In this section, you will:

- Find the derivative of a function.
- Find instantaneous rates of change.
- Find an equation of the tangent line to the graph of a function at a point.
- Find the instantaneous velocity of a particle.

12.4 DERIVATIVES

The average teen in the United States opens a refrigerator door an estimated 25 times per day. Supposedly, this average is up from 10 years ago when the average teenager opened a refrigerator door 20 times per day.[37]

It is estimated that a television is on in a home 6.75 hours per day, whereas parents spend an estimated 5.5 minutes per day having a meaningful conversation with their children. These averages, too, are not the same as they were 10 years ago, when the television was on an estimated 6 hours per day in the typical household, and parents spent 12 minutes per day in meaningful conversation with their kids.

What do these scenarios have in common? The functions representing them have changed over time. In this section, we will consider methods of computing such changes over time.

Finding the Average Rate of Change of a Function

The functions describing the examples above involve a change over time. Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

A **tangent line** to a curve is a line that intersects the curve at only a single point but does not cross it there. (The tangent line may intersect the curve at another point away from the point of interest.) If we zoom in on a curve at that point, the curve appears linear, and the slope of the curve at that point is close to the slope of the tangent line at that point.

Figure 1 represents the function $f(x) = x^3 - 4x$. We can see the slope at various points along the curve.

- slope at $x = -2$ is 8
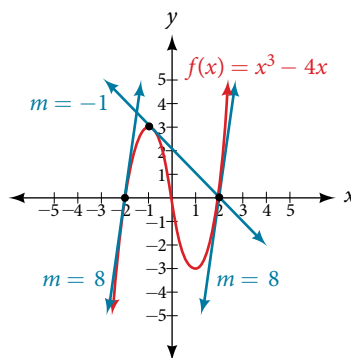- slope at $x = -1$ is –1
- slope at $x = 2$ is 8

Figure 1 Graph showing tangents to curve at –2, –1, and 2.

Let's imagine a point on the curve of function f at $x = a$ as shown in **Figure 2**. The coordinates of the point are $(a, f(a))$. Connect this point with a second point on the curve a little to the right of $x = a$, with an x-value increased by some small real number h. The coordinates of this second point are $(a + h, f(a + h))$ for some positive-value h.
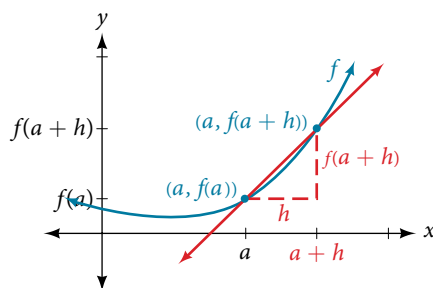
Figure 2 Connecting point a with a point just beyond allows us to measure a slope close to that of a tangent line at $x = a$.

We can calculate the slope of the line connecting the two points $(a, f(a))$ and $(a + h, f(a + h))$, called a **secant line**, by applying the slope formula, $\text{slope} = \dfrac{\text{change in } y}{\text{change in } x}$.

We use the notation m_{sec} to represent the slope of the secant line connecting two points.

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{(a + h) - (a)}$$

$$= \frac{f(a + h) - f(a)}{\cancel{a} + h - \cancel{a}}$$

The slope m_{sec} equals the *average rate of change* between two points $(a, f(a))$ and $(a + h, f(a + h))$.

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{h}$$

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the average rate of change between two points on a curve

The **average rate of change** (AROC) between two points $(a, f(a))$ and $(a + h, f(a + h))$ on the curve of f is the slope of the line connecting the two points and is given by

$$\text{AROC} = \frac{f(a + h) - f(a)}{h}$$

Example 1   Finding the Average Rate of Change

Find the average rate of change connecting the points $(2, -6)$ and $(-1, 5)$.

Solution   We know the average rate of change connecting two points may be given by

$$\text{AROC} = \frac{f(a + h) - f(a)}{h}.$$

If one point is $(2, -6)$, or $(2, f(2))$, then $f(2) = -6$.

The value h is the displacement from 2 to -1, which equals $-1 - 2 = -3$.

For the other point, $f(a + h)$ is the y-coordinate at $a + h$, which is $2 + (-3)$ or -1, so $f(a + h) = f(-1) = 5$.

$$\text{AROC} = \frac{f(a + h) - f(a)}{h}$$

$$= \frac{5 - (-6)}{-3}$$

$$= \frac{11}{-3}$$

$$= -\frac{11}{3}$$

Try It #1

Find the average rate of change connecting the points $(-5, 1.5)$ and $(-2.5, 9)$.

