[Problem 9]

Determine whether the sets U are V are subspace of R^4 defined by:

$U = \{(x, y, z) \in \mathbb{R}^4 : x + y = z\}$ *and*

$V = \{(x, y, z) \in \mathbb{R}^4 : x = 2z \text{ and } y = v + 1\}$

A subset U or V of R^4 is a subspace if it satisfies the following three properties ensure that the subset W behaves like a vector space:

A1	**Closure under addition** *For any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in the set, $\boldsymbol{u}+\boldsymbol{v}$ is also in the set*	$\vec{u} + \vec{v} \in R^4$
A2	**Existence of an additive identity** *There exists a vector $\mathbf{0}$ in the set such that for any vector $\boldsymbol{u}$ in the set, $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}$.* *Related:* *- A3:Existence of additive inverses*	$\vec{u} + 0 = \vec{u}$
M1	**Closure under scalar multiplication** *For any scalar c and any vector cu in the set, $\boldsymbol{cu}$ is also in the set.* *Implied by:* *- A1: Closure under addition*	$c\vec{u} \in R^4$

$U = \{(x, y, z) \in \mathbb{R}^4 : x + y = z\}$

A1	Let $\vec{u} = (x_1, y_1, z_1) \in U$ $\vec{v} = (x_2, y_2, z_2) \in U$ be arbitrary vectors in U where x_1 and x_2 are real numbers Then $u + v \in U$ $= (x_1, y_1, z_1) + (x_2, y_2, z_2) \in U$ $= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in U$ Therefore A1 holds.
A2	Let the zero vector in U be $0 = (0,0,0,0)$ Then $x + y = z$ $0 + 0 = 0$ $0 \in U$ Therefore A2 holds.
M1	Let $c \in U$ be an arbitrary scalar in U Let $\vec{u} = (x, y, z)$ be an arbitrary vector in U where x, y and z are real numbers Then $c \in X$ $= c.\vec{u}$ $= (cx, cy, cz)$ Then $x + y = z$ $\Rightarrow cx + cy = cz \in U$ Therefore M1 holds.

Therefore U is a subspace of R^4

$V = \{(x, y, z) \in \mathbb{R}^4 : x = 2z \text{ and } y = v + 1\}$

A2	Let the zero vector in U be $0 = (0,0,0,0)$ Then $x = 2z$ $0 = 0$ Then $y = v + 1$ $0 = 1$ $0 \in U$ *but* $1 \notin U$ Therefore A2 Fails.

Therefore U is not a subspace of R^4

[Problem 10]

Express the following as a linear combinations of
$u = (2,1,4)$ *;* $v = (1,1,3)$ *; and* $w = (3,2,5)$ *;*

Definition: linear combination

Let V *be a vector space over a field* F*, and let* $v_1, v_2 \ldots v_n$ *be vectors in* V*.*
A linear combination of $v_1, v_2 \ldots v_n$ *is any expression of the form:*

$c_1.v_1 + c_2.v_2 + \cdots c_n.v_n$
Where $c_1, c_1 \ldots c_n$ *are scalars from the field* F*.*

a) $(6,1,6)$
Let $c_1, c_1 \ldots c_n$ *be scalars from the field* F*.*
Then

$$c_1.v_1 + c_2.v_2 + \cdots c_n.v_n = (6,1,6)$$
$$\Rightarrow c_1.u + c_2.v + c_n.w = (6,1,6)$$
$$\Rightarrow c_1.\begin{bmatrix}2\\1\\4\end{bmatrix} + c_2.\begin{bmatrix}1\\1\\3\end{bmatrix} + c_n.\begin{bmatrix}3\\2\\5\end{bmatrix} = \begin{bmatrix}6\\1\\6\end{bmatrix}$$

in matrix form $Ax = b$*:*
$$\Rightarrow \begin{bmatrix}2 & 1 & 3\\1 & 1 & 2\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}6\\1\\6\end{bmatrix}$$

Initial augmented matrix:
$$\Rightarrow \begin{bmatrix}2 & 1 & 3 & 6\\1 & 1 & 2 & 1\\4 & 3 & 5 & 6\end{bmatrix}$$

