[Question 1]

*(1.1) Determine whether the system*

$\dot{x} + \dot{y} + x + y = 2e^{3t}$

$\dot{x} + \dot{y} - 3x - 3y = -e^{-t}$

*is degenerate. In the degenerate case, decide whether it has no solution or in definitely many solutions. If it has no solution, explain why, else find the general form of the solutions.*

[1] Rewrite the given system of differential equations

$$\begin{cases} \dot{x} + \dot{y} + x + y = 2e^{3t} & [1] \\ \dot{x} + \dot{y} - 3x - 3y = e^{-t} & [2] \end{cases}$$

[2] Combine the Equations

$(\dot{x} + \dot{y} + x + y) - (\dot{x} + \dot{y} - 3x - 3y) = 2e^{3t} + e^{-t}$

$\Rightarrow 4x + 4y = 2e^{3t} + e^{-t}$

$\Rightarrow x + y = \frac{e^{3t}}{2} + \frac{e^{-t}}{4}$

We have:

$\Rightarrow y = \frac{e^{3t}}{2} + \frac{e^{-t}}{4} - x$

&

$\Rightarrow x = \frac{e^{3t}}{2} + \frac{e^{-t}}{4} - y$

[3] substitute x into [2]

$\dot{x} + \dot{y} + x + y = 2e^{3t}$

$\Rightarrow \dot{\left(\frac{e^{3t}}{2} + \frac{e^{-t}}{4} - y\right)} + \dot{y} + \left(\frac{e^{3t}}{2} + \frac{e^{-t}}{4} - y\right) + y = 2e^{3t}$

$\Rightarrow \frac{3e^{3t}}{2} - \frac{e^{-t}}{4} - \dot{y} + \dot{y} + \frac{e^{3t}}{2} + \frac{e^{-t}}{4} - y + y = 2e^{3t}$

$\Rightarrow \frac{3e^{3t}}{2} - \frac{e^{-t}}{4} + \frac{e^{3t}}{2} + \frac{e^{-t}}{4} = 2e^{3t}$

$\Rightarrow \frac{3e^{3t}}{2} + \frac{e^{3t}}{2} += 2e^{3t}$

$\Rightarrow \frac{4e^{3t}}{2} = 2e^{3t}$

$\Rightarrow 2e^{3t} = 2e^{3t}$

*Therefore, equation (1) is consistent with the substitution.*

[4] substitute x into [2]

$\dot{x} + \dot{y} - 3x - 3y = e^{-t}$

$\Rightarrow \left(\frac{e^{3t}}{2} + \frac{\dot{e}^{-t}}{4} - y\right) + \dot{y} - 3\left(\frac{e^{3t}}{2} + \frac{e^{-t}}{4} - y\right) - 3y = e^{-t}$

$\Rightarrow \frac{3e^{3t}}{2} - \frac{e^{-t}}{4} - \dot{y} + \dot{y} - \frac{3e^{3t}}{2} - \frac{3e^{-t}}{4} + 3y - 3y = e^{-t}$

$\Rightarrow \frac{3e^{3t}}{2} - \frac{e^{-t}}{4} - \frac{3e^{3t}}{2} - \frac{3e^{-t}}{4} = e^{-t}$

$\Rightarrow -\frac{e^{-t}}{4} - \frac{3e^{-t}}{4} = e^{-t}$

$\Rightarrow -e^{-t} \neq e^{-t}$

*Therefore, equation (1) is* ***inconsistent*** *with the substitution.*

*Therefore, the system is* ***inconsistent &*** *no values of $x(t)$ and y(t) that can simultaneously satisfy both differential equations.*

*(1.2) Solve the system*
$\dot{x} + x - 2\dot{y} = 2t^2$
$2\dot{x} - x + \dot{y} - 2y = 2t^3$
*by using the elimination method (operator method).*
*Hint. Eliminate y first.*

[1] Rewrite the given system of differential equations

$$\begin{cases} \dot{x} + x - 2\dot{y} = 2t^2 & [1] \\ 2\dot{x} - x + \dot{y} - 2y = 2t^3 & [2] \end{cases}$$

[2] Eliminate y

$$(\dot{x} + x - 2\dot{y} = 2t^2) + (2\dot{x} - x + \dot{y} - 2y) = 2t^2 + 2t^3$$
$$\Rightarrow 3\dot{x} - \dot{y} - 2y = 2t^3 + 2t^2$$

