Optimal Seismic Reflectivity Inversion: Data-driven ℓ_p-loss-ℓ_q-regularization Sparse Regression

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***Abstract*—Seismic reflectivity inversion is widely applied to improve the seismic resolution to obtain detailed underground understandings. Based on the convolution model, seismic inversion removes the wavelet effect by solving an optimization problem. Taking advantage of the sparsity property, the ℓ_1 norm is commonly adopted in the regularization terms to overcome the noise/interference vulnerability observed in the ℓ_p-losses minimization. However, no one has provided a deterministic conclusion that ℓ_1 norm regularization is the best choice for seismic reflectivity inversion. Instead of using an unproved fixed regularization norm, we propose an optimal seismic reflectivity inversion approach. Our method adaptively adopts a ℓ_p-loss-ℓ_q-regularization (i.e. $\ell_{p,q}$ regularization) for $p = 2$, $0 < q < 1$ to estimate a more accurate and detailed reflectivity profile. In addition, we employ a K fold cross-validation based approach to obtain the optimal damping factor λ to further improve the seismic inversion results. The letter starts with the introduction of non-convex constraint for seismic inversion, and the necessity of the ℓ_q norm regularization. Then the majorization-minimization and cross validation algorithms are briefly described. The performance of the proposed seismic inversion approach is evaluated through synthetic examples and a field example from the Bohai Bay Basin, China.**

***Index Terms*—seismic reflectivity inversion, $\ell_{p,q}$ regularization, damping factor, cross-validation, optimal regularization.**

I. Introduction

RESOLUTION is an unavoidable topic in seismic interpretation, which is the ultimate goal for all seismic processing steps. High seismic resolution helps characterizing the correlation between the geological structures and geophysical images, which is important for the reservoir deposition analysis, especially thin-bed reservoirs [1].

Based on the seismic convolution model, [2] improved the seismic resolution via a wavelet scaling method. Likewise, [3] extended seismic spectrum using a non-stationary wavelet estimation. A seismic reflectivity obtained from seismic amplitude inversion has been used in the applications with little/weak well controls [4]. For regularized inversion, besides the ℓ_2 norm constraint, ℓ_1 norm regularization is more popular in sparse spike inversion [5], [6]. Theoretically, ℓ_0 norm regularization should also be a good sparsity measurement, but actual seismic signals do not always show 0 values, so finding exact p non-zero elements in signal, i.e. $\|x\|_0 = p$ is not a computationally feasible and proper constraint in the real world [7]. Signal sparsity properties show different highlights [8], so a data-driven sparsity measurement is more suitable for the seismic reflectivity inversion problem [9], [10]. The convergence of ℓ_q $(0 < q < 1)$ norm regularization has been proved by [11], and its advantages were also demonstrated. There are some well established algorithms to solve the nonconvex or concave optimization problem related to ℓ_q regularized inversion, for examples, cyclic descent algorithm [12], reweighted ℓ_1 minimization [13], iteratively reweighted algorithms [14], and so on.

However, to our best knowledge, the sparsity regularization selection for seismic reflectivity inversion has not been generally discussed. Although there are discussions with the damping factor λ selection, such as [10], they are only related to either ℓ_1 or ℓ_2 norm. In the previous work [15], a ℓ_q norm $(0 < q < 1)$ regularized optimization has been applied to the seismic reflectivity inversion. Compared to [15], this letter proposes a general data-driven seismic reflectivity inversion approach. We extend ℓ_q norm $(0 < q < 1)$ norm to a ℓ_p-loss-ℓ_q-regularization ($\ell_{p,q}$ regularization) framework, which allows us to consider a wide range of loss functions. The possible choices of loss functions include the ℓ_p-losses and prediction losses [11]. In this letter, we study the ℓ_2-loss as an illustration. Here, we also explicitly define the way to obtain the optimal q, where the majorize-minimization (MM) algorithm is described. In addition, K fold cross-validation is adopted for generality. We establish an adaptive seismic inversion approach including optimal regularization term selection as well as adaptive damping factor determination. We introduce the theories and algorithms first. Then effectiveness of the proposed method is proved through synthetic examples and a field application from Bohai Bay Basin, China.

