A perturbation-based approach for continuous network design problem with link capacity expansion

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Abstract: This paper formulates a continuous network design problem (CNDP) as a nonlinear mathematical program with complementarity constraints (NLMPCC) and then a perturbation-based approach is proposed to overcome the NLMPCC problem and the lack of constraint qualifications. This formulation permits a more general route cost structure and every stationary point of it corresponds to a global optimal solution of the perturbed problem. The contribution of this paper from the mathematical perspective is that, instead of using the conventional nonlinear programming methodology, variational analysis is taken as a tool to analyse the convergence of the perturbation-based method. From the practical point of view, a convergent algorithm is proposed for the CNDP and employs the sequential quadratic program (SQP) solver to obtain the solution of the perturbed problem. Numerical experiments are carried out in both 16 and 76-link road networks to illustrate the capability of the perturbation-based approach to the CNDP with elastic demand. Results showed that the proposed model will solve a wider class of transportation equilibrium problems than the existing ones.

Keywords: continuous network design problem; CNDP; bilevel programming; nonlinear mathematical program with complementarity constraints; MPCC; variational analysis; perturbation-based approach.

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1 Introduction

The objective of the continuous network design problem (CNDP) is to optimise the change in the capacity of particular links of a given road network under budget and/or environmental constraints. In those growing cities where urbanisation and economic activities proceed quickly, it is common to see travel demand higher than available capacity on many road links. This imbalance compromises human activities, reduces productivity due to traffic delay, increases fuel consumption and deteriorates air quality and other environmental issues. A significant amount of efforts has been made to seek the optimal capacity expansion of a set of particular roads or links with available resources in a given road network, including modelling, solving and implementation. The reader who is interested in a recent review on urban transportation network design may refer to Farahani et al. (2013).

Because of the complexity of this problem, seeking for stable and efficient algorithms for it has been a continuous effort in the past decades. Abdulaah and LeBlanc (1979) formulated a CNDP as a nonlinear optimisation problem and used the Powells and Hooke-Jeeves (H-J) methods respectively to test the formulated model on a medium-sized network. Given a convex investment function, both the Powells and H-J methods produced the same results but when a concave investment function was used, those results from the two approaches differed substantially, and the H-J method provided a better solution. Gairing et al. (2013) revisited a classical CNDP (bilevel) in which the edge based on the ratio of the flow and the capacity. Although different approximation algorithms were proposed, results regarding proposed estimate improved and guaranteed for a closed form network. The CNDP that arises when users receive information about uncertain network has been studied in comprehensively in the work of Unnikrishnan and Lin (2012). In this research, a bilevel mathematical programming network design formulation of CNDP was solved using the quantum inspired genetic algorithm (GA) and a generic GA. Sawansirikul et al. (1987) employed the equilibrium decomposition optimisation (EDO) method to solve the CNDP on several road networks, and their findings indicated that the heuristic method is more efficient than the H-J algorithm.

The effectiveness of the decomposition technique results in treating the original problem as a set of suboptimisation problems and the computational costs of the decomposed models were independent of the number of links. Marcotte and Marquis (1992), together with Marcotte (1983), presented some results of the CNDP on a small-sized network using the heuristic methods. Their approaches followed the Wardrop first principle or user equilibrium (UE) conditions. An integrated design of land use and transportation covers more than the conventional network design problem. The dimension down iterative algorithm (DDIA) for solving a mixed transportation network design problem expressed as a mathematical programming with equilibrium constraints (MPEC) studied by Liu and Chen (2016). The dimensions were developed to reduce the problem, and the findings showed that optimal solution for the problem without budget constraints achieved at little iteration and with a budget constraint, the result depends on the selection of initial values. Li et al. (2014) formulated this issue as a two-stage robust optimisation problem, in which the first stage optimises the land use and transport system by maximising a robust risk-averse objective function subject to chance constraints on system sustainability while the second stage captures residents behaviour of stochastic choice of locations and trip routes. Teng and Cheng (2015) solved EU assignment by considering the intersection delays and cost generated in the intersection contained in the upper-level model. The modified particle swarm optimisation algorithm developed to analyse the problem. A heuristic solution algorithm developed by combining the penalty function method, simulated annealing (SA) method and Gauss-Seidel decomposition approach to solve this two-stage model. An artificial bee colony algorithm (ABC) was proposed in Szeto et al. (2015) search for network design solutions of the upper-level problem while the method of successive averages (MSA) and the Frank-Wolfe algorithm were adopted to solve the lower-level time-dependent land-use transportation problem. The similar algorithm used to solve the undirected capacitated arc routing problem with profit to multi-class vehicle routing problems by Tunchan (2013). Despite the difficulties in optimising discrete problem and the new usage of ABC found to be effective, efficient, robust and comparable than other approaches. Haas and Bekhor (2016) proposed an alternative heuristic that simplifies the solution process of a discrete network design problem significantly and was tested on simple networks and applied for a real-size network. The reference also discusses the trade-offs between solution accuracy and computation time.

Comprehensive reviews concerning sensitivity analysis-based heuristic algorithms have been seen in Friesz et al. (1990) and Xu et al. (2012). Numerical results from these references revealed that the derived information could show the priority of a link needed for expansion. Yang (1997) reported that when the sensitivity analysis method was applied to solve the network equilibrium problem with elastic demand, the flow capacity of each link was bounded from above. The SA-based method was adopted by Friesz et al. (1992) to solve two different CNDPs, and their findings showed that the heuristic method performed better than the iterative optimisation assignment, H-J and EDO algorithms. Likewise, Xu et al. (2009) used the SA-based method and GA to solve the CNDP. Their numerical experiments indicated that when demand was large, the SA-based method performed better than the GA-based one in solving the CNDP and the GA-based method required more computing efforts to achieve the same optimal solution as one from the SA-based approach. At the low demand level, GA produced an excellent result but with a higher computing time. Huang and Bell (1999) applied the H-J, the equilibrium decomposed optimisation heuristic and SA-based methods to compare the numerical outputs of a continuous equilibrium network design problem for fixed and elastic demand. Their findings showed that these three algorithms generated very different results when the various values of the parameters of the algorithm were used whereas the SA-based method among them gave the best solution but its requirement on computing time was high too. Baskan (2013b) introduced the cuckoo search (CS) algorithm with Lévy flights for determining optimal link capacity expansion and found that the proposed algorithm can produce the optimal or near-optimal solution to the link capacity expansion problem. Malairajan et al. (2013) suggested refining heuristic algorithm to solve multi-commodity network flow problem with varying capacity. Results indicated that the proposed approach perform well regarding solution quality and computational time.

Bilevel programming is a method often used for the formulation of the CNDP, in which the upper level often represents the planners or authorities objective(s) aiming to minimise the total delay or transport emissions or maximise social welfare while the lower level captures traveller choice behaviour. Ban et al. (2006a, 2006b) proposed a relaxation method to solve the CNDP formulated as a bilevel program whose lower level is a link node-based nonlinear complementarity problem (NCP). The proposed algorithm produced encouraging results after a series of numerical tests on various networks. Karoonsoontawong and Waller (2006) employed the SA-based, the GA-based and the random search methods to solve the CNDP. The investigation in this reference indicated that the GA-based approach could converge faster with a better solution quality than the others. Also, they emphasised that the algorithm parameters must be specific for a particular network and optimised to achieve best results for different road networks.

Li and Lang (2014) proposed the bilevel programming model for the CNDP based on the stochastic UE model. The route-based self-regulated averaging (SRA) algorithm designed to solve the problem and the GA employed to obtained optimal solution of the upper objective function. The proposed algorithms (GA and SRA) were useful in solving simple numerical examples. Davis (1994) considered a stochastic user equilibrium (SUE) assignment procedure at the lower level of a bilevel program formulated for a CNDP and applied the generalised descent gradient and sequential quadratic programming (SQP) algorithms to solve the resulting bilevel program. It was concluded in this reference that the SUE-constrained version of the network design problem leading to differentiability offers the heuristics an opportunity to solve deterministic UE-constrained problems. Meng et al. (2014) employed a self-adaptive prediction-correction (PC) algorithm incorporated with the cost-average (CA) method to solve an asymmetric SUE problem with elastic demand and link capacity constraints. The optimal solutions of the CNDP from both SA-based and GA-based methods were investigated in Xu et al. (2009) on a small-sized network in three scenarios differing in demand level. It was noted in this reference that when the demand was significant, the SA-based method produced a better solution than the GA-based one. However, to obtain the same optimal solution to the CNDP, they suggested using SA because the GA-based method was costlier regarding the computing time. These findings are contrary to the observations in Mathew and Sharma (2009), which shows that GA was more efficient than the other heuristics for solving the CNDP. Similarly, Baskan (2013a) and Baskan and Ceylan (2013) respectively used the harmony search and differential evolution (DE) algorithms to solve the bilevel programs of the CNDP. Liu and Kachitvichyanukul (2015) employed a particle swarm optimisation algorithm for solving a capacitated location routing problem. Based on the framework of particle swarm optimisation with multiple social learning terms, a solution represented both depot and customer element. The proposed approach found to be suitable for large problem instances, and a total of nine new best solutions were obtained. Sakawa and Matsui (2016) introduced a bilevel linear programming with fuzzy random variables to fulfil the fuzzy goal. Findings showed that Stackelberg solutions could be achieved by solving the perturbed bilevel linear programming problem using the combined variable transformation method and the K^{th}-best algorithms.

