Supplementary Material for Response Surface Designs for Crossed and Nested Multi-Stratum Structures

In this document, we describe the algorithms used to construct designs and give the designs obtained for the examples illustrated in the paper and their efficiencies. A brief sensitivity study is given to explore the dependence of designs on the criterion weights.

1 Exchange Algorithms

The point and coordinate exchange algorithms for row-column designs are presented in Algorithm 1 and Algorithm 2, respectively. It should be noted that, except for the precise definition of the matrix $\mathbf{Q}$, the algorithms are essentially the same for completely randomized or blocked designs.

Algorithm 1: Point Exchange Algorithm

Input: number of factors K; candidate set CAND; number of rows r; number of columns c; model terms formula; number of random starting designs $Ntries$; weights for compound criteria $\kappa_D, \ldots, \kappa_{DF}$; weight vector for model parameters $\mathbf{W}$; significance level for pure error adjustments α_{DP} and α_{LP}; logical for multiple comparisons correction MC

Output: Optimized design matrix $\mathbf{X}$

```
n ← r · c
N ← number of rows of CAND
T ← create treatment labels for CAND rows
Q ← matrix as in equation (6)
for m ← 1 to Ntries do
    det ← 0
    while det ≤ 0 do
        X ← model matrix (no intercept) for randomly sampled design from CAND
        det ← det(X'QX)
    t ← treatment labels for the randomly sampled design
    crit ← evaluate equation (7)
    improve ← 1
    while improve == 1 do
        improve ← 0
        for i ← 1 to n do
            for j ← 1 to N do
                if t[i]! = T[j] then
                    temp_X ← X
                    temp_treat ← t
                    Exchange temp_X[i, 1 : K] ← CAND[j, ]
                    Exchange temp_treat[i] ← T[j]
                    Complete columns of temp_X[i, ] according to the model
                    crit_temp ← evaluate equation (7)
                    if crit_temp > crit then
                        improve ← 1
                        crit ← crit_temp
                        X[i, ] ← temp_X[i, ]
                        t[i] ← temp_treat[i, ]
if m == 1 then
    X_best ← X
    crit_best ← crit
else
    if crit > crit_best then
        X_best ← X
        crit_best ← crit
return X_best
```

Algorithm 2: Coordinate Exchange Algorithm

Input: number of factors K; factor $levels[K][.]$; number of rows r; number of columns c; model terms formula; number of random starting designs $Ntries$; weights for compound criteria $\kappa_D, \ldots, \kappa_{DF}$; weight vector for model parameters $\mathbf{W}$; significance level for pure error adjustments α_{DP} and α_{LP}; logical for multiple comparisons correction MC

Output: Optimized design matrix $\mathbf{X}$

```
n ← r · c
Q ← matrix as in equation (6)
for m ← 1 to Ntries do
    det ← 0
    while det ≤ 0 do
        X ← model matrix (no intercept) for randomly generated design
        det ← det(X′QX)
    t ← treatment labels for X[, 1 : K]
    crit ← evaluate equation (7)
    improve ← 1
    while improve == 1 do
        improve ← 0
        for i ← 1 to n do
            for j ← 1 to K do
                x_current ← X[i, j]
                foreach coordinate ∈ levels[j] do
                    if coordinate! = x_current then
                        temp_X ← X
                        Replace temp_X[i, j] ← coordinate
                        t ← treatment labels for temp_X[, 1 : K]
                        Complete columns of temp_X[i, ] according to the model
                        crit_temp ← evaluate equation (7)
                        if crit_temp > crit then
                            improve ← 1
                            crit ← crit_temp
                            X[i, ] ← temp_X[i, ]
if m == 1 then
    X_best ← X
    crit_best ← crit
else
    if crit > crit_best then
        X_best ← X
        crit_best ← crit
return X_best
```

2 Example 1

The designs constructed for Example 1 are presented in Table A in an unrandomized order. Randomization should be done by randomizing the Days labels to days and the Times labels to times of the day. The efficiencies of these designs, relative to the $D^{\star}$ design and for fixed levels of the ratios of variance components, are shown in Table B.

