

LEARNING OBJECTIVES

In this section, you will:

- Find function values for the sine and cosine of 30° or $\left(\frac{\pi}{6}\right)$, 45° or $\left(\frac{\pi}{4}\right)$ and 60° or $\left(\frac{\pi}{3}\right)$.
- Identify the domain and range of sine and cosine functions.
- Use reference angles to evaluate trigonometric functions.

5.2 UNIT CIRCLE: SINE AND COSINE FUNCTIONS



Figure 1 The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

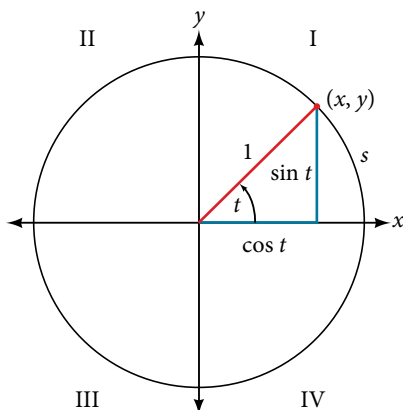
Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in **Figure 2**. The angle (in radians) that t intercepts forms an arc of length s . Using the formula $s = rt$, and knowing that $r = 1$, we see that for a unit circle, $s = t$.

Recall that the x - and y -axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t , we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y) . The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.



unit circle

A **unit circle** has a center at $(0, 0)$ and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t .

Let (x, y) be the endpoint on the unit circle of an arc of arc length s . The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the arc length and angle. The **sine function** relates a real number t to the y -coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y -value of the endpoint on the unit circle of an arc of length t . In **Figure 2**, the sine is equal to y . Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y -coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x -value of the endpoint on the unit circle of an arc of length t . In **Figure 3**, the cosine is equal to x .

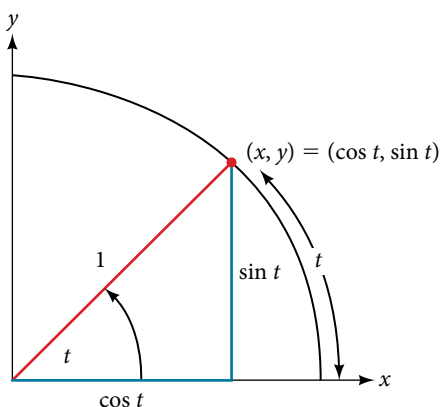


Figure 3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

sine and cosine functions

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t , then

$$\cos t = x$$

$$\sin t = y$$

How To...

Given a point $P(x, y)$ on the unit circle corresponding to an angle of t , find the sine and cosine.

1. The sine of t is equal to the y -coordinate of point P : $\sin t = y$.
2. The cosine of t is equal to the x -coordinate of point P : $\cos t = x$.

Example 1 Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t , as shown in **Figure 4**. Find $\cos(t)$ and $\sin(t)$.

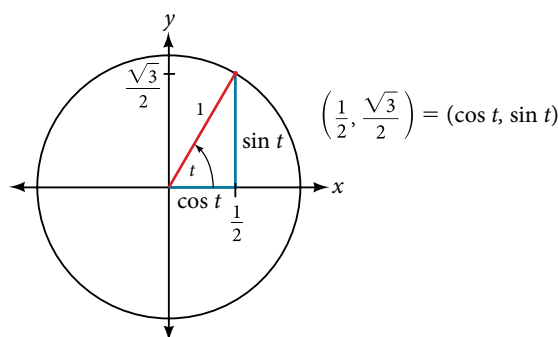


Figure 4

Solution We know that $\cos t$ is the x -coordinate of the corresponding point on the unit circle and $\sin t$ is the y -coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2}$$

$$y = \sin t = \frac{\sqrt{3}}{2}$$

Try It #1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in **Figure 5**. Find $\cos t$ and $\sin t$.

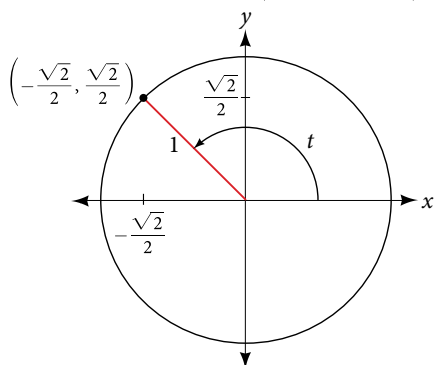


Figure 5

Finding Sines and Cosines of Angles on an Axis

For quadrantal angles, the corresponding point on the unit circle falls on the x - or y -axis. In that case, we can easily calculate cosine and sine from the values of x and y .

