[Question 1]

|  |  |  |  |
| --- | --- | --- | --- |
| Statement |  |  | If , then |
| Converse |  |  | If , then |
| Inverse |  |  | If , then |
| Contrapositive |  |  | If , then |
| Negation |  |  | It is not true that  If , then |

1.1 *Give the contrapositive of the statement.*

|  |  |  |  |
| --- | --- | --- | --- |
| Contrapositive |  |  | If , then |

1.2 *Give the converse of the statement, and determine whether the converse is true or not*

|  |  |  |  |
| --- | --- | --- | --- |
| Converse |  |  | If , then |

Let

Substitute

LHS

which is true

RHS

which is not true

Thus, the converse statement is not true.

1.3 *What is the negation of the statement?*

|  |  |  |  |
| --- | --- | --- | --- |
| Negation |  |  | It is not true that  If , then |

[Question 2]

2.1 *Prove by induction that is a multiple of 3 for all .*

**Clause**

Let be the statement:

is a multiple of 3 for all

**Basis Clause**

Let

:

, which is a multiple of 3

Therefore, basis clause is true

**Inductive Clause**

Show that

is where

Assume

, which is a multiple of 3

**Extremal Clause (Inductive Step)**

If is true

then must also be true

Assume

is a multiple of 3

Use **Inductive Clause**  is a multiple of 3

, which is a multiple of 3

2.2 *Prove that the function given by*

*is 1 - 1 on*

*is one-to-one (injective) if*

*for all , in the domain of ,*

*then .*

*OR*

*for all , in the domain of ,*

*if then.*

**Direct Proof**

*Let ​and ​ be two arbitrary real numbers such that*

Assume:

*LHS*

*RHS*

*However, this contradicts ,*

*proving that is 1 – 1.*

*and  find a formula for the inverse of*

*Let*

Inverse

*Thus,*

*2.3 Consider the functions and .*

*Indicate the domain of definition of each of the following functions: ; ; ; .*

|  |  |
| --- | --- |
|  | Domain |
|  | Domain |
|  | Domain |
|  | Domain |

[Question 3]

*2.1 Let*

*Find the infimum and supremum of .*

Hence, ,

And in the sequence, for any ​ is always a positive fraction between 0 and 1.

***supremum of (least Upper Bound):***

for any ​, approaches but never reaches 1

claim is that the supremum of is 1

***infimum of (Greatest Lower Bound):***

*for all*

for any ​, approaches but never reaches 1

Hence *infimum of* cannot be greater than 0

claim that the supremum of is 0

[Question 4]

*(4.1) Show that the sequence*

*converges to 1.*

Definition: Convergent Sequences

, , such that implies

|  |  |
| --- | --- |
|  | “for all positive numbers” |
|  | "there exists some in the set of natural numbers," |
| such that implies | if is greater than , then the absolute difference between and the limit (denoted as ) is less than |

**Given *the sequence ,***

**The limit of as approaches infinity is:**

**For every**

**There exists is a positive integer**

**such that implies**

For :

And if ,

**To verify for**

We round up to the nearest integer

and add 1 to ensure that is at least as large as

Thus satisfying

Let :

verify for

Since , we have ,

which means

*So, for*

Which holds true for every including

Thus the sequence ***,*** ​ converges to 1 as approaches infinity.

*(4.2) Suppose that is a sequence of real numbers that converges to 1 as . Prove that the converges to 2 as .*

Definition: Limits

, , such that implies

Let *be a sequence of real numbers that converges to 1 as .*

Let be

*Thus*

Let be *, which is the limit of the constant sequence 1.*

*Thus*

*Since and , we have:*

*Therefore converges to 2 as .*

*(4.3) Find two convergent subsequences of the sequence that have different limits.*

Definition: Convergent Sequences

, , such that implies

*[1] Let be the sequence of even integers, where:*

*Thus*

*Which is the sequence that converges to 1*

[2] Given *the sequence ,*

The limit of as approaches infinity is:

[3] For every

There exists is a positive integer

such that implies

For ,

Thus,

*The sequence converges to 1 as approaches infinity,*

*where*

*[1] Let be the sequence of odd integers, where:*

*Thus*

*Which is the sequence that converges to -1*

[2] Given *the sequence ,*

The limit of as approaches infinity is:

[3] For every

There exists is a positive integer

such that implies

For ,

Thus,

*The sequence converges to -1 as approaches infinity,*

*where*

*(4.4) Use the Monotone Convergence Theorem to prove that*

*converges to 0.*

**Definition: Monotone Convergence Theorem (MCT):**

If *( ​)* is non-**decreasing** and bounded **above**,

then there exists a real number such that:

or

If *( ​)* is non-**increasing** and bounded **below**,

then there exists a real number such that:

**Bounded Below:**

A sequence *( ​)* or a set of real numbers is said to be bounded below

if there exists a real number

such that *​*for all in the sequence

or for all in the set.

*There is a lower bound such that all elements of the sequence or set are greater than or equal to*

**Bounded Above:**

A sequence *( ​)* or a set of real numbers is said to be bounded above

if there exists a real number

such that *​*for all in the sequence

or for all in the set.

*There is a lower bound such that all elements of the sequence or set are less than or equal to*

*[1] For all n, we* need to prove that

Let *be a sequence of real numbers that converges to 0 as .*

*Given,*

*Then,*

*Thus*

*Therefore, is decreasing sequence*

*[2] We* need to prove that:

*For all n,* there exists a real number such that

*or*

*For all x,* there exists a real number such that

*We have that* ,

*A*nd that

Thus is bounded below by 0.

Because:

*[1] is decreasing sequence*

*[2]*  is bounded below by 0

By the definition of the Monotone Convergence Theorem (MCT),

converges

Thus,