II. Theory

A. Seismic Convolution Model

After removing the undesirable interferences, seismic data can be modeled as the convolution between the source wavelet $w(t)$ and reflectivity series $r(t)$ in the form as [10], [15], [16]:

$$s(t) = \int_{-\infty}^{\infty} w(\tau) r(t-\tau) d\tau + \epsilon(t), \tag{1}$$

where $\epsilon(t)$ is an additive random noise. Using a convolution symbol, we can rewrite it as:

$$s(t) = w(t) * r(t) + \epsilon(t). \tag{2}$$

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Then, a discrete matrix format realized at samples $t = 1, 2, \cdots, M$ can also be formulated [17]:

$$\mathbf{s} = \mathbf{W}\mathbf{r} + \boldsymbol{\epsilon}, \tag{3}$$

where $\mathbf{s} = [s_1, \cdots, s_M]^T$ is the observation, $\mathbf{W}$ is the wavelet kernel matrix which can be estimated from the seismic power spectrum [16], and $\mathbf{r} = [r_1, \cdots, r_M]^T$ is the reflectivity series.

B. Seismic Inversion

The seismic convolution model forms a linear relationship between the recorded seismic data and reflectivity series. The solution of equation (3) is nonunique, which necessitates certain constraint to achieve stable. In general, the objective function can be formulated as

$$\phi(q, \lambda) = ||\mathbf{s} - \mathbf{W}\mathbf{r}||_p^p + \lambda \|\mathbf{r}\|_q, \tag{4}$$

where $||\mathbf{s} - \mathbf{W}\mathbf{r}||_p^p = \sum_{t=1}^{M} |s(t) - w(t) * r(t)|_p^p$ and $\|\mathbf{r}\|_q = (\sum_{j=1}^{M} |r_j|^q)^{1/q}$ (q indicates the constraint condition) and λ is the damping factor [12]. The common choice for the loss function is the ℓ_2-loss, i.e. p =2.

C. ℓ_q Norm Regularization

Seismic inversion usually adopts either ℓ_1 or ℓ_2 norm regularization. This nonconvex constraint is based on the common assumption that the reflectivity series $r(t)$ is sparse. However, accurately identifying the sparsity of the signal is not easy. The ℓ_0 norm, which measures the number of non-zero of $\mathbf{r}$: $\|\mathbf{r}\|_0 = \{r_j \neq 0, j = 1, ..., N\}$, is not put into consideration. It asks to find the "best subset" among all possible sparsity candidates that makes it impractical for seismic reflectivity due to intractable computational cost and difficulty in choosing number of non-zero elements. To loose the constraint, the ℓ_1 norm regularization, i.e. Lasso regression [18], is easily calculated with sparsity constraint. [19] built the connection between ℓ_0 and ℓ_2 regularization. Because the convex ℓ_q regularization with $q > 1$ does not hold the sparsity property which is necessary for seismic inversion, we need to pick an optimal q from $(0, 1)$. As shown in Figure 1, different regularization norms produce various constraint on sparsity.