Understanding the Instantaneous Rate of Change

Now that we can find the average rate of change, suppose we make h in **Figure 2** smaller and smaller. Then $a + h$ will approach a as h gets smaller, getting closer and closer to 0. Likewise, the second point $(a + h, f(a + h))$ will approach the first point, $(a, f(a))$. As a consequence, the connecting line between the two points, called the secant line, will get closer and closer to being a tangent to the function at $x = a$, and the slope of the secant line will get closer and closer to the slope of the tangent at $x = a$. See **Figure 3**.
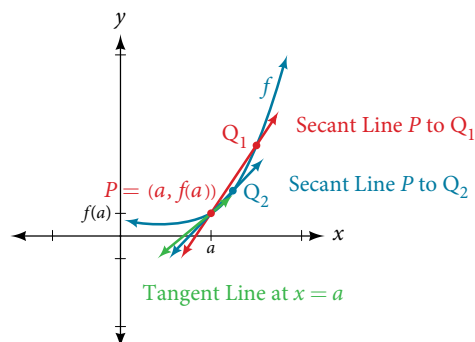

Figure 3 The connecting line between two points moves closer to being a tangent line at $x = a$.

Because we are looking for the slope of the tangent at $x = a$, we can think of the measure of the slope of the curve of a function f at a given point as the rate of change at a particular instant. We call this slope the **instantaneous rate of change**, or the **derivative** of the function at $x = a$. Both can be found by finding the limit of the slope of a line connecting the point at $x = a$ with a second point infinitesimally close along the curve. For a function f both the instantaneous rate of change of the function and the derivative of the function at $x = a$ are written as $f'(a)$, and we can define them as a two-sided limit that has the same value whether approached from the left or the right.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

The expression by which the limit is found is known as the difference quotient.

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definition of instantaneous rate of change and derivative

The **derivative**, or **instantaneous rate of change**, of a function f at $x = a$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

The expression $\dfrac{f(a + h) - f(a)}{h}$ is called the difference quotient.

We use the difference quotient to evaluate the limit of the rate of change of the function as h approaches 0.

Derivatives: Interpretations and Notation

The derivative of a function can be interpreted in different ways. It can be observed as the behavior of a graph of the function or calculated as a numerical rate of change of the function.

- The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent line to the curve $f(x)$ at $x = a$. The derivative of $f(x)$ at $x = a$ is written $f'(a)$.

- The derivative $f'(a)$ measures how the curve changes at the point $(a, f(a))$.

- The derivative $f'(a)$ may be thought of as the instantaneous rate of change of the function $f(x)$ at $x = a$.

- If a function measures distance as a function of time, then the derivative measures the instantaneous velocity at time $t = a$.

How To...

Given a function f, find the derivative by applying the definition of the derivative.

1. Calculate $f(a + h)$.
2. Calculate $f(a)$.
3. Substitute and simplify $\dfrac{f(a + h) - f(a)}{h}$.
4. Evaluate the limit if it exists: $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$.

Example 2   Finding the Derivative of a Polynomial Function

Find the derivative of the function $f(x) = x^2 - 3x + 5$ at $x = a$.

Solution   We have:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$ Definition of a derivative

Substitute $f(a + h) = (a + h)^2 - 3(a + h) + 5$ and $f(a) = a^2 - 3a + 5$.

$$f'(a) = \lim_{h \to 0} \frac{(a + h)(a + h) - 3(a + h) + 5 - (a^2 - 3a + 5)}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 3a - 3h + 5 - a^2 + 3a - 5}{h}$$ Evaluate to remove parentheses.

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 \cancel{- 3a} - 3h \cancel{+5} \cancel{-a^2} \cancel{+3a} \cancel{-5}}{h}$$ Simplify.

$$= \lim_{h \to 0} \frac{2ah + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{h(2a + h - 3)}{h}$$ Factor out an h.

$$= 2a + 0 - 3$$ Evaluate the limit.

$$= 2a - 3$$

Try It #2

Find the derivative of the function $f(x) = 3x^2 + 7x$ at $x = a$.

Finding Derivatives of Rational Functions

To find the derivative of a rational function, we will sometimes simplify the expression using algebraic techniques we have already learned.

Example 3   Finding the Derivative of a Rational Function

Find the derivative of the function $f(x) = \dfrac{3+x}{2-x}$ at $x = a$.

