MATH 623:
Partial Differential Equations II

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1 Heat Equation and Wave Equation on Finite Interval

1.1 Summary of Lecture 1

Today's lecture was brief because we spent some time introducing ourselves, discussing our various fields of interest, and asking questions about the format of the class. It was nice to see everyone again after a long Christmas break! After all the preliminaries, we dove into the content: the heat equation on the half line (intro to the "method of reflections").

One of the first things we were told to do was to review Lecture 8 (Section 8.3) from PDEs I. This information (the IVP for the heat equation, properties of solutions to the heat equation, the heat kernel, etc.) provides the foundation for the method of reflections. For the IVP on the on the whole real line, we have

$$\begin{cases} u_t - ku_{xx} = 0 & x \in (-\infty, \infty),\ t \in (0, \infty), \\ u(x,0) = f(x). \end{cases}$$

If the initial condition is

$$f(x) = Q(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0, \end{cases}$$

then the solution to the IVP is

$$S(x,t) = Q_x(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right).$$

which is the expression for the heat kernel (derived in Lecture 8).

The solution to the IVP, with the general initial condition $f(x)$, is given by the integral

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t) f(y) dy.$$

This expression for u is the "convolution" of S and f. For context, see Lecture 8 for elaboration of properties of solutions to the heat equation.

The stage has been set for the method of reflections. We now take a look at the same IVP, but with $x, t \in (0, \infty)$. Since x is bounded by 0, a boundary condition becomes necessary, so the IVP becomes an IBVP:

$$\begin{cases} v_t - kv_{xx} = 0 & x, t \in (0, \infty), \\ v(x,0) = f(x), & \\ v(0,t) = 0 & \text{(Dirichlet BC)}. \end{cases}$$

The concept of the "odd extension" of $f(x)$ is introduced. The idea is to take the initial condition on the half line and extend it as an odd function, meaning it is reflected about the origin onto the entire real line. We define $f_{\text{odd}}(x)$ as

$$f_{\text{odd}}(x) = \begin{cases} f(x) & x > 0, \\ 0 & x = 0, \\ -f(-x) & x < 0, \end{cases}$$

and we acknowledge this initial condition to solve the heat equation on the whole real line. However, we restrict the domain to $x \in (0, \infty)$. Next time, we will put everything together to solve the IBVP.

1.2 Summary of Lecture 2

Today, we discussed the method of reflections for not only the heat equation, but the wave equation as well. This method is quite useful since it can be used for more than one PDE. We pick up where we left off last time:

$$\begin{cases} v_t - kv_{xx} = 0 & x, t \in (0, \infty), \\ v(x, 0) = f(x), & \\ v(0, t) = 0 & \text{(Dirichlet BC)}. \end{cases}$$

Acknowledge the *odd extension* of the initial condition $f(x)$:

$$f_{\text{odd}}(x) = \begin{cases} f(x) & x > 0, \\ 0 & x = 0, \\ -f(-x) & x < 0. \end{cases}$$

Take $f_{\text{odd}}(x)$ as the initial condition for the "companion problem:"

$$\begin{cases} u_t - ku_{xx} = 0 & x \in (-\infty, \infty),\ t \in (0, \infty), \\ u(x, 0) = f_{\text{odd}}(x). & \end{cases}$$

We have set up the companion problem as an IVP that we know how to solve. Based on our discussion from last time, the solution $u(x, t)$ to the IVP is given by the convolution integral

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) f_{\text{odd}}(y) dy.$$

Next, we rewrite our expression for $u(x, t)$ by substituting the values of $f_{\text{odd}}(x)$. In turn, we acknowledge the original initial condition $f(x)$ for the IBVP. So

$$u(x, t) = \int_{-\infty}^{0} S(x - y, t)(-f(-y)) dy + \int_{0}^{\infty} S(x - y, t) f(y) dy.$$

Let $z = -y$. So $\frac{dz}{dy} = -1 \implies dz = -dy$. Then

$$\begin{aligned} u(x, t) &= \int_{?}^{?} S(x + z, t)(-f(z))(-dz) + \int_{0}^{\infty} S(x - y, t) f(y) dy \\ &= \int_{?}^{?} S(x + z, t) f(z) dz + \int_{0}^{\infty} S(x - y, t) f(y) dy. \end{aligned}$$

Limits of integration (first integral in sum above):

$$y = -\infty \text{ to } y = 0 \implies -z = -\infty \text{ to } -z = 0 \implies z = \infty \text{ to } z = 0.$$

So the sum of integrals can be rewritten as

$$u(x,t) = \int_{\infty}^{0} S(x+z,t)f(z)dz + \int_{0}^{\infty} S(x-y,t)f(y)dy$$

$$= -\int_{0}^{\infty} S(x+z,t)f(z)dz + \int_{0}^{\infty} S(x-y,t)f(y)dy$$

$$= -\int_{0}^{\infty} S(x+y,t)f(y)dy + \int_{0}^{\infty} S(x-y,t)f(y)dy \qquad \text{(“dummy variable” } z \text{ renamed to } y)$$

$$= \int_{0}^{\infty} [S(x-y,t)f(y) - S(x+y,t)f(y)]dy = \int_{0}^{\infty} [S(x-y,t) - S(x+y,t)]f(y)dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left[\exp\left(-\frac{(x-y)^2}{4kt}\right) - \exp\left(-\frac{(x+y)^2}{4kt}\right) \right] f(y)dy$$

$$= v(x,t) \qquad \text{when restricted to x > 0.}$$

Thus, we have solved the IBVP for the heat equation on the half line using the method of reflections.

The wave equation on the half line is more complicated. Consider the IBVP:

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & x,t \in (0,\infty), \\ v(x,0) = f(x), \\ v_t(x,0) = g(x), \\ v(0,t) = 0 & \text{(Dirichlet BC).} \end{cases}$$

Again, we take $f_{\text{odd}}(x)$ and $g_{\text{odd}}(x)$ as the initial conditions for a companion problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x,t \in (-\infty,\infty), \\ u(x,0) = f_{\text{odd}}(x), \\ u_t(x,0) = g_{\text{odd}}(x). \end{cases}$$

We know that the solution to the companion problem (IVP) is given by d'Alembert's formula:

$$u(x,t) = \frac{1}{2}[f_{\text{odd}}(x+ct) + f_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{\text{odd}}(s)ds.$$

For this problem we must consider two cases. The positive xt-plane is divided into two regions: $x > ct$ and $x < ct$ ($x - ct > 0$ and $x - ct < 0$, respectively). We can disregard $x + ct$ since for all x and t, the value of $x + ct$ will be greater than zero.

Case 1: $x > ct$. Both characteristic lines running from point (x,t) hit the positive x-axis without straying into $(-\infty, 0)$, so the domain of dependence is completely contained in $(0,\infty)$. There is no reflection taking place, so we do not need to acknowledge the odd extensions. Therefore,

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

$$= v(x,t) \text{ for } x \in (0,\infty).$$

Case 2: $x < ct$. One of the characteristic lines (running from point (x,t)) hits the negative x-axis. So the domain of dependence bleeds onto $(-\infty, 0)$. But in the IBVP, x is only in $(0,\infty)$, so we need to use the method of reflections and acknowledge $f_{\text{odd}}(x)$ and $g_{\text{odd}}(x)$:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{0} g_{\text{odd}}(s)ds + \frac{1}{2c}\int_{0}^{x+ct} g(s)ds,$$

$$u(x,t) = \frac{1}{2}[f(x+ct) - f(-(x-ct))] + \frac{1}{2c}\int_{x-ct}^{0} (-g(-s))ds + \frac{1}{2c}\int_{0}^{x+ct} g(s)ds.$$

Derive the solution reported in class. First integral:

$$-\frac{1}{2c}\int_{x-ct}^{0} g(-s)ds.$$

$$\text{Let } z = -s \qquad (\text{so } dz = -ds).$$

The limits of integration are given as

$$s = x - ct \text{ to } s = 0 \implies -z = x - ct \text{ to } -z = 0 \implies z = ct - x \text{ to } z = 0$$

$$\implies -\frac{1}{2c}\int_{ct-x}^{0} g(z)(-dz) = \frac{1}{2c}\int_{ct-x}^{0} g(z)dz = \frac{1}{2c}\int_{ct-x}^{0} g(s)ds$$

where the "dummy variable" z has been renamed to s. The solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{0} g(s)ds + \frac{1}{2c}\int_{0}^{x+ct} g(s)ds \\ &= \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds \\ &= v(x,t) \text{ for } x \in (0,\infty). \end{aligned}$$

Next time, we take a look at the IBVP for the wave equation on the finite interval (0,l). For the half line, there were two cases, but for the finite interval, there are many more. Question: are there infinite regions to consider? At any rate, when it comes to the method of reflections, the wave equation is much more complicated than the heat equation since we have to consider solutions on a case-by-case basis.

1.3 Example

As an example, we will work out the solution to Exercise 3.1-1 in Strauss. The IBVP is given on the half line:

$$\begin{cases} u_t - ku_{xx} = 0 & x,t \in (0,\infty), \\ u(x,0) = e^{-x}, & \\ u(0,t) = 0 & \text{(Dirichlet BC)}. \end{cases}$$

According to formula (6) in Strauss, the solution to the IBVP is given by

$$u(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)]e^{-y}dy.$$

We can simplify this expression a bit. We have

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[\exp\left(-\frac{(x-y)^2}{4kt}\right) - \exp\left(-\frac{(x+y)^2}{4kt}\right)\right]\exp(-y)dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[\exp\left(-\frac{(x-y)^2}{4kt} - y\right) - \exp\left(-\frac{(x+y)^2}{4kt} - y\right)\right]dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[\exp\left(-\frac{(x-y)^2 - 4kty}{4kt}\right) - \exp\left(-\frac{(x+y)^2 - 4kty}{4kt}\right)\right]dy.
\end{aligned}
$$

1.4 Exercise

For Exercise 3.1-2, we are given the the half-line Neumann problem (heat equation):

$$
\begin{cases}
w_t - kw_{xx} = 0 & x,t \in (0,\infty), \\
w(x,0) = \phi(x), \\
w_x(0,t) = 0.
\end{cases}
$$

Acknowledge the *even extension* of the initial condition $\phi(x)$:

$$
\phi_{\text{even}}(x) = \begin{cases}
\phi(x) & x > 0, \\
0 & x = 0, \\
\phi(-x) & x < 0.
\end{cases}
$$

Let $u(x,t)$ solve the heat equation on the whole real line:

$$
\begin{cases}
u_t - ku_{xx} = 0 & x,t \in (-\infty,\infty) \\
u(x,0) = \phi_{\text{even}}(x)
\end{cases}
$$

where

$$
u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)dy.
$$

The solution, $w(x,t)$, to the half-line Neumann problem is equal to $u(x,t)$ where

$$
w(x,t) = u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)dy,
$$

restricted to $x \in (0,\infty)$. Then

$$
w(x,t) = \int_{-\infty}^{0} S(x-y,t)\phi(-y)dy + \int_0^\infty S(x-y,t)\phi(y)dy.
$$

Let $z = -y$. So $\frac{dz}{dy} = -1 \implies dz = -dy$. Then

$$
w(x,t) = \int_?^? S(x+z,t)\phi(z)(-dz) + \int_0^\infty S(x-y,t)\phi(y)dy.
$$

Limits of integration (first integral in sum above):

$$
y = -\infty \text{ to } y = 0 \implies -z = -\infty \text{ to } -z = 0 \implies z = \infty \text{ to } z = 0.
$$

Then the sum of integrals can be rewritten as

$$w(x,t) = \int_{\infty}^{0} S(x+z,t)\phi(-z)(-dz) + \int_{0}^{\infty} S(x-y,t)\phi(y)dy$$

$$= -\int_{\infty}^{0} S(x+z,t)\phi(-z)dz + \int_{0}^{\infty} S(x-y,t)\phi(y)dy.$$

Since we are using the even extension of ϕ, we know that $\phi(-z) = \phi(z)$, and we have that

$$w(x,t) = \int_{0}^{\infty} S(x+z,t)\phi(z)dz + \int_{0}^{\infty} S(x-y,t)\phi(y)dy$$

$$= \int_{0}^{\infty} S(x+y,t)\phi(y)dy + \int_{0}^{\infty} S(x-y,t)\phi(y)dy \qquad \text{(“dummy variable” } z \text{ renamed to } y\text{)}$$

$$= \int_{0}^{\infty} [S(x+y,t)\phi(y) + S(x-y,t)\phi(y)]dy = \int_{0}^{\infty} [S(x+y,t) + S(x-y,t)]\phi(y)dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left[\exp\left(-\frac{(x+y)^2}{4kt}\right) + \exp\left(-\frac{(x-y)^2}{4kt}\right)\right]\phi(y)dy.$$

Therefore, $w(x,t)$ solves the heat equation on the half line ($x \in (0,\infty)$).

1.5 Summary of Lecture 3

After a brief review of the previous sections, we finished our discussion of the wave equation on the finite interval. The main idea to find the solution is actually quite elegant. In order to reduce our problem to one that we already know how to solve, the homogeneous wave equation on the entire real line, we make an extension like we did before. However, instead of taking a simple odd or even extension, we make an odd periodic extension for the Dirichlet condition, and an even periodic extension for the Neumann condition. The extension is made as follows: for the initial datum $v(x,0) = \phi(x)$ we take

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & x \in (0,l), \\ 0 & x = 0, \\ -\phi(-x) & x \in (-l,0), \\ \text{periodically extended with period } 2l, & \end{cases}$$

and the same for the initial condition $v_t(x,0) = \psi(x)$. With these extensions, we can use d'Alembert's formula to derive our solution. However, there is a small caveat that we must address. The problem occurs when the characteristics cross the boundary at either 0 or l. Because of this crossing, we must deal with our solutions on a case by case basis determined by the value of (x,t). We find this method of solutions to be incredibly tedious, and we hope that there is a more efficient way of solving this type of IBVP. We have a feeling that there will be some method that involves the use of Fourier series since we know that we can use Fourier series to make a similar kind of extensions.

After solving the wave equation, we took a quick look at the diffusion equation on the whole line. This time, instead of a homogeneous equation we looked at an initial value problem that has a forcing term. The IVP for this equation is constructed as

$$\begin{cases} u_t - ku_{xx} = f(x,t) & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \phi(x). \end{cases}$$

Though this equation seems much more difficult, the solution is actually quite simple. In order to find the total solution, we simply solve for the homogeneous case using the method of the heat kernel and add it to the solution of the inhomogeneous problem. The latter is a little bit trickier, but we have actually already encountered the method. Ultimately, we employ Duhamel's Principle to solve the inhomogeneous case. This portion is given by the expression

$$\int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s) dy ds.$$

Finally, we simply sum this solution and the homogeneous one, and we have our answer.

1.6 Example

As another example, we will work out the solution to Exercise 3.2-1 in Strauss. The question asks us to solve the Neumann problem for the wave equation on the half line. To begin, we construct our Initial Boundary Value Problem as

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & t \in \mathbb{R},\ x > 0, \\ u(x,0) = f(x), \\ u_t(x,0) = g(x), \\ u_t(0,t) = 0. \end{cases}$$

Our method will be to extend our problem to the entire real line by taking an even extension of both $f(x)$ and $g(x)$ and then use this extension in d'Alembert's formula. To begin, we construct

$$f_{\text{even}} = \begin{cases} f(x) & x \geq 0, \\ f(-x) & x \leq 0, \end{cases}$$

$$g_{\text{even}} = \begin{cases} g(x) & x \geq 0, \\ g(-x) & x \leq 0. \end{cases}$$

Using these two extensions, we construct the new IBVP as

$$\begin{cases} U_{tt} - c^2 U_{xx} = 0 & t \in \mathbb{R},\ x \in \mathbb{R}, \\ U(x,0) = f_{\text{even}}(x), \\ U_t(x,0) = g_{\text{even}}(x), \\ U_t(0,t) = 0. \end{cases}$$

Now that we have our problem on all of $\mathbb{R}$ we can employ d'Alembert's formula to obtain a solution for our extended IBVP. Then we will "unpack" our solution and restrict the values of x to be strictly positive. To begin we have that

$$U(x,t) = \frac{1}{2}\left[f_{\text{even}}(x+ct) + f_{\text{even}}(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g_{\text{even}}(s) ds.$$

Something to note is that our value of $x+ct$ will always be greater than zero. However, $x-ct$ can be either positive or negative, so we must determine what we need to do in the latter case. It is clear that in the event that $x-ct>0$, our solution to our original IBVP is given by the restriction of $U(x,t)$ to the positive x-axis:

$$u(x,t)=\frac{1}{2}\Big[f(x+ct)+f(x-ct)\Big]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds,\ \ x>0,\ \ t\in\mathbb{R}.$$

When we have that $x-ct<0$ we use the reflection method by taking the characteristic and reflecting it across the boundary at $x=0$. In this case, our solution $U(x,t)$ is given by d'Alembert, but we have

$$U(x,t)=\frac{1}{2}\Big[f_{\text{even}}(x+ct)+f_{\text{even}}(x-ct)\Big]+\frac{1}{2c}\int_{x-ct}^{x+ct}g_{\text{even}}(s)ds.$$

We begin unpacking our formula by inputting the values of $f_{\text{even}}(x)$ and $g_{\text{even}}(x)$ and breaking up our integral as

$$U(x,t)=\frac{1}{2}\Big[f(x+ct)+f(-(x-ct))\Big]+\frac{1}{2c}\int_{x-ct}^{0}g(-s)ds+\frac{1}{2c}\int_{0}^{x+ct}g(s)ds.$$

In the first integral we make the substitution $z=-s$ so we have that $dz=-ds$ and our bounds of integration become $ct-x$ and 0 respectively. Thus, our solution becomes

$$U(x,t)=\frac{1}{2}\Big[f(x+ct)+f(ct-x)\Big]+\frac{1}{2c}\int_{ct-x}^{0}g(z)(-1)dz+\frac{1}{2c}\int_{0}^{x+ct}g(s)ds.$$

Lastly, we simply flip the bounds of integration on the first integral and add a factor of negative 1. Restricting the values of x to be strictly positive, we end up with the solution to the diffusion equation for $x-ct<0$ as

$$u(x,t)=\frac{1}{2}\Big[f(x+ct)+f(ct-x)\Big]+\frac{1}{2c}\int_{0}^{ct-x}g(z)dz+\frac{1}{2c}\int_{0}^{x+ct}g(s)ds.$$

Putting this together with our solution for $x-ct>0$, we have that the solution to the Neumann problem is as follows:

$$u(x,t)=\begin{cases}\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds & x-ct>0,\\ \frac{1}{2}\left[f(x+ct)+f(ct-x)\right]+\frac{1}{2c}\int_{0}^{ct-x}g(s)ds+\frac{1}{2c}\int_{0}^{x+ct}g(s)ds & x-ct<0.\end{cases}$$

1.7 Exercise

Here is our work for Exercise 3.2-3. We have a wave given by the function $f(x+ct)$, so we can build an Initial Boundary Value Problem (IBVP) with the wave equation as follows:

$$\begin{cases}v_{tt}-c^2v_{xx}=0 & x>0,\ t\in\mathbb{R},\\ v(x,0)=f(x), & \\ v_t(x,0)=cf'(x), & \\ v(0,t)=0. & \end{cases}$$

We are able to assume that $v(0,t) = 0$ because the end of the string is fixed. Consequently, we have an IBVP with Dirichlet boundary conditions. We have already determined the general solution to the IVP for the wave equation on the infinite string (the solution of which is given by d'Alembert's Formula). However, here we have an IBVP where we are dealing with a semi-infinite string. In order to remedy this we simply make an odd extension of our initial conditions in order to apply d'Alembert's formula. We make the following extensions of our initial datum:

$$f_{\text{odd}}(x) = \begin{cases} f(x) & x > 0, \\ 0 & x = 0, \\ -f(-x) & x < 0, \end{cases}$$

and

$$cf'_{\text{odd}}(x) = \begin{cases} cf'(x) & x > 0, \\ 0 & x = 0, \\ -cf'(-x) & x < 0. \end{cases}$$

Using these extensions, we can employ our formula and we arrive at

$$u(x,t) = \frac{1}{2}\Big[f_{\text{odd}}(x+ct) + f_{\text{odd}}(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'_{\text{odd}}(s)ds.$$

Here we can cancel the common factor of c in and outside of the integral. Now, we have to consider two cases here: $x - ct > 0$ and $x - ct < 0$. We can disregard $x + ct$ because for all x and t the value of $x + ct$ will always be greater than zero. In the event that $x - ct > 0$ we can use our definition of the odd extensions, and if we evaluate the integral, we arrive at the expression

$$u(x,t) = \frac{1}{2}\Big[f(x+ct) + f(x-ct)\Big] + \frac{1}{2}\Big[f(x+ct) - f(x-ct).\Big]$$

Simplifying this equation we end up with our solution for $x - ct > 0$:

$$u(x,t) = f(x+ct).$$

In the case where $x - ct < 0$ we run into the problem where our characteristic crosses over the boundary. Using our expression for the odd extension we end up with

$$u(x,t) = \frac{1}{2}\Big[f(x+ct) - f(-(x-ct))\Big] + \frac{1}{2}\int_{x-ct}^{x+ct} f'_{\text{odd}}(s)ds.$$

Next, we must take care of our integral by breaking it up into pieces and inputting the components of our odd extension:

$$\int_{x-ct}^{x+ct} f'_{\text{odd}}(s)ds = \int_{x-ct}^{0}(-1)f'(-s)ds + \int_{0}^{x+ct} f'(s)ds.$$

Next we let $s = -s$ and change our limits of integration for the first integral and we obtain

$$\int_{x-ct}^{x+ct} f'_{\text{odd}}(s)ds = \int_{ct-x}^{0}(-1)f'(s)(-1)ds + \int_{0}^{x+ct} f'(s)ds.$$

Lastly, we flip the limits of integration for the first integral and evaluate:

$$\int_{x-ct}^{x+ct} f'_{\text{odd}}(s)ds = -f(ct-x) + f(0) + f(x+ct) - f(0) = f(x+ct) - f(ct-x).$$

Putting everything together and making the necessary combinations we arrive at our solution for $x - ct < 0$:

$$u(x,t) = \frac{1}{2}\Big[f(x+ct) - f(ct-x)\Big] + \frac{1}{2}\Big[f(x+ct) - f(ct-x)\Big] = f(x+ct) - f(ct-x).$$

Finally, the total solution for this IBVP is given by

$$u(x,t) = \begin{cases} f(x+ct) & x - ct > 0, \\ f(x+ct) - f(ct-x) & x - ct < 0. \end{cases}$$

1.8 Exercise

For Exercise 3.3-2, we are trying to solve the completely inhomogeneous IBVP given by

$$\begin{cases} v_t - kv_{xx} = f(x,t) \quad x > 0,\ t > 0, \\ v(0,t) = h(t), \\ v(x,0) = \phi(x). \end{cases}$$

Ultimately, we will develop a general solution to this problem using Duhamel's principle, but before we can do that, we must reduce this problem to have homogeneous Dirichlet boundary conditions. To begin, we use what Strauss calls the "subtraction method" by developing a new IBVP by taking $V(x,t) = v(x,t) - h(t)$ which leads us to

$$\begin{cases} V_t - kV_{xx} = f(x,t) - h'(t) = g(x) \quad x > 0,\ t > 0, \\ V(0,t) = 0, \\ V(x,0) = \phi(x) - h(0) = q(x). \end{cases}$$

We know that, on the whole real line, the inhomogeneous heat equation can be solved by taking the homogeneous solution and adding it to the solution given by Duhamel's principle. With Dirichlet boundary conditions, we have already found that the solution to the diffusion equation on the half line is given in terms of the heat kernel $S(x-y,t)$:

$$V_h(x,t) = \int_0^\infty \left[S(x-y,t) - S(x+y,t)\right] q(y)dy.$$

Next we must tackle the inhomogeneous problem using Duhamel's principle. To begin we make an odd extension of our function $g(x)$ as follows:

$$g_{\text{odd}}(x) = \begin{cases} g(x) & x > 0, \\ 0 & x = 0, \\ -g(-x) & x < 0. \end{cases}$$

By making this extension, we have effectively extended our solution to take values on all of $x \in \mathbb{R}$, so we say that for another solution $U(x,t)$ on the entire real line:

$$U_i(x,t) = \int_0^t \int_{-\infty}^\infty S(x-y,t-s) g_{\text{odd}}(y,s)dyds.$$

Next, we must "unpack" this expression and ultimately restrict our solution to only positive values of x. We begin by breaking up the inner integral as

$$\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)g_{\text{odd}}(y,s)dyds = \int_0^t \left[\int_{-\infty}^{0} S(x-y,t-s)(-1)g(-y,s)dy + \int_0^{\infty} S(x-y,t-s)g(y,s)dy\right] ds.$$

If we let $y=-y$ for the first integral and we adjust our limits of integration, we end up with the expression

$$\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)g_{\text{odd}}(y,s)dyds = \int_0^t \left[\int_{\infty}^{0} S(x+y,t-s)(-1)g(y,s)(-1)dy + \int_0^{\infty} S(x-y,t-s)g(y,s)dy\right] ds.$$

If we swap the limits of integration, we end up with one remaining negative, and we can combine our integral to obtain

$$\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)g_{\text{odd}}(y,s)dyds = \int_0^t \int_0^{\infty} \left[-S(x+y,t-s)+S(x-y,t-s)\right] g(y,s)dyds.$$

Lastly, by restricting our solution $U_i(x,t)$ to $x>0$ we end up with the solution $V_i(x,t)$. Thus, combining these two formulas for $V_h(x,t)$ and $V_i(x,t)$ then we end up with the general formula for the inhomogeneous Dirichlet boundary value problem as

$$V(x,t) = \int_0^{\infty} \left[S(x-y,t)-S(x+y,t)\right] q(y)dy + \int_0^t \int_0^{\infty} \left[-S(x+y,t-s)+S(x-y,t-s)\right] g(y,s)dyds.$$

In the context of this specific problem, we can easily solve for our solution $v(x,t)$ by adding $h(t)$ to undo our initial subtraction. Lastly, we substitute our expressions for both $q(y)$ and $g(x)$ and we have our solution to the complete inhomogeneous IBVP for the diffusion equation:

$$V(x,t) = \int_0^{\infty} \left[S(x-y,t)-S(x+y,t)\right] \left[\phi(s)-h(0)\right] dy + \int_0^t \int_0^{\infty} \left[-S(x+y,t-s)+S(x-y,t-s\right] \left[f(y,s)-h'(s)\right] dyds.$$

2 Initial Boundary Value Problems

2.1 Summary of Lecture 4

Duhamel's principle provides us with a formula for solving the inhomogeneous wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \quad x, t \in (-\infty, \infty), \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds.$$

We are interested in alternate derivations of the formula. Today, we discussed 3 different approaches.

Geometric approach:

This approach acknowledges the "past history" triangle. (formed by the characteristic lines in the xt-plane). The double integral in the formula we are deriving can be rewritten as

$$\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds = \iint\limits_{\Delta} f(x,t)dxdt$$

after renaming the "dummy variables" y to x and s to t. Characteristic lines, drawn from a point (x_0, t_0), hit the x-axis at $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ and form a triangle. This is the past history triangle. So the double integral in the formula for the solution to the inhomogeneous wave equation can be acknowledged as the integration over the past history of the "source term" $f(x,t)$.

Green's Theorem:

We reviewed some concepts from vector calculus (line integrals, divergence, etc.) and eventually wrote down Green's Theorem. We began by taking a look at the xy-plane. Given a domain in $\mathbb{R}^2$ (denoted by $\mathcal{D}$), its boundary $\partial\mathcal{D}$, and 2 continuous functions $P(x,y)$ and $Q(x,y)$ on $\bar{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$, Green's Theorem gives us the area of the region:

$$\iint\limits_{\mathcal{D}} (P_x - Q_y)dxdy = \int\limits_{\partial\mathcal{D}} Qdx + Pdy.$$

For the inhomogeneous wave equation, the domain $\mathcal{D}$ is the past history triangle Δ. Apply Green's Theorem:

$$\iint\limits_{\Delta} f(x,t)dxdt = \int\limits_{\partial\Delta} -u_t dx - c^2 u_x dt,$$

$$\iint\limits_{\Delta} f(x,t)dxdt = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx + 2cu(x_0,t_0) - c[\phi(x_0+ct_0) + \phi(x_0-ct_0)],$$

$$u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0) + \phi(x_0-ct_0)] + \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx + \iint\limits_{\Delta} f(x,t)dxdt.$$

Integrating factor:

We only scratched the surface of this approach. We looked at the second-order ODE

$$\frac{d^2u}{dt^2} + A^2u = f(t)$$

and rewrote it as a system of 2 first-order ODEs:

$$\begin{cases} \frac{du}{dt} = v, \\ \frac{dv}{dt} = -A^2u + f. \end{cases}$$

Next, we set up the ODE with vectors $\vec{U} = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\vec{F}$:

$$\frac{d\vec{U}}{dt} + M\vec{U} = \vec{F}.$$

We were encouraged to think about a few questions for next time. What's M? What's $\vec{F}$? What's the solution to the homogeneous version of the equation above? What's the integrating factor? Also, most importantly, how does this approach lead us to the derivation of the formula for the solution to the inhomogeneous wave equation?

