

# INVERTIBILITY AND ISOMORPHIC VECTOR SPACES

## Invertible

### *Definition*

A linear map  $T \in \mathcal{L}(V; W)$  is called invertible if there exists a linear map  $S \in \mathcal{L}(W; V)$  such that  $ST$  equals the identity map on  $V$  and  $TS$  equals the identity map on  $W$ .

## Inverse

### *Definition*

A linear map  $S \in \mathcal{L}(W; V)$  satisfying  $ST = I$  and  $TS = I$  is called an inverse of  $T$ .

### *Remark*

Note that the first  $I$  is the identity map on  $V$  and the second  $I$  is the identity map on  $W$

## Inverse is unique

### *Definition*

An invertible linear map has a unique inverse.

### *Proof*

Suppose  $S_1$  and  $S_2$  are both inverses of  $T$ . Then

$$S_1 = S_1I = S_1(TS_2) = (S_1T)S_2 = IS_2 = S_2$$

□

### ***Notation***

If  $T$  is invertible, then its inverse is denoted by  $T^{-1}$ . In other words, if  $T \in \mathcal{L}(V, W)$  is invertible, then  $T^{-1}$  is the unique element of  $\mathcal{L}(W, V)$  such that  $T^{-1}T = I$  and  $TT^{-1} = I$ .

### **Invertibility is equivalent to injectivity and surjectivity**

#### ***Theorem***

A linear map is invertible if and only if it is injective and surjective.

#### ***Proof***

Suppose  $T \in \mathcal{L}(V, W)$ . We need to show that  $T$  is invertible if and only if it is injective and surjective.

First suppose  $T$  is invertible. To show that  $T$  is injective, suppose  $u, v \in V$  and  $Tu = Tv$ . Then

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

so  $u = v$ . Hence  $T$  is injective.

We are still assuming that  $T$  is invertible. Now we want to prove that  $T$  is surjective. To do this, let  $w \in W$ . Then  $w = T(T^{-1}w)$ , which shows that  $w$  is in the range of  $T$ . Thus range  $T = W$ . Hence  $T$  is surjective, completing this direction of the proof.

Now suppose  $T$  is injective and surjective. We want to prove that  $T$  is invertible. For each  $w \in W$ , define  $Sw$  to be the unique element of  $V$  such that  $T(Sw) = w$  (the existence and uniqueness of such an element follow from the

surjectivity and injectivity of  $T$ ). Clearly  $T \circ S$  equals the identity map on  $W$ .

To prove that  $S \circ T$  equals the identity map on  $V$ , let  $v \in V$ . Then

$$T((S \circ T)v) = (T \circ S)(Tv) = I(Tv) = Tv$$

This equation implies that  $(S \circ T)v = v$  (because  $T$  is injective). Thus  $S \circ T$  equals the identity map on  $V$ .

To complete the proof, we need to show that  $S$  is linear. To do this, suppose  $w_1, w_2 \in W$ . Then

$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2$$

Thus  $Sw_1 + Sw_2$  is the unique element of  $V$  that  $T$  maps to  $w_1 + w_2$ . By the definition of  $S$ , this implies that

$$S(w_1 + w_2) = Sw_1 + Sw_2$$

Hence  $S$  satisfies the additive property required for linearity.

The proof of homogeneity is similar. Specifically, if  $w \in W$  and  $\lambda \in \mathbb{F}$ , then

$$T(\lambda Sw) = \lambda T(Sw) = \lambda w$$

Thus  $\lambda Sw$  is the unique element of  $V$  that  $T$  maps to  $\lambda w$ . By the definition of  $S$ , this implies that

$$S(\lambda w) = \lambda Sw$$

Hence  $S$  is linear, as desired.

□

## Isomorphisms

### ***Definition***

An isomorphism is an invertible linear map.

# Isomorphic

## ***Definition***

Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

## **Dimension shows whether vector spaces are isomorphic**

### ***Theorem***

Two finite-dimensional vector spaces over  $\mathbb{F}$  are isomorphic if and only if they have the same dimension.

### ***Proof***

( $\Rightarrow$ ) Suppose  $V \cong W$ , i.e., there exists a linear isomorphism  $T : V \rightarrow W$ . Since  $T$  is bijective, it maps a basis of  $V$  to a basis of  $W$ . Hence the number of basis vectors is the same:

$$\dim V = \dim W$$

( $\Leftarrow$ ) Suppose  $\dim V = \dim W = n$ . Let  $\{v_1, \dots, v_n\}$  be a basis of  $V$  and  $\{w_1, \dots, w_n\}$  be a basis of  $W$ . Define a linear map  $T : V \rightarrow W$  by

$$T\left(\sum_{i=1}^n \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i w_i$$

for scalars  $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ .

This map is well-defined and linear. Moreover,  $T$  is injective because the kernel is  $\{0\}$  (the linear combination  $\sum \alpha_i v_i = 0$  implies all  $\alpha_i = 0$ ). It is surjective because any  $w \in W$  can be written as a linear combination of the basis  $\{w_i\}$ . Hence  $T$  is a bijective linear map, and  $V \cong W$ .

□

## Matrix Representation Isomorphism

### *Theorem*

Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Then  $M$  is an isomorphism between  $\mathcal{L}(V, W)$  and  $\mathbb{F}^{m,n}$ .

## Dimension of $\mathcal{L}(V, W)$

### *Theorem*

Suppose  $V$  and  $W$  are finite-dimensional. Then  $\mathcal{L}(V, W)$  is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V) \cdot (\dim W)$$

## Matrix of a vector

### *Definition*

Suppose  $v \in V$  and  $v_1, \dots, v_n$  is a basis of  $V$ . The matrix of  $v$  with respect to this basis is the  $n$ -by-1 matrix

$$M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where  $c_1, \dots, c_n$  are the scalars such that

$$v = c_1 v_1 + \cdots + c_n v_n$$

## Column of a Linear Map's Matrix

### *Theorem*

Suppose  $T \in \mathcal{L}(V, W)$  and  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Let  $1 \leq k \leq n$ . Then the  $k^{\text{th}}$  column of  $M(T)$ , which is denoted by  $M(T)_{\cdot, k}$ , equals  $M(v_k)$ .

### ***Proof***

The desired result follows immediately from the definitions of  $M(T)$  and  $M(v_k)$ .

The next result shows how the notions of the matrix of a linear map, the matrix of a vector, and matrix multiplication fit together.

□

## **Linear maps act like matrix multiplication**

### ***Theorem***

Suppose  $T \in \mathcal{L}(V, W)$  and  $v \in V$ . Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Then

$$M(Tv) = M(T)M(v)$$

## **Operator**

### ***Definition***

- A linear map from a vector space to itself is called an operator.
- The notation  $\mathcal{L}(V)$  denotes the set of all operators on  $V$ . In other words,  

$$\mathcal{L}(V) = \mathcal{L}(V, V).$$

## **Injectivity is equivalent to surjectivity in finite dimensions**

### ***Theorem***

Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Then the following are equivalent:

1.  $T$  is invertible
2.  $T$  is injective
3.  $T$  is surjective