[Problem 5]

Determine whether each set equipped with the given operation is a vector space. For those that are not vector space identify the vector space axioms that fail.

In order for some set V to be a vector space the following 10 axioms must be true:

A1	**Closure under addition** *For any vectors* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *in the set,* $\boldsymbol{u}+\boldsymbol{v}$ *is also in the set*	$\vec{u}+\vec{v} \in V$
A2	**Existence of an additive identity** *There exists a vector* $\boldsymbol{0}$ *in the set such that for any vector* $\boldsymbol{u}$ *in the set,* $\boldsymbol{u}+\boldsymbol{0}=\boldsymbol{u}$. *Related:* *- A3:Existence of additive inverses*	$\vec{u}+0=\vec{u}$
A3	**Existence of additive inverses** *For every vector* $\boldsymbol{u}$ *in the set, there exists a vector* $-\boldsymbol{u}$ *in the set such that* $\boldsymbol{u}+(-\boldsymbol{u})=\boldsymbol{0}$	$\vec{u}+(-\vec{u})=0$
A4	**Associativity of addition** *For any vectors* $\boldsymbol{u}$, $\boldsymbol{v}$, *and* $\boldsymbol{w}$ *in the set,* $u+(v+w)=(u+v)+w$ *fundamental property of addition*	$\vec{u}+(v+w)=(\vec{u}+\vec{v})+w$
A5	**Commutativity of addition** *For any vectors* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *in the set,* $u+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$ *fundamental property of addition*	$\vec{u}+\vec{v} = \vec{v}+\vec{u}$
M1	**Closure under scalar multiplication** *For any scalar* c *and any vector* cu *in the set,* $c\boldsymbol{u}$ *is also in the set.* *Implied by:* *- A1: Closure under addition*	$c\vec{u} \in V$
M2	**Distributive property - vector addition** *For any scalar* c *and any vectors* u *and* v *in the set,* $c(u+v)=cu+cv$ *+cv.* *Implied by:* *- A1: Closure under addition* *- A4: Associativity of addition* *- A5: Commutativity of addition*	$c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
M3	**Distributive Property - scalar addition** *For any scalars* c_1 *and* c_2 *and any vector* u *in the set,* $(c_1+c_2)u=c_1u+c_2u$ *+cv.* *Implied by:* *- A1: Closure under addition* *- A4: Associativity of addition* *- A5: Commutativity of addition*	$(c+d)\vec{u}=c\vec{u}+d\vec{u}$
M4	**Associative Property** (Compatibility of scalar multiplication with field multiplication) *For any scalars* c_1 *and* c_2 *and any vector* u *in the set,* $(c_1c_2)u=c_1(c_2u)$ *+cv.* *Implied by:* *- A1: Closure under addition* *- A4: Associativity of addition* *- A5: Commutativity of addition*	$c(d\vec{u})=(cd)\vec{u}$
M5	**Multiplicative identity** *For any vector* $\boldsymbol{u}$ *in the set,* $1\boldsymbol{u}=\boldsymbol{u}$, *where 1 is the multiplicative identity of the underlying field*. *Implied by:* *- A1: Closure under addition* *- A4: Associativity of addition* *- A5: Commutativity of addition*	$1(\vec{u})=\vec{u}$

Axioms M1 to M5 (axioms of scalar multiplication)
are usually implied by A1-A5 (axioms of addition).

(1) The set $U = \{(x,0) \in R^2\}$ *with the standard operations on* R^2

A1	Let $u = (x_1, 0) \in U$ $v = (x_2, 0) \in U$. Then $u + v \in U$ $= (x_1, 0) + (x_2, 0) \in U$ $= (x_1 + x_2, 0) \in U$ Therefore A1 holds.
A2	Let $u = (x, 0) \in U$ the zero vector in U be $0 = (0,0)$ Then $u + 0 = u$ $= (x, 0) + 0$ $= (x, 0) \in U$ Therefore A2 holds.
A3	Let $u = (x, 0) \in U$ $-u = (-x, 0) \in U$ Then $u + (-u) = 0$ $= (x, 0) + (-x, 0)$ $= (0,0) = 0$ Therefore A3 holds.
A4	Addition in U follows the same rules as addition in R^2, so associativity holds. OR Let $u = (x_1, 0) \in U$ $v = (x_2, 0) \in U$ $w = (x_3, 0) \in U$ Then $u + (v + w) = (u + v) + w$ $= (x_1, 0) + (x_2 + x_3, 0 + 0)$ $= (x_1 + (x_2 + x_3), 0 + (0 + 0))$ $= ((x_1 + x_2) + x_3, (0 + 0) + 0)$ $= (x_1 + x_2, 0 + 0) + (x_3, 0)$ $= ((x_1, 0) + (x_2, 0)) + (x_3, 0)$
A5	Addition in U follows the same rules as addition in R^2, so commutativity holds.