2.2 Example

We will work out Exercise 3.4-1 as an example. Solve the IVP (inhomogeneous wave equation):

$$\begin{cases} u_{tt} - c^2u_{xx} = xt \quad x, t \in (-\infty, \infty), \\ u(x, 0) = 0, \\ u_t(x, 0) = 0. \end{cases}$$

We know the formula for the solution:

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds.$$

In this example, both initial conditions are zero ($\phi(x) = \psi(x) = 0$). So the formula for the solution simplifies down to

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ys\; dyds$$

$$= \frac{1}{2c}\int_0^t \frac{s}{2}[(x+c(t-s))^2 - (x-c(t-s))^2]ds = \frac{1}{2c}\int_0^t \frac{s}{2}[(x+ct-cs)^2 - (x-ct+cs)^2]ds$$

$$= \frac{1}{2c}\int_0^t \frac{s}{2}\left[x^2 + 2xct - 2xcs - 2c^2st + c^2t^2 + c^2s^2 - (x^2 - 2xct + 2xcs - 2c^2st + c^2t^2 + c^2s^2)\right] ds$$

$$= \frac{1}{2c}\int_0^t \frac{s}{2}\left[\cancel{x^2} + 2xct - 2xcs - \cancel{2c^2st} + \cancel{c^2t^2} + \cancel{c^2s^2} - \cancel{x^2} + 2xct - 2xcs + \cancel{2c^2st} - \cancel{c^2t^2} - \cancel{c^2s^2})\right] ds$$

$$= \frac{1}{2c}\int_0^t \frac{s}{2}(4xct - 4xcs)ds = \frac{1}{2c}\int_0^t 2xct\left(s - \frac{s^2}{t}\right)ds = xt\int_0^t \left(s - \frac{s^2}{t}\right)ds$$

$$= xt\int_0^t \frac{1}{t}\left(st - s^2\right)ds = x\int_0^t (ts - s^2)ds$$

$$= x\left(\frac{ts^2}{2} - \frac{s^3}{3}\right)\Big|_0^t = x\left(\frac{t^3}{2} - \frac{t^3}{3}\right)$$

$$= \frac{xt^3}{6}.$$

2.3 Exercise

Exercise 3.4-6: find another way to derive the formula for the inhomogeneous wave equation. **(a)** Rewrite the second-order PDE as a system of first-order PDEs:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \quad x, t \in (-\infty, \infty), \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases}$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = f \implies \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = f.$$

To match the proposed system, let $v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u$. Then $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)v = f$ and the system of first-order PDEs appears as proposed:

$$\begin{cases} u_t + cu_x = v, \\ v_t - cv_x = f. \end{cases}$$

(b) Solve the first equation for u in terms of v. This equation is a manifestation of the transport equation. Acknowledge the IVP:

$$\begin{cases} u_t + cu_x = v(x,t), \\ u(x,0) = \phi(x). \end{cases}$$

Let $w(s) = u(x(s), t(s))$. According to Lecture 2 (PDEs I), the expression for parametrized characteristic lines yields

$$w(s) = u(x + cs, t + s).$$

Denote $x + cs$ as a and denote $t + s$ as b. By the chain rule,

$$w'(s) = (u_a)(c) + (u_b)(1) = u_b + cu_a.$$

This expression mimics the transport equation, so

$$u_b + cu_a = w'(s) = v(a,b),$$

$$\int_{-t}^{0} w'(s)ds = \int_{-t}^{0} v(a,b)ds,$$

$$w(0) - w(-t) = \int_{-t}^{0} v(x+cs, t+s)ds,$$

$$u(x,t) - u(x-ct, 0) = \int_{-t}^{0} v(x+cs, t+s)ds,$$

$$u(x,t) = \phi(x-ct) + \int_{-t}^{0} v(x+cs, t+s)ds.$$

Let $z = t + s$ (so $dz = ds$). So the limits of integration are $s = -t$ to $s = 0$ but they change to $z = 0$ to $z = t$:

$$u(x,t) = \phi(x-ct) + \int_{0}^{t} v(x + c(z-t), z)dz$$

$$= \phi(x-ct) + \int_{0}^{t} v(x + c(s-t), s)ds = \phi(x-ct) + \int_{0}^{t} v(x - ct + cs, s)ds.$$

Note: the "dummy variable" z was changed to s. If we assume a zero initial condition, we have the requested expression for the solution:

$$u(x,t) = \int_{0}^{t} v(x - ct + cs, s)ds.$$

(c) Solve the second equation for v in terms of f:

$$\begin{cases} v_t - cu_x = f(x,t), \\ v(x,0) = \xi(x). \end{cases}$$

Let $w(s) = v(x(s), t(s))$. So

$$w(s) = v(x - cs, t + s).$$

Again, denote $x - cs$ as a and denote $t + s$ as b. By the chain rule,

$$w'(s) = (v_a)(-c) + (v_b)(1) = v_b - cv_a,$$

$$v_b - cv_a = w'(s) = f(a,b),$$

$$\int_{-t}^{0} w'(s)ds = \int_{-t}^{0} f(a,b)ds,$$

$$w(0) - w(-t) = \int_{-t}^{0} f(x - cs, t+s)ds,$$

$$v(x,t) - v(x+ct, 0) = \int_{-t}^{0} f(x - cs, t+s)ds,$$

$$v(x,t) = \xi(x+ct) + \int_{-t}^{0} f(x-cs, t+s)ds.$$

Let $z = t + s$ (so $dz = ds$). So the limits of integration are $s = -t$ to $s = 0$ but they change to $z = 0$ to $z = t$:

$$v(x,t) = \xi(x+ct) + \int_{0}^{t} f(x - c(z-t), z)dz,$$

$$= \xi(x+ct) + \int_{0}^{t} f(x - c(s-t), s)ds = \xi(x+ct) + \int_{0}^{t} f(x+ct-cs, s)ds.$$

Assume a zero initial condition again. We have the expression for v in terms of f:

$$v(x,t) = \int_{0}^{t} f(x+ct-cs, s)ds.$$

(d) The desired formula for $u(x,t)$, given as the "iterated integral" on page 79 in Strauss, is

$$u(x,t) = \frac{1}{2c}\int_{0}^{t}\int_{x-ct+cs}^{x+ct-cs} f(y,s)dyds.$$

We recognize this double integral (from Duhamel's principle). So we have an expression for u in terms of v and we have an expression for v in terms of f:

$$u(x,t) = \int_{0}^{t} v(x-ct+cs, s)ds, \qquad v(x,t) = \int_{0}^{t} f(x+ct-cs, s)ds,$$

$$u(x,t) = \int_{0}^{t}\int_{s}^{t} f((x-ct+cp)+cp-cs, s)dpds,$$

$$= \int_{0}^{t}\int_{s}^{t} f(x-ct+2cp-cs, s)dpds.$$

Let $y = x - ct + 2cp - cs$. So $dy = 2cdp \implies dp = \frac{1}{2c}dy$. The limits of integration for the inner integral are $p = s$ to $p = t$ but they change to $y = x - ct + 2cs - cs$ to $y = x - ct + 2ct - cs$. This simplifies as $y = x - ct + cs$ to $y = x + ct - cs$. So

$$u(x,t) = \int_{0}^{t}\int_{x-ct+cs}^{x+ct-cs} f(y,s)\left(\frac{1}{2c}dy\right) ds$$

$$= \frac{1}{2c}\int_{0}^{t}\int_{x-ct+cs}^{x+ct-cs} f(y,s)dyds.$$

2.4 Summary of Lecture 5

During class we spent the majority of the lecture reviewing the method of separation of variables. Though incredibly convenient in its formulation, this method does not work in complete generality. Interestingly, it is based on a "hopeful guess" that the solution to the PDE will be comprised of the multiplication of two functions of a single variable. Essentially, we take our solution $u(x,t)$ for the heat or the wave equation and equate it to $u(x,t) = X(x)T(t)$. We then use this expression in our PDE since we can calculate the partials as $u_{xx} = X''T$ and $u_{tt} = X\ddot{T}$. We then divide the entire expression by a factor of c^2TX or kTX depending on the equation we are dealing with.

Ultimately our result is that we have ODEs of one variable on each side of our expression:

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2T} \quad \text{or} \quad \frac{X''}{X} = \frac{\dot{T}}{c^2T}.$$

Since each side of the equation is a function of a different, single variable we must have that the right hand side and the left hand side are equal to a constant. Our problem then devolves into a system of two ODEs, and essentially what we have is an eigenvalue problem. The solution to this eigenvalue problem is based on the kind of boundary conditions we have.

For the remainder of the lecture, we extended upon the method of separation of variables from what we encountered last semester by looking into the mixed or Robin Boundary condition. This condition is formulated as follows:

$$\begin{cases} u_x - a_0 u = 0 & x = 0, \\ u_x + a_L u = 0 & x = L. \end{cases}$$

Essentially we just play the same game using separations of variables, however things get more interesting when we impose our boundary conditions. In the case of both the heat and wave equations, we already determined that our eigenvalues based on the spatial ODE must be strictly negative in order to avoid trivial solutions. This condition, however, no longer holds for the Robin condition. In addition, the equations that arrive are transcendental and do not have an easy solution. In class we were able to derive that in the case of strictly negative boundary conditions, the equation for the eigenvalues is given by

$$\tan \beta l = \frac{a_0 + a_l}{\beta^2 - a_0 a_l}\beta.$$

This equation is incredibly difficult to solve, and our methodology in class was to solve it graphically. The only method we could think of was to apply some of the methods we have been learning in Numerical Analysis to find the roots of a similar equation. However, we aren't trying to understand this equation in simulations, but in its generality. We hope that we will encounter some more methods to understand the solution to this eigenvalue problem.

2.5 Example

As an example of the method of separation of variables, we will re-derive the solution to the homogeneous heat equation with Dirichlet boundary conditions, and fill in all of the details about the eigenvalue problem. Since we have not gone over Fourier coefficients yet, we will exclude the consideration of the initial conditions. We formulate our Boundary Value Problem (BVP) as follows:

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < l, \ t > 0, \\ u(x,0) = u(x,l) = 0. \end{cases}$$

To begin, we assume that our solution $u(x,t)$ takes the form $u(x,t) = X(x)T(t)$. Now our partial derivatives take the form

$$u_{xx} = X''(x)T(t) \ \text{ and } \ u_t = X(x)\dot{T}(t).$$

Therefore our PDE takes the form

$$\dot{T}X - kX''T = 0.$$

Next we divide through by kXT and the result is

$$\frac{\dot{T}}{kT} - \frac{X''}{X} = 0 \implies \frac{\dot{T}}{kT} = \frac{X''}{X}.$$

Since the LHS and RHS are both functions of different single variable, each side must be equal to a constant. Let us call this constant λ, and we can break our PDE into the two separated ODES given by

$$\frac{X''}{X} = \lambda \text{ and } \frac{\dot{T}}{kT} = \lambda.$$

Since we are dealing with the boundary conditions which restrict x, we will start with the spatial ODE. We first need to consider all of the options for the value of λ, so we have three cases.

Case 1: $\lambda = 0$

In this case our ODE reads

$$\frac{X''}{X} = 0 \to X'' = 0$$

which has the linear solution

$$X = Ax + B.$$

Now we can impose our boundary conditions:

$$X(0) = B = 0 \text{ and } X(l) = 0 = Al \Rightarrow A = 0 \text{ since } l > 0.$$

This leads us to a trivial solution.

Case 2: $\lambda > 0$

In this case, we will let $\lambda = \beta^2$. In this case, our ODE is

$$X'' - \beta^2 X = 0.$$

Now, this equation has two forms of its solution. We can either use the hyperbolic or exponential form. For ease, we will use the hyperbolic functions. Therefore we have that

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

Using our boundary conditions we have that

$$X(0) = A = 0 \text{ and } X(l) = 0 = B\cosh(\beta l) \Rightarrow A = 0 \text{ since } l > 0.$$

Since the hyperbolic cosine is never zero, we must have that $B = 0$ and we are left with another trivial solution.

Case 3: $\lambda < 0$

For these values of lambda, we will let $\lambda = -\beta^2$, so our ODE is that of the simple harmonic oscillator which is given by the equation

$$X'' + \beta^2 X = 0$$

and has the solution

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

When we input our BCs, we find that

$$X(0) = A = 0 \quad X(l) = B\sin(\beta l) = 0.$$

This leads us to a family of solutions since the sine function is zero at any value of pi. Therefore, for $n \in N$, we have that our eigenvalues are $\lambda = \frac{n\pi}{l}$, and our eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{l}\right).$$

The next piece that we have to take care of is the solution to the temporal ODE, however it is much simpler since we know what our eigenvalue must be. Our ODE

$$\frac{\dot{T}}{T} = k\beta^2$$

has the simple exponential solution

$$T_n(t) = Ce^{\frac{n^2\pi^2 kt}{l^2}}.$$

These two expressions, however, do not solve our BVP, but in order to rectify this we use the linear superposition principle. Finally, we arrive at our general solution for the heat equation on the finite line with Dirichlet boundary conditions as

$$u(x,t) = \sum_{n=1}^{\infty} e^{\frac{n^2\pi^2 kt}{l^2}} \sin\left(\frac{n\pi x}{l}\right).$$

2.6 Exercise

In Exercise 4.1-4, our goal is to solve the IBVP given by

$$\begin{cases} u_{tt} + ru_t - c^2 u_{xx} = 0 & x \in (0,l), \;\; t \in \mathbb{R}, \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \phi(x), \\ u_t(x,0) = \phi(x). \end{cases}$$

To do so, we will proceed with separation of variables. We assume that our solution $u(x,t) = X(x)T(t)$ where $X(x)$ and $T(t)$ are functions of single variables. We will use the prime symbol to denote x derivatives and the dot to denote time derivatives. Thus our partial derivative reads

$$\ddot{T}X + r\dot{T}X - c^2 X'' T = 0.$$

Next, we divide both sides by $c^2 XT$ and we have

$$\frac{\ddot{T}}{c^2 T} + \frac{r\dot{T}}{c^2 T} - \frac{X''}{X} = 0.$$

Next, we can move around our expressions, and what we end up with is

$$\frac{\ddot{T}}{c^2 T} + \frac{r\dot{T}}{c^2 T} = \frac{X''}{X}.$$

Since each side is merely a function of a single variable we must have that the LHS and the RHS are both equal to a constant λ. Therefore, our problem is essentially reduced to a set of two ODEs. Given homogeneous Dirichlet boundary conditions, the only value of λ that does not lead to a trivial solution is when $\lambda < 0$ (PDEs I: Lecture 5). Hence, we will let $\lambda = -\beta^2$:

$$\frac{X''}{X} = -\beta^2 \to X'' + \beta^2 X = 0.$$

The ODE above is the simple harmonic oscillator which has the solution

$$X(x) = A\cos(\beta x) + B\sin(\beta x)$$

for some constants A, B. Now, we must impose our boundary conditions to determine the values of the constants. For the first condition $u(x,0) = 0$ we find that

$$u(x,0) = A = 0$$

and for $u(l,0)$ we have that

$$u(x,l) = B\sin(\beta l) = 0.$$

This leads us to a family of solutions for values of $n = 1, 2, \ldots$ since $sin(x)$ is zero when $x = n\pi$. Therefore, we must have that $\beta = \frac{2\pi}{l}$. We must tackle the temporal ODE next. Getting rid of our factor of c^2T in the denominator (and plugging in our value of β^2), our temporal ODE becomes the second order linear homogeneous ODE given by

$$\ddot{T} + r\dot{T} + \frac{c^2n^2\pi^2}{l^2}T = 0.$$

In order to determine the solution to this equation we must solve the characteristic equation given by

$$k^2 + rk + \frac{c^2n^2\pi^2}{l^2} = 0.$$

and determine the sign of the discriminant. By the quadratic formula,

$$k = \frac{-r \pm \sqrt{r^2 - 4\frac{c^2n^2\pi^2}{l^2}}}{2}.$$

From our given information, we know that $0 < r < \frac{2\pi c}{l}$ which clearly implies that $r^2 < \frac{4\pi^2n^2c^2}{l^2}$. This result tells us that our discriminant is negative and our solution to this ODE is given by

$$T_n(t) = e^{\frac{-r}{2}}\left[C_n\cos(\beta(d_n)t) + D_n\sin(d_nt)\right]$$

where d_n is given by

$$d_n = \frac{\sqrt{r^2 - 4\frac{c^2n^2\pi^2}{l^2}}}{2}.$$

In order to find our solutions for fixed n, we simply multiply our values of $X(x)$ and $T(t)$. Note: the constants B, C, D are all arbitrary, so we "lump" B into C and D. Now, these solutions do not solve our problem completely. In order to find the completely general solution to this equation, we use the superposition principle for linear homogeneous equations. Then we sum our individual solutions for all values of n. Our final solution is

$$\sum_{n=1}^{\infty} e^{\frac{-rt}{2}}\left[C_n\cos(\beta(d_n)t + D_n\sin(d_nt)\right]\sin\left(\frac{n\pi x}{l}\right).$$

Lastly, we must take care of our initial conditions. To do so, we use the orthogonality condition of the inner product of the L_2 norm with sines and cosines. The explicit statement of this condition can be found in the lecture notes for PDE I in Lecture 6. Imposing our initial conditions, we have that

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right),$$

$$\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right)dx = \int_0^l \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx,$$

$$= \sum_{n=1}^{\infty}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \frac{l}{2}C_m.$$

Therefore, our coefficients are given by

$$C_m = \int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right)dx.$$

Next, we must look at the second initial condition. We have that

$$u_t(x,t) = \sum_{n=1}^{\infty}\left\{\frac{-r}{2}e^{\frac{-rt}{2}}\left[C_n\cos(d_nt) + D_n sin(d_nt)\right] + e^{\frac{-rt}{2}}\left[-C_nd_n\sin(d_nt) + D_nd_n\cos(d_nt)\right]\right\}\sin\left(\frac{n\pi}{l}\right).$$

Hence,

$$u_t(x,0) = \psi(x) = \frac{-r}{2}\sum_{n=1}^{\infty}(C_n + D_nd_n)\sin\left(\frac{n\pi x}{l}\right) = \frac{-r}{2}\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi x}{l}\right) + \frac{-r}{2}\sum_{n=1}^{\infty}D_nd_n\sin\left(\frac{n\pi x}{l}\right).$$

We know that the infinite sum of our C_n and its corresponding sine terms are, by definition, the function $\phi(x)$. We replace the sum with our function and move it over to the left hand side. Then we use the orthogonality condition once again to solve for the coefficients D_n. Ultimately, our result is

$$D_n = \frac{2}{d_nl}\int_0^l \left[\psi(x) + \frac{r}{2}\phi(x)\right]\sin\left(\frac{n\pi x}{l}\right).$$

Therefore, with our equation for $u(x,t)$ above (and our expression for the coefficients C_n and D_n), we have the solution to the IBVP.

3 Fourier Series

3.1 Summary of Lecture 6

Today, we focused on the eigenvalue problem with Robin boundary conditions:

$$\begin{cases} -X'' = \lambda X & x \in (0, L), \\ X'(0) - a_0 X(0) = 0, \\ X'(L) + a_L X(L) = 0. \end{cases}$$

Overall, we are interested in three cases: *positive eigenvalues* ($\lambda = \beta^2$), *zero eigenvalues* ($\lambda = 0$), and *negative eigenvalues* ($\lambda = -\beta^2$). However, the constants a_0 and a_L from the boundary conditions complicate matters since we have to consider their signs for each case. We discussed a few different situations:

$$a_0, a_L > 0 \implies \text{radiation},$$

$$a_0 < 0, a_L > 0 \text{ and } a_0 + a_L > 0 \implies \text{radiation dominates},$$

$$a_0, a_L < 0 \implies \text{absorption},$$

$$a_0 = a_L = 0 \implies \text{homogeneous Neumann BCs emerge.}$$

Positive eigenvalues: The eigenvalue problem is $X'' = -\beta^2 X$. We recognize this ODE (harmonic oscillator) and report the solution:

$$X(x) = C\cos(\beta X) + D\sin(\beta X).$$

We impose the BCs and eventually obtain

$$\tan(\beta L) = \frac{(a_0 + a_L)\beta}{\beta^2 - a_0 a_L}.$$

We saw this implicit equation last time. After graphical analysis, we concluded that $\beta_n \to \frac{n\pi}{L}$ as $n \to \infty$. So the eigenvalues, λ_n, approach $\left(\frac{n\pi}{L}\right)^2$. Important concept for today's lecture: this situation corresponds to radiation (mentioned earlier: $a_0, a_L > 0$).

For the second situation ($a_0 < 0, a_L > 0$ and $a_0 + a_L > 0$), we study the same implicit equation, but complications arise. Denote the LHS as $f(\beta)$ and denote the RHS as $g(\beta)$. The graph for $f(\beta)$ stays the same but the graph for $g(\beta)$ changes into a curve resembling a traveling wave. When we consider the "end behavior" of this curve, we reach the same conclusion: $\beta_n \to \frac{n\pi}{L}$ as $n \to \infty$. So the eigenvalues, λ_n, approach $\left(\frac{n\pi}{L}\right)^2$.

However, we need to acknowledge two possible graphs for $g(\beta)$. Both graphs display the same end behavior ($\beta_n \to \frac{n\pi}{L}$ as $n \to \infty$) but they differ on the interval $\left(0, \frac{\pi}{L}\right)$. One graph has an eigenvalue in that interval and the other graph does not. In class, we used derivatives to explain why this difference occurs.

If $f(\beta) = \tan(\beta L)$ and $g(\beta) = \frac{(a_0+a_L)\beta}{\beta^2 - a_0 a_L}$, consider the derivatives at $\beta = 0$:

$$f'(0) = L \qquad \text{and} \qquad g'(0) = \frac{a_0 + a_L}{-a_0 a_L}.$$

If $f'(0) < g'(0)$, the two functions intersect once in the interval $\left(0, \frac{\pi}{L}\right)$. If $f'(0) > g'(0)$, the two functions do not intersect there. Therefore, in order for an eigenvalue to exist on the interval $\left(0, \frac{\pi}{L}\right)$, the following condition must be met:

$$f'(0) = L < \frac{a_0 + a_L}{-a_0 a_L} = g'(0)$$

$$\implies a_0 + a_L > -a_0 a_L L \qquad \text{(note: the product } a_0 a_L \text{ is negative).}$$

Zero eigenvalues: The eigenvalue problem becomes $X'' = 0$. The known solution to this ODE is

$$X(x) = Cx + D.$$

We impose the BCs and eventually obtain

$$a_0 + a_L = -a_0 a_L L.$$

Two situations (mentioned at the beginning) allow the possibility for the existence of zero eigenvalues. We can pay attention to the signs of a_0 and a_L and speculate on the possibility for the equality (above) to hold:

$$\text{If } a_0, a_L > 0 : \lambda = 0 \text{ not possible.}$$

$$\text{If } a_0 < 0, a_L > 0 \text{ and } a_0 + a_L > 0 : \lambda = 0 \text{ possible.}$$

$$\text{If } a_0, a_L < 0 : \lambda = 0 \text{ possible.}$$

Negative eigenvalues: The eigenvalue problem becomes $X'' = \beta^2 X$. The known solution is

$$X(x) = C\cosh(\beta X) + D\sinh(\beta X).$$

We impose the BCs and eventually obtain

$$\tanh(\beta L) = \frac{-(a_0 + a_L)\beta}{\beta^2 + a_0 a_L}.$$

$$\text{If } a_0, a_L > 0 : \lambda = -\beta^2 \text{ not possible.}$$

$$\text{If } a_0 < 0, a_L > 0 \text{ and } a_0 + a_L > 0 : \lambda = -\beta^2 \text{ possible.}$$

The latter situation is interesting. When radiation dominates, there is either a unique eigenvalue or no eigenvalue at all. If $f(\beta) = \tanh(\beta L)$ and $g(\beta) = \frac{-(a_0+a_L)\beta}{\beta^2 + a_0 a_L}$, consider the derivatives at $\beta = 0$:

$$f'(0) = L \qquad \text{and} \qquad g'(0) = \frac{a_0 + a_L}{-a_0 a_L}.$$

If $f'(0) > g'(0)$, the two functions intersect once. If $f'(0) < g'(0)$, the two functions do not intersect at all. Therefore, in order for a unique eigenvector to exist, the following condition must be met:

$$f'(0) = L > \frac{a_0 + a_L}{-a_0 a_L} = g'(0)$$

$$\implies a_0 + a_L < -a_0 a_L L \qquad \text{(note: the product } a_0 a_L \text{ is negative).}$$

3.2 Example

Exercise 4.3-3: given the same eigenvalue problem (with Robin BCs), derive the equations for the negative eigenvalues and corresponding eigenfunctions:

$$\begin{cases} -X'' = \lambda X & x \in (0, L), \\ X'(0) - a_0 X(0) = 0, \\ X'(L) + a_L X(L) = 0. \end{cases}$$

Let $\lambda = -\beta^2$ and solve the resulting second-order linear ODE:

$$X'' = \beta^2 X \implies X'' - \beta^2 X = 0.$$

Acknowledge the characteristic equation affiliated with the ODE and solve it:

$$w^2 + 0w - \beta^2 = 0,$$

$$w^2 = \beta^2 \implies w = \pm\beta.$$

Since the characteristic equation has two distinct roots, the solution to the original ODE has the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x} = \left(\frac{C+D}{2}\right) e^{\beta x} + \left(\frac{C-D}{2}\right) e^{-\beta x},$$

$$\frac{C}{2}\left(e^{\beta x} + e^{-\beta x}\right) + \frac{D}{2}\left(e^{\beta x} - e^{-\beta x}\right) = C\left(\frac{e^{\beta x} + e^{-\beta x}}{2}\right) + D\left(\frac{e^{\beta x} - e^{-\beta x}}{2}\right)$$

$$= C\cosh(\beta x) + D\sinh(\beta x).$$

Compute $X'(x)$ then impose the BCs:

$$X'(x) = C\beta\sinh(\beta x) + D\beta\cosh(\beta x),$$

$$X'(0) = C\beta\sinh(0) + D\beta\cosh(0) = D\beta,$$

$$X(0) = C\cosh(0) + D\sinh(0) = C,$$

$$X'(0) - a_0 X(0) = D\beta - a_0 C = 0 \implies D = \frac{a_0 C}{\beta},$$

$$X'(L) + a_L X(L) = C\beta\sinh(\beta L) + a_0 C\cosh(\beta L) + a_L\left[C\cosh(\beta L) + \frac{a_0 C}{\beta}\sinh(\beta L)\right] = 0,$$

$$\left(C\beta + \frac{a_0 a_L C}{\beta}\right)\sinh(\beta L) + (a_0 C + a_L C)\cosh(\beta L) = 0,$$

$$\left(\beta + \frac{a_0 a_L}{\beta}\right)\sinh(\beta L) + (a_0 + a_L)\cosh(\beta L) = 0,$$

$$\left(\beta^2 + a_0 a_L\right)\sinh(\beta L) + \beta\left(a_0 + a_L\right)\cosh(\beta L) = 0,$$

$$\left(\beta^2 + a_0 a_L\right)\tanh(\beta L) + \beta\left(a_0 + a_L\right) = 0,$$

$$\tanh(\beta L) = -\frac{\beta\left(a_0 + a_L\right)}{\beta^2 + a_0 a_L}.$$

The implicit equation for the negative eigenvalues has been derived. The corresponding eigenfunctions have the form

$$X(x) = C\cosh(\beta x) + \frac{a_0 C}{\beta}\sinh(\beta x).$$

If we choose $C = 1$, we have derived the requested expression:

$$X(x) = \cosh(\beta x) + \frac{a_0}{\beta}\sinh(\beta x).$$

3.3 Exercise

Exercise 4.3-13: a string is fixed at one end ($x = 0$) and free at the other end ($x = L$). A load of given mass is attached to the right end.

(a) We are asked to show that the string satisfies the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in (0, L), \\ u(0,t) = 0, \\ u_{tt}(L,t) = -k u_x(L,t), \end{cases}$$

for some constant k. We are told that we only need to justify the BC at $x = L$. Use Newton's Second Law to do so ($F = ma$).

At the boundary $x = L$, denote the "load of given mass" as m. Denote acceleration as $u_{tt}(x, L)$ (if $u(x,t)$ indicates the position of the string, the first derivative of u with respect to time indicates velocity and the second derivative with respect to time indicates acceleration). On the LHS of Newton's Second Law, acknowledge F as net force (the string is subject to more than one force).

In PDEs I, we were introduced to a one-dimensional model for a string. In this model, we only acknowledged one force acting on the string (tension pulling up at an angle α). This time, we need to account for the mass attached to the string, so we need to acknowledge gravity. The vertical net force acting on the string is given by

$$\left|\vec{T}_v\right| - mg,$$

$$\left|\vec{T}\right|\sin(\alpha) - mg \qquad \left(\text{for simplicity, denote } \left|\vec{T}\right| \text{ as } T \text{ from now on}\right).$$

This expression can occupy the LHS of Newton's Second Law. So

$$F = ma \implies T\sin(\alpha) - mg = m u_{tt}(L,t).$$

Last semester, we used trigonometric identities to rewrite $\sin(\alpha)$:

$$\sin(\alpha) = \frac{\tan(\alpha)}{\sec(\alpha)} = \frac{\tan(\alpha)}{\sqrt{\sec^2(\alpha)}} = \frac{\tan(\alpha)}{\sqrt{1+\tan^2(\alpha)}}.$$

One interesting fact about derivatives becomes relevant. We know that we can acknowledge $\frac{du}{dx}$ as a ratio (the change in height over the change in horizontal position). Since this model exists in the ux-plane, we can denote $\tan(\alpha)$ as u_x. So

$$\sin(\alpha) = \frac{u_x}{\sqrt{1+u_x^2}},$$

$$T\frac{u_x}{\sqrt{1+u_x^2}} - mg = mu_{tt}(L,t).$$

We made 3 assumptions for the introductory model: 1) ρ constant 2) T constant and 3) u_x negligible. For this exercise, the first assumption is not significant, but the second and third assumptions will help us derive the equation for the desired BC. If $u_x \approx 0$, then

$$Tu_x - mg = mu_{tt}(L,t),$$

$$Tu_x = mu_{tt}(L,t) + mg,$$

$$mu_{tt} = Tu_x - mg,$$

$$u_{tt} = \frac{T}{m}u_x - g,$$

$$u_{tt} = \frac{T}{m}\left(u_x - \frac{gm}{T}\right) \qquad \left(\text{note: if } T >> m, \text{ then } \frac{gm}{T} \to 0\right),$$

$$u_{tt} = \frac{T}{m}u_x.$$

Let $\frac{T}{m} = -k$ to obtain $u_{tt}(L,t) = -ku_x(L,t)$. The desired equation for the BC has been derived. So the loaded string satisfies the problem.

(b) The eigenvalue problem for the wave equation originates from the method of separation of variables. Assume the solution $u(x,t)$ has the form $u(x,t) = X(x)T(t)$. Compute the relevant derivatives:

$$u_{tt} = X(x)\frac{d^2T}{dt^2} = X\ddot{T},$$

$$u_{xx} = \frac{d^2X}{dx^2}T(t) = X''T.$$

Substitute these expressions into the wave equation:

$$X\ddot{T} = c^2X''T,$$

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2T} = -\lambda.$$

So the eigenvalue problem is

$$X'' = -\lambda X.$$

(c) For the case where the eigenvalues are positive, let $\lambda = \beta^2$:

$$X'' = -\beta^2 X.$$

This ODE (harmonic oscillator) has the known solution:

$$X(x) = C\cos(\beta X) + D\sin(\beta X).$$

Impose the boundary conditions:

$$u(0,t) = 0, \qquad u_{tt}(L,t) = -ku_x(L,t),$$

$$X(0) = C\cos(0) + D\sin(0) = C = 0 \implies X(x) = D\sin(\beta X).$$

For the second boundary condition, recall that u_{tt} is $X\ddot{T}$. Also, acknowledge u_x as $X'T$. So

$$X(L)\ddot{T}(t) = -kX'(L)T(t),$$

$$\frac{-kX'(L)}{X(L)} = \frac{\ddot{T}(t)}{T(t)} \qquad \text{(revisit shortly)}.$$

From the previous part, we have

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = -\lambda \implies \frac{\ddot{T}}{T} = -\lambda c^2.$$

Revisit and rewrite:

$$\frac{-kX'(L)}{X(L)} = -\lambda c^2,$$

$$kX'(L) = \lambda c^2 X(L).$$

From the first boundary condition, we know $X(x) = D\sin(\beta X)$. So

$$k\left[D\beta\cos(\beta L)\right] = \lambda c^2 \left[D\sin(\beta L)\right],$$

$$\frac{k\beta}{\lambda c^2} = \frac{D\sin(\beta L)}{D\cos(\beta L)} \implies \frac{k\beta}{\beta^2 c^2} = \tan(\beta L),$$

$$\tan(\beta L) = \frac{k}{\beta c^2}.$$

We have found the implicit equation for the eigenvalues. We could find the values for β_n by taking a graphical approach. Denote the LHS as $f(\beta)$ and denote the RHS as $g(\beta)$. Graph $f(\beta)$ vs. β, then superimpose the graph for $g(\beta)$ vs. β and find the intersections. Since $\lambda = \beta^2$, the values for λ_n would be equal to the square root of the values for β_n at those intersections.

