[Question 1]

Consider the linear system:

3.333x1 + 15920x2 - 10.333x3 = 15913
2.222x1 + 16.710x2 + 9.612x3 = 28.544
1.5611x1 + 5.1791x2 + 1.6852x3 = 8.4254

Solve the system using:

(a) Gaussian elimination without pivoting.

Initial augmented matrix:

$$\begin{bmatrix} 3.333 & 15920 & -10.3333 & 15913 \\ 2.222 & 16.710 & 9.612 & 28.544 \\ 1.5611 & 5.1791 & 1.6852 & 8.4254 \end{bmatrix}$$

Forward Elimination

-- iter: 1

R2: R2 - (2.2220/3.3330)*R1

R3: R3 - (1.5611/3.3330)*R1

$$\begin{bmatrix} 3.3330 & 15920 & -10.3333 & 15913 \\ 0.009 & -103596.6233 & 2.7233 & -10580.0471 \\ 0 & -7449.5076 & -3.1523 & -7438.4968 \end{bmatrix}$$

-- iter: 2

R3: R3 - (-0.0000/0.0009)*R2

$$\begin{bmatrix} 3.3330 & 15920 & -10.3330 & 15913 \\ 0.009 & -10596.6233 & 2.7233 & -10580.0471 \\ 0 & -7449.5076 & -3.1523 & -7438.4968 \end{bmatrix}$$

Back Substitution:

$x_3: -\frac{-7438.4968}{-7449.5076} = 0.9985$

x_2:

$$0.009x_2 + 2.7233 \times 0.9985 = -10580.0471$$
$$\Rightarrow 0.009x_2 = -10580.0471 - 2.7233 \times 0.9985$$
$$\Rightarrow x_2 = -\frac{10580.0471}{0.009}$$
$$\Rightarrow x_2 = -11758639.00$$

x_1:

$$3.3330x_1 + 15920 \times (-11758639.00) - 10.3330 \times 0.9985 = 15913$$
$$\Rightarrow 3.333x_1 = 15913 + 187399845018.80 + 10.3165$$
$$\Rightarrow x_1 = \frac{87399860942.12}{3.3330}$$
$$\Rightarrow x_1 = 56239811676.51$$

Thus,

$x_1 = 56239811676.51$

$x_2 = -11758639.33$

$x_3 = 0.9985$

(b) Gaussian elimination with scaled partial pivoting.

Initial augmented matrix:

$$\begin{bmatrix} 3.3330 & 15920 & -10.3330 & 15913 \\ 2.222 & 16.7100 & 9.612 & 28.544 \\ 1.5611 & 5.1791 & 1.6852 & 8.4254 \end{bmatrix}$$

Scale rows

$S_1 = max(|\ 3.3330\ |, |\ 15920\ |, |\ -10.3330\ |) = 15920$

$S_2 = max(|\ 2.2220\ |, |\ 16.7100\ |, |\ 9.6120\ |) = 16.7100$

$S_3 = max(|\ 1.5611\ |, |\ 5.1791\ |, |\ 1.6852\ |) = 15920$

Pivot row

$\frac{|3.3330|}{S_1} = \frac{3.3330}{15920} = 0.0002$

$\frac{|2.2220|}{S_2} = \frac{2.2220}{16.7100} = 0.1330$

$\frac{|1.5611|}{S_3} = \frac{5.1791}{5.1791} = 0.3014$ pivot row

Thus,

Swap R1 with R3

$$\begin{bmatrix} 1.5611 & 5.1791 & 1.6852 & 8.4254 \\ 2.2220 & 16.7100 & 9.6120 & 28.544 \\ 3.3330 & 15920 & -10.3330 & 15913 \end{bmatrix}$$

Forward Elimination

-- iter: 1

R2: R2 - (2.2220/1.5611)*R1

R3: R3 - (3.3330/1.5611)*R1

$$\begin{bmatrix} 1.5611 & 5.1791 & 1.6852 & 8.4254 \\ 0 & 9.3398 & 7.2140 & 16.5477 \\ 0 & 15908.9395 & -13.9293 & 15895.0110 \end{bmatrix}$$

-- iter: 2

Scale rows

$S_2 = max(|\ 0\ |, |\ 9.3398\ |, |\ 7.2140\ |) = 16.7100$

$S_3 = max(|\ 0\ |, |\ 15908.9395\ |, |\ -13.9293\ |) = 15908.9395$

Pivot row

$\frac{|9.3398|}{S_2} = \frac{9.3398}{9.3398} = 1$

$\frac{|15908.9395|}{S_3} = \frac{15908.9395}{15908.9395} = 1$ pivot row

R3: R3 - (15908.9395/9.3398)*R2

$$\begin{bmatrix} 1.5611 & 5.1791 & 1.6852 & 8.4254 \\ 0 & 9.3398 & 7.2140 & 16.5477 \\ 0 & 0 & -12305.2935 & -12286.1973 \end{bmatrix}$$

Back Substitution:

$x_3: -\frac{-12286.1973}{-12305.2935} = 0.9984$

x_2:

$$9.3398x_2 + 7.2140 \times 0.9984 = 16.5477$$
$$\Rightarrow 9.3398x_2 = 16.5477 - 7.2140 \times 0.9984$$
$$\Rightarrow x_2 = -\frac{9.3451}{9.3398}$$
$$\Rightarrow x_2 = 1.0006$$

x_1:

$$1.5611x_1 + 5.1791x_2 + 1.6852x_3 = 8.4254$$
$$\Rightarrow 1.5611x_1 + 5.1791(1.0006) + 1.6852(0.9984) = 8.4254$$
$$\Rightarrow x_1 = \frac{1.5602}{1.5611}$$
$$\Rightarrow x_1 = 0.9994$$

