# VECTOR SPACES

***Notations***

- $\mathbb{F}$ can be used instead of $\mathbb{C}$ or $\mathbb{R}$
- $V$ is the vector space of $\mathbb{F}$.

***Remark***

The letter $\mathbb{F}$ is used because $\mathbb{R}$ and $\mathbb{C}$ are examples of what are called fields.

## Lists and length

***Definition***

Suppose $n$ is a nonnegative integer. A list of length $n$ is an ordered collection of $n$-elements, of some mathematical objects, is separated by commas and surrounded by parentheses. A list of length $n$-looks like this:

$$(x_1, \cdots, x_n)$$

***Definition***

Two lists are equal if and only if they have the same length and the same elements in the same order.

## Field $\mathbb{F}^n$

***Definition***

$\mathbb{F}^n$ is the set of all lists of length $n$ of elements of $\mathbb{F}:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}$$

For $(x_1, \ldots, x_n) \in \mathbb{F}^n$ and $j \in \{1, \ldots, n\}$, we say that $x_j$ is the $j^{\text{th}}$ coordinate of $(x_1, \ldots, x_n)$

# Addition in $\mathbb{F}$

***Definition***

Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

# Commutativity of addition in $\mathbb{F}^n$

If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

***Proof***

Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$\begin{aligned} x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\ &= (x_1 + y_1, \ldots, x_n + y_n) \\ &= (y_1 + x_1, \ldots, y_n + x_n) \\ &= (y_1, \ldots, y_n) + (x_1, \ldots, x_n) \\ &= y + x, \end{aligned}$$

where the second and fourth equalities above hold because of the definition of addition in $\mathbb{F}^n$ and the third equality holds because of the usual commutativity of addition in $\mathbb{F}$.

$\square$

# Zero

***Definition***

Let 0 denote the list of length $n$-whose coordinates are all 0:

$$0 = (0, \ldots, 0)$$

# Additive inverse in $\mathbb{F}^n$

***Definition***

For $x \in \mathbb{F}^n$, the additive inverse of $x$, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that

$$x + (-x) = 0$$

# Definition scalar multiplication in $\mathbb{F}^n$

***Definition***

The product of a number $\lambda$ and a vector in $\mathbb{F}^n$ is computed by multiplying each coordinate of the vector by $\lambda$:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

here $\lambda \in \mathbb{F}$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

# Addition and scalar multiplication

***Definition***

- An addition on a set $V$ is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- Scalar multiplication on a set $V$ is a function that assigns an element $\lambda v \in V$ to each $\lambda \in F$ and each $v \in V$.

# Vector space

***Definition***

A vector space is a set $V$ along with an addition on $V$ and a scalar multiplication on $V$ such that the following properties hold:

- Commutativity

$$u + v = v + u \text{ for all } u, v \in V$$

- Associativity

$$(u+v)+w=u+(v+w)$$

and

$$(ab)v=a(bv) \quad \forall u,v,w \in V$$

and all $a,b \in \mathbb{F}$

- Additive identity

there exists an element $0 \in V$ such that $v+0=v$ for all $v \in V$

- additive inverse

for every $v \in V$, there exists $w \in V$ such that $v+w=0$

- Multiplicative identity

$$1v=v \quad \forall v \in V$$

- Distributive properties

$$a(u+v)=au+av$$

and

$$(a+b)v=av+bv \quad \forall a,b \in \mathbb{F} \text{ and } \forall u,v \in V.$$

# Points and vectors

***Definition***

Elements of a vector space are called points or vectors.

# Real vector space and complex vector space

***Definitions***

- A vector space over $\mathbb{R}$ is called a real vector space.

- A vector space over $\mathbb{C}$ is called a complex vector space.

# Set of functions

***Notation***

If $S$ is a set, then $\mathbb{F}^S$ denotes the set of functions from $S$ to $\mathbb{F}$.

- For $f, g \in \mathbb{F}^S$, the sum $f + g \in \mathbb{F}^S$ is the function defined by

$$(f+g)(x) = f(x) + g(x)$$

  for all $x \in S$

- For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the product $\lambda f \in \mathbb{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

  for all $x \in S$.

# Unique additive identity

***Theorem***

A vector space has a unique additive identity.

***Proof***

Suppose 0 and 0' are both additive identities for some vector space $V$. Then

$\square$

# Unique additive inverse

***Theorem***

Every element in a vector space has a unique additive inverse.

***Proof***

Suppose $v$ and $w$ are both additive inverses of some element $u \in V$. Then: $v + u = 0$ and $w + u = 0$

Adding the additive inverse of $v + u$ to both sides of the equation above gives $0 = w - v$, as desired.

$\square$

***Notation***

Let $v, W \in V$, then

- $-v$ denotes the additive inverse of $v$
- $w - v$ is defined to be $w + (-w)$

# Numbers and vectors

***Theorem***

$0v = 0$ every $v \in V$.

***Proof***

$$0v = (0 + 0)v = 0v + 0v$$

Adding the additive inverse of $0v$ to both sides of the equation above gives $0v = 0$, as desired.

$\square$

***Theorem***

$a0 = 0$ for every $a \in \mathbb{F}$.

***Proof***

$$a0 = a(0 + 0) = a0 + a0$$

Adding the additive inverse of $a0$ to both sides of the equation above gives $0 = a0$, as desired.

