

EIGENVECTORS AND EIGENVALUES

Eigenvector and Eigenvalue

Definition

Given an $(n \cdot m)$ matrix A and an n -vector \vec{r} , if $\vec{r}' = Ar$ points in the same direction as \vec{r} , i.e. $\vec{r}' = \lambda \vec{r}$ where λ is a real scalar, then r is called an eigenvector of A with real eigenvalue.

The cases $r = 0$ or $\lambda = 0$ are excluded from this definition.

Trivial Eigenvalues

Theorem

If the matrix equation $A\vec{v} = \lambda\vec{v}$, is invertible we have the trivial solutions:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

and

$$\vec{v} = \vec{0}$$

Proof

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ \Leftrightarrow (A - \lambda\mathbb{1})\vec{v} &= \vec{0} \end{aligned}$$

If A is reversible:

$$\begin{aligned} \Leftrightarrow A &= \lambda\mathbb{1} \vee \vec{v} = (A - \lambda\mathbb{1})^{-1} \vec{0} \\ \Leftrightarrow A &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vee \vec{v} = \vec{0} \end{aligned}$$

Characteristic equation

When solving for non trivial eigenvalues we call the following equation the characteristic equation:

$$\det(A - \lambda \mathbb{1}) = 0$$

Characteristic polynomial

We call $A - \lambda \mathbb{1}$, the characteristic polynomial of A, when solving for eigenvalues.

Non-trivial Eigenvalues

Theorem 1

If $(A - \lambda \mathbb{1})$ does not have an inverse, $\vec{v} \neq 0$.

Theorem 2

For the real scalar λ to be an eigenvalue of the matrix A it must be a real root of the characteristic equation:

$$\det(A - \lambda \mathbb{1}) = 0$$

Remark

This is a polynomial equation of degree n if A is an $(n \cdot n)$ matrix.

Multiple of eigenvector solutions

Theorem

For any solution of eigenvectors \vec{v} there is a multiple of the eigenvector $\alpha \in \mathbb{R}$ that still satisfies the equation:

$$A\overrightarrow{\alpha v}=\lambda\overrightarrow{\alpha v}$$