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Controllability Analysis and Optimal Controller Synthesis of Mixed Traffic Systems

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Abstract— **Connected and automated vehicles (CAVs) have a great potential to actively influence traffic systems. This has been demonstrated by large-scale numerical simulations and small-scale real experiments, whereas a comprehensive theoretical analysis is still lacking. In this paper, we focus on mixed traffic systems with one single CAV and heterogeneous human-driven vehicles, and present rigorous controllability analysis and optimal controller synthesis. Using the PBH controllability criterion, we reveal controllability properties of a linearized mixed traffic system in a ring road. It is proved that the mixed traffic flow can be stabilized by one single CAV under a very mild condition. We formulate the problem of designing CAV control strategies under a pre-specified communication topology as structured optimal controller synthesis. This formulation considers a system-level performance index that allows the CAV to actively dampen undesired perturbations in traffic flow. Numerical experiments verify the effectiveness of our results.**

I. Introduction

The emergence of connected and automated vehicles (CAVs) is expected to impact current transportation systems significantly. Compared to human-driven vehicles (HDVs), the cooperative formation of multiple CAVs, *e.g.*, adaptive cruise control (ACC) and cooperative adaptive cruise control (CACC) [1]–[3], has shown very promising effects to achieve higher traffic efficiency [4], better driving safety [5] and lower fuel consumption [6]. These technologies typically require that all the involved vehicles have autonomous capabilities.

In practice, however, as the gradual deployment of CAVs, there will have to be a transition phase of mixed traffic systems consisting of both CAVs and HDVs. Due to the interactions between neighboring vehicles, it is possible to use a few CAVs as mobile actuators to influence the motion of their surrounding vehicles, which may in turn control the global traffic flow. This notion is known as the *Lagrangian control* of traffic flow [7]. In this aspect, the first theoretical analysis in [8] and the pioneer real-world experiment in [7] both validated the potential of one single CAV to stabilize traffic flow. In this paper, we continue this direction of controlling traffic flow via CAVs and present a rigorous controllability analysis of mixed traffic systems that consist of one single CAV and multiple heterogeneous HDVs.

Understanding the dynamics of mixed traffic systems is essential to reveal the full potential of CAVs. Most previous studies are based on numerical experiments, which have demonstrated some positive effects of CAVs on improving traffic stability, capacity and throughput; see, *e.g.*, [9]–[11]. Only a few results are available with respect to theoretical analysis of mixed traffic systems that reveal fundamental properties, such as controllability and stability. For example, string stability and linear stability were examined in [4] and [12] respectively, indicating possible relationship between traffic stability and penetration rates of CAVs. More recently, theoretical controllability analysis was first carried out by Cui *et al.* [8] and then extended by Zheng *et al.* [13], where it is proved mathematically that the mixed traffic flow in a ring-road scenario can be stabilized by controlling a single CAV. This stabilizability result reveals a fundamental ability of a single CAV in smoothing traffic flow. However, the theoretical results in [8], [13] are only applicable to uniform traffic flow due to the homogeneous assumption for HDVs. In real traffic, it is necessary to consider the heterogeneous case where various types of drivers and vehicles coexist, but this also makes the controllability analysis more challenging, since the eigenvector calculation method in [8] and the block diagonalization strategy in [13] are not directly applicable.

Besides analyzing the dynamics of mixed traffic systems, several new control methods have been recently proposed for CAVs [7], [13]–[18]. One common feature of these methods is that the dynamics of other HDVs are explicitly considered in the system model. For example, a notion of connected cruise control exploits both motion and dynamics information from multiple HDVs ahead to make control decisions for the CAV [14]. In a similar setup, various topics have been addressed recently [15], [16], but the role of communication topologies in mixed traffic flow has not been addressed explicitly. It has been already shown that different communication topologies have a big influence on the performance of platoon formation; see, *e.g.*, [19], [20]. In addition, the methods in [14]–[16] typically take local traffic performance around the CAV into account for controller design. Here, we highlight a crucial transformation towards the control goal, from a local-level to a system-level. Specifically, the objectives of CAV control can involve achieving certain desired performance of the entire traffic system, instead of limiting to the CAV's behavior only. This idea is indeed supported by previous research on dampening

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traffic waves via CAVs [7], [13], [17], [18]. However, a theoretical framework is still lacking to derive system-level optimal control strategies of CAVs.

In this paper, we focus on controllability analysis and controller synthesis of a mixed traffic system in a ring road. We first use linearized microscopic car-following models to describe the mixed traffic flow as a standard linear system. Then, the PBH controllability criterion [21] is applied to address the controllability of this heterogeneous mixed traffic system. The controller synthesis problem is formulated as designing a structured optimal controller for the CAV, which allows to achieve a certain optimal performance of the entire traffic system. Precisely, our contributions are as follows.

