The Cantor Set

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Georg Cantor (1845 - 1918) was a brilliant mathematician. During his study of the real number line, he made groundbreaking discoveries that are still being discussed today in many college classrooms. He is widely regarded as the father of set theory. When discussing this topic, he was famously quoted as saying, "A set is a Many that allows itself to be thought of as a One." One such set he studied is particularly interesting. It was named after him: the Cantor Set.

This amazing set is quite complicated, so we need to take some time to make sure we fully understand what it is. There are several sources to consult. One of the best books is *Understanding Analysis* by Stephen Abbott. If we read Chapter 3, Section 1, we can gain a solid understanding of the Cantor Set, so let us go through the contents of that section and make some observations. One observation is the fact that the Cantor Set is uncountable, which we can prove. When we study the Cantor Set, *Understanding Analysis* is a gold mine because it also provides helpful exercises for its readers to try. We will attempt Exercise 4.3.12. However, we should consult as many sources as possible in order to study the various properties of the Cantor Set.

The Cantor Set extends our knowledge of the subsets of $\mathbb{R}$ (Abbott 75). We construct the set, using subsets of $\mathbb{R}$, in the following way. Consider the 3 intervals C_0, C_1 and C_2.

$$C_0 = [0, 1]$$

Form C_1 by removing the open middle third of C_0:

$$C_1 = C_0 \setminus (1/3, 2/3) = [0, 1/3] \cup [2/3, 1]$$

Form C_2 by removing the open middle thirds of the two components of C_1:

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$

Repeat this process. For each $n = 0, 1, 2, ...$ set C_n consists of 2^n components (closed intervals), each with length $1/(3^n)$. The Cantor Set, C, is defined as

$$C = \bigcap_{n=0}^{\infty} C_n$$

(Abbott 75). So the Cantor Set is an infinite intersection. In other words, C is the remainder of the interval $[0, 1]$ after open middle thirds have been removed infinitely many times.

$$C = [0, 1] \setminus (1/3, 2/3) \cup (1/9, 2/9) \cup (7/9, 8/9) \cup ...$$

Located on page 76 in *Understanding Analysis*, Figure 3.1 serves as a fantastic visualization of the Cantor Set. Let us further discuss what C actually contains.

Since we always remove open middle thirds, $0 \in C_n\ \forall n \in \mathbb{N}$, so $0 \in C$. Also, $1 \in C$. In general, endpoints such as 0 and 1 are never removed. So C contains at least the endpoints of all of the intervals that make up each of the sets C_n (Abbott 76). All endpoints are rational numbers in the form $m/(3^n)$. If we were to assume C consists of only these endpoints, then that would mean $C \subseteq \mathbb{Q}$, which would imply C is countable. But it remains to be seen if this is true or not. Later, we will actually prove that C is uncountable.

Consider the total length of the removed intervals. C_1: we remove one open interval of length 1/3. C_2: we remove two open intervals of length 1/9. Let us generalize this. C_n: we remove 2^{n-1} open intervals of length $1/(3^n)\ \forall n \in \mathbb{N}$. Recall that we started with the interval $[0, 1]$. So the length of C is 1 minus the total length removed (Abbott 76). The total length removed is

$$(1/3) + 2(1/9) + 4(1/27) + ... + 2^{n-1}(1/(3^n)) + ... = \Sigma_{n=1}^{\infty} 2^{n-1}(1/(3^n))$$

This is a geometric series. Recall the formula

$$\Sigma_{n=1}^{\infty} a_1(r)^{n-1} = a_1/(1 - r)$$

So the total length removed is

$$\Sigma_{n=1}^{\infty} 2^{n-1}(1/(3^n)) = (1/3)/(1 - (2/3)) = (1/3)/(1/3) = 1$$

The total length of C is 1 - 1 = 0. This is surprising because we know C contains at least the endpoints mentioned earlier. Yet C has no length - this is why it is famously known as "Cantor dust." When we think of C in this way, we paint a picture of a tiny set in our minds. But C is quite complicated, as mentioned earlier. We know C is uncountable, and uncountable sets are huge. This paints a completely different picture in our minds. How can the set be both tiny and huge? This paradoxical nature of the Cantor Set is one reason why it has fascinated mathematicians for years.

