Solve Symmetric TSP using ILP

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Contents

Abstract

In this report, I first formulate Traveling Salesman Problem(TSP) as Integer Linear Program(ILP), then collect data for landmarks around HangZhou, use SageMath to solve the ILP, get the exact solution.

Keywords: TSP, ILP

1 Formulation

TSP is a combinatorial optimization problem to find the shortest possible hamiltonian circuit for a complete weighted graph.

1.1 Definition

There are several key elements:

$\mathbf{s} = (s_i)$	At i step, the index of visited city is s_i, $0 \leq i \leq n-1$
$\mathbf{t} = (t_i)$	City i is visited at step t_i, $0 \leq i \leq n-1$
$\mathbf{X} = (x_{ij})$	Decision variable, $x_{ij} = \begin{cases} 1 & \text{the path goes from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$
$\pi(\cdot)$	$\pi(i) = j$ if $x_{ij} = 1$, $0 \leq i, j \leq n-1$
$\mathbf{C} = (c_{ij})$	Cost Matrix

Table 1: The notation that will be used

- Hamiltonian circuit means starting from vertex No. 0, ending at No. 0 and travel all vertexes once and only once. Note that $x_{ij}|_{i=n-1,j=0}$ will consider go back while t_i will not consider the time consumed when go back since there is *no* $t_i|_{i=n}$.
- For a real-world problem, the weights for graph are non-negative. But the weights do no necessarily be symmetric, because weights can be time or cost. I choose *unsymmetrical* TSP to solve for its generality. Thus, I should note that $\mathbf{C}$ is symmetric because the data I collect is distance while the solution $\mathbf{X}$ is unsymmetrical because it describe *directed* graph.

1.2 ILP formulation

We formulate this question step by step, first we list all sufficient constrains:

$$\text{Minmize} \quad \sum_{i=0}^{n-1} \sum_{j=0, j\neq i}^{n-1} c_{ij} x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{i=0, i\neq j}^{n-1} x_{ij} = 1 \qquad j = 0, \ldots, n-1; \tag{2}$$

$$\sum_{j=0, j\neq i}^{n-1} x_{ij} = 1 \qquad i = 0, \ldots, n-1; \tag{3}$$

$$x_{ij} \geq 0, x_{ij} \in \mathbb{Z} \qquad i, j = 0, \ldots, n-1.$$

The objective function (1) need to minimize costs, we simply drop x_{ii} throughout the problem, since $x_{ii} = 0$ is trivial. The constrains (2) means each city be arrived at from exactly one other city while equation (3) means from each city there is a departure to exactly one other city. We do not need to add $x_{ij} \leq 1$ since others are enough to conclude it.

But these constraints are not necessary conditions for TSP. This formulation in fact the formulation of Assignment Problem belong to P hard class. We still need to add constraints to avoid subtour and we will find that TSP belong to NP hard level class. I choose to add n

auxiliary variables so that I just need to add $n(n-1)$ constrain.

$$
\begin{aligned}
\text{Minmize} \quad & \sum_{i=0}^{n-1}\sum_{j=0,j\neq i}^{n-1} c_{ij}x_{ij} & & \\
\text{subject to} \quad & \sum_{i=0,i\neq j}^{n-1} x_{ij} = 1 & & j = 0,\ldots,n-1; \\
& \sum_{j=0,j\neq i}^{n-1} x_{ij} = 1 & & i = 0,\ldots,n-1; \\
& t_j \geq t_i + 1 - n(1-x_{ij})x & & 0 \leq i \leq n-1, 1 \leq j \leq n-1, i \neq j \\
& x_{ij} \geq 0, x_{ij} \in \mathbb{Z} & & i,j = 0,\ldots,n-1.
\end{aligned} \tag{4}
$$

Note that $j \geq 1$ in (4) means we do not consider salesman going back to No.0. In fact, if we let t_n denote the time step salesman go back to No.0, we have:

$$t_n = t_{n-1} + 1 = n \tag{5}$$

To prove correctness. First, Cyclic permutation without subtour conclude (4). Because (4) is equivalent to statement that if the next visited city for i is j ($x_{ij} = 1$) ,then $t_j = t_i + 1$. If $x_{ij} = 0$, then we add a trivial constraint $t_j \geq t_j + 1 - n$. The equivalent statement is satisfied. Second, (4) eliminates all subtour. We can prove it by tricky contrapositive. Suppose the feasible solution $\mathbf{X}$ contain more than one subtour. Then there exist $\mathbf{s} = (i_1,\ldots,i_r)$ not containing 0. Note that i_r in this circle will go back to i_1, so there is additional equation $t_{i_1} = t_{i_r} + 1$ quite different with (5). Go along this cycle, we find that $0 = r$, and the contradiction lies in the fact that the cycle do not containing 0.