In modelling the CNDP, different objectives have been sought in the literature. Yang and Wang (2002) compared two different objective functions, which aimed to minimise the total system cost for a limited budget and to maximise reversed capacity, respectively. Their numerical examples indicated that the use of a weighted sum of the two objective functions was more practical than an unweighted sum of them. Wang et al. (2014) investigated those models and relaxation methods for the CNDP with a tradable credit scheme and equity constraints and found that the proposed algorithm converged quickly to the optimal solution. Gao and Song (2002) used an integrated method to maximise the reserve capacity of a road network and suggested that it be important to associate the reserved capacity with the CNDP to capture the information necessary for the planner. On the other hand, Chiou (2008) proposed a projected quasi-Newton method to determine optimal link capacity expansions and increased travel demand. Her numerical experiments on signalised road networks showed that the proposed method was superior to the traditional ones. Chio (2016) considered an urban traffic network (UTN) accounting for signal setting and link capacity expansion under uncertain travel demand. A trust-region, cutting plane projection developed to solve the problem. Despite a significant role, they played in addressing different types of CNDPs, heuristic methods cannot guarantee global convergence. After discussing properties (including upper bounds) of the price of anarchy, Szeto and Wang (2015) used this conception to model the traditional UE network design problem and reliability-based network design problem. Bhavathrathan and Patil (2015) present a methodology to quantify resilience of transportation networks that are subject to recurring capacity disruptions. System-optimal total travel time at full capacities is usually adopted as a performance benchmark on networks and capacity degradation results in different capacity combinations, and thus, different network performance. A critical state was defined as an upper bound of network cost under recurring capacity degradation. This reference formulated the fundamental state link disruption problem as a minimax optimisation problem, where expected system travel time was maximised on the probability of recurrence and minimised on link flow. Chen and Chen (2013) presented a new network design problem (DNDP), where the variables were series of integer rather than 0-1. The combined branch-and-bound with Hooke-Jeeves algorithm were proposed to analyse capacity for reconstruction road networks and capacity grades of newly added roads. Numerical examples were provided to demonstrate the efficiency of the proposed algorithm.

Formulating the CNDP as one or a series of single-level optimisation problems is a way to avoid the difficulty in solving a bilevel program of the CNDP. To overcome the drawbacks observed in solving a bilevel program formulated for the CNDP, Meng et al. (2001) transformed the CNDP into a single continuous, differentiable optimisation problem and then applied the Augmented Lagrange Method to solve it. Chiou (2005) reformulated a CNDP as a bilevel programming problem and used a descent approach to solving it via the implementation of gradient-based methods. Four variant gradient-based methods were implemented in this reference, and the numerical results from three test networks showed that the proposed method outperformed regarding computational time than the others, especially on congested road networks. Similarly, Gao et al. (2007) converted a bilevel program for the CNDP into a single-level convex model and presented a globally convergent algorithm to solve it and showed that their methodology had some advantages over the existing heuristic algorithms. As noted in Wang and Lo (2010), the CNDP can be converted into a single-level optimisation problem subject to equilibrium constraints, in which the equilibrium constraints contain a set of mixed integer constraints and travel time functions are linear. Masoomeh et al. (2016) reformulated the transportation discrete designed problem (DNDP) as a mixed integer bilevel mathematical problem, based on the concept of reserved capacity. A hybrid GA with SA and an evolutionary SA algorithm was proposed to solve different numerical examples. Likewise, Sujeet and Shiv (2016) applied intuitionist fuzzy numbers to formulate the transportation problem in which costs, supplies, and demands were all triangular. The proposed approach modified the distribution method were suggested to obtain the optimal solution. Li et al. (2012) converted the CNDP into a sequence of single-level concave programs and applied global optimisation methods to solve it. This approach is capable of providing a global optimum for a large-sized network design problem. Baskan and Dell'Orco (2012) used the ABC algorithm to solve the CNDP and their numerical results obtained on small-sized example networks showed that the algorithms can generate a better solution than those from the use of SA- or GA-based algorithms in terms of the objective function value and the number of UE assignment procedures. Baskan and Ozan (2015) applied bilevel solution methodology based on DE algorithm to determine the signal timings and optimal capacity expansion for CNDP. Results indicated that the proposed algorithm could efficiently solve signalising road networks and capacity expansion problem.

Until today, most of the research on the CNDP assumes that the transportation demand is known and given. However, since realised travel demand and the resultant flow pattern depend heavily on various parameters embedded in demand and supply functions, the investment in a road network tends to influence the travel demand in the long run. Therefore, it is not suitable to use the CNDP with the assumption of fixed travel demand because the future demand itself depends partially on the network capacity for optimisation. Also, traffic assignment with elastic demand sounds more behaviourally and can easily be applied to solve general transportation problems with the assumption that the travel demand for each origin-destination (OD) pair was influenced by the level of services represented by travel costs (Gartner, 1980; Sheffi, 1985).

It is widely known that the bilevel programming of the CNDP can be reformulated as a mathematical program with equilibrium constraints (MPEC). However, MPECs fail to satisfy conventionally used constraint qualifications, such as Mangasarian-Fromovitz constraint qualification and linear independence constraint qualification (Luo et al., 1996), even for simple complementarity constraint $x \geq 0$, $y \geq 0$, $x^T y = 0$. For this reason, conventional solution techniques for optimisation problems cannot be applied directly to solve the CNDP with a guarantee of finding its optimal global solution(s) (Yang and Bell, 2001).

In this paper, the general UE assignment problem under elastic demand is formulated as a link path-based NCP and a general CNDP was further established as a mathematical program with complementarity constraints (MPCC), which fails to satisfy the constraint qualification of complementarity constraints. The smoothed Fischer-Burmeister function was adopted to relax the complementarity constraints. The proposed method relaxes the strict complementarity condition using a set of parameters. Although the smoothed Fischer-Burmeister function has been employed in many papers (see Scholtes (2001) and references herein), the convergence analysis of these smoothing methods is based on the popularly used nonlinear problem (NLP) and the convergence results are described regarding accumulation points of the generated sequences. However, variational analysis was used as a tool to analyse the convergence and the solution path to present the convergence result. The perturbed problem for the CNDP is a nice NLP, in which the linear independence constraint qualification holds at every feasible point. Therefore, the SQP method can be used to obtain its solutions efficiently. A set of numerical examples implemented to illustrate the ability of the proposed method in solving the CNDP.

The key contribution of this work includes:

a The formulation of the general CNDP as a MPCC. Such a reformulation permits a more general route cost structure and every stationary point of it corresponds to an optimal global solution of the perturbed problem.

b The variational analysis employed to analyse the convergence of the relaxation problems instead of using the conventional nonlinear programming methodology.

c Since the convergence of the proposed approach is guaranteed, a 16 link-based network and the Sioux Falls network implemented and solved to demonstrate the effectiveness and versatility of the model.

The proposed methodology, including models and algorithms, is not only applicable to a large class of network design problems (Zhang and Ge, 2004; Ekstrom et al., 2014; Wang et al., 2014) but also to those problems that can be formulated as a bilevel program [like network reserve capacity as discussed in Ge et al. (2003) and Wang et al. (2015b); integrative land use and transport modelling by Li et al. (2014)].

This paper is organised as follows. Section 2 formulates a model for the CNDP subject to link expansion. Section 3 proposes the MPCC and a perturbation-based approach for the CNDP. Section 4 presents two numerical examples to show the capability of the proposed model plus perturbation-based approach. Finally, concluding remarks plus future directions of research on the CNDP are given in Section 5.

2 Model formulation

Consider a road network $G = (N, A)$ in which a set of nodes, denoted by N is connected by a set of links, denoted by A. Let r and s be an origin and a destination in the network.

