7.9 Exercises

Conceptual

1. It was mentioned in the chapter that a cubic regression spline with one knot at ξ can be obtained using a basis of the form x, x^2, x^3, $(x-\xi)^3_+$, where $(x-\xi)^3_+ = (x-\xi)^3$ if $x > \xi$ and equals 0 otherwise. We will now show that a function of the form

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x-\xi)^3_+$$

is indeed a cubic regression spline, regardless of the values of $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$.

(a) Find a cubic polynomial

$$f_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$$

such that $f(x) = f_1(x)$ for all $x \le \xi$. Express a_1, b_1, c_1, d_1 in terms of $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$.

(b) Find a cubic polynomial

$$f_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3$$

such that $f(x) = f_2(x)$ for all $x > \xi$. Express a_2, b_2, c_2, d_2 in terms of $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$. We have now established that $f(x)$ is a piecewise polynomial.

(c) Show that $f_1(\xi) = f_2(\xi)$. That is, $f(x)$ is continuous at ξ.

(d) Show that $f_1'(\xi) = f_2'(\xi)$. That is, $f'(x)$ is continuous at ξ.

1. (a) $\because$ for all $x \le \xi$, $(x-\xi)^3_+ = 0$

$\therefore$ when $a_1 = \beta_0$, $b_1 = \beta_1$, $c_1 = \beta_2$ and $d_1 = \beta_3$, $f(x) = f_1(x)$

(b) When $x > \xi$, $f(x) = (\beta_0 - \beta_4 \xi^3) + (\beta_1 + 3\beta_4 \xi^2)x + (\beta_2 - 3\beta_4 \xi)x^2 + (\beta_3 + \beta_4)x^3$

$$\Rightarrow \begin{cases} a_2 = \beta_0 - \beta_4 \xi^3 \\ b_2 = \beta_1 + 3\beta_4 \xi^2 \\ c_2 = \beta_2 - 3\beta_4 \xi \\ d_2 = \beta_3 + \beta_4 \end{cases}$$

(c) $f_1(\xi) = \beta_0 + \beta_1 \xi + \beta_2 \xi^2 + \beta_3 \xi^3$

$f_2(\xi) = (\beta_0 - \beta_4 \xi^3) + (\beta_1 + 3\beta_4 \xi^2)\xi + (\beta_2 - 3\beta_4 \xi)\xi^2 + (\beta_3 + \beta_4)\xi^3$

$= \beta_0 + \beta_1 \xi + \beta_2 \xi^2 + (\beta_3 + \beta_4)\xi^3 - \beta_4 \xi^3 + 3\beta_4 \xi^3 - 3\beta_4 \xi^3$

$= \beta_0 + \beta_1 \xi + \beta_2 \xi^2 + \beta_3 \xi^3$

$\therefore f_1(\xi) = f_2(\xi)$

(d) $f_1'(\xi) = \beta_1 + 2\beta_2 \xi + 3\beta_3 \xi^2$

$f_2'(\xi) = (\beta_1 + 3\beta_4 \xi^2) + 2(\beta_2 - 3\beta_4 \xi)\xi + 3(\beta_3 + \beta_4)\xi^2$

$= \beta_1 + 2\beta_2 \xi + 3\beta_4 \xi^2 - 6\beta_4 \xi^2 + 3\beta_3 \xi^2 + 3\beta_4 \xi^2$

$= \beta_1 + 2\beta_2 \xi + 3\beta_3 \xi^2$

$\therefore f_1'(\xi) = f_2'(\xi)$

e) $f_1''(\xi) = 2\beta_2 + 6\beta_3 \xi$

$f_2''(\xi) = 2(\beta_2 - 3\beta_4 \xi) + 6(\beta_3 + \beta_4)\xi$

$= 2\beta_2 + 6\beta_3 \xi$

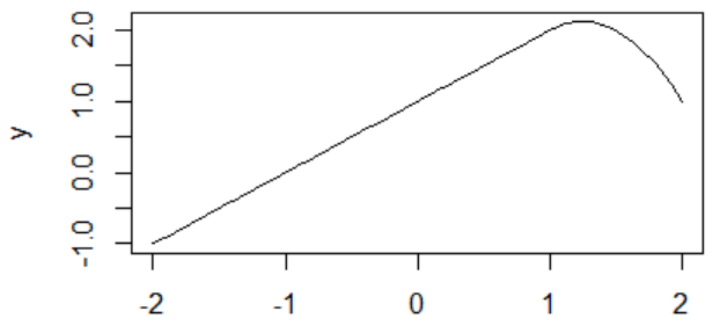
$\therefore f_1''(\xi) = f_2''(\xi)$

3. Suppose we fit a curve with basis functions $b_1(X) = X$, $b_2(X) = (X-1)^2 I(X \geq 1)$. (Note that $I(X \geq 1)$ equals 1 for $X \geq 1$ and 0 otherwise.) We fit the linear regression model

$$Y = \beta_0 + \beta_1 b_1(X) + \beta_2 b_2(X) + \epsilon,$$

and obtain coefficient estimates $\hat{\beta}_0 = 1, \hat{\beta}_1 = 1, \hat{\beta}_2 = -2$. Sketch the estimated curve between $X = -2$ and $X = 2$. Note the intercepts, slopes, and other relevant information.

```{r}
x = seq(-2,2,length=100)
y = 1+x+(-2)*(x-1)^2*I(x>1)
plot(x, y,type='l')
```



4. Suppose we fit a curve with basis functions $b_1(X) = I(0 \leq X \leq 2) - (X-1)I(1 \leq X \leq 2)$, $b_2(X) = (X-3)I(3 \leq X \leq 4) + I(4 < X \leq 5)$. We fit the linear regression model

$$Y = \beta_0 + \beta_1 b_1(X) + \beta_2 b_2(X) + \epsilon,$$

and obtain coefficient estimates $\hat{\beta}_0 = 1, \hat{\beta}_1 = 1, \hat{\beta}_2 = 3$. Sketch the estimated curve between $X = -2$ and $X = 2$. Note the intercepts, slopes, and other relevant information.

```
x = seq(-2,2,length=100)
y = 1+I(0<=x)*I(x<=2)-(x-1)*I(1<=x)*(x<=2)+3*((x-3)*I(3<=x)
*I(x<=4)+I(4<=x)*I(x<=5))
plot(x, y,type='l')
```