In this section, you will:

- Find the common difference for an arithmetic sequence.
- Write terms of an arithmetic sequence.
- Use a recursive formula for an arithmetic sequence.
- Use an explicit formula for an arithmetic sequence.
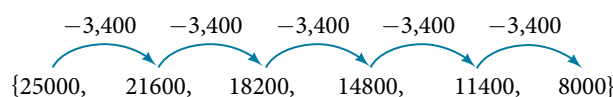
11.2   ARITHMETIC SEQUENCES

Companies often make large purchases, such as computers and vehicles, for business use. The book-value of these supplies decreases each year for tax purposes. This decrease in value is called depreciation. One method of calculating depreciation is straight-line depreciation, in which the value of the asset decreases by the same amount each year.

As an example, consider a woman who starts a small contracting business. She purchases a new truck for $25,000. After five years, she estimates that she will be able to sell the truck for $8,000. The loss in value of the truck will therefore be $17,000, which is $3,400 per year for five years. The truck will be worth $21,600 after the first year; $18,200 after two years; $14,800 after three years; $11,400 after four years; and $8,000 at the end of five years. In this section, we will consider specific kinds of sequences that will allow us to calculate depreciation, such as the truck's value.

Finding Common Differences

The values of the truck in the example are said to form an **arithmetic sequence** because they change by a constant amount each year. Each term increases or decreases by the same constant value called the **common difference** of the sequence. For this sequence, the common difference is $-3,400$.

$$-3{,}400 \quad -3{,}400 \quad -3{,}400 \quad -3{,}400 \quad -3{,}400$$
$$\{25000, \quad 21600, \quad 18200, \quad 14800, \quad 11400, \quad 8000\}$$

The sequence below is another example of an arithmetic sequence. In this case, the constant difference is 3. You can choose any term of the sequence, and add 3 to find the subsequent term.

$$+3 \quad +3 \quad +3 \quad +3$$
$$\{3, \quad 6, \quad 9, \quad 12, \quad 15, ...\}$$

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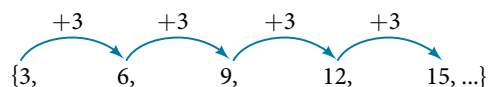
arithmetic sequence

An **arithmetic sequence** is a sequence that has the property that the difference between any two consecutive terms is a constant. This constant is called the **common difference**. If a_1 is the first term of an arithmetic sequence and d is the common difference, the sequence will be:

$$\{a_n\} = \{a_1, a_1 + d, a_1 + 2d, a_1 + 3d,...\}$$

Example 1   Finding Common Differences

Is each sequence arithmetic? If so, find the common difference.

a. $\{1, 2, 4, 8, 16, ... \}$ **b.** $\{ -3, 1, 5, 9, 13, ... \}$

Solution   Subtract each term from the subsequent term to determine whether a common difference exists.

a. The sequence is not arithmetic because there is no common difference.

$$2 - 1 = 1 \qquad 4 - 2 = 2 \qquad 8 - 4 = 4 \qquad 16 - 8 = 8$$

b. The sequence is arithmetic because there is a common difference. The common difference is 4.

$$1 - (-3) = 4 \qquad 5 - 1 = 4 \qquad 9 - 5 = 4 \qquad 13 - 9 = 4$$

Analysis The graph of each of these sequences is shown in ***Figure 1***. *We can see from the graphs that, although both sequences show growth, a is not linear whereas b is linear. Arithmetic sequences have a constant rate of change so their graphs will always be points on a line.*

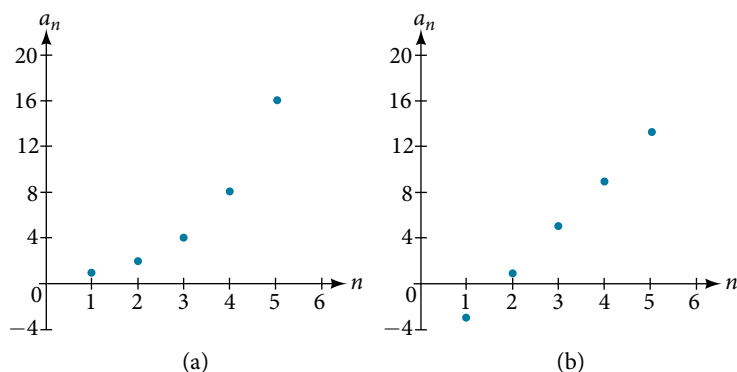(a)   (b)

Figure 1

Q & A...