The endpoints of the sets C_n form a countable set. So there must be infinitely more points in C since the set is uncountable (Abbott 77). Consider three different ways to visualize the size of the Cantor Set.

1) Length: 0. The Cantor Set is tiny.

2) Cardinality: $C \sim \mathbb{R}$. The Cantor Set is huge.

3) Dimension: 0.631. The Cantor Set is somewhere between tiny and huge.

Notice the new concept, "dimension," introduced in the third visualization. Abbott briefly explains the concept in *Understanding Analysis*. But the notion of dimension is a topic discussed in greater detail in MATH 397 Chaos and Fractals, which is a fun and interesting class. We are currently studying the Cantor Set in that class as well, so there is some overlap between MATH 491 and MATH 397. How is C affiliated with the concept of fractals? One definition of the word "fractal" is: "...Various extremely irregular curves or shapes for which any suitably chosen part is similar in shape to a given larger or smaller part when magnified or reduced to the same size" (https://www.merriam-webster.com/dictionary/fractal). Famous examples include the Koch Snowflake, the Sierpinski Triangle, and many more. Like the Cantor Set, fractals are intricate and complex. In fact, we can declare: C is a fractal. Let us discuss this subject briefly without digressing too much.

In *Chaos and Fractals: An Elementary Introduction*, by David P. Feldman, fractals are studied closely. We read and learn how they are formed, how they are useful, and how they are connected to chaos theory. About halfway through the book, the author writes, "By defining dimension in terms of the scaling properties of a shape, we will come up with a quantitative way of describing fractals" (Feldman 163). The "shape" of the Cantor Set involves line segments. Fun fact: the shape of the Sierpinski Triangle obviously involves triangles, but there is another fractal called the Sierpinski Carpet that involves squares. Pictures of these fractals can be seen in Feldman's book: pages 168 - 171. An important question is asked: "...What property of a shape determines how many small copies of it fit in a bigger copy? The answer to this question is the object's dimension" (Feldman 164). On the same page, there is a powerful formula:

$$\text{number of small copies } = (\text{magnification factor})^{\text{dimension}}$$

Using this formula, we can calculate the dimension of the Cantor Set:

$$2^n = (3^n)^{\text{dimension}}$$

$$n\ln 2 = n(\text{dimension})\ln 3$$

$$\text{dimension} = \ln 2/\ln 3 \approx 0.631$$

Now we see where the number 0.631 comes from (see visualization 3 at the top of this page). Using the formula, we find that the dimension of the Koch Snowflake is about 1.262, the dimension of the Sierpinski Triangle is about 1.585, and the dimension of the Sierpinski Carpet is about 1.893. Compare these numbers to the dimensions of well-known objects. A point is zero-dimensional, a line is one-dimensional, a square is two-dimensional, and a cube is three-dimensional. Comparing these numbers to each other helps us understand the Cantor Set in a deeper way. Notice that the dimension of C is between 1 and 0. We conclude: "The dimension between 1 and 0 indicates that the Cantor Set is in some regards line-like (one-dimensional) and in some regards point-like (zero-dimensional)" (Feldman 168). As mentioned earlier, this set is indeed quite complicated.

We have spent enough time in the MATH 397 realm. Now, let us return to the MATH 491 realm by proving that the Cantor Set is uncountable. The proof in *Understanding Analysis* is a bit confusing, so we search online for another proof. There are several places to look, but one of the best places is www.physicsforums.com. One proof found here is similar to Abbott's proof in *Understanding Analysis*, and it is a bit easier to understand.

