Question 1

Solve the following initial value problem

$$X = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} X \ , \ X(0) = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

[1] Eigenvalues of A

$$A - \lambda I \ = \begin{bmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

Thus $\lambda = 2$

[2] Eigenvectors of A

$$(A - 2I)v = 0$$

$$\begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R1:

$$v_2 + 6v_3 = 0$$
$$\Rightarrow v_2 = -6v_3$$

R2:

$$5v_3 = 0$$
$$v_3 = 0$$

Let $v_1 = 1$, $v_2 = 0$

$$(A - 2I)v_2 = v_1$$

$$\begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

R1:

$$v_2 + 6v_3 = 1$$

R2:

$$5v_3 = 0$$
$$v_3 = 0$$

Thus $v_2 = 1$

Let $v_1 = 0$, $v_2 = 1$

$$(A - 2I)v_2 = v_1$$

$$\begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

R1:

$$v_2 + 6v_3 = 0$$

R2:

$$5v_3 = 1$$

$$v_3 = \frac{1}{5}$$

Thus $v_2 = -\frac{6}{5}$

[3] General solution

$$X(t) = c_1 e^{2t} v_1 + c_2 e^{2t}(v_2 + tv_1) + c_3 e^{2t}(v_3 + tv_2 + \frac{t^2}{2} v_1)$$

$$\Rightarrow X(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{2t} \left(\begin{bmatrix} 0 \\ -\dfrac{6}{5} \\ \dfrac{1}{5} \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

[4] Apply Initial Conditions

$$\Rightarrow X(0) = c_1 e^{0} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{0} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{0} \left(\begin{bmatrix} 0 \\ -\dfrac{6}{5} \\ \dfrac{1}{5} \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{0^2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\Rightarrow X(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{bmatrix}$$

Where initial condition is $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

$$\Rightarrow X(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

[4] Solve for c_1 , c_2 , c_3

$c_1 = 2$

$c_2 - \frac{6}{5}c_3 = -3$

$\frac{1}{5}c_3 = 1$

$\Rightarrow c_3 = 5$

For c_2

$c_2 - \frac{6}{5}(5) = -3$

$\Rightarrow c_2 - 6 = -3$

$\Rightarrow c_2 = 3$

[5] Solution to the initial value problem

$$\Rightarrow X(t) = 2e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + 3e^{2t}\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 5e^{2t}\left(\begin{bmatrix}0\\-\frac{6}{5}\\\frac{1}{5}\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

$$\Rightarrow X(t) = e^{2t}\left(2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 5e^{2t}\left(\begin{bmatrix}0\\-\frac{6}{5}\\\frac{1}{5}\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\right)$$

$$\Rightarrow X(t) = e^{2t}\left(\begin{bmatrix}2\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + 3t\begin{bmatrix}1\\0\\0\end{bmatrix} + 5\begin{bmatrix}0\\-\frac{6}{5}\\\frac{1}{5}\end{bmatrix} + 5t\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{5t^2}{2}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

$$\Rightarrow X(t) = e^{2t}\left(\begin{bmatrix}2\\0\\0\end{bmatrix} + \begin{bmatrix}0\\3\\0\end{bmatrix} + t\begin{bmatrix}3\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-6\\1\end{bmatrix} + t\begin{bmatrix}0\\5\\0\end{bmatrix} + \begin{bmatrix}\frac{5t^2}{2}\\0\\0\end{bmatrix}\right)$$

$$\Rightarrow X(t) = e^{2t}\begin{bmatrix}2 + 3t + \frac{5t^2}{2}\\-3 + t\\1\end{bmatrix}$$

Question 2

We know that if Φ is a fundamental matrix of $\dot{X} = AX$ at $t_0 = 0$ and $F(t)$ is continuous in t then

$$X_p(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)F(s)ds$$

is a particular solution of $\dot{X} = AX + F(t)$.