[3] Solve for $\dot{x}$

$$3\dot{x} - \dot{y} - 2y = 2t^3 + 2t^2$$
$$\Rightarrow 3\dot{x} = 2t^3 + 2t^2 + \dot{y} + 2y$$
$$\Rightarrow \dot{x} = \frac{2t^3}{3} + \frac{2t^2}{3} + \frac{\dot{y}}{3} + \frac{2y}{3}$$

[4] Substitute $\dot{x}$ into [2]

$$2\dot{x} - x + \dot{y} - 2y = 2t^3$$
$$\Rightarrow 2\left(\frac{2t^3}{3} + \frac{2t^2}{3} + \frac{\dot{y}}{3} + \frac{2y}{3}\right) - x + \dot{y} - 2y = 2t^3$$
$$\Rightarrow \frac{4t^3}{3} + \frac{4t^2}{3} + \frac{2\dot{y}}{3} + \frac{4y}{3} - x + \dot{y} - 2y = 2t^3$$
$$\Rightarrow 4t^3 + 4t^2 + 2\dot{y} + 4y - 3x + 3\dot{y} - 6y = 6t^3$$
$$\Rightarrow 4t^3 + 4t^2 + 5\dot{y} - 3x - 2y = 6t^3$$
$$\Rightarrow 4t^2 + 5\dot{y} - 3x - 2y = 2t^3$$

[5] Solve for y

$$4t^2 + 5\dot{y} - 3x - 2y = 2t^3$$
$$\Rightarrow 5\dot{y} = 2t^3 - 4t^2 + 3x + 2y$$
$$\Rightarrow \dot{y} = \frac{2t^3}{5} - \frac{4t^2}{5} + \frac{3x}{5} + \frac{2y}{5}$$

[6] Substitute $\dot{y}$ into [1]

$$\dot{x} + x - 2\dot{y} = 2t^2$$
$$\Rightarrow \dot{x} + x - 2\left(\frac{2t^3}{5} - \frac{4t^2}{5} + \frac{2y}{5} + \frac{3x}{5}\right) = 2t^2$$
$$\Rightarrow \dot{x} + x - \frac{4t^3}{5} + \frac{8t^2}{5} - \frac{4y}{5} - \frac{6x}{5} = 2t^2$$
$$\Rightarrow 5\dot{x} - x - 4t^3 + 8t^2 - 4y = 10t^2$$

[7] Solve for x

$5\dot{x} - x - 4t^3 + 8t^2 - 4y = 10t^2$

$\Rightarrow 5\dot{x} = 4t^3 + 10t^2 - 8t^2 + 4y + x$

$\Rightarrow 5\dot{x} = 4t^3 + 10t^2 - 8t^2 + 4y + x$

$\Rightarrow 5\dot{x} \ = 4y + x + 4t^3 + 2t^2$

$\Rightarrow \dot{x} \ = \frac{4y}{5} + \frac{x}{5} + \frac{4t^3}{5} + \frac{2t^2}{5}$

$\dot{y} = \frac{2t^3}{5} - \frac{4t^2}{5} + \frac{3x}{5} + \frac{2y}{5}$

$\dot{x} \ = \frac{4y}{5} + \frac{x}{5} + \frac{4t^3}{5} + \frac{2t^2}{5}$

[Question 2]

*Show that the system:*

$(D-2)[x]+2D[y]=2-4e^{2t}$

$(2D-3)[x]+(3D-1)[y]=0$

*is equivalent to both the following triangular systems:*

$$\begin{cases}(D^2+D-2)[x]=2+20e^{2t}\\ \quad D[x]-2y=12e^{2t}-6\end{cases}$$

*and*

$$\begin{cases}(D^2+D-2)[y]=-6-4e^{2t}\\ \ x-(D+1)[y]=8e^{2t}-4\end{cases}$$

[1] Rewrite the given system of differential equations

$$\begin{cases}(D-2)[x]+2D[y]=2-4e^{2t} & [1]\\ (2D-3)[x]+(3D-1)[y]=0. & [2]\end{cases}$$

