FINITE-DIMENSIONAL VECTOR SPACES

Linear combinations

Definition

Let V be a vector space over a field $\mathbb{F}$. A linear combination of vectors $v_1, v_2, \ldots, v_n \in V$ is an expression of the form:

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n$$

where $c_1, c_2, \ldots, c_n \in \mathbb{F}$ are scalars. The set of all linear combinations of a given set of vectors is called the span of those vectors.

Span

Definition

The span of a set of vectors $v_1, v_2, \ldots, v_n \in V$ is the set of all linear combinations of those vectors.

Notation

The span of the vectors $v_1, v_2, \ldots, v_n$ is denoted by $\text{span}(v_1, v_2, \ldots, v_n)$.

Theorem

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Proof

Let W be the span of the vectors $v_1, v_2, \ldots, v_n$. By definition, W is the set of all linear combinations of these vectors. Since W is formed by taking linear combinations of the vectors in the list, it is a subspace of V.

Now, suppose there exists a subspace U of V that contains all the vectors $v_1, v_2, \ldots, v_n$. Since U is a subspace, it must also contain all linear combinations of these vectors. Therefore, we have $W \subseteq U$.

Since U was an arbitrary subspace containing the vectors in the list, we conclude that W is the smallest subspace of V containing all the vectors in the list.

□

Finite-Dimensional Vector Spaces

Definition

A vector space is called finite-dimensional if some list of vectors in it spans the space.

Polynomial

Definition

The space of polynomials with coefficients in a field $\mathbb{F}$ is denoted by $\mathcal{P}(\mathbb{F})$. A polynomial $p(x) \in \mathcal{P}(\mathbb{F})$ is an expression of the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_i \in \mathbb{F}$ for all i and n is a non-negative integer.

Degree of a polynomial

Definition

The degree of a polynomial $p(x) \in \mathcal{P}(\mathbb{F})$ is the highest power of x that appears in the polynomial with a non-zero coefficient. Formally, if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of $p(x)$ is n, denoted by $\deg(p) = n$.

Remark

The polynomial that is identically 0 is said to have degree $-\infty$.

Space of polynomials

Definition

For a nonnegative integer m, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m.

Infinite-dimensional vector space

Definition

A vector space is called infinite-dimensional if it is not finite-dimensional.

Linearly independence

Definition

A set of vectors $v_1, v_2, \ldots, v_n \in V$ is said to be linearly independent if the only linear combination of these vectors that equals the zero vector is the trivial combination where all coefficients are zero. Formally, if

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$$

implies $c_1 = c_2 = \ldots = c_n = 0$, then the vectors are linearly independent.

Remark

The empty list () is also declared to be linearly independent.

Linearly dependent

Definition

A set of vectors $v_1, v_2, \ldots, v_n \in V$ is said to be linearly dependent if there exist coefficients $c_1, c_2, \ldots, c_n \in \mathbb{F}$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$$

In other words, at least one of the vectors can be expressed as a linear combination of the others.

Linear dependence lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:

1. $v_j \in \text{span}(v_1, \ldots, v_{j-1})$
2. if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\text{span}(v_1, \ldots, v_m)$.

Proof

We will prove the linear dependence lemma by showing that if $v_1, \ldots, v_m$ is a linearly dependent list in V, then the two conditions stated in the lemma must hold for some $j \in \{1, 2, \ldots, m\}$.

Since $v_1, \ldots, v_m$ is linearly dependent, there exist coefficients $c_1, c_2, \ldots, c_m \in \mathbb{F}$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m = 0$$

Without loss of generality, assume $c_j \neq 0$ for some $j \in \{1, 2, \ldots, m\}$. We can then express v_j as a linear combination of the other vectors:

$$v_j = -\frac{c_1}{c_j} v_1 - \frac{c_2}{c_j} v_2 - \ldots - \frac{c_{j-1}}{c_j} v_{j-1} - \frac{c_{j+1}}{c_j} v_{j+1} - \ldots - \frac{c_m}{c_j} v_m$$

This shows that v_j can be expressed as a linear combination of the vectors $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m$, which proves the first condition of the lemma.

