Review for Final Exam

MATH 315 Differential Equations
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Definitions

Refer to the textbook, as well as your notes taken during class, to find the definitions of the following words/phrases.

- Ordinary differential equation (ODE)
- Mathematical modeling
- Variables and parameters
- Initial condition (IC)
- Initial value problem (IVP)
- Malthusian law for population growth
- Separable equation
- Explicit/implicit solutions
- Linear/nonlinear differential equations
- Autonomous equation
- Equilibrium solution
- Phase line
- Sinks, sources, and nodes
- Stable/unstable equilibria
- Logistic model
- Linear first-order ODE
- Integrating factor
- Compartmental analysis
- Homogeneous/nonhomogeneous ODE
- Linear second-order constant-coefficient homogeneous ODE
- Characteristic equation (AKA auxiliary equation)
- Linear independence/dependence of functions
- Linear combination of functions
- Wronskian
- Homogeneous/particular solutions
- Method of undetermined coefficients
- Superposition principle
- Inertial mass, damping coefficient, and spring stiffness (mass-spring oscillator)
- External force (mass-spring oscillator)
- Laplace transform
- Inverse Laplace transform
- Linear system of ODEs
- Eigenvalues and eigenvectors

Practice Problems

1. Determine whether the given function is a solution to the given differential equation.

$$y = 4e^{-2x} + 2e^{4x}$$

$$2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 16y = 0$$

2. Let $P(t)$ represent the population of deer, P, at time t. In 2005, the deer population in a particular region was estimated to be 112,000. That number has grown to an estimated 200,000 this year

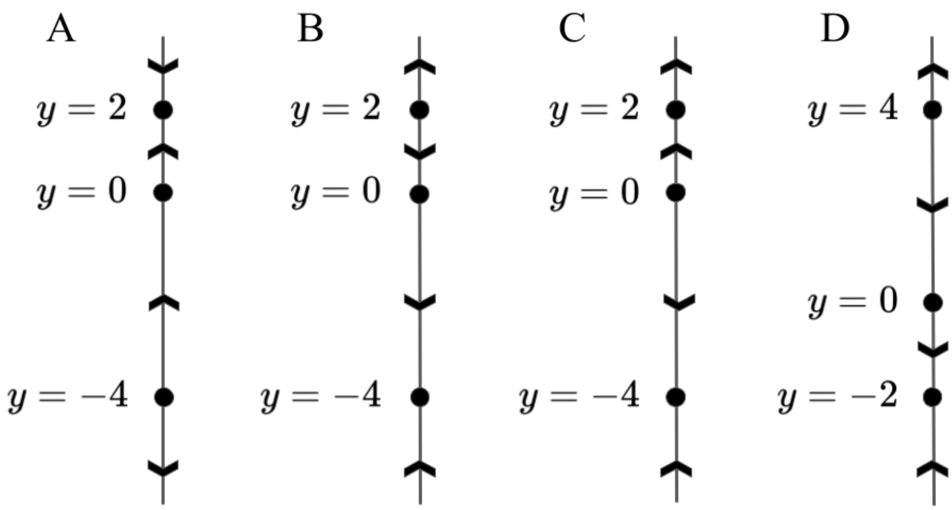
(2023). Using the Malthusian law for population growth, estimate the deer population in the region in the year 2024.

3. Solve the separable equation. The general solution is explicit, so it can be written in the form $y = f(x)$. Include $C \in \mathbb{R}$ as an arbitrary constant.

$$\frac{dy}{dx} = 2xy$$

4. Choose the correct phase line for the autonomous differential equation.

$$\frac{dy}{dt} = y^4(y+4)(y-2)$$



5. What are the equilibrium solutions to the differential equation in problem 4? Classify the equilibria as sinks, sources, or nodes. Also, determine the stability of the equilibria.

6. The logistic model is typically presented as an initial value problem (IVP). Let $P(t)$ represent a specific population, P, at time t. The parameter values for that population are given as $r = 3$ (growth rate) and $K = 22$ (carrying capacity) with the initial condition $P_0 = 5$ (initial population). How would we present the logistic model with this given information?

