Question 1

a) Let $P(n)$ be the statement

$1 + 3 + \cdots + (2n + 1) = (n + 1)^2$

Basis Clause
Show that $n = 1$
$P(n)$ is where $n = 1$

$$\begin{aligned} LHS &= 1 + 3 + \cdots + (2n + 1) \\ &= (2n + 1) \\ &= (2(1) + 1) \\ &= 3 \end{aligned}$$

$$\begin{aligned} RHS &= (n + 1)^2 \\ &= n^2 + 2n + 1 \\ &= (1)^2 + 2(1) + 1 \\ &= 3 \end{aligned}$$

$LHS = RHS = 3$.
Therefore, $P(n)$ is true

Inductive Hypothesis
Show that $n = k$.
$P(k)$ is where $n = k$

Assume k
$1 + 3 + \cdots + (2k + 1) = (k + 1)^2$

Inductive Step
If $P(k)$ is true, then $P(k+1)$ must also be true

Assume $k + 1$
$1 + 3 + \cdots + (2k + 3) = (k + 2)^2$

$$\begin{aligned} LHS &= 1 + 3 + \cdots + (2(k + 1) + 1) \\ &= 1 + 3 + \cdots + (2k + 2 + 1) \\ &= 1 + 3 + \cdots + (2k + 3) \end{aligned}$$

$$\begin{aligned} RHS &= ((k + 1) + 1)^2 \\ &= (k + 2)^2 \end{aligned}$$

$LHS = 1 + 3 + \cdots + (2k + 1) + (2k + 3)$

$RHS = (k + 2)^2$

But, $1 + 3 + \cdots + (2k + 1) = (k + 1)^2$
Therefore, by the induction hypothesis:

$$\begin{aligned} &= (k + 1)^2 + (2k + 3) \\ &= k^2 + 2k + 1 + 2k + 3 \\ &= k^2 + 4k + 4 \\ &= (k + 2)^2 \end{aligned}$$

$LHS = RHS$
Thus, $P(k + 1)$ is true
Hence, $P(k)$ is true
It then follows by mathematical induction that $P(n)$ is true.

b) Let $P(n)$ be the statement

$1 + 3^n < 4^n$

Basis Clause
Show that $n = 2$
$P(n)$ is where $n = 2$

$$LHS = 1 + 3^n = 1 + 3^2 = 10$$

$$RHS = 4^n = 4^2 = 16$$

$10 < 16$ and $LHS < RHS$
Therefore, $P(n)$ is true

Inductive Hypothesis
Show that $n = k$
$P(k)$ is where $n = k$

Assume k
$1 + 3^k < 4^k$

Inductive Step
If $P(k)$ is true, then $P(k+1)$ must also be true

Assume $k + 1$
$1 + 3^{(k+1)} < 4^{(k+1)}$

$$LHS = 1 + 3^{(k+1)} = 1 + 3.3^k$$

$$RHS = 4^{(k+1)} = 4.4^k$$

But, $1 + 3.3^k < 4.4^k$
Therefore, by the induction hypothesis:
$1 + 3.3^k < 4(1 + 3^k)$
$1 + 3.3^k < (3 + 1)(1 + 3^k)$ Re-write 4 as 3+1
$1 + 3.3^k < 3 + 3.3^k + 1 + 3^k$ Multiplying out
$1 + 3.3^k < (1 + 3.3^k) + (3 + 3^k)$ By regrouping
$0 < 3 + 3^k$ Remove $(1 + 3.3^k)$ from both sides

$0 < 3 + 3^k$ is true for all $k \geq 2$
$LHS < RHS$
Thus, $P(k+1)$ is true
Hence, $P(k)$ is true
It then follows by mathematical induction that $P(n)$ is true for $n \geq 2$

Question 2

a) $40! = 8.159152832478977343456112695961 2e+47$

b) $\binom{20}{6} = \frac{20!}{(20-6)!6!} = \frac{20!}{14!6!} = 38760$

c) $20!.20! = 5.9190122\mathrm{e}+36$

d) $\binom{20}{1}\binom{20}{1} = \frac{20!}{(20-1)!1!}.\frac{20!}{(20-1)!1!} = \frac{20!}{19!1!}.\frac{20!}{19!1!} = 20.20 = 400$

e) $\binom{20}{6}\binom{20}{10} = \frac{20!}{(20-6)!6!}.\frac{20!}{(20-10)!10!} = \frac{20!}{14!6!}.\frac{20!}{10!10!} = 38760.184756 = 7161142560$

f) $\binom{40}{15} = \frac{40!}{(40-15)!15!} = 40225345056$

g) $\binom{20}{1}\binom{20}{1} = \frac{20!}{(20-1)!1!}.\frac{20!}{(20-1)!1!} = \frac{20!}{19!1!}.\frac{20!}{19!1!} = 20.20 = 400$

h) $\left[\binom{40}{2}\binom{38}{2}\binom{36}{2}\binom{34}{2}\binom{32}{2}\binom{30}{2}\binom{28}{2}\binom{26}{2}\binom{24}{2}\binom{22}{2}\binom{20}{2}\binom{18}{2}\right] \div 24$