Solution

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3+(a+h)}{2-(a+h)} - \left(\dfrac{3+a}{2-a}\right)}{h} \qquad \text{Substitute } f(a+h) \text{ and } f(a).$$

$$= \lim_{h \to 0} \frac{(2-(a+h))(2-a)\left[\dfrac{3+(a+h)}{2-(a+h)} - \left(\dfrac{3+a}{2-a}\right)\right]}{(2-(a+h))(2-a)(h)} \qquad \begin{array}{l}\text{Multiply numerator and} \\ \text{denominator by} \\ (2-(a+h))(2-a).\end{array}$$

$$= \lim_{h \to 0} \frac{(2-\!\!\!\diagup\!\!(a+h))(2-a)\left(\dfrac{3+(a+h)}{(2-\!\!\!\diagup\!\!(a+h))}\right) - (2-(a+h))(2-\!\!\!\diagup a)\left(\dfrac{3+a}{2-\!\!\!\diagup a}\right)}{(2-(a+h))(2-a)(h)} \qquad \text{Distribute.}$$

$$= \lim_{h \to 0} \frac{6 - 3a + 2a - a^2 + 2h - ah - 6 + 3a + 3h - 2a + a2 + ah}{(2-(a+h))(2-a)(h)} \qquad \text{Multiply.}$$

$$= \lim_{h \to 0} \frac{5\cancel{h}}{(2-(a+h))(2-a)(\cancel{h})} \qquad \text{Combine like terms.}$$

$$= \lim_{h \to 0} \frac{5}{(2-(a+h))(2-a)} \qquad \text{Cancel like factors.}$$

$$= \frac{5}{(2-(a+0))(2-a)} = \frac{5}{(2-a)(2-a)} = \frac{5}{(2-a)^2} \qquad \text{Evaluate the limit.}$$

Try It #3

Find the derivative of the function $f(x) = \dfrac{10x + 11}{5x + 4}$ at $x = a$.

Finding Derivatives of Functions with Roots

To find derivatives of functions with roots, we use the methods we have learned to find limits of functions with roots, including multiplying by a conjugate.

Example 4   Finding the Derivative of a Function with a Root

Find the derivative of the function $f(x) = 4\sqrt{x}$ at $x = 36$.

Solution   We have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{4\sqrt{a+h} - 4\sqrt{a}}{h} \qquad \text{Substitute } f(a+h) \text{ and } f(a)$$

Multiply the numerator and denominator by the conjugate: $\dfrac{4\sqrt{a+h} + 4\sqrt{a}}{4\sqrt{a+h} + 4\sqrt{a}}$.

$$f'(a) = \lim_{h \to 0} \left(\frac{4\sqrt{a+h} - 4\sqrt{a}}{h} \right) \cdot \left(\frac{4\sqrt{a+h} + 4\sqrt{a}}{4\sqrt{a+h} + 4\sqrt{a}} \right)$$

$$= \lim_{h \to 0} \left(\frac{16(a+h) - 16a}{h4(\sqrt{a+h} + 4\sqrt{a})} \right) \qquad \text{Multiply.}$$

$$= \lim_{h \to 0} \left(\frac{\cancel{16a} + 16h - \cancel{16a}}{h4(\sqrt{a+h} + 4\sqrt{a})} \right) \qquad \text{Distribute and combine like terms.}$$

$$= \lim_{h \to 0} \left(\frac{16\cancel{h}}{\cancel{h}(4\sqrt{a+h} + 4\sqrt{a})} \right) \qquad \text{Simplify.}$$

$$= \lim_{h \to 0} \frac{16}{4\sqrt{a+h} + 4\sqrt{a}} \qquad \text{Evaluate the limit by letting } h = 0.$$

$$= \frac{16}{8\sqrt{a}} = \frac{2}{\sqrt{a}}$$

$$f'(36) = \frac{2}{\sqrt{36}} \qquad \text{Evaluate the derivative at } x = 36.$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

Try It #4

Find the derivative of the function $f(x) = 9\sqrt{x}$ at $x = 9$.

Finding Instantaneous Rates of Change

Many applications of the derivative involve determining the rate of change at a given instant of a function with the independent variable time—which is why the term instantaneous is used. Consider the height of a ball tossed upward with an initial velocity of 64 feet per second, given by $s(t) = -16t^2 + 64t + 6$, where t is measured in seconds and $s(t)$ is measured in feet. We know the path is that of a parabola. The derivative will tell us how the height is changing at any given point in time. The height of the ball is shown in **Figure 4** as a function of time. In physics, we call this the "*s-t* graph."
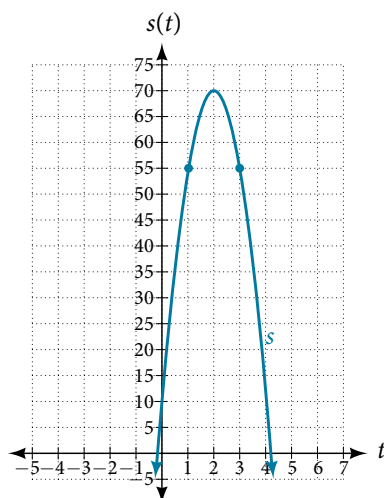

Figure 4

Example 5   Finding the Instantaneous Rate of Change

Using the function above, $s(t) = -16t^2 + 64t + 6$, what is the instantaneous velocity of the ball at 1 second and 3 seconds into its flight?

