Scaling Analysis of a Moving Guassion Heat Source in Steady State in a Semi-Infinite Solid

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Abstract

Abstract goes here

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1. Introduction

Our research focuses on developing simplified formulas with high accuracy that will substitute complex numerical calculations.

2. Governing Equation

$$T^* = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \frac{\tau^{-\frac{1}{2}}}{\tau + \sigma^{*2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau + 2\sigma^{*2}} - \frac{z^{*2}}{2\tau}} \tag{1}$$

Eq. 1 has some disadvantages. Firstly, it is a improper integral and the up limit is infinity which makes the calculation more difficult. Secondly, the integrand has two peaks. One locates at $\tau = 0$, and the other moves and is hard to determine, which may results in the omitting of second peak in integral. Use variable substitution method $t = \arctan \frac{\sqrt{\tau}}{\sigma}$,and do not consider the depth of pool, which means $z^* = 0$.

$$T^* = \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t + \frac{1}{\cos^2 t} - 2\right) + \frac{\cos^2 t\left(x^{*2}+y^{*2}\right)}{\sigma^{*2}} + 2x^*\left(1-\cos^2 t\right)\right]} dt \tag{2}$$

Eq. 2 avoids the disadvantages of Eq. 1. The integral is bounded. The integrand has one peak located at $t = \arccos\left\{\sigma^*[(\sigma^{*2} - x^*)^2 + y^{*2}]\right\}$. However, Eq. 2 can't be applied to the point-source condition.

3. the highest T^* corresponding to σ^*

To calculating the highest T^* corresponding to σ^*, y^* should be set as 0.

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3.1. $\sigma \to 0$

According the numerical calculation, x^* should be much smaller than σ^*,so $|\frac{x^*}{\sigma^*}| \sim 0$. Eq. 2 can be simplified as:

$$
\begin{aligned}
T_{mI}^* = \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\, x^{*2}}{\sigma^{*2}}+2x^*\left(1-\cos^2 t\right)\right]} dt \\
\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)} dt \\
\approx \frac{2}{\sqrt{2\pi}\sigma^*} \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}\, \sigma^{*-1}
\end{aligned} \tag{3}
$$

3.2. $\sigma \to \infty$

When σ tends to infinity, the peak locates at 0, and the integrand decreases sharply.

$$
\begin{aligned}
T_{mII}^* = \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\, x^{*2}}{\sigma^{*2}}+2x^*\left(1-\cos^2 t\right)\right]} dt \\
\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\, x^{*2}}{\sigma^{*2}}+2x^*\left(1-\cos^2 t\right)\right]} dt \\
\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\, t^4+\frac{(1-t^2)x^{*2}}{\sigma^{*2}}+2x^* t^2\right]} dt \\
= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\, t^4+(2x^*-\frac{x^{*2}}{\sigma^{*2}})t^2+\frac{x^{*2}}{\sigma^{*2}}\right]} dt \\
\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\, t^4+2x^* t^2+\frac{x^{*2}}{\sigma^{*2}}\right]} dt \\
\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left(\sigma^* t^2+\frac{x^*}{\sigma^*}\right)^2} dt \\
\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left(\sigma^* t^2+\frac{x^*}{\sigma^*}\right)^2} dt
\end{aligned} \tag{4}
$$

Where δ is infinitesimal, and $x^* \sim \sigma^* \gg 1$.

Use numerical method to find the maximum value of Eq. 4 with changes of x^*. When $x^* = -0.7650\,\sigma^*$, T^* reaches maximum value.

