MATRICES AND LINEAR MAPS

Matrix

Definition

Let m and n denote positive integers. An m-by-n matrix A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

Notation

The notation $A_{j,k}$ denotes the entry in row j, column k of A. In other words, the first index refers to the row number and the second index refers to the column number.

Matrix of a linear maps

Definition

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The matrix of T with respect to these bases is the m-by-n matrix $M(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

Notation

If the bases are not clear from the context, then the notation

$$M(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$$

is used.

Matrix addition

Definition

The sum of two matrices of the same size is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix}$$
$$= \begin{pmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} + C_{m,1} & \cdots & A_{m,n} + C_{m,n} \end{pmatrix}$$

In other words, $(A+C)_{j,k} = A_{j,k} + C_{j,k}$.

Matrix sum of linear maps

Theorem

Suppose $S, T \in \mathcal{L}(V, W)$. Then $M(S+T) = M(S) + M(T)$.

Proof

Let $S, T \in \mathcal{L}(V, W)$ and write the matrix of a linear map with respect to the bases $\mathcal{B}, \mathcal{C}$ column-wise: the j-th column of $M(S)$ is the coordinate vector $[S(v_j)]_{\mathcal{C}}$, and similarly the j-th column of $M(T)$ is $[T(v_j)]_{\mathcal{C}}$.

For each basis vector v_j of V we have by linearity of S and T that

$$(S+T)(v_j) = S(v_j) + T(v_j)$$

Taking coordinates with respect to $\mathcal{C}$ gives

$$[(S+T)(v_j)]_{\mathcal{C}} = [S(v_j) + T(v_j)]_{\mathcal{C}} = [S(v_j)]_{\mathcal{C}} + [T(v_j)]_{\mathcal{C}}$$

Therefore the j-th column of $M(S+T)$ equals the sum of the j-th columns of $M(S)$ and $M(T)$. Since this holds for every $j = 1, \ldots, n$, the two matrices are equal:

$$M(S+T) = M(S) + M(T)$$

$\square$

Scalar multiplication of a matrix

Definition

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$.

Matrix of a scalar times a linear map

Theorem

Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V; W)$. Then $M(\lambda T) = \lambda M(T)$.

Proof

Let $T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. By definition the j-th column of $M(T)$ is the coordinate vector $[T(v_j)]_{\mathcal{C}}$. For each basis vector v_j we have

$$(\lambda T)(v_j) = \lambda\big(T(v_j)\big)$$

Taking coordinates with respect to $\mathcal{C}$ yields

$$[(\lambda T)(v_j)]_{\mathcal{C}} = [\lambda T(v_j)]_{\mathcal{C}} = \lambda [T(v_j)]_{\mathcal{C}}$$

since scalar multiplication commutes with taking coordinates. Thus the j-th column of $M(\lambda T)$ is λ times the j-th column of $M(T)$. As this holds for every $j = 1, \ldots, n$, we conclude

$$M(\lambda T) = \lambda M(T)$$

□

Matrix spaces

Notation

For m and n positive integers, the set of all m-by-n matrices with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{m,n}$.

Dimensionality of $\mathbb{F}^{m,n}$

Theorem

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbb{F}^{m,n}$ is a vector space with dimension mn.

Proof

The verification that $\mathbb{F}^{m,n}$ is a vector space is left to the reader. Note that the additive identity of $\mathbb{F}^{m,n}$ is the m-by-n matrix whose entries all equal 0.
The reader should also verify that the list of m-by-n matrices that have 0 in all entries except for a 1 in one entry is a basis of $\mathbb{F}^{m,n}$. There are mn such matrices, so the dimension of $\mathbb{F}^{m,n}$ equals mn.

□

Matrix multiplication

Definition

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then AC is defined to be the m-by-p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

In other words, the entry in row j, column k, of AC is computed by taking row j of A and column k of C, multiplying together corresponding entries, and then summing.

Matrix of the product of linear maps

Theorem

If $T \in \mathcal{L}(U; V)$ and $S \in \mathcal{L}(V; W)$, then $M(ST) = M(S)M(T)$.

Proof

For each $j = 1, \dots, p$ the j-th column of $M_{\mathcal{A},\mathcal{B}}(T)$ is the coordinate vector $[T(u_j)]_{\mathcal{B}} \in \mathbb{F}^n$. Applying S to $T(u_j)$ and taking coordinates with respect to $\mathcal{C}$ gives

$$[\,S(T(u_j))\,]_{\mathcal{C}} = [\,S\,]_{\mathcal{B},\mathcal{C}}\,[T(u_j)]_{\mathcal{B}}$$

because the matrix $M_{\mathcal{B},\mathcal{C}}(S)$ sends the coordinate vector of any $v \in V$ (relative to $\mathcal{B}$) to the coordinate vector of $S(v)$ (relative to $\mathcal{C}$). But $[S(T(u_j))]_{\mathcal{C}}$ is exactly the j-th column of $M_{\mathcal{A},\mathcal{C}}(ST)$. Therefore the j-th column of $M_{\mathcal{A},\mathcal{C}}(ST)$ equals the j-th column of $M_{\mathcal{B},\mathcal{C}}(S)\,M_{\mathcal{A},\mathcal{B}}(T)$. Since this holds for every j, the matrices are equal:

$$M_{\mathcal{A},\mathcal{C}}(ST) = M_{\mathcal{B},\mathcal{C}}(S)\,M_{\mathcal{A},\mathcal{B}}(T)$$