7. Solve the IVP.

$$\begin{cases} \frac{dy}{dx} + 2y = 50e^{-10x} \\ y(0) = 40 \end{cases}$$

8. A brine solution of salt flows at a constant rate of 6 L/min into a large tank that initially held 50 L of brine solution in which was dissolved 0.5 kg of salt. The solution inside the tank is kept well-stirred and flows out of the tank at the same rate. The concentration of salt in the brine entering the tank is 0.05 kg/L. We can use a differential equation to describe how x [kg], the mass of salt in the tank, changes over time t [min]. What is the differential equation?

9. Find a general solution to the equation. Note: for problems 9 - 11, the dependent variable y is a function of the independent variable t.

$$y'' + 4y' - 21y = 0$$

10. Solve the IVP.

$$\begin{cases} y'' - 16y' + 64y = 0 \\ y(0) = 2 \\ y'(0) = 20 \end{cases}$$

11. Find a general solution to the equation.

$$y'' - 8y' + 41y = 0$$

12. Determine the Wronskian of the given functions.

 (a) $f_1(x) = \sin(5x),\ f_2(x) = 3\sin(5x)$
 (b) $f_1(t) = 2t - 4,\ f_2(t) = 3t^2$

13. Use the answers from problem 12 to determine whether the following statements are true or false.

 (a) The functions $f_1(x), f_2(x)$ are linearly independent.
 (b) The functions $f_1(t), f_2(t)$ are linearly independent.

14. Use the method of undetermined coefficients to find a particular solution (denoted as y_p) to the equation.

$$y'' - 2y' - 24y = -48e^{2t}$$

15. Use the superposition principle to find a general solution to the equation in problem 14.

16. The differential equation

$$2y'' + 3y' + y = \cos(2t)$$

 models the motion of a mass-spring oscillator. Determine the value for the damping coefficient b. What is the expression for the external force $F_{\text{ext}}(t)$?

17. Find the Laplace transform of each function.

 (a) $f(t) = t^5$
 (b) $f(t) = 4\sin(3t)$
 (c) $f(t) = 2e^{3t}\cos(4t) - 7$

18. Find the inverse Laplace transform of the given function.

$$F(s) = \frac{3s}{s^2+4} + \frac{10}{s}$$

19. Find a general solution to the linear system of ODEs.

$$\begin{aligned} x' &= 5x + 6y \\ y' &= x + 4y \end{aligned}$$

 Use the notation $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ where both x and y are functions of t.

20. Solve the IVP given by the matrix equation

$$\vec{x}\,' = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \vec{x}$$

 and the initial condition $\vec{x}(0) = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$.

Excluded Topics

The final exam will not contain problems related to the following topics.

- Existence and uniqueness of solutions (Picard-Lindelöf theorem)
- Bernoulli equations
- Exact equations
- Fundamental solution sets
- Resonance
- Defective matrices
- Complex eigenvalues
- Forward Euler

Solutions to Practice Problems

1. Yes
2. $P \approx 206,547$
3. $y = Ce^{x^2}$
4. B
5. $y = 0$ (node, unstable), $y = -4$ (sink, stable), $y = 2$ (source, unstable)
6. $\begin{cases} \frac{dP}{dt} = 3P\left(1 - \frac{P}{22}\right) \\ P(0) = 5 \end{cases}$
7. $y = \frac{185}{4}e^{-2x} - \frac{25}{4}e^{-10x}$
8. $\frac{dx}{dt} = \frac{3}{10} - \frac{3}{25}x$
9. $y = C_1e^{-7t} + C_2e^{3t}$
10. $y = 2e^{8t} + 4te^{8t}$
11. $y = e^{4t}\big[C_1\cos(5t) + C_2\sin(5t)\big]$
12. (a) $W\big[f_1, f_2\big](x) = 0$
    (b) $W\big[f_1, f_2\big](t) = 6t^2 - 24t$
13. (a) False
    (b) True
14. $y_p = 2e^{2t}$
15. $y = 2e^{2t} + C_1e^{-4t} + C_2e^{6t}$
16. $b = 3,\ F_{\text{ext}}(t) = \cos(2t)$

17. (a) $F(s) = 120/s^6$
    (b) $F(s) = 12/(s^2 + 9)$
    (c) $F(s) = 2(s-3)/\big[(s-3)^2 + 16\big] - 7/s$

18. $f(t) = 3\cos(2t) + 10$

19. $\vec{x} = C_1 e^{7t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

20. $\vec{x} = -5e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 4e^{12t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$