The set of OD pairs in G is represented by W. Each OD pair w is connected by a set of paths (routes) represented by K_w. Let q_w and u_w respectively be the demand and the minimum travel time/cost between an OD pair w. The flow and travel time/cost on link a are given by x_a and t_a, respectively, where t_a is the function of x_a and the corresponding incremental capacity y_a of link a. In addition, f_k^w and c_k^w are the flow and travel time/cost experienced by travellers along the path $k \in K_w$; $\delta_{a,k}^w = 1$ if link a is part of path k connecting OD pair w and 0 otherwise. Here is a list of additional notations used in this paper:

- $D_w(u_w)$ is the demand function between OD pair w, and $q_w = D_w(u_w)$
- OD is the origin-destination node(s)
- $D_w^{-1}(q_w)$ is the inverse of the demand function, where it is defined $\int_0^{q_w} D_w^{-1}(s)ds = h_w(q_w),\ \forall w \in W$
- q denotes the vector of q_w, $\forall w \in W$
- c_a is the capacity of link a, $\forall a \in A$
- $g_a(y_a)$ is the cost of improving link a by expanding the capacity of a link a with an incremental amount equal to y_a, where g_a is a twice continuously differentiable and non-decreasing function
- $\overline{y}_a$ is the upper bound for the link capacity expansion y_a, $\forall a \in A$
- θ is the conversion coefficient converting investment cost to travel cost
- $\delta_{\overline{K}}(\overline{E}(z,u,v))$ is the indicator function
- $B_\delta(\overline{z},\overline{u},\overline{v})$ is the open ball with centre $(\overline{z},\overline{u},\overline{v})$ and radius δ.

The objective of the CNDP of interest is to find the optimal capacity expansion for a set of selected links in a given network. The planner aims to minimise the total cost plus investment cost while taking into account the road users response to the path capacity expansion. It is assumed that travellers follow the Wardrop first principle or UE conditions while choosing their routes (Sheffi, 1985) and that the CNDP can be treated as a leader-follower or Stackelberg game, where the leader is the supplier (planner), and the follower is the road users (Fisk, 1984). In transportation systems, it is well known that the leader can only influence the road users' decision on route choice by introducing some measures such as traffic congestion charging, traveller information. As a result, it is possible to utilise the existing road capacity very efficiently regardless the growth of travel demand. In the elastic demand assumption, the road users' economic benefit (UEB) can be evaluated by

$$UEB = \sum_{w \in W} \int_0^{q_w} D_w^{-1}(s)ds - \sum_{a \in A} t_a\left(x_a, y_a\right)x_a - \theta \sum_{a \in A} a_a\left(y_a\right) \tag{1}$$

where the first term in the objective function is the value of the actual trips being made (user benefits), the second term is the cost of time spent by all users in the system (social cost), and the third term is the investment cost. The optimal solution to this problem can be achieved when the UEB is maximised. While maximising UEB, it is constrained by $0 \le y_a \le \bar{y}_a$ $(\forall a \in A)$ and the pattern (q, x) is determined by road users' response or their choice behaviour. In the leader's problem, the goal is to maximise the total social welfare (UEB), subject to the budget constraints represented by the upper bound on y_a and the relationship of the resulting (demand, flow) pattern (q, x) to the link capacity expansion y.

The pattern (q, x) can be obtained by solving the following elastic demand UE traffic assignment problem:

$$\begin{aligned}
&\min_x f = \sum_{a \in A} \int_0^{x_a} t_a(s, y_a)\,ds - \sum_{w \in W} \int_0^{q_w} D_w^{-1}(s)\,ds \\
&\text{s.t.} \sum_{k \in K_w} f_k^w - q_w, && \forall w \in W, \\
&\qquad f_k^w \ge 0, && \forall w \in W, \forall k \in K_w, \\
&\qquad q_w \ge 0, && \forall w \in W, \\
&\qquad x_a = \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w \le c_a + y_a, && \forall a \in A.
\end{aligned} \tag{2}$$

where the first set of constraints represents flow conservation, the second and third sets are non-negativity constraints respectively on path flows and O-D demand, and the last set of constraints restrains the flow on each link from above by the resulting capacity of the link.

Now a general bilevel programming problem for a CNDP can be formulated as follows:

- The leader problem

$$\begin{aligned}
&\min_y F = \sum_{a \in E} t_a(x_a, y_a) x_a + \theta \sum_{a \in A} g_a(y_a) - \sum_{w \in W} \int_0^{q_w(y)} D_w^{-1}(s)\,ds \\
&\text{s.t. } 0 \le y_a \le \bar{y}_a, \quad \forall a \in A, \\
&\qquad x = x(y).
\end{aligned}$$

where the link flow pattern $x = x(y)$ and $q_w(y)$ are determined by following UE problem.

- The follower problem

$$\min_x f = \sum_{a \in A} \int_0^{x_a} t_a(x, y_a)\,ds - \sum_{w \in W} \int_0^{q_w} D_w^{-1}(s)\,ds \tag{3}$$

$$\begin{aligned}
&\text{s.t.} \sum_{k \in W_w} f_k^w = q_w, && \forall w \in W, \\
&\qquad x_a = \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w \le c_a + y_a, && \forall a \in A, \\
&\qquad f_k^w = 0, && \forall w \in W, \quad \forall k \in K_w, \\
&\qquad q_w \ge 0, && \forall w \in W.
\end{aligned}$$

Remark 1: The leader problem is a minimisation problem whose objective function equals negative one times the objective function in the maximisation problem (1).

The lower level of the above bilevel programming problem (2)–(3) is a convex optimisation problem that can be equivalently transformed to the Karush-Kuhn-Tucker (KKT) optimality conditions as follows:

$$
\begin{aligned}
& f_k^w\left(c_k^w-\gamma_w+\sum_{a\in A}\lambda_a\delta_{a,k}^w\right)=0, && \forall w\in W,\quad \forall k\in K_w,\\
& \left(c_k^w-\gamma_w+\sum_{a\in A}\lambda_a\delta_{a,k}^w\right)\ge 0, && \forall w\in W,\quad \forall k\in K_w,\\
& q_w\left(\gamma_w-D_w^{-1}\left(q_w\right)\right)=0, && \forall w\in W,\\
& \gamma_w-D_w^{-1}\left(q_w\right)\ge 0, && \forall w\in W,\\
& \lambda_a\left(c_a+y_a-\sum_{w\in W}\sum_{k\in K_w}f_k^w\delta_{a,k}^w\right)=0, && \forall a\in A,\\
& c_a+y_a-\sum_{w\in W}\sum_{k\in K_w}f_k^w\delta_{a,k}^w\ge 0, && \forall a\in A,\\
& q_w-\sum_{k\in K_w}f_k^w=0, && \forall w\in W,\\
& q_w\ge 0, && \forall w\in W,\\
& f_k^w\ge 0, && \forall w\in W,\quad \forall k\in K_w,\\
& \lambda_a\ge 0, && \forall a\in A,\\
& \gamma_w\ge 0, && \forall w\in W.
\end{aligned}
\tag{4}
$$

which can be written in compact form as follows:

$$
\begin{aligned}
& 0\le f_k^w\perp\left(c_k^w-\gamma_w+\sum_{a\in A}\lambda_a\delta_{a,k}^w\right)\ge 0, && \forall w\in W,\quad \forall k\in K_w,\\
& 0\le q_w\perp\left(\gamma_w-D_w^{-1}\left(q_w\right)\right)\ge 0, && \forall w\in W,\\
& 0\le\lambda_a\perp\left(c_a+y_a-\sum_{w\in W}\sum_{k\in K_w}f_k^w\delta_{a,k}^w\right)\ge 0, && \forall a\in A,\\
& q_w-\sum_{k\in K_w}f_k^w=0, && \forall w\in W,\\
& q_w\ge 0, && \forall w\in W,\\
& f_k^w\ge 0, && \forall w\in W,\quad \forall k\in K_w.
\end{aligned}
\tag{5}
$$

