General Solutions to Linear Second-Order Constant-Coefficient Homogeneous ODEs: Derivations and Proofs

MATH 315 Differential Equations
Instructor: Luke Vaughan
Washington State University Vancouver, Spring 2023

Setup

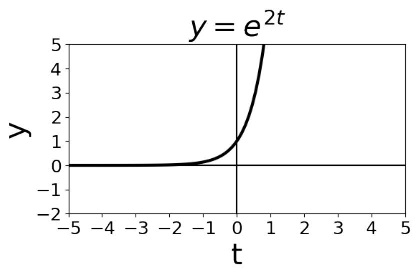
Find a function $y(t)$ that satisfies the ODE

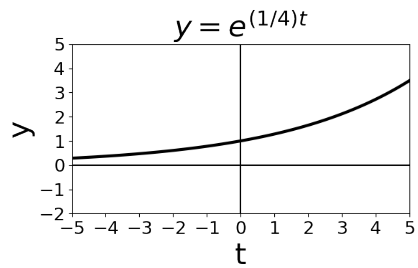
$$ay'' + by' + cy = 0 \qquad (1)$$

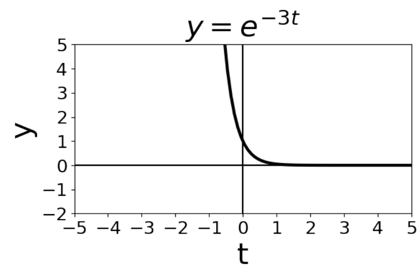
where $a, b, c \in \mathbb{R}$. Try/guess a solution of the form $y = e^{rt}$ where $r \in \mathbb{R}$. By the chain rule, we have $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Plug y and its derivatives into Eq. (1):

$$\begin{aligned} a\left(r^2e^{rt}\right) + b\left(re^{rt}\right) + c\left(e^{rt}\right) &= 0 \\ ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ \left(ar^2 + br + c\right)e^{rt} &= 0 \end{aligned}$$

We can divide both sides by e^{rt} without worrying about division by zero. The function $y = e^{rt}$ is never zero. It never touches the t-axis (for any r). See plots below showing various choices for values of r (from left to right: $r = 2$, $r = 1/4$, and $r = -3$).







Divide both sides by e^{rt}:

$$\begin{aligned} \frac{\left(ar^2 + br + c\right)e^{rt}}{e^{rt}} &= \frac{0}{e^{rt}} \\ ar^2 + br + c &= 0 \qquad (2) \end{aligned}$$

- Definition 1: Eq. (2) is the characteristic equation (AKA auxiliary equation) of Eq. (1).
- Theorem 1: r satisfies Eq. (2) $\iff$ $y = e^{rt}$ is a solution to Eq. (1).

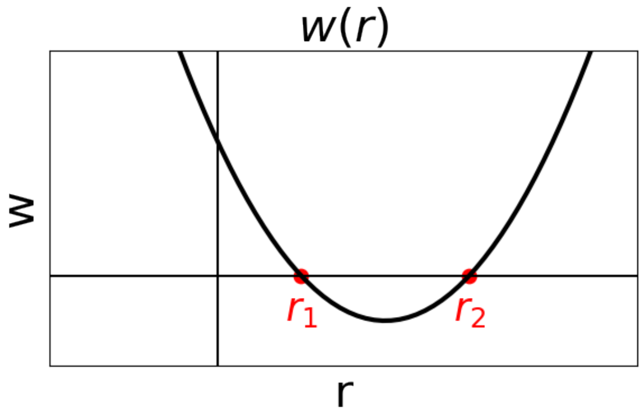
Denote the left-hand side of Eq. (2) as $w(r)$. This notation gives us the equation $w(r) = 0$. So we need to find the roots of $w(r)$, which are given by the quadratic formula:

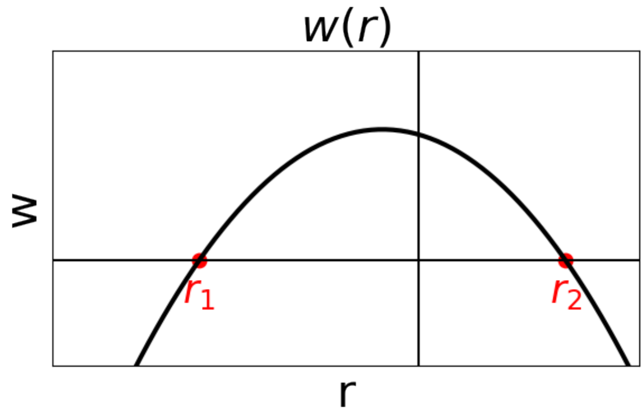
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We could have two distinct real roots (when $b^2 - 4ac > 0$), one repeated root (when $b^2 - 4ac = 0$), or complex roots (when $b^2 - 4ac < 0$). So there are three cases to acknowledge, each determined by the value of the discriminant $b^2 - 4ac$.