To prove the second condition, we note that removing v_j from the list does not change the span of the remaining vectors, since v_j can be expressed as a linear combination of them. Therefore, we have

$$\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \text{span}(v_1, \ldots, v_m)$$

This completes the proof of the linear dependence lemma.

$\square$

Length of linearly independent list and length of spanning list

Theorem

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof

Let $v_1, v_2, \ldots, v_k$ be a linearly independent list of vectors in a finite-dimensional vector space V, and let $u_1, u_2, \ldots, u_m$ be a spanning list of vectors in V. We want to show that $k \leq m$.

Since $u_1, u_2, \ldots, u_m$ is a spanning list, every vector in V can be expressed as a linear combination of the u_i's. In particular, the vectors $v_1, v_2, \ldots, v_k$ can be expressed as linear combinations of the u_i's:

$$v_j = a_{j1}u_1 + a_{j2}u_2 + \ldots + a_{jm}u_m$$

for some coefficients $a_{ji} \in \mathbb{F}$. We can form the following matrix A whose columns are the vectors $u_1, u_2, \ldots, u_m$ and whose rows correspond to the coefficients a_{ji}:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{km} \end{pmatrix}$$

Since the vectors $v_1, v_2, \ldots, v_k$ are linearly independent, the rows of the matrix A must be linearly independent as well.

However, in a finite-dimensional vector space, the maximum number of linearly independent vectors or the dimension is equal to the number of columns in the matrix A. Therefore, we must have $k \leq m$, which completes the proof.

$\square$

Finite-dimensional subspaces

Theorem

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof

Let W be a subspace of a finite-dimensional vector space V. Since V is finite-dimensional, there exists a finite basis $\{v_1, v_2, \ldots, v_k\}$ for V. We claim that the set $\{v_1, v_2, \ldots, v_k\}$ spans W.

To see this, let $w \in W$. Since W is a subspace, we can express w as a linear combination of the basis vectors of V:

$$w = a_1v_1 + a_2v_2 + \ldots + a_kv_k$$

for some coefficients $a_i \in \mathbb{F}$. However, since $w \in W$ and W is closed under linear combinations, it follows that each v_i must also be expressible in terms of the basis vectors of W. Thus, we can find a finite set of vectors in W that spans W.

Therefore, W is finite-dimensional, which completes the proof.

$\square$

Basis

Definition

A basis of V is a list of vectors in V that is linearly independent and spans V.

Theorem

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be expressed as a linear combination of the v_i's.

Proof

$(\Rightarrow)$ Suppose $v_1, \ldots, v_n$ is a basis of V. Then by definition, the v_i's are linearly independent and span V. Therefore, every $v \in V$ can be expressed as a linear combination of the v_i's.

$(\Leftarrow)$ Conversely, suppose every $v \in V$ can be expressed as a linear combination of the v_i's. We need to show that the v_i's are linearly independent. Assume for the sake of contradiction that they are not linearly independent. Then there exists a non-trivial linear combination of the v_i's that equals zero:

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$$

for some coefficients $c_i \in \mathbb{F}$, not all zero. But then we could express one of the v_i's as a linear combination of the others, contradicting the assumption that the v_i's span V.

Therefore, the v_i's must be linearly independent, and we conclude that $v_1, \ldots, v_n$ is a basis of V.

$\square$

Spanning list contains a basis

Theorem

Every spanning list in a vector space can be reduced to a basis of the vector space.

Proof

Let $v_1, v_2, \ldots, v_n$ be a spanning list in a vector space V. We will show that we can reduce this list to a basis for V.