We conclude that there is an infinite number of eigenvalues. There is also an infinite number of eigenfunctions. Recall: $X(x) = D\sin(\beta X)$ and D is an arbitrary constant. Choose $D = 1$ and acknowledge $X(x) = \sin(\beta X)$ as the simplest eigenfunction.

3.4 Summary of Lecture 7

During lecture we discussed Fourier Series in more generality than in previous lectures. We looked at the general form of the Fourier Sine, Cosine, and Full Series which, for some $f(x)$, are given by

$$\begin{cases} f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{L}) & x \in (0, L), \\ f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L}) & x \in (0, L), \\ f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi x}{L}) + B_n \sin(\frac{n\pi x}{L})\right] & x \in (-L, L), \end{cases}$$

respectively. Now, it is important to note, that the Full Fourier Series represents a function on a symmetric interval. Each one of these series, however, define periodic extensions of a function $f(x)$ to the entire real line with a period $2L$, where the sine series is an odd extension, the cosine series is an even extension, and the full series is simply periodic. In order to determine the coefficients, we use the orthogonality condition of sines and cosines

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{n,m}$$

where $\delta_{n,m}$ denotes the Kronecker delta function. We can make a similar construction for the cosine series, by exchanging sine for cosine, and also for the full series by changing $\frac{L}{2}$ to just a factor of L. Using these conditions we find that our coefficients are given by

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

for the sine series. Once again, we can make the same modifications as above to our integral for the coefficients of the cosine and full series. In order to simplify our expression for the full series, we can use formulas for the complex exponentials to simplify our complete series in terms of one complex exponential. This series takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}} \quad x \in (-L, L)$$

and we can, once again, use an orthogonality condition to determine the coefficients C_n. In this case, we use the orthogonality of the complex exponential $e^{\frac{in\pi x}{L}}$ and $e^{\frac{-im\pi x}{L}}$, and we arrive at the expression for our coefficients as

$$C_n = \int_{-L}^{L} f(x) e^{\frac{-in\pi x}{L}} dx.$$

Our discussion turned to looking at this problem in terms of an infinite dimensional vector space. The integrals that we have been taking actually define an inner product on the infinite dimensional vector space of integrable functions of one variable. We can use this fact to develop an even more simple expression for our coefficients. We simply use the notation

$$< f(x), g(x) >= \int_a^b f(x)\overline{g(x)} dx$$

to denote the inner product, where the bar represents the complex conjugate of the second function $g(x)$. Thus for our problem, the expression for the coefficients becomes

$$C_n = \left\langle f(x), e^{\frac{in\pi x}{L}} \right\rangle.$$

At the end of our discussion of Fourier series, we talked about the definition of general and symmetric boundary conditions and their implications. A boundary condition is called symmetric if $(-f'g + fg')|_a^b = 0$ for any two functions that satisfy these boundary conditions. The special fact that symmetric boundary conditions imply is that for any two distinct eigenvalues, their corresponding eigenfunctions are mutually orthogonal. This fact gives us some potentially interesting consequences, and allows us to, once again, re-express our Fourier coefficients in terms of eigenfunctions.

3.5 Example

Here we will tackle Exercise 5.2-4. The first question tells us to show that any odd function $\phi(x)$ has a full Fourier series that consists only of sine terms. Firstly, we must note that an odd function multiplied by an odd function is an even function, an even function multiplied by an even function is odd, and an even times an even is an even function. This proof is simple, and relies only on the definition of even an odd functions. Let $f(x)$ and $g(x)$ be odd functions, then by definition $f(-x) = -f(x)$, so

$$f(-x)g(-x) = -1 * -1g(x)f(x) = g(x)f(x)$$

so their product is even. If $f(x)$ is odd and $g(x)$ is even, then

$$f(-x)g(-x) = -f(x)g(x)$$

so their product is odd. Doing the exact same process with even functions, we will find that the product of two evens is also even. Now, if $\phi(x)$ is an odd function on $(-L, L)$, then the cosine coefficients are given by

$$C_n = \int_{-L}^{L} \phi(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

We can use the fact that the product of an odd function and an even function is odd, and the symmetry argument that integrals of odd functions over symmetric intervals is zero to conclude that the cosine terms are all zero. The sine terms will give us non-zero coefficients because the product of two even functions is even and symmetric integrals of even functions are not necessarily zero.

For the second part, we need to show that for an even function $\phi(x)$ the Full Fourier Series is given only by cosine terms. We simply use the same argument as before in that the product of an even function and an odd function is also odd. Therefore, the sine coefficients given by

$$C_n = \int_{-L}^{L} \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

are all zero since $\phi(x)$ is even and $\sin(x)$ is an odd function. Thus we have a odd function integrated over a symmetric interval, so the integral is always zero. Therefore, all of the sine terms are zero, and we have that the Full Fourier Series is given by only cosine terms.

3.6 Exercise

Our work for Exercise 5.3-2 is shown below.

(a) Show that $f(x) = x$ is orthogonal to any constant function $g(x) = c$ on the interval (-1,1). We simply need to show that

$$\int_{-1}^{1} cf(x)dx = \int_{-1}^{1} cxdx = 0.$$

Evaluating this integral we have that

$$\int_{-1}^{1} cxdx = \frac{c}{2}(x^2|_{-1}^{1}) = \frac{c}{2}(1^2 - (-1)^2) = 0.$$

Therefore $f(x) = x$ is orthogonal to every constant function.

(b) Here we are trying to find a quadratic polynomial $p(x) = ax^2 + bx + c$ that is orthogonal to both $f(x) = x$ and $g(x) = 1$. Now, in order to simplify our problem, we will use the symmetry argument that symmetric integrals of odd functions are zero. For the first function, we have

$$\int_{-1}^{1} x(ax^2 + bx + c)dx = \int_{-1}^{1} (ax^3 + bx^2 + cx)dx.$$

The integrals of the cubic function and the linear function will go to zero, and all that remains is

$$\int_{-1}^{1} bx^2dx = \frac{b}{3}(1^3 - (-1)^3) = \frac{2b}{3} = 0 \Rightarrow b = 0.$$

Next, we look at the integral of our quadratic and the constant function 1. Using our symmetry argument again, we have

$$\int_{-1}^{1} (ax^2 + c)dx = \frac{2a}{3} + 2c = 0 \Rightarrow a = -3c.$$

So all of the polynomials that are orthogonal to both x and 1 are given by $p(x) = -3cx^2 + c$.

(c) For this question we are trying to find a cubic polynomial that is orthogonal to every quadratic polynomial. Therefore we are trying to find $p(x) = ax^3 + bx^2 + cx + d$ that is orthogonal to $q(x) = ex^2 + fx + g$. We go by the same process as above and we have that

$$\int_{-1}^{1} (ax^3 + bx^2 + cx + d)(ex^2 + fx + g)dx = \int_{-1}^{1} \Big[aex^5 + (be + af)x^4 + (ce + bf + ag)x^3 \\ + (de + cf + bg)x^2 + (df + cg)x + dg\Big]dx.$$

We can simplify this using symmetry, and we arrive at

$$\int_{-1}^{1} \Big[(be + af)x^4 + (de + cf + bg)x^2 + dg\Big]dx = \frac{2}{5}(be + af) + \frac{2}{3}(de + cf + bg) + 2dg = 0.$$

In order to solve for the coefficients we need in $p(x)$ we need to rearrange our equation above by grouping together the terms that contain the coefficients from the arbitrary quadratic. Doing so we end up with

$$\left(\frac{2}{5}b + \frac{2}{3}d\right)e + \left(\frac{2}{5}a + \frac{2}{3}c\right)f + \left(\frac{2}{3}b + 2d\right)g = 0.$$

Since the terms e, f, g are arbitrary, their coefficients must be zero to make this expression true. The first one yields

$$\frac{2}{5}b + \frac{2}{3}d = 0 \Rightarrow b = \frac{-5}{3}d$$

and the second one yields

$$\frac{2}{5}a + \frac{2}{3}c = 0 \Rightarrow c = \frac{-3}{5}a$$

and the third one yields

$$\frac{2}{3}b + 2d = 0 \Rightarrow b = -3d.$$

The first and the third equations imply that

$$\frac{-5}{3}d = -3d.$$

Therefore $b = d = 0$. Thus all of the polynomials that are orthogonal to every quadratic polynomial are given by

$$p(x) = ax^3 - \frac{3}{5}ax.$$

4 Convergence

4.1 Summary of Lecture 8

Today, we continued our discussion on symmetric boundary conditions, considered the implications in the complex realm, acknowledged two important theorems, and ended with an introduction to three different types of convergence. Consider the eigenvalue problem:

$$\begin{cases} -X'' = \lambda X, \\ +\text{symmetric BCs.} \end{cases}$$

For boundary conditions to be symmetric, they must have the general form

$$\begin{cases} \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0, \\ \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0, \end{cases}$$

and $-f'g + fg'\big|_a^b$ must be equal to 0 for any f, g satisfying the BCs.

Last time, the lecture ended with the declaration that symmetric BCs imply that the eigenfunctions that form an orthogonal basis for the infinite dimensional Hilbert space. So $\lambda_1 \neq \lambda_2 \implies X_1 \perp X_2 \implies < X_1, X_2 >= 0$. The vector space is endowed with the inner product:

$$< f, g >\equiv \int_a^b f(x)\overline{g(x)}dx.$$

The first theorem from today's lecture was somewhat surprising: for symmetric BCs, all eigenvalues are real and the eigenfunctions can be selected to be real as well. We promptly completed a proof for the theorem. Overview:

$$-X'' = \lambda X \iff -\overline{X}'' = \overline{\lambda}\,\overline{X} \qquad \text{(both with symmetric BCs).}$$

The two equations are manipulated algebraically. After subtracting the modified equation on the right from the modified equation on the left, we obtain

$$-X''\overline{X} + \overline{X}''X = \left(\lambda - \overline{\lambda}\right)|X|^2,$$

$$\int_a^b \left(-X''\overline{X} + \overline{X}''X\right)dx = \int_a^b \left(\lambda - \overline{\lambda}\right)|X|^2 dx.$$

We can use Green's second identity on the LHS:

$$-X'\overline{X} + \overline{X}'X\Big|_a^b = \left(\lambda - \overline{\lambda}\right)\int_a^b |X|^2 dx.$$

Focusing on the LHS again, we acknowledge that the expression there is equal to 0 (as a requirement for symmetric BCs). On the RHS, we acknowledge that the integral is nonzero (assume $X \neq 0$). So we have $0 = \lambda - \overline{\lambda} \implies \overline{\lambda} = \lambda$. The only way that a pair of conjugates can be equal to each other is if the imaginary parts are both 0 ($a + bi = a - bi \implies a + 0i = a - 0i \implies a = a$). So the eigenvalues are real.

If we assume that X is complex in the eigenvalue problem $(-X'' = \lambda X)$, we have $X = U + iV$ where U and V are both real:

$$-\left(U'' + iV''\right) = \lambda\left(U + iV\right),$$

$$\begin{cases} -U'' = \lambda U, \\ -iV'' = \lambda iV \implies -V'' = \lambda V. \end{cases}$$

So the complex-valued eigenfunction (X) can be replaced with two real-valued eigenfunctions (U, V). The first theorem has been proven.

Given the same eigenvalue problem with symmetric BCs, the second theorem says that there exists an infinite sequence of positive eigenvalues. We did not prove it in class, but the theorem makes sense since we have already seen this happen with Robin BCs. Of course, an infinite sequence of mutually orthogonal eigenfunctions (corresponding to the infinite sequence of positive eigenvalues) appear in the Fourier series approximation for a target function $f(x)$:

$$\sum_{n=1}^{\infty} A_n X_n(x) \to f(x).$$

The concept of convergence was in need of acknowledgement at this point. For the conclusion of the lecture, we were introduced to three different types.

- $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f(x)$when, $\forall x \in (a, b)$, $\left|\sum_{n=1}^{N} f_n(x) - f(x)\right| \to 0$ as $N \to \infty$.
- $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to $f(x)$when, $\forall x \in [a, b]$, $\max_{x\in[a,b]} \left|\sum_{n=1}^{N} f_n(x) - f(x)\right| \to 0$ as $N \to \infty$.
- $\sum_{n=1}^{\infty} f_n(x) \to f(x)$ in the L^2 sense when, $\forall x \in (a, b)$, $\int_a^b \left|\sum_{n=1}^{N} f_n(x) - f(x)\right|^2 dx \to 0$ as $N \to \infty$.

4.2 Example

Determine which of the three types of convergence occur for the series

$$\sum_{n=1}^{\infty} (1-x)x^{n-1}$$

where the target function is $f(x) = 1$. To test for pointwise convergence, consider the series on the interval $x \in (0, 1)$:

$$\sum_{n=1}^{N} (1-x)x^{n-1} = \sum_{n=1}^{N} \left(x^{n-1} - x^n\right) \qquad \text{(telescoping series)}$$

$$= (1 \cancel{- x}) + (\cancel{x - x^2}) + (\cancel{x^2 - x^3}) + \ldots + (\cancel{x^{N-1}} + x^N) = 1 + x^N,$$

$$\left| \sum_{n=1}^{N} f_n(x) - f(x) \right| = \left| \left(1 + x^N\right) - 1 \right| = x^N \to 0 \text{ as } N \to \infty \text{ since } x \in (0,1).$$

Therefore, the series converges pointwise to the target function. To test for uniform convergence, consider the same series. But this time, the interval is $x \in [0,1]$:

$$\max_{x\in[0,1]} \left| \sum_{n=1}^{N} f_n(x) - f(x) \right| = \max_{x\in[0,1]} \left| \left(1 + x^N\right) - 1 \right|$$
$$= \max_{x\in[0,1]} \left| x^N \right| = 1 \not\to 0 \text{ as } N \to \infty$$

Therefore, the series does not converge uniformly to the target function. To test for L^2 convergence, consider the same series on the interval $x \in (0,1)$, then take the integral:

$$\int_0^1 \left| \sum_{n=1}^{N} f_n(x) - f(x) \right|^2 dx = \int_0^1 \left| \left(1 + x^N\right) - 1 \right|^2 dx = \int_0^1 \left| x^N \right|^2 dx$$
$$= \int_0^1 x^{2N} dx = \frac{1}{2N+1} x^{2N+1} \Big|_0^1 = \frac{1}{2N+1} \left[(1)^{2N+1} - (0)^{2N+1} \right]$$
$$= \frac{1}{2N+1} \to 0 \text{ as } N \to \infty.$$

Therefore, the series converges to the target function in the L^2 sense.

4.3 Exercise

Exercise 5.4-5: consider the piecewise function

$$\begin{cases} \phi(x) = 0 & x \in (0,1), \\ \phi(x) = 1 & x \in (1,3). \end{cases}$$

(a) We are asked to find the first four nonzero terms of the function's Fourier cosine series (FCS) explicitly. In general, the FCS for a given function is

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) \qquad \text{(note: the length of our interval is 3, so in this case, } L = 3\text{).}$$

Find the Fourier coefficients:

$$A_n = \frac{2}{L} \int_0^L \phi(x) \cos\left(\frac{n\pi}{L} x\right) dx = \frac{2}{3} \int_0^3 \phi(x) \cos\left(\frac{n\pi}{3} x\right) dx$$
$$= \frac{2}{3} \left[\int_0^1 (0) \cos\left(\frac{n\pi}{3} x\right) dx + \int_1^3 (1) \cos\left(\frac{n\pi}{3} x\right) dx \right]$$

$$= \frac{2}{3}\int_1^3 \cos\left(\frac{n\pi}{3}x\right)dx = \frac{2}{3}\left(\frac{3}{n\pi}\right)\sin\left(\frac{n\pi}{3}x\right)\Big|_1^3$$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{3}\cdot 3\right) - \sin\left(\frac{n\pi}{3}\cdot 1\right)\right] = \frac{2}{n\pi}\left[\sin(n\pi) - \sin\left(\frac{n\pi}{3}\right)\right] = \frac{-2}{n\pi}\sin\left(\frac{n\pi}{3}\right),$$

$$A_0 = \frac{2}{3}\int_1^3 \cos\left(\frac{0\cdot\pi}{3}x\right)dx = \frac{2}{3}\int_1^3 dx = \frac{2}{3}x\Big|_1^3 = \frac{2}{3}(3-1) = \frac{4}{3}.$$

So the FCS is

$$\begin{aligned}\phi(x) &= \frac{[4/3]}{2} + \sum_{n=1}^{\infty}\left[\frac{-2}{n\pi}\sin\left(\frac{n\pi}{3}\right)\right]\cos\left(\frac{n\pi}{3}x\right)\\ &= \frac{2}{3} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi}{3}x\right).\end{aligned}$$

For the first four nonzero terms, we have

$$\begin{aligned}\phi(x) = \frac{2}{3} - \frac{2}{\pi}\Bigg[&\frac{1}{1}\sin\left(\frac{1\pi}{3}\right)\cos\left(\frac{1\pi}{3}x\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}x\right)\\ &+ \frac{1}{3}\sin\left(\frac{3\pi}{3}\right)\cos\left(\frac{3\pi}{3}x\right) + \frac{1}{4}\sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{4\pi}{3}x\right)\Bigg]\\ = \frac{2}{3} - \frac{2}{\pi}&\left[\frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{3}x\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\cos\left(\frac{2\pi}{3}x\right) - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\cos\left(\frac{4\pi}{3}x\right)\right]\\ = \frac{2}{3} - &\frac{\sqrt{3}}{\pi}\cos\left(\frac{\pi}{3}x\right) - \frac{\sqrt{3}}{2\pi}\cos\left(\frac{2\pi}{3}x\right) + \frac{\sqrt{3}}{4\pi}\cos\left(\frac{4\pi}{3}x\right).\end{aligned}$$

(b) We are asked to find the sum of the series for each $x \in [0,3]$. Acknowledge Theorem 4. Since the target function, $\phi(x)$, is piecewise continuous and its derivative, $\phi'(x)$, is also piecewise continuous, the FCS converges pointwise $\forall x \in (-\infty,\infty)$. The value for the sum is given as

$$\sum_n A_n X_n(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right] \qquad \forall x \in (a,b)$$

where $f(x^+)$ is defined as $\lim_{a\to x^+} f(a)$ and $f(x^-)$ is defined as $\lim_{b\to x^-} f(b)$.

For every x-value in the intervals $(0,1)$ and $(1,3)$, the one-sided limits are the same. So for $x \in (0,1)$, we have $\phi(x^+) = \phi(x^-) = 0$. For $x \in (1,3)$, we have $\phi(x^+) = \phi(x^-) = 1$. However, at $x = 1$ (at the jump discontinuity), the one-sided limits are obviously different. There, we have $\phi(x^+) = 1$ and $\phi(x^-) = 0$. So

$$\sum_n A_n X_n(x) = \begin{cases}\frac{1}{2}[0+0]] & x \in (0,1),\\ \frac{1}{2}[1+1]] & x \in (1,3),\\ \frac{1}{2}[1+0] & x = 1,\end{cases} = \begin{cases}0 & x \in (0,1),\\ 1 & x \in (1,3),\\ \frac{1}{2} & x = 1.\end{cases}$$

(c) Theorem 3: the FCS converges to $\phi(x)$ in the L^2 sense if

$$\int_a^b |\phi(x)|^2 dx$$

is finite. In this case, we have

$$\int_0^3 |\phi(x)|^2 dx = \int_0^1 |0|^2 dx + \int_1^3 |1|^2 dx = \int_1^3 dx = x\Big|_1^3 = 3 - 1 = 2.$$

Therefore, the FCS does converge to $\phi(x)$ in the L^2 sense.

(d) We are asked to find the sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \dots$$

and we can use the FCS to accomplish this:

$$\phi(0) = \frac{2}{3} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi}{3}\cdot 0\right) = \frac{2}{3} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{3}\right)$$

$$= \frac{2}{3} - \frac{2}{\pi}\left[\sin\left(\frac{\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{3}\right) + \frac{1}{4}\sin\left(\frac{4\pi}{3}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{3}\right) + \dots\right]$$

$$= \frac{2}{3} - \frac{2}{\pi}\left[\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{3}(0) + \frac{1}{4}\left(\frac{-\sqrt{3}}{2}\right) + \frac{1}{5}\left(\frac{-\sqrt{3}}{2}\right) + \dots\right]$$

$$= \frac{2}{3} - \frac{2}{\pi}\left(\frac{\sqrt{3}}{2}\right)\left[1 + \frac{1}{2} + \frac{1}{4}(-1) + \frac{1}{5}(-1) + \dots\right]$$

$$= \frac{2}{3} - \frac{\sqrt{3}}{\pi}\left[1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \dots\right].$$

According to the $\phi(x)$ piecewise description, $\phi(0) = 0$. So

$$\phi(0) = \frac{2}{3} - \frac{\sqrt{3}}{\pi}\left[1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \dots\right] = 0$$

$$-\frac{\sqrt{3}}{\pi}\left[1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \dots\right] = -\frac{2}{3}$$

$$\left[1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \dots\right] = -\frac{2}{3} \cdot -\frac{\pi}{\sqrt{3}}.$$

Therefore, the sum is equal to

$$\frac{2\pi}{3\sqrt{3}}.$$

4.4 Summary of Lecture 9

Here we continued our discussion of the different types of convergence for series of functions. We acknowledged important theorems concerning uniform, pointwise and L^2 convergence. The first theorem is known as Bessel's Inequality. It is important not only because of the fact that it tells us about the nature of the sum of coefficients, but also because we use it in the proof of the theorem on uniform convergence.

Theorem 4.1. *(Bessel's Inequality) For any $g \in L^2$,*

$$\sum_{n=1}^{\infty} \frac{|<g, X_n>|^2}{||X_n||^2} \leq ||g||^2.$$

The proof of this theorem is relatively simple, but elegantly uses a specific choice of parameters. By expanding out

$$\left|\left|g - \sum_{n=1}^{\infty} C_n X_n\right|\right|^2$$

we can cleverly choose

$$C_n = \frac{|<g, X_n>|^2}{||X_n||^2}$$

to minimize this difference and ultimately this proves our claim after we pass the limit as N approaches infinity into our inequality.

We then discussed another theorem that tells us about the pointwise convergence of Fourier Series.

Theorem 4.2. *(Pointwise Convergence) All Full, Sine, and Cosine Fourier series converge pointwise to the function $f(x)$ on (a, b) provided that f is continuous on [a,b] and $f'(x)$ is continuous piecewise continuous on $[a, b]$. If f and f' are both piecewise continuous, then the series converges to the average of the left and right limits of f at x for all values of $x \in (a.b)$.*

For explicit series, this theorem gives us an easy way to prove pointwise convergence of the Fourier series.

The last theorem we introduced is similar to the previous one, but it extends beyond a finite interval.

Theorem 4.3. *If f(x) is a function on (a, b) and we extend it with period $2L$ to the entire real line. Then if both $f_{ext}(x)$ and $f'_{ext}(x)$ are both piecewise continuous on $\mathbb{R}$ then the full Fourier Series converges to*

$$\frac{1}{2}\left[f_{ext}(x^+) + f_{ext}(x^-)\right] \quad \textit{where} \quad f_{ext}(x^\pm) = \lim_{y \longrightarrow x^\pm} f_{ext}(y).$$

The proof of this theorem basically relies on the choice of good notation. We introduce a term known as the Dirichlet Kernel given by

$$K_n(\theta) = 1 + 2\sum_{n} = 1^N \cos(n\theta) = \frac{\sin\left(N + \frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}.$$

We begin by expressing the partial sums of the Fourier series, and then use the definition of the coefficients to pull out an integral. We introduce the Dirichlet Kernel by rearranging the terms in the expansion by using the angle addition formulas. Since we pull the integral out front, we can express the partial sum as a sum of inner products, and then use Bessel's Inequality to show that the sum is finite. Then we pass the limit as N goes to infinity to our partial sum to conclude that the partial sum is also finite and that the series must converge.

All of these theorems are incredibly useful for determining the properties of Fourier series. There is an additional theorem on uniform convergence as well that tells us more conditions for uniform convergence. Uniform convergence is the strongest form; as we will see in the next example, it also implies both pointwise and L^2 convergence.

4.5 Example

Let $f_n(x)$ be a sequence of functions whose sum converges uniformly to some function $f(x)$ on (a, b). By definition we must have that

$$\lim_{N\to\infty} \max_{a\leq x\leq b} |S_N(x) - f(x)| = 0$$

where

$$S_N(x) = \sum_{n=1}^{N} f_n(x).$$

In order to show that this series converges pointwise, we must show that

$$\lim_{N\to\infty} |S_N(x) - f(x)| = 0.$$

However, we know that $\forall x \in (a, b)$ we have that

$$|S_N(x) - f(x)| \leq \max_{a\leq x\leq b} |S_N(x) - f(x)|.$$

Therefore,

$$\lim_{N\to\infty} |S_N(x) - f(x)| \leq \lim_{N\to\infty} \max_{a\leq x\leq b} |S_N(x) - f(x)| = 0 \Rightarrow \lim_{N\to\infty} |S_N(x) - f(x)| = 0.$$

Hence we must have pointwise convergence! Next we deal with the case of L^2 convergence. From the definition, we need to show that

$$\lim_{N\to\infty} \int_a^b |f(x) - S_N(x)|^2 dx = 0.$$

As above, we note that

$$\int_a^b |f(x) - S_N(x)|^2 dx \leq \int_a^b \max_{a\leq x\leq b} |S_N(x) - f(x)|.$$

We know that the integral is always less than or equal to the maximum value of the integrand times the length of the integral. So

$$\int_a^b \max_{a\leq x\leq b} |S_N(x) - f(x)| \leq (b-a) \max_{a\leq x\leq b} |S_N(x) - f(x)|.$$

Therefore,

$$\lim_{N\to\infty} \int_a^b |f(x) - S_N(x)|^2 \leq \lim_{N\to\infty} (b-a) \max_{a\leq xb} |S_N(x) - f(x)|^2.$$

Since the series converges uniformly, we know that if the max tends to zero as N goes to infinity, then so does the square of the max. Consequently,

$$\lim_{N\to\infty}\int_a^b |f(x)-S_N(x)|^2 \le \lim_{N\to\infty}(b-a)\max_{a\le xb}|S_N(x)-f(x)|^2 = 0 \Rightarrow \lim_{N\to\infty}\int_a^b |f(x)-S_N(x)|^2 = 0$$

and we have L^2 convergence by definition. We remark that uniform convergence is a strong condition while pointwise and L^2 convergence are not. In fact, the latter two are on an equal level of strength since series of functions can converge pointwise but not in L^2 (and vice versa).

4.6 Exercise

For Exercise 5.5-11, our goal is to show that if $f(x)$ is a continuous function with period 2π and $f'(x)$ is piecewise continuous, then the Fourier series converges uniformly to $f(x)$. To begin, we work on the interval $[-\pi,\pi]$ without loss of generality. Next, we note that

$$f_n(x) = A_n\cos(nx) + B_n\sin(nx)$$

where the coefficients are given by

$$A_n = \int_\pi^\pi f(x)\cos(nx)dx \quad B_n = \int_{-\pi}^\pi f(x)\sin(nx)dx.$$

Since $f'(x)$ is piecewise continuous, it admits a Fourier series whose coefficients are

$$A_n' = \int_\pi^\pi f'(x)\cos(nx)dx \quad B_n' = \int_{-\pi}^\pi f'(x)\sin(nx)dx$$

We do have a relationship between these two coefficients. By integrating the coefficients A_n by parts, taking $u = f(x)$, $du = f'(x)dx$, $dv = \cos(nx)$, and $v = \frac{1}{n}\sin(nx)$, we find that

$$A_n = \int_\pi^\pi f(x)\cos(nx)dx = \frac{1}{n}f(x)\sin(nx)\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_\pi^\pi f'(x)\sin(nx)dx.$$

We know that $\sin(n\pi) = 0$ for all n. Thus,

$$A_n = -\frac{1}{n}\int_\pi^\pi f'(x)\sin(nx)dx = \frac{-1}{n}B_n'.$$

Similarly, we can devise a relationship for the B_n terms:

$$B_n = \int_{-\pi}^\pi f(x)\sin(nx)dx = \frac{1}{n}f(x)\cos(nx)\Big|_{\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^\pi f'(x)\cos(nx)dx.$$

Since $f(x)$ is periodic of period 2π, we must have that $f(-\pi) = f(-\pi+2\pi) = f(\pi)$. Using the fact that $\cos(x) = \cos(-x)$, the first term must be

$$\frac{1}{n}f(x)\cos(nx)\Big|_{\pi}^{\pi} = \frac{1}{n}\left[f(\pi)\cos(n\pi) - f(-\pi)\cos(-n\pi)\right] = \frac{1}{n}\left[f(\pi)\cos(n\pi) - f(\pi)\cos(n\pi)\right] = 0.$$

All we are left with is

$$B_n = \int_{-\pi}^\pi f(x)\sin(nx)dx = \frac{1}{n}\int_{-\pi}^\pi f'(x)\cos(nx)dx = \frac{1}{n}A_n'.$$

Since $f'(x)$ is piecewise continuous on a bounded interval, we know that $f'(x)$ must be bounded by some value $M \in \mathbb{R}$. The functions

$$|f_n(x)| = |A_n \cos(nx) + B_n \sin(nx)|.$$

By the triangle inequality, we must have that

$$|f_n(x)| \leq |A_n \cos(nx)| + |B_n \sin(nx)| = \left|\frac{1}{n} B'_n\right| + \left|\frac{-1}{n} A'_n\right| = \frac{1}{n}\left(|B'_n| + |A'_n|\right).$$

Use the fact that $|\sin(nx)|, |\cos(nx)| \leq 1$. Also use the boundedness of the derivative:

$$|B'_n| = \left|\int_{-\pi}^{\pi} f'(x)\sin(nx)dx\right| \leq \int_{-\pi}^{\pi} \left|f'(x)\sin(nx)\right| dx \leq \int_{-\pi}^{\pi} |f'(x)|dx \leq \int_{-\pi}^{\pi} |M|dx = |M|2\pi.$$

Repeating the same argument, we have that $|A'_n| \leq |M|2\pi$. Without loss of generality, we can assume that $M > 0$ to drop the $||$. Next we examine the inequality

$$\sum_{n=1}^{\infty} |A_n \cos(nx)| + |B_n \sin(nx)| \leq \sum_{n=1}^{\infty} |A_n| + |B_n| = \sum_{n=1}^{\infty} \frac{1}{n}|A'_n| + |B'_n| \leq \sum_{n=1}^{\infty} \frac{4\pi M}{n}.$$

By the Schwartz Inequality, we must have that

$$\sum_{n=1}^{\infty} \frac{1}{n}|A'_n| + |B'_n| \leq \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} 2\left(|A'_n|^2 + B'_n|^2\right)^{1/2}$$

which must be finite as well. Combining everything, we have

$$\max|f(x) - S_N(x)| \leq \max \sum_{n=N+1}^{\infty} |A_n \cos(nx) + B_n \sin(nx)| \leq \sum_{n=N+1}^{\infty} \left(|A'_n| + |B'_n|\right)$$

and we know that the last term is finite. Therefore, we have the tail end of convergent series. Since the series is convergent, we know the terms must go to zero. Therefore, as we take the limit as N goes to infinity, we must have that the sum goes to zero as well. Hence our series of functions converges uniformly by definition.