Use this result with $t_0 = 0$ to find a general solution of

$$\dot{X} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 12e^{3t} \end{bmatrix}$$

[1] Eigenvalues of A

Solve for

$\dot{X} = AX$, $\dot{X} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -\lambda & 3 \\ 3 & -\lambda \end{bmatrix}$

Thus $\lambda = 3$, $\lambda = -3$

[2] Eigenvectors of A

For $\lambda = 3$

$A - 3I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$

$(A - 3I)v = 0$

$\Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Rightarrow 3x_1 + 3x_2 = 0$

$x_1 = 0$ & $x_2 = 0$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = -3$

$A - 3I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$

$(A + 3I)v = 0$

$\Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Rightarrow 3x_1 + 3x_2 = 0$

$x_1 = -x_2$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

[3] Fundamental matrix

homogeneous solution

$X_h(t) = c_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$

$\Phi(t) = \begin{bmatrix}e^{3t} & e^{-3t}\\ e^{3t} & -e^{-3t}\end{bmatrix}$

[4] Inverse of Fundamental matrix

$\det(\Phi(t)) = \begin{bmatrix}e^{3t} & e^{-3t}\\ e^{3t} & -e^{-3t}\end{bmatrix}$

$\Rightarrow -e^{3t}e^{-3t} - e^{3t}e^{-3t} = -2$

$\mathrm{adj}(\Phi(t)) = \begin{bmatrix}-e^{3t} & -e^{-3t}\\ -e^{3t} & e^{3t}\end{bmatrix}$

Therefore,

$\Rightarrow \Phi^{-1}(t) = \frac{1}{-2}\begin{bmatrix}-e^{3t} & -e^{-3t}\\ -e^{3t} & e^{3t}\end{bmatrix}$

$\Rightarrow \Phi^{-1}(t) = \frac{1}{2}\begin{bmatrix}e^{-3t} & e^{-3t}\\ e^{3t} & -e^{3t}\end{bmatrix}$

[5] Find $X_p(t)$

$X_p(t) = \Phi(t)$

$X_p(t) = \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)F(s)ds$

Given $\begin{bmatrix}0\\12e^{3t}\end{bmatrix}$

$\Rightarrow X_p(t) = \frac{1}{2}\begin{bmatrix}e^{-3t} & e^{-3t}\\ e^{3t} & -e^{3t}\end{bmatrix}\begin{bmatrix}0\\12e^{3t}\end{bmatrix}$

$\Rightarrow X_p(t) = \frac{1}{2}\begin{bmatrix}12e^{-3t}.e^{-3t}\\ -12e^{3t}.e^{3t}\end{bmatrix}$

$\Rightarrow X_p(t) = \frac{1}{2}\begin{bmatrix}12\\ -12e^{6t}\end{bmatrix}$

$\Rightarrow X_p(t) = \begin{bmatrix}6\\ -6e^{6t}\end{bmatrix}$

[6] Integral

$\int_{t_0}^{t} \begin{bmatrix} 6 \\ -6e^{6s} \end{bmatrix} ds$

$\Rightarrow \begin{bmatrix} 6s \\ \int_{t_0}^{t} 6e^{6s}\, ds \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6t \\ -[e^{6s}]_0^t \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6t \\ -\left[\frac{1}{6} e^{6s}\right]_0^t \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6t \\ -\left[\frac{1}{6}[e^{6t} - 1]\right] \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6t \\ -e^{6t} + 1 \end{bmatrix}$

[7] Particular Solution

$X_p(t) = \begin{bmatrix} e^{3t} & e^{-3t} \\ e^{3t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} 6t \\ -e^{6t} + 1 \end{bmatrix}$

$\Rightarrow X_p(t) = \begin{bmatrix} 6te^{3t} - e^{3t} + e^{-3t} \\ 6te^{3t} + e^{3t} - e^{-3t} \end{bmatrix}$

[8] Homogeneous and Particular Solutions:

$X(t) = X_h(t) + X_p(t)$

$\Rightarrow c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 6te^{3t} - e^{3t} + e^{-3t} \\ 6te^{3t} + e^{3t} - e^{-3t} \end{bmatrix}$

Question 3

If Φ is a normalized fundamental matrix $\dot{X} = AX$ at $t_0 = 0$ and if F is continuous in t, then

$$X_p(t) = \int_{t_0}^{t} \Phi(t-s)\, F(s)\, ds$$

is a particular solution of $\dot{X} = AX + F$. Use the result above to find a general solution of

$$\dot{X} = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} X + \begin{bmatrix} te^{2t} \\ -e^{2t} \end{bmatrix}$$

