a)
$l_1 : (x, y, z) = (1,0,0) + t(1,0,1), t \in R$
$l_2 : (x, y, z) = (1,0,-1) + t(0,1,1), t \in R$
$l_3 : (x, y, z) = (0,0,0) + t(1,1,0), t \in R$

If l_1 and l_2 intersect, there is a point that lies on both lines. There must be $t_1, t_2 \in \mathbb{R}$ such that:
t_1 and t_2 are only distinguished for legibility
$l_1 = l_2$
$(1,0,0) + t_1(1,0,1) = (1,0,-1) + t_2(0,1,1)$
$(1 + t_1, 0, t_1) = (1, t_2, -1 + t_2)$
x: $1 + t_1 = 1$
$t_1 = 0$
y: $0 = t_2$
z: $t_1 = -1 + t_2$
$0 = -1$
$undefined$

If l_1 and l_3 intersect, there is a point that lies on both lines. There must be $t_1, t_2 \in \mathbb{R}$ such that:
$l_1 = l_3$
$(1,0,0) + t_1(1,0,1) = (1,0,-1) + t_2(0,1,1),$
$(1 + t_1, 0, t_1) = (1, t_2, -1 + t_2)$
x: $1 + t_1 = 1$
$t_1 = 0$
y: $0 = t_2$
z: $t_1 = -1 + t$
$0 = -1$
$undefined$

If l_2 and l_3 intersect, there is a point that lies on both lines. There must be $t_1, t_2 \in \mathbb{R}$ such that:
$l_2 = l_3$
$(1,0,-1) + t_1(0,1,1) = (0,0,0) + t_2(1,1,0)$

$(1, t_1, -1 + t_1) = (t_2, t_2, 0)$
x: $1 = t_2$
y: $t_1 = t_2$
$0 = 0$
z: $-1 + t_1 = 0$
$t_1 = 0$

Therefore, $(1,0,-1) + (0,1,1) = (1,1,0)$, which is a point on l_2 and l_3

If l_2 and l_3 are in the plane they describe, the normal to the plane must be perpendicular to l_2 and l_3

Cross product

$(1,0,-1) \times (0,1,-1)$

$= \det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$= \underline{i} \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$= \underline{i}((0)(-1) - (-1)(1)) - \underline{j}((1)(-1) - (-1)(0)) + \underline{k}((1)(1) - (0)(0))$

$= \underline{i}(0 + 1) - \underline{j}(0 - 1) + \underline{k}(0 - 1)$

$= (1,1,-1)$

$(1,1,0)$, which is a point on the plane

Dot product

$(x, y, z).(1,1,-1)$

$= (1,1,0).(1,1,-1)$

$= 1 + 1 + 0$

$= 2$

b)

$x + 2y - z - 1 = 0$

$3x - 6y + 2x + 4 = 0$

$\underline{a} \times \underline{b}$ is the vector perpendicular to the plane, where

Convert into normal vectors

$\underline{a} = (1,2,-1)$ $\qquad$ $\underline{b} = (5,-6,0)$

Cross Product

$$\underline{a} \times \underline{b} = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -1 \\ 5 & -6 & 0 \end{bmatrix}$$

$$= \underline{i}\begin{vmatrix} 2 & -1 \\ -6 & 0 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 2 \\ 5 & -6 \end{vmatrix}$$

$$= \underline{i}((2)(0) - (-1)(-6)) - \underline{j}((1)(0) - (-1)(5)) + \underline{k}((1)(-6) - (2)(5))$$

$$= \underline{i}(0 - 6) - \underline{j}(0 + 5)) + \underline{k}(-6 - 10)$$

$$= \underline{i}(-6) - \underline{j}(5)) + \underline{k}(-16)$$

$$= (-6, -5, -16)$$

Find a point on the line. Choose the arbitrary point where $z = 0$

$x + 2y = 1$

$5x - 6y = -4$

Use matrix to solve system of equations

$A\,x = b$

$$\begin{bmatrix} 1 & 2 \\ 5 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$det A = det\begin{bmatrix} 1 & 2 \\ 5 & -6 \end{bmatrix} = (1)(-6) - (2)(5) = -16$$

