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A PERTURBATION-BASED APPROACH FOR CONTINUOUS NETWORK DESIGN PROBLEM WITH EMISSIONS

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Abstract. The objective of continuous network design problem (CNDP) is to determine the optimal capacity expansion policy under a limited budget. This transportation system is formulated as a bi-level program where the upper level aims to determine the link capacity expansion vector and emission while taking into account the lower level response. This problem can be solved using various optimization algorithms and software. In this study, the CNDP with environmental considerations is designed and solved using the perturbation based approach. The lower level representing the road users subjected to user equilibrium is solved using the Frank-Wolfe algorithm. The proposed model is tested using a small hypothetical network to show the efficacy of the method. As a contribution of this paper, first it suggests a perturbation based approach for planners to design the capacity expansion, which minimize the total system cost and emission. Second the proposed method solves the nonlinear mathematical program with complementarity constraints (NLMPCC) problem, which overcomes the lack of a suitable set of constraint qualifications, such as Mangasarian Fromovitz constraint qualifications (MFCQ). Although the proposed model illustrated using the CO only and small network, the approach is not limited to large-scale network design problems and other pollutants..

1. **Introduction.** In the last decade, the sustainable development has drawn the interest of many researchers due to the adverse effects caused by the rapid population growth. In the general transportation network, urbanization and population growth possess various negative impacts such as high fuel consumption, air pollution, and traffic congestions. These elements play a significant role in evaluating the network performance of the urban transportation systems. Usually, researchers define the level of service and sustainable transport in different ways depending on the nature of the study [6]. Regardless of the definition, sustainable transportation systems are designed to respond the mobility requirements and to address the equity benefits of the society, habitat, and economic advancement in the present and future. Today as the population grows the number of vehicles increase tremendously. The increase of car ownership not only increases the traffic congestion but also compromises economic activities due to the growth of the demand exceed the

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capacity of the existing of transportation infrastructures. This increase has irreversible effects on the environmental pollution such as land pollution, air pollution, water pollution and so forth. Among those effects, the most noted in urban transportation literature is the air pollution because buses, private cars, and trucks are the primary sources of the air pollution in the cities.

Various initiatives have been proposed to improve the network performance of the transportation systems with environmental constraints. These efforts include the development of low emission vehicle engine and low fuel consumption, road operational improvements and demand management [3]. However, all these efforts have their advantages and drawbacks. The purpose of this study and many others is to develop alternative methods that can be used to reduce congestion, cut off excess fuel consumption and hence, reduce the pollution. Like other studies, the presented paper aims to alleviate congestion and transportation emissions using the alternative approach.

In this paper, we propose the perturbation based method and apply to determine the optimal capacity expansion and emissions under the elastic demand. A traffic assignment problem under study is the bi-level program. The upper level in the bi-level program usually seeks to determine the decisions about a particular policy satisfying the objective of the problem. While the lower level determines the road users behavioral assumptions and their reactions to the upper-level decisions satisfying the Wardrop principles. The emission values are computed using the emission functions defined in terms of traffic flow evaluated in a congested network. Many studies have exploited mathematical program, which aims to find the optimal capacity expansion, which reduces the vehicle emissions.

Teng and Tzeng [23] formulated traffic assignment as a bi-level decision model with the optimal condition that determines the environmental vectors. Benedek and Rilett [2] employed a system equitable traffic assignment using the generalized environmental costs. Venigalla et al.[24] proposed multi-class and multi-user equilibrium traffic algorithm with the aim to find the optimal emissions, vehicle trips and vehicle distance operating in different modes on highway links. Using three distinct paradoxical phenomena Nagurney [11] testing small hypothetical road network reported that the so-called improvements to transportation system may lead to the increase of vehicular emissions.

Nagurney [10] also used the multi-criteria traffic network model with emission as the objective function. Sugawara and Niencier [21] used the theoretical emission-optimized trip assignment model to determine the maximum carbon monoxide (CO) reduction in different congestion levels of the hypothetical network. The results from this reference show a moderate reduction of vehicle emission compared to conventional user equilibrium (UE) and system optimum (SO) models. Results also were compared with that of Benedek and Rilett [2] and Venigalla et al. [24]. Research related to emission minimization in networks by imposing emissions pricing investigated by Yin and Lu [26] and Yin and Lawphongpanich [25]. Nagurney et al. [12] argued that the environmental emission parameters obtained in the study of supply chains also can be applied in the analyzing transportation networks. Sharma and Mathew [17] formulated the transportation network design problem as a bi-level program where the road user is vehicular emissions conscious. However, in developing countries the emission pricing is less accepted by the policy makers and road users because road network expansion is at its peak. Nagurney et al. [14] examined the environmental impact assessment indices and link importance indicators during the investigation of environmental impacts on link capacity degradation in transportation networks. Recently, Sharma and Mathew [18] suggested the multi-objective network design for emission and travel time trade-off for large urban transportation network models.

This paper formulates the bi-level program for CNDP considering vehicle emission as a mathematical program with complementarity constraints (MPCC), which converge to the global or nearby global optimum. The difficulty of solving the MPCC program arises from the ground that these problems contain the non-convex and non-differentiable constraints in the complementarity constraints. However, there are plenty of optimization techniques to handle the MPCC problem. In this study, we propose the perturbation based approach for dealing with the MPCC problem. In addition, the numerical results are presented to show the ability of the proposed method to solve CNDP considering vehicle emission.

This paper is organized into five sections as follows. In section 2, a model for the continuous network design problem (CNDP) considering the environmental aspect is formulated. In section 3, the MPCC for the CNDP and a perturbation based approach for the CNDP are presented. Section 4 gives the numerical example to show the capability of the proposed model plus the perturbation based approach. Finally, some concluding remarks and the future directions of research on the CNDP with emission variable are given in section 5.

