



Digital Image Processing

Lecture #3
Ming-Sui (Amy) Lee

Announcement

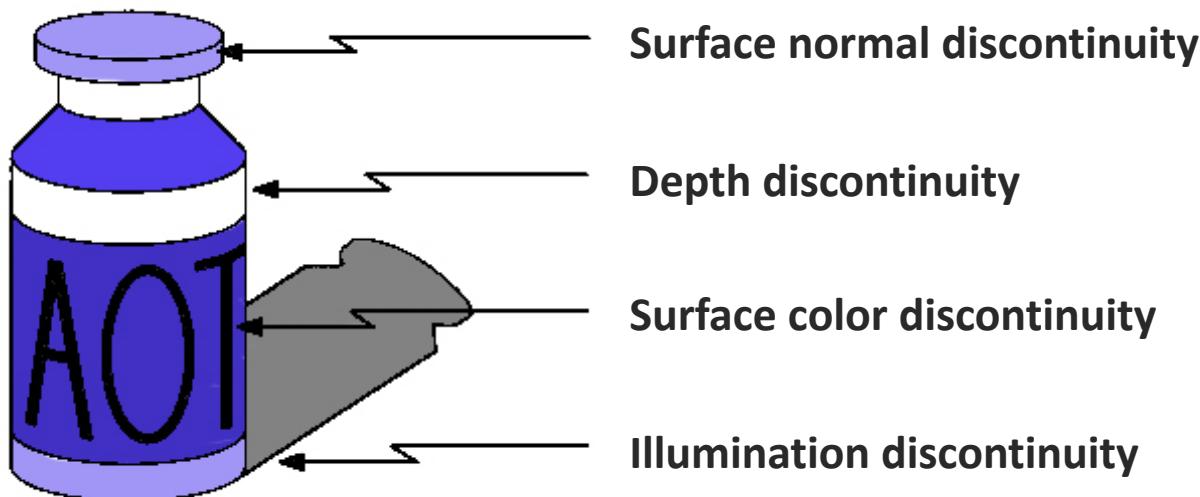
■ Class Information

- Homework #1
 - Package posted on the course website
 - Homework assignment #1
 - Submission guideline
 - Deadline: 11:59 p.m. on Mar. 24, 2021
 - Upload to
 - NTU COOL: Electronic version
 - Gradescope: Written report

Edge Crispening

Edges

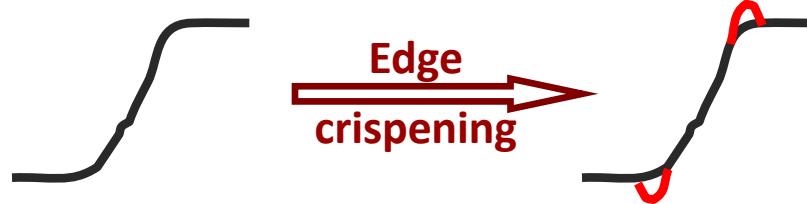
- Edges are caused by a variety of factors



[Edge Crispening]

■ Motivation

- A photograph with accentuated edges look more appealing



$$H = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Edge → high frequency
- High pass filtering

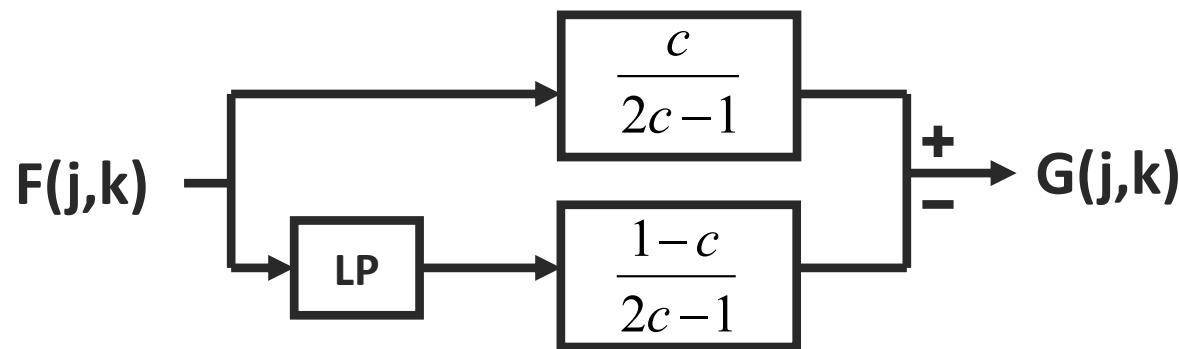
$$H = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

→ amplify the noise at the same time

Edge Crispening

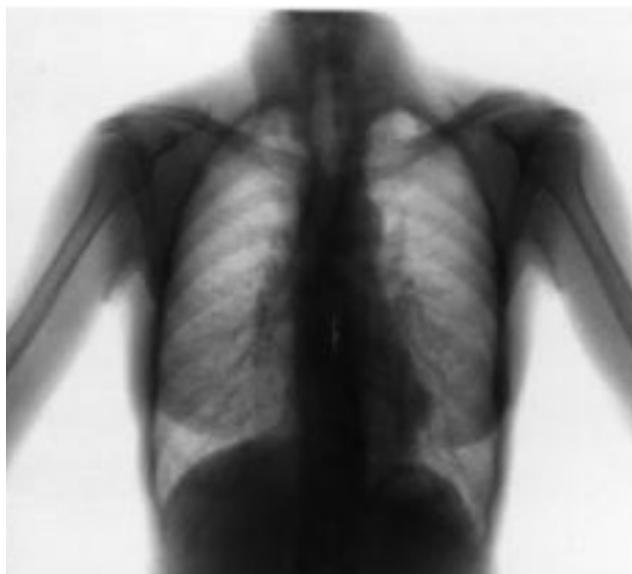
■ Unsharp Masking

- Appropriate combination of all-pass and low-pass filters

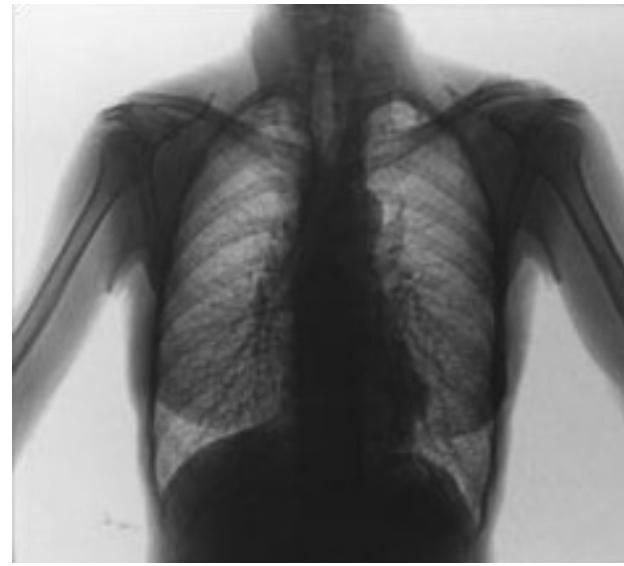


$$G(j,k) = \frac{c}{2c-1}F(j,k) - \frac{1-c}{2c-1}F_L(j,k), \text{ where } \frac{3}{5} \leq c \leq \frac{5}{6}$$

Edge Crispening



Original image



After sharpening
 $L=7, c=0.6$



(a) Normal resolution



(b) Low resolution



(c) Unsharp masking

Edge Detection

Edge Detection

■ Motivation

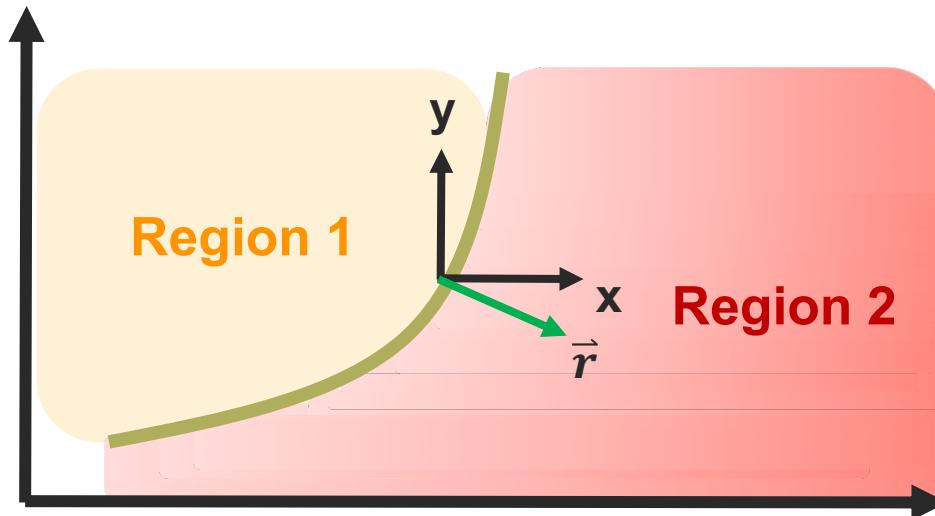
- Human eyes are more sensitive to edges
- Characterize object boundaries
- Fundamental step in image analysis
 - Segmentation, registration, identification, etc.

