Problem 37.

Let $T: R^3 \to R^3$ be a multiplication by A.

Determine whether T has an inverse.

If so, find $T^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

[1] Compute detA

$$det\,(A) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow det(A) = 1 \times det\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 2 \times det\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - 1 \times det\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$\Rightarrow det(A) =$

$$1 \times (1 \times 1 - 2 \times 2) - 2 \times (1 \times 1 - 2 \times -1) - 1 \times (1 \times 2 - 1 \times -1)$$

$$\Rightarrow det(A) = 1 \times -3 - 2 \times 3 - 1 \times 3$$

$$\Rightarrow det(A) = -12$$

[2] Compute inverse of A

$$A^{-1} = \frac{1}{\det(A)} adj\,(A)$$

$$\Rightarrow A^{-1} = \frac{1}{12} adj\,(A)$$

[3] Compute Cofactor Matrix

Cofactor of a_{11}

$$M_{11} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 1 \times 1 - 2 \times 2 = 3$$

$$C_{11} = (-1)^{1+2} \times 3 = -3$$

Cofactor of a_{12}

$$M_{12} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = 1 \times 1 - 2 \times -1 = 3$$

$$C_{12} = (-1)^{1+2} \times 3 = -3$$

Cofactor of a_{13}

$$M_{13} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = 1 \times 2 - 1 \times -1 = 3$$

$$C_{13} = (-1)^{1+3} \times 3 = 3$$

Cofactor of a_{21}

$$M_{21} = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = 2 \times 1 - 1 \times 2 = 4$$

$$C_{21} = (-1)^{2+1} \times 4 = -4$$

Cofactor of a_{22}

$$M_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1 \times 1 - 1 \times -1 = 0$$

$$C_{22} = (-1)^{2+2} \times 0 = 0$$

Cofactor of a_{23}

$M_{23} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = 1 \times 2 - 1 \times -1 = 3$

$C_{23} = (-1)^{2+3} \times 3 = -3$

Cofactor of a_{31}

$M_{31} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = 2 \times 2 - 1 \times -1 = 5$

$C_{31} = (-1)^{3+1} \times 5 = 5$

Cofactor of a_{32}

$M_{32} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = 1 \times 2 - 1 \times 1 = 3$

$C_{32} = (-1)^{3+2} \times 3 = -3$

Cofactor of a_{33}

$M_{33} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 1 \times 1 - 2 \times 1 = -1$

$C_{33} = (-1)^{3+3} \times -1 = -1$

$$C = \begin{bmatrix} -3 & -3 & 3 \\ -4 & 0 & -3 \\ 5 & -3 & -1 \end{bmatrix}$$

[4] Compute Adjugate Matrix

$$adj\,(A) = C^T = \begin{bmatrix} -3 & -4 & 5 \\ -3 & 0 & -3 \\ 3 & -3 & -1 \end{bmatrix}$$

[5] Compute the Inverse of A

$$A^{-1} = \frac{1}{\det(A)} adj\,(A)$$

$$\Rightarrow A^{-1} = \frac{1}{12} \begin{bmatrix} -3 & -4 & 5 \\ -3 & 0 & -3 \\ 3 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & -\frac{5}{12} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{3} & \frac{1}{12} \end{bmatrix}$$

[6] Find $T^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & -\frac{5}{12} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{3} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{x}{4} & \frac{y}{3} & -\frac{5z}{12} \\ \frac{x}{4} & 0 & \frac{z}{4} \\ -\frac{x}{4} & \frac{y}{3} & \frac{z}{12} \end{bmatrix}$$

Problem 38.
Let $T: P_1 \to R^2$ be defined as $T\big(p(x)\big) = (p(0), p(1))$

[1] Find $T(1 - x)$:
Let $p(x) = 1 - x$
$p(0) = 1$
$p(1) = 0$
Thus, $T\big(p(x)\big) = (p(0), p(1))$
$\Rightarrow T(1 - x) = (1, 0)$

[2] Show that T is a linear transformation.

A linear transformation (or a linear map)
is a function $T: R^{\mathrm{n}} \to R^{\mathrm{m}}$
that satisfies the following properties:

Additivity
$T(x + y) = T(x) + T(y)$
OR
$T(p(x) + q(x)) = T(p(x)) + T(q(x))$
Where $p(x)$ & $q(x)$ are polynomials

Scalar multiplication
$T(ax) = aT(x)$
OR
$T(c \times p(x)) = c \times T(p(x))$
Where $p(x)$ is a polynomial &
c is a scalar

Additivity

Let $p(x)$ & $q(x)$ be polynomials

$T(p(x) + q(x)) = T(p(x)) + T(q(x))$

$\Rightarrow\ T(p(x) + q(x)) = (p(0) + q(0), p(1) + q(1))$

And,

$T(p(x) + q(x)) = (p(0) + p(1), q(0) + q(1))$

$\Rightarrow\ T(p(x) + q(x)) = (p(0) + q(0), p(1) + q(1))$

Thus, $T(p(x) + q(x)) = T(p(x)) + T(q(x))$

Scalar multiplication

Let $p(x)$ & $q(x)$ be polynomials, and c be a scalar

$T(c \times p(x)) = c \times T(p(x))$

$\Rightarrow\ T(c \times p(x)) = (c \times p(0),\ c \times p(1)))$

And,

$c \times T\big(p(x)\big) = (c \times (p(0),\ p(1))$

$\Rightarrow c \times T\big(p(x)\big) = (c \times p(0),\quad c \times p(1)))$

Thus, $T(c \times p(x)) = c \times T(p(x))$

Therefore, both additivity and scalar multiplication hold,
T is a linear transformation.