- We prove that a mixed traffic system with multiple heterogeneous HDVs and a single CAV is not completely controllable, but is stabilizable under a mild condition. This result confirms the feasibility of traffic control via CAVs and validates the empirical observations in [7] that a single CAV is able to dampen traffic waves and stabilize traffic flow. In the case where HDVs have homogeneous dynamics, our result is consistent with [13] and generalizes the statements in [8].
- We formulate the problem of designing a control strategy for the CAV as structured optimal controller synthesis, with an explicit consideration of communication topologies. A significant distinction between our formulation and existing methods, such as CACC [3] and CCC [14], is that the CAV control directly aims at improving the performance of the entire traffic flow, indicating a system-level control objective. Recent advance in convex relaxation and sparsity invariance [22] is leveraged to compute a specific control strategy.

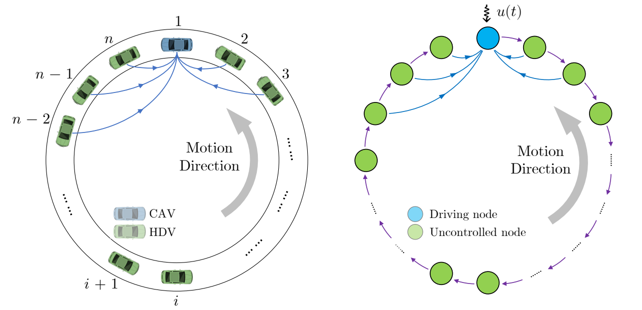
The remainder of this paper is organized as follows. Section II introduces the modeling for a mixed traffic system and the problem statement. Section III presents the controllability theorem, and a theoretical framework to obtain a system-level optimal controller is discussed in Section IV. Numerical simulations are presented in Section V. Finally, Section VI concludes this paper.

II. System Modeling and Problem Statement

In this section, we introduce the modeling of a mixed traffic system in a ring road scenario and present the problem statement.

A. Modeling the Mixed Traffic System

As shown in Fig.1, we consider a single-lane ring road with circumference L consisting of one CAV and $n-1$ HDVs. The ring road setup has been widely used since it is able to capture the dynamics in an infinite straight road with periodic traffics; see, *e.g.*, [7], [8], [13], [18], [23]. We denote the position, velocity and acceleration of vehicle i as p_i, v_i and a_i respectively. The spacing of vehicle i, *i.e.*, its relative distance from vehicle $i-1$, is defined as $s_i = p_{i-1} - p_i$. Without loss of generality, we assume that vehicle no.1 is the CAV and that the vehicle length is ignored.



(a) (b)

Fig. 1. System modeling schematic. (a) A single-lane ring road with one CAV and $n-1$ HDVs. (b) A simplified network system schematic. The blue arrows in (a)(b) illustrate the communication topology for the CAV; the purple arrows in (b) illustrate the interaction between adjacent vehicles.

The optimal velocity model (OVM) [24] and intelligent driver model (IDM) [25] are two typical models to describe car-following dynamics of human-driven vehicles. They can be written into a general form [26]

$$\dot{v}_i(t) = F_i\left(s_i(t), \dot{s}_i(t), v_i(t)\right), \tag{1}$$

where $\dot{s}_i(t) = v_{i-1}(t) - v_i(t)$, and $F_i(\cdot)$ means that the acceleration of vehicle i is determined by the relative distance, relative velocity and its own velocity. In equilibrium traffic state, each vehicle moves with a same equilibrium velocity v^*, *i.e.*, $v_i(t) = v^*, \dot{s}_i(t) = 0$, for $i = 1, \ldots, n$. Meanwhile, each vehicle has a corresponding equilibrium spacing s_i^*. According to (1), the value of s_i^* for HDVs can be obtained by solving

$$F_i\left(s_i^*, 0, v^*\right) = 0, \; i = 2, \ldots, n. \tag{2}$$

Unlike HDVs, the equilibrium spacing s_1^* of the CAV can be designed separately. Therefore, to stabilize the traffic flow at velocity v^*, (s_i^*, v^*) can be regarded as the desired state of vehicle i.

We assume that the traffic system is under small perturbations from the desired sate with velocity v^*. The error state between actual and desired state of vehicle i is defined as $x_i(t) = \left[\tilde{s}_i(t), \tilde{v}_i(t)\right]^T = \left[s_i(t) - s_i^*, v_i(t) - v^*\right]^T$. Using (2) and applying the first-order Taylor expansion to (1), we can derive a linearized model for each HDV ($i = 2, \ldots, n$)

$$\begin{cases} \dot{\tilde{s}}_i(t) = \tilde{v}_{i-1}(t) - \tilde{v}_i(t), \\ \dot{\tilde{v}}_i(t) = \alpha_{i1}\tilde{s}_i(t) - \alpha_{i2}\tilde{v}_i(t) + \alpha_{i3}\tilde{v}_{i-1}(t), \end{cases} \tag{3}$$

with $\alpha_{i1} = \frac{\partial F_i}{\partial s_i}, \alpha_{i2} = \frac{\partial F_i}{\partial \dot{s}_i} - \frac{\partial F_i}{\partial v_i}, \alpha_{i3} = \frac{\partial F_i}{\partial \dot{s}_i}$ evaluated at the equilibrium state. To reflect the real driving behavior, we have $\alpha_{i1} > 0, \alpha_{i2} > 0, \alpha_{i3} > 0$ [8]. For the CAV, indexed as $i = 1$, the acceleration signal is directly used as the control input $u(t)$, and its car-following model is

$$\begin{cases} \dot{\tilde{s}}_1(t) = \tilde{v}_n(t) - \tilde{v}_1(t), \\ \dot{\tilde{v}}_1(t) = u(t). \end{cases} \tag{4}$$

