Question 1

Consider the following data

x	f(x)
2.3	-8.1066
2.7	-17.7949
3.1	-29.7652
3.5	-40.1506

1.1) Use the following difference formulas to approximate f'(2.7)

a) the forward difference formula;

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

Where $h = 3.1 - 2.7 = 0.4$

$$\Rightarrow f'(2.7) \approx \frac{(-29.7652)-(-17.7949)}{0.4}$$

$$\Rightarrow f'(2.7) \approx \frac{1.9703}{0.4}$$

$$\Rightarrow f'(2.7) \approx -29.92575$$

b) the central difference formula

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

Where $h = 3.1 - 2.7 = 0.4$

$$\Rightarrow f'(2.7) \approx \frac{f(3.1)-f(2.3)}{(2)(0.4)}$$

$$\Rightarrow f'(2.7) \approx \frac{(-29.7652)-(-8.1066)}{0.8}$$

$$\Rightarrow f'(2.7) \approx \frac{-21.6586}{0.8}$$

$$\Rightarrow f'(2.7) \approx -27.07325$$

c) the 3-point endpoint formula.

$f'(x_0) \approx \frac{-3f(x_0)+4f(x_1)f(x_2)}{2h}$

Where $h = 3.1 - 2.7 = 0.4$

$\Rightarrow f'(2.7) \approx \frac{-3f(2.7)+4f(3.1)-f(3.5)}{2(0.4)}$

$\Rightarrow f'(2.7) \approx \frac{-3(-17.7949)+4(-29.7652)-(-40.1506)}{0.8}$

$\Rightarrow f'(2.7) \approx \frac{53.3847-119.0608+40.1506}{0.8}$

$\Rightarrow f'(2.7) \approx \frac{-25.5255}{0.8}$

$\Rightarrow f'(2.7) \approx -31.906875$

(1.2) Compute f ''(3.1) using the second derivative midpoint formula.

$f'(x_0) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$

Where $h = 3.1 - 2.7 = 0.4$

$f'(3.1) \approx \frac{f(3.5)-2f(3.1)+f(2.7)}{(0.4)^2}$

$f'(3.1) \approx \frac{-40.1506-2(-29.7652)+(-17.7949)}{0.16}$

$f'(3.1) \approx \frac{1.5853}{0.16}$

$f'(3.1) \approx 9.908125$

(1.3) The above data was generated using the function $f(x) = x^3 \cos x$. Use the Richardson's extrapolation process to determine $N_3(h)$, an approximation of f'(2.7) of order $O(h^6)$, using $N_1(h) = f'(x_0)$ is approximated by the three-point midpoint formula.

First Approximation N_1:

[1] Where $h = 0.2$

Calculate f(2.9)
$\Rightarrow f(2.9) \approx 2.9^3 \cos(2.9)$
$\Rightarrow f(2.9) \approx -23.6745$

3-point endpoint formula.

$$f'(x_0) \approx \frac{-3f(x_0)+4f(x_1)f(x_2)}{2h}$$

$$\Rightarrow f'(2.7) \approx \frac{-3f(2.7)+4f(2.9)-f(3.1)}{2(0.2)}$$
$$\Rightarrow f'(2.7) \approx \frac{-3(-17.7949)+4(-23.6745)-(-29.7652)}{0.4}$$
$$\Rightarrow f'(2.7) \approx \frac{53.3847-94.698+29.7652}{0.4}$$
$$\Rightarrow f'(2.7) \approx \frac{-11.5481}{0.4}$$
$\Rightarrow f'(2.7) \approx -28.87025$
Thus, $N_1(h) = -28.87025$

[2] Where $h = 0.4$
Given:
$\Rightarrow f'(2.7) \approx -31.906875$

First-Level Richardson's Extrapolation N_2:

[1] Richardson's extrapolation

$N_2(h) \approx \frac{4N_1\left(\frac{h}{2}\right)-N_1(h)}{3}$

$\Rightarrow N_2(0.4) \approx \frac{4N_1(0.2)-N_1(0.4)}{3}$
$\Rightarrow N_2(0.4) \approx \frac{4(-28.87025)-(-31.906875\ )}{3}$
$\Rightarrow N_2(0.4) \approx \frac{-115.481-(-31.906875)}{3}$
$\Rightarrow N_2(0.4) \approx \frac{-83.574125}{3}$
$\Rightarrow N_2(0.4) \approx -27.85804$

Second -Level Richardson's Extrapolation N_3:

[1] Find $N_2(h)$, Where $h = 0.2$

Where $h = 0.1$

Calculate f(2.6)
$\Rightarrow f(2.6) \approx 2.6^3 \cos(2.6)$
$\Rightarrow f(2.6) \approx -14.7655$

Calculate f(2.8)
$\Rightarrow f(2.8) \approx 2.8^3 \cos(2.8)$
$\Rightarrow f(2.8) \approx -21.0256$

[2] 3-point endpoint formula.

