

# **SEQUENCES**

## **Upper bound**

### *Definition*

$B \in \mathbb{R}$  is an upper bound for  $\{a_n\}_{n=n_0}^{\infty}$  if  $a_n \leq B$  for all  $n \in \mathbb{R}$ .

## **Lower bound**

### *Definition*

$B \in \mathbb{R}$  is an upper bound for  $\{a_n\}_{n=n_0}^{\infty}$  if  $a_n \geq B$  for all  $n \in \mathbb{R}$ .

## **Supremum, least upper bound**

### *Definition*

$S \in \mathbb{R}$  is the supremum, for  $\{a_n\}_{n=n_0}^{\infty}$  if  $S \leq B$  for all upper bounds  $B$ .

$$S = \min\{B \in \mathbb{R} | a_n \leq B, \forall n \in \mathbb{N}\}$$

## **Infimum, greatest lower bound**

### *Definition*

$I \in \mathbb{R}$  is the infimum, for  $\{a_n\}_{n=n_0}^{\infty}$  if  $I \geq B$  for all lower bounds  $B$ .

$$I = \max\{B \in \mathbb{R} | a_n \geq B, \forall n \in \mathbb{N}\}$$

## **Convergent sequences**

### *Theoreme*

Convergent sequences are bounded.

### *Proof*

Let  $(s_n)_{n \in \mathbb{N}}$  be a converging sequence, and let  $s = \lim_{x \rightarrow \infty} s_n$ .

Let us take  $\varepsilon = 1$  and apply the definition of a limit. We know that there exists

$N$  such that  $n > N \Rightarrow |s_n - s| < 1$ . Now by the triangle inequality  
 $n > N \Rightarrow |s_n| < |s| + 1$

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| \leq 1 + |s|$$

We can now define:  $M := \max\{|s| + 1, |s_1|, \dots, |s_N|\}$

This means  $|s_n| \leq M$  for all  $n \in \mathbb{N}$  and so  $(s_n)$  is a bounded sequence.

□

## **Monotonic sequences**

- A sequence  $\{a_n\}_{n=n_0}^{\infty}$  is called monotone increasing if:

$$a_{n+1} \geq a_n \quad \forall n \geq n_0$$

- A sequence  $\{a_n\}_{n=n_0}^{\infty}$  is called monotone decreasing if:

$$a_{n+1} \leq a_n \quad \forall n \geq n_0$$

### ***Monotone Convergence Theorem***

- If  $\{a_n\}_{n=n_0}^{\infty}$  is a monotone increasing sequence, with supremum  $S$ , then

$$a_n \rightarrow S$$

- If  $\{a_n\}_{n=n_0}^{\infty}$  is a monotone decreasing sequence, with infimum  $I$ , then

$$a_n \rightarrow I$$

### ***Proof***

Since  $S$  is the supremum we have that  $|a_n - S| = S - a_n$  for all  $n \in \mathbb{N}$ . Now let  $\epsilon > 0$  and assume that  $S - a_n \geq \epsilon$  for all  $n \in \mathbb{N}$ . Then  $a_n \leq S - \epsilon$  for all  $n \in \mathbb{N}$ , that is,  $S - \epsilon$  is an upper bound. This contradicts the fact that  $S$  is the supremum, so there must be some  $N \in \mathbb{N}$  such that  $S - a_N < \epsilon$ .

Since  $a_n$  is monotone increasing  $a_n \geq a_N$  for all  $n \geq N$ . So

$$|a_n - S| = S - a_n \leq S - a_N < \epsilon$$

for all  $n \geq N$ .

## **Arithmetic mean - geometric mean inequality**

### *Arithmetic mean - geometric mean inequality theorem*

Let  $x_1, \dots, x_n$  satisfy  $x_i \geq 0$  for  $i = 1, \dots, n$ . Then their geometric mean is at most their arithmetic mean:

$$\sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n} \leq \frac{x_1 + x_2 + \dots + x_n}{n}$$

with equality if and only if  $x_1 = x_2 = \dots = x_n$ .

### ***Proof***

We will prove it by induction over  $n \geq 1$ .

