Scaling Analysis of a Moving Guassion Heat Source in Steady State in a Semi-Infinite Solid

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Abstract

Abstract goes here

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1. Introduction

Our research focuses on developing simplified formulas with high accuracy that will substitute complex numerical calculations.

2. Governing Equation

$$T^* = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \frac{\tau^{-\frac{1}{2}}}{\tau + \sigma^{*2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau + 2\sigma^{*2}} - \frac{z^{*2}}{2\tau}} \tag{1}$$

Eq. 1 has some disadvantages. Firstly, it is a improper integral and the up limit is infinity which makes the calculation more difficult. Secondly, the integrand has two peaks. One locates at $\tau = 0$, and the other moves and is hard to determine, which may results in the omitting of second peak in integral.
Use variable substitution method $t = \arctan \frac{\sqrt{\tau}}{\sigma}$,and do not consider the depth of pool, which means $z^* = 0$.

$$T^* = \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t + \frac{1}{\cos^2 t} - 2\right) + \frac{\cos^2 t\left(x^{*2}+y^{*2}\right)}{\sigma^{*2}} + 2x^*\left(1-\cos^2 t\right)\right]} dt \tag{2}$$

Eq. 2 avoids the disadvantages of Eq. 1. The integral is bounded. The integrand has one peak located at $t = \arccos\left\{\sigma^*[(\sigma^{*2} - x^*)^2 + y^{*2}]\right\}$. However, Eq. 2 can't be applied to the point-source condition.

3. the highest temperature T^*_m corresponding to σ^*

To calculating the highest T^* corresponding to σ^*, y^* should be set as 0 because the maximum temperature $T{*}_m$ decreases as $|y^*|$ increases according to Eq. 1.

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3.1. $\sigma \to 0$

According the numerical calculation, x^* should be much smaller than σ^*,so $|\frac{x^*}{\sigma^*}| \sim 0$. Eq. 2 can be simplified as:

$$\begin{aligned}
\widehat{T^*}_{mI} &= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\, x^{*2}}{\sigma^{*2}}+2x^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}\,\sigma^{*-1}
\end{aligned} \tag{3}$$

3.2. $\sigma \to \infty$

When σ tends to infinity, the peak locates at 0, and the integrand decreases sharply.

$$\begin{aligned}
\widehat{T^*}_{mII} &= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\, x^{*2}}{\sigma^{*2}}+2x^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\, x^{*2}}{\sigma^{*2}}+2x^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+\frac{(1-t^2)x^{*2}}{\sigma^{*2}}+2x^*t^2\right]}dt \\
&= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+(2x^*-\frac{x^{*2}}{\sigma^{*2}})t^2+\frac{x^{*2}}{\sigma^{*2}}\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+2x^*t^2+\frac{x^{*2}}{\sigma^{*2}}\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left(\sigma^* t^2+\frac{x^*}{\sigma^*}\right)^2}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left(\sigma^* t^2+\frac{x^*}{\sigma^*}\right)^2}dt
\end{aligned} \tag{4}$$

Where δ is infinitesimal, and $x^* \sim \sigma^* \gg 1$.

Use numerical method to find the maximum value of Eq. 4 with changes of x^*. When $x^* = -0.7650\,\sigma^*$, T^* reaches maximum value.

