In this section, you will:

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

8.1 NON-RIGHT TRIANGLES: LAW OF SINES

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in **Figure 1** that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving non-right triangles.
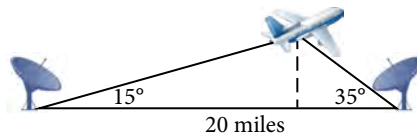
Figure 1

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

1. **ASA (angle-side-angle)** We know the measurements of two angles and the included side. See **Figure 2**.
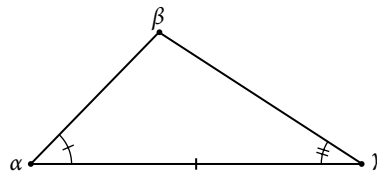
Figure 2

2. **AAS (angle-angle-side)** We know the measurements of two angles and a side that is not between the known angles. See **Figure 3**.
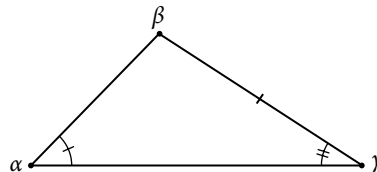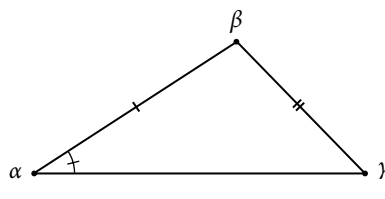
Figure 3

3. **SSA (side-side-angle)** We know the measurements of two sides and an angle that is not between the known sides. See **Figure 4**.

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in **Figure 5**.
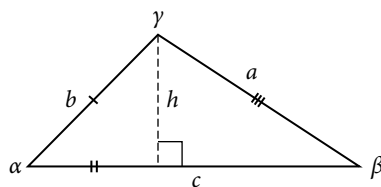

Figure 5

Using the right triangle relationships, we know that $\sin \alpha = \dfrac{h}{b}$ and $\sin \beta = \dfrac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin \alpha \text{ and } h = a\sin \beta$$

We then set the expressions equal to each other.

$$b\sin \alpha = a\sin \beta$$

$$\left(\frac{1}{ab}\right)(b\sin \alpha) = (a\sin \beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab}.$$

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$$

Similarly, we can compare the other ratios.

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c} \text{ and } \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

Collectively, these relationships are called the **Law of Sines.**

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See **Figure 6**.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.
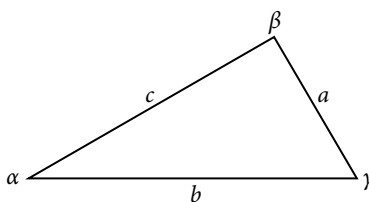

Figure 6

Law of Sines

Given a triangle with angles and opposite sides labeled as in **Figure 6**, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 1   **Solving for Two Unknown Sides and Angle of an AAS Triangle**

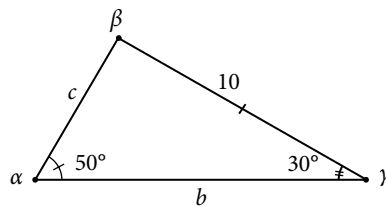Solve the triangle shown in **Figure 7** to the nearest tenth.

Figure 7

Solution   The three angles must add up to 180 degrees. From this, we can determine that

$$\beta = 180° - 50° - 30°$$
$$= 100°$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50°$ and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\frac{\sin(50°)}{10} = \frac{\sin(30°)}{c}$$

$$c\,\frac{\sin(50°)}{10} = \sin(30°) \qquad \text{Multiply both sides by } c.$$

$$c = \sin(30°)\,\frac{10}{\sin(50°)} \qquad \text{Multiply by the reciprocal to isolate } c.$$

$$c \approx 6.5$$

Similarly, to solve for b, we set up another proportion.

