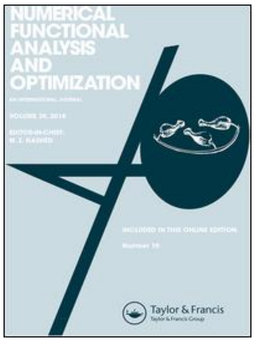

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A Decomposition Algorithm for the Sums of the Largest Eigenvalues

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ABSTRACT
In this article, we consider optimization problems in which the sums of the largest eigenvalues of symmetric matrices are involved. Considered as functions of a symmetric matrix, the eigenvalues are not smooth once the multiplicity of the function is not single; this brings some difficulties to solve. For this, the function of the sums of the largest eigenvalues with affine matrix-valued mappings is handled through the application of the $\mathcal{U}$-Lagrangian theory. Such theory extends the corresponding conclusions for the largest eigenvalue function in the literature. Inspired $\mathcal{VU}$-space decomposition, the first- and second-order derivatives of $\mathcal{U}$-Lagrangian in the space of decision variables R^m are proposed when some regular condition is satisfied. Under this condition, we can use the vectors of $\mathcal{V}$-space to generate an implicit function, from which a smooth trajectory tangent to $\mathcal{U}$ can be defined. Moreover, an algorithm framework with superlinear convergence can be presented. Finally, we provide an application about arbitrary eigenvalue which is usually a class of DC functions to verify the validity of our approach.

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1. Introduction

Optimization of the eigenvalues of real symmetric matrices is a distinguished research field in nonsmooth optimization theory, which arises in many applications such as optimization problems in physics, engineering, statistics, and finance. This spectrum extensively exists in composite materials [1], quantum computational chemistry [2], optimal system design [3, 4], shape optimization [5], pole placement in linear system theory, robotics, relaxations of combinatorial optimization problems [6], experimental design [7], and much more. A good survey on this topic is [8] with numerous references therein. One of the main hurdles with solving such problem numerically is that the eigenvalues, considered as functions of a symmetric

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matrix, fail to be smooth at those points where they are multiple, which are precisely the points of utmost interest.

The first-order algorithms for solving nonsmooth functions were developed and applied to eigenvalue optimization problems in the 1970s and 1980s. Simultaneously, various attempts were made to exploit the second-order theory for nonsmooth optimization problems. An introduction to second-order analysis was suggested by Overton [9] and developed further in [10] and [11]. Overton's method was also applied to some particular problems. In [12], Shapiro and Fan proposed a slightly different approach to the smooth minimization of the maximum eigenvalue and obtained local convergence.

The aim of this article is to present a general framework of second-order analysis for eigenvalue function. The main idea of Overton's approach can be described as follows. For the particular model problem,

$$(P)\ \inf_{x\in R^m} \lambda_1(A(x)),$$

where $\lambda_1(\cdot)$ denotes the maximum eigenvalue function, the mapping $A: R^m \to S_n$ denotes affine, which satisfies

$$A: R^m \ni x \mapsto A_0 + \mathscr{A}\cdot x,$$

and $\mathscr{A}$ is a linear operator satisfying $\mathscr{A}\cdot x = \sum_{i=1}^m A_i x_i, A_i, i = 0, ..., m$ are the given $n\times n$ symmetric matrices. Assume that the multiplicity r of $\lambda_1(A(x^*))$ at an optimal point x^* is given, then the approach consists of minimizing the maximum eigenvalue such that the constraint condition satisfies that its multiplicity is r. A local C^2-parametrization of (P) is then used for exploiting a successive quadratic programming method. In [13], the authors proposed the so-called the idea of $\mathcal{VU}$-decomposition. They displayed that while second-order information is needed in the first subspace($\mathcal{U}$), a first-order approximation is enough in the second one($\mathcal{V}$). This idea is further applied to maximum eigenvalue problems [14–17]. Here, our motivation is to extend the thought of the $\mathcal{U}$-Lagrangian to more common instance: the function of the sums of k largest eigenvalues. Given $A \in S_n$, the n eigenvalues of A are arranged in the decreasing order,

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

(We assume that the multiple eigenvalues are repeated based on their multiplicity.) Then, we can define

$$\lambda_k: A \in S_n \to \lambda_k(A) \quad (k\text{−th largest eigenvalue of } A).$$

We will be interested in another spectral function: the sum of the k largest eigenvalues of A

$$F_k(A) = \lambda_1(A) + \lambda_2(A) + \cdots + \lambda_k(A).$$

Hence, we will pay attention to the following model form

$$(P_1) \quad \min_{x \in R^m} f_k(x) := \lambda_1(A(x)) + \lambda_2(A(x)) + \cdots + \lambda_k(A(x)), 1 \leq k \leq n \tag{1.1}$$

and the constrained eigenvalue optimization problem

$$\begin{aligned} \min_{x \in R^m} \quad & c^T x \\ \text{s.t.} \quad & \sum_{i=1}^{k} \lambda_i(A(x)) \leq 0, \end{aligned} \tag{1.2}$$

where $\lambda_i(\cdot)$ is the eigenvalue function and here we denote these eigenvalues satisfy the decreasing order, that is, $\lambda_1(A(x)) \geq \lambda_2(A(x)) \geq \cdots \geq \lambda_n(A(x))$. (We assume that the multiple eigenvalues are repeated according to their multiplicity.) The mapping A is defined as in (P), which is also affine. We also use the notation $\lambda_{\max}$ for f_1 (the largest eigenvalue function); note that f_n is nothing else than the trace function. Programs of the form (1.2) may be transformed into (1.1) via exact penalization [18], even though it may be preferable to use the structure of (1.2) explicitly.

The question of the first-order sensitivity analysis of the sum of the largest eigenvalues has been solved independently by Overton and Womersley [10], as well as Hiriart-Urruty and Ye [19]. As far as the second-order analysis is concerned, Shapiro and Fan [12] have considered this problem from an algorithmic point of view, and their independent searches have result in two algorithms that adequately use the second-order information. Our contribution is mainly accounted as listed below: from a theoretical standpoint, on account of the $\mathcal{U}$-Lagrangian theory we deduce an explicit representation for a second-order operator in the case of the sum of the largest eigenvalues: $\mathcal{U}$-Hessian. In addition, our conditions are weaker than those in [17], because we study the subgradients g in $\partial f_k(x)$ not necessarily in the relative interior of $\partial f_k(x)$. Meanwhile, under the condition of transversality, an implicit function therein can be produced in view of the elements of $\mathcal{V}$-space, and a smooth track tangent to $\mathcal{U}$ as a byproduct can be defined. If this regular condition holds, f_k owns some form of second-order expansion along the corresponding smooth track. The consequent $\mathcal{VU}$-decomposition schemes execute a $\mathcal{U}$-Newton step to get access to superlinear convergence in the $\mathcal{U}$-subspace, followed by a $\mathcal{V}$ step in the $\mathcal{V}$-subspace. We also study the arbitrary eigenvalue function, which is not convex any more (the difference of two convex functions). Nevertheless, the technique that Oustry mentioned above cannot be directly extended to deal with the nonconvex case.

The rest of this work is organized as follows. Section 2 gives an overview of some useful results from the $\mathcal{U}$-Lagrangian theory. We present the main result in Section 3: the $\mathcal{U}$-Lagrangian function of the sums of the maximum eigenvalues F_k is C^∞ and convex. Meantime, we can explicitly compute the second-order derivatives. In addition, the first-order and second-order derivatives of the $\mathcal{U}$-Lagrangian function of the composite function $f_k = F_k \circ A$ in the space of decision variables R^m are studied, when the transversality condition holds. Then in Section 4, using this condition, the second-order development of the sum of the maximum eigenvalues on some smooth track that is tangent to the subspace $\mathcal{U}$ can be obtained. Moreover, we present an algorithm framework with superlinear convergence. In Section 5, we offer an application for arbitrary eigenvalue function which is a class of DC functions, and list its $\mathcal{VU}$ decomposition results. Finally, plans of our future research in eigenvalue optimization are suggested.

2. Preparation and preliminary results

2.1. Basic notation and terminology

First, we present the basic notation and term in the remainder parts. Let S_n be the space of $n \times n$ symmetric matrices, S_n^+ indicates the cone of $n \times n$ positive semidefinite symmetric matrices. A projection operator $\mathrm{proj}_{\mathcal{U}}: R^m \to \mathcal{U}$ projects R^m onto the subspace $\mathcal{U}$, and the canonical injection $\mathrm{proj}_{\mathcal{U}}^*: \mathcal{U} \mapsto R^m$ satisfies $\mathcal{U} \ni u \mapsto u \oplus 0 \in R^m$. Set $A \cdot B = \langle A, B \rangle := \mathrm{tr} AB$ be Fröbenius scalar product for $A, B \in S_n$, the sign tr stands for taking tracing arithmetic to some matrix and the Moore-Penrose inverse of A is marked as $A^\dagger$, the adjoint operator of the linear operator $\mathscr{A}: R^m \to S_n$ is recorded as $\mathscr{A}^\star: S_n \to R^m$. Let A lies on some submanifold C^∞-submanifold $\mathcal{M}(p,q)$ of S_n, defined by $\mathcal{M}(p,q) := \{A \in S_n : \lambda_p(A) > \lambda_{p+1}(A) = \cdots = \lambda_q(A) > \lambda_{q+1}(A)\}$, that is, $q - p \geq 1$ is the multiplicity of $\lambda_k(A)$ of A. The eigenvalue $\lambda_{p+1}(A)$ ranks first in the group of eigenvalues, which are equal to $\lambda_k(A)$ and is called the leading eigenvalue. The eigenspace associated with $\lambda_{p+1}, \ldots, \lambda_q$ is denoted $E_{p,q}(A)$, and the orthonormal basis of $E_{p,q}(A)$ is $Q_{p,q}(A) := Q_1(A), P_1(A)$ is an orthonormal basis associated with $\lambda_1, \ldots, \lambda_p$. The symbols $N_{\mathcal{M}}(A)$ and $T_{\mathcal{M}}(A)$ are respectively the normal and tangent spaces to the submanifold $\mathcal{M}$ at $A \in \mathcal{M}$, where $N_{\mathcal{M}}(A) = \{\Theta \in S_n \mid \langle \Theta, B - A \rangle \leq 0, \forall\ B \in \mathcal{M}\}$, $T_{\mathcal{M}}(A)$ is the polar cone of $N_{\mathcal{M}}(A)$, that is, $T_{\mathcal{M}}(A) = [N_{\mathcal{M}}(A)]^\circ = \{\Xi \in S_n \mid \langle \Theta, \Xi \rangle \leq 0, \forall\ \Theta \in N_{\mathcal{M}}(A)\}$. The rank of the matrix A is denoted by rank(A). The denotation ker Φ stands for the kernel of the linear operator Φ. The linear operator $DA(x)$ is the differential of the mapping $A(\cdot)$ at x which maps R^m into S_n written as $DA(x)z =$

$\sum_{i=1}^{m} z_i A_i(x)$, where $A_i(x) = \partial A(x)/\partial x_i$ represent $n \times n$ partial derivatives; when $A(x) = A_0 + \mathscr{A} \cdot x$ in (P), $DA(x)$ is reduced to $\mathscr{A}$. The relation $A \succ B$ and $A \preceq B$ respectively mean that, $A - B$ is positive definite and positive semidefinite. For other marks, one can refer to [20, 21].

We recall the $\mathcal{VU}$-theory developed in [13]. For a finite-valued convex function f, given a point $\bar{x} \in R^n$, take any subgradient g in $\partial f(\bar{x})$, which is the subdifferential of the function f at $\bar{x}$ in the convex sense, defined as $\partial f(\bar{x}) = \{s \in R^n | f(y) \geq f(\bar{x}) + \langle s, y - \bar{x}\rangle, \forall y \in R^n\}$. Then, for some given nonempty set S, we allow lin S indicate its linear hull, that is, the smallest linear subspace containing S. Denote aff S be the affine hull of the set S. Ref. [13] described that the orthogonal subspaces

$$\mathcal{V} := \mathrm{lin}(\partial f(\bar{x}) - g) \quad \text{and} \quad \mathcal{U} := \mathcal{V}^{\perp} \tag{2.1}$$

stated the $\mathcal{VU}$-space decomposition at $\bar{x}$. One subspace concentrates the smoothness of f about $\bar{x}$(appearing "U"-shaped and smooth) which is called the $\mathcal{U}$-subspace, and its orthogonal complement reflects all the nonsmoothness of f (behaving nonsmooth) that is named the $\mathcal{V}$-subspace. To represent f along the directions of $\mathcal{U}$, we need to find a smooth function satisfying it. The compact notation $\oplus$ is employed for this decomposition, and is composed $R^n = \mathcal{U} \oplus \mathcal{V}$, as well as

$$R^n \ni x = x_{\mathcal{U}} \oplus x_{\mathcal{V}} \in \mathcal{U} \times \mathcal{V}.$$

The relative interior $\mathrm{ri}\partial f(\bar{x})$ is the interior of $\partial f(\bar{x})$ relative to its affine hull, which is a manifold that is parallel to $\mathcal{V}$. Therefore,

$$\bar{g} \in \mathrm{ri}\partial f(\bar{x}) \quad \Rightarrow \quad \bar{g} + (B(0, \eta) \cap \mathcal{V}) \subset \partial f(\bar{x}) \quad \text{for some } \eta > 0,$$

where $B(0, \eta)$ denotes a ball in R^n centered at 0, with radius η. An extremely useful theorem on another two equivalent definition forms of $\mathcal{VU}$-decomposition follows, which is stated in [13].

