a)

$f(x,y) \in \mathbb{R}$

Critical point

$f'(c) = 0 \quad or \; f'(c) = undefined$

$\frac{\partial}{\partial x}\left(\frac{4}{3}x^3 - 4xy\right)$

$= 4(x^2 - y)$

$= 4(x^2 - y)$

$\frac{\partial}{\partial y}(4xy - 2y^2 + y + 1)$

$= -4x - 4y + 1$

$grad\, f(x,y) = (0,0)$

$(4(x^2 - y)\,, -4x - 4y + 1) = (0,0)$

$-4x - 4y + 1 = 0$

$-4y = -4x - 1$

$y = x + \frac{1}{4}$

Substitute

$4(x^2 - \left(x + \frac{1}{4}\right)) = 0$

$4x^2 - 4x - 1 \; = 0$

Solve using quadratic equation formula:

$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x_1, x_2 = \frac{-4 \pm \sqrt{(-4)^2 - 4(4)(-1)}}{2(4)}$

$x_1, x_2 = \frac{1 \pm \sqrt{2}}{2}$

$x_1 = \frac{1+\sqrt{2}}{2}$ and $x_2 = \frac{1-\sqrt{2}}{2}$

Substitute

$y = x + \frac{1}{4}$

$y_1 = \left(\frac{1+\sqrt{2}}{2}\right) + \frac{1}{4}$

$y_1 = \frac{3+\sqrt{2}}{4}$

$y_2 = \left(\frac{1-\sqrt{2}}{2}\right) + \frac{1}{4}$

$y_2 = \frac{3-\sqrt{2}}{4}$

Critical Points: $\left(\frac{1+\sqrt{2}}{2}, \frac{3+\sqrt{2}}{4}\right), \left(\frac{1-\sqrt{2}}{2}, \frac{3+\sqrt{2}}{4}\right), \left(\frac{1+\sqrt{2}}{2}, \frac{3-\sqrt{2}}{4}\right), \left(\frac{1-\sqrt{2}}{2}, \frac{3-\sqrt{2}}{4}\right)$

Second derivatives

$f_{xx} = \frac{d}{dx}(4(x^2 - y))$

$= 8x$

$f_{yy} = \frac{d}{dy}(4(x^2 - y))$

$= -4$

$f_{xy}^2 = 0$

$\Delta = f_{xx}f_{yy} - f_{xy}^2$
$= 8x(-4) - 0$
$= 32x$

Point	f_{xx}	f_{yy}	f_{xy}	Δ	
$\left(\frac{1+\sqrt{2}}{2}, \frac{3+\sqrt{2}}{4}\right)$	$4(1+\sqrt{2})$	-4	0	$16(1+\sqrt{2})$	
$\left(\frac{1-\sqrt{2}}{2}, \frac{3+\sqrt{2}}{4}\right)$	$4(1-\sqrt{2})$	-4	0	$16(1-\sqrt{2})$	Saddle point
$\left(\frac{1+\sqrt{2}}{2}, \frac{3-\sqrt{2}}{4}\right)$	$4(1+\sqrt{2})$	-4	0	$16(1+\sqrt{2})$	
$\left(\frac{1-\sqrt{2}}{2}, \frac{3-\sqrt{2}}{4}\right)$	$4(1-\sqrt{2})$	-4	0	$16(1-\sqrt{2})$	Saddle point

Therefore at $\left(\frac{1-\sqrt{2}}{2}, \frac{3+\sqrt{2}}{4}\right)$ and $\left(\frac{1-\sqrt{2}}{2}, \frac{3-\sqrt{2}}{4}\right)$ we have a saddle point

Therefore at $\left(\frac{1+\sqrt{2}}{2}, \frac{3+\sqrt{2}}{4}\right)$ and $\left(\frac{1+\sqrt{2}}{2}, \frac{3-\sqrt{2}}{4}\right)$ we have a local maximum

There are no global minima or maxima

b i)
If we have a parabola defined as $y = f(x)$, then the parametric equations are $y = f(t)$ and $x = t$.