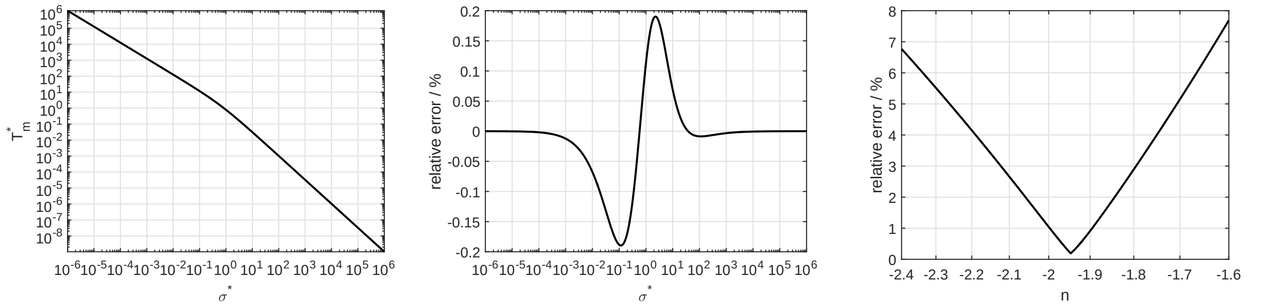
$$
T_{mII}^* = \frac{2.5596}{\sqrt{2\pi}}\, \sigma^{*-1.5} \tag{5}
$$

3.3. *blending*

Use Eq. 3 and Eq. 5 to obtaining the blending equation for all σ.

$$
T_m^* = \left[\left(\sqrt{\frac{\pi}{2}}\, \sigma^{*-1}\right)^n + \left(\frac{2.5596}{\sqrt{2\pi}}\, \sigma^{*-1.5}\right)^n\right]^{\frac{1}{n}} \tag{6}
$$

Where $n = -1.9464$, and the maximum error reaches 0.1901%.



(a) correspondence between T_m^* and σ^* (b) relative error changes with σ^* when $n = -1.9464$ (c) maximum error changes with n around -1.9464

Fig. 1: Results of the blending between T_m^* and σ^*

Eq. 6 reveals the one-to-one correspondence between σ^* and T^*. For any melting point T^*, there is a certain σ^*, below which the base substance can't melt, and vice versa. So, Eq. 3 and Eq. 5 can be rewritten as:

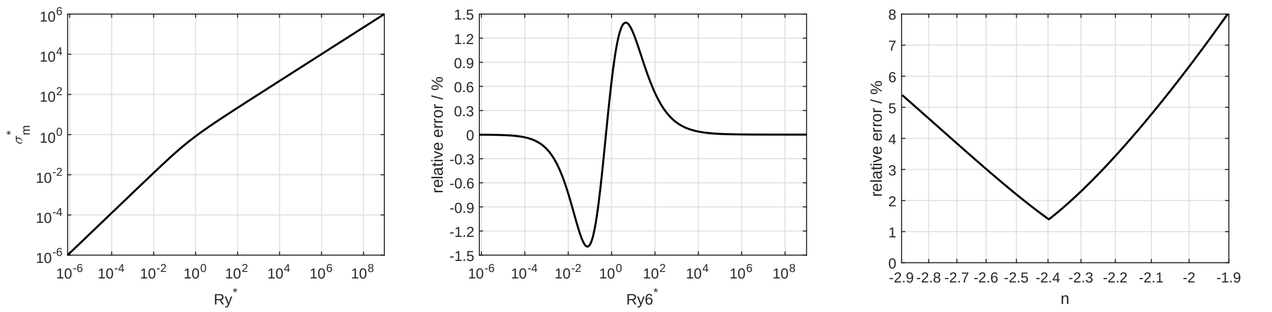
$$\sigma_{mI}^* = \sqrt{\frac{\pi}{2}} Ry^* \tag{7}$$

Where $Ry^* = \frac{1}{T^*}$.

$$\sigma_{mII}^* = 1.0140\ Ry^{*\frac{2}{3}} \tag{8}$$

$$\sigma_m^* = \left[\left(1.0140\ Ry^{*\frac{2}{3}}\right)^n + \left(\sqrt{\frac{\pi}{2}} Ry^*\right)^n \right]^{\frac{1}{n}} \tag{9}$$

Where $n = -2.3975$, and the maximum error reaches minimum, 1.39%.



(a) correspondence between σ_m^* and Ry^* (b) relative error changes with Ry^* when $n = -2.3975$ (c) maximum error changes with n around -2.3975

Fig. 2: Results of the blending between σ_m^* and Ry^*

4. $\boldsymbol{\sigma \to \sigma_m}$

When σ^* tends to σ^*_m, the welding pool vanishes, and should be axisymmetric, which means the maximum width point locates above the maximum temperature point, i.e. $x_{m,\text{corresponding to maximum width point}} = x_{m,\text{corresponding to maximum temperature point}}$.

