ENO and WENO Reconstruction

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Reconstruction from Cell Averages

ENO Reconstruction

WENO Reconstruction

Numerical examples

Reconstruction from Cell Averages I

Problem

Given the cell averages $\bar{v}_i$ of a function $v(x)$ for each cell I_i, find a polynomial $p_i(x)$ of degree at most $k-1$, such that it approximates the function $v(x)$ to k-th order accuracy within I_i:

$$p_i(x) = v(x) + O(\Delta x^k), \quad x \in I_i, \quad i = 1, \ldots, N.$$

Consider the stencil $S(i) \equiv \{I_{i-r}, \ldots, I_{i+k-1}\}$. Although the analytical expression for $p_i(x)$ can be derived, for uniform grids we have the useful result:

$$v_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} c_{rj} \bar{v}_{i-r+j} = v\left(x_{i+\frac{1}{2}}\right) + O(\Delta x^k)$$

Reconstruction from Cell Averages II

For k=3, the constants c_{rj} are given in the following table:

r	j=0	j=1	j=2
-1	$11/6$	$-7/6$	$1/3$
0	$1/3$	$5/6$	$-1/6$
1	$-1/6$	$5/6$	$1/3$
2	$1/3$	$-7/6$	$11/6$

ENO Reconstruction I

Reconstruction from cell averages is only applicable to smooth functions.
ENO adaptively selects the smoothest stencil from

$$S_r(i) = \{I_{i-r}, \dots, I_{i+k-1-r}\}, r = 0, \dots, k-1.$$

Then the reconstruction process is similar to that for smooth functions.

ENO have the following disadvantages:

1. Unnecessary computation
2. Sensitivity to perturbations
3. Inconsistent stencil pattern
4. Parallel computational inefficiency

WENO Reconstruction I

As for every $S_r(i)$, we have $v_i^{(r+\frac{1}{2})}$, WENO uses a convex combination of all of them:

$$v^-_{i+\frac{1}{2}} = \sum_{r=0}^{k-1} \omega_r v^{(r)}_{i+\frac{1}{2}}, \quad v^+_{i-\frac{1}{2}} = \sum_{r=0}^{k-1} \tilde{\omega}_r v^{(r)}_{i-\frac{1}{2}},$$

where

$$\omega_r = \frac{\alpha_r}{\sum_{s=0}^{k-1} \alpha_s}, \quad \alpha_r = \frac{d_r}{(\epsilon + \beta_r)^2} \quad r = 0, \dots k-1,$$

$$\tilde{\omega}_r = \frac{\tilde{\alpha}_r}{\sum_{s=0}^{k-1} \tilde{\alpha}_s}, \quad \tilde{\alpha}_r = \frac{\tilde{d}_r}{(\epsilon + \beta_r)^2}, \quad r = 0, \dots, k-1,$$

β is the smoothness indicator. $\epsilon = 10^{-6}$ is used to prevent division by zero.d_r are constants that satisfy the following conditions:

$$v_{i+\frac{1}{2}} = \sum_{r=0}^{k-1} d_r v^{(r)}_{i+\frac{1}{2}} = v(x_{i+\frac{1}{2}}) + O(\Delta x^{2k-1}).$$

Numerical examples

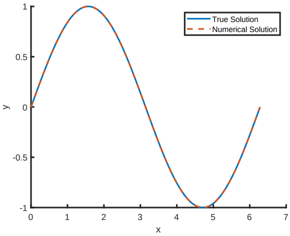

Figure: smooth

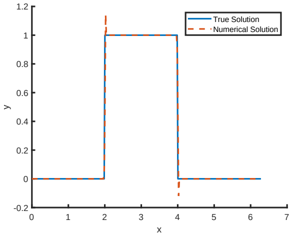

Figure: non-smooth

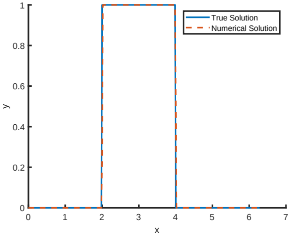

Figure: ENO

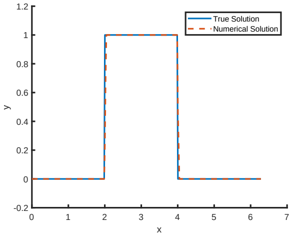

Figure: WENO

Thanks