Consider a point c in the Cantor Set. When we remove the first middle third, find out if c ends up in the left or right third. Then, when we remove the middle third from the subinterval c is in, find out if c ends up in the left or right third of that subinterval. Since the Cantor Set is an infinite intersection, we can repeat this process forever. We construct an infinite sequence of "lefts" and "rights" (for example, $c =$ L,R,L,R,R,L...) to describe where c is. By labeling L as 0 and R as 1, our construction becomes an infinite binary sequence. If we put a decimal point in front of the number, the result is a real number between 0 and 1 in binary notation (the example $c =$ L,R,L,R,R,L... becomes $c =$ 0.010110...). There is a one-to-one relationship between the points in the Cantor Set and the real numbers in the Cantor Set. In MATH 491, we have already shown that set of all real numbers between 0 and 1 is uncountable. The one-to-one relationship implies: $C \sim \mathbb{R}$. Therefore, the Cantor Set is uncountable (https://www.physicsforums.com/threads/a-way-to-count-the-uncountable-cantor-set.582139/).

Since we now possess plenty of knowledge of the Cantor Set, we can use C to solve a problem: Exercise 4.3.12 in *Understanding Analysis*. Define the function $g : [0, 1] \to \mathbb{R}$ as

$$g(x) = \begin{cases} 1 \text{ if } x \in C \\ 0 \text{ if } x \notin C \end{cases}$$

(a) Show that g fails to be continuous at any point $c \in C$.
(b) Prove that g is continuous at every point $c \notin C$.

To answer part (a), we need to review the concept of "complements." Given the set C, the complement of C is denoted by C^c. If $x \in C$ then $x \notin C^c$. Similarly, if $x \in C^c$ then $x \notin C$. This is a familiar concept, studied in depth in MATH 280 Intro to Foundations of Math. Consider the point $a \in C$. We know $g(a) = 1$ since $a \in C$. Consider the sequence (x_n) contained in C^c where $(x_n) \to a$. Note: each x_n is contained in C^c, so the limit as $n \to \infty$ of $g(x_n)$ is 0 since $x_n \notin C$. So the limit as $n \to \infty$ of $g(x_n)$ does not equal $g(a)$. This is significant because Theorem 4.3.2 (ii) in *Understanding Analysis* says: a function f is continuous at c (c is a limit point of the domain of f) if and only if the limit as $x \to c$ of $f(x)$ equals $f(c)$. Therefore, g is not continuous at a, so it fails to be continuous at any point $c \in C$.

In part (b), we acknowledge that the Cantor Set is closed. Therefore, C^c is open by

Theorem 3.2.13. According to the definition of "open," for all points $b \in C$, given $\epsilon > 0$, $\exists \delta > 0 \ni (b - \delta, b + \delta) \subseteq C^c$. Acknowledge the string of implications: $x \in (b - \delta, b + \delta) \Rightarrow x \in C^c \Rightarrow x \notin C \Rightarrow g(x) = 0$. We conclude: $g(x) \in (g(b) - \epsilon, g(b) + \epsilon)$. Therefore, g is continuous at every point $c \notin C$.

We defined the Cantor Set and discussed its size and contents. We also proved that C is uncountable. Finally, we used C to solve an interesting problem. Let us conclude our discussion of the Cantor Set by acknowledging a few of its properties, along with some fun facts affiliated with it.

1. The "Cantor Middle Fourths Set" is defined the same way as C but with middle fourths removed instead of middle thirds. The total length removed is another geometric series: $(1/4)/(1 - (3/4)) = (1/4)/(1/4) = 1$. So the Cantor Middle Fourths Set, like C, is length 0. This is discussed in MATH 397.

2. The Cantor Set contains no isolated points. Therefore, every $c \in C$ is a limit point (https://etd.ohiolink.edu). This is why C is closed.

3. C "consists of an infinite but totally disconnected set of points" (Feldman 258).

4. C is compact (https://math.dartmouth.edu). So every sequence in the Cantor Set has a subsequence that converges to a limit that is also in the Cantor Set.

Works Cited

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