Forward Elimination
$$\Rightarrow \begin{bmatrix}2 & 1 & 3 & 6\\0 & \frac{1}{2} & \frac{1}{2} & -2\\0 & 1 & -1 & -6\end{bmatrix} \quad \boxed{\begin{array}{l}R2: R2 - (1/2) * R1\\R3: R3 - (4/2) * R1\end{array}}$$
$$\Rightarrow \begin{bmatrix}2 & 1 & 3 & 6\\0 & \frac{1}{2} & \frac{1}{2} & -2\\0 & 0 & -2 & -2\end{bmatrix} \quad \boxed{R3: R3 - (1/0.5) * R2}$$

Back Substitution:
x_3*: -2/-2 = 1*
x_2*: (-2 - 0.5) /0.5 = -5*
x_1*: (6 - -2) /2 = 4*

Thus
$$x = \begin{bmatrix}4\\-5\\1\end{bmatrix}$$

b) (0,0,0)

Let $c_1, c_1 \dots c_n$ *be scalars from the field* F.

Then

$$c_1.v_1 + c_2.v_2 + \cdots c_n.v_n = (0,0,0)$$
$$\Rightarrow c_1.u + c_2.v + c_n.w = (0,0,0)$$
$$\Rightarrow 0.u + 0.v + c_n.w = (0,0,0)$$

$$\Rightarrow 0\begin{bmatrix}2\\1\\4\end{bmatrix} + 0.\begin{bmatrix}1\\1\\3\end{bmatrix} + 0.\begin{bmatrix}3\\2\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Thus

$$x = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

c) $(7, 8, 9)$

Let $c_1, c_1 \ldots c_n$ *be scalars from the field* F.

Then

$$c_1 . v_1 + c_2 . v_2 + \cdots c_n . v_n = (7, 8, 9)$$
$$\Rightarrow c_1 . u + c_2 . v + c_n . w = (7, 8, 9)$$
$$\Rightarrow c_1 . \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + c_2 . \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_n . \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

in matrix form $Ax = b$:

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Initial augmented matrix:

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 & 7 \\ 1 & 1 & 2 & 8 \\ 4 & 3 & 5 & 9 \end{bmatrix}$$

Forward Elimination

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 & 7 \\ 0 & \frac{1}{2} & \frac{1}{2} & 4.5 \\ 0 & 1 & -1 & 5 \end{bmatrix} \quad \boxed{\begin{array}{l} R2: R2 - (1/2) * R1 \\ R3: R3 - (4/2) * R1 \end{array}}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 & 7 \\ 0 & \frac{1}{2} & \frac{1}{2} & 4.5 \\ 0 & 0 & -2 & -14 \end{bmatrix} \quad \boxed{R3: R3 - (1/0.5) * R2}$$

Back Substitution:

x_3: -14/-2 = 7

x_2: (4.5 - 3.5) /0.5 = 2

x_1: (7 - 23) /2 = -8

Thus

$$x = \begin{bmatrix} -8 \\ 2 \\ 7 \end{bmatrix}$$

[Problem 11]

Which of the following sets of vectors in R^4 are linearly independent.

Definition: Linear independence

Let V be a vector space over a field F, and let $v_1, v_2 \dots v_n$ be vectors in V.
Is said to be linearly independent if the only solution to the equation:

$$a_1.v_1 + a_2.v_2 + \cdots a_n.v_n = 0$$

is the trivial solution $a_1 = a_2 = \cdots = a_n = 0$,
where 0 is the zero vector in V.

(a) (1; 2; 2; 1) ; (3; 6; 6; 3) ; (4; 2; 4; 1)
in matrix form $Ax = b$:

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 2 & 4 & 1 \end{bmatrix} \ ; \ b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ ; \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Reduce matrix A to row-echelon form (RREF):

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 2 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 4 & 1 \end{bmatrix} \quad \boxed{R2: R2 - 3 * R1}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & -4 & -3 \end{bmatrix} \quad \boxed{R3: R3 - 4 * R1}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -6 & -4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \boxed{R2: R3}$$

*Therefore, the given set of vectors is **linearly dependent**.*

(b) (2; 1; 1; -4) ; (2; -8; 9; -2) ; (0; 3; -1; 5); (0; -1; 2; 4);

in matrix form $Ax = b$*:*

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & 1 & -4 \\ 2 & -8 & 9 & -2 \\ 0 & 3 & -1 & 5 \\ 0 & -1 & 2 & 4 \end{bmatrix} \; ; \; b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Reduce matrix A to row-echelon form (RREF):