$f'(x_0) \approx \frac{-3f(x_0)+4f(x_1)f(x_2)}{2h}$

$\Rightarrow f'(2.7) \approx \frac{-3(-17.7949)+4(-21.0256)-(-14.7655)}{2(0.1)}$
$\Rightarrow f'(x_0) \approx \frac{53.3847-84.1024+14.7655}{0.2}$
$\Rightarrow f'(x_0) \approx \frac{-15.9522}{0.2}$
$\Rightarrow f'(x_0) \approx -79.761$

[3] Richardson's extrapolation

$N_2(h) \approx \frac{4N_1\left(\frac{h}{2}\right)-N_1(h)}{3}$

$\Rightarrow N_2(0.2) \approx \frac{4(-27.85804)-(-31.906875)}{3}$

$\Rightarrow N_2(0.2) \approx \frac{-111.4322+31.906875}{3}$

$\Rightarrow N_2(0.2) \approx \frac{-79.525325}{3}$

$\Rightarrow N_2(0.2) \approx -26.508$

[4] Find $N_3(h)$, Where $h = 0.4$

$N_3(h) \approx \frac{16\left(\frac{h}{2}\right)-N_1(h)}{15}$

Given:

$\Rightarrow N_2(0.2) \approx -26.508$

$\Rightarrow N_2(0.4) \approx -27.85804$

$\Rightarrow N_3(0.4) \approx \frac{16(-26.508)-(-27.85804)}{15}$

$\Rightarrow N_3(0.4) \approx \frac{-424.128+27.85804}{15}$

$\Rightarrow N_3(0.4) \approx \frac{-396.26996}{15}$

$\Rightarrow N_3(0.4) \approx -26.418$

Question 2

The integral

$I = \int_1^{1.5} x^2 \ln x \, dx$

(1.1) Use the following Newton-Cotes methods to approximate I:

a) Simpson's $\left(\frac{1}{3}\right)$ rule;

[1] Calculate no of subintervals

Let $n = 2$

$h = \frac{1.5-1}{2} = 0.25$

thus,

$x_0 = 1$

$x_1 = 1.25$

$x_2 = 1.5$

[2] Evaluate $f(x_0)$, $f(x_1)$, $f(x_2)$

$f(x) = x^2 \ln x$

$\Rightarrow f(1) = (1)^2 \ln(1)$

$\Rightarrow f(1) = 0$

$\Rightarrow f(1.25) = (1.25)^2 \ln(1.25)$

$\Rightarrow f(1.25) = 1.5625 \times 0.2231$

≈ 0.3487

$\Rightarrow f(1.5) = (1.5)^2 \ln(1.5)$

$\Rightarrow f(1.5) = 2.25 \times 0.4055$

≈ 0.9124

[3] Approximation

$I = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$

$\Rightarrow I = \frac{0.25}{3}[0 + 4 \times 0.3487 + 0.9124]$

$\Rightarrow I = \frac{0.25}{3} \times 2.3072$

$\Rightarrow I = 0.1923$

b) Simpson's $\left(\frac{3}{8}\right)$ rule;

[1] Calculate no of subintervals

Let $n = 3$

$h = \frac{1.5-1}{3} = 0.1667$

thus,

$x_0 = 1$

$x_1 = 1.1667$

$x_2 = 1.3333$

$x_3 = 1.5$

[2] Evaluate $f(x_0)$, $f(x_1)$, $f(x_2)$, $f(x_3)$

$f(x) = x^2 \ln x$

$\Rightarrow f(1) = (1)^2 \ln(1)$

$\Rightarrow f(1) = 0$

$\Rightarrow f(1.1667\ ) = (1.1667\ )^2 \ln(1.1667\ )$

$\Rightarrow f(1.1667\ ) = 1.3611 \times 0.1542$

≈ 0.2098

$\Rightarrow f(1.3333) = (1.3333)^2 \ln(1.3333)$

$\Rightarrow f(1.3333) = 1.7777 \times 0.2877$

≈ 0.5114

$\Rightarrow f(1.5) = (1.5)^2 \ln(1.5)$

$\Rightarrow f(1.5) = 2.25 \times 0.4055$

≈ 0.9124

[3] Approximation

$I = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + 3f(x_3) + \cdots + f(x_n)]$

