

# Scaling Analysis of a Moving Guassion Heat Source in Steady State in a Semi-Infinite Solid

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## Abstract

Abstract goes here

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## 1. Introduction

Our research focuses on developing simplified formulas with high accuracy that will substitute complex numerical calculations.

## 2. Governing Equation

$$T^* = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \frac{\tau^{-\frac{1}{2}}}{\tau + \sigma^{*2}} e^{-\frac{x^{*2} + 2\tau^*x^* + \tau^{*2} + y^{*2}}{2\tau + 2\sigma^{*2}} - \frac{z^{*2}}{2\tau}} \quad (1)$$

Eq. 1 has some disadvantages. Firstly, it is a improper integral and the up limit is infinity which makes the calculation more difficult. Secondly, the integrand has two peaks. One locates at  $\tau = 0$ , and the other moves and is hard to determine, which may results in the omitting of second peak in integral.

Use variable substitution method  $t = \arctan \frac{\sqrt{\tau}}{\sigma^*}$ , and do not consider the depth of pool, which means  $z^* = 0$ .

$$T^* = \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2} \left[ \sigma^{*2} (\cos^2 t + \frac{1}{\cos^2 t} - 2) + \frac{\cos^2 t (x^{*2} + y^{*2})}{\sigma^{*2}} + 2x^* (1 - \cos^2 t) \right]} dt \quad (2)$$

Eq. 2 avoids the disadvantages of Eq. 1. The integral is bounded. The integrand has one peak located at  $t = \arccos \{ \sigma^* [(\sigma^{*2} - x^*)^2 + y^{*2}] \}$ . However, Eq. 2 can't be applied to the point-source condition.

## 3. the highest temperature $T_m^*$ corresponding to $\sigma^*$

To calculating the highest  $T^*$  corresponding to  $\sigma^*$ ,  $y^*$  should be set as 0 because the maximum temperature  $T_m^*$  decreases as  $|y^*|$  increases according to Eq. 1.

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### 3.1. $\sigma \rightarrow 0$

According the numerical calculation,  $x^*$  should be much smaller than  $\sigma^*$ , so  $|\frac{x^*}{\sigma^*}| \sim 0$ . Eq. 2 can be simplified as:

$$\begin{aligned}\widehat{T}_{mI}^* &= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2} \left[ \sigma^{*2} \left( \cos^2 t + \frac{1}{\cos^2 t} - 2 \right) + \frac{\cos^2 t x^{*2}}{\sigma^{*2}} + 2x^* (1 - \cos^2 t) \right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2} \sigma^{*2} \left( \cos^2 t + \frac{1}{\cos^2 t} - 2 \right)} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \frac{\pi}{2} = \sqrt{\frac{\pi}{2}} \sigma^{*-1}\end{aligned}\quad (3)$$

### 3.2. $\sigma \rightarrow \infty$

When  $\sigma$  tends to infinity, the peak locates at 0, and the integrand decreases sharply.

$$\begin{aligned}\widehat{T}_{mII}^* &= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2} \left[ \sigma^{*2} \left( \cos^2 t + \frac{1}{\cos^2 t} - 2 \right) + \frac{\cos^2 t x^{*2}}{\sigma^{*2}} + 2x^* (1 - \cos^2 t) \right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^\delta e^{-\frac{1}{2} \left[ \sigma^{*2} \left( \cos^2 t + \frac{1}{\cos^2 t} - 2 \right) + \frac{\cos^2 t x^{*2}}{\sigma^{*2}} + 2x^* (1 - \cos^2 t) \right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^\delta e^{-\frac{1}{2} \left[ \sigma^{*2} t^4 + \frac{(1-t^2)x^{*2}}{\sigma^{*2}} + 2x^* t^2 \right]} dt \\ &= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^\delta e^{-\frac{1}{2} \left[ \sigma^{*2} t^4 + (2x^* - \frac{x^{*2}}{\sigma^{*2}}) t^2 + \frac{x^{*2}}{\sigma^{*2}} \right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^\delta e^{-\frac{1}{2} \left[ \sigma^{*2} t^4 + 2x^* t^2 + \frac{x^{*2}}{\sigma^{*2}} \right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^\delta e^{-\frac{1}{2} \left( \sigma^* t^2 + \frac{x^*}{\sigma^*} \right)^2} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2} \left( \sigma^* t^2 + \frac{x^*}{\sigma^*} \right)^2} dt\end{aligned}\quad (4)$$

Where  $\delta$  is infinitesimal, and  $x^* \sim \sigma^* \gg 1$ .

