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A smoothing majorization method for l_2^2-l_p^p matrix minimization

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In this paper, we consider the l_2^2-l_p^p (with $p \in (0, 1)$) matrix minimization for recovering the low-rank matrices. A smoothing approach for solving this non-smooth, non-Lipschitz and non-convex l_2^2-l_p^p optimization problem is developed, in which the smoothing parameter is treated as a decision variable and a majorization method is adopted to solve the smoothing problem. The convergence theorem shows that any accumulation point of the sequence generated by the proposed approach satisfies the first-order necessary optimality condition of the l_2^2-l_p^p problem. As an application, we use the proposed smoothing majorization method to solve the famous matrix completion problems. Numerical results indicate that our algorithm can solve the test problems efficiently.

Keywords: low-rank problem; l_2^2-l_p^p minimization; majorization method; lower bound analysis; smoothing method

AMS Subject Classification: 90C26; 15A83

1. Introduction

During the past decade, matrix rank minimization problems have gained remarkable attention in many engineering applications, particularly in computer vision [28], Euclidean space embedding [22] and machine learning [1–3]. The aim of the matrix rank minimization problem is to find a matrix with minimum rank that satisfies a given convex constraint, that is,

$$\begin{aligned} \min \quad & \operatorname{rank}(X) \\ \text{s.t.} \quad & X \in \mathcal{C}, \end{aligned} \tag{1}$$

where $\mathcal{C}$ is a nonempty closed convex subset of $\mathfrak{R}^{m\times n}$ and $\mathfrak{R}^{m\times n}$ represents the space of $m \times n$ matrices. Without loss of generality, we assume $m \leq n$ throughout this paper. For solving (1), Fazel *et al*. [13,14] suggested using the matrix nuclear norm to approximate the rank function and proposed the following convex optimization problem:

$$\begin{aligned} \min \quad & \|X\|_* \\ \text{s.t.} \quad & X \in \mathcal{C}, \end{aligned} \tag{2}$$

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where $\|X\|_* := \sum_{i=1}^{r} \sigma_i(X)$, $r := \text{rank}(X)$ and $\sigma_i(X)$ denotes the ith largest singular value of X. Many important problems can be formulated as (2). For example, several authors have used (2) to solve the famous matrix completion problem with the following model:

$$\begin{aligned} \min \quad & \|X\|_* \\ \text{s.t.} \quad & X_{ij} = M_{ij}, \quad (i,j) \in \Omega, \end{aligned} \tag{3}$$

where Ω is an index set of the entries of M. Various algorithms, such as the singular value thresholding algorithm [5], the fixed-point continuation algorithm [23] and the alternating-direction-type algorithm [15], have been designed to solve (3). Recently, these methods have also been applied to the nuclear norm regularized linear least square problem

$$\min_{X \in \mathfrak{R}^{m \times n}} \left\{ \frac{1}{2} \|\mathcal{A}(X) - b\|_2^2 + \tau \|X\|_* \right\}, \tag{4}$$

where $\mathcal{A}$ is a linear operator from $\mathfrak{R}^{m \times n}$ to $\mathfrak{R}^q$ and $b \in \mathfrak{R}^q$. It is worthwhile to note that (4) is regarded as a convex approximation to the regularized version of the affine rank minimization problem

$$\min_{X \in \mathfrak{R}^{m \times n}} \left\{ \frac{1}{2} \|\mathcal{A}(X) - b\|_2^2 + \tau \cdot \text{rank}(X) \right\}. \tag{5}$$

In this paper, we consider another approximation to (5), which uses the following l_2^2-l_p^p model:

$$\min_{X \in \mathfrak{R}^{m \times n}} \left\{ F(X) := \frac{1}{2} \|\mathcal{A}(X) - b\|_2^2 + \frac{\tau}{p} \|X\|_p^p \right\}, \tag{6}$$

where $\|X\|_p^p := \sum_{i=1}^{r} \sigma_i^p(X)$, $r := \text{rank}(X)$ and $p \in (0, 1)$. In fact, it is not difficult to observe that if X is a diagonal matrix whose diagonal entries are formed by the vector $x \in \mathfrak{R}^m$, problem (6) reduces to

$$\min_{x \in \mathfrak{R}^m} \left\{ \frac{1}{2} \|Cx - b\|_2^2 + \frac{\tau}{p} \|x\|_p^p \right\}, \tag{7}$$

where $C \in \mathfrak{R}^{q \times m}$ and $\|x\|_p^p := \sum_{i=1}^{m} |x_i|^p$. Note that the term $\|x\|_p^p$ in (7), in some circumstances, characterizes the sparsity of x.

The vector l_2^2-l_p^p problem (7) has been studied extensively in recent years. For instance, Chen *et al.* [10] gave a lower bound estimate of nonzero entries in local solutions of (7), while the smoothing technique to tackle the non-convex, non-Lipschitz regularized term $\|x\|_p^p$ and an SQP-type algorithm to solve (7) were introduced in [9]. In addition, Chen *et al.* [11] studied the complexity of (7) and proved that the vector l_2^2-l_p^p problem (7) is strongly NP-hard. Besides these, many numerical algorithms have been designed for recovering sparse vectors, see, for example [6–8,12]. However, only several papers have made contributions on how to determine solutions of (6). For example, Mohan *et al.* [25] used the iterative reweighted least squares technique in conjunction with the τ-null space property for the operator $\mathcal{A}$ to analyze the convergence of (6). These observations motivate us to study the l_2^2-l_p^p model in the matrix form, such theoretical analysis and efficient algorithm to solve (6) will be examined thoroughly in this study.

In this paper, we present a smoothing majorization method for solving the l_p^p regularized linear least square problem (6), which is different from the classical approaches based on the matrix nuclear norm as the regularized term. Motivated by [10], we extend the results for (7) into the matrix case and present a lower bound analysis for nonzero singular values in local solutions of (6). Our approach also borrows the concept of majorization in the algorithm design. The idea of using majorization in optimization was dated back to Ortega and Rheinboldt [27] for studying the line search strategy. Recently, Gao and Sun [16] designed a majorized penalty approach for the calibrating rank constrained correlation matrix problems.

Unlike other algorithms mentioned above, our algorithm has the following features:

(a) We treat the smoothing parameter ϵ as a decision variable and imbed the selection of ϵ in the optimization process. Compared with other algorithm [25], the above-mentioned strategy introduces an automatic update mechanism of the smoothing parameter ϵ.
(b) Our approach only needs to provide the majorization function of the l_p^p term rather than the whole objective function as Gao and Sun's [16]. Moreover, the unconstrained subproblems based on the majorization functions are solved inexactly and the corresponding optimal solutions can be obtained explicitly.
(c) Numerical experiments show that our algorithm is insensitive to the choice of the parameter p, in this regard differs from the observations of Mohan *et al.*'s [25].

The rest of the paper is organized as follows. In Section 2, we present lower bounds for nonzero singular values in local solutions of (6). Next, the corresponding approximation model to (6) is established in Section 3. In Section 4, we design a majorization algorithm for the smoothing model and prove the convergence result. As an important application, we use our proposed method to solve a large number of matrix completion problems and report numerical results in Section 5. Finally, we present some concluding remarks.

We introduce notations and definitions used in the paper to end this section.

- Given any $X, Y \in \Re^{m\times n}$, the standard inner product of X and Y is defined by $\langle X, Y\rangle := \mathrm{Tr}(XY^{\mathrm{T}})$ and the Frobenius norm of X is denoted by $\|X\|_F := \sqrt{\mathrm{Tr}(XX^{\mathrm{T}})}$, where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. The operator norm of X is written by $\|X\|$.
- Given any vector $x \in \Re^m$, let $\mathrm{Diag}(x)$ denote an $m\times m$ matrix with entries

$$(\mathrm{Diag}(x))_{ij} := \begin{cases} x_i & \text{if } i = j,\\ 0 & \text{otherwise}\end{cases}$$

  and the vector $x^\beta := (x_1^\beta, x_2^\beta, \dots, x_m^\beta)^{\mathrm{T}}$ for any scalar β. For any $X \in \Re^{m\times m}$, $\mathrm{Diag}(X)$ represents the vector whose entries are the diagonal elements of X, that is, $\mathrm{Diag}(X) := (X_{11}, X_{22}, \dots, X_{mm})^{\mathrm{T}}$, where X_{ii} denotes the ith diagonal of X.
- Given an index set $\mathcal{I} \subseteq \{1, 2, \dots, m\}$, $x_{\mathcal{I}}$ denotes the sub-vector of x indexed by $\mathcal{I}$. Similarly, $X_{\mathcal{I}}$ denotes the sub-matrix of X whose columns are indexed by $\mathcal{I}$. Denote the index $I(x) := \{j : j \in \{1, 2, \dots, m\} \text{ and } |x_j| > 0\}$ for any $x \in \Re^m$.
- Let X admit the following singular value decomposition (SVD):

$$X := U[\mathrm{Diag}(\sigma(X))\ 0_{m\times(n-m)}]V^{\mathrm{T}}, \quad (U, V) \in \mathcal{O}^{m,n}(X), \tag{8}$$

  where $\sigma(X) := (\sigma_1(X), \sigma_2(X), \dots, \sigma_m(X))^{\mathrm{T}}$, $\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_m(X) \geq 0$ denote the singular values of X (counting multiplicity) arranged in non-increasing order and the definition of $\mathcal{O}^{m,n}(X)$ is given by

$$\mathcal{O}^{m,n}(X) := \{(U, V) \in \mathcal{O}^m \times \mathcal{O}^n : X = U[\mathrm{Diag}(\sigma(X))\ 0_{m\times(n-m)}]V^{\mathrm{T}}\},$$

  where $\mathcal{O}^m$ represents the set of all $m\times m$ orthogonal matrices.
- The definitions of $\mathcal{A}(X)$ and the adjoint operator $\mathcal{A}^*(y) : \Re^q \to \Re^{m\times n}$ are given by

$$\mathcal{A}(X) := (\langle A_1, X\rangle, \langle A_2, X\rangle, \dots, \langle A_q, X\rangle)^{\mathrm{T}} \quad \text{and} \quad \mathcal{A}^*(y) := \sum_{i=1}^q y_i A_i,$$

  where $A_i \in \Re^{m\times n}$ $(i = 1, 2, \dots, q)$, $y := (y_1, y_2, \dots, y_q)^{\mathrm{T}} \in \Re^q$.

- Let $G : \Re^{m\times n} \to \Re$ and $X, H \in \Re^{m\times n}$. The directional derivative of G at X along H is expressed by the following limit (if it exists at all): (see [4] for details)

$$G'(X;H) := \lim_{t\downarrow 0} \frac{G(X+tH)-G(X)}{t}.$$

  We say that G is directionally differentiable at X if $G'(X;H)$ exists for all H. The function G is said to be Gâteaux differentiable at X if G is directionally differentiable at X and the directional derivative $G'(X;H)$ is linear and continuous in H. In other words, there exists an operator $\mathrm{D}G(X)$ such that $\langle \mathrm{D}G(X), H\rangle = G'(X;H)$. We call $\mathrm{D}G(X)$ the derivative of G. Let G be Gâteaux differentiable in a neighbourhood of X, the second-order Gâteaux derivative $\mathrm{D}^2G(X)$ at X is defined as follows:

$$\mathrm{D}^2G(X)H := \lim_{t\downarrow 0} \frac{\mathrm{D}G(X+tH)-\mathrm{D}G(X)}{t}.$$

- For a function $F : \Re \times \Re^{m\times n} \to \Re$ and a point $(\bar{t}, \bar{X}) \in \Re \times \Re^{m\times n}$, let $\mathrm{D}_X F(\bar{t}, \bar{X})$ denote the partial derivative of F with respect to X at $(\bar{t}, \bar{X})$ and $\mathrm{D}_X^2 F(\bar{t}, \bar{X})$ denote the partial second-order derivative of F with respect to X at $(\bar{t}, \bar{X})$.
- Let $\Phi : \Re^{m\times n} \to \Re$ be a continuous function. We call $\bar{\Phi} : \Re_+ \times \Re^{m\times n} \to \Re$ a smoothing function of Φ, if $\bar{\Phi}(\mu, \cdot)$ is continuously differentiable in $\Re^{m\times n}$ for any fixed $\mu > 0$, and for any $X \in \Re^{m\times n}$, we obtain that $\lim_{\mu\downarrow 0, Z\to X} \bar{\Phi}(\mu, Z) = \Phi(X)$.