Proposition 2.1. *[13, Definition 2.1,Proposition 2.2] For a finite-valued convex function f, $\bar{x} \in R^n$ and given $g \in \partial f(\bar{x})$, one has*

(i) $\mathcal{U}(\bar{x})$ is the subspace where $f'(\bar{x}; \cdot)$ is linear. In other words, $\mathcal{U}(\bar{x})$ is the subspace where $f(\bar{x} + \cdot)$ appears to be "smooth" at 0, that is,

$$\mathcal{U}(\bar{x}) = \{d \in R^n : f'(\bar{x}; d) = -f'(\bar{x}; -d)\},$$

and $\mathcal{V}(\bar{x}) = \mathcal{U}(\bar{x})^{\perp}$, where $f'(x; d) := \lim_{t \downarrow 0} \frac{f(x+td) - f(x)}{t} = \inf_{t \downarrow 0} \frac{f(x+td) - f(x)}{t}$ is the directional derivative of f at x in the direction d.

(ii) *take $g \in \mathrm{ri}\partial f(\bar{x})$ arbitrary, $\mathcal{V}(\bar{x})$ and $\mathcal{U}(\bar{x})$ are, respectively, the tangent and normal cones to $\partial f(\bar{x})$ at g, that is, $\mathcal{V}(\bar{x}) = T_{\partial f(\bar{x})}(g)$ and $\mathcal{U}(\bar{x}) = N_{\partial f(\bar{x})}(g)$.*

For a convex function f on R^n, taking any subgradient $\bar{g} = \bar{g}_{\mathcal{U}} \oplus \bar{g}_{\mathcal{V}} \in \partial f(\bar{x})$ with $\mathcal{V}$-component $\bar{g}_{\mathcal{V}}$, the $\mathcal{U}$-Lagrangian of f, relying on $\bar{g}_{\mathcal{V}}$, is named as

$$\mathcal{U}(\bar{x}) \ni u \mapsto L_{\mathcal{U}}(u) = \min_{v \in \mathcal{V}(\bar{x})} \{f(\bar{x} + u \oplus v) - \langle \bar{g}_{\mathcal{V}}, v \rangle_{\mathcal{V}}\}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ means a scalar product leaded in the subspace $\mathcal{V}$. The set of above $\mathcal{V}$-space optimal solutions is written as

$$w(u) = \{v \in \mathcal{V}(\bar{x}) : L_{\mathcal{U}}(u) = f(\bar{x} + u \oplus v) - \langle \bar{g}_{\mathcal{V}}, v \rangle_{\mathcal{V}}\}, \tag{2.3}$$

while the minimum in (2.2) exists. If $w(u)$ is not an empty set, the associated $\mathcal{U}$-Lagrangian is convex and smooth at $u = 0$, satisfying

$$\nabla L_{\mathcal{U}}(0) = \bar{g}_{\mathcal{U}} = g_{\mathcal{U}} \quad \textit{for all } g \in \partial f(\bar{x}).$$

The following theorem comes from the reference [13], where the function f is assumed to be convex for convenience.

Theorem 2.1. *[13, Theorem 3.2, Theorem 3.3, Corollary 3.5] If the function f is convex, then $L_{\mathcal{U}}$ is also convex. Furthermore, if $g \in ri\partial f(\bar{x})$, the corresponding solution set mapping w(u) is nonempty and the following results are satisfied.*

1. *Its subdifferential is*

$$\partial L_{\mathcal{U}}(u) = \mathrm{proj}_{\mathcal{U}(\bar{x})}\left[\partial f(\bar{x} + u \oplus v) \cap (g + \mathcal{U}(\bar{x}))\right], \tag{2.4}$$

where v is any element in w(u).

2. *When $u = 0$, one has $w(0) = \{0\}$ and $L_{\mathcal{U}}(0) = f(\bar{x})$. Besides, $L_{\mathcal{U}}$ is smooth at 0 and*

$$\nabla L_{\mathcal{U}}(0) = \mathrm{proj}_{\mathcal{U}(\bar{x})} g = g_{\mathcal{U}}. \tag{2.5}$$

3. *The set-valued mapping $u \mapsto \partial L_{\mathcal{U}}(u)$ is continuous at $u = 0$:*

$$\lim_{u \to 0} \partial L_{\mathcal{U}}(u) = \{\nabla L_{\mathcal{U}}(0)\}. \tag{2.6}$$

4. *For all $u \in \mathcal{U}(\bar{x})$, one obtains*

$$\partial f(\bar{x} + u \oplus v) \cap (g + \mathcal{U}(\bar{x})) = \partial L_{\mathcal{U}}(u) \oplus \{\mathrm{proj}_{\mathcal{V}(\bar{x})} g\} \textit{ for all } v \in w(u). \tag{2.7}$$

5. *The right-hand side of (2.7) is denoted by $\partial f(\bar{x} + u \oplus w(u)) \cap (g + \mathcal{U}(\bar{x}))$, the set-valued mapping $u \mapsto \partial f(\bar{x} + u \oplus w(u)) \cap (g + \mathcal{U}(\bar{x}))$ is continuous at 0:*

$$\lim_{u \to 0} \partial f(\bar{x} + u \oplus w(u)) \cap (g + \mathcal{U}(\bar{x})) = \{g\}. \tag{2.8}$$

6. *For every $u \in \mathcal{U}(\bar{x})$, w(u) is nonempty compact convex and*

$$\sup_{v \in w(u)} ||v|| = o(||u||), \tag{2.9}$$

and the set-valued mapping $u \mapsto w(u)$ is continuous at $u = 0$:

$$\lim_{u \to 0} w(u) = \{0\}. \tag{2.10}$$

Definition 2.1. [22, Definition 5.3.3] (Regular Value) We say that Z is a **regular value** of the mapping $\Phi: B(A^*, \delta_0) \to S_n$ with $\delta_0 > 0$ if for each $A \in \Phi^{-1}(Z) := \{\Xi \in B(A^*, \delta_0) : \Phi(\Xi) = Z\}$, the differential of Φ at $A \in B(A^*, \delta_0), D\Phi(A)$ is surjective.

Definition 2.2. (Local equation of a submanifold) Assume that 0 is a regular value of Φ and $\mathcal{M}$ is a manifold. If $\mathcal{M} \cap B(A^*, \delta_0) = \Phi^{-1}(0)$, we say that $\Phi(A) = 0$ is a **local equation** of the manifold $\mathcal{M}$ in $B(A^*, \delta)$.

Theorem 2.2. *[17, Theorem 3.4]*

(1) *Set $\Phi(A) = 0$ as a local equation of $\mathcal{M}$ in $B(A^*, \delta_0)$. Then, there must exist a positive number δ such that $\delta \leq \delta_0$ and a unique function*

$$v: T_{\mathcal{M}}(A^*) \cap B(0, \delta) \to N_{\mathcal{M}}(A^*)$$

satisfying

$$v = v(u), \quad \text{for all}(u, v) \in (T_{\mathcal{M}}(A^*), N_{\mathcal{M}}(A^*)).$$

Moreover, the function v is C^∞, and at $u = 0$

$$Dv(0) = 0. \tag{2.11}$$

(2) *set $A^* \in \mathcal{M}$; then there exists $\delta > 0$ satisfying*

$$\text{proj}_{N_{\mathcal{M}}(A^*)} d = v(\text{proj}_{T_{\mathcal{M}}(A^*)} d) \tag{2.12}$$

for all $d \in B(0, \delta)$ such that $A^* + d \in \mathcal{M}$.

3. $\mathcal{VU}$-space decomposition

Now we consider the function of the sums of k largest eigenvalues in matrix variable. Suppose the eigenvalues satisfy the decreasing order of $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ for $X \in S_n$. We view F_k defined by $F_k: S_n \to R$ in this section, that is, the variable X is matrix form in S_n, denoted by $F_k(X) = \sum_{i=1}^k \lambda_i(X)$. In addition, it is assumed that X is in the submanifold $\mathcal{M}(p, q)$. For the following part, the detailed structure of $\mathcal{VU}$-space for $F_k(X)$ is given. Meanwhile, in the variable space R^m, we denote $f_k(x) := (F_k \circ A)(x) = F_k(A(x))$. Under composite chain rule, similar to those for $F_k(X)$, the corresponding results for $f_k(x)$ can be obtained. Yet we will see that in order to obtain the existence of the $\mathcal{U}$-Hessian, an additional assumption is needed to satisfy. First, we give the following $\mathcal{VU}$-space. We need to notice that the details become significant, and give the detailed statements for completeness, although the proofs of the conclusions about the theoretical analysis are analogous to [17].

Theorem 3.1.

(1) *The subspaces $\mathcal{U}(X)$ and $\mathcal{V}(X)$ are respectively described by*

$$\mathcal{U}(X) = \left\{ U \in S_n : Q_1^T U Q_1 - \frac{1}{q-p} tr(Q_1^T U Q_1) I_{q-p} = 0 \right\} \tag{3.1}$$

and

$$\mathcal{V}(X) = \left\{ Q_1 Z Q_1^T : Z \in S_{q-p}^+, trZ = 0 \right\}. \tag{3.2}$$

(2) *If $k = q$, then F_k is a differentiable function, $\mathcal{U}(X) = S_n, \mathcal{V}(X) = \{0\}$.*

(3) *If the interior of $\partial F_k(X)$ is nonempty, that is, $int\ \partial F_k(X) \neq \emptyset$, then $\mathcal{U}(X) = \{0\}$, and $\mathcal{V}(X) = S_n$.*

Proof. According to Theorem 3.5 in Ref. [10], we have

$$\partial F_k(X) = \{U \in S_n : 0 \preceq \tilde{U} \preceq I, \text{tr } \tilde{U} = k - p, U = P_1 P_1^T + Q_1 \tilde{U} Q_1^T\}.$$

We take an element of $\partial F_k(X)$, for example, its center $C_k := P_1 P_1^T + \frac{k-p}{q-p} Q_1 Q_1^T$. By item (ii) of Proposition 2.1, $\mathcal{U}(X)$ is the normal cone to $\partial F_k(X)$ at C_k; then $D \in \mathcal{U}(X)$ means for all $U \in \partial F_k(X)$, it holds that

$$\begin{aligned} 0 \geq D \cdot (U - C_k) &= D \cdot \left[(P_1 P_1^T + Q_1 \tilde{U} Q_1^T) - \left(P_1 P_1^T + \frac{k-p}{q-p} Q_1 Q_1^T \right) \right] \\ &= D \cdot \left[Q_1 \left(\tilde{U} - \frac{k-p}{q-p} I_{q-p} \right) Q_1^T \right], \end{aligned}$$

which is in turn equivalent to

$$\max_{0 \preceq \tilde{U} \preceq I, \text{tr}\tilde{U} = k-p} Q_1^T D Q_1 \cdot \tilde{U} \leq \frac{k-p}{q-p} \text{tr } (Q_1^T D Q_1),$$

Divide by the positive number $k - p$ on both sides, that is,

$$\max_{0 \preceq \tilde{U} \preceq I, \text{tr}\tilde{U} = 1} Q_1^T D Q_1 \cdot \tilde{U} \leq \frac{1}{q-p} \text{tr } (Q_1^T D Q_1).$$

Because the support function of $\mathcal{S} := \{Z \mid 0 \preceq Z \preceq I, \text{tr } Z = 1\}$ is the maximum eigenvalue function [19], the inequality above is equivalent to

$$(q-p)\lambda_1(Q_1^T D Q_1) \leq \text{tr } (Q_1^T D Q_1).$$

For the above inequality, the inverse inequality sign is clearly established. So $Q_1^T D Q_1$ is a homothety:

$$Q_1^T D Q_1 - \frac{1}{q-p} \text{tr } (Q_1^T D Q_1) I_{q-p} = 0.$$

Therefore, we obtain (3.1), likewise, make use of $\mathcal{V}(X) = \mathcal{U}(X)^{\perp}$, (3.2) also holds. ■

Lemma 3.1.

(1) *At $X^* \in \mathcal{M}(p,q)$, the subspace $\mathcal{U}(X^*)$ and $\mathcal{V}(X^*)$ are respectively the tangent and normal spaces to the submanifold $\mathcal{M}(p,q)$ at X^*, that is,*

$$\mathcal{U}(X^*) = T_{\mathcal{M}(p,q)}(X^*),$$
$$\mathcal{V}(X^*) = N_{\mathcal{M}(p,q)}(X^*).$$

(2) *There exist $\delta > 0$ and a unique C^∞ mapping*

$$V : \mathcal{U}(X^*) \cap B(0,\delta) \to \mathcal{V}(X^*),$$

such that the mapping

$$\pi_{X^*} : \mathcal{U}(X^*) \cap B(0,\delta) \ni U(X^*) \mapsto X^* + U(X^*) \oplus V(U(X^*)) \tag{3.3}$$

is a C^∞ tangential parametrization of the submanifold $\mathcal{M}(p,q)$, where $D := U(X^*) \oplus V(U(X^*))$ and $X^* + D$ covers a whole neighborhood of X^* in $\mathcal{M}$.

Proof. The subspace $\mathcal{U}(X^*)$ is exactly the space of directions for which the directional derivative of F_k is linear (Prop. 2.2 (ii) of [13]), and so the normal sharpness of F_k relative to $\mathcal{M}(p,q)$ is equivalent to $T_{\mathcal{M}(p,q)}(X^*) = \mathcal{U}(X^*)$ (Note 2.9 (a) of [23]). The other equalities follow directly from the definition of $\mathcal{V}(X^*)$. The first assertion is satisfied.

For the second statement, by Theorem 2.2 and the first statement we can get the unique C^∞ mapping and obtain the tangential parametrization. ■

For decomposing S_n, there corresponds to the following decomposition of the space of vector variables R^m: at a point $x^* \in R^m$ such that $A(x^*) \in \mathcal{M}(p,q)$, that is, $x^* \in A^{-1}\mathcal{M}(p,q) := \mathcal{W}(p,q)$, we write

$$R^m = \mathcal{V}_{f_k}(x^*) \oplus \mathcal{U}_{f_k}(x^*). \tag{3.4}$$

So via the composition rule with a linear operator $\mathscr{A}$, we have the following.