4.1. $\sigma \to 0$

$$
\begin{aligned}
T^*_{I;x_0,y} = \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\ x_0^{*2}}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}\cdot e^{-\frac{\cos^2 t y^{*2}}{2\sigma^{*2}}}dt \\
\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} 1\cdot e^{-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}}dt \\
= \frac{2}{\pi}T^*_{I;x_0,y=0}\cdot\int_0^{\frac{\pi}{2}} e^{-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}}dt \\
\approx \frac{2}{\pi}T^*_{I;x_0,y=0}\cdot\int_0^{\frac{\pi}{2}} 1-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}dt \qquad \text{as } y^*\ll\sigma^* \\
= T^*_{I;x_0,y=0}\cdot\frac{2}{\pi}\left(\frac{\pi}{2}-\frac{y^{*2}\pi}{8\sigma^{*2}}\right) \\
= T^*_{I;x_0,y=0}\cdot\left(1-\frac{y^{*2}}{4\sigma^{*2}}\right) \\
= T^*_{I;x_0,y=0}\cdot\left(1-\frac{y^{*2}}{4\sigma^{*2}}\right) \\
= T^*_{I;x_0,y=0}\cdot e^{-\frac{y^{*2}}{4\sigma^{*2}}}
\end{aligned} \tag{10}
$$

Where $x_0^* = 0$, $y^* \ll \sigma^*$.

According to Eq. 10,

$$
y^*_{mI} = 2\sigma^*\sqrt{\ln\frac{T^*(\sigma^*)}{T^*}} = 2\sigma^*\sqrt{\ln\frac{Ry^*}{Ry^*_{min}(\sigma^*)}} \tag{11}
$$

4.2. $\sigma \to \infty$

When σ tends to infinity, the location of maximum temperature point $x_0^* = -0.7650\ \sigma^*$, and the integrand focuses on $t = 0$, i.e. $\cos t = 1$.

$$
\begin{aligned}
T^*_{I;x_0,y} &= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+\frac{\left(1-t^2\right)\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^* t^2\right]}dt \\
&= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+(2x_0^*-\frac{x_0^{*2}}{\sigma^{*2}})t^2+\frac{x_0^{*2}}{\sigma^{*2}}\right]}\cdot e^{-\frac{y^{*2}}{2\sigma^{*2}}}dt \\
&= T^*_{II;x_0,y=0}\cdot e^{-\frac{y^{*2}}{2\sigma^{*2}}}
\end{aligned}
\tag{12}
$$

According to Eq. 12,

$$
y^*_{mII}=\sqrt{2}\sigma^*\sqrt{\ln\frac{T^*(\sigma^*)}{T^*}}=\sqrt{2}\sigma^*\sqrt{\ln\frac{Ry^*}{Ry^*_{min}(\sigma^*)}} \tag{13}
$$

4.3. blending

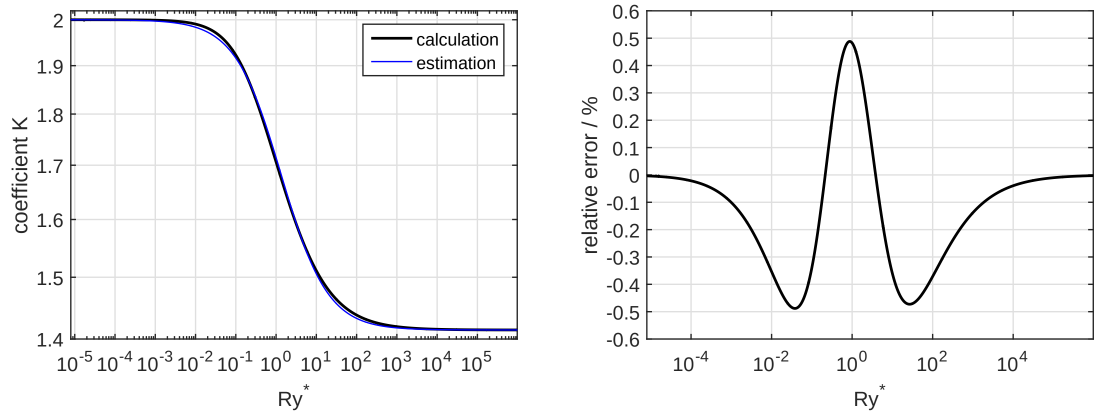
Use Eq. 11 and Eq. 13 to obtained the approximation of y^*_m when σ^* tends to σ^*_m:

$$
y^*_m=K\sigma^*\sqrt{\ln\frac{Ry^*}{Ry^*_{min}(\sigma^*)}} \tag{14}
$$

K changes with Ry.