Combining the upper level problem (2) with the KKT conditions (4), then CNDP can be reformulated as the following MPCC:

$$
\begin{aligned}
&\min_{y} F = \sum_{a \in A} t_a\left(x_a, y_a\right) x_a + \theta \sum_{a \in A} g_q\left(y_a\right) - \sum_{w \in W} \int_0^{q_w(y)} D_w^{-1}(s) ds \\
&\text{s.t.} \quad 0 \le f_k^w \perp \left(c_k^w - \gamma_w + \sum_{a \in A} \lambda_a \delta_{a,k}^w\right) \ge 0, && \forall w \in W, \quad \forall k \in K_w, \\
&\qquad 0 \le q_w \perp \left(\gamma_w - D_w^{-1}\left(q_w\right)\right) \ge 0, && \forall w \in W, \\
&\qquad 0 \le \lambda_a \perp \left(c_a - y_a - \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w\right) \ge 0, && \forall a \in A, \\
&\qquad q_w - \sum_{k \in K_w} f_k^w = 0, && \forall w \in W, \\
&\qquad x_a - \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w = 0, && \forall a \in A, \\
&\qquad 0 \le y_a \le \bar{y}_a, && \forall a \in A, \\
&\qquad \lambda_a \ge 0, && \forall a \in A, \\
&\qquad \gamma_w \ge 0, && \forall w \in W, \\
&\qquad q_w \ge 0, && \forall w \in W, \\
&\qquad f_k^w \ge 0, && \forall w \in W, \quad \forall k \in K_w.
\end{aligned}
\tag{6}
$$

Let $l(\upsilon)$ denote the number of components of a vector υ. Define

$$z = (y; q; f; x; \lambda; \gamma),$$

$$F_0(z) = \sum_{a \in A} \left(t_a\left(x_a, y_a\right) x_a + \theta g_a\left(y_a\right)\right) - \sum_{w \in W} h_w\left(q_w\right),$$

$$K = \left\{0_{l(q)+l(x)+l(\lambda)}\right\} \times \left[0, \bar{y}_a\right] \times \Re^{l(f)} + l(\gamma),$$

$$E(z) = \begin{bmatrix} q_w - \sum_{k \in K_w} f_k^w, & \forall w \in W, \\ x_a - \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w, & \forall a \in A, \\ y_a, & \forall a \in A, \\ \gamma, & \forall w \in W, \end{bmatrix},$$

$$G(z) = \begin{bmatrix} f_k^w, & \forall w \in W, \quad \forall k \in K_w \\ q_w & \forall w \in W, \\ \lambda_a & \forall a \in A, \end{bmatrix},$$

$$H(z) = \begin{bmatrix} \left(c_k^w - \gamma_w + \sum_{a \in A} \lambda_a \delta_{a,k}^w\right), & \forall w \in W, \quad \forall k \in K_w, \\ \gamma_w - D_w^{-1}\left(q_w\right), & \forall w \in W, \\ \left(c_a + y_a - \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w\right), & \forall a \in A. \end{bmatrix}$$

where $l(q)$, $l(x)$, $l(\lambda)$, $l(f)$ and $l(\gamma)$ denote the length of q, x, λ, f and γ, respectively. Then problem (6) can be put in the general framework of MPCC in the following standard form:

$$\text{(MPCC)} \qquad \begin{aligned} &\min F_0(z) \\ &\text{s.t. } \; 0 \le G(z) \perp H(z) \ge 0, \\ &\qquad E(z) \in K, \end{aligned} \tag{7}$$

where $F_0 : \mathbb{R}^n \to \mathbb{R}$, $G, H : \mathbb{R}^n \to \mathbb{R}^m$, $E : \mathbb{R}^n \to \mathbb{R}^p$ are smooth functions and $K \subset \Re^p$ is a closed convex set.

3 Perturbation-based approach

Perturbation-based approach is proposed to solve the MPCC problem (7), as follows. First, let us rewrite problem (7) in the following form:

$$\begin{aligned} &\min_{x,u,v} F_0(z) \\ &\text{s.t. } \; 0 \le u \perp v \ge 0, \\ &\qquad G(z) - u = 0, \\ &\qquad H(z) - v = 0, \\ &\qquad E(z) \in K. \end{aligned} \tag{8}$$

Define

$$\bar{K} = \{0_m\} \times \{0_m\} \times K,$$

$$\bar{E}(z, u, v) = \left[G(z) - u;\, H(z) - v;\, E(z)\right]$$

Then problem (3) is equivalent to

$$\text{(P)} \qquad \begin{aligned} &\min_{z,u,v} \bar{F}(z, u, v) \\ &\text{s.t. } 0 \le u \perp v \ge 0. \end{aligned} \tag{9}$$

where

$$\bar{F}(z, u, v) := F(x) + \delta_{\bar{K}}\left(\bar{E}(z, u, v)\right)$$

and

$$\delta_{\bar{K}}\left(\bar{E}(z, u, v)\right) = \begin{cases} 0 & (z, u, v) \in \bar{K}, \\ \infty & (z, u, v) \notin \bar{K}. \end{cases}$$

Then z minimises F over $\bar{K}$ if and only if (z, u, v) minimises $F(x) + \delta_{\bar{K}}(\bar{E}(z, u, v))$ over $\bar{K}$.

Now, the focus is on solving problem (3), which is still an MPCC problem. For such a problem, even if $\bar{F}$ is smooth, it is not suitable to treat it as a traditional nonlinear programming problem because, as explained in examples 3.1.1–3.1.2 in Luo et al. (1996), even the basic constraint qualification (namely the tangent cone is equal to the linearised cone at an optimal solution) does not hold. In formulated model (3), the Mangasarian-Fromovitz constraint qualification does not hold and the boundedness of the set of Lagrange multipliers are not guaranteed although numerical algorithms usually require this boundedness property for NLP problems. To overcome this difficulty, various relaxation approaches have been proposed to deal with the complementarity constraints. Facchinei et al. (1999) and Fukushima and Pang (1999) used $\phi_\mu(a, b) = 0$ to approximate the complementarity relation $0 \le a$, $0 \le b$, $ab = 0$, where φμ(a, b) is the smoothed Fischer-Burmeister function expressed as follows:

$$\phi_\mu(a, b) = a + b - \sqrt{a^2 + b^2 + 2\mu^2}. \tag{10}$$

Scholtes (2001) used

$$a \ge 0, b \ge 0, ab \le \mu,$$

and recently Lin and Fukushima (2005) proposed the following

$$(a + \mu)(b + \mu) \ge \mu^2 \text{ and } ab \le \mu^2$$

to relax the complementarity relationship of a and b. This paper adopted the smoothed Fischer-Burmeister function to deal with the complementarity constraints. Thus, the perturbation problem (7) is defined as follows:

$$(\mathrm{P}_\mu) \quad \begin{aligned} &\min_{z,u,v,\lambda} \bar{f}(z, u, v) \\ &\text{s.t. } \Psi_\mu(u, v) = 0, \end{aligned} \tag{11}$$

where

$$\Psi_\mu(u, v) = \begin{bmatrix} \psi_\mu(u_1, v_1) \\ \vdots \\ \psi_\mu(u_m, v_m) \end{bmatrix}$$

and ψ_μ is defined by equation (10). The difference between the methodology adopted in this paper and that of Facchinei et al. (1999) and Fukushima and Pang (1999) is that the paper considered variational analysis technique in Rockafellar and Wets (1998) to establish the convergence property of the solution set SOL(P_μ) to SOL(P). Let

$$\Omega(\mu) := \{(z, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m : \Psi_\mu(u, v) = 0\}$$

be the feasible region of problem (P_μ). Obviously, $\psi_0(a, b) = 0$ if and only if $0 \le a$, $0 \le b$, $ab = 0$. Therefore, Ω(0) is the feasible set of MPCC problem 3. Since z is not constrained in the problem (P_μ), then $\Omega(\mu)$ can be written in the following form:

$$\Omega(\mu) := \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^m : \Psi_\mu(u, v) = 0\}. \tag{12}$$

The convergence of the perturbation-based approach is analysed by demonstrating the convergence of $\Omega(\mu)$ to Ω(0) as $\mu \searrow 0$.

Proposition 1: For $\Omega(\mu)$ defined by equation (12), there is $\lim_{\mu \searrow 0} \Omega(\mu) = \Omega(0)$.

