Question 1

a i) R_1 is not a function since the input for 4 has more than one output
$R_1(4) = (4,2),(4,3)$

a ii) The function is not onto/surjective since every value in the domain cannot be mapped to a corresponding value in the range
$Range(R_1) \neq A$

a iii) The function is not one-to-one/bijective since every value in the domain cannot be mapped perfectly to a single corresponding value in the range.
$R_1(4) = (4,2),(4,3)$

a iv) R_1 is everywhere defined as value in the domain is mapped to some or other value in the range
$Domain(R_1) = A$

b i) R_2 is a function since no element in the domain maps to more than one element in the range.

b ii) The function is not onto/surjective since every value in the domain cannot be mapped to a corresponding value in the range
$Range(R_2) \neq A$

b iii) The function is not one-to-one/bijective since every value in the domain cannot be mapped to a single corresponding value in the range.
$Range(R_2) \neq Domain(R_2)$

b iv) R_3 is everywhere defined as value in the domain is mapped to some or other value in the range
$Domain(R_3) = A$

Question 2

a) $Domain(f(x)) = \mathbb{Z}$

Therefore, f is everywhere defined

b) The function $f: A \to B$ is onto/surjective if there exists an inverse function $g: B \to A$ such that the composition function $f \circ g : B \to B$ equals the identity function $1_B: B \to B$

If $f(x) = \frac{1}{10+\sqrt{x}}$

$$f^{-1}(x) \Rightarrow x = \frac{1}{\sqrt{y}+10}$$
$$\Rightarrow x(10+\sqrt{y}) = 1$$
$$\Rightarrow x10 + x\sqrt{y} = 1$$
$$\Rightarrow x\sqrt{y} = 1 - x10$$
$$\Rightarrow \sqrt{y} = \frac{1-10x}{x}$$
$$\Rightarrow y = \frac{(1-10x)^2}{x^2}$$

Then $f(x) = y$

Therefore f is onto/surjective

c) Suppose $f(a) = f(b)$

Then:

$$f(a) = f(b) \quad \Leftrightarrow \frac{1}{10+\sqrt{a}} = \frac{1}{10+\sqrt{b}}$$
$$\Rightarrow 10+\sqrt{a} = 10+\sqrt{b}$$
$$\Rightarrow \sqrt{a} = \sqrt{b}$$
$$\Rightarrow a = b$$

Therefore f is one-to-one

d) f is invertible because f is one-to-one

$$f^{-1}(x) \Rightarrow x = \frac{1}{10+\sqrt{y}}$$
$$\Rightarrow x(10+\sqrt{y}) = 1$$
$$\Rightarrow x\sqrt{y} + x10 = 1$$
$$\Rightarrow x\sqrt{y} = 1 - x10$$
$$\Rightarrow \sqrt{y} = \frac{1-10x}{x}$$
$$\Rightarrow y = \frac{(1-10x)^2}{x^2}$$

Therefore f is invertible as $\boldsymbol{f^{-1} = \frac{(1-10x)^2}{x^2}}$

Question 3

a) $f(n) = 10$
$f(2n) = 10$
Running time is constant, it's not affected by the input size.

b)
$$\begin{aligned} f(n) &= 5n + 6 \\ f(2n) &= 5(2n) + 6 \\ &= 2f(n) \end{aligned}$$

Running time is linear. When an algorithm accepts $2n$ input size, it would perform $2n$ or two times as many operations as well.

c)
$$\begin{aligned} f(n) &= 6n^2 \\ f(2n) &= 6(2n)^2 \\ f(2n) &= 6(4n)^2 \\ &= 4f(n) \end{aligned}$$
Running time is quadratic. When an algorithm accepts $2n$ input size, it would perform $4n$ or four times as many operations as well.

d)
$$\begin{aligned} f(n) &= 2^n \\ f(2n) &= 2^{(2n)} \\ f(2n) &= (2^n)^2 \\ &= f(n)^2 \end{aligned}$$
Running time is exponential. When an algorithm accepts $2n$ input size, it would perform n^2 or the number of original operations squared as well.

Question 4

a) Suppose n^2 is $O(n^2 \log n)$

Then there exist constants $k > 0$ and $C > 0$ such that:

$n^2 \leq Cn^2 \log n$ for all $n \geq k$

Now

$\frac{n^2}{n^2 \log n} \leq C$

$\frac{1}{\log n} \leq C$

$1 \leq C \log n$

Which holds true for arbitrary values like $C = 4$ and $k = 3$

Therefore, n^2 is $O(n^2 logn)$

b) Suppose $O(n^2 \log n)$ is n^2

Then there exists $n > 0$ and $C > 0$ such that:

$n^2 \log n \leq Cn^2$ for all $n \geq \mathbb{N}$

Now

$n^2 \log n \leq Cn^2$

$\log n \leq C$

But we know $\log n$ is not a bound function (of the form $|f(x)| \leq M$)

Therefore $n^2 \log n$ is not asymptotically bounded by $0(n^2)$

Therefore, the inequality is false and $n^2 \log n$ is not $O(n^2)$

Question 5

a) $Domain(f(x)) = \mathbb{Z}$

Therefore, f is everywhere defined

The function $f: A \rightarrow B$ is onto/surjective if there exists an inverse function $g: B \rightarrow A$ such that the composition function $f \circ g : B \rightarrow B$ equals the identity function $1_B: B \rightarrow B$