2. Proposed Mathematical Model with Emission Function.

Determining the link capacity expansions for continuous network design problem (CNDP) considering the environmental parameter is formulated a bi-level program under the budget constraints. The upper level minimizes the total system travel time and total emission in a given network. The lower level determine the users behavior decisions and their reactions to the planners policy decisions using the Wardrop user equilibrium principles. Nonetheless, the link cost function is modified to capture both the link travel time and vehicle emissions. The perturbation based approach is applied to solve the upper-level problem and lower level is solved using the Frank-Wolfe algorithm method [20].

In this section, we introduce the following notations to be used in the subsequent sections: A is the set of links and N is a set of nodes in the given network, W is the set of OD pairs, q_w is the trip rate (demand) of OD pairs w. x_a and t_a are the link flow and travel time while f_k^w and c_k^w are the flow and travel time on path k connecting origin r and destination s and $\delta_{a,k}^w$ is 1 if link a is part of path k connecting OD pair w and 0 otherwise.

2.1. The upper level problem.

The upper level is formulated as an optimization problem that minimizes total system travel time and total emission. The total system travel time F is the sum of product of the link flow x_a and link travel time $t_a(x_a, y_a)$ is a function of the flow and the capacity expansion y_a on link a. The total emission is the sum of the product of the link flow x_a, the emission factor $ef_a(v_a)$ which is the function of the average speed v_a on link a and the link length l_a. The upper level is presented mathematically as follows:

$$\begin{aligned} \min_{y} \quad & F(x,y,e) = \sum_{a\in A}(t_a(x_a,y_a)x_a + \theta_a g_a(y_a) + x_a ef_a(v_a)l_a) \\ \text{s.t.} \quad & 0 \le y_a \le \bar{y}_a, \qquad \forall a \in A, \end{aligned} \tag{1}$$

where the link flow patterns $x = x(y)$ is determined by solving the following network equilibrium problem. The function $g_a(y_a)$ represents the investment cost responding to the improvement of y_a. The emission function $ef_a(v_a)$ has a polynomial form with the average link speed v_a, which is the dependent variable expressed as:

$$ef_a(v_a) = \alpha_1 v_a^2 + \alpha_2 v_a + \alpha_3$$

$\alpha_1,\ \alpha_2$, and α_3 are the coefficient to be estimated from the field data and emission factor is expressed as a unit per unit length. The lower level of the bi-level program formulation assigns the trip matrix to the given network using the route choice assumption.According to Wardrop first principle at equilibrium no road users(drivers) wil benefit by unilaterally changing routes. This principle summarizes the behaviorally assumptions, possesses a unique solution and it is computationally efficient [20].

The travel time functions $t(\cdot)$ with link a proposed by the Bureau of Public Roads (BPR) function is expressed as follows:

$$t_a(x_a, y_a) = A_a + B_a \left(\frac{x_a}{c_a + y_a}\right)^4 \tag{2}$$

where A_a and B_a are link-specific constants, c_a is the capacity on link a and y_a is the link enhancement. The BPR function is a monotonically increasing convex function. The first set of constraints represents the flow conservation, the second sets are non-negative constraints on the path flow and OD demand for the transport and the last set is a constraint that restrict the flow on each link such that do not exceed its capacity.

2.2. **The lower level problem.** The lower level for user equilibrium traffic assignment problem with elastic demand can be expressed as an optimization problem (see Sheffi [20] and Patriksson [15]):

$$\begin{aligned} \min_{x} \quad & f = \sum_{a\in A} \int_0^{x_a} t_a(s, y_a) ds + \sum_{w\in W} \int_0^{q_w} D_w^{-1}(s) ds \\ \text{s.t.} \quad & \sum f_k^w = q_w, \quad \forall w \in W, \quad \forall k \in K_w, \\ & f_k^w \geq 0, \quad \forall w \in W, \quad \forall k \in K_w, \\ & x_a = \sum_{w\in W} \sum_{k\in K_w} f_k^w \delta_{a,k}^w \leq c_a + y_a, \quad \forall a \in A. \end{aligned} \tag{3}$$

where the first set of constraints represents flow conservation, the second is nonnegativity constraint on path flows and the last set of constraint is the restrain, which ensure that the flow on each link does not exceed its resulting capacity.

Remark 1. The leader problem is an minimization problem whose objective function is equal to negative one times the objective function in the maximization problem (3).

The lower level problem is a convex optimization problem and its Karush Kuhn-Tucker (KKT) optimality conditions are as follows:

$$
\begin{aligned}
& f_k^w \big(c_k^w - u_w + \sum_{a\in A} \lambda_a \delta_{a,k}^w\big) = 0, \quad \forall w \in W, \quad \forall k \in K_w, \\
& \big(c_k^w - u_w + \sum_{a\in A} \lambda_a \delta_{a,k}^w\big) \geq 0, \quad \forall w \in W, \\
& q_w \left(u_w - D_w^{-1}(q_w)\right) = 0, \quad \forall w \in W, \\
& u_w - D_w^{-1}(q_w) \geq 0, \quad \forall w \in W, \\
& \lambda_a \big(c_a + y_a - \sum_{w\in W} \sum_{k\in K_w} f_k^w \delta_{a,k}^w\big) = 0, \quad \forall a \in A, \\
& c_a + y_a - \sum_{w\in W} \sum_{k\in K_w} f_k^w \delta_{a,k}^w \geq 0, \quad \forall a \in A, \\
& \sum f_k^w - q_w = 0, \quad \forall w \in W, \quad \forall k \in K_w, \\
& q_w \geq 0, \quad \forall w \in W, \\
& f_k^w \geq 0, \quad \forall w \in W, \quad \forall k \in K_w.
\end{aligned} \tag{4}
$$

