In this section, you will:
- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

8.3 POLAR COORDINATES

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see **Figure 1**). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.
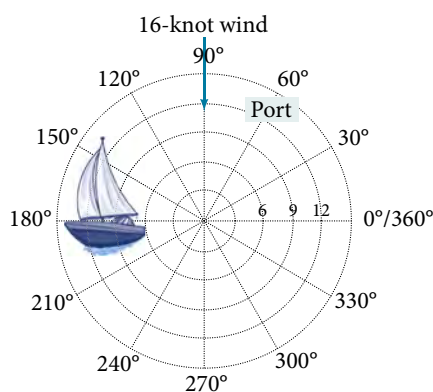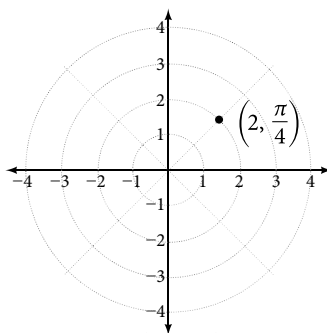

Figure 1

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of rectangular coordinates (x, y) in the Cartesian coordinate plane. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to **polar coordinates**, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the **pole**, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive x-axis now viewed as the **polar axis** and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ, and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \dfrac{\pi}{4}\right)$, we would move $\dfrac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in **Figure 2**.

Polar Grid

Figure 2

Example 1   Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2} \right)$ on the polar grid.

Solution   The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in **Figure 3**.
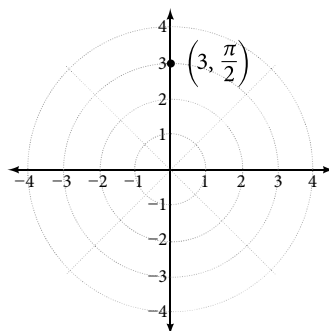

Figure 3

Try It #1

Plot the point $\left(2, \frac{\pi}{3} \right)$ in the polar grid.

Example 2   Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point $\left(-2, \frac{\pi}{6} \right)$ on the polar grid.

Solution   We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6} \right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;

2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See **Figure 4(a)**. Compare this to the graph of the polar coordinate $\left(2, \frac{\pi}{6} \right)$ shown in **Figure 4(b)**.
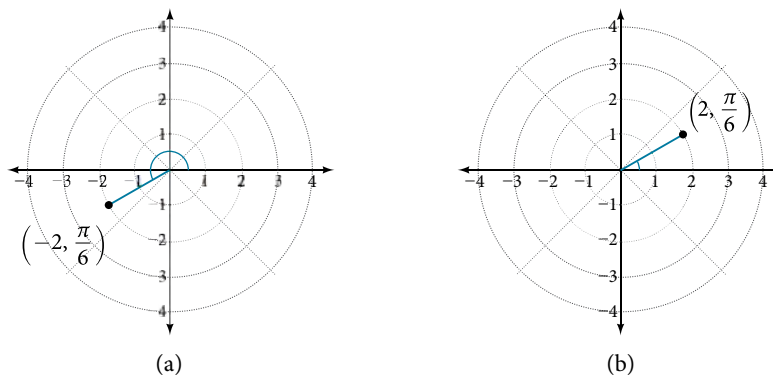

(a)   (b)

Figure 4

Try It #2

Plot the points $\left(3, -\frac{\pi}{6} \right)$ and $\left(2, \frac{9\pi}{4} \right)$ on the same polar grid.

Converting from Polar Coordinates to Rectangular Coordinates

When given a set of polar coordinates, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables x, y, r, and θ.

$$\cos \theta = \frac{x}{r} \rightarrow x = r\cos \theta$$

$$\sin \theta = \frac{y}{r} \rightarrow y = r\sin \theta$$

Dropping a perpendicular from the point in the plane to the x-axis forms a right triangle, as illustrated in **Figure 5**. An easy way to remember the equations above is to think of $\cos \theta$ as the adjacent side over the hypotenuse and $\sin \theta$ as the opposite side over the hypotenuse.
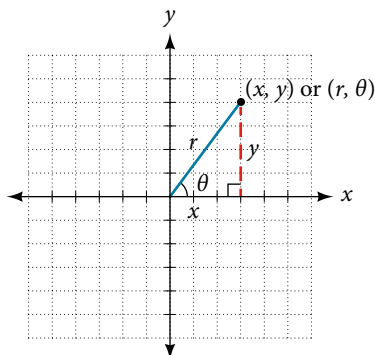