Theorem 3.2. *Let $x^* \in R^m$, then*

$$\mathcal{V}_{f_k}(x^*) = \mathscr{A}^\star \mathcal{V}_{F_k}(A(x^*)), \tag{3.5}$$

and

$$\mathcal{U}_{f_k}(x^*) = \mathscr{A}^{-1}\mathcal{U}_{F_k}(A(x^*)). \tag{3.6}$$

Proof. First, applying the chain rule presented in [20], we have

$$\partial f_k(x^*) = \mathscr{A}^\star \partial F_k(A(x^*)), \tag{3.7}$$

and take the relative interior of the left and right-hand sides in the above formula (3.7) to obtain

$$\mathrm{ri}\partial f_k(x^*) = \mathscr{A}^\star \mathrm{ri}\partial F_k(A(x^*)), \tag{3.8}$$

which is holding according to the calculus rule in Proposition III.2.1.12 of [20]. Then, we do the affine operation on both side of the formula (3.7),

$$\mathrm{aff}\partial f_k(x^*) = \mathrm{aff}\left[\mathscr{A}^\star F_k(A(x^*))\right] = \mathscr{A}^\star \mathrm{aff}\partial F_k(A(x^*)),$$

where the second equation holds, because $\mathscr{A}$ is linear. According to the formula (2.1), one has (3.5) is holding. Because of $\mathcal{U}_{f_k}(x^*) = \mathcal{V}_{f_k}(x^*)^\perp$, and we can conclude

$$\begin{aligned}\mathcal{U}_{f_k}(x^*) &= \{u \in R^m : u \in \mathcal{V}_{f_k}(x^*)^\perp\} = \{u \in R^m : \mathscr{A}(u) \in \mathcal{V}_{F_k}(A(x^*))^\perp\} \\ &= \{u \in R^m : \mathscr{A}(u) \in \mathcal{U}_{F_k}(A(x^*))\},\end{aligned}$$

so (3.6) is satisfied. ■

For $X^* \in \mathcal{M}(p,q)$, take $G^* \in \partial F_k(X^*)$. Then, we can define the $\mathcal{U}$-Lagrangian function $L_{\mathcal{U}}(X^*, G^*; U)$ and the corresponding set of minimizers $W(U)$. At the same time, the similar form for vector variable $x^* \in \mathcal{W}(p,q)$ can also be obtained, such as the Lagrangian function $L_{\mathcal{U},f_k}(x^*, g^*; u)$ and the optimal sets $w(u)$.

Proposition 3.1. *The convex function $L_{\mathcal{U}}(X^*, G^*; \cdot)$ is differentiable at $U=0$ and its gradient is written as*

$$\nabla L_{\mathcal{U}}(X^*, G^*; 0) = \mathrm{proj}_{\mathcal{U}(X^*)} G^*. \tag{3.9}$$

Set $G^* \in \partial F_k(A(x^*))$ satisfying $g^* = \mathscr{A}^\star \cdot G^*$, then,

$$\nabla L_{\mathcal{U},f_k}(x^*, g^*; 0) = \left[\mathrm{proj}_{\mathcal{U}_{f_k}(x^*)} \circ \mathscr{A}^\star \circ \mathrm{proj}^*_{\mathcal{U}_{F_k}(A(x^*))}\right] \cdot \nabla L_{\mathcal{U},F_k}(A(x^*), G^*; 0), \tag{3.10}$$

where $\mathcal{U}_{f_k}(x^*)$ is described by (3.6).

Proof. We can directly apply the second statement of Theorem 2.1 in matrix form for the first development.

The second development can be obtained by adopting the composite chain rule in [20]. In fact, according to (3.8), there exists some G^*, which satisfies the hypothesis condition. Taking advantage of (2.5),

$$\begin{aligned}\nabla L_{\mathcal{U},f_k}(x^*,g^*;0) &= \operatorname{proj}_{\mathcal{U}_{f_k}(x^*)} g^* \\ &= \operatorname{proj}_{\mathcal{U}_{f_k}(x^*)}(\mathscr{A}^\star \cdot G^*) \\ &= \operatorname{proj}_{\mathcal{U}_{f_k}(x^*)} \circ \mathscr{A}^\star \cdot (\operatorname{proj}_{\mathcal{U}_{F_k}(A(x^*))} G^* \oplus \operatorname{proj}_{\mathcal{V}_{F_k}(A(x^*))} G^*) \\ &= \operatorname{proj}_{\mathcal{U}_{f_k}(x^*)} \circ \mathscr{A}^\star \cdot (\operatorname{proj}_{\mathcal{U}_{F_k}(A(x^*))} G^* \oplus 0),\end{aligned}$$

where the last equation holds using (3.5) and (3.2), we gain

$$\mathscr{A}^\star \cdot (0 \oplus \operatorname{proj}_{\mathcal{V}_{F_k}(A(x^*))} G^*) \in \mathcal{V}_{f_k}(x^*) = \mathcal{U}_{f_k}(x^*)^\perp.$$

At last, from (3.9),

$$\operatorname{proj}_{\mathcal{U}_{F_k}(A(x^*))} G^* \oplus 0 = \operatorname{proj}^*_{\mathcal{U}_{F_k}(A(x^*))} \nabla L_{\mathcal{U},F_k}(A(x^*), G^*, 0),$$

and (3.10) follows. ■

As pointed out by M. Overton through several works [9, 11], the sum of the maximum eigenvalue function enjoys a very specific structure in a neighborhood of a point $Y \in S_n$ belonging to the set $\mathcal{M}(p,q)$, the function F_k is smooth on $\mathcal{M}(p,q)$. More explicitly, the set $\mathcal{M}(p,q)$ is a submanifold of S_n and there exists a neighborhood Ω of Y in S_n such that the function $\mathcal{M}(p,q) \cap \Omega \ni X \mapsto F_k(X)$ is C^∞. To obtain a similar result for the function f_k, some precautions must be taken: the intersection of $\mathcal{M}(p,q)$ with the image of the mapping $R^m \ni x \mapsto f_k(x)$ may have some singularities. To avoid these situations, a transversality assumption can be made, which plays a role similar to that of constraint qualification conditions in nonlinear programming.

Definition 3.1. [24, Definition 4.70], [12, Section 2] (**Transversal Map**). Let $x^* \in R^m$; the C^∞ mapping $A(\cdot)$ is said to be transversal to the submanifold $\mathcal{M}$ at x^* if $A(x^*) \in \mathcal{M}$ and the range of $DA(x^*)$ is transversal to the subspace $T_{\mathcal{M}}(A(x^*))$, that is,

$$(\mathbf{T}) \qquad \ker DA(x^*)^\star \cap N_{\mathcal{M}}(A(x^*)) = \{0\},$$

where $DA(x): R^m \to S_n$ is the linear operator as the derivative of the mapping $A: R^m \to S_n$ at the point $x \in R^m$, $DA(x^*)^\star : S_n \to R^m$ is the adjoint operator of the linear operator $DA(x^*): R^m \to S_n$, $ker[D]$ is the kernel of the linear operator D.

Due to the properties of the orthogonal complement, **(T)** can be transformed into the following form

$$\operatorname{range} DA(x^*) + T_{\mathcal{M}}(A(x^*)) = S_n. \tag{3.11}$$

So under the condition **(T)**, $A^{-1}(\mathcal{M})$ is a C^∞-submanifold in a neighborhood of x^*. According to a local version of the transversal mapping theorem in Ref. [25], we have

$$T_{A^{-1}(\mathcal{M})}(x^*) = [DA(x^*)]^{-1} T_{\mathcal{M}}(A(x^*)). \tag{3.12}$$

Then in the variable space R^m, Lemma 3.1 above can be turned into

Theorem 3.3. *Suppose* **(T)** *is holding at x^* and take $g^* \in \partial f_k(x^*)$. Then,*

(1) *the subspaces $\mathcal{V}_{f_k}(x^*)$ and $\mathcal{U}_{f_k}(x^*)$ represent respectively the normal and tangent spaces to $\mathcal{W}(p, q)$ at x^*.*
(2) *there are some positive number ρ and a C^∞-mapping $v : \mathcal{U}_{f_k}(x^*) \cap B(0, \rho) \to \mathcal{V}_{f_k}(x^*)$ satisfying the mapping*

$$p_{x^*} : \mathcal{U}_{f_k}(x^*) \cap B(0, \rho) \ni u \mapsto x^* + u \oplus v(u) \tag{3.13}$$

is a tangential parametrization of $\mathcal{W}(p, q)$.

Proof. Because the transversal condition **(T)** is satisfied, then by (3.12),

$$T_{\mathcal{W}(p,q)}(x^*) = \mathscr{A}^{-1} T_{\mathcal{M}(p,q)}(A(x^*)),$$

which coincides with the right ride of (3.6). So $\mathcal{U}_{f_k}(x^*) = T_{\mathcal{W}(p,q)}(x^*)$ and $\mathcal{V}_{f_k}(x^*) = N_{\mathcal{W}(p,q)}(x^*)$. The first statement is done.

For the second statement, we can apply the first conclusion of Theorem 2.2 to obtain the C^∞ mapping $v : B(0, \rho) \subset \mathcal{U}_{f_k}(x^*) \to \mathcal{V}_{f_k}(x^*)$. The second part of Theorem 2.2 says that the mapping

$$p_{x^*} : \mathcal{U}_{f_k}(x^*) \cap B(0, \rho) \ni u \mapsto x^* + u \oplus v(u)$$

covers a whole neighborhood of x^* in $\mathcal{M}(p, q)$, which is a tangential parametrization of the manifold $\mathcal{W}(p, q)$ at x^*. We finish the proof. ■

The proof of the following theorem is almost word by word the extension of Thm. 4.12 in [17]. Due to the similarity, we only elaborate on the details of results for statement and omit those proof process. We should note that though the ideas about the results are similar, as the reader will find, the technical details become much more involved.

Theorem 3.4. *There exists $\rho > 0$ such that the $\mathcal{U}$-Lagrangian function $L_{\mathcal{U}}(X^*, G^*; \cdot)$ is C^∞ on $B(0, \rho) \subset \mathcal{U}(X^*)$. Specially,*

(1) *the gradient at $U \in B(0, \rho)$ is*

$$\begin{cases} \nabla L_{\mathcal{U}}(X^*, G^*; U) = \mathrm{proj}_{\mathcal{U}(X^*)} Q_1(\pi_{X^*}(U)) Z(U) Q_1(\pi_{X^*}(U))^T + \\ \qquad\qquad P_1(\pi_{X^*}(U)) P_1(\pi_{X^*}(U))^T, \end{cases} \tag{3.14}$$

where Z(U) is characterized by

$$\begin{cases} Z(U) \in \{Z \in S_{q-p}, tr\ Z = k-p\}, \\ Q_1(\pi_{X^*}(U))Z(U)Q_1(\pi_{X^*}(U))^T + P_1(\pi_{X^*}(U))P_1(\pi_{X^*}(U))^T - G^* \in \mathcal{U}(X^*); \end{cases} \tag{3.15}$$

(2) *the Hessian at $U = 0$ is*

$$\nabla^2 L_{\mathcal{U}}(X^*, G^*; 0) = \mathrm{proj}_{\mathcal{U}(X^*)} \circ H(X^*, G^*) \circ \mathrm{proj}_{\mathcal{U}(X^*)}{}^*, \tag{3.16}$$

where $H(X^*, G^*)$ is the symmetric operator defined by

$$\begin{aligned} H(X^*, G^*) \cdot Y = {} & (G^* - P_1(X^*)P_1(X^*)^T)Y\left[\lambda_{p+1}^* I_n - X^*\right]^\dagger \\ & + \left[\lambda_{p+1}^* I_n - X^*\right]^\dagger Y(G^* - P_1(X^*)P_1 \\ & (X^*)^T) + P_1(X^*)P_1(X^*)^T Y\left[\lambda_1^* I_n - X^*\right]^\dagger \\ & + \left[\lambda_1^* I_n - X^*\right]^\dagger Y P_1(X^*)P_1(X^*)^T, \end{aligned} \tag{3.17}$$

and here we assume at $X^*, \lambda_1(X^*) = \ldots = \lambda_p(X^*)$.

(3) *Moreover, in the variable space R^m, if (T) is satisfied at x^* and take $g^* \in \partial f_k(x^*)$. Then, the $\mathcal{U}$-Lagrangian function $L_{\mathcal{U},f_k}(x^*, g^*; \cdot)$ is C^∞ in a neighborhood of $u = 0$. Furthermore, at $u = 0$,*

$$\nabla L_{\mathcal{U},f_k}(x^*, g^*; 0) = \mathrm{proj}^*_{\mathcal{U}_{f_k}(x^*)} g^*, \tag{3.18}$$

and

$$\nabla^2 L_{\mathcal{U},f_k}(x^*, g^*; 0) = \mathrm{proj}_{\mathcal{U}_{f_k}(x^*)} \circ \mathscr{A}^\star \circ H(A(x^*), G^*) \circ \mathscr{A} \circ \mathrm{proj}^*_{\mathcal{U}_{f_k}(x^*)}, \tag{3.19}$$

where G^* is the unique subgradient of $\partial F_k(A(x^*))$ such that $g^* = \mathscr{A}^\star G^*$ and the operator $H(A(x^*), G^*)$ is given by (3.17) (with $A^* = A(x^*)$). It also can be turned into

$$\nabla^2 L_{\mathcal{U},f_k}(x^*, g^*; 0) = B(x^*)^* \circ \nabla^2 L_{\mathcal{U},F_k}(A(x^*), G^*; 0) \circ B(x^*), \tag{3.20}$$

where $B(x^*) = \mathrm{proj}_{\mathcal{U}_{F_k}(A(x^*))} \circ \mathscr{A} \circ \mathrm{proj}^*_{\mathcal{U}_{f_k}(x^*)}$ and $\mathcal{U}_{f_k}(x^*)$ is given by (3.6).

Then, a second-order development of F_k along the manifold $\mathcal{M}(p, q)$ can be derived.

