FOURIER ANALYSIS AND ITS APPLICATIONS

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PREFACE

This book is intended for students of mathematics, physics, and engineering at the advanced undergraduate level or beyond. It is primarily a text for a course at the advanced undergraduate level, but I hope it will also be useful as a reference for people who have taken such a course and continue to use Fourier analysis in their later work. The reader is presumed to have (i) a solid background in calculus of one and several variables, (ii) knowledge of the elementary theory of linear ordinary differential equations (i.e., how to solve first-order linear equations and second-order ones with constant coefficients), and (iii) an acquaintance with the complex number system and the complex exponential function $e^{x+iy} = e^x(\cos y + i \sin y)$. In addition, the theory of analytic functions (power series, contour integrals, etc.) is used to a slight extent in Chapters 5, 6, 7, and 9 and in a serious way in Sections 8.2, 8.4, 8.6, 10.3, and 10.4. I have written the book so that lack of knowledge of complex analysis is not a serious impediment; at the same time, for the benefit of those who do know the subject, it would be a shame not to use it when it arises naturally. (In particular, the Laplace transform without analytic functions is like Popeye without his spinach.) At any rate, the facts from complex analysis that are used here are summarized in Appendix 2.

The subject of this book is the whole circle of ideas that includes Fourier series, Fourier and Laplace transforms, and eigenfunction expansions for differential operators. I have tried to steer a middle course between the mathematics-for-engineers type of book, in which Fourier methods are treated merely as a tool for solving applied problems, and the advanced theoretical treatments aimed at pure mathematicians. Since I thereby hope to please both the pure and the applied factions but run the risk of pleasing neither, I should give some explanation of what I am trying to do and why I am trying to do it.

First, this book deals almost exclusively with those aspects of Fourier analysis that are useful in physics and engineering rather than those of interest only in pure mathematics. On the other hand, it is a book on *applicable* mathematics rather than *applied* mathematics: the principal role of the physical applications herein is to illustrate and illuminate the mathematics, not the other way around. I have refrained from including many applications whose principal conceptual content comes from Subject X rather than Fourier analysis, or whose appreciation requires specialized knowledge from Subject X; such things belong more properly in a book on Subject X where the background can be more fully explained. (Many of my favorite applications come from quantum physics, but in accordance with this principle I have mentioned them only briefly.) Similarly, I have not worried too much about the physical details of the applications studied here. For example, when I think about the 1-dimensional heat equation I usually envision a long thin rod, but one who prefers to envision a 3-dimensional slab whose temperature varies only along one axis is free to do so; the mathematics is the same.

Second, there is the question of how much emphasis to lay on the theoretical aspects of the subject as opposed to problem-solving techniques. I firmly believe that theory — meaning the study of the ideas underlying the subject and the reasoning behind the techniques — is of intellectual value to everyone, applied or pure. On the other hand, I do not take "theory" to be synonymous with "logical rigor." I have presented complete proofs of the theorems when it is not too onerous to do so, but I often merely sketch the technical parts of an argument. (If the technicalities cannot easily be filled in by someone who is conversant with such things, I usually give a reference to a complete proof elsewhere.) Of course, where to draw the line is a matter of judgment, and I suppose nobody will be wholly satisfied with my choices. But those instructors who wish to include more details in their lectures are free to do so, and readers who tire of a formal argument have only to skip to the end-of-proof sign ▮. Thus, the book should be fairly flexible with regard to the level of rigor its users wish to adopt.

One feature of the theoretical aspect of this book deserves special mention. The development of Lebesgue integration and functional analysis in the period 1900–1950 has led to enormous advances in our understanding of the concepts underlying Fourier analysis. For example, the completeness of L^2 and the shift from pointwise convergence to norm convergence or weak convergence simplifies much of the discussion of orthonormal bases and the validity of series expansions. These advances have usually not found their way into application-oriented books because a rigorous development of them necessitates the building of too much machinery. However, most of this machinery can be ignored if one is willing to take a few things on faith, as one takes the intermediate value theorem on faith in freshman calculus. Accordingly, in §3.3–4 I assert the existence of an improved theory of integration, the Lebesgue integral, in the context of which one has (i) the completeness of L^2, (ii) the fact that "nice" functions are dense in L^2, and (iii) the dominated convergence theorem. I then proceed to use these facts without further ado. (The dominated convergence theorem, it should be noted, is a wonderful tool even in the context of Riemann integrable functions.) Later, in Chapter 9, I develop the theory of distributions as linear functionals on test functions, the motivation being that the value of a distribution on a test function is a smeared-out version of the value of a function at a point. Discussion of functional-analytic technicalities (which are largely irrelevant at the elementary level) is reduced to a minimum.

With the exception of the prerequisites and the facts about Lebesgue integration mentioned above, this book is more or less logically self-contained. However, certain assertions made early in the book are established only much later:

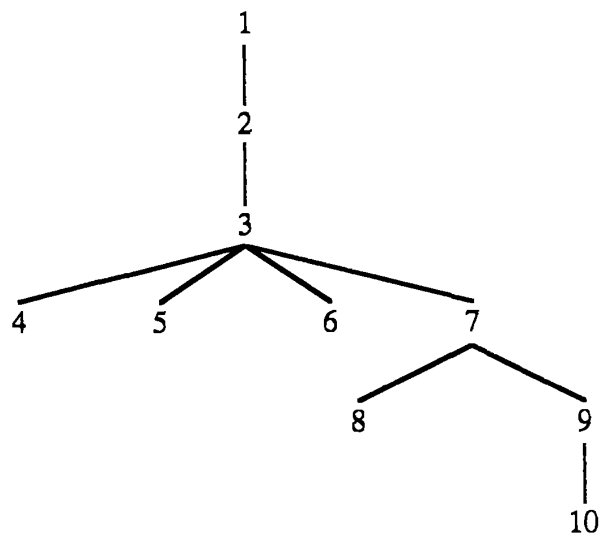
(i) The completeness of the eigenfunctions of regular Sturm-Liouville problems is stated in §3.5 and proved, in the case of separated boundary conditions, in §10.3.

