Question 1

1.1

Given $Ax = b$

$$\begin{bmatrix} 0.05 & 0.07 & 0.06 & 0.05 \\ 0.07 & 0.10 & 0.08 & 0.07 \\ 0.06 & 0.08 & 0.10 & 0.09 \\ 0.05 & 0.07 & 0.09 & 0.10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.23 \\ 0.32 \\ 0.33 \\ 0.31 \end{bmatrix}$$

1.2

A: Gaussian Elimination Without Pivoting.

Forward Elimination:

$$\tilde{A}^1|b^1 = \begin{bmatrix} 0.05 & 0.07 & 0.06 & 0.05 & : & 0.23 \\ 0.07 & 0.10 & 0.08 & 0.07 & : & 0.32 \\ 0.06 & 0.08 & 0.10 & 0.09 & : & 0.33 \\ 0.05 & 0.07 & 0.09 & 0.10 & : & 0.31 \end{bmatrix}$$

$$\tilde{A}^2|b^2 = \begin{bmatrix} 0.0500 & 0.07 & 0.0600 & 0.0500 & : & 0.2300 \\ 0.0000 & 0.002 & -0.0040 & 0.0000 & : & -0.002 \\ 0.0000 & -0.0040 & 0.0280 & 0.0300 & : & 0.0540 \\ 0.0000 & 0.0000 & 0.0300 & 0.0500 & : & -0.0800 \end{bmatrix}$$

$$\boxed{\begin{aligned} R2 &: R2 - \frac{0.07}{0.05}R1 \\ R3 &: R3 - \frac{0.06}{0.05}R1 \\ R4 &: R4 - \frac{0.05}{0.05}R1 \end{aligned}}$$

$$\tilde{A}^3|b^3 = \begin{bmatrix} 0.05 & 0.0700 & 0.0600 & 0.0500 & : & 0.2300 \\ 0.00 & 0.0020 & -0.0040 & 0.0000 & : & -0.0020 \\ 0.00 & -0.0000 & 0.0200 & 0.0300 & : & 0.0500 \\ 0.00 & 0.0000 & 0.0300 & 0.0500 & : & -0.0800 \end{bmatrix}$$

$$\boxed{R3 : R3 - \frac{-0.004}{0.002}R2}$$

$$\tilde{A}^4|b^4 = \begin{bmatrix} 0.0500 & 0.0700 & 0.0600 & 0.0500 & : & 0.2300 \\ 0.0000 & 0.0020 & -0.0040 & 0.0000 & : & -0.0020 \\ 0.0000 & 0.0000 & 0.0200 & 0.0300 & : & 0.0500 \\ 0.0000 & 0.0000 & 0.0000 & -0.0050 & : & -0.0050 \end{bmatrix}$$

$$\boxed{R4 : R4 - \frac{0.03}{0.02}R3}$$

Back Substitution:

$$x_4 = \frac{-0.0050}{-0.0050} \qquad = 1.0000$$

$$x_3 = \frac{0.0500 - 0.0300}{0.0200} \qquad = 1.0000$$

$$x_2 = \frac{-0.0020 - (-0.040)}{0.0020} \qquad = 1.0000$$

$$x_1 = \frac{0.2300 - 0.0500 - 0.0600 - 0.0700}{0.0500} \qquad = 1.0000$$

B: Gaussian Elimination With Scaled Partial Pivoting.

$$\tilde{A}^1|b^1 = \begin{bmatrix} 0.05 & 0.07 & 0.06 & 0.05 & : & 0.23 \\ 0.07 & 0.10 & 0.08 & 0.07 & : & 0.32 \\ 0.06 & 0.08 & 0.10 & 0.09 & : & 0.33 \\ 0.05 & 0.07 & 0.09 & 0.10 & : & 0.31 \end{bmatrix}$$

Iteration 1:

$a =$ [0.05 0.07 0.06 0.05]

$S =$ [0.07 0.10 0.10 0.10]

$|a_i, 1/S_i| =$ [0.7143 0.70 0.60 0.50]

Since the row in S with the largest value is $R1$, no row interchange is required

$$\tilde{A}^2|b^2 = \begin{bmatrix} 0.0500 & 0.0700 & 0.0600 & 0.0500 & : & 0.2300 \\ 0.0000 & 0.0020 & -0.0040 & 0.0000 & : & -0.0020 \\ 0.0000 & -0.0040 & 0.0280 & 0.0300 & : & 0.0540 \\ 0.0000 & 0.0000 & 0.0300 & 0.0500 & : & 0.0800 \end{bmatrix}$$

$$R3 \; : \; R3 - \frac{-0.004}{-0.004} R2$$

Iteration 2:

