Scaling Analysis of a Moving Guassion Heat Source in Steady State in a Semi-Infinite Solid

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Abstract

Abstract goes here

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1. Introduction

Our research focuses on developing simplified formulas with high accuracy that will substitute complex numerical calculations.

2. Governing Equation

$$T^* = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \frac{\tau^{-\frac{1}{2}}}{\tau + \sigma^{*2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau + 2\sigma^{*2}} - \frac{z^{*2}}{2\tau}} \tag{1}$$

Eq. 1 has some disadvantages. Firstly, it is a improper integral and the up limit is infinity which makes the calculation more difficult. Secondly, the integrand has two peaks. One locates at $\tau = 0$, and the other moves and is hard to determine, which may results in the omitting of second peak in integral. Use variable substitution method $t = \arctan \frac{\sqrt{\tau}}{\sigma}$,and do not consider the depth of pool, which means $z^* = 0$.

$$T^* = \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t + \frac{1}{\cos^2 t} - 2\right) + \frac{\cos^2 t\left(x^{*2}+y^{*2}\right)}{\sigma^{*2}} + 2x^*\left(1-\cos^2 t\right)\right]} dt \tag{2}$$

Eq. 2 avoids the disadvantages of Eq. 1. The integral is bounded. The integrand has one peak located at $t = \arccos \{\sigma^* [(\sigma^{*2} - x^*)^2 + y^{*2}]\}$. However, Eq. 2 can't be applied to the point-source condition.

3. the highest T^* corresponding to σ^*

To calculating the highest T^* corresponding to σ^*, y^* should be set as 0.

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3.1. $\sigma \to 0$

According the numerical calculation, x^* should be much smaller than σ^*,so $|\frac{x^*}{\sigma^*}| \sim 0$. Eq. 2 can be simplified as:

$$\begin{aligned} T^*_{mI} &= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t + \frac{1}{\cos^2 t} - 2\right) + \frac{\cos^2 t\, x^{*2}}{\sigma^{*2}} + 2x^*\left(1-\cos^2 t\right)\right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\sigma^{*2}\left(\cos^2 t + \frac{1}{\cos^2 t} - 2\right)} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}\, \sigma^{*-1} \end{aligned} \tag{3}$$

3.2. $\sigma \to \infty$

When σ tends to infinity, the peak locates at 0, and the integrand decreases sharply.

$$\begin{aligned} T^*_{mII} &= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t + \frac{1}{\cos^2 t} - 2\right) + \frac{\cos^2 t\, x^{*2}}{\sigma^{*2}} + 2x^*\left(1-\cos^2 t\right)\right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t + \frac{1}{\cos^2 t} - 2\right) + \frac{\cos^2 t\, x^{*2}}{\sigma^{*2}} + 2x^*\left(1-\cos^2 t\right)\right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\, t^4 + \frac{(1-t^2)x^{*2}}{\sigma^{*2}} + 2x^* t^2\right]} dt \\ &= \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\, t^4 + (2x^* - \frac{x^{*2}}{\sigma^{*2}})t^2 + \frac{x^{*2}}{\sigma^{*2}}\right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\, t^4 + 2x^* t^2 + \frac{x^{*2}}{\sigma^{*2}}\right]} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\delta} e^{-\frac{1}{2}\left(\sigma^* t^2 + \frac{x^*}{\sigma^*}\right)^2} dt \\ &\approx \frac{2}{\sqrt{2\pi}\sigma^*} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left(\sigma^* t^2 + \frac{x^*}{\sigma^*}\right)^2} dt \end{aligned} \tag{4}$$

Where δ is infinitesimal, and $x^* \sim \sigma^* \gg 1$.

Use numerical method to find the maximum value of Eq. 4 with changes of x^*. When $x^* = -0.7650\,\sigma^*$, T^* reaches maximum value.