(ii) The asymptotic formulas for Bessel functions given in §5.3 are proved via Watson's lemma in §8.6.

(iii) The proofs of completeness of Legendre, Hermite, and Laguerre polynomials in Chapter 6 rely on the Weierstrass approximation theorem and the Fourier inversion theorem, proved in Chapter 7.

(iv) The discussion of weak solutions of differential equations in §9.5 justifies many of the formal calculations with infinite series in the earlier chapters.

Thus, among the applications of the material in the later part of the book is the completion of the theory developed in the earlier part.



CHAPTER DEPENDENCE DIAGRAM

The main dependences among the chapters are indicated in the accompanying diagram, but a couple of additional comments are in order.

First, there are some minor dependences that are not shown in the diagram. For example, a few paragraphs of text and a few exercises in Sections 6.3, 7.5, 8.1, and 8.6 presuppose a knowledge of Bessel functions, but one can simply omit these bits if one has not covered Chapter 5. Also, the discussion of techniques in §4.1 is relevant to the applied problems in later chapters, particularly in §5.5.

Second, although Chapter 10 depends on Chapter 9, except in §10.2 the only part of distribution theory needed in Chapter 10 is an appreciation of delta functions on the real line and the way they arise in derivatives of functions with jump discontinuities. Hence, one could cover Sections 10.1 and 10.3–4 after an informal discussion of the delta function, without going through Chapter 9.

There is enough material in this book for a full-year course, but one can also select various subsets of it to make shorter courses. For a one-term course one could cover Chapters 1–3 and then select topics *ad libitum* from Chapters 4–7. (If one wishes to present some applications of Bessel functions without discussing the theory in detail, one could skip from the recurrence formulas in §5.2 to the statement of Theorem 5.3 at the end of §5.4 without much loss of continuity.) I have taught a one-quarter (ten-week) course from Chapters 1–5 and a sequel to it from Chapters 7–10, omitting a few items here and there.

One further point that instructors should keep in mind is the following. Most of the book deals with rather concrete ideas and techniques, but there are two places where concepts of a more general and abstract nature are discussed in a serious way: Chapter 3 (L^2 spaces, orthogonal bases, Sturm-Liouville problems) and Chapter 9 (functions as linear functionals, generalized functions). These parts are likely to be difficult for students who have had little experience with abstract mathematics, and instructors should plan their courses accordingly.

Fourier analysis and its allied subjects comprise an enormous amount of mathematics, about which there is much more to be said than is included in this book. I hope that my readers will find this fact exciting rather than dismaying. Accordingly, I have included a sizable although not exhaustive bibliography of books and papers to which the reader can refer for more information on things that are touched on lightly here. Most of these references should be reasonably accessible to the students for whom this book is primarily intended, but a few of them are of a considerably more advanced nature. This is inevitable; the topics in this book impinge on a lot of sophisticated material, and the full story on some of the things discussed here (singular Sturm-Liouville problems, for instance) cannot be told without going to a deeper level. But these advanced references should be of use to those who have the necessary background, and may at least serve as signposts to those who have yet to acquire it.

I am grateful to my colleagues Donald Marshall, Douglas Lind, Richard Bass, and James Morrow and to the students in our classes for pointing out many mistakes in the first draft of this book and suggesting a number of improvements. I also wish to thank the following reviewers for their helpful suggestions in revising the manuscript: Giles Auchmuty, University of Houston; James Herod, Georgia Institute of Technology; Raymond Johnson, University of Maryland; Francis Narcowich, Texas A & M University; Juan Carlos Redondo, University of Michigan; Jeffrey Rauch, University of Michigan; Jesus Rodriguez, North Carolina State University; and Michael Vogelius, Rutgers University.

Gerald B. Folland

CONTENTS

CHAPTER 1
OVERTURE

The subject of this book is *Fourier analysis,* which may be described as a collection of related techniques for resolving general functions into sums or integrals of simple functions or functions with certain special properties. Fourier analysis is a powerful tool for many problems, and especially for solving various differential equations of interest in science and engineering. The purpose of this introductory chapter is to provide some background concerning partial differential equations. Specifically, we introduce some of the basic equations of mathematical physics that will provide examples and motivation throughout the book, and we discuss a technique for solving them that leads directly to problems in Fourier analysis.

At the outset, let us present some notations that will be used repeatedly. The real and complex number systems will be denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively. We shall be working with functions of one or several real variables $x_1, \dots, x_n$. We shall denote the ordered n-tuple $(x_1, \dots, x_n)$ by $\mathbf{x}$ and the space of all such ordered n-tuples by $\mathbf{R}^n$.

In most of the applications, n will be 1, 2, 3, or 4, and the variables x_j will denote coordinates in one, two, or three space dimensions, together with time. In this situation we shall usually write x, y, z instead of x_1, x_2, x_3 for the spatial variables, and we shall denote the time variable by t. Moreover, we shall use the common subscript notation for partial derivatives:

$$u_x = \frac{\partial u}{\partial x}, \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \text{etc.}$$

A function f of one real variable is said to be of class $C^{(k)}$ on an interval I if its derivatives $f', \dots, f^{(k)}$ exist and are continuous on I. Similarly, a function of n real variables is said to be of class $C^{(k)}$ on a set $D \subset \mathbf{R}^n$ if all of its partial derivatives of order $\le k$ exist and are continuous on D. If the function possesses continuous derivatives of all orders, it is said to be of class $C^{(\infty)}$.

Finally, we use the common notation with square and round brackets for closed and open intervals in the real line $\mathbf{R}$:

$$[a,b] = \{x : a \le x \le b\}, \qquad (a,b) = \{x : a < x < b\},$$
$$[a,b) = \{x : a \le x < b\}, \qquad (a,b] = \{x : a < x \le b\}.$$

1.1 Some equations of mathematical physics

In order to understand the significance of the ideas as they arise, it will be useful to have a few physical applications in mind as examples of the sort of problems we are trying to solve. Accordingly, we begin with a brief and informal discussion of some of the basic partial differential equations of classical mathematical physics. These equations all involve a fundamental differential operator known as the Laplacian, which is defined as follows. If u is a function of the real variables $x_1, \ldots, x_n$ of class $C^{(2)}$, the **Laplacian** of u is the function $\nabla^2 u$ defined by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{1.1}$$

The first of the equations we shall study is the **wave equation**:

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{1.2}$$

Here u represents a wave traveling through an n-dimensional medium—where, in practice, n will usually be 1, 2, or 3. More precisely, $x_1, \ldots, x_n$ are the coordinates of a point $\mathbf{x}$ in the medium; t is the time; c is the speed of propagation of waves in the medium; and $u(\mathbf{x}, t)$ is the amplitude of the wave at position $\mathbf{x}$ and time t.