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 & -4 \\ 2 & -8 & 9 & -2 \\ 0 & 3 & -1 & 5 \\ 0 & -1 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -2 \\ 2 & -8 & 9 & -2 \\ 0 & 3 & -1 & 5 \\ 0 & -1 & 2 & 4 \end{bmatrix} \qquad \boxed{R1: \frac{1}{2} R1}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & -9 & 8 & 2 \\ 0 & 3 & -1 & 5 \\ 0 & -1 & 2 & 4 \end{bmatrix} \qquad \boxed{R2: R2 - 2R1}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & -\frac{8}{9} & -\frac{2}{9} \\ 0 & 3 & -1 & 5 \\ 0 & -1 & 2 & 4 \end{bmatrix} \qquad \boxed{R2: -\frac{2}{9} R1}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{17}{18} & -\frac{17}{9} \\ 0 & 1 & -\frac{8}{9} & -\frac{2}{9} \\ 0 & 3 & -1 & 5 \\ 0 & -1 & 2 & 4 \end{bmatrix} \qquad \boxed{R1: R1 - \frac{1}{2} R2}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{17}{18} & -\frac{17}{9} \\ 0 & 1 & -\frac{8}{9} & -\frac{2}{9} \\ 0 & 0 & \frac{5}{3} & \frac{17}{3} \\ 0 & -1 & 2 & 4 \end{bmatrix} \qquad \boxed{R3: R3 - 3R2}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{17}{18} & -\frac{17}{9} \\ 0 & 1 & -\frac{8}{9} & -\frac{2}{9} \\ 0 & 0 & \frac{5}{3} & \frac{17}{3} \\ 0 & 0 & \frac{10}{9} & \frac{34}{9} \end{bmatrix} \qquad \boxed{R4: R4 + R2}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{17}{18} & -\frac{17}{9} \\ 0 & 1 & -\frac{8}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{17}{5} \\ 0 & 0 & \frac{10}{9} & \frac{34}{9} \end{bmatrix} \qquad \boxed{R3: \frac{3}{5} R3}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{51}{10} \\ 0 & 1 & -\frac{8}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{17}{5} \\ 0 & 0 & \frac{10}{9} & \frac{34}{9} \end{bmatrix} \qquad \boxed{R1: R1 - \frac{17}{18}R3}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{51}{10} \\ 0 & 1 & 0 & \frac{14}{5} \\ 0 & 0 & 1 & \frac{17}{5} \\ 0 & 0 & \frac{10}{9} & \frac{34}{9} \end{bmatrix} \qquad \boxed{R2: R2 + \frac{8}{9}R3}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{51}{10} \\ 0 & 1 & 0 & \frac{14}{5} \\ 0 & 0 & 1 & \frac{17}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boxed{R4: R4 - \frac{10}{9}R3}$$

*Therefore, the given set of vectors is **linearly dependent.***

(c) (1; 1; 0; 0) ; (0; 1; 0; 1); (0; 0; 1; 1); (1; 0; 1; 0) ; (1; 0; 0; 1)

in matrix form $Ax = b$*:*

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \; ; \; b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \; ; \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Reduce matrix A to row-echelon form (RREF):

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \boxed{\begin{matrix} R4: R4 - R1 \\ R5: R5 - R1 \end{matrix}}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \boxed{\begin{matrix} R4: R4 + R2 \\ R5: R5 + R2 \end{matrix}}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \boxed{\begin{matrix} R4: R4 - R3 \\ R5: R5 - R3 \end{matrix}}$$

Therefore, the given set of vectors is ***linearly dependent.***

[Problem 12]

Determine whether the solution space of the system Ax = 0 is a line through the origin, a plane through the origin, or the origin only for

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}$$

Forward Elimination

$$\Rightarrow \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -6 \\ 0 & 2 & 10 \\ 3 & 10 & 6 \end{bmatrix} \qquad \boxed{R2: R2 - R1}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -6 \\ 0 & 2 & 10 \\ 0 & 4 & 24 \end{bmatrix} \qquad \boxed{R3: R3 - 3R1}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -6 \\ 0 & 2 & 10 \\ 0 & 0 & 4 \end{bmatrix} \qquad \boxed{R3: R3 - 2R2}$$

solutions in R^{n}

geometric interpretation	*solution space*	*Nullity of A*
A line through the origin	one-dimensional.	1
A plane through the origin	two-dimensional	2
The origin only	zero-dimensional	0

$Rank\ A = 3$

$nullity\ A = 0$

Thus, the solution space is The origin only