which can be written in compact form as follows:

$$
\begin{aligned}
& 0 \leq f_k^w \perp \big(c_k^w - u_w + \sum_{a\in A} \lambda_a \delta_{a,k}^w\big) \geq 0, \quad \forall w \in W, \quad \forall k \in K_w \\
& 0 \leq q_w \perp \left(u_w - D_w^{-1}(q_w)\right) \geq 0, \quad \forall w \in W, \quad \forall k \in K_w, \\
& 0 \leq \lambda_a \perp (c_a + y_a - \sum_{w\in W} \sum_{k\in K_w} f_k^w \delta_{a,k}^w) \geq 0, \quad \forall a \in A, \\
& \sum_{k\in K_w} f_k^w - q_w = 0, \quad \forall w \in W, \\
& q_w \geq 0, \quad \forall w \in W, \\
& f_k^w \geq 0, \quad \forall w \in W, \quad \forall k \in K_w.
\end{aligned} \tag{5}
$$

Combining the upper level problem (1) with the KKT conditions (5), then we can reformulate the continuous network design problem (CNDP) as the mathematical program with complementarity constraints (MPCC) as follows:

$$
\begin{aligned}
\min_{y} \quad & F(x, y, e) = \sum_{a\in A} (t_a(x_a, y_a) x_a + \theta_a g_a(y_a) + x_a e f_a(v_a) l_a) \\
\text{s.t.} \quad & 0 \leq y_a \leq \bar{y}_a, \quad \forall a \in A \\
& 0 \leq f_k^w \perp \big(c_k^w - u_w + \sum_{a\in A} \lambda_a \delta_{a,k}^w\big) \geq 0, \quad \forall w \in W, \quad \forall k \in K_w \\
& 0 \leq q_w \perp \left(u_w - D_w^{-1}(q_w)\right) \geq 0, \quad \forall w \in W, \\
& 0 \leq \lambda_a \perp (c_a + y_a - \sum_{w\in W} \sum_{k\in K} f_k^w \delta_{a,k}^w) \geq 0, \quad \forall a \in A, \\
& \sum_{k\in K_w} f_k^w - q_w = 0, \quad \forall w \in W, \\
& x_a - \sum_{w\in W} \sum_{k\in K_w} f_k^w \delta_{a,k}^w = 0, \quad \forall a \in A, \\
& x_a \geq 0, \quad \forall a \in A, \\
& u_w \geq 0, \quad \forall w \in W.
\end{aligned} \tag{6}
$$

Let $l(\upsilon)$ denote the number of components of a vector υ. Define

$$\begin{aligned}
&z = (y;\ q;\ f;\ x;\ \lambda;\ u;\ e),\\
&f_0(z) = \sum_{a\in A}\big((t_a(x_a, y_a)x_a + \theta_a g_a(y_a) + x_a e f_a(v_a) l_a\big),\\
&K = \{0_{l(q)+l(x)+l(\lambda)}\} \times [0, \bar{y}_a] \times \Re_+^{l(f)+l(u)},\\
&E(z) = \begin{bmatrix} \sum_{k\in K_w} f_k^w - q_w, \quad \forall w \in W,\\ x_a - \sum_{w\in W}\sum_{k\in K_w} f_k^w \delta_{a,k}^w, \quad \forall a \in A, \end{bmatrix},\\
&G(z) = \begin{bmatrix} f_k^w, \quad \forall w \in W, \quad \forall k \in K_w,\\ q_w, \quad \forall w \in W,\\ (x_a; u_w), \quad \forall a \in A, \quad \forall w \in W, \end{bmatrix},
\end{aligned}$$

$$H(z) = \begin{bmatrix} c_k^w - u_w + \sum_{a\in A}\lambda_a \delta_{a,k}^w, \quad \forall w \in W, \quad \forall k \in K_w,\\ u_w - D_w^{-1}(q_w), \quad \forall w \in W,\\ c_a + y_a - \sum_{w\in W}\sum_{k\in K_w} f_k^w \delta_{a,k}^w, \quad \forall a \in A, \end{bmatrix}.$$

Then problem (6) can be put in the general framework of mathematical programs with complementarity constraints (MPCC) of the following standard form:

$$\text{(MPCC)} \quad \begin{array}{ll} \min & f_0(z)\\ \text{s.t.} & 0 \le G(z) \perp H(z) \ge 0,\\ & E(z) \in K \end{array} \tag{7}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$, $G, H : \mathbb{R}^n \to \mathbb{R}^m$, $E : \mathbb{R}^n \to \mathbb{R}^p$ are smooth functions and $K \subset \Re^p$ is a closed convex set.

3. **Perturbation based Approach.** To solve MPCC problem (7), we are proposing the perturbation based approach in this section. First, let us rewrite problem (7) in the following form:

$$\text{(MPCC)} \quad \begin{array}{ll} \min\limits_{x,u,v} & f_0(z)\\ \text{s.t.} & 0 \le u \perp v \ge 0,\\ & G(z) - u = 0,\\ & H(z) - v = 0,\\ & E(z) \in K. \end{array} \tag{8}$$

Define

$$\overline{K} = \{0_m\} \times \{0_m\} \times K,\ \overline{E}(z, u, v) = [G(z) - u; H(z) - v; E(z)].$$

Then problem (8) is equivalent to

$$\text{(P)} \quad \begin{array}{ll} \min\limits_{z,u,v,\lambda} & \bar{f}(z, u, v)\\ \text{s.t.} & 0 \le u \perp v \ge 0. \end{array} \tag{9}$$

where

$$\bar{f}(z, u, v) := f(x) + \delta_{\overline{K}}(\overline{E}(z, u, v)).$$

Definition 3.1. For $\overline{K} \subseteq R^n$ define $\delta_{\overline{K}}(\overline{E}(z,u,v))$ as

$$\delta_{\overline{K}}(\overline{E}(z,u,v)) = \begin{cases} 0 & (z,u,v) \in \overline{K}, \\ \infty & (z,u,v) \notin \overline{K}. \end{cases}$$

