

## LEARNING OBJECTIVES

In this section, you will:

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

## 1.3 RATES OF CHANGE AND BEHAVIOR OF GRAPHS

Gasoline costs have experienced some wild fluctuations over the last several decades. **Table 1**<sup>[5]</sup> lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

$y$	2005	2006	2007	2008	2009	2010	2011	2012
$C(y)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

Table 1

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \$2.31 to \$3.68, an increase of \$1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*. In this section, we will investigate changes such as these.

### Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in **Table 1** did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter  $\Delta$  (delta) signifies the change in a quantity; we read the ratio as “delta- $y$  over delta- $x$ ” or “the change in  $y$  divided by the change in  $x$ .” Occasionally we write  $\Delta f$  instead of  $\Delta y$ , which still represents the change in the function’s output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by \$1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7 \text{ years}} \approx 0.196 \text{ dollars per year}$$

On average, the price of gas increased by about 19.6¢ each year.

Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by \$4,000 per quarter

### rate of change

A rate of change describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are “output units per input units.”

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

#### How To...

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values  $x_1$  and  $x_2$ .

1. Calculate the difference  $y_2 - y_1 = \Delta y$ .
2. Calculate the difference  $x_2 - x_1 = \Delta x$ .
3. Find the ratio  $\frac{\Delta y}{\Delta x}$ .

### Example 1 Computing an Average Rate of Change

Using the data in **Table 1**, find the average rate of change of the price of gasoline between 2007 and 2009.

**Solution** In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year}\end{aligned}$$

**Analysis** Note that a decrease is expressed by a negative change or “negative increase.” A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

#### Try It #1

Using the data in **Table 1** at the beginning of this section, find the average rate of change between 2005 and 2010.

### Example 2 Computing Average Rate of Change from a Graph

Given the function  $g(t)$  shown in **Figure 1**, find the average rate of change on the interval  $[-1, 2]$ .

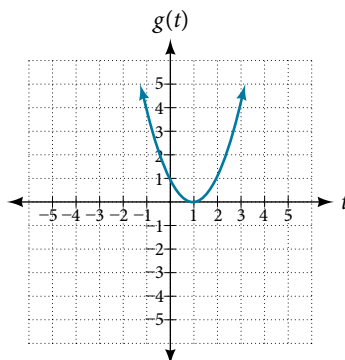


Figure 1

**Solution** At  $t = -1$ , **Figure 2** shows  $g(-1) = 4$ . At  $t = 2$ , the graph shows  $g(2) = 1$ .

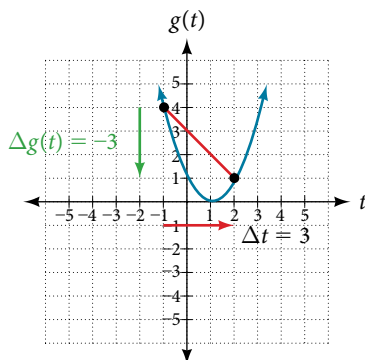


Figure 2

The horizontal change  $\Delta t = 3$  is shown by the red arrow, and the vertical change  $\Delta g(t) = -3$  is shown by the turquoise arrow. The output changes by  $-3$  while the input changes by  $3$ , giving an average rate of change of

$$\frac{1 - 4}{2 - (-1)} = \frac{-3}{3} = -1$$

*Analysis* Note that the order we choose is very important. If, for example, we use  $\frac{y_2 - y_1}{x_1 - x_2}$ , we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as  $(x_1, y_1)$  and  $(x_2, y_2)$ .

### Example 3 Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in **Table 2**. Find her average speed over the first 6 hours.

$t$ (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Table 2

**Solution** Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours, for an average speed of

$$\frac{292 - 10}{6 - 0} = \frac{282}{6} = 47$$

The average speed is 47 miles per hour.

*Analysis* Because the speed is not constant, the average speed depends on the interval chosen. For the interval  $[2, 3]$ , the average speed is 63 miles per hour.

### Example 4 Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of  $f(x) = x^2 - \frac{1}{x}$  on the interval  $[2, 4]$ .

**Solution** We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - \frac{1}{2} & f(4) &= 4^2 - \frac{1}{4} \\ &= 4 - \frac{1}{2} & &= 16 - \frac{1}{4} \\ &= \frac{7}{2} & &= \frac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned}
 \text{Average rate of change} &= \frac{f(4) - f(2)}{4 - 2} \\
 &= \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2} \\
 &= \frac{\frac{49}{4}}{2} \\
 &= \frac{49}{8}
 \end{aligned}$$

### Try It #2

Find the average rate of change of  $f(x) = x - 2\sqrt{x}$  on the interval  $[1, 9]$ .

