

## LEARNING OBJECTIVES

In this section, you will:

- Find the limit of a sum, a difference, and a product.
- Find the limit of a polynomial.
- Find the limit of a power or a root.
- Find the limit of a quotient.

## 12.2 FINDING LIMITS: PROPERTIES OF LIMITS

Consider the rational function

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

The function can be factored as follows:

$$f(x) = \frac{(x-7)(x+1)}{x-7}, \text{ which gives us } f(x) = x + 1, x \neq 7.$$

Does this mean the function  $f$  is the same as the function  $g(x) = x + 1$ ?

The answer is no. Function  $f$  does not have  $x = 7$  in its domain, but  $g$  does. Graphically, we observe there is a hole in the graph of  $f(x)$  at  $x = 7$ , as shown in **Figure 1** and no such hole in the graph of  $g(x)$ , as shown in **Figure 2**.

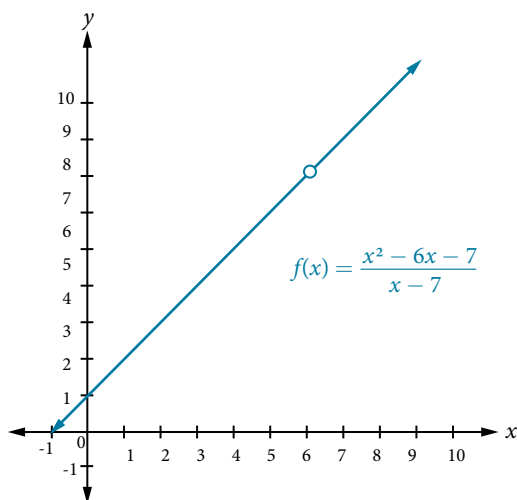


Figure 1 The graph of function  $f$  contains a break at  $x = 7$  and is therefore not continuous at  $x = 7$ .

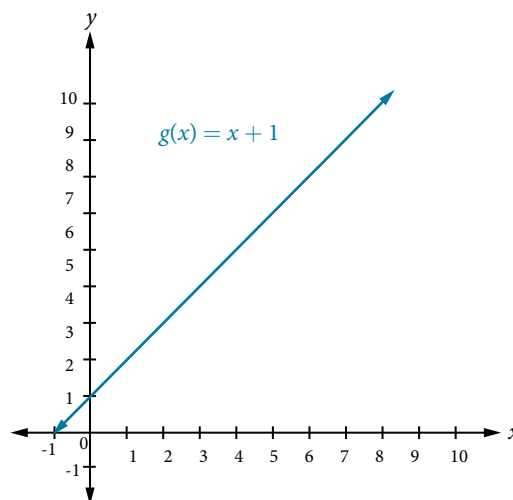


Figure 2 The graph of function  $g$  is continuous.

So, do these two different functions also have different limits as  $x$  approaches 7?

Not necessarily. Remember, in determining a limit of a function as  $x$  approaches  $a$ , what matters is whether the output approaches a real number as we get close to  $x = a$ . The existence of a limit does not depend on what happens when  $x$  equals  $a$ .

Look again at **Figure 1** and **Figure 2**. Notice that in both graphs, as  $x$  approaches 7, the output values approach 8. This means

$$\lim_{x \rightarrow 7^-} f(x) = \lim_{x \rightarrow 7^+} g(x).$$

Remember that when determining a limit, the concern is what occurs near  $x = a$ , not at  $x = a$ . In this section, we will use a variety of methods, such as rewriting functions by factoring, to evaluate the limit. These methods will give us formal verification for what we formerly accomplished by intuition.

## Finding the Limit of a Sum, a Difference, and a Product

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the **properties of limits**, which is a collection of theorems for finding limits.

Knowing the properties of limits allows us to compute limits directly. We can add, subtract, multiply, and divide the limits of functions as if we were performing the operations on the functions themselves to find the limit of the result. Similarly, we can find the limit of a function raised to a power by raising the limit to that power. We can also find the limit of the root of a function by taking the root of the limit. Using these operations on limits, we can find the limits of more complex functions by finding the limits of their simpler component functions.

### **properties of limits**

Let  $a$ ,  $k$ ,  $A$ , and  $B$  represent real numbers, and  $f$  and  $g$  be functions, such that  $\lim_{x \rightarrow a} f(x) = A$  and  $\lim_{x \rightarrow a} g(x) = B$ . For limits that exist and are finite, the properties of limits are summarized in **Table 1**.