Case 1: Two Distinct Real Roots (Discriminant > 0)

Here are two examples that show what the parabola $w(r)$ could look like.





Since $w(r)$ has two distinct roots, $r = r_1$ and $r = r_2$, we have two values for r satisfying Eq. (2). So by Theorem 1, we have two solutions to Eq. (1), given as $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$. These two solutions are linearly independent since they are not scalar multiples of one another.

<u>Claim 1:</u> The linear combination $y = C_1 y_1 + C_2 y_2$, where $C_1, C_2 \in \mathbb{R}$, is also a solution to Eq. (1).

<u>Proof 1:</u> Plug $y = C_1 y_1 + C_2 y_2$ into Eq. (1) and check that the left-hand side matches the right-hand side:

$$a\left(C_1 y_1 + C_2 y_2\right)'' + b\left(C_1 y_1 + C_2 y_2\right)' + c\left(C_1 y_1 + C_2 y_2\right) \stackrel{?}{=} 0$$

The derivative is a linear operator, so we can bring the primes inside the parentheses. For example: $\left[3x^2 + 2x\right]' = \left[3(x^2)' + 2(x)'\right] = \left[3 \cdot 2x + 2 \cdot 1\right] = 6x + 2$.

$$\begin{aligned}
a\left(C_1 y_1'' + C_2 y_2''\right) + b\left(C_1 y_1' + C_2 y_2'\right) + c\left(C_1 y_1 + C_2 y_2\right) &\stackrel{?}{=} 0 \\
aC_1 y_1'' + aC_2 y_2'' + bC_1 y_1' + bC_2 y_2' + cC_1 y_1 + cC_2 y_2 &\stackrel{?}{=} 0 \\
aC_1 y_1'' + bC_1 y_1' + cC_1 y_1 + aC_2 y_2'' + bC_2 y_2' + cC_2 y_2 &\stackrel{?}{=} 0 \\
C_1\left(ay_1'' + by_1' + cy_1\right) + C_2\left(ay_2'' + by_2' + cy_2\right) &\stackrel{?}{=} 0
\end{aligned}$$

The functions y_1 and y_2 are solutions to Eq. (1), so the expressions in parentheses are equal to zero:

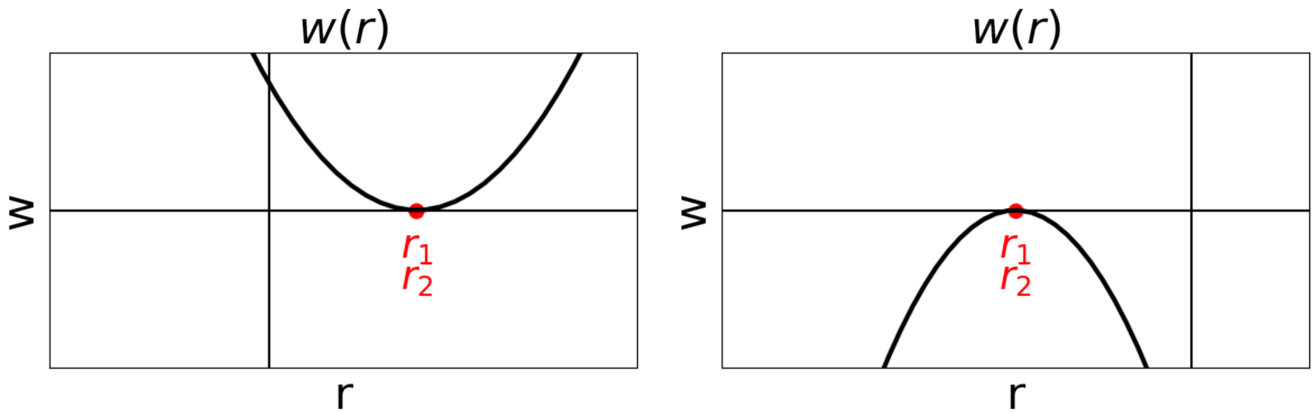
$$\begin{aligned}
C_1(0) + C_2(0) &\stackrel{?}{=} 0 \\
0 + 0 &\stackrel{?}{=} 0 \\
0 &\stackrel{\checkmark}{=} 0
\end{aligned}$$

