In this section, you will:
- Graph plane curves described by parametric equations by plotting points.
- Graph parametric equations.

8.7   PARAMETRIC EQUATIONS: GRAPHS

It is the bottom of the ninth inning, with two outs and two men on base. The home team is losing by two runs. The batter swings and hits the baseball at 140 feet per second and at an angle of approximately 45° to the horizontal. How far will the ball travel? Will it clear the fence for a game-winning home run? The outcome may depend partly on other factors (for example, the wind), but mathematicians can model the path of a projectile and predict approximately how far it will travel using parametric equations. In this section, we'll discuss parametric equations and some common applications, such as projectile motion problems.

Figure 1   Parametric equations can model the path of a projectile. (credit: Paul Kreher, Flickr)

Graphing Parametric Equations by Plotting Points

In lieu of a graphing calculator or a computer graphing program, plotting points to represent the graph of an equation is the standard method. As long as we are careful in calculating the values, point-plotting is highly dependable.

How To...

Given a pair of parametric equations, sketch a graph by plotting points.

1. Construct a table with three columns: t, $x(t)$, and $y(t)$.
2. Evaluate x and y for values of t over the interval for which the functions are defined.
3. Plot the resulting pairs (x, y).

Example 1   **Sketching the Graph of a Pair of Parametric Equations by Plotting Points**

Sketch the graph of the parametric equations $x(t) = t^2 + 1$, $y(t) = 2 + t$.

Solution   Construct a table of values for t, $x(t)$, and $y(t)$, as in **Table 1**, and plot the points in a plane.

t	$x(t) = t^2 + 1$	$y(t) = 2 + t$
-5	26	-3
-4	17	-2
-3	10	-1
-2	5	0
-1	2	1
0	1	2
1	2	3
2	5	4
3	10	5
4	17	6
5	26	7

Table 1

The graph is a parabola with vertex at the point (1, 2), opening to the right. See **Figure 2**.
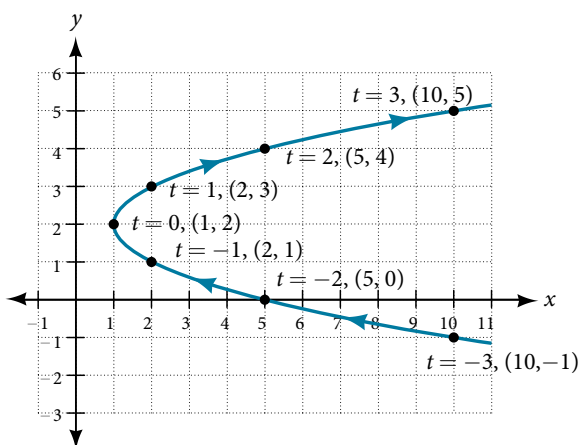

Figure 2

Analysis   *As values for t progress in a positive direction from 0 to 5, the plotted points trace out the top half of the parabola. As values of t become negative, they trace out the lower half of the parabola. There are no restrictions on the domain. The arrows indicate direction according to increasing values of t. The graph does not represent a function, as it will fail the vertical line test. The graph is drawn in two parts: the positive values for t, and the negative values for t.*

Try It #1

Sketch the graph of the parametric equations $x = \sqrt{t}, y = 2t + 3, 0 \le t \le 3$.

Example 2   Sketching the Graph of Trigonometric Parametric Equations

Construct a table of values for the given parametric equations and sketch the graph:

$$x = 2\cos t$$
$$y = 4\sin t$$

Solution   Construct a table like that in **Table 2** using angle measure in radians as inputs for t, and evaluating x and y. Using angles with known sine and cosine values for t makes calculations easier.