Use numerical method to find the maximum value of Eq. 4 with changes of  $x^*$ . When  $x^* = -0.7650 \sigma^*$ ,  $T^*$  reaches maximum value.

$$\widehat{T}_{mII}^* = \frac{2.5596}{\sqrt{2\pi}} \sigma^{*-1.5} \quad (5)$$

### 3.3. blending

Use Eq. 3 and Eq. 5 to obtaining the blending equation for all  $\sigma$ .

$$\widehat{T}_m^+(\sigma^*) = \left[ \left( \sqrt{\frac{\pi}{2}} \sigma^{*-1} \right)^n + \left( \frac{2.5596}{\sqrt{2\pi}} \sigma^{*-1.5} \right)^n \right]^{\frac{1}{n}} \quad (6)$$

Where  $n = -1.9464$ , and the maximum error reaches 0.1901%.

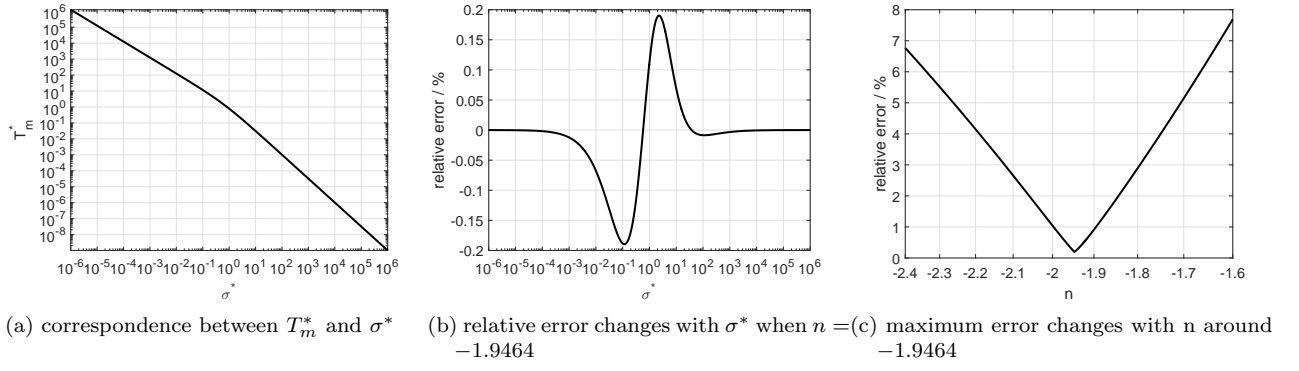


Fig. 1: Results of the blending between  $T_m^*$  and  $\sigma^*$

Eq. 6 reveals the one-to-one correspondence between  $\sigma^*$  and  $T^*$ . For any melting point  $T^*$ , there is a certain  $\sigma^*$ , below which the base substance can't melt, and vice versa. So, Eq. 3 and Eq. 5 can be rewritten as:

$$\widehat{\sigma}_{mI}^* = \sqrt{\frac{\pi}{2}} Ry \quad (7)$$

Where  $Ry^* = \frac{1}{T^*}$ .

$$\widehat{\sigma}_{mII}^* = 1.0140 Ry^{\frac{2}{3}} \quad (8)$$

$$\widehat{\sigma}_m^+ (Ry) = \left[ \left( 1.0140 Ry^{\frac{2}{3}} \right)^n + \left( \sqrt{\frac{\pi}{2}} Ry \right)^n \right]^{\frac{1}{n}} \quad (9)$$

Where  $n = -2.3975$ , and the maximum error reaches minimum, 1.39%.