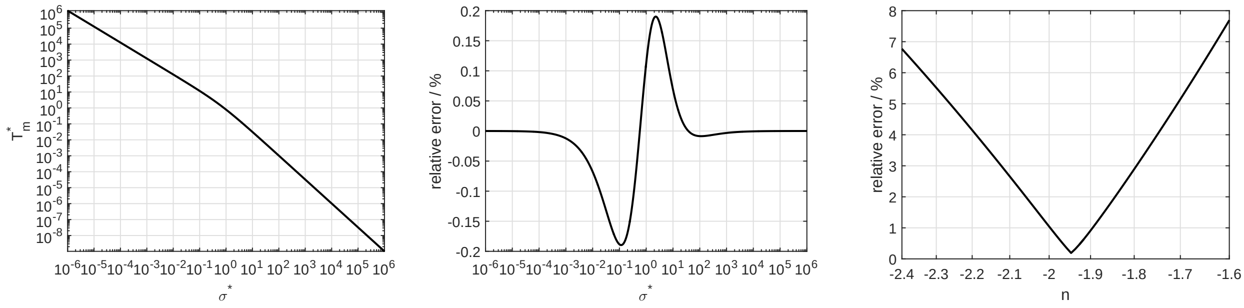
$$\widehat{T^*}_{mII} = \frac{2.5596}{\sqrt{2\pi}}\,\sigma^{*-1.5} \tag{5}$$

3.3. blending

Use Eq. 3 and Eq. 5 to obtaining the blending equation for all σ.

$$\widehat{T^*}_m^+(\sigma^*) = \left[\left(\sqrt{\frac{\pi}{2}}\,\sigma^{*-1}\right)^n + \left(\frac{2.5596}{\sqrt{2\pi}}\,\sigma^{*-1.5}\right)^n\right]^{\frac{1}{n}} \tag{6}$$

Where $n = -1.9464$, and the maximum error reaches 0.1901%.



(a) correspondence between T_m^* and σ^* (b) relative error changes with σ^* when $n = -1.9464$ (c) maximum error changes with n around -1.9464

Fig. 1: Results of the blending between T_m^* and σ^*

Eq. 6 reveals the one-to-one correspondence between σ^* and T^*. For any melting point T^*, there is a certain σ^*, below which the base substance can't melt, and vice versa. So, Eq. 3 and Eq. 5 can be rewritten as:

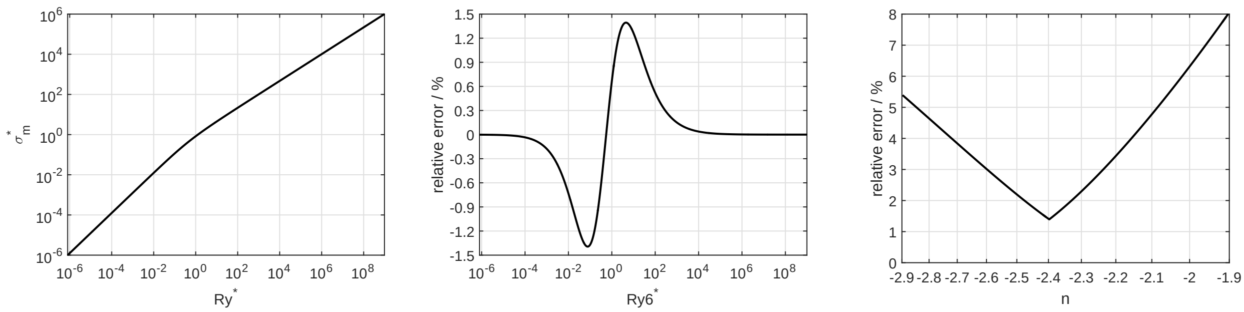
$$\widehat{\sigma^*}_{mI} = \sqrt{\frac{\pi}{2}} Ry \tag{7}$$

Where $Ry^* = \frac{1}{T^*}$.

$$\widehat{\sigma^*}_{mII} = 1.0140\ Ry^{\frac{2}{3}} \tag{8}$$

$$\widehat{\sigma^*}^{+}_{m}(Ry) = \left[\left(1.0140\ Ry^{\frac{2}{3}}\right)^n + \left(\sqrt{\frac{\pi}{2}} Ry\right)^n \right]^{\frac{1}{n}} \tag{9}$$

Where $n = -2.3975$, and the maximum error reaches minimum, 1.39%.



