Problem 29.

Show the following:

(a) If A is an orthogonal matrix, then A^{-1} is also orthogonal matrix.

(b) If A and B are orthogonal matrices, then AB is also orthogonal matrix.

Definition

A matrix A is orthogonal if $A^T \times A = I$

If A^{-1} is an orthogonal matrix, then $(A^{-1})^T \times A^{-1} = I$

Properties

[1] Inverse Property:

A^{-1} is the inverse of A,

Defined as $A \times \mathrm{A}^{-1} = I$

Where I is the identity matrix

Also

$I^{-1} = I$

[2] Inverse of Product Property:

For any two invertible matrices A and B,

the inverse of the product AB is given by:

$(AB)^{-1} = B^{-1} A^{-1}$

[3] Transpose of A^{-1}:

A^T is the transpose of A

[4] Transpose of Inverse

For any invertible matrix A, the transpose of its inverse is the same as the inverse of its transpose.

$(A^{-1})^T = (A^T)^{-1}$

$A^T \times A = I$

$\Rightarrow (A^T \times A)^{-1} = I^{-1}$

$\Rightarrow (A^T \times A)^{-1} = I^{-1}$

$\Rightarrow (A^T \times A)^{-1} = I$ [1] because $I^{-1} = I$

$\Rightarrow (A^T \times A)^{-1} = A^{-1} \times (A^T)^{-1}$ [2] because $(AB)^{-1} = B^{-1} A^{-1}$

$\Rightarrow I = A^{-1} \times (A^T)^{-1}$

$\Rightarrow I = A^{-1} \times (A^{-1})^T$ [4] because $(A^{-1})^T = (A^T)^{-1}$

Thus, A^{-1} is orthogonal

Problem 30.

What is the condition on a and b for which the matrix

$\begin{bmatrix} a + 2b & 2b - a \\ a - 2b & 2b + a \end{bmatrix}$

is orthogonal.

Definition

A matrix A is orthogonal if $A^T \times A = I$

If A^{-1} is an orthogonal matrix, then $(A^{-1})^T \times A^{-1} = I$

[1] Compute A^T

$$A = \begin{bmatrix} a + 2b & 2b - a \\ a - 2b & 2b + a \end{bmatrix}$$

$$\Rightarrow A^T = \begin{bmatrix} a + 2b & a - 2b \\ 2b - a & 2b + a \end{bmatrix}$$

[2] Compute $A^T \times A$

$$A^T \times A = \begin{bmatrix} a + 2b & a - 2b \\ 2b - a & 2b + a \end{bmatrix} \times \begin{bmatrix} a + 2b & 2b - a \\ a - 2b & 2b + a \end{bmatrix} = I$$

$$\Rightarrow A^T \times A =$$

$$\begin{bmatrix} (a+2b)(a+2b) + (a-2b)(a-2b) & (a+2b)(2b-a) + (a-2b)(2b+a) \\ (2b-a)(a+2b) + (2b+a)(a-2b) & (2b-a)(2b-a) + (2b+a)(2b+a) \end{bmatrix}$$

$$\Rightarrow A^T \times A =$$

$$\begin{bmatrix} (a+2b)^2 + (a-2b)^2 & (2ab + ab^2 - a^2) + (2ab - ab^2 + a^2) \\ (2ab + ab^2 - a^2) + (2ab - ab^2 + a^2) & 4b^2 - 4ab + a^2 + 4b^2 + 4ab + a^2 \end{bmatrix}$$

$$\Rightarrow A^T \times A = \begin{bmatrix} 2a^2 + 8b^2 & 0 \\ 0 & 2a^2 + 8b^2 \end{bmatrix} = I$$

$$\Rightarrow A^T \times A = \begin{bmatrix} 2a^2 + 8b^2 & 0 \\ 0 & 2a^2 + 8b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $2a^2 + 8b^2 = 1$

$\Rightarrow a^2 + 4b^2 = \frac{1}{2}$

Problem 31.

Find a matrix P that orthogonally diagonalizes A; and determine $P^{-1}AP$,

Where

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

[1] Eigenvalues of A

To orthogonal diagonalized matrix A

Find an orthogonal matrix P such that $P^{-1}AP = D$,

where D is a diagonal matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow detA = \begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

$$\Rightarrow detA = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} - 1 \begin{bmatrix} 0 & 1 \\ 0 & 3-\lambda \end{bmatrix}$$

$$\Rightarrow (3-\lambda)(2-\lambda)(3-\lambda) - 0$$

$$\Rightarrow\ (3-\lambda)(6-5\lambda+\lambda^2) = 0$$

$$\Rightarrow\ (3-\lambda)(\lambda-2)(\lambda-3) = 0$$

Eigenvalues : $\lambda = 3$ OR $\lambda = 2$

[2] Eigenvectors of A

Eigenvectors of $\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Thus $x = -z$

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvectors of $\lambda = 3$

$$A - 3I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Thus $x + z = 0$, $\quad -y = 0$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus,

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

[3] Compute $P^{-1}AP$

$$P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP =$$

$$\begin{bmatrix} 3 \times \left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}} & 3 \times 1 + 0 + 0 & 3 \times 0 + 0 + 0 \\ 0 & 0 & 2 \times 1 \\ 1 \times \left(-\frac{1}{\sqrt{2}}\right) + 0 + 3 \times \left(\frac{1}{\sqrt{2}}\right) & 1 \times 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} \left(-\frac{3}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} & 3 & 3 \\ 0 & 0 & 2 \\ -\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} -\sqrt{2} & 3 & 3 \\ 0 & 0 & 2 \\ \sqrt{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} -\frac{1}{\sqrt{2}} \times -\sqrt{2} + 0 + \frac{1}{\sqrt{2}}.\sqrt{2} & -\frac{1}{\sqrt{2}} \times 3 + 0 + \frac{1}{\sqrt{2}} & 0 \\ 1 \times (-\sqrt{2}) & 3 \times 1 & 0 \\ 0 & 0 & 1 \times 2 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 2 & -\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} & 0 \\ -\sqrt{2} & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 32. Find the spectral decomposition of matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

[1] Eigenvalues of A

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \det(A - \lambda I) = (1-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$\Rightarrow (1-\lambda)(1-\lambda) - (1)(1)$$

$$\Rightarrow (1-\lambda)^2 - 1$$

Thus,

$\det(A - \lambda I)$

$$\Rightarrow (1-\lambda)((1-\lambda)^2 - 1)$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 2\lambda)$$

$$\Rightarrow (1-\lambda)\lambda(\lambda - 2)$$

Eigenvalues : $\lambda = 0$ OR $\lambda = 1$ OR $\lambda = 2$

[2] Eigenvectors of A

Eigenvectors of $\lambda = 0$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $x + z = 0$, $y = 0$

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvectors of $\lambda = 2$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $-x + z = 0$, $y = 0$, $x - z = 0$

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

[3] Spectral Decomposition

Given $A = \lambda_1 \times v_1 \times v_1^T + \lambda_2 \times v_2 \times v_2^T + \lambda_3 \times v_3 \times v_3^T$

$$\Rightarrow A = 0 \times \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times [1 \quad 0 \quad -1] + 1 \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times [0 \quad 1 \quad 0] + 2 \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times [1 \quad 0 \quad 1]$$

$$\Rightarrow A = 0 \times \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + 1 \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2 \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Thus $A = 0 \times v_1 \times v_1^T + 1 \times v_2 \times v_2^T + 2 \times v_3 \times v_3^T$