Corollary 3.1. *Let $D \in S_n$ be such that $X^* + D \in \mathcal{M}(p, q)$ and $||D|| \to 0$, then*

$$F_k(X^* + D) = F_k(X^*) + G^* \cdot D + \frac{1}{2}\mathrm{proj}_{\mathcal{U}(X^*)} D \cdot \nabla^2 L_{\mathcal{U}}(X^*, G^*; 0)(\mathrm{proj}_{\mathcal{U}(X^*)} D) + o(||D||^2). \tag{3.21}$$

With the satisfactions of transversality (T), one has, in the space of decision variables R^m, the above formula becomes: for all $x^* + d \in \mathcal{W}(p, q)$ and $d \to 0$,

$$f_k(x^*+d)=f_k(x^*)+\langle g^*,d\rangle+\frac{1}{2}\langle \mathrm{proj}_{\mathcal{U}_{f_k}(x^*)}d,\nabla^2 L_{\mathcal{U},f_k}(x^*,g^*;0)\cdot(\mathrm{proj}_{\mathcal{U}_{f_k}(x^*)}d)\rangle_{\mathcal{U}_{f_k}(x^*)}+o(||d||^2). \tag{3.22}$$

Proof. We can find enough small D satisfying $X^*+D\in\mathcal{M}(p,q)$, make use of (2.12) and let $U=\mathrm{proj}_{\mathcal{U}(A^*)}D$, $V=V(U)=\mathrm{proj}_{\mathcal{U}(X^*)}D$. Exploiting Theorem 3.4 in the second-order Taylor decomposition, we obtain

$$\begin{aligned}L_{\mathcal{U}}(X^*,G^*;U)&=F_k(X^*)+\nabla L_{\mathcal{U}}(X^*,G^*;0)\cdot U\\&\quad+\frac{1}{2}U\cdot\nabla^2 L_{\mathcal{U}}(X^*,G^*;0)\cdot U+o(||U||^2)\\&=F_k(X^*+U\oplus V(U))-\mathrm{proj}_{\mathcal{V}(X^*)}G^*\cdot V(U).\end{aligned}$$

Recall (2.9), we have $V=O(||U||^2)=O(||D||^2)$, and the first development is holding. Finally, the second development is obtained by $D=\mathscr{A}(d)$ and by noticing that $v=O(||u||^2)=O(||d||^2)$, the proofs are finished. ■

4. Smooth trajectories and second-order algorithm

4.1. Smooth trajectories and second-order expansion

To recognize the tracks of $f_k(\cdot)$ in a smooth manner, we recur to the concept of the transversality condition (**T**). The condition (**T**) is always the necessary assumption for many scholars on optimization to get the high order advance (see [11, 12]), when they study the eigenvalue problems. And next we will explain how to get a smooth track that is tangent to $\mathcal{U}$ exploiting the structure of f_k. An implicit function theorem is also devoted to parameterize the track which is on account of u.

Denote $e_1(x),\ldots,e_n(x)$ be a set of orthonormal eigenvectors of $A(x)$ associated with the eigenvalues $\lambda_1(x),\ldots,\lambda_n(x)$. The mapping $A:R^m\to S_n$ are associated symmetric matrices, $A_l=\frac{\partial A(x)}{\partial x_l}, l=1,\ldots,m$, are the partial derivative for x_l and m-dimensional vectors

$$v_{ij}(x)=(e_i(x)^TA_1e_j(x),\ldots,e_i(x)^TA_me_j(x))^T,\quad i,j=1,\ldots,n.$$

Let

$$I_1:=\{p+1,p+2,\ldots,q\}\quad and\quad I_2:=\{(k,l)\in I_1\times I_1:k<l\},$$

thus, the number of elements in the set $|I_1|=q-p,|I_2|=\frac{(q-p)(q-p-1)}{2}$. Letting $\bar{U}$ be an orthonormal basis matrix for $\mathcal{U}$, and $\bar{V}=[\{v_{kl}(x)\}\cup\{v_{ii}(x)-v_{qq}(x)\}],p+1\le k<l\le q,i=p+1,\ldots,q-1,$ which is the corresponding basis matrix for $\mathcal{V}$. In the light of the transversality condition, for

the mapping A the following lemma is holding concerning the smooth manifold $\mathcal{W}(p,q)$.

Lemma 4.1. *[12, Theorem 2.2] Assume $A(x) \in \mathcal{W}(p,q)$. Then, the transversality mapping is satisfied if and only if the vectors $v_{kl}(x), v_{ii}(x) - v_{qq}(x)$ are linearly independent, $p+1 \le k < l \le q, i = p+1, ..., q-1$.*

The following theorem shows that, to present formulas for the $\mathcal{U}$-Lagrangian and its derivatives, and to obtain a second-order expansion for f_k, we need to identify smooth tracks that are in $\bar{x} + \bar{U}u + w(u)$.

Theorem 4.1. *Suppose the transversality condition holds at $\bar{x}$. For all sufficiently small u,*

(i) *the solution of the following nonlinear equations in variables (u, v)*

$$\begin{cases} e_i^T(\bar{x} + \bar{U}u + \bar{V}v)A(\bar{x} + \bar{U}u + \bar{V}v)e_i(\bar{x} + \bar{U}u + \bar{V}v) \\ \quad -e_{p+1}^T(\bar{x} + \bar{U}u + \bar{V}v)A(\bar{x} + \bar{U}u + \bar{V}v)e_{p+1}(\bar{x} + \bar{U}u + \bar{V}v) = 0, & i \in I_1 \\ e_k^T(\bar{x} + \bar{U}u + \bar{V}v)A(\bar{x} + \bar{U}u + \bar{V}v)e_l(\bar{x} + \bar{U}u + \bar{V}v) = 0, & (k,l) \in I_2 \end{cases} \tag{4.1}$$

is unique with $v = v(u)$, where $v\colon R^{\dim \mathcal{U}} \to R^{\dim \mathcal{V}}$ is C^2;

(ii) *the Jacobian of the track $\mathcal{X}(u) := \bar{x} + \bar{U}u + \bar{V}v(u)$ exists and is continuous,*

$$Jv(u) := -\bar{V}(V(u)^T\bar{V})^{-1}V(u)^T\bar{U}$$

and

$$J\mathcal{X}(u) := \bar{U} - \bar{V}(V(u)^T\bar{V})^{-1}V(u)^T\bar{U},$$

where

$$V(u) := [\nabla(e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u)) - e_{p+1}^T(\mathcal{X}(u))A(\mathcal{X}(u))e_{p+1}(\mathcal{X}(u)))_i \\ \cup \nabla(e_k^T(\mathcal{X}(u))A(\mathcal{X}(u))e_l(\mathcal{X}(u)))_{(k,l)}];$$

$\nabla(\cdot)$ is the differential about u;

(iii) *especially, $v(0) = 0, \mathcal{X}(0) = \bar{x}, V(0) = \bar{V}, Jv(0) = 0, J\mathcal{X}(0) = \bar{U}$;*

(iv) *$v(u) = O(|u|^2)$ and the track $\mathcal{X}(u)$ is tangential to $\mathcal{U}$ at $\mathcal{X}(0) = \bar{x}$.*

Proof. (i) We Differentiate the left hand of (4.1) in regard to v which gives

$$\begin{cases} \left[v_{ii}(\bar{x} + \bar{U}u + \bar{V}v) - v_{p+1,p+1}(\bar{x} + \bar{U}u + \bar{V}v)\right]^T\bar{V}, & i \in I_1; \\ v_{kl}^T(\bar{x} + \bar{U}u + \bar{V}v)\bar{V}, & (k,l) \in I_2. \end{cases}$$

The Jacobian matrix at $(u,v)=(0,0)$ is just $\bar{V}^T\bar{V}$, which is nondegenerate as a result of transversality qualification and Lemma 4.1. Likewise, there also exists a Jacobian concerning the variable u. Therefore, in the vicinity of $u=0$, we can find a C^1 function $v(u)$, according to the theorem of implicit function, such that $v(0)=0$.

(ii) Via the item (i), we know that $v(u)$ is C^1, so there must exist the Jacobian matrices $Jv(u)$ and $J\mathcal{X}(u)$, and they are continuous. We differentiate the following equations in reference to u

$$\begin{cases} e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u)) - e_{p+1}^T(\mathcal{X}(u))A(\mathcal{X}(u))e_{p+1}(\mathcal{X}(u)) = 0, & i \in I_1; \\ e_k^T(\mathcal{X}(u))A(\mathcal{X}(u))e_l(\mathcal{X}(u)) = 0, & (k,l) \in I_2. \end{cases}$$

and one has

$$\begin{cases} \left[v_{ii}(\mathcal{X}(u)) - v_{p+1,p+1}(\mathcal{X}(u))\right]^T J(\mathcal{X}(u)) = 0, & i \in I_1; \\ v_{kl}^T(\mathcal{X}(u))J(\mathcal{X}(u)) = 0, & (k,l) \in I_2, \end{cases}$$

or in equivalent form of matrix, $V(u)^T J(\mathcal{X}(u)) = 0$. Utilizing the representation $J(\mathcal{X}(u)) = \bar{U} + \bar{V}Jv(u)$, one can conclude that

$$V(u)^T(\bar{U} + \bar{V}Jv(u)) = 0.$$

Because $V(u)$ is continuous, $V(u)^T\bar{V}$ is nondegenerate, that is, $V(u)^T\bar{V}$ is reversible. Hence,

$$Jv(u) = -(V(u)^T\bar{V})^{-1}V(u)^T\bar{U}.$$

Additionally, both of $e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u))$ and $e_k^T(\mathcal{X}(u))A(\mathcal{X}(u))$ $e_l(\mathcal{X}(u))$ are C^1, so $V(u)$ is C^1. thereupon $Jv(u)$ is C^1. Thus, $\mathcal{X}(u)$ and $v(u)$ are C^2.

(iii) By the definition of $\mathcal{VU}$ space decomposition, we have $\mathcal{V}\perp\mathcal{U}$. Consequently, $\bar{V}^T\bar{U}=0$. Then $Jv(0)=0$ and $J\mathcal{X}(0)=\bar{U}$.

(iv) We use $Jv(0)=0$ with the help of the Taylor expansion to obtain

$$v(u) = v(0) + Jv(0)u + o(||u||) = 0 + 0\cdot u + o(||u||) = o(||u||).$$

thus, the track is

$$\mathcal{X}(u) = \mathcal{X}(0) + J\mathcal{X}(0)u + o(||u||) = \bar{x} + \bar{U}u + o(||u||),$$

that is to say, $\mathcal{X}(u)$ is tangential to $\mathcal{U}$ at $\mathcal{X}(0)=\bar{x}$. ■

For the sake of representing the gradient and Hessian of $L_{\mathcal{U}}$ which can be unions of the gradients and Hessians of the eigenvalue functions, it is necessary to give an explicit expression for the combination coefficients. Proposition 4.1 below reveals these multipliers are smooth functions of u.

Proposition 4.1. *Assume the transversality mapping (T) satisfies at $\bar{x}$ relative to $\bar{g} \in \partial f_k(\bar{x})$ with the track $\mathcal{X}(u) = \bar{x} + \bar{U}u + \bar{V}v(u)$, then, for every sufficiently small u, the solution of the linear system concerning the variables $\alpha \in R^{\frac{(q-p)(q-p+1)}{2}}$*

$$\begin{cases} \bar{V}^T\left[\sum_{i\in I_1}\alpha_i v_{ii}(\mathcal{X}(u)) + \sum_{(k,l)\in I_2}\alpha_j v_{kl}(\mathcal{X}(u))\right] = \bar{V}^T\bar{g} \in R^{m-1}, \\ \sum_{i\in I_1}\alpha_i = 1, \end{cases}$$

exists and is unique, $\alpha = \alpha(u)$, represented as

$$\begin{aligned} \{\alpha_j(u)\} &= (\bar{V}^T V(u))^{-1}\bar{V}^T(\bar{g} - v_{p+1,p+1}(\mathcal{X}(u)), \\ \alpha_{p+1}(u) &= 1 - \sum_{p+1\neq j\in I_1}\alpha_j(u). \end{aligned}$$

Especially, $\alpha(0) = \bar{\alpha}$, where $\bar{\alpha}$ is the solution of the linear system about the variables α

$$\begin{cases} \sum_{i\in I_1}\alpha_i v_{ii}(\bar{x}) + \sum_{(k,l)\in I_2}\alpha_j v_{kl}(\bar{x}) = \bar{g} \in R^{m-1}, \\ \sum_{i\in I_1}\alpha_i = 1. \end{cases}$$

Moreover, the following holds:

$$\sum_{i\in I_1}\alpha_j(u)v_{ii}(\mathcal{X}(u))J\alpha_j(u) + \sum_{(k,l)\in I_2}\alpha_j(u)v_{kl}(\mathcal{X}(u))J\alpha_j(u) = -V(u)(\bar{V}V(u))^{-1}\bar{V}^T M(u)J\mathcal{X}(u).$$

Proof. Toward the linear system about the variable $\alpha(u)$ in the condition, we can conclude $\alpha_1(u) = 1 - \sum_{i\neq p+1,\, i\in I_1}\alpha_i(u)$ from the second equation and replume items in the first one, to acquire

$$\begin{aligned} &\bar{V}^\top\left(\sum_{i\neq p+1,\, i\in I_1}\alpha_i(u)v_{ii}(\mathcal{X}(u)) + \sum_{(k,l)\in I_2}\alpha_j(u)v_{kl}(\mathcal{X}(u))\right) \\ &= -\bar{V}^\top\left(\left(1 - \sum_{i\neq p+1,\, i\in I_1}\alpha_i(u))v_{11}(\mathcal{X}(u)\right) - \bar{g}\right). \end{aligned}$$

Substituting the same $\bar{V}^\top \sum_{i\neq p+1,\, i\in I_1}\alpha_i(u)v_{ii}(\mathcal{X}(u))$ for both sides of the above expression, and combining with the definition of $V(u)$ in Theorem 4.1 one has

$$(\bar{V}^\top V(u))\left[\{\alpha_i(u)\}_{i\neq p+1}\cup\{\alpha_j(u)\}\right]=-\bar{V}^\top(v_{p+1,p+1}(\mathcal{X}(u))-\bar{g}),$$

and the result holds. Especially, when $u=0$, in view of $\mathcal{X}(0)=\bar{x},\alpha=\bar{\alpha}$ therein is the unique solution of the above linear system; hence, $\alpha(0)=\bar{\alpha}$. ■

These two smooth functions $\mathcal{X}(u)$ and $\alpha(u)$ in above theorems provide us with a constructive way to express the $\mathcal{U}$-Hessian of f_k in the light of the Hessians of $e_i(x)^T A(x)e_i(x)$. A second-order expansion for f_k can be obtained, for the gradients and the multipliers are both C^1. Now we will present some detailed forms of $\mathcal{U}$-Lagrangian and its derivatives. We can prove the existence of a $\mathcal{U}$-Hessian if the transversality condition holds.