Figure 5

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converting from polar coordinates to rectangular coordinates
To convert polar coordinates (r, θ) to rectangular coordinates (x, y), let

$$\cos \theta = \frac{x}{r} \rightarrow x = r\cos \theta$$

$$\sin \theta = \frac{y}{r} \rightarrow y = r\sin \theta$$

How To...
Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ), write $x = r\cos \theta$ and $y = r\sin \theta$.
2. Evaluate $\cos \theta$ and $\sin \theta$.
3. Multiply $\cos \theta$ by r to find the x-coordinate of the rectangular form.
4. Multiply $\sin \theta$ by r to find the y-coordinate of the rectangular form.

Example 3   **Writing Polar Coordinates as Rectangular Coordinates**

Write the polar coordinates $\left(3, \frac{\pi}{2} \right)$ as rectangular coordinates.

Solution   Use the equivalent relationships.

$$x = r\cos \theta$$

$$x = 3\cos \frac{\pi}{2} = 0$$

$$y = r\sin \theta$$

$$y = 3\sin \frac{\pi}{2} = 3$$

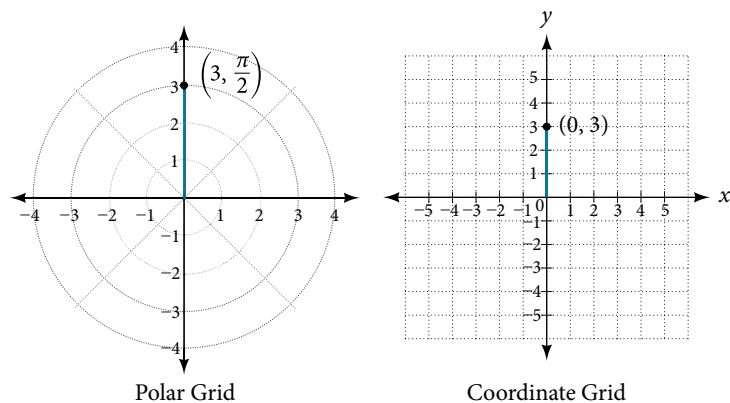The rectangular coordinates are $(0, 3)$. See **Figure 6**.

Polar Grid   Coordinate Grid

Figure 6

Example 4   **Writing Polar Coordinates as Rectangular Coordinates**

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

Solution   See **Figure 7**. Writing the polar coordinates as rectangular, we have

$$x = r\cos\theta$$
$$x = -2\cos(0) = -2$$
$$y = r\sin\theta$$
$$y = -2\sin(0) = 0$$
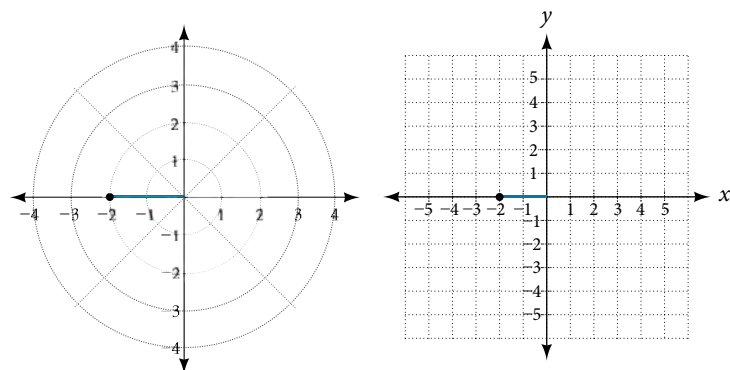
The rectangular coordinates are also $(-2, 0)$.

Figure 7

Try It #3

Write the polar coordinates $\left(-1, \dfrac{2\pi}{3}\right)$ as rectangular coordinates.

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in **Figure 8**.

$$\cos \theta = \frac{x}{r} \text{ or } x = r\cos \theta$$

$$\sin \theta = \frac{y}{r} \text{ or } y = r\sin \theta$$
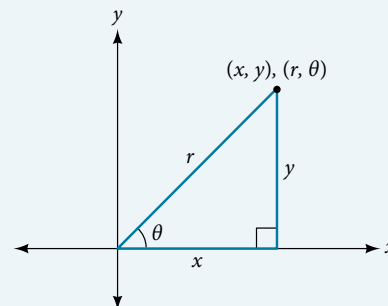
$$r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

Figure 8

Example 5   Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates (3, 3) to polar coordinates.