### Example 5 Finding the Average Rate of Change of a Force

The electrostatic force  $F$ , measured in newtons, between two charged particles can be related to the distance between the particles  $d$ , in centimeters, by the formula  $F(d) = \frac{2}{d^2}$ . Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

**Solution** We are computing the average rate of change of  $F(d) = \frac{2}{d^2}$  on the interval  $[2, 6]$ .

$$\begin{aligned}
 \text{Average rate of change} &= \frac{F(6) - F(2)}{6 - 2} \\
 &= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{6 - 2} && \text{Simplify.} \\
 &= \frac{\frac{2}{36} - \frac{2}{4}}{4} \\
 &= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms.} \\
 &= -\frac{1}{9} && \text{Simplify.}
 \end{aligned}$$

The average rate of change is  $-\frac{1}{9}$  newton per centimeter.

### Example 6 Finding an Average Rate of Change as an Expression

Find the average rate of change of  $g(t) = t^2 + 3t + 1$  on the interval  $[0, a]$ . The answer will be an expression involving  $a$ .

**Solution** We use the average rate of change formula.

$$\begin{aligned}
 \text{Average rate of change} &= \frac{g(a) - g(0)}{a - 0} && \text{Evaluate.} \\
 &= \frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0} && \text{Simplify.} \\
 &= \frac{a^2 + 3a + 1 - 1}{a} && \text{Simplify and factor.} \\
 &= \frac{a(a + 3)}{a} && \text{Divide by the common factor } a. \\
 &= a + 3
 \end{aligned}$$

This result tells us the average rate of change in terms of  $a$  between  $t = 0$  and any other point  $t = a$ . For example, on the interval  $[0, 5]$ , the average rate of change would be  $5 + 3 = 8$ .

### Try It #3

Find the average rate of change of  $f(x) = x^2 + 2x - 8$  on the interval  $[5, a]$ .

## Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. **Figure 3** shows examples of increasing and decreasing intervals on a function.

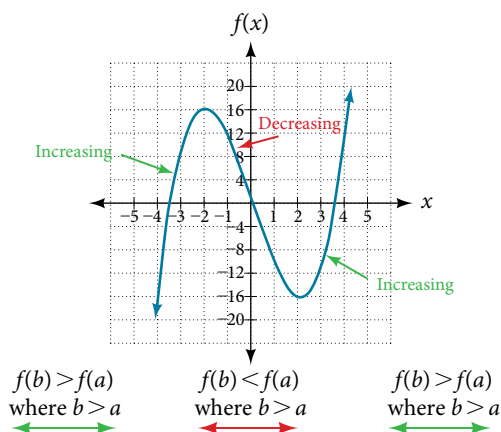


Figure 3 The function  $f(x) = x^3 - 12x$  is increasing on  $(-\infty, -2) \cup (2, \infty)$  and is decreasing on  $(-2, 2)$ .

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is “local minima.” Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is “extremum.”) Often, the term *local* is replaced by the term *relative*. In this text, we will use the term *local*.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function’s entire domain.

For the function whose graph is shown in **Figure 4**, the local maximum is 16, and it occurs at  $x = -2$ . The local minimum is  $-16$  and it occurs at  $x = 2$ .

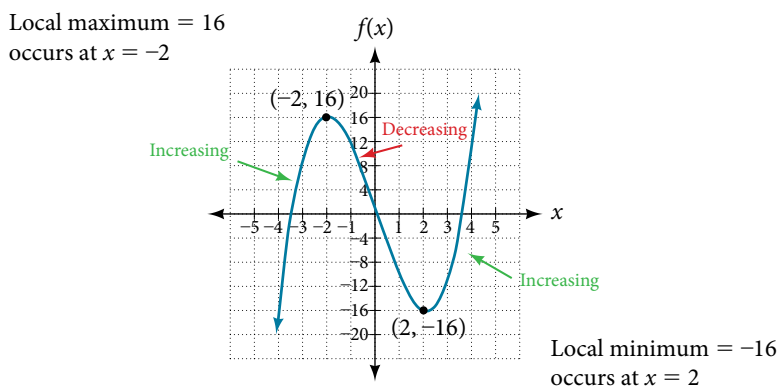


Figure 4

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. **Figure 5** illustrates these ideas for a local maximum.

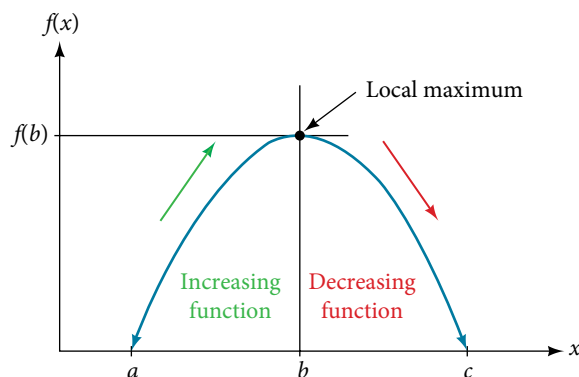


Figure 5 Definition of a local maximum

These observations lead us to a formal definition of local extrema.