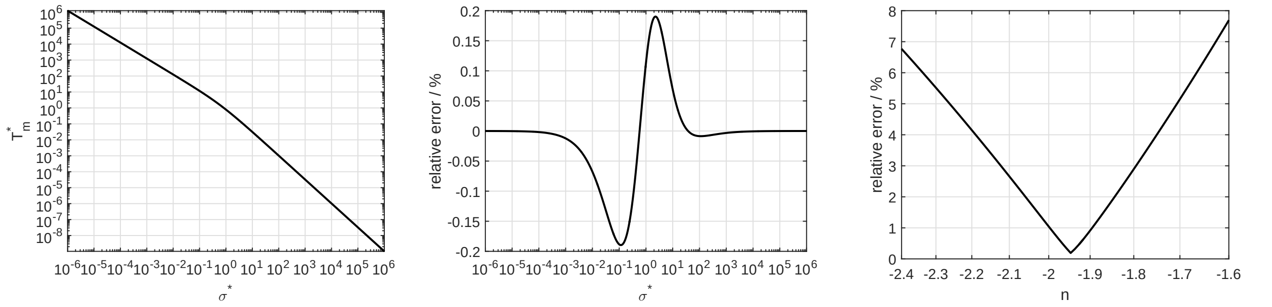
$$T^*_{mII} = \frac{2.5596}{\sqrt{2\pi}}\, \sigma^{*-1.5} \tag{5}$$

3.3. *blending*

Use Eq. 3 and Eq. 5 to obtaining the blending equation for all σ.

$$T^*_m = \left[\left(\sqrt{\frac{\pi}{2}}\, \sigma^{*-1}\right)^n + \left(\frac{2.5596}{\sqrt{2\pi}}\, \sigma^{*-1.5}\right)^n\right]^{\frac{1}{n}} \tag{6}$$

Where $n = -1.9464$, and the maximum error reaches 0.1901%.



(a) correspondence between T_m^* and σ^* (b) relative error changes with σ^* when $n = -1.9464$ (c) maximum error changes with n around -1.9464

Fig. 1: Results of the blending between T_m^* and σ^*

Eq. 6 reveals the one-to-one correspondence between σ^* and T^*. For any melting point T^*, there is a certain σ^*, below which the base substance can't melt, and vice versa. So, Eq. 3 and Eq. 5 can be rewritten as:

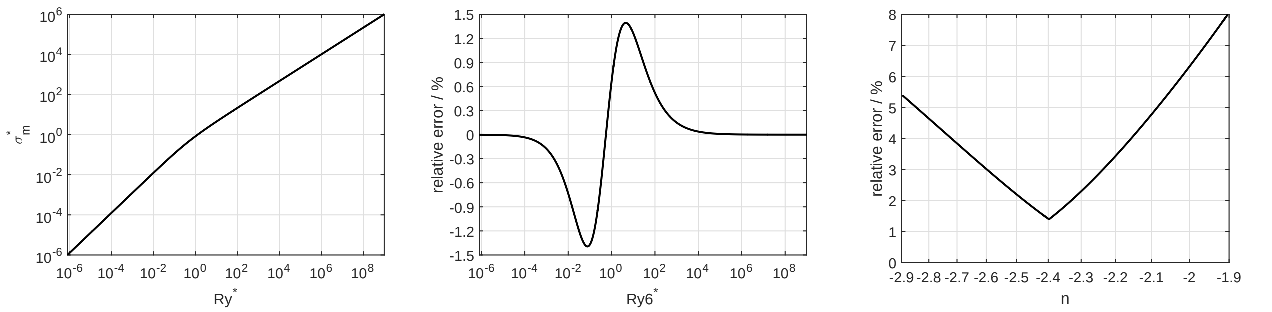
$$\sigma_{mI}^* = \sqrt{\frac{\pi}{2}} Ry^* \tag{7}$$

Where $Ry^* = \frac{1}{T^*}$.

$$\sigma_{mII}^* = 1.0140 \; Ry^{*\frac{2}{3}} \tag{8}$$

$$\sigma_m^* = \left[\left(1.0140 \; Ry^{*\frac{2}{3}} \right)^n + \left(\sqrt{\frac{\pi}{2}} Ry^* \right)^n \right]^{\frac{1}{n}} \tag{9}$$

Where $n = -2.3975$, and the maximum error reaches minimum, 1.39%.



