Problem 17.
Find the coordinate vectors for p relative to the basis
$S = \{p_1; p_2; p_3\}$ in $P2$;
where $p = 3 + 4x + 2x^2$;
$p1 = 1 + x$,
$p2 = 1 + x^2$,
$p3 = x + x^2$

[1] Find the scalars a, b and c such that:
$p = ap_1 + bp_2 + cp_3$

$\Rightarrow 3 + 4x + 2x^2 = a(1 + x) + b(1 + x^2) + c(x + x^2)$
$\Rightarrow 3 + 4x + 2x^2 = a + ax + b + bx^2 + cx + cx^2$
$\Rightarrow 3 + 4x + 2x^2 = (a + b) + (a + c)x + (b + c)x^2$

Thus,
$3 = a + b$

$4x = (a + c)x$
$\Rightarrow 4 = a + c$

$2x^2 = (b + c)x^2$
$\Rightarrow 2 = b + c$

[2] Solve system of linear equations
$4 - 3 = (a + c) - (a + b)$
$\Rightarrow 1 = c - b$
$\Rightarrow c = b + 1$

$2 = b + c$
$\Rightarrow 2 = b + (b + 1)$
$\Rightarrow 2 = 2b + 1$
$\Rightarrow b = \frac{1}{2}$

$3 = a + b$
$\Rightarrow 3 = a + \left(\frac{1}{2}\right)$
$\Rightarrow a = 3 - \frac{1}{2} =$
$\Rightarrow a = \frac{5}{2}$

[3] Coordinate vector

$$[p]s = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow [p]s = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

Problem 18. Discuss how the rank of A varies with t:

$$A = \begin{bmatrix} 1 & -1 & t \\ 1 & t & -1 \\ t^2 & 1 & -1 \end{bmatrix}$$

[1] Gaussian elimination

Forward Elimination

-- iter: 1

R2: R2 - R1

$$A = \begin{bmatrix} 1 & -1 & t \\ 0 & t+1 & -1-t \\ t^2 & 1 & -1 \end{bmatrix}$$

-- iter: 2

R3: R3 - tR1

$$A = \begin{bmatrix} 1 & -1 & t \\ 0 & t+1 & -1-t \\ 0 & 1+t^2 & -1-t^2 \end{bmatrix}$$

-- iter: 3

R3: R3 - R2

$$A = \begin{bmatrix} 1 & -1 & t \\ 0 & t+1 & -1-t \\ 0 & 0 & (-1-t^2)-(-1-t) \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & -1 & t \\ 0 & t+1 & -1-t \\ 0 & 0 & t(t-1) \end{bmatrix}$$

[2] Rank

$t = 0$

$$\Rightarrow A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$rankA = 2$

$t = 1$

$$\Rightarrow A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$rankA = 2$

$$\Rightarrow A = \begin{bmatrix} 1 & -1 & t \\ 0 & t+1 & -1-t \\ 0 & 0 & t(t-1) \end{bmatrix}$$

$rankA = 3$

Problem 18. Discuss how the rank of A varies with t:

$$A = \begin{bmatrix} 1 & 1 & -t \\ t & 3 & -1 \\ 3 & 6 & -2 \end{bmatrix}$$

[1] Gaussian elimination

Forward Elimination

-- iter: 1

R2: R2 - tR1

$$A = \begin{bmatrix} 1 & 1 & -t \\ 0 & 3-t & t-1 \\ 3 & 6 & -2 \end{bmatrix}$$

-- iter: 2

R3: R3 - 3R1

$$A = \begin{bmatrix} 1 & 1 & -t \\ 0 & 3-t & t-1 \\ 0 & 3 & -2+3t \end{bmatrix}$$

-- iter: 3

R3: R3 -(3R2/(3-t))

$$A = \begin{bmatrix} 1 & 1 & -t \\ 0 & 3-t & t-1 \\ 0 & 0 & -2+3t-\frac{3\,.\,3\,(t-1)}{3-t} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & -t \\ 0 & 3-t & t-1 \\ 0 & 0 & 1 \end{bmatrix}$$

[2] Rank

$t = 3$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $rankA = 2$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & -t \\ 0 & 3-t & t-1 \\ 0 & 0 & 1 \end{bmatrix}$$

$rankA = 3$

Problem 19. Let U and V be two subspaces of R4 defined by
$U = \{(x_1;\ x_2;\ x_3;\ x_4) \in R^4 :\ x_1 = x_2\ and\ x_3 = 2x_4\}$

And
$V = \{(x_1;\ x_2;\ x_3;\ 0) \in R^4 :\ x_1 + x_2 = 0\ and\ x_3 = x_1 + x_2\}$
Find the dimensions of U and V .

[1] Subspace U:
Any vector in U can be expressed as:
$\left(x_1;\ x_2;\ x_3;\frac{x_3}{2}\right)$

$\Rightarrow x_1(1,1,0,0)$ and $x_3\left(0,0,1,\frac{1}{2}\right)$

[2] Basis vectors for U
$\{(1,1,0,0),\left(0,0,1,\frac{1}{2}\right)\}$

Thus $\dim U = 2$

[1] Subspace V:
Any vector in U can be expressed as:
$(x_1, -x_1, 0, 0)$

$\Rightarrow x_1(1,-1,0,0)$

[2] Basis vectors for U
$\{(1,-1,0,0)\}$

Thus $\dim V = 1$

Problem 20.