LINEAR MAPS

Linear Map

Definition

A linear map from V to W is a function $T : V \to W$ with the following properties:

- additivity: $T(u + v) = T(u) + T(v)$ for all $u, v \in V$
- homogeneity: $T(c \cdot v) = c \cdot T(v)$ for all $v \in V$ and all scalars c

Notation

The set of all linear maps from V to W is denoted $\mathcal{L}(V; W)$.

Linear maps and basis of domain

Theorem

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$T(v_i) = w_i$ for all $i = 1, \ldots, n$.

Proof

Since $\{v_1, \ldots, v_n\}$ is a basis for V, every $v \in V$ can be written uniquely as

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n,$$

where $c_1, c_2, \ldots, c_n$ are scalars. Define $T : V \to W$ by

$$T(v) = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n.$$

This is well-defined due to the uniqueness of the representation. We verify that T is linear:

- Let $v, u \in V$, with $v = \sum_{i=1}^n c_i v_i$ and $u = \sum_{i=1}^n d_i v_i$. Then

$$v + u = \sum_{i=1}^n (c_i + d_i) v_i,$$

so

$$T(v+u) = \sum_{i=1}^n (c_i + d_i) w_i = \sum_{i=1}^n c_i w_i + \sum_{i=1}^n d_i w_i = T(v) + T(u).$$

- For any scalar a,

$$av = \sum_{i=1}^n (ac_i) v_i,$$

so

$$T(av) = \sum_{i=1}^n (ac_i) w_i = a \sum_{i=1}^n c_i w_i = aT(v).$$

Thus, T is linear. Moreover, for each basis vector v_j,

$$v_j = 0 \cdot v_1 + \cdots + 1 \cdot v_j + \cdots + 0 \cdot v_n,$$

so

$$T(v_j) = 0 \cdot w_1 + \cdots + 1 \cdot w_j + \cdots + 0 \cdot w_n = w_j.$$

Therefore, T satisfies $T(v_i) = w_i$ for all i.

Suppose $T : V \to W$ is a linear map such that $T(v_i) = w_i$ for all i. For any $v \in V$, write

$$v = \sum_{i=1}^n c_i v_i.$$

Then by linearity of T,

$$T(v) = T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i T(v_i) = \sum_{i=1}^{n} c_i w_i.$$

This shows that $T(v)$ is completely determined by the values $T(v_i) = w_i$. Hence, T is unique.

$\square$

Addition and scalar multiplication on $\mathcal{L}(V;W)$

Definition

Let $T_1, T_2 \in \mathcal{L}(V;W)$ and $c \in \mathbb{F}$. We define addition and scalar multiplication on $\mathcal{L}(V;W)$ as follows:

- Addition: $(T_1 + T_2)(v) = T_1(v) + T_2(v)$ for all $v \in V$.
- Scalar multiplication: $(c \cdot T)(v) = c \cdot T(v)$ for all $v \in V$.

Vector space $\mathcal{L}(V;W)$

Theorem

The set $\mathcal{L}(V;W)$ is a vector space over the field $\mathbb{F}$.

Proof

To show that $\mathcal{L}(V;W)$ is a vector space, we need to verify the following properties:

- Closure under addition: Let $T_1, T_2 \in \mathcal{L}(V;W)$. Then $(T_1 + T_2)(v) = T_1(v) + T_2(v)$ is a linear map, so $T_1 + T_2 \in \mathcal{L}(V;W)$.
- Closure under scalar multiplication: Let $T \in \mathcal{L}(V;W)$ and $c \in \mathbb{F}$. Then $(c \cdot T)(v) = c \cdot T(v)$ is a linear map, so $c \cdot T \in \mathcal{L}(V;W)$.

- Existence of zero vector: The zero map $0 : V \to W$ defined by $0(v) = 0$ for all $v \in V$ is in $\mathcal{L}(V; W)$.
- Existence of additive inverses: For each $T \in \mathcal{L}(V; W)$, the map $-T$ defined by $(-T)(v) = -T(v)$ is in $\mathcal{L}(V; W)$.

Since all vector space axioms are satisfied, we conclude that $\mathcal{L}(V; W)$ is a vector space over the field $\mathbb{F}$.

$\square$

Product of Linear Maps

Definition

Let $T_1 : V \to W$ and $T_2 : W \to U$ be linear maps. The product of T_1 and T_2, denoted $T_2 \circ T_1$, is defined by

$$(T_2 \circ T_1)(v) = T_2(T_1(v))$$

for all $v \in V$.

Algebraic properties of products of linear maps

- associativity

$$(T_3 \cdot T_2) \cdot T_1 = T_3 \cdot (T_2 \cdot T_1)$$

  Whenever T_1, T_2, and T_3 are linear maps such that the products make sense.

- identity

$$T \cdot I = T \cdot I = T$$

  Whenever $T \in \mathcal{L}(V; W)$.

- distributive properties

$$(S_1 + S_2)T = S_1T + S_2T \quad \text{and} \quad S(T_1 + T_2) = ST_1 + ST_2$$

Whenever $T, T_1, T_2 \in \mathcal{L}(U; V)$ and $S, S_1, S_2 \in \mathcal{L}(V; W)$.

Linear maps take 0 to 0

Theorem

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Proof

Let $v \in V$. Then by linearity, we have:

$$T(0) = T(0 \cdot v) = 0 \cdot T(v) = 0.$$

$\square$