(a) correspondence between σ_m^* and Ry^* (b) relative error changes with Ry^* when $n = -2.3975$ (c) maximum error changes with n around -2.3975

Fig. 2: Results of the blending between σ_m^* and Ry^*

4. $\boldsymbol{\sigma \to \sigma_m}$

When σ^* tends to σ_m^*, the welding pool vanishes, and should be axisymmetric, which means the maximum width point locates above the maximum temperature point, i.e. $x_{m,\text{corresponding to maximum width point}} = x_{m,\text{corresponding to maximum temperature point}}$.

4.1. $\sigma \to 0$

$$
\begin{aligned}
T^*_{I;x_0,y} &= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
&= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\ x_0^{*2}}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}\cdot e^{-\frac{\cos^2 t y^{*2}}{2\sigma^{*2}}}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} 1\cdot e^{-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}}dt \\
&= \frac{2}{\pi}T^*_{I;x_0,y=0}\cdot\int_0^{\frac{\pi}{2}} e^{-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}}dt \\
&\approx \frac{2}{\pi}T^*_{I;x_0,y=0}\cdot\int_0^{\frac{\pi}{2}} 1-\frac{\cos^2 t\ y^{*2}}{2\sigma^{*2}}dt \qquad \text{as } y^*\ll\sigma^* \\
&= T^*_{I;x_0,y=0}\cdot\frac{2}{\pi}\left(\frac{\pi}{2}-\frac{y^{*2}\pi}{8\sigma^{*2}}\right) \\
&= T^*_{I;x_0,y=0}\cdot\left(1-\frac{y^{*2}}{4\sigma^{*2}}\right) \\
&= T^*_{I;x_0,y=0}\cdot\left(1-\frac{y^{*2}}{4\sigma^{*2}}\right) \\
&= T^*_{I;x_0,y=0}\cdot e^{-\frac{y^{*2}}{4\sigma^{*2}}} \qquad (10)
\end{aligned}
$$

Where $x_0^*=0$, $y^*\ll\sigma^*$.

According to Eq. 10,

$$
y^*_{mI} = 2\sigma^*\sqrt{\ln\frac{T^*(\sigma^*)}{T^*}} = 2\sigma^*\sqrt{\ln\frac{Ry^*}{Ry^*_{min}(\sigma^*)}} \qquad (11)
$$

4.2. $\sigma \to \infty$

When σ tends to infinity, the location of maximum temperature point $x_0^* = -0.7650\ \sigma^*$, and the integrand focuses on $t=0$, i.e. $\cos t = 1$.

$$
\begin{aligned}
T^*_{I;x_0,y} &= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos^2 t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\left(\cos t+\frac{1}{\cos^2 t}-2\right)+\frac{\cos^2 t\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^*\left(1-\cos^2 t\right)\right]}dt \\
&\approx \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+\frac{\left(1-t^2\right)\left(x_0^{*2}+y^{*2}\right)}{\sigma^{*2}}+2x_0^* t^2\right]}dt \\
&= \frac{2}{\sqrt{2\pi}\sigma^*}\int_0^{\delta} e^{-\frac{1}{2}\left[\sigma^{*2}\ t^4+(2x_0^*-\frac{x_0^{*2}}{\sigma^{*2}})t^2+\frac{x_0^{*2}}{\sigma^{*2}}\right]}\cdot e^{-\frac{y^{*2}}{2\sigma^{*2}}}dt \\
&= T^*_{II;x_0,y=0}\cdot e^{-\frac{y^{*2}}{2\sigma^{*2}}}
\end{aligned}
\tag{12}
$$