$x = A^{-1}b$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-16}\begin{bmatrix} -6 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-16}\begin{bmatrix} (-6)(1) + (-2)(-4) \\ (-5)(1) + (1)(-4) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-16}\begin{bmatrix} 2 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{-16} \\ \frac{-9}{-16} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{-8} \\ \frac{9}{16} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A}. \textit{Adjoint of matrix}$$

$$A^{-1} = \frac{1}{|A|}.\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{-4}.\begin{bmatrix} -6 & -2 \\ -5 & 1 \end{bmatrix}$$

Therefore, a point that passes through the plane is
$\underline{a} = \left(-\frac{1}{8}, \frac{9}{16}, 0\right)$

Therefore, the equation for the line of intersection of two planes
$(x, y, z) = \underline{a} + t\underline{b}$

$(x, y, z) = \left(-\frac{1}{8}, \frac{9}{16}, 0\right) + t(-6, -5, -16)$

c i)

$f(x, y) = 2\sqrt{x^2 + y^2}$ $\qquad g(x, y) = 1 + x^2 + y^2$ $\qquad h(x, y) = \sqrt{1 - x^2 - y^2}$

f is a Cone (top portion)
g is an Elliptic Paraboloid
h is a Hyperbolic Hyperboloid

c ii)
Intersection where $f = h$
$2\sqrt{x^2 + y^2} = \sqrt{1 - x^2 - y^2}$
$2\sqrt{x^2} + 2\sqrt{y^2} = \sqrt{1 - x^2 - y^2}$
$4x^2 + 4y^2 = 1 - x^2 - y^2$
$5x^2 + 5y^2 = 1$
$x^2 + y^2 = \frac{1}{5}$

The intersection of f and h is a circle

d)

Given $f(x,y) = 4x + y - 2$,
Prove from first principles
that $\lim_{(x,y)\to(1,-1)} f(x,y) = 1$

Epsilon-Delta definition
Multi variable

If $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$
Then $|f(x,y) - L| < \epsilon$
Because $r = \sqrt{(x-a)^2 + (y-b)^2}$. This will always be a positive number
$\lim_{(x,y)\to(a,b)} f(x,y) = L$
As the coordinate system x,y approaches some random coordinate point a,b the limit is L

If $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$
Then $|f(x,y) - L| < \epsilon$

Substitute L=-1, $a = 1$, $b = -1$
If $0 < \sqrt{(x-1)^2 + (y-(-1))^2} < \delta$
Then $|4x + y - 2 - (1)| < \epsilon$

Substitute $x = (x-1)$, $y = (y+1)$.
Add +4 and +1 for extra constants created by substitution
If $0 < \sqrt{(x-1)^2 + (y+1)^2} < \delta$
Then $|4(x-1) + 4 - (y+1) + 1 - 3)| < \epsilon$

If $0 < \sqrt{(x-1)^2 + (y+1)^2} < \delta$
Then $|4(x-1) - (y+1) - 3| < \epsilon$

Now we can start with the calculation using the function $f(x,y)$
$|f(x,y) - (-1)| = 4|x-1| - 2|y+1|$

From the Epsilon-Delta definition above, we now must find some relationship between ϵ and δ

$|x-1| \leq \sqrt{(x-1)^2}$
$|x-1| \leq \sqrt{(x-1)^2 + (y+1)^2}$
$|x-1| = \left||x-1, y+1|\right|$

$|y+1| \leq \sqrt{(y+1)^2}$
$|y+1| \leq \sqrt{(x-1)^2 + (y+1)^2}$
$|y+1| = \left||(x,y) - (1,-1)|\right|$ or $\left||x-1, y+1|\right|$

triangle inequality
$|f(x,y) - (-1)| \leq 4|x-1| + 2|y+1|$
Substitute $|x-1| = ||x-1, y+1||$, $|y+1| = ||x-1, y+1||$
$|f(x,y) - (-1)| \leq 4\left||(x,y) - (1,-1)|\right| + 2\left||(x,y) - (1,-1)|\right|$
$\leq 6\left||(x,y) - (1,-1)|\right|$