5 Jump Discontinuities

5.1 Summary of Lecture 10

The topic of pointwise convergence of Fourier series needs more acknowledgment. Today's lecture began with a discussion of the case where f and f' are piecewise continuous (jump discontinuities exist). So we wish to prove Theorem 4.3 (pg 39).

For the proof of Theorem 4.2, we showed that $S_N(x) - f(x) \to 0$ as $N \to \infty$ where $S_N(x)$ is the "truncated" Fourier series. In doing so, we showed that the Fourier series converged pointwise to $f(x)$. We had

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\left(f(x+\theta) - f(x)\right) d\theta$$

$$\text{where } K_N(\theta) = \frac{\sin\left((N+\frac{1}{2})\theta\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

We modified this expression for our proof of Theorem 4.3: show that $S_N(x) - \frac{1}{2}[f(x^+) + f(x^-)] \to 0$ as $N \to \infty$ where

$$S_N(x) - \frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2\pi}\int_0^{\pi} K_N(\theta)\left(f(x+\theta) - f(x^+)\right) d\theta + \frac{1}{2\pi}\int_{-\pi}^{0} K_N(\theta)\left(f(x+\theta) - f(x^-)\right) d\theta.$$

Of course, we can acknowledge the integrals as inner products. In Bessel's Inequality, we can replace g with

$$g_{\pm}(\theta) = \frac{f(x+\theta) - f(x^{\pm})}{\sin\left(\frac{\theta}{2}\right)}$$

and we can aim for the same goal as last time (show that $\|g_{\pm}\|$ is finite). By doing this, the proof is completed because we are showing that

$$\lim_{N\to\infty} < g_{\pm}, X_n >= 0.$$

During our analysis of $\|g_{\pm}\|$, we took the limit

$$\lim_{\theta\to 0^+} \frac{f(x+\theta) - f(x^{\pm})}{\sin\left(\frac{\theta}{2}\right)} = \lim_{\theta\to 0^+} \frac{f(x+\theta) - f(x^{\pm})}{\theta} \cdot \frac{\theta}{\sin\left(\frac{\theta}{2}\right)}.$$

The fraction on the left resembles the difference quotient. Also, we can use L'Hospital's Rule for the fraction on the right. Thus, we have $f'(x^{\pm}) \cdot 2$. If $f'(x^{\pm})$ exists, then $\|g_{\pm}\|$ is finite and the proof is complete. If $f'(x^{\pm})$ does not exist, then it matters not since the derivative is bounded above, so we reach the same conclusion. Therefore, Theorem 4.3 is proven.

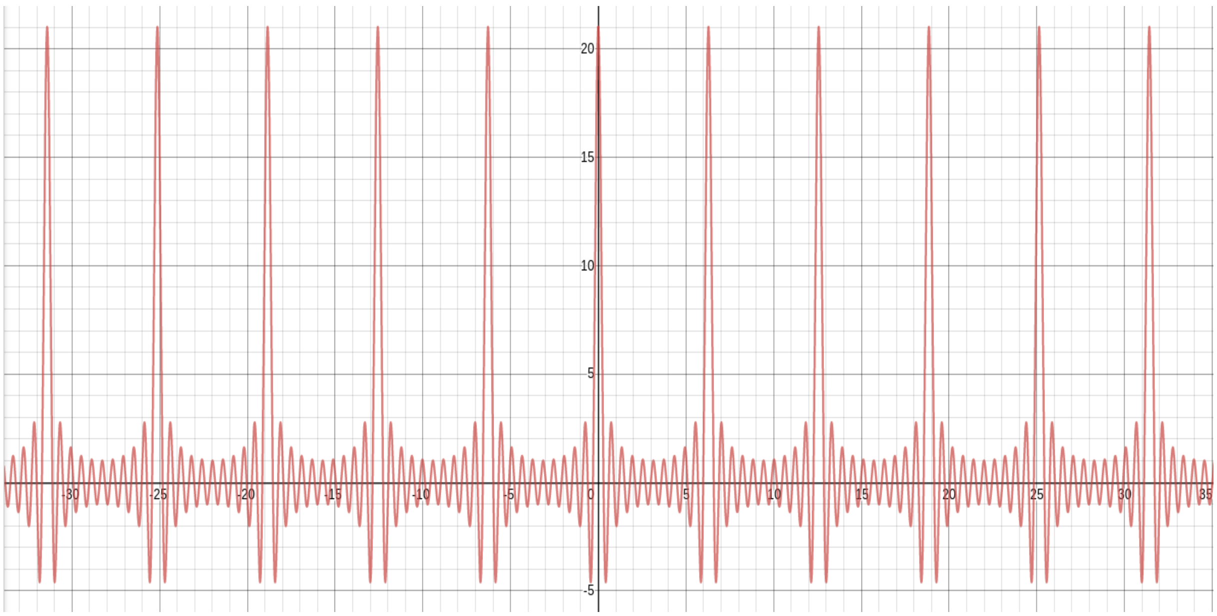
Today's lecture ended with a brief discussion of the Gibbs Phenomenon. We examined a function with jump discontinuities (a square wave) and we made an observation concerning the Fourier series approximation. No matter how high the N-value, the function was always "overshot" near the jump discontinuities. We were able to quantify this (we calculated a 9% overshoot).

5.2 Example

Dirichlet Kernel:

$$K_N(\theta) = \frac{\sin\left((N+\frac{1}{2})\theta\right)}{\sin(\frac{1}{2}\theta)}.$$

Sketch the graph for $N = 10$ (Desmos).



5.3 Exercise

Exercise 5.5-13: explain Chernoff's proof for the pointwise convergence of Fourier series.

(a) Let $f(x)$ be a C' function of period 2π. Show that we may as well assume that $f(0) = 0$ and we need only show that the Fourier series converges to 0 at $x = 0$. Acknowledge the difference function $a(x) = f(x) - f(0)$. Let A_n denote the complex Fourier coefficients for the Fourier series approximation for $a(x)$. Then the Fourier series approximation for $a(x)$ is

$$a(x) = \sum_{n=-\infty}^{\infty} A_n \exp\left(\frac{in\pi}{L}x\right).$$

Since $f(x)$ is periodic of period 2π, $L = \pi$ here:

$$a(x) = \sum_{n=-\infty}^{\infty} A_n \exp(inx).$$

Let C_n denote the complex Fourier coefficients for the Fourier series approximation for $f(x)$:

$$a(x) = \sum_{n=-\infty}^{\infty} C_n \exp(inx) - f(0) \qquad \text{since } a(x) = f(x) - f(0).$$

At $x = 0$, we have $a(0) = f(0) - f(0) = 0$. So

$$a(0) = \sum_{n=-\infty}^{\infty} C_n \exp(in \cdot 0) - f(0) = 0,$$

$$\sum_{n=-\infty}^{\infty} C_n \exp(in \cdot 0) = f(0).$$

The complex Fourier series approximation for $f(x)$ is equal to $f(0)$ at $x = 0$. If we assume that $f(0) = 0$, then we have

$$\sum_{n=-\infty}^{\infty} C_n \exp(in \cdot 0) \to 0.$$

Let $b(x) = f(x - c)$. Then $b(c) = f(c - c) = f(0) = 0$. So we need only show that the Fourier series converges to 0 at $x = 0$ because we can translate our function vertically and horizontally.

(b) Let $g(x) = f(x)/(e^{ix} - 1)$ and show that $g(x)$ is continuous. Set the denominator of $g(x)$ equal to 0 to look for potential problems:

$$\text{Solve } \; e^{ix} - 1 = 0 \implies e^{ix} = 1.$$

$$\text{Euler: } \; e^{i\pi} = -1 \implies (e^{i\pi})^n = (-1)^n \implies e^{in\pi} = (-1)^n.$$

$$\text{Even integers: } \; e^{i(2n)\pi} = (-1)^{2n} = 1.$$

$$\text{Problems: } \; x = 2n\pi.$$

So there is cause for concern at the locations $x = 2n\pi$. But we can analyze one of these locations ($n = 0 \implies x = 0$), determine continuity, and generalize our conclusion for the other locations since $f(x)$ is periodic of period 2π. For $g(x)$ to be continuous at $x = 0$, the left-sided limit must agree with the right-sided limit:

$$\lim_{x \to 0} \frac{f(x)}{e^{ix} - 1} = \lim_{x \to 0} \frac{f'(x)}{ie^{ix}}$$

$$= \frac{f'(0)}{ie^{i \cdot 0}} = \frac{f'(0)}{i}.$$

The limit exists, so the left-sided limit does agree with the right-sided limit. Assuming $f(x)$ is differentiable at 0, $g(x)$ is a continuous function.

(c) Let D_n denote the complex Fourier coefficients for $g(x)$ and show that $D_n \to 0$.

$$\text{Bessel: } \; \sum_{n=1}^{\infty} \frac{| < g, X_n > |^2}{\|X_n\|^2} \leq \|g\|^2,$$

$$\sum_{n=1}^{\infty} \frac{\left[\int_{-\pi}^{\pi} g \cdot \exp(inx)dx\right]^2}{\int_{-\pi}^{\pi} (\exp(inx))^2 dx} \leq \|g\|^2,$$

$$\sum_{n=1}^{\infty} \frac{\left[\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} D_n \exp(inx) \cdot \exp(inx)dx\right]^2}{\int_{-\pi}^{\pi} (\exp(inx))^2 dx} \leq \|g\|^2.$$

During the proof we completed in class for Theorem 4.3 (pg 39), we declared $\|g\|^2$ to be finite. We also acknowledged the implication this makes: the inner product in the numerator (on the LHS) approaches 0. So

$$| < g, X_n > | \to 0 \text{ as } n \to \infty$$

$$\implies D_n \to 0 \text{ as } n \to \infty.$$

(d) Show that $C_n = D_{n-1} - D_n$ so that the series $\sum_{n=-\infty}^{\infty} C_n$ is telescoping:

$$g(x) = \frac{f(x)}{e^{ix} - 1} \implies f(x) = g(x)\left(e^{ix} - 1\right),$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \exp(-inx)dx \implies C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)(\exp(ix) - 1)\exp(-inx)dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \Big[\exp(ix - inx) - \exp(-inx)\Big] dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big[g(x)\exp(ix - inx) - g(x)\exp(-inx)\Big] dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\exp(ix - inx)dx - \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\exp(-inx)dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\exp(ix(1 - n))dx - \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\exp(-inx)dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\exp(-ix(n - 1))dx - \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\exp(-inx)dx = D_{n-1} - D_n.$$

Therefore, $\sum_{n=-\infty}^{\infty} C_n$ is telescoping.

(e) At $x = 0$, we have

$$f(0) = \sum_{n=-\infty}^{\infty} C_n \cancel{\exp(in \cdot 0)} = \sum_{n=-\infty}^{\infty} C_n$$

$$= \sum_{n=-\infty}^{\infty} (D_{n-1} - D_n) = \sum_{n=-\infty}^{\infty} D_{n-1} - \sum_{n=-\infty}^{\infty} D_n$$

$$= 0 - 0 \text{ (since } D_n \to 0 \text{ as } n \to \infty) = 0.$$

5.4 Summary of Lecture 11

We began by discussing the principle of causality in one and two dimensions. For the wave equation in one dimension, if we are looking at a point (x_0, t_0) in the xt-plane, then the characteristic lines passing through it actually define the past and future history. This idea means that all of the points within these triangles are those that have been or will be affected by the original point (x_0, t_0). For higher dimensions, the idea is exactly the same. Instead, however, the characteristics form a cone in two dimensions, and a hypercone in higher dimensions. Everything contained within the cone is in the area of influence for the point $\vec{x}_0, t$.

Along with the principle of causality, we also derived the principle of conservation of energy for the wave equation. The concept of the derivation is simple. We begin by multiplying through by u_t (to express the wave equation as a total derivative) and we have that

$$u_{tt}u_t - c^2\Delta u u_t = 0.$$

We can see that $u_{tt}u_t$ is a perfect derivative so we can re-express this equation as

$$\frac{1}{2}(u_t)_t^2 - c^2\Delta u u_t = 0.$$

The second term is a little trickier, but if we unfold the terms we can get a better picture:

$$\Delta u u_t = u_{xx}u_t + u_{yy}u_t + u_{zz}u_t.$$

Let's just consider the x term since the rest are exactly the same with different variables:

$$u_{xx}u_t = (u_t u_x)_x - u_{tx}u_x = (u_t u_x)_x - \frac{1}{2}(u_x^2)_t.$$

Substituting this expression and the corresponding ones for y and z, we can move the spatial partial derivatives to the right hand side. We end up with

$$\left(\frac{1}{2}u_t^2 + c^2\left(\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2\right)\right)_t = c^2\left[(u_t u_x)_x + (u_t u_y)_y + (u_t u_z)_z\right].$$

If we recall the divergence operation, we can re-express this as

$$\frac{1}{2}(||\nabla u||^2 - c^2u_t^2)_t = \nabla \cdot (u_t \nabla u)c^2.$$

If we integrate over our domain $(\mathbb{R}^3)$, we can get rid of the $\vec{x}$-dependencies and extract the t derivative. So we arrive at

$$\frac{d}{dt}\iiint_{\mathbb{R}^3} \frac{1}{2}(||\nabla u||^2 - c^2u_t^2)d\vec{x} = \iiint_{\mathbb{R}^3} \nabla \cdot (u_t \nabla u)c^2 d\vec{x} = \iiint_{\mathbb{R}^3} (u_t u_{\vec{x}})_{\vec{x}} c^2 d\vec{x} = u_t u_{\vec{x}}.$$

However, if we impose our boundary conditions, which were that the partials of initial conditions go to zero as the length of $\vec{x}$ approaches infinity, then this quantity must also go to zero. In addition, from a physical standpoint, the first term on the left side of the integral can be thought of as the of the potential and kinetic energies. Thus

$$\frac{d}{dt}E(t) = \frac{d}{dt}\iiint_{\mathbb{R}^3} \frac{1}{2}(||\nabla u||^2 - c^2u_t^2)d = 0.$$

By simple integration, we conclude that the energy must be a constant, which is exactly what conservation of energy implies. If the energy is constant, that means that the energy must equal the energy at the initial time, therefore the energy is determined by the initial conditions.

The last concept that we introduced was the spherical mean. If we consider a function $u(\vec{x}, t)$ we can derive what we call the spherical mean, where we integrate over a sphere of radius r. We compute it by taking the average of the function over the sphere S_r:

$$\bar{u}(r,t) = \frac{1}{4\pi r^2} \iint_{S_r} \vec{u}(\vec{x}, t) dS.$$

We can easily simplify this formula by converting to spherical coordinates, and we find that the spherical mean can explicitly be expressed as

$$\bar{u}(r,t) \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \vec{u}(\vec{x}, t) \sin(\theta) d\theta d\phi$$

for $\vec{x}(r, \theta, \phi)$.

5.5 Example

As an example, we will look at Exercise 9.1-7 to show that conservation of energy does not always hold. In this case, we are working with a certain boundary condition, and the derivation of conservation is exactly the same until we take our integral. In this case, because our initial domain in space-time is restricted to a surface S, we have that the derivative of the energy is given by

$$\frac{d}{dt}E(t) = \frac{d}{dt} \iiint_S \frac{1}{2}(||\nabla u||^2 - c^2 u_t^2) d\vec{x} = \iiint_S \nabla \cdot (u_t \nabla u) c^2 d\vec{x}.$$

If we apply the divergence theorem, we can rewrite our second integral as

$$\iiint_S \nabla \cdot (u_t \nabla u) c^2 d\vec{x} = \iint_{\partial S} c^2 u_t \nabla u \cdot \vec{n}\, dA$$

where $\vec{n}$ is the outward pointing unit normal vector. However, $\nabla u \cdot \vec{n}$ is the directional derivative $\frac{\partial u}{\partial \vec{n}}$, so we can use our boundary condition

$$\frac{\partial u}{\partial n} = -b \frac{\partial u}{\partial t}$$

to re-express the integral as

$$\iint_{\partial S} c^2 u_t \nabla u \cdot \vec{n}\, dA = -bc^2 \iint_{\partial S} u_t^2 dA.$$

Thus, we have that

$$\frac{d}{dt}E(t) = -bc^2 \iint_{\partial S} u_t^2 dA.$$

Since $b, c^2, u_t^2 > 0$, the derivative of the energy must be negative, which means that the energy decreases over time. Thus, as we can see, the energy is not always a constant, and conservation of energy does not hold.

5.6 Exercise

Our work for Exercise 9.1-4 is shown below.

(a) Let L, M be Lorentz matrices. So $L^{-1} = \Gamma L^t \Gamma$ and $M^{-1} = \Gamma M^t \Gamma$. Our goal is to show that the product of these two matrices is also Lorentz. Since Γ is diagonal, the product $\Gamma\Gamma$ is simply the product of the diagonal elements. By definition, it is clear that this product is the identity matrix. Moreover, the transpose of gamma is equal to itself. Thus, we have that

$$(LM)^{-1} = M^{-1}L^{-1} = \Gamma M^t \Gamma\Gamma L^t \Gamma = \Gamma M^t L^t \Gamma = \Gamma (LM)^t \Gamma.$$

So $(LM)^{-1}$ is also Lorentz. In addition, we need to show that the inverse of a Lorentz Matrix is also Lorentz. We consider

$$L^{-1} = \Gamma L^t \Gamma \Rightarrow L^t = \Gamma L^{-1} \Gamma \Rightarrow L = (\Gamma L{-}1 L\Gamma)^t.$$

We arrive at

$$L = \left(L^{-1}\right)^{-1} = \Gamma L^{-1} \Gamma$$

so the inverse of a Lorentz matrix is also Lorentz.

(b) For the forward direction, we assume that L is a Lorentz matrix. We need to show that $m(Lv) = m(v)$ where $m(v) = x^2 + y^2 + z^2 - t^2$. Since L is Lorentz, we have that

$$\Gamma = \Gamma I = \Gamma L^{-1} L = \Gamma\Gamma L^T \Gamma L = L^t \Gamma L.$$

We can also see that the column vector $v = (x, y, z, t)$. So we can express the Lorentz metric as

$$m(v) = (\Gamma v) \cdot v = v^t \Gamma v = (v^t L^t \Gamma L) \cdot v = \Gamma (vL)^t Lv.$$

However, this is simply equal to

$$m(v) = \Gamma (Lv) Lv = m(Lv).$$

For the reverse direction, we assume that $m(v) = L(v)$. So we have that

$$\Gamma v \cdot v = v^t \Gamma v = \Gamma (Lv) Lv = v^t L^t \Gamma.$$

We can cancel the factors of v and v^t so we end up with

$$\Gamma = L^t \Gamma L \Rightarrow L^{-1} = \Gamma L^t \Gamma.$$

So L is Lorentz as desired.

(c) Here, we assume that $U(\vec{x}) = u(L\vec{x})$ where L is a Lorentz matrix. Using summation notation, we can re-express the expression for the partial derivatives in terms of a product with the matrix Γ as

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} = \sum_{j=1}^{4} \sum_{i=1}^{4} \Gamma_{ij} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} u(L\vec{x})$$

where Γ_{ij} is the element in the ith row and the jth column and $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = t$. If we apply the chain rule for multivariate functions, we introduce two more summations and we have that

$$\sum_{j=1}^{4} \sum_{i=1}^{4} \Gamma_{ij} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} u(L\vec{x}) = \sum_{j=1}^{4} \sum_{i=1}^{4} \Gamma_{ij} \left(\sum_{k=1}^{4} \sum_{l=1}^{4} \frac{\partial^2 u(L\vec{x})}{\partial x_k \partial x_l} \left(\frac{\partial}{\partial x_i} (L\vec{x})_k \frac{\partial}{\partial x_j} (L\vec{x})_l \right) \right).$$

However, if we consider L_i as the ith row of L, then we can unpack

$$\frac{\partial}{\partial x_j}(L\vec{x})_l = L_1\frac{\partial \vec{x}}{\partial x_j} + L_2\frac{\partial \vec{x}}{\partial x_j} + \dots$$

and we know that $\frac{\partial \vec{x}_i}{\partial x_j} = 0$ for $j \neq i$, so all we are left with is the element of L in the kth row and the jth column. We can use our assumption about U (along with the fact that we can reorder the terms in finite sums) and our expression becomes

$$\sum_{j=1}^{4}\sum_{i=1}^{4}\Gamma_{ij}\left(\sum_{k=1}^{4}\sum_{l=1}^{4}\frac{\partial^2 u(L\vec{x})}{\partial x_k \partial x_l}\left(\frac{\partial}{\partial x_i}(L\vec{x})_k\frac{\partial}{\partial x_j}(L\vec{x})_l\right)\right) =$$
$$\sum_{i=i}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4}\frac{\partial^2 U}{\partial x_k \partial x_l}L_{kj}\Gamma_{ij}L_{lj} = \sum_{i=i}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4}\frac{\partial^2 U}{\partial x_k \partial x_l}L_{kj}\Gamma_{ij}L^t_{jl}.$$

From part (b), we know that

$$\Gamma^t = \Gamma = L^t\Gamma L = (L^t\Gamma L)^t = L\Gamma L^t$$

and since Γ is diagonal, we arrive at

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} = \sum_{k=1}^{4}\sum_{l=1}^{4}\Gamma_{kl}\frac{\partial^2 U}{\partial x_i^2} = U_{xx} + U_{yy} + U_{zz} - U_{tt}$$

as desired.

(d) The Lorentz transformation represents a transformation from one frame in space-time to another. We can think of it in terms of rotations in space time that preserves velocities.

6 Wave Equation in Higher Dimensions

6.1 Summary of Lecture 12

Today, we explored the method of using the spherical mean to solve the three-dimensional wave equation. The spherical mean is given by the surface integral

$$\overline{u}(r,t) = \frac{1}{4\pi r^2} \iint\limits_{S_r} u(\vec{x},t)dS = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi,t)\sin(\theta)d\theta d\phi$$

where $r = \|\vec{x}\|$. If we take the spherical mean throughout the IVP in three dimensions, we can solve an associated IVP in one dimension (eventually), thereby simplifying the computations:

$$\text{3D WE:} \begin{cases} u_{tt} - c^2\Delta u = 0, \\ u(\vec{x},0) = f(\vec{x}), \\ u_t(\vec{x},0) = g(\vec{x}). \end{cases}$$

For the wave equation itself, taking the spherical mean throughout gives us $\overline{u_{tt}} - c^2\overline{\Delta u} = 0 \implies \overline{u}_{tt} - c^2\Delta\overline{u} = 0$. Note: $\overline{u_{tt}} = \overline{u}_{tt}$ is trivial while $\overline{\Delta u} = \Delta\overline{u}$ is not. We completed a proof for the latter equality in PDEs I (Exercise 7.2).

We still want to shrink from three dimensions to one. It is important to note that the Laplacian operator for $\Delta\overline{u}$ has changed from $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ to $\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$ ("spherical Laplacian"). So $\overline{u}_{tt} - c^2\Delta\overline{u} = 0 \implies \overline{u}_{tt} - c^2\left(\overline{u}_{rr} + \frac{2}{r}\overline{u}_r\right) = 0$.

Define $v(r,t) = r\overline{u}(r,t)$. In PDEs I (Exercise 6.9), we showed that $v(r,t)$ solves the one-dimensional wave equation where $v_t = r\overline{u}_t$, $v_{tt} = r\overline{u}_{tt}$; $v_r = (1)(\overline{u}) + (\overline{u}_r)(r) = \overline{u} + \overline{u}_r r$, $v_{rr} = \overline{u}_r + (\overline{u}_{rr})(r) + (1)(\overline{u}_r) = \overline{u}_{rr}r + 2\overline{u}_{rr} = r\left(\overline{u}_{rr} + \frac{2}{r}\overline{u}_r\right)$ by the product rule. Multiply throughout the three-dimensional IVP to obtain

$$r\left[\overline{u}_{tt} - c^2\left(\overline{u}_{rr} + \frac{2}{r}\overline{u}_r\right)\right] = \Big[0\Big]r \implies r\overline{u}_{tt} - c^2 r\left(\overline{u}_{rr} + \frac{2}{r}\overline{u}_r\right) = 0 \implies v_{tt} - c^2 v_{rr} = 0.$$

$$\text{1D WE:} \begin{cases} v_{tt} - c^2 v_{rr} = 0, \\ v(r,0) = r\overline{f}(r), \\ v_t(r,0) = r\overline{g}(r). \end{cases}$$

We know that the solution to the associated one-dimensional IVP is given by the d'Alembert formula as

$$v(r,t) = \frac{1}{2}\Big[(r+ct)\overline{f}(r+ct) + (r-ct)\overline{f}(r-ct)\Big] + \frac{1}{2c}\int_{r-ct}^{r+ct} s\overline{g}(s)ds$$

but an issue arises for the case where $r < ct$. The arguments in the initial conditions $\overline{f}$ and $\overline{g}$ must be positive. To handle this case, use the method of reflections. Consider the odd extensions $-(ct-r)\overline{f}(ct-r)$ and $-(-s)\overline{g}(-s)$. The solution to the one-dimensional IVP becomes

$$v(r,t) = \frac{1}{2}\Big[(r+ct)\overline{f}(r+ct) - (ct-r)\overline{f}(ct-r)\Big] + \frac{1}{2c}\int_{ct-r}^{r+ct} s\overline{g}(s)ds$$

$$= \frac{1}{2c}\frac{\partial}{\partial t}\int_{ct-r}^{r+ct} s\overline{f}(s)ds + \frac{1}{2c}\int_{ct-r}^{r+ct} s\overline{g}(s)ds \qquad \text{(“compact expression” via the Leibniz rule).}$$

Recall: $v(r,t) = r\overline{u}(r,t)$. So $\overline{u}(r,t) = v(r,t)/r$. Since we derived a solution expression for the case $r < ct$, we become interested in small values for the radius. These values exist near the origin, so $\overline{u}(r,t) \to u(\vec{0},t)$ as $r \to 0^+$. Also, it is apparent (see the limits of integration for the "compact expression" above) that $v(r,t) = 0$ for $r = 0$. So

$$u(\vec{0},t) = \lim_{r\to 0^+}\frac{v(r,t)}{r} = \lim_{r\to 0^+}\frac{v(r,t)-0}{r} = \lim_{r\to 0^+}\frac{v(r,t)-v(0,t)}{r} = \frac{\partial v}{\partial r}$$

$$= \frac{1}{2c}\frac{\partial}{\partial t}\left[(r+ct)\overline{f}(r+ct) + (ct-r)\overline{f}(ct-r)\right] + \frac{1}{2c}\left[(r+ct)\overline{g}(r+ct) + (ct-r)\overline{g}(ct-r)\right].$$

For $r = 0$, we have

$$u(\vec{0},t) = \frac{1}{\cancel{2c}}\frac{\partial}{\partial t}\left(\cancel{2c}t\overline{f}(ct)\right) + \frac{1}{\cancel{2c}}\left(\cancel{2c}t\overline{g}(ct)\right) = \frac{\partial}{\partial t}\left(t\overline{f}(ct)\right) + t\overline{g}(ct).$$

Use the first double integral expression given for the spherical mean. Then the value for the solution (sampled at the origin) can be written as

$$u(\vec{0},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint\limits_{\|\vec{x}\|=ct} f(\vec{x})dS\right) + \frac{1}{4\pi c^2 t}\iint\limits_{\|\vec{x}\|=ct} g(\vec{x})dS.$$

We need to generalize. Translation invariance implies that if we find the expression for the solution in one place (in this case, the origin), we have automatically found the expression for the solution everywhere else. We can slightly modify what we have above and report the value for u at any point:

$$u(\vec{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint\limits_{\|\vec{\xi}-\vec{x}\|=ct} f(\vec{\xi})dS\right) + \frac{1}{4\pi c^2 t}\iint\limits_{\|\vec{\xi}-\vec{x}\|=ct} g(\vec{\xi})dS.$$

6.2 Example

In Exercise 9.2-2, we are asked to verify the general solution expression for the case where $u(\vec{x},t) \equiv t$. We can speculate that $u(\vec{x},0) = 0$ and $u_t = 1$ for all time $\implies u_t(\vec{x},0) = 1$. So the initial conditions are simple:

$$f(\vec{\xi}) = 0 \;\text{ and }\; g(\vec{\xi}) = 1$$

$$\implies u(\vec{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint\limits_{\|\vec{\xi}-\vec{x}\|=ct} 0\cdot dS\right) + \frac{1}{4\pi c^2 t}\iint\limits_{\|\vec{\xi}-\vec{x}\|=ct} 1\cdot dS$$

$$= \frac{1}{4\pi c^2 t} \iint\limits_{\|\vec{\xi}-\vec{x}\|=ct} dS = \frac{1}{4\pi c^2 t} \int_0^{2\pi} \int_0^{\pi} r^2 \sin(\theta) d\theta d\phi$$

$$= \frac{r^2}{4\pi c^2 t} \int_0^{2\pi} \int_0^{\pi} \sin(\theta) d\theta d\phi = \frac{r^2}{4\pi c^2 t} \int_0^{2\pi} \left(-\cos(\theta) \Big|_0^{\pi} \right) d\phi$$

$$= \frac{r^2}{4\pi c^2 t} \int_0^{2\pi} \left(-\cos(\pi) - (-\cos(0)\right) d\phi = \frac{r^2}{4\pi c^2 t} \int_0^{2\pi} (2) d\phi = \frac{r^2}{2\pi c^2 t} \int_0^{2\pi} d\phi$$

$$= \frac{r^2}{2\pi c^2 t} \left(\phi \Big|_0^{2\pi} \right) = \frac{r^2}{\cancel{2\pi} c^2 t} \cdot \cancel{2\pi} = \frac{r^2}{c^2 t}.$$

The surface integral was computed on a sphere with radius ct, so

$$u(\vec{x}, t) = \frac{r^2}{c^2 t} = \frac{(ct)^2}{c^2 t} = \frac{\cancel{c^2} t^2}{\cancel{c^2} t} = t.$$

Therefore, the general solution expression is verified for the case where $u(\vec{x}, t) \equiv t$.

