CONTINUITY

Definition

Let $a \in \mathbb{R}$. A function $f : D \to \mathbb{R}$ is continuous at a if there exists an open interval $I \subset D$ containing a and if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

Definition

If a function is continuous at every point in its domain, we say that the function is continuous.

Sum of functions

Theorem

The sum of two continuous functions is continuous.

Proof

Let $f, g : D \to \mathbb{R}$ be two continuous functions at $a \in D$. Then for all $\epsilon > 0$ there exist $\delta_f, \delta_g > 0$ such that if $|x - a| < \delta_f$ then $|f(x) - f(a)| < \epsilon/2$ and if $|x - a| < \delta_g$ then $|g(x) - g(a)| < \epsilon/2$. Let $\delta = \min(\delta_f, \delta_g)$. Then if $|x - a| < \delta$ we have

$$\begin{aligned}
|(f+g)(x) - (f+g)(a)| &= |f(x) + g(x) - f(a) - g(a)| \\
&\leq |f(x) - f(a)| + |g(x) - g(a)| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

$\square$

Product of functions

Theorem

The product of two continuous functions is continuous.

Proof

Let $f, g : D \to \mathbb{R}$ be two continuous functions at $a \in D$. Then for all $\epsilon > 0$ there exist $\delta_f, \delta_g > 0$ such that if $|x - a| < \delta_f$ then $|f(x) - f(a)| < \epsilon/2$ and if $|x - a| < \delta_g$ then $|g(x) - g(a)| < \epsilon/2$. Let $\delta = \min(\delta_f, \delta_g)$. Then if $|x - a| < \delta$ we have

$$
\begin{gathered}
|(f \cdot g)(x) - (f \cdot g)(a)| = |f(x) \cdot g(x) - f(a) \cdot g(a)| \\
\leq |f(x) \cdot g(x) - f(x) \cdot g(a)| + |f(x) \cdot g(a) - f(a) \cdot g(a)| \\
= |f(x)| \cdot |g(x) - g(a)| + |g(a)| \cdot |f(x) - f(a)| \\
< |f(x)| \cdot \epsilon/2 + |g(a)| \cdot \epsilon/2 \\
< \epsilon/2 + \epsilon/2 \\
= \epsilon.
\end{gathered}
$$

□

Intermediate Value Theorem

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and N is a number between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ such that $f(c) = N$.

Proof

Let $f : [a, b] \to \mathbb{R}$ be continuous and N a number between $f(a)$ and $f(b)$. Without loss of generality, assume $f(a) < N < f(b)$. Then the set $S = \{x \in [a, b] : f(x) < N\}$ is non-empty and bounded above by b. Let $c = \sup S$. Then $f(c) \leq N$.

If $f(c) = N$, we are done. If $f(c) < N$, then by continuity of f at c, there exists a $\delta > 0$ such that for all $x \in [c-\delta, c+\delta] \cap [a,b]$, we have $|f(x) - f(c)| < N - f(c)$. This implies $f(x) < N$ for all such x, contradicting the definition of c as the least upper bound.

$\square$

Uniform continuity

Definition

A function $f : D \to \mathbb{R}$ is uniformly continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x, y \in D$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Theorem

Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

Proof

Let $\epsilon > 0$. Since f is continuous on the compact interval $[a,b]$, it is uniformly continuous on $[a,b]$. Therefore, there exists a $\delta > 0$ such that for all $x, y \in [a,b]$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Brouwers' fixed point theorem in dimension 1

Theorem

Let $f : [a,b] \to [a,b]$ be a continuous function. Then there exists a point $c \in [a,b]$ such that $f(c) = c$.

Proof

We will use the Brouwer fixed point theorem in a more general setting. Consider the function $g : [a,b] \to [a,b]$ defined by $g(x) = f(x) - x$. Then g is continuous and $g(a) \leq 0$, $g(b) \geq 0$. By the intermediate value theorem, there exists a point $c \in [a, b]$ such that $g(c) = 0$, also $f(c) = c$.

$\square$

Bolzano-Weierstrass Theorem

Theorem

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof

Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\mathbb{R}^n$. By the Bolzano-Weierstrass theorem, every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. Therefore, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ that converges to some limit $L \in \mathbb{R}^n$.

Maxima and minima

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f admits a maximum and a minimum on $[a, b]$.

Proof

Let $f : [a, b] \to \mathbb{R}$ be continuous. Since $[a, b]$ is compact, f is uniformly continuous on $[a, b]$. Therefore, by the extreme value theorem, f attains a maximum and a minimum on $[a, b]$.