LEARNING OBJECTIVES

In this section, you will:

- Decompose $P(x)Q(x)$, where $Q(x)$ has only nonrepeated linear factors.
- Decompose $P(x)Q(x)$, where $Q(x)$ has repeated linear factors.
- Decompose $P(x)Q(x)$, where $Q(x)$ has a nonrepeated irreducible quadratic factor.
- Decompose $P(x)Q(x)$, where $Q(x)$ has a repeated irreducible quadratic factor.

9.4 PARTIAL FRACTIONS

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions.

Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

Decomposing $\frac{P(x)}{Q(x)}$ Where $Q(x)$ Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at partial fraction decomposition, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified rational expression to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2}$$

We would first need to find a common denominator, $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right)=\frac{2x+4-x+3}{(x+2)(x-3)}=\frac{x+7}{x^2-x-6}$$

Partial fraction decomposition is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

$$\underset{\text{Simplified sum}}{\frac{x+7}{x^2-x-6}} \qquad = \qquad \underset{\text{Partial fraction decomposition}}{\frac{2}{x-3}+\frac{-1}{x+2}}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of x^2-x-6 are $(x-3)(x+2)$, the denominators of the decomposed rational expression.

So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

partial fraction decomposition of $\frac{P(x)}{Q(x)}$: Q(x) has nonrepeated linear factors

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1 x + b_1)} + \frac{A_2}{(a_2 x + b_2)} + \frac{A_3}{(a_3 x + b_3)} + \ldots + \frac{A_n}{(a_n x + b_n)}.$$

How To...

Given a rational expression with distinct linear factors in the denominator, decompose it.

1. Use a variable for the original numerators, usually A, B, or C, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use A_n for each numerator
$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1 x + b_1)} + \frac{A_2}{(a_2 x + b_2)} + \ldots + \frac{A_n}{(a_n x + b_n)}.$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 1 Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)}$$

Solution We will separate the denominator factors and give each numerator a symbolic label, like A, B, or C.

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right] = \cancel{(x+2)}(x-1)\left[\frac{A}{\cancel{(x+2)}}\right] + (x+2)\cancel{(x-1)}\left[\frac{B}{\cancel{(x-1)}}\right]$$

The resulting equation is

$$3x = A(x-1) + B(x+2)$$

Expand the right side of the equation and collect like terms.

$$3x = Ax - A + Bx + 2B$$
$$3x = (A+B)x - A + 2B$$

Set up a system of equations associating corresponding coefficients.

$$3 = A + B$$
$$0 = -A + 2B$$

Add the two equations and solve for B.

$$\begin{array}{l} 3 = \mathrm{A} + B \\ \underline{0 = -A + 2B} \\ 3 = 0 + 3B \\ 1 = B \end{array}$$

Substitute $B = 1$ into one of the original equations in the system.

$$3 = A + 1$$
$$2 = A$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Another method to use to solve for A or B is by considering the equation that resulted from eliminating the fractions and substituting a value for x that will make either the A- or B-term equal 0. If we let $x = 1$, the A-term becomes 0 and we can simply solve for B.

$$\begin{aligned} 3x &= A(x-1) + B(x+2) \\ 3(1) &= A[(1)-1] + B[(1)+2] \\ 3 &= 0 + 3B \\ 1 &= B \end{aligned}$$

Next, either substitute $B = 1$ into the equation and solve for A, or make the B-term 0 by substituting $x = -2$ into the equation.

$$\begin{aligned} 3x &= A(x-1) + B(x+2) \\ 3(-2) &= A[(-2)-1] + B[(-2)+2] \\ -6 &= -3A + 0 \\ \frac{-6}{-3} &= A \\ 2 &= A \end{aligned}$$

We obtain the same values for A and B using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the Heaviside method, named after Charles Heaviside, a pioneer in the study of electronics.

Try It #1

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$

Decomposing $\frac{P(x)}{Q(x)}$ Where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

partial fraction decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ has repeated linear factors

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has repeated linear factor occurring n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \ldots + \frac{A_n}{(ax+b)^n}$$

Write the denominator powers in increasing order.

