[Question 1]

Statement	$P \Rightarrow Q$	$x > 2 \Rightarrow x^2 > 4$	If $x > 2$, then $x^2 > 4$
Converse	$Q \Rightarrow P$	$x^2 > 4 \Rightarrow x > 2$	If $x^2 > 4$, then $x > 2$
Inverse	$\neg P \Rightarrow \neg Q$	$x \leq 2 \Rightarrow x^2 \leq 4$	If $x \leq 2$, then $x^2 \leq 4$
Contrapositive	$\neg Q \Rightarrow \neg P$	$x \leq 4 \Rightarrow x \leq 2$	If $x \leq 4$, then $x \leq 2$
Negation	$\neg(P \Rightarrow Q)$	$\neg(x > 2 \Rightarrow x^2 > 4)$	It is not true that If $x > 2$, then $x^2 > 4$

1.1 *Give the contrapositive of the statement.*

Contrapositive	$\neg Q \Rightarrow \neg P$	$x \leq 4 \Rightarrow x \leq 2$	If $x \leq 4$, then $x \leq 2$

1.2 *Give the converse of the statement, and determine whether the converse is true or not*

Converse	$Q \Rightarrow P$	$x^2 > 4 \Rightarrow x > 2$	If $x^2 > 4$, then $x > 2$

Let $x = -5$

Substitute $x = -5$

LHS

$\Rightarrow x^2 > 4$

$\Rightarrow (-5)^2 > 4$

$\Rightarrow 25 > 4$ which is true

RHS

$\Rightarrow x > 2$

$\Rightarrow -5 > 2$ which is not true

Thus, the converse statement is not true.

1.3 *What is the negation of the statement?*

Negation	$\neg(P \Rightarrow Q)$	$\neg(x > 2 \Rightarrow x^2 > 4)$	It is not true that If $x > 2$, then $x^2 > 4$

[Question 2]

2.1 Prove by induction that $11^n - 8^n$ *is a multiple of 3 for all* $n \in N$*.*

Clause
Let $P(n)$ be the statement:
$11^n - 8^n$ is a multiple of 3 for all $n \in \mathbb{N}$

Basis Clause
Let $n = 1$

$P(1)$:
$11^n - 8^n$
$\Rightarrow 11^1 - 8^1$
$\Rightarrow 3$, which is a multiple of 3
Therefore, basis clause is true

Inductive Clause
Show that $n = k$
$P(k)$ is where $n = k$

Assume k
$\Rightarrow 11^k - 8^k$, which is a multiple of 3

Extremal Clause (Inductive Step)
If $P(k)$ is true
then $P(k+1)$ must also be true

Assume $k+1$
$\Rightarrow 11^{k+1} - 8^{k+1}$ is a multiple of 3
$\Rightarrow 11 \times 11^k - 8 \times 8^k$
$\Rightarrow 11 \times 11^k - 11^k + 11^k - 8 \times 8^k$
$\Rightarrow 11^k(11-1) + 8^k(11-8)$

Use **Inductive Clause** $11^k - 8^k$ is a multiple of 3
$\Rightarrow 11^k(11-1) + 8^k(11-8)$
$\Rightarrow 11^k(10) + 8^k(3)$
$\Rightarrow 3 \times (3 \times 11^k + 8^k \times 3)$, which is a multiple of 3

2.2 Prove that the function given by

$$f(x) = \frac{x-3}{x+2}$$

is 1 - 1 on $\mathbb{R}$

$f(x)$ *is one-to-one (injective) if*
for all a, b *in the domain of* $f(x)$*,*
$f(a) = f(b)$ *then* $a = b$*.*

OR
for all a, b *in the domain of* $f(x)$*,*
if $a \neq b$ *then.* $f(a) \neq f(b)$

Direct Proof
Let x_1 *and* x_2 *be two arbitrary real numbers such that* $x_1 \neq x_2$

Assume $f(x_1) = f(x_2)$:
$\Rightarrow \frac{x_1-3}{x_1+2} = \frac{x_2-3}{x_2+2}$
$\Rightarrow (x_1 - 3)(x_2 + 2) = (x_2 - 3)(x_1 + 2)$

LHS
$\Rightarrow x_1x_2 + 2x_1 - 3x_2 - 6$

RHS
$\Rightarrow x_1x_2 + 2x_2 - 3x_1 - 6$

$\therefore LHS = RHS$
$\Rightarrow x_1x_2 + 2x_1 - 3x_2 - 6 = x_1x_2 + 2x_2 - 3x_1 - 6$
$\Rightarrow x_1x_2 + 2x_1 - 3x_2 = x_1x_2 + 2x_2 - 3x_1$
$\Rightarrow 2x_1 - 3x_2 = 2x_2 - 3x_1$
$\Rightarrow 2x_1 + \ 3x_1 = 2x_2 + 3x_2$
$\Rightarrow 5x_1 = 5x_2$
$\Rightarrow x_1 = x_2$