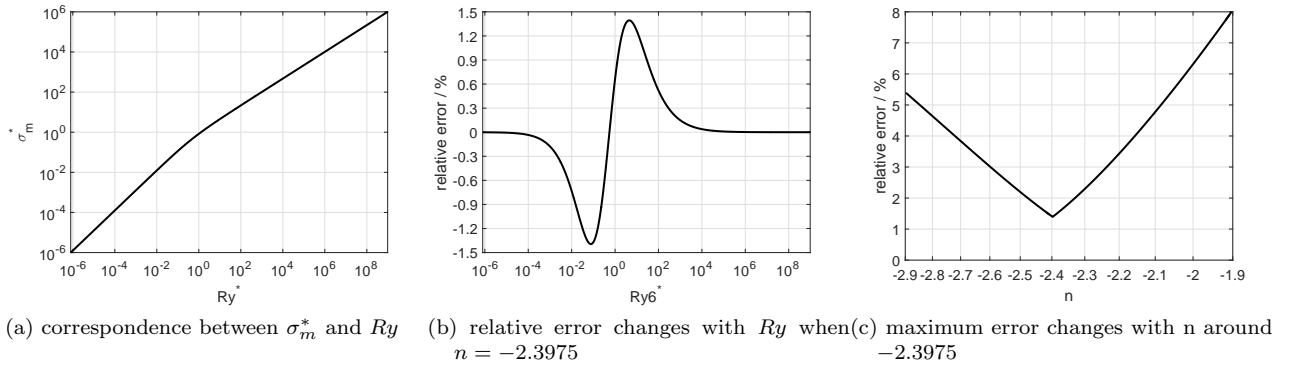


Fig. 2: Results of the blending between  $\sigma_m^*$  and  $Ry$

#### 4. $\sigma \rightarrow \sigma_m$

When  $\sigma^*$  tends to  $\sigma_m^*$ , the welding pool vanishes, and should be axisymmetric, which means the maximum width point locates above the maximum temperature point, i.e.  $x_{m,\text{corresponding to maximum width point}} = x_{m,\text{corresponding to maximum temperature point}}$ .

##### 4.1. $\sigma \rightarrow 0$

$$\begin{aligned}
T_{I;x_0,y}^* &= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2} \left[ \sigma^{*2} \left( \cos^2 t + \frac{1}{\cos^2 t} - 2 \right) + \frac{\cos^2 t (x_0^{*2} + y^{*2})}{\sigma^{*2}} + 2x_0^* (1 - \cos^2 t) \right]} dt \\
&= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2} \left[ \sigma^{*2} \left( \cos^2 t + \frac{1}{\cos^2 t} - 2 \right) + \frac{\cos^2 t x_0^{*2}}{\sigma^{*2}} + 2x_0^* (1 - \cos^2 t) \right]} \cdot e^{-\frac{\cos^2 t y^{*2}}{2\sigma^{*2}}} dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} 1 \cdot e^{-\frac{\cos^2 t y^{*2}}{2\sigma^{*2}}} dt \\
&= \frac{2}{\pi} T_{I;x_0,y=0}^* \cdot \int_0^{\frac{\pi}{2}} e^{-\frac{\cos^2 t y^{*2}}{2\sigma^{*2}}} dt \\
&\approx \frac{2}{\pi} T_{I;x_0,y=0}^* \cdot \int_0^{\frac{\pi}{2}} 1 - \frac{\cos^2 t y^{*2}}{2\sigma^{*2}} dt \quad \text{as } y^* \ll \sigma^* \\
&= T_{I;x_0,y=0}^* \cdot \frac{2}{\pi} \left( \frac{\pi}{2} - \frac{y^{*2}\pi}{8\sigma^{*2}} \right) \\
&= T_{I;x_0,y=0}^* \cdot \left( 1 - \frac{y^{*2}}{4\sigma^{*2}} \right) \\
&= T_{I;x_0,y=0}^* \cdot \left( 1 - \frac{y^{*2}}{4\sigma^{*2}} \right) \\
&= T_{I;x_0,y=0}^* \cdot e^{-\frac{y^{*2}}{4\sigma^{*2}}} \tag{10}
\end{aligned}$$

Where  $x_0^* = 0$ ,  $y^* \ll \sigma^*$ .

According to [Eq. 10](#),

$$\hat{y}_{mI}^* = 2\sigma^* \sqrt{\ln \frac{T_m^*(\sigma^*)}{T^*}} = 2\sigma^* \sqrt{\ln \frac{Ry}{Ry_{min}(\sigma^*)}} \tag{11}$$

##### 4.2. $\sigma \rightarrow \infty$

When  $\sigma$  tends to infinity, the location of maximum temperature point  $x_0^* = -0.7650 \sigma^*$ , and the integrand focuses on  $t = 0$ , i.e.  $\cos t = 1$ .