How To...

Given a rational expression with repeated linear factors, decompose it.

1. Use a variable like A, B, or C for the numerators and account for increasing powers of the denominators.
$$\frac{P(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 2 Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}$$

Solution The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: x, $(x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}=\frac{A}{x}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}$$

Next, we multiply both sides by the common denominator.

$$x(x-2)^2\left[\frac{-x^2+2x+4}{x(x-2)^2}\right]=\left[\frac{A}{x}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}\right]x(x-2)^2$$
$$-x^2+2x+4=A(x-2)^2+Bx(x-2)+Cx$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned}-x^2+2x+4&=A(x^2-4x+4)+B(x^2-2x)+Cx\\&=Ax^2-4Ax+4A+Bx^2-2Bx+Cx\\&=(A+B)x^2+(-4A-2B+C)x+4A\end{aligned}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$-x^2+2x+4=(A+B)x^2+(-4A-2B+C)x+4A$$
$$\begin{aligned}A+B&=-1 &&(1)\\-4A-2B+C&=2 &&(2)\\4A&=4 &&(3)\end{aligned}$$

Solving for A, we have

$$\begin{aligned}4A&=4\\A&=1\end{aligned}$$

Substitute $A=1$ into equation (1).

$$\begin{aligned}A+B&=-1\\(1)+B&=-1\\B&=-2\end{aligned}$$

Then, to solve for C, substitute the values for A and B into equation (2).

$$\begin{aligned}-4A-2B+C&=2\\-4(1)-2(-2)+C&=2\\-4+4+C&=2\\C&=2\end{aligned}$$

Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}=\frac{1}{x}-\frac{2}{(x-2)}+\frac{2}{(x-2)^2}$$

Try It #2

Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x-11}{(x-1)^2}$$

Decomposing $\frac{P(x)}{Q(x)}$ Where $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators A, B, or C representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax + B$, $Bx + C$, etc.

> ***decomposition of $\frac{P(x)}{Q(x)}$: Q(x) has a nonrepeated irreducible quadratic factor***
>
> The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the ***degree of*** $P(x)$ is less than the degree of $Q(x)$, is ***written as***
>
> $$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{(a_1x^2 + b_1x + c_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \dots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)}$$
>
> The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: A, B, C, and so on.

How To...

Given a rational expression where the factors of the denominator are distinct, irreducible quadratic factors, decompose it.

1. Use variables such as A, B, or C for the constant numerators over linear factors, and linear expressions such as $A_1 x + B_1$, $A_2x + B_2$, etc., for the numerators of each quadratic factor in the denominator.

$$\frac{P(x)}{Q(x)} = \frac{A}{ax + b} + \frac{A_1x + B_1}{(a_1x^2 + b_1x + c_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \dots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 3 **Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor**

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)}$$

Solution We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)} = \frac{A}{(x + 3)} + \frac{Bx + C}{(x^2 + x + 2)}$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$(x + 3)(x^2 + x + 2)\left[\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)}\right] = \left[\frac{A}{(x + 3)} + \frac{Bx + C}{(x^2 + x + 2)}\right](x + 3)(x^2 + x + 2)$$

$$8x^2 + 12x - 20 = A(x^2 + x + 2) + (Bx + C)(x + 3)$$

Notice we could easily solve for A by choosing a value for x that will make the $Bx + C$ term equal 0. Let $x = -3$ and substitute it into the equation.

$$8x^2 + 12x - 20 = A(x^2 + x + 2) + (Bx + C)(x + 3)$$
$$8(-3)^2 + 12(-3) - 20 = A((-3)^2 + (-3) + 2) + (B(-3) + C)((-3) + 3)$$
$$16 = 8A$$
$$A = 2$$

Now that we know the value of A, substitute it back into the equation. Then expand the right side and collect like terms.