Then z minimizes f over $\overline{K}$ if and only if z minimizes $f(x) + \delta_{\overline{K}}(\overline{E}(z,u,v))$ over $\mathbb{R}^n$. Now, we focus on solving problem (9), which is still an MPCC problem. For such a problem, even if $\bar{f}$ is smooth, it is not suitable to treat it as a traditional nonlinear programming problems because, as explained in example 3.1.1 and 3.1.2 in Luo, Pang and Ralph [8], even the basic constraint qualification (namely the tangent cone is equal to the linearized cone at an optimal solution) does not hold. In these examples, the Mangasarian-Fromovitz constraint qualification does not hold and the boundedness of the set of Lagrange multipliers are not guaranteed, whereas a numerical algorithm usually requires this boundedness property for NLP problems. To overcome this difficulty, various relaxation approaches have been proposed to deal with the complementarity constraints. Facchinei, Jiang and Qi [4] and Fukushima and Pang [5] used $\phi_\mu(a,b) = 0$ to approximate the complementarity relation $0 \leq a, 0 \leq b, ab = 0$, where $\phi_\mu(a,b)$ is the smoothed Fischer-Burmeister function.

$$\phi_\mu(a,b) = a + b - \sqrt{a^2 + b^2 + 2\mu^2}. \tag{10}$$

Scholtes [22] used

$$a \geq 0, b \geq 0, ab \leq \mu,$$

and recently Lin and Fukushima [7] proposed the following

$$(a+\mu)(b+\mu) \geq \mu^2 \text{ and } ab \leq \mu^2$$

to relax the complementarity relationship of a and b. In this paper, we adopt the smoothed Fischer-Burmeister function to deal with the complementarity constraints, so the perturbation problem (9) is defined as follows:

$$(P_\mu) \qquad \begin{array}{cl} \min\limits_{z,u,v,\lambda} & \bar{f}(z,u,v) \\ \text{s.t.} & \Psi_\mu(u,v) = 0. \end{array} \tag{11}$$

where

$$\Psi_\mu(u,v) = \begin{bmatrix} \psi_\mu(u_1,v_1) \\ \vdots \\ \psi_\mu(u_m,v_m) \end{bmatrix}$$

and ψ_μ is defined by (10). The difference between our methodology and that of Facchinei, Jiang and Qi [4] and Fukushima and Pang [5] is that we use *variational analysis technique* in Rockafellar and Wets [16] to establish the convergence property of the solution set SOL(P_μ) to SOL(P).
Let

$$\Omega(\mu) := \Big\{ (z,u,v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m : \Psi_\mu(u,v) = 0 \Big\}$$

be the feasible region of problem (P_μ). Obviously $\psi_0(a,b) = 0$ if and only if $0 \leq a, 0 \leq b, ab = 0$. Therefore $\Omega(0)$ is the feasible set of MPCC problem. Since z is not constrained in the problem (P_μ), we can rewrite $\Omega(\mu)$ in the following form:

$$\Omega(\mu) := \Big\{ (u,v) \in \mathbb{R}^m \times \mathbb{R}^m : \Psi_\mu(u,v) = 0 \Big\}. \tag{12}$$

We first give the convergence analysis for the perturbation approach. For this purpose, we demonstrate the convergence of $\Omega(\mu)$ to $\Omega(0)$ as $\mu \searrow 0$.

Lemma 3.2. *For $\Omega(\mu)$ defined by (12), we have*

$$\lim_{\mu\searrow 0} \Omega(\mu) = \Omega(0).$$

Proof. For any $(u,v) \in \limsup_{\mu\searrow 0} \Omega(\mu)$, there exist $\mu_k \searrow 0$ and $(u^k, v^k) \in \Omega(\mu_k)$ such that $(u^k, v^k) \to (u,v)$. The inclusion $(u^k, v^k) \in \Omega(\mu_k)$ implies

$$u^k + v^k - \sqrt{(u^k)^2 + (v^k)^2 + 2\mu_k^2} = 0.$$

Then, letting $k \to \infty$, we have

$$u + v - \sqrt{u^2 + v^2} = 0,$$

namely $\Psi_0(u,v) = 0$ and $(u,v) \in \Omega(0)$. Therefore we have

$$\limsup_{\mu\searrow 0} \Omega(\mu) \subset \Omega(0).$$

For any $(u,v) \in \Omega(0)$, let

$$I_+ = \{i : u_i > 0\}, J_+ = \{i : v_i > 0\}, I_0 = \{1, \dots, m\} \setminus \Big(I_+ \cup J_+\Big).$$

For any $\mu > 0$ defined $(u(\mu), v(\mu))$ by

$$(u_i(\mu), v_i(\mu)) = \begin{cases} (u_i, \mu^2/u_i) & \text{if } i \in I_+, \\ (\mu^2/v_i, v_i) & \text{if } i \in J_+, \\ (\mu, \mu) & \text{if } i \in I_0, \end{cases}$$

Then $\psi_\mu(u_i(\mu), v_i(\mu)) = 0$ for $i = 1, \dots, m$ or equivalently $\Psi_\mu(u(\mu), v(\mu)) = 0$ or $(u(\mu), v(\mu) \in \Omega(\mu)$. Obviously $(u(\mu), v(\mu) \to (u,v)$ and this implies that

$$\liminf_{\mu\searrow 0} \Omega(\mu) \supset \Omega(0).$$

Therefore $\Omega(\mu) \to \Omega(0)$ as $\mu \searrow 0$. □

Let us introduce following notations:

$$\kappa(\mu) := \inf\{\bar{f}(z,u,v) \,|\, (u,v) \in \Omega(\mu)\},$$

$$S(\mu) := \operatorname{Argmin}\{\bar{f}(z,u,v) \,|\, (u,v) \in \Omega(\mu)\}.$$

□

The following theorem shows the convergence of the smoothing approach for solving problem, which is characterized by using the terminology in variational analysis.