[1] Eigenvalues of A

Solve for

$\dot{X} = AX$, $\dot{X} = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ -4 & 2-\lambda \end{bmatrix}$$
$$\Rightarrow (2-\lambda)^2 + 4$$
$$\Rightarrow \lambda^2 - 4\lambda + 8 = 0$$

Thus $\lambda = 2 \pm 2i$

[2] Eigenvectors of A

For $\lambda = 2 + 2i$

$$(A - \lambda I)v = 0$$

$$\Rightarrow \begin{bmatrix} 2-(2+2i) & 1 \\ -4 & 2-(2+2i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$
$$\Rightarrow \begin{bmatrix} -2i & 1 \\ -4 & -2i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$
$$v_1 = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

For $\lambda = 2 - 2i$

$$(A - \lambda I)v = 0$$

$$\Rightarrow \begin{bmatrix} 2-(2-2i) & 1 \\ -4 & 2-(2-2i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$
$$\Rightarrow \begin{bmatrix} 2i & 1 \\ -4 & 2i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$
$$v_1 = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

[3] Fundamental matrix

Given by e^{At}, here A is the coefficient matrix of the homogeneous system $\dot{X} = AX$

$$A = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}$$

$$\Phi(t) = e^{At} = Pe^{Jt}P^{-1}$$

where P is the matrix of eigenvectors

J is the Jordan form

[4] Jordan form

$$J = \begin{bmatrix} 2+2i & 0 \\ 0 & 2-2i \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix}$$

[5] matrix of eigenvectors

Thus, $P^{-1} = \frac{1}{\det(P)}\begin{bmatrix} -2i & -1 \\ -2i & 1 \end{bmatrix}$

$$\det(P) = 1 \times -2\mathrm{i} - 1 \times 2i$$

$$\Rightarrow -4i$$

$$P^{-1} = \frac{1}{-4i}\begin{bmatrix} -2i & -1 \\ -2i & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4i} \\ \frac{1}{2} & -\frac{1}{4i} \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{i}{4} \\ \frac{1}{2} & \frac{1}{4i} \end{bmatrix}$$

[6] compute e^{Jt}

$$e^{Jt} = \begin{bmatrix} e^{(2+2i)t} & 0 \\ 0 & e^{(2-2i)t} \end{bmatrix}$$

$$\Rightarrow e^{Jt} = \begin{bmatrix} e^{(\cos(2t)+\mathrm{i}\sin(2t))} & 0 \\ 0 & e^{2t}(\cos(2t) - \mathrm{i}\sin(2t)) \end{bmatrix}$$

[7] compute $Pe^{Jt}P^{-1}$

$$\Phi(t) = Pe^{Jt}P^{-1}$$

$$\Rightarrow \Phi(t) = \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix} \begin{bmatrix} e^{(\cos(2t) + \text{ i} \sin(2t))} & 0 \\ 0 & e^{2t}(\cos(2t) - \text{ i} \sin(2t)) \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{4} \\ \frac{1}{2} & \frac{1}{4i} \end{bmatrix}$$

Let $a = e^{2t}(\cos(2t) + \text{ i} \sin(2t))$, $b = e^{2t}(\cos(2t) + \text{ i} \sin(2t))$

$$\Rightarrow \Phi(t) = \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{4} \\ \frac{1}{2} & \frac{1}{4i} \end{bmatrix}$$

$$\Rightarrow \Phi(t) = \begin{bmatrix} a & b \\ 2ia & -2ib \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{4} \\ \frac{1}{2} & \frac{1}{4i} \end{bmatrix}$$

If $a = e^{2t}(\cos(2t) + \text{ i} \sin(2t))$, $b = e^{2t}(\cos(2t) - \text{ i} \sin(2t))$, *then*

(1,1)

$\Rightarrow \Phi(t) = a \times -\frac{1}{2} + b \times -\frac{1}{2}$

$\Rightarrow -\frac{1}{2}(a + b)$

$\Rightarrow -\frac{1}{2}(e^{2t}(\cos(2t) + \text{ i} \sin(2t)) + e^{2t}(\cos(2t) - \text{ i} \sin(2t)))$

$\Rightarrow 2e^{2t} \cos(2t)$

(1,2)

$\Rightarrow \Phi(t) = a \times -\frac{1}{4i} + b \times \frac{1}{4i}$

$\Rightarrow \frac{1}{4i}(e^{2t}(\cos(2t) + \text{ i} \sin(2t)) - e^{2t}(\cos(2t) - \text{ i} \sin(2t)))$