- For  $n = 1$ : we get an equality

$$x = \sqrt{x} = \frac{x}{1} = x$$

- Suppose its true for any  $n$ -tuple of non-negative numbers. Consider an  $(n + 1)$ -tuple  $x_1, \dots, x_{n+1}$  and let  $\alpha$  be their arithmetic mean, that is:

$$\alpha = \frac{x_1 + x_2 + \dots + x_{n+1}}{n + 1}$$

We now have

$$(n + 1)\alpha = x_1 + \dots + x_{n+1} = \sum_{k=1}^{n+1} x_k$$

Examining different cases:

1. One of the numbers is 0, that is there exists some  $i$  such that  $x_i = 0$ . The inequality here is obvious as one side is 0. And for the other side to also be 0 we need that  $x_k = 0$  for  $k = 1, \dots, n + 1$ , so equality only occurs in the case when all terms are equal.
2. When we have  $x_i = \alpha$ ,  $i = 1, \dots, n + 1$ , then we get equality by a simple computation:

$$\sum_{k=1}^{n+1} x_k = (n + 1)\alpha$$

so

$$\frac{\sum_{k=1}^{n+1} x_k}{n + 1} = \alpha$$

and

$$\sqrt[n+1]{x_1 \cdot \dots \cdot x_{n+1}} = \sqrt[n+1]{\alpha^{n+1}} = \alpha$$

3. Everything else! That is  $x_k > 0$  for  $k = 1, \dots, n + 1$  and not all the terms are equal. In this case there is an  $x_i$  strictly greater than  $\alpha$  and one that is strictly smaller (otherwise  $\sum_{k=1}^{n+1} x_k \neq (n + 1)\alpha$ ). Up to reordering, let's suppose that these numbers are  $x_n$  and  $x_{n+1}$ , that is:  $x_n > \alpha$  and  $x_{n+1} < \alpha$ . In particular,  $x_n - \alpha > 0$  and  $\alpha - x_{n+1} > 0$  so we have

$$(x_n - \alpha)(\alpha - x_{n+1}) > 0. \quad (\star)$$

We define a new real number  $y$  as

$$y = x_n + x_{n+1} - \alpha \geq x_n - \alpha > 0$$

now

$$(n+1)\alpha = x_1 + \cdots + \underbrace{x_n + x_{n+1}}_{y+\alpha} = x_1 + \cdots + x_{n-1} + y + \alpha.$$

so

$$n \cdot \alpha = x_1 + \cdots + x_{n-1} + y$$

And thus  $\alpha$  is also the arithmetic mean of  $x_1, \dots, x_{n-1}, y$ .

By the induction hypothesis we have

$$\underbrace{\frac{x_1 + \cdots + x_{n-1} + y}{n}}_{\alpha} \geq \sqrt[n]{x_1 \cdots x_{n-1} y}$$

and so

$$\alpha^n \geq x_1 \cdots x_{n-1} y$$

now

$$\alpha^{n+1} = \alpha^n \cdot \alpha \geq x_1 \cdots x_{n-1} y \alpha$$

But  $y = x_n + x_{n+1} - \alpha$  and so

$$\begin{aligned} y \cdot \alpha - x_n \cdot x_{n+1} &= (x_n + x_{n+1} - \alpha)\alpha - x_n x_{n+1} \\ &= x_n \alpha + x_{n+1} \alpha - \alpha^2 - x_n x_{n+1} \\ &= (\alpha - x_{n+1})(x_n - \alpha) > 0 \end{aligned}$$

by the inequality  $(\star)$  above.

We thus have that  $y \cdot \alpha > x_n x_{n+1}$ .

In conclusion:  $\alpha^{n+1} > x_1 \cdots x_{n-1} \cdot x_n \cdot x_{n+1}$ , that is

$$\left( \frac{x_1 + \cdots + x_{n+1}}{n+1} \right)^{n+1} > x_1 \cdots x_{n+1}$$

as wanted. And that proves the theorem!

□

### ***Lemma***

The sequences  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=2}^{\infty}$  have the following monotonicity properties:

1.  $a_{n+1} > a_n$  for  $n \geq 1$
2.  $b_{n+1} < b_n$  for  $n \geq 2$

### ***Proofs***

1. Let  $x_1 = 1, x_2 = \dots = x_k = \dots = x_{n+1} = 1 + \frac{1}{n}$ . We apply the AM-GM inequality to these numbers. Their arithmetic mean satisfies:

$$\frac{1 + \left(1 + \frac{1}{n}\right) + \dots + \left(1 + \frac{1}{n}\right)}{n+1} = \frac{n+1 + n \cdot \frac{1}{n}}{n+1} = 1 + \frac{1}{n+1} \quad (**)$$

Their geometric mean is

$$\sqrt[n+1]{x_1 \dots x_{n+1}} = \sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n} \quad (***)$$

The  $x_i$ s are not all equal, so we know that  $(**) > (***)$ . From this:

$$1 + \frac{1}{n+1} > \sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n}$$

And so:

$$a_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n = a_n$$

2. This case is very similar to what we just did. We set  $y_1 = 1$  and  $y_2 = y_3 = \dots = y_{n+1} = 1 - \frac{1}{n} > 0$ . Then:

$$\begin{aligned}
\frac{\sum_{i=1}^{n+1} y_i}{n+1} &= \frac{1 + n \left(1 - \frac{1}{n}\right)}{n+1} \\
&= \frac{1 + n - 1}{n+1} \\
&= 1 - \frac{1}{n+1} \\
&> \sqrt[n+1]{\left(1 - \frac{1}{n}\right)^n}
\end{aligned}$$

by the arithmetic mean - geometric mean inequality applied to  $y_1, \dots, y_{n+1}$ . We deduce

$$\left(1 - \frac{1}{n+1}\right)^{n+1} > \left(1 - \frac{1}{n}\right)^n$$

By taking the inverse on both sides, we reverse the inequality and we get:

$$b_{n+1} = \left(1 - \frac{1}{n+1}\right)^{-(n+1)} < \left(1 - \frac{1}{n}\right)^{-n} = b_n$$

as required.