$$
K=k_0-A*\tanh\left(B\ln\frac{Ry}{C}\right) \tag{15}
$$

Where $k_0=\frac{2+\sqrt{2}}{2}$,$A=\frac{2-\sqrt{2}}{2}$,$B=0.3775$,$C=1.0690$.The maximum error reaches 0.5%. Eq. 14 can be



(a) coefficient changes with Ry^*

(b) relative error changes with Ry^*

Fig. 3: Results of approximation of coefficient K against Ry^*

written as a function depicting the near field temperature distribution around the maximum temperature point:

$$Ry^* = Ry^*_{min}(\sigma^*)\, e^{\frac{y_m^{*2}}{K^2\sigma^{*2}}} \tag{16}$$

5. quasi point source

When $\sigma^* = 0$, the Eq. 1 describes the point heat source.

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \quad \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2}+y^{*2}}{2\tau}} = \frac{1}{r^*}e^{-r^*-x^*} \tag{17}$$

Do derivations on Eq. 17 with respect to y:

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \quad \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2}+y^{*2}}{2\tau}} = \frac{1}{r^*}e^{-r^*-x^*} \tag{18}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \quad \tau^{-\frac{5}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2}+y^{*2}}{2\tau}} = -\frac{1}{y^*}\frac{\partial}{\partial y^*}\left(\frac{1}{r^*}e^{-r^*-x^*}\right) = e^{-r^*-x^*}\left(\frac{1}{r^{*2}}+\frac{1}{r^{*3}}\right) \tag{19}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \quad \tau^{-\frac{7}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2}+y^{*2}}{2\tau}} = \frac{1}{y^*}\left[\frac{1}{y^*}\frac{\partial}{\partial y^*}\left(\frac{1}{r^*}e^{-r^*-x^*}\right)\right] = e^{-r^*-x^*}\left(\frac{1}{r^{*3}}+\frac{3}{r^{*4}}+\frac{3}{r^{*5}}\right) \tag{20}$$

When $\frac{\sigma^*}{\sigma^*_m}$ tends to zero, the Gaussian heat source can be treated as point source, with little error. So, use Eq. 1 rather than Eq. 2.

$$\begin{aligned} T^* &= \frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau \frac{\tau^{-\frac{1}{2}}}{\tau+\sigma^{*2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2}+y^{*2}}{2\tau+2\sigma^{*2}}} \\ &\approx \frac{1}{\sqrt{2\pi}}\int_0^\infty d\tau\, \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2}+y^{*2}}{2\tau}} \cdot \left[\left(1+\frac{\sigma^{*2}}{2}\right) + \sigma^{*2}(x-1)\frac{1}{\tau} + \frac{x^{*2}+y^{*2}}{2}\sigma^{*2}\frac{1}{\tau^2}\right] \\ &= e^{-r^*-x^*}\left[\left(1+\frac{\sigma^{*2}}{2}\right)\frac{1}{r^*} + \sigma^{*2}(x-1)\left(\frac{1}{r^{*2}}+\frac{1}{r^{*3}}\right) + \frac{\sigma^{*2}}{2}\left(\frac{1}{r^*}+\frac{3}{r^{*2}}+\frac{3}{r^{*3}}\right)\right] \end{aligned} \tag{21}$$

This process uses the first two terms of Taylor series of integrand with respect to σ^*. Eq. 21 describes the temperature distribution of far field.

