[Problem 5]

*Determine whether each set equipped with the given operation is a vector space. For those that are not vector space identify the vector space axioms that fail.*

In order for some set V to be a vector space the following 10 axioms must be true:

|  |  |  |
| --- | --- | --- |
| **A1** | **Closure under addition**  *For any vectors and in the set, is also in the set* |  |
| **A2** | **Existence of an additive identity**  *There exists a vector in the set such that for any vector in the set, .*  *Related:*  *- A3:Existence of additive inverses* |  |
| **A3** | **Existence of additive inverses**  *For every vector in the set, there exists a vector in the set such that* |  |
| **A4** | **Associativity of addition**  *For any vectors , , and in the set,*  *fundamental property of addition* |  |
| **A5** | **Commutativity of addition**  *For any vectors and* *in the set,*  *fundamental property of addition* |  |
| **M1** | **Closure under scalar multiplication**  *For any scalar and any vector in the set, is also in the set.*  *Implied by:*  *- A1:**Closure under addition* |  |
| **M2** | **Distributive property - vector addition**  *For any scalar and any vectors and in the set, +cv.*  *Implied by:*  *- A1:**Closure under addition*  *- A4:* *Associativity of addition*  *- A5:**Commutativity of addition* |  |
| **M3** | **Distributive Property - scalar addition**  *For any scalars and and any vector in the set, +cv.*  *Implied by:*  *- A1:**Closure under addition*  *- A4:* *Associativity of addition*  *- A5:**Commutativity of addition* |  |
| **M4** | **Associative Property**  (Compatibility of scalar multiplication with field multiplication)  *For any scalars and and any vector in the set, +cv.*  *Implied by:*  *- A1:**Closure under addition*  *- A4:* *Associativity of addition*  *- A5:**Commutativity of addition* |  |
| **M5** | **Multiplicative identity**  *For any vector in the set, , where 1 is the multiplicative identity of the underlying field*.  *Implied by:*  *- A1:**Closure under addition*  *- A4:* *Associativity of addition*  *- A5:**Commutativity of addition* |  |

Axioms M1 to M5 (axioms of scalar multiplication)

are usually implied by A1-A5 (axioms of addition).

*(1) The set with the standard operations on*

|  |  |
| --- | --- |
| A1 | Let .  Then      Therefore A1 holds. |
| A2 | Let  the zero vector in be  Then      Therefore A2 holds. |
| A3 | Let  Then      Therefore A3 holds. |
| A4 | Addition in follows the same rules as addition in , so associativity holds.  OR  Let  Then |
| A5 | Addition in follows the same rules as addition in , so commutativity holds. |

All 5 fundamental axioms for vector spaces are satisfied,

*with the standard operations on*  is a vector space.

Axioms M1 to M5 (axioms of scalar multiplication)

are usually implied by A1-A5 (axioms of addition).

*(2) The set with the standard operations on*

|  |  |
| --- | --- |
| A1 | Let  be arbitrary vectors in  where and are real numbers  Then      Therefore A1 holds. |
| A2 | Let  be an arbitrary vector in  where is a real number  Let the zero vector in be  But if (as is the case for the zero vector)  then is not satisfied,  violating the definition of  Then  *V* does not have an additive identity,  and thus, it is not a vector space. |
| A3 | Let  be arbitrary vectors in  But if  then is not satisfied,  violating the definition of  Then  *V* does not have additive inverses,  and thus, it is not a vector space. |
| A4 | Addition in follows the same rules as addition in , so associativity holds.  OR  Let  be arbitrary vectors in  where , and are real numbers  Then            Therefore A4 holds. |
| A5 | Addition in follows the same rules as addition in , so commutativity holds. |

*fails to satisfy A2* Existence of an additive identity

*fails to satisfy A3* Existence of additive inverses

Thus is not a vector space

Axioms M1 to M5 (axioms of scalar multiplication)

are usually implied by A1-A5 (axioms of addition).

*(3) The set with the standard operations on*

|  |  |
| --- | --- |
| A1 | Let  be arbitrary vectors in  where and are real numbers    Then      Therefore A1 holds. |
| A2 | Let  be arbitrary vectors in  where is a real number  Let the zero vector in be  Then      Therefore A2 holds. |
| A3 | Let  be arbitrary vectors in  where is a real number  Then      Therefore A3 holds. |
| A4 | Addition in follows the same rules as addition in , so associativity holds.  OR  Let  be arbitrary vectors in  where , and are real numbers  Then            Therefore A4 holds. |
| A5 | Addition in follows the same rules as addition in , so commutativity holds. |

All 5 fundamental axioms for vector spaces are satisfied,

*with the standard operations on*  is a vector space.

