CHAPTER 4 REVIEW

Key Terms

annual percentage rate (APR) the yearly interest rate earned by an investment account, also called *nominal rate*

carrying capacity in a logistic model, the limiting value of the output

change-of-base formula a formula for converting a logarithm with any base to a quotient of logarithms with any other base.

common logarithm the exponent to which 10 must be raised to get x; $\log_{10}(x)$ is written simply as $\log(x)$.

compound interest interest earned on the total balance, not just the principal

doubling time the time it takes for a quantity to double

exponential growth a model that grows by a rate proportional to the amount present

extraneous solution a solution introduced while solving an equation that does not satisfy the conditions of the original equation

half-life the length of time it takes for a substance to exponentially decay to half of its original quantity

logarithm the exponent to which b must be raised to get x; written $y = \log_b(x)$

logistic growth model a function of the form $f(x) = \frac{c}{1 + ae^{-bx}}$ where $\frac{c}{1 + a}$ is the initial value, c is the carrying capacity, or limiting value, and b is a constant determined by the rate of growth

natural logarithm the exponent to which the number e must be raised to get x; $\log_e(x)$ is written as $\ln(x)$.

Newton's Law of Cooling the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

nominal rate the yearly interest rate earned by an investment account, also called *annual percentage rate*

order of magnitude the power of ten, when a number is expressed in scientific notation, with one non-zero digit to the left of the decimal

power rule for logarithms a rule of logarithms that states that the log of a power is equal to the product of the exponent and the log of its base

product rule for logarithms a rule of logarithms that states that the log of a product is equal to a sum of logarithms

quotient rule for logarithms a rule of logarithms that states that the log of a quotient is equal to a difference of logarithms

Key Equations

definition of the exponential function	$f(x) = b^x$, where $b > 0$, $b \neq 1$
definition of exponential growth	$f(x) = ab^x$, where $a > 0$, $b > 0$, $b \neq 1$
compound interest formula	$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, where $A(t)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A(t) = ae^{rt}$, where t is the number of unit time periods of growth a is the starting amount (in the continuous compounding formula a is replaced with P, the principal) e is the mathematical constant, $e \approx 2.718282$
General Form for the Translation of the Parent Function $f(x) = b^x$	$f(x) = ab^{x+c} + d$
Definition of the logarithmic function	For $x > 0$, $b > 0$, $b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$.
Definition of the common logarithm	For $x > 0$, $y = \log(x)$ if and only if $10^y = x$.

Definition of the natural logarithm	For $x > 0$, $y = \ln(x)$ if and only if $e^y = x$.
General Form for the Translation of the Parent Logarithmic Function $f(x) = \log_b(x)$	$f(x) = a\log_b(x + c) + d$
The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b} \quad n > 0, n \neq 1, b \neq 1$
One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where $b^S = b^T$ if and only if $S = T$.
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$.
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$.
Half-life formula	If $A = A_0 e^{kt}$, $k < 0$, the half-life is $t = -\frac{\ln(2)}{k}$. $t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$
Carbon-14 dating	A_0 is the amount of carbon-14 when the plant or animal died, A is the amount of carbon-14 remaining today, t is the age of the fossil in years
Doubling time formula	If $A = A_0 e^{kt}$, $k > 0$, the doubling time is $t = \frac{\ln(2)}{k}$
Newton's Law of Cooling	$T(t) = Ae^{kt} + T_s$, where T_s is the ambient temperature, $A = T(0) - T_s$, and k is the continuous rate of cooling.

Key Concepts

4.1 Exponential Functions

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See **Example 1**.
- A function is evaluated by solving at a specific value. See **Example 2** and **Example 3**.
- An exponential model can be found when the growth rate and initial value are known. See **Example 4**.
- An exponential model can be found when the two data points from the model are known. See **Example 5**.
- An exponential model can be found using two data points from the graph of the model. See **Example 6**.
- An exponential model can be found using two data points from the graph and a calculator. See **Example 7**.
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See **Example 8**.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See **Example 9**.
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key $[e^x]$ or $[\exp(x)]$ for calculating powers of e. See **Example 10**.
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See **Example 11** and **Example 12**.

4.2 Graphs of Exponential Functions

- The graph of the function $f(x) = b^x$ has a y-intercept at (0, 1), domain $(-\infty, \infty)$, range $(0, \infty)$, and horizontal asymptote $y = 0$. See **Example 1**.
- If $b > 1$, the function is increasing. The left tail of the graph will approach the asymptote $y = 0$, and the right tail will increase without bound.
- If $0 < b < 1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the parent function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the parent function $f(x) = b^x$. See **Example 2**.
- Approximate solutions of the equation $f(x) = b^{x+c} + d$ can be found using a graphing calculator. See **Example 3**.
- The equation $f(x) = ab^x$, where $a > 0$, represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the parent function $f(x) = b^x$. See **Example 4**.
- When the parent function $f(x) = b^x$ is multiplied by -1, the result, $f(x) = -b^x$, is a reflection about the x-axis. When the input is multiplied by -1, the result, $f(x) = b^{-x}$, is a reflection about the y-axis. See **Example 5**.
- All translations of the exponential function can be summarized by the general equation $f(x) = ab^{x+c} + d$. See **Table 3**.
- Using the general equation $f(x) = ab^{x+c} + d$, we can write the equation of a function given its description. See **Example 6**.

