Question 1

a) Let $P(n)$ be the statement

$1 + 3 + \cdots + (2n + 1) = (n + 1)^2$

Basis Clause
Show that $n = 1$
$P(n)$ is where $n = 1$

$LHS = 1 + 3 + \cdots + (2n + 1)$
$= (2n + 1)$
$= (2(1) + 1)$
$= 3$

$RHS = (n + 1)^2$
$= n^2 + 2n + 1$
$= (1)^2 + 2(1) + 1$
$= 3$

$LHS = RHS = 3$.
Therefore, $P(n)$ is true

Inductive Hypothesis
Show that $n = k$.
$P(k)$ is where $n = k$

Assume k
$1 + 3 + \cdots + (2k + 1) = (k + 1)^2$

Inductive Step
If $P(k)$ is true, then $P(k + 1)$ must also be true

Assume $k + 1$
$1 + 3 + \cdots + (2k + 3) = (k + 2)^2$

$LHS = 1 + 3 + \cdots + (2(k + 1) + 1)$
$= 1 + 3 + \cdots + (2k + 2 + 1)$
$= 1 + 3 + \cdots + (2k + 3)$

$RHS = ((k + 1) + 1)^2$
$= (k + 2)^2$

$LHS = 1 + 3 + \cdots + (2k + 1) + (2k + 3)$
But, $1 + 3 + \cdots + (2k + 1) = (k + 1)^2$
Therefore, by the induction hypothesis:
$= (k + 1)^2 + (2k + 3)$
$= k^2 + 2k + 1 + 2k + 3$
$= k^2 + 4k + 4$
$= (k + 2)^2$

$RHS = (k + 2)^2$

$LHS = RHS$
Thus, $P(k + 1)$ is true
Hence, $P(k)$ is true
It then follows by mathematical induction that $P(n)$ is true.

b) Let $P(n)$ be the statement

$1 + 3^n < 4^n$

Basis Clause
Show that $n = 2$
$P(n)$ is where $n = 2$

$$
\begin{aligned}
LHS &= 1 + 3^n \\
&= 1 + 3^2 \\
&= 10
\end{aligned}
\qquad
\begin{aligned}
RHS &= 4^n \\
&= 4^2 \\
&= 16
\end{aligned}
$$

$10 < 16$ and $LHS < RHS$
Therefore, $P(n)$ is true

Inductive Hypothesis
Show that $n = k$
$P(k)$ is where $n = k$

Assume k
$1 + 3^k < 4^k$

Inductive Step
If $P(k)$ is true, then $P(k+1)$ must also be true

Assume $k + 1$
$1 + 3^{(k+1)} < 4^{(k+1)}$

$$
\begin{aligned}
LHS &= 1 + 3^{(k+1)} \\
&= 1 + 3.3^k
\end{aligned}
\qquad
\begin{aligned}
RHS &= 4^{(k+1)} \\
&= 4.4^k
\end{aligned}
$$

But, $1 + 3.3^k < 4.4^k$
Therefore, by the induction hypothesis:
$1 + 3.3^k < 4(1 + 3^k)$
$1 + 3.3^k < (3 + 1)(1 + 3^k)$ Re-write 4 as 3+1
$1 + 3.3^k < 3 + 3.3^k + 1 + 3^k$ Multiplying out
$1 + 3.3^k < (1 + 3.3^k) + (3 + 3^k)$ By regrouping
$0 < 3 + 3^k$ Remove $(1 + 3.3^k)$ from both sides

$0 < 3 + 3^k$ is true for all $k \geq 2$
$LHS < RHS$
Thus, $P(k + 1)$ is true
Hence, $P(k)$ is true
It then follows by mathematical induction that $P(n)$ is true for $n \geq 2$

Question 2

a) $40! = \ 8.1591528324789773434561126959612e + 47$

b) $\binom{20}{6} = \frac{20!}{(20-6)!6!} = \frac{20!}{14!6!} = 38760$

c) $20!.20! = \ 5.9190122\text{e} + 36$

d) $\binom{20}{1}\binom{20}{1} = \frac{20!}{(20-1)!1!}.\frac{20!}{(20-1)!1!} = \frac{20!}{19!1!}.\frac{20!}{19!1!} = 20.20 = 400$

e) $\binom{20}{6}\binom{20}{10} = \ \frac{20!}{(20-6)!6!}.\frac{20!}{(20-10)!10!} = \frac{20!}{14!6!}.\frac{20!}{10!10!} = 38760.184756 = 7161142560$

f) $\binom{40}{15} = \ \frac{40!}{(40-15)!15!} = 40225345056$

g) $\binom{20}{1}\binom{20}{1} = \frac{20!}{(20-1)!1!}.\frac{20!}{(20-1)!1!} = \frac{20!}{19!1!}.\frac{20!}{19!1!} = 20.20 = 400$

h) $\left[\binom{40}{2}\binom{38}{2}\binom{36}{2}\binom{34}{2}\binom{32}{2}\binom{30}{2}\binom{28}{2}\binom{26}{2}\binom{24}{2}\binom{22}{2}\binom{20}{2}\binom{18}{2}\right] \div 24$

$= [780 \times 703 \times 630 \times 561 \times 496 \times 435 \times 378 \times 325 \times 276 \times 231 \times 190 \times 153] \div 24$