2 Experiments

The steps to solve the ILP include: prepare location data for landmarks around HangZhou, solve the ILP by MILP class and visualize the final results on Map. The equivalent code for (4) is shown below:

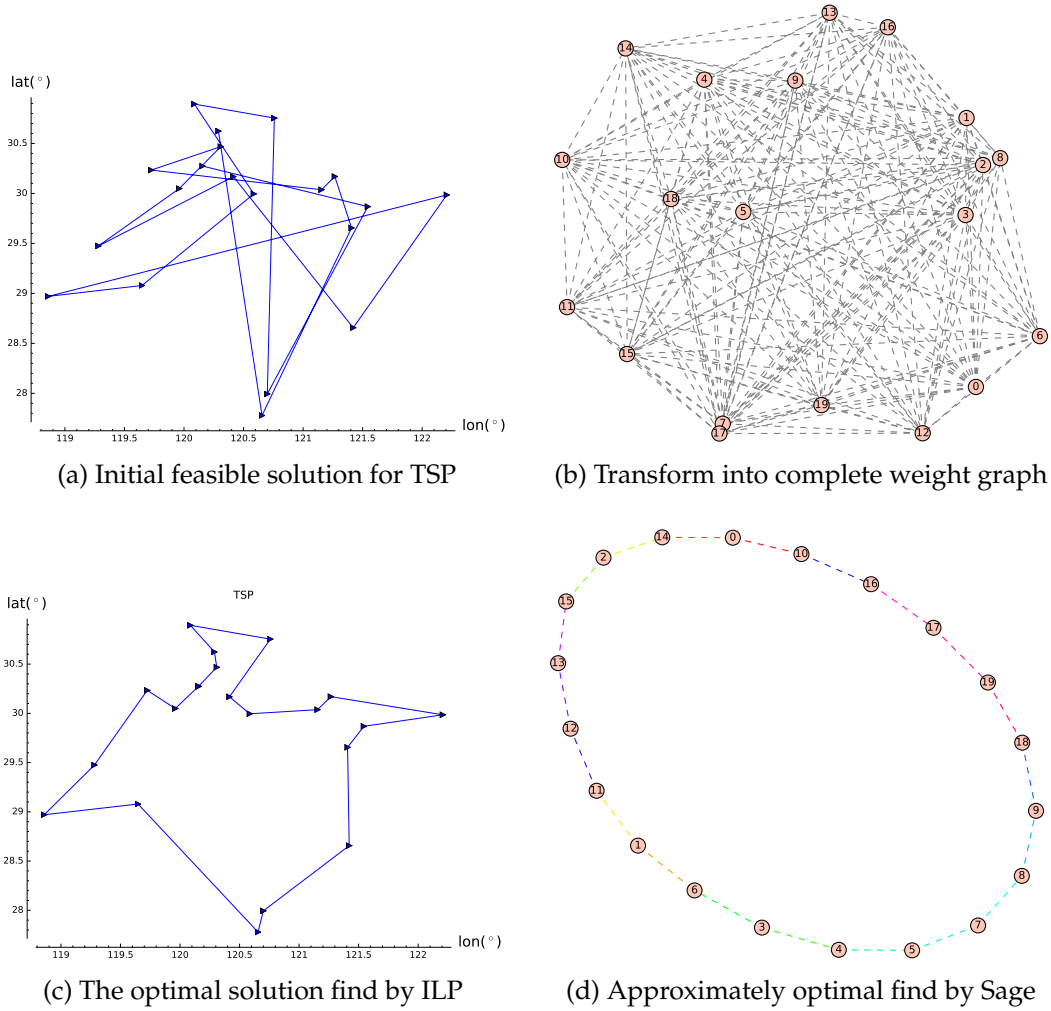
```
p=MixedIntegerLinearProgram(maximization=False)
# Define Varibles:
x=p.new_variable(nonnegative=True,integer=True)
t=p.new_variable(nonnegative=True,integer=True)
n=dist.nrows()
obj_func=0
# Define Obj Function:
for i in range(n):
    for j in range(n):
        obj_func+=x[i,j]*dist[i,j] if i!=j else 0
p.set_objective(obj_func)
# Add Constrains
for i in range(n):
    p.add_constraint(sum([x[i,j] for j in range(n) if i!=j ])==1)
for j in range(n):
    p.add_constraint(sum([x[i,j] for i in range(n) if i!=j ])==1)
for i in range(n):
    for j in range(1,n):
        if i==j :
```

```
            continue
        p.add_constraint(t[j]>=t[i]+1-n*(1-x[i,j]))
for i in range(n):
    p.add_constraint(t[i]<=n-1)

#p.show()
p.solve()
```

Apply ILP to a toy case with 20 landmarks around HangZhou, we get a optimal strategy to travel around HangZhou, which is quite interesting. Ref. to Fig 1c. Then I also use TSP solver in Sage to partially verify that my code is correct. The approximate optimal solution got by some heuristic algorithm is the same as the exact optimal solution. The I try to not just travelling Hangzhou, and travel JiangZheHu Region(Yangtze River delta). Visualized on the map, Ref. to Fig 2b on the next page.