2. Lower bound analysis

In this section, we first introduce the definition of necessary conditions of (6) and establish the theorem that every local minimizer of (6) satisfies these necessary conditions. Then, we present lower bounds for nonzero singular values in local solutions of (6). Finally, we will show how to choose the parameter τ for obtaining a low-rank solution of (6).

Definition 2.1 *For $X \in \Re^{m\times n}$ and $p \in (0,1)$, X is said to satisfy the first-order necessary condition of* (6) *if*

$$\mathcal{A}(X)^{\mathrm{T}}(\mathcal{A}(X)-b) + \tau\|X\|_p^p = 0. \tag{9}$$

Also, X is said to satisfy the second-order necessary condition of (6) *if*

$$\|\mathcal{A}(X)\|_2^2 + \tau(p-1)\|X\|_p^p \geq 0. \tag{10}$$

Remark 1 It is not difficult to see that if $X := \mathrm{Diag}(x)$ and the conditions of (9) and (10) hold at X, then there exists a $q \times m$ matrix C satisfying

$$x^{\mathrm{T}}C^{\mathrm{T}}(Cx-b) + \tau\|x\|_p^p = 0, \quad \|Cx\|_2^2 + \tau(p-1)\|x\|_p^p \geq 0, \tag{11}$$

which are weaker than the necessary conditions of (7) introduced in [10, Definition 3.2].

Now, we are in the position to prove an important lemma used in the analysis below.

Lemma 2.2 *Let $X^\star$ be a local minimizer of* (6). *Then, for any pair $(U^\star, V^\star) \in \mathcal{O}^{m,n}(X^\star)$, the vector $z^\star := \sigma(X^\star) \in \Re^m$ is a local minimizer of the following problem:*

$$\begin{aligned} \min \quad & \varphi(z) := F(U^\star[Diag(z)\ 0_{m\times(n-m)}](V^\star)^{\mathrm{T}}) \\ s.t. \quad & z \in \Re^m. \end{aligned} \tag{12}$$

Proof It follows from $(U^\star, V^\star) \in \mathcal{O}^{m,n}(X^\star)$ that

$$X^\star = U^\star[\mathrm{Diag}(\sigma(X^\star))\ 0_{m\times(n-m)}](V^\star)^{\mathrm{T}}.$$

Since $X^\star$ is a local minimizer of (6), there exists a positive scalar δ such that

$$F(X^\star) \le F(X), \quad \forall X \in \mathcal{N}(X^\star, \delta) := \{X \in \mathfrak{R}^{m\times n} : \|X - X^\star\|_F \le \delta\}.$$

Therefore, for any $z \in \mathcal{N}(\sigma(X^\star), \delta) := \{z \in \mathfrak{R}^m : \|z - \sigma(X^\star)\|_2 \le \delta\}$, we have

$$\hat{X}(z) := U^\star[\mathrm{Diag}(z)\ 0_{m\times(n-m)}](V^\star)^{\mathrm{T}} \in \mathcal{N}(X^\star, \delta), \quad F(X^\star) \le F(\hat{X}(z)).$$

From the above observations and the definition of φ, the following relationships hold at any $z \in \mathcal{N}(\sigma(X^\star), \delta) \subseteq \mathfrak{R}^m$:

$$\begin{aligned}\varphi(z^\star) &= F(U^\star[\mathrm{Diag}(\sigma(X^\star))\ 0_{m\times(n-m)}](V^\star)^{\mathrm{T}}) \\ &= F(X^\star) \\ &\le F(\hat{X}(z)) \\ &= F(U^\star[\mathrm{Diag}(z)\ 0_{m\times(n-m)}](V^\star)^{\mathrm{T}}) = \varphi(z),\end{aligned}$$

which implies that $z^\star$ is a local minimizer of (12). ■

Given a pair $(U^\star, V^\star) \in \mathcal{O}^{m,n}(X^\star)$, we split $(U^\star)^{\mathrm{T}} A_i (V^\star)$ into two parts,

$$(U^\star)^{\mathrm{T}} A_i V^\star := [\tilde{A}_i^\star\ \bar{A}_i^\star], \quad \tilde{A}_i^\star \in \mathfrak{R}^{m\times m}, \quad \bar{A}_i^\star \in \mathfrak{R}^{m\times(n-m)} \ \ (i = 1, 2, \ldots, q)$$

and denote $A^{\mathrm{T}} := (\mathrm{Diag}(\tilde{A}_1^\star), \mathrm{Diag}(\tilde{A}_2^\star), \ldots, \mathrm{Diag}(\tilde{A}_q^\star)) \in \mathfrak{R}^{m\times q}$. One can see that

$$\mathcal{A}(U^\star[\mathrm{Diag}(z)\ 0_{m\times(n-m)}](V^\star)^{\mathrm{T}}) = Az \tag{13}$$

and (12) reduces to the vector l_2^2-l_p^p problem (7) studied in [10].

From the above observations, we can easily obtain the following corollary, which is similar to [10, Theorems 2.1 and 2.3].

Corollary 2.3 *Let $z^\star \in \mathfrak{R}^m$ be any local minimizer of (12). Then, the following conditions hold at $z^\star$:*

$$A^{\mathrm{T}}(Az^\star - b) + \tau(z^\star)^{p-1} = 0$$

and the matrix $A_{\mathcal{I}}^{\mathrm{T}} A_{\mathcal{I}} + \tau(p-1)\,\mathrm{Diag}((z_{\mathcal{I}}^\star)^{p-2})$ is positive semi-definite, where $\mathcal{I} := I(z^\star)$.

Combining Lemma 2.2 with Corollary 2.3, we can show that any local minimizer of (6) satisfies the necessary conditions (9) and (10).

Theorem 2.4 *Let $X^\star$ be any local minimizer of (6). Then $X^\star$ satisfies the conditions (9) and (10).*

Proof From Lemma 2.2 and Corollary 2.3, we deduce that the following conditions hold at the vector $z^\star = \sigma(X^\star)$:

$$(z^\star)^{\mathrm{T}} A^{\mathrm{T}}(Az^\star - b) + \tau\|z^\star\|_p^p = 0 \quad \text{and} \quad (z^\star)^{\mathrm{T}} A^{\mathrm{T}} A z^\star + \tau(p-1)\|z^\star\|_p^p \ge 0.$$

It follows from the SVD of $X^\star$ in (8) and the relationship (13) that

$$\mathcal{A}(X^\star)^{\mathrm{T}}(\mathcal{A}(X^\star) - b) + \tau\|\sigma(X^\star)\|_p^p = 0 \quad \text{and} \quad \mathcal{A}(X^\star)^{\mathrm{T}}\mathcal{A}(X^\star) + \tau(p-1)\|\sigma(X^\star)\|_p^p \ge 0.$$

Utilizing the definition of $\mathcal{O}^{m,n}(X^\star)$, for any pair $(U^\star, V^\star) \in \mathcal{O}^{m,n}(X^\star)$, we obtain

$$\|\sigma(X^\star)\|_p^p = \|U^\star[\mathrm{Diag}(\sigma(X^\star))\ 0_{m\times(n-m)}](V^\star)^{\mathrm{T}}\|_p^p,$$

which implies that (9) and (10) hold at $X^\star$. ■

Next, we use Corollary 2.3 to derive a first-order lower bound for nonzero singular values in local solutions of (6).

THEOREM 2.5 *Let $X^\star$ be any local minimizer of (6) satisfying $F(X^\star) \le F(X_0)$ for any given point $X_0 \in \mathfrak{R}^{m\times n}$ and $\mu_A := \sqrt{q}\max_{1\le i\le q}\|A_i\|_F$. Then, for any $i \in \{1, 2, \ldots, m\}$, we have*

$$\sigma_i(X^\star) < L(\tau, \mu_A, X_0, p) := \left(\frac{\tau}{\mu_A\sqrt{2F(X_0)}}\right)^{1/(1-p)} \quad \Rightarrow \quad \sigma_i(X^\star) = 0.$$

In addition, the rank of $X^\star$ is bounded by $\min(m, pF(X_0)/\tau L(\tau, \mu_A, X_0, p)^p)$.

Proof Since $F(X^\star) \le F(X_0)$, we have

$$\|A^{\mathrm{T}}(\mathcal{A}(X^\star) - b)\|_2^2 \le \|A\|^2\left(\|\mathcal{A}(X^\star) - b\|_2^2 + \frac{2\tau}{p}\|X^\star\|_p^p\right) \le 2\|A\|^2 F(X_0).$$

From the relationship between A and A_i ($i = 1, 2, \ldots, q$), we obtain

$$\begin{aligned}
\|A\| &= \max_{\|z\|_2=1}\|Az\|_2 \\
&= \max_{\|z\|_2=1}\|\mathcal{A}(U^\star[\mathrm{Diag}(z)\ 0_{m\times(n-m)}](V^\star)^{\mathrm{T}})\|_2 \\
&\le \max_{\|z\|_2=1}\sqrt{q}\max_{1\le i\le q}\|(U^\star)^{\mathrm{T}} A_i V^\star\|_F \cdot \|[\mathrm{Diag}(z)\ 0_{m\times(n-m)}]\|_F \\
&= \max_{\|z\|_2=1}\sqrt{q}\max_{1\le i\le q}\|A_i\|_F \cdot \|z\|_2 \\
&= \sqrt{q}\max_{1\le i\le q}\|A_i\|_F = \mu_A,
\end{aligned} \tag{14}$$

where the second equality uses Equation (13) and the third inequality is deduced using the Cauchy–Schwarz inequality. Hence, it follows from (14) that

$$\|A^{\mathrm{T}}(\mathcal{A}(X^\star) - b)\|_2^2 \le 2\mu_A^2 F(X_0). \tag{15}$$

By Corollary 2.3 and (15), for any $j \in I(\sigma(X^\star))$, we have

$$\begin{aligned}
\tau\sigma_j^{p-1}(X^\star) &\le \tau\left(\sum_{j=1}^{m}[\sigma_j^{p-1}(X^\star)]^2\right)^{1/2} = \tau\|(z^\star)^{p-1}\|_2 \\
&= \|A^{\mathrm{T}}(Az^\star - b)\|_2 = \|A^{\mathrm{T}}(\mathcal{A}(X^\star) - b)\|_2 \le \mu_A\sqrt{2F(X_0)}.
\end{aligned}$$

Note that $p \in (0,1)$, we find

$$\sigma_j(X^\star) \geq \left(\frac{\tau}{\mu_A\sqrt{2F(X_0)}}\right)^{1/(1-p)} = L(\tau, \mu_A, X_0, p).$$

Hence, all nonzero singular values of $X^\star$ are no less than $L(\tau, \mu_A, X_0, p)$. That is, if $\sigma_i(X^\star) < L(\tau, \mu_A, X_0, p)$, we have $\sigma_i(X^\star) = 0$.

