Tutorial Letter 201/0/2024

Ordinary Differential Equations

APM3706

Year module

Department of Mathematical Sciences

This tutorial letter contains
Assignment 01.

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ASSIGNMENT 01

STUDY GUIDE: CHAPTERS 1, 2 and 3

Answer All Questions

In All assignments we may mark all or few of those questions.

Question 1

(1.1) Determine whether the system

$$\begin{aligned} \dot{x} + \dot{y} + x + y &= 2e^{3t}, \\ \dot{x} + \dot{y} - 3x - 3y &= -e^{-t}, \end{aligned}$$

is degenerate. In the degenerate case, decide whether it has no solution or infinitely many solutions. If it has no solution, explain why, else find the general form of the solutions.

(1.2) Solve the system:

$$\begin{aligned} \ddot{x} + x - 2\dot{y} &= 2t^2, \\ 2\dot{x} - x + \dot{y} - 2y &= 2t^3, \end{aligned}$$

by using the **elimination method (operator method)**.
Hint. Eliminate y first.

Question 2

Show that the system:

$$\begin{aligned} (D-2)[x] + 2D[y] &= 2 - 4\mathrm{e}^{2t} \\ (2D-3)[x] + (3D-1)[y] &= 0 \end{aligned}$$

is equivalent to both the following triangular systems:

$$\begin{cases} (D^2 + D - 2)[x] = 2 + 20\mathrm{e}^{2t} \\ D[x] - 2y = 12\mathrm{e}^{2t} - 6 \end{cases} \quad \text{and} \quad \begin{cases} (D^2 + D - 2)[y] = -6 - 4\mathrm{e}^{2t} \\ x - (D+1)[y] = 8\mathrm{e}^{2t} - 4 \end{cases}.$$

Given the three systems above, separately, state (with complete justifications) your strategy in solving them completely

Question 3

Solve the system:

$$\begin{aligned}(D^2+1)[x_1]-2D[x_2]&=2t,\\(2D-1)[x_1]-(2-D)[x_2]&=5,\end{aligned}$$

by using the elimination method (operator method).

Question 4

(4.1) Use the **eigenvalue-eigenvectors** to solve the initial value problem

$$\dot{\mathbf{X}}=\begin{bmatrix}0&-1\\9&0\end{bmatrix}\mathbf{X},\qquad \mathbf{X}(0)=\begin{bmatrix}1\\1\end{bmatrix}.$$

(4.2) Use the **eigenvalue-eigenvectors** to solve the initial value problem

$$\dot{\mathbf{X}}=\begin{bmatrix}2&0&0\\-1&1&0\\-2&3&2\end{bmatrix}\mathbf{X},\qquad \mathbf{X}(0)=\begin{bmatrix}1\\2\\-2\end{bmatrix}.$$

Question 5

Consider the system of linear differential equations:

$$\dot{\mathbf{X}}=\mathbf{A}\mathbf{X}.$$

(i) Suppose λ is an eigen value of $\mathbf{A}$ of multiplicity $k>1$. Define a root vector corresponding to the eigen value λ.

(ii) Show that each eigen value λ of $\mathbf{A}$ of multiplicity k has k linearly independent root vectors.

(iii) Suppose that λ is an eigen value of $\mathbf{A}$ of multiplicity k and $\mathbf{U}_{\lambda,1}$, $\mathbf{U}_{\lambda,2}$, …, $\mathbf{U}_{\lambda,k}$ are root vectors corresponding to λ of orders 1, 2, …, k, respectively. Further, let:

$$\mathbf{X}_i(t)=e^{\lambda t}\big[\mathbf{U}_{\lambda,i}+t\mathbf{U}_{\lambda,i-1}+\frac{t^2}{2!}\mathbf{U}_{\lambda,i-2}+\dots\frac{t^{i-1}}{(i-1)!}\mathbf{U}_{\lambda,1}\big],\ \text{for } i=1,2,\dots k.$$

Show that $\left\{\mathbf{X}_1(t),\mathbf{X}_2(t),\dots,\mathbf{X}_k(t)\right\}$ is a set of linearly independent solutions of the system of linear differential equations and every solution of the system is a linear combination of the solutions in this set.

Question 6

Solve the system:

$$\dot{\mathbf{X}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Why is there a unique solution to the above system?

Question 7

Solve the system:

$$\dot{\mathbf{X}} = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix} \mathbf{X}$$

completely. Find the unique solution that passes through the point $(11, 3, -7)$ when $t = 0$.