If we are told that a sequence is arithmetic, do we have to subtract every term from the following term to find the common difference?

No. If we know that the sequence is arithmetic, we can choose any one term in the sequence, and subtract it from the subsequent term to find the common difference.

Try It #1

Is the given sequence arithmetic? If so, find the common difference.

$$\{18, 16, 14, 12, 10, \dots \}$$

Try It #2

Is the given sequence arithmetic? If so, find the common difference.

$$\{1, 3, 6, 10, 15, \dots \}$$

Writing Terms of Arithmetic Sequences

Now that we can recognize an arithmetic sequence, we will find the terms if we are given the first term and the common difference. The terms can be found by beginning with the first term and adding the common difference repeatedly. In addition, any term can also be found by plugging in the values of n and d into formula below.

$$a_n = a_1 + (n-1)d$$

How To...

Given the first term and the common difference of an arithmetic sequence, find the first several terms.

1. Add the common difference to the first term to find the second term.
2. Add the common difference to the second term to find the third term.
3. Continue until all of the desired terms are identified.
4. Write the terms separated by commas within brackets.

Example 2   Writing Terms of Arithmetic Sequences

Write the first five terms of the arithmetic sequence with $a_1 = 17$ and $d = -3$.

Solution   Adding -3 is the same as subtracting 3. Beginning with the first term, subtract 3 from each term to find the next term.

The first five terms are $\{17, 14, 11, 8, 5\}$
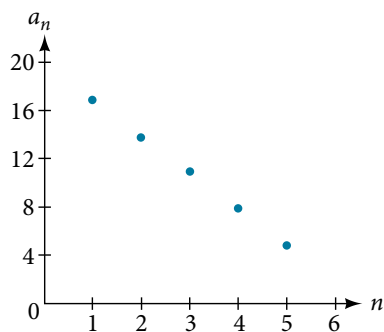
Figure 2

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Try It #3

List the first five terms of the arithmetic sequence with $a_1 = 1$ and $d = 5$.

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How To...

Given any first term and any other term in an arithmetic sequence, find a given term.

1. Substitute the values given for a_1, a_n, n into the formula $a_n = a_1 + (n-1)d$ to solve for d.
2. Find a given term by substituting the appropriate values for a_1, n, and d into the formula $a_n = a_1 + (n-1)d$.

Example 3   **Writing Terms of Arithmetic Sequences**

Given $a_1 = 8$ and $a_4 = 14$, find a_5.

Solution   The sequence can be written in terms of the initial term 8 and the common difference d.

$$\{8, 8 + d, 8 + 2d, 8 + 3d\}$$

We know the fourth term equals 14; we know the fourth term has the form $a_1 + 3d = 8 + 3d$.

We can find the common difference d.

$$a_n = a_1 + (n-1)d$$
$$a_4 = a_1 + 3d$$
$$a_4 = 8 + 3d \qquad \text{Write the fourth term of the sequence in terms of } a_1 \text{ and } d.$$
$$14 = 8 + 3d \qquad \text{Substitute 14 for } a_4.$$
$$d = 2 \qquad \text{Solve for the common difference.}$$

Find the fifth term by adding the common difference to the fourth term.

$$a_5 = a_4 + 2 = 16$$

Analysis Notice that the common difference is added to the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by adding the common difference to the first term nine times or by using the equation $a_n = a_1 + (n-1)d$.

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Try It #4

Given $a_3 = 7$ and $a_5 = 17$, find a_2.

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Using Recursive Formulas for Arithmetic Sequences

Some arithmetic sequences are defined in terms of the previous term using a recursive formula. The formula provides an algebraic rule for determining the terms of the sequence. A recursive formula allows us to find any term of an arithmetic sequence using a function of the preceding term. Each term is the sum of the previous term and the

common difference. For example, if the common difference is 5, then each term is the previous term plus 5. As with any recursive formula, the first term must be given.

$$a_n = a_{n-1} + d \quad n \geq 2$$

> **recursive formula for an arithmetic sequence**
> The recursive formula for an arithmetic sequence with common difference d is:
> $$a_n = a_{n-1} + d \quad n \geq 2$$

Given an arithmetic sequence, write its recursive formula.