Theorem 4.2. *Assume the transversality mapping (T) satisfies at $\bar{x},\bar{g}\in\partial f_k(\bar{x})$, the corresponding track $\mathcal{X}(u)$ comes from Theorem 4.1 and basic matrix $\bar{V}$, and the $\mathcal{U}$-Lagrangian function is deduced from (2.2), and the multiplier functions $\alpha_i(u)$ results from Proposition 4.1. Then for all sufficiently small u, the following statements hold:*

(i) *the vector $\bar{V}v(u)$ is in the set w(u), equivalently, the trajectory vector $\mathcal{X}(u)$ belongs to the set $\bar{x}+\bar{U}u+w(u)$;*
(ii) *the expression of the $\mathcal{U}$-Lagrangian function can be written as*

$$\begin{aligned}L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}}) &= f_k(\mathcal{X}(u))-\bar{g}^T\bar{V}v(u)\\ &= \sum_{i=1}^{p}e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u))+\frac{k-p}{q-p}\sum_{i=p+1}^{q}e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u))-\bar{g}^T\bar{V}v(u)\end{aligned}$$

and

$$e_k^T(\mathcal{X}(u))A(\mathcal{X}(u))e_l(\mathcal{X}(u))=0;$$

(iii) *the differential of $L_{\mathcal{U}}$ is given by*

$$\nabla L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}})=\bar{U}^T g(u), \tag{4.2}$$

where g(u) is named as

$$g(u)=\sum_{i\in I_1}\alpha_j v_{ii}(\mathcal{X}(u))+\sum_{(k,l)\in I_2}\alpha_j v_{kl}(\mathcal{X}(u));$$

(iv) *the representation of the Hessian of $L_{\mathcal{U}}$ is*

$$\nabla^2 L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}})=J\mathcal{X}(u)^T M(u)J\mathcal{X}(u),$$

where the $n\times n$ matrix function M(u) is written as

$$\begin{aligned}M(u)=&\sum_{i\in I_1}\alpha_j(u)\nabla^2\left[e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u))\right]\\ &+\sum_{(k,l)\in I_2}\alpha_j(u)\nabla^2\left[e_k^T(\mathcal{X}(u))A(\mathcal{X}(u))e_l(\mathcal{X}(u))\right];\end{aligned}$$

(v) *specifically,* $L_{\mathcal{U}}(0;\bar{g}_{\mathcal{V}}) = f_k(\bar{x})$, *and the concrete expression of the* $\mathcal{U}$*-gradient and* $\mathcal{U}$*-Hessian at* $\bar{x}$ *are respectively*

$$\nabla L_{\mathcal{U}}(0;\bar{g}_{\mathcal{V}}) = \bar{U}^T\bar{g} = \bar{U}^T g, \quad \textit{for all } g \in \partial f_k(\bar{x})$$

and

$$\nabla^2 L_{\mathcal{U}}(0;\bar{g}_{\mathcal{V}}) = \bar{U}^T M(0)\bar{U}.$$

Proof. (i) It is easy to get the assertion by the definition of $w(u)$ and the transversality mapping (T).

(ii) Combing $v(u)$ in item (i) of Theorem 4.1, with the transversality mapping (T), one has

$$\begin{aligned} f_k(\mathcal{X}(u)) &= \max\{\sum_{i=1}^{k} e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u))\} \\ &= \sum_{i=1}^{p} e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u)) + \frac{k-p}{q-p}\sum_{i=p+1}^{q} e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u)), \end{aligned}$$

thus,

$$\begin{aligned} L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}}) &= f_k(\mathcal{X}(u)) - \bar{g}^T\bar{V}v(u) \\ &= \sum_{i=1}^{p} e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u)) + \frac{k-p}{q-p}\sum_{i=p+1}^{q} e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u)) - \bar{g}^T\bar{V}v(u). \end{aligned}$$

(iii) According to the chain rule, we differentiate the following equation system in regard to u,

$$\begin{cases} L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}}) = \sum_{i=1}^{p} e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u)) + \frac{k-p}{q-p}\sum_{i=p+1}^{q} e_i^T(\mathcal{X}(u))A(\mathcal{X}(u))e_i(\mathcal{X}(u)) - \bar{g}^T\bar{V}v(u), & i \in I_1 \\ e_k^T(\mathcal{X}(u))A(\mathcal{X}(u))e_l(\mathcal{X}(u)) = 0, & (k,l) \in I_2. \end{cases}$$

This gives

$$\begin{cases} \nabla L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}}) = J\mathcal{X}(u)^T v_{ii}(\mathcal{X}(u)) - Jv(u)^T\bar{V}^T\bar{g}, & i \in I_1 \\ J\mathcal{X}(u)^T v_{kl}(\mathcal{X}(u)) = 0, & (k,l) \in I_2. \end{cases}$$

We attempt to multiply the appropriate multiplicators $\alpha_j(u)$, on both sides of the above equations, respectively, and add these formulas in the index, together with the fact that $\sum_j \alpha_j(u) = 1$. So we conclude

$$\nabla L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}}) = J\mathcal{X}(u)^T g(u) - Jv(u)^T\bar{V}^T\bar{g},$$

where $g(u) = \sum_{i\in I_1} \alpha_j v_{ii}(\mathcal{X}(u)) + \sum_{(k,l)\in I_2} \alpha_j v_{kl}(\mathcal{X}(u))$. Taking the transpose to $J\mathcal{X}(u)$ given in Theorem 4.1 (ii), it is obtained that

$$\nabla L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}}) = \bar{U}^T g(u) - Jv(u)^T \bar{V}^T (g(u) - \bar{g}).$$

This deduces the requested consequence along with the fact that $\bar{V}^T\,(g(u) - \bar{g}) = 0$.

(iv) We take the differential to the Equation (4.2) in regard to u, and obtain

$$\nabla^2 L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}}) = \bar{U}^T M(u) J\mathcal{X}(u) + \bar{U}^T \Bigg[\sum_{i\in I_1} \alpha_j(u) v_{ii}(\mathcal{X}(u)) J\alpha_j(u)$$
$$+ \sum_{(k,l)\in I_2} \alpha_j(u) v_{kl}(\mathcal{X}(u)) J\alpha_j(u) \Bigg],$$

where

$$M(u) = \sum_{i\in I_1} \alpha_j(u) \nabla^2 \left[e_i^T(\mathcal{X}(u)) A(\mathcal{X}(u)) e_i(\mathcal{X}(u)) \right]$$
$$+ \sum_{(k,l)\in I_2} \alpha_j(u) \nabla^2 \left[e_k^T(\mathcal{X}(u)) A(\mathcal{X}(u)) e_l(\mathcal{X}(u)) \right].$$

According to Proposition 4.1, this proves

$$\sum_{i\in I_1} \alpha_j(u) v_{ii}(\mathcal{X}(u)) J\alpha_j(u) + \sum_{(k,l)\in I_2} \alpha_j(u) v_{kl}(\mathcal{X}(u)) J\alpha_j(u)$$
$$= -V(u)(\bar{V}V(u))^{-1} \bar{V}^T M(u) J\mathcal{X}(u).$$

Accordingly, in view of the expression of $Jv(u)$ and $J\mathcal{X}(u)$ in Theorem 4.1,

$$\begin{aligned}
\nabla^2 L_{\mathcal{U}}(u;\bar{g}_{\mathcal{V}}) &= \bar{U}^T M(u) J\mathcal{X}(u) - \bar{U}^T V(u)(\bar{V}V(u))^{-1} \bar{V}^T M(u) J\mathcal{X}(u) \\
&= \bar{U}^T M(u) J\mathcal{X}(u) + Jv(u)^T \bar{V}^T M(u) J\mathcal{X}(u) \\
&= \left[\bar{U}^T + Jv(u)^T \bar{V}^T \right] M(u) J\mathcal{X}(u) \\
&= J\mathcal{X}(u)^T M(u) J\mathcal{X}(u).
\end{aligned}$$

(v) Since $v(0) = 0, \mathcal{X}(0) = \bar{x}, \mathcal{X}(0) = \bar{U}, \sum \alpha_j(0) = 1$,

$$\begin{aligned}
L_{\mathcal{U}}(0;\bar{g}_{\mathcal{V}}) &= f_k(\bar{x}) - \langle \bar{g}_{\mathcal{V}}, v(0) \rangle = f_k(\bar{x}) - \langle \bar{g}_{\mathcal{V}}, 0 \rangle = \lambda_1(\bar{x}) \\
\nabla L_{\mathcal{U}}(0;\bar{g}_{\mathcal{V}}) &= \bar{U}^T g(0) = \bar{U}^T \bar{g} \\
\nabla^2 L_{\mathcal{U}}(0;\bar{g}_{\mathcal{V}}) &= J\mathcal{X}(0)^T M(0) J\mathcal{X}(0) = \bar{U}^T M(0) \bar{U}.
\end{aligned}$$

We have finished the proofs. ■

The function of sum of the maximum eigenvalue has good configurable properties which consider the structure of smooth tracks on which the function is C^2. the following conclusion is true:

Theorem 4.3. *Assume the transversality mapping (T) satisfies, the corresponding smooth track $\mathcal{X}(u)$ comes from Theorem 4.1 $\mathcal{X}(u) = \bar{x} + \bar{U}u + \bar{V}v(u)$. Then for every sufficiently small u,*

$$f_k(\mathcal{X}(u)) = f_k(\bar{x}) + \bar{g}_{\mathcal{U}}^T u + \bar{g}^T \bar{V} v(u) + \frac{1}{2} u^T \nabla^2 L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}) u + o(||u||^2). \quad (4.3)$$

Or the above formula can be equivalently turned into

$$f_k(\mathcal{X}(u)) = f_k(\bar{x}) + \bar{g}^T(\mathcal{X}(u) - \bar{x}) + \frac{1}{2}(\mathcal{X}(u) - \bar{x})^T \bar{U} \nabla^2 L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}) \bar{U}(\mathcal{X}(u) - \bar{x}) + o(||(\mathcal{X}(u) - \bar{x})||^2).$$

Proof. Because $L_{\mathcal{U}} \in C^2$, one gets

$$\begin{aligned} L_{\mathcal{U}}(u; \bar{g}_{\mathcal{V}}) &= L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}) + \langle \nabla L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}), u \rangle_{\mathcal{U}} + \frac{1}{2} u^T \nabla^2 L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}) u + o(||u||^2) \\ &= f_k(\bar{x}) + \langle \bar{g}_{\mathcal{U}}, u \rangle_{\mathcal{U}} + \frac{1}{2} u^T \nabla^2 L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}) u + o(||u||^2), \end{aligned} \quad (4.4)$$

where $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ denotes a scalar product induced in the subspace $\mathcal{U}$. By the structure of $L_{\mathcal{U}}$, one concludes

$$L_{\mathcal{U}}(u; \bar{g}_{\mathcal{V}}) = f_k(\mathcal{X}(u)) - \langle \bar{g}_{\mathcal{V}}, v(u) \rangle_{\mathcal{V}}.$$

As a result,

$$\begin{aligned} f_k(\mathcal{X}(u)) &= L_{\mathcal{U}}(u; \bar{g}_{\mathcal{V}}) + \langle \bar{g}_{\mathcal{V}}, v(u) \rangle_{\mathcal{V}} \\ &= f_k(\bar{x}) + \langle \bar{g}_{\mathcal{U}}, u \rangle_{\mathcal{V}} + \langle \bar{g}_{\mathcal{V}}, v(u) \rangle_{\mathcal{V}} + \frac{1}{2} u^T \nabla^2 L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}) u + o(||u||^2) \\ &= f_k(\bar{x}) + \langle \bar{g}, u \oplus v(u) \rangle + \frac{1}{2} u^T \nabla^2 L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}) u + o(||u||^2) \\ &= f_k(\bar{x}) + \langle \mathcal{X}(u) - \bar{x} \rangle + \frac{1}{2}(\mathcal{X}(u) - \bar{x})^T \bar{U} \nabla^2 L_{\mathcal{U}}(0; \bar{g}_{\mathcal{V}}) \bar{U}^T (\mathcal{X}(u) - \bar{x}) + o(||\mathcal{X}(u) - \bar{x}||^2). \end{aligned}$$

We finish the proof. ■

Remark 4.1. In [17], the existence of a $\mathcal{U}$-Hessian is proved under the condition that $\bar{g} \in ri\ \partial f(\bar{x})$. However, the above theorems show that, less restrictive conditions $\bar{g} \in \partial f(\bar{x})$ are considered which also ensure the existence of a $\mathcal{U}$-Hessian for the function of sum of maximum eigenvalue f_k.

We find that, second-order $\mathcal{U}$-derivatives, as described above, is helpful for appointing second-order forms for f_k. The corresponding Hessian of $L_{\mathcal{U}}$ at $u = 0$ is named by a $\mathcal{U}$-Hessian for f_k at $\bar{x}$ and labeled with $H_{\mathcal{U}}(f_k(\bar{x})) := \nabla^2 L_{\mathcal{U}}(0; 0)$.

4.2. $\mathcal{VU}$-decomposition scheme

Throughout the developments of the theory, we have a conclusion that we should try to minimize the second-order advance of the $\mathcal{U}$-Lagrangian of f_k near an optimal solution of (P_1). Based on this idea, we propose a notional scheme.

Given a minimum point x^*, we demand $q-p$ the multiplicity of $\lambda_k(A(x^*))$. Taking $x \in B(x^*, \rho)$ for some positive number ρ, we wish to compute some x_+ which is close to x^* superlinearly. We consider the coming notional scheme.