Table A: Designs for Example 1, a row×column structure Days(7)*Times(4), with three 3-level factors

Design $D^{\star}$

	Days																				
Times	1			2			3			4			5			6			7		
1	0	-1	-1	-1	0	-1	0	0	0	-1	-1	1	1	1	0	1	-1	1	-1	1	1
2	-1	1	0	-1	-1	1	1	-1	-1	1	1	1	-1	0	-1	0	1	-1	1	-1	1
3	1	1	-1	1	-1	0	1	1	1	-1	1	-1	0	-1	1	-1	0	1	-1	-1	-1
4	1	0	1	0	1	1	-1	1	-1	0	0	0	1	-1	-1	-1	-1	0	1	1	-1

Design MSS_{D_S}

	Days																				
Times	1			2			3			4			5			6			7		
1	1	1	1	-1	1	1	0	-1	-1	1	1	-1	1	-1	1	0	0	0	-1	0	1
2	-1	-1	1	0	-1	1	-1	1	0	-1	-1	-1	1	0	-1	1	-1	-1	1	1	1
3	-1	1	-1	-1	0	-1	1	1	-1	0	0	0	0	1	1	-1	-1	1	1	-1	0
4	1	-1	-1	1	1	0	1	0	1	-1	1	1	-1	-1	0	-1	0	1	0	1	-1

Design $\mathrm{MSS}_{(DP)_S}$

	Days																				
Times	1			2			3			4			5			6			7		
1	-1	0	0	-1	-1	-1	-1	-1	1	0	1	0	1	-1	-1	-1	1	1	1	-1	1
2	1	1	1	1	-1	0	-1	1	-1	1	1	1	-1	1	-1	0	0	1	-1	-1	-1
3	1	-1	-1	0	0	1	0	1	0	-1	0	0	-1	-1	1	1	1	-1	-1	1	1
4	-1	-1	1	0	0	1	-1	0	0	1	-1	-1	0	1	0	1	-1	1	1	1	-1

Design $\mathrm{MSS}_{CP_{\kappa}}$

Days

Table A: Designs for Example 1 (Continued)

Times	1			2			3			4			5			6			7		
1	-1	0	-1	1	0	0	-1	-1	1	0	1	0	1	1	1	-1	1	1	0	-1	-1
2	0	1	0	-1	1	-1	1	1	1	-1	0	-1	1	-1	-1	1	-1	1	-1	-1	1
3	-1	1	1	0	-1	-1	0	0	1	1	1	-1	-1	1	-1	-1	-1	0	1	0	0
4	-1	-1	0	1	1	1	1	-1	-1	1	-1	1	0	0	1	1	1	-1	-1	1	-1

Table B: D_S- and A_w-efficiencies, relative to the $D^\star$ design, of the row×column designs for Example 1

		Criterion							
		D_S Designs				A_w Designs			
η_{Days}	η_{Times}	D_S	$(DP)_S$	$CP_{\boldsymbol{\kappa}}$	M_w[1]	D_S	$(DP)_S$	$CP_{\boldsymbol{\kappa}}$	M_w[1]
1	1	99.87	84.49	93.23	82.75	100.46	78.46	90.67	75.23
10	1	99.97	83.02	92.19	82.11	100.52	76.76	89.36	74.94
100	1	99.98	82.83	92.06	82.03	100.53	76.54	89.18	74.90
1	10	99.70	83.65	93.04	82.52	100.27	77.41	90.41	74.89
10	10	99.80	82.18	91.99	81.88	100.33	75.75	89.07	74.61
100	10	99.81	81.99	91.86	81.80	100.34	75.53	88.89	74.57
1	100	99.68	83.55	93.01	82.49	100.25	77.28	90.37	74.85
10	100	99.78	82.08	91.97	81.86	100.31	75.62	89.04	74.57
100	100	99.79	81.89	91.83	81.77	100.32	75.40	88.86	74.53

[1]: Design I from Table 3 of Gilmour and Trinca (2003).

3 Example 2

The designs constructed for Example 2 are presented in Table C. Treatments are presented in an unrandomized order. To execute the experiment appropriate randomization is required. For that, randomize the Days labels to days and randomize the Periods labels to periods of the day.