□

***Theorem***

$(-1)v = -v$ for every $v \in V$.

***Proof***

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0$$

This equation says that $(-1)v$, when added to $v$, gives $0$. Thus $(-1)v$ is the additive inverse of $v$, as desired.

□

# Subspaces

***Definition***

A subset $W$ of a vector space $V$ is called a subspace if:

- It contains the zero vector.
- It is closed under vector addition.
- It is closed under scalar multiplication.

# Sum of subsets

***Definition***

Let $U$ and $W$ be subsets of a vector space $V$. The sum of $U$ and $W$, denoted $U + W$, is the set defined by:

$$U + W = \{u + w \mid u \in U, w \in W\}$$

# Sum of subspaces is the smallest containing subspace

***Theorem***

Suppose $U_1, \dots, U_m$ are subspaces of $V$. Then $U_1 + \dots + U_m$ is the smallest subspace of $V$ containing $U_1, \dots, U_m$.

***Proof***

Let $W$ be the smallest subspace of $V$ containing $U_1, \dots, U_m$. We need to show that $W$ is contained in $U_1 + \dots + U_m$.
Since $W$ is a subspace, it is closed under addition and scalar multiplication. Therefore, if $w \in W$, then for any $u_i \in U_i$ and $\lambda \in \mathbb{F}$, we have $w + u_i \in W$ and $\lambda w \in W$.
Now, consider any element $z \in U_1 + \dots + U_m$. By definition, $z$ can be written as $z = u_1 + \dots + u_m$ for some $u_i \in U_i$. Since each $U_i$ is contained in $W$, it follows that $z \in W$.
Thus, we have shown that $W \subseteq U_1 + \dots + U_m$, which proves the theorem.

$\square$

# Direct Sum

***Definition***

Suppose $U_1, \dots, U_m$ are subspaces of $V$.

- The sum $U_1 + \dots + U_m$ is called a direct sum if each element of $U_1 + \dots + U_m$ can be written in only one way as a sum $u_1 + \dots + u_m$, where each $u_j$ is in $U_j$.
- If $U_1 + \dots + U_m$ is a direct sum, then $U_1 \oplus \dots \oplus U_m$ denotes $U_1 + \dots + U_m$, with the $\oplus$ notation serving as an indication that this is a

direct sum.

# Condition for a direct sum

***Theorem***

Suppose $U_1, \ldots, U_m$ are subspaces of $V$. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the intersection $U_i \cap U_j = \{0\}$ for all $i \neq j$.

***Proof***

Suppose $U_1, \ldots, U_m$ are subspaces of $V$. We will show that $U_1 + \cdots + U_m$ is a direct sum if and only if the intersection $U_i \cap U_j = \{0\}$ for all $i \neq j$.
($\Rightarrow$) Assume $U_1 + \cdots + U_m$ is a direct sum. Let $v \in U_i \cap U_j$ for some $i \neq j$. Then we can write

$$v = u_i + \cdots + u_m$$

for some $u_k \in U_k$. However, since $v \in U_i$ and $v \in U_j$, this representation is not unique, contradicting the assumption that the sum is direct. Thus, we must have $v = 0$, proving that $U_i \cap U_j = \{0\}$.
($\Leftarrow$) Now assume $U_i \cap U_j = \{0\}$ for all $i \neq j$. Let $v \in U_1 + \cdots + U_m$. Then we can write

$$v = u_1 + \cdots + u_m$$

for some $u_k \in U_k$. Suppose there is another representation

$$v = w_1 + \cdots + w_m$$

for some $w_k \in U_k$. Then we have

$$u_1 + \cdots + u_m = w_1 + \cdots + w_m.$$

Rearranging gives

$$(u_1 - w_1) + \cdots + (u_m - w_m) = 0.$$

Since the $U_i$ are pairwise disjoint, it follows that each $u_k - w_k = 0$, proving the uniqueness of the representation. Thus, $U_1 + \cdots + U_m$ is a direct sum.

$\square$

# Direct sum of two subspaces

***Theorem***

Suppose $U$ and $W$ are subspaces of $V$. Then $U \oplus W$ is a direct sum if and only if $U \cap W = \{0\}$.

***Proof***

Suppose $U$ and $W$ are subspaces of $V$. We will show that $U \oplus W$ is a direct sum if and only if $U \cap W = \{0\}$.
($\Rightarrow$) Assume $U \oplus W$ is a direct sum. Let $v \in U \cap W$. Then we can write

$$v = u + w$$

for some $u \in U$ and $w \in W$. However, since $v \in U$ and $v \in W$, this representation is not unique, contradicting the assumption that the sum is direct. Thus, we must have $v = 0$, proving that $U \cap W = \{0\}$.
($\Leftarrow$) Now assume $U \cap W = \{0\}$. Let $v \in U \oplus W$. Then we can write

$$v = u + w$$

for some $u \in U$ and $w \in W$. Suppose there is another representation

$$v = w_1 + w_2$$

for some $w_k \in W$. Then we have

$$u + w = w_1 + w_2.$$

Rearranging gives

$$(u - w_1) + (w - w_2) = 0.$$

Since $U \cap W = \{0\}$, it follows that each $u - w_1 = 0$ and $w - w_2 = 0$, proving the uniqueness of the representation. Thus, $U \oplus W$ is a direct sum.

$\square$