Constant, $k$	$\lim_{x \rightarrow a} k = k$
Constant times a function	$\lim_{x \rightarrow a} [k \cdot f(x)] = k \lim_{x \rightarrow a} f(x) = kA$
Sum of functions	$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = A + B$
Difference of functions	$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = A - B$
Product of functions	$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = A \cdot B$
Quotient of functions	$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{A}{B}, B \neq 0$
Function raised to an exponent	$\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n = A^n$ , where $n$ is a positive integer
$n$ th root of a function, where $n$ is a positive integer	$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} [f(x)]} = \sqrt[n]{A}$
Polynomial function	$\lim_{x \rightarrow a} p(x) = p(a)$

Table 1

### **Example 1** Evaluating the Limit of a Function Algebraically

Evaluate  $\lim_{x \rightarrow 3} (2x + 5)$ .

**Solution**

$$\begin{aligned}
 \lim_{x \rightarrow 3} (2x + 5) &= \lim_{x \rightarrow 3} (2x) + \lim_{x \rightarrow 3} (5) && \text{Sum of functions property} \\
 &= 2 \lim_{x \rightarrow 3} (x) + \lim_{x \rightarrow 3} (5) && \text{Constant times a function property} \\
 &= 2(3) + 5 && \text{Evaluate} \\
 &= 11
 \end{aligned}$$

*Try It #1*

Evaluate the following limit:  $\lim_{x \rightarrow -12} (-2x + 2)$ .

## Finding the Limit of a Polynomial

Not all functions or their limits involve simple addition, subtraction, or multiplication. Some may include polynomials. Recall that a polynomial is an expression consisting of the sum of two or more terms, each of which consists of a constant and a variable raised to a nonnegative integral power. To find the limit of a polynomial function, we can find the limits of the individual terms of the function, and then add them together. Also, the limit of a polynomial function as  $x$  approaches  $a$  is equivalent to simply evaluating the function for  $a$ .

### How To...

Given a function containing a polynomial, find its limit.

1. Use the properties of limits to break up the polynomial into individual terms.
2. Find the limits of the individual terms.
3. Add the limits together.
4. Alternatively, evaluate the function for  $a$ .

### Example 2 Evaluating the Limit of a Function Algebraically

Evaluate  $\lim_{x \rightarrow 3} (5x^2)$ .

**Solution**

$$\begin{aligned}\lim_{x \rightarrow 3} (5x^2) &= 5 \lim_{x \rightarrow 3} (x^2) && \text{Constant times a function property} \\ &= 5(3^2) && \text{Function raised to an exponent property} \\ &= 45\end{aligned}$$

### Try It #2

Evaluate  $\lim_{x \rightarrow 4} (x^3 - 5)$ .

### Example 3 Evaluating the Limit of a Polynomial Algebraically

Evaluate  $\lim_{x \rightarrow 5} (2x^3 - 3x + 1)$ .

**Solution**

$$\begin{aligned}\lim_{x \rightarrow 5} (2x^3 - 3x + 1) &= \lim_{x \rightarrow 5} (2x^3) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} (1) && \text{Sum of functions} \\ &= 2 \lim_{x \rightarrow 5} (x^3) - 3 \lim_{x \rightarrow 5} (x) + \lim_{x \rightarrow 5} (1) && \text{Constant times a function} \\ &= 2(5^3) - 3(5) + 1 && \text{Function raised to an exponent} \\ &= 236 && \text{Evaluate}\end{aligned}$$

### Try It #3

Evaluate the following limit:  $\lim_{x \rightarrow -1} (x^4 - 4x^3 + 5)$ .

## Finding the Limit of a Power or a Root

When a limit includes a power or a root, we need another property to help us evaluate it. The square of the limit of a function equals the limit of the square of the function; the same goes for higher powers. Likewise, the square root of the limit of a function equals the limit of the square root of the function; the same holds true for higher roots.

### Example 4 Evaluating a Limit of a Power

Evaluate  $\lim_{x \rightarrow 2} (3x + 1)^5$ .

**Solution** We will take the limit of the function as  $x$  approaches 2 and raise the result to the 5<sup>th</sup> power.

$$\begin{aligned}\lim_{x \rightarrow 2} (3x + 1)^5 &= \left( \lim_{x \rightarrow 2} (3x + 1) \right)^5 \\ &= (3(2) + 1)^5 \\ &= 7^5 \\ &= 16,807\end{aligned}$$

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*Try It #4*

Evaluate the following limit:  $\lim_{x \rightarrow -4} (10x + 36)^3$ .