The efficiencies of the designs, relative to the $D^\star$ design and for known values of the variance component ratios, are shown in Table D.

Table C: Designs for Example 2, a split-row×column structure Days(26)*Periods(2), with 1 HS and 4 ES three-level factors

$D^{\star}$

		Morning				Afternoon			
Day	X_1	X_2	X_3	X_4	X_5	X_2	X_3	X_4	X_5
1	−1	−1	−1	−1	−1	1	1	1	−1
2	−1	−1	0	1	1	−1	1	−1	−1
3	−1	−1	1	1	−1	1	−1	1	1
4	−1	−1	0	0	−1	1	1	−1	0
5	−1	0	−1	−1	1	−1	−1	1	−1
6	−1	0	−1	1	0	1	0	−1	1
7	−1	1	−1	−1	−1	−1	1	−1	1
8	−1	1	−1	0	1	0	0	−1	−1
9	−1	1	−1	1	−1	−1	1	1	1
10	−1	1	1	−1	−1	0	−1	1	0
11	−1	1	1	1	1	−1	−1	−1	1
12	0	−1	−1	1	1	1	−1	−1	0
13	0	0	1	0	0	1	−1	1	−1
14	0	0	1	1	−1	−1	0	0	0
15	0	1	1	−1	1	−1	0	−1	−1
16	1	−1	−1	−1	1	1	1	−1	−1
17	1	−1	−1	1	−1	1	1	0	−1
18	1	−1	0	1	0	1	−1	−1	1
19	1	−1	1	−1	−1	1	1	1	1
20	1	−1	1	−1	1	0	−1	0	−1
21	1	1	−1	0	−1	−1	1	1	−1
22	1	1	−1	−1	0	0	1	0	1
23	1	1	−1	1	1	−1	−1	−1	−1
24	1	1	0	1	−1	−1	−1	1	1
25	1	1	1	−1	1	−1	1	1	1
26	1	1	1	1	−0.11	0	0	−1	1

MSS_{D_S}

		Morning				Afternoon			
Day	X_1	X_2	X_3	X_4	X_5	X_2	X_3	X_4	X_5
1	−1	−1	−1	−1	−1	1	1	1	−1
2	−1	−1	−1	0	1	1	−1	−1	−1
3	−1	−1	−1	1	−1	1	−1	−1	1
4	−1	−1	0	−1	−1	−1	−1	1	1
5	−1	−1	1	1	1	1	−1	1	−1
6	−1	1	−1	−1	0	−1	1	−1	1
7	−1	1	1	−1	−1	1	−1	1	1
8	−1	1	1	−1	1	−1	1	1	−1
9	−1	1	1	1	1	−1	−1	−1	1
10	0	−1	1	1	0	0	0	0	−1
11	0	0	−1	1	1	1	0	0	0
12	0	0	0	1	0	−1	1	0	−1
13	0	0	1	0	0	−1	0	1	1
14	0	1	−1	0	−1	−1	0	−1	0
15	0	1	−1	1	−1	0	1	−1	−1
16	0	1	0	0	−1	1	1	−1	1
17	0	1	1	1	−1	0	0	0	0
18	1	−1	−1	1	1	0	1	1	−1
19	1	−1	1	−1	−1	0	1	1	1
20	1	−1	1	−1	1	−1	−1	1	−1
21	1	−1	1	1	−1	1	−1	−1	−1
22	1	0	−1	1	−1	1	1	1	1
23	1	1	−1	−1	1	−1	1	1	1
24	1	1	−1	1	1	−1	−1	−1	−1
25	1	1	0	−1	1	1	−1	1	0
26	1	1	1	−1	−1	−1	−1	−1	1

Table C: Designs for Example 2 (Continued)