Example 2 Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Solution Moving 90° counterclockwise around the unit circle from the positive x -axis brings us to the top of the circle, where the (x, y) coordinates are $(0, 1)$, as shown in **Figure 6**.

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^\circ) = 0$$

$$y = \sin t = \sin(90^\circ) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

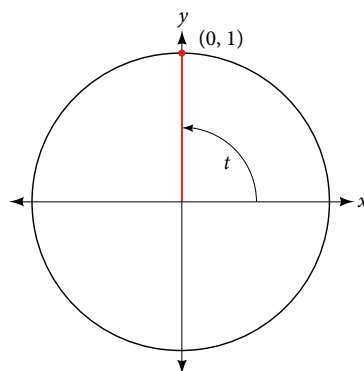


Figure 6

Try It #2

Find cosine and sine of the angle π .

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See **Figure 7**.

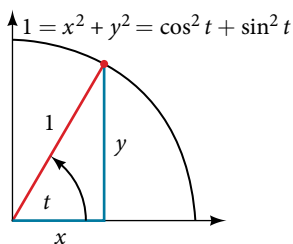


Figure 7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

Pythagorean Identity

The **Pythagorean Identity** states that, for any real number t ,

$$\cos^2 t + \sin^2 t = 1$$

How To...

Given the sine of some angle t and its quadrant location, find the cosine of t .

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
 2. Solve for $\cos(t)$.
 3. Choose the solution with the appropriate sign for the x -values in the quadrant where t is located.
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Example 3 Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution If we drop a vertical line from the point on the unit circle corresponding to t , we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See **Figure 8**.

Substituting the known value for sine into the Pythagorean Identity,

$$\cos^2(t) + \sin^2(t) = 1$$

$$\cos^2(t) + \frac{9}{49} = 1$$

$$\cos^2(t) = \frac{40}{49}$$

$$\cos(t) = \pm \sqrt{\frac{40}{49}} = \pm \frac{\sqrt{40}}{7} = \pm \frac{2\sqrt{10}}{7}$$

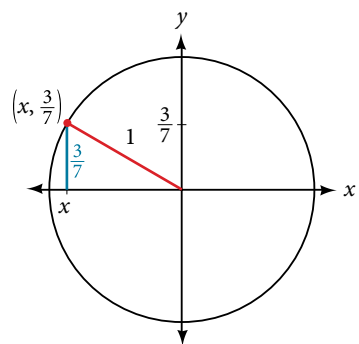


Figure 8

Because the angle is in the second quadrant, we know the x -value is a negative real number, so the cosine is also negative. So $\cos(t) = -\frac{2\sqrt{10}}{7}$.

Try It #3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the Pythagorean Identity and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in **Figure 9**. A $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles triangle, so the x - and y -coordinates of the corresponding point on the circle are the same. Because the x - and y -values are the same, the sine and cosine values will also be equal.

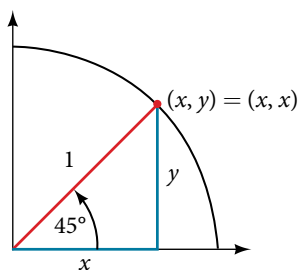


Figure 9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first quadrantal angle. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y ($y = x$), and a radius $= 1$. See **Figure 10**.

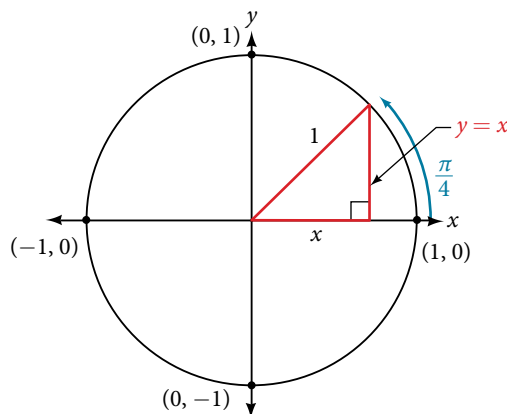


Figure 10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x , we get

$$x^2 = \frac{1}{2}$$
$$x = \pm \frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$$

$$\cos t = \frac{1}{\sqrt{2}}, \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\cos t = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

$$\sin t = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30° , or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30° , and another at an angle of -30° , as shown in **Figure 11**. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60° , as shown in **Figure 12**.