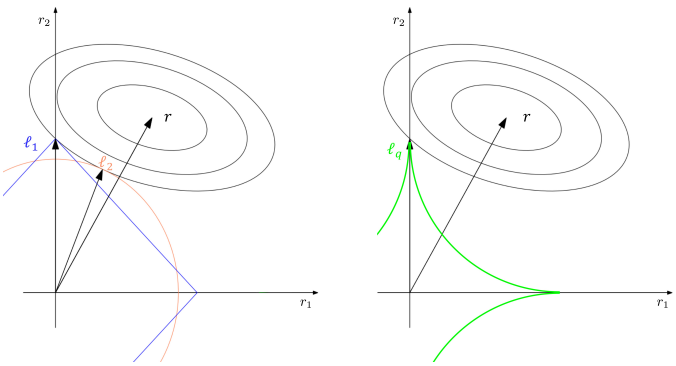

Fig. 1. Geometric interpretation of different regularizations. Left: ℓ_1 regularization (blue), ℓ_2 regularization (red); Right: ℓ_q (green) regularization with $q = 0.5$. The ellipses are contours of the ℓ_p-loss errors of the toy signal $\mathbf{r} = [r_1, r_2]^T$.

In order to obtain the optimal choice of q, for a given λ, we can formulate the objective function as:

$$\hat{q} = \arg\min_{q \in \mathcal{Q}} \phi(q, \lambda) \Rightarrow \hat{q} = \arg\min_{q \in \mathcal{Q}} \phi(q), \tag{5}$$

where $\mathcal{Q}$ is a predefined nonempty set containing possible q values. This problem can be solved through a coordinate-wise optimization method.

D. Majorize-Minimization Algorithm

We propose to use the majorize-minimization (MM) algorithm [20] to solve the nonconvex optimization problem in equation (4) with $0 < q < 1$. As an iterative optimization method, the MM algorithm consists of two steps in one iteration. In the first majorization step, a surrogate function $g(\cdot|q_k)$ is employed to locally approximate the objective function with their difference minimized at the current point. In other words, the surrogate upperbounds the objective function up to a constant c_k. Then in the minimization step, the surrogate function is minimized. The sequence $\phi(q_k)$ is non-increasing since $\phi(q_{k+1}) \leq g(q_{k+1}|q_k) - c_k \leq g(q_k|q_k) - c_k = \phi(q_k)$. The procedure is shown in Algorithm 1. The computational complexity of the MM algorithm is $O(M^3)$.

Algorithm 1: Majorization-Minimization Algorithm

1: Initialize $q_0 \in \mathcal{Q}$ and set $k = 0$
2: **repeat**
3: **Majorization step:**
4: Construct $g(\cdot|q_k)$ satisfying the upperbound property: $g(q|q_k) \geq \phi(q) + c_k$, $\forall q \in \mathcal{Q}$
5: **Minimization step:**
6: Update x as $q_{k+1} = \arg\min_{q \in \mathcal{Q}} g(q|q_k)$
7: $k \leftarrow k + 1$
8: **until** certain convergence criterion is met

E. Damping Factor Selection

The damping factor λ in equation (4) controls the sparsity of the recoverd signal. We propose to use the cross-validation (CV) [21] approach to choose the optimal value of damping factor, which works for parameter selection according to its statistical performance. Cross-validation provides the optimal bias-variance tradeoff under the setting of equation (4). The K-fold CV is adopted roughly as: First, randomly split the data into K equal clusters; Second, the kth cluster ($k = 1, 2, \cdots, K$) is used for testing the optimization model fitted using the data from other cluster; Third, the CV prediction error for each λ is calculated; Then, repeat the random selection; In the end, combine all the K prediction errors together to obtain the CV estimate:

$$CV(\lambda) = \frac{1}{K} \sum_{k=1}^{K} \left(\mathbf{s}_k - \mathbf{W}_k \hat{\mathbf{r}}^{(-k)}(\lambda) \right)^2, \tag{6}$$

leading to the chosen optimal $\hat{\lambda} = \arg\min_{\lambda \in [a,b]} CV(\lambda)$, where $[a, b], a, b \in \mathbb{R}_+$ is the given range of λ.

III. Experiments

A. Synthetic Example

We first test the proposed method in numerical experiments. Compared with synthetic experiments in [10] and [15], we designed the sparse reflectivity series randomly instead of using fixed positions, which makes more sense for the non-stationary seismic applications.