I just take 20 landmarks from 49 collected coordinate data. The reason lies in I find 50 points consume seemingly endless time. I also have some basic test on running time, and find it grow explosively. Ref. to Fig 3 on the following page



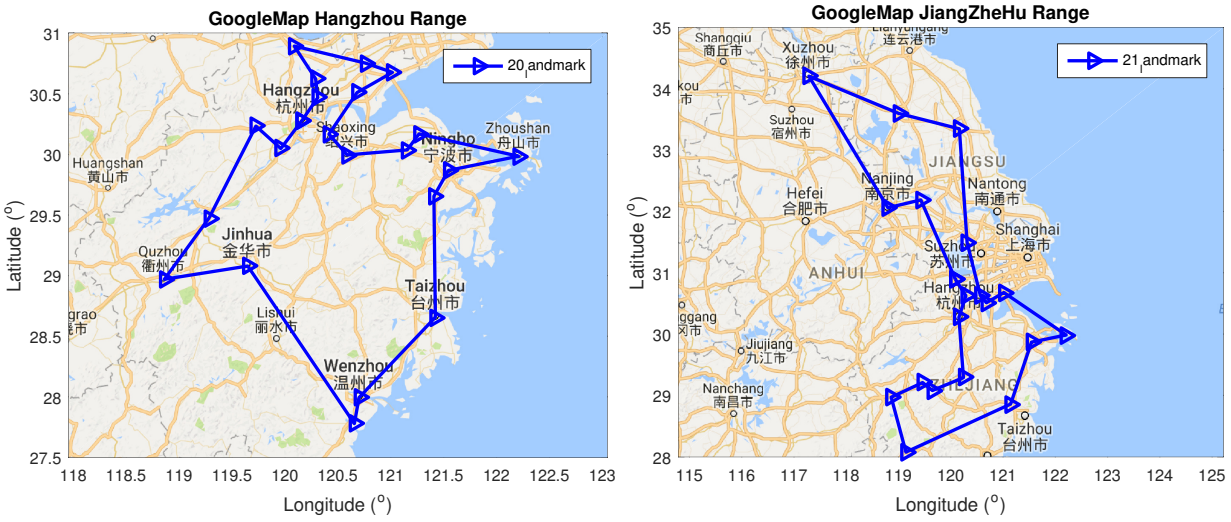
(a) Initial feasible solution for TSP

(b) Transform into complete weight graph

(c) The optimal solution find by ILP

(d) Approximately optimal find by Sage

Figure 1: The plot of TSP class 1d only keep topological relations, but after simple transformation, we find for 20-pts, two solutions $\mathbf{X}$ are equivalent. (Here the transformation means change from undirected graph's $\mathbf{X} = (x_{ij})$ with $i < j$ to directed graph's $\mathbf{X}$)



(a) TSP on Hangzhou (b) TSP on JiangZheHu

Figure 2: JiangZheHu is a wider range than Hangzhou. We limit number of landmarks to about 20 for speed.

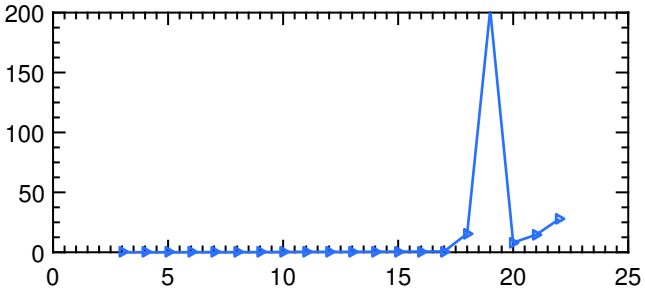


Figure 3: X-axis: number of landmarks, Y-axis: elapsed time in seconds. Running time starts to grow explosively at $n = 17$ and becomes $200s$ at $n = 19$ suddenly

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Appendices

Appendix A Lab Code

My codes are somewhat lengthy, so I just put some important code, hope not to impact the compactness of this report.