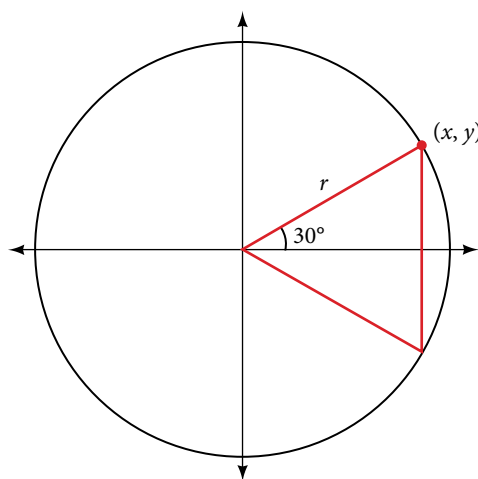


Figure 11

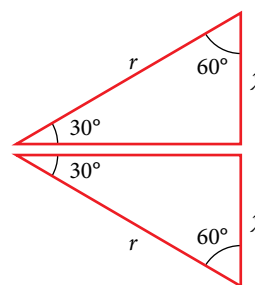


Figure 12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}r$$

And since $r = 1$ in our unit circle,

$$\begin{aligned} \sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1) \\ &= \frac{1}{2} \end{aligned}$$

— —

$$\cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 = 1$$

Use the square root property.

Since y is positive, choose the positive root.

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At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$$

$$\cos t = \frac{1}{2}, \sin t = \frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. **Table 1** summarizes these values.

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Table 1

Figure 14 shows the common angles in the first quadrant of the unit circle.

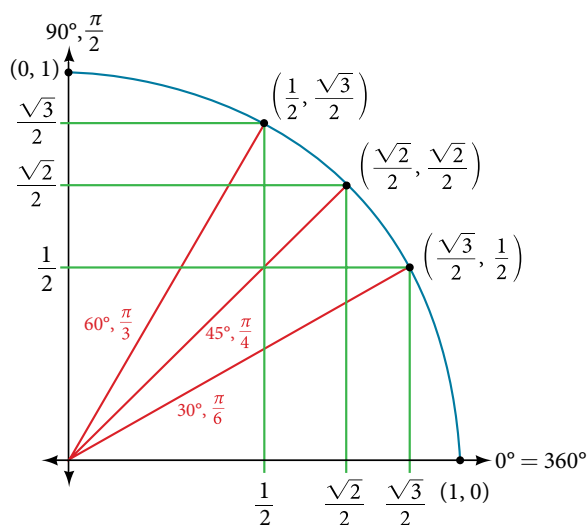


Figure 14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the special angles, we turn to a computer or calculator. **Be aware:** Most calculators can be set into “degree” or “radian” mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To...

Given an angle in radians, use a graphing calculator to find the cosine.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the **COS** key.
3. Enter the radian value of the angle and press the close-parentheses key “)”.
4. Press **ENTER**.

Example 4 Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution Enter the following keystrokes:

COS(5 × π ÷ 3) ENTER

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20° , for example, by including the conversion factor to radians as part of the input:

SIN(20 × π ÷ 180) ENTER

Try It #4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x , y , and r , there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x -axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown in **Figure 15**. The bounds of the x -coordinate are $[-1, 1]$.

The bounds of the y -coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

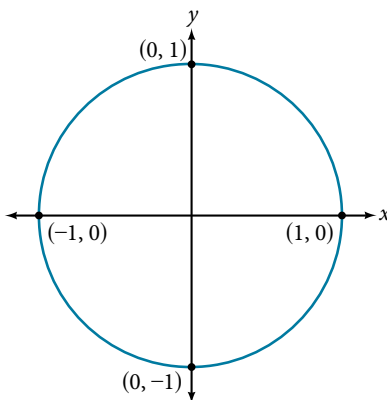


Figure 15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y -coordinate on the unit circle, the other angle with the same sine will share the same y -value, but have the opposite x -value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x -value but will have the opposite y -value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in **Figure 16**, angle α has the same sine value as angle t ; the cosine values are opposites. Angle β has the same cosine value as angle t ; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha)$$