Proof: For any $(u, v) \in \limsup_{\mu \searrow 0} \Omega(\mu)$, there exist $\mu_k \searrow 0$ and $(u^k, v^k) \in \Omega(\mu_k)$ such that $(u^k, v^k) \to$ (u, v). The inclusion $(u^k, v^k) \in \Omega(\mu_k)$ implies

$$u^k + v^k - \sqrt{\left(u^k\right)^2 + \left(v^k\right)^2 + 2\mu_k^2} = 0.$$

Then, letting $k \to \infty$, we have

$$u + v - \sqrt{u^2 + v^2} = 0,$$

namely $\Psi_0(u, v) = 0$ and $(u, v) \in \Omega(0)$. Therefore, we have

$$\limsup_{\mu \searrow 0} \Omega(\mu) \subset \Omega(0).$$

For any $(u, v) \in \Omega(0)$, let

$$I_+ = \{i : u_i > 0\}, J_+ = \{i : v_i > 0\},$$

$$I_0 = \{1, \ldots, m\} \setminus (I_+ \cup J_+).$$

For any $\mu > 0$ defined $(u(\mu), v(\mu))$ by

$$\left(u_i(\mu), v_i(\mu)\right) = \begin{cases} \left(u_i, \mu^2/u_i\right) & \text{if } i \in I_+, \\ \left(\mu^2/v_i, v_i\right) & \text{if } i \in J_+, \\ (\mu, \mu) & \text{if } i \in I_0, \end{cases}$$

Then $\psi_\mu(u_i(\mu), v_i(\mu)) = 0$ for $i = 1, \ldots, m$ or equivalently $\Psi_\mu(u(\mu), v(\mu)) = 0$ or $(u(\mu), v(\mu)) \in \Omega(\mu)$. Obviously, $(u(\mu), v(\mu)) \to (u, v)$ and this implies that

$$\liminf_{\mu \searrow 0} \Omega(\mu) \supset \Omega(0).$$

Therefore, $\Omega(\mu) \to \Omega(0)$ as $\mu \searrow 0$. □

Let us introduce following notations:

$$\kappa(\mu) := \inf\left\{\bar{f}(z, v, v) \middle| (u, v) \in \Omega(\mu)\right\},$$

$$S(\mu) := \arg\min\left\{\bar{f}(z, u, v) \middle| (u, v) \in \Omega(\mu)\right\}.$$

The following theorem shows the convergence of the smoothing approach for solving the MPCC problem, which is characterised by using the terminology in variational analysis.

Theorem 2: Assume that $\bar{f}$ is level-bounded and then the function $\kappa(\mu)$ is continuous at 0 with respect to $\mathbb{R}_+$ and the set-valued mapping $S(\mu)$ is outer semi-continuous at 0 with respect to $\mathbb{R}_+$.

Proof: As $\overline{f}$ is level-bounded, $\kappa(\mu)$ is finite and $S(\mu) \neq \varnothing$ for any $\mu \geq 0$. Let

$$\hat{f}_\mu(z, u, v) = \overline{f}(z, u, v) + \delta_{\Omega(\mu)}(u, v),$$

where $\delta_{\Omega(\mu)}$ is the indicator function of $\Omega(\mu)$. From Proposition 1, $\Omega(\mu) \to \Omega(0)$ as $\mu \searrow 0$, $\hat{f}_\mu$ epi-converges to $\hat{f}_0$. The level-boundedness of $\hat{f}_\mu$ is easily verified for $\mu \geq 0$. Therefore, as it was reported from Theorem 7.41 of Rockafellar and Wets (1998) that the function $\kappa(\mu)$ is continuous at 0 with respect to $\mathbb{R}_+$ and the set-valued mapping $S(\mu)$ is outer semi-continuous at 0 with respect to $\mathbb{R}_+$. □

Now, this paper discussed the computational issue for problem (P_μ) when $\mu > 0$ is small enough. For any $\mu > 0$ and $x \in \mathbf{R}^n$, we have

$$\mathcal{J}_{u,v}\Psi_\mu(u, v) = \begin{bmatrix} \mathcal{J}_u\Psi_\mu(u, v) & \mathcal{J}_v\Psi_\mu(u, v) \end{bmatrix},$$

where

$$\mathcal{J}_u\Psi_\mu(u, v) = \begin{bmatrix} 1 - \dfrac{u_1}{\sqrt{u_1^2 + v_1^2 + 2\mu^2}} & & \\ & \ddots & \\ & & 1 - \dfrac{u_m}{\sqrt{u_m^2 + v_m^2 + 2\mu^2}} \end{bmatrix}$$

and

$$\mathcal{J}_v\Psi_\mu(u, v) = \begin{bmatrix} 1 - \dfrac{v_1}{\sqrt{u_1^2 + v_1^2 + 2\mu^2}} & & \\ & \ddots & \\ & & 1 - \dfrac{v_m}{\sqrt{u_m^2 + v_m^2 + 2\mu^2}} \end{bmatrix}$$

Obviously, for any $\mu > 0$ and (u, v) $\in \mathbf{R}^{2m}$, both $\mathcal{J}_u\Psi_\mu(u, v)$ and $\mathcal{J}_v\Psi_\mu(u, v)$ are non-singular matrices, we can easily obtain the following conclusion.

Proposition 3: Let $\mu > 0$. Then for any $(u, v) \in \Omega(\mu)$ the linear independence constraint qualification (LICQ) holds and the tangent cone of $\Omega(\mu)$ at (u, v) is

$$T_{\Omega(\mu)}(u, v) = \left\{(\Delta u, \Delta v) \in R^{2m} : \mathcal{J}_{u,v}\Psi_\mu(u, v)(\Delta u, \Delta v) = 0\right\}$$

and the normal cone of $\Omega(\mu)$ at (u, v) is

$$N_{\Omega(\mu)}(u, v) = \mathcal{J}_{u,v}\Psi_\mu(u, v)^T R^m.$$

Now the problem P_μ is written as follows:

$$\begin{aligned} &\min_{z,u,v} f_0(z) \\ &\text{s.t. } \bar{E}(z,u,v) \in \bar{K} \\ &\quad\ \Psi_\mu(u,v) = 0. \end{aligned} \tag{13}$$

The Lagrangian for problem P_μ is defined as

$$L(u,v,\lambda) = f_0(z)\langle \lambda, \Psi_\mu(u,v)\rangle + \langle \chi, \bar{E}(z,u,v)\rangle.$$

If $(\bar{z},\bar{u},\bar{v})$ is local minimiser for problem P_μ and basic constraint qualification (from Rockafellar and Wets, 1998) holds, namely

$$\left.\begin{aligned} 0 \in \mathcal{J}_{z,u,v}\bar{E}(\bar{z},\bar{u},\bar{v})^T \chi + \{0_n\} \times N_{\Omega(\mu)}(\bar{u},\bar{v}) \\ \chi \in N_{\bar{E}}(\bar{z},\bar{u},\bar{v}) \end{aligned}\right\} \Rightarrow \chi = 0, \tag{14}$$

then KKT conditions for P_μ are satisfied, namely

$$\mathcal{J}_{z,u,v} L(\bar{z},\bar{u},\bar{v},\bar{\lambda},\bar{\chi}) = 0, \quad \Psi_\mu(\bar{u},\bar{v}) = 0, \quad \bar{\chi} \in N_{\bar{E}}(\bar{z},\bar{u},\bar{v}), \tag{15}$$

namely,

$$\mathcal{J}_{z,u,v} L(\bar{z},\bar{u},\bar{v},\bar{\lambda},\bar{\chi}) = 0, \quad \Psi_\mu(\bar{u},\bar{v}) = 0, \quad \bar{\chi} \in N_{\bar{E}}(\bar{z},\bar{u},\bar{v}), \tag{16}$$

As the linear independence constraint qualification holds at any feasible solution of P_μ, we have that if there exists $\bar{\lambda}$ such that the above KKT condition holds, then $\bar{\lambda}$ is unique. The following proposition gives the second order sufficient conditions at a KKT point of P_μ.

