# Controllability Analysis and Optimal Controller Synthesis of Mixed Traffic Systems

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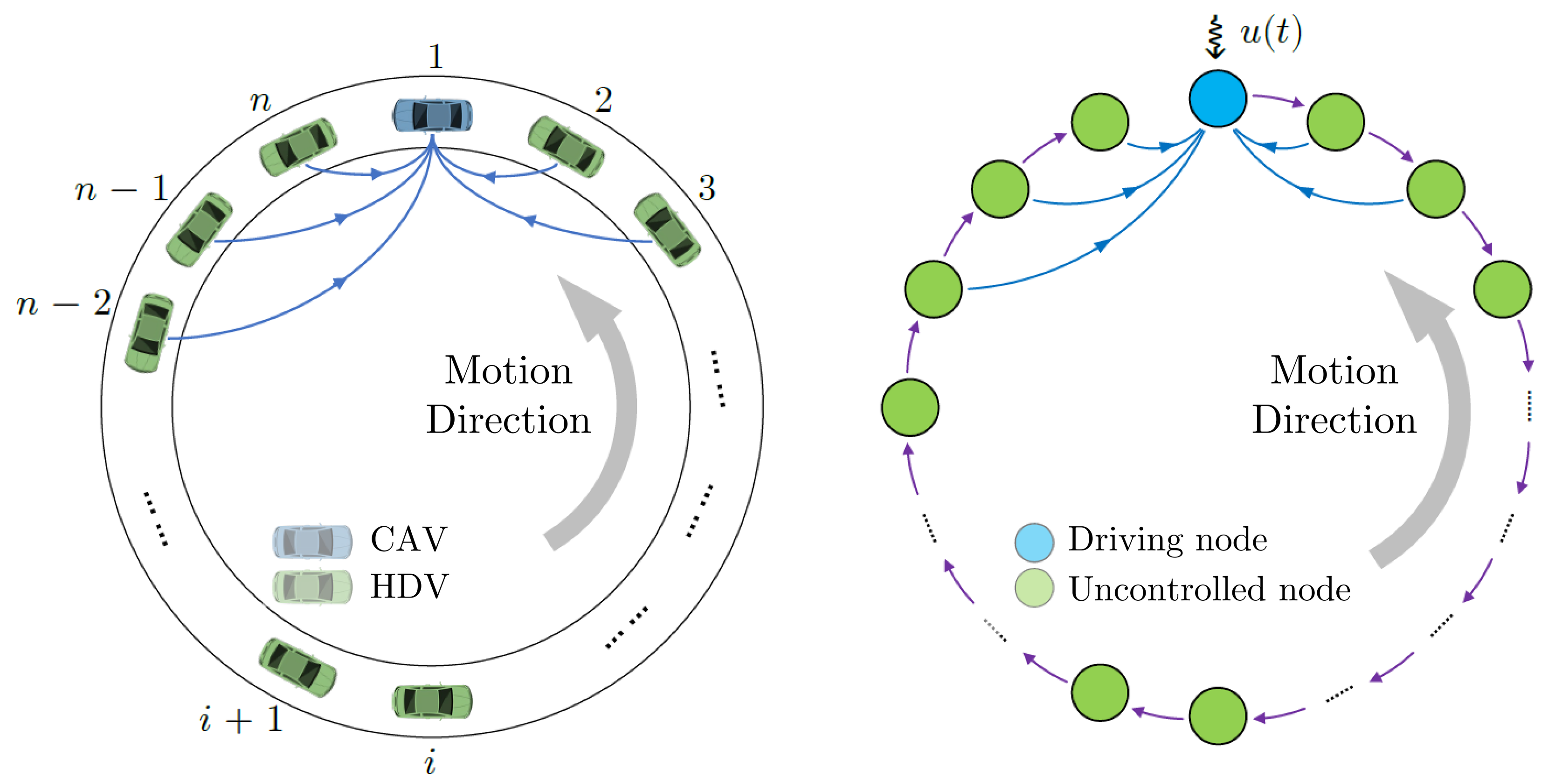
## 1. Main Contributions

We focus on a ring-road mixed traffic system with one CAV and multiple heterogeneous human-driven vehicles (HDVs).

- **Controllability analysis**: we prove that the ring-road mixed traffic system is stabilizable under a mild condition.
- **Optimal controller synthesis**: we consider a system-level control objective and the communication topology.