Theorem 3.3. *Assume that $\bar{f}$ is level-bounded.Then the function $\kappa(\mu)$ is continuous at 0 with respect to $\mathbb{R}_+$ and the set-valued mapping $S(\mu)$ is outer semi-continuous at 0 with respect to $\mathbb{R}_+$.*

Proof. As $\bar{f}$ is level-bounded, we have $\kappa(\mu)$ is finite and $S(\mu) \neq \emptyset$ for any $\mu \geq 0$. Let

$$\widehat{f}_\mu(z,u,v) = \bar{f}(z,u,v) + \delta_{\Omega(\mu)}(u,v),$$

where $\delta_{\Omega(\mu)}$ is the indicator function of $\Omega(\mu)$. From Lemma 3.2, $\Omega(\mu) \to \Omega(0)$ as $\mu \searrow 0$, $\widehat{f}_\mu$ epi-converges to $\widehat{f}_0$. The level-boundedness of $\widehat{f}_\mu$ is easily verified for $\mu \geq 0$. Therefore, we have from Theorem 7.41 of Rockafellar and Wets [16] that the function $\kappa(\mu)$ is continuous at 0 with respect to $\mathbb{R}_+$ and the set-valued mapping $S(\mu)$ is outer semi-continuous at 0 with respect to $\mathbb{R}_+$. □

From Theorem 3.3, we know that for any sequence $\mu_k \searrow 0$, for any $(u^k, v^k) \in S(\mu_k)$, any accumulation point of $\{(u^k, v^k)\}$ is an global optimal solution to Problem (9).

Now we discuss the computational issue for problem (P_μ) when $\mu > 0$ is small enough. For any $\mu > 0$ and $x \in \mathbf{R}^n$, we have

$$\mathcal{J}_{u,v}\Psi_\mu(u,v) = [\mathcal{J}_u\Psi_\mu(u,v) \quad \mathcal{J}_v\Psi_\mu(u,v)]$$

where

$$\mathcal{J}_u\Psi_\mu(u,v) = \begin{bmatrix} 1 - \dfrac{u_1}{\sqrt{u_1^2 + v_1^2 + 2\mu^2}} & & \\ & \ddots & \\ & & 1 - \dfrac{u_m}{\sqrt{u_m^2 + v_m^2 + 2\mu^2}} \end{bmatrix}$$

and

$$\mathcal{J}_v\Psi_\mu(u,v) = \begin{bmatrix} 1 - \dfrac{v_1}{\sqrt{u_1^2 + v_1^2 + 2\mu^2}} & & \\ & \ddots & \\ & & 1 - \dfrac{v_m}{\sqrt{u_m^2 + v_m^2 + 2\mu^2}} \end{bmatrix}.$$

Obviously for any $\mu > 0$ and $(u,v) \in \mathbf{R}^{2m}$, both $\mathcal{J}_u\Psi_\mu(u,v)$ and $\mathcal{J}_v\Psi_\mu(u,v)$ are nonsingular matrices, we can easily obtain the following conclusion.

Lemma 3.4. *Let $\mu > 0$. Then for any $(u,v) \in \Omega(\mu)$ the linear independence constraint qualification (LICQ) holds and the tangent cone of $\Omega(\mu)$ at (u,v) is*

$$T_{\Omega(\mu)}(u,v) = \left\{(\triangle u, \triangle v) \in \mathbf{R}^{2m} : \mathcal{J}_{u,v}\Psi_\mu(u,v)(\triangle u, \triangle v) = 0\right\},$$

and the normal cone of $\Omega(\mu)$ at (u,v) is

$$N_{\Omega(\mu)}(u,v) = \mathcal{J}_{u,v}\Psi_\mu(u,v)^T\mathbf{R}^m.$$

Now we rewrite problem P_μ as follows:

$$\begin{array}{ll} \min\limits_{z,u,v} & f_0(z) \\ \text{s.t.} & \overline{E}(z,u,v) \in \overline{K} \\ & \Psi_\mu(u,v) = 0. \end{array} \tag{13}$$

The Lagrangian for problem P_μ is defined as

$$L(u,v,\lambda) = f_0(z) + \langle \lambda, \Psi_\mu(u,v)\rangle + \langle \chi, \overline{E}(z,u,v)\rangle.$$

If $(\bar{z}, \bar{u}, \bar{v})$ is a local minimizer for problem P_μ and basic constraint qualification (from Rockafellar and Wets [16]) holds, namely

$$\left.\begin{array}{r} 0 \in \mathcal{J}_{z,u,v}\overline{E}(\bar{z},\bar{u},\bar{v})^T\chi + \{0_n\} \times N_{\Omega(\mu)}(\bar{u},\bar{v}) \\ \chi \in N_{\overline{E}}(\bar{z},\bar{u},\bar{v}) \end{array}\right\} \Longrightarrow \chi = 0, \tag{14}$$

then Karush-Kuhn-Tucker (KKT)conditions for P_μ are satisfied, namely

$$\mathcal{J}_{z,u,v}L(\bar{z},\bar{u},\bar{v},\bar{\lambda},\bar{\chi}) = 0,\ \Psi_\mu(\bar{u},\bar{v}) = 0, \bar{\chi} \in N_{\overline{E}}(\bar{z},\bar{u},\bar{v}). \tag{15}$$

As the linear independence constraint qualification holds at any feasible solution of P_μ if there exists $\bar{\lambda}$ such that the above KKT condition holds, then $\bar{\lambda}$ is unique. The following Lemma gives the second order sufficient conditions at KKT point of P_μ.