$\Rightarrow \frac{1}{4}(2e^{2t} \, si\,n(2t))$

$\Rightarrow e^{2t} sin\,(2t)$

(2,1)

$\Rightarrow \Phi(t) = 2ia \times -\frac{1}{2} + 2ib \times -\frac{1}{2}$

$\Rightarrow -ia + ib$

$\Rightarrow 2e^{2t} sin\,(2t)$

(2,2)

$\Rightarrow \Phi(t) = 2ia \times -\frac{1}{4i} - 2ib \times \frac{1}{4i}$

$\Rightarrow -\frac{1}{2}(a + b)$

$\Rightarrow -\frac{1}{2}(2e^{2t} \cos(2t))$

$\Rightarrow -e^{2t} \cos(2t)$

Thus,

$$\Phi(t) = \begin{bmatrix} 2e^{2t} \cos(2t) & e^{2t} sin\,(2t \\ 2e^{2t} sin\,(2t) & -e^{2t} \cos(2t) \end{bmatrix}$$

$$\Rightarrow \Phi(t) = e^{2t} \begin{bmatrix} \cos(2t) & -sin\,(2t \\ sin\,(2t) & \cos(2t) \end{bmatrix}$$

Question 4

(4.1) Write the companion system for the equation given below.

$\Rightarrow y^{(3)} - y^{(2)} + 2y^{(1)} - 5y = \sin t$

Write your final answer in terms of the appropriate trigonometric functions.

[1] Re-write as First-order differential equation

$$y^{(3)} - y^{(2)} + 2y^{(1)} - 5y = \sin t$$
$$y^{(3)} = y^{(2)} - 2y^{(1)} + 5y + \sin t$$

And

$$y^{(1)} = (0 \quad 1 \quad 0)$$
$$y^{(2)} = (0 \quad 0 \quad 1)$$

Thus,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -2 & 1 \end{bmatrix}$$

[2] Re-write in matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}$$

(4.2) Find the series solution of the blow system of first order differential equations using the power series method

$$\dot{X} = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} X \quad \textit{with } X(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Write your final answer in terms of the appropriate trigonometric functions.

[1] Power series

$$X(t) = \sum_{n=0}^{\infty} X_n t^n$$

$$\Rightarrow \dot{X}(t) = \sum_{n=1}^{\infty} n X_n t^{n-1}$$

$$\Rightarrow \dot{X}(t) = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \sum_{n=0}^{\infty} X_n t^n$$

$$\Rightarrow \sum_{n=1}^{\infty} n X_n t^{n-1} = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} (X_0 + X_1 t + X_2 t^2 + X_3 t^3 \dots)$$

$$\Rightarrow \sum_{n=1}^{\infty} n X_n t^{n-1} = X_0 \begin{bmatrix} -1 & 2 & -2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} + X_1 t \begin{bmatrix} -1 & 2 & -2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} + X_2 t^2 \begin{bmatrix} -1 & 2 & -2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} + X_3 t^3 \begin{bmatrix} -1 & 2 & -2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \dots$$

[2] Recurrence Relation

$$\Rightarrow nX_n = X_{n-1}\begin{bmatrix}-1 & 2 & -2\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$$

$$X_0 = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$X_1 = X_0\begin{bmatrix}-1 & 2 & -2\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}-1 & 2 & -2\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$X_2 = \frac{1}{2}X_1\begin{bmatrix}-1 & 2 & -2\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$$

$$\Rightarrow X_2 = \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}-1 & 2 & -2\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$$

$$\Rightarrow X_2 = \begin{bmatrix}-1.5\\0\\0\end{bmatrix}$$

$$X_3 = \frac{1}{3}X_2\begin{bmatrix}-1 & 2 & -2\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$$

$$\Rightarrow X_3 = \frac{1}{3}\begin{bmatrix}-1.5\\0\\0\end{bmatrix}\begin{bmatrix}-1 & 2 & -2\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$$

$$\Rightarrow X_3 = \begin{bmatrix}0.5\\-0.5\\-0.5\end{bmatrix}$$

Thus

$$X(t) = \begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix}t + \begin{bmatrix}-1.5\\0\\0\end{bmatrix}t^2 + \begin{bmatrix}0.5\\-0.5\\-0.5\end{bmatrix}t^3$$