Algorithm 1

$\mathcal{V}$-Step. Solve $\hat{x} \in \mathcal{W}(p,q)$,

$$\min \hat{x}\{||\hat{x}-x|| : \hat{x} \in \mathcal{W}(p,q)\}. \tag{4.5}$$

Dual-Step. Solve

$$g(\hat{x}) := \mathrm{proj}_{\partial f_k(\hat{x})}(0). \tag{4.6}$$

$\mathcal{U}$-Step. Calculate

$$\min_{u\in\mathcal{U}(\hat{x})}\langle u, \nabla L_{\mathcal{U},f}(\hat{x}, g(\hat{x}); 0)\rangle_{\mathcal{U}_{f_k}(\hat{x})} + \frac{1}{2}\langle u, \nabla^2 L_{\mathcal{U},f}(\hat{x}, g(\hat{x}); 0)u\rangle_{\mathcal{U}_{f_k}(\hat{x})}. \tag{4.7}$$

Update. Set $x_+ = \hat{x} + u$.

Definition 4.1. (Strict Complementarity) If $0 \in ri\partial f_k(x^*)$, the strict complementarity (SC) is satisfied at x^*.

Definition 4.2. (strict second-order condition) If (T) and (SC) hold at x^*, and the Hessian $H_{\mathcal{U}}(f_k(\bar{x}))$ of $L_{\mathcal{U},f_k}(x^*, 0; \cdot)$ is positive definite at $u=0$, the strict second-order condition (SSOC) is satisfied at x^*.

Proposition 4.2. *Assume $\hat{x} \in \mathcal{W}(p,q) \cup B(x^*, \rho)$, then $g(\hat{x})$ derived from (4.6) has the following results.*

1. $g(\hat{x}) = \mathscr{A}^\star \cdot (Q_1(A(\hat{x}))ZQ_1(A(\hat{x}))^T + P_1(A(\hat{x}))P_1(A(\hat{x}))^T)$, *where Z is a solution of the following problem:*
$$\min_{Z\in S^+_{q-p},\, trZ=k-p} ||\mathscr{A}^\star \cdot (Q_1(A(\hat{x}))ZQ_1(A(\hat{x}))^T + P_1(A(\hat{x}))P_1(A(\hat{x}))^T)||^2. \tag{4.8}$$
2. *When (T) holds at x^*, then, for sufficiently small ρ, the solution of the following minimization problem*
$$\min_{Z\in S_{q-p},\, trZ=k-p} ||\mathscr{A}^\star \cdot (Q_1(A(\hat{x}))ZQ_1(A(\hat{x}))^T + P_1(A(\hat{x}))P_1(A(\hat{x}))^T)||^2 \tag{4.9}$$

exists and unique, denoted by $Z(\hat{x})$; additionally, the mapping

$$\mathcal{W}(p,q) \cup B(x^*, \rho) \ni \hat{x} \mapsto Z(\hat{x}) \in \mathcal{S}_{q-p}$$

is C^∞.

3. *When (T) and (SC) hold, then, for sufficiently small ρ, $Z(\hat{x}) \succ 0$. As a result, the solution of (4.8is unique and just happened to be $Z(\hat{x})$, and*

$$g(\hat{x}) \in \mathrm{ri}\partial f_k(\hat{x}). \tag{4.10}$$

Proof. In view of the chain rule of Ref. [10], we have

$$\partial f_k(\hat{x}) = \mathscr{A}^\star \partial F_k(A(\hat{x})),$$

where $\partial F_k(A(\hat{x})) = \{Z(\hat{x}) \in S_n : 0 \preceq \tilde{Z}(\hat{x}) \preceq I, \mathrm{tr}\tilde{Z}(\hat{x}) = k-p, Z(\hat{x}) = P_1(\hat{x}) P_1(\hat{x})^T + Q_1(\hat{x})\tilde{Z}(\hat{x})Q_1(\hat{x})^T\}$. Utilizing the above formula, we rearrange the projection problem (4.6) to obtain (4.8). The first assertion is satisfied.

For the second statement, combining the transversality mapping and the theorem of implicit function, we obtain the uniqueness and requested regularity.

Concerning the third conclusion, because (SC) holds, that is, $0 \in \mathrm{ri}\partial f_k(x^*)$, together with (3.8), this means

$$\exists Z^* \in S_n, \text{s.t.} 0 \prec \tilde{Z}^* \prec I, \mathrm{tr}\ \tilde{Z}^* = k-p, Z^* = P_1(x^*)P_1(x^*)^T + Q_1(x^*)\tilde{Z}^*Q_1(x^*)^T \textit{ and } \mathscr{A}^\star(Z^*) = 0.$$

It is easy to find that Z^* is also feasible for (4.9) and $\mathscr{A}^\star(Z^*) = 0$. Since (4.9) in the second description has a unique solution, then $Z^* \succ 0$ is also the optimum solution of (4.9). On account of the continuity of the mapping $Z(\cdot)$, $Z(\hat{x}) \succ 0$ for $\hat{x}$ in a neighborhood of x^*. Due to the dual theorem, because $Z(\hat{x})$ is an optimum solution of (4.9) and a feasible solution of (4.8), it is also an optimum solution of (4.10). For (4.8), we utilize (3.8) again to obtain it. ∎

The above proposition is used as a reformulation of the Dual-step of Algorithm 1 to obtain the following Algorithm 2. In Algorithm 1, the computation of $\hat{x}$ is very tricky. To solve this problem, we present the following.

Proposition 4.3. *Suppose (SSOC) satisfies at x^*. Then, for given sufficiently small ρ, the Dual-step, the $\mathcal{U}$-step and the Update of Algorithm 1 can be equivalently turned into the following forms.*

Dual-Step. *Calculate (4.9) to obtain a solution $Z \in S_{q-p}$, and denote*

$$G(\hat{x}) := Q_1(A(\hat{x}))ZQ_1(A(\hat{x}))^T + P_1(A(\hat{x}))P_1(A(\hat{x}))^T.$$

$\mathcal{U}$-Step. *Solve the following, the optimization in $d \in R^m$*

$$\min_d \mathscr{A}(d) \cdot G(\hat{x}) + \mathscr{A}(d) \cdot H(A(\hat{x}), G(\hat{x})) \cdot V(x) + \frac{1}{2}\mathscr{A}(d) \cdot H(A(\hat{x}), G(\hat{x})) \cdot \mathscr{A}(d),$$
$$V(x) + \mathscr{A}(d) \in \mathcal{U}_{F_k}(A(\hat{x})),$$

where $V(x) := A(x) - A(\hat{x})$.

Update. *Set $x_+ = x + d$.*

Proof. In fact, according to Proposition 4.2, we reformulate the Dual-step in S_n:

$$g(\hat{x}) = \mathscr{A}^{\star} G(\hat{x}) \text{ for } \hat{x} \text{ near } x^{*}.$$

For the $\mathcal{U}$-step and Update-step, we can employ (3.10) and (3.19), together with the substitution of variable $d := \hat{x} - x + u$ in (4.7) to obtain the requested results. ■

For computational convenience, we try to take some approximation of $A(\hat{x})$ that does not impact on the eigenvectors. We do it in this way by arranging from $p+1$th to qth eigenvalues equal to $\lambda_{p+1}(A(x))$. The subspace spanned by the $p - q$ corresponding to eigenvectors of A is called the total eigenspace for the set from $p+1$th to qth largest eigenvalues of $A(x)$, denoted by $E_{\text{tot}}(A(x))$. Given $x \in B(x^{*}, \rho)$ and sufficiently small positive number ρ, set $Q_{\text{tot}}(A(x))$ be some orthonormal basis of $E_{\text{tot}}(A(x))$ and $\Lambda_{\text{tot}}(A(x)) = Q_{\text{tot}}(A(x))^{T} A(x) Q_{\text{tot}}(A(x))$; then one looks upon the following matrix $\hat{A}(x) \in \mathcal{M}(p, q)$ as an approximation of $A(\hat{x})$:

$$\hat{A}(x) = \lambda_{p+1}(A) Q_{\text{tot}}(A(x)) Q_{\text{tot}}(A(x))^{T} + A(x) - Q_{\text{tot}}(A(x)) \Lambda_{\text{tot}}(A(x)) Q_{\text{tot}}(A(x))^{T},$$

which fulfills

$$A(x) - \hat{A}(x) \in \operatorname{span} \partial F_k(\hat{A}(x)) \subset \ker H(\hat{A}(x), \hat{G}(x)).$$

So a more implementable version of Algorithm 1 is stated in the following.

Algorithm 2. Given $x \in B(x^{*}, \rho)$.

$\mathcal{V}$-Step. Calculate $Q_{\text{tot}}(A(x)), \Lambda_{\text{tot}}(A(x)), \hat{A}(x)$ and

$$\hat{V}(x) := A(x) - \hat{A}(x).$$

Dual-Step Calculate $Z \in S_{q-p}$ of the following problem

$$\min_{Z \in S_{q-p},\, trZ = k-p} \| \mathscr{A}^{\star} \cdot (Q_1(\hat{A}(x)) Z Q_1(\hat{A}(x))^{T} + P_1(\hat{A}(x)) P_1(\hat{A}(x))^{T}) \|^2,$$

and set

$$\hat{G}(x) := Q_1(\hat{A}(x)) Z Q_1(\hat{A}(x))^{T} + P_1(\hat{A}(x)) P_1(\hat{A}(x))^{T}.$$

$\mathcal{U}$-Step. Solve the following, the problem in variable $d \in R^{m}$

$$\begin{aligned} &\min_{d} \mathscr{A}(d) \cdot \hat{G}(x) + \frac{1}{2} \mathscr{A}(d) \cdot H(\hat{A}(x), \hat{G}(x)) \cdot \mathscr{A}(d), \\ &\hat{V}(x) + \mathscr{A}(d) \in \mathcal{U}_{F_k}(\hat{A}(x)). \end{aligned} \tag{4.11}$$

Update. Set $x_{+} = x + d$.

Next, we present our main convergence result about Algorithm 2, which enjoys the quadratic convergent rate.

Theorem 4.4. *Assume (SSOC) satisfies at x^*, then there are $\delta>0$ and $L>0$ so that, for every $x \in B(x^*, \delta), x_+$ derived by Algorithm 2 meets:*

$$||x_+ - x^*|| \leq L||x - x^*||^2.$$

Proof. The course is similar to Section 6 in [12], we briefly give the framework of the proof. To minimize some given differentiable function $\varphi(y)$ on some differentiable manifold $S \subset R^m$ first. Assume we can find a neighborhood of x^* so that S can be described by some differentiable equations $\phi_i(y) = 0, i = 1, \ldots, l$. Here, the classical Newton method can be employed. A current iteration is denoted by y^k as and the corresponding Lagrange multipliers vector is labeled θ^k. So the next iteration point is $y^{k+1} = y^k + s^{k+1}$, where s^{k+1} solves the following quadratic programming problem

$$\begin{cases} \min & s^T \nabla \varphi(y^k) + \frac{1}{2} s^T H^k s \\ \text{s.t.} & \phi_i(y^k) + s^T \nabla \phi_i(y^k) = 0, \quad i = 1, \ldots, l, \end{cases} \tag{4.12}$$

and $H^k = \nabla^2_{yy} L(y^k, \theta^k)$ is the Hessian of the Lagrangian function

$$L(y, \theta) = \varphi(y) + \sum_{i=1}^{l} \theta_i \phi_i(y).$$

It is not hard to prove that a solution of the following linear equation system in the variable t is just y^{k+1} and the matching Lagrangian multipliers θ^{k+1} can be obtained as

$$F(t^k) + \nabla F(t^k)(t - t^k) = 0,$$

where $F(t) = (\nabla_y L(y, \theta), \phi(y)), \phi(y) = (\phi_1(y), \ldots, \phi_l(y))$ and $t = (y, \theta)$.

If the initial point of the algorithm (sequential quadratic programming algorithm, SQP) is close to optimum point x^* enough and the second-order sufficiency condition satisfies at x^*, then it is obtained that the Newton scheme has quadratic convergence. For the above Algorithm 2, the $\mathcal{U}$-Hessian $H(\hat{A}(x), \hat{G}(x))$ coheres with the matrix H^k. Therefore, the $\mathcal{U}$-step (4.11) can be turned into the problem (4.12). We can acquire Algorithm 2 converges quadratically by the (SSOC) condition. ∎

5. An application

Next we study a class of particular problem for arbitrary eigenvalue based on the $\mathcal{VU}$ decomposition, whose structure is

$$(P_2) \quad \lambda_{l+1}(A) = h_1(A) - h_2(A) = \sum_{i=1}^{l+1} \lambda_i(A) - \sum_{i=1}^{l} \lambda_i(A),$$

where, $h_2(A) \in C^2$ and $h_1(A)$ is finite convex, the eigenvalues of A satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > \lambda_{l+1} = \lambda_{l+2} = \cdots = \lambda_r > \lambda_{r+1} \geq \cdots \geq \lambda_n$.

Evidently, the function λ_{l+1} of the above problem is a kind of special functions, we call the difference of two convex functions as DC function.

Eigenvalue programs (P_2) have been intensely studied since 1990s. They arise in many applications in automatic control, finance, statistics, and design engineering. The readers can see the references [8, 26] and [27].

Definition 5.1. [28] A vector $\xi \in R^n$ is referred to as a proximal subgradient (or P-subgradient)of f at $x \in \text{dom}f$ if $(\xi, -1) \in N^P_{epif}(x, f(x))$, where $N^P_{epif}(x, f(x))$ is the proximal normal cone to *epif* at $(x, f(x))$, or in other words, if there are positive numbers σ and η satisfying

$$f(y) \geq f(x) + \langle \xi, y - x \rangle - \sigma ||y - x||^2, \forall\ y \in B(x, \eta),$$

The set of all such ξ, denoted by $\partial_P f(x)$, is called the proximal subdifferential, or P-subdifferential of f at x.