5.1. $\sigma \to 0$

When $\sigma \to 0$, $x^* \ll y^* \ll 1$. Eq. 21 can be simplified as

$$\begin{aligned} T^* &\approx e^{-r^*-x^*}\left[\left(1+\frac{\sigma^{*2}}{2}\right)\frac{1}{r^*} + \sigma^{*2}(x-1)\left(\frac{1}{r^{*2}}+\frac{1}{r^{*3}}\right) + \frac{\sigma^{*2}}{2}\left(\frac{1}{r^*}+\frac{3}{r^{*2}}+\frac{3}{r^{*3}}\right)\right] \\ &\approx 1\cdot\left[\left(1+\frac{\sigma^{*2}}{2}\right)\frac{1}{y^*} - \sigma^{*2}\frac{1}{y^{*3}} + \frac{\sigma^{*2}}{2}\frac{3}{y^{*3}}\right] \\ &\approx \frac{1}{y^*} + \frac{\sigma^{*2}}{2}\frac{1}{y^{*3}} \end{aligned}$$

Use perturbation method, $y^*_{m,gauss} = y^*_{m,point}\left(1 + a\sigma^{*2}\right)$, $a\sigma^{*2} \ll 1$, $y^*_{m,point} = Ry^*$.

$$\begin{aligned}
\frac{1}{Ry^*} \approx \frac{1}{y^*} + \frac{\sigma^{*2}}{2}\frac{1}{y^{*3}} &\approx \frac{1}{y^*_{m,point}\left(1 + a\sigma^{*2}\right)} + \frac{\sigma^{*2}}{2}\frac{1}{y^{*3}_{m,point}\left(1 + 3a\sigma^{*2}\right)} \\
&\approx \frac{1}{Ry^*\left(1 + a\sigma^{*2}\right)} + \frac{\sigma^{*2}}{2}\frac{1}{Ry^{*3}\left(1 + 3a\sigma^{*2}\right)} \\
&\Rightarrow a = \frac{1}{2Ry^{*2}} && (22) \\
y^*_{m,gauss,0} &= y^*_{m,point}\left(1 + \frac{1}{2Ry^{*2}}\sigma^{*2}\right) && (23)
\end{aligned}$$

5.2. $\sigma \to \infty$

When $\sigma \to \infty$, $1 \ll \sigma^* \ll y^* \ll x^*$. Eq. 21 can be simplified as

$$\begin{aligned}
T^* &\approx e^{-r^*-x^*}\left[\left(1 + \frac{\sigma^{*2}}{2}\right)\frac{1}{r^*} + \sigma^{*2}(x-1)\left(\frac{1}{r^{*2}} + \frac{1}{r^{*3}}\right) + \frac{\sigma^{*2}}{2}\left(\frac{1}{r^*} + \frac{3}{r^{*2}} + \frac{3}{r^{*3}}\right)\right] \\
&\approx e^{-r^*-x^*}\left[\frac{1}{r^*} + \frac{\sigma^{*2}\left(r^* + x^{*2}\right)}{r^{*2}} - \frac{\sigma^{*2}}{2r^{*2}}\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}}\left(-\frac{y^{*2}}{2x^*} - 0.5\right)\right] && (24)
\end{aligned}$$

Use perturbation method, $y^*_{m,gauss} = y^*_{m,point}\left(1 + b\sigma^{*2}\right)$, $x^*_{m,gauss} = x^*_{m,point}\left(1 + c\sigma^{*2}\right)$, $b\sigma^{*2} \ll 1$,$c\sigma^{*2} \ll 1$, $y^*_{m,point} = \sqrt{\frac{2}{e}Ry^*}$,$x^*_{m,point} = -\frac{Ry^*}{e}$.

$$\begin{aligned}
\frac{1}{Ry^*} &\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}}\left(-\frac{y^{*2}}{2x^*} - 0.5\right)\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}}\left(0.5 + 2b\sigma^{*2} - c\sigma^{*2}\right)\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left(-\frac{1}{x^*} + 0.5\frac{\sigma^{*2}}{x^{*2}}\right) \\
y^* &= \sqrt{2x^*\ln\frac{x^{*2}/Ry^*}{-x^* + 0.5\sigma^{*2}}} \\
\frac{dy^*}{dx^*} &= \frac{\sqrt{2}\left(2\ln\left(-\frac{2tx^2}{-s^2+2x}\right) + \frac{4x-4s^2}{-s^2+2x}\right)}{4\sqrt{x\ln\left(-\frac{2tx^2}{2x-s^2}\right)}} = 0 \\
&\Rightarrow x^*_{m,gauss} = x^*_{m,point} \\
&\Rightarrow b = \frac{e}{4Ry^*} \\
y^*_{m,gauss,infinity} &= y^*_{m,point}\left(1 + \frac{e}{4Ry^*}\sigma^{*2}\right) && (25)
\end{aligned}$$

