Question 1

a.

i.

Converse	$Q \Rightarrow P$	I go shopping when I take plastic bags.
Contrapositive	$\neg Q \rightarrow \neg P$	I do not go shopping when I do not take plastic bags
Negation	$\neg Q$	I take plastic bags and I do not go shopping.

ii.

Converse	$Q \Rightarrow P$	$x \notin X$ or $x \notin Y$ implies $x \in B$
Contrapositive	$\neg Q \rightarrow \neg P$	$x \in X$ or $x \in Y$ implies $x \notin B$
Negation	$\neg Q$	$x \in B$ implies $x \in X$ or $x \in Y$

iii.

Converse	$Q \Rightarrow P$	$x \in A$ and $x \in B$ then $x \in A \cap B$
Contrapositive	$\neg Q \rightarrow \neg P$	$x \notin A$ and $x \notin B$ then $x \notin A \cap B$
Negation	$\neg Q$	$x \in A \cap B$ and $x \notin A$ or $x \notin B$

b.

Indirect proof: proof of the contrapositive states that:

$P \Rightarrow Q \Longleftrightarrow \neg Q \rightarrow \neg Q$

Let the statement $P \Rightarrow Q$ be

"if the average of this set of test scores is greater than 90, then at least one of the scores is greater than 90"

Then the contrapositive, $\neg Q \rightarrow \neg P$ will be

"if none of the scores will be greater than 90, then the average of this set of test scores is not greater than 90"

Then we have that for every $a_n < 90 \in S$

And if the sum of these is $\sum_{a \in S} a$, and

the number of elements in S is n ,

then we have

$$\frac{\sum_{a \in S} a}{n} < \frac{90n}{n}$$

Therefore, the average test score will be less than 90.

Question 2
i.
hypothesis:
Let $P(n)$ be the statement $3^n \geq 2n^2 + 1$ for all $n \in \mathbb{N}$

Basis clause
$P(n)$ is where $n = 1$

$$LHS = 3^{(1)}$$
$$= 3$$

$$RHS = 2(1)^2 + 1$$
$$= 3$$

$LHS = RHS = 3$.
Therefore, $P(1)$ is true
Inductive hypothesis
$P(k)$ is where $n = k$

Assume $n = k$. Assume $P(k)$ is true
Then also assume that $3^k \geq 2k^2 + 1$

Inductive Step
If $P(k)$ is true, then $P(k+1)$ must also be true

$$LHS = 3^{(k+1)}$$
$$= 3^k.3$$

$$RHS = 2(k+1)^2 + 1$$
$$= 2\left(k^2 + k + 1\right) + 1$$
$$= 2k^2 + 2k + 3$$

But
$3^k \geq 2k^2 + 1$
$3^k.3 \geq 2k^2.3 + 1.3$
$2k^2 + 2k + 3 \geq 2k^2.3 + 3$ by the induction hypothesis
$2k^2 + 2k + 3 \geq 6k^2 + 3$
$0 \geq 4k^2 + 2k$
$0 \geq k$

$LHS \geq RHS$
Thus, $P(k+1)$ is true
Hence, $P(k)$ is true
It then follows by mathematical induction that $P(n)$ is true.

ii.
A function is bijective if it is injective and surjective

Injection (one-to-one)
A function is injective where $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

Let $g(x_1) = g(x_2)$,
$\frac{x_1-1}{2x_1-4} = \frac{x_2-1}{2x_2-4}$
$x_1 = x_2$

$x_1 - 1 = x_2 - 1$
$x_1 = x_2$

$2x_1 - 4 = 2x_2 - 4$
$2x_1 = 2x_2$
$x_1 = x_2$

Therefore, $g(x)$ is injective

Surjection (onto)
A function is surjective where for each $x_1 \in \mathbb{R}$ there exists $x_2 \in \mathbb{R}$ such that $g(x_2) = x_1$

Let $g(x) = y$, then
$y = \frac{x-1}{2x-4}$
$y(2x-4) = x - 1$
$2xy - 4y = x - 1$
$2xy - x = -1 - 4y$
$x(2y - 1) = -1 - 4y$
$x = \frac{-1-4y}{2y-1}$
$\Rightarrow f(x) = \frac{-1-4x}{2x-1}$
$\therefore f\left(\frac{1}{2}\right) = \frac{-1-4\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)-1} = \frac{-3}{1-1} = -\frac{3}{0} = undefined$
Thus, the function g is not surjective as not every element of the codomain maps onto the domain.

Inverse
A function has an inverse where it is bijective (injective and surjective). The function g is injective and not surjective, and is not bijective. Thus, the function g has no inverse.

Question 3

a.

$S = \{x | -x^2 + 6x - 3 > 0\}$

$S = \{x | \, x^2 - 6x + 3 < 0\}$

$S = \{x | \, (x - (3 + \sqrt{6}))(x - (3 - \sqrt{6})) < 0\}$

$S = \{x | \, 3 - \sqrt{6} < x < \, 3 + \sqrt{6} \,\}$

Thus, $\inf S = 3 - \sqrt{6}$ and $\sup S = 3 + \sqrt{6}$

b.

