[Problem 1]

(a) Give an example of a set A such that there is a set B with $B \in A$ but $B \not\subseteq A$

Let $A = \{\{X\}, \{Y\}, \{Z\}\}$
Let $B = \{X\}$

We claim that there exists a set B such that $B \in A$ but $B \not\subseteq A$.

B is an element of A
- If $B = \{X\}$, we have $B \in A$

B is not a subset of A
$A \subseteq B \Leftrightarrow \exists x(x \in A \cap x \notin B)$
- there exists at least one element in set A that is not in set B, being $\{Y\}$ or $\{Z\}$, thus $B \not\subseteq A$

Therefore, there exists a set B with $B \in A$ but $B \not\subseteq A$

(b) Give an example of a set A such that there is a set B with $B \subseteq A$ but $B \notin A$

Let $A = \{\{X\}, \{Y\}, \{Z\}\}$
Let $B = \{\{X\}\}$

We claim that there exists a set B such that $B \subseteq A$ but $B \notin A$.

B is a subset of A
$A \subseteq B$ if and only if $x \in A \rightarrow x \in B$
- iff $B = \{\{X\}\}$, it means every element of set B is also an element of set A, we have $B \subseteq A$

B is not an element of A
- The element of B is a set itself
- B has the set $\{X\}$, which is an element of A,
  but B itself is not an element of A.
- The elements of A are a sets themselves

Therefore, there exists a set B with $B \subseteq A$ but $B \notin A$

[Problem 2]

Calculate the following powersets:

The cardinality (number of elements) of the power set of a set with n elements is 2^n, including the empty set and the set itself.

$P(A)$ is the set of all subsets of A

[0] $P(\emptyset)$

i.e. the powerset of the empty set

- has only one element

  $\emptyset$ (the empty set)

- the empty set has no elements, so the only subset it can have is the empty set.

$\therefore P(\emptyset) = \{\emptyset\}$

[1] $P(\{\emptyset\})$

i.e. the power set of the set containing only the empty set

- has only one element

  $\emptyset$ (the empty set)

- powerset is the empty set and the set itself.

$\therefore P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

Cardinality: $|P(\{\emptyset\})| = 2^1 = 2$

[2] $P(\{\emptyset, \{\emptyset\}\})$

i.e. the power set of the set containing the empty set & the set containing the empty set

- has two elements,

  $\emptyset$ (the empty set) &

  $\{\emptyset\}$ (the set containing the empty set)

- all possible subsets + the empty set and the set itself.

$\therefore P(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Cardinality: $|P(\{\emptyset, \{\emptyset\}\})| = 2^2 = 4$

[3] $P(\{\{\emptyset\}\}) = \{\emptyset, \{\emptyset\}\}$

i.e. the power set of the set containing the empty set

- has only one element

  $\{\emptyset\}$ (the set containing the empty set)

- powerset is the empty set and the set itself.

$= \{\emptyset, \{\emptyset\}\}$

$\therefore P(\{\{\emptyset\}\}) = \{\emptyset, \{\emptyset\}\}$

Cardinality: $|P(\{\{\emptyset\}\})\ | = 2^1 = 2$

[4] $P\big(P(\emptyset)\big) = \{\emptyset, \{\emptyset\}\}$
i.e. the powerset of the powerset of the empty set

FROM [0] ABOVE
$P(\emptyset)$ i.e. the powerset of the empty set
- has only one element
  $\emptyset$ (the empty set)
- the empty set has no elements, so the only subset it can have is the empty set.
$= \{\emptyset\}$

substitute $P(\emptyset) = \{\emptyset\}$ into $P(P(\emptyset))$
$P(P(\emptyset))$
$\Longrightarrow P(\{\emptyset\})$

FROM [1] ABOVE
$P(\{\emptyset\})$ i.e. the power set of the set containing only the empty set
- has only one element
  $\emptyset$ (the empty set)
- powerset is the empty set and the set itself.
$\therefore P\big(P(\emptyset)\big) = \{\emptyset, \{\emptyset\}\}$
Cardinality: $|P(\{\emptyset\})| = 2^1 = 2$