6.3 Exercise

Exercise 9.2-4: solve the 3D wave equation with the given initial data

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, \\ u(\vec{x}, 0) = \phi(\vec{x}) = 0, \\ u_t(\vec{x}, 0) = \psi(\vec{x}) = x^2 + y^2 + z^2. \end{cases}$$

We derived the expression for the solution in class:

$$u(\vec{x}, t) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \iint\limits_{\|\vec{\xi}-\vec{x}\|=ct} \phi(\vec{\xi}) dS \right) + \frac{1}{4\pi c^2 t} \iint\limits_{\|\vec{\xi}-\vec{x}\|=ct} \psi(\vec{\xi}) dS.$$

Since the initial condition ϕ is identically zero, the first double integral in the solution expression vanishes. So we have

$$u(\vec{x}, t) = \frac{1}{4\pi c^2 t} \iint\limits_{\|\vec{\xi}-\vec{x}\|=ct} \psi(\vec{\xi}) dS.$$

Strauss gives us a hint: use

$$(\overline{u})_{tt} = c^2 (\overline{u})_{rr} + \frac{2c^2}{r} (\overline{u})_r .$$

Take the spherical mean throughout the wave equation to obtain

$$\begin{cases} \overline{u_{tt}} - c^2\overline{\Delta u} = 0 \implies (\overline{u})_{tt} - c^2\Delta\overline{u} = 0 \implies (\overline{u})_{tt} - c^2\left[(\overline{u})_{rr} + \frac{2}{r}(\overline{u})_r\right] = 0, \\ \overline{u}(\vec{x},0) = \overline{\phi}(\vec{x}) = 0, \\ (\overline{u})_t(\vec{x},0) = \overline{\psi}(\vec{x}) = \overline{x^2+y^2+z^2}. \end{cases}$$

In general, for some function w, the computation for its spherical mean is given as

$$\overline{w}(r,t) = \frac{1}{4\pi r^2}\iint\limits_{S_r} w(\vec{x},t)dS = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} w(r,\theta,\phi,t)\sin(\theta)d\theta d\phi \qquad (\text{where } r = \|\vec{x}\|)$$

$$\implies \overline{\psi}(\vec{x}) = \frac{1}{4\pi r^2}\iint\limits_{S_r}\left(x^2+y^2+z^2\right)dS = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(r^2\right)\sin(\theta)d\theta d\phi$$

$$= \frac{r^2}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin(\theta)d\theta d\phi = \frac{r^2}{4\pi}\int_0^{2\pi}\left(-\cos(\theta)\Big|_0^{\pi}\right)d\phi$$

$$= \frac{r^2}{4\pi}\int_0^{2\pi}\left(-\cos(\pi) - (-\cos(0)\right)d\phi = \frac{r^2}{4\pi}\int_0^{2\pi}(2)d\phi = \frac{r^2}{2\pi}\int_0^{2\pi}d\phi$$

$$= \frac{r^2}{2\pi}\left(\phi\Big|_0^{2\pi}\right) = \frac{r^2}{2\pi}(2\pi - 0) = r^2.$$

The spherical mean of the initial condition is r^2 (so $\overline{\psi} = \psi$). Acknowledge the IVP we set up by taking the spherical mean throughout the wave equation. The problem has been reduced from 3D to 1D since the solution has the form $\overline{u}(r,t)$. For that solution expression, acknowledge the d'Alembert solution:

$$v(r,t) = \frac{1}{2}\left[(r+ct)\overline{\phi}(r+ct) - (ct-r)\overline{\phi}(ct-r)\right] + \frac{1}{2c}\int_{ct-r}^{r+ct} s\overline{\psi}(s)ds$$

$$= \frac{1}{2c}\int_{ct-r}^{r+ct} s\overline{\psi}(s)ds \qquad (\text{since } \overline{\phi}(r) = 0)$$

where $v(r,t) = r\overline{u}(r,t)$. Since $\overline{\psi}(r) = r^2$, we have

$$v(r,t) = \frac{1}{2c}\int_{ct-r}^{r+ct} s(s^2)ds = \frac{1}{2c}\int_{ct-r}^{r+ct} s^3 ds$$

$$= \frac{1}{2c}\left(\frac{s^4}{4}\Big|_{ct-r}^{r+ct}\right) = \frac{1}{8c}\left[(r+ct)^4 - (ct-r)^4\right]$$

$$= \frac{1}{8c}\left[(r^2+2rct+c^2t^2)(r^2+2rct+c^2t^2) - (c^2t^2-2rct+r^2)(c^2t^2-2rct+r^2)\right]$$

$$= \frac{1}{8c}\Big[\cancel{r^4} + 2r^3ct + \cancel{r^2c^2t^2} + 2r^3ct + \cancel{4r^2c^2t^2} + 2rc^3t^3 + \cancel{r^2c^2t^2} + 2rc^3t^3 + \cancel{c^4t^4}$$

$$- \left(\cancel{c^4t^4} - 2rc^3t^3 + \cancel{r^2c^2t^2} - 2rc^3t^3 + \cancel{4r^2c^2t^2} - 2r^3ct + \cancel{r^2c^2t^2} - 2r^3ct + \cancel{r^4}\right)\Big]$$

$$= \frac{1}{8c}\left[4r^3ct + 4rc^3t^3 - \left(-4rc^3t^3 - 4r^3ct\right)\right] = \frac{1}{8c}\left(8r^3ct + 8rc^3t^3\right)$$

$$= \frac{1}{\cancel{8c}}\cancel{8c}\left(r^3t + rc^2t^3\right) = rt\left(r^2 + c^2t^2\right),$$

$$v(r,t) = r\overline{u}(r,t) \implies \overline{u}(r,t) = \frac{v(r,t)}{r} = \frac{\cancel{r}t\left(r^2+c^2t^2\right)}{\cancel{r}} = t\left(r^2+c^2t^2\right).$$

So $u(\vec{x},t) = t\left(x^2+y^2+z^2+c^2t^2\right) = x^2t + y^2t + z^2t + c^2t^3$ solves the 3D wave equation with the given initial data.

6.4 Summary of Lecture 13

We began with a brief review of what we discussed in the last lecture by talking about Kirchoff's formula for the solution of the wave equation in 3 dimensions. Using Kirchoff's equation we are able to develop a new method of solving the wave equation that actually generalizes to higher dimensions. The process is known as the method of descent. We begin by examining the case of the wave equation in three dimensions. For simplicity we take

$$u(\vec{x},0) = 0$$

and we consider u to be a solution to the wave equation that does not depend on the z. Then the solution to the wave equation at the origin is given by

$$u(0,t) = \frac{1}{4\pi c}\iint_{||\vec{x}||=ct} g(x,y)dS$$

where $g(x,y) = u_t(\vec{x},0)$. Here we are integrating over the sphere but we can use symmetry to integrate only over the upper hemisphere given by

$$\left\{(x,y,z)\in\mathbb{R}\;\middle|\; z = \sqrt{c^2t^2 - x^2 - y^2}\right\}.$$

Essentially, we have eliminated one of the dimensions by solving the equation of the sphere for z. Now we have a parameterized surface in 3D. By taking the partials of z with respect to x and y we can compute the area element. Take the cross product of the two derivatives given by

$$\frac{\partial}{\partial x}(x,y,z) = \left(1,0,\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial y}(x,y,z) = \left(0,1,\frac{\partial z}{\partial y}\right).$$

Taking the length of the cross product we have that the area element dS is given by

$$\left\|(1,0,\frac{\partial z}{\partial x}) \times (0,1,\frac{\partial z}{\partial y})\right\| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

Upon expanding this out, we have

$$dS = \frac{ct}{\sqrt{c^2t^2 - x^2 - y^2}}.$$

So the solution to the wave equation at $\vec{x} = 0$ is given by

$$u(0,0,t) = \frac{1}{2\pi c} \iint\limits_{x^2+y^2 \leq c^2t^2} \frac{g(x,y)}{\sqrt{c^2t^2 - x^2 - y^2}}.$$

We can repeat the same process for $f \neq 0$, but we note that we end up with an extra time derivative out front. If we are looking at an arbitrary initial point in space, we can make a simple translation and end up with the general solution

$$u(\vec{x}_0, t) = \frac{1}{2\pi c} \frac{\partial}{\partial t} \iint\limits_{D} \frac{f(x,y)}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}} + \iint\limits_{D} \frac{g(x,y)}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}}$$

where D is the sphere of radius ct.

6.5 Example

Our goal is to use Kirchoff's formula to find all solutions to the wave equation that depend only on the radial coordinate. First we need the Laplacian in spherical coordinates, which is given by

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2} + \frac{1}{r^2}\sin^2(\theta)\frac{\partial^2 u}{\partial \phi^2}.$$

Since we are only looking for functions that depend on r we can assume that all of the theta and phi derivatives are zero, so our expression for the Laplacian simplifies to

$$\Delta u = u_{rr} + \frac{2}{r}u_r.$$

Therefore, we can rewrite the wave equation as

$$u_t t = c^2 \left(u_{rr} + \frac{2}{r}u_r \right).$$

If we multiply through by r we have that

$$ru_{tt} = c^2 \left(ru_{rr} + u_r \right).$$

We can see that the right hand side is a perfect derivative, and the left hand side can be rewritten as follows:

$$(ru)_{tt} = c^2 \left(ru_r + u \right)_r.$$

Again, we can see that the right hand side is another perfect derivative. Thus we have that

$$(ru)_{tt} = c^2 \left(ru \right)_{rr}.$$

If we let v = (ru) then we have exactly the one dimensional wave equation which we know the solution. Thus for any two continuous functions f, g the solution is given by

$$v = ru = f(r+ct) + g(r-ct) \Rightarrow u(r,t) = \frac{1}{r}f(r+ct) + g(r-ct).$$

Therefore all solutions that depend only on r are given by this expression above.

6.6 Exercise

Here, we begin by taking equation (66) from the lecture notes and integrating. We are given that $f = 0$ so the solution is given by

$$u(x,t) = \frac{1}{2\pi c} \iint\limits_{||\vec{x}-\vec{x}_0||=ct} \frac{g(x)}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}} dxdy$$

which we can rearrange as

$$u(x,t) = \frac{1}{2\pi c} \int g(x) \int \frac{1}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}} dydx.$$

We need to understand the bounds of integration. We can clearly see that we can integrate the x coordinate over the domain of dependence $(x_0 - ct, x_0 + ct)$. The y coordinate is a bit trickier. If we consider the circle of radius $c_0 t$ we have that the y values are given by

$$y = y_0 \pm \sqrt{c^2t^2 - (x - x_0)}.$$

So our integral is taken over

$$u(x,t) = \frac{1}{2\pi c} \int_{x_0-ct}^{x_0+ct} g(x) \int_{y_0 - \sqrt{c^2t^2-(x-x_0)^2}}^{y_0 + \sqrt{c^2t^2-(x-x_0)^2}} \frac{1}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}} dydx.$$

We can make a substitution by taking $r = y + y_0$. Clearly there is no change in the differentials, but our solution becomes

$$u(x,t) = \frac{1}{2\pi c} \int_{x_0-ct}^{x_0+ct} g(x) \int_{-\sqrt{c^2t^2-(x-x_0)^2}}^{\sqrt{c^2t^2-(x-x_0)^2}} \frac{1}{\sqrt{c^2t^2 - (x-x_0)^2 - (r)^2}} drdx.$$

Since we are integrating with respect to r the quantity $c^2t^2 - (x - x_0)^2$ is just a constant. Therefore, this is just the antiderivative of arccos. Therefore we have that

$$u(x,t) = \frac{1}{2\pi c} \int_{x_0-ct}^{x_0+ct} g(x) \arccos\left(\frac{r}{\sqrt{c^2t^2 - (x-x_0)^2}} \right) \Bigg|_{-\sqrt{c^2t^2-(x-x_0)^2}}^{\sqrt{c^2t^2-(x-x_0)^2}} dx$$
$$= \frac{1}{2\pi c} \int_{x_0-ct}^{x_0+ct} g(x)(\arccos(1) - \arccos(-1))dx = \frac{1}{2\pi c} \int_{x_0-ct}^{x_0+ct} g(x)(0 - (-\pi))dx$$

which leads us to the solution

$$u(x,t) = \frac{1}{2c} \int_{x_0-ct}^{x_0+ct} g(x)dx.$$

This is exactly the solution given by d'Alembert's formula.

7 Fourier Methods in Higher Dimensions

7.1 Summary of Lecture 14

We began by discussing Fourier methods in two and three dimensions. By looking at the diffusion equation inside of a cube of length π. We construct the initial boundary value problem

$$\begin{cases} u_t - \kappa\Delta u = \quad \text{for} \quad x \in D = \{\vec{x} \in \mathbb{R}^3 | 0 < x < \pi, 0 < y < \pi, 0 < z < \pi\}, \\ u = 0 \quad \text{on} \quad \partial D, \\ u(\vec{x}, 0) = \phi(\vec{x}). \end{cases}$$

In order to solve this IBVP we use separation of variables. Assuming that $u(\vec{x}, t) = V(\vec{x})T(t)$ makes our PDE look like

$$V\dot{T} - \kappa\Delta VT = 0.$$

Then we proceed as normal, by dividing both sides by κXT. By letting $\vec{V}(x) = X(x)Y(y)Z(z)$ we arrive at

$$\frac{\dot{T}}{\kappa T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}$$

and we can conclude that each side is equal to a constant. So we have the following eigenvalue problems:

$$\begin{aligned} \frac{\dot{T}}{\kappa T} &= -\lambda \to \dot{T} = -\kappa\lambda T, \\ \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} &= -\lambda. \end{aligned}$$

For the latter equation, we use the same argument that allows us to let the spatial and temporal ODES both equal to a constant. So we have that each of the individual derivatives in the sum is also equal to some constant. So we arrive at the following equations

$$\begin{aligned} \frac{X''}{X} &= -a^2, \\ \frac{Y''}{Y} = -\lambda + a^2 - \frac{Z''}{Z} \Rightarrow \frac{Y''}{Y} &= -b^2, \\ \frac{Z''}{Z} = -\lambda + a^2 + b^2 \Rightarrow \frac{Y''}{Y} &= -c^2. \end{aligned}$$

Since we are dealing with Dirichlet Boundary Conditions, we know that for $l \in \mathbb{N}$ the solution to these equations are given by

$$\begin{aligned} X_l(x) &= \sin(lx), \\ Y_m(y) &= \sin(my), \\ Z_n(z) &= \sin(nz), \end{aligned}$$

with

$$\lambda_{lmn} = l^2 + m^2 + n^2,$$

and the spatial solution is given by the product of these equations. As for the temporal ODE, we must have that

$$\dot{T} = -\lambda_{lmn}\kappa T \quad \Rightarrow \quad T(t) = A_{lmn}e^{-\lambda\kappa t}.$$

However, we also need to satisfy the initial condition, so we take the triple sum over l, m, and n and impose the condition that

$$u(0,t) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn}\sin(lx)\sin(my)\sin(nz) = \phi(x).$$

We can use the orthogonality of sines and cosines to conclude that we must have

$$A_{lmn} = \frac{8}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\phi(\vec{x})\sin(lx)\sin(my)\sin(nz)dxdydz.$$

We can also extend this to general eigenfunctions for symmetric boundary conditions. We take the definitions of our eigenfunctions as

$$\begin{aligned}\Delta u &= \lambda_1 u,\\ \Delta v &= -\lambda_2 v.\end{aligned}$$

If we subtract these two equations and integrate both sides, we can use the divergence theorem to conclude that with symmetric boundary conditions the eigenfunctions are mutually orthogonal.

We concluded our discussion by setting up the general problem for the vibration on a drum head. By using the Laplacian in polar coordinates and considering the circular domain of radius a centered at the origin, we have that our initial boundary value problem is given by

$$\begin{cases} u_{tt} - c^2(u_r r + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) = 0 \quad \text{for} \quad 0 < r < a, 0 < \theta < 2\pi,\\ u(a,\theta,t) = 0,\\ u(r,\theta,0) = \phi(r,\theta) \quad u_t(r,\theta,0) = \psi(r,\theta).\end{cases}$$

7.2 Example

As an example problem, Exercise 10.1-2 will be worked out here. Our goal is to solve the wave equation in two dimensions in a rectangle. The initial boundary value problem is given by

$$\begin{cases} u_t - c^2\Delta u = 0,\\ u(o,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0,\\ u(x,y,0) = xy(b-y)(a-x) \quad u_t(x,y,0) = 0.\end{cases}$$

We can use separation of variables to determine the solution, but the argument is exactly the same as what we discussed in lecture, except there is no variable z. The biggest difference is the temporal ODE. From separation of variables, we determine that the eigenvalues must be negative, and they are given by

$$\lambda = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}.$$

Therefore, for the temporal ODE given by

$$\ddot{T} = -\lambda T$$

and the solution is given by

$$T_{nm}(t) = A_n m \cos\left(c\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}t\right) + B_{cm} \sin\left(c\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\right).$$

For the condition on the initial velocity, we find that

$$\dot{T}(0) = -A_{nm}c\sqrt{\lambda}\sin(0) + B_{nm}c\sqrt{\lambda}\cos(0) = 0 \Rightarrow B_{nm} = 0.$$

Imposing the first initial condition, and the fact that the solution $u(x,t)$ is given by the summations of the product of the solution to the spatial ODES given by

$$X_n(x) = \sin\left(\frac{n\pi}{a}y\right),$$
$$Y_m(y) = \sin\left(\frac{m\pi}{b}y\right),$$

and for the temporal ODE, we find that by evaluating $u(x,t)$ we must have

$$\phi(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right).$$

Therefore, we can use the orthogonality of sines to determine the value of A_{nm} as

$$A_{nm} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b \phi(x,y)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)dydx.$$

Plugging in our value for the initial condition we find that

$$A_{nm} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b (xy(b-y)(a-x))\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)dydx$$

which we can break apart as

$$A_{nm} = \frac{2}{a}\frac{2}{b}\int_0^a (x(a-x))\sin\left(\frac{n\pi}{a}x\right)dx\int_0^b y(b-y)\sin\left(\frac{m\pi}{b}y\right)dy.$$

We can integrate each by parts. The coefficients are given by

$$A_{mn} = \frac{4a^2}{n^3\pi^3}((-1)^n - 1)\frac{4b^3}{m^3\pi^3}((-1)^m - 1).$$

We find that when n or m is even, $A_{nm} = 0$. When they are both odd ($n = 2k+1$ and $m = 2l+1$), the coefficients are

$$\frac{64a^2b^2}{(2k+1)^2(2l+1)^2\pi^6}$$

and we arrive at the final solution which is given by

$$u(x,t) = \frac{64a^2b^2}{\pi^6}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2(2l+1)^2}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\cos\left(\sqrt{\frac{(2k+1)^2\pi^2}{a^2} + \frac{(2l+1)^2\pi^2}{b^2}}\,t\right).$$

7.3 Exercise

Our work for Exercise 10.1-4 is shown below.

(a) - (b) Here, we are examining the eigenvalue problem in two dimensions given by

$$-v_x x - v_y y = \lambda v.$$

So we use separation of variables to attempt a solution. To begin we let the solution $v(x, y) = X(x)Y(y)$ so our PDE can be given by

$$-X^{''}Y - XY'' = \lambda XY.$$

If we move the negative to the right side and divide through by XT, we are left with

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda.$$

If we move the Y term to the right hand side we can conclude that each side must be equal to some constant, so we conclude that

$$\frac{X''}{X} = -\gamma,$$
$$\frac{Y''}{Y} - \lambda = -\gamma.$$

So if we look at the first ODE, we will have three cases based on the sign of γ. However, for the ODE in X we have Dirichlet boundary conditions, so we know that $\gamma > 0$ and the solution is

$$X(x) = b \sin(n\pi).$$

Next we examine the ODE in y. If we move the constant terms over to the right hand side we have that

$$Y'' = -(\lambda - n^2\pi^2)Y.$$

Depending on the value of the constant term, the solution is given by either trigonometric functions or hyperbolic functions. Consider the case where $\lambda \leq 0$, and let $\lambda = -\beta^2$, then the solution is given by

$$Y(y) = A \cosh\left(\sqrt{\beta^2 - n^2\pi^2 y}\right) + B \sinh\left(\sqrt{\beta^2 + n^2\pi^2 y}\right).$$

The boundary conditions for y are Robin, so if we look at the first one we have that, at $y = 0$,

$$Y(0) = A = 0.$$

The second BC tells us that $dv = -v$ at 1, so we have that

$$B\left[\sqrt{\beta^2 + n^2\pi^2} \cosh\left(\sqrt{\beta^2 + n^2\pi^2}\right) + \sinh\left(\sqrt{\beta^2 + n^2\pi^2}\right)\right] = 0.$$

In both cases, we must have that $B = 0$ because the sum of the hyperbolic sine and cosine for positive values will never be zero, so we have the trivial solution. In the second case, we assume that $\lambda > 0$, and the solution is given by

$$Y(y) = A \cos\left(\sqrt{\lambda - n^2\pi^2}\right) + B \sin\left(\sqrt{\lambda - n^2\pi^2}\right).$$

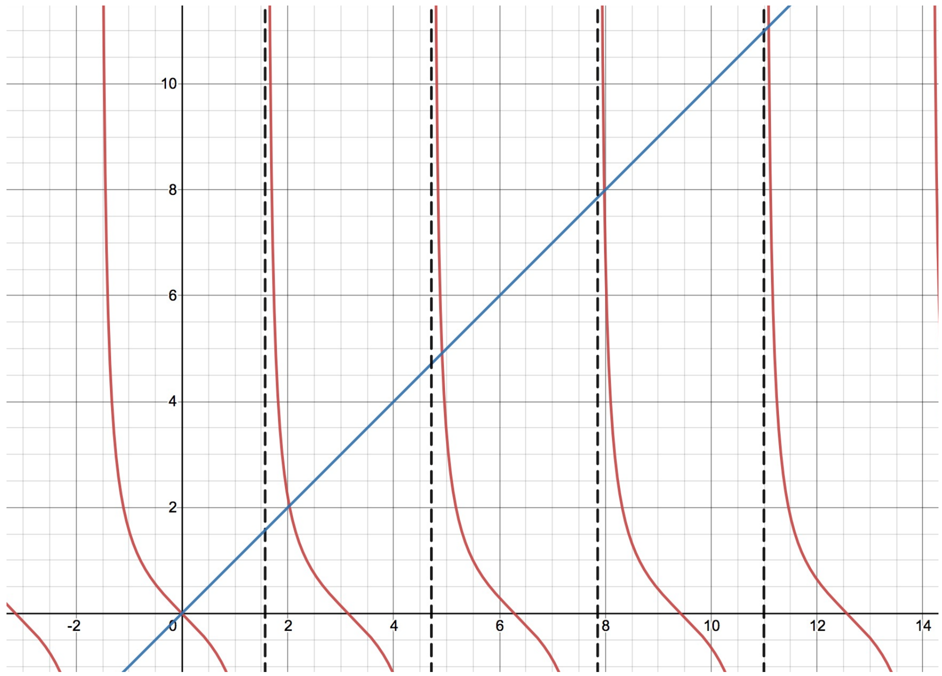


Figure 1: graph of $y = s$ and $y = -\tan(s)$ with vertical asympototes on the left at $s = (2n+1)\pi/2$

By imposing the first boundary condition, we see that

$$Y(0) = A = 0.$$

If we look at the second boundary condition at $y = 1$ we have that

$$B\left[\sqrt{\lambda + n^2\pi^2}\cos\left(\sqrt{\lambda + n^2\pi^2}\right) + \sin\left(\sqrt{\lambda + n^2\pi^2}\right)\right] = 0.$$

Clearly if $B = 0$ we must have a trivial solution. We can see that if we let $s = \sqrt{\lambda + n^2\pi^2}$, then the eigenvalues can be found by looking at the roots of the equation given by

$$s\cos(s) + \sin(s) = 0.$$

If we divide both sides by the cosine we have that the eigenvalues can be determined by the roots of the equation

$$s + \tan(s) = 0.$$

(c) If we want to solve this graphically, we can look at two functions $y = s$ and $y = -\tan(s)$ and find the intersections. We can see that, for larger values of m and n, the intersections lie close to the vertical asymptotes on the left of each period. These occur at values of $s = \frac{(2m+1)\pi}{2}$. So we can say that

$$s \approx \frac{(2m+1)\pi}{2}$$

but using our expression for s, we have that

$$\sqrt{\lambda - n^2\pi^2} \approx \frac{(2m+1)\pi}{2}.$$

If we solve for λ, we find that

$$\lambda \approx= \frac{(2m+1)^2\pi^2}{4} + n^2\pi^2.$$

7.4 Summary of Lecture 15

Today, we continued our discussion of the drum head problem and Bessel functions. We are given the IBVP in two dimensions:

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{in disk } D \text{ with radius } a, \\ u = 0 & \text{on } \partial D, \\ u = \phi \quad u_t = \psi & \text{at } t = 0. \end{cases}$$

To solve this IBVP, it becomes necessary to use the polar Laplacian:

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right).$$

Use the method of separation of variables. Assume

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t).$$

When we compute the derivatives of the separated solution and use those expressions in the wave equation, we have

$$R\Theta\ddot{T} = c^2 \left(R''\Theta T + \frac{1}{r} R'\Theta T + \frac{1}{r^2} R\Theta'' T \right)$$

$$= c^2 R''\Theta T + \frac{c^2}{r} R'\Theta T + \frac{c^2}{r^2} R\Theta'' T.$$

$$\text{Divide both sides by } R\Theta c^2 T: \qquad \frac{\ddot{T}}{c^2 T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda.$$

The LHS is a function of t only. Since the RHS is a function of r and θ, both sides must be equal to some constant $(-\lambda)$. By similar logic, we can set Θ''/Θ equal to the constant $-n^2$. So we have

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}(-n^2) = -\lambda$$

$$\implies R'' + \frac{1}{r} R' + \frac{1}{r^2}(-n^2) R = -\lambda R$$

$$\implies R'' + \frac{1}{r} R' + \frac{1}{r^2}(-n^2) R + \lambda R = 0 \implies R'' + \frac{1}{r} R' + \left(\lambda - \frac{n^2}{r^2} \right) R = 0.$$

We have converted the PDE (with three variables r, θ, t) into a second-order ODE (with one variable r). Require: $R(a) = 0$ and $\lim_{r \to 0^+} R(r) < \infty$. Let $\rho = \sqrt{\lambda} r$ and use

$$\frac{d}{dr} = \frac{d\rho}{dr}\frac{d}{d\rho} = \sqrt{\lambda}\frac{d}{d\rho}.$$

The second-order ODE becomes

$$\sqrt{\lambda}\sqrt{\lambda}R_{\rho\rho} + \frac{\sqrt{\lambda}}{\rho}\sqrt{\lambda}R_\rho + \left(\lambda - \frac{n^2}{\frac{\rho^2}{\sqrt{\lambda}^2}}\right)R = 0$$

$$\implies \lambda R_{\rho\rho} + \frac{\lambda}{\rho}R_\rho + \left(\lambda - \frac{\lambda n^2}{\rho^2}\right)R = 0 \implies R_{\rho\rho} + \frac{1}{\rho}R_\rho + \left(1 - \frac{n^2}{\rho^2}\right)R = 0.$$

Compare to Euler's equation

$$\frac{d^2u}{dt^2} + \frac{A}{t}\frac{du}{dt} + \frac{B}{t^2}u = 0$$

with the "singular point" $t = 0$. Since the two equations are similar, we use the same approach to solve the second-order ODE with respect to ρ with the singular point $\rho = 0$.

$$\text{"Educated guess:"}\ R(\rho) = \rho^\alpha \sum_{k=0}^{\infty} a_k \rho^k \qquad \text{where } \alpha = 1/\rho.$$

We determined the coefficients (a_k) in class:

$$R_n(\rho) = J_n(p) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{1}{2}\rho\right)^{n+2j}}{j!(n+j)!}.$$

where $J_n(\rho)$ are Bessel functions "of first kind." This summation solves one of three eigenvalue problems (two spatial and one time). We will solve the time problem next lecture.

7.5 Example

Exercise 10.2-4 is worked out here as an example. We are asked to find all solutions of the wave equation of the form $u = e^{-i\omega t}f(r)$ that are finite at the origin where $r = \sqrt{x^2 + y^2}$. Use the polar Laplacian again:

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right).$$

Solutions of the form $u = e^{-i\omega t}f(r)$ imply that the angular derivative vanishes. Compute the relevant derivatives:

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right),$$

$$f(r)(-i\omega)^2 e^{-i\omega t} = c^2\left[e^{-i\omega t}f''(r) + \frac{1}{r}e^{-i\omega t}f'(r)\right],$$

$$(-i\omega)^2 e^{-i\omega t}f(r) - c^2 e^{-i\omega t}f''(r) - \frac{c^2}{r}e^{-i\omega t}f'(r) = 0,$$

$$\left[\frac{1}{-c^2e^{-i\omega t}}\right]\left[-c^2e^{-i\omega t}f''(r) - \frac{c^2}{r}e^{-i\omega t}f'(r) + (-i\omega)^2e^{-i\omega t}f(r)\right] = \left[0\right]\left[\frac{1}{-c^2e^{-i\omega t}}\right],$$

$$f''(r) + \frac{1}{r}f'(r) + \frac{(-i\omega)^2}{-c^2}f(r) = 0 \qquad \text{(note: } (-i\omega)^2 = (i\omega)^2 = i^2\omega^2 = -\omega^2\text{)},$$

$$f''(r) + \frac{1}{r}f'(r) + \frac{\omega^2}{c^2}f(r) = 0.$$

This second-order ODE resembles Bessel's differential equation. However, we need to make a substitution in order to match equation 10 (pg 266 in Strauss) exactly. Let $\rho = \frac{\omega}{c}r$ and use

$$\frac{d}{dr} = \frac{d\rho}{dr}\frac{d}{d\rho} = \frac{\omega}{c}\frac{d}{d\rho},$$

$$\frac{d^2}{d\rho^2}f + \frac{1}{r}\frac{d}{dr}f + \frac{\omega^2}{c^2}f = 0 \implies \left(\frac{\omega}{c}\right)^2\frac{d^2}{d\rho^2}f + \frac{1}{r}\frac{\omega}{c}\frac{d}{d\rho}f + \frac{\omega^2}{c^2}f = 0$$

$$\implies \frac{d^2}{d\rho^2}f + \frac{1}{r}\frac{c}{\omega}\frac{d}{d\rho}f + f = 0 \implies f''(\rho) + \frac{1}{\rho}f'(\rho) + f(\rho) = 0 \qquad \text{since } \rho = \frac{\omega r}{c}.$$

This matches Bessel's differential equation exactly. The order is $n = 0$ since the coefficient for $f(\rho)$ is $\left(1 - \frac{n^2}{\rho^2}\right) = 1$ in this case. So the solution is given by $f(\rho) = J_0(\rho)$. We were asked to find all solutions of the wave equation of the form $u = e^{-i\omega t}f(r)$. Since f is a function of r and we have $\rho = \frac{\omega r}{c}$, we should report the solution as $u = Ae^{-i\omega t}J_0(\frac{\omega r}{c})$ where A is a constant.

