Question 1

1.1.

Let $z = |z|e^{i\theta}$

Exponential form

$\Rightarrow \ln(z) = \ln|z| + i\theta$

Alternate exponential form

$\Rightarrow \ln(z) = \ln|1| + i\left(\frac{\pi}{2} + 2n\pi\right)$ where $-\pi < \theta < \pi$; $n \in Z$

principal of arg (z)

$\Rightarrow \ln(z) = -\frac{\pi}{2} + 2n\pi$ where $-\pi < \theta < \pi$; $n \in Z$

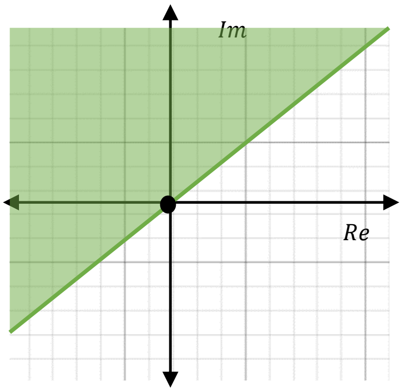
$\Rightarrow z = e^{-\frac{\pi}{2}} + 2n\pi$ where $-\pi < \theta < \pi$; $n \in Z$

Therefore, we have an infinite number of values, all differing by integral multiples of 2πi.

It's exponential form:

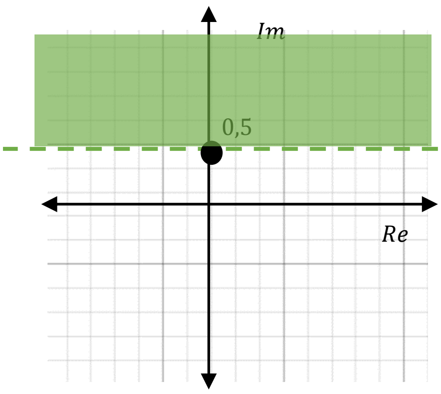
$-1 - i = \sqrt{2}\exp\left[i\left(\frac{3}{4}\right)\right]$ *also written as* $-1 - i = \sqrt{2}\, e^{-\frac{i3\pi}{4}}$

1.2.1. $A = \left\{z \in \mathbb{C} \middle| \arg(z) \geq \frac{\pi}{4}\right\}$



The region is $\{x + yi \mid x \in \mathbb{R}, y = x\}$, which is the region above and including the line $y = x$. The region is closed since it contains all its boundary points.

1.2.2. $A = \{z \in \mathbb{C} \mid \ |z-1| < 1 \ and \ |z| > 1\}$
Also, $|z-1| < |z|$
Let $z = x + iy$ where $x, y \in \mathbb{R}$, then
$\Rightarrow |z-1| < |z|$
$\Rightarrow |x + (y \ -1)i| < |x + iy \ |$
$\Rightarrow x^2 + (y \ -1)^2 < x^2 + y^2$
$\Rightarrow x^2 + y^2 - 2y + 1 < x^2 + y^2$
$\Rightarrow -2y < -1$
$\Rightarrow y < \frac{1}{2}$
The region is $\{x + yi \mid x \in \mathbb{R}, y < \frac{1}{2}\}$, which is the region below and not including the line $y = \frac{1}{2}$. The region is closed since it contains all its boundary points.



Question 2

2.1.

Suppose that $g = x + yi$. The complex-valued function $g(z)$ is differentiable at any point z in the complex plane:

$g(z) = (x^3 - y^2x) + i(x^2y + y^3)$

The real part $u(x,y)$ and the imaginary part $v(x,y)$ are

$u(x,y) = x^3 - y^2x$ $\quad\quad$ $v(x,y) = x^2y + y^3$

And their partial derivatives are

$u_x = 3x^2 - y^2$ $\quad\quad$ $v_x = 2xy$

$u_y = -2yx$ $\quad\quad$ $v_y = x^2 + 3y^2$

All the partial derivatives are polynomial and thus continuous. Therefore, the function g is differentiable wherever the Cauchy-Rieman equations are satisfied

If the Cauchy–Riemann equations are to hold at a point (x, y), it follows that $2x = 0$ and $2y = 0$, or that $x = y = 0$.

We have that:

$u_y = -v_x$ holds

Thus, $v_y = u_x$ holds:

$v_y = u_x$

$\Rightarrow x^2 + 3y^2 = 3x^2 - y^2$

$\Rightarrow 4y^3 = 2x^2$

$\Rightarrow 2y^3 = x^2$

$\Rightarrow y = \frac{1}{\sqrt{2}}x$ or $-\frac{1}{\sqrt{2}}x$

Therefore, the Cauchy-Rieman equations are satisfied at the lines $y = \frac{1}{\sqrt{2}}x$ and $y = -\frac{1}{\sqrt{2}}x$.

2.2.

In order to be analytic, a function $f = u + vi$ needs the partial derivatives of the real part $u(x,y)$ and the imaginary part $v(x,y)$ should:

- satisfy Cauchy-Rieman equations $u_y = -v_x$ and $v_y = u_x$, and
- be continuous

There is no neighbourhood of any point throughout which g is analytic as Cauchy-Rieman does not hold for an open set. Every neighbourhood of any point will have points which are not on the lines $y = \frac{1}{\sqrt{2}}x$ and $y = -\frac{1}{\sqrt{2}}x$. Therefore, g is nowhere analytic.