We lump the error states of all the vehicles into one global state, *i.e.*, $x(t) = \left[x_1^T(t), x_2^T(t), \ldots, x_n^T(t)\right]^T$, to derive the global dynamics of the mixed traffic system. Based on (3) and (4), a linearized state-space model for the mixed traffic system is then obtained

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5}$$

where

$$A = \begin{bmatrix} C_1 & 0 & \dots & \dots & 0 & C_2 \\ A_{22} & A_{21} & 0 & \dots & \dots & 0 \\ 0 & A_{32} & A_{31} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & A_{n2} & A_{n1} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \\ B_2 \\ \vdots \\ B_2 \end{bmatrix}$$

with each block matrix defined as

$$A_{i1} = \begin{bmatrix} 0 & -1 \\ \alpha_{i1} & -\alpha_{i2} \end{bmatrix}, A_{i2} = \begin{bmatrix} 0 & 1 \\ 0 & \alpha_{i3} \end{bmatrix}, \; i = 1, 2, \dots, n$$
$$C_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that unlike [8], [13] which focused on homogenous dynamics only, we allow HDVs to have heterogeneous car-following dynamics. Thus, the equilibrium spacing s_i^* and the blocks A_{i1}, A_{i2} are in general different for different vehicles. This heterogeneity consideration is more suitable for practical scenarios, but also brings more challenges for theoretical analysis.

B. Problem Statement

Before proceeding with the problem of designing an optimal control input $u(t)$ for the mixed traffic system (5), we introduce two standard notions.

Definition 1 (Controllability): The dynamical system $\dot{x} = Ax + Bu$, or the pair (A, B), is controllable if, for any initial state $x(0) = x_0$, any time $t_f > 0$ and any final state x_f, there exists an input $u(t)$ such that $x(t_f) = x_f$.

For uncontrollable systems, there exist some uncontrollable modes that cannot be moved. If the uncontrollable modes are all stable, we can still design a feedback controller to stabilize the closed-loop system. This leads to the notion of stabilizability.

Definition 2 (Stabilizability): A system is stabilizable if its uncontrollable modes are all stable.

Controllability and stabilizability are two fundamental properties of linear systems, which guarantee the existence of a stabilizing feedback controller. We note that many previous studies in mixed traffic systems mostly considered a specific CAV controller and focused on traffic stability, but have not discussed these notions explicitly. Two notable exceptions are in [8], [13] where a homogeneous HDV model is utilized. In this paper, our first objective is to address the controllability and stabilizability of the mixed traffic system (5) where HDV models are heterogeneous.

Problem 1 (Controllability and Stabilizability): Investigate whether or under what circumstances the mixed traffic system (5) is controllable or stabilizable.

When designing the control input $u(t)$ in (5), it is important to consider the local available information of the neighboring vehicles. This leads to the notion of structured

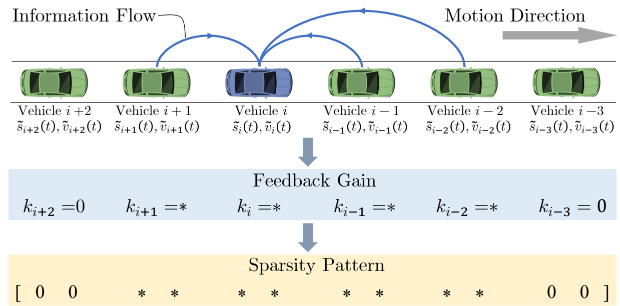


Fig. 2. Illustration for structured constraints. The feedback gain is zero for those vehicles of which vehicle i cannot receive the information.

optimal control, which is in general challenging for large-scale network systems [27], [28]. As depicted in Fig.1b, the mixed traffic flow can be viewed as a network system with the only CAV as a single driving node [29]. Due to the limit of communication range in practice, the CAV can only receive partial information of the global traffic system. Here, we define $\mathcal{E}^c \subseteq \{1, 2, \dots, n\} \times \{1\}$ as the communication network between the CAV and HDVs, where $(i, 1) \in \mathcal{E}^c$ means that the CAV can receive information from vehicle i.