From our earlier definition

If $0 < \sqrt{(x-1)^2 + (y+1)^2} < \delta$

Then $6\left|\left|(x, y) - (1, -1)\right|\right| \epsilon$

But for any $\epsilon > 0$, if $6\left|\left|(x, y) - (1, -1)\right|\right| < \epsilon$, then $|f(x, y) - (-1)| < \epsilon$

Therefore

Substitute $\sqrt{(x-1)^2 + (y+1)^2} = ||x-1, y+1||$

If $|f(x, y) - (-1)| < \left|\left|(x, y) - (1, -1)\right|\right| < \delta$

Then $6\left|\left|(x, y) - (1, -1)\right|\right| < \epsilon$

Multiply all expressions by 6

If $|f(x, y) - (-1)| < 6\left|\left|(x, y) - (1, -1)\right|\right| < 6\delta$

Then $6\left|\left|(x, y) - (1, -1)\right|\right| < \epsilon$

If $0 < \left|\left|(x, y) - (1, -1)\right|\right| < \delta$

Then $|f(x, y) - (-1)| < \epsilon$

We can see that for any $\epsilon > 0$, $\delta > 0$. Therefore $\lim_{(x,y)\to(1,-1)} f(x, y) = -1$

e i)

$$\lim_{(x,y)\to(0,0)} \frac{y}{x^2}$$

For the limit to exist, $\frac{y}{x^2}$ must approach the same value L, irrespective of the curve along we approach the origin $(0,0)$

Curve 1: along the x-axis that approaches origin

$$f(x,0) = \frac{0}{x^2} = 0 = L$$

Curve 2: parabola that passes through origin

$$f(\sqrt{y}, y) = \frac{y}{\sqrt{y}^2} = 1 = L$$

Since approaching the origin along these two different curves leads to different limits, the limit does not exist

e ii)

$$\lim_{(x,y)\to(0,0)} \frac{y}{x}$$

For the limit to exist, $\frac{y}{x}$ must approach the same value L, irrespective of the curve along we approach the origin $(0,0)$

Curve 1:

$$\lim_{(x,y)\to(0,0)} f(x,0) = \frac{0}{x} = 0 = L$$

Curve 2:

$$\lim_{(x,y)\to(0,0)} f(y,y) = \frac{y}{y} = 1 = L$$

Since approaching the origin along these two different curves leads to different limits, the limit does not exist

e iii)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

For the limit to exist, $\frac{x^2-y^2}{x^2+y^2}$ must approach the same value L, irrespective of the curve along we approach the origin $(0,0)$

Curve 1:

$$\lim_{(x,y)\to(0,0)} f(x,-1) = \frac{x^2-(-1)^2}{x^2+(-1)^2} = \frac{x^2-1}{x^2+1} = -\frac{1}{+1} = -1 = L$$

Curve 2:

$$\lim_{(x,y)\to(0,0)} f(y,y) = \frac{y^2-y^2}{y^2+y^2} = \frac{0}{y^4} = 0 = L$$

Since approaching the origin along these two different curves leads to different limits, the limit does not exist

e iv)

$\lim_{(x,y)\to(0,0)} x^2 sinx + y^2 siny$

For the limit to exist, $x^2 sinx + y^2 siny$ must approach the same value L, irrespective of the curve along we approach the origin $(0,0)$.