All 5 fundamental axioms for vector spaces are satisfied,
$U = \{(x,0) \in R^2\}$ *with the standard operations on* R^2 is a vector space.

Axioms M1 to M5 (axioms of scalar multiplication)
are usually implied by A1-A5 (axioms of addition).

(2) The set $V = \{(x,0) \in R^2 : y \geq 0\}$ *with the standard operations on* R^2

<table>
<tr><td>A1</td><td>Let $u = (x_1, 0) \in V$ $v = (x_2, 0) \in V$
be arbitrary vectors in V
where x_1 and x_2 are real numbers

Then $u + v \in V$
$= (x_1, 0) + (x_2, 0) \in V$
$= (x_1 + x_2, 0) \in V$
Therefore A1 holds.</td></tr>
<tr><td>A2</td><td>Let $u = (x, 0) \in V$
be an arbitrary vector in V
where x is a real number
Let the zero vector in V be $0 = (0,0)$

But if $x = 0$ (as is the case for the zero vector)
then $y \geq 0$ is not satisfied,
violating the definition of V

Then $-u \notin V$
V does not have an additive identity,
and thus, it is not a vector space.</td></tr>
<tr><td>A3</td><td>Let $u = (x, 0) \in V$ $-u = (-x, 0) \in V$
be arbitrary vectors in V

But if $x < 0$
then $y \geq 0$ is not satisfied,
violating the definition of V

Then $-u \notin V$
V does not have additive inverses,
and thus, it is not a vector space.</td></tr>
<tr><td>A4</td><td>Addition in V follows the same rules as addition in R^2, so associativity holds.

OR
Let $u = (x_1, 0) \in V$ $v = (x_2, 0) \in V$ $w = (x_3, 0) \in V$
be arbitrary vectors in V
where x_1 , x_2 and x_3 are real numbers

Then $u + (v + w) = (u + v) + w$
$= (x_1, 0) + (x_2 + x_3, 0 + 0)$
$= (x_1 + (x_2 + x_3), 0 + (0 + 0))$
$= ((x_1 + x_2) + x_3, (0 + 0) + 0)$
$= (x_1 + x_2, 0 + 0) + (x_3, 0)$
$= ((x_1, 0) + (x_2, 0)) + (x_3, 0)$
Therefore A4 holds.</td></tr>
<tr><td>A5</td><td>Addition in V follows the same rules as addition in R^2, so commutativity holds.</td></tr>
</table>

V fails to satisfy A2 Existence of an additive identity
V fails to satisfy A3 Existence of additive inverses
Thus V is not a vector space

Axioms M1 to M5 (axioms of scalar multiplication)
are usually implied by A1-A5 (axioms of addition).

(3) The set $W = \{(x, 0) \in R^2 : x + y = 0\}$ *with the standard operations on* R^2

<table>
<tr><td>A1</td><td>Let $u = (x_1, 0) \in W$ $v = (x_2, 0) \in W$
be arbitrary vectors in W
where x_1 and x_2 are real numbers

Then $u + v \in W$
$= (x_1, 0) + (x_2, 0) \in W$
$= (x_1 + x_2, 0) \in W$
Therefore A1 holds.</td></tr>
<tr><td>A2</td><td>Let $u = (x, 0) \in W$
be arbitrary vectors in W
where x is a real number

Let the zero vector in W be $0 = (0,0)$
Then $u + 0 = u$
$= (x, 0) + 0$
$= (x, 0) \in W$
Therefore A2 holds.</td></tr>
<tr><td>A3</td><td>Let $u = (x, 0) \in W$ $-u = (-x, 0) \in W$
be arbitrary vectors in W
where x is a real number
Then $u + (-u) = 0$
$= (x, 0) + (-x, 0)$
$= (0,0) = 0$
Therefore A3 holds.</td></tr>
<tr><td>A4</td><td>Addition in W follows the same rules as addition in R^2, so associativity holds.
OR
Let $u = (x_1, 0) \in W$ $v = (x_2, 0) \in W$ $w = (x_3, 0) \in W$
be arbitrary vectors in W
where x_1 , x_2 and x_3 are real numbers