(a) correspondence between σ_m^* and Ry (b) relative error changes with Ry when $n = -2.3975$ (c) maximum error changes with n around -2.3975

Fig. 2: Results of the blending between σ_m^* and Ry

4. $\sigma \to \sigma_m$

When σ^* tends to σ^*_m, the welding pool vanishes, and should be axisymmetric, which means the maximum width point locates above the maximum temperature point, i.e. $x_{m,\text{corresponding to maximum width point}} = x_{m,\text{corresponding to maximum temperature point}}$.

4.1. $\sigma \to 0$

$$\begin{aligned}
\widehat{T^*}_{I;x_0,y} &= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
&= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\ x_0^{*2}}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}\cdot e^{-\frac{\cos^2 t y^{*2}}{2\sigma^{*2}}}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} 1\cdot e^{-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}}dt \\
&= \frac{2}{\pi}T^*_{I;x_0,y=0}\cdot\int_0^{\frac{\pi}{2}} e^{-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}}dt \\
&\approx \frac{2}{\pi}T^*_{I;x_0,y=0}\cdot\int_0^{\frac{\pi}{2}} 1-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}dt \qquad\qquad \text{as } y^*\ll\sigma^* \\
&= T^*_{I;x_0,y=0}\cdot\frac{2}{\pi}\left(\frac{\pi}{2}-\frac{y^{*2}\pi}{8\sigma^{*2}}\right) \\
&= T^*_{I;x_0,y=0}\cdot\left(1-\frac{y^{*2}}{4\sigma^{*2}}\right) \\
&= T^*_{I;x_0,y=0}\cdot\left(1-\frac{y^{*2}}{4\sigma^{*2}}\right) \\
&= T^*_{I;x_0,y=0}\cdot e^{-\frac{y^{*2}}{4\sigma^{*2}}}
\end{aligned} \tag{10}$$

Where $x_0^* = 0$, $y^* \ll \sigma^*$.

According to Eq. 10,

$$\widehat{y^*}_{mI} = 2\sigma^*\sqrt{\ln\frac{T^*_m(\sigma^*)}{T^*}} = 2\sigma^*\sqrt{\ln\frac{Ry}{Ry_{min}(\sigma^*)}} \tag{11}$$

4.2. $\sigma \to \infty$

When σ tends to infinity, the location of maximum temperature point $\widehat{x^*}_0 = -0.7650\ \sigma^*$, and the integrand focuses on $t = 0$, i.e. $\cos t = 1$.

$$
\begin{aligned}
\widehat{T^*}_{II;x_0,y} &= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+\frac{\left(1-t^2\right)\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^* t^2\right]}dt \\
&= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+(2x_0^*-\frac{x_0^{*2}}{\sigma^{*2}})t^2+\frac{x_0^{*2}}{\sigma^{*2}}\right]}\cdot e^{-\frac{y^{*2}}{2\sigma^{*2}}}dt \\
&= T^*_{II;x_0,y=0}\cdot e^{-\frac{y^{*2}}{2\sigma^{*2}}}
\end{aligned}
\tag{12}
$$

According to Eq. 12,

$$
\widehat{y^*}_{mII} = \sqrt{2}\sigma^*\sqrt{\ln\frac{T^*(\sigma^*)}{T^*}} = \sqrt{2}\sigma^*\sqrt{\ln\frac{Ry}{Ry_{min}(\sigma^*)}} \tag{13}
$$

