Mixing Model: Carbon Monoxide

MATH 315 Differential Equations
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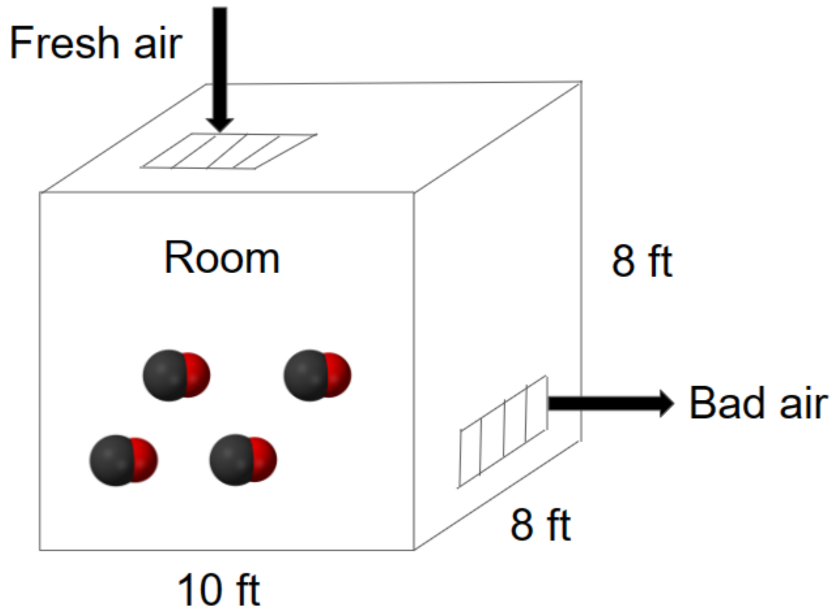
The air in a room 10 ft by 8 ft by 8 ft is 2% carbon monoxide (CO). Starting at $t = 0$, fresh air containing no CO is blown into the room at a rate of 75 ft^3/min. If air in the room flows out through a vent at the same rate, when will the air in the room be 0.01% CO?



Define $x(t)$:= volume of carbon monoxide [ft^3] in room at time t [min]. To formulate the mathematical model, use the following differential equation.

$$
\begin{aligned}
\frac{dx}{dt} &= \text{input rate} - \text{output rate} \\
\left[\frac{\text{ft}^3}{\text{min}}\right] &= \left[\frac{\text{ft}^3}{\text{min}}\right] - \left[\frac{\text{ft}^3}{\text{min}}\right] \\
\left[\frac{\text{ft}^3}{\text{min}}\right] &= \left[\frac{\text{ft}^3}{\text{min}}\right]\left[\frac{\text{ft}^3}{\text{ft}^3}\right] - \left[\frac{\text{ft}^3}{\text{min}}\right]\left[\frac{\text{ft}^3}{\text{ft}^3}\right] \\
\left[\frac{\text{ft}^3}{\text{min}}\right] &= \text{flow rate in} \cdot \text{CO level for incoming air} - \text{flow rate out} \cdot \text{CO level for outgoing air}
\end{aligned}
$$

In this example, we have flow rate in = flow rate out. So the volume of the air in the room, 640 ft^3, remains constant for all time. We can define CO levels as fractions between 0 and 1, then use those fractions to convert between volumes and percentages. Since fresh air is being blown into the room, we can set the CO level for incoming air equal to zero. The CO level for outgoing air is unknown, so we should put x in the numerator, and the volume of the air in the room (640 ft^3) in the denominator. Plug in the given values to solve the differential equation.

$$\frac{dx}{dt} = \left(\frac{75 \text{ ft}^3}{1 \text{ min}}\right)\left(\frac{0 \text{ ft}^3}{640 \text{ ft}^3}\right) - \left(\frac{75 \text{ ft}^3}{1 \text{ min}}\right)\left(\frac{x \text{ ft}^3}{640 \text{ ft}^3}\right)$$

Get rid of the units.

$$\frac{dx}{dt} = -\frac{75}{640}x$$

The general solution to the differential equation is given as

$$x = Ce^{-75t/640}.$$

Since x is defined as a volume [ft^3], we need our initial condition $x(0) = ?$ to be expressed as a volume as well. We are told that the air in the room starts out at 2% CO. To convert the percentage to a volume, use the following equation.

$$\frac{\text{volume of CO}}{\text{volume of air in room}} \cdot 100\% = \text{percentage of room filled by CO}$$

$$\frac{? \text{ ft}^3}{640 \text{ ft}^3} \cdot 100\% = 2\%$$

$$? \text{ ft}^3 = 2\% \cdot \frac{640 \text{ ft}^3}{100\%}$$

$$? \text{ ft}^3 = 12.8 \text{ ft}^3$$

So our initial condition is $x(0) = 12.8$. Plug in these values for x and t and solve for C.

$$\begin{aligned} 12.8 &= Ce^{-75 \cdot 0/640} \\ 12.8 &= Ce^0 \end{aligned}$$

We have $C = 12.8$. The solution to the IVP, which is the equation for x, the volume of carbon monoxide [ft^3] at time t [min], is given below.

$$x = 12.8e^{-75t/640} \qquad (1)$$

We can use Eq. (1) to answer the question. However, we have to convert the percentage 0.01% to a volume first. Use the same percentage/volume equation again.

$$\frac{? \text{ ft}^3}{640 \text{ ft}^3} \cdot 100\% = 0.01\%$$

$$? \text{ ft}^3 = 0.01\% \cdot \frac{640 \text{ ft}^3}{100\%}$$

$$? \text{ ft}^3 = 0.064 \text{ ft}^3$$

Set the left-hand side of Eq. (1) equal to 0.064 and solve for t.

$$
\begin{aligned}
0.064 &= 12.8e^{-75t/640} \\
0.005 &= e^{-75t/640}
\end{aligned}
$$

$$
\begin{aligned}
\ln(0.005) &= -\frac{75}{640}t \\
-\frac{640}{75}\ln(0.005) &= t
\end{aligned}
$$

This value is approximated as $t \approx 45.21$. So the answer to the question is: after about 45 min.