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*Q & A...*

**If we can't directly apply the properties of a limit, for example in  $\lim_{x \rightarrow 2} \left( \frac{x^2 + 6x + 8}{x - 2} \right)$ , can we still determine the limit of the function as  $x$  approaches  $a$ ?**

Yes. Some functions may be algebraically rearranged so that one can evaluate the limit of a simplified equivalent form of the function.

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## Finding the Limit of a Quotient

Finding the limit of a function expressed as a quotient can be more complicated. We often need to rewrite the function algebraically before applying the properties of a limit. If the denominator evaluates to 0 when we apply the properties of a limit directly, we must rewrite the quotient in a different form. One approach is to write the quotient in factored form and simplify.

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*How To...*

Given the limit of a function in quotient form, use factoring to evaluate it.

1. Factor the numerator and denominator completely.
  2. Simplify by dividing any factors common to the numerator and denominator.
  3. Evaluate the resulting limit, remembering to use the correct domain.
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### Example 5 Evaluating the Limit of a Quotient by Factoring

Evaluate  $\lim_{x \rightarrow 2} \left( \frac{x^2 - 6x + 8}{x - 2} \right)$ .

**Solution** Factor where possible, and simplify.

$$\begin{aligned}\lim_{x \rightarrow 2} \left( \frac{x^2 - 6x + 8}{x - 2} \right) &= \lim_{x \rightarrow 2} \left( \frac{(x - 2)(x - 4)}{x - 2} \right) && \text{Factor the numerator.} \\ &= \lim_{x \rightarrow 2} \left( \frac{\cancel{(x - 2)}(x - 4)}{\cancel{x - 2}} \right) && \text{Cancel the common factors.} \\ &= \lim_{x \rightarrow 2} (x - 4) && \text{Evaluate.} \\ &= 2 - 4 = -2\end{aligned}$$

*Analysis* When the limit of a rational function cannot be evaluated directly, factored forms of the numerator and denominator may simplify to a result that can be evaluated.

Notice, the function

$$f(x) = \frac{x^2 - 6x + 8}{x - 2}$$

is equivalent to the function

$$f(x) = x - 4, x \neq 2.$$

Notice that the limit exists even though the function is not defined at  $x = 2$ .

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**Try It #5**

Evaluate the following limit:  $\lim_{x \rightarrow 7} \left( \frac{x^2 - 11x + 28}{7 - x} \right)$ .

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**Example 6** Evaluating the Limit of a Quotient by Finding the LCD

Evaluate  $\lim_{x \rightarrow 5} \left( \frac{\frac{1}{x} - \frac{1}{5}}{x - 5} \right)$ .

**Solution** Find the LCD for the denominators of the two terms in the numerator, and convert both fractions to have the LCD as their denominator.

$$\begin{aligned} \lim_{x \rightarrow 5} \left( \frac{\frac{1}{x} - \frac{1}{5}}{x - 5} \right) &= \lim_{x \rightarrow 5} \left( \frac{5x \left( \frac{1}{x} - \frac{1}{5} \right)}{5x(x - 5)} \right) && \text{Multiply numerator and denominator by LCD.} \\ &= \lim_{x \rightarrow 5} \left( \frac{5x \left( \frac{1}{x} \right) - 5x \left( \frac{1}{5} \right)}{5x(x - 5)} \right) && \text{Apply distributive property.} \\ &= \lim_{x \rightarrow 5} \left( \frac{5 - x}{5x(x - 5)} \right) && \text{Simplify.} \\ &= \lim_{x \rightarrow 5} \left( \frac{-1(x - 5)}{5x(x - 5)} \right) && \text{Factor the numerator} \\ &= \lim_{x \rightarrow 5} -\frac{1}{5x} && \text{Cancel out like fractions} \\ &= -\frac{1}{5(5)} && \text{Evaluate for } x = 5 \\ &= -\frac{1}{25} \end{aligned}$$

**Analysis** When determining the limit of a rational function that has terms added or subtracted in either the numerator or denominator, the first step is to find the common denominator of the added or subtracted terms; then, convert both terms to have that denominator, or simplify the rational function by multiplying numerator and denominator by the least common denominator. Then check to see if the resulting numerator and denominator have any common factors.

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**Try It #6**

Evaluate  $\lim_{x \rightarrow -5} \left( \frac{\frac{1}{5} + \frac{1}{x}}{10 + 2x} \right)$ .

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**How To...**

Given a limit of a function containing a root, use a conjugate to evaluate.