According to Eq. 12,

$$
y^*_{mII} = \sqrt{2}\sigma^*\sqrt{\ln\frac{T^*(\sigma^*)}{T^*}} = \sqrt{2}\sigma^*\sqrt{\ln\frac{Ry^*}{Ry^*_{min}(\sigma^*)}}
\tag{13}
$$

4.3. blending

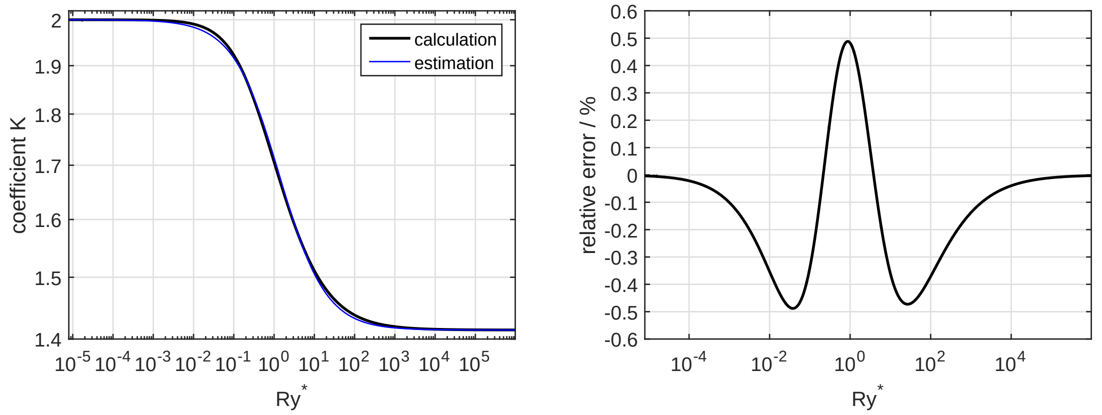
Use Eq. 11 and Eq. 13 to obtained the approximation of y^*_m when σ^* tends to σ^*_m:

$$
y^*_m = K\sigma^*\sqrt{\ln\frac{Ry^*}{Ry^*_{min}(\sigma^*)}}
\tag{14}
$$

K changes with Ry.

$$
K = k_0 - A * \tanh\left(B\ln\frac{Ry}{C}\right)
\tag{15}
$$

Where $k_0 = \frac{2+\sqrt{2}}{2}$,$A = \frac{2-\sqrt{2}}{2}$,$B = 0.3775$,$C = 1.0690$.The maximum error reaches 0.5%. Eq. 14 can be



(a) coefficient changes with Ry^*

(b) relative error changes with Ry^*

Fig. 3: Results of approximation of coefficient K against Ry^*

written as a function depicting the near field temperature distribution around the maximum temperature point:

$$Ry^* = Ry^*_{min}\left(\sigma^*\right) e^{\frac{y_m^{*2}}{K^2\sigma^{*2}}} \tag{16}$$

5. quasi point source

When $\sigma^* = 0$, the Eq. 1 describes the point heat source.

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \quad \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} = \frac{1}{r^*} e^{-r^* - x^*} \tag{17}$$

Do derivations on Eq. 17 with respect to y:

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \quad \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} = \frac{1}{r^*} e^{-r^* - x^*} \tag{18}$$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \quad \tau^{-\frac{5}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} = -\frac{1}{y^*} \frac{\partial}{\partial y^*} \left(\frac{1}{r^*} e^{-r^* - x^*} \right) = e^{-r^* - x^*} \left(\frac{1}{r^{*2}} + \frac{1}{r^{*3}} \right) \tag{19}$$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \quad \tau^{-\frac{7}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} = \frac{1}{y^*} \left[\frac{1}{y^*} \frac{\partial}{\partial y^*} \left(\frac{1}{r^*} e^{-r^* - x^*} \right) \right] = e^{-r^* - x^*} \left(\frac{1}{r^{*3}} + \frac{3}{r^{*4}} + \frac{3}{r^{*5}} \right) \tag{20}$$