Proposition 4: Let $(\bar{z},\bar{u},\bar{v},\bar{\lambda},\bar{\chi})$ be a KKT point for P_μ. Suppose the following condition hold:

$$\left\langle d, \nabla^2_{(z,u,v)} L(\bar{z},\bar{u},\bar{v},\bar{\lambda}) d \right\rangle > 0 \tag{17}$$

for $\forall d \neq 0$ satisfying $\mathcal{J}_{z,u,v}\Psi_\mu(\bar{u},\bar{v})(d_u; d_v) = 0$ and $\mathcal{J}_{z,u,v}E(\bar{z},\bar{u},\bar{v})d \in T_{\bar{K}}(\bar{E}(\bar{z},\bar{u},\bar{v}))$. Then the second order growth condition holds at $(\bar{u},\bar{v})$, namely there exist positive numbers $\gamma > 0$ and $\delta > 0$ such that

$$\begin{aligned} &f_0(z) - f_0(\bar{z}) \geq \gamma \left\| (z,u,v) - (\bar{z},\bar{u},\bar{v}) \right\|^2, \\ &\forall (z,u,v) \in [\Re^n \times \Omega(\mu)] \cap B_\delta(\bar{z},\bar{u},\bar{v}). \end{aligned}$$

where

$$\mathrm{B}_\delta(\bar{z},\bar{u},\bar{v}) = \left\{ (z,u,v) \in R^n \,\middle|\, \left\| (z,u,v) - (\bar{z},\bar{u},\bar{v}) \right\| < \delta. \right\}$$

Based on the above analysis, it is evident that the assumptions that govern the link travel cost function, demand function and the investment function are smooth. The MPCC for the CNDP can be converted to an NLP which also lacks desired mathematical properties due to the complementarity slackness constraints. However, using the proposed relaxation method the NLP for the CNDP can be solved. For that reason, applying a similar technique proposed above for analysis we can relax the MPCC of the CNDP and transform it to an NLP problem. One of the perturbations is based on the F-B function and the perturbed NLP (P_μ) for the CNDP is presented as:

$$\min Z = \sum_{a \in A} t_a\left(x_a, y_a\right) x_a + \theta \sum_{a \in A} g_a\left(y_a\right) - \sum_{w \in W} \int_0^{q_w(y)} D_w^{-1}(s) ds$$

$$\begin{aligned}
\text{s.t.}\quad & 0 \le y_a \le \bar{y}_a, && \forall a \in A, \\
& \psi_{FB}\left(f_k^w, \eta_a, \mu\right) = 0, && \forall a \in A, \forall w \in W, \forall k \in K_w, \\
& \psi_{FB}\left(\lambda_a, \zeta_a, \mu\right) = 0, && \forall a \in A, \\
& q_w = \sum_{k \in K_w} f_k^w, && \forall w \in W, \\
& x_a = \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w, && \forall a \in A, \\
& x_a, y_a, \lambda_a, \eta_a, \zeta_a, f_k^w, \gamma_w, q_w \ge 0, && \forall a \in A, \forall w \in W, \forall k \in K_w.
\end{aligned}$$

where

$$\eta_a = c_k^w - \gamma_w + \sum_{a \in A} \lambda_a \delta_{a,k}^w, \quad \zeta_a = c_a + y_a - \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w.$$

4 Numerical examples

This section uses two numerical examples to illustrate the capability of the perturbation-based approach to solving a CNDP. The first example is a road network with 16 links, six nodes and tow OD pairs, as shown in Figure 1 and the second one is a 76-link, 24-nodes network shown in Figure 3, i.e., an abstract road network of the city of Sioux Falls that is often used in the literature. The results were compared with the perturbation-based approach with those from the EDO, Hooke-Jeeves algorithm (H-J) and SA. All results are set up in the same scenario, i.e., CNDP with elastic demand.

The following typical Bureau of Public Roads (BPR) link travel time function and the exponential OD demand function (the same as that used in Sawansirikul et al. (1987) and Huang and Bell (1999), etc.) are used in both examples:

$$\begin{aligned}
& t_a\left(x_a, y_a\right) = A_a + B_a \left(\frac{x_a}{c_a + y_a}\right)^4, && \forall a \in A, \\
& D_w(\gamma_w) = \bar{q}_w \exp\left(-\alpha_w \gamma_w\right), && \forall w \in W.
\end{aligned}$$

where A_a, x_a and B_a respectively denote the free flow travel time, link flow and link specific constant, $\bar{q}_w$ represents the maximum level of travel demand between OD pair w and α_w is a positive parameter specific to OD pair w. $\bar{y}_a$ is set to 20 for all links in the experiments throughout this paper. In addition, all experiments are carried out using MATLAB 8.3.0 (2014a), on a 64-bit desktop computer with the Intel(R), Core(TM) 2 of 3.3 GHz CPU and 4 GB RAM running using Windows 7.

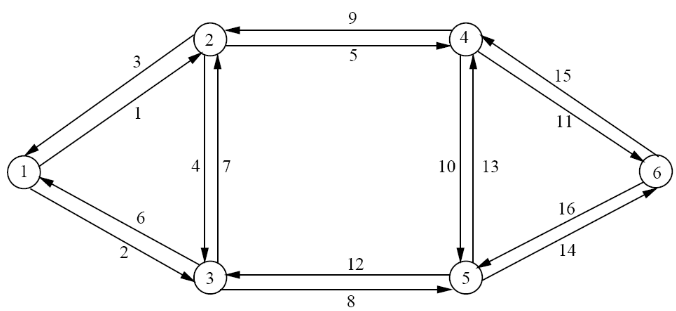
The following variables are used for interpretation of the numerical results presented in this paper:

- F is the objective function value at initial point
- EUE is the elastic demand UE with link capacity improvement y
- T is the temperature at which the algorithm stopped [this one only applied to SA-based methods]
- CPU time is the time required to achieve equilibrium between the shortest path cost and the value of inverse demand function.

4.1 Example 1: A 16-link, 6-node and 2-OD pair network

The network for this example is shown in Figure 1. The linear construction cost function for all links considered for improvement is given by $g_a(y_a) = \psi_a y_a$ and the link construction coefficient is set to 1.0 (i.e., $\theta = 1.0$). The maximum level of travel demand for OD pairs (1, 6) and (6, 1) is 10 and 20 and the parameters α_w in the demand function are 0.03 and 0.01, respectively. Table 1 presents the characteristics of these links, where a link a from node r to node s is denoted by (r, s).

Figure 1 A 16-link network



The NLP (ψ, μ) for the numerical presented as follows:

$$\min \sum_{a \in A} \left(A_a x_a + B_a x_a \left(\frac{x_a}{c_a + y_a} \right)^4 + \psi_a y_a \right) - \bar{q}_w e^{-\alpha_w}(s) ds$$

$$\begin{aligned}
\text{s.t.} \quad & 0 \le y_a \le \bar{y}_a, && \forall a \in A, \\
& \psi_{FB}\left(f_k^w, \eta_a, \mu\right) = 0, && \forall a \in A, \forall w \in W, \forall k \in K_w, \\
& \psi_{FB}\left(\lambda_a, \zeta_a, \mu\right) = 0, && \forall a \in A, \\
& q_w = \sum_{k \in K_w} f_k^w, && \forall w \in W, \\
& x_a = \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w, && \forall a \in A, \\
& \lambda_a, \eta_w, \zeta_a, f_k^w, \gamma_w, q_w \ge 0, && \forall a \in A, \forall w \in W, \forall k \in K_w.
\end{aligned}$$

where

$$\eta_a = c_k^w - \gamma_w + \sum_{a \in A} \lambda_a \delta_{a,k}^w$$

$$\zeta_a = c_a + y_a - \sum_{w \in W} \sum_{k \in K_w} f_k^w \delta_{a,k}^w$$

Results in Table 2 showed that the proposed method was able to solve the CNDP with elastic demand satisfactorily. It can be seen that, regarding the computational time, the SA-based approach was the most expensive one among all tested algorithms, at the different initial values of y. Furthermore, the solutions from these algorithms are different, but the general pattern of capacity expansion is nearly the same. For example, links (1, 3), (2, 3), (2, 4) and (3, 5) are expanded to the nearly same scale for all compared algorithms; and almost all the remaining links each has no or only a small amount of increase in capacity. Therefore, these results from the different algorithms are comparable. Thirdly, was changed the parameters in each method and found that this may lead to a big variation to the resulting pattern of capacity expansion, which shows that the parameter values may play a significant role in the resulting capacity expansion. Fourthly, the demand patterns from all algorithms are nearly the same. The fifth point tends to show that both EDO and SQP methods produced capacity expansion for link a = (6, 5) significantly different from the other two approaches, which implies that these algorithms do not always provide a solution at the same scale for all links.

Table 1 Data for a 16-link network

Link	*A*	*B*	*C*	ψ
(1, 3)	1	10	3	2
(1, 4)	2	5	10	3
(2, 5)	5	5	1	6
(2, 6)	6	1	4.5	1
(3, 1)	3	3	9	5
(3, 4)	4	20	4	4
(3, 5)	5	50	3	9
(4, 1)	2	20	2	1
(4, 3)	1	10	1	4
(4, 6)	1	1	10	3
(5, 2)	9	2	2	6
(5, 3)	2	8	45	2
(5, 6)	3	3	3	5
(6, 2)	2	33	20	3
(6, 4)	4	10	6	8
(6, 5)	4	25	44	5

Note that F represents the minus one time the social welfare. Therefore, it can be seen that the SA-based algorithm produced a better solution than the other algorithms found in the literature including the perturbation-based approach for solving the CNDP but required much more computational effort to achieve the best solution. The computational of EDO algorithm is more efficient than that obtained from the H-J algorithm. The SQP method had nearly the same performance as the SA-based method, but it took much less CPU computing times performance as the SA-based method, but it took much less CPU computing times. Figure 2 presents the value of the objective function at each iteration. It is observed that an actual value which is very close to the optimal value can be obtained even when the value of the auxiliary parameter is no trivial. For example, at iteration #25, the objective value is close to the final one (–1,896.57), at iteration #50. The results imply that to obtain the approximate solution to the original bilevel model, we do not need to satisfy the lower level UE conditions strictly. Similar findings can be observed in scenario 2.