$$\sin(t) = -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta)$$

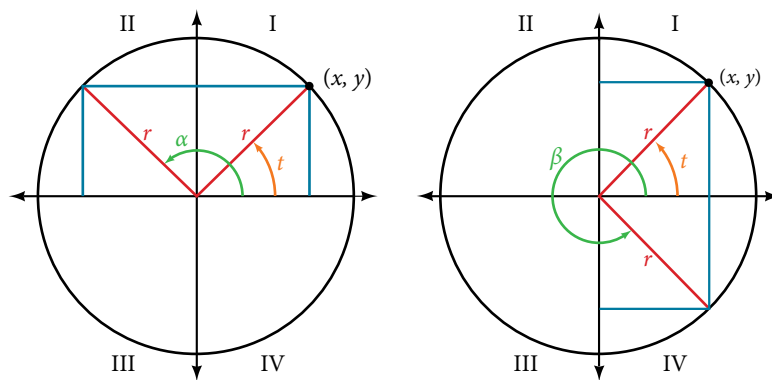


Figure 16

Recall that an angle's reference angle is the acute angle, t , formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90° , or 0 and $\frac{\pi}{2}$ radians. As we can see from **Figure 17**, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

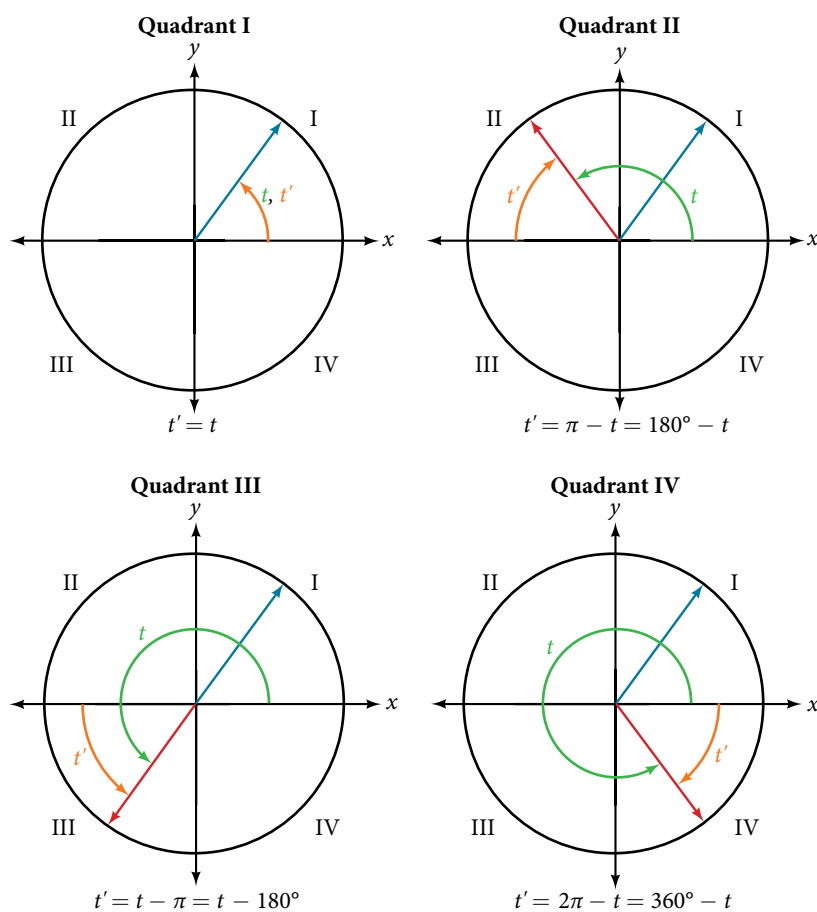


Figure 17

How To...

Given an angle between 0 and 2π , find its reference angle.

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π , add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π .

Example 5 Finding a Reference Angle

Find the reference angle of 225° as shown in **Figure 18**.

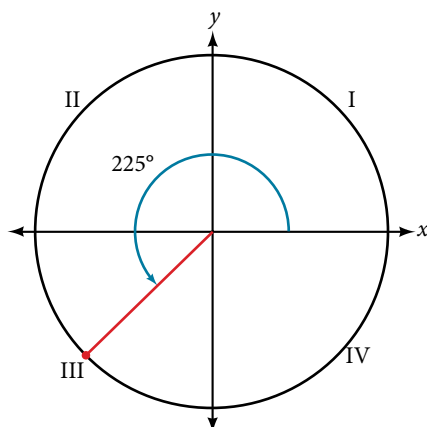


Figure 18

Solution Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

Try It #5

Find the reference angle of $\frac{5\pi}{3}$.