Solution   The velocity at $t = 1$ and $t = 3$ is the instantaneous rate of change of distance per time, or velocity. Notice that the initial height is 6 feet. To find the instantaneous velocity, we find the derivative and evaluate it at $t = 1$ and $t = 3$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{-16(t+h)^2 + 64(t+h) + 6 - (-16t^2 + 64t + 6)}{h} \qquad \text{Substitute } s(t+h) \text{ and } s(t).$$

$$= \lim_{h \to 0} \frac{-16t^2 - 32ht - h^2 + 64t + 64h + 6 + 16t^2 - 64t - 6}{h} \qquad \text{Distribute.}$$

$$= \lim_{h \to 0} \frac{-32ht - h^2 + 64h}{h} \qquad \text{Simplify.}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-32t - h + 64)}{\cancel{h}} \qquad \text{Factor the numerator.}$$

$$= \lim_{h \to 0} -32t - h + 64 \qquad \text{Cancel out the common factor } h.$$

$$s'(t) = -32t + 64 \qquad \text{Evaluate the limit by letting } h = 0.$$

For any value of t, $s'(t)$ tells us the velocity at that value of t.

Evaluate $t = 1$ and $t = 3$.

$$s'(1) = -32(1) + 64 = 32$$
$$s'(3) = -32(3) + 64 = -32$$

The velocity of the ball after 1 second is 32 feet per second, as it is on the way up.

The velocity of the ball after 3 seconds is -32 feet per second, as it is on the way down.

....

Try It #5

The position of the ball is given by $s(t) = -16t^2 + 64t + 6$. What is its velocity 2 seconds into flight?

....

Using Graphs to Find Instantaneous Rates of Change

We can estimate an instantaneous rate of change at $x = a$ by observing the slope of the curve of the function $f(x)$ at $x = a$. We do this by drawing a line tangent to the function at $x = a$ and finding its slope.

How To…
Given a graph of a function $f(x)$, find the instantaneous rate of change of the function at $x = a$.

1. Locate $x = a$ on the graph of the function $f(x)$.
2. Draw a tangent line, a line that goes through $x = a$ at a and at no other point in that section of the curve. Extend the line far enough to calculate its slope as
$$\frac{\text{change in } y}{\text{change in } x}.$$

Example 6   Estimating the Derivative at a Point on the Graph of a Function

From the graph of the function $y = f(x)$ presented in **Figure 5**, estimate each of the following:
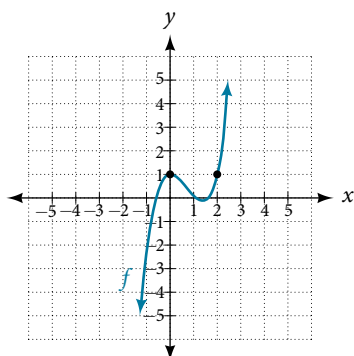
$$f(0); f(2); f'(0); f'(2)$$

Figure 5

Solution   To find the functional value, $f(a)$, find the y-coordinate at $x = a$.

To find the derivative at $x = a$, $f'(a)$, draw a tangent line at $x = a$, and estimate the slope of that tangent line. See **Figure 6**.
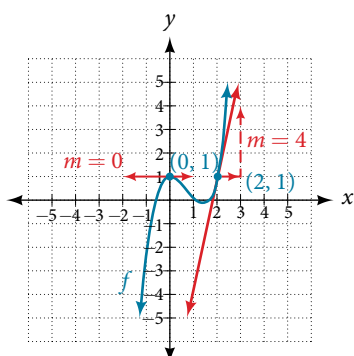

Figure 6

a. $f(0)$ is the y-coordinate at $x = 0$. The point has coordinates $(0, 1)$, thus $f(0) = 1$.

b. $f(2)$ is the y-coordinate at $x = 2$. The point has coordinates $(2, 1)$, thus $f(2) = 1$.

c. $f'(0)$ is found by estimating the slope of the tangent line to the curve at $x = 0$. The tangent line to the curve at $x = 0$ appears horizontal. Horizontal lines have a slope of 0, thus $f'(0) = 0$.

d. $f'(2)$ is found by estimating the slope of the tangent line to the curve at $x = 2$. Observe the path of the tangent line to the curve at $x = 2$. As the x value moves one unit to the right, the y value moves up four units to another point on the line. Thus, the slope is 4, so $f'(2) = 4$.