## 2. System Modeling

Core idea: traffic system ⇔ network system




- Consider heterogeneous dynamics of HDVs
- Corresponding relationship between equilibrium spacing $s_i^*$ and equilibrium velocity $v^*$
  $\dot{v}_i(t) = F_i\left(s_i(t), \dot{s}_i(t), v_i(t)\right) \Rightarrow F_i\left(s_i^*, 0, v^*\right) = 0,\ i = 2, \ldots, n$
- Linearized state-space model for the mixed traffic system

$$\dot{x}(t) = Ax(t) + Bu(t). \quad (1)$$

$$x(t) = \left[s_1(t) - s_1^*, v_1(t) - v^*, \ldots, s_i(t) - s_n^*, v_n(t) - v^*\right]^T$$

$$\alpha_{i1} = \partial F_i/\partial s_i, \alpha_{i2} = \partial F_i/\partial \dot{s}_i - \partial F_i/\partial v_i, \alpha_{i3} = \partial F_i/\partial \dot{s}_i$$

$$A = \begin{bmatrix} C_1 & 0 & \ldots & \ldots & 0 & C_2 \\ A_{22} & A_{21} & 0 & \ldots & \ldots & 0 \\ 0 & A_{32} & A_{31} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & A_{n2} & A_{n1} \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \\ 0_2 \\ \vdots \\ 0_2 \end{bmatrix}$$

## 3. Controllability and Stabilizability

### Theorem 1

1) System (1) is not completely controllable; there exists an uncontrollable mode corresponding to a zero eigenvalue.
2) If we have
$$\alpha_{j1}^2 - \alpha_{i2}\alpha_{j1}\alpha_{j3} + \alpha_{i1}\alpha_{j3}^2 \neq 0,\ \forall i, j \in \{1, 2, \ldots, n\}, \quad (2)$$
then, system (1) is stabilizable.

Proof: PBH test + eigenvalue-eigenvector analysis

**Remark**

- The uncontrollable mode ⇒ the ring-road constraint
- To stabilize traffic flow at exactly $v^*$, $s_1^* = L - \sum_{i=2}^n s_i^*$
- No requirement on system size $n$ or the stability property of the original traffic system with HDVs only.
- The heterogeneous mixed traffic system (1) is stabilizable with probability one.

## 4. Optimal Controller Synthesis

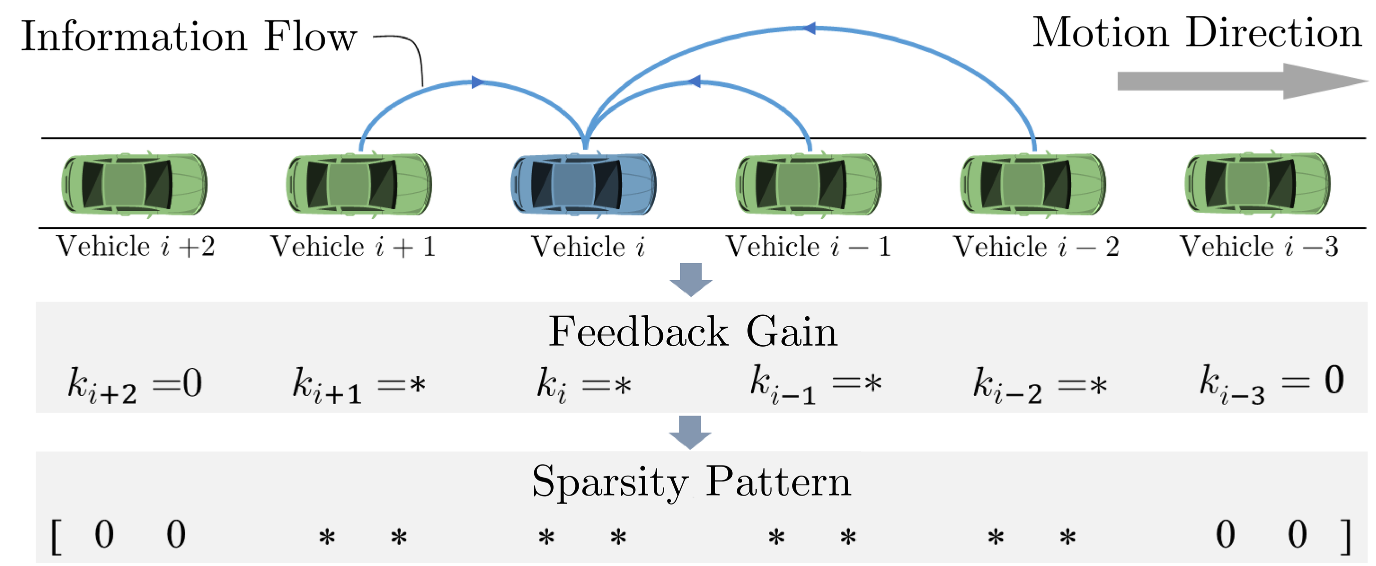
Local-level (focus on CAVs only) ⇒ system-level (improve the entire traffic flow).