t	$x = 2\cos t$	$y = 4\sin t$
0	$x = 2\cos(0) = 2$	$y = 4\sin(0) = 0$
$\dfrac{\pi}{6}$	$x = 2\cos\left(\dfrac{\pi}{6}\right) = \sqrt{3}$	$y = 4\sin\left(\dfrac{\pi}{6}\right) = 2$
$\dfrac{\pi}{3}$	$x = 2\cos\left(\dfrac{\pi}{3}\right) = 1$	$y = 4\sin\left(\dfrac{\pi}{3}\right) = 2\sqrt{3}$
$\dfrac{\pi}{2}$	$x = 2\cos\left(\dfrac{\pi}{2}\right) = 0$	$y = 4\sin\left(\dfrac{\pi}{2}\right) = 4$
$\dfrac{2\pi}{3}$	$x = 2\cos\left(\dfrac{2\pi}{3}\right) = -1$	$y = 4\sin\left(\dfrac{2\pi}{3}\right) = -2\sqrt{3}$
$\dfrac{5\pi}{6}$	$x = 2\cos\left(\dfrac{5\pi}{6}\right) = -\sqrt{3}$	$y = 4\sin\left(\dfrac{5\pi}{6}\right) = 2$
π	$x = 2\cos(\pi) = -2$	$y = 4\sin(\pi) = 0$
$\dfrac{7\pi}{6}$	$x = 2\cos\left(\dfrac{7\pi}{6}\right) = -\sqrt{3}$	$y = 4\sin\left(\dfrac{7\pi}{6}\right) = -2$
$\dfrac{4\pi}{3}$	$x = 2\cos\left(\dfrac{4\pi}{3}\right) = -1$	$y = 4\sin\left(\dfrac{4\pi}{3}\right) = -2\sqrt{3}$
$\dfrac{3\pi}{2}$	$x = 2\cos\left(\dfrac{3\pi}{2}\right) = 0$	$y = 4\sin\left(\dfrac{3\pi}{2}\right) = -4$
$\dfrac{5\pi}{3}$	$x = 2\cos\left(\dfrac{5\pi}{3}\right) = 1$	$y = 4\sin\left(\dfrac{5\pi}{3}\right) = -2\sqrt{3}$
$\dfrac{11\pi}{6}$	$x = 2\cos\left(\dfrac{11\pi}{6}\right) = \sqrt{3}$	$y = 4\sin\left(\dfrac{11\pi}{6}\right) = -2$
2π	$x = 2\cos(2\pi) = 2$	$y = 4\sin(2\pi) = 0$

Table 2
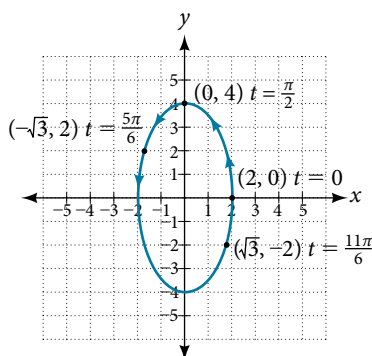
Figure 3 shows the graph.

Figure 3

By the symmetry shown in the values of x and y, we see that the parametric equations represent an ellipse.

The ellipse is mapped in a counterclockwise direction as shown by the arrows indicating increasing t values.

Analysis   *We have seen that parametric equations can be graphed by plotting points. However, a graphing calculator will save some time and reveal nuances in a graph that may be too tedious to discover using only hand calculations.*

*Make sure to change the mode on the calculator to parametric (**PAR**). To confirm, the **Y=** window should show*

$$X_{1T} =$$

$$Y_{1T} =$$

instead of $\mathbf{Y_1}=$.

Try It #2

Graph the parametric equations: $x = 5\cos t$, $y = 3\sin t$.

Example 3   Graphing Parametric Equations and Rectangular Form Together

Graph the parametric equations $x = 5\cos t$ and $y = 2\sin t$. First, construct the graph using data points generated from the parametric form. Then graph the rectangular form of the equation. Compare the two graphs.

Solution   Construct a table of values like that in **Table 3**.

t	$x = 5\cos t$	$y = 2\sin t$
0	$x = 5\cos(0) = 5$	$y = 2\sin(0) = 0$
1	$x = 5\cos(1) \approx 2.7$	$y = 2\sin(1) \approx 1.7$
2	$x = 5\cos(2) \approx -2.1$	$y = 2\sin(2) \approx 1.8$
3	$x = 5\cos(3) \approx -4.95$	$y = 2\sin(3) \approx 0.28$
4	$x = 5\cos(4) \approx -3.3$	$y = 2\sin(4) \approx -1.5$
5	$x = 5\cos(5) \approx 1.4$	$y = 2\sin(5) \approx -1.9$
−1	$x = 5\cos(-1) \approx 2.7$	$y = 2\sin(-1) \approx -1.7$
−2	$x = 5\cos(-2) \approx -2.1$	$y = 2\sin(-2) \approx -1.8$
−3	$x = 5\cos(-3) \approx -4.95$	$y = 2\sin(-3) \approx -0.28$
−4	$x = 5\cos(-4) \approx -3.3$	$y = 2\sin(-4) \approx 1.5$
−5	$x = 5\cos(-5) \approx 1.4$	$y = 2\sin(-5) \approx 1.9$