Therefore, $y = C_1 y_1 + C_2 y_2$ is a solution to Eq. (1). Furthermore, we can call it a general solution. $\square$

<u>Case 1:</u> If Eq. (2) has two distinct real roots $r = r_1$ and $r = r_2$, a general solution to Eq. (1) is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Case 2: One Repeated Root (Discriminant = 0)



Since $w(r)$ has one repeated root, $r = r_1 = r_2$, we have one value for r satisfying Eq. (2). So by Theorem 1, we have one solution to Eq. (1), given as $y_1 = e^{rt}$.

Claim 2: The function $y_2 = te^{rt}$ is also a solution to Eq. (1).

Proof 2: To check that the function is a solution, we need to plug y_2 and its derivatives y_2', y_2'' into Eq. (1). However, y_2 is equal to the product of two expressions in t, so finding the derivatives is a nontrivial process. Recall the product rule: $z = a(t)b(t) \implies z' = a'(t)b(t) + b'(t)a(t)$. Use the product rule a few times to find y_2', y_2'':

$$\begin{aligned} y_2 &= te^{rt} \\ y_2' &= (1)(e^{rt}) + (re^{rt})(t) \\ &= e^{rt} + rte^{rt} \\ y_2'' &= re^{rt} + (r)(e^{rt}) + (re^{rt})(rt) \\ &= re^{rt} + re^{rt} + r^2te^{rt} \\ &= 2re^{rt} + r^2te^{rt} \end{aligned}$$

Plug y_2 and its derivatives into Eq. (1) and check that the left-hand side matches the right-hand side:

$$\begin{aligned} a\left(2re^{rt} + r^2te^{rt}\right) + b\left(e^{rt} + rte^{rt}\right) + c\left(te^{rt}\right) &\stackrel{?}{=} 0 \\ 2are^{rt} + ar^2te^{rt} + be^{rt} + brte^{rt} + cte^{rt} &\stackrel{?}{=} 0 \\ \left(2ar + ar^2t + b + brt + ct\right)e^{rt} &\stackrel{?}{=} 0 \end{aligned}$$

Divide both sides by e^{rt} to get

$$\begin{aligned} 2ar + ar^2t + b + brt + ct &\stackrel{?}{=} 0 \\ 2ar + b + \left(ar^2 + br + c\right)t &\stackrel{?}{=} 0 \end{aligned}$$

Note: r satisfies Eq. (2), so the expression in parentheses is equal to zero.

$$2ar + b + (0)\,t \stackrel{?}{=} 0$$

$$2ar + b \stackrel{?}{=} 0$$

The discriminant $b^2 - 4ac$ is equal to zero in Case 2.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b}{2a}$$

$$2a\left(\frac{-b}{2a}\right) + b \stackrel{?}{=} 0$$

$$-b + b \stackrel{?}{=} 0$$

$$0 \stackrel{\checkmark}{=} 0$$

Therefore, $y_2 = te^{rt}$ is a solution to Eq. (1). □

The two solutions $y_1 = e^{rt}$ and $y_2 = te^{rt}$ are linearly independent.

Claim 3: The linear combination $y = C_1y_1 + C_2y_2$ is also a solution to Eq. (1).

