EIGENVECTORS AND EIGENVALUES

Eigenvector and Eigenvalue

Definition

Given an $(n \cdot m)$ matrix A and an n-vector $\overrightarrow{r}$, if $\overrightarrow{r'} = Ar$ points in the same direction as $\overrightarrow{r}$, i.e. $\overrightarrow{r'} = \lambda\overrightarrow{r}$ where λ is a real scalar, then r is called an eigenvector of A with real eigenvalue.

The cases $r = 0$ or $\lambda = 0$ are excluded from this definition.

Trivial Eigenvalues

Theorem

If the matrix equation $A\overrightarrow{v} = \lambda\overrightarrow{v}$, is inversible we have the trivial solutions:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

and

$$\overrightarrow{v} = \overrightarrow{0}$$

Proof

$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

$$\Leftrightarrow (A - \lambda\mathbb{1})\overrightarrow{v} = \overrightarrow{0}$$

If A is reversible:

$$\Leftrightarrow A = \lambda\mathbb{1} \vee \overrightarrow{v} = (A - \lambda\mathbb{1})^{-1}\overrightarrow{0}$$

$$\Leftrightarrow A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vee \overrightarrow{v} = \overrightarrow{0}$$

$\square$

Characteristic equation

When solving for non trivial eigenvalues we call the following equation the characteristic equation:

$$\det(A - \lambda \mathbb{1}) = 0$$

Characteristic polynomial

We call $A - \lambda\mathbb{1}$, the characteristic polynomial of A, when solving for eigenvalues.

Non-trivial Eigenvalues

Theorem 1

If $(A - \lambda\mathbb{1})$ does not have an inverse, $\overrightarrow{v} \neq 0$.

Theorem 2

For the real scalar λ to be an eigenvalue of the matrix A it must be a real root of the characteristic equation:

$$\det(A - \lambda \mathbb{1}) = 0$$

Remark

This is a polynomial equation of degree n if A is an $(n \cdot n)$ matrix.

Multiple of eigenvector solutions

Theorem

For any solution of eigenvectors $\overrightarrow{v}$ there is a multiple of the eigenvector $\alpha \in \mathbb{R}$ that still satisfies the equation:

$$A\alpha\overrightarrow{v} = \lambda\alpha\overrightarrow{v}$$