■ Edge description

- Model-based methods
 - Rarely used
- Non-parametric approaches
 - 1st and 2nd order derivatives

Edge Detection (1st order)

Orthogonal gradient generation



$$\frac{dF}{dr} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial r} = \left(\frac{\partial F}{\partial x} \frac{\partial F}{\partial y} \right) \cdot \begin{pmatrix} \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial r} \end{pmatrix} = \left(\frac{\partial F}{\partial x} \frac{\partial F}{\partial y} \right) \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

Edge Detection (1st order)

Orthogonal gradient generation

- When $\begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{pmatrix}$ (gradient direction) is parallel to $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$
 $\left\| \frac{dF}{dr} \right\|$ has maximum value

$$\Rightarrow \left\| \frac{dF}{dr} \right\| = \left\| \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{pmatrix} \right\| = \sqrt{\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2}$$

Magnitude

$$\theta = \tan^{-1} \left(\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} \right)$$

Orientation

Edge Detection (1st order)

Discrete case

Approximation I – 3 points

Row gradient

$$\frac{\partial F}{\partial x}(j,k) \cong F(j,k) - F(j,k-1) = G_R(j,k)$$

Column gradient

$$\frac{\partial F}{\partial y}(j,k) \cong F(j,k) - F(j+1,k) = G_C(j,k)$$

$$\Rightarrow G(j,k) = \sqrt{G_R^2(j,k) + G_C^2(j,k)}$$

A_0	A_1	A_2
A_7	$F(j,k)$	A_3
A_6	A_5	A_4

$$\theta(j,k) = \tan^{-1} \left(\frac{G_C(j,k)}{G_R(j,k)} \right)$$

[Edge Detection (1st order)]

■ Example

$$G_R(j,k) = F(j,k) - F(j,k-1)$$

$$G_C(j,k) = F(j,k) - F(j+1,k)$$



Original image



Horizontal magnitude



Vertical magnitude

Edge Detection (1st order)

Discrete case

- Approximation II – 4 points
- Roberts cross differentiation (0~90→45~135)
 - Row gradient
 - Column gradient

$$G_1(j, k) = F(j, k) - F(j + 1, k + 1)$$

■ Column gradient

$$G_2(j, k) = F(j, k + 1) - F(j + 1, k)$$

$$\Rightarrow G(j, k) = \sqrt{G_1^2(j, k) + G_2^2(j, k)} \quad \theta(j, k) = \tan^{-1} \left(\frac{G_2(j, k)}{G_1(j, k)} \right) + \frac{\pi}{4}$$

A_0	A_1	A_2
A_7	$F(j, k)$	A_3
A_6	A_5	A_4

Edge Detection (1st order)

Discrete case

Approximation III – 9points

Row gradient

$$G_R(j,k) = \frac{1}{K+2} [(A_2 + KA_3 + A_4) - (A_0 + KA_7 + A_6)]$$

Column gradient

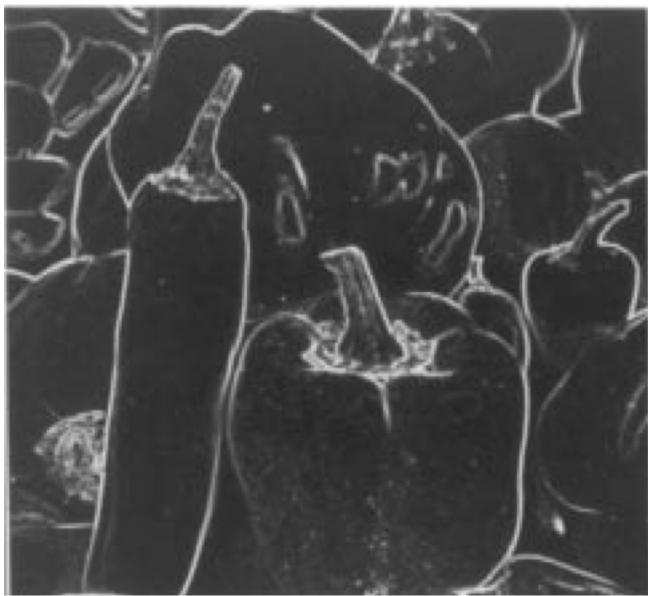
$$G_C(j,k) = \frac{1}{K+2} [(A_0 + KA_1 + A_2) - (A_6 + KA_5 + A_4)]$$

$$\Rightarrow G(j,k) = \sqrt{G_R^2(j,k) + G_C^2(j,k)} \quad \theta(j,k) = \tan^{-1} \left(\frac{G_C(j,k)}{G_R(j,k)} \right)$$

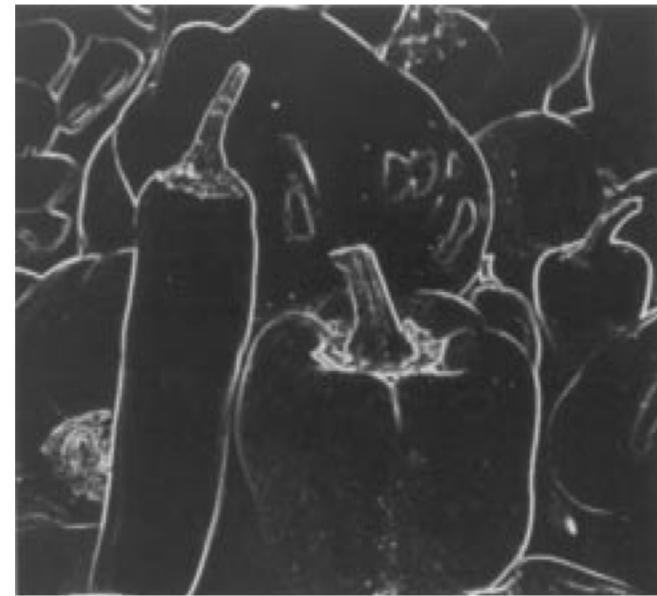
K=1: Prewitt Mask ; K=2: Sobel Mask

A_0	A_1	A_2
A_7	$F(j,k)$	A_3
A_6	A_5	A_4

[Edge Detection (1st order)]



Roberts magnitude



Roberts square root

Edge Detection (1st order)



- Compute row and column gradients
- ◆ Analyze the statistics of the magnitude (histogram)
- Pick a threshold T
- If $G(j,k) \geq T$
 - set it as an edge point
 - otherwise (If $G(j,k) < T$)
 - non-edge point
- Q: How to select T ?
- ◆ Examine the cumulative distribution function

Canny Edge Detector

■ Why Canny?

- Good Detection

- The optimal detector must minimize the probability of false positives as well as false negatives

- Good Localization

- The edges detected must be as close as possible to the true edges

- Single Response Constraint

- The detector must return one point only for each edge point

Canny Edge Detector

■ Five Steps:

- Noise reduction
- Compute gradient magnitude and orientation
- Non-maximal suppression
- Hysteretic thresholding
- Connected component labeling method



Smooth the image
with a Gaussian filter



Example:
5x5 Gaussian filter with $\sigma = 1.4$

$$F_{NR} = \frac{1}{159} \begin{bmatrix} 2 & 4 & 5 & 4 & 2 \\ 4 & 9 & 12 & 9 & 4 \\ 5 & 12 & 15 & 12 & 5 \\ 4 & 9 & 12 & 9 & 4 \\ 2 & 4 & 5 & 4 & 2 \end{bmatrix} * F$$



Canny Edge Detector

■ Five Steps:

- Noise reduction
- Compute gradient magnitude and orientation
- Non-maximal suppression
- Hysteretic thresholding
- Connected component labeling method

$$G(j,k) = \sqrt{G_R^2(j,k) + G_C^2(j,k)}$$

$$\theta(j,k) = \tan^{-1} \left(\frac{G_C(j,k)}{G_R(j,k)} \right)$$



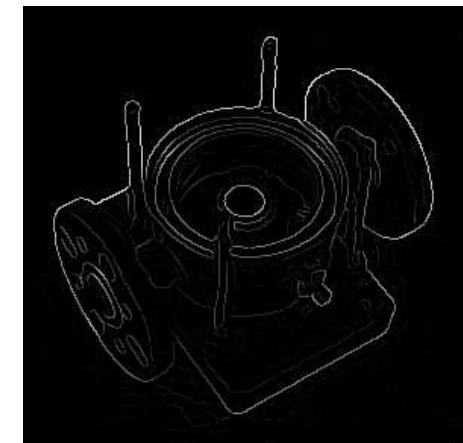
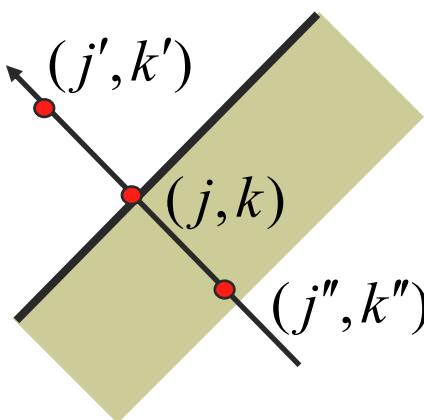
Canny Edge Detector

■ Five Steps:

- Noise reduction
- Compute gradient magnitude and orientation
- **Non-maximal suppression**
- Hysteretic thresholding
- Connected component labeling method

Search the nearest neighbors (j', k') and (j'', k'') along the edge normal

$$G_N(j, k) = \begin{cases} G(j, k) & \text{if } G(j, k) > G(j', k') \\ & \text{and } G(j, k) > G(j'', k'') \\ 0 & \text{otherwise} \end{cases}$$



Canny Edge Detector

■ Five Steps:

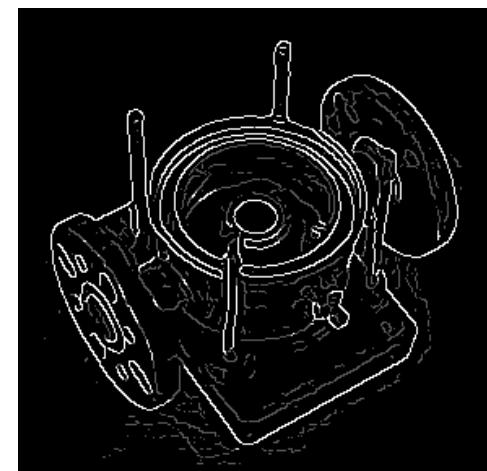
- Noise reduction
- Compute gradient magnitude and orientation
- Non-maximal suppression
- **Hysteretic thresholding**
- Connected component labeling method