Given:

$a \in R$, $A \subset \mathbb{R}$ and $a = \sup A$

$\forall \epsilon > 0$

If $a = supA$ then

$x \leq a$ for all $x \in A$.

Suppose $\forall \epsilon > 0$, $\exists a \in A$ such that $a - \epsilon < x$, then assume

$\exists \epsilon > 0$, $\forall x \in A$, $a - \epsilon \geq x$

Thus, $a - \varepsilon \geq \sup A$

$\Rightarrow \epsilon \leq 0$

Which is a contradiction

Therefore, $a - \epsilon < x \leq a$

Question 4

a.

i.

$$\lim_{n\to\infty}\left(\sqrt{2n+1}-\sqrt{2n}\right)$$

$$= \lim_{n\to\infty}\frac{\left(\sqrt{2n+1}-\sqrt{2}\sqrt{n}\right)\left(\sqrt{2n+1}+\sqrt{2}\sqrt{n}\right)}{\left(\sqrt{2n+1}+\sqrt{2}\sqrt{n}\right)}$$

conjugate: $\left(\sqrt{2n+1}+\sqrt{2n}\right)$

$$= \lim_{n\to\infty}\frac{1}{\left(\sqrt{2n+1}+\sqrt{2}\sqrt{n}\right)}$$

$$= \frac{1}{\lim_{n\to\infty}\left(\sqrt{2n+1}+\sqrt{2}\sqrt{n}\right)}$$

reciprocal: $\lim_{x\to a}\frac{1}{f(x)} = \frac{1}{\lim_{x\to a} f(x)}$

$$= \frac{1}{\lim_{n\to\infty}\sqrt{2n+1} + \lim_{n\to\infty}\sqrt{2}\,.\,\sqrt{n}}$$

$$= \frac{1}{\lim_{n\to\infty}\sqrt{2n+1} + \sqrt{2}\,.\,\lim_{n\to\infty}\sqrt{n}}$$

$$= \frac{1}{\sqrt{\lim_{n\to\infty}(2n+1)} + \sqrt{2}\,.\,\sqrt{\lim_{n\to\infty} n}}$$

power rule: $\lim_{x\to a}\left(f(x)\right)^n = \left(\lim_{x\to a} f(x)\right)^n$

$$= \frac{1}{\sqrt{\lim_{n\to\infty}(2n)} + \sqrt{2}\,.\,\infty}$$

$$= \frac{1}{\sqrt{2\infty} + \sqrt{2}\,.\,\infty}$$

$$= 0$$

ii.

$$\lim_{n\to\infty}\frac{3n^3-n+8}{4n(n-1)(n-2)}$$

$$=\lim_{n\to\infty}\frac{1}{4}\cdot\frac{3n^3-n+8}{n(n-1)(n-2)}$$

$$=\lim_{n\to\infty}\frac{1}{4}\cdot\frac{3n^3-n+8}{n^3-3n^2+2n}$$

$$=\lim_{n\to\infty}\frac{1}{4}\cdot\frac{\frac{3n^3}{n^3}-\frac{n}{n^3}+\frac{8}{n^3}}{\frac{n^3}{n^3}-\frac{3n^2}{n^3}+\frac{2n}{n^3}}$$

divide by leading term n^3

$$=\lim_{n\to\infty}\frac{1}{4}\cdot\frac{3-\frac{1}{n^2}+\frac{8}{n^3}}{1-\frac{3}{n}+\frac{2}{n^2}}$$

$$=\frac{1}{4}\cdot\frac{3-\lim_{n\to\infty}\left(\frac{1}{n^2}\right)+\lim_{n\to\infty}\left(\frac{8}{n^3}\right)}{1-\lim_{n\to\infty}\left(\frac{3}{n}\right)+\lim_{n\to\infty}\left(\frac{2}{n^2}\right)}$$

$$=\frac{1}{4}\cdot\frac{3-\lim_{n\to\infty}\left(\frac{1}{n^2}\right)+8\lim_{n\to\infty}\left(\frac{1}{n^3}\right)}{1-3\lim_{n\to\infty}\left(\frac{1}{n}\right)+2\lim_{n\to\infty}\left(\frac{1}{n^2}\right)}$$

product rule: $\lim_{x\to a}\left(f(x).g(x)\right)=\lim_{x\to a}f(x).\lim_{x\to a}g(x)$

$$=\frac{1}{4}\cdot\frac{3-\left(\frac{1}{\lim_{n\to\infty}n^2}\right)+\left(\frac{8}{\lim_{n\to\infty}n^3}\right)}{1-\left(\frac{3}{\lim_{n\to\infty}n}\right)+\left(\frac{2}{\lim_{n\to\infty}n^2}\right)}$$

reciprocal: $\lim_{x\to a}\frac{1}{f(x)}=\frac{1}{\lim_{x\to a}f(x)}$

$$=\frac{1}{4}\cdot\frac{3-\left(\frac{1}{\left(\lim_{n\to\infty}n\right)^2}\right)+\left(\frac{8}{\left(\lim_{n\to\infty}n\right)^3}\right)}{1-\left(\frac{3}{\lim_{n\to\infty}n}\right)+\left(\frac{2}{\left(\lim_{n\to\infty}n\right)^2}\right)}$$

