Problem 13.

Find the coordinate vectors of v relative to the basis of set $S = \{v_1, v_2\}$, where

(a) $v = (5, -3)$; $v_1 = (1, 2)$; $v_2 = (1, 0)$

(b) $v = (a, b)$; $v_1 = (0, 2)$; $v_2 = (1, 1)$

To find the coordinate vectors of v relative to the basis set $S = \{v_1, v_2\}$, we can use the formula:

$$[v]_S = [a_1\ a_2]^{-1}\ [v_1\ v_2]\ [v]$$

(a) Thus,

$$[v_1\ v_2] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$[v] = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Therefore,

$$[v]_S = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$[v]_S = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$[v]_S = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

(b) Thus,

$$[v_1\ v_2] = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$[v] = \begin{bmatrix} a \\ b \end{bmatrix}$$

Therefore,

$$[v]_S = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$[v]_S = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$[v]_S = \begin{bmatrix} -a + b \\ 2a \end{bmatrix}$$

Problem 14.

Let U and V be two subspaces of R^5 defined by

$U = \{(x_1, x_2, x_3, x_4, 0) \in R^5 : x_1 = 2x_2 \; and \; x_3 + x_4 = 0\}$ and

$V = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 : x_1 + x_2 = 2x_3 \; and \; x_4 = x_5\}$ and

Find the bases of U and V

Bases of U

[1] Rewrite as system of equations:

$$\begin{cases} x_1 - 2x_2 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

[2] Express U as the span of a set of vectors:

$U = span\{(2,1,0,0,0), (0,0,1,-1,0)\}$

$(2,1,0,0,0)$ is linearly independent

$(0,0,1,-1,0)$ is linearly independent

Thus, both vectors form a basis for U.

Bases of V

[1] Rewrite as system of equations:

$$\begin{cases} x_1 + x_2 - 2x_3 = 0 \\ \quad x_4 - x_5 = 0 \end{cases}$$

[2] Express U as the span of a set of vectors:

$V = span\{(1,1,-1,0,0), (0,0,0,1,-1)\}$

$(1,1,-1,0,0)$ is linearly independent

$(0,0,0,1,-1)$ is linearly independent

Thus, both vectors form a basis for U.

ASS4: Problem 15.

Determine whether the following form basis for P_2:

(a) $1 + 2x - x^2$; $x + 4x^2$; $1 - x + 2x^2$

(b) $1 + x$; $1 + x^2$; $x + x^2$

To determine whether a set of polynomials forms a basis for P_2, we need to check two conditions:

1. Linear Independence: The polynomials in the set must be linearly independent.
2. Spanning: The set must span P_2, meaning that any polynomial in P_2, can be expressed as a linear combination of the polynomials in the set.

(a) $1 + 2x - x^2$; $x + 4x^2$; $1 - x + 2x^2$

[1] Express in terms of $ax^2 + bx + c$

Rewrite as system of equations:

$$\begin{cases} -x^2 + 2x + 1 \\ 4x^2 + x + 0 \\ 2x^2 - x + 1 \end{cases}$$

Thus,

$-x^2 + 2x + 1$

$$\begin{cases} a = -1 \\ b = 2 \\ c = 1 \end{cases}$$

And,

$4x^2 + x + 0$

$$\begin{cases} a = 4 \\ b = 1 \\ c = 0 \end{cases}$$

And,

$2x^2 - x + 1$

$$\begin{cases} a = 2 \\ b = -1 \\ c = 1 \end{cases}$$

[2] Linearly independence

find constants k_1, k_2, and k_3 such that:

$k_1(1 + 2x - x^2) + k_2(4x^2 + x) + k_3(2x^2 - x + 1) = 0$

$\Rightarrow (k_1 + k_3) + (2k_1 + k_2 - k_3)x + (-k_1 + 4k_2 + 2k_3)x^2 = 0$

Rewrite as system of equations:

$$\begin{cases} k_1 + k_3 = 0 & [1] \\ 2k_1 + k_2 - k_3 = 0 & [2] \\ -k_1 + 4k_2 + 2k_3 = 0 & [3] \end{cases}$$

[1] $k_3 = -k_1$

[2] $2k_1 + k_2 - (-k_1) = 0$

$\Rightarrow 3k_1 + k_2 = 0$

$\Rightarrow k_2 = -3k_1$

[3] $-k_1 + 4k_2 + 2k_3 = 0$

$\Rightarrow -k_1 + 4(-3k_1) + 2(-k_1) = 0$

$\Rightarrow -k_1 - 12k_1 - 2k_1 = 0$

$\Rightarrow -15k_1 = 0$

$\Rightarrow k_1 = 0$

Thus,

$k_1 = k_2 = k_3 = 0$

Therefore, the polynomials are linearly independent.

