

LIMITS OF FUNCTIONS

Limits of functions at infinity

Definition

For a function f , we say that $\lim_{x \rightarrow +\infty} f(x) = L$ if for all $\epsilon > 0$ there is a number $N > 0$ such that $|f(x) - L| < \epsilon$ for all $x > N$.

Definition

For a function f , we say that $\lim_{x \rightarrow -\infty} f(x) = L$ if for all $\epsilon > 0$ there is a number $N < 0$ such that $|f(x) - L| < \epsilon$ for all $x < N$.

Definition

For a function f , we say that $\lim_{x \rightarrow +\infty} f(x) = \infty$ if for all $M > 0$ there is a number $N > 0$ such that $f(x) > M$ for all $x > N$.

Limits of functions at real numbers

Epsilon delta definition

For a function f , we say that $\lim_{x \rightarrow a} f(x) = L$ if for all $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all x with $0 < |x - a| < \delta$.

Limits on the right and left of a point

Definitions

- We say that

$$\lim_{x \rightarrow a^-} f(x) = L$$

if for all $\varepsilon > 0$ there exists some $\delta > 0$ such that if $a - x < \delta$ for all $x < a$ then $|f(x) - L| < \varepsilon$.

- We say that

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for all $\varepsilon > 0$ there exists some $\delta > 0$ such that if $x - a < \delta$ for all $x > a$ then $|f(x) - L| < \varepsilon$.

Theorem

A function f admits a limit at a point $a \in \mathbb{R}$ if and only if f admits a left limit and right limit at a and they coincide. That is,

$$\lim_{x \rightarrow a} f(x) \Leftrightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

Sum rule

Theorem

If f and g are functions such that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

$$\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$$

Proof

Let $\epsilon > 0$ and let $\epsilon_1 > 0$ and $\epsilon_2 > 0$ be values so that $\epsilon = \epsilon_1 + \epsilon_2$. Since $\lim_{x \rightarrow a} f(x)$ exists and is equal to L we know that there exists some $\delta_1 > 0$ such that

$$\text{if } 0 < |x - a| < \delta_1 \text{ then } |f(x) - L| < \epsilon_1$$

So if we define $\delta = \min\{\delta_1, \delta_2\}$, then if $0 < |x - a| < \delta$ we have that $|x - a|$ is also smaller than both δ_1 and δ_2 , so

$$|f(x) - L| < \epsilon_1 \quad \text{and} \quad |g(x) - M| < \epsilon_2$$

We can now apply the triangle inequality to see that

$$\begin{aligned}
|(f+g)(x) - (L+M)| &= |f(x) + g(x) - L - M| \\
&= |(f(x) - L) + (g(x) - M)| \\
&\leq |f(x) - L| + |g(x) - M| \\
&< \epsilon_1 + \epsilon_2 \\
&= \epsilon
\end{aligned}$$

So for any $\epsilon > 0$ we have found a value $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |(f+g)(x) - (L+M)| < \epsilon$$

□

Product rule

Theorem

If f and g are functions such that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

$$\lim_{x \rightarrow a} (f(x) \cdot g(x)) = L \cdot M$$

Proof

Let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow a} f(x) = L$, we can find a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then $|f(x) - L| < \frac{\epsilon}{2(|M|+1)}$. Similarly, since $\lim_{x \rightarrow a} g(x) = M$, we can find a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$.

So if we define $\delta = \min\{\delta_1, \delta_2\}$, then if $0 < |x - a| < \delta$ we have that $|x - a|$ is also smaller than both δ_1 and δ_2 , so

$$|f(x) - L| < \frac{\epsilon}{2(|M|+1)} \quad \text{and} \quad |g(x) - M| < \frac{\epsilon}{2(|L|+1)}$$

We can now apply the product limit theorem to see that

$$|(f \cdot g)(x) - (L \cdot M)| = |f(x) \cdot g(x) - L \cdot M|$$

$$\begin{aligned}
&= |f(x) \cdot g(x) - f(x) \cdot M + f(x) \cdot M - L \cdot M| \\
&= |f(x) \cdot (g(x) - M) + (f(x) - L) \cdot M| \\
&\leq |f(x)| \cdot |g(x) - M| + |f(x) - L| \cdot |M| \\
&< (|L| + 1) \cdot \frac{\epsilon}{2(|L| + 1)} + \frac{\epsilon}{2(|M| + 1)} \cdot |M| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

So for any $\epsilon > 0$ we have found a value $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |(f \cdot g)(x) - (L \cdot M)| < \epsilon$$

□

Squeeze theorem, sandwich theorem

Theorem

Let f , g , and h be functions such that

$$f(x) \leq g(x) \leq h(x) \text{ for all } x \in \mathbb{R}$$

if

$$\lim_{x \rightarrow a} f(x) = L \text{ and } \lim_{x \rightarrow a} h(x) = L \text{ then } \lim_{x \rightarrow a} g(x) = L$$

Proof

For all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } |x - a| < \delta \text{ then } |f(x) - L| < \epsilon, \text{ and}$$

$$\text{if } |x - a| < \delta \text{ then } |h(x) - L| < \epsilon$$

Recall that $|f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$ is equivalent to writing

$$-\epsilon < f(x) - L < \epsilon \quad \text{and} \quad -\epsilon < h(x) - L < \epsilon$$

So if $|x - a| < \delta$ then

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$$

hence

$$-\epsilon \leq g(x) - L \leq \epsilon$$

So $|g(x) - L| < \epsilon$

□