[2] Solve for y

$(2D-3)[x]+(3D-1)[y]=0$

$\Rightarrow (3D-1)[y]=-(2D-3)[x]$

$\Rightarrow [y]=-[x]\frac{(2D-3)}{(3D-1)}$

[3] Solve for x

$(2D-3)[x]+(3D-1)[y]=0$

$\Rightarrow (2D-3)[x]=-(3D-1)[y]$

$\Rightarrow [x]=-[y]\frac{(3D-1)}{(2D-3)}$

[4] substitute y into [2]

$$D[x]-2y=12e^{2t}-6$$

$$\Rightarrow D[x]-2\left(-[x]\frac{(2D-3)}{(3D-1)}\right)=12e^{2t}-6$$

$$\Rightarrow D[x]+[x]\frac{2(2D-3)}{(3D-1)}=12e^{2t}-6$$

$$\Rightarrow D[x]+[x]\frac{4D-6}{(3D-1)}=12e^{2t}-6$$

$$\Rightarrow [x]\left(D+\frac{4D-6}{(3D-1)}\right)=12e^{2t}-6$$

$$\Rightarrow [x]\left(\frac{D(3D-1)+4D-6}{(3D-1)}\right)=12e^{2t}-6$$

$$\Rightarrow [x]\left(\frac{3D^2-D+4D-6}{(3D-1)}\right)=12e^{2t}-6$$

$$\Rightarrow [x]\left(\frac{3D^2+3D-6}{(3D-1)}\right)=12e^{2t}-6$$

$$\Rightarrow [x]\,(3D^2+3D-6)=(3D-1)(12e^{2t}-6)$$

$$\Rightarrow [x]\,(3D^2+3D-6)=36De^{2t}-18D-\ 12e^{2t}+6$$

$$\Rightarrow [x]\,(3D^2+3D-6)=72e^{2t}-\ 12e^{2t}+6$$

$$\Rightarrow [x]\,(3D^2+3D-6)=60e^{2t}+6$$

$$\Rightarrow [x]3\,(D^2+D-2)=60e^{2t}+6$$

$$\Rightarrow [x]\,(D^2+D-2)=20e^{2t}+2$$

| |
|---|
| $36De^{2t}$<br>$\Rightarrow D.36e^{2t}$<br>$\Rightarrow \frac{d}{dt}.36e^{2t}$<br>$\Rightarrow 36.\frac{d}{dt}(e^{2t})$<br>$\Rightarrow 36.2e^{2t}$<br>$\Rightarrow 74e^{2t}$ |

| |
|---|
| $-18D$<br>$\Rightarrow D.(-18)$<br>$\Rightarrow \frac{d}{dt}.(-18)$<br>$\Rightarrow 0$ |

Which matches $(D^2+D-2)[x]=2+20e^{2t}$ in the first triangular system.

[Question 3]

*Solve the system:*

$(D^2+1)[x_1]-2D[x_2]=2t$

$(2D-1)[x_1]-(2-D)[x_2]=0$

*by using the elimination method (operator method).*

[1] Rewrite the given system of differential equations

$$\begin{cases}(D^2+1)[x_1]-2D[x_2]=2t & [1]\\ (2D-1)[x_1]-(2-D)[x_2]=0 & [2]\end{cases}$$

[2] Solve for $x_1$

$(2D-1)[x_1]-(2-D)[x_2]=0$

$\Rightarrow -(2-D)[x_2]=-(2D-1)[x_1]$

$\Rightarrow [x_1]=\frac{(2-D)[x_2]}{(2D-1)}$

[3] Solve for $x_2$

$(2D-1)[x_1]-(2-D)[x_2]=0$

$\Rightarrow -(2-D)[x_2]=-(2D-1)[x_1]$

$\Rightarrow [x_2]=\frac{(2D-1)[x_1]}{(2-D)}$

[4] substitute $x_2$ into [1]