[5] $P(P(\{\emptyset\}))$
i.e. the power set of the powerset of the set containing only the empty set

FROM [1] ABOVE
$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
- has only one element
  $\{\emptyset\}$ (the set containing the empty set)
- all possible subsets + the empty set and the set itself.
$\therefore P(P(\{\emptyset\})) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Cardinality: $|P(P(\{\emptyset\}))| = 2^2 = 4$

[Problem 3]

For each of the following functions determine the image of $S = \{x \in \mathbb{R} : 4 \leq x^2\}$

bounds of x for which $4 \leq x^2$

x^2 *will always be positive*

$\therefore 4 \leq x^2$

$\pm\sqrt{4} \leq \sqrt{x^2}$

$\pm 2 \leq x$

$x \geq 2$ *or* $x \geq -2$

$S = \{x \in \mathbb{R} : 4 \leq x^2\}$

$\Rightarrow S = \{x \in \mathbb{R} : x \leq -2, x \geq 2\}$

(a) $f: \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = 3x + 1$.

Interval where $x \in \mathbb{R} :, x \geq 2$*:*

$f(x) = 3x + 1$

$\Rightarrow f(2) \geq 3(2) + 1$

$\Rightarrow f(2) \geq 7$

Interval where $x \in \mathbb{R} :, x \leq -2$*:*

$f(x) = 3x + 1$

$\Rightarrow f(-2) \leq 3(-2) + 1$

$\Rightarrow f(-2) \leq -5$

Therefore, the Image of f is $\{x \in \mathbb{R}: f(x) \geq 7, f(x) \leq -5\}$

(b) $g: \mathbb{R} \to \mathbb{R}$ *defined by* $g(x) = 4x^2$.

Interval where $x \in \mathbb{R} :, x \geq 2$*:*

$f(x) = 4(2)^2 = 16$

Interval where $x \in \mathbb{R} :, x \leq -2$*:*

$f(x) = 4(-2)^2 = 16$

$4x^2$ *will always be positive, as* x^2 *will always be positive*

Therefore, the Image of f is $\{x \in \mathbb{R}: f(x) \geq 0, f(x) \leq 0\}$

i.e. set of non-negative real numbers

[Problem 4]

Consider the following two functions.
(1) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 4$.
(2) $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 4x^2$.
Determine whether the given functions are one-to-one correspondences.

A function $f: A \to B$ is said to be one-to-one correspondence
iff f is both:
Injective (one-to-one): $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ and,
Surjective (ONTO): for all $b \in B$ there is some $a \in A$ such that $f(a) = b$

(1) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 4$.

INJECTIVE
Take $x_1, x_2 \in \mathbb{R}$ and assume that:
$f(x_1) = f(x_2)$
$\Rightarrow 3x_1 - 4 = 3x_2 - 4$
$\Rightarrow 3x_1 = 3x_2$
$\Rightarrow x_1 = x_2$
Therefore f is one-to-one, by definition of one-to-one.

SURJECTIVE
We need to find an x that maps to y.
$3x - 4 = y$
$3x = y + 4$
$x = \frac{y+4}{3}$s

(2) $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 4x^2$.

INJECTIVE
Take $x_1, x_2 \in \mathbb{R}$ and assume that:
$f(x_1) = f(x_2)$
$\Rightarrow 4x_1^2 = 4x_2^2$
$\Rightarrow \sqrt{4x_1^2} = \sqrt{4x_2^2}$
$\Rightarrow |2x_1| = |2x_2|$
$\Rightarrow |x_1| = |x_2|$
The values for x_1 & x_2 could be the same, with different signs.

e.g.
$47 \neq -47$
$x_1 \neq -x_2$

Therefore f is not one-to-one, by definition of one-to-one.