(b) $1 + x$; $1 + x^2$; $x + x^2$

[1] Express in terms of $ax^2 + bx + c$

Rewrite as system of equations:

$$\begin{cases} 1 + x \\ 1 + x^2 \\ x + x^2 \end{cases}$$

Thus,

$1 + x$

$$\begin{cases} a = 0 \\ b = 1 \\ c = 1 \end{cases}$$

And,

$1 + x^2$

$$\begin{cases} a = 1 \\ b = 0 \\ c = 1 \end{cases}$$

And,

$x + x^2$

$$\begin{cases} a = 1 \\ b = 1 \\ c = 0 \end{cases}$$

[2] Linearly independence

find constants k_1, k_2, and k_3 such that:

$k_1(1+x) + k_2(1+x^2) + k_3(x+x^2) = 0$

$\Rightarrow (k_1+k_3) + (k_1+k_3)x + (-k_1+4k_2+2k_3)x^2 = 0$

Rewrite as system of equations:

$$\begin{cases} k_1 + k_2 = 0 & [1] \\ k_1 + k_3 = 0 & [2] \\ k_2 + k_3 = 0 & [3] \end{cases}$$

[1] $k_1 = -k_2$

[2] $-k_2 + k_3 = 0$

$\Rightarrow k_3 = \ k_2$

[3] $k_2 + k_3 = 0$

$\Rightarrow k_2 + k_2 = 0$

$\Rightarrow k_2 = 0$

Thus,

$k_1 = k_2 = k_3 = 0$

Therefore, the polynomials are linearly independent.

Problem 16.

Find the basis and dimension of the solution space of given homogeneous linear system.

$x_1 + 3x_2 - x_3 + x_4 = 0$
$2x_1 + x_2 - 3x_3 + x_4 = 0$
$3x_1 + x_2 - x_3 + 2x_4 = 0$

[1] Rewrite as system of equations:

$$\begin{cases} x_1 + 3x_2 - x_3 + x_4 = 0 & [1] \\ 2x_1 + x_2 - 3x_3 + x_4 = 0 & [2] \\ 3x_1 + x_2 - x_3 + 2x_4 = 0 & [3] \end{cases}$$

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & -3 & 1 \\ 3 & 1 & -1 & 2 \end{bmatrix} \ ; \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Forward Elimination
-- iter: 1
R2: R2 - 2R1

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -5 & -1 & -1 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

Forward Elimination
-- iter: 2
R3: R3 - 3R1

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -5 & -1 & -1 \\ 0 & -8 & 2 & -1 \end{bmatrix}$$

Forward Elimination
-- iter: 3
R2: (1/-5)R2

$$A = \begin{bmatrix} 1 & 0 & -\frac{8}{5} & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{18}{5} & \frac{3}{5} \end{bmatrix}$$

Forward Elimination

-- iter: 4

R3: (5/18)R3

$$A = \begin{bmatrix} 1 & 0 & -\frac{8}{5} & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}$$

Forward Elimination

-- iter: 5

R1: R1 + (8/5)R3

R2: R2 – (1/5)R3

$$A = \begin{bmatrix} 0 & 1 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & \frac{6}{30} \end{bmatrix}$$

[2] Rewrite as system of equations:

$$\begin{cases} x_3 + \frac{1}{6}x_4 = 0 & [1] \\ x_2 + \frac{1}{30}x_4 = 0 & [2] \\ x_1 + 3x_2 + \frac{7}{6}x_4 \ = 0 & [3] \end{cases}$$

$$x_3 = -\ \frac{1}{6}x_4$$

$$x_2 = -\ \frac{1}{30}x_4$$

$$x_1 = -\ 3x_2 - \frac{7}{6}x_4$$