Label each pixels according to two threshold: T_H, T_L

$G_N(x, y) \geq T_H$ Edge Pixel

$T_H > G_N(x, y) \geq T_L$ Candidate Pixel

$G_N(x, y) < T_L$ Non-edge Pixel

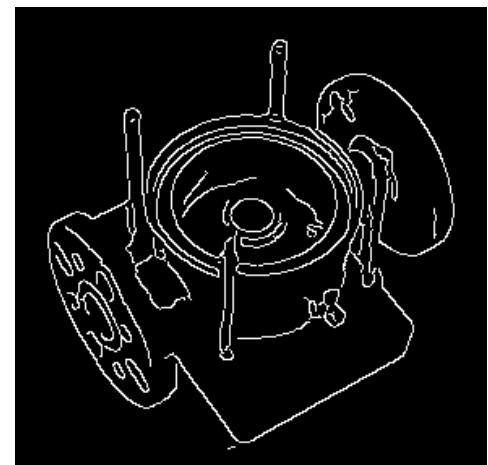


Canny Edge Detector

■ Five Steps:

- Noise reduction
- Compute gradient magnitude and orientation
- Non-maximal suppression
- Hysteretic thresholding
- Connected component labeling method

If a candidate pixel is connected to an edge pixel directly or via another candidate pixel then it is declared as an edge pixel



Edge Map

Canny Edge Detector



Input



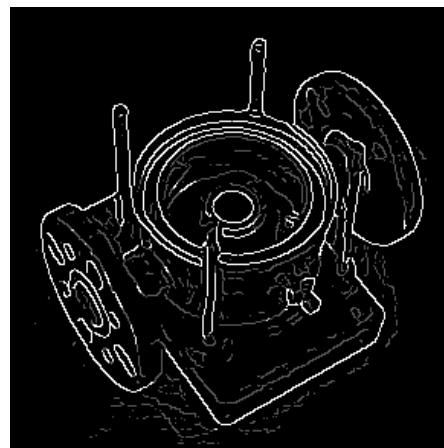
Step 1



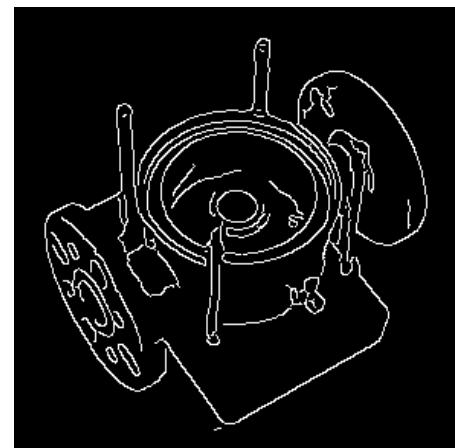
Step 2



Step 3



Step 4



Step 5

Edge Detection – Part II

Edge Detection (2nd order)

■ Why 2nd order?

- Significant spatial change occurs

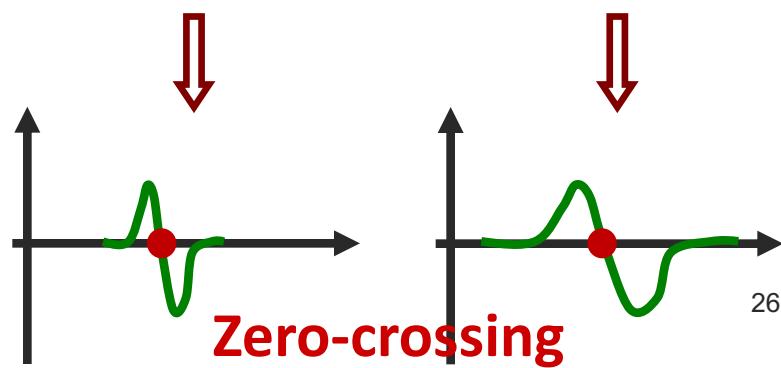
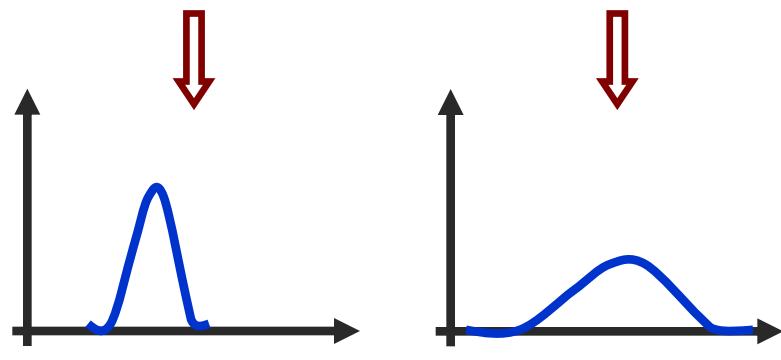
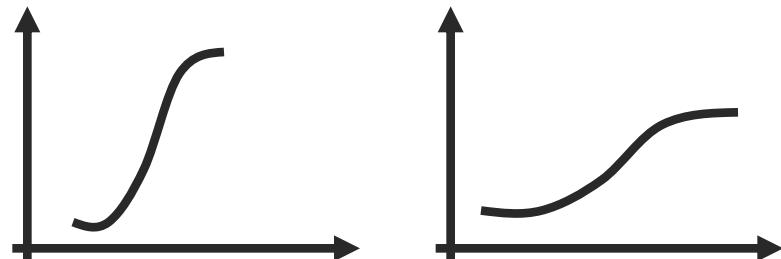
1st order derivative

$$\frac{\partial F}{\partial x}$$

2nd order derivative

$$-\frac{\partial^2 F}{\partial x^2}$$

1D data



Edge Detection (2nd order)

■ Laplacian Generation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \Rightarrow \quad \nabla^2 F(x, y) = \frac{\partial^2 F(x, y)}{\partial x^2} + \frac{\partial^2 F(x, y)}{\partial y^2}$$

■ Discrete Approximation

$$-\frac{\partial^2}{\partial x^2} \approx \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \quad \Rightarrow \quad 2f(x) - (f(x+h) + f(x-h))$$

○ By Taylor series expansion

$$\begin{aligned} & 2f(x) - (f(x+h) + f(x-h)) \\ = & 2f(x) - \left[f(x) + hf'(x) + \frac{h^2}{2} f''(x) + f(x) + (-h)f'(x) + \frac{h^2}{2} f''(x) + \dots \right] \\ \approx & -h^2 f''(x) \end{aligned}$$

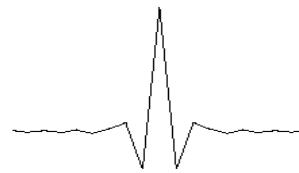
[Edge Detection (2nd order)]

■ Discrete Approximation

$$-\frac{\partial^2}{\partial x^2} \cong [-1 \ 2 \ -1] \quad -\frac{\partial^2}{\partial y^2} \cong [-1 \ 2 \ -1]^T$$

- combine together

$$-\nabla^2 \cong \frac{1}{4} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}; \quad \nabla^2 \cong \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

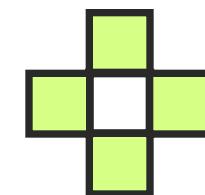


gain-normalized²⁸

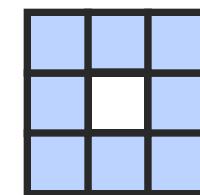
[Edge Detection (2nd order)]

- Laplacian impulse response
 - four-neighbor

$$H = \frac{1}{4} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$



4-neighbor



8-neighbor

- eight-neighbor

$$H = \frac{1}{8} \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

non-separable

$$H = \frac{1}{8} \begin{bmatrix} -2 & 1 & -2 \\ 1 & 4 & 1 \\ -2 & 1 & -2 \end{bmatrix}$$

separable

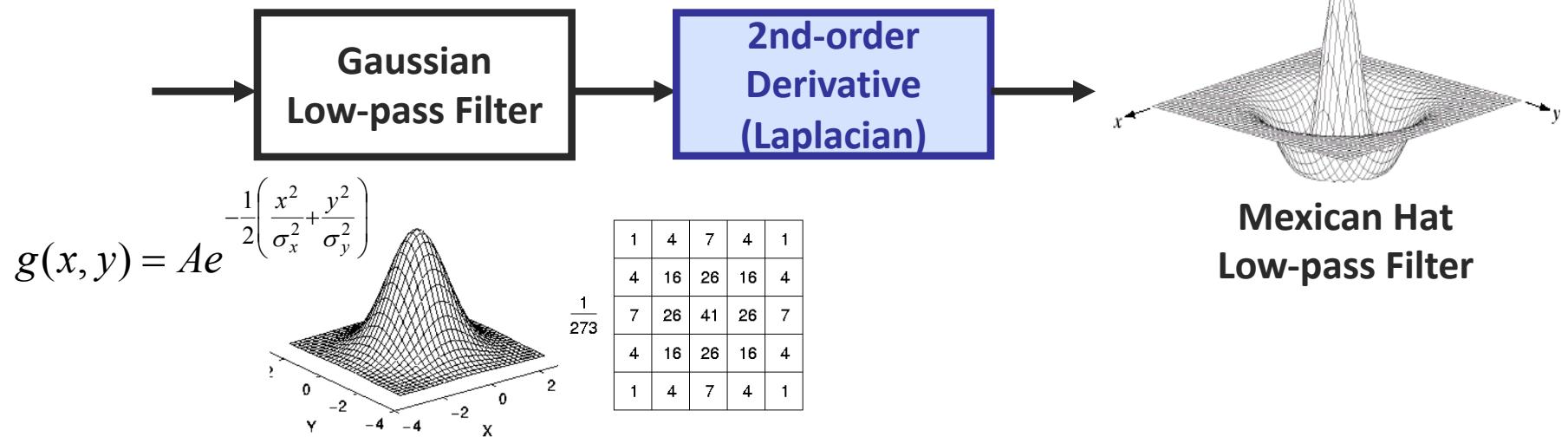
$$H_1 = \frac{1}{8} \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{bmatrix}$$

$$H_2 = \frac{1}{8} \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$$

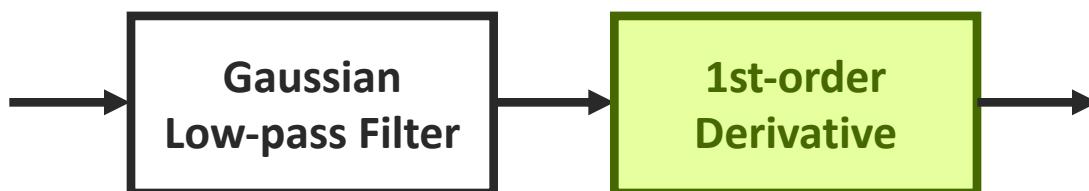
²⁹

Edge Detection (2nd order)