Table 3

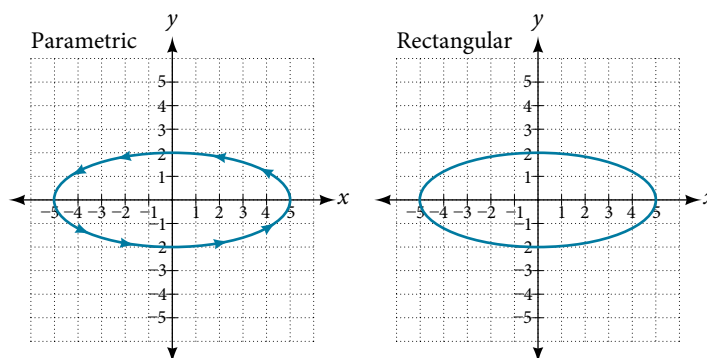Plot the (x, y) values from the table. See **Figure 4**.

Figure 4

Next, translate the parametric equations to rectangular form. To do this, we solve for t in either $x(t)$ or $y(t)$, and then substitute the expression for t in the other equation. The result will be a function $y(x)$ if solving for t as a function of x, or $x(y)$ if solving for t as a function of y.

$$x = 5\cos t$$

$$\frac{x}{5} = \cos t \qquad \text{Solve for } \cos t.$$

$$y = 2\sin t \qquad \text{Solve for } \sin t.$$

$$\frac{y}{2} = \sin t$$

Then, use the Pythagorean Theorem.

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Analysis   In **Figure 5**, the data from the parametric equations and the rectangular equation are plotted together. The parametric equations are plotted in blue; the graph for the rectangular equation is drawn on top of the parametric in a dashed style colored red. Clearly, both forms produce the same graph.
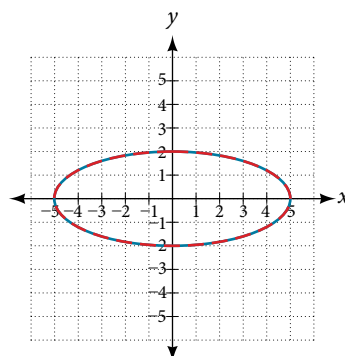

Figure 5

Example 4   Graphing Parametric Equations and Rectangular Equations on the Coordinate System

Graph the parametric equations $x = t + 1$ and $y = \sqrt{t}, t \geq 0$, and the rectangular equivalent $y = \sqrt{x-1}$ on the same coordinate system.

Solution   Construct a table of values for the parametric equations, as we did in the previous example, and graph $y = \sqrt{t}, t \geq 0$ on the same grid, as in **Figure 6**.
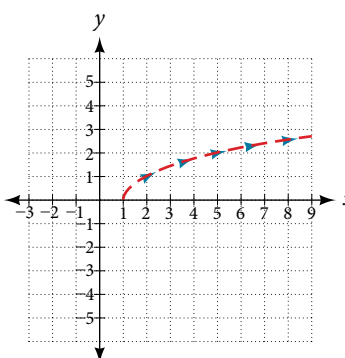

Figure 6

Analysis   With the domain on t restricted, we only plot positive values of t. The parametric data is graphed in blue and the graph of the rectangular equation is dashed in red. Once again, we see that the two forms overlap.

Try It #3

Sketch the graph of the parametric equations $x = 2\cos \theta$ and $y = 4\sin \theta$, along with the rectangular equation on the same grid.

Applications of Parametric Equations

Many of the advantages of parametric equations become obvious when applied to solving real-world problems. Although rectangular equations in x and y give an overall picture of an object's path, they do not reveal the position of an object at a specific time. Parametric equations, however, illustrate how the values of x and y change depending on t, as the location of a moving object at a particular time.