$$\begin{aligned}
T_{I;x_0,y}^* &= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2} \left[ \sigma^{*2} \left( \cos^2 t + \frac{1}{\cos^2 t} - 2 \right) + \frac{\cos^2 t (x_0^{*2} + y^{*2})}{\sigma^{*2}} + 2x_0^* (1 - \cos^2 t) \right]} dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^\delta e^{-\frac{1}{2} \left[ \sigma^{*2} \left( \cos t + \frac{1}{\cos t} - 2 \right) + \frac{\cos^2 t (x_0^{*2} + y^{*2})}{\sigma^{*2}} + 2x_0^* (1 - \cos^2 t) \right]} dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^\delta e^{-\frac{1}{2} \left[ \sigma^{*2} t^4 + \frac{(1-t^2)(x_0^{*2} + y^{*2})}{\sigma^{*2}} + 2x_0^* t^2 \right]} dt \\
&= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^\delta e^{-\frac{1}{2} \left[ \sigma^{*2} t^4 + (2x_0^* - \frac{x_0^{*2}}{\sigma^{*2}}) t^2 + \frac{x_0^{*2}}{\sigma^{*2}} \right]} \cdot e^{-\frac{y^{*2}}{2\sigma^{*2}}} dt \\
&= T_{II;x_0,y=0}^* \cdot e^{-\frac{y^{*2}}{2\sigma^{*2}}}
\end{aligned} \tag{12}$$

According to Eq. 12,

$$\hat{y}_{mII}^* = \sqrt{2}\sigma^* \sqrt{\ln \frac{T^*(\sigma^*)}{T^*}} = \sqrt{2}\sigma^* \sqrt{\ln \frac{Ry}{Ry_{min}(\sigma^*)}} \tag{13}$$

#### 4.3. blending

Use Eq. 11 and Eq. 13 to obtained the approximation of  $y_m^*$  when  $\sigma^*$  tends to  $\sigma_m^*$ :

$$\hat{y}_{m,near}^* = K\sigma^* \sqrt{\ln \frac{Ry}{Ry_{min}(\sigma^*)}} \tag{14}$$

$K$  changes with  $Ry$ .

$$K = k_0 - A * \tanh \left( B \ln \frac{Ry}{C} \right) \tag{15}$$

Where  $k_0 = \frac{2+\sqrt{2}}{2}$ ,  $A = \frac{2-\sqrt{2}}{2}$ ,  $B = 0.3775$ ,  $C = 1.0690$ . The maximum error reaches 0.5%. Eq. 14 can be

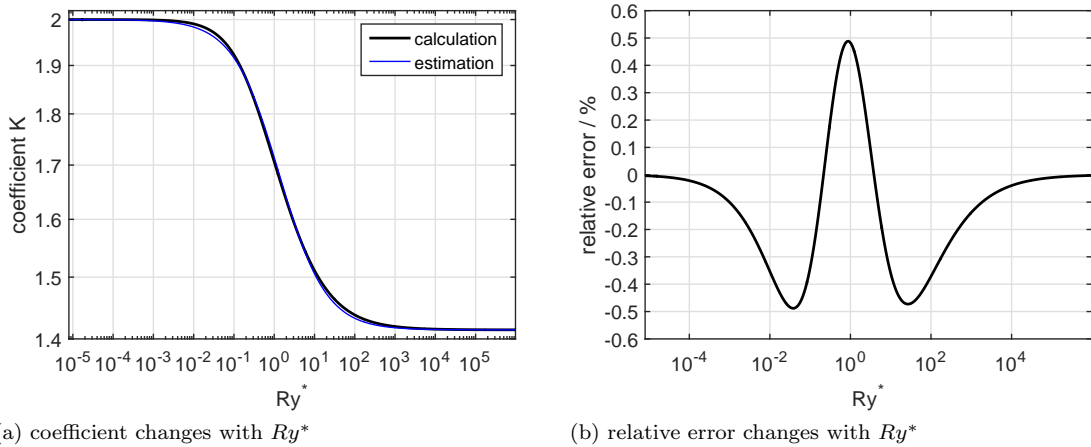


Fig. 3: Results of approximation of coefficient  $K$  against  $Ry^*$

written as a function depicting the near field temperature distribution around the maximum temperature

point:

$$Ry = Ry_{min}(\sigma^*) e^{\frac{y_m^{*2}}{\kappa^2 \sigma^{*2}}} \quad (16)$$

## 5. quasi point source

When  $\sigma^* = 0$ , the Eq. 1 describes the point heat source.