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90° or at a negative angle. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the reference angle of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x -values in that quadrant. The sine will be positive or negative depending on the sign of the y -values in that quadrant.

using reference angles to find cosine and sine

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To...

Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
 2. Determine the values of the cosine and sine of the reference angle.
 3. Give the cosine the same sign as the x -values in the quadrant of the original angle.
 4. Give the sine the same sign as the y -values in the quadrant of the original angle.
-

Example 6 Using Reference Angles to Find Sine and Cosine

- Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.
- Using the reference angle, find $\cos \frac{5\pi}{4}$ and $\sin \frac{5\pi}{4}$.

Solution

- 150° is located in the second quadrant. The angle it makes with the x -axis is $180^\circ - 150^\circ = 30^\circ$, so the reference angle is 30° .

This tells us that 150° has the same sine and cosine values as 30° , except for the sign. We know that

$$\cos(30^\circ) = \frac{\sqrt{3}}{2} \text{ and } \sin(30^\circ) = \frac{1}{2}.$$

Since 150° is in the second quadrant, the x -coordinate of the point on the circle is negative, so the cosine value is negative. The y -coordinate is positive, so the sine value is positive.

$$\cos(150^\circ) = -\frac{\sqrt{3}}{2} \text{ and } \sin(150^\circ) = \frac{1}{2}$$

- $\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \text{ and } \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

Try It #6

- Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
 - Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$.
-

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in **Figure 19**. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

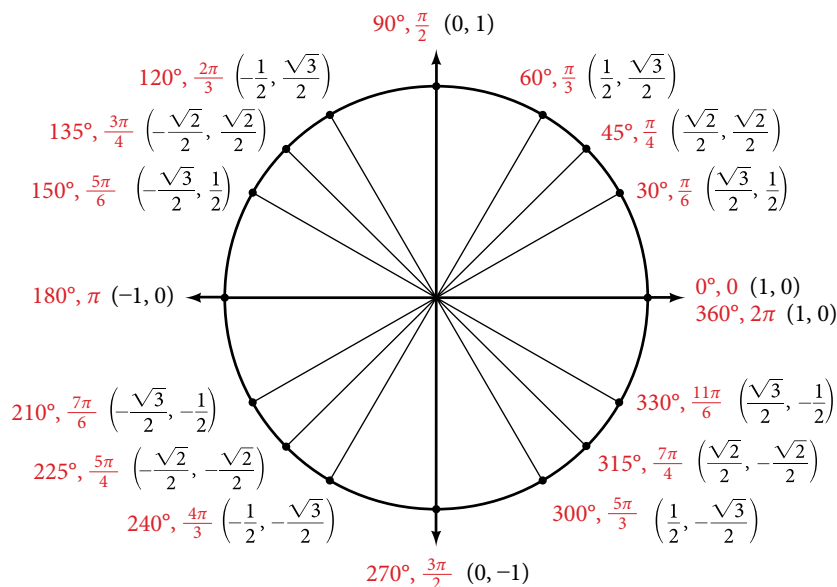


Figure 19 Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the identities

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the reference angle, and give them the signs corresponding to the y - and x -values of the quadrant.

How To...

Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.

1. Find the reference angle by measuring the smallest angle to the x -axis.
 2. Find the cosine and sine of the reference angle.
 3. Determine the appropriate signs for x and y in the given quadrant.
-

Example 7 Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x -axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π .

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos \theta$ and $y = \sin \theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It #7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Access these online resources for additional instruction and practice with sine and cosine functions.

- [Trigonometric Functions Using the Unit Circle \(http://openstaxcollege.org/l/trigunitcir\)](http://openstaxcollege.org/l/trigunitcir)
- [Sine and Cosine from the Unit Circle \(http://openstaxcollege.org/l/sincosuc\)](http://openstaxcollege.org/l/sincosuc)
- [Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six \(http://openstaxcollege.org/l/sincosmult\)](http://openstaxcollege.org/l/sincosmult)
- [Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four \(http://openstaxcollege.org/l/sincosmult4\)](http://openstaxcollege.org/l/sincosmult4)
- [Trigonometric Functions Using Reference Angles \(http://openstaxcollege.org/l/trigrefang\)](http://openstaxcollege.org/l/trigrefang)