Problem 33

Suppose $S = \{v_1, v_2\}$ *is a basis of* R^2*;*
where $v_1 = (2,1)$ *and* $v_2 = (-1,3)$
Let $T : R^2 \to R^3$ *be the linear operator for which*
$T(v_1) = (1,3,2)$ *and* $T(v_2) = (-1,0,4)$
Find a formula for $T(x;y)$ *; and use that formula to find* $T = (1,-4)$.

[1] x,y as linear combination of v_1 and v_2
Let $(x;y) = c_1v_1 + c_2v_2$

$\Rightarrow (x;y) = c_1v_1 + c_2v_2$
$\Rightarrow (x;y) = c_1(2,1) + c_2(-1,3)$
$\Rightarrow (x;y) = (2c_1 - c_2, c_1 + 3c_2)$

1. $x = 2c_1 - c_2$
2. $y = c_1 + 3c_2$

[2] Solve for system

$y = c_1 + 3c_2$
3. $\Rightarrow c_1 = y - 3c_2$

$x = 2c_1 - c_2$
$\Rightarrow x = 2(y - 3c_2) - c_2$
$\Rightarrow x = 2y - 6c_2 - c_2$
$\Rightarrow x = 2y - 7c_2$
4. $\Rightarrow c_2 = \frac{2y-x}{7}$

$c_1 = y - 3c_2$
$\Rightarrow c_1 = y - 3\left(\frac{2y-x}{7}\right)$
$\Rightarrow c_1 = y + \left(\frac{-6y+3x}{7}\right)$
$\Rightarrow c_1 = \left(\frac{7y-6y+3x}{7}\right)$
$\Rightarrow c_1 = \left(\frac{y+3x}{7}\right)$

[3] Formula for $T\,(x;\,y)$

Given $T\,(v_1) = (1,3,2)$ *and* $T\,(v_2) = (-1,0,4)$

$$T\,(x;\,y) = T\,(c_1 v_1 + c_2 v_2)$$
$$\Rightarrow T\,(x;\,y) = Tc_1 v_1 + Tc_2 v_2$$
$$\Rightarrow T\,(x;\,y) = \left(\frac{y+3x}{7}\right)(1,3,2) + \frac{2y-x}{7}(-1,0,4)$$
$$\Rightarrow T\,(x;\,y) = \left(\frac{y+3x}{7}, \frac{3y+6x}{7}, \frac{2y+6x}{7}\right) + \left(\frac{-2y+x}{7}, 0, \frac{8y-4x}{7}\right)$$
$$\Rightarrow T\,(x;\,y) = \left(\frac{4x-y}{7}, \frac{3y+6x}{7}, \frac{10y+2x}{7}\right)$$

[4] Find $T = (1,-4)$.

$$T\,(x;\,y) = \left(\frac{4x-y}{7}, \frac{3y+6x}{7}, \frac{10y+2x}{7}\right)$$
$$\Rightarrow T\,(1,-4) = \left(\frac{4(1)-(-4)}{7}, \frac{3(-4)+6(1)}{7}, \frac{10(-4)+2(1)}{7}\right)$$
$$\Rightarrow T\,(1,-4) = \left(\frac{8}{7}, -\frac{3}{7}, -\frac{38)}{7}\right)$$

Problem 34

Let v_1; v_2; and v_3 be vectors in a vector space v;
and let $T: V \to R^4$ be a linear transformation for which
$T(v_1) = (1,0,2,-1)$
$T(v_2) = (0,2,1,-1)$
$T(v_3) = (1,-1,0,1)$

Find $T\,(v_1 - 2v_2 + 3v_3)$

[1] Linearity Property

$$T(av_1 + bv_2 + cv_3) = aT(v_1) + bT(v_2) + cT(v_3)$$

Thus,

$$\Rightarrow T\,(v_1 - 2v_2 + 3v_3) = (1,0,2,-1) - 2(0,2,1,-1) + 3(1,-1,0,1)$$
$$\Rightarrow T\,(v_1 - 2v_2 + 3v_3) = (1,0,2,-1) + (0,-4,-2,2) + (3,-3,0,3)$$
$$\Rightarrow T\,(v_1 - 2v_2 + 3v_3) = (4,-7,0,4)$$

Problem 35

Let A be a 7x6 matrix such that Ax = 0 has only the trivial solution.
If $T: R^6 \to R^7$ is multiplication by A; then find the nullity and rank of T.

[1] Nullity of T
Zero

[2] Rank of T
6

Problem 36
Let T be multiplication by the matrix

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix}$$

1. Find a basis for the kernel of T.
2. Find a basis for the range of T.

[1] Solve for system

$$-x_1 + 2x_2 + 4x_3 = 0$$
$$3x_1 + x_3 = 0$$
$$2x_1 + 2x_2 + 5x_3 = 0$$

$$\begin{bmatrix} -1 & 2 & 4 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 2 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ -1 & 2 & 4 & 0 \\ 2 & 2 & 5 & 0 \end{bmatrix}$$

R2: R2 + 1/3R1

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & \frac{13}{3} & 0 \\ 2 & 2 & 5 & 0 \end{bmatrix}$$

R3: R3 + 2/3R1

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & \frac{13}{3} & 0 \\ 0 & 2 & \frac{13}{3} & 0 \end{bmatrix}$$

R3: R3 - R2

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & \frac{13}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R2: R2/2

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & \frac{13}{6} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R1: R1-13/6R2

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & \frac{13}{6} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R1: R1/3

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{13}{6} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for the kernel:

$$\left\{ \begin{bmatrix} 0 \\ -\frac{13}{6} \\ 1 \end{bmatrix} \right\}$$

Basis for the range:

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$$