For r_p we have
$r_p(x, y) = (t, x^2 + 1)$
$\Longrightarrow r_p(x, y) = (t, t^2 + 1)$

The parameterisation of $y = x^2 + 1$ is $r_p(t)$

For r_ℓ we have
$r_\ell(x, y) = (s, x)$
$\Longrightarrow r_\ell(x, y) = (s, s)$

The parameterisation of $y = x$ is $r_\ell(t)$

b ii)

$$\begin{aligned} \left\|\boldsymbol{r_p(t)} - \boldsymbol{r_\ell(s)}\right\| &= \left\|(\boldsymbol{t}, \boldsymbol{t^2} + \boldsymbol{1}) - (\boldsymbol{s}, \boldsymbol{s})\right\| \\ &= \left\|\boldsymbol{t} - \boldsymbol{s}, \boldsymbol{t^2} + \boldsymbol{1} - \boldsymbol{s}\right\| \\ &= \sqrt{(\boldsymbol{t} - \boldsymbol{s})^2 + (\boldsymbol{t^2} + \boldsymbol{1} - \boldsymbol{s})^2} \end{aligned}$$

b iii)

$d = (t-s)^2 + (t^2+1-s)^2$

$grad\ d(s,t) = \left(\frac{\partial d}{\partial s}, \frac{\partial d}{\partial t}\right)$

$\frac{\partial}{\partial s}((t-s)^2 + (t^2+1-s)^2)$

$= 2(t-s)\frac{\partial}{\partial s}(t-y) + 2(t^2+1-s) + \frac{\partial}{\partial s}(t^2+1-s)$

$= -2(t-s) - 2(t^2+1-s)$

$= -2t + 2s - 2t^2 + 2 + 2s$

$= 4s - 2t - 2t^2 + 2$

$\frac{\partial}{\partial t}((t-s)^2 + (t^2+1-s)^2)$

$= 2(t-s)\frac{\partial}{\partial t}(t-s) + 2(t^2+1-s)\frac{\partial}{\partial t}(t^2+1-s)$

$= 2(t-s) + 4t(t^2+1-s)$

$= 2t - 2s + 4t^3 + 4t - 4ts$

$= (4s - 2t - 2t^2 + 2, 2t - 2s + 4t^3 + 4t - 4ts)$

$= (2(2s - t - t^2 + 1), 2(2t^3 + 3t - 2ts - s))$

Critical points are where $grad\ d(s,t) = \left(\frac{\partial d}{\partial s}, \frac{\partial d}{\partial t}\right) = (0,0)$

Therefore $(0,0) = (2(2s - t - t^2 + 1), 2(2t^3 + 3t - 2ts - s))$

$2(2s - t - t^2 + 1) = 0$

$2(2t^3 + 3t - 2ts - s) = 0$

Combining the above:

$\Rightarrow 2(2s - t - t^2 + 1) = 2(2t^3 + 3t - 2ts - s)$

$\Rightarrow 2s - t - t^2 + 1 = 2t^3 + 3t - 2ts - s$

$\Rightarrow 2t^3 + t^2 + 4t - 2ts - 3s - 1 = 0$

$\Rightarrow t(2t^2 + t + 4 - 2s) - 3s - 1 = 0$

Therefore $-3s - 1 = 0$ and $t(2t^2 + t + 4 - 2s) = 0$

$-3s - 1 = 0$

$s = -\frac{1}{3}$

$t(2t^2 + t + 4 - 2s) = 0$

$2t^2 + t + 4 - 2s = 0$

$2t^2 + t = 2s - 4$

$2t(t+1) = 2s - 4$

$t = 0$

c)

Let x be the length of the box
Let y be the width of the box
Let z be the height of the box
The objective function is $R(x,y,z) = xyz$

The restrictions for the dimensions $g(x,y,z) = x + 2y + z = 300$
$\Rightarrow g(x,y,z) = x + 2y + z - 300$

$y = 150 - \frac{x}{2} - \frac{z}{2}$

The volume function $V = x^2 y$
Substitute restriction into V $V = x^2\left(150 - \frac{x}{2} - \frac{z}{2}\right)$
$= 150x^2 - \frac{x^3}{2} - \frac{x^2 z}{2}$

Gradients:
$\nabla R(x,y,z) = (yz, xz, xy)$
$\nabla g(x,y,z) = (1,2,1)$

Determine Critical points
$\frac{dV}{dx} = 150x^2 - \frac{x^3}{2} - \frac{x^2 z}{2}$
$= 300x - \frac{3x^2}{2} - \frac{2xz}{2}$