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^*x^*+\tau^{*2}+y^{*2}}{2\tau}} = \frac{1}{r^*} e^{-r^*-x^*} \quad (17)$$

Do derivations on Eq. 17 with respect to y:

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^*x^*+\tau^{*2}+y^{*2}}{2\tau}} = \frac{1}{r^*} e^{-r^*-x^*} \quad (18)$$

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \tau^{-\frac{5}{2}} e^{-\frac{x^{*2}+2\tau^*x^*+\tau^{*2}+y^{*2}}{2\tau}} &= -\frac{1}{y^*} \frac{\partial}{\partial y^*} \left( \frac{1}{r^*} e^{-r^*-x^*} \right) \\ &= e^{-r^*-x^*} \left( \frac{1}{r^{*2}} + \frac{1}{r^{*3}} \right) \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \tau^{-\frac{7}{2}} e^{-\frac{x^{*2}+2\tau^*x^*+\tau^{*2}+y^{*2}}{2\tau}} &= \frac{1}{y^*} \left[ \frac{1}{y^*} \frac{\partial}{\partial y^*} \left( \frac{1}{r^*} e^{-r^*-x^*} \right) \right] \\ &= e^{-r^*-x^*} \left( \frac{1}{r^{*3}} + \frac{3}{r^{*4}} + \frac{3}{r^{*5}} \right) \end{aligned} \quad (20)$$

When  $\frac{\sigma^*}{\sigma_m^*}$  tends to zero, the Gaussian heat source can be treated as point source, with little error. So, use Eq. 1 rather than Eq. 2.

$$\begin{aligned} \widehat{T}_{far}^* &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \frac{\tau^{-\frac{1}{2}}}{\tau + \sigma^{*2}} e^{-\frac{x^{*2}+2\tau^*x^*+\tau^{*2}+y^{*2}}{2\tau+2\sigma^{*2}}} \\ &\approx \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^*x^*+\tau^{*2}+y^{*2}}{2\tau}} \cdot \left[ \left( 1 + \frac{\sigma^{*2}}{2} \right) + \sigma^{*2} (x-1) \frac{1}{\tau} + \frac{x^{*2}+y^{*2}}{2} \sigma^{*2} \frac{1}{\tau^2} \right] \\ &= e^{-r^*-x^*} \left[ \left( 1 + \frac{\sigma^{*2}}{2} \right) \frac{1}{r^*} + \sigma^{*2} (x-1) \left( \frac{1}{r^{*2}} + \frac{1}{r^{*3}} \right) + \frac{\sigma^{*2}}{2} \left( \frac{1}{r^*} + \frac{3}{r^{*2}} + \frac{3}{r^{*3}} \right) \right] \end{aligned} \quad (21)$$

This process uses the first two terms of Taylor series of integrand with respect to  $\sigma^*$ . Eq. 21 describes the temperature distribution of far field.

### 5.1. $\sigma \rightarrow 0$

When  $\sigma \rightarrow 0$ ,  $x^* \ll y^* \ll 1$ . Eq. 21 can be simplified as

$$\begin{aligned} T^* &\approx e^{-r^*-x^*} \left[ \left(1 + \frac{\sigma^{*2}}{2}\right) \frac{1}{r^*} + \sigma^{*2} (x-1) \left( \frac{1}{r^{*2}} + \frac{1}{r^{*3}} \right) + \frac{\sigma^{*2}}{2} \left( \frac{1}{r^*} + \frac{3}{r^{*2}} + \frac{3}{r^{*3}} \right) \right] \\ &\approx 1 \cdot \left[ \left(1 + \frac{\sigma^{*2}}{2}\right) \frac{1}{y^*} - \sigma^{*2} \frac{1}{y^{*3}} + \frac{\sigma^{*2}}{2} \frac{3}{y^{*3}} \right] \\ &\approx \frac{1}{y^*} + \frac{\sigma^{*2}}{2} \frac{1}{y^{*3}} \end{aligned}$$

Use perturbation method,  $y_{m,gauss}^* = y_{m,point}^* (1 + a\sigma^{*2})$ ,  $a\sigma^{*2} \ll 1$ ,  $y_{m,point}^* = Ry$ .

