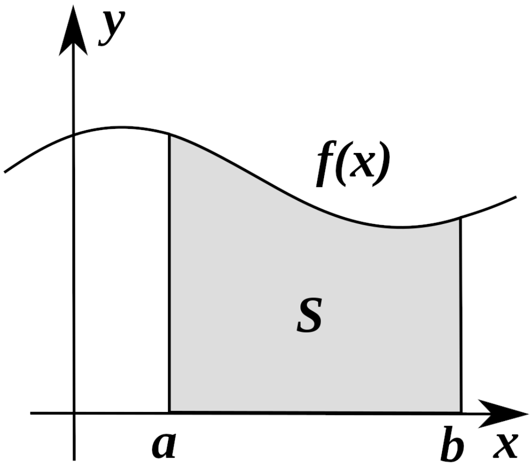
Numerical Integration

Assignment 2: MAT2319 – Introduction to Julia

October 15, 2021

Numerical integration is used to calculate a numerical approximation for the value S, the area under the curve defined by $f(x)$:



In this assignment, you are asked to approximate an integral of one dimensional function. Compute the value of the following integral

$$I = \int_0^5 xe^{-x/3}dx.$$

Display the difference between your numerical answer and the analytical answer

$$-24e^{-5/3} + 9.$$

Numerical integration methods can generally be described as combining evaluations of the integrand to get an approximation to the integral. The integrand is evaluated at a finite set of points called *integration points* and a weighted sum of these values is used to approximate the integral. The integration points and weights depend on the specific method used and the accuracy required from the approximation.[1]

An important part of the analysis of any numerical integration method is to study the behavior of the approximation error as a function of the number of integrand evaluations. A method that yields a small error for a small number of

[1] `https://en.wikipedia.org/wiki/Numerical_integration`

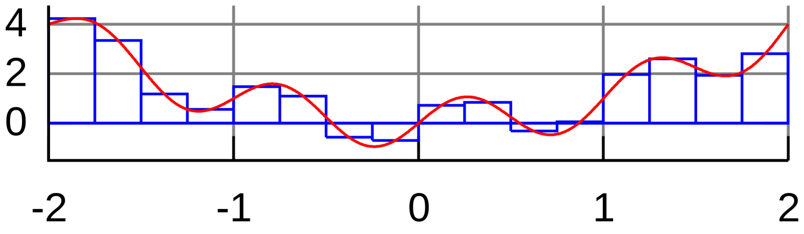


Figure 1: Illustration of the midpoint/rectangle rule

evaluations is usually considered superior. Reducing the number of evaluations of the integrand reduces the number of arithmetic operations involved, and therefore reduces the total round-off error. Also, each evaluation takes time, and the integrand may be arbitrarily complicated.

A large class of quadrature rules can be derived by constructing interpolating functions that are easy to integrate. Typically these interpolating functions are polynomials. In practice, since polynomials of very high degree tend to oscillate wildly, only polynomials of low degree are used, typically linear and quadratic.

Rectangle rule

The simplest method of this type is to let the interpolating function be a constant function (a polynomial of degree zero) that passes through the point $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$. This is called the *midpoint rule* or *rectangle rule*.

$$\int_a^b f(x)\,dx \approx (b-a) f\left(\frac{a+b}{2}\right). \tag{1}$$

Trapezoidal rule

The interpolating function may be a straight line (an affine function, i.e. a polynomial of degree 1) passing through the points $(a, f(a))$ and $(b, f(b))$. This

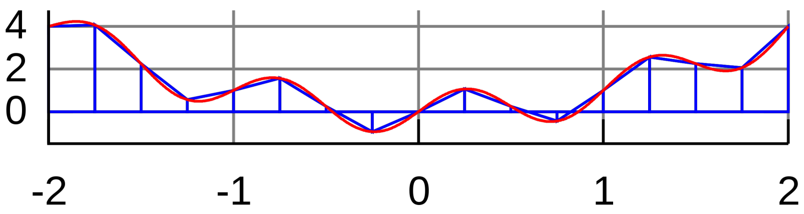


Figure 2: Illustration of the trapezoidal rule

is called the *trapezoidal rule*.

$$\int_a^b f(x)\,dx \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right).$$

For either one of these rules, we can make a more accurate approximation by breaking up the interval $[a,b]$ into some number n of subintervals, computing an approximation for each subinterval, then adding up all the results. This is called a *composite rule*, *extended rule*, or *iterated rule*. For example, the composite trapezoidal rule can be stated as

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{f(a)}{2}+\sum_{k=1}^{n-1}\left(f\left(a+k\frac{b-a}{n}\right)\right)+\frac{f(b)}{2}\right),$$

where the subintervals have the form $[a+kh, a+(k+1)h] \subset [a,b]$, with $h=\frac{b-a}{n}$ and $k=0,1,2,\ldots,n-1$.

Write two functions with arguments n to compute the given integral using two rules as described above.

```
function rectangle(n)
  ...
end

function trapezoidal(n)
  ...
end
```

Show on the same plot two error curves with respect to different n.