Table I shows the comparisons of the similarity measurement between ground truth and inverted reflectivity with different SNRs. ℓ_2, ℓ_1 and $\ell_{p,q}$ regularizations are applied to invert the random reflectivity profile, respectively. SNR varies from 20 dB to -10 dB, which covers possible SNR situations of field applications. And Monte-Carlo experiments are conducted 100 times for every different situation. Note that the proposed $\ell_{p,q}$ method achieves the highest inversion accuracies at different noise levels with relatively stable performances. Specifically, when the data have a low SNR, the proposed regularization constraint produces the most robust results.

TABLE I. Similarity between the ground truth and inverted reflectivity profile from 100 times Monte-Carlo experiments of every situation. ('std' stands for standard deviation.)

SNR (dB)	Regularization					
	ℓ_2		ℓ_1		$\ell_{p,q}$	
	mean	std	mean	std	mean	std
20	0.5038	0.0625	0.8602	0.0402	0.8624	0.0424
10	0.4991	0.0726	0.8362	0.0426	0.8541	0.0453
3	0.4701	0.0670	0.8081	0.0492	0.8279	0.0473
0	0.4292	0.0650	0.7896	0.0564	0.7981	0.0556
-3	0.3914	0.0694	0.7417	0.0738	0.7840	0.0584
-10	0.3685	0.0690	0.5156	0.1865	0.7385	0.0737

To visualize the results, Figure 2(a) shows one of synthetic Monte-Carlo reflectivity models consisting of randomly located, sparse reflectivity series. Based on the convolution model, we obtain the seismic data by convolving the reflectivity profile with a 30 Hz Ricker wavelet. The synthetic seismic trace is shown in Figure 2(b). Figure 2(c) displays a noisy seismic trace with 10 dB random noise. Figures 2(d)-2(f) demonstrate the inverted results from the noisy seismic data based on the regularizations of ℓ_2, ℓ_1 and proposed ℓ_q norms. Our proposed ℓ_q method and ℓ_1 norm successfully recovered the sparsity of the reflectivity, where ℓ_2 norm failed to do so. Compared to ℓ_1 norm, the proposed ℓ_q method captured more detailed information, such as small-amplitude picks, successive picks, than ℓ_1 norm method. Due to the highly non-convexity of the ℓ_q regularization, our proposed method distinguishes the true non-zero reflectivity components from noises by projected observed data onto a ℓ_q-ball. By searching the ℓ_q-ball using MM algorithm, the true reflectivity can be efficiently recovered from the noisy data. The results demonstrated that, under the noise contamination, the ℓ_q method outperforms the popular seismic inversion techniques such as ℓ_1 or ℓ_2 norm regularized inversion.

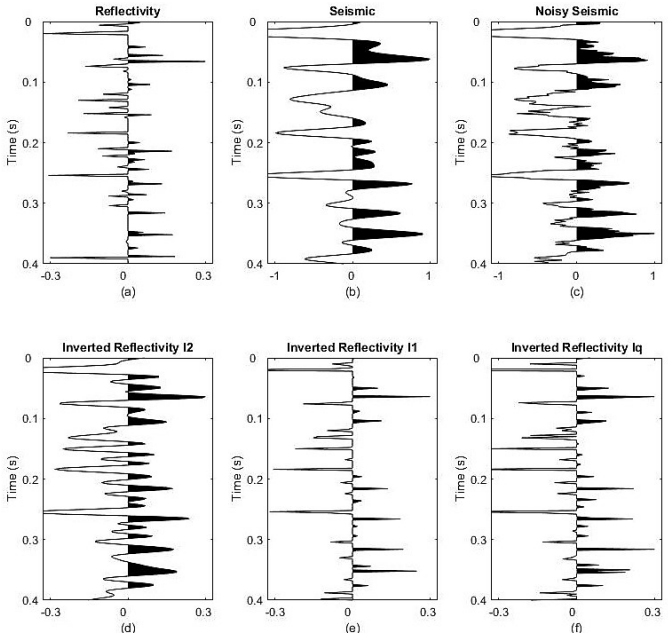

Fig. 2. Synthetic example. (a) Random reflectivity; (b) Synthetic seismic trace; (c) Noisy seismic trace with 10 dB noise; And Inverted reflectivity from (d) ℓ_2 norm, (e) ℓ_1 norm and (f) the proposed ℓ_q method.

