A type theory for cellular spaces

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Main Idea

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The interval

We add to the syntax a collection of shapes

$$\bullet \longrightarrow \bullet$$

$$\mathbb{2}$$

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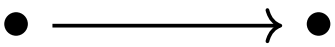

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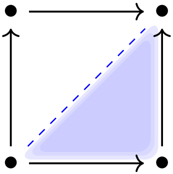

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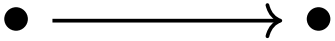

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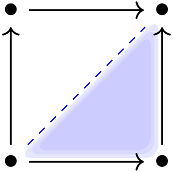

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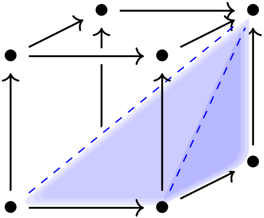

Δ^3

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S shape, and X type:

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So each type give rise to a *simplicial type !*

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$$\mathsf{isiso}(f) :\equiv \left(\sum_{g:\mathsf{hom}_A(y,x)} g \circ f = \mathsf{id}_x \right) \times \left(\sum_{h:\mathsf{hom}_A(y,x)} f \circ h = \mathsf{id}_y \right)$$

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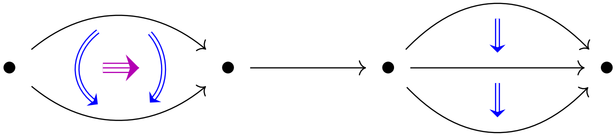
Reminiscent to Rezk-completes/univalent categories in HoTT.

The category Θ

- Objects: Pasting schemes.



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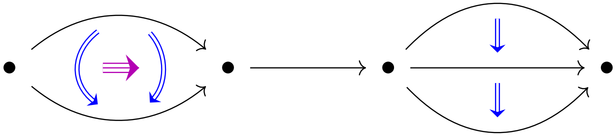

- Morphisms: Morphisms of strict ω-categories.

Some objects of Θ

$$\bullet$$

$$\mathcal{O}_0 = *$$

Some objects of Θ

$$\bullet \qquad\qquad \bullet \longrightarrow \bullet$$

$$\mathcal{O}_0 = * \qquad\qquad \mathcal{O}_1 = \Delta^1$$

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$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

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$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

$$\mathcal{O}_2$$

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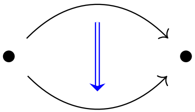

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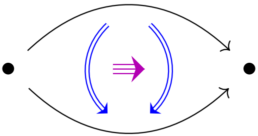

$$\mathcal{O}_3$$

Some objects of Θ

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$\mathcal{O}_0 = *$

$\bullet \longrightarrow \bullet$

$\mathcal{O}_1 = \Delta^1$

$$\begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

$\mathcal{O}_2$

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, bend right=60, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, bend right, ""{name=L}] \arrow[Rightarrow, from=U, to=D, bend left, ""'{name=R}] \arrow[Rrightarrow, from=L, to=R] \end{tikzcd}$$

$\mathcal{O}_3$

$\bullet \longrightarrow \bullet \dashrightarrow \bullet \longrightarrow \bullet$

Δ^n

A combinatorial description of Θ

- Objects are lists of objects.

$$\bullet \xrightarrow{P_1} \bullet \xrightarrow{P_2} \bullet \cdots\cdots\rightarrow \bullet \xrightarrow{P_n} \bullet$$

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$$\begin{tikzcd} \bullet \arrow[r, "P_1"] & \bullet \arrow[r, "P_2"] & \bullet \arrow[r, dotted] & \bullet \arrow[r, "P_n"] & \bullet \end{tikzcd}$$

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$* = []$ $\qquad \mathcal{O}_1 = [*]$ $\qquad \mathcal{O}_2 = [\mathcal{O}_1] = [[*]]$

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$$\Delta^n = [*, *, \cdots, *]$$

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\end{tikzcd}
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- Morphisms:

$$
\begin{tikzcd}
 & \bullet \arrow[rr, "P_1"] \arrow[dr, blue] & & \bullet \arrow[rr, "P_2", name=p2] \arrow[dl, blue] & & \bullet \arrow[rr, "P_3", name=p3] \arrow[dr, blue] & & \bullet \arrow[dr, blue] & & {[3]} \arrow[d, blue, "f"] \\
\bullet \arrow[rr, "Q_1"'] & & \bullet \arrow[rr, "Q_2"', name=q2] & & \bullet \arrow[rr, "Q_3"', name=q3] & & \bullet \arrow[rr, "Q_4"', name=q4] & & \bullet & {[4]}
\arrow[from=p2, to=q2, squiggly, magenta, "\delta_{2,2}"']
\arrow[from=p2, to=q3, squiggly, magenta, "\delta_{2,3}"]
\arrow[from=p3, to=q4, squiggly, magenta, "\delta_{3,4}"]
\end{tikzcd}
$$