Question 5

We know that if Φ is a normalized fundamental matrix of $\dot{X} = AX$ at $t_0 = 0$; then

$$\Phi(s+t) = \Phi(s)\,\Phi(t)$$

for all real numbers s and t. Use this hypothesis and

(5.1) Show that for all real numbers s and t

$$\Phi(t) + \Phi(s) = \Phi(s)\Phi(t)$$

[1] Initial Condition

If I is the identity matrix, at $t_0 = 0$ we have:

$$\Phi(0) = I$$

[2] Fundamental matrix

Given by e^{At}, here A is the coefficient matrix of the homogeneous system $\dot{X} = AX$

$\Phi(t) = e^{At}$

[3] Matrix Exponential

Given a homogeneous system $\dot{X} = AX$, *the* fundamental matrix $\Phi(t)$ can be expressed as e^{At}, where e^{At} is the matrix exponential.

$\Phi(t) = e^{At}$
$\Rightarrow \Phi(s+t) = e^{A(s+t)}$
$\Rightarrow \Phi(s+t) = e^{As+At}$
$\Rightarrow \Phi(s+t) = e^{A(s)}e^{A(t)}$
$\Rightarrow \Phi(s+t) = \Phi(s)\Phi(t)$

(5.2) *Show that if Φ is a normalized fundamental matrix at $t_0 = 0$; then*

$\Phi^{-1}(t) = \Phi(-t)$ for all t

[1] Initial Condition

If I is the identity matrix, at $t_0 = 0$ we have:

$$\Phi(0) = I$$

[2] Fundamental matrix

Given by e^{At}, here A is the coefficient matrix of the homogeneous system $\dot{X} = AX$

$\Phi(t) = e^{At}$

[3] Inverse of the Fundamental Matrix

$\Phi(t) = e^{At}$

$\Rightarrow \Phi^{-1}(t) = (e^{At})^{-1}$

$\Rightarrow \Phi^{-1}(t) = e^{-At}$

$\Rightarrow \Phi(-t) = e^{-At}$

Thus, $\Phi^{-1}(t) = \Phi(-t) = e^{-At}$

Question 6

Classify the critical points of the plane autonomous system corresponding to the second order non-linear differential equation:

$\ddot{x} + 2(1 - x^3)\dot{x} - 3x = 0$

[1] Re-write as First-order differential equation

$\ddot{x} + 2(1 - x^3)\dot{x} - 3x = 0$

$\Rightarrow \ddot{x} = -2(1 - x^3)\dot{x} + 3x$

Thus $\dot{x} = 0$ and $\ddot{x} = 0$

$\Rightarrow 0 = -2(1 - x^3) \times 0 + 3x$

$\Rightarrow 0 = 3x$

$\Rightarrow x = 0$

[2] Jacobian matrix

$\dot{x} = y$

(1,1)

$$\frac{\partial \dot{x}}{\partial x} = 0$$

(1,2)

$$\frac{\partial \dot{x}}{\partial y} = 1$$

(2,1)

$\ddot{x} = -2(1 - x^3)\dot{x} + 3x$

$\Rightarrow \dot{y} = -2(1 - x^3)y + 3x$

$\Rightarrow \frac{\partial \dot{y}}{\partial x} = -2(-3x^2)y + 3$ *derivative*

$\Rightarrow \frac{\partial \dot{y}}{\partial x} = 3$

(2,2)

$\frac{\partial \dot{y}}{\partial y} = -2(1 - x^3)$

$\Rightarrow \frac{\partial \dot{y}}{\partial y} = -2$

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

[3] Eigenvalues of J

$\det(J - \lambda I) = 0$

$\Rightarrow \det \begin{bmatrix} -\lambda & 1 \\ 3 & -2-\lambda \end{bmatrix} = 0$

Thus $\lambda^2 + 2\lambda + 3 = 0$

[4] Solve for λ

$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 3}}{2 \times 1}$

$\Rightarrow \lambda = \frac{-2 \pm \sqrt{-8}}{2}$

$\Rightarrow \lambda = \frac{-2 \pm 2i\sqrt{2}}{2}$

$\Rightarrow \lambda = -1 \pm i\sqrt{2}$