$$\frac{\sin(50°)}{10} = \frac{\sin(100°)}{b}$$

$$b\sin(50°) = 10\sin(100°) \qquad \text{Multiply both sides by } b.$$

$$b = \frac{10\sin(100°)}{\sin(50°)} \qquad \text{Multiply by the reciprocal to isolate } b.$$

$$b \approx 12.9$$

Therefore, the complete set of angles and sides is

$$\alpha = 50° \qquad a = 10$$
$$\beta = 100° \qquad b \approx 12.9$$
$$\gamma = 30° \qquad c \approx 6.5$$

Try It #1

Solve the triangle shown in **Figure 8** to the nearest tenth.
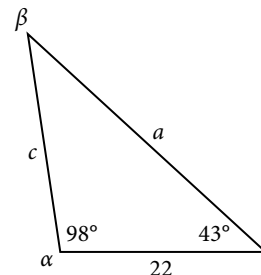

Figure 8

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an **ambiguous case**. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Oblique triangles in the category SSA may have four different outcomes. **Figure 9** illustrates the solutions with the known sides a and b and known angle α.

No triangle, $a < h$ Right triangle, $a = h$ Two triangles, $a > h$, $a < b$ One triangle, $a \geq b$

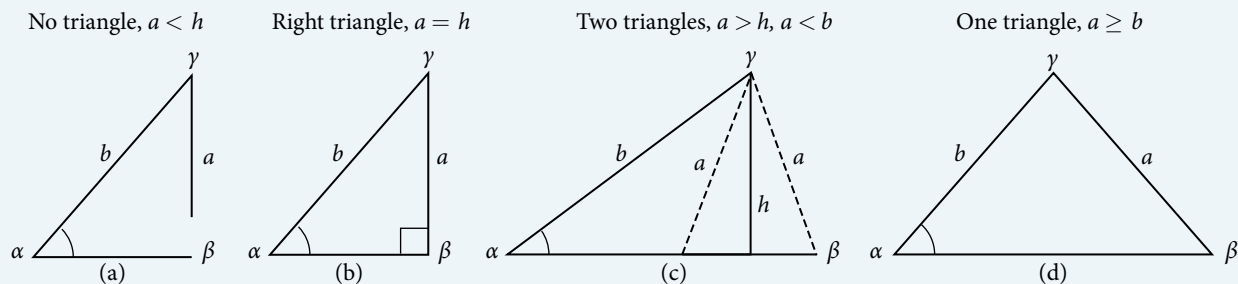(a)   (b)   (c)   (d)

Figure 9

Example 2   Solving an Oblique SSA Triangle

Solve the triangle in **Figure 10** for the missing side and find the missing angle measures to the nearest tenth.
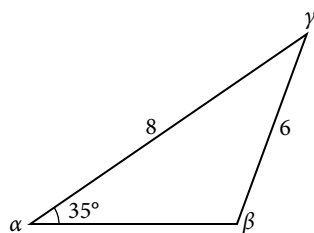

Figure 10

Solution   Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$$

$$\frac{\sin(35°)}{6} = \frac{\sin \beta}{8}$$

$$\frac{8\sin(35°)}{6} = \sin \beta$$

$$0.7648 \approx \sin \beta$$

$$\sin^{-1}(0.7648) \approx 49.9°$$

$$\beta \approx 49.9°$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have **Figure 11**. It appears that there may be a second triangle that will fit the given criteria.
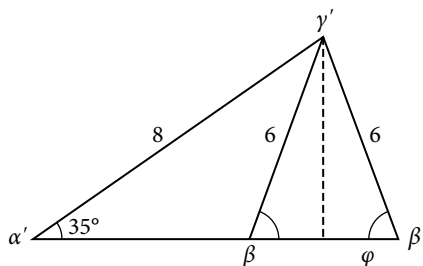

Figure 11

The angle supplementary to β is approximately equal to 49.9°, which means that $\beta = 180° - 49.9° = 130.1°$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\gamma = 180° - 35° - 130.1° \approx 14.9°$$

We can then use these measurements to solve the other triangle. Since γ' is supplementary to α and β, we have

$$\gamma' = 180° - 35° - 49.9° \approx 95.1°$$

Now we need to find c and c'.