$$8x^2 + 12x - 20 = 2(x^2 + x + 2) + (Bx + C)(x + 3)$$
$$8x^2 + 12x - 20 = 2x^2 + 2x + 4 + Bx^2 + 3B + Cx + 3C$$
$$8x^2 + 12x - 20 = (2 + B)x^2 + (2 + 3B + C)x + (4 + 3C)$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

$$2 + B = 8 \qquad (1)$$
$$2 + 3B + C = 12 \qquad (2)$$
$$4 + 3C = -20 \qquad (3)$$

Solve for B using equation (1) and solve for C using equation (3).

$$2 + B = 8 \qquad (1)$$
$$B = 6$$
$$4 + 3C = -20 \qquad (3)$$
$$3C = -24$$
$$C = -8$$

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)} = \frac{2}{(x + 3)} + \frac{6x - 8}{(x^2 + x + 2)}$$

Q & A…

Could we have just set up a system of equations to solve Example 3?

Yes, we could have solved it by setting up a system of equations without solving for A first. The expansion on the right would be:

$$8x^2 + 12x - 20 = Ax^2 + Ax + 2A + Bx^2 + 3B + Cx + 3C$$
$$8x^2 + 12x - 20 = (A + B)x^2 + (A + 3B + C)x + (2A + 3C)$$

So the system of equations would be:

$$A + B = 8$$
$$A + 3B + C = 12$$
$$2A + 3C = -20$$

Try It #3

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2 - 6x + 7}{(x - 1)(x^2 + 1)}$$

Decomposing $\dfrac{P(x)}{Q(x)}$ Where $Q(x)$ Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

decomposition of $\dfrac{P(x)}{Q(x)}$: when Q(x) has a repeated irreducible quadratic factor

The partial fraction decomposition of $\dfrac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{(ax^2+bx+c)^n} = \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Write the denominators in increasing powers.

How To...

Given a rational expression that has a repeated irreducible factor, decompose it.

1. Use variables like A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as

$$\frac{P(x)}{Q(x)} = \frac{A}{ax+b} + \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 4 Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$$

Solution The factors of the denominator are x, (x^2+1), and $(x^2+1)^2$. Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B$. So, let's begin the decomposition.

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

We eliminate the denominators by multiplying each term by $x(x^2+1)^2$. Thus,

$$x^4+x^3+x^2-x+1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$

Expand the right side.

$$\begin{aligned} x^4+x^3+x^2-x+1 &= A(x^4+2x^2+1) + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \\ &= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \end{aligned}$$

Now we will collect like terms.

$$x^4+x^3+x^2-x+1 = (A+B)x^4 + (C)x^3 + (2A+B+D)x^2 + (C+E)x + A$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.

$$\begin{aligned} A + B &= 1 \\ C &= 1 \\ 2A + B + D &= 1 \\ C + E &= -1 \\ A &= 1 \end{aligned}$$

We can use substitution from this point. Substitute $A = 1$ into the first equation.

$$\begin{aligned} 1 + B &= 1 \\ B &= 0 \end{aligned}$$

Substitute $A = 1$ and $B = 0$ into the third equation.

$$\begin{aligned} 2(1) + 0 + D &= 1 \\ D &= -1 \end{aligned}$$

Substitute $C = 1$ into the fourth equation.

$$\begin{aligned} 1 + E &= -1 \\ E &= -2 \end{aligned}$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A = 1$, $B = 0$, $C = 1$, $D = -1$, and $E = -2$. We can write the decomposition as follows:

$$\frac{x^4 + x^3 + x^2 - x + 1}{x(x^2 + 1)^2} = \frac{1}{x} + \frac{1}{(x^2 + 1)} - \frac{x + 2}{(x^2 + 1)^2}$$

Try It #4

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2}$$

Access these online resources for additional instruction and practice with partial fractions.

- Partial Fraction Decomposition (http://openstaxcollege.org/l/partdecomp)
- Partial Fraction Decomposition With Repeated Linear Factors (http://openstaxcollege.org/l/partdecomprlf)
- Partial Fraction Decomposition With Linear and Quadratic Factors (http://openstaxcollege.org/l/partdecomlqu)