$= 9.5206265\text{e} + 30 \div (12!)$

$= 1.9875981\text{e} + 22$

i) $\binom{40}{3} = \frac{40!}{(40-3)!3!} = \frac{40!}{37!3!} = 9880$

j)

k)

l)

m)

n)

o)

p)

Question 3

Arrangement with unlimited repetition

$5.5.5.5.5.5.5.5.5.5 = 5^{10} = 9765625$

Question 4

a)

- If no student got less than 10 out of 20, there are eleven possible marks that the students could have gotten.
- Each mark will represent a student (pigeon)
- Each container will be a pair or marks (pigeonhole)

  [10,10] [11,11] [12,12] [13,13] [14,14] [15,15] [16,16] [17,17] [18,18] [19,19] [20,20]

- We note that where each container has two students, the total number of students is 22.
- We have three remaining students, that need to be assigned to one pigeonhole each.
- Each pigeonhole already contains two students.
- If we add the three remaining students to any three pigeonholes. At least three will have the same mark

b)

- Group consecutive numbers into pairs (pigeonholes):
  [1,2] [3,4] [5,6]… [$2n$ -1, $2n$]
  Where $n > 1$
- If we chose $n + 1$ integers, by the pigeonhole principle, we should get a two that are from one of the pairs mentioned above.
- The pairs are already consecutive integers so two of the numbers chosen will also be consecutive

Question 5

By the extended pigeonhole principle, at least one pigeonhole will contain $\left|\frac{n-1}{m}\right| + 1$ pigeon(s).

If no student got less than 20% there are 81 possible marks that the students could have gotten.

- Each mark will represent a student (pigeon)
- Each container will be a pair or marks (pigeonhole)

$\left|\frac{165-1}{81}\right| + 1 = \left|\frac{164}{81}\right| + 1 = 3.02469\ldots.$

Therefore, at least 3 students obtained the same mark

Question 6

$R = \{(a,b) | \ a \ modulo \ b \leq 1 \ \}$

	1	2	3	4	5	6
1	x	x	x	x	x	x
2	x	x				
3	x	x	x			
4	x	x	x	x		
5	x	x	x	x	x	
6	x	x	x	x	x	x

a) yes. R is reflexive

b) no. R is not irreflexive

c) no. R is not symmetric

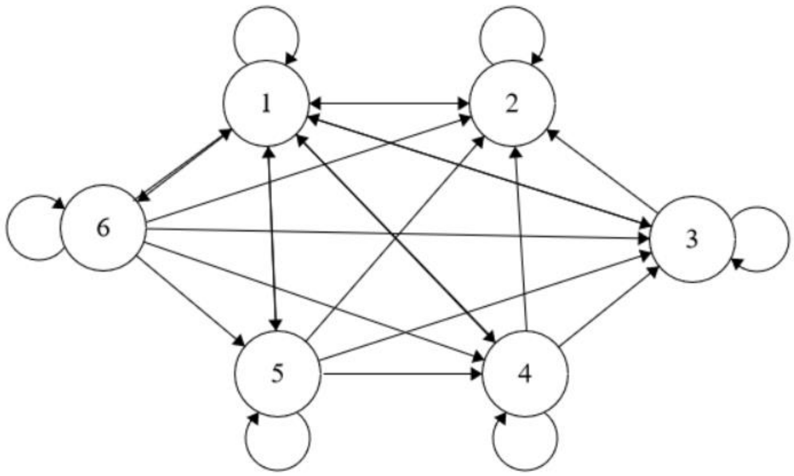
d) no. R is not asymmetric

e) yes. R is antisymmetric

f) no. R is not transitive

Question 7

a)



b)

2: in 5, out 1

3: in 4, out 2

c) $Dom(R) = A$ $\quad$ $Ran(R) = A$

d) 2-1-5

e) $R(2) = \{1,2,3,4,5,6\}$

f) $M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ $\quad$ $M_{R^2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$M_{R^2} = \begin{bmatrix} 6 & 6 & 5 & 4 & 3 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 1 & 1 & 1 \\ 4 & 4 & 3 & 2 & 1 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 5 & 4 & 3 & 2 \end{bmatrix}$ $\quad$ $= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

all positions non-zero

g)

- M_{R^2} shows the possible pairs that transivity can be tested against
- In M_{R^2}, if, for every position (a,b) and (b,c) that each have a 1, there is a 1 at (a,c), then the relation is true.
- Also, for all the positions in M_{R^2} that are non-zero (or 1), if M_R already has a 1 in the corresponding position, R is transitive

Question 8

a) no. R is not reflexive.
   The centre (main) diagonal has all 0's

b) yes. R is irreflexive.
   The centre (main) diagonal has all 0's

c) no. R is not symmetric.
   For every value, the value in the transposed position is not equal.

d) yes. R is asymmetric
   The centre (main) diagonal has all 0's
   For every value, the value in the transposed position is not equal.

e) yes. R is antisymmetric
   It does not matter what values the centre (main) diagonal has
   For every value and the value in the transposed position, they are both not 1

f) no. R is not transitive
   M_{R^2} has 1's in positions which M_R does not have

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 5 & 2 & 2 & 4 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 2 & 1 & 0 & 1 \\ 4 & 2 & 2 & 3 & 1 & 0 & 0 \\ 4 & 2 & 2 & 4 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Question 9
a)
b)

Question 10
a)
b)
c)

Question 11

Question 12
a)
b)