$\Rightarrow I = \frac{3\times1.667}{8} \times [0 + 3 \times 0.2098 + 3 \times 0.5114 + 0.9124]$

$\Rightarrow I = \frac{0.5001}{8} \times [0 + 0.6294 + 1.5342 + 0.9124]$

$\Rightarrow I = 0.0625 \times 3.076$

$\Rightarrow I = 0.1923$

c) composite trapezoidal rule with h = 0.1

[1] Calculate no of subintervals

Let $n = 3$

$h = \frac{1.5-1}{0.1} = 5$

thus,

$x_0 = 1$

$x_1 = 1.1$

$x_2 = 1.2$

$x_3 = 1.3$

$x_4 = 1.4$

$x_5 = 1.5$

[2] Evaluate $f(x_0)$, $f(x_1)$, $f(x_2)$, $f(x_3)$

$f(x) = x^2 \ln x$

$\Rightarrow f(1) = (1)^2 \ln(1)$

$\Rightarrow f(1) = 0$

$\Rightarrow f(1.1) = (1.1)^2 \ln(1.1)$

$\Rightarrow f(1.1\,) = 1.21 \times 0.0953$

≈ 0.1153

$\Rightarrow f(1.2) = (1.2)^2 \ln(1.2)$

$\Rightarrow f(1.2\,) = 1.44 \times 0.1823$

≈ 0.2625

$\Rightarrow f(1.3) = (1.3)^2 \ln(1.3)$

$\Rightarrow f(1.3\,) = 1.69 \times 0.2624$

≈ 0.4434

$\Rightarrow f(1.4) = (1.4)^2 \ln(1.4)$

$\Rightarrow f(1.1\,) = 1.96 \times 0.3365$

≈ 0.6595

$\Rightarrow f(1.5) = (1.5)^2 \ln(1.5)$

$\Rightarrow f(1.1\,) = 2.25 \times 0.4055$

≈ 0.9124

[3] Approximation

$$I = \frac{0.1}{2}[0 + 2(0.1153 + 0.2625 + 0.4434 + 0.6595) + 0.9124]$$
$$\Rightarrow I = 0.05 \times [0 + 2 \times 1.4807 + 0.9124]$$
$$\Rightarrow I = 0.05 \times [2.9614 + 0.9124]$$
$$\Rightarrow I = 0.05 \times 3.8738$$
$$\Rightarrow I = 0.1937$$

(1.2) Compute the integral I analytically and determine the actual error in the approximations obtained in (1.1) above.

[1] Integration by Parts

$$\int u\, dv = uv - \int v\, du$$
$$u = \ln(x)$$
$$du = \frac{1}{x}\, dx$$

$$dv = x^2 dx$$
$$v = \frac{x^3}{3}$$

$$I = \int_1^{1.5} x^2\, ln(x)\, dx$$
$$\Rightarrow \left[\frac{x^3}{3}\, ln(x)\right]_1^{1.5} - \int_1^{1.5} x^2\, ln\left(\frac{x^3}{3}\right) \times \frac{1}{x}\, dx$$
$$\Rightarrow \left[\frac{x^3}{3}\, ln(x)\right]_1^{1.5} - \frac{1}{3}\int_1^{1.5} x^2\ dx$$
$$\Rightarrow \left(\frac{(1.5)^3}{3} ln(1.5)\right) - \left(\frac{1^3}{3} ln(1.5)\right) - \frac{1}{3}\int_1^{1.5} x^2\ dx$$
$$\Rightarrow \frac{3.375}{3} ln(1.5) - \frac{1}{3}\int_1^{1.5} x^2\ dx$$
$$\Rightarrow 1.125\ ln(1.5) - \frac{1}{3}\int_1^{1.5} x^2\ dx$$
$$\Rightarrow 1.125\ ln(1.5) - \frac{1}{3}\left(\frac{(1.5)^3}{3} - \frac{1^3}{3}\right)$$
$$\Rightarrow 1.125\ ln(1.5) - \frac{1}{3}\left(\frac{3.375}{3} - \frac{1}{3}\right)$$
$$\Rightarrow 1.125\ ln(1.5) - \frac{1}{3}\left(1.125 - \frac{1}{3}\right)$$
$$\Rightarrow 1.125\ ln(1.5) - 0.2639$$
$$\Rightarrow 1.125 \times 0.4055 - 0.2639$$
$$\Rightarrow 0.1923$$

