f is differentiable.

Differentiation and Continuity

Theorem

If $f : D \to \mathbb{R}$ is differentiable at $a \in D$ then f is continuous at a.

Proof

We want to show that $\lim_{x\to a} f(x) = f(a)$. But if $x = a + h$ then this is the same as $\lim_{h\to 0} f(a+h) = f(a)$. We will show that this is true by proving $\lim_{h\to 0}(f(a+h) - f(a)) = 0$.

$$\begin{aligned}\lim_{h\to 0}(f(a+h) - f(a)) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot h \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h\to 0} h \\ &= f'(a) \cdot 0 \\ &= 0\end{aligned}$$

□

Remarks

- Note that this proof only makes sense when $f'(a)$ exists, that is, when f is differentiable.
- The converse to this theorem is not true.

Absolute value function

Theorem

The function $f(x) = |x|$ is continuous, but not differentiable.

Proof

At the point $a = 0$ we have

$$\lim_{x\to 0^-} |x| = 0 = \lim_{x\to 0^+} |x| \quad \text{so} \quad \lim_{x\to 0} f(x) = 0 = f(0)$$

So the function is continuous.

We will now show that $f(x)$ is not differentiable. Unsurprisingly, the interesting thing happens when $x = 0$.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} -1 & \text{if } h < 0 \\ 1 & \text{if } h > 0 \end{cases}$$

The expression $\frac{|h|}{h}\Big|$ is not defined when $h = 0$ (but we don't care about this as h never equals zero in our limit). So

$$\lim_{h \to 0^-} \frac{|h|}{h} = -1 \qquad \text{and} \qquad \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

These limits are not equal, so $\lim_{h \to 0} \frac{|h|}{h}$ does not exist. So $f(x) = |x|$ is not differentiable when $x = 0$.

$\square$

Higher derivatives

Definition

Deriving a function n times is called taking the n-th derivative.

Notation

As for the prime notation we denote higher derivatives:

$$f'''^{\cdots}(x) \quad \text{or} \quad f^{(n)}(x)$$

As for the Leibnizian notation we write:

$$\frac{d^n f(x)}{dx^n}$$

Remark

For some functions it may not be possible or useless to take the derivative more than a few times.

Differentiation of common functions

See the proof of many common functions under démonstration des fonctions dérivées.

Sum rule of differentiation

Theorem

If f and g are differentiable at x then

$$(f+g)'(x) = f'(x) + g'(x)$$

Proof

We want to show that $(f+g)'(x)$ exists and is equal to $f'(x) + g'(x)$. Recall that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

so

$$\begin{aligned}
(f+g)'(x) &= \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}
\end{aligned}$$

$$= f'(x) + g'(x)$$

$\square$

Product rule of differentiation

Theorem

If f and g are differentiable at x then

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$$

Proof

As for the sum rule, we want to show that $(fg)'(x)$ exists and is equal to $f(x)g'(x) + f'(x)g(x)$. Recall that

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x) \\
&= f(x)g'(x) + f'(x)g(x)
\end{aligned}$$

$\square$

Chain rule of differentiation

Theorem

If f and g are differentiable then

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Proof

$$
\begin{aligned}
(f \circ g)'(x) &= \lim_{h \to 0} \frac{(f \circ g)(x+h) - (f \circ g)(x)}{h} \\
&= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\
&= \left(\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right) \left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right) \\
&= f'(g(x)) g'(x)
\end{aligned}
$$

To justify this, we define $H = g(x+h) - g(x)$, and equivalently $g(x+h) = g(x) + H$. Note that

$$
\lim_{h \to 0} H = \lim_{h \to 0} g(x+h) - g(x) = g(x) - g(x) = 0
$$

By making the substitutions we have that

$$
\begin{aligned}
\left(\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right) g'(x) &= \left(\lim_{H \to 0} \frac{f(g(x) + H) - f(g(x))}{H} \right) g'(x) \\
&= f'(g(x)) g'(x)
\end{aligned}
$$

$\square$

Quotient rule of differentiation

Theorem

If f and g are differentiable at x, then

$$
\left(\frac{f}{g} \right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}
$$

Proof

Since $f/g = f \cdot (1/g)$ we have

$$\left(\frac{f}{g}\right)'(a) = \left(f \cdot \frac{1}{g}\right)'(a)$$

$$= f'(a) \cdot \left(\frac{1}{g}\right)(a) + f(a) \cdot \left(\frac{1}{g}\right)'(a)$$

$$= \frac{f'(a)}{g(a)} + \frac{f(a)(-g'(a))}{[g(a)]^2}$$

$$= \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{[g(a)]^2}$$

□

Inverse functions

Definition

We say that $g : D_g \to \mathbb{R}$ is the inverse function of $f : D_f \to \mathbb{R}$ if for all $x \in D_g$

$$(f \circ g)(x) = x = (g \circ f)(x)$$

and if for all $x \in D_f$ we have

$$(g \circ f)(x) = x$$

Remark

Note that in certain cases we might be interested in functions that are not defined over all of $\mathbb{R}$.

Differentiation of inverse functions

Proposition

If f^{-1} is differentiable at y, then

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Proof

By the rule of composing functions, the derivative of $f^{-1}(f(x))$ is

$$f'(x)(f^{-1})'(f(x))$$

but because $f^{-1}(f(x)) = x$ this is also equal to 1. Hence

$$f'(x)(f^{-1})'(f(x)) = 1$$

and so

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Now setting $f(x) = y$ we have $x = f^{-1}(y)$ and so

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

□