Relativity Notes

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Contents

1 Review

- Minkowski Space

  4 dimensions

  4-vector ($\vec{R}$)

$$(x, y, z, ct); \qquad ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

  ds^2 is called "**metric**"

- Euclidean Space

  Distance between two points

$$dr^2 = dx^2 + dy^2 + dz^2$$

  3 vectors ($\vec{r}$), anything that transform like a displacement vector is define to be a vector.

  Scalar = scalar ($S_1 = S_2 \to S_1' = S_2'$)

  Vector = vector ($\vec{a} = \vec{b} \to \vec{a}' = \vec{b}'$)

$$\frac{d\vec{a}}{ds} \quad \text{is a } \textbf{vector}$$

$$\begin{aligned} a &= |\vec{a}| \\ \vec{a} &= (a_1, a_2, a_3) \\ a^2 &= a_1^2 + a_2^2 + a_3^2 \quad (\text{must be a } \textbf{scalar}) \\ \vec{a} \cdot \vec{b} &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

1.1 Lorentz Transformation

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \\ ct' &= \gamma(ct - \beta x) \end{aligned}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

2 4-vector

$$\begin{aligned}
\vec{A} &= (A_1, A_2, A_3, A_4) \\
\vec{A}^2 &= (A_4^2 - A_1^2 - A_2^2 - A_3^2) \\
\vec{A} \cdot \vec{B} &= A_4B_4 - A_1B_1 - A_2B_2 - A_3B_3 \quad \textbf{scalar} \\
f &= (\vec{A} + \vec{B})^2 - \vec{A}^2 - \vec{B}^2 \\
f &= (A_4 + B_4)^2 - (A_1 + B_1)^2 - (A_2 + B_2)^2 - (A_3 + B_3)^2 - (A_4^2 - A_3^2 - A_2^2 - A_1^2) - (B_4^2 - B_3^2 - B_2^2 \\
f &= 2(A_4B_4 - A_1B_1 - A_2B_2 - A_3B_3)
\end{aligned}$$

2.1 Minkowski Space *Cont.*

Define $d\tau^2 \equiv \frac{dx^2}{c^2} = dt^2 - \frac{(dx^2+dy^2+dz^2)}{c^2}$

- $d\tau^2$ is the interval of proper time $\rightarrow$ Particle moving at $\vec{u}$ $dt = \gamma(u)d\tau$

2.1.1 4-velocity

$$\underset{\text{4-space velocity}}{\vec{U}} = \frac{d\vec{R}}{d\tau} = \text{4-vector} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, \frac{d(ct)}{d\tau}\right)$$

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau}$$

$$\vec{U} = \gamma(u)(\vec{u}, c) = \gamma(u)(u_x, u_y, u_z, c)$$

$$\vec{U} = (\gamma(u)\vec{U}, \gamma(u)c)$$

2.1.2 4-acceleration

$$\vec{A} = \frac{d\vec{U}}{d\tau} = \frac{d\vec{U}}{dt}\frac{dt}{d\tau}$$

$$\vec{A} = \gamma\frac{d\vec{U}}{dt} = \gamma(u)\left[\frac{d}{dt}(\gamma\vec{u}), \frac{d}{dt}(\gamma c)\right]$$

$$\vec{A} = \gamma\left(\frac{d\gamma}{dt}\vec{u} + \gamma\frac{d\vec{u}}{dt}, c\frac{d\gamma}{dt}\right)$$

$$\vec{A} = \gamma(\gamma'\vec{u} + \gamma\vec{a}, \gamma' c)$$

$$\vec{A} = \gamma' = \frac{d\gamma}{dt}$$

2.1.3 Instantaneous rest frame

$$\vec{u} = 0 \to \gamma(u) = 1$$
$$\vec{U} = (\vec{0}, c)$$

$$\vec{A} = ?$$
$$\gamma' = \frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{3}{2}} \frac{u}{c^2}\frac{du}{dt} = 0$$
$$\boxed{\vec{A} = (\vec{a}, 0)}$$

$$\vec{U}^2 = \gamma^2(c^2 - u^2)$$
$$\vec{U}^2 = \gamma^2 c^2 \left(1 - \frac{u^2}{c^2}\right)$$
$$\vec{U}^2 = \gamma^2 c^2 \frac{1}{\gamma^2}$$
$$\vec{U}^2 = \cancel{\gamma^2} c^2 \frac{1}{\cancel{\gamma^2}}$$
$$\vec{U}^2 = c^2$$

$$\boxed{\vec{U}^2 = c^2 \ \textbf{(scalar)}}$$

- Define $\vec{A}^2 = -\alpha^2$ where α^2 is the "**proper acceleration**"

3 Sample problems

The dot product between $\vec{U}$ and $\vec{A}$, given $\vec{U} = (\vec{0}, c)$ and $\vec{A} = (\vec{a}, 0)$, is:

$$\vec{U} \cdot \vec{A} = c * 0 - 0 * a = 0$$

3.1 Problem 1

Use the fact that $U = \gamma(\vec{u}, c)$ transforms as a 4-vector to rederive the velocity addition formula and the following transformation formula for $\gamma : \gamma(u') = \gamma(u)\gamma(v)(1 - u_x v/c^2)$

$$\vec{U} = (\gamma(u)u_x, \gamma(u)u_y, \gamma(u)u_z, \gamma(u)c)$$
$$\vec{U}' = (\gamma(u')u'_x, \gamma(u')u'_y, \gamma(u')u'_z, \gamma(u')c)$$

$$
\begin{aligned}
\vec{A} &= (A_1, A_2, A_3, A_4) \\
\vec{A_1}' &= \gamma(v)(A_1 - \beta A_4) \\
\vec{A_2}' &= A_2 \\
\vec{A_3}' &= A_3 \\
\vec{A_4}' &= \gamma(v)(A_4 - \beta A_1)
\end{aligned}
$$