Now, we turn to the remaining part. Assume that the rank of $X^\star$ is equal to r, by the definition of $F(X)$, we have

$$\frac{r\tau}{p}L(\tau, \mu_A, X_0, p)^p \leq \frac{\tau}{p}\|X^\star\|_p^p \leq \frac{1}{2}\|\mathcal{A}(X^\star) - b\|_2^2 + \frac{\tau}{p}\|X^\star\|_p^p = F(X^\star) \leq F(X_0),$$

which implies

$$\operatorname{rank}(X^\star) = r \leq \frac{pF(X_0)}{\tau L(\tau, \mu_A, X_0, p)^p} \quad \text{or} \quad \operatorname{rank}(X^\star) \leq \min\left(m, \frac{pF(X_0)}{\tau L(\tau, \mu_A, X_0, p)^p}\right).$$

■

It is not difficult to see that the lower bound in Theorem 2.5 depends on the parameters τ, p, the matrices A_i $(i = 1, 2, \ldots, q)$ in the operator $\mathcal{A}$ and the given point X_0. Hence, we can compute this lower bound in advance. Note that the process of establishing the lower bound $L(\tau, \mu_A, X_0, p)$ is similar to [10, Theorem 2.3].

In order to avoid the null matrix (the trivial solution of (6)), we always consider (6) in the level set $\{X : F(X) \leq F(0) = \frac{1}{2}\|b\|_2^2\}$. Hence, if $X_0 = 0$ and $\|A_i\|_F = 1$ $(i = 1, 2, \ldots, q)$, we obtain the following corollary:

COROLLARY 2.6 *Let $X^\star$ be any local minimizer of (6). Then, for any $i \in \{1, 2, \ldots, m\}$, we have*

$$\sigma_i(X^\star) < L_1(\tau, p) := \left(\frac{\tau}{\sqrt{q}\|b\|_2}\right)^{1/(1-p)} \Rightarrow \sigma_i(X^\star) = 0.$$

In addition, the rank of $X^\star$ is bounded by $\min(m, p\|b\|_2^2/2\tau L_1(\tau, p)^p)$.

Similar to [10, Theorem 2.1], we derive a second-order lower bound for nonzero singular values in local solutions of (6) in the next theorem.

THEOREM 2.7 *Let $X^\star$ be any local minimizer of (6) and $\mu_A := \sqrt{q}\max_{1\leq i\leq q}\|A_i\|_F$. Then, for any $i \in \{1, 2, \ldots, m\}$, we have*

$$\sigma_i(X^\star) < L_2(\tau, \mu_A, p) := \left(\frac{\tau(1-p)}{\mu_A^2}\right)^{1/(2-p)} \Rightarrow \sigma_i(X^\star) = 0.$$

Proof By Corollary 2.3, we deduce that the matrix $A_{\mathcal{I}}^{\mathrm{T}}A_{\mathcal{I}} + \tau(p-1)\operatorname{Diag}((z_{\mathcal{I}}^\star)^{p-2})$ is positive semi-definite, where $z^\star = \sigma(X^\star)$ and $\mathcal{I} = I(z^\star)$. Therefore, we obtain

$$(e_i)^{\mathrm{T}}A_{\mathcal{I}}^{\mathrm{T}}A_{\mathcal{I}}e_i + \tau(p-1)\sigma_i^{p-2}(X^\star) \geq 0, \quad i \in \mathcal{I},$$

where e_i is the ith column of the identity matrix $I_{|\mathcal{I}|} \in \mathfrak{R}^{|\mathcal{I}|\times|\mathcal{I}|}$ and $|\mathcal{I}|$ denotes the size of $\mathcal{I}$. From the definitions of μ_A and $\|A\|$, the above inequality implies that for all $i \in \mathcal{I}$,

$$\mu_A^2 + \tau(p-1)\sigma_i^{p-2}(X^\star) \geq 0 \Rightarrow \sigma_i(X^\star) \geq \left(\frac{\tau(1-p)}{\mu_A^2}\right)^{1/(2-p)} = L_2(\tau, \mu_A, p).$$

Hence, the conclusion holds at $X^\star$. ■

Before ending this section, we give a sufficient condition on the parameter τ of (6) to obtain a desirable low-rank solution, which is a natural extension of that introduced in [11, Theorem 2] for (7).

THEOREM 2.8 *Let $X^\star$ be any local minimizer of (6) satisfying $F(X^\star) \leq F(X_0)$ for any given point $X_0 \in \Re^{m\times n}$ and $\mu_A := \sqrt{q}\,\max_{1\leq i\leq q} \|A_i\|_F$. Let*

$$\tau(\mu_A, s, X_0, p) := \left(\frac{p}{s}\right)^{1-p} (F(X_0))^{1-p/2} 2^{p/2} \mu_A^p.$$

If $\tau \geq \tau(\mu_A, s, X_0, p)$, then rank$(X^\star) < s$ for $s \geq 1$.

Proof From Theorem 2.5, for any local minimizer of (6) in the level set $\{X : F(X) \leq F(X_0)\}$, we obtain

$$F(X^\star) = \frac{1}{2}\|\mathcal{A}(X^\star) - b\|_2^2 + \frac{\tau}{p}\|X^\star\|_p^p > \frac{\tau}{p}\sum_{i\in I(\sigma(X^\star))} \sigma_i^p(X^\star) \geq \frac{\tau}{p}\,\mathrm{rank}(X^\star) L(\tau, \mu_A, X_0, p)^p.$$

If $\mathrm{rank}(X^\star) \geq s \geq 1$, then

$$F(X^\star) > \frac{\tau}{p} s L(\tau, \mu_A, X_0, p)^p = \frac{\tau}{p} s \left(\frac{\tau}{\mu_A\sqrt{2F(X_0)}}\right)^{p/(1-p)} \geq F(X_0),$$

which contradicts with the fact that $X^\star$ lies in the level set $\{X : F(X) \leq F(X_0)\}$. ■

Similar to Corollary 2.6, if $X_0 = 0$ and $\|A_i\|_F = 1$ $(i = 1, 2, \ldots, q)$, the following corollary holds at $X^\star$.

COROLLARY 2.9 *Let $X^\star$ be any local minimizer of (6). Let*

$$\tau_1(s, p) := \left(\frac{p}{2s}\right)^{1-p} \|b\|_2^{2-p} q^{p/2}.$$

If $\tau \geq \tau_1(s, p)$, then rank$(X^\star) < s$ for $s \geq 1$.

3. The smoothing model

In this section, we establish the smooth approximation model to (6) and analyze the properties of the sequence of the corresponding optimal solutions. We define the smoothing model as follows:

$$\begin{aligned} \min \quad & \bar{F}(\epsilon, X) \\ \text{s.t.} \quad & X \in \Re^{m\times n}, \end{aligned} \tag{16}$$

where $\bar{F}(\epsilon, X)$ is defined by

$$\bar{F}(\epsilon, X) = \frac{1}{2}\|\mathcal{A}(X) - b\|_2^2 + \frac{\tau}{p}\sum_{i=1}^{m} (\sigma_i^2(X) + \epsilon^2)^{p/2}. \tag{17}$$

It follows from the definition of the smoothing function that $\bar{F}(\epsilon, X)$ is indeed a smoothing function of $F(X)$. Notice that (16) is an unconstrained optimization problem. Let $X_\epsilon^\star$ be a local minimizer of (16) for a given $\epsilon > 0$. Then, for any $H \in \Re^{m\times n}$, the following conditions hold at $X^\star_\epsilon$:

$$\langle \mathrm{D}_X F(\epsilon, X^\star_\epsilon), H\rangle = 0, \tag{18}$$

$$\langle \mathrm{D}^2_X F(\epsilon, X^\star_\epsilon)H, H\rangle \geq 0. \tag{19}$$

According to the definitions of $F(X)$ and $\bar{F}(\epsilon, X)$, we obtain

$$0 \leq \bar{F}(\epsilon, X) - F(X) \leq \frac{\tau m|\epsilon|^p}{p}. \tag{20}$$

The inequality (20) gives a bound of the difference between the original objective function $F(X)$ and the smoothing function $\bar{F}(\epsilon, X)$.

Now, we show the relationship between (6) and its smooth approximation (16) in the following theorems.

THEOREM 3.1 *Let $\{\epsilon^k\}$ denote a sequence of positive scalars such that $\epsilon^k \to 0$ as $k \to \infty$. The following conclusions hold:*

(1) *Let $\{X^\star_{\epsilon^k}\}$ be a sequence of matrices satisfying (18) with $\epsilon = \epsilon^k$. Then any accumulation of $\{X^\star_{\epsilon^k}\}$ satisfies the first-order necessary condition of (6).*
(2) *Let $\{X^\star_{\epsilon^k}\}$ be a sequence of matrices satisfying (19) with $\epsilon = \epsilon^k$. Then any accumulation of $\{X^\star_{\epsilon^k}\}$ satisfies the second-order necessary condition of (6).*
(3) *Let $\{X^\star_{\epsilon^k}\}$ be a sequence of matrices being global minimizer of (16) with $\epsilon = \epsilon^k$. Then any accumulation of $\{X^\star_{\epsilon^k}\}$ is the global minimizer of (6).*

Proof Let $X^\star$ be an accumulation point of $\{X^\star_{\epsilon^k}\}$. Without loss of generality, we assume that $X^\star_{\epsilon^k} \to X^\star$ as $k \to \infty$. To prove part 1 of the conclusions, we employ (18) and Theorem 1.1 in [20] that

$$\mathcal{A}(X^\star_{\epsilon^k})^{\mathrm{T}}(\mathcal{A}(X^\star_{\epsilon^k}) - b) + \tau\sigma(X^\star_{\epsilon^k})^{\mathrm{T}} \begin{pmatrix} \sigma_1(X^\star_{\epsilon^k})(\sigma_1^2(X^\star_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-1} \\ \sigma_2(X^\star_{\epsilon^k})(\sigma_2^2(X^\star_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-1} \\ \vdots \\ \sigma_m(X^\star_{\epsilon^k})(\sigma_m^2(X^\star_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-1} \end{pmatrix} = 0.$$

The above equation is equivalent to

$$(\mathcal{A}(X^\star_{\epsilon^k}) - b)^{\mathrm{T}}\mathcal{A}(X^\star_{\epsilon^k}) + \tau\sum_{i=1}^m \sigma_i^2(X^\star_{\epsilon^k})(\sigma_i^2(X^\star_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-1} = 0.$$

When $k \to \infty$, we obtain that

$$\mathcal{A}(X^\star_{\epsilon^k}) \to \mathcal{A}(X^\star) \quad \text{and} \quad \sum_{i=1}^m \sigma_i^2(X^\star_{\epsilon^k})(\sigma_i^2(X^\star_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-1} \to \|\sigma(X^\star)\|_p^p.$$

Hence, $\mathcal{A}(X^\star)^{\mathrm{T}}(\mathcal{A}(X^\star) - b) + \tau\|X^\star\|_p^p = 0$ and $X^\star$ satisfies the first-order necessary condition of (6).