Question 3

We can solve Laplace's equation in any domain simply by taking the real part of any analytic function in that domain. Suppose that $z = x + yi$. The complex-valued function $g(z)$ is differentiable at any point z in the complex plane and the Cauchy-Rieman equations hold at any point $z \in \mathbb{C}$,

We have that:
$u_y = -v_x$
$v_y = u_x$
$u_{yx} = v_{xy}$

Where the Cauchy-Rieman equations hold, and
$v_{xx} = v_{yx}$
$u_{yy} = -v_{xy}$
And also
$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$

Question 4

Question 5

When $w = e^z$, the image of the infinite strip $0 \leq y \leq \pi$ is the upper half $v \geq 0$ of the w plane.

Let $z = x + iy$, where $z \in \mathbb{C}$

Under the transformation $w = e^z$,

we have:

$|w| = e^x$

Under transformation $w = e^z$ where $x \geq 0$ we have:

$w = e^x e^{i\pi}$

Under transformation $w = e^z$ where $y \geq 0$ we have:

$w = e^{iy}$

Which is a semi-circle in the upper half $v \geq 0$ of the w plane. The final sketch will not include points inside the circle.

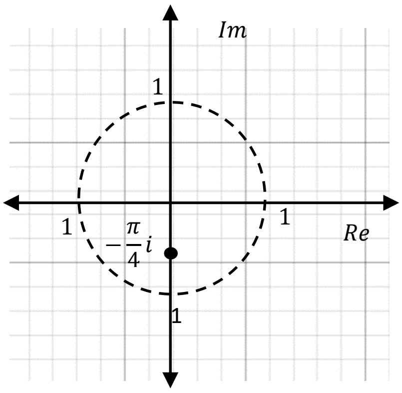
So if $x \geq 0$ and $0 \leq y < \pi$, then:

$|w| = e^x \geq e^0 = 1$ where $0 \leq \arg w < \pi$

Question 6

6.1.

Example: $\oint_{|z|=1} \frac{ze^z}{(4z+\pi i)^2} dz$



$z = -\frac{\pi i}{4}$ *is the singularity, inside of* γ

$$\oint_{|z|=1} \frac{ze^z}{(4z+\pi i)^2} dz$$

$$= \frac{2\pi i}{1!} . ze^z . \frac{1}{16} . dz \;\Big|_{z=-\frac{\pi i}{4}}$$

Evaluate the function at $z = -\frac{\pi i}{4}$

$$= \frac{2\pi i}{1!} . (1+z) . e^z . \frac{1}{16} . \;\Big|_{z=-\frac{\pi i}{4}}$$

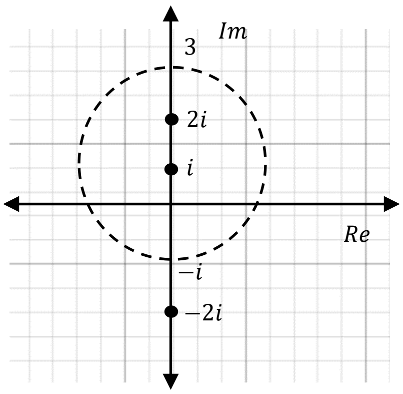
$$= \frac{2\pi i}{16} . \left(1 - \frac{\pi i}{4}\right) . e^{-\frac{\pi i}{4}}$$

$$= \frac{\pi i}{8} . \frac{1-i}{\sqrt{2}} . \left(1 - \frac{\pi i}{4}\right)$$

$$= \frac{\pi}{8\sqrt{2}} . \left[\left(1 + \frac{\pi}{4}\right) + i\left(1 + \frac{\pi}{4}\right)\right]$$

6.2.

Example: $\oint_{|z-i|=2} \frac{e^{z+1}}{(z^2+4)^2} dz$



Singularities:

$z^2 + 4 = (z - 2i)(z + 2i)$

$z = 2i$ *is the singularity, inside of* γ

$z = -2i$ *is the singularity, outside of* γ

Irrelevant: we only want to look at analytic/inside γ

$$\oint_{|z-i|=2} \frac{\frac{e^{z+1}}{(z+2i)^2}}{(z-2i)^2} dz$$

$$= \frac{2\pi i}{1!} . \frac{e^{z+1}}{(z+2i)^2} . dz \;\Big|_{z=2i}$$

Evaluate the function at $z = 2i$

$$= \frac{2\pi i}{1} . \frac{(z+2i)^2 . e^{z+1} - e^{z+1} . 2(z+2i)}{(z+2i)^4} \Big|_{z=2i}$$

$$= \frac{2\pi i}{1} . \frac{(z+2i) . e^{z+1}\,(z+2i-2)}{(z+2i)^4} \Big|_{z=2i}$$

$$= \frac{2\pi i}{1} . \frac{e^{z+1}\,(z+2i-2)}{(z+2i)^3} \Big|_{z=2i}$$

$$= \frac{2\pi i}{1} . \frac{e^{z+1}\,(2i+2i-2)}{(2i+2i)^3}$$

$$= \frac{2\pi i}{1} . \frac{e^{2i+1}\,(4i-2)}{(4i)^3}$$

$$= \frac{2\pi i . e^{2i+1}\,(4i-2)}{64\, i^3}$$

$$= \frac{\pi . e^{2i+1}\,(4i-2)}{32\, i^2}$$

$$= \frac{\pi . e^{2i+1}\,(4i-2)}{-32}$$

$$= \frac{\pi . e^{2i+1}\,(1-2i)}{16}$$

Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{V\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) \; if\ v \neq 0$$

$v = (z + 2i)^2 \quad u = e^{z+1}$

$v' = 2(z + 2i) \quad u' = e^{z+1}$

$$\frac{e^{z+1}}{(z+2i)^2} . dz = \left(\frac{(z+2i)^2 . e^{z+1} - e^{z+1} . 2(z+2i)}{(z+2i)^4}\right)$$