$$\begin{aligned} \frac{1}{Ry} &\approx \frac{1}{y^*} + \frac{\sigma^{*2}}{2} \frac{1}{y^{*3}} \\ &\approx \frac{1}{y_{m,point}^* (1 + a\sigma^{*2})} + \frac{\sigma^{*2}}{2} \frac{1}{y_{m,point}^{*3} (1 + 3a\sigma^{*2})} \\ &\approx \frac{1}{Ry (1 + a\sigma^{*2})} + \frac{\sigma^{*2}}{2} \frac{1}{Ry^3 (1 + 3a\sigma^{*2})} \\ &\Rightarrow a = \frac{1}{2Ry^2} \end{aligned} \tag{22}$$

$$y_{m,gauss,0}^* = y_{m,point}^* \left( 1 + \frac{1}{2Ry^2} \sigma^{*2} \right) \tag{23}$$

### 5.2. $\sigma \rightarrow \infty$

When  $\sigma \rightarrow \infty$ ,  $1 \ll \sigma^* \ll y^* \ll x^*$ . Eq. 21 can be simplified as

$$\begin{aligned} T^* &\approx e^{-r^*-x^*} \left[ \left(1 + \frac{\sigma^{*2}}{2}\right) \frac{1}{r^*} + \sigma^{*2} (x-1) \left( \frac{1}{r^{*2}} + \frac{1}{r^{*3}} \right) + \frac{\sigma^{*2}}{2} \left( \frac{1}{r^*} + \frac{3}{r^{*2}} + \frac{3}{r^{*3}} \right) \right] \\ &\approx e^{-r^*-x^*} \left[ \frac{1}{r^*} + \frac{\sigma^{*2} (r^* + x^{*2})}{r^{*2}} - \frac{\sigma^{*2}}{2r^{*2}} \right] \\ &\approx e^{\frac{1}{2} \frac{y^{*2}}{x^*}} \left[ -\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}} \left( -\frac{y^{*2}}{2x^*} - 0.5 \right) \right] \end{aligned} \tag{24}$$

Use perturbation method,  $y_{m,gauss}^* = y_{m,point}^* (1 + b\sigma^{*2})$ ,  $x_{m,gauss}^* = x_{m,point}^* (1 + c\sigma^{*2})$ ,  $b\sigma^{*2} \ll 1$ ,  $c\sigma^{*2} \ll 1$ ,  $y_{m,point}^* = \sqrt{\frac{2}{e} Ry}$ ,  $x_{m,point}^* = -\frac{Ry}{e}$ .

$$\begin{aligned}
\frac{1}{Ry} &\approx e^{\frac{1}{2} \frac{y^{*2}}{x^{*2}}} \left[ -\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}} \left( -\frac{y^{*2}}{2x^*} - 0.5 \right) \right] \\
&\approx e^{\frac{1}{2} \frac{y^{*2}}{x^{*2}}} \left[ -\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}} (0.5 + 2b\sigma^{*2} - c\sigma^{*2}) \right] \\
&\approx e^{\frac{1}{2} \frac{y^{*2}}{x^{*2}}} \left( -\frac{1}{x^*} + 0.5 \frac{\sigma^{*2}}{x^{*2}} \right) \\
y^* &= \sqrt{2x^* \ln \frac{x^{*2}/Ry}{-x^* + 0.5\sigma^{*2}}} \\
\frac{dy^*}{dx^*} &= \frac{\sqrt{2} \left( 2 \ln \left( -\frac{2tx^2}{-s^2+2x} \right) + \frac{4x-4s^2}{-s^2+2x} \right)}{4\sqrt{x \ln \left( -\frac{2tx^2}{2x-s^2} \right)}} = 0 \\
&\Rightarrow x_{m,gauss}^* = x_{m,point}^* \\
&\Rightarrow b = \frac{e}{4Ry} \\
y_{m,gauss,infinity}^* &= y_{m,point}^* \left( 1 + \frac{e}{4Ry} \sigma^{*2} \right)
\end{aligned} \tag{25}$$

### 5.3. blending

Use the following equation to blending:

$$y_{m,gauss}^* = y_{m,point}^* (1 + P * \sigma^{*2}) \tag{26}$$

$$P = \left[ \left( \frac{1}{2Ry^2} \right)^n + \left( \frac{e}{4Ry} \right)^n \right]^{\frac{1}{n}} \tag{27}$$

Where  $n = 0.8655$ , maximum error reaches 1.45%.