Then $u + (v + w) = (u + v) + w$
$= (x_1, 0) + (x_2 + x_3, 0 + 0)$
$= (x_1 + (x_2 + x_3), 0 + (0 + 0))$
$= ((x_1 + x_2) + x_3, (0 + 0) + 0)$
$= (x_1 + x_2, 0 + 0) + (x_3, 0)$
$= ((x_1, 0) + (x_2, 0)) + (x_3, 0)$
Therefore A4 holds.</td></tr>
<tr><td>A5</td><td>Addition in W follows the same rules as addition in R^2, so commutativity holds.</td></tr>
</table>

All 5 fundamental axioms for vector spaces are satisfied,
$W = \{(x, 0) \in R^2 : x + y = 0\}$ *with the standard operations on* R^2 is a vector space.

(4) The set $X=\{(x,y)\in R^2\}$ with the standard vector addition but with scalar multiplication defined by $k(x,y)=(k^2x,k^2y)$.

A1	Let $u=(x_1,y_1)\in X$ $v=(x_2,y_2)\in X$ be arbitrary vectors in X where x_1 , x_2 , y_1 and y_2 are real numbers Then $u+v\in X$ $=(x_1,y_1)+(x_2,y_2)\in X$ $=(x_1+x_2,y_1+y_2)\in X$ Therefore A1 holds.
A2	Let $u=(x,y)\in X$ be an arbitrary vector in X where x and y are real numbers the zero vector in X be $0=(0{,}0)$ Then $u+0=u$ $=(x,y)+0$ $=(x,y)\in X$ Therefore A2 holds.
A3	Let $u=(x,y)\in X$ be an arbitrary vector in X where x and y are real numbers The additive inverse of u is $-u=(-x,-y)\in X$ $C(x,y)$ $=(c^2x,c^2y)$ *the given scalar multiplication* $c(x,y)=(c^2x,c^2y)$ $c(-x,-y)$ $=((-1)^2x,(-1)^2y)$ *the given scalar multiplication* $c(x,y)=(c^2x,c^2y)$ $=(x,y)$ Therefore A3 fails.
A4	Addition in X follows the same rules as addition in R^2, so associativity holds. OR Let $u=(x_1,y_1)\in X$ $v=(x_2,y_2)\in X$ $w=(x_3,y_3)\in X$ be arbitrary vectors in X where x_1 , x_2 , x_3, y_1 , y_2 and y_3 are real numbers Then $u+(v+w)=(u+v)+w$ $=(x_1,y_1)+(x_2+x_3,y_2+y_3)$ $=(x_1+(x_2+x_3),y_1+(y_2+y_3))$ $=((x_1+x_2)+x_3,(y_1+y_2)+y_3)$ $=(x_1+x_2,y_1+y_2)+(x_3,y_3)$ $=((x_1,y_1)+(x_2,y_2))+(y_3,y_3)$ Therefore A4 holds.
A5	Addition in X follows the same rules as addition in R^2, so commutativity holds.

M1	Let $c \in X$ be an arbitrary scalar in X Let $u = (x, y) \in X$ be an arbitrary vector in X where x and y are real numbers Then $c \in X$ $= c.u(x, y)$ $= c(x, y)$ $= (c^2x, c^2y)$ *the given scalar multiplication* $c(x, y) = (c^2x, c^2y)$. $= (c^2x, c^2y) \in X$ Therefore M1 holds.
M2	Let $c \in X$ be an arbitrary scalar in X Let $u = (x_1, y_1) \in X$ $v = (x_2, y_2) \in X$ be arbitrary vectors in X where x_1 , x_2 , y_1 and y_2 are real numbers Then $c(u + v) = cu + cv$ Since $u + v \in X$, through **Closure under addition** Then $u + v \in X$ $= (x_1 + x_2, y_1 + y_2) \in X$ LHS $c(u + v)$ $= c(x_1 + x_2, y_1 + y_2)$ $= c^2(x_1 + x_2),\ c^2(y_1 + y_2)$ *the given scalar multiplication* $c(x, y) = (c^2x, c^2y)$. $= (c^2x_1 + c^2x_2,\ c^2y_1 + c^2y_2)$ RHS $cu + cv$ $= (c^2x_1, c^2y_1) + (c^2x_2, c^2y_2)$ *the given scalar multiplication* $c(x, y) = (c^2x, c^2y)$. $= (c^2x_1 + c^2x_2,\ c^2y_1 + c^2y_2)$ *And LHS = RHS or* $c(u + v) = cu + cv$ Therefore M2 holds.
M3	Let $c \in X$ $d \in X$ be arbitrary scalars in X Let $u = (x, y) \in X$ be an arbitrary vector in X Then $(c + d)u = cu + du$ LHS $(c + d)u$ $= ((c + d)^2x, (c + d)^2y)$ RHS $cu + du$ $= (c^2x, c^2y) + (d^2x, d^2y)$ *the given scalar multiplication* $c(x, y) = (c^2x, c^2y)$. $= ((c^2 + d^2)x, (c^2 + d^2)y)$ Therefore M3 holds.