The wave equation provides a reasonable mathematical model for a number of physical processes, such as the following:

(a) Vibrations of a stretched string, such as a guitar string.
(b) Vibrations of a column of air, such as an organ pipe or clarinet.
(c) Vibrations of a stretched membrane, such as a drumhead.
(d) Waves in an incompressible fluid, such as water.
(e) Sound waves in air or other elastic media.
(f) Electromagnetic waves, such as light waves and radio waves.

The number n of spatial dimensions is 1 in examples (a) and (b), 2 in examples (c) and (d) (since the waves appear on the *surface* of the water), and 3 in examples (e) and (f). In (a), (c), and (d), u represents the transverse displacement of the string, membrane, or fluid surface; in (b) and (e), u represents the longitudinal displacement of the air; and in (f), u is any of the components of the electromagnetic field.

We shall not attempt to derive the wave equation from physical principles here, since each of the preceding examples involves different physics. Examples (a) and (f) are explained in Appendix 1; discussions of the others may be found, for example, in Ingard [32]* and Taylor [51]. We should point out, however, that in most cases the derivation involves making some simplifying assumptions. Hence, the wave equation gives only an approximate description of the actual physical process, and the validity of the approximation will depend on whether

* Numbers in brackets refer to the bibliography at the end of the book.

certain physical conditions are satisfied. For instance, in example (a) the vibrations should be small enough so that the string is not stretched beyond its limits of elasticity. In example (f) it follows from Maxwell's equations, the fundamental equations of electromagnetism, that the wave equation is satisfied *exactly* in regions containing no electric charges or currents — but of course the assumption of no charges or currents can only be approximately valid in the real world. (Of course, it is precisely the fact that the wave equation is only an approximation that allows it to be a useful model in so many different situations!)

The next basic differential equation on our list is the **heat equation**:

$$u_t = k\nabla^2 u. \tag{1.3}$$

This equation describes the diffusion of thermal energy in a homogeneous material (that is, one whose composition does not change from point to point). As in the wave equation, the variables x_j are spatial coordinates and t is time, but now $u(\mathbf{x}, t)$ is the *temperature* at a position $\mathbf{x}$ and time t, and k is a constant called the "thermal diffusivity" of the material. A brief derivation is given in Appendix 1. As for the number of spatial variables, the case $n = 3$ is the most fundamental from the physical point of view, but the cases $n = 1$ and $n = 2$ are also of interest as models of situations where the heat flow is practically all in one or two directions. For example, the heat equation with $n = 1$ can be used to describe heat flow along a wire or rod, provided that heat flow in directions perpendicular to the axis of the rod can be neglected. It can also be used to describe heat flow in a slab of material, such as a wall separating two rooms, where only the heat flow from one room toward the other (as opposed to flow in directions parallel to the wall) is significant.

Two warnings: (i) The heat equation can be used to model heat flow in both solids and fluids (liquids and gases), but in the latter case it does *not* take any account of the phenomenon of convection; that is, it will provide a reasonable model only if conditions are such as to exclude any macroscopic currents in the fluid. (ii) The heat equation is not a fundamental law of physics, and it does not give reliable answers at very low or very high temperatures. In particular, it is obvious that if u is a solution then so is $u + c$ for any constant c; thus the heat equation does not recognize the existence of absolute zero!

The heat equation can also be used to model other diffusion processes. For example, if a drop of red dye is placed in a body of water, the dye will gradually spread out and permeate the entire body. If convection effects are negligible, equation (1.3) will describe the diffusion of the dye through the water ($u(\mathbf{x}, t)$ now being the concentration of dye at position $\mathbf{x}$ and time t).

Next, we come to the **Laplace equation**:

$$\nabla^2 u = 0. \tag{1.4}$$

Laplace's equation arises in a number of different contexts. It is satisfied by the electrostatic potential in any region containing no electric charge, and by the gravitational potential in any region containing no mass. It is also the equation that governs standing waves and steady-state heat distributions — that is, solutions of the wave equation and the heat equation that are independent of time. We shall meet other applications of it later on.

Partial differential equations such as the ones discussed above typically have solutions in such great abundance that there is no reasonable way of giving an explicit description of all of them. The most common way of pinning down a particular solution is to impose some boundary conditions. Different types of differential equations require different types of boundary conditions, and the particular conditions that are appropriate for a given physical problem will depend on the particular physical situation. The physics is generally a good guide to the mathematics: "reasonable" physical conditions usually lead to "reasonable" mathematical problems.

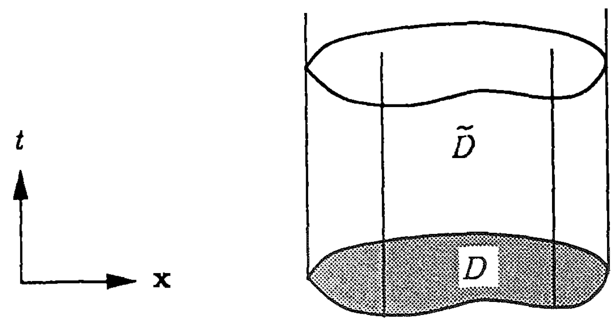


FIGURE 1.1. The region D in $\mathbf{x}$-space and the region $\widetilde{D}$ in $\mathbf{x}t$-space.