In this paper, we consider a static state-feedback controller, *i.e.*, $u(t) = -Kx(t)$, where $K = \left[k_1, k_2, \dots, k_n\right] \in \mathbb{R}^{1 \times 2n}$. Each block $k_i \in \mathbb{R}^{1\times 2}$ represents the feedback gain of the state of vehicle i, *i.e.*, $x_i(t)$. To reflect the information topology, we require $k_i = 0$, if the CAV cannot obtain the information of vehicle i, *i.e.*, $(i, 1) \notin \mathcal{E}^c$. Hence, the information topology requirement can be naturally formulated as a pre-specified sparsity pattern on K; see Fig.2 for illustration. For simplicity, we define a block-sparsity pattern

$$\mathcal{K} := \{K \in \mathbb{R}^{1\times 2n} | k_i = 0, \text{if } (i, 1) \notin \mathcal{E}^c, k_i \in \mathbb{R}^{1\times 2}\}.$$

We are now ready to present our second objective.

Problem 2 (Optimal controller synthesis): compute a strutured optimal controller $K \in \mathcal{K}$ for the CAV to dampen undesired perturbations in traffic flow, where the constraint K is determined by the communication topology.

III. Controllability and Stabilizability

In this section, we analyze the controllability and stabilizability of the mixed traffic system (5) and introduce our first main result. First, the following lemmas are needed.

Lemma 1 (PBH controllability test [21]): System (A, B) is controllable, if and only if there exists no left eigenvector of A orthogonal to the columns of B, *i.e.*, $\rho^T B \neq 0$, for any $\rho \neq 0$ that satisfies $\rho^T A = \lambda \rho^T$.

Lemma 2 (Invariance under state feedback [21]): The pair (A, B) shares the same controllability, stabilizability and uncontrollable modes with $(A - BK, B)$ for every K with compatible dimension.

According to Definition 2, in order to prove the stabilizability of a system, we need to show that all uncontrollable modes are stable, or equivalently that all unstable modes are controllable. Our first main result is as follows.

Theorem 1: Consider the mixed traffic system in a ring road with 1 CAV and $n - 1$ heterogeneous HDVs given by (5), and then we have the following statements.

1) System (5) is not controllable, and there exists at least one uncontrollable mode corresponding to a zero eigenvalue.

2) If we have

$$\alpha_{j1}^2 - \alpha_{i2}\alpha_{j1}\alpha_{j3} + \alpha_{i1}\alpha_{j3}^2 \neq 0,\ \forall i,j \in \{1,2,\ldots,n\}. \tag{6}$$

then, system (5) is stabilizable.

Proof: For convenience of the proof, we first assume that the CAV has an external input $\hat{u}(t)$, defined as

$$\hat{u}(t) = u(t) - (\alpha_{11}\tilde{s}_i(t) - \alpha_{12}\tilde{v}_i(t) + \alpha_{13}\tilde{v}_{i-1}(t)).$$

Then, the original system (A,B) can be transformed into $(\hat{A},B)$, described by the following dynamics

$$\dot{x}(t) = \hat{A}x(t) + B\hat{u}(t), \tag{7}$$

where

$$\hat{A} = \begin{bmatrix} A_{11} & 0 & \ldots & \ldots & 0 & A_{12} \\ A_{22} & A_{21} & 0 & \ldots & \ldots & 0 \\ 0 & A_{32} & A_{31} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & A_{n2} & A_{n1} \end{bmatrix}.$$

Lemma 2 guarantees the preservation of controllability, stabilizability and uncontrollable modes between $(\hat{A},B)$ and (A,B). Thus in the following, we focus on system $(\hat{A},B)$.

Denote $\rho_0 = [1,0,1,0,\ldots,1,0]^T \in \mathbb{R}^{2n\times 1}$. Then it is easy to verify that $\rho_0^T\hat{A} = 0\cdot\rho_0^T$ and $\rho_0^T B = 0$. According to Lemma 1, we know that ρ_0 is an uncontrollable mode corresponding to a zero eigenvalue. This result confirms the first statement of Theorem 1. Furthermore, we can show ρ_0 is the only mode corresponding to the zero eigenvalue and thus it is stable (but not asymptotically stable) by proving that the algebraic multiplicity of the zero eigenvalue is one. Due to page limit, a detailed proof will be presented in an extended version.

We next prove the second statement of Theorem 1. Our main idea is to show that all other modes corresponding to nonzero eigenvalues of $\hat{A}$ are controllable. Based on Lemma 1, we need to show that $\rho^T B \neq 0$ for all $\rho \neq 0$ satisfying $\rho^T\hat{A} = \lambda\rho^T (\lambda \neq 0)$. We note that condition (6) in Theorem 1 leads to the following fact (the proof will be presented in an extended version): for each non-zero eigenvalue λ of $\hat{A}$, it satisfies

$$\lambda^2 + \alpha_{i2}\lambda + \alpha_{i1} \neq 0,\ \alpha_{i3}\lambda + \alpha_{i2} \neq 0,\ \forall i \in \{1,2,\ldots,n\}, \tag{8}$$