*(4) The set with the standard vector addition*

*but with scalar multiplication defined by .*

|  |  |
| --- | --- |
| A1 | Let  be arbitrary vectors in  where , , and are real numbers  Then      Therefore A1 holds. |
| A2 | Let  be an arbitrary vector in  where and are real numbers  the zero vector in be  Then      Therefore A2 holds. |
| A3 | Let  be an arbitrary vector in  where and y are real numbers  The additive inverse of is  *C*    *the given scalar multiplication c*  *c*    *the given scalar multiplication c*    Therefore A3 fails. |
| A4 | Addition in follows the same rules as addition in , so associativity holds.  OR  Let  be arbitrary vectors in  where , , , , and are real numbers  Then            Therefore A4 holds. |
| A5 | Addition in follows the same rules as addition in , so commutativity holds. |

|  |  |
| --- | --- |
| **M1** | Let  be an arbitrary scalar in    Let  be an arbitrary vector in  where and are real numbers  Then        *the given scalar multiplication c.*    Therefore M1holds. |
| **M2** | Let  be an arbitrary scalar in    Let  be arbitrary vectors in  where , , and are real numbers  Then  Since , through **Closure under addition**  Then    LHS      *the given scalar multiplication c.*    RHS    *the given scalar multiplication c.*    *And LHS = RHS or*  Therefore M2holds. |
| **M3** | Let  be arbitrary scalars in  Let  be an arbitrary vector in  Then  LHS    RHS    *the given scalar multiplication c.*    Therefore M3holds. |
| **M4** | Let  be arbitrary scalars in      be an arbitrary vector in  Then  LHS      *the given scalar multiplication c.*  RHS    *the given scalar multiplication c.*        *And LHS = RHS or*  Therefore M4holds. |
| **M5** | Let  Then        Therefore M5holds. |

*fails to satisfy A3* Existence of additive inverses

Thus is not a vector space

*(5) The set 2x2 matrices with the standard matrix addition and scalar multiplication.*

|  |  |
| --- | --- |
| A1 | Let  be arbitrary matrices in  where , , , , , are real numbers  Then    Therefore A1 holds. |
| A2 | Let  be an arbitrary matrix in  where , , are real numbers  Let the zero matrix in be  Then      Therefore A2 holds. |
| A3 | Let  be an arbitrary matrix in  where , , are real numbers  The additive inverse of is  Therefore A3 holds. |
| A4 | Addition in follows the same rules as addition of 2x2 matrices, so associativity holds. |
| A5 | Addition in follows the same rules as addition of 2x2 matrices, so commutativity holds. |

|  |  |
| --- | --- |
| **M1** | Let  be an arbitrary scalar in    Let  be an arbitrary matrix in  where , , are real numbers  Then      Therefore M1holds. |
| **M2** | Let  be an arbitrary scalars in    Let  be arbitrary matrices in  where , , , , , are real numbers  Then  LHS      RHS        *And LHS = RHS or*  Therefore M2holds. |
| **M3** | Let l  be arbitrary scalars in  Let  be an arbitrary matrix in  where , , are real numbers  Then  LHS        RHS        *And LHS = RHS or*  Therefore M3holds. |
| **M4** | Let  be arbitrary scalars in    Let  be an arbitrary matrix in  where , , are real numbers  Then  LHS      RHS        *And LHS = RHS or*  Therefore M4holds. |
| **M5** | Let  be an arbitrary matrix in  where , , are real numbers  Then      Therefore M5holds. |

All 10 axioms for vector spaces are satisfied,

*The set 2x2 matrices* with the standard matrix addition and scalar multiplication

is a vector space.

[Problem 6]

*Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on and ) :*

*(1) Compute and for , and*

*(2) Determine whether the Axioms 7, 8, 9 and 10 hold.*

|  |  |
| --- | --- |
| **M2** | IfThen  LHS        RHS        *LHS RHS or*  Therefore M2fails |
| **M3** | Let  be anarbitrary vector  where and are arbitrary scalars  Then  LHS    *the given scalar multiplication*    RHS    *the given scalar multiplication*    *And LHS = RHS or*  Therefore M3holds. |
| **M4** | Let  be anarbitrary vector  Let and be arbitrary scalars  Then  LHS    *the given scalar multiplication*  RHS    *the given scalar multiplication*      *And LHS = RHS or*  Therefore M4holds. |
| **M5** | Let  be anarbitrary vector  Then    *the given scalar multiplication*    Therefore M5holds. |

*Axiom 7 (M2) Distributive property of vector addition over scalar addition*

*fails Thus is not a vector space*

[Problem 7]

*Let be a vector space,*

*a vector in ;*

*and a scalar.*

*Then show that if , then or*

***Proof by cases:***

Case 1:

If

Then

Thus

Case 2:

If

Then

Thus

Thus If Then

Therefore *if ,* thenor

***Proof by contrapositive:***

*: if ,* then *or*

*: if* and

Assume *negation of the conclusion of*

Then and

if *,* then

Thus

Thusimplies

[Problem 8]

*Let and denote two distinct objects, neither of which is in .*

*Define an addition and scalar multiplication on*

*Specifically, the sum and product of two real numbers is as usual,*

*and for define:*

*1.*

*2.*

*3.*

*4.*

*5.*

*Show that is not a vector space over .*

Given

Implied

|  |  |
| --- | --- |
| A1 | Let and be arbitrary vectors  Then      Given  But  Therefore A1 fails. |
| A2 | Let be the additive identity  Let be an arbitrary vector , where  Then    Given *1*  But , and  Therefore A2 fails. |
| A3 | Let be the additive identity to  Then    Given *5.*  But , and  Therefore A3 fails. |

Axioms of addition *A1, A2 & A3 all faill*

*Thus is not a vector space over .*