Next, we prove the second part. It follows from (19) and Theorem 4.1 in [21] that

$$\mathcal{A}(X^{\star}_{\epsilon^k})^{\mathrm{T}}\mathcal{A}(X^{\star}_{\epsilon^k}) + \tau\sigma(X^{\star}_{\epsilon^k})^{\mathrm{T}}\Lambda_{\epsilon^k}\sigma(X^{\star}_{\epsilon^k}) \geq 0,$$

where Λ_{ϵ^k} is an $m \times m$ diagonal matrix whose diagonal entries are of the form $(\sigma_i^2(X^{\star}_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-2}((p-1)\sigma_i^2(X^{\star}_{\epsilon^k}) + (\epsilon^k)^2)$, $i \in \{1, 2, \ldots, m\}$. The above inequality emphasizes that

$$\mathcal{A}(X^{\star}_{\epsilon^k})^{\mathrm{T}}\mathcal{A}(X^{\star}_{\epsilon^k}) + \tau\sum_{i=1}^{m}\sigma_i^2(X^{\star}_{\epsilon^k})(\sigma_i^2(X^{\star}_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-2}((p-1)\sigma_i^2(X^{\star}_{\epsilon^k}) + (\epsilon^k)^2) \geq 0,$$

which implies that $\|\mathcal{A}(X^{\star})\|_2^2 + \tau(p-1)\|X^{\star}\|_p^p \geq 0$. Hence, $X^{\star}$ satisfies the second-order necessary condition of (6).

To end this theorem, let us prove the remaining part. Let $\hat{X}^{\star}$ be any global minimizer of (6), we have

$$F(X^{\star}_{\epsilon^k}) \leq \bar{F}(\epsilon^k, X^{\star}_{\epsilon^k}) \leq \bar{F}(\epsilon^k, \hat{X}^{\star}) \leq F(\hat{X}^{\star}) + \frac{\tau m|\epsilon^k|^p}{p}.$$

It follows from the above relationships that $F(X^{\star}) \leq F(\hat{X}^{\star})$. Hence, $X^{\star}$ is the global minimizer of (6). ■

The following theorem presents a first-order lower bound for nonzero singular values in local solutions of the smoothing model (16).

Theorem 3.2 *Let $X^{\star}_{\epsilon^k}$ be any local minimizer of* (16) *satisfying $\bar{F}(\epsilon^k, X^{\star}_{\epsilon^k}) \leq F(X_0)$ for any given point $X_0 \in \Re^{m\times n}$ and $\mu_A := \sqrt{q}\max_{1\leq i\leq q}\|A_i\|_F$. Then, for any $i \in \{1, 2, \ldots, m\}$ and any scalar $\lambda \in (0, +\infty)$, we have*

$$\sigma_i(X^{\star}_{\epsilon^k}) < \bar{L}(\tau, \mu_A, X_0, p, \lambda) := \left(\frac{\lambda^2}{1+\lambda^2}\right)^{(2-p)/2(1-p)}\left(\frac{\tau}{\mu_A\sqrt{2F(X_0)}}\right)^{1/(1-p)} \Rightarrow \sigma_i(X^{\star}_{\epsilon^k}) \leq \lambda|\epsilon^k|.$$

Proof Set $H_i^{\epsilon^k} := U_{\epsilon^k}[E(i,i)\ 0]V_{\epsilon^k}^{\mathrm{T}}$ $(i = 1, 2, \ldots, m)$, where $(U_{\epsilon^k}, V_{\epsilon^k}) \in \mathcal{O}^{m,n}(X^{\star}_{\epsilon^k})$ and $E(i,i)$ denotes an $m \times m$ matrix whose (i,i) entry equals 1 and all others are 0. Similar to (13), we also show that

$$\mathcal{A}(U_{\epsilon^k}[\mathrm{Diag}(z)\ 0_{m\times(n-m)}]V_{\epsilon^k}^{\mathrm{T}}) = A_{\epsilon^k}z \quad \text{and} \quad \|A_{\epsilon^k}\| \leq \mu_A, \tag{21}$$

where the matrix A_{ϵ^k} satisfies the following relationship:

$$A_{\epsilon^k}^{\mathrm{T}} := (\mathrm{Diag}(\tilde{A}_1^{\epsilon^k}), \mathrm{Diag}(\tilde{A}_2^{\epsilon^k}), \ldots, \mathrm{Diag}(\tilde{A}_q^{\epsilon^k})) \in \Re^{m\times q}$$

and $\tilde{A}_i^{\epsilon^k}$ is the first m columns of $(U_{\epsilon^k})^{\mathrm{T}}A_iV_{\epsilon^k}$ $(i = 1, 2, \ldots, m)$. Hence, we obtain

$$\mathcal{A}(H_i^{\epsilon^k}) = A_{\epsilon^k}e_i^m \quad (i = 1, 2, \ldots, m), \tag{22}$$

where e_i^m is the ith column of the identity matrix $I_m \in \Re^{m\times m}$. Since $X^{\star}_{\epsilon^k}$ is a local minimizer of (16), similar to the proof of Theorem 3.1, Equation (18) implies that

$$\mathcal{A}(H_i^{\epsilon^k})^{\mathrm{T}}(\mathcal{A}(X^{\star}_{\epsilon^k}) - b) + \tau(e_i^m)^{\mathrm{T}}\begin{pmatrix}\sigma_1(X^{\star}_{\epsilon^k})(\sigma_1^2(X^{\star}_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-1}\\ \sigma_2(X^{\star}_{\epsilon^k})(\sigma_2^2(X^{\star}_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-1}\\ \vdots\\ \sigma_m(X^{\star}_{\epsilon^k})(\sigma_m^2(X^{\star}_{\epsilon^k}) + (\epsilon^k)^2)^{p/2-1}\end{pmatrix} = 0.$$

It follows from the above relationship and (22) that

$$\begin{aligned}
&\tau\sigma_i(X^\star_{\epsilon^k})(\sigma_i^2(X^\star_{\epsilon^k})+(\epsilon^k)^2)^{p/2-1}\\
&\quad\le \|\mathcal{A}(H_i^{\epsilon^k})\|_2\cdot\sqrt{\|\mathcal{A}(X^\star_{\epsilon^k})-b\|_2^2+\frac{\tau}{p}\sum_{i=1}^m(\sigma_i^2(X^\star_{\epsilon^k})+(\epsilon^k)^2)^{p/2}}\\
&\quad=\|A_{\epsilon^k}e_i^m\|_2\cdot\sqrt{2\bar F(\epsilon^k,X^\star_{\epsilon^k})}\le\|A_{\epsilon^k}\|\sqrt{2F(X_0)}\le\mu_A\sqrt{2F(X_0)}.
\end{aligned}\tag{23}$$

Suppose $\sigma_i(X^\star_{\epsilon^k})>\lambda|\epsilon^k|$, we have

$$\frac{1+\lambda^2}{\lambda^2}\sigma_i^2(X^\star_{\epsilon^k})>\sigma_i^2(X^\star_{\epsilon^k})+(\epsilon^k)^2\Rightarrow\left(\frac{1+\lambda^2}{\lambda^2}\sigma_i^2(X^\star_{\epsilon^k})\right)^{p/2-1}<(\sigma_i^2(X^\star_{\epsilon^k})+(\epsilon^k)^2)^{p/2-1}.$$

Therefore, the above relationship and (23) tell us that

$$\sigma_i(X^\star_{\epsilon^k})\ge\left(\frac{\lambda^2}{1+\lambda^2}\right)^{(2-p)/2(1-p)}\left(\frac{\tau}{\mu_A\sqrt{2F(X_0)}}\right)^{1/(1-p)}=\bar L(\tau,\mu_A,X_0,p,\lambda).$$

Hence we can claim that, for $i\in\{1,2,\ldots,m\}$, if $\sigma_i(X^\star_{\epsilon^k})<\bar L(\tau,\mu_A,X_0,p,\lambda)$, then $\sigma_i(X^\star_{\epsilon^k})\le\lambda|\epsilon^k|$. ■

A second-order lower bound for nonzero singular values in local solutions of (16) is presented in the following theorem.

THEOREM 3.3 *Let $X^\star_{\epsilon^k}$ be any local minimizer of* (16) *and $\mu_A:=\sqrt{q}\max_{1\le i\le q}\|A_i\|_F$. Then, for any $i\in\{1,2,\ldots,m\}$ and any scalar $\lambda\in(0,+\infty)$, we have*

$$\sigma_i(X^\star_{\epsilon^k})<\bar L_2^k(\tau,\mu_A,p,\lambda):=\left(\frac{\lambda^2}{1+\lambda^2}\right)^{(4-p)/2(2-p)}\left(\frac{\tau(1-p)}{\mu_A^2+\tau(1+\lambda^2)^{p/2-2}(\epsilon^k)^{p-2}}\right)^{1/(2-p)}$$
$$\Rightarrow\sigma_i(X^\star_{\epsilon^k})\le\lambda|\epsilon^k|.$$

Proof Similar to the proof of Theorems 3.1 and 3.2, the inequality (19) implies that

$$(e_i^m)^{\mathrm T}A^{\mathrm T}_{\epsilon^k}A_{\epsilon^k}e_i^m+\tau(e_i^m)^{\mathrm T}\Lambda_{\epsilon^k}e_i^m\ge0,$$

where A_{ϵ^k} is defined as above and Λ_{ϵ^k} is an $m\times m$ diagonal matrix whose diagonal entries are of the form $(\sigma_i^2(X^\star_{\epsilon^k})+(\epsilon^k)^2)^{p/2-2}((p-1)\sigma_i^2(X^\star_{\epsilon^k})+(\epsilon^k)^2)$, $i\in\{1,2,\ldots,m\}$. By the definitions of $\|A_{\epsilon^k}\|$ and μ_A, the above inequality reveals that

$$\mu_A^2+\tau(\sigma_i^2(X^\star_{\epsilon^k})+(\epsilon^k)^2)^{p/2-2}((p-1)\sigma_i^2(X^\star_{\epsilon^k})+(\epsilon^k)^2)\ge0.$$

Suppose $\sigma_i(X^\star_{\epsilon^k})>\lambda|\epsilon^k|$, we obtain

$$\mu_A^2+\tau(p-1)\left(\frac{1+\lambda^2}{\lambda^2}\right)^{p/2-2}\sigma_i^{p-2}(X^\star_{\epsilon^k})+\tau(1+\lambda^2)^{p/2-2}(\epsilon^k)^{p-2}\ge0,$$

which implies that

$$\sigma_i(X^\star_{\epsilon^k})\ge\left(\frac{\lambda^2}{1+\lambda^2}\right)^{(4-p)/2(2-p)}\left(\frac{\tau(1-p)}{\mu_A^2+\tau(1+\lambda^2)^{p/2-2}(\epsilon^k)^{p-2}}\right)^{1/(2-p)}=\bar L_2^k(\tau,\mu_A,p,\lambda).$$

Hence, the conclusion holds at $X^\star_{\epsilon^k}$. ■

THEOREM 3.4 *Let $X^\star_{\epsilon^k}$ be any local minimizer of* (16) *satisfying* $\bar{F}(\epsilon^k, X^\star_{\epsilon^k}) \le F(X_0)$ *for any given point* $X_0 \in \mathfrak{R}^{m\times n}$ *and* $\mu_A := \sqrt{q}\max_{1\le i\le q}\|A_i\|_F$. *Let* $\{X^\star_{\epsilon^k}\}$ *be a convergent sequence. Then, for any scalar* $\lambda \in (0, +\infty)$, *there exist an integer* $K > 0$ *and a local minimizer* $X^\star$ *of* (6) *such that for any* $k \ge K$, *we have*

$$\begin{aligned}\Gamma_{\epsilon^k} &:= \{i \in \{1,2,\dots,m\} \mid \sigma_i(X^\star_{\epsilon^k}) \le \lambda|\epsilon^k|\}\\ &= \{i \in \{1,2,\dots,m\}|\sigma_i(X^\star) = 0\} =: \Gamma.\end{aligned} \tag{24}$$

Proof Since the level set $\{X : \bar{F}(\epsilon, X) \le F(X_0)\}$ is bounded, the sequence $\{X^\star_{\epsilon^k}\}$ is bounded. Similar to (1) of Theorem 3.1, let $X^\star$ be an accumulation point of $\{X^\star_{\epsilon^k}\}$. Without loss of generality, we assume that $X^\star_{\epsilon^k} \to X^\star$ as $k \to \infty$ and there exists an integer $K > 0$ such that for $k \ge K$, $|\epsilon^k| < \bar{L}(\tau, \mu_A, X_0, p, \lambda)/2\lambda$, $\mathrm{dist}(X^\star_{\epsilon^k}, X^\star) = \|X^\star_{\epsilon^k} - X^\star\| \le \bar{L}(\tau, \mu_A, X_0, p, \lambda)/2$ and $F(X^\star) \le F(X_0)$ hold. Then we have $\sigma_i(X^\star) - \sigma_i(X^\star_{\epsilon^k}) \le |\sigma_i(X^\star) - \sigma_i(X^\star_{\epsilon^k})| \le \|X^\star - X^\star_{\epsilon^k}\| \le \bar{L}(\tau, \mu_A, X_0, p, \lambda)/2$.