$\mathrm{MSS}_{(DP)_S}$

		Morning				Afternoon			
Day	X_1	X_2	X_3	X_4	X_5	X_2	X_3	X_4	X_5
1	−1	−1	−1	−1	1	1	−1	1	−1
2	−1	−1	0	−1	−1	−1	1	0	1
3	−1	−1	0	−1	−1	−1	1	0	1
4	−1	0	−1	0	1	−1	1	1	0
5	−1	0	−1	0	1	−1	1	1	0
6	−1	1	−1	−1	0	−1	0	1	1
7	−1	1	1	0	−1	0	−1	1	0
8	−1	1	1	1	0	−1	1	−1	1
9	−1	1	1	1	1	−1	1	1	−1
10	0	−1	0	−1	1	0	1	−1	−1
11	0	−1	−1	1	1	0	0	0	−1
12	0	1	0	0	−1	−1	−1	0	−1
13	0	1	0	0	−1	−1	−1	0	−1
14	0	1	0	0	−1	−1	−1	0	−1
15	0	1	1	−1	1	1	−1	−1	−1
16	0	0	1	0	0	1	−1	−1	1
17	0	0	1	1	−1	1	0	0	0
18	1	−1	1	−1	−1	−1	−1	1	−1
19	1	−1	1	1	−1	−1	−1	−1	1
20	1	0	−1	−1	−1	1	1	1	−1
21	1	1	−1	1	−1	−1	1	1	1
22	1	1	1	−1	−1	1	−1	1	1
23	1	1	1	−1	−1	1	−1	1	1
24	1	1	1	−1	−1	1	−1	1	1
25	1	1	1	0	1	−1	0	0	0
26	1	1	1	0	1	−1	0	0	0

$\mathrm{MSS}_{CP_{\kappa}}$

		Morning				Afternoon			
Day	X_1	X_2	X_3	X_4	X_5	X_2	X_3	X_4	X_5
1	−1	−1	1	−1	−1	1	−1	1	−1
2	−1	−1	1	1	−1	1	1	−1	1
3	−1	−1	1	1	1	0	−1	−1	−1
4	−1	0	0	0	1	1	1	0	−1
5	−1	0	0	0	1	1	1	0	−1
6	−1	1	−1	−1	1	0	0	0	−1
7	−1	1	1	−1	1	−1	−1	1	1
8	−1	1	1	1	1	−1	1	−1	1
9	−1	1	1	1	1	−1	1	−1	1
10	0	−1	−1	−1	−1	0	1	0	−1
11	0	−1	−1	−1	−1	0	1	0	−1
12	0	−1	−1	0	1	1	−1	1	1
13	0	−1	−1	0	1	1	−1	1	1
14	0	−1	−1	1	−1	1	0	−1	0
15	0	0	1	1	0	1	−1	0	1
16	0	1	−1	−1	−1	1	1	1	−1
17	0	1	−1	−1	−1	1	1	1	−1
18	1	−1	0	1	1	0	−1	−1	−1
19	1	−1	1	−1	−1	1	1	−1	1
20	1	−1	1	−1	1	−1	−1	1	0
21	1	0	−1	1	1	−1	0	1	−1
22	1	0	0	−1	0	0	1	1	1
23	1	0	0	−1	0	0	1	1	1
24	1	1	0	1	−1	−1	1	−1	0
25	1	1	1	−1	−1	1	−1	0	0
26	1	1	1	0	0	0	−1	−1	1

Table D: D_S- and A_w-efficiencies, relative to the $D^\star$ design, given η's, of the split-row×column designs for Example 2

		Criterion							
		D_S				A_w			
		Designs				Designs			
η_{Days}	$\eta_{Periods}$	MSS$_{D_S}$	MSS$_{(DP)_S}$	MSS$_{CP_\kappa}$	CP^1	MSS$_{D_S}$	MSS$_{(DP)_S}$	MSS$_{CP_\kappa}$	CP^1
1	1	97.67	81.67	86.56	80.80	96.62	76.92	84.21	77.49
10	1	100.97	79.02	86.30	79.07	110.01	89.59	98.61	90.44
100	1	101.81	78.26	86.27	78.67	117.13	111.76	114.46	112.07
1	10	97.66	81.56	86.51	80.52	96.61	76.81	84.16	76.78
10	10	100.95	78.89	86.24	78.62	110.00	89.49	98.55	89.29
100	10	101.80	78.13	86.20	78.16	117.13	111.72	114.45	111.67
0	100	97.66	81.55	86.50	80.49	96.61	76.80	84.15	76.70
10	100	100.95	78.88	86.23	78.57	110.00	89.47	98.55	89.15
100	100	101.80	78.11	86.20	78.10	117.13	111.72	114.45	111.62

[1]: Design labeled as CP in Table 1 of Trinca and Gilmour (2017) with post-crossing.