7.6 Exercise

Exercise 10.2-6: solve the diffusion equation on the annulus $A = \{a^2 < x^2 + y^2 < b^2\}$. We have

$$\begin{cases} u_t = k\Delta u, \\ u = B \qquad \text{on } \partial A, \\ u = 0 \qquad \text{at } t = 0. \end{cases}$$

Hint from 5 (same problem but on a full disk instead of an annulus): the answer is radial. There is no θ-dependence for that problem, so we can make the same assumption here. So the solution has the form $u(r,t)$. Using the polar Laplacian again, we have

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r + \cancel{\frac{1}{r^2}u_{\theta\theta}}\right) = k\left(u_{rr} + \frac{1}{r}u_r\right).$$

Dirichlet boundary conditions are desirable on ∂A. Take the difference function

$$v(r,t) = u(r,t) - B,$$

$$v(a,t) = u(a,t) - B = 0 \qquad \text{since } a \text{ is the inner radius length of the annulus,}$$

$$v(b,t) = u(b,t) - B = 0 \qquad \text{since } b \text{ is the outer radius length of the annulus,}$$

$$v(r,0) = u(r,0) - B = -B \qquad \text{since } u = 0 \text{ at t = 0.}$$

Solve the "companion" IBVP. Compute the relevant derivatives in the polar Laplacian:

$$v_t = R(r)T'(t), \qquad v_r = R'(r)T(t), \qquad v_{rr} = R''(r)T(t).$$

These expressions would give us 2 eigenvalue problems if we used them in the diffusion equation. According to equation (16) in Strauss, the solution to the spatial part is given by

$$J_n\left(\sqrt{\lambda} r\right)\Big[A_n \cos(n\theta) + B_n \sin(n\theta)\Big].$$

We no longer need the Bessel functions to be finite at $r = 0$. There are two kinds of Bessel functions we need to acknowledge: J_n (finite at 0) and Y_n (not finite at 0). Since the answer to the original problem has no θ-dependence (mentioned earlier), we must have

$$J_0\left(\sqrt{\lambda} r\right)\Big[A_0 \cos(0 \cdot \theta) + B_0 \sin(0 \cdot \theta)\Big]$$

which is simply $A_0 J_0(\sqrt{\lambda} r)$. From equation (5) in Section 10.1, we know that the solution to the diffusion equation is given by

$$v(\vec{x}, t) = \sum_n C_n e^{-\lambda_n k t} w_n(\vec{x})$$

where $w_n(\vec{x})$ denotes the eigenfunctions $A_0 J_0(\sqrt{\lambda} r)$ and Y_n. In this case, we have

$$v(r, t) = \sum_n C_n e^{-\lambda_n k t}\Big[A_0 J_0(\beta_{nm} r) + Y_n\Big]$$

where $\beta_{nm} = \sqrt{\lambda}$. Use the initial condition to find C_n:

$$v(r, 0) = \sum_n C_n \Big[A_0 J_0(\beta_{nm} r) + Y_n\Big] = -B,$$

$$\sum_n C_n \Big[A_0 J_0(\beta_{nm} r) + Y_n\Big]\Big[A_0 J_0(\beta_{np} r) + Y_n\Big] r = -B\Big[A_0 J_0(\beta_{np} r) + Y_n\Big] r,$$

$$\int_0^a \sum_n C_n \Big[A_0 J_0(\beta_{nm} r) + Y_n\Big]\Big[A_0 J_0(\beta_{np} r) + Y_n\Big] r dr = \int_0^a (-B)\Big[A_0 J_0(\beta_{np} r) + Y_n\Big] r dr,$$

$$\sum_n C_n \int_0^a \Big[A_0 J_0(\beta_{nm} r) + Y_n\Big]\Big[A_0 J_0(\beta_{np} r) + Y_n\Big] r dr = -B \int_0^a \Big[A_0 J_0(\beta_{np} r) + Y_n\Big] r dr.$$

Orthogonality: the integral on the LHS is equal to 0 unless $m = p$:

$$\sum_n C_n \int_0^a \Big[A_0 J_0(\beta_{nm} r) + Y_n\Big]^2 r dr = -B \int_0^a \Big[A_0 J_0(\beta_{nm} r) + Y_n\Big] r dr,$$

$$C_n = \frac{-B\int_0^a \left[A_0 J_0(\beta_{nm} r) + Y_n\right] r dr}{\int_0^a \left[A_0 J_0(\beta_{nm} r) + Y_n\right]^2 r dr}.$$

The difference function is $v(r,t) = u(r,t) - B$ so $u(r,t) = v(r,t) + B$. Then

$$u(r,t) = \sum_n C_n e^{-\lambda_n k t}\left[A_0 J_0(\beta_{nm} r) + Y_n\right] + B$$

given the formula for the coefficients C_n above. We also have

$$A_0 = \frac{1}{2\pi j_{0m}} \int_0^a \int_{-\pi}^{\pi} (-B)\left[J_0(\beta_{nm} r) + Y_n\right] r d\theta dr.$$

8 Bessel Functions

8.1 Summary of Lecture 16

Today, we began the lecture by taking a closer look at the Bessel functions. We declared that each J_n has the following properties:

- Each J_n has infinitely many zeros z_{jn}.

- $J_n'(z_{jn}) \neq 0$ (the curves hit the ρ-axis transversally).

- The zeros of J_n, J_{n+1} are intertwined.

- Recursion relations: (1) $J_{n+1}(\rho) + J_{n-1}(\rho) = \dfrac{2n}{\rho} J_n(\rho)$, (2) $J_{n\pm 1}(\rho) = \dfrac{n}{\rho} J_n(\rho) \mp J_n'(\rho)$.

- Decay of functions: $J_n(\rho) \sim \dfrac{1}{\rho} \cos\left(\rho - \dfrac{n\pi}{2} - \dfrac{\pi}{4}\right) + \mathcal{O}\left(\left(\dfrac{1}{p}\right)^{3/2}\right)$ as $\rho \to \infty$.

These 5 properties (able to be proven) should help us when we try to solve the drum head problem.

During the last lecture, we created three eigenvalue problems:

$$\begin{cases} -\Theta'' = n^2\Theta, \\ \frac{\ddot{T}}{T} = -c^2\lambda, \\ \frac{R''}{R} + \frac{R'}{rR} = -\lambda + n^2. \end{cases}$$

We solved both spatial problems (R and Θ) last time. The solution $R_n(r)$ is given as the Bessel functions summation and the solution $\Theta_n(\theta)$ is given as

$$\Theta_n(\theta) = A\cos(n\theta) + B\sin(n\theta) \qquad \text{(harmonic oscillator)}.$$

When we join the spatial solutions together, we obtain

$$X_{nm}(r,\theta) = J_n\left(\sqrt{\lambda_{nm}}\, r\right)\Big[A\cos(n\theta) + B\sin(n\theta)\Big]$$

where n indexes the Bessel function we are referring to and m indexes the zero we are referring to. There is only one eigenvalue problem left to solve:

$$\ddot{T} = -c^2\lambda_{nm}T,$$

$$T_n(t) = A\cos\left(c\sqrt{\lambda_{nm}}\, t\right) + B\sin\left(c\sqrt{\lambda_{nm}}\, t\right) \qquad \text{(harmonic oscillator)}.$$

Finally, we can combine our results and report the solution to the drum head problem:

$$
\begin{aligned}
u(r,\theta,t) = \sum_{m=1}^{\infty} J_0\left(\sqrt{\lambda_{n0}}\, r\right)\left[A_{n0}\cos\left(c\sqrt{\lambda_{n0}}\, t\right) + C_{n0}\sin\left(c\sqrt{\lambda_{n0}}\, t\right)\right] \\
+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n\left(\sqrt{\lambda_{nm}}\, r\right)\Big\{\Big[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\Big]\cos\left(c\sqrt{\lambda_{nm}}\, t\right) \\
+\Big[C_{nm}\cos(n\theta) + D_{nm}\sin(n\theta)\Big]\sin\left(c\sqrt{\lambda_{nm}}\, t\right)\Big\}.
\end{aligned}
$$

Today's lecture concluded with a discussion of orthogonality. We have $X_{nm} \perp X_{n'm'}$ if $n \neq n'$ and if $n = n'$ and $m \neq m'$. So the inner product is zero:

$$
< X_{nm}, X_{n'm'} > \;\; = \iint_{D_a} X_{nm} X_{n'm'}\, dS = 0.
$$

The Laplacian has orthogonality for Dirichlet boundary conditions. The Bessel functions themselves are also orthogonal:

$$
\int_0^a J_n\left(\sqrt{\lambda_{nm}}\, r\right)\; J_n\left(\sqrt{\lambda_{np}}\, r\right)\; rdr = 0 \qquad m \neq p.
$$

9 Spherical Harmonics

9.1 Summary of Lecture 17

Our goal was to set up the problem for looking at the solid vibrations inside of a ball. We begin by considering the wave equation in three dimensions in the ball defined by

$$B = \{(x,y,z) | \|\vec{x}\| < a\}, \qquad \partial B = \{(x,y,z) | \|\vec{x}\| = a\},$$

so the initial boundary value problem is given as

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

We examine this equation by changing to the usual spherical coordinates given by

$$\begin{aligned} x &= r\sin(\theta)\cos(\phi), \\ y &= r\sin(\theta)\sin(\phi), \\ z &= r\cos(\theta), \end{aligned}$$

and use the spherical Laplacian given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{\sin^2(\theta) r^2}\frac{\partial}{\partial \phi^2} + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right).$$

By separating our solution as $u = T(t)v(r,\theta,\phi)$, we have the eigenvalue problem given by

$$\begin{cases} -\Delta v = \lambda v & \text{in } B, \\ v = 0 & \text{on } \partial B. \end{cases}$$

By separating v into the radial and angular portions by $v = R(r)Y(\theta,\phi)$ and plugging this into our problem, we have

$$YR_{rr} + \frac{2}{r}YR_r + \frac{R}{r^2\sin(\theta)}Y_{\phi\phi} + \frac{R}{r^2\sin(\theta)}\left(\sin(\theta)Y_\theta\right) + \lambda RY = 0.$$

Multiply both sides by $\frac{r^2}{RY}$:

$$\frac{r^2}{2R}\left(R_{rr} + \frac{2}{r}R_r\right) + \lambda R) + \frac{1}{Y}\frac{1}{\sin(\theta)}\left(\frac{1}{\sin^2(\theta)}Y_{\phi\phi} + \frac{1}{\sin(\theta)}(\sin(\theta)Y_{\theta\theta})\right) = 0$$

Since the first terms in this expression are functions of r and the latter are functions of the angular variables, we conclude that each is equal to some constant γ. The radial component must be equal to γ and the angular component must be equal to $-\gamma$. The radial ODE is given by

$$R_{rr} + \frac{2}{r}R_r + \frac{(\lambda - \gamma)}{r^2}R = 0.$$

where we must have the conditions that at the origin, the solution R must be finite and that on the boundary, $R(a) = 0$. If we define the function $w(r) = \sqrt{r}R(r)$, then we arrive at the ODE

$$w_{rr} + \frac{1}{r}w_r + \left(\lambda - \frac{\gamma + \frac{1}{4}}{r^2}\right)w = 0$$

which is exactly Bessel's ODE. Therefore, if we solve for w and use this in the definition of w to solve for R, we find that

$$R(r) = J_{\sqrt{\gamma+\frac{1}{4}}}\left(\sqrt{\lambda}r\right)\frac{1}{\sqrt{r}}.$$

Next we look at the angular equation, which is given by

$$\frac{1}{\sin^2(\theta)}Y_{\phi\phi} + \frac{1}{\sin(\theta)}(\sin(\theta)Y_\theta)_\theta + \gamma Y = 0$$

with the conditions that our solution Y must be finite at 0 and 2π and that Y must be periodic in ϕ. By multiplying through by $\sin^2(\theta)$ to simplify, and separating our angular portion into $p(\theta)q(\phi)$, we have that

$$q_{\phi\phi}p + \sin(\theta p_\theta q)_\theta + \gamma\sin^2(\theta)pq = 0.$$

Divide by pq and our equation becomes

$$\frac{q_{\phi\phi}}{q} + \frac{\sin(\theta p_\theta)_\theta}{p} + \gamma\sin^2(\theta) = 0.$$

Again, we have two ODEs that are both equal to some constant α. Thus we have the two ODEs given by

$$\begin{cases} q_{\phi\phi} + \alpha q = 0, \\ q(\phi) = q(\phi + 2\pi), \end{cases}$$

and

$$\begin{cases} \frac{(\sin(\theta)p_\theta)_\theta}{\sin(\theta)} + \left(\gamma - \frac{\alpha}{\sin^2(\theta)}\right)p = 0, \\ p(0), p(\phi) < \infty. \end{cases}$$

We know that the first equation is the harmonic oscillator.

9.2 Example

Here we will work out the solution to Exercise 10.3-5. We are trying to solve the heat equation with constant Dirichlet boundary conditions. The IBVP is given by

$$\begin{cases} u_t - k\Delta u = 0 \quad x \in D, \\ u(\vec{x}, 0) = A, \\ u(\vec{x}, t) = B \quad x \in \partial D, \end{cases}$$

where D is the sphere of radius a. If we define the function $v(\vec{x}, t) = u - B$, then we actually have an IBVP with homogeneous Dirichlet boundary conditions given by

$$\begin{cases} v_t - k\Delta v = 0 \quad x \in D, \\ v(\vec{x}, 0) = A - B, \\ v(\vec{x}, t) = 0 \quad x \in \partial D. \end{cases}$$

We can use separation of variables to solve this IBVP in spherical coordinates. Since the time portion of the heat equation does not change, we have that the general solution to this IBVP is given by

$$v(\vec{x}, t) = \sum_{j=1}^{\infty}\sum_{m=-l}^{l}\sum_{l=0}^{\infty} A_{lmj}J_{l+\frac{1}{2}}\left(\sqrt{\lambda_{lj}}\right)P_l^{|m|}(\cos(\theta))e^{im\phi}e^{-k\lambda_{lj}t}.$$

If we impose our initial condition, we must have that

$$A - B = \sum_{j=1}^{\infty} \sum_{m=-l}^{l} \sum_{l=0}^{\infty} A_{lmj} \frac{1}{\sqrt{r}} J_{l+\frac{1}{2}} \left(\sqrt{\lambda_{lj}}\right) P_l^{|m|}(\cos(\theta)) e^{im\phi}.$$

We know that the spherical harmonics are mutually orthogonal, and the harmonic for $l = 0, m = 0$ is simply the constant one. Thus we must have that

$$A - BY_0^0 = \sum_{j=1}^{\infty} \sum_{m=-l}^{l} \sum_{l=0}^{\infty} A_{lmj} J_{l+\frac{1}{2}} \left(\sqrt{\lambda_{lj}}\right) P_l^{|m|}(\cos(\theta)) e^{im\phi}.$$

If we use the orthogonality condition, we determine that all of the constants with $l \neq 0$ and $m \neq 0$ must be zero. Therefore our solution can be written as

$$v = \sum_{j=1}^{\infty} A_{0,0,j} \frac{1}{\sqrt{r}} J_{\frac{1}{2}} \left(\sqrt{\lambda_0 j} r\right).$$

Next we impose the boundary condition that at $r = a$ we must have that the solution is zero. So we have that

$$0 = \sum_{j=1}^{\infty} A_{0,0,j} \frac{1}{\sqrt{r}} J_{\frac{1}{2}} \left(\sqrt{\lambda_0 j} a\right).$$

We do not want a trivial solution so the lambda values must correspond to the roots of the equation

$$J_{\frac{1}{2}} \left(\sqrt{\lambda_0 j} a\right) = 0.$$

From section 10.5, we can find the explicit formula for the Bessel function

$$J_{\frac{1}{2}} \left(\sqrt{\lambda_0 j} r\right) = \sqrt{\frac{2}{\pi \sqrt{\lambda_{0j}} r}} \sin\left(\sqrt{\lambda_{0j}} r\right)$$

which tells us that $\lambda_{0j} = \frac{j^2\pi^2}{a^2}$. Lastly, if we go back to our initial condition, we have that

$$A - B = \sum_{j=1} A_{00j} \sqrt{\frac{2a}{\pi^2 j}} \frac{\sin(\frac{j\pi r}{a})}{r}.$$

So if we multiply both sides by $r \sin(\frac{n\pi}{a})$ and integrate from 0 to π, we can use the orthogonality of sines and cosines to deduce that the coefficients are given by the relationship

$$A_{00j} \sqrt{\frac{2a}{\pi j}} = A - B \frac{2a}{\pi j} (-1)^{j+1}.$$

We arrive at the solution for $v(r, t)$ as

$$v(\vec{x}, t) = (A - B) \frac{2a}{j\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \frac{sin(\frac{j\pi}{a})r}{r} e^{-\frac{j^2\pi^2}{a^2}kt}.$$

To recover $u(r, t)$, we simply add a factor of B to both sides. So our IBVP is given by

$$u(r, t) = (A - B) \frac{2a}{j\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \frac{sin(\frac{j\pi}{a})r}{r} e^{-\frac{j^2\pi^2}{a^2}kt}.$$

9.3 Exercise

In Exercise 10.3-7, our goal is to derive the solution to the IBVP given by

$$\begin{cases} u_t = k\Delta u & \vec{x} \in D, \\ u_r = B & \vec{x} \in \partial D, \\ u(\vec{x}, 0) = C. \end{cases}$$

From the hint, we know that the solution must be radial, so the spherical Laplacian is given by

$$\Delta u = u_{rr} + \frac{2}{r} u_r.$$

Now, let us consider the function

$$v = u - \frac{3kBt}{a} - \frac{Br^2}{2a}.$$

Then if we differentiate with respect to t we have that

$$v_t = u_t - \frac{3kB}{a}.$$

But from the heat equation, we know that $u_t = k\Delta u$, so

$$v_t = ku_{rr} + \frac{2k}{r} u_r - \frac{3kB}{a}.$$

Now, we can also derive

$$v_r = u_r - \frac{Br}{a}, \qquad v_{rr} = u_{rr} - \frac{B}{a},$$

and use this in

$$\begin{aligned} k\Delta v &= k\left(v_{rr} + \frac{2}{r} v_r\right) \\ &= k\left(u_{rr} - \frac{B}{a} + \frac{2}{r} u_r - \frac{2B}{a}\right) \\ &= v_t. \end{aligned}$$

Therefore, v satisfies the diffusion equation and

$$v_r|_a = u_r - \frac{Ba}{a} = 0.$$

so we have homogeneous Neumann Boundary Conditions. Now, if we use separation of variables on v by separating v into a radial part and a temporal part we will arrive at the following system of ODEs

$$\begin{aligned} R'' + \frac{2}{r} R' + \lambda R &= 0, \\ \dot{T} + k\lambda T &= 0. \end{aligned}$$

We know that the temporal part will be given by the exponential decay

$$T(t) = e^{-k\lambda t}$$

but the radial solution is a little bit tricker. In this case we make the substitution $R = \sqrt{r}w(r)$ then we our ODE becomes

$$w'' + \frac{1}{r}w' + \left(\lambda - \frac{\frac{1}{4}}{r^2}\right) w = 0.$$

This is Bessel's ODE and the solution is given by the Bessel function

$$J_{\frac{1}{2}}\left(\sqrt{\lambda}r\right)$$

so the radial solution is given by

$$R(r) = J_{\frac{1}{2}}\left(\sqrt{\lambda}r\right)\frac{1}{\sqrt{r}}.$$

Imposing the Neumann boundary condition we have that

$$\sqrt{\lambda}\frac{J'_{\frac{1}{2}}\left(\sqrt{\lambda}a\right)}{\sqrt{a}} - \frac{J_{\frac{1}{2}}\left(\sqrt{\lambda}a\right)}{a^{-3/2}} = \frac{1}{\sqrt{a}}\left(J'_{\frac{1}{2}}\left(\sqrt{\lambda}a\right) - \frac{J'_{\frac{1}{2}}\left(\sqrt{\lambda}a\right)}{2a}\right) = 0.$$

Thus we must have that λ_j are the roots of the equation given by

$$J'_{\frac{1}{2}}\left(\sqrt{\lambda}a\right)2a\sqrt{\lambda} = J_{\frac{1}{2}}\left(\sqrt{\lambda}a\right).$$

Now, if we put this together with our solution to the temporal ODE, we arrive at our solution for v

$$v(x,t) = A_0 + \sum_{j=1}^{\infty}\frac{J_{1/2}\left(\sqrt{\lambda_j}r\right)}{\sqrt{r}}e^{-k\lambda_j t}$$

and we can recover $u(x,t)$ as

$$u(x,t) = \frac{3kBt}{a} + \frac{Br^2}{2a} + A_0 + \sum_{j=1}^{\infty}A_j\frac{J_{1/2}\left(\sqrt{\lambda_j}r\right)}{\sqrt{r}}e^{-k\lambda_j t}.$$

Now, if we impose our initial condition we have that our solution must satisfy

$$C = \frac{Br^2}{2a} + A_0 + \sum_{j=1}^{\infty}A_j\frac{J_{1/2}\left(\sqrt{\lambda_j}r\right)}{\sqrt{r}}e^{-k\lambda_j t}$$

Now, if we multiply both sides by r^2 and integrate from 0 to a then we can use the orthogonality condition for Bessel functions to compute A_0 as

$$\frac{Ca^3}{3} = \frac{Ba^4}{10} + \frac{A_0 a^3}{3} \implies A_0 = C - \frac{3Ba}{10}.$$

Now, by multiplying by

$$J_{1/2}\left(\sqrt{\lambda_n}r^{3/2}\right)$$

and integrating from 0 to a on r, we use the orthogonality of Bessel functions to conclude that

$$\int_0^a\left(\frac{3Ba}{10} - \frac{Br^2}{2a}\right)J_{1/2}\left(\sqrt{\lambda_n}r\right)r^{-3/2}dr = A_n\int_0^a\left[J_{1/2}\left(\sqrt{\lambda_n}r\right)\right]^2 rdr.$$

Therefore the non-decaying terms of this solution are given by

$$\frac{3kBt}{a} + \frac{Br^2}{2a} + C - \frac{3Ba}{10}$$

and the decaying terms are given by

$$\sum_{j=1}^{\infty} A_j \frac{J_{1/2}\left(\sqrt{\lambda_j} r\right)}{\sqrt{r}} e^{-k\lambda_j t},$$
$$A_j = \frac{\int_0^a \left(\frac{3Ba}{10} - \frac{Br^2}{2a}\right) J_{1/2}\left(\sqrt{\lambda_n} r\right) r^{-3/2} dr}{\int_0^a \left[J_{1/2}\left(\sqrt{\lambda_n} r\right)\right]^2 r dr}.$$

9.4 Summary of Lecture 18

Given the wave equation (with Dirichlet BCs) with the spherical Laplacian, we can, of course, arrive at a solution through separation of variables. Today's lecture began with an acknowledgement of 3 steps for this process. Assume that $u(r,\theta,\phi,t) = T(t)v(r,\theta,\phi)$. Furthermore, assume that $v(r,\theta,\phi) = R(r)p(\theta)q(\phi)$. Then we have

$$\text{I. } \ddot{T} + \lambda c^2 T = 0.$$

$$\text{II. } R_{rr} + \frac{2}{r} R_r + \left(\lambda - \frac{\gamma}{r^2}\right) R = 0.$$

$$\text{III. } q_{\phi\phi} + \alpha q = 0 \qquad \text{and} \qquad \frac{1}{\sin\theta}\left(\sin\theta p_\theta\right)_\theta + \left(\gamma - \frac{\alpha}{\sin^2\theta}\right) p = 0.$$

We derived the solution to the radial portion (II) last time. The solution is

$$R(r) = \frac{J_{\sqrt{\gamma + \frac{1}{4}}}\left(\sqrt{\lambda}\, r\right)}{\sqrt{r}}.$$

We scratched the surface of the angular portion (III) last time. Today, we dove deeper. Let $\alpha = m^2$ in both equations:

$$q_{\phi\phi} + m^2 q = 0 \implies q(\phi) = A_m \cos(m\phi) + B_m \sin(m\phi) \qquad \text{(harmonic oscillator)},$$

$$\frac{1}{\sin\theta}\left(\sin\theta p_\theta\right)_\theta + \left(\gamma - \frac{m^2}{\sin^2\theta}\right) p = 0.$$

Last time, we let $s = \cos\theta$ in order to convert the form to the "associated Legendre equation:"

$$\frac{d}{ds}\left((1-s^2)\frac{dp}{ds}\right) + \left(\gamma - \frac{m^2}{1-s^2}\right) p = 0.$$

The solution is given by the "associated Legendre function"

$$P_l^m(s) = \frac{(-1)^m}{2^l l!} \left(1-s^2\right)^{m/2} \frac{d^{l+m}}{ds^{l+m}} \left((s^2-1)\right)^l$$

where $\gamma = l(l+1)$, $l \in \mathbb{N}$, and $l \geq m$.

We finally have all of the pieces needed to construct the expression for the spatial solution $v(r,\theta,\phi)$. For the radial portion, we acknowledge that $\sqrt{\gamma + \frac{1}{4}} = \sqrt{l(l+1)+\frac{1}{4}} = l + \frac{1}{2}$. We also acknowledge the boundary condition at $r = a$. We have $\sqrt{\lambda a} = z_{lj}$ where z_{lj} are the roots of the Bessel functions (l is the number of the Bessel function and j is the number of the root). The spatial solution is

$$v_{lmj}(r,\theta,\phi) = \frac{J_{l+\frac{1}{2}}\left(\sqrt{\lambda_{lj}}\, r\right)}{\sqrt{r}}\, P_l^m(\cos\theta) \left[A_{lmj}\cos(m\phi) + B_{lmj}\sin(m\phi)\right].$$

These are the eigenfunctions of the spherical Laplacian. We can provide a more compact expression for them (using complex notation):

$$v_{lmj}(r,\theta,\phi) = \frac{J_{l+\frac{1}{2}}\left(\sqrt{\lambda_{lj}}\, r\right)}{\sqrt{r}}\, P_l^{|m|}(\cos\theta)\, e^{im\phi}.$$

Regarding orthogonality, we have

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} v_{lmj}(r,\theta,\phi)\, v_{l'm'j'}(r,\theta,\phi)\, r^2 \sin\theta dr d\theta d\phi = 0$$

for all $(l,m,j) \neq (l',m',j')$.

We concluded today's lecture with a discussion of spherical harmonics. The family

$$Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)e^{im\phi} \qquad (l = 0,1,2,... \qquad m = -l,...,0,...l)$$

is a group of eigenfunctions for the Laplacian on the two-dimensional spherical surface. They define the spherical harmonics.