[3] Show that T is one-to-one

Injective

A transformation $T: R^{\mathrm{n}} \to R^{\mathrm{m}}$ is one-to-one if
for every vector b in R^{m}
the equation $T(x) = b$ has at most one solution x in R^{n}

In the case of a transformation defined by polynomials,
T, is one-to-one if:
$T\big(p(x)\big) = T\big(q(x)\big)$ implies $p(x) = q(x)$
Where $p(x)$ & $q(x)$ are polynomials

Let $T\big(p(x)\big) = T\big(q(x)\big)$

$T\big(p(x)\big) = T\big(q(x)\big)$

$\Rightarrow \big(p(0),\ p(1)\big) = \big(q(0),\ q(1)\big)$

Thus, $p(0) = q(0)$ and $p(1) = q(1)$
Therefore, T is one-to-one

Problem 39.

Suppose $T: R^3 \to R^3$ is a linear operator defined by
$T\,(x, y, z) = (2x - y, 2y - z, 2z - x)$

Find the matrix for T with respect to the basis $B = \{v_1, v_2, v_3\}$ where
$v_1 = (1, -1, 0)$
$v_2 = (-1, 0, -1)$
$v_3 = (0, 1, -1)$

[1] Compute $T(v_1)$

$T(v_1)$
$\Rightarrow\ T(x, y, z) =\ (2x - y, 2y - z, 2z - x)$
$\Rightarrow\ T(1, -1, 0) =\ (2(1) - (-1), 2(-1) - 0, 2(0) - 1)$
$\Rightarrow\ T(1, -1, 0) = (3, -2, -1)$

As a linear combination
$T(v_1) = (3, -2, -1)$
$\Rightarrow T(v_1) = a_1 v_1 + a_2 v_2 + a_3 v_3$

System of equations
$3 = a_1(1) + a_2(-1) + a_3(0)$
$-2 = a_1(-1) + a_2(0) + a_3(-1)$
$-1 = a_1(0) + a_2(1) + a_3(-1)$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

Thus , $a_1 = 2$, $a_2 = -1$, $a_3 = 0$

[2] Compute $T(v_2)$

$T(v_2)$
$\Rightarrow\ T(x, y, z) =\ (2x - y, 2y - z, 2z - x)$
$\Rightarrow\ T(-1, 0, -1) = (2(-1) - 0, 2(0) - (-1), 2(-1) - (-1))$
$\Rightarrow\ T(-1, 0, -1) = (-2, 1, -1)$

As a linear combination
$T(v_2)$ = $(-2, 1, -1)$
$\Rightarrow T(v_2) = b_1 v_1 + b_2 v_2 + b_3 v_3$

System of equations
$-2 = b_1(1) + b_2(-1) + b_3(0)$
$1 = b_1(-1) + b_2(0) + b_3(-1)$
$-1 = b_1(0) + b_2(1) + b_3(-1)$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

Thus , $b_1 = -2$, $b_2 = 0$, $b_3 = 1$

[3] Compute $T(v_3)$

$T(v_3)$

$\Rightarrow\ T(x, y, z) = \ (2x - y, 2y - z, 2z - x)$

$\Rightarrow\ T(0,1,-1) = (2(0) - 1, 2(1) - (-1), 2(-1) - 0)$

$\Rightarrow\ T(0,1,-1) = (-1,3,-2)$

As a linear combination

$T(v_3)\ = (-1,3,-2)$

$\Rightarrow T(v_3) = c_1v_1 + c_2v_2 + c_3v_3$

System of equations

$-1 = c_1(1) + c_2(-1) + c_3(0)$

$3 = c_1(-1) + c_2(0) + c_3(-1)$

$-2 = c_1(0) + c_2(1) + c_3(-1)$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

Thus , $c_1 = -3$, $c_2 = -2$, $c_3 = 0$

[4] Basis $B = \{v_1, v_2, v_3\}$

$$[T]_B \begin{bmatrix} 2 & -2 & -3 \\ -1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 40.

Find $(T_3 \circ T_2 \circ T_1)(x, y)$ where

$T_1(x, y) = (x, -y, x - y)$

$T_2(x, y, z) = (3x, 0, x - y + z)$

$T_3(x, y, z) = (x + y - z, x + 2y)$

Given

$T_1(x, y) = (x, -y, x - y)$

[1] Compute $T_2 \circ T_1$

$T_2(x, y, z) = (3x, 0, x - y + z)$

$\Rightarrow T_2(T_1) = (3x, 0, x - y + z)$

$\Rightarrow T_2(x, -y, x - y) = (3x, 0, x - (-y) + (x - y))$

$\Rightarrow T_2(x, -y, x - y) = (3x, 0,2x)$

[2] Compute $T_3 \circ T_2$

$T_3(x, y, z) = (x + y - z, x + 2y)$

$\Rightarrow T_3(T_2) = (x + y - z, x + 2y)$

$\Rightarrow T_3(3x, 0,2x) = (3x + 0 - 2x,\ 3x + 2(0))$

$\Rightarrow T_3(3x, 0,2x) = (x, 3x)$

Thus, $(T_3 \circ T_2 \circ T_1)(x, y) = (x, 3x)$