4 Example 3

The designs constructed for Example 3 are presented in Table E. Treatments are presented in unrandomized order. To execute the experiment appropriate randomization is required. For that, randomize the Ovens labels to ovens, the Batches labels to batches and the Runs labels to runs within each combination of Oven×Batch. The efficiencies of the designs, relative to MSS_{D_S} and for known values of the variance components, are given in Table F.

Table E: Designs for Example 3, a strip-split-plot structure (Ovens(10)*Batches(3))/Runs(2), with two HS and three ES 3-level factors

Design $D^{\star}$

Oven		Batch								
		1			2			3		
X_1	X_2	X_3	X_4	X_5	X_3	X_4	X_5	X_3	X_4	X_5
-1	-1	-1	-1	1	1	-1	-1	0	0	-1
		-1	1	-1	1	-1	0	1	1	1
-1	-1	1	1	-1	-1	1	0	1	1	1
		1	-1	1	-1	0	1	-1	-1	-1
-1	1	-1	-1	-1	-1	1	1	-1	1	1
		0	1	-1	1	0	0	1	-1	1
-1	1	1	1	1	0	-1	1	1	-1	-1
		1	1	-1	-1	1	-1	-1	-1	0
0	1	1	1	-1	-1	-1	-1	-1	1	-1
		-1	0	1	1	-1	1	0	1	0
1	-1	-1	1	1	1	-1	1	0	-1	-1
		-1	-1	-1	1	1	-1	-1	0	1
1	-1	1	-1	0	-1	-1	1	-1	1	-1
		1	1	1	-1	-1	-1	1	0	-1
1	0	-1	1	-1	1	1	-1	1	0	1
		-1	-1	0	0	1	1	1	-1	-1
1	1	0	0	0	1	1	1	-1	-1	1
		1	-1	-1	-1	1	1	-1	1	-1
1	1	1	-1	1	-1	0	-1	1	1	0
		-1	1	1	0	-1	-1	-1	-1	1

Table E: Designs for Example 3 (Continued)

Design MSS_{D_S}

Oven		Batch 1			Batch 2			Batch 3		
X_1	X_2	X_3	X_4	X_5	X_3	X_4	X_5	X_3	X_4	X_5
-1	-1	-1	1	0	-1	-1	-1	-1	-1	-1
		1	-1	1	-1	1	1	1	1	-1
-1	0	-1	-1	1	1	1	1	-1	1	-1
		1	-1	-1	1	-1	-1	1	0	0
-1	1	-1	1	1	-1	-1	0	0	-1	1
		1	1	-1	1	1	1	-1	1	-1
0	-1	0	1	-1	1	-1	1	-1	-1	1
		-1	0	1	1	1	-1	1	1	1
0	0	1	-1	0	0	1	1	1	1	0
		0	0	-1	1	0	0	0	0	-1
0	1	0	1	0	-1	0	0	1	-1	1
		-1	-1	-1	1	-1	-1	0	0	-1
1	-1	1	1	1	0	-1	1	-1	1	1
		-1	-1	-1	-1	1	-1	1	-1	-1
1	0	1	1	-1	1	-1	-1	1	0	-1
		-1	0	0	-1	1	-1	-1	-1	0
1	1	1	-1	1	0	-1	1	-1	1	1
		-1	1	1	1	1	-1	0	-1	-1
1	1	1	0	1	-1	-1	1	1	1	1
		-1	-1	-1	0	1	0	-1	1	-1

Table E: Designs for Example 3 (Continued)