We have

$$\frac{c}{\sin(14.9°)} = \frac{6}{\sin(35°)}$$

$$c = \frac{6\sin(14.9°)}{\sin(35°)} \approx 2.7$$

Finally,

$$\frac{c'}{\sin(95.1°)} = \frac{6}{\sin(35°)}$$

$$c' = \frac{6\sin(95.1°)}{\sin(35°)} \approx 10.4$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in **Figure 12**.

Figure 12
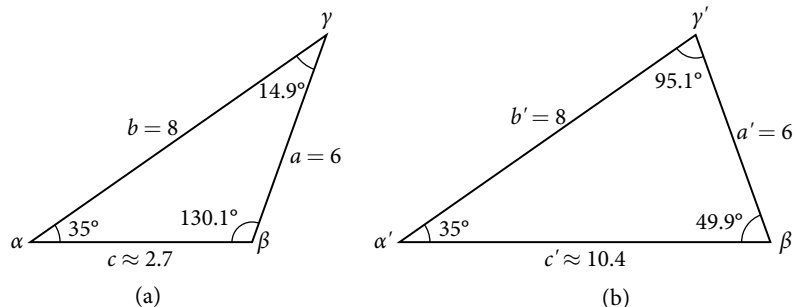
However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (*a*) in **Figure 12**.

Try It #2

Given $\alpha = 80°$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Example 3   Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in **Figure 13**, solve for the unknown side and angles. Round your answers to the nearest tenth.
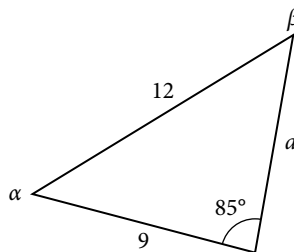

Figure 13

Solution   In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle $\gamma = 85°$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85°)}{12} = \frac{\sin \beta}{9} \qquad \text{Isolate the unknown.}$$

$$\frac{9\sin(85°)}{12} = \sin \beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85°)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3°$$

In this case, if we subtract β from 180°, we find that there may be a second possible solution. Thus, $\beta = 180° - 48.3° \approx 131.7°$. To check the solution, subtract both angles, 131.7° and 85°, from 180°. This gives

$$\alpha = 180° - 85° - 131.7° \approx -36.7°,$$

which is impossible, and so $\beta \approx 48.3°$.

To find the remaining missing values, we calculate $\alpha = 180° - 85° - 48.3° \approx 46.7°$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85°)}{12} = \frac{\sin(46.7°)}{a}$$
$$a\frac{\sin(85°)}{12} = \sin(46.7°)$$
$$a = \frac{12\sin(46.7°)}{\sin(85°)} \approx 8.8$$

The complete set of solutions for the given triangle is

$$\alpha \approx 46.7° \qquad a \approx 8.8$$
$$\beta \approx 48.3° \qquad b = 9$$
$$\gamma = 85° \qquad c = 12$$

Try It #3

Given $\alpha = 80°$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Example 4   Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution   Using the given information, we can solve for the angle opposite the side of length 10. See **Figure 14**.
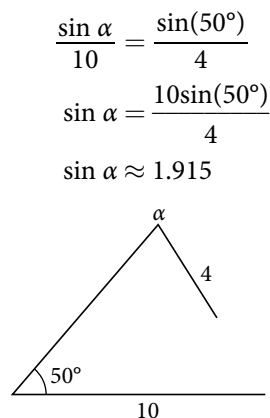
$$\frac{\sin \alpha}{10} = \frac{\sin(50°)}{4}$$
$$\sin \alpha = \frac{10\sin(50°)}{4}$$
$$\sin \alpha \approx 1.915$$

Figure 14

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$ in a graphing calculator generates an **ERROR DOMAIN**. Therefore, no triangles can be drawn with the provided dimensions.