Laplacian of Gaussian (LOG)

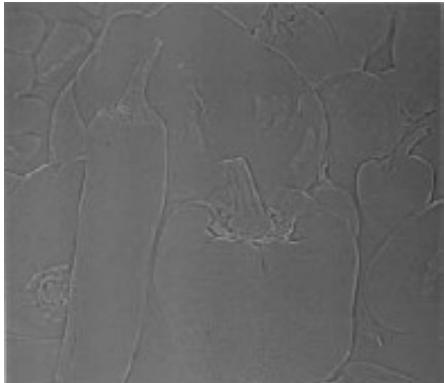


Difference of Gaussians (DOG)



[Edge Detection (2nd order)]

■ Examples



Four-neighbor



Eight-neighbor



Separable
eight-neighbor



Laplacian of
Gaussian (LOG)

Are we done yet?

Edge Detection (2nd order)

■ 2nd Order Edge Detection



- How to detect zero-crossing?
 - many ways

Edge Detection (2nd order)

■ Zero-crossing



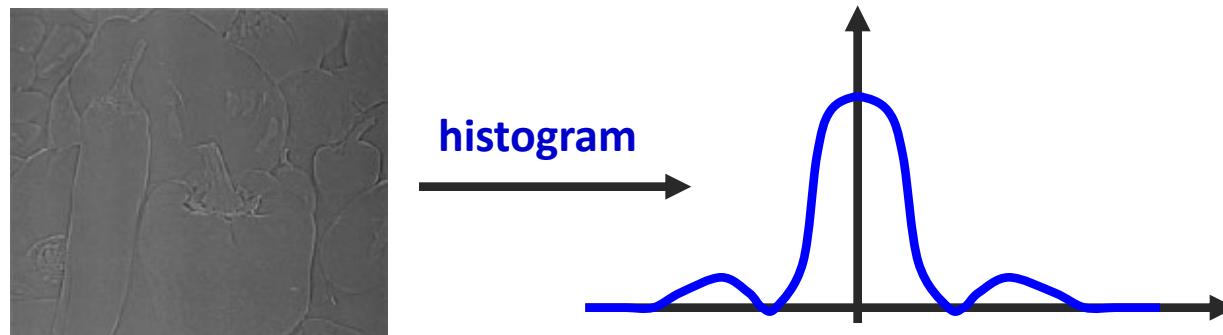
3 steps:

- Generate the histogram of G
- Set up a threshold to separate zero and non-zero, output as G'
- For $G'(j,k)=0$, decide whether (j,k) is a zero-crossing point

Edge Detection (2nd order)

■ Zero-crossing

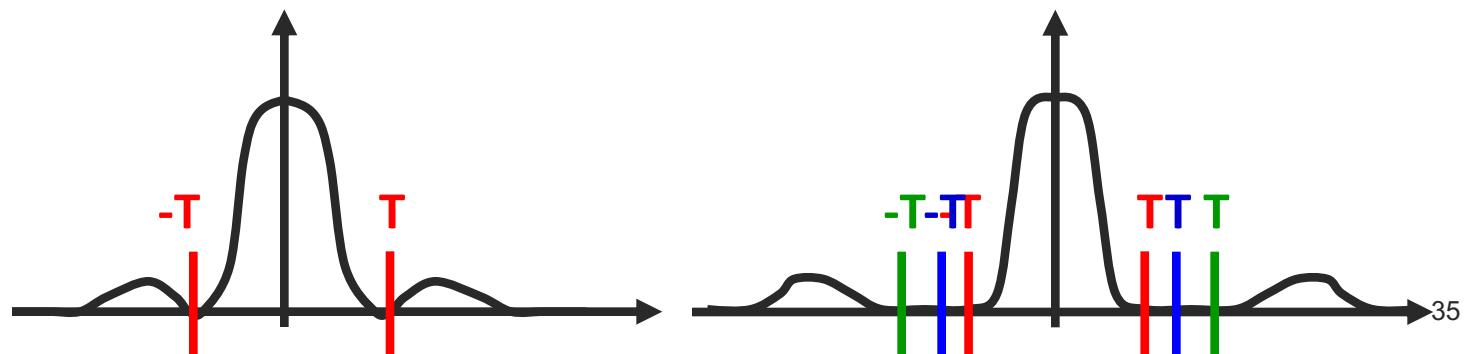
- 3 steps:
 - Generate the histogram of G
 - Set up a threshold to separate zero and non-zero to get G'
 - For $G'(j,k)=0$, decide whether (j,k) is a zero-crossing point



Edge Detection (2nd order)

■ Zero-crossing

- 3 steps:
 - Generate the histogram of G
 - Set up a threshold to separate zero and non-zero to get G' $|G(j,k)| \leq T \Rightarrow G'(j,k) = 0$
 - For $G'(j,k)=0$, decide whether (j,k) is a zero-crossing point

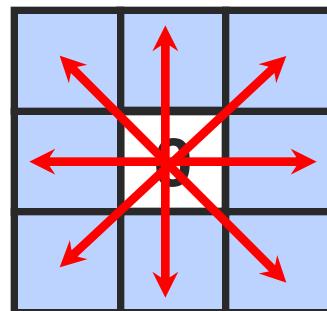


Edge Detection (2nd order)

■ Zero-crossing

- 3 steps:
 - Generate the histogram of G
 - Set up a threshold to separate zero and non-zero to get G'
 - **For $G'(j,k)=0$, decide whether (j,k) is a zero-crossing point → edge map**

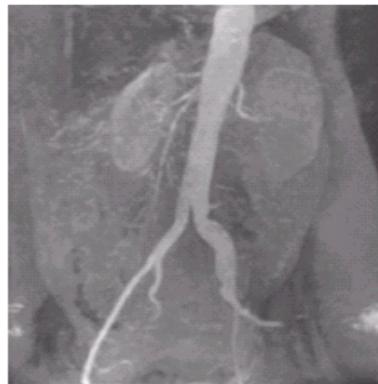
$$G'(j, k) = 0$$



e.g. $\{-1, 0, 1\}$

Edge Detection (2nd order)

Example



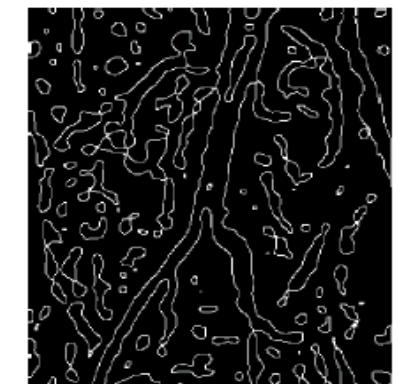
Original



Laplacian



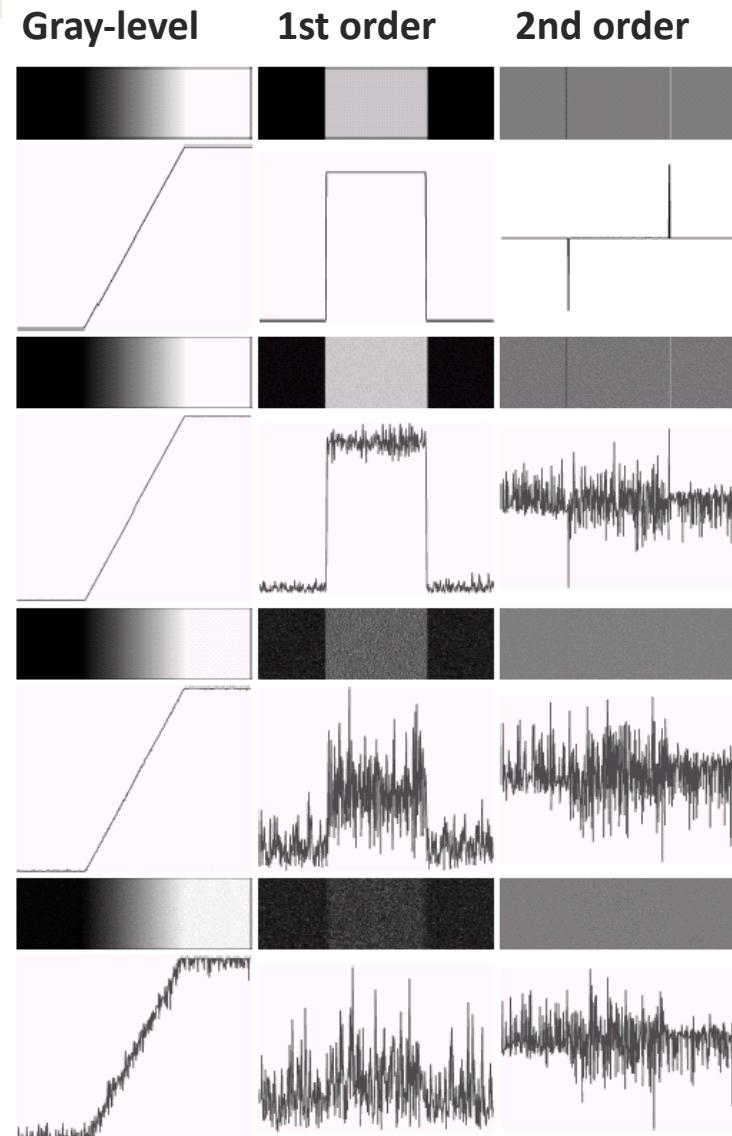
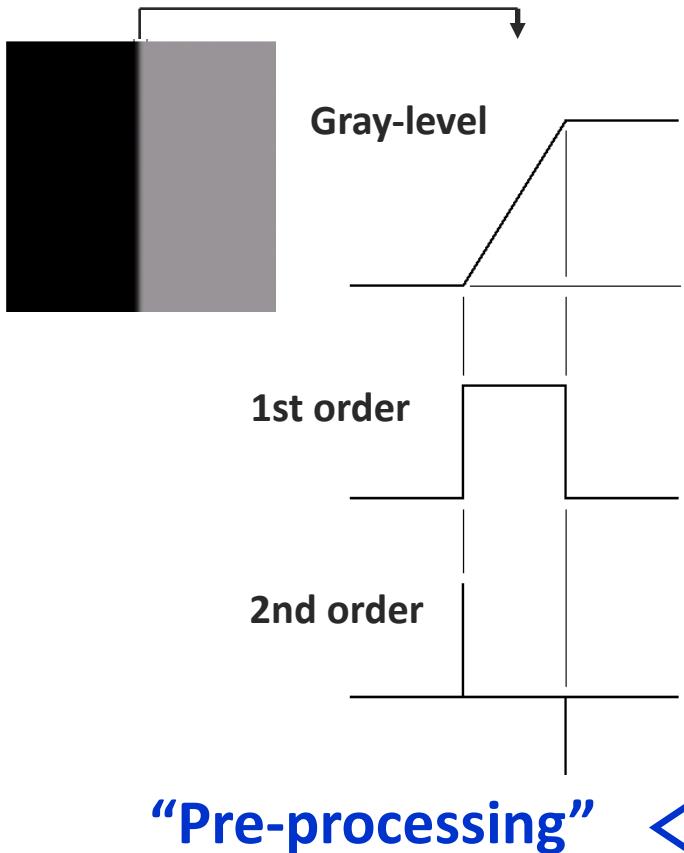
Thresholded