If $f(a) = a + 2$

$f^{-1}(a) \Rightarrow a = b + 2$

Then $f(a) = b$

Therefore f is onto/surjective

Suppose $f(a) = f(b)$

$\Rightarrow a + 2 = b + 2$

$\Rightarrow a = b$

Therefore f is one-to-one

Therefore, f is a permutation

b) $Domain(f(x)) = \mathbb{Z}$

Therefore, f is everywhere defined

The function $f: A \rightarrow B$ is onto/surjective if there exists an inverse function $g: B \rightarrow A$ such that the composition function $f \circ g : B \rightarrow B$ equals the identity function $1_B: B \rightarrow B$

If $f(a) = a^2 - 2a$

$f^{-1}(a) \Rightarrow a = b^2 + 2b$

$\Rightarrow 0 = b^2 + 2b - a$

Then $Range(f) = \{a \in \mathbb{Z} \mid a \geq -1\}$

Solve using quadratic equation formula:

$y_1, y_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y_1, y_2 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-x)}}{2(1)}$

$y_1, y_2 = \pm\sqrt{x + 1} - 1$

$y_1 = \sqrt{x + 1} - 1$ and $y_2 = -\sqrt{x + 1} - 1$

Therefore f is not onto/surjective

Therefore, f is not a permutation

Question 6

a) $(2\quad 3) \circ (4\quad 5\quad 6) \circ (1\quad 3\quad 6\quad 7)$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 & 8 \end{pmatrix}$

b) $(1\quad 2) \circ (5\quad 2\quad 3) \circ (3\quad 4\quad 6\quad 7)$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 6 & 2 & 7 & 5 & 8 \end{pmatrix}$

Question 7

$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 7 & 3 & 4 & 8 & 1 \end{pmatrix}$

a) $p(1) = 5$ $\quad p(2) = 6$ $\quad p(3) = 2$ $\quad p(4) = 7$

$p(5) = 3$ $\quad p(6) = 4$ $\quad p(7) = 8$ $\quad p(8) = 1$

b) $p = (15326478)$

c) $p = (18)(17)(14)(16)(12)(13)(15)$

d) p is odd with an odd number of transpositions

e) $p \circ p$

f) Using transpositions: $p = (18)(17)(14)(16)(12)(13)(15)$

Unordered	
(18)	$1 \rightarrow 8$
(17)	$8 \rightarrow 7$
(14)	$7 \rightarrow 4$
(16)	$4 \rightarrow 6$
(12)	$6 \rightarrow 2$
(13)	$2 \rightarrow 3$
(15)	$3 \rightarrow 5$

Ordered	
(18)	$1 \rightarrow 8$
(13)	$2 \rightarrow 3$
(15)	$3 \rightarrow 5$
(16)	$4 \rightarrow 6$
Missing	$5 \rightarrow 1$
(12)	$6 \rightarrow 2$
(14)	$7 \rightarrow 4$
(17)	$8 \rightarrow 7$

$p^{-1} = (8\ 3\ 5\ 6\ 1\ 2\ 4\ 7)$

g) $p = (15326478)$

$Period(p) = 8$

Question 8

Let A be a permutation on $(1,2,3,4,5,6)$ with period 5

Therefore $A = (2\ \ 3\ \ 4\ \ 5\ 1)$

Question 9

a) $R_1 = \{(a,b) |\ a\, modulo\, b \leq 1\ \}$

	1	2	3	4	5
1	X	X			
2	X	x	X		
3		X	x		
4					
5					

R_1 is not reflexive as $(4,4),(5,5) \notin R_1$

Therefore R_1 is not a partial order

b) $R_2 = \{(a,b) |\ a\, modulo\, b \leq 1\ \}$

	1	2	3	4	5
1	X	X			X
2		x	X		
3		X	x		
4				X	
5		x			x

R_2 is reflexive as $(a,a) \in R_2$

R_2 is not symmetric as for every value, the value in the transposed position is not equal

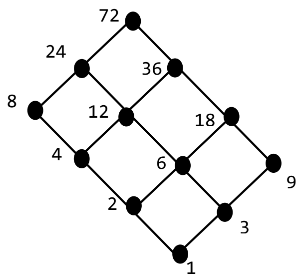
R_2 is antisymmetric as it is reflexive and not symmetric

R_2 is transitive: as $(5,2) \in R_2$ and $(1,2) \in R_2$ and $(1,5) \in R_2$

Therefore R_1 is a partial order

Question 10

$R_{72} = \{1,2,3,4,6,8,9,12,18,24,36,72\}$

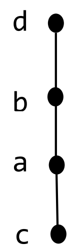


Question 11

Question 12

$R = \{(a, b) |\ a\ modulo\ b \leq 1\ \}$

	a	b	c	d
a	X	X	x	x
b		x	X	X
c			x	
d				X



Question 13

Question 14

Question 15

Question 16

	y'		y	
x'	1	1	1	0
x	1	0	0	0