Try It #6

Using the graph of the function $f(x) = x^3 - 3x$ shown in **Figure 7**, estimate: $f(1), f'(1), f(0)$, and $f'(0)$.
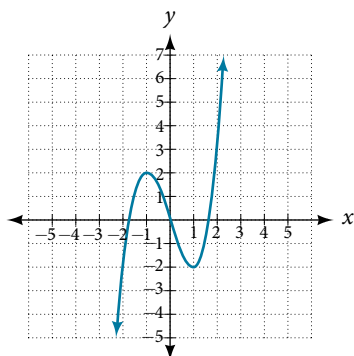

Figure 7

Using Instantaneous Rates of Change to Solve Real-World Problems

Another way to interpret an instantaneous rate of change at $x = a$ is to observe the function in a real-world context. The unit for the derivative of a function $f(x)$ is

$$\frac{\text{output units}}{\text{input unit}}$$

Such a unit shows by how many units the output changes for each one-unit change of input. The instantaneous rate of change at a given instant shows the same thing: the units of change of output per one-unit change of input.

One example of an instantaneous rate of change is a marginal cost. For example, suppose the production cost for a company to produce x items is given by $C(x)$, in thousands of dollars. The derivative function tells us how the cost is changing for any value of x in the domain of the function. In other words, $C'(x)$ is interpreted as a marginal cost, the additional cost in thousands of dollars of producing one more item when x items have been produced. For example, $C'(11)$ is the approximate additional cost in thousands of dollars of producing the 12th item after 11 items have been produced. $C'(11) = 2.50$ means that when 11 items have been produced, producing the 12th item would increase the total cost by approximately $2,500.00.

Example 7   Finding a Marginal Cost

The cost in dollars of producing x laptop computers is $f(x) = x^2 - 100x$. At the point where 200 computers have been produced, what is the approximate cost of producing the 201st unit?

Solution   If $f(x) = x^2 - 100x$ describes the cost of producing x computers, $f'(x)$ will describe the marginal cost. We need to find the derivative. For purposes of calculating the derivative, we can use the following functions:

$$f(a + b) = (x + h)^2 - 100(x + h)$$

$$f(a) = a^2 - 100a$$

$f'(x) = \dfrac{f(a + h) - f(a)}{h}$	Formula for a derivative
$= \dfrac{(x + h)^2 - 100(x + h) - (x^2 - 100x)}{h}$	Substitute $f(a + h)$ and $f(a)$.
$= \dfrac{x^2 + 2xh + h^2 - 100x - 100h - x^2 + 100x}{h}$	Multiply polynomials, distribute.
$= \dfrac{2xh + h^2 - 100h}{h}$	Collect like terms.
$= \dfrac{\cancel{h}(2x + h - 100)}{\cancel{h}}$	Factor and cancel like terms.
$= 2x + h - 100$	Simplify.
$= 2x - 100$	Evaluate when $h = 0$.
$f'(x) = 2x - 100$	Formula for marginal cost
$f'(200) = 2(200) - 100 = 300$	Evaluate for 200 units.

The marginal cost of producing the 201st unit will be approximately $300.

Example 8   Interpreting a Derivative in Context

A car leaves an intersection. The distance it travels in miles is given by the function $f(t)$, where t represents hours. Explain the following notations:

$$f(0) = 0 \,; f'(1) = 60; f(1) = 70; f(2.5) = 150$$

Solution   First we need to evaluate the function $f(t)$ and the derivative of the function $f'(t)$, and distinguish between the two. When we evaluate the function $f(t)$, we are finding the distance the car has traveled in t hours. When we evaluate the derivative $f'(t)$, we are finding the speed of the car after t hours.

a. $f(0) = 0$ means that in zero hours, the car has traveled zero miles.

b. $f'(1) = 60$ means that one hour into the trip, the car is traveling 60 miles per hour.

c. $f(1) = 70$ means that one hour into the trip, the car has traveled 70 miles. At some point during the first hour, then, the car must have been traveling faster than it was at the 1-hour mark.

d. $f(2.5) = 150$ means that two hours and thirty minutes into the trip, the car has traveled 150 miles.

Try It #7

A runner runs along a straight east-west road. The function $f(t)$ gives how many feet eastward of her starting point she is after t seconds. Interpret each of the following as it relates to the runner.

$$f(0) = 0; f(10) = 150; f'(10) = 15; f'(20) = -10; f(40) = -100$$

Finding Points Where a Function's Derivative Does Not Exist

To understand where a function's derivative does not exist, we need to recall what normally happens when a function $f(x)$ has a derivative at $x = a$. Suppose we use a graphing utility to zoom in on $x = a$. If the function $f(x)$ is **differentiable**, that is, if it is a function that can be differentiated, then the closer one zooms in, the more closely the graph approaches a straight line. This characteristic is called *linearity*.