Since $sinx$, $cosx$ and $tanx$ are continuous, we use the sum and product rules

$$\lim_{(x,y)\to(0,0)} x^2 sinx + y^2 siny = \lim_{x\to 0} x^2 . \lim_{x\to 0} sinx + \lim_{y\to 0} y^2 . \lim_{y\to 0} siny$$
$$= 0.0 + 0.0$$
$$= 0$$

f i)

$$f(x,y) = \begin{cases} \frac{2x^2+2xy+y^2}{x^2+xy} & \text{if } x \neq -y \\ 0 & \text{if } (x,y) = (0,0) \text{ or } (x,y) = (2,-2) \end{cases}$$

$D_f = \{(x,y) : x \neq -y\} \cup \{(0,0),(2,-2)\}$

f ii)

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+2xy+y^2}{x^2+xy} = \lim_{(x,y)\to(0,0)} \frac{2x^2+2xy+y^2}{x(x+y)} = \frac{2(0)^2+2(0)(0)+(0)^2}{(0)(0+0)} = \frac{0}{0} = 0$$

$$\lim_{(x,y)\to(2,-2)} \frac{2x^2+2xy+y^2}{x^2+xy} = \lim_{(x,y)\to(2,-2)} \frac{2(2)^2+2(2)(-2)+(-2)^2}{2(2-2)} = \frac{4}{0} = undefined$$

f iii)

$f(0,0) = 0$

$f(2,-2) = 0$

f iv)

yes.

The value of the function: $f(0,0) = 0$.

The limit of the function: $\lim_{(x,y)\to(2,-2)} f(x,y) = 0$

The function value and the limit is the same at $(x,y) = (0,0)$

Therefore, the function is continuous at $(x,y) = (0,0)$

f iv)

No.

The value of the function: $f(2,-2) = 0$.

The limit of the function: $\lim_{(x,y)\to(2,-2)} f(x,y) = 1$

The function value and the limit are not the same at $(x,y) = (2,-2)$

Therefore, the function is not continuous at $(x,y) = (2,-2)$

f vi)

No.

The function is not continuous at $(x, y) = (2, -2)$
Therefore, f is not continuous at every point of its domain

g i)
$f(x, y, z) = e^{-x^2-y^2-2z^2}$

Chain rule $\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$
$\frac{\partial f}{\partial x}\left(e^{-x^2-y^2-2z^2}\right)$
$= e^{-x^2-y^2-2z^2} \cdot \frac{\partial f}{\partial x}\left(e^{-x^2-y^2-2z^2}\right)$

$\frac{\partial f}{\partial x}\left(e^{-x^2-y^2-2z^2}\right)$
$= e^{-x^2-y^2-2z^2} \cdot \frac{\partial f}{\partial x}\left(e^{-x^2-y^2-2z^2}\right)$
$= -2x. e^{-x^2-y^2-2z^2}$

$\frac{\partial f}{\partial y}\left(e^{-x^2-y^2-2z^2}\right)$
$= e^{-x^2-y^2-2z^2} \cdot \frac{\partial f}{\partial y}\left(e^{-x^2-y^2-2z^2}\right)$
$= -2y. e^{-x^2-y^2-2z^2}$

$\frac{\partial f}{\partial z}\left(e^{-x^2-y^2-2z^2}\right)$
$= e^{-x^2-y^2-2z^2} \cdot \frac{\partial f}{\partial z}\left(e^{-x^2-y^2-2z^2}\right)$
$= -4z. e^{-x^2-y^2-2z^2}$

The gradient of a function $w = f(x,y,z)$ is described by the vector function:
$\nabla f = grad\ f \quad = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right)$
$\quad = \left(-2x. e^{-x^2-y^2-2z^2}, -2y. e^{-x^2-y^2-2z^2}, -4z. e^{-x^2-y^2-2z^2}\right)$

Therefore
$grad\ f(-2,2,-1) \quad = \left(-2(-2). e^{-(-2)^2-(2)^2-2(-1)^2}, -2(2). e^{-(-2)^2-(2)^2-2(-1)^2}, -4(-1). e^{-(-2)^2-(2)^2-2(-1)^2}\right)$
$\quad = (4. e^{-10}, -2. e^{-10}, 4. e^{-10})$

To calculate u in the direction of v, we just need to divide by its magnitude.
$u = v \cdot \frac{1}{||v||}$
$= (1, -4, 5)\frac{1}{||(1,-4,5)||}$
$= (1, -4, 5)\frac{1}{\sqrt{1+(-4)^2+5^2}}$
$= \frac{1}{\sqrt{42}}(1, -4, 5)$