For completeness, an equivalent entry-wise argument: if $M_{\mathcal{A},\mathcal{B}}(T) = (t_{kj})$ (with $1 \le k \le n$, $1 \le j \le p$) and $M_{\mathcal{B},\mathcal{C}}(S) = (s_{ik})$ (with $1 \le i \le m$, $1 \le k \le n$), then the (i,j)-entry of the product is $\sum_{k=1}^{n} s_{ik}t_{kj}$. This equals the i-th coordinate (relative to $\mathcal{C}$) of $S(T(u_j))$, i.e. the (i,j)-entry of $M_{\mathcal{A},\mathcal{C}}(ST)$. Thus the two matrices have the same entries and are equal.

$\square$

Notation

Suppose A is an m-by-n matrix.

- If $1 \le j \le m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
- If $1 \le k \le n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

Entry of matrix product equals row times column

Theorem

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{j,k} = A_{j,\cdot}C_{\cdot,k}$$

for $1 \le j \le m$ and $1 \le k \le p$.

Proof

Write the matrices in entry form:

$$A = (A_{j,i})_{1\le j\le m,\ 1\le i\le n}, \qquad C = (C_{i,k})_{1\le i\le n,\ 1\le k\le p}$$

By the definition of matrix multiplication, the (j,k)-entry of AC is the sum of products of corresponding entries in the j-th row of A and the k-th column of C

:

$$(AC)_{j,k} = \sum_{i=1}^{n} A_{j,i}\, C_{i,k}$$

Interpreting $A_{j,\cdot}$ as the row vector $(A_{j,1}, \ldots, A_{j,n})$ and $C_{\cdot,k}$ as the column vector $(C_{1,k}, \ldots, C_{n,k})^T$, their matrix or dot product equals the same sum:

$$A_{j,\cdot}\, C_{\cdot,k} = (A_{j,1}, \ldots, A_{j,n}) \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{n,k} \end{pmatrix} = \sum_{i=1}^{n} A_{j,i}\, C_{i,k}$$

Thus for every $1 \leq j \leq m$ and $1 \leq k \leq p$ we have $(AC)_{j,k} = A_{j,\cdot}\, C_{\cdot,k}$, as required.

□

Column of matrix product equals matrix times column

Theorem

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k}$$

for $1 \leq k \leq p$.

Proof

Let $C_{\cdot,k}$ denote the k-th column of C,

$$C_{\cdot,k} = \begin{pmatrix} C_{1,k} \\ C_{2,k} \\ \vdots \\ C_{n,k} \end{pmatrix} \in \mathbb{F}^n$$

By the definition of matrix multiplication, the j-th entry of $(AC)_{\cdot,k}$ is

$$(AC)_{j,k} = \sum_{i=1}^{n} A_{j,i}\, C_{i,k}, \qquad 1 \le j \le m$$

On the other hand, the j-th entry of the product $A\, C_{\cdot,k}$ is

$$\left(A\, C_{\cdot,k}\right)_j = \sum_{i=1}^{n} A_{j,i}\, (C_{\cdot,k})_i = \sum_{i=1}^{n} A_{j,i}\, C_{i,k}$$

Thus for each j, the j-th entry of $(AC)_{\cdot,k}$ equals the j-th entry of $A\, C_{\cdot,k}$. Therefore

$$(AC)_{\cdot,k} = A\, C_{\cdot,k}$$

$\square$

Linear combination of columns

Theorem

Suppose A is an m-by-n matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n-by-1 matrix. Then

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$$

In other words, Ac is a linear combination of the columns of A, with the scalars that multiply the columns coming from c.

Proof

Write A in terms of its columns:

$$A = \begin{bmatrix} A_{\cdot,1} & A_{\cdot,2} & \cdots & A_{\cdot,n} \end{bmatrix}$$

where each $A_{\cdot,i}$ is the i-th column of A (an $m \times 1$ vector). Multiplying A by c yields the matrix product

$$Ac = \begin{bmatrix} A_{\cdot,1} & A_{\cdot,2} & \cdots & A_{\cdot,n} \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

By the rules of block (or column) multiplication this equals the linear combination of the columns of A with coefficients c_i:

$$Ac = c_1 A_{\cdot,1} + c_2 A_{\cdot,2} + \cdots + c_n A_{\cdot,n}$$

Equivalently, checking entries: for each $1 \leq j \leq m$ the j-th entry of Ac is

$$(Ac)_j = \sum_{i=1}^{n} A_{j,i}\, c_i$$

while the j-th entry of the right-hand side is

$$\left(c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}\right)_j = \sum_{i=1}^{n} c_i A_{j,i}$$

which is the same sum. Hence the two sides are equal.

$\square$