Set $U_1 = (e_1, \ldots, e_l), U_2 = (e_{l+1}, \ldots, e_r)$, here $e_i, i = 1, \ldots, r$ represent eigenvectors for corresponding eigenvalues λ_i.

The following proposition provides an explicit description of the first-order sensitivity of the arbitrary function λ_{l+1}.

Proposition 5.1. *For the function λ_{l+1} rooted by (P_2) and $\bar{A} \in S_n$, one has*

(1) $\lambda\prime_{l+1}(\bar{A}; H) = h_1'(\bar{A}; H) - \langle \nabla h_2(\bar{A}), H \rangle = \lambda_1(U_2^T H U_2), \forall H \in S_n;$
(2) $G \in \partial_P \lambda_{l+1}(\bar{A}) \Longleftrightarrow G + \nabla h_2(\bar{A}) \in \partial h_1(\bar{A}) \Longleftrightarrow G \in \partial h_1(\bar{A}) - \nabla h_2(\bar{A});$
where

$$\begin{aligned} \partial_P \lambda_{l+1}(\bar{A}) &= \partial h_1(\bar{A}) - \nabla h_2(\bar{A}) \\ &= \{U \in S_n : 0 \preceq \tilde{U} \preceq I, tr\tilde{U} = 1, U = U_1 U_1^T + U_2 \tilde{U} U_2^T\} - U_1 U_1^T \\ &= \{U_2 \tilde{U} U_2^T : 0 \preceq \tilde{U} \preceq I, tr\tilde{U} = 1\}. \end{aligned}$$

Proof. (1) It is easy to obtain the result on account of the form of the function in (P_1).

(2) In the light of Proposition 2.11 of [28], one gets

$$\partial h_1(\bar{A}) - \nabla h_2(\bar{A}) \supseteq \partial_P \lambda_{l+1}(\bar{A}).$$

For completing the demonstration, it suffices to show that the opposite inclusion relation is true. Selecting $\Xi \in \partial h_1(\bar{A}) - \nabla h_2(\bar{A})$ arbitrary, we obtain $\Xi + \nabla h_2(\bar{A}) \in \partial h_1(\bar{A})$, and thus, due to the convexity of h_1,

$$h_1(Y) \geq h_1(\bar{A}) + \langle \xi + \nabla h_2(\bar{A}), Y - \bar{A} \rangle, \forall \ Y \in S_n. \tag{5.1}$$

Because $h_2 \in C^2$, we can find $\eta>0$ and $\sigma>0$ so that $||\nabla^2 h_2(Y)|| \leq \sigma, \ \forall \ Y \in B(\bar{A}, \eta)$. Set $\bar{D} = Y - \bar{A}$, and

$$\begin{array}{rcl} h_2(Y) & = & h_2(\bar{A}) + \nabla h_2(\bar{A})^T \bar{D} + \frac{1}{2} \bar{D}^T \nabla^2 h_2(\bar{A} + \theta \bar{D}) \bar{D} 0 \leq \theta \leq 1 \\ & \leq & h_2(\bar{A}) + \nabla h_2(\bar{A})^T \bar{D} + \sigma ||\bar{D}||^2, \forall \bar{D} \in B(0, \eta). \end{array} \tag{5.2}$$

On the basis of (5.1) and (5.2), one derives $\lambda_{l+1}(Y) \geq \lambda_{l+1}(\bar{A}) + \Xi^T \bar{D} - \sigma ||\bar{D}||^2, \ \forall \ \bar{D} \in B(0, \eta)$. By the meaning of proximal subgradient this shows $\Xi \in \partial_P \lambda_{l+1}(\bar{A})$. ■

Some fundamental results of the space decomposition for the function λ_{l+1} are collected in the next proposition.

Proposition 5.2. *Assume the function λ_{l+1} is rooted in (P_2). The $\mathcal{VU}$-space decompositions are described as:*

$$\mathcal{U}_{\lambda_{l+1}}(\bar{A}) = \{H \in S_n : U_2^T H U_2 = \frac{1}{r-l} tr(U_2^T H U_2) I_{r-l}\},$$
$$\mathcal{V}_{\lambda_{l+1}}(\bar{A}) = \{U_2 Z U_2^T : Z \in S_{r-l}^+, trZ = 0\}.$$

Proof. According to the previous analysis, λ_{l+1} can be equivalently turned into a difference of a convex function and a C^1 convex function, which implies it is regular. Accordingly, one needs to know that the directional derivative is replaced by subderivate, which is defined by $df(\bar{x})(\bar{w}) := \liminf_{\tau \downarrow 0, w \to \bar{w}} \frac{f(\bar{x} + \tau w) - f(\bar{x})}{\tau}$.

The rest results follow from the proof of Theorem 3.1, we omit the detailed statement. ■

In the following, we consider the $\mathcal{U}$-Lagrangian function of the function λ_{l+1} and its winner set.

Set $\nabla h_2(\bar{A}) := \tilde{G} = \tilde{G}_{\mathcal{U}} \oplus \tilde{G}_{\mathcal{V}}$ and $H_2 := \nabla^2 h_2(\bar{A}) \succcurlyeq 0$ (positive semidefinite), and $\bar{\mathbb{U}}$ stands for a basis of the subspace $\mathcal{U}$. Given $\bar{G} \in \partial h_1(\bar{A})$, the corresponding $\mathcal{U}$-Lagrangian of λ_{l+1} is denoted as

$$L_{\mathcal{U}}(U;\bar{G}) = \inf_{v\in\mathcal{V}}\{h_1(\bar{A}+U\oplus V)-\langle\bar{G}_{\mathcal{V}},V\rangle_{\mathcal{V}}\}-(h_2(\bar{A})+\langle\tilde{G}_{\mathcal{U}},U\rangle_{\mathcal{U}} + \frac{1}{2}\langle\bar{\mathbb{U}}^T H_2\bar{\mathbb{U}}U,U\rangle_{\mathcal{U}}), \tag{5.3}$$

and the associated optimal solution set in $\mathcal{V}$-spaces is described as

$$W(U) = \arg\min_{v\in\mathcal{V}}\{h_1(\bar{A}+U\oplus V)-\langle\bar{G}_{\mathcal{V}},V\rangle_{\mathcal{V}}\}. \tag{5.4}$$

Remark 5.1. The $\mathcal{U}$-Lagrangian is usually a nonconvex function. When $h_2\equiv 0$, that is, $l=0$, (5.3) gets the classic form in the sense of Lemaréchal, Oustry, and Sagastizábal [13], where the structure of the $\mathcal{U}$-Lagrangian $L_{\mathcal{U}}(U,\bar{G})$ therein is actually defined by

$$L^1_{\mathcal{U}}(U;\bar{G}) := \inf_{V\in\mathcal{V}}\{h_1(\bar{A}+U\oplus V)-\langle\bar{G}_{\mathcal{V}},V\rangle_{\mathcal{V}}\}. \tag{5.5}$$

The set $W(U)$ and the one endowed in [13] are identical, while $h_2\neq 0$.

The following proposition describes the main result of $L_{\mathcal{U}}(U;\bar{G})$.

Proposition 5.3. *Assume the function λ_{l+1} is deduced by (P_2). the following statements hold:*

(1) *The function $L_{\mathcal{U}}(U;\bar{G})$ given by (5.3) is finite-valued DC.*
(2) *For every minimum point W in the set $W(U)$ of (5.4), it has the feature: there exists some $G\in\partial_P\lambda_{l+1}(\bar{A}+U\oplus W)$ so that $G_{\mathcal{V}}=\bar{G}_{\mathcal{V}}-\tilde{G}_{\mathcal{V}}$.*
(3) *Specially, $0\in W(0)$ and $L_{\mathcal{U}}(0;\bar{G})=\lambda_{l+1}(\bar{A})$.*
(4) *If $G\in ri\partial h_1(\bar{A})$, then $W(U)\neq\emptyset$ for all $U\in\mathcal{U}$ and $W(0)=\{0\}$.*

Proof. (1) The function $L^1_{\mathcal{U}}(U;\bar{G})$ rooted by (5.5) is convex and finite-valued everywhere via The. 3.2 (i) in [13], therefore, $L_{\mathcal{U}}(U;\bar{G})$, the difference of two convex functions, is also finite valued everywhere.

(2) The inner function for infimum of (5.3) is labeled as $h(\cdot)=h_1(\bar{A}+U\oplus\cdot)-\langle\bar{G}_{\mathcal{V}},\cdot\rangle_{\mathcal{V}}$. By the subdifferential operation in convex sense one obtains

$$\begin{aligned}
W\in W(U) \quad &\Longleftrightarrow \quad 0\in\partial h(W)=\partial h_1(\bar{A}+U\oplus W)\cap\mathcal{V}-\bar{G}_{\mathcal{V}}\\
&\Longleftrightarrow \quad \bar{G}_{\mathcal{V}}\in\partial h_1(\bar{A}+U\oplus W)\cap\mathcal{V}\\
&\Longleftrightarrow \quad \bar{G}_{\mathcal{V}}-\tilde{G}_{\mathcal{V}}\in\partial h_1(\bar{A}+U\oplus W)\cap\mathcal{V}-\tilde{G}_{\mathcal{V}}\\
&\Longleftrightarrow \quad \exists\ G=G_{\mathcal{U}}\oplus G_{\mathcal{V}}\in\partial_P f(\bar{A}+U\oplus W)\text{s.t.}G_{\mathcal{V}}=\bar{G}_{\mathcal{V}}-\tilde{G}_{\mathcal{V}}.
\end{aligned}$$

(3) When $U=0$, let $W=0$ and $G=\bar{G}-\tilde{G}\in\partial_P\lambda_{l+1}(\bar{A})=\partial_P\lambda_{l+1}(\bar{A}+0\oplus 0)$ in the above conclusion, thus, it follows that $0\in W(0)$ and $L_{\mathcal{U}}(0;\bar{G})=\lambda_{l+1}(\bar{A})$.

(4) One can clearly conclude that the inner infimum function $h(\cdot)$ is inf-compact in $\mathcal{V}$ and the set $W(U) \neq \emptyset$, through The. 2.4 in [13]. When $U = 0$,

$$h_1(\bar{A} + 0 \oplus V) - \langle \bar{G}_{\mathcal{V}}, V \rangle \geq h_1(\bar{A}) + \eta ||V||_{\mathcal{V}} + o(||V||).$$

This shows that $V = 0$ is the unique solution. ■

Property 5.1.

(1) *When W(U) is nonempty,*

$$\partial_P L_{\mathcal{U}}(U; \bar{G}) = \{G_{\mathcal{U}} : G_{\mathcal{U}} \oplus (\bar{G}_{\mathcal{V}} - \tilde{G}_{\mathcal{V}}) \in \partial_P \lambda_{l+1}(\bar{A} + U \oplus W), W \in W(U)\}. \tag{5.6}$$

(2) *When $\bar{G} \in ri\partial h_1(\bar{A})$, one has $W(U) = o(||U||_{\mathcal{U}})$.*

(3) *If $U \in \mathcal{U}$ satisfies $W(U) \neq \emptyset$, then*

$$\lambda_{l+1}(\bar{A} + U \oplus W) = \lambda_{l+1}(\bar{A}) + \langle \bar{G} - \tilde{G}, U \oplus W \rangle + o(||U||_{\mathcal{U}}), \forall \ W \in W(U). \tag{5.7}$$

We have the following second-order expansion of λ_{l+1} along the smooth manifold $\mathcal{M}_{q-p}$:

Theorem 5.1. *If $D \in S_n, \bar{G} \in ri\partial h_1(\bar{A})$ and $||D|| \to 0$, then*

$$\lambda_{l+1}(\bar{A} + D) = \lambda_{l+1}(\bar{A}) + \langle \bar{G} - \tilde{G}, D \rangle + \frac{1}{2} \text{proj}_{\mathcal{U}}\ D \cdot (H_1 - \bar{\mathbb{U}}^T H_2 \bar{\mathbb{U}})(\text{proj}_{\mathcal{U}}\ D) + o(||D||^2), \tag{5.8}$$

where $H_1 = \nabla^2 L^1_{\mathcal{U}}(0, \bar{G})$ described as

$$\nabla^2 L^1_{\mathcal{U}}(0, \bar{G}) = \text{proj}_{\mathcal{U}(\bar{A})} \circ H(\bar{A}, \bar{G}) \circ \text{proj}_{\mathcal{U}(\bar{A})}{}^*, \tag{5.9}$$

and $H(\bar{A}, \bar{G})$ is the symmetric operator whose structure is

$$\begin{aligned} H(\bar{A}, \bar{G}) \cdot Y \quad = \quad & (\bar{G} - U_1(\bar{A})U_1(\bar{A})^T)Y[\lambda^*_{l+1}I_n - \bar{A}]^\dagger + [\lambda^*_{l+1}I_n - \bar{A}]^\dagger Y(\bar{G} - U_1(\bar{A})U_1 \\ & (\bar{A})^T) + U_1(\bar{A})U_1(\bar{A})^T Y[\lambda^*_1 I_n - \bar{A}]^\dagger + [\lambda^*_1 I_n - \bar{A}]^\dagger Y U_1(\bar{A})U_1(\bar{A})^T, \end{aligned}$$

and we request at $\bar{A}, \lambda_1(\bar{A}) = \cdots = \lambda_l(\bar{A})$ for convenience.

Proof. Combining (5.3) with (5.5), one has $L_{\mathcal{U}}(U; \bar{G}) = L^1_{\mathcal{U}}(U; \bar{G}) - (h_2(\bar{A}) + \langle \tilde{G}_{\mathcal{U}}, U \rangle_{\mathcal{U}} + \frac{1}{2} \langle \bar{\mathbb{U}}^T H_2 \bar{\mathbb{U}} U, U \rangle_{\mathcal{U}})$. On the basis of Theorem 3.4, $L^1_{\mathcal{U}}(U; \bar{G})$ is C^∞, this is also true for $L_{\mathcal{U}}(U; \bar{G})$. Let $k = l + 1$,

$$\nabla^2 L^1_{\mathcal{U}}(0, \bar{G}) = \text{proj}_{\mathcal{U}(\bar{A})} \circ H(\bar{A}, \bar{G}) \circ \text{proj}_{\mathcal{U}(\bar{A})}{}^*,$$

where $H(\bar{A}, \bar{G})$ is rooted in (3.9), so (5.9) is holding.