$= [780 \times 703 \times 630 \times 561 \times 496 \times 435 \times 378 \times 325 \times 276 \times 231 \times 190 \times 153] \div 24$

$= 9.5206265\mathrm{e}+30 \div (12!)$

$= 1.9875981\mathrm{e}+22$

i) $\binom{40}{3} = \frac{40!}{(40-3)!3!} = \frac{40!}{37!3!} = 9880$

j) $\left[\binom{40}{20}\binom{40}{20}\right] \div 3 = 9.5008328\mathrm{e}+21$

k) $\binom{40}{20} - \binom{20}{0}\binom{20}{20} + \binom{20}{1}\binom{20}{19} = 137846528419$

l) $2^{39} - 1 = 549755813887$

m) $6^{40} = 1.3367495\mathrm{e}+31$

n) $\binom{40}{6} = \frac{40!}{(40-6)!6!} = \frac{40!}{(34)!6!} = 3838380$

o)

p)

Question 3

Arrangement with unlimited repetition

$5.5.5.5.5.5.5.5.5.5 = 5^{10} = 9765625$

Question 4

a)

- If no student got less than 10 out of 20, there are eleven possible marks that the students could have gotten.
- Each mark will represent a student (pigeon)
- Each container will be a pair or marks (pigeonhole)

  [10,10] [11,11] [12,12] [13,13] [14,14] [15,15] [16,16] [17,17] [18,18] [19,19] [20,20]

- We note that where each container has two students, the total number of students is 22.
- We have three remaining students, that need to be assigned to one pigeonhole each.
- Each pigeonhole already contains two students.
- If we add the three remaining students to any three pigeonholes. At least three will have the same mark

b)

- Group consecutive numbers into pairs (pigeonholes):
  [1,2] [3,4] [5,6]… $[2n - 1, 2n]$
  Where $n > 1$
- If we chose $n + 1$ integers, by the pigeonhole principle, we should get a two that are from one of the pairs mentioned above.
- The pairs are already consecutive integers so two of the numbers chosen will also be consecutive

Question 5

By the extended pigeonhole principle, at least one pigeonhole will contain $\left|\frac{n-1}{m}\right| + 1$ pigeon(s).

If no student got less than 20% there are 81 possible marks that the students could have gotten.

- Each mark will represent a student (pigeon)
- Each container will be a pair or marks (pigeonhole)

$\left|\frac{165-1}{81}\right| + 1 = \left|\frac{164}{81}\right| + 1 = 3.02469\ldots.$

Therefore, at least 3 students obtained the same mark

Question 6

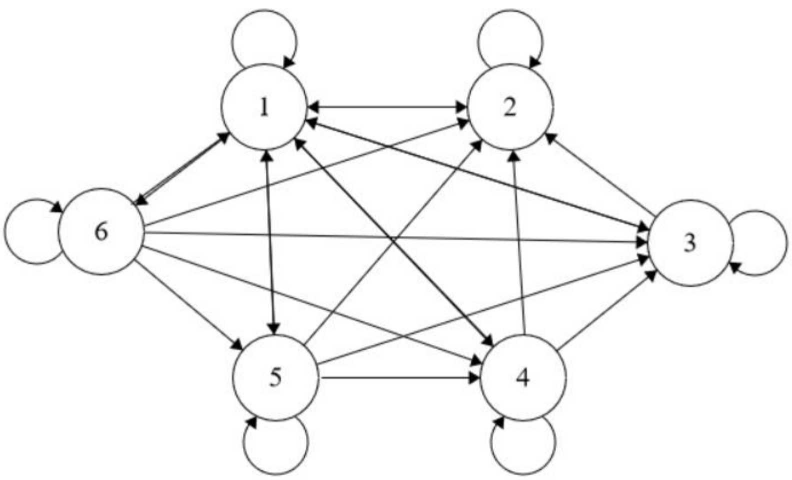
$R = \{(a, b) |\ a\ modulo\ b \leq 1\ \}$

