SEQUENCES

Upper bound

Definition

$B \in \mathbb{R}$ is an upper bound for $\{a_n\}_{n=n_0}^{\infty}$ if $a_n \leq B$ for all $n \in \mathbb{R}$.

Lower bound

Definition

$B \in \mathbb{R}$ is an upper bound for $\{a_n\}_{n=n_0}^{\infty}$ if $a_n \geq B$ for all $n \in \mathbb{R}$.

Supremum, least upper bound

Definition

$S \in \mathbb{R}$ is the supremum, for $\{a_n\}_{n=n_0}^{\infty}$ if $S \leq B$ for all upper bounds B.

$$S = \min\{B \in \mathbb{R} | a_n \leq B, \forall n \in \mathbb{N}\}$$

Infimum, greatest lower bound

Definition

$I \in \mathbb{R}$ is the infimum, for $\{a_n\}_{n=n_0}^{\infty}$ if $I \geq B$ for all lower bounds B.

$$I = \max\{B \in \mathbb{R} | a_n \geq B, \forall n \in \mathbb{N}\}$$

Convergent sequences

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Convergent sequences are bounded.

Proof

Let $(s_n)_{n \in \mathbb{N}}$ be a converging sequence, and let $s = \lim_{x \to \infty} s_n$.

Let us take $\varepsilon = 1$ and apply the definition of a limit. We know that there exists

N such that $n > N \Rightarrow |s_n - s| < 1$. Now by the triangle inequality $n > N \Rightarrow |s_n| < |s| + 1$

$$|s_n| = |s_n - s + s| \leq |s_n| + |s| \leq 1 + |s|$$

We can now define: $M := \max\{|s| + 1, |s_1|, \cdots, |s_N|\}$

This means $|s_n| \leq M$ for all $n \in \mathbb{N}$ and so (s_n) is a bounded sequence.

$\square$

Monotonic sequences

- A sequence $\{a_n\}_{n=n_0}^{\infty}$ is called monotone increasing if:

$$a_{n+1} \geq a_n \quad \forall n \geq n_0$$

- A sequence $\{a_n\}_{n=n_0}^{\infty}$ is called monotone decreasing if:

$$a_{n+1} \leq a_n \quad \forall n \geq n_0$$

Monotone Convergence Theorem

- If $\{a_n\}_{n=n_0}^{\infty}$ is a monotone increasing sequence, with supremum S, then

$$a_n \to S$$

- If $\{a_n\}_{n=n_0}^{\infty}$ is a monotone decreasing sequence, with infimum I, then

$$a_n \to I$$

Proof

Since S is the supremum we have that $|a_n - S| = S - a_n$ for all $n \in \mathbb{N}$. Now let $\epsilon > 0$ and assume that $S - a_n \geq \epsilon$ for all $n \in \mathbb{N}$. Then $a_n \leq S - \epsilon$ for all $n \in \mathbb{N}$, that is, $S - \epsilon$ is an upper bound. This contradicts the fact that S is the supremum, so there must be some $N \in \mathbb{N}$ such that $S - a_N < \epsilon$.

Since a_n is monotone increasing $a_n \geq a_N$ for all $n \geq N$. So

$$|a_n - S| = S - a_n \leq S - a_N < \epsilon$$

for all $n \geq N$.

Arithmetic mean - geometric mean inequality

Arithmetic mean - geometric mean inequality theorem

Let $x_1, \ldots, x_n$ satisfy $x_i \geq 0$ for $i = 1, \ldots, n$. Then their geometric mean is at most their arithmetic mean:

$$\sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

Proof

We will prove it by induction over $n \geq 1$.

- For $n = 1$: we get an equality

$$x = \sqrt{x} = \frac{x}{1} = x$$

- Suppose its true for any n-tuple of non-negative numbers. Consider an $(n+1)$-tuple $x_1, \ldots, x_{n+1}$ and let α be their arithmetic mean, that is:

$$\alpha = \frac{x_1 + x_2 + \ldots + x_{n+1}}{n+1}$$

  We now have

$$(n+1)\alpha = x_1 + \cdots + x_{n+1} = \sum_{k=1}^{n+1} x_k$$

Examining different cases:

1. One of the numbers is 0, that is there exists some i such that $x_i = 0$. The inequality here is obvious as one side is 0. And for the other side to also be 0 we need that $x_k = 0$ for $k = 1, \ldots, n+1$, so equality only occurs in the case when all terms are equal.

