LEARNING OBJECTIVES

In this section, you will:

- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

3.9 MODELING USING VARIATION

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for \$4,600, she will earn \$736. She wants to evaluate the offer, but she is not sure how. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

Solving Direct Variation Problems

In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, e, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See **Table 1**.

***s*, sales prices**	$\boldsymbol{e = 0.16s}$	**Interpretation**
\$4,600	$e = 0.16(4{,}600) = 736$	A sale of *a* \$4,600 vehicle results in \$736 earnings.
\$9,200	$e = 0.16(9{,}200) = 1{,}472$	A sale of *a* \$9,200 vehicle results in \$1472 earnings.
\$18,400	$e = 0.16(18{,}400) = 2{,}944$	A sale of *a* \$18,400 vehicle results in \$2944 earnings.

Table 1

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from \$4,600 to \$9,200, and we double the earnings from \$736 to \$1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value k is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$.

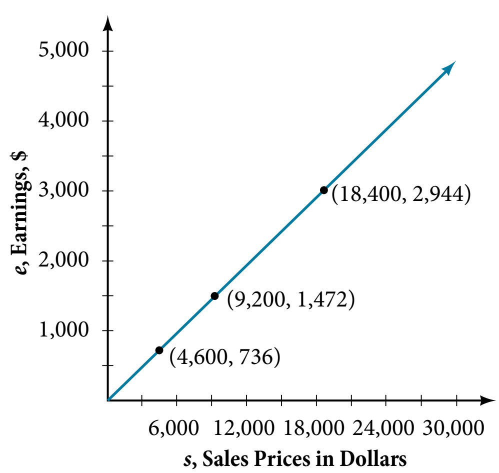

Figure 1

direct variation

If x and y are related by an equation of the form

$$y = kx^n$$

then we say that the relationship is **direct variation** and y **varies directly** with the nth power of x. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where k is called the **constant of variation**, which help defines the relationship between the variables.

How To...

Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to divide y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example 1 Solving a Direct Variation Problem

The quantity y varies directly with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Solution The general formula for direct variation with a cube is $y = kx^3$. The constant can be found by dividing y by the cube of x.

$$\begin{aligned} k &= \frac{y}{x^3} \\ &= \frac{25}{2^3} \\ &= \frac{25}{8} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \frac{25}{8}(6)^3 \\ &= 675 \end{aligned}$$

Analysis *The graph of this equation is a simple cubic, as shown in **Figure 2**.*

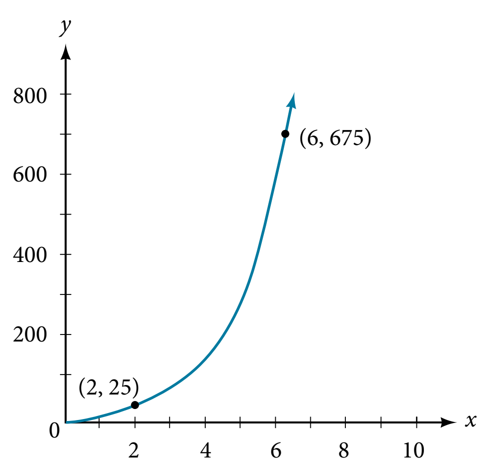

Figure 2

Q & A…

Do the graphs of all direct variation equations look like Example 1?

No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through (0,0).

Try It #1

The quantity y varies directly with the square of x. If $y = 24$ when $x = 3$, find y when x is 4.

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. Between the depths of 250 feet and 500 feet, the formula $T = \frac{14,000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F. If we create **Table 2**, we observe that, as the depth increases, the water temperature decreases.

***d*, depth**	$T = \frac{14,000}{d}$	**Interpretation**
500 ft	$\frac{14,000}{500} = 28$	At a depth of 500 ft, the water temperature is 28° F.
350 ft	$\frac{14,000}{350} = 40$	At a depth of 350 ft, the water temperature is 40° F.
250 ft	$\frac{14,000}{250} = 56$	At a depth of 250 ft, the water temperature is 56° F.