B. Field Application

We apply the proposed method to seismic field data from Bohai Bay Basin, China. The target reservoir in this area is the siliciclastic reservoir, where the thin bed structure develops, so the high resolution interpretation, especially the reflectivity inversion is of great importance.

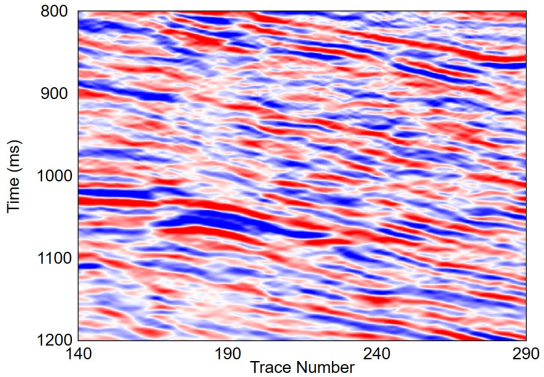

Fig. 3. Field seismic data from Bohai Bay Basin, China.

We display the original seismic amplitude in Figure 3. Figure 4 is the inverse result of ℓ_2 norm regularization with 5-fold CV. We see that it fails to remove the noise from observations or recover the sparse reflectivity series. Compared to ℓ_2 method, ℓ_1 and ℓ_q methods shown in Figure 5-7 provide more sparse recovered reflectivity series. The optimal ℓ_q inverse result with $q = 0.7$ is shown in Figure 6. We see that our proposed ℓ_q method recovered more detailed information when comparing to the l_1 method, and more thin beds can

be characterized now. Note that the damping factors for both methods are chosen using 5-fold CV.

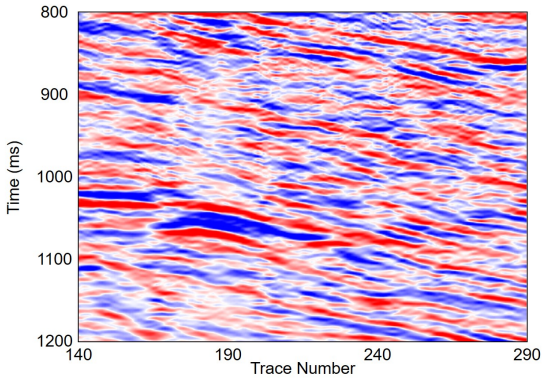

Fig. 4. Seismic inversion using ℓ_2 norm.

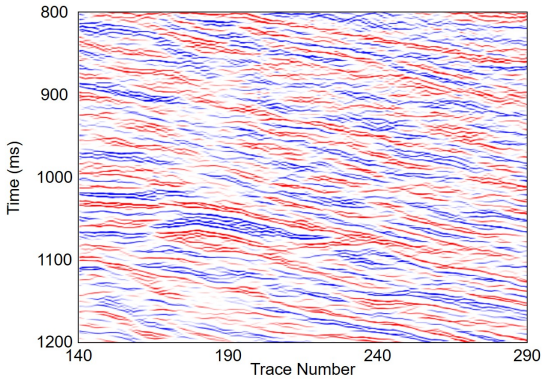

Fig. 5. Seismic inversion using ℓ_1 norm.