M4	Let $c \in X$ $d \in X$ be arbitrary scalars in X $u = (x, y) \in X$ be an arbitrary vector in X Then $c(du) = (cd)u$ LHS $c(du)$ $= c(dx, dy)$ $= (c^2dx, c^2dy)$ *the given scalar multiplication* $c(x, y) = (c^2x, c^2y)$. RHS $(cd)u$ $= ((cd)^2x, (cd)^2y)$ *the given scalar multiplication* $c(x, y) = (c^2x, c^2y)$. $= (c^2d^2x, c^2d^2y)$ $= (c^2dx, c^2dy)$ *And LHS = RHS or* $c(du) = (cd)u$ Therefore M4 holds.
M5	Let $u = (x, y) \in X$ Then $1(u) = u$ $= 1(x, y)$ $= (1x, y) \in X$ $= (x, y) \in X$ Therefore M5 holds.

X fails to satisfy A3 Existence of additive inverses

Thus X is not a vector space

(5) The set 2x2 matrices $Y = \{\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b, c \in R^2\}$ *with the standard matrix addition and scalar multiplication.*

A1	Let $u = \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} \in Y$ $v = \begin{bmatrix} a_2 & b_2 \\ c_2 & 0 \end{bmatrix} \in Y$ be arbitrary matrices in Y where a_1 , b_1 , c_1 , a_2 , b_2 , c_2 are real numbers Then $u + v \in X$ $= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{bmatrix}$ Therefore A1 holds.
A2	Let $u = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in Y$ be an arbitrary matrix in Y where a , b , c are real numbers Let the zero matrix in X be $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Then $u + 0 = u$ $= \begin{bmatrix} a + 0 & b + 0 \\ c + 0 & 0 + 0 \end{bmatrix}$ $= \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ Therefore A2 holds.
A3	Let $u = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in Y$ be an arbitrary matrix in Y where a , b , c are real numbers The additive inverse of u is $-u = \begin{bmatrix} -a & -b \\ -c & 0 \end{bmatrix} \in Y$ Therefore A3 holds.
A4	Addition in Y follows the same rules as addition of 2x2 matrices, so associativity holds.
A5	Addition in Y follows the same rules as addition of 2x2 matrices, so commutativity holds.

<table>
<tr>
<td>M1</td>
<td>Let $k \in X$
be an arbitrary scalar in X

Let $u = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in Y$
be an arbitrary matrix in Y
where a , b , c are real numbers

Then $k \in X$
$= k \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$
$= \begin{bmatrix} ka & kb \\ kc & 0 \end{bmatrix}$
Therefore M1 holds.</td>
</tr>
<tr>
<td>M2</td>
<td>Let $k \in X$
be an arbitrary scalars in X

Let $u = \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} \in Y$ $v = \begin{bmatrix} a_2 & b_2 \\ c_2 & 0 \end{bmatrix} \in Y$
be arbitrary matrices in Y
where a_1 , b_1 , c_1 , a_2 , b_2 , c_2 are real numbers

Then $k(u + v) = ku + kv$

LHS $k(u + v)$
$= k \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{bmatrix}$
$= \begin{bmatrix} ka_1 + ka_2 & kb_1 + kb_2 \\ kc_1 + kc_2 & 0 \end{bmatrix}$

