Integrating Factor: Example Problem

MATH 315 Differential Equations
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Find the general solution to the linear first-order ODE

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1$$

Put the equation in standard form:

$$\frac{dy}{dx} - \frac{1}{x}y = 2x + 1$$

The integrating factor is

$$\mu(x) = e^{\int(-1/x)dx}$$

Evaluating the integral in the exponent, we obtain

$$\begin{aligned} \mu(x) &= e^{-\int(1/x)dx} \\ &= e^{-\ln|x|} \end{aligned}$$

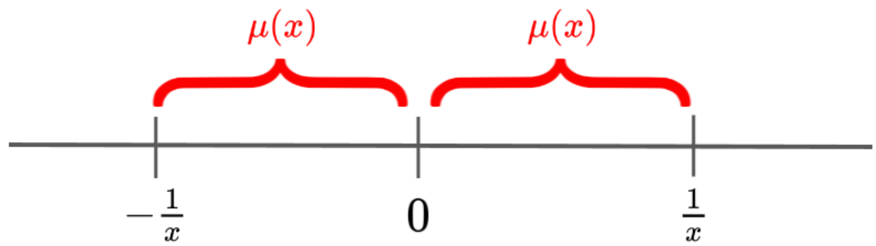
Using laws of logarithms from college algebra / precalculus, we can rewrite this as

$$\begin{aligned} \mu(x) &= e^{\ln|x|^{-1}} \\ &= |x|^{-1} \end{aligned}$$

The integrating factor is $\mu(x) = |x|^{-1}$ but we want to simplify this expression as much as possible. We want the absolute value bars to be gone when we incorporate the integrating factor into the general solution. Here's how we can get rid of the absolute value bars:

$$\begin{aligned} \mu(x) &= |x|^{-1} \\ &= \frac{1}{|x|} \\ &= \frac{|1|}{|x|} \\ &= \left|\frac{1}{x}\right| \end{aligned}$$

We have the equation $\mu(x) = \left|\frac{1}{x}\right|$. But an absolute value can always be interpreted as a distance away from 0. So we can restate the equation verbally as, "$\mu(x)$ is a distance of $\frac{1}{x}$ away from 0." See figure below.



We can see that we must have either $\mu(x) = -\frac{1}{x}$ or $\mu(x) = \frac{1}{x}$. So the figure serves as a visual aid for getting rid of the absolute value bars and arriving at the equation

$$\mu(x) = \pm\frac{1}{x}$$

Either choice works (they both lead to the same answer). The general solution is found below using the choice $\mu(x) = \frac{1}{x}$:

$$\begin{aligned} y &= \frac{1}{1/x} \int \frac{1}{x}(2x+1)dx \\ &= x \int \frac{2x+1}{x} dx \\ &= x \int \left(\frac{2x}{x} + \frac{1}{x} \right) dx \\ &= x \int \left(2 + \frac{1}{x} \right) dx \\ &= x \left(2x + \ln|x| + C \right) \end{aligned}$$

Therefore, the general solution to the ODE

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1$$

is given as

$$y = 2x^2 + x\ln|x| + Cx$$