Design $\text{MSS}_{(DP)_S}$

Oven		Batch 1			Batch 2			Batch 3		
X_1	X_2	X_3	X_4	X_5	X_3	X_4	X_5	X_3	X_4	X_5
-1	-1	1	1	-1	1	-1	-1	0	1	0
		-1	0	1	-1	1	-1	-1	-1	-1
-1	-1	-1	-1	-1	1	-1	-1	-1	1	-1
		0	1	0	1	1	1	1	1	1
-1	1	1	1	-1	1	1	1	1	0	-1
		1	-1	1	-1	-1	0	-1	1	1
-1	1	1	1	1	-1	1	1	1	-1	1
		-1	-1	0	1	0	-1	1	1	-1
0	0	0	-1	-1	-1	0	0	0	-1	-1
		-1	-1	1	0	-1	1	-1	-1	1
0	1	-1	-1	-1	1	1	-1	-1	0	1
		1	1	-1	-1	-1	-1	1	1	0
1	-1	1	1	-1	-1	1	1	-1	-1	0
		1	-1	1	1	0	-1	1	1	1
1	-1	-1	1	1	1	1	1	1	1	-1
		1	0	-1	-1	-1	0	1	-1	1
1	1	1	-1	-1	0	1	-1	-1	1	-1
		-1	1	0	1	-1	0	0	0	1
1	1	0	0	1	-1	-1	1	1	-1	-1
		-1	1	-1	1	1	0	-1	1	0

Table E: Designs for Example 3 (Continued)

Design $\text{MSS}_{CP_{\boldsymbol{\kappa}}}$

Oven		Batch 1			Batch 2			Batch 3		
X_1	X_2	X_3	X_4	X_5	X_3	X_4	X_5	X_3	X_4	X_5
-1	-1	1	0	1	-1	-1	1	-1	1	1
		-1	-1	-1	0	1	-1	1	-1	1
-1	-1	-1	-1	1	-1	1	-1	1	0	-1
		0	1	-1	1	-1	-1	0	-1	1
-1	1	-1	-1	-1	1	-1	1	-1	-1	-1
		-1	1	1	-1	1	-1	1	1	-1
-1	1	0	-1	0	-1	-1	-1	1	-1	1
		1	1	1	-1	1	1	-1	1	-1
0	-1	-1	1	-1	1	1	1	1	1	1
		0	-1	0	-1	0	-1	-1	0	-1
0	0	0	0	1	0	0	1	0	1	0
		1	-1	0	1	-1	0	1	0	-1
1	-1	-1	1	1	-1	-1	-1	1	1	-1
		1	1	-1	-1	1	1	-1	-1	-1
1	0	1	-1	1	1	-1	1	1	-1	1
		-1	-1	-1	0	1	0	0	1	0
1	1	1	-1	-1	1	0	0	0	1	1
		-1	0	0	-1	1	-1	-1	-1	0
1	1	-1	-1	1	0	-1	-1	1	0	0
		-1	1	-1	1	1	1	-1	-1	1

Table F: D_S- and A_w-efficiencies, relative to the best fixed-effects D design obtained, given η's, for the strip-split-plot designs for Example 3.

			D_S Designs			A_w Designs		
η_{Oven}	η_{Batch}	$\eta_{Oven*Batch}$	D^*	$\text{MSS}_{(DP)_S}$	$\text{MSS}_{CP_{\boldsymbol{\kappa}}}$	D^*	$\text{MSS}_{(DP)_S}$	$\text{MSS}_{CP_{\boldsymbol{\kappa}}}$
1	1	1	94.57	95.91	99.35	81.55	68.41	90.83
1	1	10	83.01	94.21	99.77	74.80	64.60	90.43
1	1	100	79.90	93.77	99.84	75.51	62.69	90.03
1	10	1	94.55	95.91	99.33	81.54	68.41	90.83
1	10	10	83.00	94.21	99.76	74.80	64.60	90.43
1	10	100	79.90	93.77	99.84	75.51	62.69	90.03
1	100	1	94.55	95.91	99.33	81.54	68.41	90.83
1	100	10	82.99	94.21	99.76	74.80	64.60	90.43
1	100	100	79.90	93.77	99.84	75.51	62.69	90.03
10	1	1	94.83	96.24	99.33	77.08	63.51	90.13
10	1	10	83.05	94.30	99.76	75.57	63.19	90.13
10	1	100	79.90	93.78	99.84	75.62	62.62	90.02
10	10	1	94.82	96.23	99.33	77.08	63.51	90.13
10	10	10	83.04	94.30	99.76	75.57	63.19	90.13
10	10	100	79.90	93.78	99.84	75.62	62.62	90.02
10	100	1	94.82	96.23	99.34	77.08	63.51	90.13
10	100	10	83.04	94.30	99.76	75.57	63.19	90.13
10	100	100	79.90	93.78	99.84	75.62	62.62	90.02
100	1	1	94.87	96.29	99.32	76.14	62.50	89.98
100	1	10	83.07	94.34	99.76	75.96	62.49	89.99
100	1	100	79.91	93.79	99.84	75.89	62.46	89.98
100	10	1	94.86	96.28	99.33	76.14	62.50	89.98
100	10	10	83.06	94.34	99.76	75.96	62.49	89.99
100	10	100	79.91	93.79	99.84	75.89	62.46	89.98
100	100	1	94.86	96.28	99.32	76.14	62.50	89.98
100	100	10	83.06	94.34	99.75	75.96	62.49	89.99
100	100	100	79.90	93.79	99.84	75.89	62.46	89.98