Consider a left eigenvector ρ of $\hat{A}$ corresponding to a non-zero eigenvalue λ. Denote

$$\rho = \left[\rho_1^T, \rho_2^T, \ldots, \rho_n^T\right]^T,$$

where $\rho_i = [\rho_{i1}, \rho_{i2}]^T \in \mathbb{R}^{1\times 2}$. Assume that ρ corresponds to an uncontrollable mode, *i.e.*, $\rho^T B = 0$. Considering that only the second element in B is non-zero, this assumption leads to $\rho_{12} = 0$. In addition, due to $\rho^T A = \lambda\rho^T$, we know for $i = 1,2,\ldots,n$

$$\rho_i^T(\lambda I - A_{i1}) = \rho_{i+1}^T A_{(i+1)2}$$

Since $\lambda I - A_{i1}$ is invertible according to (8), we then have

$$\rho_i^T = \rho_{i+1}^T A_{(i+1)2}(\lambda I - A_{i1})^{-1}, \tag{9}$$

for $i = 1,2,\ldots,n$. In (9), due to the circulant property, we denote $i = 1$ when i takes the value of $n+1$. Upon denoting $D_{i1} = (\lambda I - A_{i1})^{-1}$, and considering the recursive result of (9), we have

$$\begin{aligned} \rho_1^T &= \rho_2^T A_{22} D_{11} \\ &= \rho_3^T A_{32} D_{21} A_{22} D_{11} \\ &= \cdots \\ &= \rho_n^T A_{n2} D_{(n-1)1} A_{(n-1)2} \cdots D_{21} A_{22} D_{11} \\ &= \rho_1^T A_{12}(D_{n1}A_{n2}) \cdots (D_{21}A_{22}) D_{11}, \end{aligned} \tag{10}$$

where $D_{i1}A_{i2}$ is equal to

$$D_{i1}A_{i2} = \frac{1}{a_i}\begin{bmatrix} 0 & c_i \\ 0 & b_i \end{bmatrix}, i = 1,2,\ldots,n, \tag{11}$$

with $a_i = \lambda^2 + \alpha_{i2}\lambda + \alpha_{i1}$, $b_i = \alpha_{i3}\lambda + \alpha_{i2}$, and $c_i = \lambda + \alpha_{i2} - \alpha_{i3}$. According to (8), $a_i \neq 0$ and $b_i \neq 0$. Then substituting (11) into (10) yields

$$\begin{bmatrix} \rho_{11} & \rho_{12} \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \lambda \\ \alpha_{11}\alpha_{13} & \lambda\alpha_{13} \end{bmatrix} \frac{\Pi_{i=2}^n b_i}{\Pi_{i=1}^n a_i}. \tag{12}$$

Since $\rho_{12} = 0$, we know $\lambda\rho_{11}\Pi_{i=2}^n b_i = 0$. Because $\lambda \neq 0$ and $b_i \neq 0$, we have $\rho_{11} = 0$, leading to $\rho_1 = 0$. Using (9) to iterate recursively, we can obtain $\rho = \left[\rho_1^T, \rho_2^T, \ldots, \rho_n^T\right]^T = 0$, which contradicts with $\rho \neq 0$. Consequently, the assumption that ρ corresponds to an uncontrollable mode does not hold. Therefore, all the modes corresponding to non-zero eigenvalues are controllable.

We have shown that there is only one mode corresponding to the zero eigenvalue. This mode is not asymptotically stable but is stable. Thus, the mixed traffic system (5) is stabilizable. This completes the proof of the second statement in Theorem 1. ■

The physical interpretation for the uncontrollable mode $\rho_0 = [1,0,1,0,\ldots,1,0]^T$ is that it reflects the ring-road constraint on system (5). One can verify that $\rho_0^T x(t) = \sum_{i=1}^n \tilde{s}_i(t) = \sum_{i=1}^n s_i(t) - \sum_{i=1}^n s_i^*$, indicating that the sum of each vehicle's spacing must remain constant, *i.e.*, $\sum_{i=1}^n s_i(t) = L$. In order to regulate the mixed traffic system to the desired velocity v^*, the adjustable spacing s_1^* for the CAV should satisfy $s_1^* = L - \sum_{i=2}^n s_i^*$, where $s_2^*, \ldots, s_n^*$ are given by (2).

Remark 1: Note that (6) is a sufficient condition for the mixed traffic system to be stabilizable. This condition restricts the locations of the closed-loop poles as shown in (8), and allows us to reveal the characteristic of their corresponding eigenvectors. Note that when choosing α_{i1}, α_{i2}, α_{i3} randomly, condition (6) is satisfied with probability one; accordingly, the heterogeneous mixed traffic system (5) is stabilizable with probability one. In the case of homogeneous traffic flow, Theorem 1 is consistent with that in [13] and generalizes the results in [8].

IV. Optimal Controller Synthesis

After revealing the stabilizability of the mixed traffic system (5), we proceed to design an optimal control input $u(t)$ to dampen undesired perturbations. We formulate this task as a structured optimal control problem to achieve a certain system-level control objective.