	1	2	3	4	5	6
1	x	x	x	x	x	x
2	x	x				
3	x	x	x			
4	x	x	x	x		
5	x	x	x	x	x	
6	x	x	x	x	x	x

a) yes. R is reflexive
b) no. R is not irreflexive
c) no. R is not symmetric
d) no. R is not asymmetric
e) yes. R is antisymmetric
f) no. R is not transitive

Question 7

a)



b)
2: in 5, out 1
3: in 4, out 2

c) $Dom(R) = A$ $Ran(R) = A$

d) 2-1-5
e) $R(2) = \{1,2,3,4,5,6\}$

f) $M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ $M_{R^2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$M_{R^2} = \begin{bmatrix} 6 & 6 & 5 & 4 & 3 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 1 & 1 & 1 \\ 4 & 4 & 3 & 2 & 1 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 5 & 4 & 3 & 2 \end{bmatrix}$ $= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

all positions non-zero

g)

- M_{R^2} shows the possible pairs that transitivity can be tested against
- In M_{R^2}, if, for every position (a,b) and (b,c) that each have a 1, there is a 1 at (a,c), then the relation is true.
- Also, for all the positions in M_{R^2} that are non-zero (or 1), if M_R already has a 1 in the corresponding position, R is transitive

Question 8

a) no. R is not reflexive.
 The centre (main) diagonal has all 0's

b) yes. R is irreflexive.
 The centre (main) diagonal has all 0's

c) no. R is not symmetric.
 For every value, the value in the transposed position is not equal.

d) yes. R is asymmetric
 The centre (main) diagonal has all 0's
 For every value, the value in the transposed position is not equal.

e) yes. R is antisymmetric
 It does not matter what values the centre (main) diagonal has
 For every value and the value in the transposed position, they are both not 1

f) no. R is not transitive
 M_{R^2} has 1's in positions which M_R does not have

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 5 & 2 & 2 & 4 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 2 & 1 & 0 & 1 \\ 4 & 2 & 2 & 3 & 1 & 0 & 0 \\ 4 & 2 & 2 & 4 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Question 9

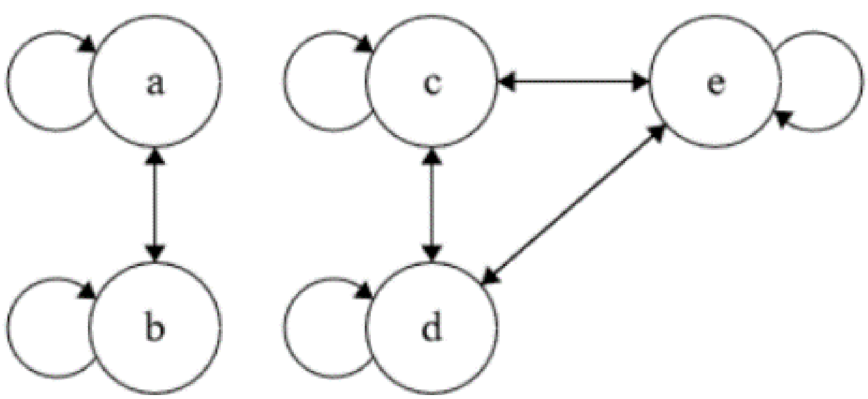
$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$[a] = \{a, b\}$
$[c] = \{c, d, e\}$

a)

$$\begin{aligned} A/R = &\{(a,b),(b,a)\} \\ &\{(c,d),(d,c),(c,e),(e,c),(d,e),(e,d)\} \end{aligned}$$

b)



Question 10

a) Let $X = \{x_1, x_2, \dots x_n\}$ be a finite set with n number of elements.

- Using the cartesian product of sets, we have $X \times X = \{(x, y): x, y \ \in X\}$
- The cartesian product $X \times X$ contains pairs of elements from X
- For each pair (x, y), we have x of the n elements from X
- For each pair (x, y), we also have y of the n elements from X
- Thus, there are n^2 possible ordered pairs where $(x, y) \in X$

Let $R \subseteq X \times X$, where R is a relation on our cartesian product

- For each of the n^2 possible ordered pairs (x, y), we have two possibilities: either $(x, y) \in R$ or $(x, y) \notin R$
- To account for both possibilities, we have 2^{n^2}
- Thus, the number of distinct relations R on X is 2^{n^2}

b) Let us assume $X = \{x_1, x_2, \dots x_n\}$ be a finite set with n number of elements.