4.3. blending

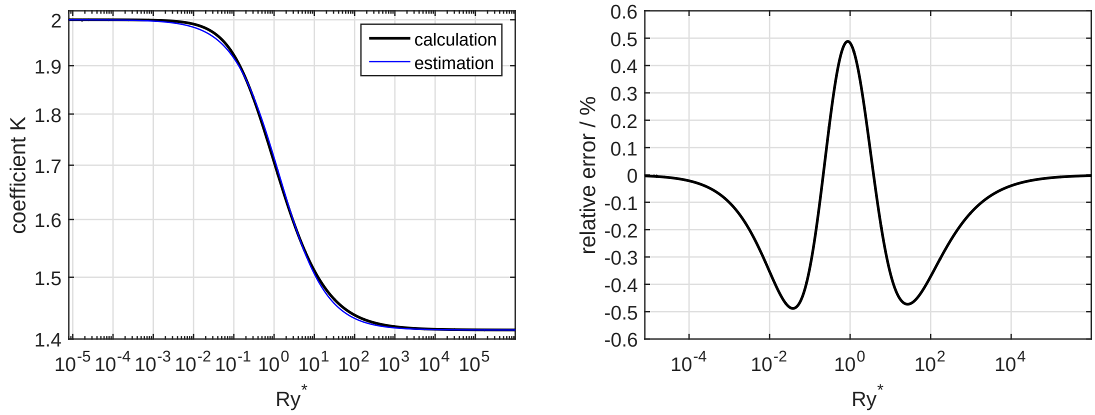
Use Eq. 11 and Eq. 13 to obtained the approximation of y_m^* when σ^* tends to σ_m^*:

$$
\widehat{y^*}^{+}_{m,near} = K\sigma^*\sqrt{\ln\frac{Ry}{Ry_{min}(\sigma^*)}} \tag{14}
$$

K changes with Ry.

$$
K = k_0 - A * \tanh\left(B\ln\frac{Ry}{C}\right) \tag{15}
$$

Where $k_0 = \frac{2+\sqrt{2}}{2}$,$A = \frac{2-\sqrt{2}}{2}$,$B = 0.3775$,$C = 1.0690$.The maximum error reaches 0.5%. Eq. 14 can be



(a) coefficient changes with Ry^*

(b) relative error changes with Ry^*

Fig. 3: Results of approximation of coefficient K against Ry^*

written as a function depicting the near field temperature distribution around the maximum temperature point:

$$Ry = Ry_{min}\left(\sigma^*\right) e^{\frac{y_m^{*2}}{K^2\sigma^{*2}}} \tag{16}$$

5. quasi point source

When $\sigma^* = 0$, the Eq. 1 describes the point heat source.

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \quad \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} = \frac{1}{r^*} e^{-r^* - x^*} \tag{17}$$

Do derivations on Eq. 17 with respect to y:

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \quad \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} = \frac{1}{r^*} e^{-r^* - x^*} \tag{18}$$

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \quad \tau^{-\frac{5}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} &= -\frac{1}{y^*}\frac{\partial}{\partial y^*}\left(\frac{1}{r^*} e^{-r^* - x^*}\right) \\ &= e^{-r^* - x^*}\left(\frac{1}{r^{*2}} + \frac{1}{r^{*3}}\right)\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \quad \tau^{-\frac{7}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} &= \frac{1}{y^*}\left[\frac{1}{y^*}\frac{\partial}{\partial y^*}\left(\frac{1}{r^*} e^{-r^* - x^*}\right)\right] \\ &= e^{-r^* - x^*}\left(\frac{1}{r^{*3}} + \frac{3}{r^{*4}} + \frac{3}{r^{*5}}\right)\end{aligned} \tag{20}$$

When $\frac{\sigma^*}{\sigma_m^*}$ tends to zero, the Gaussian heat source can be treated as point source, with little error. So, use Eq. 1 rather than Eq. 2.