If $i \in \Gamma_{\epsilon^k}$, we have

$$\begin{aligned}\sigma_i(X^\star) &\le \sigma_i(X^\star_{\epsilon^k}) + \frac{\bar{L}(\tau, \mu_A, X_0, p, \lambda)}{2} \le \lambda|\epsilon^k| + \frac{\bar{L}(\tau, \mu_A, X_0, p, \lambda)}{2}\\ &< \bar{L}(\tau, \mu_A, X_0, p, \lambda) \le L(\tau, \mu_A, X_0, p).\end{aligned}$$

Suppose $\sigma_i(X^\star) \ne 0$, from Corollary 2.3, we have

$$\tau L(\tau, \mu_A, X_0, p)^{p-1} < \tau\sigma_i^{p-1}(X^\star) \le |A^{\mathrm{T}}(Az^\star - b)|_i \le \|A\| \cdot \|Az^\star - b\|_2,$$

where $z^\star = \sigma(X^\star)$ and $\mathcal{A}(X^\star) = Az^\star$. Hence, we obtain

$$\tau L(\tau, \mu_A, X_0, p)^{p-1} < \|A\| \cdot \|Az^\star - b\|_2 = \|A\| \cdot \|\mathcal{A}(X^\star) - b\|_2 \le \|A\|\sqrt{2F(X_0)},$$

which leads to the following contradiction:

$$L(\tau, \mu_A, X_0, p) > \left(\frac{\tau}{\|A\|\sqrt{2F(X_0)}}\right)^{1/(1-p)} \ge \left(\frac{\tau}{\mu_A\sqrt{2F(X_0)}}\right)^{1/(1-p)} = L(\tau, \mu_A, X_0, p).$$

Therefore, we obtain that $\sigma_i(X^\star) = 0$, which means that $\Gamma_{\epsilon^k} \subset \Gamma$.

On the other hand, if $i \in \Gamma$, then $\sigma_i(X^\star) = 0$ and

$$\sigma_i(X^\star_{\epsilon^k}) = \sigma_i(X^\star_{\epsilon^k}) - \sigma_i(X^\star) \le \|X^\star_{\epsilon^k} - X^\star\| \le \frac{\bar{L}(\tau, \mu_A, X_0, p, \lambda)}{2} < \bar{L}(\tau, \mu_A, X_0, p, \lambda).$$

From Theorem 3.2, we deduce that $\sigma_i(X^\star_{\epsilon^k}) \le \lambda|\epsilon^k|$. Hence, $\Gamma \subset \Gamma_{\epsilon^k}$, that is, the conclusion (24) is true. ■

Note that the above presented theorems extend Theorems 3.3 and 3.4 in [10] for (7) into the l_2^2-l_p^p matrix minimization problem (6).

4. The majorization algorithm for the smoothing model

The purpose of this section is to introduce the majorization algorithm for solving (16). For simplicity, we define $W(\epsilon, X)$ by

$$W(\epsilon, X) := U\,\mathrm{Diag}((\sigma_1^2(X)+\epsilon^2)^{p/2-1}, \dots, (\sigma_m^2(X)+\epsilon^2)^{p/2-1})U^{\mathrm{T}},$$

where U is a left singular matrix of X such that there exists an orthogonal matrix $V \in \mathcal{O}^n$ satisfying $(U, V) \in \mathcal{O}^{m,n}(X)$. Denote

$$\bar{F}_1(X) := \frac{1}{2}\|\mathcal{A}(X)-b\|_2^2, \quad \bar{F}_2(\epsilon, X) := \frac{\tau}{p}\sum_{i=1}^{m}(\sigma_i^2(X)+\epsilon^2)^{p/2},$$
$$\bar{F}(\epsilon, X) = \bar{F}_1(X) + \bar{F}_2(\epsilon, X).$$

Using [20, Theorem 1.1] again, we derive the explicit formula of the partial derivative of $\bar{F}(\epsilon, X)$ with respect to X as follows:

$$\begin{aligned} \mathrm{D}_X\bar{F}_1(X) &= \mathcal{A}^*(\mathcal{A}(X)-b), \quad \mathrm{D}_X\bar{F}_2(\epsilon, X) = \tau W(\epsilon, X)X, \\ \mathrm{D}_X\bar{F}(\epsilon, X) &= \mathrm{D}_X\bar{F}_1(X) + \mathrm{D}_X\bar{F}_2(\epsilon, X). \end{aligned} \tag{25}$$

In the sequel, we treat ϵ as a decision variable of $\bar{F}(\epsilon, X)$ and imbed the selection of ϵ in the optimization process. Similarly, the partial derivative of $\bar{F}(\epsilon, X)$ with respect to ϵ is given by

$$\mathrm{D}_\epsilon\bar{F}(\epsilon, X) = \tau\epsilon\,\mathrm{Tr}(W(\epsilon, X)) \quad \text{if } \epsilon > 0. \tag{26}$$

Next, we construct an approximation subproblem to (16) at (ϵ^k, X^k) in the following form:

$$\begin{aligned} \min \quad & \hat{F}^k(\epsilon, X) \\ \text{s.t.} \quad & (\epsilon, X) \in \Re \times \Re^{m\times n}, \end{aligned} \tag{27}$$

where the objective function $\hat{F}^k(\epsilon, X)$ is a majorization function of $\bar{F}(\epsilon, X)$ at (ϵ^k, X^k), that is,

$$\hat{F}^k(\epsilon, X) := \bar{F}_1(X) + \tilde{F}_2^k(\epsilon, X, \eta^k) + \frac{\tau\rho^k}{2}[\|X - X^k\|_F^2 + (\epsilon-\epsilon^k)^2],$$
$$\tilde{F}_2^k(\epsilon, X, \eta^k) := \frac{\tau}{2}\sum_{i=1}^{m}\left[(\sigma_i^2(X)+\epsilon^2)(\eta^k)_i - \frac{p-2}{p}(\eta^k)_i^{p/(p-2)}\right],$$

where η^k is a vector in $\Re^m$ such that

$$\eta^k := ((\sigma_1^2(X^k)+(\epsilon^k)^2)^{p/2-1}, \dots, (\sigma_m^2(X^k)+(\epsilon^k)^2)^{p/2-1})^{\mathrm{T}} \tag{28}$$

and $\rho^k > 0$ denotes the proximal parameter.

Remark 2 By the first-order necessary condition of (27), we solve the following system of equations:

$$\mathrm{D}_X \bar{F}_1(X) + \tau W(\epsilon^k, X^k)X + \tau\rho^k(X - X^k) = 0, \quad \epsilon\,\mathrm{Tr}(W(\epsilon^k, X^k)) + \rho^k(\epsilon - \epsilon^k) = 0.$$

Instead of solving the above equations exactly, we consider the underlying equations:

$$\mathrm{D}_X \bar{F}_1(X) + \tau W(\epsilon^k, X^k)X^k + \tau\rho^k(X - X^k) = 0, \quad \epsilon\,\mathrm{Tr}(W(\epsilon^k, X^k)) + \rho^k(\epsilon - \epsilon^k) = 0. \tag{29}$$

There are two reasons why we solve (27) inexactly. First, (29) has an explicit solution such that

$$\begin{aligned} \hat{X}^k &= \mathcal{G}^{-1}(\tau\rho^k X^k - \tau W(\epsilon^k, X^k)X^k + \mathcal{A}(b)), \\ \hat{\epsilon}^k &= \frac{\rho^k}{\rho^k + \mathrm{Tr}(W(\epsilon^k, X^k))}\epsilon^k, \end{aligned} \tag{30}$$

where $\mathcal{G}(X) := \mathcal{A}^*(\mathcal{A}(X)) + \tau\rho^k X$ and $\mathcal{G}^{-1}(\cdot)$ is the inverse operator of $\mathcal{G}$. Second, we can obtain a reasonable bound for ρ^k under the update rule (30), which can be found in Lemma 4.2 for further details.

Now we propose our algorithm to solve (27).

Algorithm: Smajor (Smoothing majorization algorithm).

Step 0: Choose the initial pair (ϵ^0, X^0) and set the counter $k := 0$.
Step 1: Set the parameter $\rho^k \geq 0$. Construct the problem (27) at (ϵ^k, X^k).
Step 2: Set $(\epsilon^{k+1}, X^{k+1}) := (\hat{\epsilon}^k, \hat{X}^k)$, $k := k + 1$ and go to Step 1.

Before proceeding, we establish the following lemmas that will be used subsequently.