Hom Types in STT

In STT:

$$\mathsf{hom}_A(x,y) = \sum_{f : \mathsf{I} \to A} (f\,0 = x) \times (f\,1 = y)$$

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$$\begin{tikzcd} x \arrow[rr, dash] & & y \end{tikzcd}$$

Then:

$$\mathsf{hom}_{\mathsf{hom}_A(x,y)}(f, g) = \sum_{H\,:\,\mathsf{I}^2 \to A} \cdots$$

$$\begin{tikzcd} x \arrow[rr, dash, "f" description] \arrow[d, equal] & & y \arrow[d, equal] \\ x \arrow[rr, dash, "g" description] & & y \end{tikzcd}$$

Does not generalize to Θ

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Hence I^2 is 1-categorical

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3ex, "="] \arrow[dr, phantom, shift right=3ex, "="] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3ex, "\Swarrow"] \arrow[dr, phantom, shift right=3ex, "\Swarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

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Another approach

Workaround:

$$\begin{tikzcd}
\mathbf{Psh}(\Theta) \arrow[rr, bend left, "\$"] \arrow[rr, phantom, "\perp"] & & \mathbf{Psh}(\Theta)_{\bullet\bullet} \arrow[ll, bend left, "\mathsf{hom}"]
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Two subgoals:

- Defining a suspension \$.
- Postulating the adjunction.

($\$ \dashv \mathsf{hom}$) adjunction

$$(A \to \mathsf{hom}_B(x, y)) \overset{?}{\simeq} (\$A \to_{\bullet\bullet} (B, x, y))$$

($\$ \dashv \mathsf{hom}$) adjunction

$$\underbrace{(A \to \mathsf{hom}_B(x, y))}_{\text{internal hom}} \neq \underbrace{(\$A \to_{\bullet\bullet} (B, x, y))}_{\text{internal hom}}$$

$(\$ \dashv \mathsf{hom})$ adjunction

$$\flat(A \to \mathsf{hom}_B(x, y)) = \flat(\$A \to_{\bullet\bullet} (B, x, y))$$

The $\flat$ modality

$$A :: \mathcal{U} \quad \rightsquigarrow \quad [\![A]\!] \in \mathbf{Psh}(\mathbf{\Theta})$$

$$\flat A : \mathcal{U} \quad \rightsquigarrow \quad [\![\flat A]\!] = [\![A]\!](*)$$

Semantic of $\flat$

$$\begin{tikzcd} \mathcal{S} \arrow[rr, bend left, "\delta"] \arrow[rr, phantom, "\perp"] & & \mathbf{Psh}(\Theta) \arrow[ll, bend left, "F \mapsto F(*)"] \end{tikzcd}$$

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$$\frac{\Gamma | \cdot \vdash X : \mathcal{U}}{\Gamma | \cdot \vdash \flat X : \mathcal{U}}$$

CellTT

$$\text{CellTT} = \underbrace{\text{HoTT} \; + \; \text{Idempotent comodality}}_{\text{Crisp Type Theory}} + \text{Axioms}$$

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$$\sigma : P \to_{\mathsf{PS}} Q \quad \rightsquigarrow \quad \sigma^* : X_Q \to X_P$$

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Corollary 5.1.6.11. *Let $\mathcal{C}$ be an ∞-category which admits small colimits. Let S be a small simplicial set and $F : \mathcal{P}(S) \to \mathcal{C}$ a colimit preserving functor. Then F is an equivalence if and only if the following conditions are satisfied:*

(1) *The composition $f = F \circ j : S \to \mathcal{C}$ is fully faithful.*

(2) *For every vertex $s \in S$, the object $f(s) \in \mathcal{C}$ is completely compact.*

(3) *The set of objects $\{f(s) : s \in S_0\}$ generates $\mathcal{C}$ under colimits.*

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- $\left(\sum_{P:\mathsf{PS}} \sum_{c:X_P} よP\right) \to X$ is an effective epi ((-1)-truncated)
- $\flat$-discreteness is *cellular discreteness*:

$$\text{is-equiv}(\flat X \to X) \leftrightarrow \prod_{P:\mathsf{PS}} \text{is-equiv}(X \to (よ(P) \to X))$$

Levelwiseness of level

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  HoTT book: $\text{is-}n\text{-type}(X) \leftrightarrow \text{is-equiv}\left(X^{\mathbb{S}^{n+1}} \to X\right)$

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  $\leftrightarrow \prod_{P::PS} \textsf{is-equiv}\left((X^{\mathbb{S}^{n+1}})_P \to X_P\right)$

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$$\text{is-}n\text{-type}(X) \leftrightarrow \prod_{P:\mathsf{PS}} \text{is-}n\text{-type}(X_P)$$