$$\begin{aligned}\widehat{T^*}_{far} &= \frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \frac{\tau^{-\frac{1}{2}}}{\tau + \sigma^{*2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau + 2\sigma^{*2}}} \\ &\approx \frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau\, \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} \cdot \left[\left(1 + \frac{\sigma^{*2}}{2}\right) + \sigma^{*2}\left(x - 1\right)\frac{1}{\tau} + \frac{x^{*2} + y^{*2}}{2}\sigma^{*2}\frac{1}{\tau^2}\right] \\ &= e^{-r^* - x^*}\left[\left(1 + \frac{\sigma^{*2}}{2}\right)\frac{1}{r^*} + \sigma^{*2}\left(x - 1\right)\left(\frac{1}{r^{*2}} + \frac{1}{r^{*3}}\right) + \frac{\sigma^{*2}}{2}\left(\frac{1}{r^*} + \frac{3}{r^{*2}} + \frac{3}{r^{*3}}\right)\right]\end{aligned} \tag{21}$$

This process uses the first two terms of Taylor series of integrand with respect to σ^*. Eq. 21 describes the temperature distribution of far field.

5.1. $\sigma \to 0$

When $\sigma \to 0$, $x^* \ll y^* \ll 1$. Eq. 21 can be simplified as

$$\begin{aligned}
\widehat{T^*}_{far,0} &\approx e^{-r^*-x^*}\left[\left(1+\frac{\sigma^{*2}}{2}\right)\frac{1}{r^*}+\sigma^{*2}\left(x-1\right)\left(\frac{1}{r^{*2}}+\frac{1}{r^{*3}}\right)+\frac{\sigma^{*2}}{2}\left(\frac{1}{r^*}+\frac{3}{r^{*2}}+\frac{3}{r^{*3}}\right)\right] \\
&\approx 1\cdot\left[\left(1+\frac{\sigma^{*2}}{2}\right)\frac{1}{y^*}-\sigma^{*2}\frac{1}{y^{*3}}+\frac{\sigma^{*2}}{2}\frac{3}{y^{*3}}\right] \\
&\approx \frac{1}{y^*}+\frac{\sigma^{*2}}{2}\frac{1}{y^{*3}}
\end{aligned}$$

Use perturbation method, $\widehat{y^*}_{m,gauss,0} \approx \widehat{y^*}_{m,point,0}\left(1+a\sigma^{*2}\right)$, $a\sigma^{*2} \ll 1$, $\widehat{y^*}_{m,point,0} = Ry$.

$$\begin{aligned}
\frac{1}{Ry} &\approx \frac{1}{y^*}+\frac{\sigma^{*2}}{2}\frac{1}{y^{*3}} \\
&\approx \frac{1}{y^*_{m,point}\left(1+a\sigma^{*2}\right)}+\frac{\sigma^{*2}}{2}\frac{1}{y^{*3}_{m,point}\left(1+3a\sigma^{*2}\right)} \\
&\approx \frac{1}{Ry\left(1+a\sigma^{*2}\right)}+\frac{\sigma^{*2}}{2}\frac{1}{Ry^3\left(1+3a\sigma^{*2}\right)} \\
&\Rightarrow a = \frac{1}{2Ry^2} \qquad (22)
\end{aligned}$$

$$\widehat{y^*}_{m,gauss,0} = \widehat{y^*}_{m,point}\left(1+\frac{1}{2Ry^2}\sigma^{*2}\right) \qquad (23)$$

5.2. $\sigma \to \infty$

When $\sigma \to \infty$, $1 \ll \sigma^* \ll y^* \ll x^*$. Eq. 21 can be simplified as

$$\begin{aligned}
\widehat{T^*}_{far,\infty} &\approx e^{-r^*-x^*}\left[\left(1+\frac{\sigma^{*2}}{2}\right)\frac{1}{r^*}+\sigma^{*2}\left(x-1\right)\left(\frac{1}{r^{*2}}+\frac{1}{r^{*3}}\right)+\frac{\sigma^{*2}}{2}\left(\frac{1}{r^*}+\frac{3}{r^{*2}}+\frac{3}{r^{*3}}\right)\right] \\
&\approx e^{-r^*-x^*}\left[\frac{1}{r^*}+\frac{\sigma^{*2}\left(r^*+x^{*2}\right)}{r^{*2}}-\frac{\sigma^{*2}}{2r^{*2}}\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*}+\frac{\sigma^{*2}}{x^{*2}}\left(-\frac{y^{*2}}{2x^*}-0.5\right)\right] \qquad (24)
\end{aligned}$$

