MAT3701 - Linear Algebra III **ASSIGNMENT 01**

Opens: 1 March 2025 **Due: Friday, 30 April 2025**

Instructions for the Assignment

(1) Carefully explain all your arguments.

(2) Only hand written PDF files will be accepted.

(3) Late submissions will not be marked.

(4) Write your name, surname and student number on the first page.

Question 1 (10 marks)

(1.1) Find the equation of the line through the points $(3, -2, 4)$ and $(-5, 7, 1)$.

(1.2) Find the equation of the plane containing the following points in space:

$$(2, -5, -1), \qquad (0, 4, 6), \qquad \text{and} \qquad (-3, 7, 1).$$

Question 2 (8 marks)

Let $S = \{0, 1\}$. Let $f, g, h \in \mathcal{F}(S, \mathbb{R})$, where

$$f(t) = 2t + 1, \qquad g(t) = 1 + 4t - 2t^2, \qquad \text{and} \qquad h(t) = 5^t + 1.$$

Prove that

(2.1) $f = g$, and

(2.2) $f + g = h$.

Question 3 (8 marks)

Let $\mathbb{V}$ denote the set of ordered pairs of real numbers. If $(a_1, a_2), (b_1, b_2) \in \mathbb{V}$ and $c \in \mathbb{R}$ we define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2), \qquad \text{and} \qquad c(a_1, a_2) = (ca_1, a_2).$$

Is $\mathbb{V}$, together with the above operations, a vector space over $\mathbb{R}$? Justify your answer.

Question 4 (4 marks)

Let A be a square matrix. Prove that $A + A^t$ is symmetric.

Question 5 (6 marks)

Let S be a nonempty set and $\mathbb{F}$ a field. Let $s_0 \in S$. Let

$$\mathbb{V} = \{f \in \mathcal{F}(S, \mathbb{F}) : f(s_0) = 0\}.$$

Prove that $\mathbb{V}$ is a subspace of $\mathcal{F}(S, \mathbb{F})$.

Question 6 (4 marks)

Suppose that W is a subspace of a vector space $\mathbb{V}$. Suppose that $w_1, w_2, \cdots, w_n \in W$. Prove that

$$\sum_{i=1}^{n} a_i w_i \in W$$

for all $a_i \in \mathbb{F}$.

Definition:

Let S_1 and S_2 be nonempty subsets of a vector space $\mathbb{V}$. We define the **sum of** S_1 **and** S_2 as

$$S_1 + S_2 = \{x + y : x \in S_1 \text{ and } y \in S_2\}.$$

Question 7 (12 marks)

Let W_1 and W_2 be subspaces of a vector space $\mathbb{V}$.

(7.1) Prove that $W_1 + W_2$ is a subspace of $\mathbb{V}$.

(7.2) Prove that $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

(7.3) Suppose that $\mathbb{V}'$ is a subspace of $\mathbb{V}$ with the property that $W_1 \subseteq \mathbb{V}'$ and $W_2 \subseteq \mathbb{V}'$. Prove that $W_1 + W_2 \subseteq \mathbb{V}'$.

Question 8 (8 marks)

Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ and let $x \in \mathbb{V}$.

(8.1) Prove that $\text{span}\ (\{x\}) = \{ax : a \in \mathbb{F}\}$.

(8.2) Suppose that $\mathbb{F} = \mathbb{R}$ in (8.1). Interpret (8.1) geometrically.

Question 9 (6 marks)

Do the polynomials

$$x^3 - 2x + 1, \qquad 4x^2 - x + 3 \qquad \text{and} \qquad 3x - 2$$

generate $\mathcal{P}_3(\mathbb{R})$? Justify your answer.

Question 10 (8 marks)

Suppose that $\mathbb{V}$ is a vector space and $S_1 \subseteq \mathbb{V}$ and $S_2 \subseteq \mathbb{V}$. Prove that

$$\text{span}\ (S_1 \cup S_2) = \text{span}\ (S_1) + \text{span}\ (S_2).$$

Question 11 (6 marks)

Let u and v be distinct vectors in a vector space $\mathbb{V}$. Show that $\{u, v\}$ is linearly dependent if and only if u is a multiple of v or v is a multiple of u.

Question 12 (4 marks)

Let $S = \{u_1, u_2, \cdots, u_n\}$ be a linearly independent subset of a vector space $\mathbb{V}$ over $\mathbb{Z}_2$.
How many vectors are there in $\text{span}\ (S)$? Justify your answer.

Question 13 (8 marks)

The set of solutions to the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

Question 14 (8 marks)

Use the Lagrange interpolation formula to construct the polynomial of smallest degree whose graph contains the following points:

$$(-2, -6), \qquad (-1, 5) \qquad \text{and} \qquad (1, 3).$$

(Total = 100 marks)