Question 1

a) Let $P(n)$ be the statement

$1 + 3 + \cdots + (2n + 1) = (n + 1)^2$

Basis Clause
Show that $n = 1$
$P(n)$ is where $n = 1$

$$\begin{aligned} LHS &= 1 + 3 + \cdots + (2\text{n} + 1) \\ &= (2\text{n} + 1) \\ &= (2(1) + 1) \\ &= 3 \end{aligned}$$

$$\begin{aligned} RHS &= (\text{n} + 1)^2 \\ &= n^2 + 2n + 1 \\ &= (1)^2 + 2(1) + 1 \\ &= 3 \end{aligned}$$

$LHS = RHS = 3$.
Therefore, $P(n)$ is true

Inductive Hypothesis
Show that $n = k$.
$P(k)$ is where $n = k$

Assume k
$1 + 3 + \cdots + (2k + 1) = (k + 1)^2$

Inductive Step
If $P(k)$ is true, then $P(k+1)$ must also be true

Assume $k + 1$
$1 + 3 + \cdots + (2\text{k} + 3) = (k + 2)^2$

$$\begin{aligned} LHS &= 1 + 3 + \cdots + (2(\text{k} + 1) + 1) \\ &= 1 + 3 + \cdots + (2\text{k} + 2 + 1) \\ &= 1 + 3 + \cdots + (2\text{k} + 3) \end{aligned}$$

$$\begin{aligned} RHS &= ((\text{k} + 1) + 1)^2 \\ &= (k + 2)^2 \end{aligned}$$

$LHS = 1 + 3 + \cdots + (2\text{k} + 1) + (2\text{k} + 3)$

$RHS = (k + 2)^2$

But, $1 + 3 + \cdots + (2\text{k} + 1) = (k + 1)^2$
Therefore, by the induction hypothesis:

$$\begin{aligned} &= (k + 1)^2 + (2\text{k} + 3) \\ &= k^2 + 2k + 1 + 2k + 3 \\ &= k^2 + 4k + 4 \\ &= (k + 2)^2 \end{aligned}$$

$LHS = RHS$
Thus, $P(k+1)$ is true
Hence, $P(k)$ is true
It then follows by mathematical induction that $P(n)$ is true.

b) Let $P(n)$ be the statement

$1 + 3^n < 4^n$

Basis Clause
Show that $n = 2$
$P(n)$ is where $n = 2$

$$LHS = 1 + 3^n = 1 + 3^2 = 10$$

$$RHS = 4^n = 4^2 = 16$$

$10 < 16$ and $LHS < RHS$
Therefore, $P(n)$ is true

Inductive Hypothesis
Show that $n = k$
$P(k)$ is where $n = k$

Assume k
$1 + 3^k < 4^k$

Inductive Step
If $P(k)$ is true, then $P(k+1)$ must also be true

Assume $k + 1$
$1 + 3^{(k+1)} < 4^{(k+1)}$

$$LHS = 1 + 3^{(k+1)} = 1 + 3.3^k$$

$$RHS = 4^{(k+1)} = 4.4^k$$

But, $1 + 3.3^k < 4.4^k$
Therefore, by the induction hypothesis:
$1 + 3.3^k < 4(1 + 3^k)$
$1 + 3.3^k < (3 + 1)(1 + 3^k)$ Re-write 4 as 3+1
$1 + 3.3^k < 3 + 3.3^k + 1 + 3^k$ Multiplying out
$1 + 3.3^k < (1 + 3.3^k) + (3 + 3^k)$ By regrouping
$0 < 3 + 3^k$ Remove $(1 + 3.3^k)$ from both sides

$0 < 3 + 3^k$ is true for all $k \geq 2$
$LHS < RHS$
Thus, $P(k+1)$ is true
Hence, $P(k)$ is true
It then follows by mathematical induction that $P(n)$ is true for $n \geq 2$

Question 2
a)