Table 2 Solution from different algorithms

Links	*H-J*	*EDO*	*SA*	*SQP*
y(1, 3)	0.009	0.005	0.012	0.039
y(1, 4)	1.172	0.827	0.521	0.505
y(2, 5)	0.009	0.005	0.000	0.016
y(2, 6)	13.978	19.990	18.958	17.028
y(3, 1)	5.503	7.605	5.964	5.210
y(3, 4)	0.000	0.005	0.001	0.000
y(3, 5)	0.000	0.005	0.000	0.209
y(4, 1)	8.772	7.664	6.626	4.730
y(4, 3)	0.000	0.005	0.031	0.018
y(4, 6)	0.000	0.014	0.075	0.000
y(5, 2)	0.000	0.005	0.000	0.000
y(5, 3)	0.000	0.105	0.000	0.066
y(5, 6)	0.000	0.005	0.003	0.091
y(6, 2)	0.000	0.014	0.000	0.000
y(6, 4)	0.000	0.025	0.000	0.427
y(6, 5)	0.000	12.318	0.000	2.610
F	–1,882.93	–1,835.04	–1,897.86	–1,896.57
q(1, 2)	7.968	7.934	7.907	7.975
EUE	622	33	7,817	7,453
q(2, 1)	16.829	16.995	16.885	16.874
CPU(m)	5.417	1.433	50.916	3.621

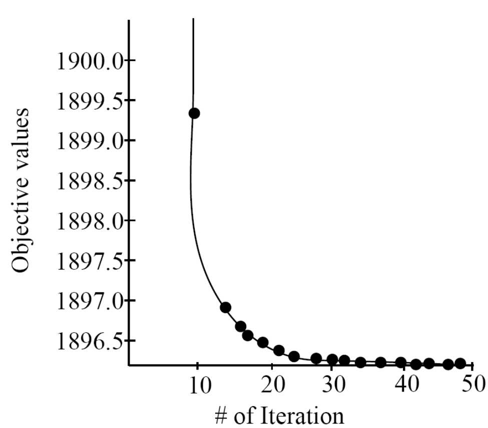
Notes: [a]$\eta = 0.01$ and $\alpha = 3$ for H-J, $\eta = 0.01$ and $d(6, 5) = 5.0$ for EDO, $\eta = 0.01$, $k_B = 0.001$, $T_f = 250$ and $T = 204.80$ for SA while for SQP, $\alpha = 3$ for each.
[b]$q(1, 6)$ and $q(6, 1)$ are the resulting demand between OD pairs (1, 6) and (6, 1), respectively.

Table 3 Solution from different algorithms

Links	*HJ*	*EDO*	*SA*	*SQP*
y(1, 3)	0.122	0.005	0.123	0.031
y(1, 4)	1.959	0.560	0.234	0.389
y(2, 5)	2.259	0.005	0.048	0.059
y(2, 6)	9.947	19.995	14.502	11.548
y(3, 1)	5.409	7.333	6.714	7.086
y(3, 4)	0.000	0.005	0.000	0.000
y(3, 5)	0.000	0.005	0.000	0.027
y(4, 1)	8.419	7.673	6.878	4.730
y(4, 3)	0.000	0.005	0.000	0.002
y(4, 6)	0.000	0.005	0.000	0.024
y(5, 2)	0.000	0.005	0.000	0.000
y(5, 3)	0.000	0.005	0.000	0.238
y(5, 6)	0.000	0.005	0.000	0.020
y(6, 2)	0.000	0.005	0.000	0.038
y(6, 4)	0.000	0.005	0.036	0.016
y(6, 5)	0.000	0.005	0.000	0.079
F	–1,897.51	–1,896.11	–1,899.26	–1,898.83
q(1, 2)	8.017	7.911	7.780	8.120
EUE	456	33	8,769	8,115
q(2, 1)	16.748	16.960	16.843	16.669
CPU(m)	5.950	1.467	44.718	7.877

Notes: [a]η =0.01 and α = 5 for H-J, η = 0.02 and $d(6, 5)$ = 30 for EDO, η = 0.02, k_B = 0.001, T_f = 250 and T = 204.80 for SA while for SQP, α = 5 for each.
[b]$q(1, 6)$ and $q(6, 1)$ are the resulting demand between OD pairs (1, 6) and (6, 1), respectively.

Figure 2 Convergence of a 16-link network



4.2 Example 2: Sioux Falls network

The link characteristics for the road network of Example 2 were adopted from Sawansirikul et al. (1987). The link construction cost function in this example is $g_a(y_a) = \psi_a y_a^2$ for all links and the expansion cost coefficient is set to $\theta = 0.001$. There are ten links (i.e., link 16, 17, 19, 20, 25, 26, 29, 39, 48 and 74), which is equivalent to five sets of the two-way links marked with dashed lines in Figure 3 as candidates considered for capacity improvements. All nodes in these links serves as both origin and destination (OD) and the sets are (16, 19), (17, 20), (25, 26), (29, 40) and (39, 74). The values of the parameter γ_w in the demand function are set to 2.05 for all OD pairs. The link data and the O-D travel demands for 552 OD pairs are the same as those used by Sawansirikul et al. (1987). All results for this example are given in Table 5.

The NLP(ψ, μ) for Example 2 is presented as follows:

$$\min \sum_{a\in A}\left(A_a x_a + B_a x_a \left(\frac{x_a}{c_a + y_a} \right)^4 + 0.001\psi_a y_a^2 \right) - \bar{q}_w e^{-\alpha_w}(s)ds$$

$$\begin{aligned}
\text{s.t.}\quad & 0 \le y_a \le \bar{y}_a, && \forall a \in A, \\
& \psi_{FB}\left(f_k^w, \eta_a, \mu\right) = 0, && \forall a \in A, \forall w \in W, \forall k \in K_w, \\
& \psi_{FB}\left(\lambda_a, \zeta_a, \mu\right) = 0, && \forall a \in A, \\
& q_w = \sum_{k\in K_w} f_k^w, && \forall w \in W, \\
& x_a = \sum_{w\in W}\sum_{k\in K_w} f_k^w \delta_{a,k}^w, && \forall a \in A, \\
& \lambda_a, \eta_w, \zeta_a, f_k^w, \gamma_w, q_w \ge 0, && \forall a \in A, \forall w \in W, \forall k \in K_w.
\end{aligned}$$

where

$$\eta_a = c_k^w - \gamma_w + \sum_{a\in A} \lambda_a \delta_{a,k}^w$$

$$\zeta_a = c_a + y_a - \sum_{w\in W}\sum_{k\in K_w} f_k^w \delta_{a,k}^w.$$

The Sioux Falls network solved 100 times, and the best average results given in Table 5 showed how the proposed method can be applied to determine the optimal solution for the CNDP associated with the Sioux Falls network. The results from the approach recommended in this paper are compared with those given in Huang and Bell (1999) from the other three different algorithms, as presented in Table 5. It is clear that the perturbation-based approach can produce a better solution among the four algorithms except the SA-based. Although the SA-based algorithm outperformed than the perturbation-based one, the objective function values (F) from both algorithms are quite close to each other.

In addition, the perturbation-based approach achieved good results with less computation efforts (CPU time) in solving the CNDP when compared to SA. It can also be seen that the perturbation-based method yielded an encouraging set of results for solving the CNDP. Although the SA-based algorithm outperformed over the proposed method, it requires more computational efforts in solving the traffic assignment problem.