$a =$ [0.05 0.00 0.00 0.00]

$S =$ [0.07 0.10 0.10 0.10]

$|a_i, 1/S_i| =$ [0.02 0.0 0.00 0.00]

$$\tilde{A}^3|b^3 = \begin{bmatrix} 0.0500 & 0.0700 & 0.0600 & 0.0500 & : & 0.2300 \\ 0.0000 & -0.0040 & 0.0280 & 0.0300 & : & 0.0540 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & : & 0.0000 \\ 0.0000 & 0.0000 & 0.0300 & 0.0500 & : & 0.0800 \end{bmatrix}$$

$$R4 \; : \; R3 - \frac{-0.004}{-0.004} R2$$

Iteration 3:

$a =$ [0.05 0.00 0.00 0.00]

$S =$ [0.07 0.10 0.10 0.10]

$|a_i, 1/S_i| =$ [0.02 0.00 0.00 0.00]

$$\tilde{A}^4|b^4 = \begin{bmatrix} 0.0500 & 0.0700 & 0.0600 & 0.0500 & : & 0.2300 \\ 0.0000 & -0.0040 & 0.0280 & 0.0300 & : & 0.0540 \\ 0.0000 & 0.0000 & 0.0300 & 0.0500 & : & 0.0800 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & : & 0.0000 \end{bmatrix}$$

No unique solution

C: LU Deompisition

$$A = \begin{bmatrix} 0.05 & 0.07 & 0.06 & 0.05 \\ 0.07 & 0.10 & 0.08 & 0.07 \\ 0.06 & 0.08 & 0.10 & 0.09 \\ 0.05 & 0.07 & 0.09 & 0.10 \end{bmatrix}$$

Step 1:

$$L = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.4000 & 0.0000 & 0.0000 & 0.0000 \\ 1.2000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \quad U = \begin{bmatrix} 0.0500 & 0.0700 & 0.0600 & 0.0500 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

Step 2:

$$L = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.4000 & 1.0000 & 0.0000 & 0.0000 \\ 1.2000 & -2.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \quad U = \begin{bmatrix} 0.0500 & 0.0700 & 0.0600 & 0.0500 \\ 0.0000 & 0.0020 & -0.0040 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

Step 3:

$$L = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.4000 & 1.0000 & 0.0000 & 0.0000 \\ 1.2000 & -2.0000 & 1.0000 & 0.0000 \\ 1.0000 & 0.0000 & 1.5000 & 0.0000 \end{bmatrix} \quad U = \begin{bmatrix} 0.0500 & 0.0700 & 0.0600 & 0.0500 \\ 0.0000 & 0.0020 & -0.0040 & 0.0000 \\ 0.0000 & 0.0000 & 0.0200 & 0.0300 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

Step 4:

$$L = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.4000 & 1.0000 & 0.0000 & 0.0000 \\ 1.2000 & -2.0000 & 1.0000 & 0.0000 \\ 1.0000 & 0.0000 & 1.5000 & 0.0000 \end{bmatrix} \quad U = \begin{bmatrix} 0.0500 & 0.0700 & 0.0600 & 0.0500 \\ 0.0000 & 0.0020 & -0.0040 & 0.0000 \\ 0.0000 & 0.0000 & 0.0200 & 0.0300 \\ 0.0000 & 0.0000 & 0.0000 & 0.0050 \end{bmatrix}$$

1.3

Question 2
2.1

A

B: Jacobi Method
We have $x^{(0)} = (0{,}0{,}0{,}0)^t$ as the initial point, hence we can perform three iterations of the algorithm.

Iteration 1:

$$x_1^{(1)} = \frac{1}{0.05}\ [0.23\ -\ 0.07\ \times\ 0.0\ -\ 0.06\ \times\ 0.0\ -\ 0.05\ \times\ 0.0\,]\ =\ 4.6$$

$$x_2^{(1)} = \frac{1}{0.1}\ [0.32\ -\ 0.07\ \times\ 0.0\ -\ 0.08\ \times\ 0.0\ -\ 0.07\ \times\ 0.0\,]\ =\ 3.2$$

$$x_3^{(1)} = \frac{1}{0.1}\ [0.33\ -\ 0.06\ \times\ 0.0\ -\ 0.08\ \times\ 0.0\ -\ 0.09\ \times\ 0.0\,]\ =\ 3.3$$

$$x_4^{(1)} = \frac{1}{0.1}\ [0.31\ -\ 0.05\ \times\ 0.0\ -\ 0.07\ \times\ 0.0\ -\ 0.09\ \times\ 0.0\,]\ =\ 3.1$$