Try It #4

Determine the number of triangles possible given $a = 31$, $b = 26$, $\beta = 48°$.

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as Area $= \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in **Figure 15**, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin \alpha = \dfrac{\text{opposite}}{\text{hypotenuse}}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin \alpha = \dfrac{h}{c}$ or $c \sin \alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.
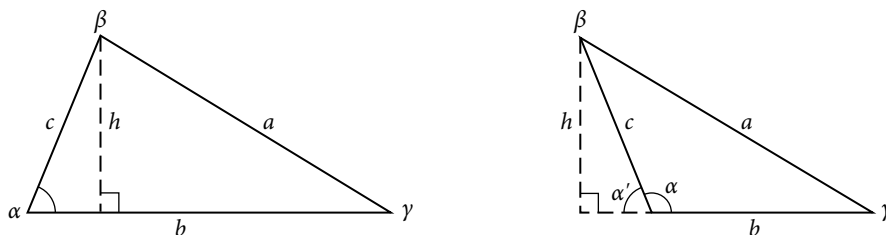

Figure 15

Thus,
$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}b(c\sin \alpha)$$

Similarly,
$$\text{Area} = \frac{1}{2}a(b\sin \gamma) = \frac{1}{2}a(c\sin \beta)$$

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area of an oblique triangle

The formula for the area of an oblique triangle is given by
$$\text{Area} = \frac{1}{2}bc\sin \alpha$$
$$= \frac{1}{2}ac\sin \beta$$
$$= \frac{1}{2}ab\sin \gamma$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Example 5 Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102°$. Round the area to the nearest integer.

Solution Using the formula, we have
$$\text{Area} = \frac{1}{2}ab\sin \gamma$$
$$\text{Area} = \frac{1}{2}(90)(52)\sin(102°)$$
$$\text{Area} \approx 2289 \text{ square units}$$

Try It #5

Find the area of the triangle given $\beta = 42°$, $a = 7.2$ ft, $c = 3.4$ ft. Round the area to the nearest tenth.

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

<u>Example 6</u>   **Finding an Altitude**

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in **Figure 16**. Round the altitude to the nearest tenth of a mile.
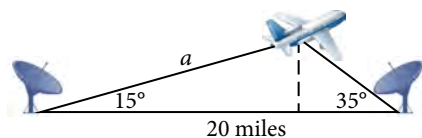

Figure 16

Solution   To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side *a,* and then use right triangle relationships to find the height of the aircraft, *h.*

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180° - 15° - 35° = 130°$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\frac{\sin(130°)}{20} = \frac{\sin(35°)}{a}$$

$$a\sin(130°) = 20\sin(35°)$$

$$a = \frac{20\sin(35°)}{\sin(130°)}$$

$$a \approx 14.98$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know *a,* we can use right triangle relationships to solve for *h.*

$$\sin(15°) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin(15°) = \frac{h}{a}$$

$$\sin(15°) = \frac{h}{14.98}$$

$$h = 14.98\sin(15°)$$

$$h \approx 3.88$$

The aircraft is at an altitude of approximately 3.9 miles.

Try It #6

The diagram shown in **Figure 17** represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point *A,* is 70°, the angle of elevation from the northern end zone, point *B,* is 62°, and the distance between the viewing points of the two end zones is 145 yards.
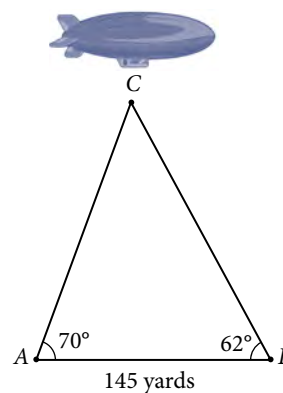

Figure 17

Access the following online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics (http://openstaxcollege.org/l/sinesbasic)
- Law of Sines: The Ambiguous Case (http://openstaxcollege.org/l/sinesambiguous)