Lemma 3.5. *Let $(\bar{z}, \bar{u}, \bar{v}, \bar{\lambda}, \bar{\chi})$ be a Karush-Kuhn-Tucker (KKT) point for P_μ. Suppose the following condition hold:*

$$\left\langle d, \nabla^2_{(z,u,v)} L(\bar{z}, \bar{u}, \bar{v}, \bar{\lambda}) d \right\rangle > 0 \tag{16}$$

for

$\forall d \neq 0$ satisfying $\mathcal{J}_{z,u,v} \Psi_\mu(\bar{u}, \bar{v})(d_u; d_v) = 0$ and $\mathcal{J}_{z,u,v} \overline{E}(\bar{z}, \bar{u}, \bar{v}) d \in T_{\overline{K}}(\overline{E}(\bar{z}, \bar{u}, \bar{v}))$.

Then the second order growth condition holds at $(\bar{u}, \bar{v})$, namely there exist positive numbers $\gamma > 0$ and $\delta > 0$ such that

$$f_0(z) - f_0(\bar{z}) \geq \gamma \|(z, u, v) - (\bar{z}, \bar{u}, \bar{v})\|^2, \forall (z, u, v) \in [\Re^n \times \Omega(\mu)] \cap \boldsymbol{B}_\delta(\bar{z}, \bar{u}, \bar{v}).$$

From the above analysis and the assumption that the link travel cost function, demand function, emission function and the investment function are smooth. Based on the convergence of perturbed problem 9 and 11 the traffic equilibrium problem can be reformulated as nonlinear complementarity problem (NCP) and offers the flexibility of relaxing the assumption of additive route costs. The MPCC for CNDP converted to NLP also lacks mathematical properties due to the complementarity slackness constraints. Fortunately, there are exist plenty of optimization techniques to handle the problem such as NLP solver. Therefore, in this paper optimal capacity expansion considering the environmental parameters is determined using the perturbation based approach. The proposed perturbation based on F-B function and the perturbed NLP (P_μ) to solve CNDP with emission function is expressed as:

$$\begin{aligned}
\min \quad & F = \sum_{a \in A} (t_a(x_a, y_a) x_a + \theta_a g_a(y_a) + x_a e f_a(v_a) l_a) \\
\text{s.t.} \quad & 0 \leq y_a \leq \bar{y}_a \\
& x_a, y_a, \lambda_a, \zeta_a, f^w_k, u_w, q_w \geq 0, \quad \forall a \in A, \quad \forall w \in W, \quad \forall k \in K_w, \\
& \psi_{FB}(f^w_k, \eta_a, \mu) = 0, \quad \forall a \in A, \quad \forall w \in W, \quad \forall k \in K_w, \\
& \psi_{FB}(\lambda_a, \zeta_a, \mu) = 0, \quad \forall a \in A, \\
& q_w = \textstyle\sum f^w_k, \quad \forall w \in W, \quad \forall k \in K_w, \\
& x_a = \textstyle\sum \sum f^w_k \delta^w_{a,k}, \quad \forall a \in A, \quad \forall w \in W, \quad \forall k \in K_w.
\end{aligned}$$

where

$$\eta_a = c^w_k - u_w + \sum_{a \in A} \lambda_a \delta^w_{a,k} \qquad \zeta_a = c_a + y_a - \sum_{w \in W} \sum_{k \in K_w} f^w_k \delta^w_{a,k}.$$

4. **Numerical example and computation results.**

We present the numerical example to show the ability of the proposed model in determining the optimal solution of the capacity expansion problem considering emission. The difficulty of solving the proposed model is due to the presence of non-convex and non-differentiable constraints found in the complementarity constraints in the formulated problem. Fortunately, there exist plenty of optimization tools such as NLMPCC to handle the MPCC problem. NLMPCC reformulates the complementarity constraints of the MPCC model with a user specified reformulation option that circumvent the failure of the problem to satisfy constraints qualification, such as Mangasarian

Fromovitz constraints qualification (MFCQ). The results from the nonlinear programming solver can converge to the global or nearly global of the original MPCC and the complementarity constraints are checked for violation.

In this study, we use the road network (see, Figure. 1) with four nodes and five links to show the ability of the proposed model and perturbation based approach for the CNDP with CO emission. The speed depending on emission functions are rarely available in the emission analysis. In this study, we use the polynomial emission function and parameters proposed by Anusha [1] and Sharma [19]. The pollutants are presented in gm/km, and the speed is expressed in km/hr. The emission and travel parameters of the road network are shown in Table 1. The five-link network is formulated as MPCC, converted to NLP and solved using proposed perturbation approach.

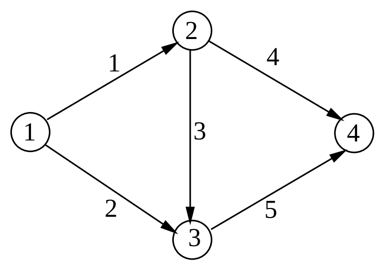

FIGURE 1. Five link network

TABLE 1. Parameters for the 5-link network

link	A_a	B_a	C_a	l_a	θ_a
1-2	1	10	3	4	0.5
1-3	2	5	10	5	0.5
2-3	4	20	4	0.5	0.5
2-4	5	50	3	5	0.5
3-4	1	1	10	4	0.5
	$\alpha_1 = 0.002038$	$\alpha_2 = -0.22270$	$\alpha_3 = 8.8100$	$\Psi = 10$	

The upper objective function for the five link network is defined as:

$$\begin{aligned} \min_{y} \quad F(x,y,e) &= \sum_{a\in A}(t_a(x_a,y_a)x_a + \theta_a g_a(y_a) + x_a e f_a(v_a) l_a) \\ s.t. \quad & 0 \le y_a \le \bar{y}_a, \quad \forall a \in \mathcal{A}. \end{aligned} \tag{17}$$

where θ_a is the investment cost coefficient and $\bar{y}$ represents the link capacity expansion. The function $g_a(y_a) = y_a$ is set to 20.