When $\frac{\sigma^*}{\sigma^*_m}$ tends to zero, the Gaussian heat source can be treated as point source, with little error. So, use Eq. 1 rather than Eq. 2.

$$\begin{aligned} T^* &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \frac{\tau^{-\frac{1}{2}}}{\tau + \sigma^{*2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau + 2\sigma^{*2}}} \\ &\approx \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \, \tau^{-\frac{3}{2}} e^{-\frac{x^{*2}+2\tau^* x^* + \tau^{*2} + y^{*2}}{2\tau}} \cdot \left[\left(1 + \frac{\sigma^{*2}}{2}\right) + \sigma^{*2} \left(x - 1\right) \frac{1}{\tau} + \frac{x^{*2} + y^{*2}}{2} \sigma^{*2} \frac{1}{\tau^2} \right] \\ &= e^{-r^* - x^*} \left[\left(1 + \frac{\sigma^{*2}}{2}\right) \frac{1}{r^*} + \sigma^{*2} \left(x - 1\right) \left(\frac{1}{r^{*2}} + \frac{1}{r^{*3}} \right) + \frac{\sigma^{*2}}{2} \left(\frac{1}{r^*} + \frac{3}{r^{*2}} + \frac{3}{r^{*3}} \right) \right] \end{aligned} \tag{21}$$

This process uses the first two terms of Taylor series of integrand with respect to σ^*. Eq. 21 describes the temperature distribution of far field.

5.1. $\sigma \to 0$

When $\sigma \to 0$, $x^* \ll y^* \ll 1$. Eq. 21 can be simplified as

$$\begin{aligned} T^* &\approx e^{-r^* - x^*} \left[\left(1 + \frac{\sigma^{*2}}{2}\right) \frac{1}{r^*} + \sigma^{*2} \left(x - 1\right) \left(\frac{1}{r^{*2}} + \frac{1}{r^{*3}} \right) + \frac{\sigma^{*2}}{2} \left(\frac{1}{r^*} + \frac{3}{r^{*2}} + \frac{3}{r^{*3}} \right) \right] \\ &\approx 1 \cdot \left[\left(1 + \frac{\sigma^{*2}}{2}\right) \frac{1}{y^*} - \sigma^{*2} \frac{1}{y^{*3}} + \frac{\sigma^{*2}}{2} \frac{3}{y^{*3}} \right] \\ &\approx \frac{1}{y^*} + \frac{\sigma^{*2}}{2} \frac{1}{y^{*3}} \end{aligned}$$

Use perturbation method, $y^*_{m,gauss} = y^*_{m,point}\left(1 + a\sigma^{*2}\right)$, $a\sigma^{*2} \ll 1$, $y^*_{m,point} = Ry^*$.

$$\begin{aligned}
\frac{1}{Ry^*} \approx \frac{1}{y^*} + \frac{\sigma^{*2}}{2}\frac{1}{y^{*3}} &\approx \frac{1}{y^*_{m,point}\left(1+a\sigma^{*2}\right)} + \frac{\sigma^{*2}}{2}\frac{1}{y^{*3}_{m,point}\left(1+3a\sigma^{*2}\right)} \\
&\approx \frac{1}{Ry^*\left(1+a\sigma^{*2}\right)} + \frac{\sigma^{*2}}{2}\frac{1}{Ry^{*3}\left(1+3a\sigma^{*2}\right)} \\
&\Rightarrow a = \frac{1}{2Ry^{*2}} \qquad (22)\\
y^*_{m,gauss,0} &= y^*_{m,point}\left(1+\frac{1}{2Ry^{*2}}\sigma^{*2}\right) \qquad (23)
\end{aligned}$$