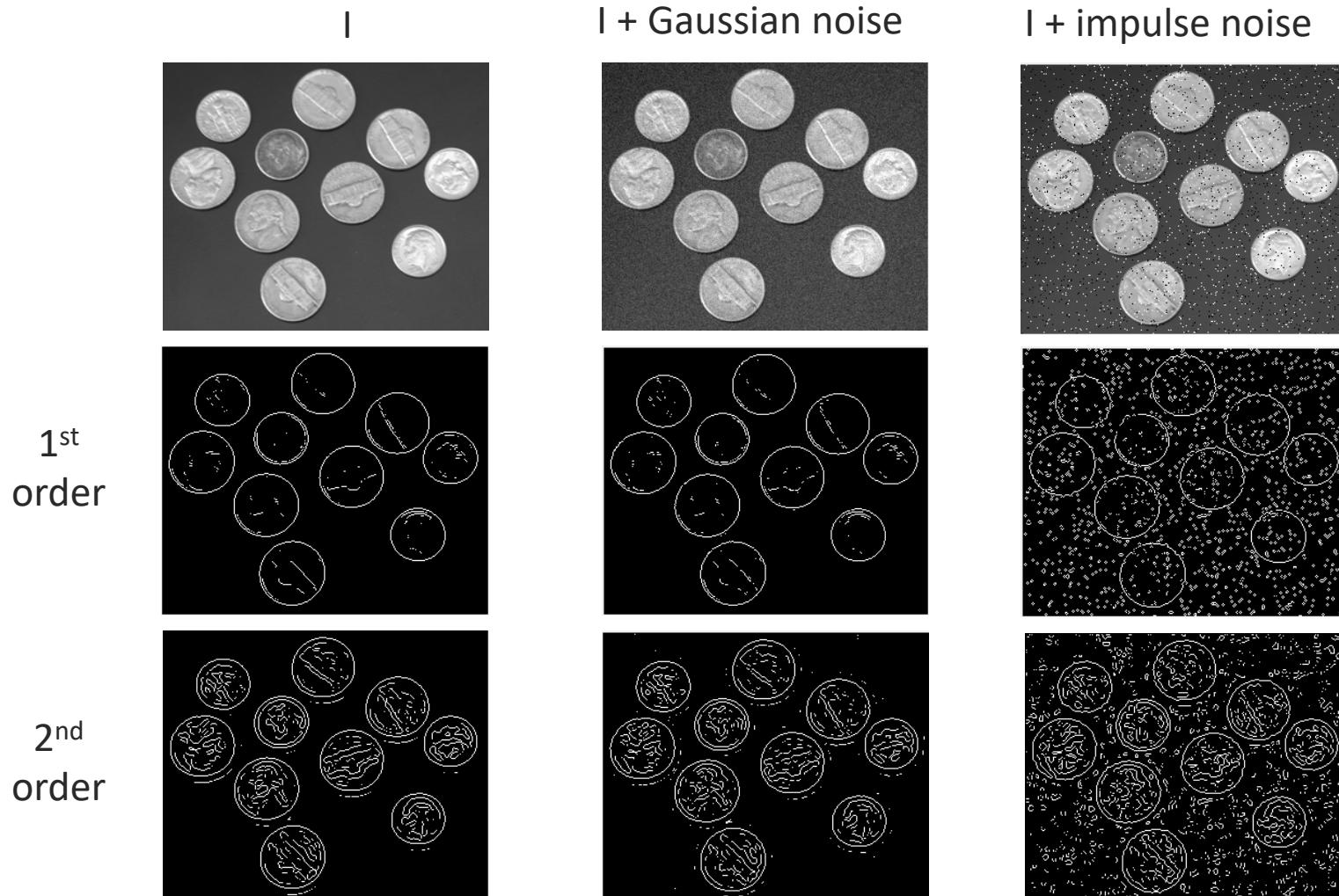
Edge map

Edge Detection

Noisy image



Edge Detection

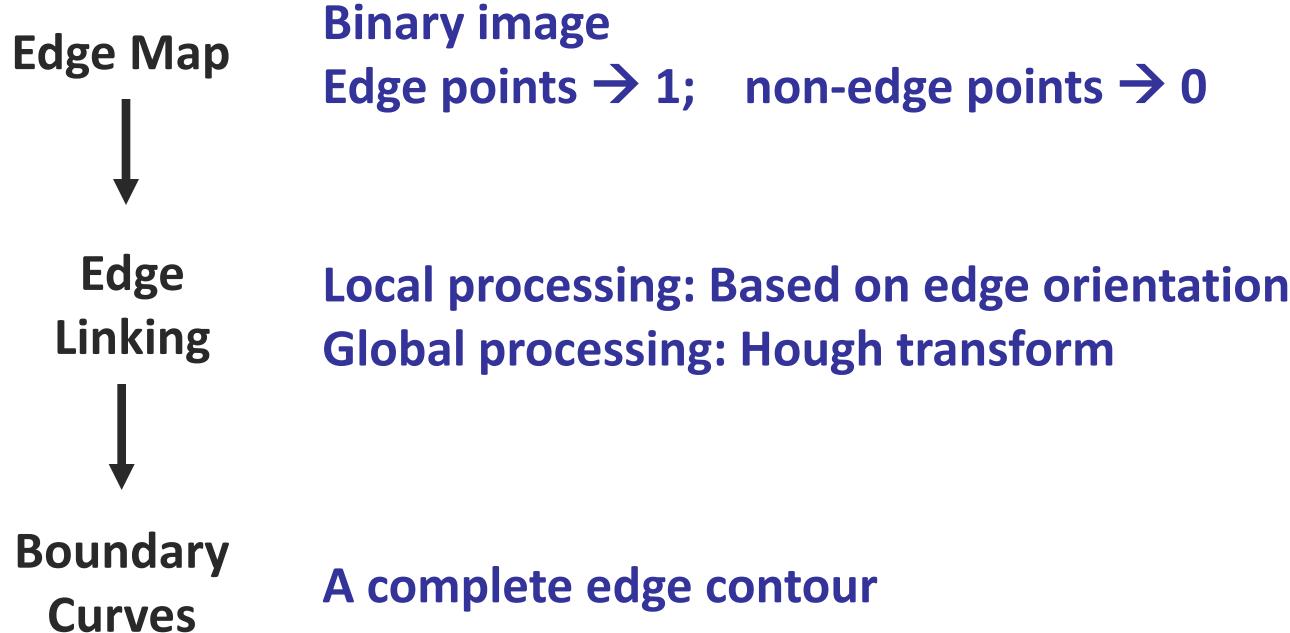


[

Edge Detection

]

■ Post-Processing



How can we detect lines?