Therefore, the rate of increase of f at the point $(-2,2,-1)$ in the direction of the vector $v=(1,-4,5)$:

$$
\begin{aligned}
grad\, f(-2,2,-1).u &= u.(4.e^{-10},-2.e^{-10},4.e^{-10}) \\
&= \frac{1}{\sqrt{42}}(1,-4,5)\ .(4.e^{-10},-2.e^{-10},4.e^{-10}) \\
&= \frac{1}{\sqrt{42}}\ ((1)(4.e^{-10})+(-4)(-2.e^{-10})+(5)(4.e^{-10})) \\
&= \frac{1}{\sqrt{42}}\ (4.e^{-10}+8.e^{-10}+20.e^{-10}) \\
&= \left(\frac{4.e^{-10}}{\sqrt{42}}+\frac{8.e^{-10}}{\sqrt{42}}+\frac{20.e^{-10}}{\sqrt{42}}\right) \\
&= \frac{32.e^{-10}}{\sqrt{42}}
\end{aligned}
$$

g ii)

Therefore, the rate of increase of f at the point $(-2,2,-1)$ in the direction of the negative z-axis u= $(0,0,-1)$:

$$
\begin{aligned}
grad\, f(-2,2,-1).u &= u.(4.e^{-10},-2.e^{-10},4.e^{-10}) \\
&= (0,0,-1)\,.(4.e^{-10},-2.e^{-10},4.e^{-10}) \\
&= (0)(4)e^{-10},(0)(-2).e^{-10},(-1)(4).e^{-10} \\
&= -4.e^{-10}
\end{aligned}
$$

g iii)

The maximum rate of change at a given point:

$$
\begin{aligned}
||\ grad\, f(x,y,z)\ || &= ||(4.e^{-10},-2.e^{-10},4.e^{-10})|| \\
&= \sqrt{(4.e^{-10})^2,(-2.e^{-10})^2,(4.e^{-10})^2} \\
&= \sqrt{(4.e^{-10})^2+(-2.e^{-10})^2+(4.e^{-10})^2} \\
&= \sqrt{4^2.e^{-20}+(-2)^2.e^{-20}+4^2.e^{-20}} \\
&= \sqrt{16.e^{-20}+4.e^{-20}+16.e^{-20}} \\
&= 4.e^{-10}+2.e^{-10}+4.e^{-10} \\
&= 10.e^{-10}
\end{aligned}
$$

h i)

C is a Helix (spiral)

h ii)

Given that $x^2+y^2=1$

Also given: $x=\cos t$ $\quad y=\sin t$ $\quad z=t$

Pythagorean Identity
$\sin^2 x+\cos^2 x=1$

Substitute the parametric equations of the curve into equation
Apply Pythagorean identity

$x^2+y^2=\sin^2 x+\cos^2 x=1$

Every point of the parametric curve satisfies the equation of the cylinder, and so the curve lies on the cylinder.

h iii)
The gradient of the tangent line r is r'
Therefore, the gradient at $r'(1)$:

$$r'(t) = \frac{\mathrm{d}}{dt}(\cos t, \sin t,\ t)$$
$$= (-\sin t, \cos t, 1)$$

$$r'(1) = (-\sin(1), \cos(1), 1)$$
$$= (-\sin(1), \cos(1), 1)$$

h iv)
The velocity vector at instant t is given by $r'(t) = (-\sin t, \cos t, 0)$
Therefore, the speed at instant t is given by

$$||r'(t)|| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$
$$= \sqrt{\sin^2 t + \cos^2 t + 1}$$
$$= \sqrt{1 + 1^2}$$
$$= 2$$

Pythagorean Identity
$\sin^2 x + \cos^2 x = 1$

h v)

At the point $P = (-1,0,\pi)$, on the curve $r(t)$ we have:
$\cos t = -1$
$\sin t = -1$
$t = \pi$
Resulting in the vector $(-1, -1, \pi)$