Lemma 4.1 *Let $\{(\epsilon^k, X^k)\}$ be the pairs of sequence generated by algorithm Smajor. Then, we have*

(1) *For any positive integer k, $\hat{F}^k(\epsilon^k, X^k) = \bar{F}(\epsilon^k, X^k)$.*
(2) *For any positive integer k, we have*

$$\hat{F}^k(\epsilon^{k+1}, X^{k+1}) \geq \bar{F}(\epsilon^{k+1}, X^{k+1}) + \frac{\tau\rho^k}{2}[\|X^{k+1} - X^k\|_F^2 + (\epsilon^{k+1} - \epsilon^k)^2].$$

Proof Let us denote

$$\tilde{F}_2(\epsilon, X, \eta) := \frac{\tau}{2}\sum_{i=1}^m \left[(\sigma_i^2(X) + \epsilon^2)\eta_i - \frac{p-2}{p}\eta_i^{p/(p-2)}\right].$$

It is easy to see that for any positive integer k,

$$\arg\min_{\eta\geq 0} \tilde{F}_2(\epsilon^k, X^k, \eta) = \eta^k, \quad \tilde{F}_2(\epsilon^k, X^k, \eta^k) = \bar{F}_2(\epsilon^k, X^k),$$

where η^k is defined as (28). It follows from the above relationships that

$$\tilde{F}_2(\epsilon^{k+1}, X^{k+1}, \eta^k) \geq \tilde{F}_2(\epsilon^{k+1}, X^{k+1}, \eta^{k+1}) = \bar{F}_2(\epsilon^{k+1}, X^{k+1}). \tag{31}$$

Combining the definitions of $\hat{F}^k$ and $\bar{F}$ with the relationship (31), we obtain

$$\hat{F}^k(\epsilon^{k+1}, X^{k+1}) \geq \bar{F}(\epsilon^{k+1}, X^{k+1}) + \frac{\tau\rho^k}{2}[\|X^{k+1} - X^k\|_F^2 + (\epsilon^{k+1} - \epsilon^k)^2]$$

and $\hat{F}^k(\epsilon^k, X^k) = \bar{F}(\epsilon^k, X^k)$. Hence, the above conclusions are true for any positive integer k. ■

LEMMA 4.2 *Let* $\{(\epsilon^k, X^k)\}$ *be the pairs of sequence generated by algorithm Smajor. The parameter* ρ^k *satisfies the following condition:*

$$\rho^k \geq \max_{1\leq i\leq m} \eta_i^k, \tag{32}$$

where η^k *is defined as* (28). *Then*

$$\hat{F}^k(\epsilon^k, X^k) \geq \hat{F}^k(\epsilon^{k+1}, X^{k+1}). \tag{33}$$

Proof Let us denote

$$G^k(\epsilon, X) = \bar{F}_1(X) + \langle \tau W(\epsilon^k, X^k)X^k, X\rangle + \frac{\tau\rho^k}{2}[\|X - X^k\|_F^2 + (\epsilon - \epsilon^k)^2] + \frac{\tau}{2}\epsilon^2 \operatorname{Tr}(W(\epsilon^k, X^k)).$$

It is not difficult to find that $G^k(\epsilon, X)$ is a separable strongly convex function of (ϵ, X) and $(\epsilon^{k+1}, X^{k+1})$ is a unique minimizer of $G^k(\epsilon, X)$. Hence, by using [26, Theorem 2.1.8], we obtain

$$G^k(\epsilon^k, X^k) \geq G^k(\epsilon^{k+1}, X^{k+1}) + \frac{\tau\rho^k}{2}[\|X^{k+1} - X^k\|_F^2 + (\epsilon^{k+1} - \epsilon^k)^2] + \frac{\tau \operatorname{Tr}(W(\epsilon^k, X^k))}{2}(\epsilon^{k+1} - \epsilon^k)^2.$$

From the definition of $G^k(\epsilon, X)$, we have

$$G^k(\epsilon^k, X^k) = \bar{F}_1(X^k) + \tau\sum_{i=1}^m \sigma_i^2(X^k)(\eta^k)_i + \frac{\tau}{2}(\epsilon^k)^2 \operatorname{Tr}(W(\epsilon^k, X^k))$$

and

$$\begin{aligned} G^k(\epsilon^{k+1}, X^{k+1}) &= \bar{F}_1(X^{k+1}) + \langle \tau W(\epsilon^k, X^k)X^{k+1}, X^{k+1}\rangle + \langle \tau W(\epsilon^k, X^k)(X^k - X^{k+1}), X^{k+1}\rangle \\ &\quad + \frac{\tau\rho^k}{2}[\|X^{k+1} - X^k\|_F^2 + (\epsilon^{k+1} - \epsilon^k)^2] + \frac{\tau}{2}(\epsilon^{k+1})^2 \operatorname{Tr}(W(\epsilon^k, X^k)) \\ &\geq \bar{F}_1(X^{k+1}) + \tau\sum_{i=1}^m \sigma_i^2(X^{k+1})(\eta^k)_i + \langle \tau W(\epsilon^k, X^k)(X^k - X^{k+1}), X^{k+1}\rangle \\ &\quad + \frac{\tau\rho^k}{2}[\|X^{k+1} - X^k\|_F^2 + (\epsilon^{k+1} - \epsilon^k)^2] + \frac{\tau}{2}(\epsilon^{k+1})^2 \operatorname{Tr}(W(\epsilon^k, X^k)), \end{aligned} \tag{34}$$

where the inequality (34) follows from the Von Neumann trace inequality. Combining the above relationships with the definition of $\hat{F}^k$, we obtain that

$$2\hat{F}^k(\epsilon^k, X^k) \geq 2\hat{F}^k(\epsilon^{k+1}, X^{k+1}) + L_X^k + L_\epsilon^k, \tag{35}$$

where the definitions of L_X^k and L_ϵ^k are given by

$$\begin{aligned} L_X^k &:= \bar{F}_1(X^k) - \bar{F}_1(X^{k+1}) + \langle \tau W(\epsilon^k, X^k)(X^k - X^{k+1}), X^{k+1}\rangle, \\ L_\epsilon^k &:= \tau\epsilon^k \operatorname{Tr}(\epsilon^k, X^k)(\epsilon^k - \epsilon^{k+1}). \end{aligned}$$

It follows from the first equation in (29) and the definition of $\mathrm{D}_X\bar{F}_1(X)$ that

$$(\mathcal{A}(X^k)-\mathcal{A}(X^{k+1}))^{\mathrm{T}}(\mathcal{A}(X^{k+1})-b)+\langle\tau W(\epsilon^k,X^k)(X^k-X^{k+1}),X^k-X^{k+1}\rangle$$
$$+\langle\tau W(\epsilon^k,X^k)(X^k-X^{k+1}),X^{k+1}\rangle-\tau\rho^k\|X^{k+1}-X^k\|_F^2=0.$$

From the above equation, we deduce that

$$\begin{aligned}L_X^k &= \frac{1}{2}\|\mathcal{A}(X^k)-b\|_2^2-\frac{1}{2}\|\mathcal{A}(X^{k+1})-b\|_2^2-(\mathcal{A}(X^k)-\mathcal{A}(X^{k+1}))^{\mathrm{T}}(\mathcal{A}(X^{k+1})-b)\\ &\quad+\tau\rho^k\|X^k-X^{k+1}\|_F^2-\langle\tau W(\epsilon^k,X^k)(X^k-X^{k+1}),X^k-X^{k+1}\rangle\\ &\geq\frac{1}{2}\|\mathcal{A}(X^k)-\mathcal{A}(X^{k+1})\|_2^2+\tau(\rho^k-\max_{1\leq i\leq m}\eta_i^k)\|X^k-X^{k+1}\|_F^2.\end{aligned}$$

Similarly, from the second part of (29), we obtain

$$L_\epsilon^k=\tau\rho^k\frac{\epsilon^k}{\epsilon^{k+1}}(\epsilon^k-\epsilon^{k+1})^2=\tau(\rho^k+\mathrm{Tr}(W(\epsilon^k,X^k)))(\epsilon^k-\epsilon^{k+1})^2.$$

Combining these inequalities with (35), we have

$$\begin{aligned}\hat{F}^k(\epsilon^k,X^k)&\geq\hat{F}^k(\epsilon^{k+1},X^{k+1})+\frac{1}{4}\|\mathcal{A}(X^k)-\mathcal{A}(X^{k+1})\|_2^2\\ &\quad+\frac{1}{2}\tau(\rho^k-\max_{1\leq i\leq m}\eta_i^k)\|X^k-X^{k+1}\|_F^2+\frac{1}{2}\tau(\rho^k+\mathrm{Tr}(W(\epsilon^k,X^k)))(\epsilon^k-\epsilon^{k+1})^2,\end{aligned}$$

which implies that the inequality (33) holds at (ϵ^k,X^k) under the condition (32). ∎

The following theorem shows that any accumulation point of the sequence $\{(\epsilon^k,X^k)\}$ generated by algorithm Smajor satisfies the first-order necessary condition of (6).

THEOREM 4.3 *Let $\{(\epsilon^k,X^k)\}$ be the pairs of sequence generated by algorithm Smajor. The parameter ρ^k satisfies the condition* (32). *Then*

(1) *$\{\bar{F}(\epsilon^k,X^k)\}$ is a monotonically decreasing sequence satisfying the following inequality*:

$$\bar{F}(\epsilon^k,X^k)-\frac{\tau\rho^k}{2}[\|X^{k+1}-X^k\|_F^2+(\epsilon^{k+1}-\epsilon^k)^2]\geq\bar{F}(\epsilon^{k+1},X^{k+1}).$$

(2) *The sequence $\{(\epsilon^k,X^k)\}$ contained in the level set $\{(\epsilon,X):\bar{F}(\epsilon,X)\leq F(X_0)\}$ for some $X_0\in\mathfrak{R}^{m\times n}$ is bounded. Let $(\epsilon^\star,X^\star)$ be any accumulation point of the sequence $\{(\epsilon^k,X^k)\}$. Then $X^\star$ satisfies the first-order necessary condition of (6).*

Proof It follows from Lemmas 4.1 and 4.2 that

$$\begin{aligned}\bar{F}(\epsilon^{k+1},X^{k+1})+\frac{\tau\rho^k}{2}[\|X^{k+1}-X^k\|_F^2+(\epsilon^{k+1}-\epsilon^k)^2]\leq\hat{F}^k(\epsilon^{k+1},X^{k+1})&\leq\hat{F}^k(\epsilon^k,X^k)\\ &=\bar{F}(\epsilon^k,X^k).\end{aligned}$$

Combining the definition of $\bar{F}(\epsilon,X)$ with the assumption on (ϵ^k,X^k), we obtain

$$\sum_{i=1}^m(\sigma_i^2(X^k)+(\epsilon^k)^2)^{p/2}=\frac{p}{\tau}\bar{F}_2(\epsilon^k,X^k)\leq\frac{p}{\tau}\bar{F}(\epsilon^k,X^k)\leq\frac{pF(X_0)}{\tau}. \tag{36}$$

The inequality (36) implies that the sequence of $\{(\epsilon^k,X^k)\}$ is bounded. To prove the remaining part of this theorem, we assume that $(\epsilon^\star,X^\star)$ is an accumulation point of $\{(\epsilon^k,X^k)\}$. There exists an index set $\{k_s\} \subseteq \{0, 1, 2, \ldots, \}(k_0 := 0)$ such that $\lim_{s\to+\infty}(\epsilon^{k_s}, X^{k_s}) = (\epsilon^\star, X^\star)$. From the first part of the theorem, we obtain

$$\lim_{s\to+\infty} \sum_{i=0}^{s} \frac{\tau\rho^{k_i}}{2} \left(\|X^{k_i+1} - X^{k_i}\|_F^2 + (\epsilon^{k_i+1} - \epsilon^{k_i})^2\right)$$
$$\leq \liminf_{s\to+\infty} \left(\bar{F}(\epsilon^0, X^0) - \bar{F}(\epsilon^{k_s+1}, X^{k_s+1})\right) \leq \bar{F}(\epsilon^0, X^0) \leq F(X_0) < +\infty,$$

which implies that

$$\lim_{i\to\infty} \|X^{k_i+1} - X^{k_i}\|_F = 0, \quad \lim_{i\to\infty} |\epsilon^{k_i+1} - \epsilon^{k_i}| = 0.$$

Furthermore, from the second equation of (30) and the boundedness of $\{X^k\}$, we have

$$\lim_{i\to\infty} X^{k_i+1} = \lim_{i\to\infty} X^{k_i} = X^\star \quad \text{and} \quad \lim_{i\to\infty} \epsilon^{k_i} = \epsilon^\star = 0.$$

It follows from the first equation of (29) that

$$\langle \mathcal{A}^*(\mathcal{A}(X^{k_i+1}) - b), X^{k_i}\rangle + \langle \tau\rho^{k_i}(X^{k_i+1} - X^{k_i}), X^{k_i}\rangle = \langle -\tau W(\epsilon^{k_i}, X^{k_i})X^{k_i}, X^{k_i}\rangle.$$

Hence, when $i \to +\infty$, we have $\mathcal{A}(X^\star)^{\mathrm{T}}(\mathcal{A}(X^\star) - b) + \tau \sum_{j=1}^{m} \sigma_j^p(X^\star) = 0$. Namely, $X^\star$ satisfies the first-order necessary condition of (6). ■

Remark 3 From the above theorem, we define the termination criterion of our algorithm as follows:

$$e(\epsilon, X) := \max\{\mathcal{A}(X)^{\mathrm{T}}(\mathcal{A}(X) - b) + \tau\|X\|_p^p,\ \epsilon^2\} \leq \mathrm{tol}, \tag{37}$$

where $e(\epsilon, X)$ is the residual function and tol represents the corresponding tolerance.