These matters may best be explained by examining a few examples. Let us consider the heat equation: suppose we are interested in studying the diffusion of heat in a body that occupies a bounded region D of $\mathbf{x}$-space, given the initial temperature distribution in the body. That is, we wish to solve the heat equation (1.3) in the region

$$\widetilde{D} = \left\{(\mathbf{x}, t) : \mathbf{x} \in D, \quad t > 0\right\}$$

of $(\mathbf{x}, t)$-space subject to the initial condition

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \tag{1.5}$$

where $f(\mathbf{x})$ is the temperature distribution at time $t = 0$. (See Figure 1.1.) Equation (1.5) is a condition on u on the "horizontal" part of the boundary of $\widetilde{D}$, but it is not enough to specify u completely; we also need a boundary condition on the "vertical" part of the boundary to tell what happens to the heat when it reaches the boundary surface S of the spatial region D. Here the particular physical conditions at hand must be our guide. One reasonable assumption is that S is held at a constant temperature u_0 (for example, by immersing the body in a bath of ice water), thus:

$$u(\mathbf{x}, t) = u_0 \quad \text{for } \mathbf{x} \in S, \ t > 0. \tag{1.6}$$

Another reasonable assumption is that D is insulated, so that no heat can flow in or out across S. Mathematically, this amounts to requiring the *normal derivative* of u along the boundary S to vanish:

$$(\nabla u \cdot \mathbf{n})(\mathbf{x}, t) = 0 \quad \text{for } \mathbf{x} \in S,\ t > 0. \tag{1.7}$$

Here $\mathbf{n}$ is the unit outward normal vector to S (and we are implicitly assuming that the surface S is smooth, so that $\mathbf{n}$ is well-defined). A more realistic assumption than either (1.6) or (1.7) is that the region outside D is held at a constant temperature u_0, and the rate of heat flow across the boundary S is proportional to the difference in temperatures on the two sides:

$$(\nabla u \cdot \mathbf{n})(\mathbf{x}, t) + a\Big(u(\mathbf{x}, t) - u_0\Big) = 0 \quad \text{for } \mathbf{x} \in S,\ t > 0. \tag{1.8}$$

This is **Newton's law of cooling,** and $a > 0$ is the proportionality constant. The conditions (1.6) and (1.7) may be regarded as the limiting cases of (1.8) as $a \to \infty$ or $a \to 0$.

At any rate, it turns out that the initial condition (1.5) together with any one of the boundary conditions (1.6), (1.7), or (1.8) leads to a *well-posed* problem: one having a unique solution that depends continuously (in some appropriate sense) on the initial data f. The same discussion is also valid for the heat equation in one or two space dimensions. (In one space dimension, the "region" D is just an interval in the x-axis, and the "normal derivative" $\nabla u \cdot \mathbf{n}$ is just u_x at the right endpoint and $-u_x$ at the left endpoint.)

A similar analysis applies to boundary value problems for the wave equation (1.2), with one significant difference: the wave equation is second-order in the time variable t, whereas the heat equation is only first-order in t. For this reason, in solving the wave equation it is appropriate to specify not only the initial values of u as in (1.5) but also the initial velocity u_t:

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \qquad u_t(\mathbf{x}, 0) = g(\mathbf{x}) \quad \text{for } \mathbf{x} \in D. \tag{1.9}$$

The imposition of the initial conditions (1.9) together with a boundary condition of the form (1.6), (1.7), or (1.8) leads to a unique solution of the wave equation. For example, to analyze the motion of a vibrating string of length l that is fixed at both endpoints, we take the "region" D to be the interval $[0, l]$ on the x-axis and solve the one-dimensional wave equation with boundary conditions (1.6) (where $u_0 = 0$) and (1.9):

$$\begin{aligned} u_{tt} = c^2 u_{xx}, \qquad & u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \quad \text{for } 0 < x < l, \\ & u(0,t) = u(l,t) = 0 \quad \text{for } t > 0. \end{aligned}$$

Remark: The "velocity" u_t is not the same as the constant c in the wave equation. c is the speed of propagation of the wave along the string, whereas u_t is the rate of change of the displacement of a particular point on the string. (The same is true for waves in media other than strings.)

The Laplace equation (1.4) is of a rather different character, as it does not involve time. The most important boundary value problem for this equation, the so-called **Dirichlet problem**, consists in specifying the values of u on the boundary of the region in question. That is, we solve $\nabla^2 u = 0$ in a region D subject to the condition that u agrees with a given function f on the boundary S of D. This is a well-posed problem when D is bounded and S is smooth (except perhaps for corners and edges). Another useful boundary value problem for Laplace's equation is the **Neumann problem**, which consists of specifying the values of the normal derivative $\nabla u \cdot \mathbf{n}$ on S:

$$\nabla^2 u = 0 \text{ in } D, \qquad (\nabla u \cdot \mathbf{n})(\mathbf{x}) = g(\mathbf{x}) \quad \text{for } \mathbf{x} \in S.$$

Here we do not quite have uniqueness, for if u is a solution, then so is $u + C$ for any constant C. Moreover, the boundary data g must satisfy the condition $\iint_S g = 0$ in order for a solution to exist, because by the divergence theorem,

$$\iint_S (\nabla u \cdot \mathbf{n})\, dS = \iiint_D \nabla^2 u\, dV = 0$$

for any u such that $\nabla^2 u = 0$. However, there are no other obstructions to existence and uniqueness; and since there is only one constant to be specified to obtain uniqueness, and only one linear equation to be satisfied to obtain existence, the Neumann problem is still regarded as well behaved.

There is one more point that should be mentioned in connection with the interpretation of boundary conditions. Suppose, for example, that we are interested in the initial value problem for the heat equation:

$$u_t = k\nabla^2 u \quad \text{for } t > 0, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}).$$

If one interprets this absolutely literally, one obtains a solution by defining $u(\mathbf{x}, t)$ to be $f(\mathbf{x})$ when $t = 0$ and 0 when $t > 0$, but clearly this is not what is really wanted unless f is identically zero! Rather, in such boundary value problems there is always an implicit continuity assumption: we ask not only that $u(\mathbf{x}, 0) = f(\mathbf{x})$ but that $u(\mathbf{x}, t)$ should approach $f(\mathbf{x})$ as $t \to 0$. The precise way in which this approach is achieved (pointwise convergence, uniform convergence, mean square convergence, etc.) will depend on the particular problem at hand. This is not a matter that requires a lot of deep thought — merely a little care to avoid making silly mistakes.