Look at the graph in **Figure 8**. The closer we zoom in on the point, the more linear the curve appears.
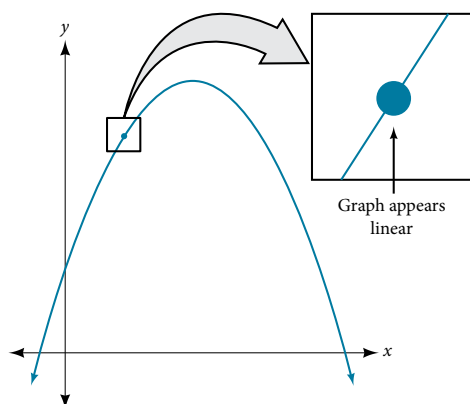

Graph appears linear

Figure 8

We might presume the same thing would happen with any continuous function, but that is not so. The function $f(x) = |x|$, for example, is continuous at $x = 0$, but not differentiable at $x = 0$. As we zoom in close to 0 in **Figure 9**, the graph does not approach a straight line. No matter how close we zoom in, the graph maintains its sharp corner.
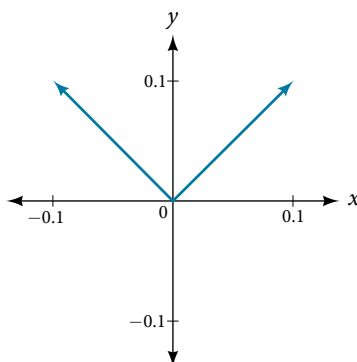

Figure 9 Graph of the function $f(x) = |x|$, with x-axis from −0.1 to 0.1 and y-axis from −0.1 to 0.1.

We zoom in closer by narrowing the range to produce **Figure 10** and continue to observe the same shape. This graph does not appear linear at $x = 0$.
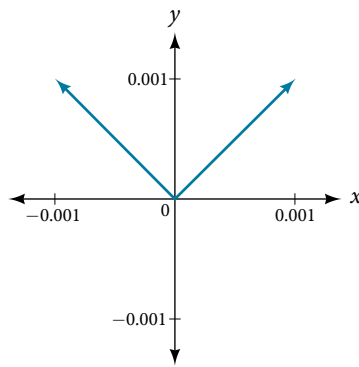
Figure 10 Graph of the function $f(x) = |x|$, with x-axis from -0.001 to 0.001 and y-axis from -0.001 to 0.001.

What are the characteristics of a graph that is not differentiable at a point? Here are some examples in which function $f(x)$ is not differentiable at $x = a$.

In **Figure 11**, we see the graph of

$$f(x) = \begin{cases} x^2, & x \le 2 \\ 8 - x, & x > 2 \end{cases}.$$

Notice that, as x approaches 2 from the left, the left-hand limit may be observed to be 4, while as x approaches 2 from the right, the right-hand limit may be observed to be 6. We see that it has a discontinuity at $x = 2$.
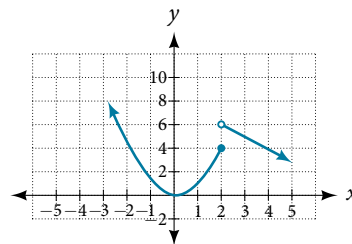
Figure 11 The graph of $f(x)$ has a discontinuity at $x = 2$.

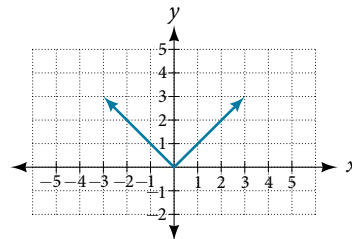In **Figure 12**, we see the graph of $f(x) = |x|$. We see that the graph has a corner point at $x = 0$.

Figure 12 The graph of $f(x) = |x|$ has a corner point at $x = 0$.

In **Figure 13**, we see that the graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$. A cusp has a unique feature. Moving away from the cusp, both the left-hand and right-hand limits approach either infinity or negative infinity. Notice the tangent lines as x approaches 0 from both the left and the right appear to get increasingly steeper, but one has a negative slope, the other has a positive slope.
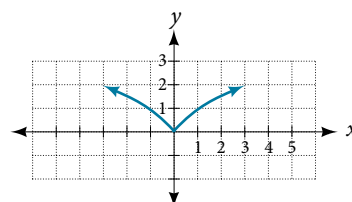

Figure 13 The graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$.

In **Figure 14**, we see that the graph of $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$. Recall that vertical tangents are vertical lines, so where a vertical tangent exists, the slope of the line is undefined. This is why the derivative, which measures the slope, does not exist there.
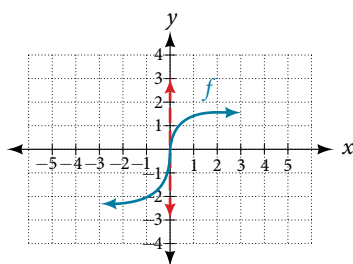

Figure 14 The graph of $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$.

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differentiability

A function $f(x)$ is differentiable at $x = a$ if the derivative exists at $x = a$, which means that $f'(a)$ exists. There are four cases for which a function $f(x)$ is not differentiable at a point $x = a$.