5. Numerical experiments

In this section, we report numerical results for solving a series of matrix completion problems of the form:

$$\begin{aligned} \min \quad & \frac{1}{2}\|(X - X_R)_\Omega\|_2^2 + \frac{\tau}{p}\|X\|_p^p \\ \text{s.t.} \quad & X \in \mathfrak{R}^{m\times n}, \end{aligned} \tag{38}$$

where Ω is an index set of the original matrix X_R and $(X - X_R)_\Omega \in \mathfrak{R}^q$ is obtained from $(X - X_R)$ by selecting entries whose indices are in Ω. From (30), we present the update formulas of X and

ϵ for (38) as follows:

$$\mathcal{P}_\Omega(X^{k+1}) = \mathcal{P}_\Omega\left(\frac{\tau\rho^k}{1+\tau\rho^k}X^k - \frac{\tau}{1+\tau\rho^k}W(\epsilon^k, X^k)X^k\right) + \frac{1}{1+\tau\rho^k}\mathcal{P}_\Omega(X_R),$$
$$\mathcal{P}_{\Omega^c}(X^{k+1}) = \mathcal{P}_{\Omega^c}\left(X^k - \frac{1}{\rho^k}W(\epsilon^k, X^k)X^k\right), \tag{39}$$
$$\epsilon(\rho^k) = \frac{\rho^k}{\rho^k + \mathrm{Tr}(W(\epsilon^k, X^k))}\epsilon^k,$$

where Ω^c denotes all index pairs (i,j) not contained in Ω, that is, the complement of Ω. The operator $\mathcal{P}_\Omega$ is defined as follows:

$$(\mathcal{P}_\Omega(X))_{ij} = \begin{cases} 0 & \text{if } (i,j) \notin \Omega, \\ X_{ij} & \text{otherwise.} \end{cases}$$

All tests were implemented in MATLAB 2010a executed on a PC with Intel (R) Core(TM) 2 of 3.3 GHz CPU and 3.23 GB RAM under Ubuntu 12.04 LTS operating system.

Some technical details of our algorithm Smajor are summarized as follows:

- Generating the original matrix X_R. The matrix X_R was exactly of low-rank with the form $X_R = M_L M_R^{\mathrm{T}}$ as [25], where M_L, M_R are $m \times s$ and $n \times s$ matrices with i.i.d. standard Gaussian entries and s is the pre-defined rank. In this paper, we set $m = n$.
- Initial point choice. Our algorithm starts their iterations with the random initial iterate X^0 generated by MATLAB built-in function 'randn' and $\epsilon^0 = 10$.
- The stopping criterion. Our algorithm Smajor stops when the condition (37) is met, where $\mathrm{tol} = 10^{-4}$.
- The choice of the parameter τ. In order to recover a rank s matrix, we view τ as a target parameter and use the following continuation technique for τ: First, from Corollary 2.9, we set the parameter τ^0 as follows:

$$\left(\frac{p}{2(s+1)}\right)^{1-p} \|(X_R)_\Omega\|_2^{2-p} q_0^{p/2} \le \tau^0 \le \left(\frac{p}{2s}\right)^{1-p} \|(X_R)_\Omega\|_2^{2-p} q_0^{p/2}, \tag{40}$$

  where q_0 is the size of Ω. Then, the parameter τ is updated as [23, Section 4.1]

$$\tau^{k+1} = \max\{\gamma_\tau \tau^k, \bar{\tau}\}, \tag{41}$$

  where γ_τ controls the rate of reduction for τ^k and $\bar{\tau}$ is the target value of τ^k. Here, we set $\gamma_\tau = 0.1$ and $\bar{\tau} = 10^{-6}$.
- The choice of the parameter ρ^k. In this paper, the update rule of ρ^k is defined as follows:

$$\rho^k = \nu_\rho \,\mathrm{Tr}(W(\epsilon^k, X^k)),$$

  where ν_ρ controls the rate of reduction for ϵ^k according to the third equation in (39). Here, we set $\nu_\rho = \max(6.5 - 0.0001(n-1000)p, 5.5)$.
- Computing the SVD is a time-consuming task. Hence, the PROPACK package [19] will be used in our algorithm for computing the singular values and singular vectors directly, which is an efficient package for computing SVD of large and sparse or structured matrices. The SVD routines in PROPACK are based on the Lanczos bidiagonalization algorithm with partial reorthogonalization.

- For an $m \times n$ matrix of rank s, the degrees of freedom is $s(m+n-s)$. The degrees of freedom ratio (FR) and the sampling ratio (SR) are defined as follows:

$$\text{FR} := \frac{s(m+n-s)}{q_0}, \quad \text{SR} := \frac{q_0}{(mn)}.$$

To measure the accuracy of our solution X_S, we define the following relative error:

$$\text{Res} := \frac{\|X_S - X_R\|_F}{\|X_R\|_F}.$$

In all the tables below, the labels r, rr, it., time and Res denote the rank of X_R, the rank of the final iterate, the number of iterations, the total computing time in seconds and the residual at the final iterate, respectively.

5.1 *Numerical experiments on random matrix completion problems*

In this subsection, we begin by examining the behaviour of Smajor on random matrix completion problems and its sensitivity to the parameters p, SR and X^0.

5.1.1 *Sensitivity to the parameter p*

Now, we report the sensitivity of the Smajor approach to the involved parameter p. For succinctness, we only focus on random matrix completion problems with five groups of different sizes $n = 1000, 1500, 2000, 2500$ and 3000. In this test, we set $r = 10$ and $\text{SR} = 0.39$.

In Figure 1, we plot the computing time (in seconds) and the residual with respect to different choices of p. It is noticed that our algorithm is insensitive to the value of p. Hence, we set $p = 0.5$ in the following tests.

5.1.2 *Sensitivity to the parameter SR*

Table 1 presents the numerical results for random matrix completion problems under $\text{SR} = 0.39$ and SR$= 0.57$. In each test, we choose 10 groups of matrices with different sizes to test our algorithms.

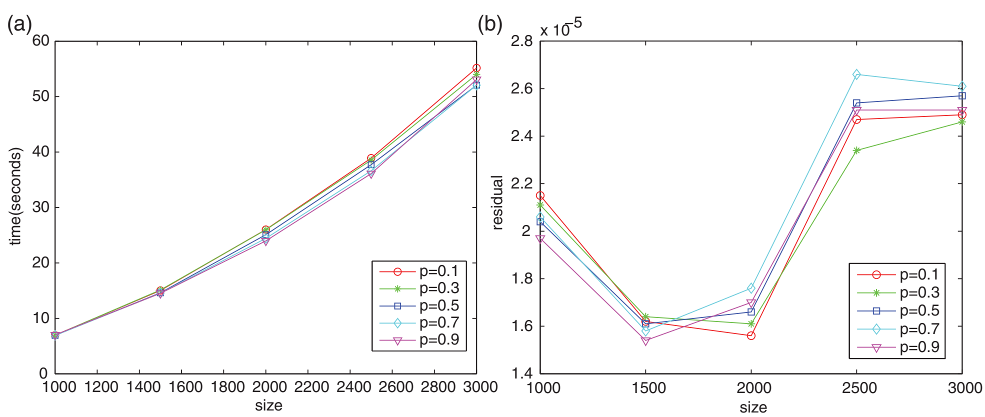

Figure 1. (a) The total computing time for different p. (b) The residual for different p.

Table 1. Numerical results for different SRs.

		SR = 0.39					SR = 0.57				
n	r	FR	rr	it.	time	Res	FR	rr	it.	time	Res
1200	5	0.021	5	47	7.48	7.73e−6	0.015	5	47	7.54	2.60e−6
1400	5	0.018	5	46	9.77	8.05e−6	0.013	5	46	9.69	2.51e−6
1600	5	0.016	5	46	12.66	8.41e−6	0.011	5	46	12.73	2.52e−6
1800	5	0.014	5	45	15.19	9.21e−6	0.010	5	45	15.34	2.39e−6
2000	5	0.013	5	44	18.94	9.80e−6	0.009	5	44	19.36	2.53e−6
2200	5	0.012	5	44	23.49	1.27e−5	0.008	5	44	23.51	2.44e−6
2400	5	0.011	5	43	27.01	1.39e−5	0.007	5	43	27.18	2.50e−6
2600	5	0.010	5	42	31.03	1.65e−5	0.007	5	42	32.07	2.42e−6
2800	5	0.009	5	42	36.26	1.96e−5	0.006	5	42	36.41	2.42e−6
3000	5	0.009	5	41	40.38	2.28e−5	0.006	5	41	40.60	2.50e−6

5.1.3 *Sensitivity to the initial point X^0*

Numerical results for random matrix completion problems under different initial points X^0 are displayed in Table 2, where A-time, Std-time, A-Res and Std-Res denote the average computing time, the standard deviation of the computing time, the average residual and the standard deviation of the residual, respectively. In each test, we generate 10 groups of different X^0 randomly. It follows from numerical results presented in Table 2 that our algorithm is also insensitive to the choice of X^0.

5.2 *Numerical experiments on different algorithms.*

In this section, we describe in detail all steps for implementing our proposed algorithm to solve random matrix completion problems. To compare the performance of finding the low-rank matrix solutions, some other competitive algorithms such as the singular value thresholding algorithm (SVT) [5] and the smoothing iterative reweighted least squares algorithm (sIRLS) [25] have also been demonstrated together with our algorithm Smajor. In each test, we consider a number of scenarios of examples with different n and r, where $n = 1000, 1500, 2000, 2500, 3000$ and $r = 5, 10, 20$. For simplicity, we set SR $= 0.39$ and run five times in each test problem.

5.2.1 *Comparison of algorithms for matrix completion problems.*

Now, we report numerical results from two groups of experiments. In the first group of test, we compare our algorithm Smajor with SVT and sIRLS under the assumption that the information of rank(X_R) is known in advance. In the second group, the rank of X_R is assumed to be unknown.

Before proceeding, we present the usage of the lower bound for estimating the true rank of the solution to (38) when the information of rank(X_R) is missing.