5.2. $\sigma \to \infty$

When $\sigma \to \infty$, $1 \ll \sigma^* \ll y^* \ll x^*$. Eq. 21 can be simplified as

$$\begin{aligned}
T^* &\approx e^{-r^*-x^*}\left[\left(1+\frac{\sigma^{*2}}{2}\right)\frac{1}{r^*} + \sigma^{*2}\left(x-1\right)\left(\frac{1}{r^{*2}}+\frac{1}{r^{*3}}\right) + \frac{\sigma^{*2}}{2}\left(\frac{1}{r^*}+\frac{3}{r^{*2}}+\frac{3}{r^{*3}}\right)\right] \\
&\approx e^{-r^*-x^*}\left[\frac{1}{r^*} + \frac{\sigma^{*2}\left(r^*+x^{*2}\right)}{r^{*2}} - \frac{\sigma^{*2}}{2r^{*2}}\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}}\left(-\frac{y^{*2}}{2x^*} - 0.5\right)\right] \qquad (24)
\end{aligned}$$

Use perturbation method, $y^*_{m,gauss} = y^*_{m,point}\left(1 + b\sigma^{*2}\right)$, $x^*_{m,gauss} = x^*_{m,point}\left(1 + c\sigma^{*2}\right)$, $b\sigma^{*2} \ll 1$,$c\sigma^{*2} \ll 1$, $y^*_{m,point} = \sqrt{\frac{2}{e}Ry^*}$,$x^*_{m,point} = -\frac{Ry^*}{e}$.

$$\begin{aligned}
\frac{1}{Ry^*} &\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}}\left(-\frac{y^{*2}}{2x^*} - 0.5\right)\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left[-\frac{1}{x^*} + \frac{\sigma^{*2}}{x^{*2}}\left(0.5 + 2b\sigma^{*2} - c\sigma^{*2}\right)\right] \\
&\approx e^{\frac{1}{2}\frac{y^{*2}}{x^*}}\left(-\frac{1}{x^*} + 0.5\frac{\sigma^{*2}}{x^{*2}}\right) \\
y^* &= \sqrt{2x^*\ln\frac{x^{*2}/Ry^*}{-x^* + 0.5\sigma^{*2}}} \\
\frac{dy^*}{dx^*} &= \frac{\sqrt{2}\left(2\ln\left(-\frac{2tx^2}{-s^2+2x}\right) + \frac{4x-4s^2}{-s^2+2x}\right)}{4\sqrt{x\ln\left(-\frac{2tx^2}{2x-s^2}\right)}} = 0 \\
&\Rightarrow x^*_{m,gauss} = x^*_{m,point} \\
&\Rightarrow b = \frac{e}{4Ry^*} \\
y^*_{m,gauss,infinity} &= y^*_{m,point}\left(1+\frac{e}{4Ry^*}\sigma^{*2}\right) \qquad (25)
\end{aligned}$$

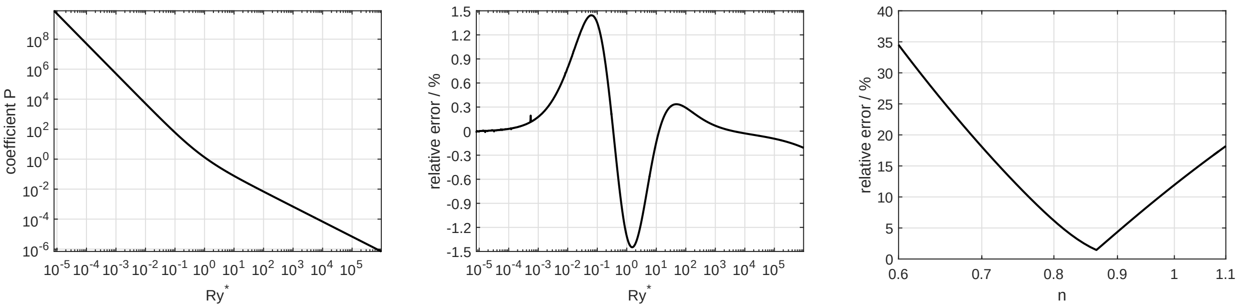
5.3. blending

Use the following equation to blending:

$$y^*_{m,gauss} = y^*_{m,point}\left(1 + P*\sigma^{*2}\right) \qquad (26)$$

$$P = \left[\left(\frac{1}{2Ry^{*2}} \right)^n + \left(\frac{e}{4Ry^*} \right)^n \right]^{\frac{1}{n}} \tag{27}$$

Where $n = 0.8655$, maximum error reaches 1.45%.