Use perturbation method, $\widehat{y^*}_{m,gauss,\infty} = \widehat{y^*}_{m,point,\infty}\left(1 + b\sigma^{*2}\right)$,$\widehat{x^*}_{m,gauss,\infty} = \widehat{x*}_{m,point,\infty}\left(1 + c\sigma^{*2}\right)$, $b\sigma^{*2} \ll 1$,$c\sigma^{*2} \ll 1$, $\widehat{y^*}_{m,point,\infty} = \sqrt{\frac{2}{e}Ry}$, $\widehat{x^*}_{m,point,\infty} = -\frac{Ry}{e}$.

$$
\begin{aligned}
\frac{1}{Ry} &\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}}\left(-\frac{y^{*2}}{2x^*} - 0.5\right)\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}}\left(0.5 + 2b\sigma^{*2} - c\sigma^{*2}\right)\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left(-\frac{1}{x^*} + 0.5\frac{\sigma^{*2}}{x^{*2}}\right) \\
y^* &= \sqrt{2x^* \ln \frac{x^{*2}/Ry}{-x^* + 0.5\sigma^{*2}}} \\
\frac{dy^*}{dx^*} &= \frac{\sqrt{2}\left(2\ln\left(-\frac{2tx^2}{-s^2+2x}\right) + \frac{4x-4s^2}{-s^2+2x}\right)}{4\sqrt{x\ln\left(-\frac{2tx^2}{2x-s^2}\right)}} = 0 \\
&\Rightarrow \widehat{x^*}_{m,gauss,\infty} = \widehat{x^*}_{m,point,\infty} \\
&\Rightarrow b = \frac{e}{4Ry} \\
\widehat{y^*}_{m,gauss,infinity} &= \widehat{y^*}_{m,point}\left(1 + \frac{e}{4Ry}\sigma^{*2}\right) \qquad (25)
\end{aligned}
$$

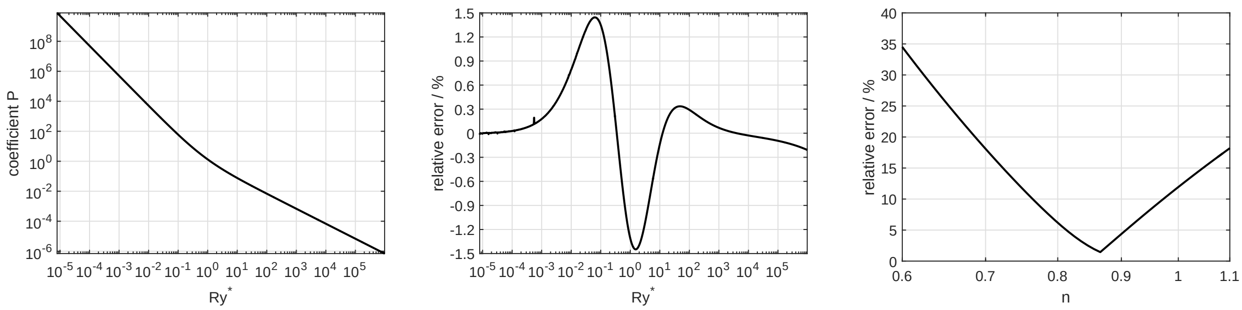
5.3. blending

Use the following equation to blending:

$$
\widehat{y^*}^{+}_{m,far} = \widehat{y^*}_{m,point}\left(1 + P * \sigma^{*2}\right) \qquad (26)
$$

$$
P = \left[\left(\frac{1}{2Ry^2}\right)^n + \left(\frac{e}{4Ry}\right)^n\right]^{\frac{1}{n}} \qquad (27)
$$

Where $n = 0.8655$, maximum error reaches 1.45%.