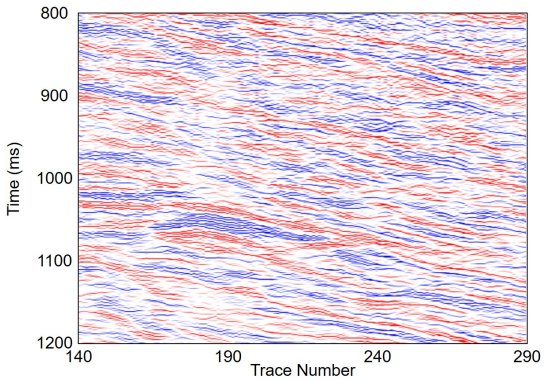

Fig. 6. Seismic inversion using ℓ_q norm.

To demonstrate the effectiveness of choice of damping factors, we compare the optimal ℓ_q inversion (Figure 6) with an over regularized ℓ_q inversion (Figure 7). We see that a proper regularization, i.e. finding optimal damping factor using CV, will help keeping the continuity of the inversion result while maintaining the sparsity.

To further evaluate the inverted results, we compare the ℓ_1 and ℓ_q norm inversion reflectivity profiles with the reflectivity calculated from the sonic logs, shown in Figure 8. The results from the proposed ℓ_q norm inversion show more details and have better correspondence with the reflectivity from well

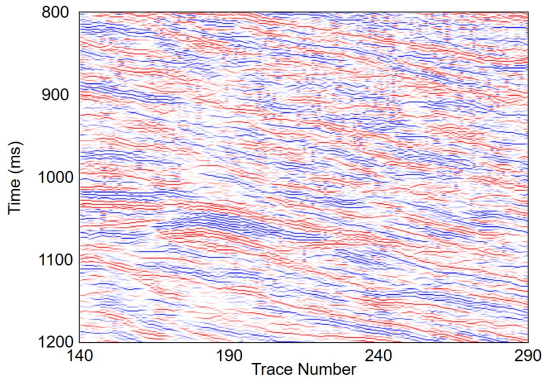

Fig. 7. Seismic inversion using ℓ_q norm without optimal regularization parameter selection.

logs. In addition, we also calculate the relative similarity coefficient, $\rho(\cdot, \text{Refl}_{log})/\rho(\text{Seismic}, \text{Refl}_{log})$, between the inverted reflectivity and the sonic reflectivity with $\rho(\cdot)$ as the cross correlation function, which are 0.5843 with the ℓ_q norm and 0.4347 with the ℓ_1 norm. So from both qualitative and quantitative comparisons, the proposed method is better than the existing methods.

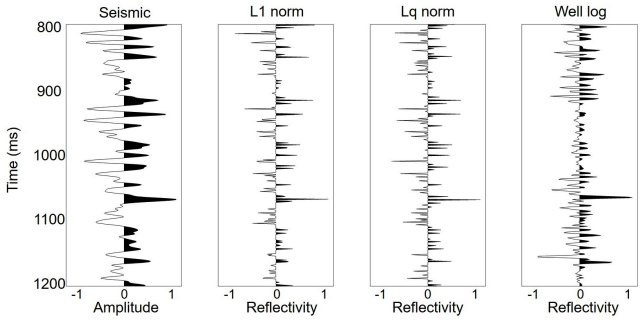

Fig. 8. Comparison among the reflectivity calculated from the sonic logs, the inverted reflectivity based on ℓ_1 and ℓ_q norms. (The trace number is 209.)

IV. Conclusions

We propose a novel sparse reflectivity inversion method with $\ell_{p,q}$ norm regularization and optimal damping factor selection. The non-convex constraints have drawn a lot of attentions, because it satisfies the mathematical assumptions of seismic reflectivity series. Illustrating the case of $p = 2$, we adaptively adopt the ℓ_q $(0 < q < 1)$ norm constraint. In addition, because the estimates of q and λ value are fully data-driven, we believe the inverted results are relatively optimal. The synthetic examples verify that the proposed method can eliminate the effects of wavelet effectively and can obtain the stable reflectivity series even with moderate random noises. Furthermore, the derived reflectivity in the field example shows high correspondence with the reflectivity calculated from well logs.

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