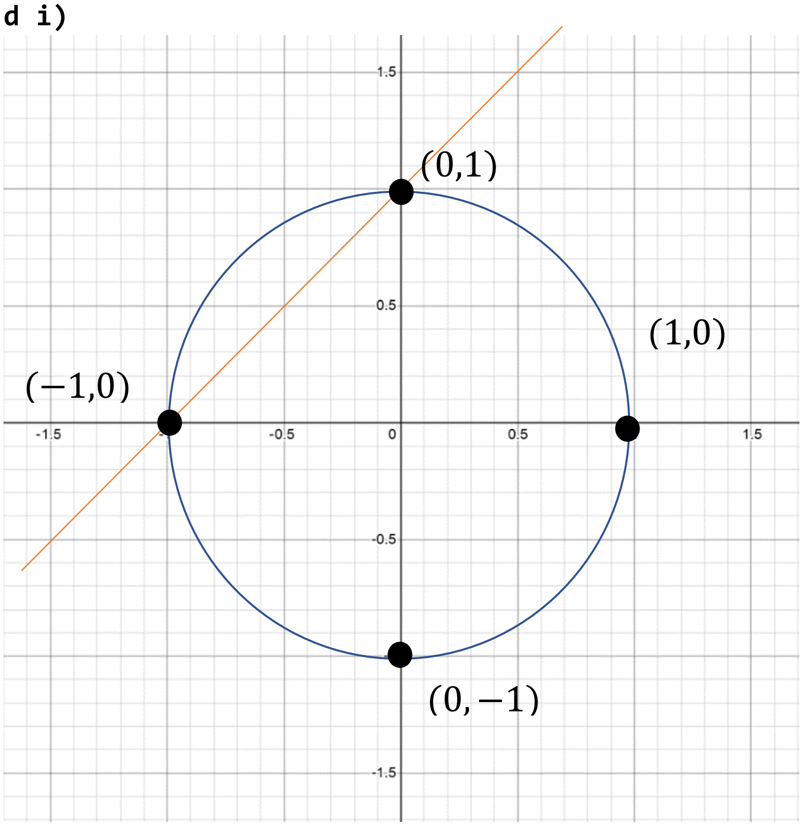
Language method: Optimum is at $grad\, R = \lambda grad\, f$
$grad\, R = \lambda grad\, f$
$\Rightarrow (yz, xz, xy) = \lambda(1,2,1)$

Therefore
$\lambda = yz$
$\lambda = \frac{xz}{2}$
$\lambda = xy$

Combining the equations
$yz = \frac{xz}{2} = xy$
$\Rightarrow x = 2y$ and
$\Rightarrow x = z$

Substitute back into constraint
$x + x + x = 300$
$\Rightarrow 3x = 300$
$\Rightarrow x = 100$
Therefore $x = 100$, $z = 100$ and $y = 50$

d i)



d ii)

No. R is not a type I region

As a union of type I regions $R = R_1 \cup R_1$

$$R_1 = \{(x, y): 0 \leq x \leq 1, -(x + 1) < y < x + 1\}$$

$$R_2 = \{(x, y): -1 \leq y \leq 0, -\sqrt{1 - y^2} < x < \sqrt{1 - y^2}\}$$

d iii)

Yes. R is a type II region

$R =: \quad -1 \leq y \leq 1,\ \sqrt{1 - x^2} \leq y \leq\ x + 1\ \}$

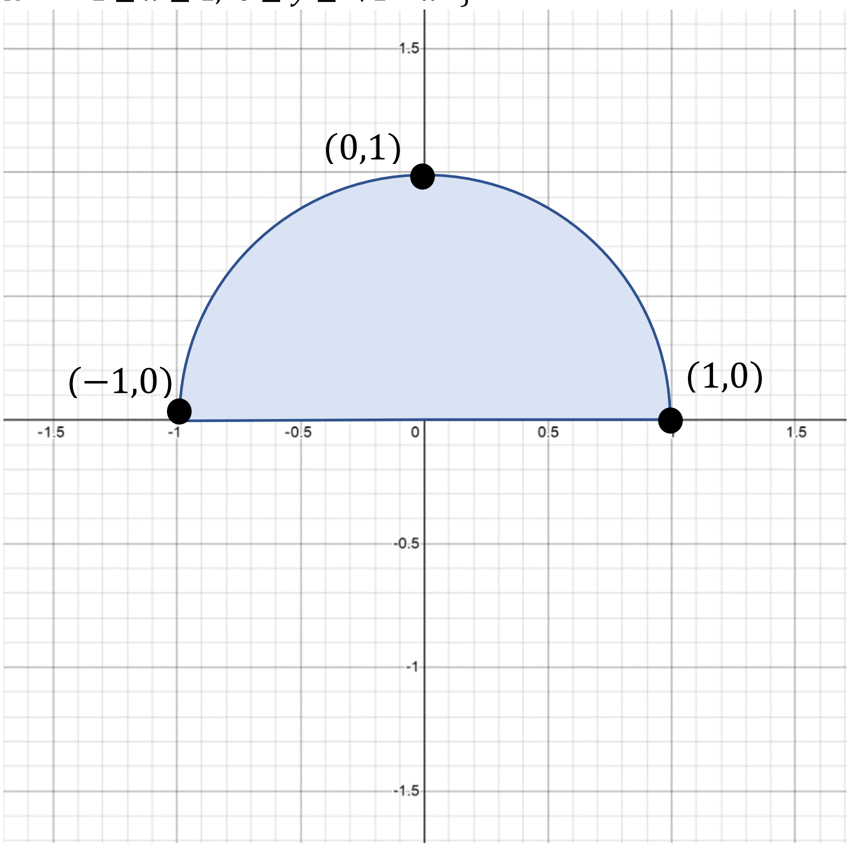
d iv)