- Introduce disturbances
  $\dot{x}(t) = Ax(t) + Bu(t) + H\omega(t)$
- Linear controller
  $u = -Kx$
- System-level output
  $z(t) = \left[Q^{\frac{1}{2}}, 0\right]^T x(t) + \left[0, R^{\frac{1}{2}}\right]^T u(t)$
- $\mathcal{H}_2$ optimal control
  $\min_K \quad \|G_{wz}\|^2$

The pre-specified communication topology is imposed as a structured constraint on feedback gain: $K \in \mathcal{K}$ .

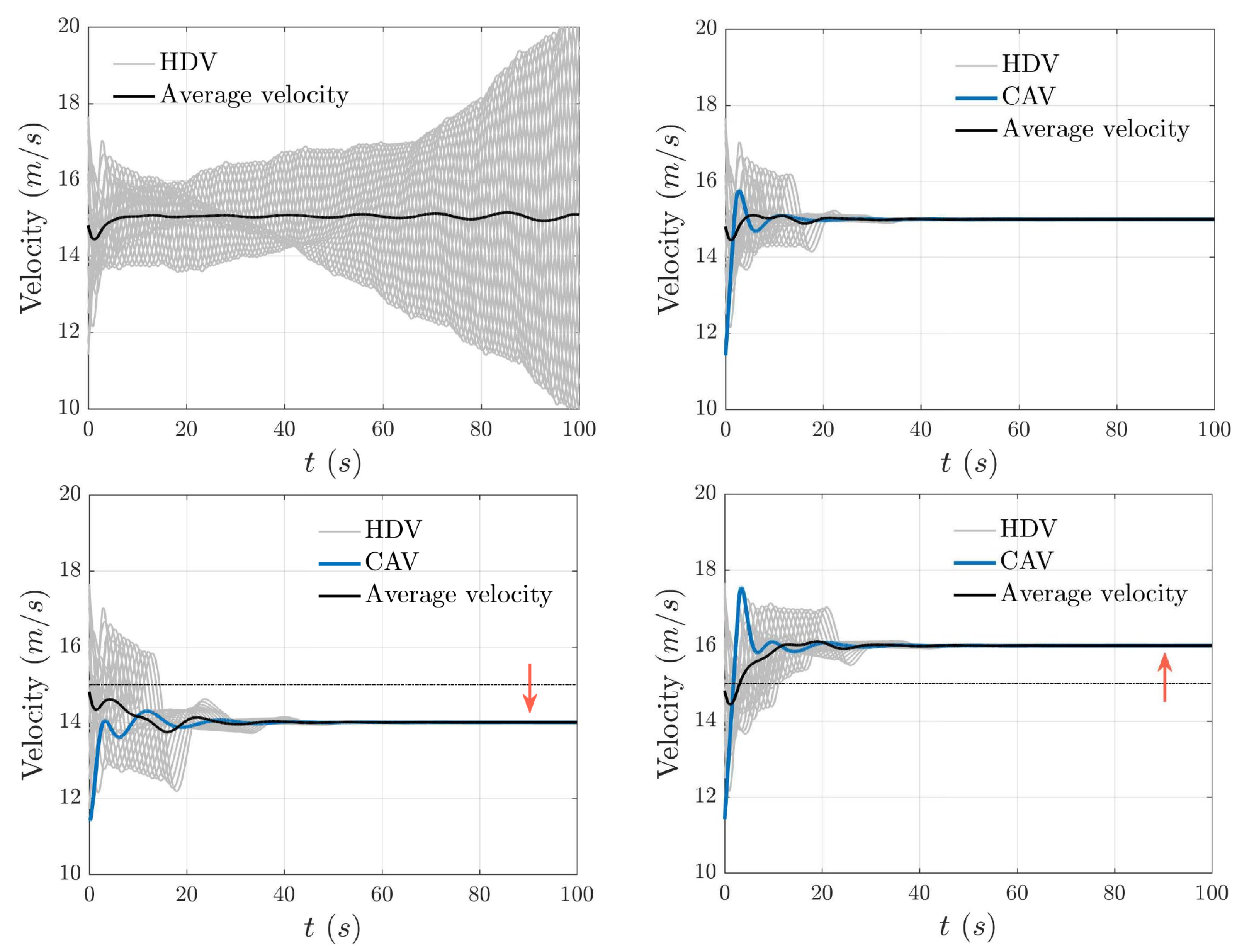
- Sparsity invariance: $Z \in \mathcal{T}, X \in \mathcal{S} \Rightarrow K = ZX^{-1} \in \mathcal{K}$
- Convex relaxation formulation