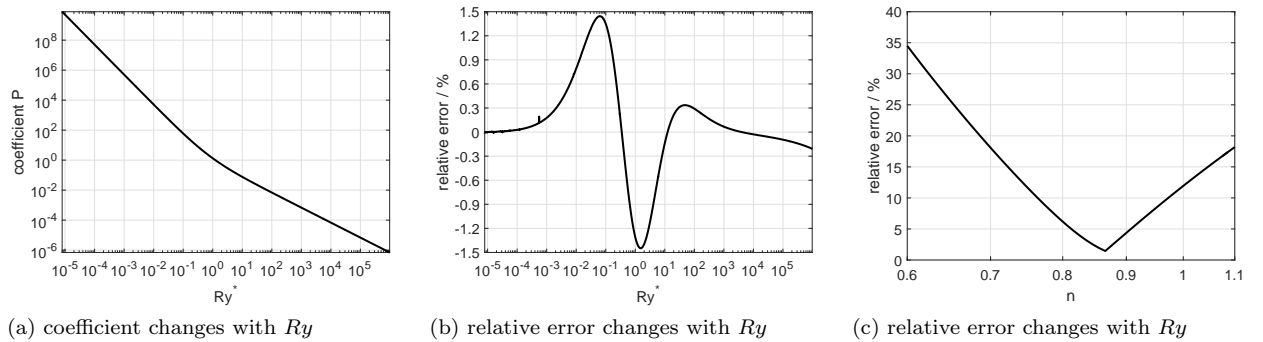


Fig. 4: Results of approximation of coefficient  $K$  against  $Ry$



## 6. Combination

The maximum width of welding pool is a combination of far-field and near-field. To cover the middle range of  $\frac{\sigma}{\sigma_m}$ , the correction of both equations is needed.

### 6.1. near-field

$$L = 0.93175 - 0.06825 \tanh \left( -0.6571 \ln \frac{Ry}{15.926} \right) - 0.0132 \sin [3\pi \tanh (0.2485 Ry^{0.3718})] \quad (28)$$

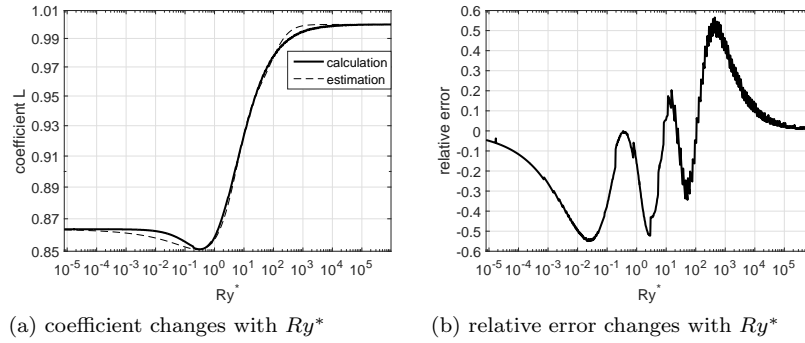


Fig. 5: Results of approximation of coefficient  $L$  against  $Ry$

The maximum error reaches 0.55%.

### 6.2. far-field

The maximum width can be written in form of step function, consisting of far-field expression and near-field expression. However, lines of  $y_{m,1}^*$  and  $y_{m,12}^*$  cross. A correction of far-field expression is applied, which is a confinement with the range of  $\frac{\sigma^*}{\sigma_m}$ . It's difficult to calculate the cross points directly, so a back step is took that the point between two cross points is found with any  $Ry$ .

When  $Ry \leq 5 \times 10^3$ ,  $lin = 0.4$ .

When  $Ry > 5 \times 10^3$ ,

$$y_{m,1}^* - y_{m,12}^* = \sqrt{\frac{2}{e} Ry} + 0.5993 \left( \frac{\sigma^*}{\sigma_m} \right)^2 Ry^{\frac{5}{6}} - 2.4838 \frac{\sigma^*}{\sigma_m} \sqrt{\log \frac{\sigma_m^*}{\sigma^*}} Ry^{\frac{2}{3}} < 0$$

When  $\frac{\sigma^*}{\sigma_m} = Ry^{-\frac{1}{6}}$ , the in-equation is satisfied. So:

$$lin = Ry^{-\frac{1}{6}} \cdot (Ry > 5 \times 10^3) + 0.4 (Ry \leq 5 \times 10^3); \quad (29)$$