9.5 Example

Example 2 in Strauss (pg 276) is worked out here. We are asked to solve the IVP

$$\begin{cases} \Delta u = 0 & \text{in the ball } D, \\ u = g & \text{on } \partial D. \end{cases}$$

Separation of variables would give us

$$R_{rr} + \frac{2}{r}R_r - \left(\frac{\gamma}{r^2}\right)R = 0$$

for the radial portion. So $\lambda = 0$ here. In turn, we can make the "educated guess" for the solution to the ODE since it resembles Euler's equation:

$$R(r) = r^\alpha \implies \alpha(\alpha-1)r^{\alpha-2} + \frac{2}{r}\alpha r^{\alpha-1} - \left(\frac{\gamma}{r^2}\right) r^\alpha = 0$$

$$\implies \alpha(\alpha-1)r^{\alpha-2} + 2\alpha r^{\alpha-2} - \gamma r^{\alpha-2} = 0 \implies \left[\alpha(\alpha-1) + 2\alpha - \gamma\right] r^{\alpha-2} = 0$$

$$\implies \alpha(\alpha-1) + 2\alpha - \gamma = 0 \implies \alpha^2 - \alpha + 2\alpha - \gamma = 0$$

$$\implies \alpha^2 + \alpha - \gamma = 0.$$

From the angular portion, we had $\gamma = l(l+1)$:

$$\alpha^2 + \alpha - \left[l(l+1)\right] = 0 \implies \alpha^2 + \alpha - l^2 - l = 0$$

$$\implies (\alpha - l)(\alpha + l + 1) = 0 \implies \alpha = l, -(l+1).$$

We "toss out" the latter root to avoid negative powers for r^α (we want $R(0)$ to be finite). So $\alpha = l$. So the spatial solution is

$$v_{lmj}(r,\theta,\phi) = r^l \, P_l^m(\cos\theta)\, e^{im\phi}.$$

In class, we pointed out that these eigenfunctions are known as solid spherical harmonics. Strauss uses these to report what he refers to as the "complete solution:"

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l P_l^m(\cos\theta) e^{im\phi}.$$

9.6 Exercise

Exercise 10.3-12: solve the IVP given the diffusion equation $u_t = k\Delta u$ with the initial condition

$$\phi(r) = \begin{cases} 1 & r < a, \\ 0 & r > a. \end{cases}$$

We are told that the solution $u(r,t)$ is radial, so we can reduce the spherical Laplacian to

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

in this case. The IVP becomes

$$\begin{cases} u_t = k\left(u_{rr} + \frac{2}{r}u_r\right), \\ u(r,0) = \phi(r). \end{cases}$$

Let $v(r,t) = ru(r,t)$. The relevant derivatives are

$$v_t = ru_t, \qquad v_r = (1)(u) + (u_r)(r) = u + ru_r,$$

$$v_{rr} = u_r + \Big[(1)(u_r) + (u_{rr})(r)\Big] = 2u_r + ru_{rr}.$$

Acknowledge companion IVP **(1)**:

$$\begin{cases} ru_t = rk\left(u_{rr} + \frac{2}{r}u_r\right) = k\left(ru_{rr} + 2u_r\right), \\ ru(r,0) = r\phi(r), \end{cases} \qquad \Longrightarrow \begin{cases} v_t = kv_{rr}, \\ v(r,0) = r\phi(r). \end{cases}$$

The initial condition for companion IVP **(1)** is

$$r\phi(r) = \begin{cases} r & r < a, \\ 0 & r > a. \end{cases}$$

We have the one-dimensional diffusion equation on the whole real line. The known solution is

$$v(r,t) = \int_{-\infty}^{\infty} S(r-y,t)y\phi(y)dy.$$

But the solution only makes sense (geometrically) for $r > 0$. Consider companion IVP **(2)**:

$$\begin{cases} w_t = kw_{rr}, \\ w(r,0) = r\phi_{\text{odd}}(r), \end{cases}$$

where $\phi_{\text{odd}}(r) = \phi(r)$ for $r > 0$ and $\phi_{\text{odd}}(r) = -\phi(-r)$ for $r < 0$. The solution to companion IVP **(2)** is given by

$$w(r,t) = \int_{-\infty}^{\infty} S(r-y,t)y\phi_{\text{odd}}(y)dy$$

$$= \int_{-\infty}^{0} S(r-y,t)(-y)\phi(-y)dy + \int_{0}^{\infty} S(r-y,t)y\phi(y)dy.$$

Let $z = -y \implies dz = -dy$. Limits of integration: $y = -\infty$ to $y = 0 \implies z = \infty$ to $z = 0$. The first integral in the sum is

$$\int_{\infty}^{0} S(r+z,t)z\phi(z)(-dz) = -\int_{\infty}^{0} S(r+z,t)z\phi(z)dz = \int_{0}^{\infty} S(r+y,t)y\phi(y)dy$$

after renaming the dummy variable $z \to y$. So

$$w(r,t) = \int_{0}^{\infty} S(r+y,t)y\phi(y)dy + \int_{0}^{\infty} S(r-y,t)y\phi(y)dy$$

$$= \int_{0}^{\infty} \Big[S(r+y,t) + S(r-y,t)\Big] y\phi(y)dy.$$

Let $v(r,t) = w(r,t)$ for $r > 0$. This is the solution to companion IVP **(1)**. Retrieve $u(r,t)$:

$$v(r,t) = ru(r,t) \implies u(r,t) = \frac{v(r,t)}{r},$$

$$u(r,t) = \frac{1}{r}\int_0^\infty \Big[S(r+y,t) + S(r-y,t)\Big] y\phi(y)dy \qquad r > 0.$$

10 Eigenfunctions

10.1 Summary of Lecture 19

During today's lecture, we discussed a useful tool for understanding the behavior of eigenfunctions. We took a look at the nodal set defined by

$$\mathscr{N} = \{x \in D \mid v(x) = 0\}$$

where D is the open domain, and $v(x)$ is any eigenfunction. In one dimension, we have can easily graph these eigenfunctions and determine when they will be positive or negative. By adding the time portion of the differential equation, we can see that the nodal set will remain unchanged which gives us a nice illustration of the behavior of solutions. We can also generalize this idea to higher dimensions. We can have nodal regions in the plane as well as nodal surfaces in three dimensions. Both of these can lend some useful information as to the behavior of solutions of different eigenvalue problems.

We also began to discuss the theory of distributions during today's lecture. First, we developed some intuition. Recall the solution to the heat equation which is given by

$$E(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4\pi t}}.$$

This function tends to 0 for $x \neq 0$ as t approaches zero from the right. But if $x = 0$, then as t goes to 0, the solution blows up. However, the integral over the entire real line is equal to one when $t = 0$. What these ideas imply is that we are dealing with a point source. Attributed to Schwartz, the theory of distributions was developed in order to handle point sources effectively.

The first example of a distribution for us to study is the Dirac δ function:

$$\lim_{t->0^+} E(x,t) = \delta(x), \qquad \begin{cases} \delta(0) = \infty, \\ \delta(x) = 0, \\ \int_{-\infty}^{\infty} \delta(x)dx = 1. \end{cases}$$

We acknowledged that we can extend this to take solutions of arbitrary initial value functions by introducing the convolution integral. Next, we proved the following theorem.

Theorem 10.1. *If* $\sup |\phi(y)| < \infty$, *then*

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} E(y,t)\phi(y)dy = \phi(0).$$

We can use this theorem to describe Dirac's function as the function that takes ϕ as its input and produces $\phi(0)$. We can consider $\delta(x)$ to be a "function of functions." We generalize this to the following definition: a distribution is a continuous, real-valued function defined on the space of test functions.

10.2 Example

Exercise 12.1-1 is worked out here as an example. Our goal is to show that the distribution defined by

$$\phi \mapsto \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

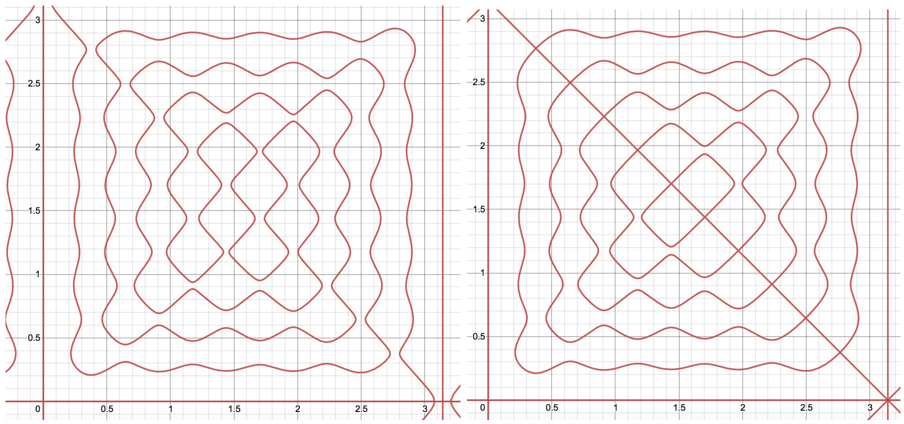


Figure 2: Graph of nodal set with $v = .9$ (left) and $v = .1$ (right)

is a distribution by definition. Therefore, we need to show that this function is both continuous and linear. The property of linearity follows directly from the linearity of the integral. For continuity, we only need to show that, for some sequence of test functions $\{\phi_n\}$ converging uniformly to 0, the sequence of real numbers (T_f, ϕ_n) also converges to 0. Use the following property of integrals:

$$\left|(T_f, \phi_n)\right| = \left|\int_{-\infty}^{\infty} f(x)\phi_n(x)dx\right| \leq \int_{-\infty}^{\infty} |f(x)\phi_n(x)|dx.$$

Since $\{\phi_n\}$ is a convergent sequence of test functions, we know that for some $R \in \mathbb{R}$, if $|x| > R$ then each ϕ_n is zero. Moreover, we know that $|\phi_n| \leq \sup(|\phi_n(x)|) \leq P_{Rk}$. So we have

$$\left|(T_f, \phi_n)\right| \leq P_{Rk} \int_{-R}^{R} f(x)dx.$$

However, we know that P_{Rk} goes to zero as n goes to infinity, so we must have that $|(T_f, \phi_n)|$ also goes to zero. Therefore, this function is continuous and linear, hence it is a distribution.

10.3 Exercise

In Exercise 10.4-3, our goal is to graph the nodal set for the two eigenfunctions given by

$$\sin(12x)\sin(y) + v\sin(x)\sin(12y).$$

We look at the graphs of these functions for $v = .9$ and $v = 1$ to see the different nodal sets given above. We can see in Figure 2 that, for v close to 1, the region is divided into two halves with the two main graphs defining the division. For the graph of $v = 1$, we can see that there are 5 square-like lines that divide the region into 6 parts. But the diagonal line that splits these actually gives us 12 total regions. These figures match Figure 2 in Strauss (pg 282).

10.4 Summary of Lecture 20

At the end of the last lecture, we only scratched the surface of distributions. So for today's lecture, our goal was to dive much deeper into the topic. A definition for distributions (also known as generalized functions) was provided. A distribution is a continuous, real-valued linear "function" (better words to use would be "rule," "functional," or "transformation") defined on the space of test functions. For example, the Dirac delta function is $\delta(\phi) = \phi(0)$ where $\phi(0)$ is the value of a test function at 0.

To discuss linearity and continuity, we first introduced the notation (f, ϕ). Interpretation: the test function ϕ is plugged into the distribution f. Linearity means that

$$(f, c_1\phi_1 + c_2\phi_2) = c_1(f, \phi_1) + c_2(f, \phi_2)$$

for all $c_1, c_2 \in \mathbb{R}$. Continuity has a more complicated meaning. If a sequence of test functions $\{\phi_n\}$ vanish outside a common interval and converge uniformly to a test function ϕ (and if the same is true for all the derivatives of the sequence), then

$$(f, \phi_n) \to (f, \phi) \qquad \text{as } n \to \infty.$$

The most common test function space is $\mathcal{C}_0^\infty$. We say $\phi \in \mathcal{C}_0^\infty$ if $\phi \in \mathcal{C}^\infty$ and ϕ is "compactly supported," meaning that supp(ϕ) is a compact (closed and bounded) set. Define

$$\text{supp}(\phi) = \left\{ x \in \mathbb{R} \,\middle|\, \phi(x) \neq 0 \right\}.$$

The following 3 statements are equivalent:

- supp(ϕ) is a compact set,
- $\phi(x) = 0$ for $|x| > R$ where R is some constant,
- $\text{supp}(\phi) \subseteq B_R$ where $B_R = \left\{ x \,\middle|\, |x| \leq R \right\}$.

We examined the function

$$\phi(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and we showed why it provides us with a good foundation for building functions in $\mathcal{C}_0^\infty$ ("bump functions"). They should have the form

$$\phi(x) = \begin{cases} 0 & x \leq a, \\ \exp\left(-\frac{1}{(x-a)(x-b)}\right) & a \leq x \leq b, \\ 0 & x \geq b. \end{cases}$$

After we discussed bump functions, we spent some time on the concept of convergence in $\mathcal{C}_0^\infty$. Given the sequence $\{\phi_n\}$ (where $\phi_n \in \mathcal{C}_0^\infty$), we have that $\{\phi_n\} \to 0$ as $n \to \infty$ if 2 conditions are satisfied:

1. $\exists R > 0 \text{ s.t. } \phi_n(x) = 0 \text{ when } |x| > R \text{ for all } n.$
2. $P_{R,k}(\phi_n) = \max\limits_{j \leq k} \left(\sup\limits_{|x|<R} \left| \dfrac{\partial^j \phi_n}{\partial x^j} \right| \right) \text{ and } P_{R,k}(\phi_n) \to 0 \text{ as } n \to \infty \text{ for all fixed } k.$

The non-example we saw during the lecture satisfied the first condition but clearly violated the second condition. Next, we proved that the two conditions held for a valid example, given as

$$T_f(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx.$$

We used $T_f(\phi)$ as a guinea pig to discuss derivatives of distributions. We began with

$$T_{f'}(\phi) = \int_{-\infty}^{\infty} f'(x)\phi(x)dx$$

$$= f\phi\Big|_{\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)dx \qquad \text{(integration by parts)}$$

$$= -\int_{-\infty}^{\infty} f(x)\phi'(x)dx.$$

In general, if T is a distribution, we have

$$T'(\phi) = -T(\phi'),\ T''(\phi) = T(\phi''),\ T'''(\phi) = -T(\phi''')$$

and so on and so forth.

10.5 Example

Exercise 12.1-3 is worked out here as an example. We are asked to verify that the derivative is a linear operator on the vector space of distributions. If this is the case, we should have

$$\big((af+bg)', \phi\big) = a(f', \phi) + b(g', \phi)$$

for any two distributions f, g and any two constants a, b. Generally, the derivative of a distribution h is defined as $(h', \phi) = -(h', \phi)$ for all test functions ϕ. In integral form, we have

$$\big((af+bg)', \phi\big) = \int_{-\infty}^{\infty} \Big[af(x) + bg(x)\Big]' \phi(x)dx = -\int_{-\infty}^{\infty} \Big[af(x) + bg(x)\Big]\phi'(x)dx$$

$$= -\int_{-\infty}^{\infty} \Big[af(x)\phi'(x) + bg(x)\phi'(x)\Big]dx = -a\int_{-\infty}^{\infty} f(x)\phi'(x)dx \ - b\int_{-\infty}^{\infty} g(x)\phi'(x)dx$$

$$= -a(f, \phi') \ - b(g, \phi') = a(f', \phi) + b(g', \phi).$$

Therefore, the derivative is a linear operator on the vector space of distributions.

10.6 Exercise

A sequence of L^2 functions $f_n(x)$ converge to a function $f(x)$ in the mean-square sense:

$$\int_a^b \left| \sum_{n=1}^{N} f_n(x) - f(x) \right|^2 dx \to 0 \qquad \text{as } N \to \infty.$$

In Exercise 12.1-7, we need to show that the sequence "converges weakly" in the sense of distributions. Weak convergence definition: if f_n is a sequence of distributions and f is another distribution, then f_n converges weakly to f if $(f_n, \phi) \to (f, \phi)$ as $n \to \infty$ for all test functions ϕ. This definition can be restated: $|(f_n, \phi) - (f, \phi)| \to 0$ as $n \to \infty$. Acknowledge the difference as integrals:

$$\left| \int_a^b f_n(x)\phi(x)dx \right| - \left| \int_a^b f(x)\phi(x)dx \right| = \left| \int_a^b \left[f_n(x) - f(x) \right] \phi(x)dx \right|.$$

Schwarz inequality: $|(g,h)| \leq \|g\| \, \|h\|$. Equivalently, we can use $|(g,h)|^2 \leq \|g\|^2 \, \|h\|^2$ to avoid square roots with the norms. Then the difference is

$$\left| \int_a^b \left[f_n(x) - f(x) \right] \phi(x)dx \right|^2 \leq \int_a^b \left[f_n(x) - f(x) \right]^2 dx \, \int_a^b \left[\phi(x) \right]^2 dx.$$

Recall the definition for convergence in the mean-square sense. We can make the acknowledgement for the first integral on the right-hand side:

$$\int_a^b \left[f_n(x) - f(x) \right]^2 dx = 0.$$

So the difference is

$$\left| \int_a^b \left[f_n(x) - f(x) \right] \phi(x)dx \right|^2 \leq 0.$$

Note: the left-hand side must be positive since it is an absolute value squared. Therefore, by the squeeze theorem, we must have

$$\left| \int_a^b \left[f_n(x) - f(x) \right] \phi(x)dx \right|^2 = 0$$

$$\implies \int_a^b \left[f_n(x) - f(x) \right] \phi(x)dx = 0 \implies \int_a^b f_n(x)\phi(x)dx \; - \int_a^b f(x)\phi(x)dx = 0$$

$$\implies \int_a^b f_n(x)\phi(x)dx = \int_a^b f(x)\phi(x)dx$$

$$\implies (f_n, \phi) \to (f, \phi) \qquad \text{as } n \to \infty.$$

So the sequence converges weakly in the sense of distributions.

11 Distributions

11.1 Summary of Lecture 21

After a brief review of the definition of continuity for distributions, today's lecture began with a discussion of the definition of the derivative in distributional sense. We define the derivative of a distribution as

$$(T', \phi) := -(T, \phi').$$

As an example, we looked at the Heaviside function which is given by

$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$$

Applying this to a test function using the natural associated distribution, we have that

$$(H', \phi) = -(H, \phi') = -\int_0^\infty \phi' dx = -\phi(x)\Big|_0^\infty$$

for $\phi \in C_0^\infty$. Since ϕ is a test function, we know that it must tend to zero as x approaches infinity. Therefore, we conclude that

$$(H', \phi) = \phi(0)$$

which is precisely the definition of the Dirac delta function. In the distributional sense, we have that the derivative of the Heaviside function is given as the Dirac delta function.

As another example, take the Fourier cosine series for $f(x) = |x|$ whose sequence of partial sums is given by

$$S_N(x) = \frac{\pi}{4} - \sum_{n=1}^{N} \frac{4}{n^2\pi} \cos(nx)$$

and converges uniformly to $|x|$ on $(-\pi, \pi)$. If we apply this to a test function and integrate, it converges "weakly." If we apply the definition of the distributional derivative, we find that the derivative of this function is actually given by $2H(x) - 1$ where $H(x)$ is the Heaviside function. Moreover, distributions are infinitely differentiable, so we can take the derivative again. We find that the second derivative simply equals two times the Dirac delta function.

The next two ideas that we discussed were the distributional product rule and the product of a distribution with an arbitrary smooth function. Begin with the latter: if $f \in C^\infty$ and T is a distribution, then

$$(fT, \phi) := (T, f\phi).$$

This idea simply comes from looking at the integral notation and moving around the order of the multiplication. In addition, it is quite useful in practice. If we consider an arbitrary smooth function given by $a(x)$ together with its product with the Dirac delta function, we can use the definition above to deduce

$$a(x)\delta(x) \to (a(x)\delta(x), \phi(x)) = (\delta(x), a(x)\phi(x)) = a(0)\phi(0).$$

With this, we arrive at the following conclusion:

$$a(x)\delta(x) = a(0)\delta(x).$$

Lastly, we looked at the concept of a distributional product rule. If we consider the product of any arbitrary smooth function $f(x)$ and any distribution T, we can deduce that the product rule for distributions can be given by

$$(Tf)' = T'f + Tf'.$$

To do so, we simply apply the definition of the derivative and the definition of products of distributions. Combined with the linearity of distributions, we can easily arrive at the conclusion above.

11.2 Addendum

If we use the expression for $\phi(x)$ provided and plug in the given value we arrive at

$$\frac{1}{n}\phi\left(\frac{x}{n}\right) = \begin{cases} \frac{1}{n}\exp\left(\frac{1}{1-\frac{x^2}{n^2}}\right) & \frac{|x|}{n} < 1, \\ 0 & \frac{|x|}{n} \geq 1. \end{cases}$$

To show that this does not satisfy the first part of the definition of a distribution, we will show a contradiction. Let $R \in \mathbb{R}$ be such that $\phi_n(x) = 0$ for $|x| > R$ for all n. If we can find an $N \in \mathbb{N}$ such that $N > R$, then for $|x| < N$, we have that $\phi_N(x) \neq 0$ which contradicts the first condition of the definition. Therefore, this sequence of test functions cannot converge to zero since it fails to satisfy condition a) of the Definition of Convergence.

11.3 Summary of Lecture 22

Last time, we derived a product rule for distributions. Given the distribution T and the function $f \in \mathcal{C}^\infty$, we have $(fT)' = f'T + fT'$. This is more accurately expressed as

$$\left((fT)', \phi\right) = \left(f'T, \phi\right) + \left(fT', \phi\right)$$

where the result of the right-hand side is a distribution with f' as a "standard derivative" and T' as the derivative of a distribution. To see this product rule in action, we were asked to find the derivative $(a(x)H(x))'$ given $a(x) \in \mathcal{C}^\infty$ and the Heaviside function $H(x)$:

$$(a(x)H(x))' = a'(x)H(x) + a(x)H'(x) = a'(x)H(x) + a(0)\delta(x).$$

After our discussion of the product rule, we revisited the Dirichlet Kernel and analyzed it in the context of distributions. We wrote down the relevant expressions as follows.

$$\text{Series: } \sum_{k=-\infty}^{\infty} e^{ikx} \qquad \text{for } x \in (-\pi, \pi).$$

$$\text{Partial sums: } S_N(x) = \sum_{k=-N}^{N} e^{ikx}$$

$$= 1 + \sum_{k=1}^{N} \cos(kx) = \frac{\sin\left((N+\frac{1}{2})x\right)}{\sin\left(\frac{1}{2}x\right)} = K_N(x),$$

$$\lim_{x\to 0} K_N(x) = 2N + 1,$$

$$\lim_{N\to\infty}\left(\lim_{x\to 0} K_N(x)\right) = \infty.$$

$$\text{Distribution context: } \lim_{N\to\infty} K_N(x) = 2\pi\delta(x).$$

Punchline from our discussion of the Dirichlet Kernel: many types of sequences can represent the Dirac delta function.

The lecture concluded with a discussion of three-dimensional distributions. As an example, we are given the sphere S_a^2 centered at the origin with radius a. So

$$S_a^2 = \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \|\vec{x}\| = a \right\}.$$

We define the distribution as the surface integral

$$(\delta(\|\vec{x}\| - a), \phi) := \iint_{S_a^2} \phi(\vec{x}) dS.$$

Strauss has some "abuse of notation" in his book, but we excuse it for now to restate the definition as

$$\iiint_{\mathbb{R}^3} \delta\left(\|\vec{x}\| - a\right)\phi(\vec{x}) d^3\vec{x} = \int_0^\infty \int_0^{2\pi} \int_0^\pi \delta(r-a)\phi(\vec{x})\, r^2 \sin\theta d\theta d\phi dr$$

$$= \int_0^{2\pi} \int_0^\pi \phi(\vec{x})\, a^2 \sin\theta d\theta d\phi = \iint_{S_a^2} \phi\, dS$$

where $\vec{x} = (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)$ and $a^2\sin\theta d\theta d\phi$ is the area element dS.

11.4 Addendum

Prove that, if $\phi \in \mathcal{C}_0^\infty$ and $g \in \mathcal{C}^\infty$, then $\phi g \in \mathcal{C}_0^\infty$. In general, for a given test function ψ, we say that $\psi \in \mathcal{C}_0^\infty$ if (i) $\psi \in \mathcal{C}^\infty$ and (ii) ψ is compactly supported. This means that supp(ψ) is a compact (closed and bounded) set where

$$\text{supp}(\psi) = \left\{ x \in \mathbb{R} \;\middle|\; \psi(x) \neq 0 \right\}$$

on $[-R, R]$ for some $R \in \mathbb{R}$.

(i) We need to show that $\phi g \in \mathcal{C}^\infty$. By the definition of $\mathcal{C}_0^\infty$, we have $\phi \in \mathcal{C}^\infty$. It is given that $g \in \mathcal{C}^\infty$, so showing that $\phi g \in \mathcal{C}^\infty$ equates to showing that the product of two smooth functions is smooth. Two known limit rules are

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M$$

$$\implies \lim_{x \to a} \Big[f(x) + g(x) \Big] = L + M$$

$$\text{and } \lim_{x \to a} \Big[f(x) \cdot g(x) \Big] = L \cdot M.$$

Therefore, the product of two smooth functions must be smooth since derivatives are defined as limits and must obey the same rules. So $\phi g \in \mathcal{C}^\infty$.

(ii) The statement "supp(ψ) is compactly supported" is equivalent to the statement "$\psi = 0 \ \ (\forall |x| > R)$." Since $\phi \in \mathcal{C}_0^\infty,$ we know that supp(ϕ) is compactly supported. So $\phi = 0 \ \ (\forall |x| > R)$. So we must have $\phi g = 0 \ \ (\forall |x| > R)$ as well. So ϕg is compactly supported. Therefore,

$$\phi g \in \mathcal{C}_0^\infty.$$

12 Green's Functions

12.1 Summary of Lecture 23

Our discussion today covered Green's functions and some associated examples. The first example we discussed was that of the ODE given by

$$\begin{cases} -c^2u'' = f(x) & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

The Green's function $G(x;\xi)$ is defined as the solution to the ODE given by

$$\begin{cases} -c^2u'' = \delta(x-\xi) & \xi \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Additionally, if we consider any $f \in C^\infty$ such that f vanishes outside of the interval $(0,1)$, meaning f is a test function that vanishes smoothly, we know that

$$f(x) = \int_0^1 \delta(x-\xi)f(\xi)d\xi.$$

Then the solution to our ODE can be expressed as

$$u(x) = \int_0^1 G(x;\xi)f(\xi)d\xi.$$

Finding the Green's function is a bit trickier. We use what we know about the delta function to show that

$$u'' = \frac{-1}{c^2}\delta(x-\xi) \Rightarrow u'\frac{-1}{c^2}H(x-\xi) + a$$

where H denotes the Heaviside function. If we integrate again, we find that the integral of the Heaviside function is given by

$$u(x) = \begin{cases} \frac{-1}{c^2}(x-\xi) + ax + b & x > \xi, \\ ax & x < \xi. \end{cases}$$

If we impose the boundary conditions, we have that

$$u(0) = \frac{-1}{c^2}0 + a \cdot 0 + b = 0 \Rightarrow b = 0, \qquad u(1) = \frac{-1}{c^2}(1-\xi) + a = 0 \Rightarrow a = \frac{1-\xi}{c^2}.$$

Using these values for our constants, we have that our Green's function is given by

$$G(x,\xi) = \begin{cases} \frac{x-\xi}{c^2} + x\frac{1-\xi}{c^2} & x > \xi, \\ x\frac{1-\xi}{c^2} & x < \xi. \end{cases}$$

If we go back to our original ODE, we have that the solution must be given by

$$u(x) = \int_0^1 G(x,\xi)d\xi = \int_0^x \xi\frac{1-\xi}{c^2}f(\xi) + \int_x^1 \frac{1-\xi}{c^2}f(\xi)d\xi.$$

Simplifying, we arrive at the final expression

$$u(x) = \frac{1-x^2}{c^2}\int_0^x \xi f(\xi)d\xi + \frac{x}{c^2}\int_x^1 (1-\xi)f(\xi)d\xi.$$

The next example we discussed was the Poisson Equation:

$$\begin{cases} \Delta u = f & D \subset \mathbb{R}^3, \\ u = 0 & \text{on } \partial D. \end{cases}$$

In this case, the Green's function is given by the solution to the associated problem:

$$\begin{cases} \Delta u = \delta(\vec{x} - \vec{\xi}) & D \subset \mathbb{R}^3, \\ u = 0 & \text{on } \partial D. \end{cases}$$

The function itself is

$$G(\vec{x}; \xi) = \frac{1}{4\pi\|\vec{x} - \vec{\xi}\|}.$$

In the spherical case, we take $\vec{\xi} = 0$ and $v(\vec{x}) = \frac{1}{\|\vec{\xi}\|}$. We can show that $\Delta v = 0$ everywhere except at the origin. We must check that the inverse of r is a harmonic function.