2. When we have $x_i = \alpha$, $i = 1, \ldots, n+1$, then we get equality by a simple computation:

$$\sum_{k=1}^{n+1} x_k = (n+1)\alpha$$

so

$$\frac{\sum_{k=1}^{n+1} x_k}{n+1} = \alpha$$

and

$$\sqrt[n+1]{x_1 \cdot \ldots \cdot x_{n+1}} = \sqrt[n+1]{\alpha^{n+1}} = \alpha$$

3. Everything else! That is $x_k > 0$ for $k = 1, \ldots, n+1$ and not all the terms are equal. In this case there is an x_i strictly greater than α and one that is strictly smaller (otherwise $\sum_{k=1}^{n+1} x_k \neq (n+1)\alpha$). Up to reordering, let's suppose that these numbers are x_n and x_{n+1}, that is: $x_n > \alpha$ and $x_{n+1} < \alpha$. In particular, $x_n - \alpha > 0$ and $\alpha - x_{n+1} > 0$ so we have

$$(x_n - \alpha)(\alpha - x_{n+1}) > 0. \qquad (\star)$$

We define a new real number y as

$$y = x_n + x_{n+1} - \alpha \geq x_n - \alpha > 0$$

now

$$(n+1)\alpha = x_1 + \cdots + \underbrace{x_n + x_{n+1}}_{y+\alpha} = x_1 + \cdots + x_{n-1} + y + \alpha.$$

so

$$n \cdot \alpha = x_1 + \cdots + x_{n-1} + y$$

And thus α is also the arithmetic mean of $x_1, \ldots, x_{n-1}, y$.

By the induction hypothesis we have

$$\underbrace{\frac{x_1 + \cdots + x_{n-1} + y}{n}}_{\alpha} \geq \sqrt[n]{x_1 \ldots x_{n-1} y}$$

and so

$$\alpha^n \geq x_1 \ldots x_{n-1} y$$

now

$$\alpha^{n+1} = \alpha^n \cdot \alpha \geq x_1 \ldots x_{n-1} y \alpha$$

But $y = x_n + x_{n+1} - \alpha$ and so

$$\begin{aligned} y \cdot \alpha - x_n \cdot x_{n+1} &= (x_n + x_{n+1} - \alpha)\alpha - x_n x_{n+1} \\ &= x_n \alpha + x_{n+1}\alpha - \alpha^2 - x_n x_{n+1} \\ &= (\alpha - x_{n+1})(x_n - \alpha) > 0 \end{aligned}$$

by the inequality $(\star)$ above.

We thus have that $y \cdot \alpha > x_n x_{n+1}$.

In conclusion: $\alpha^{n+1} > x_1 \ldots x_{n-1} \cdot x_n \cdot x_{n+1}$, that is

$$\left(\frac{x_1 + \cdots + x_{n+1}}{n+1}\right)^{n+1} > x_1 \cdot \ldots \cdot x_{n+1}$$

as wanted. And that proves the theorem!

$\square$

Lemma

The sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=2}^{\infty}$ have the following monotonicity properties:

1. $a_{n+1} > a_n$ for $n \geq 1$
2. $b_{n+1} < b_n$ for $n \geq 2$

Proofs

1. Let $x_1 = 1, x_2 = \ldots = x_k = \ldots = x_{n+1} = 1 + \frac{1}{n}$. We apply the AM-GM inequality to these numbers. Their arithmetic mean satisfies:

$$\frac{1 + \left(1 + \frac{1}{n}\right) + \cdots + \left(1 + \frac{1}{n}\right)}{n+1} = \frac{n + 1 + n \cdot \frac{1}{n}}{n+1} = 1 + \frac{1}{n+1} \quad (\star\star)$$

Their geometric mean is

$$\sqrt[n+1]{x_1 \ldots x_{n+1}} = \sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n} \quad (\star\star\star)$$

The x_is are not all equal, so we know that $(\star\star) > (\star\star\star)$. From this:

$$1 + \frac{1}{n+1} > \sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n}$$