The wave, heat, and Laplace equations can be generalized by adding in an extra term, as follows:

$$u_{tt} - c^2\nabla^2 u = F(\mathbf{x}, t), \tag{1.10}$$

$$u_t - k\nabla^2 u = F(\mathbf{x}, t), \tag{1.11}$$

$$\nabla^2 u = F(\mathbf{x}). \tag{1.12}$$

These equations are called the **inhomogeneous wave**, **heat**, and **Laplace equations**; equation (1.12) is also called the **Poisson equation**. Here F is a function that is given in advance, and the original equations (1.2), (1.3), and (1.4) are the special cases where $F = 0$. The interpretation of F will vary with the particular situation considered. In the wave equation (1.10), F may represent a force that is driving the waves; in the case of electromagnetic fields, it represents the effect of charges or currents (see Appendix 1). In the heat equation (1.11), F may represent a source (or sink) of heat within the material in which the heat is flowing. The Poisson equation (1.12) is satisfied by electrostatic potential in a region when F is interpreted as -4π times the charge density in the region, or by the graviatational potential when F is interpreted as 4π times the mass density. (See Appendix 1. The difference in signs occurs because positive masses attract each other, whereas positive charges repel.) The boundary conditions appropriate for these inhomogeneous equations are much the same as for the corresponding homogeneous equations.

Finally, we mention one other basic equation of physics, the **Schrödinger equation**:

$$i\hbar u_t = -\frac{\hbar^2}{2m}\nabla^2 u + V(\mathbf{x})u.$$

In this equation u is the quantum-mechanical wave function for a particle of mass m moving in a potential $V(\mathbf{x})$, $\hbar$ is Planck's constant, and $i = \sqrt{-1}$. When the particle has a definite energy E, the time dependence drops out and one obtains the steady-state equation

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(\mathbf{x})u = Eu.$$

For the physics behind these equations we refer the reader to books on quantum mechanics such as Messiah [39] and Landau-Lifshitz [35]. Readers who are not familiar with this subject can safely ignore the occasional references to the Schrödinger equation, but those who are will find the solutions to some important special cases in later chapters.

EXERCISES

1. Show that $u(x,t) = t^{-1/2}\exp(-x^2/4kt)$ satisfies the heat equation $u_t = ku_{xx}$ for $t > 0$.
2. Show that $u(x,y,t) = t^{-1}\exp[-(x^2+y^2)/4kt]$ satisfies the heat equation $u_t = k(u_{xx} + u_{yy})$ for $t > 0$.
3. Show that $u(x,y) = \log(x^2+y^2)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$ for $(x,y) \neq (0,0)$.
4. Show that $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ satisfies Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ for $(x,y,z) \neq (0,0,0)$.
5. Proportionality constants in the equations of physics can often be eliminated by a suitable choice of units of measurement. Mathematically, this amounts to rewriting the equation in terms of new variables that are constant multiples of the original ones. Show that the substitutions $\tau = kt$ and $\tau = ct$ reduce the heat and wave equations, respectively, to $u_\tau = \nabla^2 u$ and $u_{\tau\tau} = \nabla^2 u$.

6. The object of this exercise is to derive *d'Alembert's formula* for the general solution of the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$.
   a. Show that if $u(y, z) = f(y) + g(z)$ where f and g are $C^{(2)}$ functions of one variable, then u satisfies $u_{yz} = 0$. Conversely, show that every $C^{(2)}$ solution of $u_{yz} = 0$ is of this form. (Hint: If $v_y = 0$, then v is independent of y.)
   b. Let $y = x - ct$ and $z = x + ct$. Use the chain rule to show that $u_{tt} - c^2 u_{xx} = -4c^2 u_{yz}$.
   c. Conclude that the general $C^{(2)}$ solution of the wave equation $u_{tt} = c^2 u_{xx}$ is $u(x, t) = f(x - ct) + g(x + ct)$ where f and g are $C^{(2)}$ functions of one variable. (Observe that $f(x - ct)$ represents a wave traveling to the right with speed c, and $g(x + ct)$ represents a wave traveling to the left with speed c.)
   d. Show that the solution of the initial value problem

$$u_{tt} = c^2 u_{xx}, \qquad u(x, 0) = \phi(x), \qquad u_t(x, 0) = \psi(x)$$

   is

$$u(x, t) = \tfrac{1}{2}\Big[\phi(x - ct) + \phi(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy.$$

7. The voltage v and current i in an electrical cable along the x-axis satisfy the coupled equations

$$i_x + Cv_t + Gv = 0, \qquad v_x + Li_t + Ri = 0,$$

   where C, G, L, and R are the capacitance, (leakage) conductance, inductance, and resistance per unit length in the cable. Show that v and i both satisfy the **telegraph equation**

$$u_{xx} = LCu_{tt} + (RC + LG)u_t + RGu.$$

8. Set $u(x, t) = f(x, t)e^{at}$ in the telegraph equation of Exercise 7. What is the differential equation satisfied by f? Show that a can be chosen so that this equation is of the form $f_{xx} = Af_{tt} + Bf$ (with no first-order term), provided that $LC \neq 0$.

1.2 Linear differential operators

The partial differential equations considered in the preceding section can all be written in the form $L(u) = F$, where $L(u)$ stands for $u_{tt} - c^2\nabla^2 u$, $u_t - k\nabla^2 u$, or $\nabla^2 u$. In each case $L(u)$ is a function obtained from u by performing certain operations involving partial derivatives, which we regard as the result of applying the *operator* L to the function u.