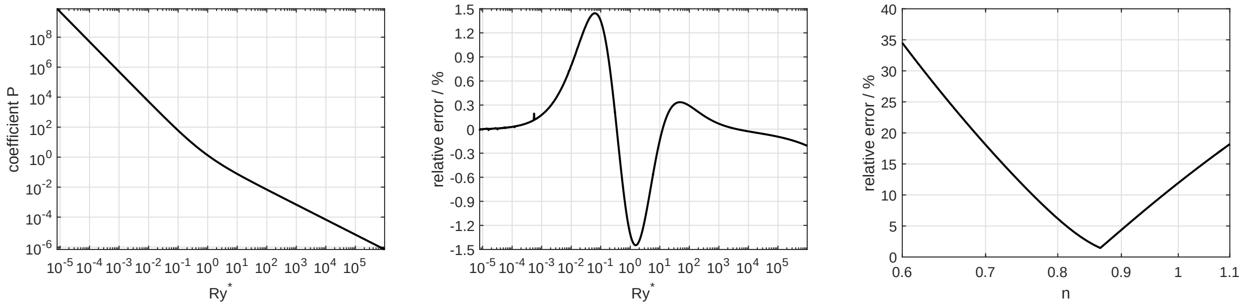
5.3. blending

Use the following equation to blending:

$$y^*_{m,gauss} = y^*_{m,point}\left(1 + P * \sigma^{*2}\right) \tag{26}$$

$$P = \left[\left(\frac{1}{2Ry^{*2}} \right)^n + \left(\frac{e}{4Ry^*} \right)^n \right]^{\frac{1}{n}} \tag{27}$$

Where $n = 0.8655$, maximum error reaches 1.45%.



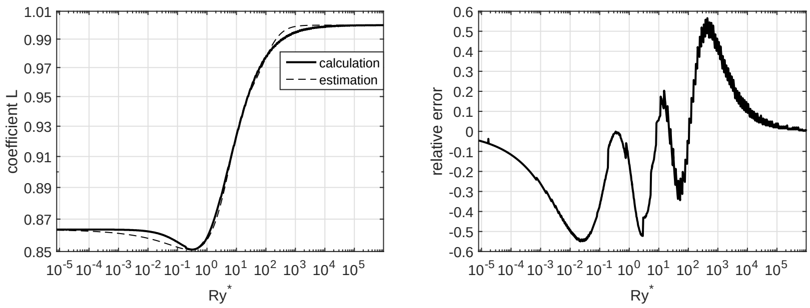
(a) coefficient changes with Ry^* (b) relative error changes with Ry^* (c) relative error changes with Ry^*

Fig. 4: Results of approximation of coefficient K against Ry^*

6. Combination

The maximum width of welding pool is a combination of far-field and near-field. To cover the middle range of $\frac{\sigma}{\sigma_m}$, the correction of near-field equation is needed.

$$L = 0.93175 - 0.06825 \tanh \left(-0.6571 \ \ln \frac{Ry}{15.926} \right) - 0.0132 \sin \left[3\pi \tanh \left(0.2485 \ Ry^{0.3718} \right) \right] \tag{28}$$



(a) coefficient changes with Ry^* (b) relative error changes with Ry^*

Fig. 5: Results of approximation of coefficient L against Ry^*

The maximum error reaches 0.55%.