Solution   We see that the original point (3, 3) is in the first quadrant. To find θ, use the formula $\tan \theta = \frac{y}{x}$. This gives

$$\tan \theta = \frac{3}{3}$$

$$\tan \theta = 1$$

$$\theta = \tan^{-1}(1)$$

$$\theta = \frac{\pi}{4}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\frac{\pi}{4}$ is in the first quadrant. Thus

$$r = \sqrt{3^2 + 3^2}$$

$$r = \sqrt{9 + 9}$$

$$r = \sqrt{18} = 3\sqrt{2}$$

So, $r = 3\sqrt{2}$ and $\theta = \frac{\pi}{4}$, giving us the polar point $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. See **Figure 9**.
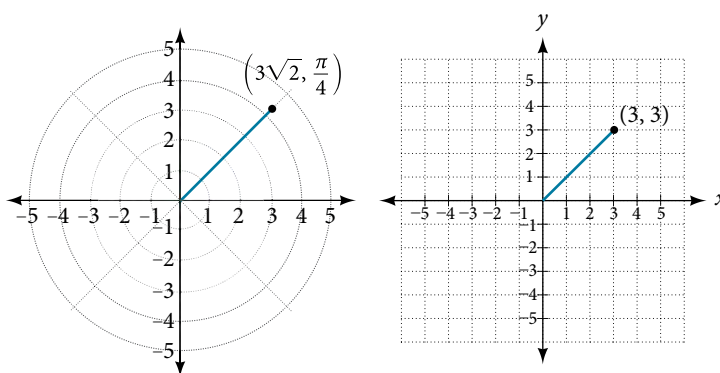

Figure 9

Analysis *There are other sets of polar coordinates that will be the same as our first solution. For example, the points* $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ *and* $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ *and will coincide with the original solution of* $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. *The point* $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ *indicates a move further counterclockwise by* π, *which is directly opposite* $\frac{\pi}{4}$. *The radius is expressed as* $-3\sqrt{2}$. *However, the angle* $\frac{5\pi}{4}$ *is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as* $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. *The point* $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ *is a move further clockwise by* $-\frac{7\pi}{4}$, *from* $\frac{\pi}{4}$. *The radius, $3\sqrt{2}$,, is the same.*

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How To...

Given an equation in polar form, graph it using a graphing calculator.

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: $r_1, r_2, ... , r_6$.
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example 6   **Writing a Cartesian Equation in Polar Form**

Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

Solution   The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x, y) and (r, θ). Since $x = r\cos \theta$ and $y = r\sin \theta$, we can substitute and solve for r.

$$(r\cos \theta)^2 + (r\sin \theta)^2 = 9$$
$$r^2\cos^2 \theta + r^2 \sin^2 \theta = 9$$
$$r^2(\cos^2 \theta + \sin^2 \theta) = 9$$
$$r^2(1) = 9 \qquad\qquad \text{Substitute } \cos^2 \theta + \sin^2 \theta = 1.$$
$$r = \pm 3 \qquad\qquad \text{Use the square root property.}$$

Thus, $x^2 + y^2 = 9$, $r = 3$, and $r = -3$ should generate the same graph. See **Figure 10**.

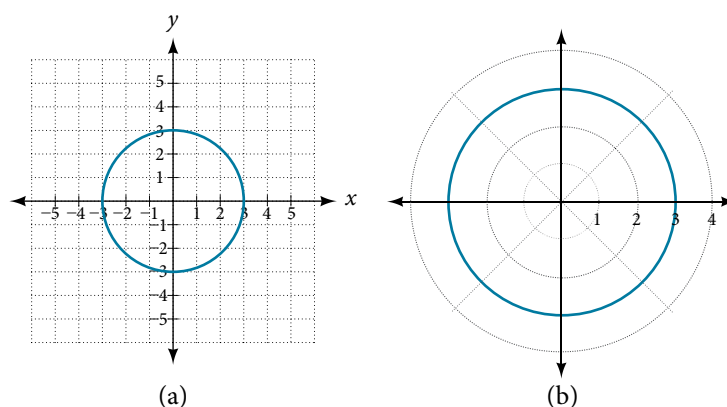

(a)   (b)

Figure 10   (a) Cartesian form $x^2 + y^2 = 9$ (b) Polar form $r = 3$

To graph a circle in rectangular form, we must first solve for y.

$$x^2 + y^2 = 9$$
$$y^2 = 9 - x^2$$
$$y = \pm \sqrt{9 - x^2}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1 = \sqrt{9 - x^2}$ and $Y_2 = -\sqrt{9 - x^2}$. Press **GRAPH**.