□

### ***Lemma***

For  $n \geq 2$ :

$$0 < b_n - a_n < \frac{4}{n}$$

### ***Proof***

We already know that  $b_n - a_n > 0$ . Let's check the other inequality:

$$b_n - a_n = b_n \left(1 - \frac{a_n}{b_n}\right)$$

We have:

$$a_n = \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$$

and

$$b_n = \left(1 - \frac{1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^n$$

so

$$\begin{aligned} b_n - a_n &= b_n \left(1 - \left(\frac{n+1}{n}\right)^n \left(\frac{n-1}{n}\right)^n\right) \\ &= b_n \left(1 - \left(\frac{n^2-1}{n^2}\right)^n\right) \end{aligned}$$

Set  $q = \frac{n^2-1}{n^2} < 1$ . We have:

$$b_n - a_n = b_n(1 - q^n)$$

and so

$$1 - q^n = (1 - q)(1 + q + q^2 + \cdots + q^{n-1})$$

As  $q < 1$ , we also have that  $q^k < 1$  and we can deduce that

$$1 - q^n < (1 - q) \cdot n$$

We've thus shown that

$$b_n - a_n < b_n(1 - q) \cdot n$$

Now as

$$1 - q = \left(1 - \frac{n^2-1}{n^2}\right) = \frac{n^2 - (n^2-1)}{n^2} = \frac{1}{n^2}$$

and  $b_n$  is decreasing so  $b_n \leq b_2 = 4$ . We can conclude:



$$b_n - a_n < 4 \cdot \frac{1}{n^2} \cdot n = \frac{4}{n}$$

□

## **Squeeze theorem, Sandwich theorem**

### ***Squeeze Theorem***

If  $(x_n)_{n=n_0}^{\infty}$ ,  $(y_n)_{n=n_0}^{\infty}$  and  $(z_n)_{n=n_0}^{\infty}$  are sequences that satisfy

$$x_n \leq y_n \leq z_n$$

and if

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = \ell$$

for  $\ell \in \mathbb{R}$  then

$$\lim_{n \rightarrow \infty} y_n = \ell.$$

## **Euler's number**

### ***Definition***

We define the real number  $e$  as the limit

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{-n} = e$$

## **Approximating factorials**

### ***Theorem***

For  $n \geq 2$  we have

$$\frac{n^n}{e^{n-1}} < n! < n^n$$

### ***Proof***

We have

$$n! = n(n-1)(n-2) \dots 2 \cdot 1 \quad \text{so} \quad n! < \underbrace{n \cdot n \dots n}_{n \text{ times}} = n^n$$

Let us establish the second inequality. As  $e$  is the limit of an increasing sequence  $a_k = \left(1 + \frac{1}{k}\right)^k$ , we have that, for all  $k \geq 1$ ,  $e > a_k = \left(1 + \frac{1}{k}\right)^k$ .

Thus

$$e^{n-1} > \prod_{k=1}^{n-1} a_k (= a_1 \cdot \dots \cdot a_{n-1})$$

and from this

$$e^{n-1} > a_1 \cdot a_2 \dots a_{n-1} = \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \dots \left(\frac{n}{n-1}\right)^{n-1}$$

Many terms simplify through telescoping:

$$a_1 \cdot a_2 \dots a_{n-1} = \frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{(n-1)!} \cdot \frac{n}{n} = \frac{n^n}{n!}$$

Therefore

$$e^{n-1} > \frac{n^n}{n!} \quad \text{and} \quad n! > \frac{n^n}{e^{n-1}}$$

This completes the proof.

□

***Stirlings formula theorem***

$$\lim_{n \rightarrow \infty} \frac{n!}{\left(\frac{n^n}{e^n}\right) \sqrt{2\pi n}} = 1$$

***Proof***

We will use the result of the previous theorem. We have

$$n! \sim \left( \frac{n^n}{e^n} \right) \sqrt{2\pi n}$$

Thus

$$\frac{n!}{\left( \frac{n^n}{e^n} \right) \sqrt{2\pi n}} \rightarrow 1$$

as  $n \rightarrow \infty$ .

□