- Then again, the cartesian product of sets, we have $X \times X$
- The cartesian product $X \times X$ contains pairs of elements from X
- For each pair (x, y), we have x of the n elements from X
- For each pair (x, y), we also have y of the n elements from X
- Thus, there are n^2 possible ordered pairs where $(x, y) \in X$

Let $R \subseteq X \times X$, where R is a reflexive relation on X

- For each of the n^2 possible ordered pairs (x, y), we have two possibilities: either $(x, y) \in R$ or $(x, y) \notin R$
- But to be reflexive, the main (centre) diagonal of the matrix $X \times X$ needs to be all 1's, we can remove these from our ordered pairs
- After removing the main diagonal elements, we have $n^2 - n$ ordered pairs
- To account for both possibilities, we have 2^{n^2-n}
- Thus, the number of distinct relations the reflexive relation R on X is 2^{n^2-n}

c) c) Let $X = \{x_1, x_2, \dots x_n\}$ be a finite set with n number of elements.

- Using the cartesian product of sets, we have $X \times X = \{(x, y): x, y \ \in X\}$
- The cartesian product $X \times X$ contains pairs of elements from X
- For each pair (x, y), we have x of the n elements from X
- For each pair (x, y), we also have y of the n elements from X
- Thus, there are n^2 possible ordered pairs where $(x, y) \in X$

Let $R \subseteq X \times X$, where R is a asymmetric relation on X

- For each of the n^2 possible ordered pairs (x, y), we have two possibilities: either $(x, y) \in R$ or $(x, y) \notin R$
- But to be asymmetric, every value and its value in the transposed position in the matrix $X \times X$ should not be equal.
- To account for this, we have $\frac{n(n-1)}{2}$ ordered pairs
- Thus, the number of distinct relations the asymmetric relation R on X is $2^{\frac{n(n+1)}{2}}$

Question 11

If R is a symmetric relation on A, then $(a,b) \in R \Longrightarrow (b,a) \in R$.

If R is a symmetric relation on A,
then a related to b, and subsequently b related to a

Using the cartesian product of sets, we can compute R^2. This will help us identify elements to show transitivity in R

By doing so, we create have the pair (a,a), where $(a,a) \in R^2$, $\forall a \in A$

We create the element in R^2 where a is related to itself.
All a's are elements of the set A

If we suppose that $(a,b) \in R^2$, then $\exists c$, where $c \in A$, $(a,c) \in R$ and $(c,b) \in R$

assume a related to b.
then there exists some c that exists in R,
where a is related to c
and where c is related to b

And if $(a,c) \in R$ and $(c,b) \in R$, $\exists c \in A$, then $(a,b) \in R^2 \Longrightarrow (b,a) \in R^2$.

a related to b (in R^2) and subsequently b related to a

It then follows that if R is a symmetric relation on A, then R^2 is symmetric.

Question 12
To be an equivalence relation on a set, a relation R or S would need to be reflexive, symmetric, and transitive.

a) yes, the relation R is an equivalence relation.

Reflexivity
The relation R contains pairs in the form (a, a), where $(a, a) \in Ra$.
These pairs are $(0,0), (1,1), (2,2), (3,3)$.
R contains all these pairs therefore it is reflexive.

Symmetry
The relation R contains pairs in the form (a, b) and (b, a) where $(a, b) \in R \Longrightarrow (b, a) \in R$
These pairs are $(2,1), (1,2), (2,3), (3,2)$.
Since R contains these pairs, and the only other pairs it contains are the ones explained above in its reflexivity property, R is symmetrical

Transitivity
The relation R contains pairs in the form (a, b), (b, c) and (a, c) where $(a, b) \in R \wedge (b, c) \in R$.
Examples of these pairs are $\{(1,1), (1,2), (2,1)\}$, $\{(2,3), (3,2), (2,1)\}$, $\{(1,2), (2,3), (3,2)\}$,
R contains these pairs and can compute many others, therefore it is transitive.

b) no, the relation S represented by the matrix is not an equivalence relation.

Reflexivity
The main centre (main) diagonal is not only 1's, so the relation S is not reflexive

Symmetry
For every value, it is not equal to the value in the transposed position, so the relation is not symmetric

Transitivity
Let the Relation S, be represented by the matrix M_R. Using cartesian product of sets, we have M_{R^2}

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad M_{R^2} = \begin{bmatrix} 2 & 4 & 3 & 3 \\ 1 & 3 & 2 & 2 \\ 1 & 3 & 3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

M_{R^2} has 1's in positions which M_R does not have. Therefore, the Relation S is not transitive