Example 7   Rewriting a Cartesian Equation as a Polar Equation

Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

Solution   This equation appears similar to the previous example, but it requires different steps to convert the equation. We can still follow the same procedures we have already learned and make the following substitutions:

$$r^2 = 6y \qquad \text{Use } x^2 + y^2 = r^2.$$
$$r^2 = 6r\sin\theta \quad \text{Substitute } y = r\sin\theta.$$
$$r^2 - 6r\sin\theta = 0 \qquad \text{Set equal to 0.}$$
$$r(r - 6\sin\theta) = 0 \qquad \text{Factor and solve.}$$
$$r = 0 \qquad \text{We reject } r = 0, \text{ as it only represents one point, } (0, 0).$$
$$\text{or } r = 6\sin\theta$$

Therefore, the equations $x^2 + y^2 = 6y$ and $r = 6\sin\theta$ should give us the same graph. See **Figure 11**.
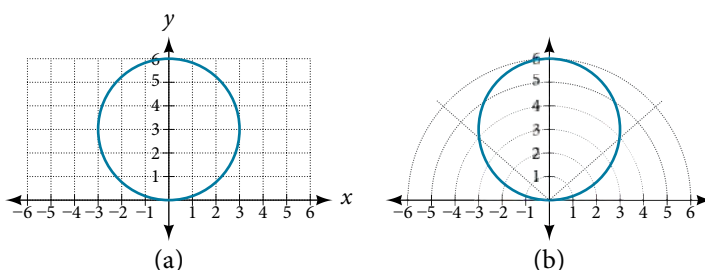

(a)   (b)

Figure 11   (a) Cartesian form $x^2 + y^2 = 6y$ (b) polar form $r = 6\sin\theta$

The Cartesian or rectangular equation is plotted on the rectangular grid, and the polar equation is plotted on the polar grid. Clearly, the graphs are identical.

Example 8   Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

Solution   We will use the relationships $x = r\cos\theta$ and $y = r\sin\theta$.

$$y = 3x + 2$$
$$r\sin\theta = 3r\cos\theta + 2$$
$$r\sin\theta - 3r\cos\theta = 2$$
$$r(\sin\theta - 3\cos\theta) = 2 \qquad\qquad \text{Isolate } r.$$
$$r = \frac{2}{\sin\theta - 3\cos\theta} \qquad\qquad \text{Solve for } r.$$

Try It #4

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 9   Graphing a Polar Equation by Converting to a Rectangular Equation

Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Solution   The conversion is

$$r = 2\sec\theta$$

$$r = \frac{2}{\cos\theta}$$

$$r\cos\theta = 2$$

$$x = 2$$

Notice that the equation $r = 2\sec\theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see **Figure 12**). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\theta$ is the standard form for a vertical line in polar form.
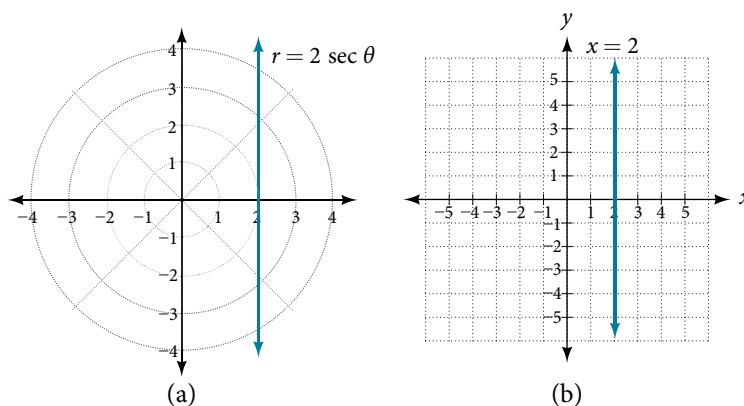

Figure 12   (a) Polar grid (b) Rectangular coordinate system

A similar discussion would demonstrate that the graph of the function $r = 2\csc\theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example 10   Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \dfrac{3}{1 - 2\cos\theta}$ as a Cartesian equation.