The lecture concluded with a discussion of Green's Second Identity and the Representation theorem. The latter is given by the following:

Theorem 12.1. *If $\Delta u = 0$ in D, then for every $x_0 \in D$,*

$$u(x_0) = \iint_{\partial D} \left[-u(\vec{x})\frac{\partial}{\partial \vec{n}}\left(\frac{1}{\|\vec{x} - \vec{x}_0\|}\right) + \frac{1}{\|\vec{x} - \vec{x}_0\|}\frac{\partial u}{\partial \vec{n}}\right]\frac{dS}{4\pi}$$

and at $x_0 = 0$ for $0 \in D$,

$$u(\vec{0}) = \iint_{\partial D} \left[-u(\vec{x})\frac{\partial}{\partial \vec{n}}\left(\frac{-1}{4\pi\|\vec{x}\|}\right) + \frac{1}{4\pi\|\vec{x}\|}\frac{\partial u}{\partial \vec{n}}\right]dS.$$

Green's Second Identity is given by

$$\iiint_D (u\Delta v - v\Delta u)d\vec{x} = \iint_{\partial D}\left(u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}}\right)dS.$$

Take u to be a harmonic function in our domain. Let

$$v(\vec{x}) = \frac{1}{4\pi\|\vec{x}\|}$$

which solves the Laplace equation everywhere except at the origin. Create a hollow domain by subtracting off a ball of radius ϵ about the origin. Ultimately, by splitting up our integral and taking the limit as ϵ goes to zero, we find that

$$\iiint_D \Delta\left(\frac{1}{4\pi\|\vec{x}\|}\right)\phi(\vec{x})d\vec{x} = \phi(0)$$

which tells us that, in the distributional sense,

$$\Delta\left(\frac{1}{4\pi\|\vec{x}\|}\right) = \delta(\vec{x}).$$

12.2 Example

Here, we will work out the solution to Exercise 12.2-11. To begin, we need to define the composition of a function with the Dirac delta function. The composition is defined as

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

where x_i are the roots of the the function $g(x_i)$. For our problem, we have the function defined by

$$g(\lambda) = (\lambda - a)(\lambda - b)$$

so the roots are clearly given by a and b. The derivative of the function is

$$g'(\lambda) = 2\lambda - (a + b).$$

Putting all of this together, we have that

$$\begin{aligned} g\left((\lambda - a)(\lambda - b)\right) &= \frac{1}{|2a - (a+b)|}(\lambda - a) + \frac{1}{|2b - (a+b)|}(\lambda - b) \\ &= \frac{1}{|a-b|}(\lambda - a) + \frac{1}{|b-a|}(\lambda - b). \end{aligned}$$

We can use the symmetry of the absolute value to combine these two fractions and arrive at the identity

$$g((\lambda - a)(\lambda - b)) = \frac{1}{|a-b|}((\lambda - a) + (\lambda - b)).$$

12.3 Exercise

In Exercise 12.2-13, our goal is to calculate the distribution given by $\Delta \log(r)$ in two dimensions. To begin, let $\phi(x) \in \mathcal{C}_0^\infty$ and consider

$$(\Delta \log(r), \phi).$$

From the properties of distributions, we can rewrite this expression as

$$(\log(r), \Delta\phi) = \iint\limits_{\mathbb{R}^2} \log(r)\Delta\phi dxdy.$$

We know that there is a singularity at the origin, so let us consider a ball of radius ϵ centered at zero given by $B_\epsilon(0)$. We can re-express the integral as

$$\iint\limits_{\mathbb{R}^2} \log(r)\Delta\phi dxdy = \lim_{\epsilon \to 0^+} \iint\limits_{\mathbb{R}^2 \setminus B_\epsilon(0)} \log(r)\Delta\phi dxdy.$$

Green's second identity tells us that

$$\iint\limits_{\mathbb{R}^2 \setminus B_\epsilon(0)} \log(r)\Delta dxdy + \iint\limits_{\mathbb{R}^2 \setminus B_\epsilon(0)} \Delta \log(r)\phi dxdy = \int\limits_{\partial\mathbb{R}^2 \setminus B_\epsilon(0)} \left[\log(r)\frac{\partial\phi}{\partial\vec{n}} + \frac{\partial \log(r)}{\partial\vec{n}}\phi\right] dL.$$

Using the Laplacian in polar coordinates, we can calculate

$$\Delta(\log(r)) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial \log(r)}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{r}\right) = 0.$$

So the expression reduces to

$$\iint\limits_{\mathbb{R}^2\setminus B_\epsilon(0)} \log(r)\Delta\phi dxdy = \int\limits_{\partial\mathbb{R}^2\setminus B_\epsilon(0)} \left[\log(r)\frac{\partial\phi}{\partial\vec{n}} + \frac{\partial \log(r)}{\partial\vec{n}}\phi\right] dL.$$

If we convert the line integral along the boundary to polar coordinates, we know that the boundary occurs when the radius is equal to the radius of our ball $B_\epsilon(0)$ and we integrate the angular coordinate from 0 to 2π. We also know that in polar coordinates, for a ball, the normal direction along the circle is given by the radial direction. So we can re-express the integral as

$$\iint\limits_{\mathbb{R}^2\setminus B_\epsilon(0)} \log(r)\Delta\phi dxdy = \int_0^{2\pi} \left[log(r)\frac{\partial\phi}{\partial r} + \frac{\partial \log(r)}{\partial r}\phi r\right] d\theta.$$

Evaluate the derivatives and let $r = \epsilon$. We arrive at the expression

$$\iint\limits_{\mathbb{R}^2\setminus B_\epsilon(0)} \log(r)\Delta\phi dxdy = \epsilon\log(\epsilon)\int_0^{2\pi} \phi_r d\theta + \int_0^{2\pi} \phi d\theta.$$

To recover the desired expression, we take the limit as ϵ tends to zero from the right. Consider

$$\lim_{\epsilon\to 0^+} \epsilon\log(\epsilon) = \frac{\log(\epsilon)}{1/\epsilon}.$$

We can use L'Hospital's rule to calculate the first term in the limit as

$$\lim_{\epsilon\to 0^+} \frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon\to 0^+} -\epsilon = 0.$$

For the second limit, since we assume that ϕ is a test function, we know ϕ must be continuous. Therefore, we can interchange the integral and the limit. As ϵ tends to zero, $\phi(\epsilon,\theta)$ tends to the value at the origin, which is $\phi(0)$. Note: $\phi(0)$ is a constant, so all that remains in the integral is

$$\lim_{\epsilon\to 0^+} \int_0^{2\pi} \phi d\theta = 2\pi\phi(0).$$

We know that the distribution that maps a test function to its value at the origin is simply the Dirac delta function. Therefore, in distributional sense,

$$\Delta\log(r) = 2\pi\delta.$$

12.4 Summary of Lecture 24

Today, we officially derived the expression for the Fourier transform. After its introduction, we were able to use its properties to work on a few different examples. We began by acknowledging the complex Fourier series representation of a function $f(x)$ as well as the expression for its Fourier coefficients (seen before: pg 116 in Strauss). To derive the Fourier transform, substitute that expression for C_n into the Fourier series. Let $k = n\pi/L$. If we take the limit as L approaches infinity, the summation becomes an integral with k as the variable of integration. Eventually, we arrive at

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx.$$

This is the expression for the Fourier transform. The purpose of the Fourier transform is to make a transition from the spatial domain to the frequency domain. This is the reason why k was defined as the "wave number" $n\pi/L$. If we need to reverse the process, we can use the inverse Fourier transform, defined as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}dk.$$

We discussed a few examples (table: pg 345 in Strauss) showing the Laplace transform in action. Of the seven examples given in the table, we spent the most time on three in particular:

$$f(x) = \delta(x) \iff F(k) = 1,$$

$$f(x) = 1 \iff F(k) = 2\pi\delta(k),$$

$$f(x) = e^{-x^2/2} \iff F(k) = \sqrt{2\pi}e^{-k^2/2}.$$

The examples from that table begin to show us why the Fourier transform will be convenient for future problems. We also spent some time on different table (pg 346 in Strauss) detailing some of the properties of the Fourier transform. We completed a proof for the first property:

$$\text{Function: } \frac{df}{dx} \iff \text{Transform: } ikF(k).$$

We also completed a proof for a property called Plancherel's theorem. This theorem states:

$$\int_{-\infty}^{\infty}\left|f(x)\right|^2 dx = \int_{-\infty}^{\infty}\left|F(k)\right|^2 \frac{dk}{2\pi}.$$

After our discussion of examples and properties of the Fourier transform, we studied an application from quantum mechanics. We acknowledged Heisenberg's uncertainty principle:

$$\overline{x} \cdot \overline{k} \geq \frac{1}{2}.$$

Simply stated, this principle asserts that the product of the uncertainty in the measurement of position and the uncertainty in the measurement of momentum is always at least $\frac{1}{2}$ (never 0). The more precisely the position is known, the more uncertain the momentum is (and vice versa). We used the Fourier transform, the Schwartz inequality, Plancherel's theorem, and a few additional tactics to successfully prove this principle.

To close the lecture, we proved a useful property of the Fourier transform in the context of convolution. Recall the definition for the convolution integral:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

If we take the Fourier transform of a convolution, we have

$$F\Big[(f * g)(x)\Big] = F(k)G(k).$$

12.5 Example

Exercise 12.3-9 is worked out here as an example. We are asked to use Fourier transforms to solve the ODE $-u_{xx} + a^2 u = \delta(x)$. Take the Fourier transform of both sides:

$$F\Big(- u_{xx} + a^2 u\Big) = F\Big(\delta(x)\Big),$$

$$-F\Big(u_{xx}\Big) + a^2 F\Big(u\Big) = F\Big(\delta(x)\Big) \qquad \text{(the Fourier transform can be shown to be linear),}$$

$$-F\Big(u_{xx}\Big) + a^2 F\Big(u\Big) = 1 \qquad \text{by property (v).}$$

Denote $F\Big(u(x)\Big)$ as $u(k)$. Then we have

$$-(i^2 k^2)u(k) + a^2 u(k) = 1 \qquad \text{by property (i),}$$

$$k^2 u(k) + a^2 u(k) = 1 \qquad \text{since } i^2 = -1,$$

$$u(k)\Big(k^2 + a^2\Big) = 1 \implies u(k) = \frac{1}{k^2 + a^2}.$$

According to our notation, we have

$$u(x) = F^{-1}\Big(u(k)\Big)$$

$$= F^{-1}\Big(\frac{1}{k^2 + a^2}\Big)$$

$$= \frac{1}{2a} e^{-a|x|} \qquad \text{by table entry (7) on pg 345 in Strauss.}$$

12.6 Exercise

Our work for Exercise 12.3-7 is shown below.

(a) Let $f(x)$ be a continuous function on the line $(-\infty, \infty)$ that vanishes for large $|x|$. Show that the function

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + 2n\pi)$$

is periodic with period 2π. Note: $2n$ is always even. The function $g(x)$ is periodic with period 2π if $g(x + 2\pi) = g(x)$. We have

$$g(x + 2\pi) = \sum_{n=-\infty}^{\infty} f(x + 2n\pi + 2\pi)$$

$$= \sum_{n=-\infty}^{\infty} f\Big(x + 2\pi(n+1)\Big) = \sum_{n=-\infty}^{\infty} f\Big(x + (2n+2)\pi\Big).$$

Note: $2n + 2$ is always even as well. So we can rewrite $g(x)$ and $g(x + 2\pi)$ as

$$g(x) = g(x + 2\pi) = \sum_{m=-\infty}^{\infty} f(x + m\pi) \qquad \text{for even } m.$$

Therefore, $g(x)$ is periodic with period 2π.

(b) Show that the Fourier coefficients C_m of $g(x)$ on the interval $(-\pi, \pi)$ are $F(m)/(2\pi)$, where $F(k)$ is the Fourier transform of $f(x)$. Since $L = \pi$, the complex Fourier series representation for $g(x)$ is

$$g(x) = \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} = \sum_{m=-\infty}^{\infty} C_m e^{imx}.$$

The Fourier coefficients are given as

$$C_m = \frac{1}{2L} \int_{-L}^{L} g(x) e^{-im\pi x/L} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-imx} dx,$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} f(x + 2n\pi) \right] e^{-imx} dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} f(x + 2n\pi) e^{-imx} dx.$$

Let $y = x + 2n\pi \implies dy = dx$. Limits of integration: $x = -\pi$ to $x = \pi \implies y = 2n\pi - \pi$ to $y = 2n\pi + \pi$. We have

$$C_m = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2n\pi-\pi}^{2n\pi+\pi} f(y) e^{-im(y-2n\pi)} dy = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2n\pi-\pi}^{2n\pi+\pi} f(y) e^{-imy} e^{im\cdot 2n\pi} dy.$$

Note: $e^{im\cdot 2n\pi} = e^{ip\pi}$ where p is an even integer. In this journal, on page 45, we showed that e raised to an even integer multiple of π equals 1. So

$$C_m = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2n\pi-\pi}^{2n\pi+\pi} f(y) e^{-imy} dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-imy} dy = F(m)/(2\pi).$$

(c) In the Fourier series of $g(x)$ on $(-\pi, \pi)$, let $x = 0$ to obtain the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(2n\pi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} F(n).$$

Let $x = 0$ in the original expression for the function $g(x)$:

$$g(0) = \sum_{n=-\infty}^{\infty} f(2n\pi).$$

We have already shown that the complex Fourier series representation for $g(x)$ can be written as

$$g(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}.$$

So if we let $x = 0$ again and proceed to use the result from part (b), we have

$$g(0) = \sum_{n=-\infty}^{\infty} C_n = \sum_{n=-\infty}^{\infty} F(n)/(2\pi).$$

We have two expressions set equal to $g(0)$. Therefore, we can set the two expressions equal to each other to obtain the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(2n\pi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} F(n).$$

13 Fourier Transform and Laplace Transform

13.1 Summary of Lecture 25

Today's discussion revolved around using the Fourier transform to derive source functions and Green's functions. We started with an example where we rederived the source function for the diffusion equation. We took the heat conductivity to be equal to 1 for simplicity. Our initial value problem is given by

$$\begin{cases} S_t - S_{xx} = 0 \qquad x \in \mathbb{R},\ t > 0, \\ S(x,0) = \delta(x). \end{cases}$$

Transforming the entire system into the frequency domain, our IVP reduces to the ODE given by

$$\begin{cases} \hat{S}_t + k^2\hat{S} = 0, \\ \hat{S}(k,0) = 1. \end{cases}$$

This is a simple ODE whose solution is given by the equation

$$\hat{S}(k,t) = e^{-k^2 t}.$$

In order to solve our original problem, we transform back to the space domain by using the inverse Fourier transform. We can either evaluate the integral that defines the inverse Fourier transform or use the known properties of the Fourier transform to pick our solution. Picking the correct constants to make our solution resemble one of the known Fourier transforms, we arrive at the source function for the diffusion equation which is given by

$$S(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

The next question we chose to answer was how to derive the source function for the wave equation given by the IVP

$$\begin{cases} S_{tt} - c^2 S_{xx} = 0 \quad x \in \mathbb{R},\ t > 0, \\ S(x,0) = 0, \\ S_t(x,0) = \delta(x). \end{cases}$$

Once again, we transform this IVP into the frequency domain and it becomes

$$\begin{cases} \hat{S}_{tt} + c^2k^2\hat{S} = 0 \quad x \in \mathbb{R},\ t > 0, \\ S(x,0) = 0, \\ S_t(x,0) = 1. \end{cases}$$

This is the ODE for the harmonic oscillator. We know that this has the solution given by

$$\hat{S}(k,t) = A\cos(ckt) + B\sin(ckt).$$

Imposing the initial conditions, we find that

$$\hat{S}(k,0) = A = 0, \qquad S_t(k,0) = 1 = Bck \Rightarrow B = \frac{1}{ck}.$$

Therefore, in the frequency domain, we have that the solution to the IVP is given by

$$\hat{S}(k,t) = \frac{1}{ck}\sin(ckt).$$

If we take the exponential form of the sine function, we can use the fact that the sign function has a Fourier transform given by

$$\operatorname{sgn}(x+ct) \to \frac{e^{ickt}}{ik}.$$

Therefore, we have that

$$\hat{S}(k,t) = \frac{1}{ck}\sin(ckt) = \frac{e^{ickt} - e^{-ickt}}{2ick} \quad \to \quad \frac{\operatorname{sgn}(x+ct) - \operatorname{sgn}(x-ct)}{4c}.$$

If we simplify the expression containing the sign functions, we have that our source function is given by

$$\begin{cases} 0 & |x| < ct,\ t > 0, \\ \frac{1}{2c} & |x| > ct. \end{cases}$$

The last problem we discussed was how to derive the source function for the Laplace equation in the upper half plane. This IVP is given by

$$\begin{cases} u_{xx} + u_{yy} = 0 & y > 0,\ x \in \mathbb{R}, \\ u(x,0) = \delta(x). \end{cases}$$

In this case, we take the Fourier transform with respect to the x variable and we arrive at the following:

$$\begin{cases} -k^2\hat{u} + \hat{u}_{yy} = 0 & y > 0,\ x \in \mathbb{R}, \\ \hat{u}(k,0) = 1. \end{cases}$$

This is, once again, simply an ODE whose solution is given by

$$\hat{u}(k,y) = Ae^{ky} + Be^{-ky}.$$

To avoid blow-up, we take the solution as

$$\hat{u}(k,y) = Ae^{-|k|y}$$

and if we impose our boundary condition, we find that the constant must be equal to 1. Now we take the inverse Fourier transform and split up the integral as

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|y}e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{0} e^{k(y+ix)} + \frac{1}{2\pi}\int_{0}^{\infty} e^{k(ix-y)}dk = \frac{1}{2\pi}\left(\frac{e^{k(y+ix)}}{y+ix}\bigg|_{-\infty}^{0} + \frac{e^{k(ix-y)}}{ix-y}\bigg|_{0}^{\infty}\right).$$

Evaluating and simplifying, we have that the source function for the Laplace equation in the upper half plane is given by

$$u(x,y) = \frac{2y}{\pi(x^2+y^2)}.$$

If we had a non-constant boundary condition $u(x,0) = f(x)$, we would have that our solution is given by the convolution integral

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(z-x)^2 + y^2} f(z)dz.$$

If we were looking at the equation in three dimensions with $f(x,y,z)$, we would have

$$\hat{f}(k_x, k_y, k_z) = \iiint\limits_{\mathbb{R}^3} f(\vec{x}) e^{i\vec{k}\vec{x}} d^3\vec{x}$$

with the given vectors:

$$\vec{x} = (x,y,z),$$
$$\vec{k} = (k_x, k_y, k_z).$$

13.2 Example

Exercise 12.4-6 is worked out here as an example. Our goal is to use the Fourier transform to solve the IBVP given by

$$\begin{cases} u_{xx} + u_{yy} = 0 \qquad x \in \mathbb{R},\ y \in (0,1), \\ u(x,0) = 0, \\ u(x,1) = f(x). \end{cases}$$

Applying the Fourier transform, the problem is given in the frequency domain by the following ODE:

$$\begin{cases} -k^2\hat{u} + \hat{u}_{yy} = 0, \\ \hat{u}(x,0) = 0, \\ \hat{u}(x,1) = \hat{f}(x). \end{cases}$$

The solution to the ODE is given by

$$\hat{u}(k,y) = Ae^{-ky} + Be^{ky}$$

where A and B are functions depending on k. Imposing our boundary conditions, we find that $A = -B$ and that $A = \frac{\hat{f}(k)}{\sinh(k)}$. Therefore, our solution in the frequency domain is given by

$$\hat{u}(x,y) = \hat{f}(k)\frac{\sinh(ky)}{\sinh(k)}.$$

We can use the convolution property of the Fourier transform by letting $F(k) = \hat{f}(k)$ and $G(k) = \frac{\sinh(ky)}{\sinh(k)}$. We can easily find the inverse transform of $F(k)$, but $G(k)$ is tricky. Apply the inverse Fourier transform and apply the convolution property by taking the solution as

$$g(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh(ky)}{\sinh(k)} e^{iky} dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh(kx)}{\sinh(k)}(\cos(ky)) + \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh(ky)}{\sinh(k)} \sin(ky)dk.$$

We can use the symmetry argument to say that $G(k)\cos(kx)$ is an even function and $G\sin(kx)$ is an odd function. Therefore, our expression becomes

$$g(x,y) = \frac{1}{\pi}\int_0^\infty \frac{\sinh(ky)}{\sinh(k)}\cos(kx)dk.$$

Putting this together with the convolution property, we have that the solution to the IBVP is given by

$$\frac{1}{\pi}\int_{-\infty}^\infty f(y)\int_0^\infty \frac{\sinh(kx-ky)}{\sinh(k)}\cos(kx)dkdy = \int_0^\infty f(y)\frac{\sinh(kx-ky)}{\sinh(k)}dydk.$$

13.3 Exercise

Here, we will work out the solution to Exercise 12.4-2. We know that solutions to Laplace's equation are harmonic functions, so we are trying to solve the IVP given by

$$\begin{cases} u_{xx} + u_{yy} = 0, \\ u_y(x,0) = h(x). \end{cases}$$

Apply the Fourier transform throughout:

$$\begin{cases} -k^2\hat{u} + \hat{u}_{yy} = 0, \\ \hat{u}_y(k,0) = \hat{k}(x). \end{cases}$$

Once again, we have reduced the PDE into an ODE. We know that the solution is given by

$$\hat{u}(k,y) = Ae^{ky} + Be^{-ky}.$$

Since we do not want our solution to blow up, we take the solution

$$\hat{u}(k,y) = Ae^{-|k|y}.$$

If we impose the boundary conditions, we have that

$$-A|k| = \hat{h}*(k) \quad \Rightarrow \quad A = -\frac{h(\hat{k})}{|k|}.$$

To recover the solution to the original problem, we employ the inverse Fourier transform as well as the convolution property to assert that the harmonic function we are looking for is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty h(z)\frac{e^{-|k|y}}{|k|}e^{ik(x-z)}dkdz.$$

13.4 Summary of Lecture 26

Today, we began and concluded a discussion of the Laplace transform. We noted that the Laplace transform is closely related to the Fourier transform (seen last week). This becomes clear when we write down the definition. $F(s)$ is the Laplace transform of $f(t)$ where

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

As we can see, this integral expression looks quite similar to that of the Fourier transform. Unlike the Fourier transform, however, we only focus on the time domain. Another difference between the two is that the integral expression for the inverse Laplace transform is not as straightforward as Fourier. To compute the integral expression, techniques from complex analysis are required. To avoid this, we rely on the Laplace transform properties.

We can use the properties of the Laplace transform to navigate freely between $f(t)$ and $F(s)$ by manipulating expressions to mimic forms given in property tables. The two tables in Strauss (on pages 353 & 354) show us several of these useful properties. During the lecture, we derived the following.

$$\text{Function: } \frac{f(t)}{t}. \qquad \text{Transform: } \int_0^\infty F(s')ds'.$$

$$\text{Function: } \frac{df}{dt}. \qquad \text{Transform: } sF(s) - f(0).$$

Of course, several additional properties are given in the tables. We spent the most time on these two in particular because they are among the most useful. For example, we used the second property to prove that the Laplace transform of the second derivative is $s^2F(s) - sf(0) - f'(0)$. After our discussion of properties, we proceeded to solve three problems via the usage of the Laplace transform.

For the first problem (ODEs application), we were given the IVP

$$\begin{cases} u'' + \omega^2 u = f(t), \\ u(0) = u'(0) = 0. \end{cases}$$

If we take the Laplace transform of both sides of the equation, we obtain

$$s^2U(s) - su(0) - u'(0) + \omega^2 U(s) = F(s),$$

$$s^2U(s) + \omega^2 U(s) = F(s) \implies U(s) = \frac{F(s)}{s^2 + \omega^2}.$$

We retrieve the solution $u(t)$ not by taking the inverse Laplace transform but by using property (ix) in Strauss. We eventually obtain

$$u(t) = \frac{1}{\omega}\int_0^t \sin\Big(\omega(t - t')\Big) f(t')dt'.$$

For the second problem (PDEs application), we were given the IBVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = u_t(x,0) = 0, \\ u(0,t) = f(t) \qquad u \to 0 \text{ as } x \to \infty. \end{cases}$$

Take the Laplace transform throughout again. Eventually, we are left with

$$\begin{cases} U_{xx} - \frac{s^2}{c^2}U = 0, \\ U(0,s) = F(s). \end{cases}$$

Impose the boundary conditions:

$$U(x,s) = A(s)e^{-sx/c} + B(s)e^{sx/c} \qquad \text{(assert that } B(s) = 0 \text{ to avoid blow-up),}$$

$$F(s) = A(s)e^0 \implies F(s) = A(s) \implies U(x,s) = F(s)e^{-sx/c}.$$

Retrieve $u(x,t)$ by using property (vii) in Strauss. The solution is

$$u(x,t) = H\left(t - \frac{x}{c}\right) f\left(t - \frac{x}{c}\right).$$

For the third problem (another PDEs application), we were given the IBVP

$$\begin{cases} u_t - ku_{xx} = 0, \\ u(x,0) = 0, \\ u(0,t) = f(t) \qquad u \to 0 \text{ as } x \to \infty. \end{cases}$$

If we take the Laplace transform throughout, we are left with

$$\begin{cases} U_{xx} - \frac{s}{k}U = 0, \\ U(0,s) = F(s). \end{cases}$$

Impose the boundary conditions (same as the previous problem):

$$U(x,s) = F(s)e^{-x\sqrt{s/k}}.$$

This time, we use property (10) in Strauss. The algebra becomes messy but we can manipulate the expressions carefully to obtain a familiar result. The solution is

$$u(x,t) = \frac{x}{2\sqrt{k\pi}} \int_0^t \frac{e^{-x^2/(4k(t-t'))}}{(t-t')^{3/2}} f(t')dt'.$$

If we were to take $f \equiv 1$, after a change of variables, the solution would become

$$u(x,t) = 1 - \mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right).$$

13.5 Example

Exercise 12.5-3 is worked out here as an example. We are asked to find $f(t)$ if its Laplace transform is given as

$$F(s) = \frac{1}{s(s^2+1)}.$$

Use partial fraction decomposition:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \implies 1 = A(s^2+1) + (Bs+C)s$$

$$\implies 1 = As^2 + A + Bs^2 + Cs \implies 1 = (A+B)s^2 + Cs + A$$

$$\implies A+B=0, \qquad C=0, \qquad A=1 \implies B=-1.$$

So the Laplace transform of the desired function $f(t)$ is

$$F(s) = \frac{1}{s} + \frac{-s}{s^2+1}.$$

The desired function $f(t)$ is given as the inverse Laplace transform of $F(s)$ so we have

$$f(t) = F^{-1}(F(s)) = F^{-1}\left(\frac{1}{s} + \frac{-s}{s^2+1}\right)$$

$$= F^{-1}\left(\frac{1}{s}\right) - F^{-1}\left(\frac{s}{s^2+1}\right) \qquad \text{(the inverse Laplace transform can be shown to be linear).}$$

Firstly, use property (2) in Strauss. Here, we have $a = 0$.

$$\text{Function: } e^{at}. \qquad \text{Transform: } \frac{1}{s-a}.$$

$$\text{Function: } e^{0}. \qquad \text{Transform: } \frac{1}{s}.$$

$$F^{-1}\left(\frac{1}{s}\right) = 1.$$

Secondly, use property (3). We have $\omega = 1$.

$$\text{Function: } \cos(\omega t). \qquad \text{Transform: } \frac{s}{s^2+\omega^2}.$$

$$\text{Function: } \cos(t). \qquad \text{Transform: } \frac{s}{s^2+1^2}.$$

$$F^{-1}\left(\frac{s}{s^2+1}\right) = \cos(t).$$

Therefore, the desired function $f(t)$ is

$$f(t) = F^{-1}\left(\frac{1}{s}\right) - F^{-1}\left(\frac{s}{s^2+1}\right) = 1 - \cos(t).$$

13.6 Exercise

Exercise 12.5-6: use the Laplace transform to solve

$$u_{tt} = c^2 u_{xx} + \cos(\omega t)\sin(\pi x),$$

$$u(0,t) = u(1,t) = u(x,0) = u_t(x,0) = 0.$$

Take the Laplace transform of both sides:

$$s^2 U(x,s) - su(x,0) - u_t(x,0) = c^2 U_{xx} + \sin(\pi x)\left[\frac{s}{s^2+\omega^2}\right],$$

$$s^2 U(x,s) = c^2 U_{xx} + \sin(\pi x)\left[\frac{s}{s^2+\omega^2}\right],$$

$$U_{xx} - \frac{s^2}{c^2}U(x,s) = -\frac{\sin(\pi x)}{c^2}\left[\frac{s}{s^2+\omega^2}\right],$$

$$U(x,s) = U_h + U_p.$$

Find the homogeneous solution U_h:

$$U_{xx} - \frac{s^2}{c^2}U(x,s) = 0$$

,

$$U_h = A\cosh\left(\frac{s}{c}x\right) + B\sinh\left(\frac{s}{c}x\right).$$

Impose the boundary conditions after taking the Laplace transform of both sides:

$$U(0,s) = 0 \implies A\cosh(0) + B\sinh(0) = 0 \implies A = 0,$$

$$U(1,s) = 0 \implies B\sinh\left(\frac{s}{c}\right) = 0 \implies B = 0,$$

$$U_h = 0.$$

To find U_p, take the educated guess $U_p = C\cos(\pi x) + D\sin(\pi x)$:

$$-C\pi^2\cos(\pi x) + -D\pi^2\sin(\pi x) - \frac{s^2}{c^2}\Big[C\cos(\pi x) + D\sin(\pi x)\Big] = -\frac{\sin(\pi x)}{c^2}\left[\frac{s}{s^2+\omega^2}\right],$$

$$C\bigg(-\pi^2 - \frac{s^2}{c^2}\bigg)\cos(\pi x) + D\bigg(-\pi^2 - \frac{s^2}{c^2}\bigg)\sin(\pi x) = -\frac{1}{c^2}\left[\frac{s}{s^2+\omega^2}\right]\sin(\pi x)$$

$$\implies C = 0 \text{ and } D\bigg(-\pi^2 - \frac{s^2}{c^2}\bigg) = -\frac{1}{c^2}\left[\frac{s}{s^2+\omega^2}\right]$$

$$\implies D = -\frac{1}{c^2}\left[\frac{s}{s^2+\omega^2}\right]\left(\frac{1}{-\pi^2-\frac{s^2}{c^2}}\right) = -\frac{1}{c^2}\left[\frac{s}{s^2+\omega^2}\right]\left(\frac{1}{\frac{-\pi^2c^2-s^2}{c^2}}\right)$$
$$= -\frac{sc^2}{c^2(s^2+\omega^2)(-\pi^2c^2-s^2)} = \frac{s}{(s^2+\omega^2)(\pi^2c^2+s^2)}.$$

The particular solution U_p is

$$U_p = \frac{s}{(s^2+\omega^2)(\pi^2c^2+s^2)}\sin(\pi x)$$

so the Laplace transform of the desired solution $u(x,t)$ is

$$U = U_h + U_p = 0 + \frac{s}{(s^2+\omega^2)(\pi^2c^2+s^2)}\sin(\pi x) = \frac{s\sin(\pi x)}{(s^2+\omega^2)(\pi^2c^2+s^2)}.$$

Note: the boundary conditions actually should have been imposed at this stage. Coincidentally, however, the same conclusion would be reached if we were to repeat the process: $A = B = 0$. So we can proceed with the retrieval of the desired solution $u(x,t)$. Use (3) and (4) from the table in Strauss (pg 353):

$$f(t) = \cos(\omega t) \iff F(s) = \frac{s}{s^2+\omega^2} \qquad (3),$$
$$f(t) = \sin(\omega t) \iff F(s) = \frac{\omega}{s^2+\omega^2} \qquad (4).$$

Disregard $\sin(\pi x)$ temporarily. Use partial fraction decomposition on the remaining expression:

$$\frac{s}{(s^2+\omega^2)(\pi^2c^2+s^2)} = \frac{As+B}{s^2+\omega^2} + \frac{Cs+D}{\pi^2c^2+s^2},$$

$$s = (As+B)(\pi^2c^2+s^2) + (Cs+D)(s^2+\omega^2),$$

$$s = A\pi^2c^2s + As^3 + B\pi^2c^2 + Bs^2 + Cs^3 + C\omega^2s + Ds^2 + D\omega^2,$$

$$s = (A+C)s^3 + (B+D)s^2 + (A\pi^2c^2 + C\omega^2)s + (B\pi^2c^2 + D\omega^2)$$

$$\implies A+C=0, \qquad B+D=0, \qquad A\pi^2c^2 + C\omega^2 = 1, \qquad B\pi^2c^2 + D\omega^2 = 0$$

$$\implies C=-A, \qquad D=-B, \qquad A\pi^2c^2 - A\omega^2 = 1, \qquad B\pi^2c^2 - B\omega^2 = 0$$

$$\implies A(\pi^2c^2-\omega^2) = 1, \qquad B(\pi^2c^2-\omega^2) = 0$$

$$\implies A = \frac{1}{\pi^2c^2-\omega^2}, \qquad B = D = 0 \ \text{(if } \omega \neq \pi c).$$

Therefore, we have

$$u(x,t) = \left[\frac{\left(\frac{1}{\pi^2c^2-\omega^2}\right) t}{t^2+\omega^2} + \frac{\left(\frac{1}{\omega^2-\pi^2c^2}\right) t}{\pi^2c^2+t^2} \right] \sin(\pi x)$$

$$= \left[\frac{t}{(\pi^2c^2-\omega^2)(t^2+\omega^2)} + \frac{t}{(\omega^2-\pi^2c^2)(\pi^2c^2+t^2)} \right] \sin(\pi x).$$

If $\omega = \pi c$, we should use (ix) from the other table in Strauss (pg 354) to obtain

$$U = \frac{s\sin(\pi x)}{(s^2+\omega^2)^2} = \sin(\pi x)\left[\frac{s}{s^2+\omega^2}\,\frac{1}{s^2+\omega^2}\right]$$

$$= \frac{\sin(\pi x)}{\omega}\left[\frac{s}{s^2+\omega^2}\,\frac{\omega}{s^2+\omega^2}\right] = \frac{\sin(\pi x)}{\omega}\Big[F(s)G(s)\Big]$$

where $F(s)$ and $G(s)$ are the Laplace transforms of $\cos(\omega t)$ and $\sin(\omega t)$, respectively, by properties (3) and (4) from the previous table. Therefore, the solution is

$$u(x,t) = \frac{\sin(\pi x)}{\omega}\int_0^t \sin(\omega(t-t'))\cos(\omega t')dt'.$$

References

[Str07] Walter A. Strauss. *Partial Differential Equations: An Introduction*. 2nd ed. John Wiley, 2007.