AXIOM OF COMPLETENESS

Axiom of completeness

Every nonempty set of real numbers that is bounded above has a least upper bound.

Upper bound

Definition

A set $A \subseteq \mathbb{R}$ is bounded above if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an upper bound for A.

Lower bound

Definition

The set A is bounded below if there exists a lower bound $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Supremum, Least upper bound

Definition

A real number s is the least upper bound for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

- s is an upper bound for A
- if b is any upper bound for A, then $s \leq b$

Notation

The Supremum s of a subset A is written as:

$$s = \sup(A)$$

Remark

A less common notation is: $s = \text{lub}\,(A)$

Meaning *least upper bound*.

Corollary

A set can have a lot of upper bounds but it can only have at least one least upper bound.

Proof

If s_1 and s_2 are both least upper bounds for a set A, then we can assert $s_1 \leq s_2$ and $s_2 \leq s_1$. The conclusion is that $s_1 = s_2$ and least upper bounds are unique.

Minimum and Maximum

Definition

A real number a_0 is a maximum of the set A if a_0 is an element of A and $a_0 \geq a \quad \forall a \in A$.

Similarly, a number a_1 is a minimum of A if $a_1 \in A$ and $a_1 \leq a \quad \forall a \in A$.

Lemma

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup(A)$ if and only if, for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $s - \varepsilon < a$.

Proof

Firstly, we need to prove that if $s = \sup(A)$, then for every choice of $\varepsilon > 0$, there exists an element $a \in A$ such that $s - \varepsilon < a$.

Secondly, we need to prove that if for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $s - \varepsilon < a$, then $s = \sup(A)$.

1. Assume $s = \sup(A)$. We need to show that for every $\varepsilon > 0$, there exists an element $a \in A$ such that $s - \varepsilon < a$.

   Since $s = \sup(A)$, by definition, s is the least upper bound of A. This means:

   - s is an upper bound for A, i.e., $a \leq s$ for all $a \in A$.
   - For any $t < s$, there exists some element $a_t \in A$ such that $a_t > t$.

   Let $\varepsilon > 0$. Consider $t = s - \varepsilon$. Since $t < s$, by the property of the supremum, there must exist an element $a_\varepsilon \in A$ such that $a_\varepsilon > t$. Therefore, we have:

   $$a_\varepsilon > s - \varepsilon.$$

   Thus, for every $\varepsilon > 0$, there exists an element $a_\varepsilon \in A$ such that $s - \varepsilon < a_\varepsilon$.

2. Assume that for every $\varepsilon > 0$, there exists an element $a \in A$ satisfying $s - \varepsilon < a$. We need to show that $s = \sup(A)$.

   First, we show that s is an upper bound of A. Assume for contradiction that s is not an upper bound. Then there exists some $b \in A$ such that $b > s$, which contradicts the assumption that s is an upper bound. Hence, s must be an upper bound.

   Next, we show that s is the least upper bound. Assume for contradiction that there exists some $u < s$ which is also an upper bound of A. Choose $\varepsilon = s - u > 0$. By assumption, there exists an element $a_\varepsilon \in A$ such that:

   $$s - \varepsilon < a_\varepsilon.$$

Substituting $\varepsilon = s - u$, we get:

$$s - (s - u) < a_\varepsilon, \quad u < a_\varepsilon.$$

This contradicts the assumption that u is an upper bound of A. Therefore, no such $u < s$ can be an upper bound, and s must be the least upper bound.

Thus,

$$s = \sup(A)$$

Combining both parts, we conclude that $s = \sup(A)$ if and only if for every $\varepsilon > 0$, there exists an element $a \in A$ such that $s - \varepsilon < a$.

$\square$