### ***local minima and local maxima***

A function  $f$  is an **increasing function** on an open interval if  $f(b) > f(a)$  for every two input values  $a$  and  $b$  in the interval where  $b > a$ .

A function  $f$  is a **decreasing function** on an open interval if  $f(b) < f(a)$  for every two input values  $a$  and  $b$  in the interval where  $b > a$ .

A function  $f$  has a local maximum at a point  $b$  in an open interval  $(a, c)$  if  $f(b) \geq f(x)$  for every point  $x$  ( $x$  does not equal  $b$ ) in the interval.  $f$  has a local minimum at a point  $b$  in the interval  $(a, c)$  if  $f(b) \leq f(x)$  for every point  $x$  ( $x$  does not equal both) in the interval.

### **Example 7 Finding Increasing and Decreasing Intervals on a Graph**

Given the function  $p(t)$  in **Figure 6**, identify the intervals on which the function appears to be increasing.

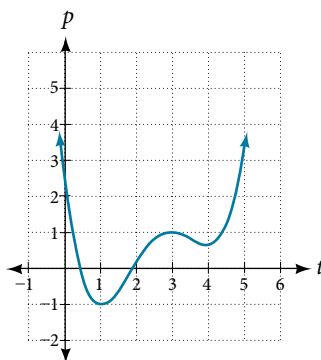


Figure 6

**Solution** We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from  $t = 1$  to  $t = 3$  and from  $t = 4$  on.

In interval notation, we would say the function appears to be increasing on the interval  $(1, 3)$  and the interval  $(4, \infty)$ .

*Analysis* Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at  $t = 1$ ,  $t = 3$ , and  $t = 4$ . These points are the local extrema (two minima and a maximum).

### Example 8 Finding Local Extrema from a Graph

Graph the function  $f(x) = \frac{2}{x} + \frac{x}{3}$ . Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

**Solution** Using technology, we find that the graph of the function looks like that in **Figure 7**. It appears there is a low point, or local minimum, between  $x = 2$  and  $x = 3$ , and a mirror-image high point, or local maximum, somewhere between  $x = -3$  and  $x = -2$ .

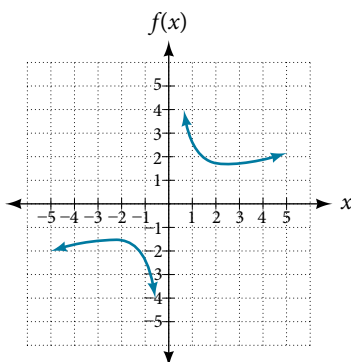


Figure 7

*Analysis* Most graphing calculators and graphing utilities can estimate the location of maxima and minima. **Figure 8** provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

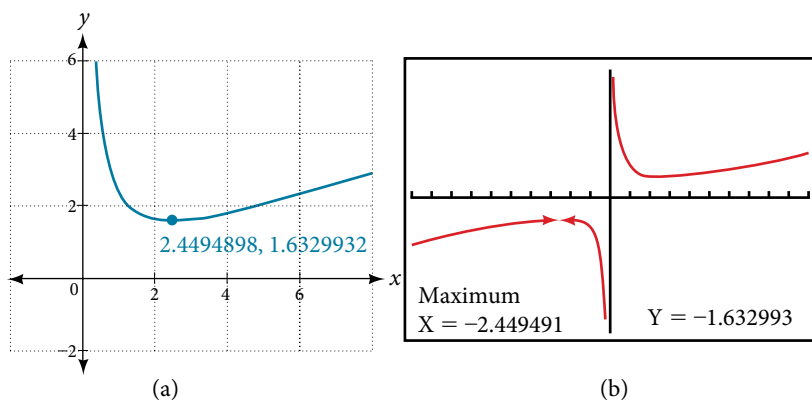


Figure 8

Based on these estimates, the function is increasing on the interval  $(-\infty, -2.449)$  and  $(2.449, \infty)$ . Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at  $\pm\sqrt{6}$ , but determining this requires calculus.)

### Try It #4

Graph the function  $f(x) = x^3 - 6x^2 - 15x + 20$  to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

### Example 9 Finding Local Maxima and Minima from a Graph

For the function  $f$  whose graph is shown in **Figure 9**, find all local maxima and minima.