■ Option 1:

- Search for the line at every possible position/orientation

■ Option 2:

- Hough transform
 - performed after edge detection
- Advantages
 - tolerant of gaps in the edges
 - relatively unaffected by noise
 - unaffected by occlusion in the image

Hough Transform

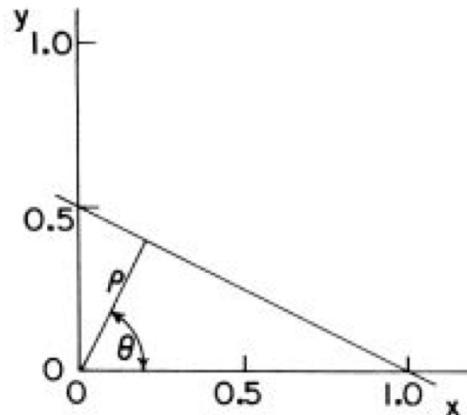
- A straight line can be represented as

$$y = mx + b$$

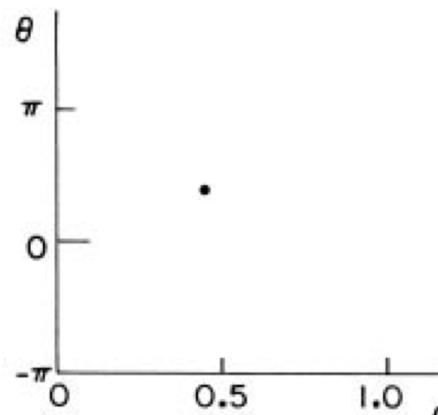
- fails in case of vertical lines

- A more useful representation is

$$\rho = x\cos\theta + y\sin\theta$$



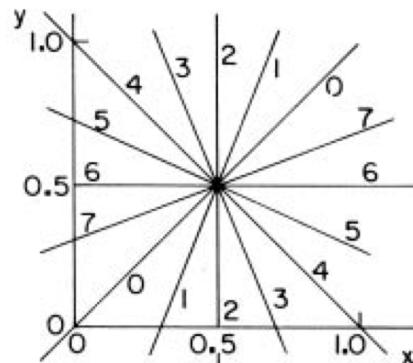
(a) Parametric line



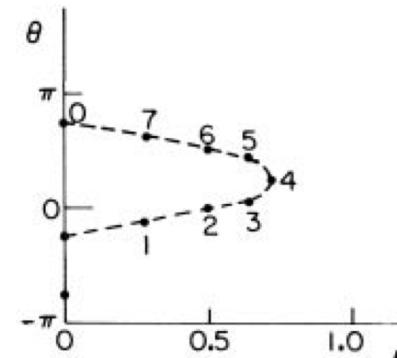
(b) Hough transform of (a)

Hough Transform

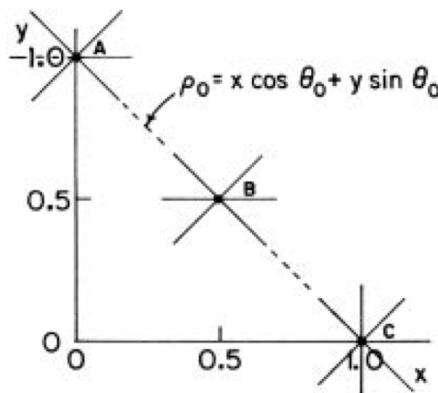
$$\rho = x \cos \theta + y \sin \theta$$



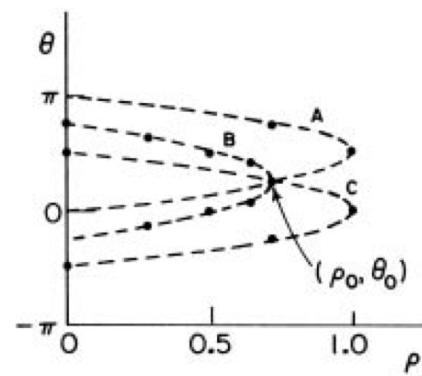
(c) Family of lines, common point



(d) Hough transform of (c)



(e) Colinear points



(f) Hough transform of (e)

Review

Review

- Noise Cleaning
 - Uniform noise → low-pass filtering
 - Impulse noise → non-linear filtering
 - Mixed noise → ?
- Edge Crispening
 - Unsharp masking
- Edge Detection
 - 1st-order edge detection -- threshold
 - 2nd-order edge detection -- zero-crossing

Geometrical Modification

Geometrical Modification

■ Goal

- Translate, scale, rotate, reflect or nonlinear warp an image

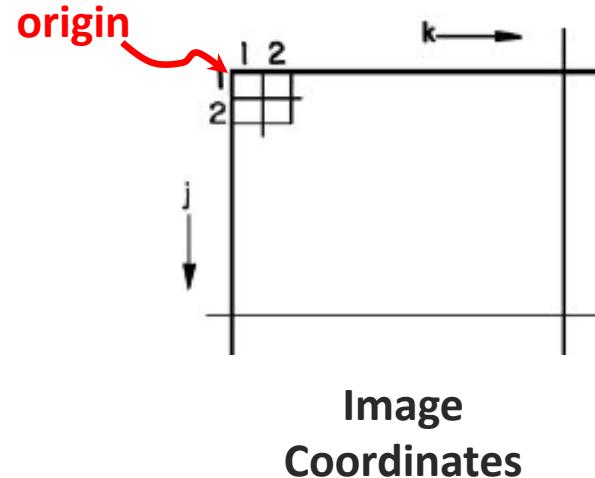
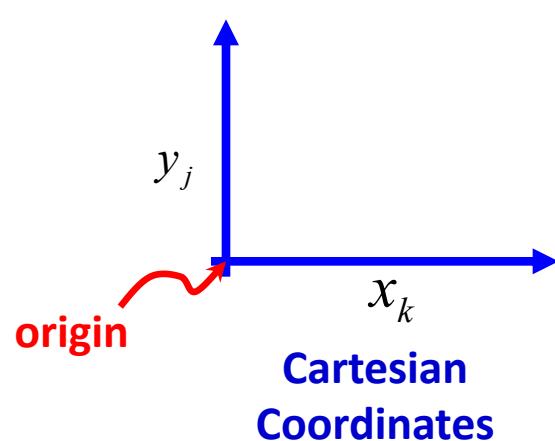
■ Applications

- Zoom-in/zoom-out
- Image registration
- Image mosaicking
 - <https://www.youtube.com/watch?v=wzEfkQ6zHZA>
 - <http://www.vision.huji.ac.il/dynmos/>
- Special effects
 - Use 2D image to simulate the 3D environment
 - <http://www.kula3d.com/how-to-use-the-cross-eyed-method.html>

Geometrical Modification

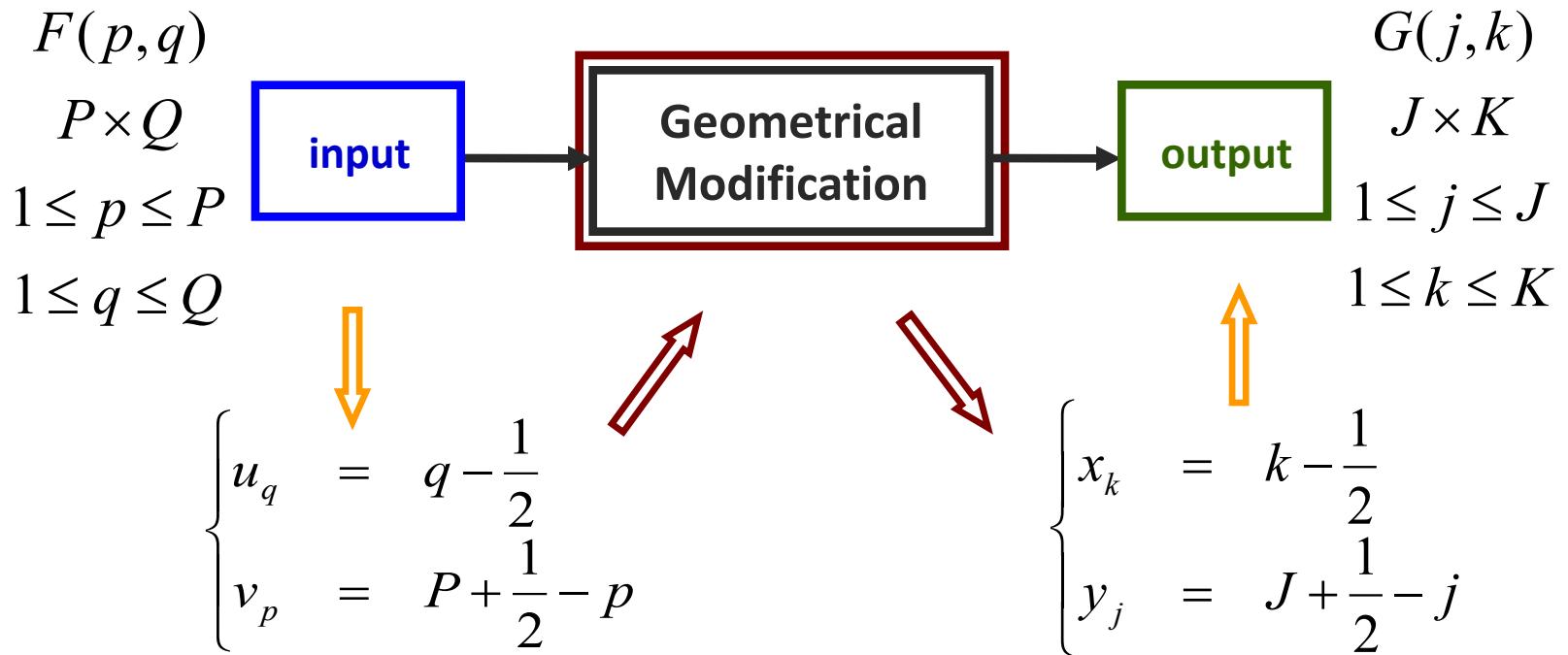
■ Coordinates

- Geometrical transformations
 - Cartesian coordinates
- Discrete image
 - Cartesian coordinates v.s. Image coordinates



Geometrical Modification

Linear/Affine coordinates transformation



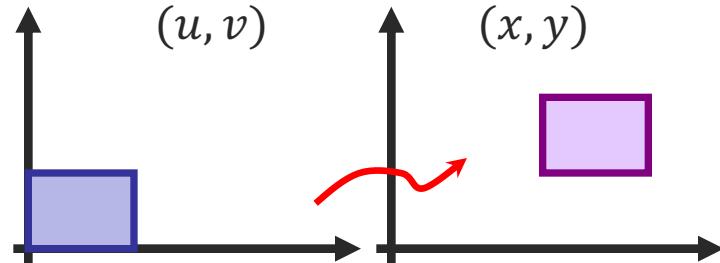
Geometrical Modification

$$F(p, q) \rightarrow G(j, k)$$

■ Translation (Shift)

$$\begin{cases} x_k &= u_q + t_x \\ y_j &= v_p + t_y \end{cases}$$

substitute $\begin{cases} u_q &= q - \frac{1}{2} \\ v_p &= P + \frac{1}{2} - p \end{cases}$ and $\begin{cases} x_k &= k - \frac{1}{2} \\ y_j &= J + \frac{1}{2} - j \end{cases}$



$$\Rightarrow \begin{cases} k' &= q + t_x \\ j' &= p - (P - J) - t_y \end{cases} \quad \begin{cases} k &= q' + t_x \\ j &= p' - (P - J) - t_y \end{cases}$$