1. When there is a discontinuity at $x = a$.
2. When there is a corner point at $x = a$.
3. When there is a cusp at $x = a$.
4. Any other time when there is a vertical tangent at $x = a$.

Example 9   Determining Where a Function Is Continuous and Differentiable from a Graph

Using **Figure 15**, determine where the function is

    a. continuous   **b.** discontinuous   **c.** differentiable   **d.** not differentiable

At the points where the graph is discontinuous or not differentiable, state why.
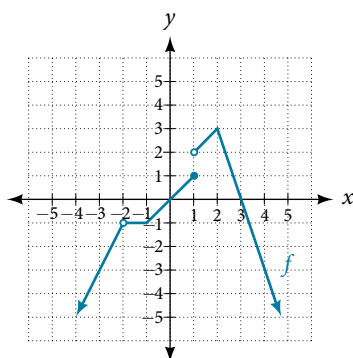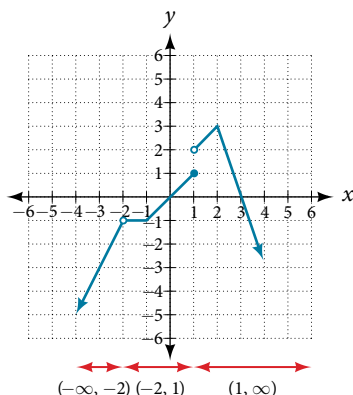

Figure 15

Solution The graph of $f(x)$ is continuous on $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$. The graph of $f(x)$ has a removable discontinuity at $x = -2$ and a jump discontinuity at $x = 1$. See **Figure 16**.

$(-\infty, -2)$ $(-2, 1)$ $(1, \infty)$

The graph of $f(x)$ is differentiable on $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$. The graph of $f(x)$ is not differentiable at $x = -2$ because it is a point of discontinuity, at $x = -1$ because of a sharp corner, at $x = 1$ because it is a point of discontinuity, and at $x = 2$ because of a sharp corner. See **Figure 17**.
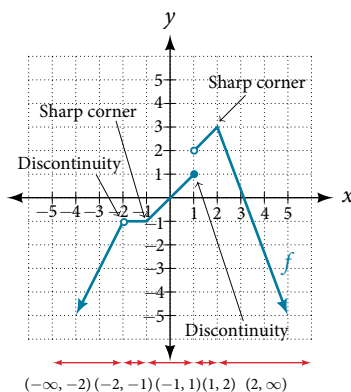

Figure 17 Five intervals where the function is differentiable

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Try It #8

Determine where the function $y = f(x)$ shown in **Figure 18** is continuous and differentiable from the graph.
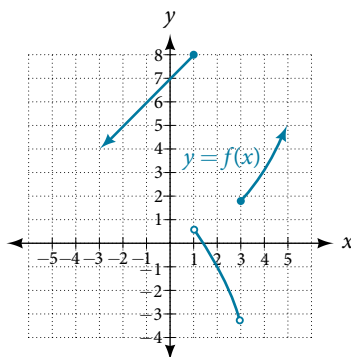

Figure 18

Finding an Equation of a Line Tangent to the Graph of a Function

The equation of a tangent line to a curve of the function $f(x)$ at $x = a$ is derived from the point-slope form of a line, $y = m(x - x_1) + y_1$. The slope of the line is the slope of the curve at $x = a$ and is therefore equal to $f'(a)$, the derivative of $f(x)$ at $x = a$. The coordinate pair of the point on the line at $x = a$ is $(a, f(a))$.

If we substitute into the point-slope form, we have

$$m = f'(a) \qquad x_1 = a \qquad y_1 = f(a)$$
$$y = m(x - x_1) + y_1$$
$$\uparrow \qquad \uparrow \qquad \uparrow$$
$$f'(a) \qquad a \qquad f(a)$$

The equation of the tangent line is

$$y = f'(a)(x - a) + f(a)$$

---

the equation of a line tangent to a curve of the function f

The equation of a line tangent to the curve of a function f at a point $x = a$ is

$$y = f'(a)(x - a) + f(a)$$

Given a function f, find the equation of a line tangent to the function at $x = a$.

1. Find the derivative of $f(x)$ at $x = a$ using $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$.
2. Evaluate the function at $x = a$. This is $f(a)$.
3. Substitute $(a, f(a))$ and $f'(a)$ into $y = f'(a)(x - a) + f(a)$.
4. Write the equation of the tangent line in the form $y = mx + b$.

Example 10   Finding the Equation of a Line Tangent to a Function at a Point

Find the equation of a line tangent to the curve $f(x) = x^2 - 4x$ at $x = 3$.