5 Example 4

The designs for this example are stored, in an unrandomized order, in the Supp `designs.zip` file (`https://anonymous.4open.science/r/Crossed-Multi-Stratum-Designs-1670`). To randomize, we should randomize the Batches labels to batches, the Occasions labels to occasions and the Runs labels to runs within each combination of Batch$\times$Occasion. The efficiencies of designs, relative to the MSS_{D_S} design and for known values of the variance component ratios, are shown in Table G.

Table G: D_S- and A_w-efficiencies, relative to the MSS_{D_S} optimum design, given η's, for designs in Example 4

			Criterion			
			D_S Designs		A_w Designs	
$\eta_{Batches}$	η_{Occs}	$\eta_{Batches\star Occs}$	$\text{MSS}_{(DP)_S}$	$\text{MSS}_{CP_{\boldsymbol{\kappa}}}$	$\text{MSS}_{(DP)_S}$	$\text{MSS}_{CP_{\boldsymbol{\kappa}}}$
1	1	1	86.11	90.96	75.42	96.44
10	1	1	86.10	90.95	75.78	95.75
100	1	1	86.10	90.95	77.21	93.32
1	10	1	86.11	90.96	88.50	98.55
10	10	1	86.10	90.95	87.83	98.11
100	10	1	86.10	90.95	84.07	95.61
1	100	1	86.11	90.96	98.18	99.79
10	100	1	86.10	90.95	97.96	99.71
100	100	1	86.10	90.95	96.03	99.01
1	1	10	85.80	90.73	77.63	93.03
10	1	10	85.79	90.72	77.67	92.94
100	1	10	85.79	90.72	77.90	92.44
1	10	10	85.80	90.73	84.69	95.51
10	10	10	85.79	90.72	84.43	95.36
100	10	10	85.79	90.72	82.69	94.31
1	100	10	85.80	90.73	96.31	99.01
10	100	10	85.79	90.72	96.14	98.95
100	100	10	85.79	90.72	94.53	98.36
1	1	100	85.75	90.69	79.16	91.30
10	1	100	85.75	90.69	79.15	91.30
100	1	100	85.75	90.69	79.08	91.32
1	10	100	85.75	90.69	80.46	91.92
10	10	100	85.75	90.69	80.44	91.92
100	10	100	85.75	90.69	80.28	91.89
1	100	100	85.75	90.69	87.97	95.28
10	100	100	85.75	90.69	87.93	95.27
100	100	100	85.75	90.69	87.47	95.10

6 Sensitivity Study

The literature on compound criteria recommends running the search for several weight patterns, calculating efficiencies of each property of interest and choosing the pattern that best suits the priorities of the experimenter (see, for example, Atkinson et al. (2007, Chapter 21), McGree et al. (2008); Gilmour and Trinca (2012); da Silva et al. (2017); de Oliveira et al. (2022); Egorova and Gilmour (2023)). Here we investigated the sensitivity of the designs on the weights for Example 3. Figure A shows D_S and $(DP)_S$ efficiencies and pure error degrees of freedom, for each stratum, as the values of κ_{DP} were varied between 0 and 1. The design for $\kappa_D = 1$ is also included. For $0 < \kappa_{DP} < 1$ a compound criterion was used with κ_A fixed to 1/3 and $\kappa_{DF} = 1 - (\kappa_{DP} + \kappa_A)$. Results show that small changes in the weights do not affect the properties of the design too much, except at zero weight for $(DP)_S$-optimality, where the requirement to have pure error degrees of freedom disappears.