4.3 Logarithmic Functions

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See **Example 1**.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See **Example 2**.
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of b. See **Example 3** and **Example 4**.
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10. See **Example 5**.
- When common logarithms cannot be evaluated mentally, a calculator can be used. See **Example 6**.
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See **Example 7**.
- Natural logarithms can be evaluated using a calculator **Example 8**.

4.4 Graphs of Logarithmic Functions

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See **Example 1** and **Example 2**
- The graph of the parent function $f(x) = \log_b(x)$ has an x-intercept at (1, 0), domain $(0, \infty)$, range $(-\infty, \infty)$, vertical asymptote $x = 0$, and
  - if $b > 1$, the function is increasing.
  - if $0 < b < 1$, the function is decreasing.

  See **Example 3**.
- The equation $f(x) = \log_b(x + c)$ shifts the parent function $y = \log_b(x)$ horizontally
  - left c units if $c > 0$.
  - right c units if $c < 0$.

  See **Example 4**.
- The equation $f(x) = \log_b(x) + d$ shifts the parent function $y = \log_b(x)$ vertically
  - up d units if $d > 0$.
  - down d units if $d < 0$.

- For any constant $a > 0$, the equation $f(x) = a\log_b(x)$
  - stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| > 1$.
  - compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| < 1$.

  See **Example 6** and **Example 7**.
- When the parent function $y = \log_b(x)$ is multiplied by -1, the result is a reflection about the x-axis. When the input is multiplied by -1, the result is a reflection about the y-axis.
  - The equation $f(x) = -\log_b(x)$ represents a reflection of the parent function about the x-axis.
- The equation $f(x) = \log_b(-x)$ represents a reflection of the parent function about the y-axis.

  See **Example 8**.
  - A graphing calculator may be used to approximate solutions to some logarithmic equations See **Example 9**.
- All translations of the logarithmic function can be summarized by the general equation $f(x) = a\log_b(x + c) + d$. See **Table 4**.
- Given an equation with the general form $f(x) = a\log_b(x + c) + d$, we can identify the vertical asymptote $x = -c$ for the transformation. See **Example 10**.
- Using the general equation $f(x) = a\log_b(x + c) + d$, we can write the equation of a logarithmic function given its graph. See **Example 11**.

4.5 Logarithmic Properties

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See **Example 1**.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See **Example 2**.
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See **Example 3**, **Example 4**, and **Example 5**.
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See **Example 6**, **Example 7**, and **Example 8**.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See **Example 9**, **Example 10**, **Example 11**, and **Example 12**.
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See **Example 13**.
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and e as the quotient of natural or common logs. That way a calculator can be used to evaluate. See **Example 14**.

4.6 Exponential and Logarithmic Equations

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See **Example 1**.
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See **Example 2**, **Example 3**, and **Example 4**.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See **Example 5**.
- We can solve exponential equations with base e, by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See **Example 6** and **Example 7**.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See **Example 8**.

- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See **Example 9** and **Example 10**.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the x-value of the intersecting point. See **Example 11**.
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See **Example 12**.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See **Example 13**.

4.7 Exponential and Logarithmic Models

- The basic exponential function is $f(x) = ab^x$. If $b > 1$, we have exponential growth; if $0 < b < 1$, we have exponential decay.
- We can also write this formula in terms of continuous growth as $A = A_0e^{kx}$, where A_0 is the starting value. If A_0 is positive, then we have exponential growth when $k > 0$ and exponential decay when $k < 0$. See **Example 1**.
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay. See **Example 2**.
- We can find the age, t, of an organic artifact by measuring the amount, k, of carbon-14 remaining in the artifact and using the formula $t = \dfrac{\ln(k)}{-0.000121}$ to solve for t. See **Example 3**.
- Given a substance's doubling time or half-life we can find a function that represents its exponential growth or decay. See **Example 4**.
- We can use Newton's Law of Cooling to find how long it will take for a cooling object to reach a desired temperature, or to find what temperature an object will be after a given time. See **Example 5**.
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors. See **Example 6**.
- We can use real-world data gathered over time to observe trends. Knowledge of linear, exponential, logarithmic, and logistic graphs help us to develop models that best fit our data. See **Example 7**.
- Any exponential function with the form $y = ab^x$ can be rewritten as an equivalent exponential function with the form $y = A_0e^{kx}$ where $k = \ln b$. See **Example 8**.

4.8 Fitting Exponential Models to Data

- Exponential regression is used to model situations where growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero.
- We use the command "ExpReg" on a graphing utility to fit function of the form $y = ab^x$ to a set of data points. See **Example 1**.
- Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time.
- We use the command "LnReg" on a graphing utility to fit a function of the form $y = a + b\ln(x)$ to a set of data points. See **Example 2**.
- Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows as the function approaches an upper limit.
- We use the command "Logistic" on a graphing utility to fit a function of the form $y = \dfrac{c}{1 + ae^{-bx}}$ to a set of data points. See **Example 3**.