$$\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{x+1} x^2 y^2 \, dydx$$

e i)

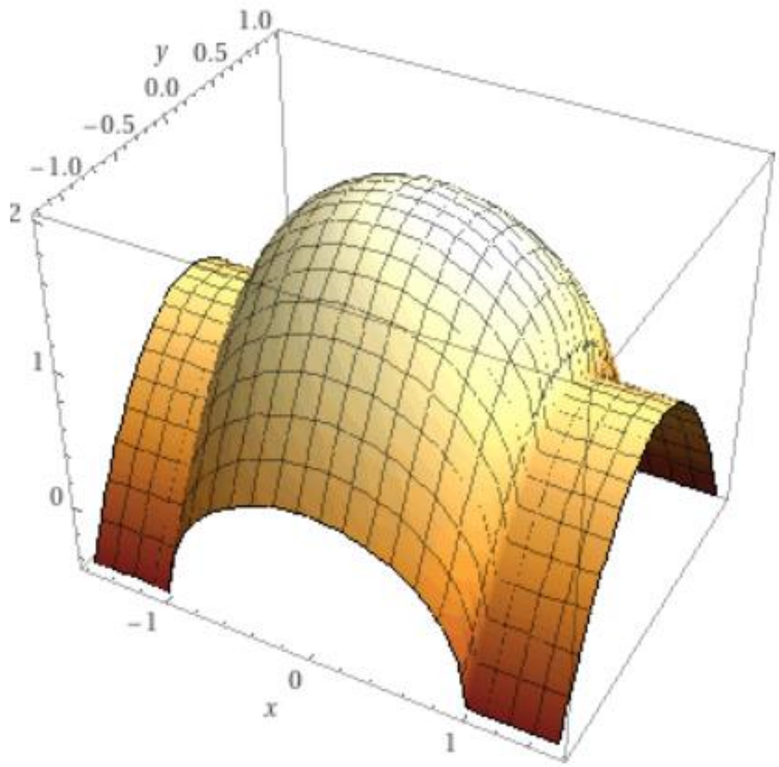
$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 1-y^2\,dydx$$

$R = \ -1 \le x \le 1,\ 0 \le y \le \ \sqrt{1-x^2}\,\}$



e ii)

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}} + (\mathbf{1}-\boldsymbol{y}^{\mathbf{2}})$$



e iii)

f)

$C: \quad r(t) = (1+3t, 2t), \qquad 0 \le t \le 1$

$f(x,y) = \ x(1+y^2)$

Therefore

$\int_C \ x(1+y^2)$

$= \int_0^1 F(r)(t)||\, r'(t)||dt$

$= \int_0^1 t(1+2t^2)\, \sqrt{3^2+2^2}\ dt$

$= \int_0^1 t\sqrt{13}(1+2t^2)\ \ dt$

$= \sqrt{13} \int_0^1 t + 2t^3 \ \ dt$

$= \sqrt{13} \left[\int_0^1 t\ dt \ + \ 2\int_0^1 t^3 \ \ dt\right]$

$= \sqrt{13} \left[\frac{t^2}{2}\right]_0^1 + \ 2\left[\frac{t^4}{4}\right]_0^1$

$= \sqrt{13}\ \left[\frac{1}{2} \ + \ 2\frac{1}{4}\right]$

$= \sqrt{13}$

g)