Solution

Using:
$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Substitute $f(a + h) = (a + h)^2 - 4(a + h)$ and $f(a) = a^2 - 4a$.

$$f'(a) = \lim_{h \to 0} \frac{(a + h)(a + h) - 4(a + h) - (a^2 - 4a)}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 4a - 4h - a^2 + 4a}{h} \qquad \text{Remove parentheses.}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 \cancel{-4a} - 4h \cancel{-a^2} \cancel{+4a}}{h} \qquad \text{Combine like terms.}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 - 4h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2a + h - 4)}{\cancel{h}} \qquad \text{Factor out } h.$$

$$= 2a + 0 - 4$$

$$f'(a) = 2a - 4 \qquad \text{Evaluate the limit.}$$

$$f'(3) = 2(3) - 4 = 2$$

Equation of tangent line at $x = 3$:

$$y = f'(a)(x - a) + f(a)$$
$$y = f'(3)(x - 3) + f(3)$$
$$y = 2(x - 3) + (-3)$$
$$y = 2x - 9$$

Analysis   *We can use a graphing utility to graph the function and the tangent line. In so doing, we can observe the point of tangency at $x = 3$ as shown in **Figure 19**.*
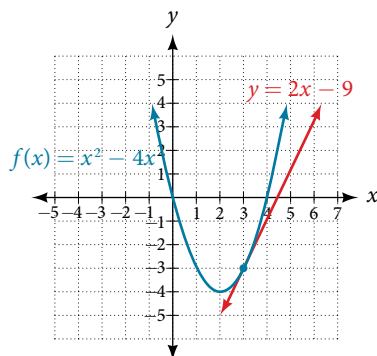

Figure 19 Graph confirms the point of tangency at $x = 3$.

Finding the Instantaneous Speed of a Particle

If a function measures position versus time, the derivative measures displacement versus time, or the speed of the object. A change in speed or direction relative to a change in time is known as velocity. The velocity at a given instant is known as **instantaneous velocity**.

In trying to find the speed or velocity of an object at a given instant, we seem to encounter a contradiction. We normally define speed as the distance traveled divided by the elapsed time. But in an instant, no distance is traveled, and no time elapses. How will we divide zero by zero? The use of a derivative solves this problem. A derivative allows us to say that even while the object's velocity is constantly changing, it has a certain velocity at a given instant. That means that if the object traveled at that exact velocity for a unit of time, it would travel the specified distance.

> ***instantaneous velocity***
> Let the function $s(t)$ represent the position of an object at time t. The **instantaneous velocity** or velocity of the object at time $t = a$ is given by
> $$s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Example 11 Finding the Instantaneous Velocity

A ball is tossed upward from a height of 200 feet with an initial velocity of 36 ft/sec. If the height of the ball in feet after t seconds is given by $s(t) = -16t^2 + 36t + 200$, find the instantaneous velocity of the ball at $t = 2$.

Solution First, we must find the derivative $s'(t)$. Then we evaluate the derivative at $t = 2$, using

$$s(a+h) = -16(a+h)^2 + 36(a+h) + 200 \text{ and } s(a) = -16a^2 + 36a + 200.$$

$$s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

$$= \lim_{h \to 0} \frac{-16(a+h)^2 + 36(a+h) + 200 - (-16a^2 + 36a + 200)}{h}$$

$$= \lim_{h \to 0} \frac{-16(a^2 + 2ah + h^2) + 36(a+h) + 200 - (-16a^2 + 36a + 200)}{h}$$

$$= \lim_{h \to 0} \frac{-16a^2 - 32ah - 16h^2 + 36a + 36h + 200 + 16a^2 - 36a - 200}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-16a^2} - 32ah - 16h^2 \cancel{+36a} + 36h \cancel{+200} \cancel{+16a^2} \cancel{-36a} \cancel{-200}}{h}$$

$$= \lim_{h \to 0} \frac{-32ah - 16h^2 + 36h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-32a - 16h + 36)}{\cancel{h}}$$

$$= \lim_{h \to 0} (-32a - 16h + 36)$$

$$= -32a - 16 \cdot 0 + 36$$

$$s'(a) = -32a + 36$$

$$s'(2) = -32(2) + 36$$

$$= -28$$

Analysis *This result means that at time $t = 2$ seconds, the ball is dropping at a rate of 28 ft/sec.*

A fireworks rocket is shot upward out of a pit 12 ft below the ground at a velocity of 60 ft/sec. Its height in feet after t seconds is given by $s = -16t^2 + 60t - 12$. What is its instantaneous velocity after 4 seconds?

Access these online resources for additional instruction and practice with derivatives.

- Estimate the Derivative (http://openstaxcollege.org/l/estimatederiv)
- Estimate the Derivative Ex. 4 (http://openstaxcollege.org/l/estimatederiv4)