Table 2. Numerical results for different initial point X^0.

n	rr	it.	A-time	Std-time	A-Res	Std-Res
1000	10	48	6.20	1.20e−2	1.69e−5	9.82e−7
1500	10	46	12.79	4.09e−2	1.62e−5	3.91e−7
2000	10	44	22.15	6.03e−2	1.90e−5	2.79e−7
2500	10	43	33.64	5.71e−2	2.52e−5	2.58e−7
3000	10	41	46.49	4.60e−2	3.77e−5	4.03e−7

The procedure for estimating the true rank has the following steps:

Step 0: Initialize the rank $s := 1$ and set the step $s_{\rm inc} := 3$. Choose τ^0 to satisfy the condition (40). Set $\tau := \tau^0$, $k = 0$ and run the s-truncated SVD of X^k using the package PROPACK, that is,

$$X^k := \bar{U}^k \operatorname{Diag}(\sigma_1(X^k), \ldots, \sigma_s(X^k))(\bar{V}^k)^{\rm T}, \quad \bar{U}^k \in \mathfrak{R}^{m\times s} \quad \text{and} \quad \bar{V}^k \in \mathfrak{R}^{n\times s}.$$

Step 1: Calculate the lower bound $L_1(\tau^0, p)$ as follows:

$$L_1(\tau^0, p) := \left(\frac{\tau^0}{\sqrt{q_0}\|(X_R)_\Omega\|_2}\right)^{1/(1-p)}.$$

Step 2: Compute $W(\epsilon^k, X^k)$ and $\operatorname{Tr}(W(\epsilon^k, X^k))$ as follows:

$$W(\epsilon^k, X^k) := \bar{U}^k \operatorname{Diag}((\sigma_1^2(X^k) + (\epsilon^k)^2)^{p/2-1}, \ldots, (\sigma_s^2(X^k) + (\epsilon^k)^2)^{p/2-1})(\bar{U}^k)^{\rm T},$$

$$\operatorname{Tr}(W(\epsilon^k, X^k)) := \sum_{i=1}^{s}(\sigma_i^2(X^k) + (\epsilon^k)^2)^{p/2-1} + (m-s)(\epsilon^k)^{p-2}.$$

Step 3: Using (39) to obtain the iteration $(\epsilon^{k+1}, X^{k+1})$ and run the $(s + s_{\rm inc})$-truncated SVD of X^{k+1}, that is, $X^{k+1} := \bar{U}^{k+1} \operatorname{Diag}(\sigma_1(X^{k+1}), \ldots, \sigma_{(s+s_{\rm inc})}(X^{k+1}))(\bar{V}^{k+1})^{\rm T}$.
Step 4: Set the estimated rank s as follows:

$$s := \max\{i \in \{1, \ldots, (s + s_{\rm inc})\} \mid \sigma_i(X^{k+1}) > L_1(\tau^0, p)\}$$

and set

$$X^{k+1} := \hat{U}^{k+1} \operatorname{Diag}(\sigma_1(X^{k+1}), \ldots, \sigma_s(X^{k+1}))(\hat{V}^{k+1})^{\rm T},$$

where $\hat{U}^{k+1}$ and $\hat{V}^{k+1}$ are the sub-matrix of $\bar{U}^{k+1}$ and $\bar{V}^{k+1}$ whose columns are the first s columns of $\bar{U}^{k+1}$ and $\bar{V}^{k+1}$, respectively. Set $\bar{U}^{k+1} := \hat{U}^{k+1}$ and $\bar{V}^{k+1} := \hat{V}^{k+1}$.
Step 5: If the termination criterion (37) holds at $(\epsilon^{k+1}, X^{k+1})$, stop; otherwise, update the parameter τ as (41), choose τ^0 satisfy (40), set $k := k + 1$ and go to Step 1.

Table 3 reports the results for random matrix completion problems when the rank of X_R is known. Numerical results show that our proposed algorithm requires less than 48 iterations to solve all problems analyzed in this paper and the residuals are smaller than 8.57×10^{-5}. All three algorithms can solve the test problems successfully. From Table 3, Smajor performs better than other algorithms in terms of the total computing time and the corresponding residual.

Comparing the performance of all tested algorithms with or without the information of $\operatorname{rank}(X_R)$ on random matrix completion problems, we can observe from Table 4 that all three algorithms are more time consuming when $\operatorname{rank}(X_R)$ is unknown. Our algorithm can solve each test problem in less than 65 s and the residuals with less than 1.02×10^{-4}.

5.3 *Numerical experiments on Movielens 100k data sets*

In this subsection, we implement Smajor, sIRLS, IHT [24] and Optspace [18] to tackle the matrix completion problem whose data is taken from the well-known MovieLens data sets. In our numerical experiments, we consider MovieLens 100k data sets, which is available on the website http://www.grouplens.org/node/73. The MovieLens 100k data sets include four small data pairs (u1.base,u1.test), (u2.base,u2.test), (u3.base,u3.test) and (u4.base,u4.test). For each data set, we train Smajor sIRLS, IHT and Optspace on the training set and compare their performance on the corresponding test set. To measure the accuracy of the completed matrix, as in

Table 3. Numerical results for random matrix completion problems when the rank of X_R is known.

			Smajor				SVT				sIRLS			
n	r	FR	rr	it.	time	Res	rr	it.	time	Res	rr	it.	time	Res
1000	5	0.026	5	48	5.18	7.66e−6	5	50	6.06	3.28e−5	5	80	6.33	3.39e−4
1500	5	0.017	5	46	10.60	8.45e−6	5	47	12.97	3.08e−5	5	80	14.07	3.33e−4
2000	5	0.013	5	44	18.77	9.73e−6	5	43	25.03	2.95e−5	5	80	25.66	3.28e−4
2500	5	0.010	5	43	28.99	1.54e−5	5	41	34.03	2.79e−5	5	80	43.14	3.19e−4
3000	5	0.009	5	41	40.25	2.43e−5	5	39	43.96	3.07e−5	5	80	60.21	3.11e−4
1000	10	0.051	10	48	7.34	2.04e−5	10	54	8.49	3.60e−5	10	80	7.41	3.85e−4
1500	10	0.034	10	46	15.04	1.60e−5	10	47	18.75	3.21e−5	10	80	15.48	3.60e−4
2000	10	0.026	10	44	25.13	1.67e−5	10	43	31.61	2.92e−5	10	80	27.55	3.47e−4
2500	10	0.020	10	43	38.46	2.50e−5	10	41	43.70	2.57e−5	10	80	45.75	3.40e−4
3000	10	0.017	10	41	52.99	2.62e−5	10	39	68.28	2.72e−5	10	80	63.51	3.33e−4
1000	20	0.102	20	48	8.97	4.86e−5	20	67	12.75	1.01e−4	20	80	9.81	4.80e−4
1500	20	0.068	20	46	17.94	5.09e−5	20	56	24.87	9.24e−5	20	80	19.15	4.17e−4
2000	20	0.051	20	44	27.95	5.37e−5	20	51	42.73	9.22e−5	20	80	32.95	3.97e−4
2500	20	0.041	20	43	43.28	6.22e−5	20	47	59.84	8.07e−5	20	80	52.98	3.62e−4
3000	20	0.034	20	41	61.16	8.57e−5	20	45	82.15	8.83e−5	20	80	72.85	3.58e−4

Table 4. Numerical results for random matrix completion problems when the rank of X_R is unknown.

			Smajor				SVT				sIRLS			
n	r	FR	rr	it.	time	Res	rr	it.	time	Res	rr	it.	time	Res
1000	5	0.026	5	48	6.19	7.67e−6	5	54	6.80	5.07e−5	5	80	10.73	3.49e−4
1500	5	0.017	5	46	14.12	8.62e−6	5	50	14.36	4.83e−5	5	80	34.77	3.38e−4
2000	5	0.013	5	44	22.94	9.80e−6	5	46	27.78	4.81e−5	5	80	79.95	3.30e−4
2500	5	0.010	5	43	37.79	1.68e−5	5	44	38.46	4.66e−5	5	80	141.24	3.24e−4
3000	5	0.009	5	41	52.75	2.66e−5	5	42	59.02	5.23e−5	5	80	254.01	3.16e−4
1000	10	0.051	10	48	7.55	2.15e−5	10	64	9.89	1.17e−4	10	80	11.59	4.14e−4
1500	10	0.034	10	46	15.57	1.61e−5	10	56	19.79	1.11e−4	10	80	35.94	3.72e−4
2000	10	0.026	10	44	25.68	1.90e−5	10	51	33.97	9.70e−5	10	80	81.81	3.50e−4
2500	10	0.020	10	43	41.86	2.67e−5	10	48	47.31	9.52e−5	10	80	155.47	3.44e−4
3000	10	0.017	10	41	57.11	3.89e−5	10	44	69.57	9.84e−5	10	80	256.12	3.35e−4
1000	20	0.102	20	48	9.01	6.18e−5	20	69	13.45	1.22e−4	20	80	16.10	4.80e−4
1500	20	0.068	20	46	17.58	7.72e−5	20	58	27.62	1.16e−4	20	80	40.97	4.26e−4
2000	20	0.051	20	44	29.35	8.30e−5	20	52	47.27	1.05e−4	20	80	85.71	4.02e−4
2500	20	0.041	20	43	46.31	9.80e−5	20	49	64.69	1.04e−4	20	80	161.66	3.75e−4
3000	20	0.034	20	41	64.91	1.02e−4	20	46	93.27	1.07e−4	20	80	264.12	3.66e−4

Goldberg *et al*. [17], we define the mean absolute error (MAE) of the output matrix X generated by the algorithm as follows:

$$\text{MAE} := \frac{\sum_{(i,j)\in\Omega} |X_{ij} - M_{ij}|}{|\Omega|},$$

where Ω is the support set of M, that is, $\Omega := \{(i,j) : M_{ij} \neq 0\}$, and $|\Omega|$ is the cardinality of Ω. The matrices M_{ij} and X_{ij} are the original and computed ratings of movie j by user i, respectively. The normalized mean absolute error (NMAE) is used to measure the accuracy of the approximated completion X

$$\text{NMAE} := \frac{\text{MAE}}{r_{\max} - r_{\min}},$$

Table 5. NMAE for different algorithms.

Data sets	Smajor	sIRLS	IHT	Optspace
(u1.base, u1.test)	0.1924	0.1924	0.1925	0.1887
(u2.base, u2.test)	0.1871	0.1872	0.1884	0.1877
(u3.base, u3.test)	0.1883	0.1873	0.1874	0.1882
(u4.base, u4.test)	0.1888	0.1898	0.1897	0.1883

where $r_{\max}$ and $r_{\min}$ are upper and lower bounds for the ratings of movies. For the MovieLens 100k data sets, all ratings are scaled to the range $[1, 5]$. Hence, we have $r_{min} = 1$, $r_{max} = 5$ and set the rank of the unknown matrix to be equal to 5 in our test.

Table 5 shows that the NMAE for Smajor is as good as sIRLS, IHT and Optspace across different data sets.

6. Concluding remarks

In this paper, we propose a smoothing majorization method for solving the l_2^2-l_p^p matrix minimization problem, which can be viewed as an approximation optimization model for the low-rank recovery problems. The lower bound theory for nonzero singular values in local solutions of the l_2^2-l_p^p problem is established. Because our objective function is a non-smooth, non-Lipschitz and non-convex function, the smoothing techniques and the concept of majorization are applied to alleviate these difficulties. We design a smoothing majorization method for solving the l_2^2-l_p^p problem in which the smoothing parameter is treated as a variable and the optimal solutions of smoothing subproblems can be obtained explicitly. The convergence theorem indicates that any accumulation point of the sequence generated by our method satisfies the first-order necessary optimality condition of the l_2^2-l_p^p problem. Numerical experiments for solving random matrix completion problems show that the proposed algorithm can provide high-quality recovery solutions for the test problems.

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