(a) coefficient changes with Ry^* (b) relative error changes with Ry^* (c) relative error changes with Ry^*

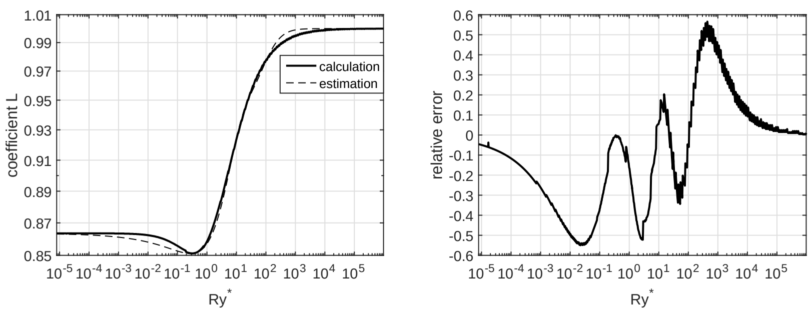
Fig. 4: Results of approximation of coefficient K against Ry^*

6. Combination

The maximum width of welding pool is a combination of far-field and near-field. To cover the middle range of $\frac{\sigma}{\sigma_m}$, the correction of both equations is needed.

6.1. near-field

$$L = 0.93175 - 0.06825 \tanh\left(-0.6571 \ \ln \frac{Ry}{15.926} \right) - 0.0132 \sin\left[3\pi \tanh\left(0.2485 \ Ry^{0.3718} \right) \right] \tag{28}$$



(a) coefficient changes with Ry^* (b) relative error changes with Ry^*

Fig. 5: Results of approximation of coefficient L against Ry^*

The maximum error reaches 0.55%.

6.2. far-field

The maximum width can be written in form of step function, consisting of far-field expression and near-field expression. However, lines of $y^*_{m,1} and y^*_{m,12}$ cross. A correction of far-field expression is applied, which is a confinement with the range of $\frac{\sigma^*}{\sigma^*_m}$. It's difficult to calculate the cross points directly, so a back step is took that the point between two cross points is found with any Ry.
When $Ry \leqslant 5 \times 10^3$, $lin = 0.4$.
When $Ry > 5 \times 10^3$,

$$y^*_{m,1} - y^*_{m,12} = \sqrt{\frac{2}{e}Ry} + 0.5993\left(\frac{\sigma^*}{\sigma^*_m}\right)^2 Ry^{\frac{5}{6}} - 2.4838\frac{\sigma^*}{\sigma^*_m}\sqrt{\log\frac{\sigma^*_m}{\sigma^*}}\, Ry^{\frac{2}{3}} < 0$$

When $\frac{\sigma^*}{\sigma^*_m} = Ry^{-\frac{1}{6}}$,the in-equation is satisfied. So:

$$lin = Ry^{-\frac{1}{6}} \cdot \left(Ry > 5 \times 10^3\right) + 0.4\left(Ry \leqslant 5 \times 10^3\right); \tag{29}$$

7. whole plane

$$\begin{aligned} y^*_m &= MAX\{ \quad y^*_{m,1}\,, \quad y^*_{m,2} \quad \} \\ y^*_{m,1} &= y^*_{m,point}\left(1 + P\sigma^{*2}\right) \cdot (\sigma^*/\sigma^*_m < lin) \\ y^*_{m,2} &= K\left[L\left(\sigma^* - \sigma^*_m\right) + \sigma^*_m\right]\sqrt{\ln\frac{Ry^*}{Ry^*_{min}(\sigma^*)}} \end{aligned} \tag{30}$$