Forward treatment

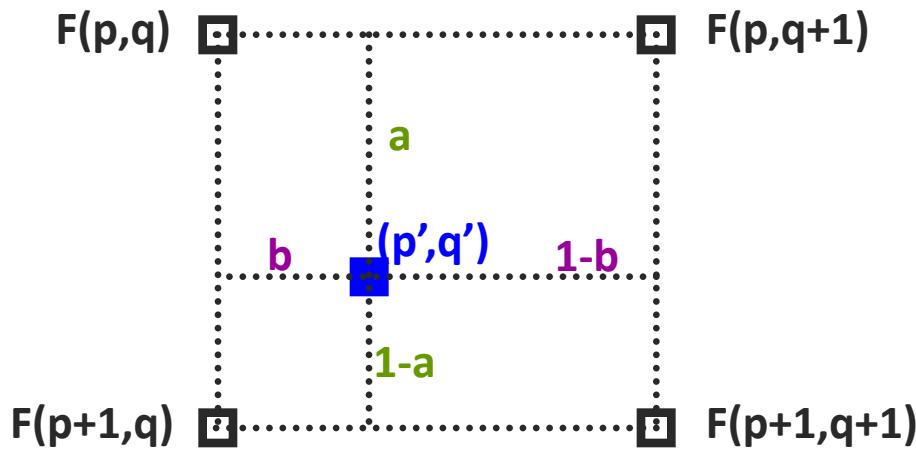
Backward treatment

Better

Geometrical Modification

■ Translation (Shift)

- Non-integer pixel positions
- i.e. How to compute p' and q' ?
 - Bilinear interpolation

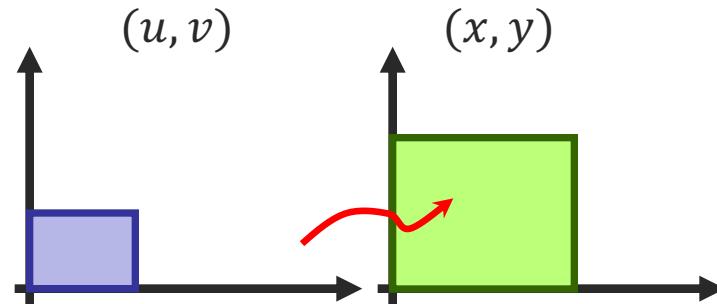


$$\begin{aligned} F(p', q') = & (1-a)(1-b)F(p, q) \\ & + (1-a)bF(p, q+1) \\ & + a(1-b)F(p+1, q) \\ & + abF(p+1, q+1) \end{aligned}$$

Geometrical Modification

Scaling

$$\begin{cases} x_k &= s_x u_q \\ y_j &= s_y v_p \end{cases}$$



where s_x & s_y are scaling parameters, and s_x & $s_y > 0$

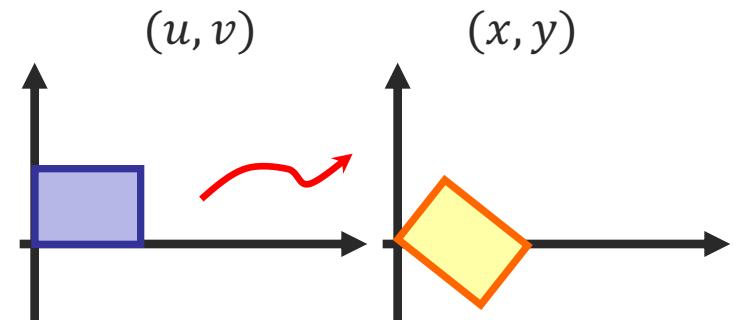
$\begin{cases} s_x \text{ & } s_y > 1: \text{ magnification} \\ s_x \text{ & } s_y < 1: \text{ minification} \end{cases}$

$$\Rightarrow \begin{cases} p' &= \frac{1}{s_y} \left(j - J - \frac{1}{2} \right) + P + \frac{1}{2} \\ q' &= \frac{1}{s_x} \left(k - K - \frac{1}{2} \right) + Q + \frac{1}{2} \end{cases}$$

Geometrical Modification

■ Rotation

$$\begin{cases} x_k &= u_q \cos \theta - v_p \sin \theta \\ y_j &= u_q \sin \theta + v_p \cos \theta \end{cases}$$



Rotate by an angle with respect to the origin of the
Cartesian coordinates

What if the reference point is not the origin of the
Cartesian coordinate?

Geometrical Modification

■ Generalized Linear Geometrical Transformations

➤ translation

$$\begin{bmatrix} x_k \\ y_j \end{bmatrix} = \begin{bmatrix} u_q \\ v_p \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

➤ scaling

$$\begin{bmatrix} x_k \\ y_j \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} u_q \\ v_p \end{bmatrix}$$

➤ rotation

$$\begin{bmatrix} x_k \\ y_j \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_q \\ v_p \end{bmatrix}$$

Geometrical Modification

- Generalized Linear Geometrical Transformations
 - Compound operator

$$\begin{bmatrix} u_q \\ v_p \end{bmatrix} \rightarrow \text{translation} \rightarrow \text{scaling} \rightarrow \text{rotation} \rightarrow \begin{bmatrix} x_k \\ y_j \end{bmatrix}$$

$$\begin{bmatrix} x_k \\ y_j \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \left[\begin{pmatrix} u_q \\ v_p \end{pmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \right]$$

How to convert the above affine system to a linear one?

Geometrical Modification

■ Generalized Linear Geometrical Transformations

- Expand the system from 2D to 3D

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_k \\ y_j \\ 1 \end{bmatrix} = R(\theta)S(s_x, s_y)T(t_x, t_y) \begin{bmatrix} u_q \\ v_p \\ 1 \end{bmatrix} \quad \xrightarrow{\text{red arrow}} \quad \begin{bmatrix} u_q \\ v_p \\ 1 \end{bmatrix} = T^{-1}(t_x, t_y)S^{-1}(s_x, s_y)R^{-1}(\theta) \begin{bmatrix} x_k \\ y_j \\ 1 \end{bmatrix}$$

Geometrical Modification

Exercise

- Write down a linear system which represents the following operation:
 - Rotate an image by an angle of θ w.r.t. a pivot point (x_c, y_c)

$$H = \begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix}$$

Geometrical Modification

Part II

Geometrical Modification

■ Non-linear Coordinates Transformation and Spatial Warping

○ Non-linear address mapping

■ Forward $\begin{cases} x = X\{u, v\} \\ y = Y\{u, v\} \end{cases}$

■ Backward (reverse) $\begin{cases} u = U\{x, y\} \\ v = V\{x, y\} \end{cases}$

$$\begin{cases} u_q = q - \frac{1}{2} \\ v_p = P + \frac{1}{2} - p \end{cases} \quad \boxed{\text{input}} \quad \Rightarrow \quad \boxed{\text{output}} \quad \begin{cases} x_k = k - \frac{1}{2} \\ y_j = J + \frac{1}{2} - j \end{cases}$$

Geometrical Modification

■ Polynomial Warping (2nd-order)

$$(u, v) \rightarrow (x, y) \quad \begin{cases} u = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 \\ v = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}$$

Geometrical Modification

■ Polynomial Warping

- Rubber-sheet stretching

- Identify spatial distortion

- Calibration → test patterns

- Two steps:

- Based on 'known' input and output pairs (control points), compute the coefficients 'a' and 'b' (either exact or least squares solution),

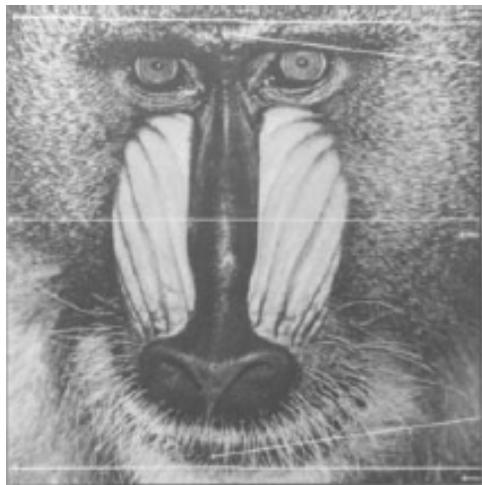
- i.e. the spatial warping matrix

- Use the spatial warping matrix to compute all the output (input) points from their corresponding input (output) points

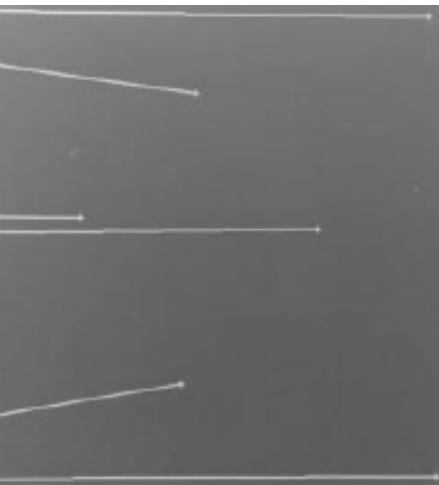
→ proper interpolation is necessary

Geometrical Modification

- Polynomial Warping
 - Example



input



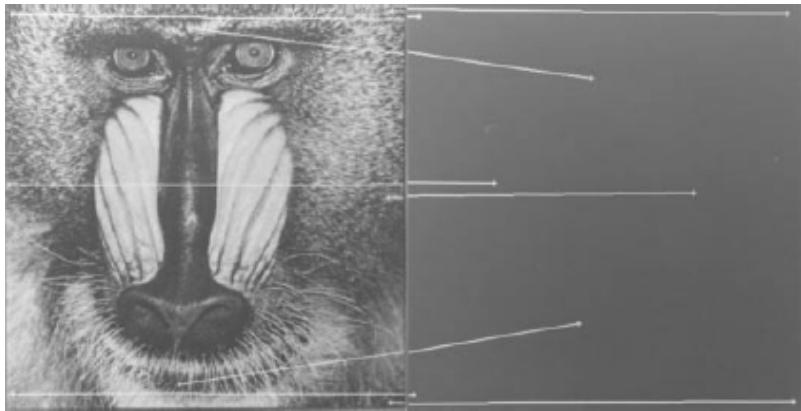
control points



desired output

Geometrical Modification

Example



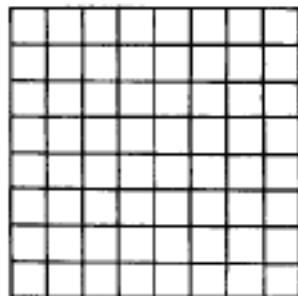
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_k \\ v_1 & v_2 & \cdots & v_k \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ y_1 & y_2 & \cdots & y_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ x_1y_1 & x_2y_2 & \cdots & x_ky_k \\ y_1^2 & y_2^2 & \cdots & y_k^2 \end{bmatrix}^{63}$$