[2] Simpson's $\left(\frac{1}{3}\right)$ rule;
Error $= |0.1923 - 0.1923| = 0$

[3] Simpson's $\left(\frac{3}{8}\right)$ rule;
Error $= |0.1923 - 0.1923| = 0$

[4] Composite Trapezoidal Rule;
Error $= |0.1923 - 0.1937| = 0.0014$

(1.3) Approximate I using the three-point Gaussian quadrature scheme

[1] Abscissas

Legendre polynomial of degree 3
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
$\Rightarrow 0 = \frac{1}{2}(5x^3 - 3x)$
$\Rightarrow 0 = 5x^3 - 3x$
$\Rightarrow 3x = 5x^3$

$x_1 = -\sqrt{\frac{3}{5}}$
$x_2 = 0$
$x_3 = \sqrt{\frac{3}{5}}$

[2] Weights
$w_1 = \frac{5}{9}$
$w_2 = \frac{8}{9}$
$w_3 = \frac{5}{9}$

[3] Change of Interval
Over Interval [1, 1.5]

$x = \frac{1.5-1}{2}t + \frac{1.5+1}{2} = 0.25t + 1.15$

Thus, $dx = \frac{1.5-1}{2}dt = 0.25dt$

[4] Transformed Interval

$I = \int_{1}^{1.5} x^2 \ln x \, dx$

$\Rightarrow I = \int_{1}^{1.5} (0.25t + 1.25)^2 \times \ln(0.25t + 1.25) \times 0.25 \, dx$

[5] Three-point Gaussian quadrature rule

$\sum_{i=1}^{3} (0.25x_i + 1.25)^2 \ln\left(0.25x_i + 1.25\right) \times 0.25$

$\Rightarrow \sum_{i=1}^{3} (0.25x_i + 1.25)^2 \ln(0.25x_i + 1.25) \times 0.25$

[6] Where $x_1 = -\sqrt{\frac{3}{5}}$

$\Rightarrow \sum_{i=1}^{3} \left(0.25 \times \left(-\sqrt{\frac{3}{5}}\right) + 1.25\right)^2 \ln\left(0.25 \times \left(-\sqrt{\frac{3}{5}}\right) + 1.25\right) \times 0.25$

$\Rightarrow \sum_{i=1}^{3} (1.0563)^2 \times \ln(1.0563) \times 0.25$

$\Rightarrow \sum_{i=1}^{3} 1.1158 \times 0.0548 \times 0.25$

$\Rightarrow \sum_{i=1}^{3} 0.0153$

[7] Where $x_2 = 0$

$\Rightarrow \sum_{i=1}^{3} (0.25 \times (0) + 1.25)^2 \ln(0.25 \times (0) + 1.25) \times 0.25$

$\Rightarrow \sum_{i=1}^{3} (1.25)^2 \times \ln(1.25) \times 0.25$

$\Rightarrow \sum_{i=1}^{3} 1.5625 \times 0.2231 \times 0.25$

$\Rightarrow \sum_{i=1}^{3} 0.0872$

[8] Where $x_3 = \sqrt{\frac{3}{5}}$

$\Rightarrow \sum_{i=1}^{3} \left(0.25 \times \left(\sqrt{\frac{3}{5}}\right) + 1.25\right)^2 \ln\left(0.25 \times \left(\sqrt{\frac{3}{5}}\right) + 1.25\right) \times 0.25$

$\Rightarrow \sum_{i=1}^{3} (1.4437)^2 \times \ln(1.4437) \times 0.25$

$\Rightarrow \sum_{i=1}^{3} 2.0833 \times 0.3664 \times 0.25$

$\Rightarrow \sum_{i=1}^{3} 0.1908$

[9] Apply weights

$I = \frac{5}{9} \times 0.0153 + \frac{8}{9} \times 0.0872 + \frac{5}{9} \times 0.1908$

$\Rightarrow 0.0085 + 0.0775 + 0.1060$

$\Rightarrow 0.1920$