VECTOR SPACES

Notations

- $\mathbb{F}$ can be used instead of $\mathbb{C}$ or $\mathbb{R}$

- V is the vector space of $\mathbb{F}$.

Remark

The letter $\mathbb{F}$ is used because $\mathbb{R}$ and $\mathbb{C}$ are examples of what are called fields.

Lists and length

Definition

Suppose n is a nonnegative integer. A list of length n is an ordered collection of n-elements, of some mathematical objects, is separated by commas and surrounded by parentheses. A list of length n-looks like this:

$$(x_1, \cdots, x_n)$$

Definition

Two lists are equal if and only if they have the same length and the same elements in the same order.

Field $\mathbb{F}^n$

Definition

$\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}$$

For $(x_1, \ldots, x_n) \in \mathbb{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the jth coordinate of $(x_1, \ldots, x_n)$

Addition in $\mathbb{F}$

Definition

Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

Commutativity of addition in $\mathbb{F}^n$

If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

Proof

Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$\begin{aligned}
x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\
&= (x_1 + y_1, \ldots, x_n + y_n) \\
&= (y_1 + x_1, \ldots, y_n + x_n) \\
&= (y_1, \ldots, y_n) + (x_1, \ldots, x_n) \\
&= y + x,
\end{aligned}$$

where the second and fourth equalities above hold because of the definition of addition in $\mathbb{F}^n$ and the third equality holds because of the usual commutativity of addition in $\mathbb{F}$.

$\square$

Zero

Definition

Let 0 denote the list of length n-whose coordinates are all 0:

$$0 = (0, \ldots, 0)$$

Additive inverse in $\mathbb{F}^n$

For $x \in \mathbb{F}^n$, the additive inverse of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that

$$x + (-x) = 0$$

Definition scalar multiplication in $\mathbb{F}^n$

Definition

The product of a number λ and a vector in $\mathbb{F}^n$ is computed by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

here $\lambda \in \mathbb{F}$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Addition and scalar multiplication

Definition

- An addition on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.

- Scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in F$ and each $v \in V$.

Vector space

Definition

A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- Commutativity

$$u + v = v + u \text{ for all } u, v \in V$$

- Associativity

$$(u + v) + w = u + (v + w)$$

and

$$(ab)v = a(bv) \quad \forall u, v, w \in V$$

and all $a, b \in \mathbb{F}$

- Additive identity

there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$

- additive inverse

for every $v \in V$, there exists $w \in V$ such that $v + w = 0$

- Multiplicative identity

$$1v = v \quad \forall v \in V$$

- Distributive properties

$$a(u + v) = au + av$$

and

$$(a + b)v = av + bv \quad \forall a, b \in \mathbb{F} \text{ and } \forall u, v \in V.$$

Points and vectors

Definition

Elements of a vector space are called points or vectors.

Real vector space and complex vector space

Definitions

- A vector space over $\mathbb{R}$ is called a real vector space.

- A vector space over $\mathbb{C}$ is called a complex vector space.

Set of functions

Notation

If S is a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$.

- For $f, g \in \mathbb{F}^S$, the sum $f + g \in \mathbb{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

  for all $x \in S$

- For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the product $\lambda f \in \mathbb{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

  for all $x \in S$.

Unique additive identity

Theorem

A vector space has a unique additive identity.

Proof

Suppose 0 and 0' are both additive identities for some vector space V. Then

$\square$

Unique additive inverse

Theorem

Every element in a vector space has a unique additive inverse.

Proof

Suppose v and w are both additive inverses of some element $u \in V$. Then:

$v + u = 0$ and $w + u = 0$

Adding the additive inverse of $v + u$ to both sides of the equation above gives

$0 = w - v$, as desired.

☐

Notation

Let $v, W \in V$, then

- $-v$ denotes the additive inverse of v

- $w - v$ is defined to be $w + (-w)$

Numbers and vectors

Theorem

$0v = 0$ every $v \in V$.

Proof

$$0v = (0 + 0)v = 0v + 0v$$

Adding the additive inverse of $0v$ to both sides of the equation above gives

$0v = 0$, as desired.

☐

Theorem

$a0 = 0$ for every $a \in \mathbb{F}$.

Proof

$$a0 = a(0 + 0) = a0 + a0$$

Adding the additive inverse of $a0$ to both sides of the equation above gives $0 = a0$, as desired.

$\square$

Theorem

$(-1)v = -v$ for every $v \in V$.