However, this contradicts $x_1 \neq x_2$*,*
proving that $f(x) = \frac{x-3}{x+2}$ *is 1 - 1.*

and find a formula for the inverse of f

$f(x) = \frac{x-3}{x+2}$

Let $y = \frac{x-3}{x+2}$

Inverse

$\Rightarrow x = \frac{y-3}{y+2}$

$\Rightarrow x(y+2) = y-3$

$\Rightarrow xy + 2x = y-3$

$\Rightarrow xy - y = -2x-3$

$\Rightarrow y(x-1) = -2x-3$

$\Rightarrow y = \frac{-2x-3}{(x-1)}$

Thus, $f^{-1}(x) = \frac{-2x-3}{(x-1)}$

2.3 Consider the functions $f(x) = 2x + 1$ and $g(x) = x + 1$. Indicate the domain of definition of each of the following functions: f; g; $f \circ g$; $g \circ f$.

f	$f(x) = 2x + 1$ Domain $x \in \mathbb{R}$
g	$g(x) = x + 1$ Domain $x \in \mathbb{R}$
$f \circ g$	$f(x) = 2x + 1$ $\Rightarrow f(g(x)) = 2(x + 1) + 1$ $\Rightarrow f \circ g = 2x + 3$ Domain $x \in \mathbb{R}$
$g \circ f$	$g(x) = x + 1$ $\Rightarrow g(f(x)) = (2x + 1) + 1$ $\Rightarrow g \circ f = 2x + 2$ Domain $x \in \mathbb{R}$

[Question 3]

2.1 Let $S = \left\{\frac{n}{n+1} : n = 1,2,3 \ldots\right\}$

Find the infimum and supremum of S*.*

$S = \left\{\frac{n}{n+1} : n = 1,2,3 \ldots\right\}$
Hence, $n \in \mathbb{Z}^+$,
And in the sequence, for any $\frac{n}{n+1}$ is always a positive fraction between 0 and 1.

supremum of S (Least Upper Bound):
$\Rightarrow S = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$
for any $\frac{n}{n+1}$, approaches but never reaches 1
claim is that the supremum of S is 1

infimum of S (Greatest Lower Bound):
$\frac{n}{n+1} > 0 \;\; \textit{for all } n$
for any $\frac{n}{n+1}$, approaches but never reaches 1
Hence *infimum of* S cannot be greater than 0
claim that the supremum of S is 0

[Question 4]

(4.1) Show that the sequence $a_n = \frac{n}{n+1}$ converges to 1.

Definition: Convergent Sequences

$\forall\epsilon > 0$, $\exists\, N \in \mathbb{N}$, such that $\mathrm{n} > N$ implies $|a_n - a| < \epsilon$

$\forall\epsilon > 0$	"for all positive numbers"
$\exists\, N \in \mathbb{N}$	"there exists some N in the set of natural numbers,"
such that $\mathrm{n} > N$ implies $\lvert a_n - a\rvert < \epsilon$	if n is greater than N, then the absolute difference between a_n and the limit a(denoted as $\lvert a_n - a\rvert$) is less than ϵ

Given *the sequence* $a_n = \frac{n}{n+1}$,

The limit of a_n as n approaches infinity is:

$\lim_{n\to\infty} \frac{n}{n+1}$

$\Rightarrow \lim_{n\to\infty} \frac{n}{n}$

$\Rightarrow 1$

For every $\epsilon > 0$

There exists is a positive integer N

such that $\mathrm{n} > N$ implies $|a_n - 1| < \epsilon$

For $\mathrm{n} > N$:

$|a_n - 1|$

$\Rightarrow \left|\frac{n}{n+1} - 1\right|$

$\Rightarrow \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right|$

$\Rightarrow \frac{1}{n+1}$

And if $|a_n - 1| < \epsilon$,

$|a_n - 1| < \epsilon$

$\Rightarrow \frac{1}{n+1} < \epsilon$

$\Rightarrow n + 1 > \frac{1}{\epsilon}$

$\Rightarrow n > \frac{1}{\epsilon} - 1$

To verify $|a_n - 1| < \epsilon$ for $\mathbf{n} > N$

We round up $\frac{1}{\epsilon}$ to the nearest integer

and add 1 to ensure that N is at least as large as $\frac{1}{\epsilon}$

Thus satisfying $\mathrm{n} > N$

Let $N = 2$:
verify for $\mathrm{n} > N$
$\Rightarrow n > 2$
$\Rightarrow n + 1 > 3$

Since $n > 2$, we have $n + 1 > 3$,
which means $\frac{1}{n+1} < \frac{1}{3}$
So, $|a_n - 1| < \frac{1}{3}$ *for* $n > 2$
Which holds true for every $\epsilon > 0$ including $\epsilon = \frac{1}{3}$

Thus the sequence $\boldsymbol{a_n} = \frac{\boldsymbol{n}}{\boldsymbol{n+1}}$, converges to 1 as n approaches infinity.

(4.2) Suppose that x_n is a sequence of real numbers that converges to 1 as $n \to \infty$. Prove that the lim_{1+x_n} converges to 2 as $n \to \infty$.