## 7. whole plane

$$\begin{aligned}
y_m^* &= MAX\{ y_{m, far}^*, y_{m, near}^* \} \\
y_{m, far}^* &= y_{m, point}^* (1 + P\sigma_m^{*2}) \cdot (\sigma^*/\sigma_m^* < lin) \\
y_{m, near}^* &= K [L(\sigma^* - \sigma_m^*) + \sigma_m^*] \sqrt{\ln \frac{Ry^*}{Ry_{min}^*(\sigma^*)}}
\end{aligned} \tag{30}$$

The parameters in equations are as follows:

$$\begin{aligned}
y_{m, point}^* &= \left[ (Ry)^n + \left( \sqrt{\frac{2}{e}} Ry \right)^n \right]^{\frac{1}{n}} \\
\text{Where } n &= -1.7312 \\
P &= \left[ \left( \frac{1}{2Ry^2} \right)^n + \left( \frac{e}{4Ry} \right)^n \right]^{\frac{1}{n}} \\
\text{Where } n &= 0.8655 \\
lin &= Ry^{-\frac{1}{6}} \cdot (Ry > 5 \times 10^3) + 0.4 (Ry \leq 5 \times 10^3); \\
K &= k_0 - A * \tanh \left( B \ln \frac{Ry}{C} \right) \\
\text{Where } k_0 &= \frac{2+\sqrt{2}}{2}, A = \frac{2-\sqrt{2}}{2}, B = 0.3775, C = 1.0690. \\
L &= 0.93175 - 0.06825 \tanh \left( -0.6571 \ln \frac{Ry}{15.926} \right) - 0.0132 \sin [3\pi \tanh (0.2485 Ry^{0.3718})] \\
\sigma_m^* &= \left[ \left( 1.0140 Ry^{\frac{2}{3}} \right)^n + \left( \sqrt{\frac{\pi}{2}} Ry \right)^n \right]^{\frac{1}{n}} \\
\text{Where } n &= -2.3975 \\
Ry_{min}(\sigma^*) &= \left[ \left( \sqrt{\frac{\pi}{2}} \sigma^{*-1} \right)^n + \left( \frac{2.5596}{\sqrt{2\pi}} \sigma^{*-1.5} \right)^n \right]^{-\frac{1}{n}} \\
\text{Where } n &= -1.9464
\end{aligned}$$

The maximum error reaches 5.25%. There is a limit that  $\frac{\sigma^*}{\sigma_m^*} < 98\%$ , because when  $\frac{\sigma^*}{\sigma_m^*}$  tends to 1,  $y_m^*$  tends to 0, the approximation (means that it's not accurate) of  $Ry_{min}, \sigma_m^*$  leads to a large relative error. If the high-precision value is obtained these equations still work.

## 8. Results

Results

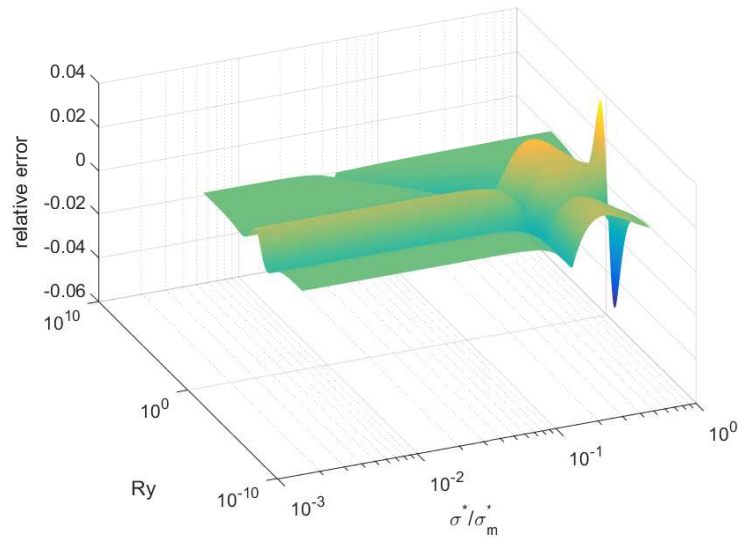


Fig. 6: relative error over whole plane.

## 9. Discussion

## 10. Conclusions

Conclusions Section

## 11. Conclusions

Conclusions Section

## 12. Acknowledgement

This study has been supported by...

## 13. References