7. whole plane

$$
\begin{aligned}
y_m^* &= MAX\{ \quad y_{m,1}^* \ , \quad y_{m,2}^* \quad \} \\
y_{m,1}^* &= \begin{cases} y_{m,point}^* \left(1 + P\sigma^{*2}\right) & \dfrac{\sigma^*}{\sigma_m^*} < \ lin \\ 0 & \dfrac{\sigma^*}{\sigma_m^*} \geq \ lin \end{cases} \\
y_{m,2}^* &= K \left[L \left(\sigma^* - \sigma_m^*\right) + \sigma_m^* \right] \sqrt{\ln \frac{Ry^*}{Ry_{min}^*(\sigma^*)}}
\end{aligned}
\tag{29}
$$

The parameters in equations are as follows:

$$y_{m,point}^* = \left[(Ry^*)^n + \left(\sqrt{\frac{2}{e}} Ry^* \right)^n \right]^{\frac{1}{n}}$$

Where n= -1.7312

$$P = \left[\left(\frac{1}{2Ry^{*2}} \right)^n + \left(\frac{e}{4Ry^*} \right)^n \right]^{\frac{1}{n}}$$

Where n= 0.8655

$$lin = 0.1 \left(Ry^* > 5 \times 10^3\right) + 0.4 \left(Ry \leqslant 5 \times 10^3\right);$$

$$K = k_0 - A * \tanh \left(B \ln \frac{Ry}{C} \right)$$

Where $k_0 = \frac{2+\sqrt{2}}{2}$,$A = \frac{2-\sqrt{2}}{2}$,$B = 0.3775$,$C = 1.0690$.

$$L = 0.93175 - 0.06825 \tanh \left(-0.6571 \ \ln \frac{Ry}{15.926} \right) - 0.0132 \sin \left[3\pi \tanh \left(0.2485 \ Ry^{0.3718} \right) \right]$$

$$\sigma_m^* = \left[\left(1.0140 \ Ry^{*\frac{2}{3}} \right)^n + \left(\sqrt{\frac{\pi}{2}} Ry^* \right)^n \right]^{\frac{1}{n}}$$

Where $n = -2.3975$

$$Ry_{min}^*(\sigma^*) = \left[\left(\sqrt{\frac{\pi}{2}} \ \sigma^{*-1} \right)^n + \left(\frac{2.5596}{\sqrt{2\pi}} \ \sigma^{*-1.5} \right)^n \right]^{-\frac{1}{n}}$$

Where $n = -1.9464$

The maximum error reaches 5.25%.There is a limit that $\frac{\sigma^*}{\sigma_m^*} < 98\%$, because when $\frac{\sigma^*}{\sigma_m^*}$ tends to 1, y_m^* tends to 0, the approximation (means that it's not accurate) of Ry_{min}^*, σ_m^* leads to a large relative error. If the high-precision value is obtained these equations still work.

8. Results

Results

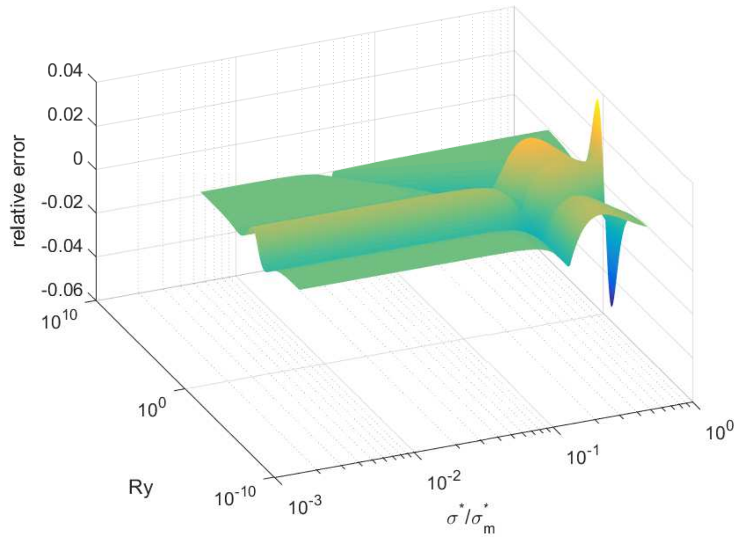


Fig. 6: relative error over whole plane.

9. Discussion

10. Conclusions

Conclusions Section

11. Conclusions

Conclusions Section

12. Acknowledgement

This study has been supported by...

13. References

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