Proof

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0$$

This equation says that $(-1)v$, when added to v, gives 0. Thus $(-1)v$ is the additive inverse of v, as desired.

$\square$

Subspaces

Definition

A subset W of a vector space V is called a subspace if:

- It contains the zero vector.

- It is closed under vector addition.

- It is closed under scalar multiplication.

Sum of subsets

Definition

Let U and W be subsets of a vector space V. The sum of U and W, denoted $U + W$, is the set defined by:

$$U + W = \{u + w \mid u \in U, w \in W\}$$

Sum of subspaces is the smallest containing subspace

Theorem

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Proof

Let W be the smallest subspace of V containing $U_1, \ldots, U_m$. We need to show that W is contained in $U_1 + \cdots + U_m$.

Since W is a subspace, it is closed under addition and scalar multiplication. Therefore, if $w \in W$, then for any $u_i \in U_i$ and $\lambda \in \mathbb{F}$, we have $w + u_i \in W$ and $\lambda w \in W$.

Now, consider any element $z \in U_1 + \cdots + U_m$. By definition, z can be written as $z = u_1 + \cdots + u_m$ for some $u_i \in U_i$. Since each U_i is contained in W, it follows that $z \in W$.

Thus, we have shown that $W \subseteq U_1 + \cdots + U_m$, which proves the theorem.

$\square$

Direct Sum

Definition

Suppose $U_1, \ldots, U_m$ are subspaces of V.

- The sum $U_1 + \cdots + U_m$ is called a direct sum if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j.

- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a

direct sum.

Condition for a direct sum

Theorem

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the intersection $U_i \cap U_j = \{0\}$ for all $i \neq j$.

Proof

Suppose $U_1, \ldots, U_m$ are subspaces of V. We will show that $U_1 + \cdots + U_m$ is a direct sum if and only if the intersection $U_i \cap U_j = \{0\}$ for all $i \neq j$.

($\Rightarrow$) Assume $U_1 + \cdots + U_m$ is a direct sum. Let $v \in U_i \cap U_j$ for some $i \neq j$. Then we can write

$$v = u_i + \cdots + u_m$$

for some $u_k \in U_k$. However, since $v \in U_i$ and $v \in U_j$, this representation is not unique, contradicting the assumption that the sum is direct. Thus, we must have $v = 0$, proving that $U_i \cap U_j = \{0\}$.

($\Leftarrow$) Now assume $U_i \cap U_j = \{0\}$ for all $i \neq j$. Let $v \in U_1 + \cdots + U_m$. Then we can write

$$v = u_1 + \cdots + u_m$$

for some $u_k \in U_k$. Suppose there is another representation

$$v = w_1 + \cdots + w_m$$

for some $w_k \in U_k$. Then we have

$$u_1 + \cdots + u_m = w_1 + \cdots + w_m.$$

Rearranging gives

$$(u_1 - w_1) + \cdots + (u_m - w_m) = 0.$$

Since the U_i are pairwise disjoint, it follows that each $u_k - w_k = 0$, proving the uniqueness of the representation. Thus, $U_1 + \cdots + U_m$ is a direct sum.

$\square$

Direct sum of two subspaces

Theorem

Suppose U and W are subspaces of V. Then $U \oplus W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof

Suppose U and W are subspaces of V. We will show that $U \oplus W$ is a direct sum if and only if $U \cap W = \{0\}$.

($\Rightarrow$) Assume $U \oplus W$ is a direct sum. Let $v \in U \cap W$. Then we can write

$$v = u + w$$

for some $u \in U$ and $w \in W$. However, since $v \in U$ and $v \in W$, this representation is not unique, contradicting the assumption that the sum is direct. Thus, we must have $v = 0$, proving that $U \cap W = \{0\}$.

($\Leftarrow$) Now assume $U \cap W = \{0\}$. Let $v \in U \oplus W$. Then we can write

$$v = u + w$$

for some $u \in U$ and $w \in W$. Suppose there is another representation

$$v = w_1 + w_2$$

for some $w_k \in W$. Then we have

$$u + w = w_1 + w_2.$$

Rearranging gives

$$(u - w_1) + (w - w_2) = 0.$$

Since $U \cap W = \{0\}$, it follows that each $u - w_1 = 0$ and $w - w_2 = 0$, proving the uniqueness of the representation. Thus, $U \oplus W$ is a direct sum.

$\square$