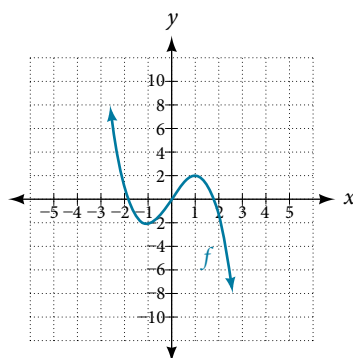


Figure 9

**Solution** Observe the graph of  $f$ . The graph attains a local maximum at  $x = 1$  because it is the highest point in an open interval around  $x = 1$ . The local maximum is the  $y$ -coordinate at  $x = 1$ , which is 2.

The graph attains a local minimum at  $x = -1$  because it is the lowest point in an open interval around  $x = -1$ .

The local minimum is the  $y$ -coordinate at  $x = -1$ , which is  $-2$ .

### Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in **Figure 10**, **Figure 11**, and **Figure 12**.

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 10



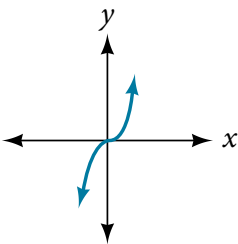
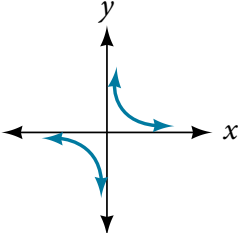
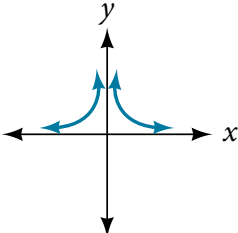
Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

Figure 11

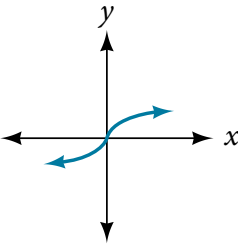
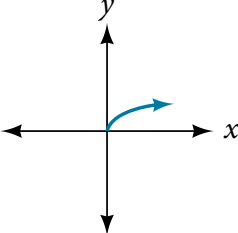
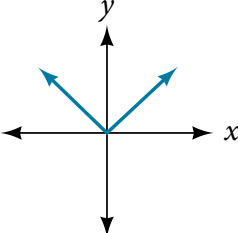
Function	Increasing/Decreasing	Example
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) =  x $	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	

Figure 12

## Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The  $y$ -coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively.

To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points on the domain of the function. See **Figure 13**.

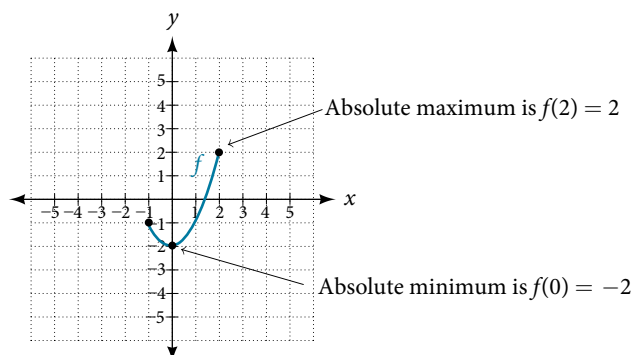


Figure 13

Not every function has an absolute maximum or minimum value. The toolkit function  $f(x) = x^3$  is one such function.

### ***absolute maxima and minima***

The **absolute maximum** of  $f$  at  $x = c$  is  $f(c)$  where  $f(c) \geq f(x)$  for all  $x$  in the domain of  $f$ .

The **absolute minimum** of  $f$  at  $x = d$  is  $f(d)$  where  $f(d) \leq f(x)$  for all  $x$  in the domain of  $f$ .

### Example 10 Finding Absolute Maxima and Minima from a Graph

For the function  $f$  shown in **Figure 14**, find all absolute maxima and minima.

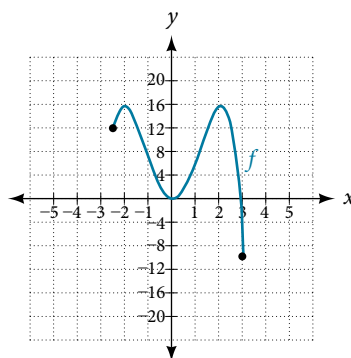


Figure 14

**Solution** Observe the graph of  $f$ . The graph attains an absolute maximum in two locations,  $x = -2$  and  $x = 2$ , because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the  $y$ -coordinate at  $x = -2$  and  $x = 2$ , which is 16.

The graph attains an absolute minimum at  $x = 3$ , because it is the lowest point on the domain of the function's graph. The absolute minimum is the  $y$ -coordinate at  $x = 3$ , which is  $-10$ .

Access this online resource for additional instruction and practice with rates of change.

- [Average Rate of Change \(http://openstaxcollege.org/l/aroc\)](http://openstaxcollege.org/l/aroc)