$$\begin{aligned} \min_{X,Y,Z} \quad & \mathrm{Trace}(QX) + \mathrm{Trace}(RY) \\ \text{s.t.} \quad & AX + XA^T - BZ - Z^T B^T + HH^T \preceq 0, \\ & \begin{bmatrix} Y & Z \\ Z^T & X \end{bmatrix} \succeq 0,\ X \succ 0,\ Z \in \mathcal{T},\ X \in \mathcal{S} \end{aligned}$$

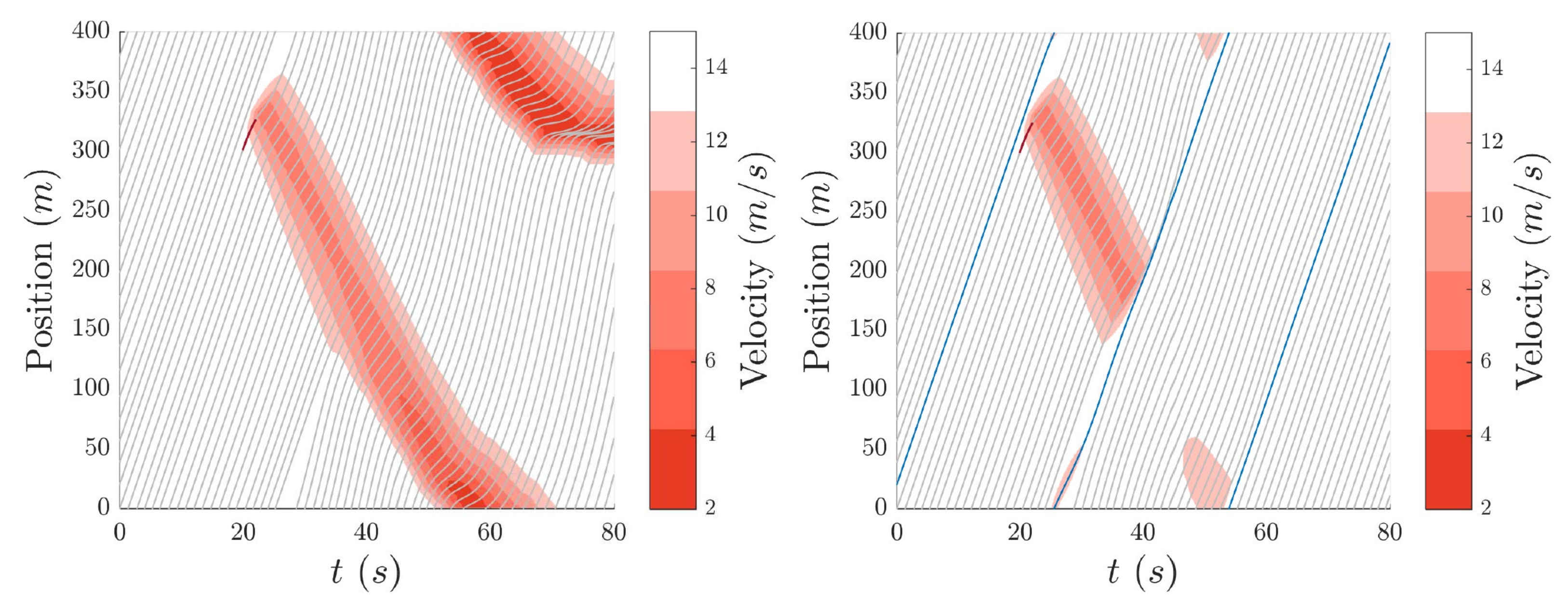



## 5. Numerical Experiments & Conclusions

**Conclusion 1:** one single CAV can stabilize traffic flow, and can also regulate traffic flow to a higher or lower velocity.




**Conclusion 2:** the proposed controller enables one single CAV with limited communication ability to dampen traffic waves actively.

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