1. If the quotient as given is not in indeterminate  $\left( \frac{0}{0} \right)$  form, evaluate directly.
  2. Otherwise, rewrite the sum (or difference) of two quotients as a single quotient, using the least common denominator (LCD).
  3. If the numerator includes a root, rationalize the numerator; multiply the numerator and denominator by the conjugate of the numerator. Recall that  $a \pm \sqrt{b}$  are conjugates.
  4. Simplify.
  5. Evaluate the resulting limit.
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### Example 7 Evaluating a Limit Containing a Root Using a Conjugate

Evaluate  $\lim_{x \rightarrow 0} \left( \frac{\sqrt{25-x} - 5}{x} \right)$ .

**Solution**

$$\lim_{x \rightarrow 0} \left( \frac{\sqrt{25-x} - 5}{x} \right) = \lim_{x \rightarrow 0} \left( \frac{\sqrt{25-x} - 5}{x} \cdot \frac{\sqrt{25-x} + 5}{\sqrt{25-x} + 5} \right)$$

Multiply numerator and denominator by the conjugate.

$$= \lim_{x \rightarrow 0} \left( \frac{(25-x) - 25}{x(\sqrt{25-x} + 5)} \right)$$

$$\begin{aligned} \text{Multiply: } (\sqrt{25-x} - 5) \cdot (\sqrt{25-x} + 5) \\ = (25-x) - 25. \end{aligned}$$

$$= \lim_{x \rightarrow 0} \left( \frac{-x}{x(\sqrt{25-x} + 5)} \right)$$

Combine like terms.

$$= \lim_{x \rightarrow 0} \left( \frac{\cancel{-x}}{\cancel{x}(\sqrt{25-x} + 5)} \right)$$

$$\text{Simplify } \frac{-x}{x} = -1.$$

$$= \frac{-1}{\sqrt{25-0} + 5}$$

Evaluate.

$$= \frac{-1}{5+5} = -\frac{1}{10}$$

**Analysis** When determining a **limit** of a function with a root as one of two terms where we cannot evaluate directly, think about multiplying the numerator and denominator by the conjugate of the terms.

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*Try It #7*

Evaluate the following limit:  $\lim_{h \rightarrow 0} \left( \frac{\sqrt{16-h} - 4}{h} \right)$ .

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### Example 8 Evaluating the Limit of a Quotient of a Function by Factoring

Evaluate  $\lim_{x \rightarrow 4} \left( \frac{4-x}{\sqrt{x}-2} \right)$ .

**Solution**

$$\lim_{x \rightarrow 4} \left( \frac{4-x}{\sqrt{x}-2} \right) = \lim_{x \rightarrow 4} \left( \frac{(2+\sqrt{x})(2-\sqrt{x})}{\sqrt{x}-2} \right)$$

Factor.

$$= \lim_{x \rightarrow 4} \left( \frac{(2+\sqrt{x})(\cancel{2-\sqrt{x}})}{-(\cancel{2-\sqrt{x}})} \right)$$

Factor  $-1$  out of the denominator. Simplify.

$$= \lim_{x \rightarrow 4} -(2+\sqrt{x})$$

Evaluate.

$$= -(2+\sqrt{4})$$

$$= -4$$

**Analysis** Multiplying by a conjugate would expand the numerator; look instead for factors in the numerator. Four is a perfect square so that the numerator is in the form

$$a^2 - b^2$$

and may be factored as

$$(a+b)(a-b).$$

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*Try It #8*

Evaluate the following limit:  $\lim_{x \rightarrow 3} \left( \frac{x-3}{\sqrt{x}-\sqrt{3}} \right)$ .

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*How To...*

Given a quotient with absolute values, evaluate its limit.

1. Try factoring or finding the LCD.
  2. If the **limit** cannot be found, choose several values close to and on either side of the input where the function is undefined.
  3. Use the numeric evidence to estimate the limits on both sides.
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**Example 9** Evaluating the Limit of a Quotient with Absolute Values

Evaluate  $\lim_{x \rightarrow 7} \frac{|x-7|}{x-7}$ .

**Solution** The function is undefined at  $x = 7$ , so we will try values close to 7 from the left and the right.

Left-hand limit:  $\frac{|6.9-7|}{6.9-7} = \frac{|6.99-7|}{6.99-7} = \frac{|6.999-7|}{6.999-7} = -1$

Right-hand limit:  $\frac{|7.1-7|}{7.1-7} = \frac{|7.01-7|}{7.01-7} = \frac{|7.001-7|}{7.001-7} = 1$

Since the left- and right-hand limits are not equal, there is no limit.

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*Try It #9*

Evaluate  $\lim_{x \rightarrow 6^+} \frac{6-x}{|x-6|}$ .

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Access the following online resource for additional instruction and practice with properties of limits.

- **Determine a Limit Analytically** (<http://openstaxcollege.org//limitanalytic>)