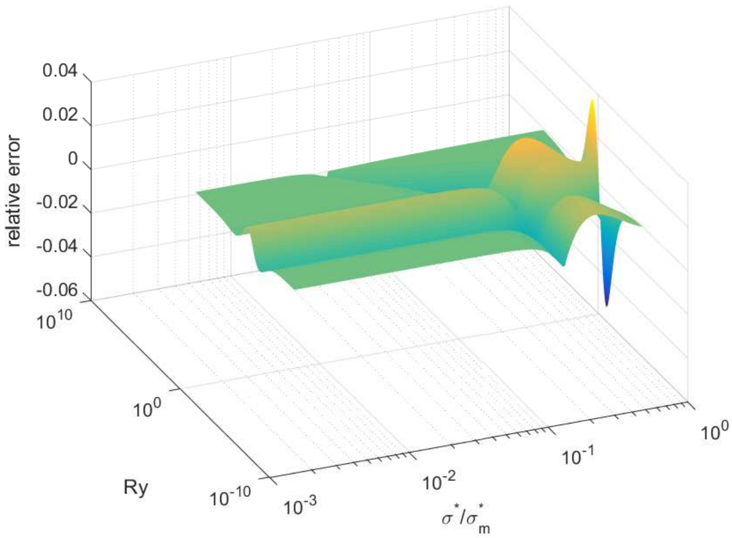


Fig. 6: relative error over whole plane.

The parameters in equations are as follows:

$$y^*_{m,point} = \left[(Ry^*)^n + \left(\sqrt{\frac{2}{e}} Ry^* \right)^n \right]^{\frac{1}{n}}$$

Where n= -1.7312

$$P = \left[\left(\frac{1}{2Ry^{*2}} \right)^n + \left(\frac{e}{4Ry^*} \right)^n \right]^{\frac{1}{n}}$$

Where n= 0.8655

$$lin = Ry^{-\frac{1}{6}} \cdot \left(Ry > 5 \times 10^3 \right) + 0.4 \left(Ry \leqslant 5 \times 10^3 \right);$$

$$K = k_0 - A * \tanh \left(B \ln \frac{Ry}{C} \right)$$

Where $k_0 = \frac{2+\sqrt{2}}{2}$,$A = \frac{2-\sqrt{2}}{2}$,$B = 0.3775$,$C = 1.0690.$

$$L = 0.93175 - 0.06825 \tanh \left(-0.6571 \ \ln \frac{Ry}{15.926} \right) - 0.0132 \sin \left[3\pi \tanh \left(0.2485 \ Ry^{0.3718} \right) \right]$$

$$\sigma^*_m = \left[\left(1.0140 \ Ry^{*\frac{2}{3}} \right)^n + \left(\sqrt{\frac{\pi}{2}} Ry^* \right)^n \right]^{\frac{1}{n}}$$

Where $n = -2.3975$

$$Ry^*_{min}(\sigma^*) = \left[\left(\sqrt{\frac{\pi}{2}} \ \sigma^{*-1} \right)^n + \left(\frac{2.5596}{\sqrt{2\pi}} \ \sigma^{*-1.5} \right)^n \right]^{-\frac{1}{n}}$$

Where $n = -1.9464$

The maximum error reaches 5.25%.There is a limit that $\frac{\sigma^*}{\sigma^*_m} < 98\%$, because when $\frac{\sigma^*}{\sigma^*_m}$ tends to 1, y^*_m tends to 0, the approximation (means that it's not accurate) of Ry^*_{min}, σ^*_m leads to a large relative error. If the high-precision value is obtained these equations still work.

8. Results

Results

9. Discussion

10. Conclusions

Conclusions Section

11. Conclusions

Conclusions Section

12. Acknowledgement

This study has been supported by...

13. References