In general, a **linear partial differential operator** L is an operation that transforms a function u of the variables $\mathbf{x} = (x_1, \ldots, x_n)$ into another function $L(u)$ given by

$$L(u) = a(\mathbf{x})u + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} + \sum_{i,j=1}^{n} c_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \cdots .$$

(Here the dots at the end indicate higher-order terms, but it is understood that the whole sum contains only *finitely* many terms.) In other words, $L(u)$ is obtained by taking a finite collection of partial derivatives of u, multiplying them by the *coefficients* a, b_i, c_{ij}, etc., and adding them up. We may describe the operator L by itself, without reference to an input function u, by writing

$$L = a(\mathbf{x}) + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} c_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \cdots . \tag{1.13}$$

The term *linear* in the phrase "linear partial differential operator" refers to the following fundamental property: if L is given by (1.13), $u_1, \ldots, u_k$ are any functions possessing the requisite derivatives, and $c_1, \ldots, c_k$ are any constants, then

$$L(c_1 u_1 + \cdots + c_k u_k) = c_1 L(u_1) + \cdots + c_k L(u_k). \tag{1.14}$$

This is an immediate consequence of the fact that the derivative of a sum is the sum of the derivatives, and the derivative of a constant multiple of a function is the constant multiple of the derivative. Any function of the form $c_1 u_1 + \cdots + c_k u_k$ (where the c_j's are constants) is called a **linear combination** of $u_1, \ldots, u_k$. Thus, (1.14) says that L takes every linear combination of u_j's into the corresponding linear combination of $L(u_j)$'s.

More generally, any operator L, differential or otherwise, that satisfies (1.14) is called **linear**; here the inputs u and the outputs $L(u)$ can be any sort of objects for which linear combinations make sense, such as functions, vectors, numbers, etc. For instance, the formula $L(f) = \int_a^b f(t)\,dt$ defines a linear operation taking continuous functions on the interval $[a, b]$ to numbers; and if $\mathbf{x}_0$ is a fixed 3-dimensional vector, the formula $L(\mathbf{x}) = \mathbf{x} \times \mathbf{x}_0$ (the cross product of $\mathbf{x}$ with $\mathbf{x}_0$) defines a linear operation on 3-dimensional vectors.

A **linear partial differential equation** is simply an equation of the form

$$L(u) = F,$$

where L is a linear partial differential operator and F is a function of $\mathbf{x}$. Such an equation is called **homogeneous** if $F = 0$ and **inhomogeneous** if $F \neq 0$. The boundary conditions we associate to a differential equation are usually of a similar form themselves; that is, they are of the form "$B(u) = f$ on the boundary" where B is another linear differential operator and f is a function on the boundary. (We shall often omit the phrase "on the boundary" and write the boundary conditions simply as $B(u) = f$. Here also, the terms *homogeneous* and *inhomogeneous* refer to the cases $f = 0$ and $f \neq 0$.) The linearity of the operators L and B can be restated in the following way.

The Superposition Principle. *If $u_1, \ldots, u_k$ satisfy the linear differential equations $L(u_j) = F_j$ and the boundary conditions $B(u_j) = f_j$ for $j = 1, \ldots, k$, and $c_1, \ldots, c_k$ are any constants, then $u = c_1 u_1 + \cdots + c_k u_k$ satisfies*

$$L(u) = c_1 F_1 + \cdots + c_k F_k, \qquad B(u) = c_1 f_1 + \cdots + c_k f_k.$$

The importance of the superposition principle can hardly be overestimated. We shall use it repeatedly in a number of different ways, of which the most important are the following.

Suppose we want to find all solutions of a differential equation subject to one or more boundary conditions, say

$$L(u) = F, \qquad B(u) = f. \tag{1.15}$$

If we can find all solutions of the corresponding *homogenous* problem

$$L(u) = 0, \qquad B(u) = 0 \tag{1.16}$$

which is often simpler to handle, then it suffices to obtain just *one* solution, say v, of the original problem (1.15). Indeed, if u is any other solution of (1.15), then $w = u - v$ satisfies (1.16), for $L(w) = F - F = 0$ and $B(w) = f - f = 0$. Hence we obtain the general solution of (1.15) by adding the general solution w of (1.16) to any particular solution of (1.15).

In the same spirit, the superposition principle can be used to break down a problem involving several inhomogeneous terms into (presumably simpler) problems in which these terms are dealt with one at a time. For instance, suppose we want to find a solution to (1.15). It suffices to find solutions u_1 and u_2 to the problems

$$L(u_1) = F, \qquad B(u_1) = 0;$$
$$L(u_2) = 0, \qquad B(u_2) = f,$$

for we can then take $u = u_1 + u_2$.

Perhaps most important, if $u_1, u_2, \ldots$ are any solutions to a *homogeneous* differential equation $L(u) = 0$ that satisfy *homogeneous* boundary conditions $B(u) = 0$, then any linear combination of the u_j's will satisfy the same differential equation and the same boundary conditions. Thus, starting out with a sequence of solutions u_j, we can generate many more solutions by taking linear combinations. If we then take appropriate *limits* of such linear combinations, we arrive at solutions defined by infinite series or integrals — and this is where things get interesting!

Of course, there are also *nonlinear* differential equations involving *nonlinear* operations such as $L(u) = u_{xx} - \sin u$ or $L(u) = uu_x + (u_y)^3$. Indeed, many of the important equations of physics and engineering, including most of the refinements of the wave equation to describe waves and vibrations, are nonlinear.

However, nonlinear equations are, on the whole, much more difficult to solve than linear ones, and their study is beyond the scope of this book.

One final note: the reader will have observed that all the differential equations we discussed in §1.1 involve the Laplacian ∇^2. The reason for this is that the Laplacian commutes with all rigid motions of Euclidean space; that is, if $\mathscr{T}$ denotes any translation or rotation of n-space, then $\nabla^2(f \circ \mathscr{T}) = (\nabla^2 f) \circ \mathscr{T}$ for all functions f. Moreover, the *only* linear differential operators of order ≤ 2 that have this property are the operators $a\nabla^2 + b$ where a and b are constants. Hence, the differential equation describing any process that is spatially symmetric (i.e., unaffected by translations and rotations) is likely to involve the Laplacian.