$(D^2+1)[x_1]-2D[x_2]=2t$

$\Rightarrow (D^2+1)[x_1]-2D\left(\frac{(2D-1)[x_1]}{(2-D)}\right)=2t$

$\Rightarrow (D^2+1)[x_1]-2D\left(\frac{(2D-1)[x_1]}{(2-D)}\right)=2t$

$\Rightarrow (D^2+1)[x_1]-\left(\frac{2D(2D-1)[x_1]}{(2-D)}\right)=2t$

$\Rightarrow (D^2+1)[x_1]-\left(\frac{(4D^2-2D)}{(2-D)}\right)[x_1]=2t$

$\Rightarrow [x_1]\left(D^2+1-\frac{(4D^2-2D)}{(2-D)}\right)=2t$

$\Rightarrow [x_1]\left(D^2+1-\frac{(4D^2-2D)}{(2-D)}\right)=2t$

$\Rightarrow [x_1]\big(D^2(2-D)+1(2-D)-(4D^2-2D)\big)=2t\,(2-D)$

$\Rightarrow [x_1](2D^2-D^3+2-D-4D^2+2D)=4t-2Dt$

$\Rightarrow [x_1](-D^3-2D^2+D+\ 2)=4t-2Dt$

$\Rightarrow [x_1](D^3+2D^2-D-2)=-4t+2Dt$

$\Rightarrow [x_1](D^3+2D^2-D-2)=-4t$

$2Dt$

$\Rightarrow D.(2t)$

$\Rightarrow \frac{d}{dt}.(2t)$

$\Rightarrow 0$

[5] solve for $[x_1]$

Find complementary function (homogeneous solution)

$[x_1](D^3 + 2D^2 - D - 2) = -4t$

$\Rightarrow m^3 + 2m^2 - m - 2 = 0$ *(characteristic polynomial)*

$\Rightarrow (m - 1)(m + 1)(m + 2) = 0$

$\Rightarrow m = 1 \quad or \ m = -1 \quad or \ m = -2$

complimentary function using roots:

$x_1C.F.(t) = c_1e^{-2t} + c_2e^{-t} + c_3e^{t}$

where c1, c2 and c3 are arbitrary constants

[6] solve for $[x_1]$

Find particular function (particular solution)

Assume $x_1P(t) = At + B$

$[x_1](D^3 + 2D^2 - D - 2) = -4t$

$\Rightarrow (At + B)(D^3 + 2D^2 - D - 2) = -4t$

$\Rightarrow D^3(At + B) + 2D^2(At + B) - D(At + B) - 2(At + B) = -4t$

$\Rightarrow -A - 2(At + B) = -4t$

$\Rightarrow -A - 2B - 2At + 4t = 0$

So,

$\Rightarrow 2At + 4t = 0$

$\Rightarrow 2At = -4t$

$\Rightarrow 2A = -4$

$\Rightarrow A = -2$

Then,

$\Rightarrow -A - 2B = 0$

$\Rightarrow 2B = -A$

$\Rightarrow B = -\frac{-2}{2}$

$\Rightarrow B = 1$

Thus, particular solution is

$x_1P(t) = -2t + B$

Therefore, the general solution for $x_1$ is:

$x_1 = \ homogeneous\ solution + particular\ solution$

$\Rightarrow x_1 = x_1C.F.(t) + x_1P(t)$

$\Rightarrow x_1 = c_1e^{-2t} + c_2e^{-t} + c_3e^{t} - 2t + B$

[6] solve for $[x_2]$

$(D^2+1)[x_1]-2D[x_2]=2t$
$\Rightarrow -2D[x_2]=2t-(D^2+1)[x_1]$
$\Rightarrow D[x_2]=\frac{1}{2}(D^2+1)[x_1]-\frac{2t}{2}$
$\Rightarrow D[x_2]=\frac{1}{2}(D^2+1)[x_1]-t$

Substitute [1] into $[x_1](t)$
$D[x_2]=\frac{1}{2}(D^2+1)[x_1]-t$
$\Rightarrow D[x_2]=\frac{1}{2}(D^2+1)(c_1e^{-2t} + c_2e^{-t} + c_3e^{t} - 2t + B) - t$
$\Rightarrow D[x_2]\frac{5}{2}c_1e^{-2t} + c_2e^{-t} + c_3e^{t} - t$

Thus,
$x_1(t) = c_1e^{-2t} + c_2e^{-t} + c_3e^{t} - 2t + B$
ands
$x_2(t) = -\frac{5}{4}c_1e^{-2t} + c_2e^{-t} + c_3e^{t} - t$

[Question 4]

*(4.1) Use the eigenvalue-eigenvectors to solve the initial value problem*

$$\dot{X} = \begin{bmatrix} 0 & -1 \\ 9 & 0 \end{bmatrix} X \quad , \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} X$$

[1] find *eigenvalues*

$$A = \begin{pmatrix} 0 & -1 \\ 9 & 0 \end{pmatrix}$$

$$\Rightarrow detA = \begin{pmatrix} 0 - \lambda & -1 \\ 9 & 0 - \lambda \end{pmatrix}$$

$$\Rightarrow \lambda^2 + 9 = 0$$

$$\Rightarrow (\lambda + 3)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 3i \ \ or \ \ \lambda = -3i$$

