INVERTIBILITY AND ISOMORPHIC VECTOR SPACES

Invertible

Definition

A linear map $T \in \mathcal{L}(V;W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W;V)$ such that ST equals the identity map on V and TS equals the identity map on W.

Inverse

Definition

A linear map $S \in \mathcal{L}(W;V)$ satisfying $ST = I$ and $TS = I$ is called an inverse of T.

Remark

Note that the first I is the identity map on V and the second I is the identity map on W

Inverse is unique

Definition

An invertible linear map has a unique inverse.

Proof

Suppose S_1 and S_2 are both inverses of T. Then

$S_1 = S_1 I = S_1(TS_2) = (S_1 T)S_2 = IS_2 = S_2$

$\square$

Notation

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

Invertibility is equivalent to injectivity and surjectivity

Theorem

A linear map is invertible if and only if it is injective and surjective.

Proof

Suppose $T \in \mathcal{L}(V, W)$. We need to show that T is invertible if and only if it is injective and surjective.

First suppose T is invertible. To show that T is injective, suppose $u, v \in V$ and $Tu = Tv$. Then

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

so $u = v$. Hence T is injective.

We are still assuming that T is invertible. Now we want to prove that T is surjective. To do this, let $w \in W$. Then $w = T(T^{-1}w)$, which shows that w is in the range of T. Thus range $T = W$. Hence T is surjective, completing this direction of the proof.

Now suppose T is injective and surjective. We want to prove that T is invertible. For each $w \in W$, define Sw to be the unique element of V such that $T(Sw) = w$ (the existence and uniqueness of such an element follow from the surjectivity and injectivity of T). Clearly $T \circ S$ equals the identity map on W.

To prove that $S \circ T$ equals the identity map on V, let $v \in V$. Then

$$T\big((S \circ T)v\big) = (T \circ S)(Tv) = I(Tv) = Tv$$

This equation implies that $(S \circ T)v = v$ (because T is injective). Thus $S \circ T$ equals the identity map on V.

To complete the proof, we need to show that S is linear. To do this, suppose $w_1, w_2 \in W$. Then

$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2$$

Thus $Sw_1 + Sw_2$ is the unique element of V that T maps to $w_1 + w_2$. By the definition of S, this implies that

$$S(w_1 + w_2) = Sw_1 + Sw_2$$

Hence S satisfies the additive property required for linearity.

The proof of homogeneity is similar. Specifically, if $w \in W$ and $\lambda \in \mathbb{F}$, then

$$T(\lambda Sw) = \lambda T(Sw) = \lambda w$$

Thus λSw is the unique element of V that T maps to λw. By the definition of S, this implies that

$$S(\lambda w) = \lambda Sw$$

Hence S is linear, as desired.

$\square$

Isomorphisms

Definition

An isomorphism is an invertible linear map.

Isomorphic

Definition

Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

Dimension shows whether vector spaces are isomorphic

Theorem

Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Proof

($\Rightarrow$) Suppose $V \cong W$, i.e., there exists a linear isomorphism $T : V \to W$. Since T is bijective, it maps a basis of V to a basis of W. Hence the number of basis vectors is the same:

$$\dim V = \dim W$$

($\Leftarrow$) Suppose $\dim V = \dim W = n$. Let $\{v_1, \dots, v_n\}$ be a basis of V and $\{w_1, \dots, w_n\}$ be a basis of W. Define a linear map $T : V \to W$ by

$$T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i w_i$$

for scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$.

This map is well-defined and linear. Moreover, T is injective because the kernel is $\{0\}$ (the linear combination $\sum \alpha_i v_i = 0$ implies all $\alpha_i = 0$). It is surjective because any $w \in W$ can be written as a linear combination of the basis $\{w_i\}$. Hence T is a bijective linear map, and $V \cong W$.

$\square$

Matrix Representation Isomorphism

Theorem

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then M is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.

Dimension of $\mathcal{L}(V, W)$

Theorem

Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite- dimensional and

$$\dim L(V, W) = (\dim V) \cdot (\dim W)$$

Matrix of a vector

Definition

Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. The matrix of v with respect to this basis is the n-by-1 matrix

$$M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_1, \ldots, c_n$ are the scalars such that

$$v = c_1 v_1 + \cdots + c_n v_n$$

Column of a Linear Map's Matrix

Theorem

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Let $1 \leq k \leq n$. Then the k^{th} column of $M(T)$, which is denoted by $M(T)_{\cdot,k}$, equals $M(v_k)$.

Proof

The desired result follows immediately from the definitions of $M(T)$ and $M(v_k)$.

The next result shows how the notions of the matrix of a linear map, the matrix of a vector, and matrix multiplication fit together.

□

Linear maps act like matrix multiplication

Theorem

Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$M(Tv) = M(T)M(v)$$

Operator

Definition

- A linear map from a vector space to itself is called an operator.
- The notation $\mathcal{L}(V)$ denotes the set of all operators on V. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

Injectivity is equivalent to surjectivity in finite dimensions

Theorem

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. T is invertible
2. T is injective
3. T is surjective