The NLP(ψ, ε) for example 1 is presented as

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{5}\left(x_i(A_i + B_i\left(\frac{x_i}{c_i+y_i}\right)^4) + \theta_i y_i\right) + \alpha_1\left(\sum_{i=1}^{5}\frac{x_i l_i^3}{(A_i + B_i(\frac{x_i}{c_i+y_i})^4)^2}\right) \\
& +\alpha_2\left(\sum_{i=1}^{5}\frac{x_i l_i^2}{A_i + B_i(\frac{x_i}{c_i+y_i})^4}\right) + \alpha_3\left(\sum_{i=1}^{5} x_i l_i\right) \\
s.t. \quad & x_i, y_i, \lambda_i, \eta_i, \zeta_i, f_j^{14}, u_{14}, q_{14} \geq 0,\ \forall i = 1, \cdots, 5,\ j = 1, 2, 3 \\
& \psi_{FB}(f_1^{14}, \eta_1, \varepsilon) = 0, \quad \psi_{FB}(f_2^{14}, \eta_2, \varepsilon) = 0, \quad \psi_{FB}(f_3^{14}, \eta_3, \varepsilon) = 0, \\
& \psi_{FB}(q_{14}, \eta_4, \varepsilon) = 0, \quad \psi_{FB}(\lambda_1, \zeta_1, \varepsilon) = 0, \quad \psi_{FB}(\lambda_2, \zeta_2, \varepsilon) = 0, \\
& \psi_{FB}(\lambda_3, \zeta_3, \varepsilon) = 0, \quad \psi_{FB}(\lambda_4, \zeta_4, \varepsilon) = 0, \quad \psi_{FB}(\lambda_5, \zeta_5, \varepsilon) = 0, \\
& q_{14} = f_1^{14} + f_2^{14} + f_3^{14}, \quad x_1 = f_1^{14} + f_2^{14}, \quad x_2 = f_3^{14}, \\
& x_3 = f_2^{14}, \quad x_4 = f_1^{14}, \quad x_5 = f_2^{14} + f_3^{14}
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
& \eta_1 = A_1 + B_1\left(\tfrac{x_1}{c_1+y_1}\right)^4 + A_4 + B_4\left(\tfrac{x_4}{c_4+y_4}\right)^4 + \lambda_1 + \lambda_4 - u_{14} \\
& \eta_2 = A_1 + B_1\left(\tfrac{x_1}{c_1+y_1}\right)^4 + A_3 + B_3\left(\tfrac{x_3}{c_3+y_3}\right)^4 + A_5 + B_5\left(\tfrac{x_5}{c_5+y_5}\right)^4 - u_{14} \\
& \qquad +\lambda_1 + \lambda_3 + \lambda_5 \\
& \eta_3 = A_2 + B_2\left(\tfrac{x_2}{c_2+y_2}\right)^4 + A_5 + B_5\left(\tfrac{x_5}{c_5+y_5}\right)^4 - u_{14} + \lambda_2 + \lambda_5 \\
& \eta_4 = u_{14} - D_{14}^{-1}(q_{14}), \\
& \zeta_i = c_i + y_i - x_i, \quad i = 1 \cdots 5.
\end{aligned}
$$

The perturbation method to solve the CNDP problem considering the environmental parameter has been executed in Matlab "fmincon" by using the SQP solver for approximating the solution of P_μ. The SQP solver is a search method and the results obtained from this algorithm is selected as the best output as shown in Table 2 and Table 3.

Table 2 shows that the optimal link capacity expansion value (y_a) obtained from the perturbation based approach and the optimal value is the optimal percentage of the capacity expansion found for each link. The perturbation based approach allows capacity expansions only for links with minimum travel cost helps to relief congestion and minimize emission. The proposed method takes the links on path 1-2-3-4 with link 1, 3, and 5 and expands them optimally (see, y_1, y_3 and y_5 as presented in Table 2). It also considers the user's response during the capacity expansion process in which the reduction of link travel time does not cause significant shift of the traffic flow from the expanded links, while retaining the congestion level.

Table 3 shows the average travel time and the total emission using the perturbation based approach. While the minimum emission of CO, minimum total emission and minimum total system travel time is given in Table 3. The values of the total system travel time and total emission obtained using the perturbation based approach confirms the ability of the method to minimize both travel time and vehicle emissions. Although the results obtained using this method is not very significant when compared to the report presented by Mathew and Sharma [9], its impact over the entire transportation networks will be substantial to the overall reduction of vehicle emissions. Since the minimization of total system travel time does not necessarily minimize emissions in transportation networks, it calls to treat the emission separately from the total system travel time.

Table 2. Optimal link capacity-expansion values for proposed method

Link	SQP
y_1	0.928
y_2	0.000
y_3	0.386
y_4	0.000
y_5	0.865

Table 3. Results for small test network using the proposed approach

Network performance measure	SQP
Average speed(km/hr)	26.83
Average travel time(min)	20.25
Total system travel time	97218
CO emission(Kg)	341.2
Maximum speed(Km/hr)	31.26
Minimum speed(Km/hr)	21.56

5. **Conclusions.** In this paper, we incorporate emission function in the traffic flow function in order to estimate the emission amount in the congested network accurately. Using the proposed emission function, we develop the perturbation based approach to determine the optimal capacity expansions and emissions. We observe that under the capacity enhancement strategy, the increase of the link capacity decreases the travel time on that particular link, but increases the travel demand and the emission. Despite the implementation to limit the capacity expansion, it is possible to obtain the optimal solution after improvements even if the demand is increasing. It is worth to note that the determination of the links to be enhanced are important in the formulation of the policies.