Definition: Limits

$\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$, such that $\mathrm{n} > N$ implies $|x_n - a| < \epsilon$

Let x_n *be a sequence of real numbers that converges to 1 as* $n \to \infty$.

Let L be $\lim_{n\to\infty}(x_n)$

Thus

$\Rightarrow L = \lim_{n\to\infty} x_n$

$\Rightarrow L = 1$

Let M be $\lim_{n\to\infty}(1)$, *which is the limit of the constant sequence 1.*

Thus

$\lim_{n\to\infty}(1) = 1$

Since $\lim_{n\to\infty}(x_n) = 1$ *and* $\lim_{n\to\infty}(1) = 1$, *we have:*

$\Rightarrow \lim_{n\to\infty} x_n + \lim_{n\to\infty}(1)$

$\Rightarrow\ L + M$

$\Rightarrow\ 1 + 1 = 2$

Therefore lim_{1+x_n} converges to 2 as $n \to \infty$.

(4.3) Find two convergent subsequences of the sequence $(-1)^n$ that have different limits.

Definition: Convergent Sequences
$\forall\epsilon > 0$, $\exists\, N \in \mathbb{N}$, such that $\mathrm{n} > N$ implies $|a_n - a| < \epsilon$

[1] *Let a_{2n} be the sequence of even integers, where:*
$\{n \in Z \mid n = 2k \, for \, some \, k \in Z\}$
Thus
$\Rightarrow a_{2n} = 1$
Which is the sequence that converges to 1

[2] Given *the sequence* $a_{2n} = 1$*,*
The limit of a_{2n} as n approaches infinity is:
$\lim_{n\to\infty}(1) = 1$

[3] For every $\epsilon > 0$
There exists is a positive integer N
such that $\mathrm{n} > N$ implies $|a_{2n} - 1| < \epsilon$
For $\mathrm{n} > N$*,*
$\Rightarrow |1 - 1| = 0$
Thus, $\boldsymbol{0 < \epsilon}$
The sequence a_{2n} converges to 1 as n approaches infinity,
where $\lim_{n\to\infty}(a_{2n}) = 1$

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[1] *Let a_{2n+1} be the sequence of odd integers, where:*
$\{n \in Z \mid n = 2k + 1 \, for \, some \, k \in Z\}$
Thus
$\Rightarrow a_{2n+1} = -1$
Which is the sequence that converges to -1

[2] Given *the sequence* $a_{2n+1} = -1$*,*
The limit of a_{2n+1} as n approaches infinity is:
$\lim_{n\to\infty}(-1) = -1$

[3] For every $\epsilon > 0$
There exists is a positive integer N
such that $\mathrm{n} > N$ implies $|a_{2n+1} - (-1)| < \epsilon$
For $\mathrm{n} > N$*,*
$\Rightarrow |-1 - -1| = 0$
Thus, $0 < \epsilon$
The sequence a_{2n+1} converges to -1 as n approaches infinity,
where $\lim_{n\to\infty}(a_{2n+1}) = -1$

(4.4) Use the Monotone Convergence Theorem to prove that
$x_n = \frac{1}{\sqrt{n}}$ converges to 0.

Definition: Monotone Convergence Theorem (MCT):
If (x_n) is non-**decreasing** and bounded **above**,
then there exists a real number L such that:
$\lim_{n\to\infty} x_n = L$
or
If (x_n) is non-**increasing** and bounded **below**,
then there exists a real number L such that:
$\lim_{n\to\infty} x_n = L$

Bounded Below:
A sequence (x_n) or a set of real numbers is said to be bounded below
if there exists a real number L
such that $x_n \geq L$ for all n in the sequence
or $x \geq L$ for all x in the set.

There is a lower bound L such that all elements of the sequence or set are greater than or equal to L

Bounded Above:
A sequence (x_n) or a set of real numbers is said to be bounded above
if there exists a real number U
such that $x_n \leq U$ for all n in the sequence
or $x \leq U$ for all x in the set.

There is a lower bound U such that all elements of the sequence or set are less than or equal to U

[1] *For all n, we* need to prove that
$x_{n+1} \leq x_n$

Let x_n *be a sequence of real numbers that converges to 0 as* $n \to \infty$.

Given, $x_n = \frac{1}{\sqrt{n}}$
Then, $x_{n+1} = \frac{1}{\sqrt{n+1}}$
Thus
$\Rightarrow x_{n+1} \leq x_n$
$\Rightarrow \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$

Therefore, x_n *is decreasing sequence*

[2] *We* need to prove that:
For all n, there exists a real number L such that $x_n \geq L$
or
For all x, there exists a real number L such that $x \geq L$

We have that $\sqrt{n} \in \mathbb{Z}^+$,

And that $\frac{1}{\sqrt{n}} < 1$

Thus x_n is bounded below by 0.

Because:
[1] x_n *is decreasing sequence*
[2] x_n is bounded below by 0

By the definition of the Monotone Convergence Theorem (MCT),
x_n converges
Thus, $\lim_{n\to\infty}(x_n) = 0$