Proof 3: Identical to Proof 1. Plug $y = C_1y_1 + C_2y_2$ into Eq. (1) and check that the left-hand side matches the right-hand side:

$$a\left(C_1y_1 + C_2y_2\right)'' + b\left(C_1y_1 + C_2y_2\right)' + c\left(C_1y_1 + C_2y_2\right) \stackrel{?}{=} 0$$

$$\vdots$$

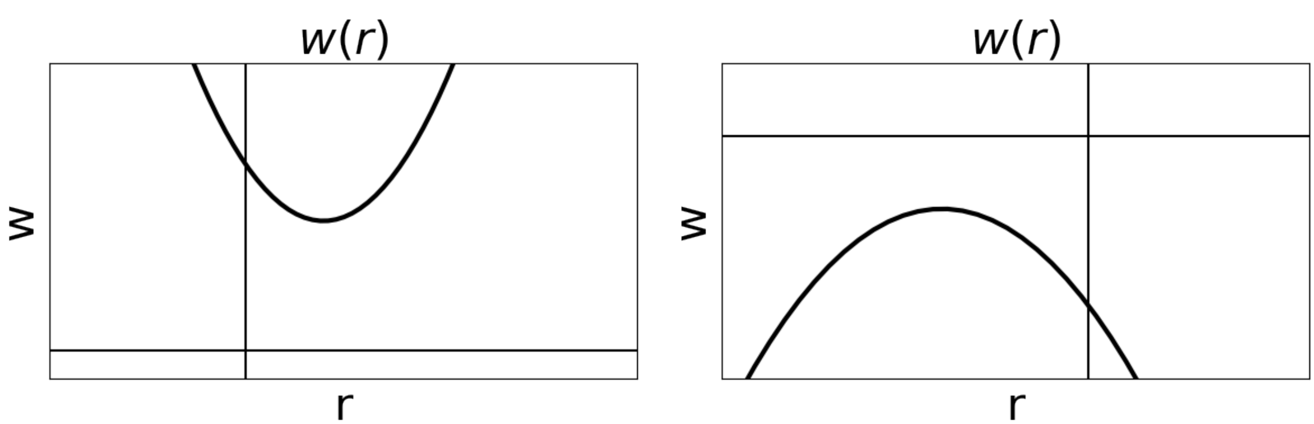
$$0 \stackrel{\checkmark}{=} 0$$

Therefore, $y = C_1y_1 + C_2y_2$ is a solution to Eq. (1). Furthermore, we can call it a general solution. □

Case 2: If Eq. (2) has one repeated root $r = r_1 = r_2$, a general solution to Eq. (1) is

$$y = C_1e^{rt} + C_2te^{rt}$$

Case 3: Complex Roots (Discriminant < 0)



Since $w(r)$ does not cross the r-axis, we have roots occurring in complex conjugate pairs: $r = \alpha \pm \beta i$ (by the complex conjugate root theorem). Recall Theorem 1: r satisfies Eq. (2) $\iff$ $y = e^{rt}$ is a solution to

Eq. (1). This theorem is true not only for $r \in \mathbb{R}$, but for $r \in \mathbb{C}$ as well. So Theorem 1 is true for all 3 cases. Therefore, we have two solutions to Eq. (1), given as $y_1 = e^{(\alpha+\beta i)t}$ and $y_2 = e^{(\alpha-\beta i)t}$. However, we are not satisfied with these solutions since they involve i. So we will use y_1 to derive two real-valued solutions. We can ignore y_2 for the sake of brevity, but using y_2 to derive the real-valued solutions is valid as an alternative approach (it leads to the same answers).

We have $y_1 = e^{(\alpha+\beta i)t} = e^{\alpha t+\beta it} = e^{\alpha t}e^{\beta it} = e^{\alpha t}e^{i(\beta t)}$. We can use the following theorem to deal with the complex exponential.

- Theorem 2 (Euler's Formula): $e^{i\theta} = \cos\theta + i\sin\theta$

Use Theorem 2 to rewrite the solution y_1 as

$$\begin{aligned} y_1 &= e^{\alpha t}e^{i(\beta t)} \\ &= e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t)\right] \\ &= e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t) \end{aligned}$$

The solution y_1 is complex with the following real and imaginary parts:

$$\mathrm{Re}(y_1) = e^{\alpha t}\cos(\beta t) \qquad \mathrm{Im}(y_1) = e^{\alpha t}\sin(\beta t)$$