A common application of parametric equations is solving problems involving projectile motion. In this type of motion, an object is propelled forward in an upward direction forming an angle of θ to the horizontal, with an initial speed of v_0, and at a height h above the horizontal.

The path of an object propelled at an inclination of θ to the horizontal, with initial speed v_0, and at a height h above the horizontal, is given by

$$x = (v_0 \cos \theta)t$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin \theta)t + h$$

where g accounts for the effects of gravity and h is the initial height of the object. Depending on the units involved in the problem, use $g = 32$ ft/s² or $g = 9.8$ m/s². The equation for x gives horizontal distance, and the equation for y gives the vertical distance.

Example 5 — Finding the Parametric Equations to Describe the Motion of a Baseball

Solve the problem presented at the beginning of this section. Does the batter hit the game-winning home run? Assume that the ball is hit with an initial velocity of 140 feet per second at an angle of 45° to the horizontal, making contact 3 feet above the ground.

    a. Find the parametric equations to model the path of the baseball.

    b. Where is the ball after 2 seconds?

    c. How long is the ball in the air?

    d. Is it a home run?

Solution

    a. Use the formulas to set up the equations. The horizontal position is found using the parametric equation for x. Thus,

$$x = (v_0 \cos \theta)t$$
$$x = (140\cos(45°))t$$

The vertical position is found using the parametric equation for y. Thus,

$$y = -16t^2 + (v_0 \sin \theta)t + h$$
$$y = -16t^2 + (140\sin(45°))t + 3$$

    b. Substitute 2 into the equations to find the horizontal and vertical positions of the ball.

$$x = (140\cos(45°))(2)$$
$$x = 198 \text{ feet}$$
$$y = -16(2)^2 + (140\sin(45°))(2) + 3$$
$$y = 137 \text{ feet}$$

After 2 seconds, the ball is 198 feet away from the batter's box and 137 feet above the ground.

c. To calculate how long the ball is in the air, we have to find out when it will hit ground, or when $y = 0$. Thus,

$$y = -16t^2 + (140\sin(45°))t + 3$$

$$y = 0 \qquad\qquad\qquad \text{Set } y(t) = 0 \text{ and solve the quadratic.}$$

$$t = 6.2173$$

When $t = 6.2173$ seconds, the ball has hit the ground. (The quadratic equation can be solved in various ways, but this problem was solved using a computer math program.)

d. We cannot confirm that the hit was a home run without considering the size of the outfield, which varies from field to field. However, for simplicity's sake, let's assume that the outfield wall is 400 feet from home plate in the deepest part of the park. Let's also assume that the wall is 10 feet high. In order to determine whether the ball clears the wall, we need to calculate how high the ball is when $x = 400$ feet. So we will set $x = 400$, solve for t, and input t into y.

$$x = (140\cos(45°))t$$

$$400 = (140\cos(45°))t$$

$$t = 4.04$$

$$y = -16(4.04)^2 + (140\sin(45°))(4.04) + 3$$

$$y = 141.8$$

The ball is 141.8 feet in the air when it soars out of the ballpark. It was indeed a home run. See **Figure 7**.
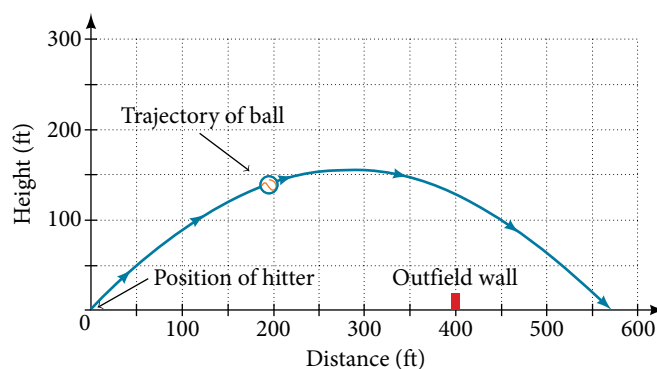

Figure 7

Access the following online resource for additional instruction and practice with graphs of parametric equations.

- Graphing Parametric Equations on the TI-84 (http://openstaxcollege.org/l/graphpara84)