EXERCISES

1. Suppose u_1 and u_2 are both solutions of the linear differential equation $L(u) = f$, where $f \neq 0$. Under what conditions is the linear combination $c_1u_1 + c_2u_2$ also a solution of this equation?
2. Consider the nonlinear (ordinary) differential equation $u' = u(1-u)$.
   a. Show that $u_1(x) = e^x/(1+e^x)$ and $u_2(x) = 1$ are solutions.
   b. Show that $u_1 + u_2$ is not a solution.
   c. For which values of c is cu_1 a solution? How about cu_2?
3. Give examples of linear differential operators L and M for which it is not true that $L(M(u)) = M(L(u))$ for all u. (Hint: At least one of L and M must have nonconstant coefficients.)
4. What form must G have for the differential equation $u_{tt} - u_{xx} = G(x,t,u)$ to be linear? Linear and homogeneous?
5. a. Show that for $n = 1, 2, 3, \ldots$, $u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(1,y) = u(x,0) = 0.$$

   b. Find a linear combination of the u_n's that satisfies $u(x,1) = \sin 2\pi x - \sin 3\pi x$.
   c. Show that for $n = 1, 2, 3, \ldots$, $\widetilde{u}_n(x,y) = \sin(n\pi x)\sinh n\pi(1-y)$ satisfies

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(1,y) = u(x,1) = 0.$$

   d. Find a linear combination of the $\widetilde{u}_n$'s that satisfies $u(x,0) = 2\sin \pi x$.
   e. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = 0,$$
$$u(x,0) = 2\sin \pi x, \quad u(x,1) = \sin 2\pi x - \sin 3\pi x.$$

1.3 Separation of variables

In this section we discuss a very useful technique for solving certain linear partial differential equations, known as *separation of variables.* This technique works only for very special sorts of equations, but fortunately the equations for which it works include many of the most important ones.

The idea is as follows. Suppose, for simplicity, that we have a homogeneous partial differential equation $L(u) = 0$ involving just two independent variables x and y, with some homogeneous boundary conditions $B(u) = 0$. We try to find solutions u of the form

$$u(x, y) = X(x)Y(y).$$

If the method is to work, when we substitute this formula for u into the equation $L(u) = 0$, the terms can be rearranged so that the left side of the equation involves only the variable x and the right side involves only the variable y, say $P(x) = Q(y)$. But since x and y are independent, a quantity that depends on x alone and also on y alone must be a constant. Hence we have $P(x) = C$ and $Q(y) = C$, and these equations will be *ordinary* differential equations for the functions X and Y whose product is u. With luck, these equations can be solved subject to the boundary conditions on X and Y that are implied by the original conditions on u, and we thus obtain a whole family of solutions by varying the constant C. By the superposition principle, all linear combinations of these will also be solutions; and if we are lucky, we will obtain *all* solutions of the original problem by taking appropriate limits of these linear combinations.

The same procedure can be used for equations for functions of more than two variables. If there are three independent variables involved, say x, y, and z, we look for solutions of the form u of the form

$$u(x, y, z) = X(x)v(y, z).$$

If the variables can be separated, we obtain an *ordinary* differential equation for X and a *partial* differential equation for v, but now involving only the *two* variables y and z. We can then try to write $v(y, z) = Y(y)Z(z)$ and obtain ordinary differential equations for Y and Z. In other words, we use separation of variables to "peel off" the independent variables one at a time, thereby reducing the original problem to some simpler ones.

Of course, once one has reduced the problem to some ordinary differential equations, one must be able to solve them! For the time being all our examples will involve homogeneous equations with real constant coefficients, whose solutions we now briefly review. (See, for example, Boyce-DiPrima [10] for a more extensive discussion.) For first-order equations the situation is very simple:

$$f' = af \quad \Longrightarrow \quad f(x) = Ce^{ax}.$$

For second-order equations, the basic fact is as follows.

Theorem 1.1. *The general solution of $f'' + af' + bf = 0$ is $f(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$, where r_1, r_2 are the roots of the equation $r^2 + ar + b = 0$ and C_1, C_2 are arbitrary complex numbers. If $r_1 = r_2$, the general solution is $(C_1 + C_2 x)e^{r_1 x}$.*

Here r_1 and r_2 may, of course, be complex; see Appendix 2 for a discussion of the complex exponential function. In certain cases it may be more convenient to express the solution in terms of trigonometric or hyperbolic functions. In particular:

(i) *If $r_1 = \rho + i\sigma$ and $r_2 = \rho - i\sigma$, the general solution is $e^{\rho x}(C_1 \cos \sigma x + C_2 \sin \sigma x)$.*

(ii) *If $\alpha > 0$, the general solution of $f'' + \alpha^2 f = 0$ is $C_1 \cos \alpha x + C_2 \sin \alpha x$, and the general solution of $f'' - \alpha^2 f = 0$ is $C_1 \cosh \alpha x + C_2 \sinh \alpha x$.*

Enough generalities; let us look at a couple of specific examples.

Consider the problem of 1-dimensional heat flow: we may think of a circular metal rod of length l, insulated along its curved surface so that heat can enter or leave only at the ends. Suppose, moreover, that both ends are held at temperature zero. (Zero in which temperature scale? It doesn't matter: the mathematics is the same.) Ignoring the question of initial conditions for the moment, we then have the boundary value problem

$$u_t = k u_{xx}, \qquad u(0,t) = u(l,t) = 0. \tag{1.17}$$

If we substitute $u(x,t) = X(x)T(t)$ into (1.17), we obtain

$$X(x)T'(t) = kX''(x)T(t), \tag{1.18}$$

$$X(0) = X(l) = 0. \tag{1.19}$$

The variables in (1.18) may be separated by dividing both sides by $kX(x)T(t)$, yielding

$$T'(t)/kT(t) = X''(x)/X(x).$$

Now the left side depends only on t, whereas the right side depends only on x; since they are equal, they must both be equal to a constant A:

$$T'(t) = AkT(t), \qquad X''(x) = AX(x).$$

These are simple ordinary differential equations for T and X that can be solved by elementary methods — indeed, almost by inspection. The general solution of the equation for T is

$$T(t) = C_0 e^{Akt},$$

and the general solution of the equation for X is

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x, \qquad \lambda = \sqrt{-A}. \tag{1.20}$$

(If A is positive, one might prefer to avoid imaginary numbers by rewriting (1.20) as

$$X(x) = C_1' \cosh \mu x + C_2' \sinh \mu x = \frac{C_1' + C_2'}{2} e^{\mu x} + \frac{C_1' - C_2'}{2} e^{-\mu x}, \qquad \mu = \sqrt{A}.$$

But so far A is just an arbitrary (possibly complex) constant, so there is no reason yet to choose one form over the other.) However, we must now take account of the boundary conditions (1.19). The condition $X(0) = 0$ forces $C_1 = 0$ in (1.20), and the condition $X(l) = 0$ then becomes $C_2 \sin \lambda l = 0$. If we take $C_2 = 0$, then our solution $u(x,t)$ vanishes identically, which is of no interest: we are looking for *nontrivial* solutions. So we take $C_2 \neq 0$; hence $\sin \lambda l = 0$, which means that $\lambda l = n\pi$ for some integer n; in other words, $A = -(n\pi/l)^2$. (So A is negative after all!) We may take $n > 0$, since the case $n = 0$ gives the zero solution and replacing n by $-n$ merely amounts to replacing C_2 by $-C_2$.