1. Subtract any term from the subsequent term to find the common difference.
2. State the initial term and substitute the common difference into the recursive formula for arithmetic sequences.

Example 4   Writing a Recursive Formula for an Arithmetic Sequence

Write a recursive formula for the arithmetic sequence.

$$\{-18, -7, 4, 15, 26, \ldots\}$$

Solution   The first term is given as -18. The common difference can be found by subtracting the first term from the second term.

$$d = -7 - (-18) = 11$$

Substitute the initial term and the common difference into the recursive formula for arithmetic sequences.

$$a_1 = -18$$
$$a_n = a_{n-1} + 11, \text{ for } n \geq 2$$

Analysis   *We see that the common difference is the slope of the line formed when we graph the terms of the sequence, as shown in **Figure 3**. The growth pattern of the sequence shows the constant difference of 11 units.*
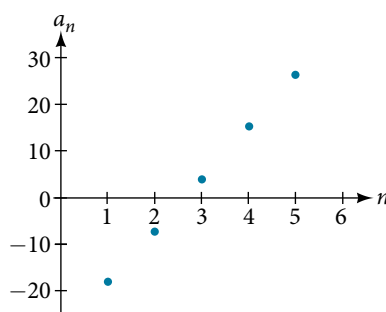

Figure 3

Do we have to subtract the first term from the second term to find the common difference?

No. We can subtract any term in the sequence from the subsequent term. It is, however, most common to subtract the first term from the second term because it is often the easiest method of finding the common difference.

Try It #5

Write a recursive formula for the arithmetic sequence.

$$\{25, 37, 49, 61, \ldots\}$$

Using Explicit Formulas for Arithmetic Sequences

We can think of an arithmetic sequence as a function on the domain of the natural numbers; it is a linear function because it has a constant rate of change. The common difference is the constant rate of change, or the slope of the function. We can construct the linear function if we know the slope and the vertical intercept.

$$a_n = a_1 + d(n - 1)$$

To find the y-intercept of the function, we can subtract the common difference from the first term of the sequence. Consider the following sequence.

$$\{200, \quad 150, \quad 100, \quad 50, \quad 0, ...\}$$

with -50, -50, -50, -50 between terms.
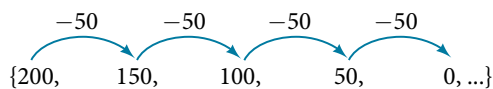
The common difference is -50, so the sequence represents a linear function with a slope of -50. To find the y-intercept, we subtract -50 from 200: $200 - (-50) = 200 + 50 = 250$. You can also find the y-intercept by graphing the function and determining where a line that connects the points would intersect the vertical axis. The graph is shown in **Figure 4**.
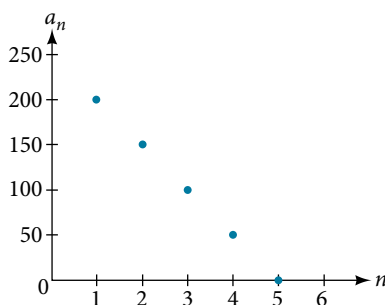

Figure 4

Recall the slope-intercept form of a line is $y = mx + b$. When dealing with sequences, we use a_n in place of y and n in place of x. If we know the slope and vertical intercept of the function, we can substitute them for m and b in the slope-intercept form of a line. Substituting -50 for the slope and 250 for the vertical intercept, we get the following equation:

$$a_n = -50n + 250$$

We do not need to find the vertical intercept to write an explicit formula for an arithmetic sequence. Another explicit formula for this sequence is $a_n = 200 - 50(n - 1)$, which simplifies to $a_n = -50n + 250$.

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explicit formula for an arithmetic sequence

An explicit formula for the nth term of an arithmetic sequence is given by

$$a_n = a_1 + d\,(n - 1)$$

How To...

Given the first several terms for an arithmetic sequence, write an explicit formula.

1. Find the common difference, $a_2 - a_1$.
2. Substitute the common difference and the first term into $a_n = a_1 + d(n - 1)$.