Previous works on the control of CAVs usually focused on local-level performance, *i.e.*, to achieve a better driving behavior of a single CAV or a CAV platoon [3], [5]. Here, we first define a system-level performance index for the entire traffic system (5). In particular, we aims to minimize the influence of undesired perturbations on traffic flow by controlling one CAV. To model this scenario, we assume that there exist certain disturbances $\omega(t) = \left[\omega_1(t), \omega_2(t), \dots, \omega_n(t)\right]^T$, where $\omega_i(t)$ is a scalar disturbance signal with finite energy in the acceleration of vehicle i. Then, the system model (5) becomes

$$\dot{x}(t) = Ax(t) + Bu(t) + H\omega(t), \tag{13}$$

where $H \in \mathbb{R}^{2n\times n}$ is

$$H = \begin{bmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & H_1 \end{bmatrix},$$

with the block entry denoting $H_1 = [0,1]^T$. We use $z(t) = \left[\gamma_s \tilde{s}_1(t), \gamma_v \tilde{v}_1(t), \dots, \gamma_s \tilde{s}_n(t), \gamma_v \tilde{v}_n(t), \gamma_u u(t)\right]^T$ to denote a system-level performance output, with weight coefficients $\gamma_s, \gamma_v, \gamma_u > 0$ representing the penalty for spacing error, velocity error and control input, respectively. Alternatively, $z(t)$ can be written as a compact form

$$z(t) = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix} u(t), \tag{14}$$

with $Q^{\frac{1}{2}} = \mathrm{diag}(\gamma_s, \gamma_v, \dots, \gamma_s, \gamma_v)$, $R^{\frac{1}{2}} = \gamma_u$. We use $G_{\omega z}$ to denote the transfer function from disturbance ω to performance output z. Upon denoting $\|G_{wz}\|$ as the $\mathcal{H}_2$ norm that reflects the influence of disturbances, the design of an optimal control input $u(t)$ for the CAV under a pre-specified communication topology can be formulated as

$$\begin{aligned} \min_{K} \quad & \|G_{wz}\|^2 \\ \text{subject to} \quad & u = -Kx,\ K \in \mathcal{K}, \end{aligned} \tag{15}$$

where $K \in \mathcal{K}$ reflects the information that is available for the CAV. Note that the formulation (15) is known as the structured optimal control problem [27], [30]. This problem is in general non-convex and computationally hard to find a globally optimal solution. In the following, we highlight that one particular difficulty lies in the structural constraint $K \in \mathcal{K}$.

Lemma 3: Consider a closed-loop stable system with the dynamics $x(t) = Ax(t) + H\omega(t)$, and the output $z(t) = Cx(t)$. The $\mathcal{H}_2$ norm of the transfer function from $w(t)$ to $z(t)$ can be computed by

$$\|G_{wz}\|^2 = \inf_{X \succ 0} \{\mathrm{Trace}(CXC^T) | AX + XA^T + HH^T \preceq 0\}.$$

Lemma 3 offers a standard technique to compute the $\mathcal{H}_2$ norm of a linear system. We next transform (13) and (14) into the closed-loop model using $u = -Kx$

$$\begin{aligned} \dot{x}(t) &= (A - BK)x(t) + H\omega(t), \\ z(t) &= \begin{bmatrix} Q^{\frac{1}{2}} \\ -R^{\frac{1}{2}}K \end{bmatrix} x(t). \end{aligned} \tag{16}$$

According to Lemma 3, problem (15) can be equivalently reformulated as

$$\begin{aligned} \min_{X,K} \quad & \mathrm{Trace}(QX) + \mathrm{Trace}(K^T RKX) \\ \text{subject to} \quad & (A - BK)X + X(A - BK)^T + HH^T \preceq 0, \\ & X \succ 0,\ K \in \mathcal{K}. \end{aligned} \tag{17}$$

Using a standard change of variables [30]

$$K = ZX^{-1}, \tag{18}$$

pre-and post-multiplying $AX + XA^T + HH^T \preceq 0$ by $P = X^{-1}$, and using the Schur complement, we can obtain the following equivalent form of (17)

$$\begin{aligned} \min_{X,Y,Z} \quad & \mathrm{Trace}(QX) + \mathrm{Trace}(RY) \\ \text{subject to} \quad & AX + XA^T - BZ - Z^T B^T + HH^T \preceq 0, \\ & \begin{bmatrix} Y & Z \\ Z^T & X \end{bmatrix} \succeq 0,\ X \succ 0,\ ZX^{-1} \in \mathcal{K}. \end{aligned} \tag{19}$$

In the absence of $ZX^{-1} \in \mathcal{K}$, problem (19) is convex and can be solved via existing conic solvers, *e.g.*, Mosek [31]. However, this constraint appears naturally since the CAV can only use its neighboring vehicles' information for feedback control in mixed traffic flow.