RHS $ku + kv$
$= k \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} + k \begin{bmatrix} a_2 & b_2 \\ c_2 & 0 \end{bmatrix}$
$= \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & 0 \end{bmatrix} + \begin{bmatrix} ka_2 & kb_2 \\ kc_2 & 0 \end{bmatrix}$
$= \begin{bmatrix} ka_1 + ka_2 & kb_1 + kb_2 \\ kc_1 + kc_2 & 0 \end{bmatrix}$

And LHS = RHS or $k(u + v) = ku + kv$

Therefore M2 holds.</td>
</tr>
<tr>
<td>M3</td>
<td>Let $k \in X$ l∈ X
be arbitrary scalars in X

Let $u = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in Y$
be an arbitrary matrix in Y
where a , b , c are real numbers

Then $(k + l)u = ku + lu$

LHS $(k + l)u$
$= (k + l) \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$
$= \begin{bmatrix} a(k + l) & b(k + l) \\ c(k + l) & 0 \end{bmatrix}$
$= \begin{bmatrix} ka + la & kb + lb \\ kc + lc & 0 \end{bmatrix}$</td>
</tr>
</table>

	RHS $ku + lu$ $= k\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + l\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ $= \begin{bmatrix} ka & kb \\ kc & 0 \end{bmatrix} + \begin{bmatrix} la & lb \\ lc & 0 \end{bmatrix}$ $= \begin{bmatrix} ka + la & kb + lb \\ kc + lc & 0 \end{bmatrix}$ *And LHS = RHS or* $(k + l)u = ku + lu$ Therefore M3 holds.
M4	Let $k \in X$ $l \in X$ be arbitrary scalars in X Let $u = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in Y$ be an arbitrary matrix in Y where a , b , c are real numbers Then $c(du) = (cd)u$ LHS $k(lu)$ $= k\begin{bmatrix} la & lb \\ lc & 0 \end{bmatrix}$ $= \begin{bmatrix} kla & klb \\ klc & 0 \end{bmatrix}$ RHS $(kl)u$ $= (kl)\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ $= \begin{bmatrix} kla & klb \\ klc & 0 \end{bmatrix}$ *And LHS = RHS or* $c(du) = (cd)u$ Therefore M4 holds.
M5	Let $u = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in Y$ be an arbitrary matrix in Y where a , b , c are real numbers Then $1(u) = u$ $= 1\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ $= \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ Therefore M5 holds.

All 10 axioms for vector spaces are satisfied,
The set 2x2 matrices $Y = \{\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b, c \in R^2\}$ with the standard matrix addition and scalar multiplication
is a vector space.

[Problem 6]

Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $u = (u_1, u_2, u_3)$ *and* $v = (v_1, v_2, v_3))$ *:*

$$u + v = (u_1 + v_1, \quad u_2 + v_2, \quad u_3 + v_3);\ ku = (ku_1, ku_2, 0)$$

(1) Compute $u + v$ *and* ku *for* $u = (-1, 2, -3)$ *,* $v = (2, -3, 1)$ *and* $k = -2$

$u + v$
$= \big((-1+2), (2+(-3)), (-3+1)\big)$
$= (1, -1, -2)$

ku
$= (-2)(-1, 2, -3)$
$= (-1(-2), 2(-2), -3(-2))$
$= (2, -4, 6)$

(2) Determine whether the Axioms 7, 8, 9 and 10 hold.

M2	If $k(u+v) = ku + kv$ Then LHS $k(u+v)$ $= k(u_1 + v_1, u_2 + v_2, u_3 + v_3)$ $= (k(u_1 + v_1), k(u_2 + v_2), k(u_3 + v_3))$ $= (ku_1 + kv_1, ku_2 + kv_2, ku_3 + kv_3)$ RHS $ku + kv$ $= (ku_1, ku_2, 0) + (kv_1, kv_2, 0)$ $= (ku_1 + kv_1, ku_2 + kv_2, 0)$ *LHS* $\neq$ *RHS or* $c(du) \neq (cd)u$ Therefore M2 fails
M3	Let $u = (k, l)$ be an arbitrary vector where k and l are arbitrary scalars Then $(k+l)u = ku + lu$ LHS $(k+l)u$ $= ((k+l)u_1, (k+l)u_2, 0)$ *the given scalar multiplication* $ku = (ku_1, ku_2, 0)$ $= (ku_1 + lu_1, ku_2 + lu_2, 0)$ RHS $ku + lu$ $= ((k+l)u_1, (k+l)u_2, 0)$ *the given scalar multiplication* $ku = (ku_1, ku_2, 0)$ $= (ku_1 + lu_1, ku_2 + lu_2, 0)$ *And LHS = RHS or* $c(u+v) = cu + cv$ Therefore M3 holds.