At the tangent's gradient $r'(\pi)$, we have:
$-\sin(\pi) = 0$
$\cos(\pi) = -1$
$t = 1$
Resulting in the vector $(0, -1, 1)$

The vector equation of the line l:
$l = (-1,0,\pi) + t(0, -1, 1)$

The tangent t (in terms of x, y and z):
$x = -1$
$0t = -x - 1$

$y = -t$
$t = -y$

$z = \pi + t$
$t = z - \pi$

Therefore, the re-written cartesian form:
$-y = z - \pi$

h vi)

i i)
$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$ $r(t) = (t \cos t \, , t \sin t \,)$

The composite function $f\big(f(t)\big)$:
$$f(r(t)) = \frac{1}{\sqrt{(\cos x)^2 + (\sin x)^2}}$$
$$= \frac{1}{\sqrt{\cos^2 x + \sin^2 x}}$$

i ii)
The gradient of a function $w = f(x, y)$ is described by the vector function:
$$grad\, f = \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)\right)$$

i iii A)
i iii B)

j i)
j ii)
j iii)
j iv)

k i)

$F(x,y,z) = (2xyz,\ x^2z + 2yz^2,\ x^2y + 2y^2z + e^z)$

$$J_f = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 2yz & 2xz & 2xy \\ 2xz & 2z^2 & x^2 + 4yz \\ 2xy & x^2 + 4yz & 2y^2 + e^z \end{bmatrix}$$

k ii)

$divF = 2yz + 2xz + 6yz$

k iii)

$$curl\,F = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & x^2z + 2yz^2 & x^2y + 2y^2z + e^z \end{bmatrix}$$

$$= \underline{i} \begin{vmatrix} \partial_y & \partial_z \\ x^2z + 2yz^2 & x^2y + 2y^2z + e^z \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \underline{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \underline{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= [(x^2 + 4yz) - (x^2 + 4yz)] - [(2xy - 2xy)] + [(2xz) - (2xz)]$$

$$= (0,0,0)$$

k iv)

F is a Conservative vector field when:

Second order derivatives of F are continuous

The scalar curl of F is zero *(as* $curl\,F(x,y,z) = (0,0,0)$*)*

F is defined on all $\mathbb{R}^3$

Therefore, F is a conservative vector field and has a potential function

k v)

1)

First, second and third order derivatives

$$f_x \qquad = e^{x^2+y^2}.2x$$
$$f_{xx} \qquad = 2x\left(e^{x^2+y^2}.2x\right) + e^{x^2+y^2}(x)$$
$$= 2\left[e^{x^2+y^2} + 2x^2e^{x^2+y^2}\right]$$

$$f_{xxx} \qquad = 2\left[\frac{\partial}{\partial y}\left(e^{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(2e^{x^2+y^2}\right) + \frac{\partial}{\partial y}(x^2)\right]$$
$$= 2\left[2x.e^{x^2+y^2} + 4x.e^{x^2+y^2} + 2x\right]$$
$$= 2\left[6x.e^{x^2+y^2} + 2x\right]$$

$$f_y \qquad = e^{x^2+y^2}.2y$$
$$f_{yy} \qquad = 2y\left(e^{x^2+y^2}.2y\right) + e^{x^2+y^2}(y)$$
$$= 2\left[e^{x^2+y^2} + 2y^2e^{x^2+y^2}\right]$$
$$f_{yyy} \qquad = 2\left[\frac{\partial}{\partial y}\left(e^{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(2e^{x^2+y^2}\right) + \frac{\partial}{\partial y}(y^2)\right]$$
$$= 2\left[2y.e^{x^2+y^2} + 4y.e^{x^2+y^2} + 2y\right]$$
$$= 2\left[6y.e^{x^2+y^2} + 2y\right]$$

$$f_{xy} \qquad = (e^{x^2+y^2}.2x)(e^{x^2+y^2}.2y)$$
$$= 4xy.e^{x^2+y^2}$$

$$f_{xyy} \qquad = \left[e^{x^2+y^2}.2y\right].2\left[e^{x^2+y^2} + 2y^2e^{x^2+y^2}\right]$$
$$= 4y.e^{2x^2+2y^2} + 8y^3.e^{2x^2+2y^2}$$

$$f_{xxy} \qquad = 2\left[e^{x^2+y^2} + 2x^2e^{x^2+y^2}\right].\left[e^{x^2+y^2}.2y\right]$$
$$= 4x.e^{2x^2+2y^2} + 8x^3.e^{2x^2+2y^2}$$