Iteration 2:

$$x_1^{(2)} = \frac{1}{0.05}\ [0.23\ -\ 0.07\ \times\ 3.2\ -\ 0.06\ \times\ 3.3\ -\ 0.05\ \times\ 3.1\,]\ =\ -6.94$$

$$x_2^{(2)} = \frac{1}{0.1}\ [0.32\ -\ 0.07\ \times\ 4.6\ -\ 0.08\ \times\ 3.3\ -\ 0.07\ \times\ 3.1\,]\ =\ -4.83$$

$$x_3^{(2)} = \frac{1}{0.1}\ [0.33\ -\ 0.06\ \times\ 4.6\ -\ 0.08\ \times\ 3.2\ -\ 0.09\ \times\ 3.1\,]\ =\ -4.81$$

$$x_4^{(2)} = \frac{1}{0.1}\ [0.31\ -\ 0.05\ \times\ 4.6\ -\ 0.07\ \times\ 3.2\ -\ 0.09\ \times\ 3.3\,]\ =\ -4.41$$

Iteration 3:

$$x_1^{(3)} = \frac{1}{0.05}\ [0.23\ -\ 0.07\ \times\ -4.83\ -\ 0.06\ \times\ -4.81\ -\ 0.05\ \times\ -4.41\,]\ =\ 21.544$$

$$x_2^{(3)} = \frac{1}{0.1}\ [0.32\ -\ 0.07\ \times\ -6.94\ -\ 0.08\ \times\ -4.81\ -\ 0.07\ \times\ -4.41\,]\ =\ 14.993$$

$$x_3^{(3)} = \frac{1}{0.1}\ [0.33\ -\ 0.06\ \times\ -6.94\ -\ 0.08\ \times\ -4.83\ -\ 0.09\ \times\ -4.41\,]\ =\ 15.297$$

$$x_4^{(3)} = \frac{1}{0.1}\ [0.31\ -\ 0.05\ \times\ -6.94\ -\ 0.07\ \times\ -4.83\ -\ 0.09\ \times\ -4.81\,]\ =\ 14.28$$

c

2.2

a

B: Gauss-Siedel Method

We have $x^{(0)} = (0,0,0,0)^t$ as the initial point, hence we can perform three iterations of the algorithm.

iteration 1:

$x_1^{(1)} = \frac{1}{0.05} \times (0.23 - 0.07 \times 0.0 - 0.06 \times 0.0 - 0.05 \times 0.0) = 4.6$

$x_2^{(1)} = \frac{1}{0.1} \times (0.32 - 0.07 \times 4.6 - 0.08 \times 0.0 - 0.07 \times 0.0) = -0.02$

$x_3^{(1)} = \frac{1}{0.1} \times (0.33 - 0.06 \times 4.6 - 0.08 \times -0.02 - 0.09 \times 0.0) = 0.556$

$x_4^{(1)} = \frac{1}{0.1} \times (0.31 - 0.05 \times 4.6 - 0.07 \times -0.02 - 0.09 \times 0.556) = 0.3136$

iteration 2:

$x_1^{(2)} = \frac{1}{0.05} \times (0.23 - 0.07 \times -0.02 - 0.06 \times 0.556 - 0.05 \times 0.3136) = 3.6472$

$x_2^{(2)} = \frac{1}{0.1} \times (0.32 - 0.07 \times 3.6472 - 0.08 \times 0.556 - 0.07 \times 0.3136) = -0.01736$

$x_3^{(2)} = \frac{1}{0.1} \times (0.33 - 0.06 \times 3.6472 - 0.08 \times -0.01736 - 0.09 \times 0.3136) = 0.84333$

$x_4^{(2)} = \frac{1}{0.1} \times (0.31 - 0.05 \times 3.6472 - 0.07 \times -0.01736 - 0.09 \times 0.84333) = 0.52956$

iteration 3:

$x_1^{(3)} = \frac{1}{0.05} \times (0.23 - 0.07 \times -0.01736 - 0.06 \times 0.84333 - 0.05 \times 0.52956) = 3.08275$

$x_2^{(3)} = \frac{1}{0.1} \times (0.32 - 0.07 \times 3.08275 - 0.08 \times 0.84333 - 0.07 \times 0.52956) = -0.00328$