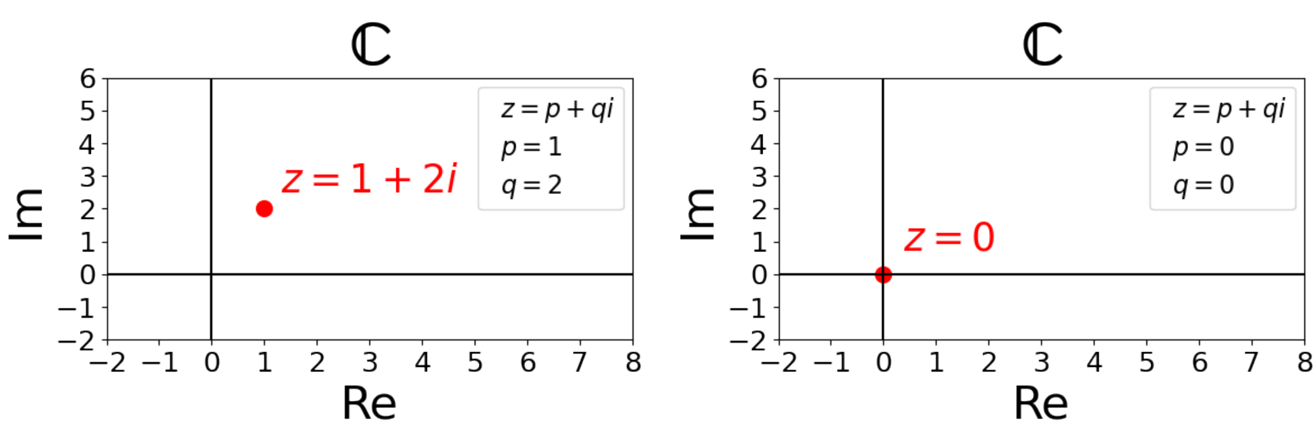
Claim 4: Assume that $z = u(t) + iv(t)$ is a solution to Eq. (1). Then the real part $u(t)$ and the imaginary part $v(t)$ are also solutions to Eq. (1).

Proof 4: In this claim, we are assuming that if we plug z into Eq. (1), we get zero.

$$\begin{aligned} az'' + bz' + cz &= 0 \\ a(u+iv)'' + b(u+iv)' + c(u+iv) &= 0 \\ a(u''+iv'') + b(u'+iv') + c(u+iv) &= 0 \\ au'' + iav'' + bu' + ibv' + cu + icv &= 0 \\ au'' + bu' + cu + i(av'' + bv' + cv) &= 0 \end{aligned}$$

The expression on the left-hand side is complex with real part $au''+bu'+cu$ and imaginary part $av''+bv'+cv$.

- Theorem 3: If a complex number $z = p + qi$ is set equal to zero, the real part $\mathrm{Re}(z) = p$ and the imaginary part $\mathrm{Im}(z) = q$ must both be set equal to zero as well. So we have $p = 0$ and $q = 0$.



By Theorem 3, the real part $au'' + bu' + cu$ and the imaginary part $av'' + bv' + cv$ must both be set equal to zero.

$$\begin{aligned} au'' + bu' + cu &= 0 \\ av'' + bv' + cv &= 0 \end{aligned}$$

Therefore, u and v are both solutions to Eq. (1). $\square$

The complex solution to Eq. (1) is given as $y_1 = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t)$. Use the notation $y_1 = u(t) + iv(t)$ where $u(t) = e^{\alpha t}\cos(\beta t)$ and $v(t) = e^{\alpha t}\sin(\beta t)$. Since Claim 4 has been proven, we can say that u and v are both solutions to Eq. (1). So u and v are the two real-valued solutions we have been looking for. Note that these two functions are linearly independent.

<u>Claim 5:</u> The linear combination $y = C_1 u + C_2 v$ is also a solution to Eq. (1).

<u>Proof 5:</u> Same process shown in Proof 1.

$$\begin{aligned} a\left(C_1 u + C_2 v\right)'' + b\left(C_1 u + C_2 v\right)' + c\left(C_1 u + C_2 v\right) &\stackrel{?}{=} 0 \\ &\vdots \\ 0 &\stackrel{\checkmark}{=} 0 \end{aligned}$$

Therefore, $y = C_1 u + C_2 v$ is a solution to Eq. (1). Furthermore, we can call it a general solution. $\square$

$$\begin{aligned} y &= C_1 u + C_2 v \\ &= C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t) \\ &= e^{\alpha t}\big[C_1\cos(\beta t) + C_2\sin(\beta t)\big] \end{aligned}$$

<u>Case 3:</u> If Eq. (2) has complex roots $r = \alpha \pm \beta i$, a general solution to Eq. (1) is

$$y = e^{\alpha t}\big[C_1\cos(\beta t) + C_2\sin(\beta t)\big]$$