Table 4 Data for the Sioux Falls network

Link	A_a	B_a	C_a	ψ_a
1–3	0.06	0.0090	25.9002	
2–5	0.04	0.0060	23.4035	
4–14	0.05	0.0075	4.9582	
6–8	0.04	0.0060	17.1105	
7–35	0.04	0.0060	23.4035	
9–11	0.02	0.0030	17.7828	
10–31	0.06	0.0090	4.9088	
12–15	0.04	0.0060	4.9480	
13–23	0.05	0.0075	10.000	
16–19	0.02	0.0030	4.8986	26.00
17–20	0.03	0.0045	7.8418	40.00
18–54	0.02	0.0030	23.4035	
21–24	0.10	0.0150	5.0502	
22–47	0.05	0.0075	5.0458	
25–26	0.03	0.0045	13.9158	25.00
27–32	0.05	0.0075	10.0000	
28–43	0.06	0.0090	13.5120	
29–48	0.05	0.0075	5.1335	48.00
30–51	0.08	0.0120	4.9935	
33–36	0.06	0.0090	4.9088	
34–40	0.04	0.0060	4.8765	
37–38	0.03	0.0045	25.9002	
39–74	0.04	0.0060	5.0913	34.00
41–44	0.05	0.0075	5.1275	
42–71	0.04	0.0060	4.9248	
45–57	0.04	0.0060	15.6508	
46–67	0.04	0.0060	10.3150	
49–52	0.02	0.0030	5.2299	
50–55	0.03	0.0045	19.6799	
53–58	0.02	0.0030	48,240	
56–60	0.04	0.0060	23.4035	
59–61	0.04	0.0060	5.0026	
62–64	0.06	0.0090	5.0599	
63–68	0.05	0.0075	5.0757	
65–69	0.02	0.0030	5.2299	
66–75	0.03	0.0045	4.8854	
70–72	0.04	0.0060	5.0000	
73–76	0.020	0.0030	5.0785	

Figure 3 Sioux Falls network

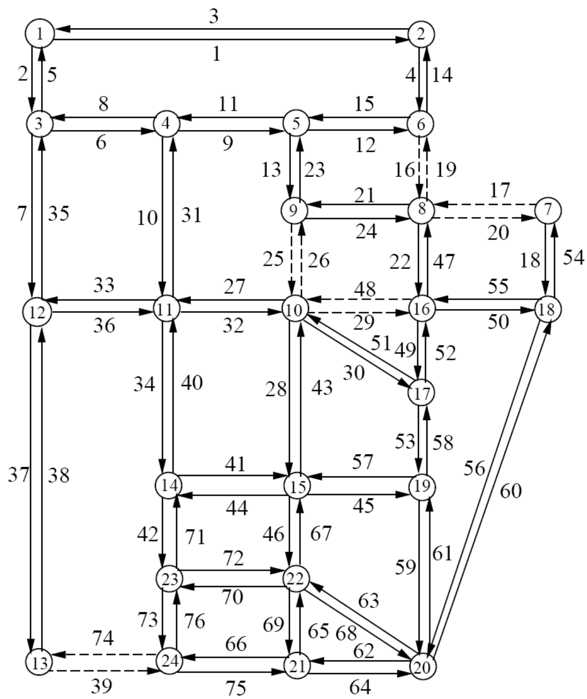


Table 5 Solutions from different algorithms

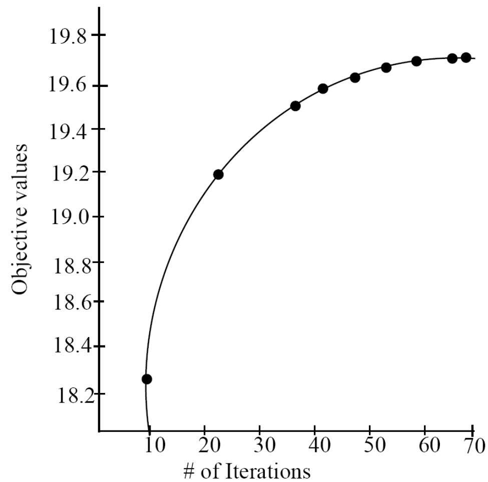
Link	*HJ*	*EDO*	*SA*	*SQP*
y(6, 8)	3.088	2.399	4.184	2.638
y(7, 8)	1.758	1.189	1.173	1.235
y(8, 6)	2.989	2.225	4.643	3.305
y(8, 7)	0.961	1.044	1.302	1.398
y(9, 10)	1.805	1.577	1.703	1.488
y(10, 9)	1.939	2.300	2.016	2.219
y(10, 16)	2.145	2.121	1.952	1.722
y(13, 24)	2.202	2.605	3.051	2.573
y(16, 10)	2.003	1.438	1.926	1.722
y(24, 13)	2.302	2.764	2.686	2.293
F	19.822	20.492	19.330	19.447
Demand	357.03	357.13	357.346	364.1995
EUE	345	45	14,822	1,869
CPU(m)	95.650	12.05	3,985.08	23.97

Notes: [a] $\eta = 0.01$ and $\alpha = 5.0$ for H-J, and $\eta = 0.02$ for EDO, $\eta = 0.02$, $k_B = 0.01$ and $T = 2.577$ while for SQP, $\alpha = 5$ for each.
[b] The total demand implemented in each EUE is solved using the 100 F-W iterations.

Figure 4 shows the convergence of perturbation-based approach on the Sioux Falls network. In this figure, the x-axis shows the number of iterations and y-axis is the objective value. It is observed that the perturbation-based approach converges faster especially when the optimal solution is approached. The perturbation-based approach only requires about 45 iterations to reach the accuracy level of the optimal solution.

It can be concluded that the findings on this Sioux Falls network are consistent with those on a 16-link network in Example 1.

Figure 4 Convergence of Sioux Falls network



5 Concluding remarks

This paper formulated a CNDP as a bilevel program, where the upper level aims to optimise the link capacity expansion vector to maximise the social welfare while the lower level determines the (demand, flow) pattern satisfying the variable demand UE conditions. To show the applicability of the proposed model to determine the link capacity expansions on a road network, the CNDP is reformulated as an MPCC model. A perturbation-based approach offered to solve this model has proven to be a rigorous method in solving MPCC problem which lacks a suitable set of constraint qualifications and convergence property. As revealed in this paper that the optimal global solution of the perturbed problem converges to one of the solutions to the MPCC problem.

The proposed model and perturbation-based approach were tested on both 16-link and 76-link networks; the latter one is the Sioux Falls network widely used in the literature on transportation network analysis. The numerical experiments showed that an optimal value of existing link capacities for proposed model can be obtained and it can minimise the sum of total impedance under constrained budget, which is useful reference for the policy makers (decision makers).

This model can be extended to the design of multi-modal urban transportation networks with an aim to improve the environment and/or human health related to transport emissions. For example, to introduce a transport emissions cost into the objective function as an extra cost at the upper level of the bilevel program for the CNDP and use a multi-modal traffic assignment model at the lower level. Jiang and Szeto (2015) proposed ‘a bilevel optimisation framework for time-dependent discrete road network design’ that “simultaneously considers the health impacts of road traffic emissions, traffic noise, and accidents due to network expansion”. The work in this reference does consider not only social and economic factors but also the environmental factor in a network design problem, which may support sustainable use of a road network. To examine the effect of road network design on landowner inequity and intergeneration inequity, Szeto et al. (2015) proposed a multi-objective bilevel optimisation model for capturing social, economic and environmental factors related to sustainability plus interaction between land use and transportation. The sustainable interaction between land use and transportation is achieved in an integrative model formulated in Li et al. (2014), another highlight of which is the consideration of uncertainty in future population. The proposed model with the proposed solution method can be used by the planner to allocate the budgets available and prioritise the links within a road network for expansion. Although the proposed model may be computationally time demanding and may take the time to find the optimal solution for large-sized network design, yet it can easily be converted to a smaller dimensional problem and solved. The numerical examples show that the proposed model can produce better solution of CNDP after solving the model several times with different values of the parameters. If all numerical results presented in this paper will be reproduced, different CPU computing times may be found because of different computer uses. As long as all algorithms test is made on the same computer, the resulting results from the different algorithms should be comparable.

A limitation of the perturbation-based approach proposed in this paper is that the need for path enumeration prevents the implementation of the algorithm on real-sized road networks. The formulated model in this article is a bilevel one, which upper and lower levels both are link-based. In the lower level, the definitional constraints on link flows involve path flows. As the first attempt of the efforts to solve this bilevel model, a perturbation-based approach is proposed on path flow. The work is still made on seeking for an algorithm for this problem on link flow. Another limitation of the work is that the link capacity expansion is treated as a continuous variable while it is discrete in real life because it is dependent on the number of lanes on a link (Wang et al., 2015a). How to extend the newly proposed approach for the discrete network design problem is another issue we are investigating. Although CNDP can be reformulated and solved using MPCC techniques, the approach is more efficient in identifying the local solutions and fails to guarantee a global optimum. The difficulty of determining exactly solution of CNDP stems from the reason that even if an optimum solution is obtained, there is no sure way of understanding whether the solution is the global optimum or not due to non-convexity.

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