(a) coefficient changes with Ry (b) relative error changes with Ry (c) relative error changes with Ry

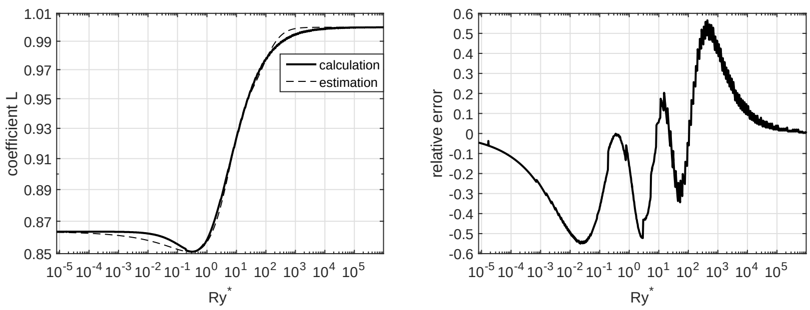
Fig. 4: Results of approximation of coefficient K against Ry

6. Combination

The maximum width of welding pool is a combination of far-field and near-field. To cover the middle range of $\frac{\sigma}{\sigma_m}$, the correction of both equations is needed.

6.1. near-field

$$L = 0.93175 - 0.06825 \tanh\left(-0.6571\ \ln \frac{Ry}{15.926}\right) - 0.0132 \sin\left[3\pi \tanh\left(0.2485\ Ry^{0.3718}\right)\right] \tag{28}$$



(a) coefficient changes with Ry^* (b) relative error changes with Ry^*

Fig. 5: Results of approximation of coefficient L against Ry

The maximum error reaches 0.55%.

6.2. far-field

The maximum width can be written in form of step function, consisting of far-field expression and near-field expression. However, lines of $\widehat{y^*}^+_{m,far}$ and $\widehat{y^*}^+_{m,near}$ cross. A correction of far-field expression is applied, which is a confinement with the range of $\frac{\sigma^*}{\sigma^*_m}$. It's difficult to calculate the cross points directly, so a back step is took that the point between two cross points is found with any Ry.

When $Ry \leqslant 5 \times 10^3$, $lin = 0.4$.

When $Ry > 5 \times 10^3$,

$$\widehat{y^*}^+_{m,far} - \widehat{y^*}^+_{m,near} = \sqrt{\frac{2}{e} Ry} + 0.5993 \left(\frac{\sigma^*}{\sigma^*_m}\right)^2 Ry^{\frac{5}{6}} - 2.4838 \frac{\sigma^*}{\sigma^*_m} \sqrt{\log \frac{\sigma^*_m}{\sigma^*}}\ Ry^{\frac{2}{3}} < 0$$

When $\frac{\sigma^*}{\sigma^*_m} = Ry^{-\frac{1}{6}}$,the in-equation is satisfied. So:

$$lin = Ry^{-\frac{1}{6}} \cdot \left(Ry > 5 \times 10^3\right) + 0.4 \left(Ry \leqslant 5 \times 10^3\right);\tag{29}$$

7. whole plane

$$
\begin{aligned}
\widehat{y^*}^{+}_{m} &= MAX\{\quad \widehat{y^*}^{+}_{m,near}\ ,\quad \widehat{y^*}^{+}_{m,far}\quad \} \\
\widehat{y^*}^{+}_{m,far} &= y^*_{m,point}\left(1+P\sigma^{*2}\right)\cdot\left(\sigma^*/\sigma^*_m\ < lin\right) \\
\widehat{y^*}^{+}_{m,near} &= K\left[L\left(\sigma^*-\sigma^*_m\right)+\sigma^*_m\right]\sqrt{\ln\frac{Ry^*}{Ry^*_{min}(\sigma^*)}}
\end{aligned}
\tag{30}
$$