And so:

$$a_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n = a_n$$

2. This case is very similar to what we just did. We set $y_1 = 1$ and $y_2 = y_3 = \cdots = y_{n+1} = 1 - \frac{1}{n} > 0$. Then:

$$\begin{aligned}\frac{\sum_{i=1}^{n+1} y_i}{n+1} &= \frac{1+n\left(1-\frac{1}{n}\right)}{n+1} \\ &= \frac{1+n-1}{n+1} \\ &= 1-\frac{1}{n+1} \\ &> \sqrt[n+1]{\left(1-\frac{1}{n}\right)^n}\end{aligned}$$

by the arithmetic mean - geometric mean inequality applied to $y_1, \ldots, y_{n+1}$. We deduce

$$\left(1-\frac{1}{n+1}\right)^{n+1} > \left(1-\frac{1}{n}\right)^n$$

By taking the inverse on both sides, we reverse the inequality and we get:

$$b_{n+1} = \left(1-\frac{1}{n+1}\right)^{-(n+1)} < \left(1-\frac{1}{n}\right)^{-n} = b_n$$

as required.

□

Lemma

For $n \geq 2$:

$$0 < b_n - a_n < \frac{4}{n}$$

Proof

We already know that $b_n - a_n > 0$. Let's check the other inequality:

$$b_n - a_n = b_n\left(1-\frac{a_n}{b_n}\right)$$

We have:

$$a_n = \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$$

and

$$b_n = \left(1 - \frac{1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^n$$

so

$$\begin{aligned} b_n - a_n &= b_n \left(1 - \left(\frac{n+1}{n}\right)^n \left(\frac{n-1}{n}\right)^n\right) \\ &= b_n \left(1 - \left(\frac{n^2-1}{n^2}\right)^n\right) \end{aligned}$$

Set $q = \frac{n^2-1}{n^2} < 1$. We have:

$$b_n - a_n = b_n(1 - q^n)$$

and so

$$1 - q^n = (1-q)(1 + q + q^2 + \cdots + q^{n-1})$$

As $q < 1$, we also have that $q^k < 1$ and we can deduce that

$$1 - q^n < (1-q) \cdot n$$

We've thus shown that

$$b_n - a_n < b_n(1-q) \cdot n$$

Now as

$$1 - q = \left(1 - \frac{n^2-1}{n^2}\right) = \frac{n^2 - (n^2-1)}{n^2} = \frac{1}{n^2}$$

and b_n is decreasing so $b_n \leq b_2 = 4$. We can conclude:

$$b_n - a_n < 4 \cdot \frac{1}{n^2} \cdot n = \frac{4}{n}$$

$\square$

Squeeze theorem, Sandwhich theorem

Squeeze Theorem

If $(x_n)_{n=n_0}^{\infty}$, $(y_n)_{n=n_0}^{\infty}$ and $(z_n)_{n=n_0}^{\infty}$ are sequences that satisfy

$$x_n \le y_n \le z_n$$

and if

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$$

for $\ell \in \mathbb{R}$ then

$$\lim_{n\to\infty} y_n = \ell.$$

Euler's number

Definition

We define the real number e as the limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{-n} = e$$

Approximating factorials

Theorem

For $n \ge 2$ we have

$$\frac{n^n}{e^{n-1}} < n! < n^n$$

Proof

We have

$$n! = n(n-1)(n-2)\dots 2\cdot 1 \quad \text{so} \quad n! < \underbrace{n\cdot n\dots n}_{n \text{ times}} = n^n$$

Let us establish the second inequality. As e is the limit of an increasing sequence $a_k = \left(1+\frac{1}{k}\right)^k$, we have that, for all $k \geq 1$, $e > a_k = \left(1+\frac{1}{k}\right)^k$. Thus

$$e^{n-1} > \prod_{k=1}^{n-1} a_k (= a_1 \cdot \dots \cdot a_{n-1})$$

and from this

$$e^{n-1} > a_1 \cdot a_2 \cdots a_{n-1} = \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n}{n-1}\right)^{n-1}$$

Many terms simplify through telescoping:

$$a_1 \cdot a_2 \cdots a_{n-1} = \frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{(n-1)!} \cdot \frac{n}{n} = \frac{n^n}{n!}$$

Therefore

$$e^{n-1} > \frac{n^n}{n!} \quad \text{and} \quad n! > \frac{n^n}{e^{n-1}}$$

This completes the proof.

$\square$

Stirlings formula theorem

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n^n}{e^n}\right)\sqrt{2\pi n}} = 1$$

Proof

We will use the result of the previous theorem. We have

$$n! \sim \left(\frac{n^n}{e^n}\right)\sqrt{2\pi n}$$

Thus

$$\frac{n!}{\left(\frac{n^n}{e^n}\right)\sqrt{2\pi n}} \to 1$$

as $n \to \infty$.

$\square$