M4	Let u be an arbitrary vector Let k and l be arbitrary scalars Then $k(lu) = (kl)u$ LHS $k(lu)$ $= (ku_1, ku_2, 0)$ *the given scalar multiplication* $ku = (ku_1, ku_2, 0)$ RHS $(kl)u$ $= ((kl)u_1, (kl)u_2, 0)$ *the given scalar multiplication* $ku = (ku_1, ku_2, 0)$ $= (ku_1, ku_2, 0)$ *And LHS = RHS or* $k(lu) = (kl)u$ Therefore M4 holds.
M5	Let u be an arbitrary vector Then $1(u) = u$ $= (1u_1, 1u_2, 0)$ *the given scalar multiplication* $ku = (ku_1, ku_2, 0)$ $= (u_1, u_2, 0)$ Therefore M5 holds.

Axiom 7 (M2) Distributive property of vector addition over scalar addition fails Thus V is not a vector space

[Problem 7]

Let V be a vector space,
u a vector in V;
and k a scalar.
Then show that if $ku = 0$, then $ku = 0$ or $u = 0$

Proof by cases:
Case 1: $k = 0$
If $k = 0$
Then $ku = 0$
Thus $u = 0$

Case 2: $k \neq 0$
If $k \neq 0$
Then $ku/k = 0/k$
Thus $u = 0$

Thus If $k \neq 0$ Then $ku = 0 \Longrightarrow u = 0$
Therefore *if* $ku = 0$, then $ku = 0$ or $u = 0$

Proof by contrapositive:
P *: if* $ku = 0$, then $(ku = 0)$ *or* $(u = 0)$
P' *: if* $\neg(u = 0)$ and $\neg(k = 0) \Longrightarrow \neg(ku = 0)$

Assume $\neg((ku = 0) or\ (u = 0))$ *negation of the conclusion of P*
Then $k \neq 0$ and $u \neq 0$
if $u \neq 0$, then $ku \neq 0$
Thus $ku \neq 0$
Thus P implies P'

[Problem 8]

Let $-\infty$ and ∞ denote two distinct objects, neither of which is in R.
Define an addition and scalar multiplication on $R \cup \{\infty\} \cup \{-\infty\}$
Specifically, the sum and product of two real numbers is as usual,
and for $k \in R$ define:

$$k\infty = \begin{cases} -\infty & if\ k < 0 \\ 0 & if\ k = 0 \\ \infty & if\ k > 0 \end{cases} \qquad k(-\infty) = \begin{cases} \infty & if\ k < 0 \\ 0 & if\ k = 0 \\ -\infty & if\ k > 0 \end{cases}$$

1. $k + \infty = \infty + k = \infty$
2. $k + (-\infty) = -\infty + k = -\infty$
3. $\infty + \infty = \infty$
4. $(-\infty) + (-\infty) = -\infty$
5. $\infty + (-\infty) = 0$

Show that $R \cup \{\infty\} \cup \{-\infty\}$ is not a vector space over R.

Given
$\infty \notin R$
$-\infty \notin R$
$\infty + (-\infty) = 0$

Implied
$-\infty \neq 0$
$\infty = 0$
$0 \notin R$

A1	Let $u = \infty$ and $v = -\infty$ be arbitrary vectors Then $u + v \in R$ $= u + v$ $= 0$ Given $\infty + (-\infty) = 0$ But $0 \notin R$ Therefore A1 fails.
A2	Let 0 be the additive identity ∞ Let u be an arbitrary vector k, where $k \in R$ Then $u + 0 = u$ $\Rightarrow k + \infty = \infty$ Given *1* $k + \infty = \infty + k = \infty$ But $0 \notin R$, $\infty \notin R$ and $-\infty \notin R$ Therefore A2 fails.
A3	Let ∞ be the additive identity to $-\infty$ Then $u + (-u) = 0$ $\Rightarrow \infty + (-\infty) = 0$ Given *5.* $\infty + (-\infty) = 0$ But $0 \notin R$, $\infty \notin R$ and $-\infty \notin R$ Therefore A3 fails.

Axioms of addition *A1, A2 & A3 all faill*
Thus $R \cup \{\infty\} \cup \{-\infty\}$ is not a vector space over R.