Numerical Analysis Homework #1

due 2021 OCT 12, 9:50 a.m.

1 Assignments

Caution:

- To get full credit, *you must write down sufficient intermediate steps*, only giving the final answer earns you no credit!
- Please make sure that your handwriting is recognizable, otherwise you only get partial credit for the recognizable part.

I. Consider the bisection method starting with the initial interval $[1.5, 3.5]$. In the following questions "the interval" refers to the bisection interval whose width changes across different loops.

- What is the width of the interval at the nth step?
- What is the maximum possible distance between the root r and the midpoint of the interval?

II. In using the bisection algorithm with its initial interval as $[a_0, b_0]$, we want to determine the root with its *relative* error no greater than ϵ. Assume $a_0 > 0$. Prove that the number of steps n must satisfy

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2} - 1.$$

III. Perform four iterations of Newton's method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

IV. Consider a variation of Newton's method in which only the derivative at x_0 is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

Find C and s such that

$$e_{n+1} = Ce_n^s.$$

V. Within $(-\frac{\pi}{2}, \frac{\pi}{2})$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

VI. Let $p > 1$. What is the value of the following continued fraction?

$$x = \frac{1}{p + \frac{1}{p + \frac{1}{p + \dots}}}$$

Prove that the sequence of values converges. (Hint: this can be interpreted as $x = \lim_{n \to \infty} x_n$, where $x_1 = \frac{1}{p}$, $x_2 = \frac{1}{p + \frac{1}{p}}$, $x_3 = \frac{1}{p + \frac{1}{p + \frac{1}{p}}}$, ..., and so forth. Formulate x as a fixed point of some function.)

VII. What happens in problem II if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in II. In this case, is the relative error still an appropriate measure?

VIII. $(*)$ Consider solving $f(x) = 0$ by Newton's method with the starting point x_0 close to a root of multiplicity k. We assume that $f \in \mathcal{C}^{k+1}$. Note that α is a zero of multiplicity k of the function f iff

$$f^{(k)}(\alpha) \neq 0; \qquad \forall i < k, \quad f^{(i)}(\alpha) = 0.$$

- How can a multiple zero be detected by examining the behavior of the points $(x_n, f(x_n))$?
- Prove that if r is a zero of multiplicity k of the function f, then quadratic convergence in Newton's iteration will be restored by making this modification:

$$x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}.$$

Each of the first six problems weighs 5 points. Each of the other problems weighs 10 points. In particular, the last problem is for extra credit and you do not have to solve it. However, *special students* such as my graduate students who audit this class have to solve all problems.

2 C++ programming

All programming assignments in Section 1.8.2.

3 Extra credits

Additional 10% credits will be given to you if you typeset your solutions in LaTeX. You are welcome to use the LaTeX template. You can also get partial extra credit for typesetting solutions of *some* problems.

Note: If you choose to typeset your solutions in LaTeX, you still need to turn in a hard copy in class. In addition, please upload your latex source (.tex), supporting files, and C++ program in a single zip file (**format**: `YourName_Homework1.zip`) to the course email `NumApproximation@163.com`.

4 Reading

Download the electronic version of the following book from the library website:

- Numerical Analysis by W. Gautschi, 2nd Edition. ISBN: 978-0-8176-8259-0.

The above book will be referred to as NAG2012 in the rest of this class. Students who are eager to learn are encouraged to read pages 55-112 and 159-195.

Note: If you are a special student, you have to print out these pages and read the text before this semester ends.