Table 2

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, **Figure 3** depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14,000$.

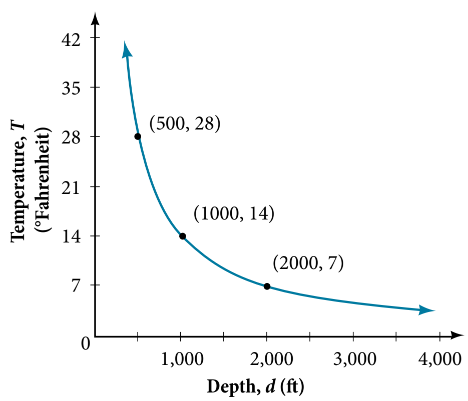

Figure 3

inverse variation

If x and y are related by an equation of the form

$$y = \frac{k}{x^n}$$

where k is a nonzero constant, then we say that y **varies inversely** with the nth power of x. In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

Example 2 Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

Solution Recall that multiplying speed by time gives distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which the tourist drives, then $vt =$ distance. Because the distance is fixed at 100 miles, $vt = 100$. Solving this relationship for the time gives us our function.

$$\begin{aligned} t(v) &= \frac{100}{v} \\ &= 100v^{-1} \end{aligned}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a fraction.

How To...

Given a description of an indirect variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to multiply y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example 3 Solving an Inverse Variation Problem

A quantity y varies inversely with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Solution The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying y by the cube of x.

$$\begin{aligned} k &= x^3 y \\ &= 2^3 \cdot 25 \\ &= 200 \end{aligned}$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x^3},\ k = 200$$

$$y = \frac{200}{x^3}$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \frac{200}{6^3} \\ &= \frac{25}{27} \end{aligned}$$

Analysis *The graph of this equation is a rational function, as shown in* ***Figure 4***.

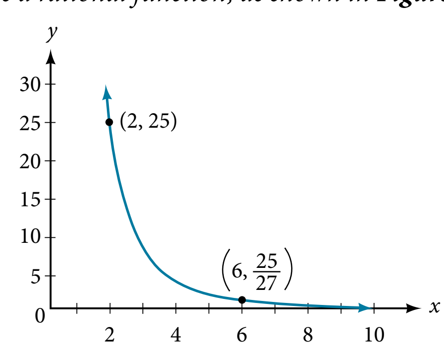

Figure 4

Try It #2

A quantity y varies inversely with the square of x. If $y = 8$ when $x = 3$, find y when x is 4.

Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called **joint variation**. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable c, cost, varies jointly with the number of students, n, and the distance, d.

> ***joint variation***
>
> Joint variation occurs when a variable varies directly or inversely with multiple variables.
>
> For instance, if x varies directly with both y and z, we have $x = kyz$. If x varies directly with y and inversely with z, we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

Example 4 **Solving Problems Involving Joint Variation**

A quantity x varies directly with the square of y and inversely with the cube root of z. If $x = 6$ when $y = 2$ and $z = 8$, find x when $y = 1$ and $z = 27$.

Solution Begin by writing an equation to show the relationship between the variables.

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6$, $y = 2$, and $z = 8$ to find the value of the constant k.

$$6 = \frac{k2^2}{\sqrt[3]{8}}$$

$$6 = \frac{4k}{2}$$

$$3 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find x when $y = 1$ and $z = 27$, we will substitute values for y and z into our equation.

$$x = \frac{3(1)^2}{\sqrt[3]{27}}$$

$$= 1$$

Try It #3

x varies directly with the square of y and inversely with z. If $x = 40$ when $y = 4$ and $z = 2$, find x when $y = 10$ and $z = 25$.

Access these online resources for additional instruction and practice with direct and inverse variation.

- Direct Variation (http://openstaxcollege.org/l/directvariation)
- Inverse Variation (http://openstaxcollege.org/l/inversevariatio)
- Direct and Inverse Variation (http://openstaxcollege.org/l/directinverse)