Solution   The goal is to eliminate θ and r, and introduce x and y. We clear the fraction, and then use substitution. In order to replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

$$r = \frac{3}{1 - 2\cos\theta}$$

$$r(1 - 2\cos\theta) = 3$$

$$r\left(1 - 2\left(\frac{x}{r}\right)\right) = 3 \qquad\qquad \text{Use } \cos\theta = \frac{x}{r} \text{ to eliminate } \theta.$$

$$r - 2x = 3$$

$$r = 3 + 2x \qquad\qquad \text{Isolate } r.$$

$$r^2 = (3 + 2x)^2 \qquad\qquad \text{Square both sides.}$$

$$x^2 + y^2 = (3 + 2x)^2 \qquad\qquad \text{Use } x^2 + y^2 = r^2.$$

The Cartesian equation is $x^2 + y^2 = (3 + 2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

$$x^2 + y^2 = (3 + 2x)^2$$

$$y^2 = (3 + 2x)^2 - x^2$$

$$y = \pm\sqrt{(3 + 2x)^2 - x^2}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See **Figure 13**.
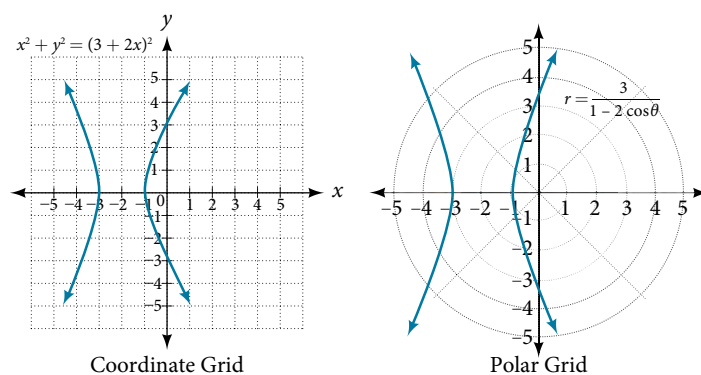
Figure 13

The "hour-glass" shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in **Analytic Geometry**.

Analysis   In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola's standard form. To do this, we can start with the initial equation.

$$x^2 + y^2 = (3 + 2x)^2$$
$$x^2 + y^2 - (3 + 2x)^2 = 0$$
$$x^2 + y^2 - (9 + 12x + 4x^2) = 0$$
$$x^2 + y^2 - 9 - 12x - 4x^2 = 0$$
$$-3x^2 - 12x + y^2 = 9 \qquad \text{Multiply through by } -1.$$
$$3x^2 + 12x - y^2 = -9$$
$$3(x^2 + 4x + \ \ ) - y^2 = -9 + 12 \quad \text{Organize terms to complete the square for } x.$$
$$3(x^2 + 4x + 4) - y^2 = -9 + 12$$
$$3(x + 2)^2 - y^2 = 3$$
$$(x + 2)^2 - \frac{y^2}{3} = 1$$

Try It #5

Rewrite the polar equation $r = 2\sin\theta$ in Cartesian form.

Example 11   **Rewriting a Polar Equation in Cartesian Form**

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

Solution
$$r = \sin(2\theta) \qquad \text{Use the double angle identity for sine.}$$
$$r = 2\sin\theta\cos\theta \qquad \text{Use } \cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}.$$
$$r = 2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) \qquad \text{Simplify.}$$
$$r = \frac{2xy}{r^2} \qquad \text{Multiply both sides by } r^2.$$
$$r^3 = 2xy$$
$$\left(\sqrt{x^2 + y^2}\right)^3 = 2xy \qquad \text{As } x^2 + y^2 = r^2, r = \sqrt{x^2 + y^2}.$$

This equation can also be written as

$$(x^2 + y^2)^{\frac{3}{2}} = 2xy \text{ or } x^2 + y^2 = (2xy)^{\frac{2}{3}}.$$

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates (http://openstaxcollege.org/l/intropolar)
- Comparing Polar and Rectangular Coordinates (http://openstaxcollege.org/l/polarrect)