$x_3^{(3)} = \frac{1}{0.1} \times (0.33 - 0.06 \times 3.08275 - 0.08 \times -0.00328 - 0.09 \times 0.52956) = 0.97637$

$x_4^{(3)} = \frac{1}{0.1} \times (0.31 - 0.05 \times 3.08275 - 0.07 \times -0.00328 - 0.09 \times 0.97637) = 0.68219$

C

2.3

Successive Over-Relaxation Method

We have $x^{(0)} = (0,0,0,0)^t$ as the initial point, with $w = 0.5$, hence we can perform three iterations of the algorithm.

iteration 1 :

$$x_1^{(1)} = (1 - 0.5) \times 0.0 + \left(\frac{0.5}{0.05}\right)[0.23 - 0.07 \times 0.0 - 0.06 \times 0.0 - 0.05 \times 0.0] = 2.3$$

$$x_2^{(1)} = (1 - 0.5) \times 0.0 + \left(\frac{0.5}{0.1}\right)[0.32 - 0.07 \times 2.3 - 0.08 \times 0.0 - 0.07 \times 0.0] = 0.795$$

$$x_3^{(1)} = (1 - 0.5) \times 0.0 + \left(\frac{0.5}{0.1}\right)[0.33 - 0.06 \times 2.3 - 0.08 \times 0.795 - 0.09 \times 0.0] = 0.642$$

$$x_4^{(1)} = (1 - 0.5) \times 0.0 + \left(\frac{0.5}{0.1}\right)[0.31 - 0.05 \times 2.3 - 0.07 \times 0.795 - 0.09 \times 0.642] = 0.40785$$

iteration 2 :

$$x_1^{(2)} = (1 - 0.5) \times 2.3 + \left(\frac{0.5}{0.05}\right)[0.23 - 0.07 \times 0.795 - 0.06 \times 0.642 - 0.05 \times 0.40785] = 2.30438$$

$$x_2^{(2)} = (1 - 0.5) \times 0.795 + \left(\frac{0.5}{0.1}\right)[0.32 - 0.07 \times 2.30438 - 0.08 \times 0.642 - 0.07 \times 0.40785] = 0.79142$$

$$x_3^{(2)} = (1 - 0.5) \times 0.642 + \left(\frac{0.5}{0.1}\right)[0.33 - 0.06 \times 2.30438 - 0.08 \times 0.79142 - 0.09 \times 0.40785] = 0.77959$$

$$x_4^{(2)} = (1 - 0.5) \times 0.40785 + \left(\frac{0.5}{0.1}\right)[0.31 - 0.05 \times 2.30438 - 0.07 \times 0.79142 - 0.09 \times 0.77959] = 0.55002$$

iteration 3 :

$$x_1^{(3)} = (1 - 0.5) \times 2.30438 + \left(\frac{0.5}{0.05}\right)[0.23 - 0.07 \times 0.79142 - 0.06 \times 0.77959 - 0.05 \times 0.55002] = 2.15543$$

$$x_2^{(3)} = (1 - 0.5) \times 0.79142 + \left(\frac{0.5}{0.1}\right)[0.32 - 0.07 \times 2.15543 - 0.08 \times 0.77959 - 0.07 \times 0.55002] = 0.73697$$

$$x_3^{(3)} = (1 - 0.5) \times 0.77959 + \left(\frac{0.5}{0.1}\right)[0.33 - 0.06 \times 2.15543 - 0.08 \times 0.73697 - 0.09 \times 0.55002] = 0.85087$$

$$x_4^{(3)} = (1 - 0.5) \times 0.55002 + \left(\frac{0.5}{0.1}\right)[0.31 - 0.05 \times 2.15543 - 0.07 \times 0.73697 - 0.09 \times 0.85087] = 0.64532$$

2.4

Question 3

3.1

3.2

3.3

Question 4

```
x(1) = 1; y(1) = 1; z(1) = 1; dxyz = [1;1;1]; tol = 10^(-4);

    while (sqrt(dxyz(1)^2+dxyz(2)^2+dxyz(3)^2)>tol)
    jac = [2*x(i),2*y(i),2*z(i);2*x(i),0,3*z(i)^2;2*x(i),2*y(i),-4];
    b = [x(i)^2+y(i)^2+z(i)^2-1;x(i)^2+z(i)^3-0.25;x(i)^2+y(i)^2- 4*x(i)];
    dxyz = -inv(jac)*b;
    x(i+1) = x(i)+dxyz(1);
    y(i+1) = y(i)+dxyz(2);
    z(i+1) = z(i)+dxyz(3);
    end
```