The hypothetical network with 5-link was tested using the proposed model. The numerical result shows that the proposed model and perturbation-based approach can significantly handle complex continuous network design problem under investigation. Also, the proposed model has ability to locate links to be enhanced and in turn minimizes the emission. The proposed model can be used directly by transportation planners to evaluate the effects of various policies that focus on reducing carbon emission on the overall performance of the road network. As a future research, the road types, such as belt lines and highways can be investigated because traffic flow in those areas has significant emission. Finally, we intend to use the proposed approach to solve the large scale real-life problems by including the multiclass or dynamic traffic assignment in the lower level.

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REFERENCES

[1] S. P. Anusha, *Study of Influence of Lane Restrictions on Vehicular Emissions under the Heterogeneous Traffic Flow*, MS thesis, Department of Civil Engineering, Indian Institute of Technology, Madras, 2007.

[2] C. M. Benedek and L. R. Rillet, Equitable traffic assignment with environmental cost functions, *Journal Transportation Engineering*, **124** (1998), 16–24.

[3] E. Deakin, *Sustainable develoment and sustaionable transportation: Strategies for economic prosperity environmental quality and equity*, Technical Report, Institute of Urban and Region Development, University of California, Berkely, 2001.

[4] F. Facchinei, H. Jiang and L. Qi, A smoothing method for mathematical programs with equilibrium constraints, *Mathematical Programming*, **85** (1999), 107–134.

[5] M. Fukushima and J. S. Pang, *Convergence of a Smoothing Continuation Method for Mathematical Problems with Complementarity Constraints, Ill-posed Variational Problems and Regularization Techniques*, *Lecture Notes in Economics and Mathematical Systems* (eds. M. Théra and R. Tichatschke), 477, Springer-Verlag,

[6] C. M. Jeon and A. Amekudzi, Addressing sustainability in transportation systems, (French) [Definitions, Indicators and Metrics], *Journal of Infrastucture Systems*, **11** (2005), 31–50. Berlin/Heidelberg, 1999.

[7] G. H. Lin and M. Fukushima, A modified relaxation scheme for mathematical programs with complementarity constraints, *Annals of Operations Research*, **133** (2005), 63–84.

[8] Z. Q. Luo, J. S. Pang and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, 1996.

[9] T. V. Mathew and S. Sharma, Capacity expansion problem for large urban transportation networks, *Journal of Transportation Engineering*, **135** (2009), 406–4015.

[10] A. Nagurney, *Sustainable Transportation Networks*, Edward Elgar, Cheltenham, Glos,2000b.

[11] A. Nagurney, Congested urban transportation networks and emission paradoxes, *Transportation Research Part D*, **5** (2000a), 145–151.

[12] A. Nagurney, Q. Qiang and L. S. Nagurney, Environmental impact assessment of transportation networks with degradable links in an era of climate change, *International Journal of Sustainable Transportation*, **1** (2007), 29–51.

[13] A. Nagurney, Z. Liu and T. Woolley, Sustainable supply chain and transportation networks, *International Journal of Transportation*, **4** (2010), 154–171.

[14] A. Nagurney, Z. Liu and T. Woolley, Sustainable supply chain and transportation networks, *International Journal of Transportation*, **4** (2010), 154–171.

[15] M. Patriksson, *The traffic assignment problem: Models and Methods*, VSP, Utrecht, The Netherlands, 1994.

[16] R. T. Rockafellar and R. J. B. Wets, *Variational Analysis*, Springer-Verlag, New York, 1998.

[17] S. Sharma and T. V. Mathew, *Transportation network design considering emissions as bi-level optimization problem, in TBR 86th Annual Meeting Compendium of the Paper CD-ROM*, Transportation Research Board, Washington, DC, 2007.

[18] S. Sharma and S. Mishra, *Optimal emission pricing models for containing carbon footprints due to vehicular pollution in a city network*, Proceedings of Transportation Research Board 90th Annual Meeting, 2011.

[19] S. Sharma, *Transportation Network Design Considering Environmental Parameters and Demand Uncertainity*, PhD thesis, Indian Institute of Technology, Bombay, India, 2009.

[20] Y. Sheffi, *Urban Transportation Networks*, First edition, Mathematical Models [Equilibrium Analysis with Mathematical programming Methods], 416, Prentice-Hall, Englewood Cliffs, 1985.

[21] S. Sugawara and D.A. Niemeier, How much can vehicle emissions be reduced?, (French) [exploratory analysis of an upper boundary using an emissions optimized trip assignment], *Transportation Research Record*, **1815** (2003), 29–37.

[22] S. Scholtes, Convergence properties of a regularization scheme for mathematical programs with complementarity constraints, *SIAM J. Optim*, **11** (2001), 918–936.

[23] J. Y. Teng and G. H. Tzeng, A multiobjective programming approach for selecting non-independent transportation investment alternatives, *Transportation Research Part B*, **30** (1996), 291–307.

[24] M. M. Venigalla, A. Chatterjee and M. S. Bronzini, A specialized equilibrium assignment algorithm for air quality modeling, *Transportation Research Part D*, **4** (1999), 29–44.

[25] Y. Yin and S. Lawphongpanich, Internalizing emission externality on road networks, *Transportation Research Part D*, **11** (2006), 292–301.

[26] Y. Yin and H. Lu, Traffic equilibrium problems with environmental concerns, *Journal of Eastern Asia Society for Transportation Study*, **3** (1999), 195–206.

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