Thus,

$x_1 = 0.9994$

$x_2 = 1.0006$

$x_3 = 0.9984$

c) Basic LU decomposition.

$$A = \begin{bmatrix} 3.3330 & 15920 & -10.3330 \\ 2.222 & 16.7100 & 9.612 \\ 1.5611 & 5.1791 & 1.6852 \end{bmatrix}$$

Forward Elimination

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.6667 & 1 & 0 \\ 0.4682 & 5.1791 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3.3330 & 15920 & -10.3330 \\ 0 & -10613.3333 & 9.612 \\ 0 & 0 & -3.1523 \end{bmatrix}$$

Back Substitution: Solve for Y

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.6667 & 1 & 0 \\ 0.4682 & 5.1791 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 15920 \\ -10580.0471 \\ -7438.4968 \end{bmatrix}$$

Thus,

$y_1 = 15913$

y_2:

$$0.6667 \times y_1 + y_2 = -10580.0471$$
$$\Rightarrow 0.6667 \times (15913) + y_2 = -10580.0471$$
$$\Rightarrow y_2 = \frac{-10580.0471}{0.6667 \times (15913)}$$
$$\Rightarrow y_2 = -21188.6382$$

y_3:

$$0.4682 y_1 + 0 \times y_2 + y_3 = -7438.4968$$
$$\Rightarrow 0.4682 \times (15913) + y_3 = -7438.4968$$
$$\Rightarrow y_3 = -14895.419$$

Thus,

$$Y = \begin{bmatrix} 15913 \\ -21188.6382 \\ -14895.419 \end{bmatrix}$$

Back Substitution: Solve for X

x_3 :

$$-3.1523x_3 = -14895.419$$
$$\Rightarrow x_3 = \frac{-14895.419}{-3.1523}$$
$$\Rightarrow x_3 = 4717.1651$$

x_2 :

$$-10613.3333x_2 + 2.7233x_3 = -21188.6382$$
$$\Rightarrow -10613.3333x_2 + 2.7233 \times (3898.9752) = -21188.6382$$

$$\Rightarrow -10613.3333x_2 = -21188.6382 - 2.7233 \times (3898.9752)$$
$$\Rightarrow x_2 = \frac{-21188.6382 - 2.7233 \times 3898.9752}{-10613.3333}$$
$$\Rightarrow x_2 = 3.0975$$

x_1:

$$3.3330x_1 + 15920x_2 - 10.3330x_3 = 15913$$
$$\Rightarrow 3.3330x_1 + 15920 \times (3.0975) - 10.3330 \times (4717.1651) = 15913$$
$$\Rightarrow 3.3330x_1 = 15913 - 15920 \times (3.0975) + 10.3330 \times (4717.1651)$$
$$\Rightarrow x_1 = \frac{15913 - 15920 \times (3.0975) + 10.3330 \times (4717.1651)}{3.3330}$$
$$\Rightarrow x_1 = 4638.4423$$

Thus,

$$X = \begin{bmatrix} 4717.1651 \\ 3.0975 \\ 3898.9752 \end{bmatrix}$$

[Question 2]

Consider the linear system Ax = b in which

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

(2.1) Use Gauss-Jordan method to solve the system Ax = b.

```
Initial augmented matrix:
   2  -3   1   0
   1   1  -1   1
  -1   1  -3  -3

Forward Elimination
-------------------------------------------------------- iter: 1
R2: R2 - (1/2)*R1
R3: R3 - (-1/2)*R1

   2.0000  -3.0000   1.0000        0
        0   2.5000  -1.5000   1.0000
        0  -0.5000  -2.5000  -3.0000
-------------------------------------------------------- iter: 2
R3: R3 - (-0.5/2.5)*R2

   2.0000  -3.0000   1.0000        0
        0   2.5000  -1.5000   1.0000
        0        0  -2.8000  -2.8000

Back Substitution:
x_3: -2.8/-2.8 = 1
x_2: (1 - -1.5) /2.5  = 1
x_1: (0 - -2) /2  = 1

   1
   1
   1
```

Thus,

$x_1 = 1$

$x_2 = 1$

$x_3 = 1$

(2.2) Use Gauss-Jordan method to compute the inverse A−1 exactly.

```
Initial augmented matrix (with the identity matrix):
   2  -3   1   0   1   0   0
   1   1  -1   1   0   1   0
  -1   1  -3  -3   0   0   1

Forward Elimination
-------------------------------------------------------- iter: 1
R2: R2 - (1/2)*R1
R3: R3 - (-1/2)*R1

   2.0000  -3.0000   1.0000   1.0000    1.0000    0
        0   2.5000  -1.5000   -0.5000   1.0000    0
        0  -0.5000  -2.5000   0.5000    1.0000    0
-------------------------------------------------------- iter: 2
R2:R2×(2/7)
```

2.3) Use Cramer’s rule to compute the inverse A−1.

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{bmatrix}$$

$$\det(A) = 2 \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} - (-3) \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} + 1 \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} = (1 \times -3) - (-1 \times 1) = -2$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} = (1 \times -3) - (-1 \times -1) = -4$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} = (1 \times 1) - (-1 \times -1) = 2$$

Thus,

$$\det(A) = 2\,(-2) - (-3)(-4) + 1(2) = -14$$

[Question 3]

[Question 4]