Phase Diagrams

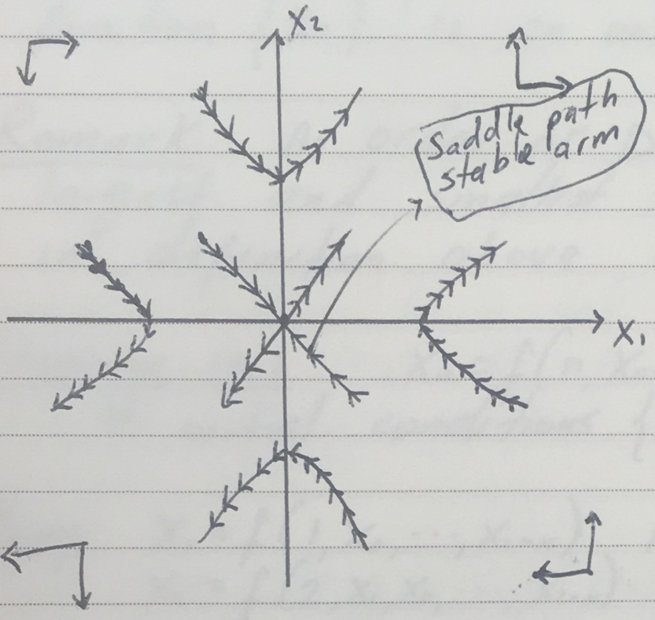
$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

Example: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 \end{cases} \Leftrightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\dot{x}_1 = 0 \iff x_2 = 0$

$\dot{x} > 0 \iff x_2 > 0$

$\dot{x}_2 > 0 \iff x_1 > 0$



$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$

$\lambda_1 = 1 \quad \lambda_2 = -1$

$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} -\lambda x_1 + x_2 = 0 \\ x_1 - \lambda x_2 = 0 \end{matrix}$

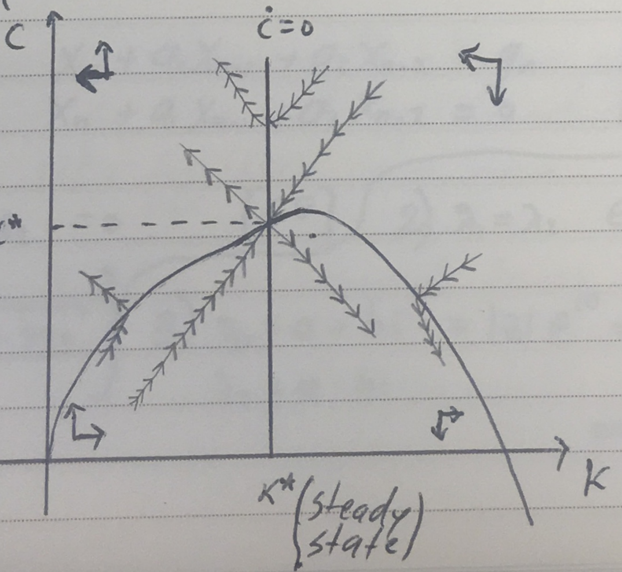
$\lambda = 1 \quad -x_1 + x_2 = 0 \iff x_1 = x_2 \Rightarrow c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1 \quad x_1 + x_2 = 0 \iff x_1 = -x_2 \Rightarrow c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Remark. Saddle path corresponds to an eigenvector with eigen value that has negative real part.

Exercises: a) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$ b) $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 \end{cases}$

Example $\begin{cases} \dot{k} = k^\alpha - c - \delta k \\ \dot{c} = \frac{c}{\sigma}\left(\alpha k^{\alpha - 1} - \delta - \rho\right) \end{cases}$ $\quad \alpha, \delta, \rho \in (0,1)$, $\sigma \in (0, \infty)$



10. Difference equations: basic concepts

Def 10.1 Let $x: \mathbb{Z} \to \mathbb{C}$ be a complex sequence $\{x_n\}$. A difference equation (DE) in explicit form is an equation $x_n = f(n, x_{n-1}, \dots, x_{n-k})$ where f is a known function $\{x_n\}$ is an unknown sequence.

Remark: An order of DE is the difference between the largest and smallest sequence index, for example in definition above $n-(n-k) = k$

Theorem 10.1: $x_n = f(n, x_{n-1}, \dots, x_{n-k})$ has a unique solution $\forall$ initial conditions $\{x_0, \dots, x_{1-k}\}$

Proof: $x_1 = f(1, x_0, \dots, x_{1-k})$ is uniquely determined
$x_2 = f(2, x_1, x_0, \dots, x_{2-k})$ is uniquely determined
and so on $\square$

Remark: For a k-th order equation you need k initial conditions

11. Linear Difference Equations: Let $\varphi_n(L) = 1 + a_{1,n} L + \dots + a_{k,n} L^k$ (lag polynomial)

$$\begin{matrix} \varphi_n(L)\, x_n = g_n & [C] \\ \varphi_n(L)\, x_n = 0 & [H] \end{matrix} \iff \left[\begin{matrix} x_n + a_{1,n} x_{n-1} + \dots + a_{k,n} x_{n-k} = g_n & [C] \\ x_n + a_{1,n} x_{n-1} + \dots + a_{k,n} x_{n-k} = 0 & [H] \end{matrix}\right.$$

Example:
$$x_n + a_1 x_{n-1} + a_2 x_{n-2} = g_n \quad [C]$$
$$x_n + a_1 x_{n-1} + a_2 x_{n-2} = 0 \quad [H]$$

$$\lambda^2 + a_1 \lambda + a_2 = 0 \quad [CE]$$
$$D = a_1^2 - 4a_2$$
$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

1) $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ $\Rightarrow x_{h,n} = C_1 \lambda_1^n + C_2 \lambda_2^n$

2) $\lambda_1 = \lambda_2 \in \mathbb{R}$ $\Rightarrow x_{h,n} = C_1 \lambda_1^n + C_2 n \lambda_2^n$

3) $\lambda_1 = a + bi = |z| e^{i\theta} \Rightarrow \lambda_1^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i \sin(n\theta))$
$\lambda_2 = a - bi$

$\Rightarrow x_{h,n}(t) = r^n(C_1 \cos(n\theta) + C_2 \sin(n\theta))$