power rule: $\lim_{x\to a}\left(f(x)\right)^n=\left(\lim_{x\to a}f(x)\right)^n$

$$=\frac{1}{4}\cdot\frac{3-\left(\frac{1}{\infty^2}\right)+\left(\frac{8}{\infty^3}\right)}{1-\left(\frac{3}{\infty}\right)+\left(\frac{2}{\infty^2}\right)}$$

$$=\frac{1}{4}\cdot\frac{3-0+0}{1-0+0}$$

$$=\frac{1}{4}\cdot\frac{3}{1}$$

$$=\frac{3}{4}$$

b.
Given the sequence a_n
$a_n = (-1)^n - 2n$; $\lim_{n\to\infty}(a_n) = -\infty$

Let $n = 1$ as the first term in the sequence, then

$a_1 = -3$	first term
$a_2 = -3$	second term
$a_3 = -7$	third term
$a_4 = -7$	fourth term

…
$-a_n \geq a_{n+1}$ for every n

Lower bound (bounded below)
Let $m = 1$
then, $\exists m \leq a_n$, for every n

Upper bound (bounded above)
Let $M = 0$,
then $a_n \leq \exists M$, for every n

Therefore, the given sequence a_n is monotone decreasing

Question 5

a.

For the sequence $a_n = \frac{3n^2-2n+1}{2n^2-4}$

Let $\ell = \frac{3}{2}$ be given.

Let $\varepsilon > 0$ be given.

$|a_n - \ell\,|$

$= \left|\frac{3n^2-2n+1}{2n^2-4} - \frac{3}{2}\right|$

$= \left|\frac{3n^2-2n+1}{2n^2-4} - \frac{3\,.\,(n^2-2)}{2\,.\,(n^2-2)}\right|$

$= \left|\frac{3n^2-2n+1-3n^2+6}{2n^2-4}\right|$

$= \left|\frac{-2n+7}{2n^2-4}\right|$

If $\epsilon > 0$, then

$\left|\frac{-2n+7}{2n^2-4}\right| < \left|\frac{-2n}{2n^2}\right|$

$= \left|\frac{-2n}{2n^2}\right|$

$= \left|-\frac{1}{n}\right| < \frac{1}{n} < \epsilon$, then

$N > \frac{1}{\epsilon}$

$\forall\epsilon > 0, \exists N(\epsilon) \in N$ such that

$|a_n - \ell\,| < \epsilon$ for all $n \geq N$

Therefore, $\lim_{n\to\infty} \frac{3n^2-2n+1}{2n^2-4} = \frac{3}{2}$

b.

For the function $f(x) = \begin{cases} x^2 - 5x - 5, \ x \geq -1 \\ x^2 + x + 1\ , \ x < -1 \end{cases}$

Definition: Let $f : [a, b] \rightarrow R$ and $x_0 \in [a, b]$.
f is continuous at x_0 if for every $\varepsilon > 0$
there exists $\delta > 0$ such that $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Let $\varepsilon > 0$ be given.

Where $x \geq -1$,
$f(-1) = (-1)^2 \ -5(-1) - 5 = 1$

Where $x < -1$,
$f(-1) = (-1)^2 - 1 + 1 = 1$

Then for $|x - (-1)| < \delta$ we have
$|f(x) - f(-1)|$
$= |x^2 - 5x - 5 - 1| < |(x+1)^2|$
$= |(x+1)^2| < \epsilon$

If $\epsilon > 0$, then
$= |x^2 - 5x - 6| < |x^2 + x + 1| < \epsilon$
$= |x^2 - 5x - 6| < |(x+1)^2| < \epsilon$, then
$|(x+1)^2| < \epsilon$
$|x + 1| < \sqrt{\epsilon}$

Since $\delta < 1$, and $|x - 1| < \delta$ we have $x \in (0, 2)$,
so $|x^2 + x + 3| < 9$.
Therefore,
$|f(x) - f(1)| < \delta < \varepsilon$

Thus, $f(x)$ is continuous at $x = 1$

Question 6

For the function $f(x) = \begin{cases} \frac{x^2+8x+15}{x+3}, & x < -3 \\ x^2 - 7, & x \geq -3 \end{cases}$

If $x < -3$, then the limit from the left-hand side (LHS) will be:

LHS: $\lim_{x \to -3^-} \frac{x^2+8x+15}{x+3}$

$\Rightarrow \lim_{x \to -3^-} \frac{x^2+8x+15}{x+3}$

$\Rightarrow \lim_{x \to -3^-} \frac{(x+3)(x+5)}{(x+3)}$

$\Rightarrow \lim_{x \to -3^-} x + 5 = -3 + 5 = 2$

If $x \geq -3$, then the limit from the right-hand side (RHS) will be:

RHS: $\lim_{x \to -3^+} x^2 - 7 = (-3)^2 - 7 = 9 - 7 = 2$

LHS = RHS. Therefore, f is continuous at $x = -3$