Evaluate derivatives at the point $(0,0)$

$f\ (0,0) \ = \ e^{0^2+0^2} = 1$

$f_x\ (0,0) \ = \ e^{0^2+0^2}.2(0)$

$= \ 0$

$f_{xx}\ (0,0) \ = \ 2\left[e^{0^2+0^2} + 2e^{0^2+0^2}.0^2\right]$

$= 2$

$f_{xxx}\ (0,0) = \ 2\left[6(0).e^{0^2+0^2} + 2(0)\right]$

$= 0$

$f_y\ (0,0) \ = \ e^{0^2+0^2}.2(0)$

$= \ 0$

$f_{yy}\ (0,0) \ = \ 2\left[e^{0^2+0^2} + 2e^{0^2+0^2}.0^2\right]$

$= 2$

$f_{yyy}\ (0,0) = \ 2\left[6(0).e^{0^2+0^2} + 2(0)\right]$

$= 0$

$f_{xy} \quad = (0)(0)$

$= 0$

$f_{xxy} \quad = (2)(0)$

$= 0$

$f_{xyy} \quad = (0)(2)$

$= 0$

First Order:

General formula for 1st degree Taylor polynomial

$$P(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

2D vector

$$Df(x,y) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

Using matrices:

$$= f(a,b) + Df(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$= f(0,0) + Df(0,0)\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$= 1 + [0 \quad 0]\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1$$

Second Order:

General formula for 2nd degree Taylor polynomial

$$P_2(x,y) = L(x,y) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

Using matrices:

$$= L(x,y) + \frac{1}{2!}[x-a \quad y-b]DDf(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$= 1 + \frac{1}{2}[x-0 \quad y-0]DDf(0,0)\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$= 1 + \frac{1}{2}[x \quad y]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 + \frac{1}{2}[x \quad y]\begin{bmatrix} 2x+0y \\ 0x+2y \end{bmatrix}$$

$$= 1 + \frac{1}{2}[2x^2 \quad 2y^2]$$

$$= 1 + x^2 + y^2$$

2x2 symmetric matrix

Hessian Matrix

$$DDf(x) = Hf(x,y)$$

$$= \begin{bmatrix} f_x \\ f_y \end{bmatrix}\begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$= \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

General formula for 3rd degree Taylor polynomial

$$P_3(x,y) = P_2(x,y) + \frac{f_{yyy}(a,b)}{0!3!}(y-b)^3 + \frac{f_{xyy}(a,b)}{1!2!}(x-a)^1(y-b)^2 + \frac{f_{xxy}(a,b)}{2!1!}(x-a)^2(y-b)^1 + \frac{f_{xxx}(a,b)}{3!0!}(x-a)^3$$

2x2x2 symmetric tensor

$$DDDf(x) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}\begin{bmatrix} f_x \\ f_y \end{bmatrix}\begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$= \begin{bmatrix} f_{xxx} & f_{xyx} & f_{xxy} & f_{xyy} \\ f_{yxx} & f_{yyx} & f_{yxy} & f_{yyy} \end{bmatrix}$$

Using matrices:

$$= P_2(x,y) + \frac{1}{3!}[x-a \quad y-b]DDDf(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$= 1 + x^2 + y^2 + \frac{1}{6}[x-0 \quad y-0]DDDf(0,0)\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$= 1 + x^2 + y^2 + \frac{1}{6}[x \quad y]\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 + x^2 + y^2$$

Therefore P_3 or $T_3(x,y) = 1 + x^2 + y^2$