We then utilize a convex approximation method recently proposed in [22] to deal with the non-convex constraint, $ZX^{-1} \in \mathcal{K}$. The underlying idea is based on the sparsity invariance property, which allows one to replace this constraint by assuming Z and X to have certain sparsity patterns. More specifically, if we have $Z \in \mathcal{T}$, $X \in \mathcal{S}$, such that $ZX^{-1} \in \mathcal{K}$, problem (19) can be converted to a convex relaxation form

$$\begin{aligned} \min_{X,Y,Z} \quad & \mathrm{Trace}(QX) + \mathrm{Trace}(RY) \\ \text{subject to} \quad & AX + XA^T - BZ - Z^T B^T + HH^T \preceq 0, \\ & \begin{bmatrix} Y & Z \\ Z^T & X \end{bmatrix} \succeq 0,\ X \succ 0,\ Z \in \mathcal{T},\ X \in \mathcal{S}. \end{aligned} \tag{20}$$

where $\mathcal{T}$ and $\mathcal{S}$ are convex sets describing certain sparsity patterns. Then the optimal value of Z and X can be accordingly computed based on problem (20). One structured optimal controller is therefore recovered as $K = ZX^{-1} \in \mathcal{K}$, which naturally satisfies the structural constraint. We refer the interested reader to [22] for details about the notion of sparsity invirance and a full characterization of $\mathcal{T}$ and $\mathcal{S}$.

Remark 2: Without the constraint $K \in \mathcal{K}$, (15) is a standard $\mathcal{H}_2$ optimal control problem, for which efficient methods are available to compute an optimal solution. Indeed, due to the wide industrial applications, the structured

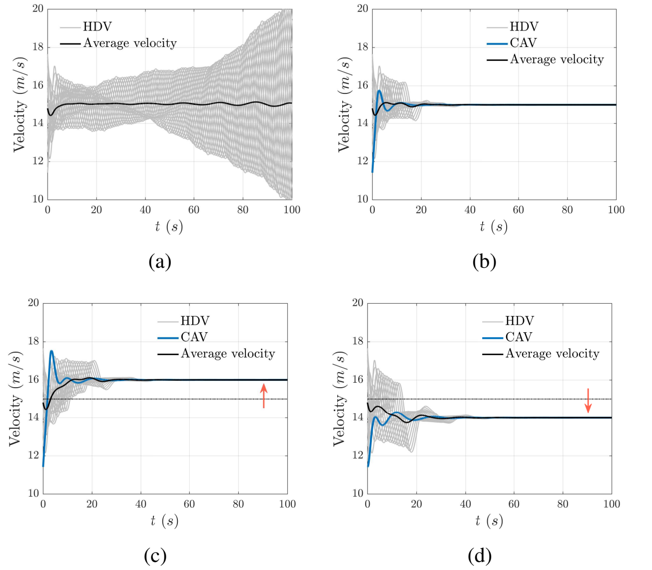


Fig. 3. Velocity profile of each vehicle (Experiment A). (a) All the vehicles are HDVs. (b)-(d) correspond to the cases where one vehicle is the CAV with different values of v^*: $15m/s, 16m/s, 14m/s$, respectively.

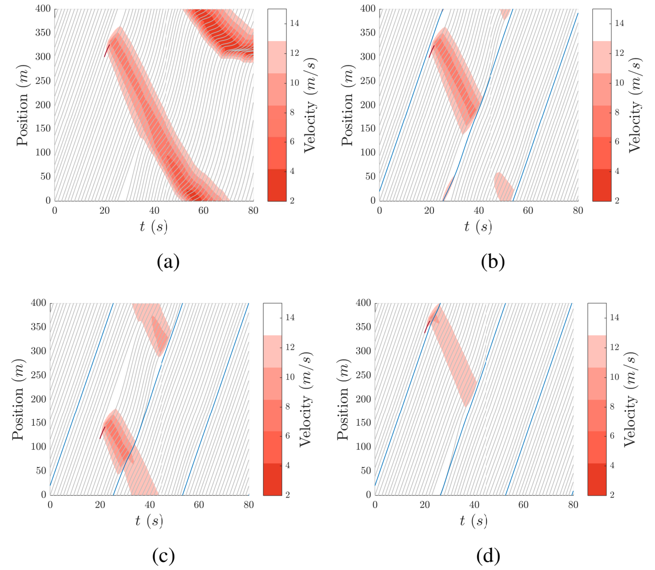


Fig. 4. Vehicle trajectories (Experiment B). The darker the color, the lower the velocity. (a) All the vehicles are human-driven. (b)-(d) correspond to the cases where vehicle 2,11,20 is under the perturbation, respectively.

optimal control problem (15), or its variants, have attracted some attention. A few methods have been proposed to find an approximation solution, such as using convex approximations [27], [28], or directly employing non-convex optimization techniques [32]. However, many existing methods require that the system is controllable, and therefore they are not applicable to our problem since the mixed traffic system is not completely controllable, as proved in Theorem 1.