[2] find *eigenvectors for* $\lambda = 3i$

$$detA = \begin{pmatrix} 0 - \lambda & -1 \\ 9 & 0 - \lambda \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 - 3i & -1 \\ 9 & 0 - 3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -3i & -1 \\ 9 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R1$

$$-3i \, . v_1 - v_2 = 0$$

$$\Rightarrow v_2 = -3i \, . v_1$$

*Thus,*

$$v_1 = \begin{pmatrix} 1 \\ -3i \end{pmatrix}$$

[3] find $eigenvectors\ for\ \lambda = -3i$

$$detA = \begin{pmatrix} 0 - \lambda & -1 \\ 9 & 0 - \lambda \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 + 3i & -1 \\ 9 & 0 + 3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3i & -1 \\ 9 & 3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R1$

$$3i\,.v_1 - v_2 = 0$$

$$\Rightarrow v_2 = 3i\,.v_1$$

*Thus,*

$$v_2 = \begin{pmatrix} 1 \\ 3i \end{pmatrix}$$

[4] general solution

function using roots:
$\Rightarrow \lambda = 3i\ \ or\ \lambda = -3i$
Thus,
$X(t) = c_1 e^{3it} v_1 + c_2 e^{-3it} v_2$
where c1 and c2 are arbitrary constants

Substitute eigenvalues and eigenvectors:

$$X(t) = c_1 e^{3it} v_1 + c_2 e^{-3it} v_2$$

$$\Rightarrow X(t) = c_1 e^{3it} \begin{pmatrix} 1 \\ -3i \end{pmatrix} + c_2 e^{-3it} \begin{pmatrix} 1 \\ 3i \end{pmatrix}$$

[5] use Initial conditions

$$X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} X$$

$$X(0) = c_1 \begin{pmatrix} 1 \\ -3i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

*Thus, we have*

$$c_1 + c_2 = 1$$

$$\Rightarrow -3ic_1 + 3ic_2 = 1$$

And also,

$$\Rightarrow c_2 - c_1 = -\frac{i}{3}$$

[6] find $c_1$ & $c_2$

Rewrite the given system of differential equations

$$\begin{cases} c_1 + c_2 = 1 & [1] \\ c_2 - c_1 = -\frac{i}{3} & [2] \end{cases}$$

Add [1] + [2]

$$(c_1 + c_2 - 1) + (c_2 - c_1 + \tfrac{i}{3}) = 0$$

$$\Rightarrow c_1 + c_2 - 1 + c_2 - c_1 + \frac{i}{3} = 0$$

$$\Rightarrow 2c_2 - 1 + \frac{i}{3} = 0$$

$$\Rightarrow 2c_2 = 1 - \frac{i}{3}$$

$$\Rightarrow c_2 = \frac{1}{2} - \frac{i}{6}$$

Substitute $c_2$ into [1]

$$c_1 + c_2 = 1$$

$$\Rightarrow c_1 + \frac{1}{2} - \frac{i}{6} = 1$$

$$\Rightarrow c_1 = \frac{1}{2} + \frac{i}{6}$$

[7] general solution

Substitute $c_1$ & $c_2$ into general solution

$$X(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -3i \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 3i \end{pmatrix}$$

$$\Rightarrow X(t) = \left(\frac{1}{2} + \frac{i}{6}\right) e^{3it} \begin{pmatrix} 1 \\ -3i \end{pmatrix} + \left(\frac{1}{2} - \frac{i}{6}\right) e^{-3it} \begin{pmatrix} 1 \\ 3i \end{pmatrix}$$

Thus,

$$\Rightarrow X(t) = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{6}\right) e^{3it} + \left(\frac{1}{2} - \frac{i}{6}\right) e^{-3it} \\ -3\left(\frac{1}{2} + \frac{i}{6}\right) e^{3it} + 3\left(\frac{1}{2} - \frac{i}{6}\right) e^{-3it} \end{bmatrix}$$

*(4.2) Use the eigenvalue-eigenvectors to solve the initial value problem*

$$\dot{X} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 2 \end{bmatrix} X \quad , \quad X(0) = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} X$$

[1] find *eigenvalues*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 2 \end{pmatrix}$$

$$\Rightarrow detA = \begin{pmatrix} 2-\lambda & 0 & 0 \\ -1 & 1-\lambda & 0 \\ -2 & 3 & 2-\lambda \end{pmatrix}$$

