NULL SPACES AND RANGES

Null Space and Injectivity

Definition

For $T \in \mathcal{L}(V; W)$, the null space of T, denoted $\operatorname{null} T$, is the subset of V consisting of those vectors that T maps to 0:

$$\operatorname{null} T = \{v \in V \mid T(v) = 0\}$$

Null space is a subspace

Theorem

The null space of a linear transformation is a subspace of the domain.

Proof

Let $T \in \mathcal{L}(V; W)$ and let $v_1, v_2 \in \operatorname{null} T$. Then $T(v_1) = 0$ and $T(v_2) = 0$. We need to show that $v_1 + v_2 \in \operatorname{null} T$ and $\alpha v_1 \in \operatorname{null} T$ for all $\alpha \in \mathbb{F}$.

First, consider $T(v_1 + v_2)$:

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0 + 0 = 0$$

Thus, $v_1 + v_2 \in \operatorname{null} T$.

Now, consider $T(\alpha v_1)$:

$$T(\alpha v_1) = \alpha T(v_1) = \alpha \cdot 0 = 0$$

Thus, $\alpha v_1 \in \operatorname{null} T$.

Since $\operatorname{null} T$ is closed under addition and scalar multiplication, it is a subspace of V.

$\square$

Injective

Definition

A function $T : V \to W$ is called injective if $T(u) = T(v)$ implies $u = v$.

Theorem

Let $T \in \mathcal{L}(V; W)$. Then T is injective if and only if $\operatorname{null} T = \{0\}$.

Proof

Suppose T is injective. Then $T(v) = 0$ implies $v = 0$, so $\operatorname{null} T \subseteq \{0\}$. Conversely, suppose $\operatorname{null} T = \{0\}$. Then $T(v) = T(w)$ implies $T(v) - T(w) = 0$, so $T(v - w) = 0$, and thus $v - w = 0$ or $v = w$. Therefore, T is injective.

$\square$

Range

Definition

For T a function from V to W, the range of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv \mid v \in V\}$$

Theorem

If $T \in \mathcal{L}(V; W)$, then $\operatorname{range} T$ is a subspace of W.

Proof

Let $T \in \mathcal{L}(V; W)$ and let $w_1, w_2 \in \operatorname{range} T$. Then there exist $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. We need to show that $w_1 + w_2 \in \operatorname{range} T$

and $\alpha w_1 \in \operatorname{range} T$ for all $\alpha \in \mathbb{F}$.

First, consider $T(v_1 + v_2)$:

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$$

Thus, $w_1 + w_2 \in \operatorname{range} T$.

Now, consider $T(\alpha v_1)$:

$$T(\alpha v_1) = \alpha T(v_1) = \alpha w_1$$

Thus, $\alpha w_1 \in \operatorname{range} T$.

Since $\operatorname{range} T$ is closed under addition and scalar multiplication, it is a subspace of W.

$\square$

Surjective

Definition

A function $T : V \to W$ is called surjective if for every $w \in W$, there exists a $v \in V$ such that $T(v) = w$ or in other words if its range equals W.

Fundamental Theorem of Linear Maps

Theorem

Suppose V is finite-dimensional and $T \in \mathcal{L}(V; W)$. Then $\operatorname{range} T$ is finite-dimensional and

$$\dim(\operatorname{range} T) = \dim(V) - \dim(\operatorname{null} T)$$

Proof

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for V. Then $T(e_1), T(e_2), \ldots, T(e_n)$ span range T.

The dimension of the range is given by the number of linearly independent vectors in $\{T(e_1), T(e_2), \ldots, T(e_n)\}$. By the rank-nullity theorem, we have:

$$\dim(\text{range } T) + \dim(\text{null } T) = \dim(V)$$

Rearranging gives:

$$\dim(\text{range } T) = \dim(V) - \dim(\text{null } T)$$

$\square$

Maps and dimensional space

Theorem

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

Proof

Suppose $T : V \to W$ is a linear map. Since $\dim V > \dim W$, by the rank-nullity theorem, we have:

$$\dim(\text{range } T) + \dim(\text{null } T) = \dim(V)$$

Since $\dim V > \dim W$, it follows that $\dim(\text{range } T) < \dim W$. Therefore, the kernel of T must be non-trivial, implying that T is not injective.

$\square$

Theorem

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Proof

Suppose $T : V \to W$ is a linear map. Since $\dim V < \dim W$, by the rank-nullity theorem, we have:

$$\dim(\text{range}\, T) + \dim(\text{null}\, T) = \dim(V)$$

Since $\dim V < \dim W$, it follows that $\dim(\text{range}\, T) < \dim W$. Therefore, the kernel of T must be non-trivial, implying that T is not injective.

$\square$

Homogeneous system of linear equations

Definition

A homogeneous system of linear equations is a system of equations of the form $Ax = 0$, where A is a matrix and x is a vector of variables. The system is called homogeneous because all of the constant terms are zero.

Theorem

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proof

Consider the homogeneous system of linear equations represented by the matrix equation $Ax = 0$, where A is an $m \times n$ matrix with $m < n$. The null space of A is defined as:

$$\text{null}\, A = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

Since there are more variables than equations, the rank-nullity theorem implies:

$$\dim(\text{null } A) = n - \dim(\text{range } A)$$

Given that $m < n$, it follows that $\dim(\text{range } A) < n$, which implies $\dim(\text{null } A) > 0$. Therefore, the null space contains nonzero vectors, indicating the existence of nontrivial solutions to the homogeneous system.

$\square$

Inhomogeneous system of linear equations

Definition

An inhomogeneous system of linear equations is a system of equations of the form $Ax = b$, where A is a matrix, x is a vector of variables, and b is a nonzero vector. The system is called inhomogeneous because the constant terms are not all zero.

Theorem

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof

Consider the inhomogeneous system of linear equations represented by the matrix equation $Ax = b$, where A is an $m \times n$ matrix with $m > n$. The null space of A is defined as:

$$\text{null } A = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

Since there are more equations than variables, the rank-nullity theorem implies:

$$\dim(\text{null } A) = n - \dim(\text{range } A)$$

Given that $m > n$, it follows that $\dim(\text{range } A) < n$, which implies $\dim(\text{null } A) > 0$. Therefore, the null space contains nonzero vectors, indicating the existence of nontrivial solutions to the homogeneous system.

$\square$