The parameters in equations are as follows:

$$
\begin{aligned}
y^*_{m,point} &= \left[(Ry)^n+\left(\sqrt{\frac{2}{e}Ry}\right)^n\right]^{\frac{1}{n}} \\
&\quad \text{Where}\quad \text{n=}\quad \text{-1.7312} \\
P &= \left[\left(\frac{1}{2Ry^2}\right)^n+\left(\frac{e}{4Ry}\right)^n\right]^{\frac{1}{n}} \\
&\quad \text{Where}\quad \text{n=}\quad \text{0.8655} \\
lin &= Ry^{-\frac{1}{6}}\cdot\left(Ry>5\times10^3\right)+0.4\left(Ry\leqslant5\times10^3\right); \\
K &= k_0-A*\tanh\left(B\ln\frac{Ry}{C}\right) \\
&\quad \text{Where } k_0=\tfrac{2+\sqrt{2}}{2}, A=\tfrac{2-\sqrt{2}}{2}, B=0.3775, C=1.0690. \\
L &= 0.93175-0.06825\tanh\left(-0.6571\ \ln\frac{Ry}{15.926}\right)-0.0132\sin\left[3\pi\tanh\left(0.2485\ Ry^{0.3718}\right)\right] \\
\sigma^*_m &= \left[\left(1.0140\ Ry^{\frac{2}{3}}\right)^n+\left(\sqrt{\frac{\pi}{2}Ry}\right)^n\right]^{\frac{1}{n}} \\
&\quad \text{Where } n=-2.3975 \\
Ry_{min}(\sigma^*) &= \left[\left(\sqrt{\frac{\pi}{2}}\ \sigma^{*-1}\right)^n+\left(\frac{2.5596}{\sqrt{2\pi}}\ \sigma^{*-1.5}\right)^n\right]^{-\frac{1}{n}} \\
&\quad \text{Where } n=-1.9464
\end{aligned}
$$

The maximum error reaches 5.25%.There is a limit that $\frac{\sigma^*}{\sigma^*_m} < 98\%$, because when $\frac{\sigma^*}{\sigma^*_m}$ tends to 1, y^*_m tends to 0, the approximation (means that it's not accurate) of Ry_{min}, σ^*_m leads to a large relative error. If the high-precision value is obtained these equations still work.

8. Results

Results

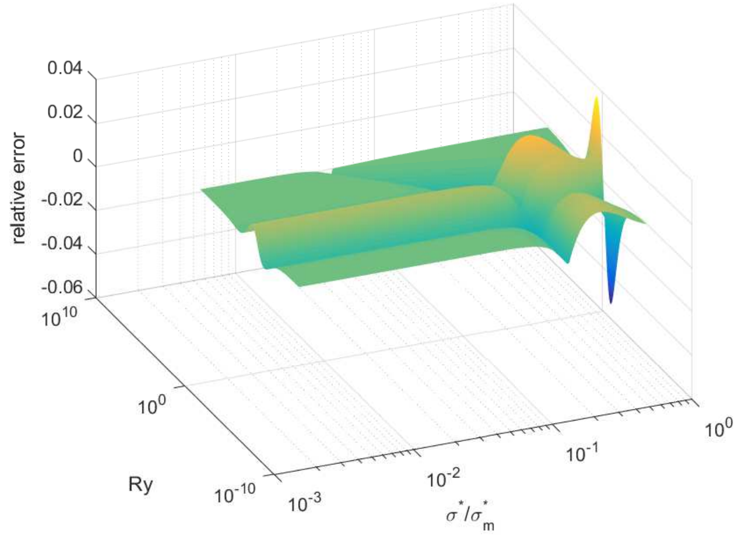


Fig. 6: relative error over whole plane.

9. Discussion

10. Conclusions

Conclusions Section

11. Conclusions

Conclusions Section

12. Acknowledgement

This study has been supported by...

13. References