First, we can apply the process of Gaussian elimination to the vectors $v_1, v_2, \ldots, v_n$ to obtain a set of linearly independent vectors $u_1, u_2, \ldots, u_k$ (where $k \leq n$) that still spans V. This is possible because the original vectors span V, and we can remove any linear dependencies among them.

Next, we claim that the set $\{u_1, u_2, \ldots, u_k\}$ is a basis for V. To see this, we need to show that it is linearly independent and spans V.

Since we obtained $u_1, u_2, \ldots, u_k$ from $v_1, v_2, \ldots, v_n$ through a process that removes linear dependencies, it follows that the u_i's are linearly independent.

Furthermore, because the v_i's span V, any vector $v \in V$ can be expressed as a linear combination of the v_i's:

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$$

Since the u_i's are obtained from the v_i's, we can also express v as a linear combination of the u_i's:

$$v = b_1 u_1 + b_2 u_2 + \ldots + b_k u_k$$

for some coefficients $b_i \in \mathbb{F}$. This shows that the u_i's span V.

Therefore, we conclude that $\{u_1, u_2, \ldots, u_k\}$ is a basis for V, and we have successfully reduced the spanning list $v_1, v_2, \ldots, v_n$ to a basis.

$\square$

Basis of finite-dimensional vector space

Theorem

Every finite-dimensional vector space has a basis.

Proof

Let V be a finite-dimensional vector space. By definition, this means that there exists a finite spanning list $v_1, v_2, \ldots, v_n$ of vectors in V. We will show that we can reduce this spanning list to a basis for V.

By the previous theorem, we know that every spanning list can be reduced to a basis. Therefore, we can apply this result to our spanning list $v_1, v_2, \ldots, v_n$ to obtain a basis $u_1, u_2, \ldots, u_k$ for V.

Thus, we conclude that every finite-dimensional vector space has a basis.

$\square$

Linearly independent list extends to a basis

Theorem

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof

Let V be a finite-dimensional vector space, and let $\{v_1, v_2, \ldots, v_k\}$ be a linearly independent list of vectors in V. Since V is finite-dimensional, it has a basis $\{u_1, u_2, \ldots, u_n\}$ with n vectors.

We can extend the list $\{v_1, v_2, \ldots, v_k\}$ to a basis of V by adding vectors from the basis $\{u_1, u_2, \ldots, u_n\}$ that are not in the span of $\{v_1, v_2, \ldots, v_k\}$.

Specifically, we can take a vector u_i from the basis $\{u_1, u_2, \ldots, u_n\}$ that is not in the span of $\{v_1, v_2, \ldots, v_k\}$ and add it to our list. This new list $\{v_1, v_2, \ldots, v_k, u_i\}$ will still be linearly independent, as the addition of u_i does not introduce any linear dependencies.

We can repeat this process until we have added enough vectors to form a basis for V. Since V is finite-dimensional, this process must terminate, and we will obtain a basis for V that extends the original linearly independent list $\{v_1, v_2, \ldots, v_k\}$.

Therefore, we conclude that every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

$\square$

Every subspace of V is part of a direct sum equal to V

Theorem

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof

Let V be a finite-dimensional vector space and U be a subspace of V. Since V is finite-dimensional, we can choose a basis $\{u_1, u_2, \ldots, u_k\}$ for U and extend it to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$ for V.

We claim that $V = U \oplus W$, where $W = \text{span}\{v_1, v_2, \ldots, v_m\}$. To show this, we need to verify two things:

1. $V = U + W$: Any vector $v \in V$ can be expressed as a linear combination of the basis vectors, so we can write

$$v = a_1u_1 + a_2u_2 + \ldots + a_ku_k + b_1v_1 + b_2v_2 + \ldots + b_mv_m$$

   for some scalars a_i and b_j. This shows that V is the sum of U and W.