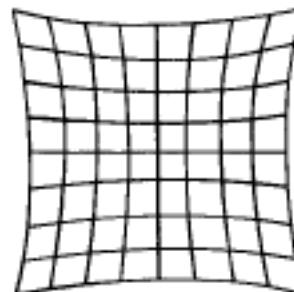
Geometrical Modification

■ Polynomial Warping

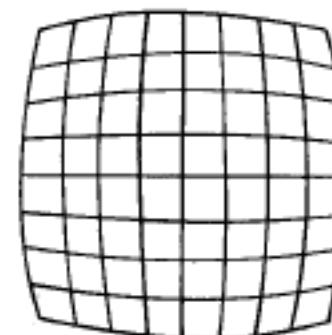
- Useful to compensate the spatial distortion caused by the limitation of a physical imaging system



Original



Pincusion
distortion



Barrel
distortion

Geometrical Modification

■ Examples



Original



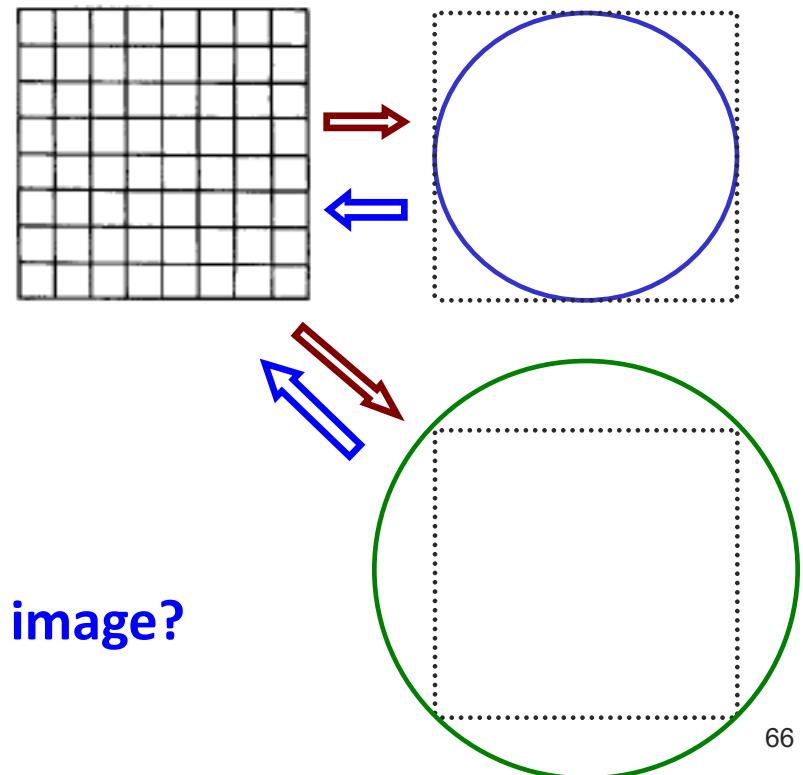
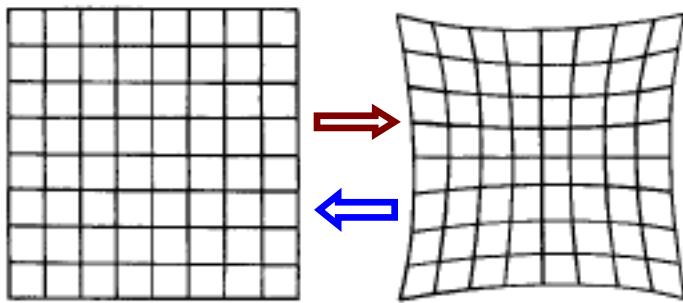
Barrel
distortion



Pincusion
distortion

Geometrical Modification

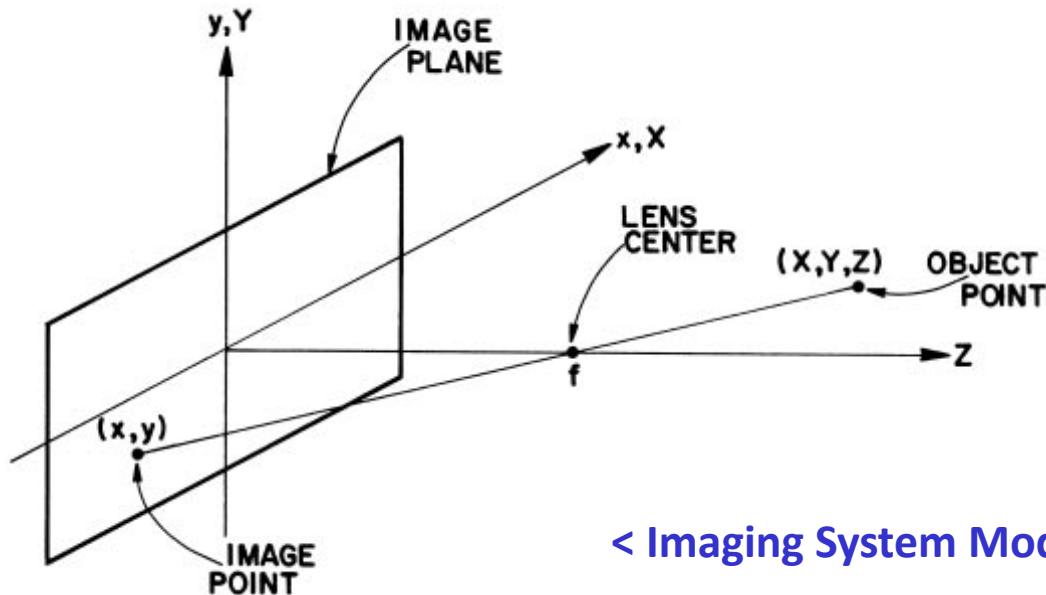
Example



Can we warp back to the original image?

Geometrical Modification

- Perspective Transformation
 - Imaging in the 3D space
 - Fundamentals of computer graphics



< Imaging System Model >

Geometrical Modification

Perspective Transformation

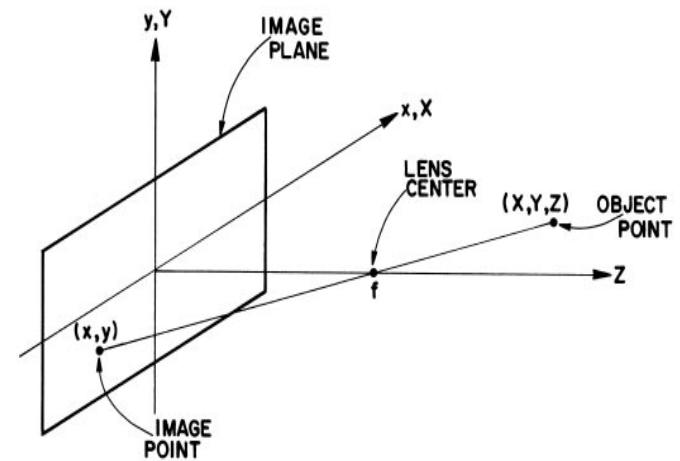
Cartesian to image coordinates

Similar triangle property

$$\frac{X}{-x} = \frac{Z-f}{f} \Rightarrow x = \frac{fX}{f-Z};$$

$$y = \frac{fY}{f-Z}$$

→ Many-to-one mapping



Geometrical Modification

■ Perspective Transformation

○ Image to Cartesian coordinates

■ Need another degree of freedom

$$X = \frac{fx_i}{f + z_i}; \quad Y = \frac{fy_i}{f + z_i}; \quad Z = \frac{fz_i}{f + z_i} \quad z_i \text{ is a free variable}$$

■ Given Z, we may compute z_i and then X & Y via

$$X = \frac{x_i}{f}(f - Z) \quad Y = \frac{y_i}{f}(f - Z)$$

Geometrical Modification

Perspective Transformation

P is a perspective transformation matrix,

$$\tilde{v} = s \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

homogeneous vector
3D object
s: scaling factor

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/f & 1 \end{bmatrix}$$

$$\tilde{w} = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$

homogeneous image position vector

$$\tilde{w} = P\tilde{v} = \begin{bmatrix} sX \\ sY \\ sZ \\ s - sz/f \end{bmatrix} \Rightarrow s = \frac{f}{f - z}$$

Geometrical Modification

■ Camera Imaging Model

- Camera is supported by a gimbal (X_G, Y_G, Z_G)
- Gimbal can do 3D movements
 - panning (θ) /tilting (ϕ)
- Offset between the gimbal support and the image plane center is (X_0, Y_0, Z_0)
- The complete camera imaging model can be derived by sequentially operating on the homogeneous vector

$$\tilde{w} = PT_c RT_G \tilde{v}$$