- Case $n = -2$: $X \simeq \mathbb{1} \longleftrightarrow \prod_{P:\mathsf{PS}}(X_P \simeq \mathbb{1})$
- Case $n \geq -1$: LR implication: $\flat\|X\|_n \simeq \|\flat X\|_n$
  HoTT book: $\text{is-}n\text{-type}(X) \leftrightarrow \text{is-equiv}\left(X^{\mathbb{S}^{n+1}} \to X\right)$
  $\leftrightarrow \prod_{P::PS} \text{is-equiv}\left((X^{\mathbb{S}^{n+1}})_P \to X_P\right)$
  $(X^{\mathbb{S}^{n+1}})_P \simeq \flat(\text{よ}P \times \mathbb{S}^{n+1} \to X) \simeq \flat(\mathbb{S}^{n+1} \to X_P)$

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Suspension

We extend $\$$ to $\mathcal{U}$.

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- If $P = [P_1, P_2, \cdots, P_n]$:

$$(\$X)_P \cong \mathbb{1} + \left(\sum_{i : \mathsf{Fin}_n} X_{P_i} \right) + \mathbb{1}$$

$$\begin{tikzcd}
\bullet \arrow[r, dotted, "P_1, \cdots, P_{i-1}"] \arrow[dr] & \bullet \arrow[r, "P_i"] \arrow[d] \arrow[dr, phantom, "\rightsquigarrow"] & \bullet \arrow[r, dotted, "P_{i+1}, \cdots, P_n"] \arrow[d] & \bullet \arrow[dl] \\
& \bullet \arrow[r, "X"'] & \bullet &
\end{tikzcd}$$

Hom Types

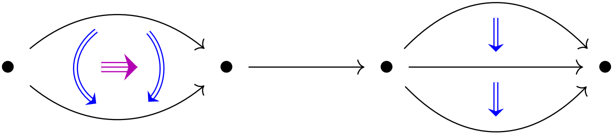
We postulate an adjunction

$$\flat(A \to \mathsf{hom}_B(x, y)) \cong \flat(\$A \to_{\bullet\bullet} (B, x, y))$$

Cellular Realization

$$\langle [P_1, \cdots, P_m] \rangle :\equiv$$

$$\mathrm{colim} \quad \begin{tikzcd}[column sep=small] & \mathbb{1} \arrow[dl] \arrow[dr] & & \mathbb{1} \arrow[dl] \arrow[dr] & & \mathbb{1} \arrow[dl] \arrow[dr] & \\ \$\langle P_1 \rangle & & \$\langle P_2 \rangle & & \cdots & & \$\langle P_m \rangle \end{tikzcd}$$



Segal Types

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Complete Types

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$$\begin{tikzcd} \mathsf{D}_1 + \mathsf{D}_1 \arrow[r, "\alpha, \beta"] \arrow[d] & よ[3] \arrow[d] \\ \mathbb{1} + \mathbb{1} \arrow[r] & \mathsf{E} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

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$$

representing equivalences

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We define is-complete:

$$\flat(\mathsf{D}_n \to X) \xrightarrow{\sim} \flat(\mathsf{E}_{n+1} \to X)$$

(∞, ω)-categories

As in Riehl-Shulman STT, there is a completeness condition.
$\mathsf{is\text{-}complete}(X)$

$$(\infty, \omega)\text{-categories} = \text{complete Segal Types.}$$

$$\mathsf{Cat}_{\infty,\omega} = \sum_{X:\mathcal{U}} \mathsf{is\text{-}Segal}(X) \times \mathsf{is\text{-}complete}(X)$$

Some elementary results

- (∞, ω)-categories are stable by pullback and sum.

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Main goal: Proving a Yoneda Lemma.

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- Crans-Gray tensor product $\boxtimes$.

Towards a bunched-affine variant of HoTT

$$\Gamma, \Delta \quad ::= \quad \diamond \mid \Gamma * \Delta \mid \Gamma, x : A$$

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- A non-symmetric affine product $\otimes$.
- New abstractions ∂ and ϱ for adjoints to $\otimes$.

Some rules

$$(\otimes\text{-}\textsc{Intro})\ \frac{\underline{\Lambda} \mid \Gamma \vdash a : A \qquad \underline{\Lambda} \mid \Delta \vdash b : B}{\underline{\Lambda} \mid \Gamma \otimes \Delta \vdash a \otimes b : A \otimes B}$$

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Some rules (2)

$$(\multimap\text{-}\textsc{Intro})\ \frac{\begin{array}{c}\underline{\Lambda} \mid \diamond \vdash A : \mathcal{U} \\ \underline{\Lambda} \mid \diamond \vdash B : \mathcal{U} \qquad \underline{\Lambda} \mid \Gamma \otimes (x : A) \vdash b : B\end{array}}{\underline{\Lambda} \mid \Gamma \vdash \varrho x.b : A \multimap B}$$

Some rules (2)

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The end

Thank you !

Questions ?