Note that as the design in the Runs stratum is conditional on the design obtained in the Ovens stratum, it is possible to get a compromise design that is more D_S-efficient in the Runs stratum than the design obtained for $\kappa_D = 1$.

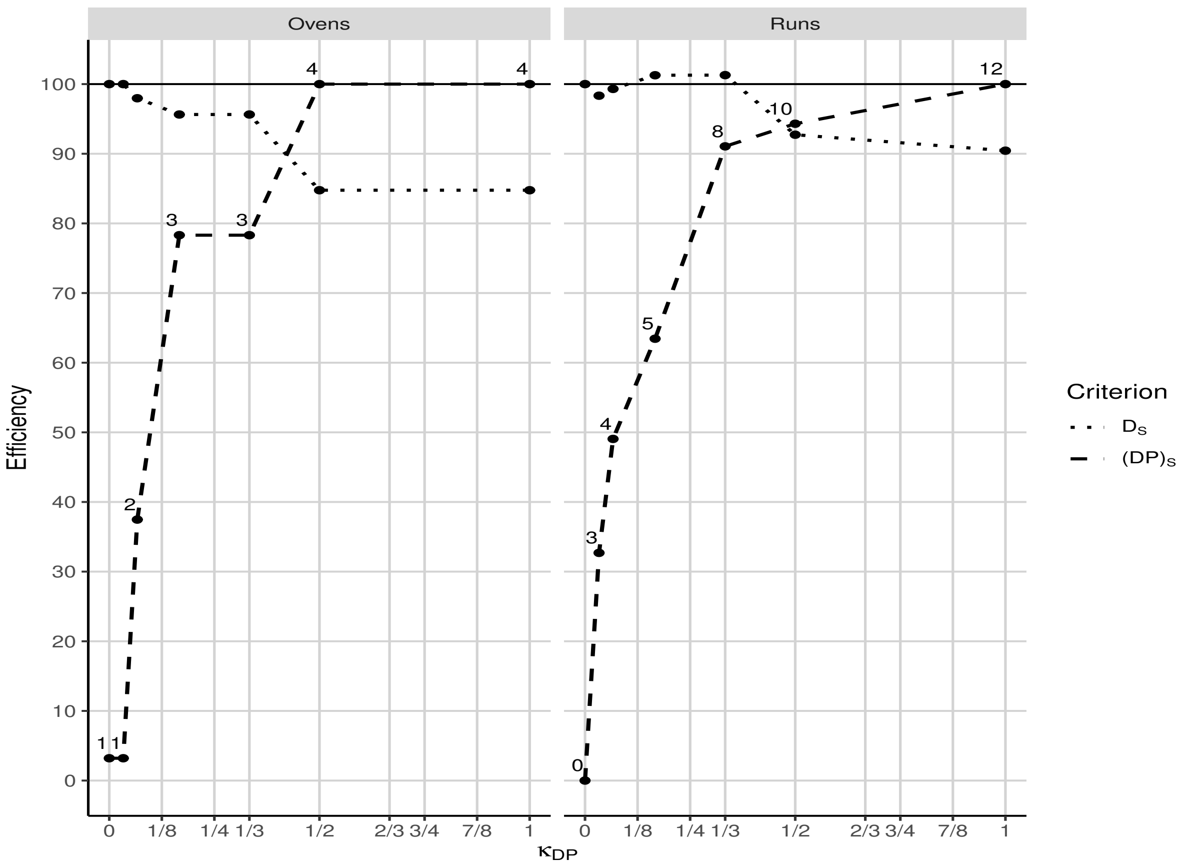

Figure A: Efficiencies for designs in each stratum, as a function of the weight κ_{DP} in the compound criterion for Example 3.

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