A.1 Crawling Data of CHINA Landmarks

```
import urllib,json,cPickle

f=open('city_name_pinyin.txt')
city_name=f.readlines()
city_name=[name.strip(' \t\r\n') for name in city_name]
f.close()
print city_name

key='AIzaSyDw12ZWkEypDDOJMbT3UPVouVQLLppeimM'
```

```
# address='Beijing'
location=[]
for address   in city_name:
    my_url = (
        'https://maps.googleapis.com/maps/api/geocode/json?address=%s&key=%s'
        ) % (address, key)
    print my_url
    try:
        response=urllib.urlopen(my_url)
    except urllib.error.HTTPError:
        print "Check Internet"
    else:
        print "Unkown"
    json_data=json.loads(response.read())
    print json_data

    loc=[json_data['results'][0]['formatted_address'],
        json_data['results'][0]['geometry']['location']]
    print loc
    location.append(loc)

print len(location),location
with open("location.pkl",'w') as f:
    cPickle.dump(location,f)
```

A.2 Solve By Sage

```
import numpy as np
import matplotlib.pyplot as plt
import sys, os, \
    glob, cPickle, \
    argparse, errno, json,\
    copy, re,time,datetime
import numpy as np
import os.path as osp
import scipy.io as sio
from pprint import pprint
def lldistkm(latlon1,latlon2):
    '''
     Distance:
     d1km: distance in km based on Haversine formula
     (Haversine: http://en.wikipedia.org/wiki/Haversine_formula)
    '''

    radius=6371
    lat1=latlon1[0]*pi/180
    lat2=latlon2[0]*pi/180
    lon1=latlon1[1]*pi/180
    lon2=latlon2[1]*pi/180
    deltaLat=lat2-lat1
    deltaLon=lon2-lon1
    a=sin((deltaLat)/2)^2 + cos(lat1)*cos(lat2) * sin(deltaLon/2)^2
    c=2*atan2(sqrt(a),sqrt(1-a))
    d1km=radius*c    #Haversine distance
    return d1km
def lldistkm_o(l1,l2):
    l1=vector(l1)
    l2=vector(l2)
```

```
    return norm(l2-l1)
root='/home/xlwang/l-opt'
os.chdir(root)
print os.getcwd()
with open('location.pkl','r') as f:
    locations =cPickle.load(f)

for m_limits  in range(3,23):
    old_t = time.time()

    location=locations[:m_limits]

    latlons=[ [latlon[1]['lat'],latlon[1]['lng']] for latlon in location ]
    names=[latlon[0].split(',')[0] for latlon in location]
    n_pts=len(latlons)
    tab=table( \
    rows=[(latlons[i][0],latlons[i][1]) for i in range(n_pts)],\
    header_column=names,\
    frame=True\
    )
    str=latex(tab)
    with open("city_data.tex",'w') as f:
        f.write(str)
    dist=np.zeros((n_pts,n_pts))
    for i in range(n_pts):
        for j in range(n_pts):
            dist[i,j]=N(lldistkm(latlons[i],latlons[j]),32)
    dist

    # For small  problem:
    #dist=Matrix([[0,1,2],[2,0,1],[1,2,0]])
    dist=Matrix(dist)
    p=MixedIntegerLinearProgram(maximization=False,solver="GLPK") #,solver="PPL"
    x=p.new_variable(nonnegative=True,integer=True)
    t=p.new_variable(nonnegative=True,integer=True)
    n=dist.nrows()
    obj_func=0
    for i in range(n):
        for j in range(n):
            obj_func+=x[i,j]*dist[i,j] if i!=j else 0
    p.set_objective(obj_func)
    for i in range(n):
        p.add_constraint(sum([x[i,j] for j in range(n) if i!=j ])==1)
    for j in range(n):
        p.add_constraint(sum([x[i,j] for i in range(n) if i!=j ])==1)
    for i in range(n):
        for j in range(1,n):
            if i==j :
                continue
            p.add_constraint(t[j]>=t[i]+1-n*(1-x[i,j]))
    for i in range(n):
        p.add_constraint(t[i]<=n-1)

    #p.show()
    p.solve()
    elaspe=time.time() - old_t
    print m_limits, elaspe
```

A.3 Draw on GMap

```
ccc
addpath(genpath('..\'));
all_pnt=load('res_latlon.txt');
figure,

lat=all_pnt(:,1);
lon=all_pnt(:,2);
lat_new=[];
lon_new=[];

for i=1:length(lat)
    longitude=lon(i);
    latitude=lat(i);
%     if ~(longitude<120.13 || longitude>120.21)
%         if ~(latitude<30.26 || latitude>30.3)
            lat_new(end+1)=lat(i);
            lon_new(end+1)=lon(i);
%             continue;
%         end
%     end

end
lat=lat_new;
lon=lon_new;

plot(lon,lat,'MarkerSize',8,'Marker','>',...
    'LineWidth',2,...
    'Color',[0 0 1])

plot_google_map
set(gcf,'Color','White');
legend('21_landmark')

% create xlabel
xlabel('Longitude (^o)');

% create ylabel
ylabel('Latitude (^o)');
title('GoogleMap JiangZheHu Range')
export_fig .\l-opt\GMap_Jzh.pdf
```