*upper triangular matrix, thus*

$$(2-\lambda)\begin{bmatrix} 1-\lambda & 0 \\ 3 & 2-\lambda \end{bmatrix}$$

$$\Rightarrow (2-\lambda)(1-\lambda)(2-\lambda)$$

$$\Rightarrow \lambda = 2 \;\; or \;\; \lambda = 1$$

[2] find *eigenvectors for* $\lambda = 2$

$$detA = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R1$

$$0\,.v_1 + 0.v_2 + 0.v_3 = 0$$

$$\Rightarrow 0 = 0$$

$R2$

$$-v_1 - v_2 += 0$$

$$\Rightarrow \; v_1 = -sv_2$$

$R3$

$$-2v_1 - 3v_2 = 0$$

$$\Rightarrow \; v_2 = \frac{2}{3}v_1$$

*Thus,*

$$v_1 = v_2 = 0$$

*Therefore,*

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

[3] find *eigenvectors for* $\lambda = 1$

$$detA = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & 3 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R1$

$$v_1 = 0$$

$R2$

$$-v_1 = 0$$

$R3$

$$-2v_1 - 3v_2 + v_3 = 0$$

$$\Rightarrow\ -2v_1 - 3v_2 + v_3 = 0$$

*Thus,*

$$v_1 = v_2 = 0$$

*Therefore,*

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

[4] general solution

function using roots:

$\Rightarrow \lambda = 2 \ \ or \ \lambda = 1$

Thus,

$X(t) = c_1 e^{2t} v_1 + c_2 e^{2t} v_2 + c_3 e^t v_3$

where c1, c2 and c3 are arbitrary constants

Substitute eigenvalues and eigenvectors:

$$X(t) = c_1 e^{2t} v_1 + c_2 e^{2t} v_2 + c_3 e^t v_3$$

$$\Rightarrow X(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

$$\Rightarrow X(t) = \begin{pmatrix} 1 \,.\, c_1 e^{2t} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \,.\, c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \,.\, c_3 e^t \\ -3 \,.\, c_3 e^t \end{pmatrix}$$

$$\Rightarrow X(t) = \begin{pmatrix} c_1 e^{2t} \\ c_3 e^t \\ c_2 e^{2t} - c_3 e^t \end{pmatrix}$$

[5] use Initial conditions to find $c_1$, $c_2$ & $c_3$

$$X(0) = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} X$$

$$\Rightarrow X(0) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

*Thus, we have*

$c_1 = 1$

$c_3 = 2$

$c_2 - 3c_3 = -2$

And also,

$c_2 - 3c_3 = -2$

$\Rightarrow c_2 - 3(2) = -2$

$\Rightarrow c_2 = 4$

[7] general solution

Rewrite the given system of differential equations

$$\begin{cases} c_1 = 1 & [1] \\ c_2 = 4 & [2] \\ c_3 = 2 & [3] \end{cases}$$

Substitute $c_1$ , $c_2$ & $c_3$ into general solution

$$\Rightarrow X(t) = \begin{pmatrix} e^{2t} \\ 2e^{t} \\ 4e^{2t} - 6e^{t} \end{pmatrix}$$

[Question 5]

[Question 6]

Solve the system:

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} X \quad , \; X(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

*Why is there a unique solution to the above system?*

[1] find *eigenvalues*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

$$\Rightarrow detA = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 2 & 1 & -2-\lambda \end{pmatrix}$$

*thus*

$$(-\lambda) \begin{bmatrix} -\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

$$\Rightarrow (-\lambda)(-2-\lambda) - 1 = 0$$

$$\Rightarrow (-\lambda)(\lambda^2 + 2\lambda - 1) = 0$$

$$\Rightarrow \lambda^2 + 2\lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{-2+\sqrt{4+4}}{2} \quad or\ \lambda = \frac{-2-\sqrt{4+4}}{2}$$

$$\Rightarrow \lambda = \frac{-2+\sqrt{8}}{2} \quad or\ \lambda = \frac{-2-\sqrt{8}}{2}$$

$$\Rightarrow \lambda = \frac{-2+2\sqrt{2}}{2} \quad or\ \lambda = \frac{-2+2\sqrt{2}}{2}$$

$$\Rightarrow \lambda = -1+\sqrt{2} \quad or\ \lambda = -1-\sqrt{2}$$

[3] find *eigenvectors*

[Question 7]