Example 5 Writing the nth Term Explicit Formula for an Arithmetic Sequence

Write an explicit formula for the arithmetic sequence.

$$\{2, 12, 22, 32, 42, ...\}$$

Solution The common difference can be found by subtracting the first term from the second term.

$$d = a_2 - a_1$$
$$= 12 - 2$$
$$= 10$$

The common difference is 10. Substitute the common difference and the first term of the sequence into the formula and simplify.

$$a_n = 2 + 10(n - 1)$$
$$a_n = 10n - 8$$

Analysis The graph of this sequence, represented in **Figure 5**, shows a slope of 10 *and a vertical intercept of* -8.
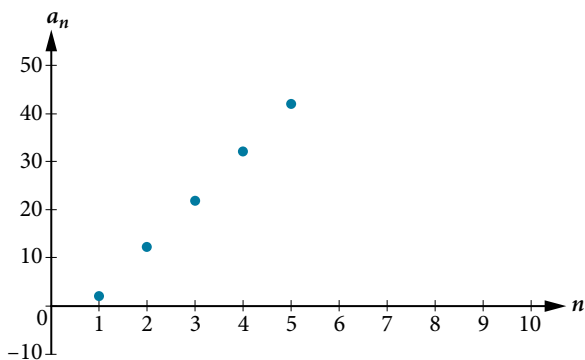

Figure 5

Try It #6

Write an explicit formula for the following arithmetic sequence.

$$\{50, 47, 44, 41, \dots\}$$

Finding the Number of Terms in a Finite Arithmetic Sequence

Explicit formulas can be used to determine the number of terms in a finite arithmetic sequence. We need to find the common difference, and then determine how many times the common difference must be added to the first term to obtain the final term of the sequence.

How To...

Given the first three terms and the last term of a finite arithmetic sequence, find the total number of terms.

1. Find the common difference d.
2. Substitute the common difference and the first term into $a_n = a_1 + d(n - 1)$.
3. Substitute the last term for a_n and solve for n.

Example 6   Finding the Number of Terms in a Finite Arithmetic Sequence

Find the number of terms in the finite arithmetic sequence.

$$\{8, 1, -6, \dots, -41\}$$

Solution The common difference can be found by subtracting the first term from the second term.

$$1 - 8 = -7$$

The common difference is -7. Substitute the common difference and the initial term of the sequence into the nth term formula and simplify.

$$a_n = a_1 + d(n-1)$$
$$a_n = 8 + (-7)(n-1)$$
$$a_n = 15 - 7n$$

Substitute -41 for a_n and solve for n

$$-41 = 15 - 7n$$
$$8 = n$$

There are eight terms in the sequence.

Try It #7

Find the number of terms in the finite arithmetic sequence.

$$\{6, 11, 16, \ldots, 56\}$$

Solving Application Problems with Arithmetic Sequences

In many application problems, it often makes sense to use an initial term of a_0 instead of a_1. In these problems, we alter the explicit formula slightly to account for the difference in initial terms. We use the following formula:

$$a_n = a_0 + dn$$

Example 7   Solving Application Problems with Arithmetic Sequences

A five-year old child receives an allowance of \$1 each week. His parents promise him an annual increase of \$2 per week.

    a. Write a formula for the child's weekly allowance in a given year.

    b. What will the child's allowance be when he is 16 years old?

Solution

    a. The situation can be modeled by an arithmetic sequence with an initial term of 1 and a common difference of 2.

        Let A be the amount of the allowance and n be the number of years after age 5. Using the altered explicit formula for an arithmetic sequence we get:

$$A_n = 1 + 2n$$

    b. We can find the number of years since age 5 by subtracting.

$$16 - 5 = 11$$

        We are looking for the child's allowance after 11 years. Substitute 11 into the formula to find the child's allowance at age 16.

$$A_{11} = 1 + 2(11) = 23$$

        The child's allowance at age 16 will be \$23 per week.

Try It #8

A woman decides to go for a 10-minute run every day this week and plans to increase the time of her daily run by 4 minutes each week. Write a formula for the time of her run after n weeks. How long will her daily run be 8 weeks from today?

Access this online resource for additional instruction and practice with arithmetic sequences.

- Arithmetic Sequences (http://openstaxcollege.org/l/arithmeticseq)