V. Numerical Experiments

Our main results are based on the linearized model of the mixed traffic system. To evaluate their effectiveness in the presence of nonlinear car-following dynamics, in this section we consider a realistic nonlinear HDV model and conduct two types of simulation experiments using MATLAB.

We consider 19 HDVs and one CAV in a ring road with circumference $L = 400m$. Note that vehicle no.1 is the CAV and the penetration rate is only 5%. The information from five vehicles in front and five vehicles behind is available to the CAV for feedback control. For the parameters in the performance output (14), we choose $\gamma_s = 0.03$, $\gamma_v = 0.15$, $\gamma_u = 1$. Based on the approach in Section IV, a structured linear feedback gain K is obtained using Mosek.

In our simulations, a nonlinear optimal velocity model (OVM) [24] is used to describe the car-following dynamics of HDVs. The specific expression for (1) becomes

$$F_i(\cdot) = \alpha_i \left(V_i\left(s_i(t)\right) - v_i(t)\right) + \beta_i \dot{s}_i(t), \tag{21}$$

where

$$V_i(s) = \begin{cases} 0, & s \leq s_{i,\mathrm{st}}, \\ f_{i,v}(s), & s_{i,\mathrm{st}} < s < s_{i,\mathrm{go}}, \\ v_{i,\max}, & s \geq s_{i,\mathrm{go}}, \end{cases}$$

and

$$f_{i,v}(s) = \frac{v_{i,\max}}{2}\left(1 - \cos(\pi \frac{s - s_{i,\mathrm{st}}}{s_{i,\mathrm{go}} - s_{i,\mathrm{st}}})\right).$$

Due to the heterogeneity setting, partial parameters are set to the following values, which is similar to [16]: $\alpha_i = 0.6 + U[-0.1, 0.1]$, $\beta_i = 0.9 + U[-0.1, 0.1]$, $s_{i,go} = 35 + U[-5, 5]$, with $U[\cdot]$ denoting the uniform distribution. For other parameters, we use homogeneous values: $v_{i,\max} = 30$, $s_{i,\mathrm{st}} = 5$. As stated in [13], this set of values describes an unstable ring-road traffic system. It is not difficult to verify that the stabilizability condition (6) is guaranteed. To avoid crashes, we also assume that all the vehicles are equipped with a standard automatic emergency braking system: $\dot{v}(t) = a_{\min}$, if $\frac{v_i^2(t) - v_{i-1}^2(t)}{2s_i(t)} \geq |a_{\min}|$, where each vehicle's maximum acceleration and deceleration rate is set to $a_{\max} = 2m/s^2$, $a_{\min} = -5m/s^2$, respectively.

Experiment A aims to examine whether the mixed heterogeneous traffic flow can be stabilized by the CAV, which is to confirm the result from controllability analysis. At the beginning, all the vehicles are distributed randomly on the road, with the initial velocity following the distribution $15 + U\left[-4, 4\right]$. If all the vehicles are under human control, it can be observed in Fig.3a that multiple perturbations arise inside the traffic flow, and they are amplified gradually. In contrast, if there is one CAV using the proposed control method, the traffic flow can be stabilized to the original average velocity within a short time (Fig.3b). Moreover, by adjusting the desired equilibrium velocity v^*, *i.e.*, the desired system state, the CAV can regulate the entire traffic flow to a higher or lower velocity, through influencing other vehicles (Fig.3c-d). This finding confirms the stabilizability of the mixed traffic system via control of one single CAV.

Experiment B aims at evaluating the effectiveness of the proposed system-level control strategy. The traffic flow has an initial velocity of $15m/s$. We then reproduce the traffic wave phenomenon by assuming that one vehicle is under a sudden perturbation. At $t = 20s$, one HDV brakes at $-3m/s^2$ for $3s$. It can be observed that the perturbed vehicle's action results in a traffic wave propagating upstream; the wave persists and does not vanish (Fig.4a). When one CAV with the proposed method is included in the traffic system, it can prevent the propagation of the traffic wave and dampen the perturbation quickly (Fig.4b-d). Here we only show three cases with respect to the position of the perturbation, *i.e.*, which vehicle is the source of the traffic wave. Indeed, we observe that the proposed method allows the CAV to dissipate the traffic wave whatever the perturbed vehicle is.

VI. Conclusions

In this paper, we have proved that the mixed traffic system with one single CAV and heterogeneous HDVs is not completely controllable, but is stabilizable under a very mild condition. This theoretical result confirms the potential of traffic control via CAVs. We have also established a mathematical framework to design a system-level optimal controller for mixed traffic systems, which utilizes structured optimal control to address the issue of arbitrary communication topologies and allows the CAV to stabilize traffic flow actively. Extensive numerical results have validated the effectiveness of our method. One future direction is to consider drivers' reaction time in the problem formulation. Considering more than one CAVs may coexist in traffic systems, another interesting topic is to design cooperative strategies for multiple CAVs to smooth traffic flow efficiently.

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