2. $U \cap W = \{0\}$: If a vector x is in both U and W, it can be expressed as a linear combination of the basis vectors for U and W. However, since the basis vectors for W are not in U, the only vector that can be in both subspaces is the zero vector. Thus, $U \cap W = \{0\}$.

Since both conditions are satisfied, we conclude that $V = U \oplus W$.

$\square$

Basis length does not depend on basis

Theorem

Any two bases of a finite-dimensional vector space have the same length.

Proof

Let $\{u_1, u_2, \ldots, u_n\}$ be a basis for the finite-dimensional vector space V, and let $\{v_1, v_2, \ldots, v_m\}$ be another basis for V. We need to show that $n = m$.

Since $\{v_1, v_2, \ldots, v_m\}$ is a basis for V, it is linearly independent and spans V. Therefore, each vector u_i can be expressed as a linear combination of the vectors v_j:

$$u_i = a_{i1}v_1 + a_{i2}v_2 + \ldots + a_{im}v_m$$

for some scalars a_{ij}. This means that the set $\{u_1, u_2, \ldots, u_n\}$ is also linearly dependent on the set $\{v_1, v_2, \ldots, v_m\}$.

Conversely, since $\{u_1, u_2, \ldots, u_n\}$ is a basis for V, each vector v_j can be expressed as a linear combination of the vectors u_i:

$$v_j = b_{j1}u_1 + b_{j2}u_2 + \ldots + b_{jn}u_n$$

for some scalars b_{ji}. This means that the set $\{v_1, v_2, \ldots, v_m\}$ is also linearly dependent on the set $\{u_1, u_2, \ldots, u_n\}$.

Since both sets are linearly dependent on each other, we conclude that they must have the same number of vectors, $n = m$.

$\square$

Dimension

Definition

The dimension of a vector space V is defined as the length of any basis of V.

Notation

The dimension of a vector space V is denoted by $\dim(V)$.

Dimension of a subspace

Theorem

If V is finite-dimensional and U is a subspace of V, then $\dim(U) \leq \dim(V)$.

Proof

Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for the subspace U and extend it to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$ for V. Since the basis for U has k vectors and the basis for V has $k+m$ vectors, we have

$$\dim(U) = k \leq k + m = \dim(V).$$

$\square$

Linearly independent list of the right length is a basis

Theorem

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim(V)$ is a basis of V.

Proof

Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent list of vectors in V with length $n = \dim(V)$. We need to show that this list is a basis for V.

Since $n = \dim(V)$, any basis for V must also have n vectors. We can extend the list $\{v_1, v_2, \ldots, v_n\}$ to a basis for V by adding vectors from V that are not in the span of the v_i's.

However, since $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, it cannot be expressed as a linear combination of any other vectors in V. Therefore, the only way to extend this list to a basis is to include all n vectors.

Thus, we conclude that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.

$\square$

Spanning list of the right length is a basis

Theorem

Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

Proof

Let $\{v_1, v_2, \ldots, v_n\}$ be a spanning list of vectors in V with length $n = \dim(V)$. We need to show that this list is a basis for V.

Since $n = \dim(V)$, any basis for V must also have n vectors. We can extend the list $\{v_1, v_2, \ldots, v_n\}$ to a basis for V by adding vectors from V that are not in the span of the v_i's.

However, since $\{v_1, v_2, \ldots, v_n\}$ is spanning, it must be able to express any vector in V as a linear combination of the v_i's. Therefore, the only way to extend this list to a basis is to include all n vectors.

Thus, we conclude that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.

$\square$

Dimension of a sum

Theorem

If U and W are finite-dimensional subspaces of a vector space V, then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Proof

Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for U and $\{w_1, w_2, \ldots, w_m\}$ be a basis for W. Then $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_m\}$ spans $U + W$.

To show that this set is linearly independent, we need to consider the intersection $U \cap W$. Let $\{z_1, z_2, \ldots, z_p\}$ be a basis for $U \cap W$. Then we can express the dimensions as follows:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

$\square$