Taking sufficiently small D satisfying $\bar{A} + D \in \mathcal{M}_{q-p}$, and setting $U = \text{proj}_{\mathcal{U}(\bar{A})} D$, $V = V(U) = \text{proj}_{\mathcal{V}(\bar{A})} D$, we employ the second-order Taylor expansion with the third conclusion of Proposition 5.3:

$$\begin{aligned} L_{\mathcal{U}}(U;G) &= L^1_{\mathcal{U}}(U;\bar{G}) - (h_2(\bar{A}) + \langle \tilde{G}_{\mathcal{U}}, U\rangle_{\mathcal{U}} + \frac{1}{2}\langle \bar{\mathbb{U}}^T H_2 \bar{\mathbb{U}} U, U\rangle_{\mathcal{U}}) \\ &= L^1_{\mathcal{U}}(0;\bar{G}) + \langle \nabla L^1_{\mathcal{U}}(0;G); U\rangle_{\mathcal{U}} + \frac{1}{2}\langle U, H_1 U\rangle_{\mathcal{U}} \\ &\quad - (h_2(\bar{A}) + \langle \tilde{G}_{\mathcal{U}}, U\rangle_{\mathcal{U}} + \frac{1}{2}\langle \bar{\mathbb{U}}^T H_2 \bar{\mathbb{U}} U, U\rangle_{\mathcal{U}}) + o(||U||^2) \\ &= \lambda_{l+1}(\bar{A}) + \langle \bar{G}_{\mathcal{U}} - \tilde{G}_{\mathcal{U}}, U\rangle_{\mathcal{U}} + \frac{1}{2}\langle U, (H_1 - \bar{\mathbb{U}}^T H_2 \bar{\mathbb{U}})U\rangle_{\mathcal{U}} + o(||U||^2), \end{aligned}$$

that is to say, in virtue of $W(U) = V(U)$, so we acquire

$$\begin{aligned} L_{\mathcal{U}}(U;G) &= h_1(\bar{A} + U \oplus V) - \langle \bar{G}_{\mathcal{V}}, V\rangle_{\mathcal{V}} - (h_2(\bar{A}) + \langle \tilde{G}_{\mathcal{U}}, U\rangle_{\mathcal{U}} + \frac{1}{2}\langle \bar{\mathbb{U}}^T H_2 \bar{\mathbb{U}} U, U\rangle_{\mathcal{U}}) \\ &= \lambda_{l+1}(\bar{A} + U \oplus V) - \langle \tilde{G}_{\mathcal{U}} \oplus \bar{G}_{\mathcal{V}}, U \oplus V\rangle + h_2(\bar{A} + U \oplus V) - h_2(\bar{A}) - \frac{1}{2}\langle \bar{\mathbb{U}}^T H_2 \bar{\mathbb{U}} U, U\rangle_{\mathcal{U}} \\ &= \lambda_{l+1}(\bar{A} + U \oplus V) - \langle \tilde{G}_{\mathcal{U}} \oplus \bar{G}_{\mathcal{V}}, U \oplus V\rangle + h_2(\bar{A}) + \langle \tilde{G}, U \oplus V\rangle + o(||U \oplus V||) \\ &\quad - h_2(\bar{A}) - \frac{1}{2}\langle \bar{\mathbb{U}}^T H_2 \bar{\mathbb{U}} U, U\rangle_{\mathcal{U}} \\ &= \lambda_{l+1}(\bar{A} + U \oplus V) - \langle \tilde{G}_{\mathcal{V}} - \bar{G}_{\mathcal{V}}, U \oplus V\rangle + o(||U||^2). \end{aligned}$$

Finally, think of Theorem 3.4 of [17], one concludes $V = O(||U||^2) = O(||D||^2)$. Connect the above two expressions, (5.8) is derived. ■

We label $f(A) := \lambda_{l+1}(A)$ for convenience of statement. Assume $\bar{A}$ is some local minimizer of (P_2), and we have accomplished the $\mathcal{VU}$-structure of $f(\bar{A})$ at $\bar{A}$, there exists $\mathcal{U}$-Hessian $H_{\mathcal{U}}f(\bar{A})$. In the following, we design a superlinearly convergent conceptual DC $\mathcal{VU}$-decomposition scheme for solving (P_2).

Algorithm 3. DC $\mathcal{VU}$-decomposition scheme: computing (P_2).

Step 0. (Initialization)

Select some initial point $A_0 \in S_n$, sufficiently close to $\bar{A}$. let $k=0$.

Step 1. ($\mathcal{V}$-Step) Compute V_k in the following problem

$$\min_{V \in \mathcal{V}} h_1(A_k + 0 \oplus V).$$

Set $A^c_k = A_k + 0 \oplus V_k$, and $\bar{G}^c \in \partial h_1(A^c_k)$ meets $\bar{G}^c_{\mathcal{V}} = \tilde{G}^c_{\mathcal{V}}$, where $\tilde{G}^c = \nabla h_2(A^c_k)$, $G^c = \bar{G}^c - \tilde{G}^c = (\bar{G}^c_{\mathcal{U}} - \tilde{G}^c_{\mathcal{U}}) \oplus 0 \in \partial_P f(A^c_k)$.

Once $\bar{G}^c_{\mathcal{U}} = \tilde{G}^c_{\mathcal{U}}$, then quit, and A^c_k is the estimated optimal solution. Elsewise, return to Step 2.

Step 2. ($\mathcal{U}$-Step)

Calculate the equation

$$H_{\mathcal{U}}f(\bar{A})U = -(\bar{G}^c_{\mathcal{U}} - \tilde{G}^c_{\mathcal{U}}) \tag{5.10}$$

to acquire the solution U_k.

Step 3. (Corrector-Step)

Set $A_{k+1} = A_k^c + U_k \oplus 0 = A_k + U_k \oplus V_k$, substitute $k+1$ for k, and return to Step 1.

Now the convergence of DC $\mathcal{VU}$-decomposition scheme is presented, which supports our idea.

Theorem 5.2. *If the following conditions satisfy:*

1. $\bar{A}$ *is some local optimal solution of* (P_2)*;*
2. $\mathcal{U}$*-Hessian* $H_{\mathcal{U}}(\lambda_{l+1}(\bar{A})) \succ 0$*;*
3. $\tilde{G} = \nabla(h_2(\bar{A})) \in ri\partial(h_1(\bar{A}))$ *and* $\tilde{G}_{\mathcal{V}} = 0$.

Then, the iterate points $\{A_k^c\}_{k=1}^{\infty}$ *generated by the Algorithm 3 converge to* $\bar{A}$ *superlinearly, namely,*

$$||A_{k+1}^c - \bar{A}|| = o(||A_k^c - \bar{A}||).$$

Proof. We will do this in two parts. First, one indicates that $||(A_{k+1}^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}} = o(||(A_k^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}})$. Because $\bar{A}$ is an optimal solution of (P_2), we obtain $\tilde{G} = \nabla h_2(\bar{A}) \in \partial h_1(\bar{A})$. Selecting $\bar{G} = \tilde{G}$ in (5.3), it can be acquired that $\bar{G}_{\mathcal{U}} - \tilde{G}_{\mathcal{U}} = 0$. For sufficiently small $U \in \mathcal{U}$, it follows that, combing (5.6) with Step 1 of Algorithm 3

$$\begin{array}{rl} & \{\bar{G}_{\mathcal{U}}^c - \tilde{G}_{\mathcal{U}}^c | (\bar{G}_{\mathcal{U}}^c - \tilde{G}_{\mathcal{U}}^c) \oplus (\bar{G}_{\mathcal{V}} - \tilde{G}_{\mathcal{V}}) \in \partial_P \lambda_{l+1}(\bar{A} + U \oplus W), W \in W(U)\} \\ \subset & \bar{G}_{\mathcal{U}} - \tilde{G}_{\mathcal{U}} + H_{\mathcal{U}} \lambda_{l+1}(\bar{A}) U + o(||U||_{\mathcal{U}}) B_{\mathcal{U}} \\ = & H_{\mathcal{U}} \lambda_{l+1}(\bar{A}) U + o(||U||_{\mathcal{U}}) B_{\mathcal{U}}. \end{array} \tag{5.11}$$

Since U_k solves (5.10) in Step 2, utilizing $\bar{G}_{\mathcal{V}}^c = \tilde{G}_{\mathcal{V}}^c$ in Step 1 we obtain that

$$-H_{\mathcal{U}} \lambda_{l+1}(\bar{A}) U_k = \bar{G}_{\mathcal{U}}^c - \tilde{G}_{\mathcal{U}}^c \in H_{\mathcal{U}} \lambda_{l+1}(\bar{A})(A_k^c - \bar{A})_{\mathcal{U}} + o(||(A_k^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}}) B_{\mathcal{U}}.$$

Thus,

$$-H_{\mathcal{U}} \lambda_{l+1}(\bar{A})(U_k + (A_k^c - \bar{A})_{\mathcal{U}}) \in o(||(A_k^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}}) B_{\mathcal{U}}.$$

Because $H_{\mathcal{U}} \lambda_{l+1}(\bar{A})$ is positive definite by the second condition, the relation $||U_k + (A_k^c - \bar{A})_{\mathcal{U}}|| = o(||(A_k^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}})$ holds. As a result, $A_{k+1}^c = A_{k+1} + 0 \oplus V_{k+1} = A_k^c + U_k \oplus V_{k+1}$. We can conclude

$$\begin{array}{rcl} ||(A_{k+1}^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}} & = & ||(A_k^c - \bar{A} + U_k \oplus V_{k+1})_{\mathcal{U}}||_{\mathcal{U}} \\ & = & ||U_k + (A_k^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}} \\ & = & o(||(A_k^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}}). \end{array} \tag{5.12}$$

Now let us deduce the second part: $||(A_{k+1}^c - \bar{A})_{\mathcal{U}}||_{\mathcal{V}} = o(||(A_k^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}})$. Since

$$A_{k+1} + (0 \oplus V) = \bar{A} + ((A_k^c - \bar{A})_{\mathcal{U}} + U_k) \oplus ((A_k^c - \bar{A})_{\mathcal{V}} + V),$$

and

$$\begin{aligned} V_{k+1} &\in \operatorname{argmin}_{V \in \mathcal{V}} h_1(A_{k+1} + (0 \oplus V)) \\ &= \operatorname{argmin}_{V \in \mathcal{V}} h_1(\bar{A} + ((A_k^c - \bar{A})_{\mathcal{U}} + U_k) \oplus ((A_k^c - \bar{A})_{\mathcal{V}} + V)), \end{aligned}$$

it follows that

$$\begin{aligned} V_{k+1} + ((A_k^c - \bar{A})_{\mathcal{V}}) &\in \operatorname{argmin}_{V \in \mathcal{V}} h_1(\bar{A} + ((A_k^c - \bar{A})_{\mathcal{U}} + U_k) \oplus V) \\ &= \operatorname{argmin}_{V \in \mathcal{V}} \{h_1(\bar{A} + ((A_k^c - \bar{A})_{\mathcal{U}} + U_k) \oplus V) - \langle \bar{G}_{\mathcal{V}}, V \rangle_{\mathcal{V}}\}, \end{aligned}$$

where $\bar{G}_{\mathcal{V}} = \tilde{G}_{\mathcal{V}} = 0$. Hence,

$$V_{k+1} + (A_k^c - \bar{A})_{\mathcal{V}} \in W((A_k^c - \bar{A})_{\mathcal{U}} + U_k).$$

In view of the given condition, $W(U)$ is nonempty, we obtain

$$V_{k+1} + (A_k^c - \bar{A})_{\mathcal{V}} = o(||(A_k^c - \bar{A})_{\mathcal{U}} + U_k||_{\mathcal{U}}). \tag{5.13}$$

Because $(A_{k+1}^c - \bar{A})_{\mathcal{V}} = (A_k^c - \bar{A})_{\mathcal{V}} + V_{k+1}$, associating (5.11) with (5.13), one has

$$||(A_{k+1}^c - \bar{A})_{\mathcal{V}}||_{\mathcal{V}} = o(||(A_k^c - \bar{A})_{\mathcal{U}} + U_k||_{\mathcal{U}}) = o(||(A_k^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}}). \tag{5.14}$$

It follows from (5.12) and (5.14) that

$$||A_{k+1}^c - \bar{A}|| = ||(A_{k+1}^c - \bar{A})_{\mathcal{U}}||_{\mathcal{U}} + ||(A_{k+1}^c - \bar{A})_{\mathcal{V}}||_{\mathcal{V}} = o(||A_k^c - \bar{A}||_{\mathcal{U}}) = o(||A_k^c - \bar{A}||).$$

We finish the proof. □

6. Conclusions

This article mainly considers the $\mathcal{VU}$ decomposition for a special class of eigenvalue function: the sum of the largest eigenvalues. Utilizing $\mathcal{U}$-Lagrangian theory, with the satisfaction of the transversality condition, the first- and second-order derivatives of the $\mathcal{U}$-Lagrangian function can be acquired. Along some smooth track, we can derive the second-order expansion of f_k. Moreover, we provide the optimality conditions and a conceptual scheme with local superlinear convergence. In addition, we apply our results to some practical optimization problem: the arbitrary eigenvalue problem.

In this article, the conceptual algorithm for solving this special class of eigenvalue optimization problems is provided, and the next work we should consider is to investigate the performance of its fast convergent performable algorithm for solving large scale semidefinite programming problems and eigenvalue optimization, and study how to employ bundle algorithms to approximate $\mathcal{V}$-step and other $\mathcal{VU}$-related attachments. Another topic is the nonconvex SDP problems, which may be related to the study of a class of structured eigenvalue optimization problems with constraints. Moreover, as for the study of arbitrary eigenvalue problem, we only analyze the eigenvalue which ranks first in a group of equal eigenvalues, and we do not discuss other cases. These are also worth studying in development.

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