In short, for every positive integer n we have obtained a solution $u_n(x,t)$ of (1.17), namely,

$$u_n(x,t) = \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin \frac{n\pi x}{l} \qquad (n = 1, 2, 3, \ldots).$$

(We have taken $C_0 = C_2 = 1$; other choices of C_0 and C_2 give constant multiples of u_n.) We obtain more solutions by taking linear combinations of the u_n's, and then passing to *infinite* linear combinations — that is, infinite series

$$u = \sum_1^\infty a_n u_n = \sum_1^\infty a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin \frac{n\pi x}{l}. \tag{1.21}$$

Of course, there are questions to be answered about the convergence of such series, but for the moment we shall not worry about that.

Finally, we bring the initial conditions into the picture: can we solve (1.17) subject to the initial condition $u(x,0) = f(x)$, where f is a given function on the interval $(0,l)$? The solution (1.21) will do the job, provided that

$$f(x) = \sum_1^\infty a_n \sin \frac{n\pi x}{l}. \tag{1.22}$$

We have now arived at one of the main subjects of this book: the study of series expansions like (1.22). Before setting foot in this new territory, however, let us look at a couple of other boundary value problems.

Consider the problem of heat flow in a rod, as before, but now assume that the ends of the rod are insulated. Thus, instead of (1.17) we consider

$$u_t = k u_{xx}, \qquad u_x(0,t) = u_x(l,t) = 0. \tag{1.23}$$

The technique we used to solve (1.17) also works here, with only the following differences. The conditions (1.19) are replaced by

$$X'(0) = X'(l) = 0, \tag{1.24}$$

which force $C_2 = 0$ (rather than $C_1 = 0$) and $\lambda l = n\pi$ in (1.20). Again, we may assume that $n \geq 0$ since $\cos(n\pi x/l) = \cos(-n\pi x/l)$, but now we must include $n = 0$. We thus obtain the sequence of solutions

$$u_n(x,t) = \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)\cos\frac{n\pi x}{l} \qquad (n = 0, 1, 2, \ldots),$$

which can be combined to form the series

$$u = \sum_0^\infty a_n u_n = \sum_0^\infty a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)\cos\frac{n\pi x}{l}.$$

This series will solve the problem (1.23) subject to the initial condition $u(x,0) = f(x)$ provided that

$$f(x) = \sum_0^\infty a_n \cos\frac{n\pi x}{l}. \tag{1.25}$$

Thus we have arrived at another series expansion problem, different from but similar to (1.22).

For yet another variation on the same theme, consider heat flow in a rod that is bent into the shape of a circle, with the ends joined together. We may specify the position of a point on the circle by its angular coordinate θ, measured from some fixed base point. Since linear distance on a circle is proportional to angular distance ($\Delta x = r\Delta\theta$ where r is the radius), the heat equation $u_t = k_0 u_{xx}$ can be rewritten as

$$u_t = k u_{\theta\theta}$$

where $k = k_0/r^2$. We try to find solutions of the form $u(\theta, t) = \Theta(\theta)T(t)$, and just as before we find that

$$T(t) = C_0 e^{Akt}, \qquad \Theta(\theta) = C_1 \cos\theta\sqrt{-A} + C_2 \sin\theta\sqrt{-A} \tag{1.26}$$

for some constant A. Here there are no boundary conditions like (1.19) or (1.24) because the rod has no ends. Instead, since the angular coordinate θ is well-defined only up to multiples of 2π, we have the requirement that $\Theta(\theta)$ must be periodic with period 2π. This condition does not kill off either of the coefficients C_1 or C_2 in (1.26), but it does force $\sqrt{-A}$ to be an integer n. The upshot is that we obtain series solutions of the form

$$u(\theta, t) = \sum_0^\infty (a_n \cos n\theta + b_n \sin n\theta)e^{-n^2 kt},$$

and such a series will satisfy the initial condition $u(\theta, 0) = f(\theta)$ provided that

$$f(\theta) = \sum_0^\infty (a_n \cos n\theta + b_n \sin n\theta). \tag{1.27}$$

Finally, we present an illustration of these techniques involving something other than the heat equation. Consider the problem of a vibrating string of length l, fixed at both endpoints. The mathematical problem to be solved is

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(l,t) = 0. \tag{1.28}$$

If we take $u(x,t) = X(x)T(t)$, (1.28) becomes

$$X(x)T''(t) = c^2 X''(x)T(t), \tag{1.29}$$

$$X(0) = X(l) = 0 \tag{1.30}$$

On dividing (1.29) through by $c^2 X(x)T(t)$, we get

$$X''(x)/X(x) = T''(t)/c^2 T(t),$$

and both sides of this equation must be equal to a constant that we call $-\lambda^2$. (As before, λ might be any complex number until we pin it down further.) Hence,

$$X''(x) = -\lambda^2 X(x), \qquad T''(t) = -\lambda^2 c^2 T(t).$$

The general solutions of these ordinary differential equations are

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x, \qquad T(t) = C_3 \cos \lambda ct + C_4 \sin \lambda ct.$$

As with the heat equation, the boundary conditions (1.30) imply that $C_1 = 0$ and $\lambda = n\pi/l$ where n is a (positive) integer. We therefore obtain the series solutions

$$u(x,t) = \sum_1^\infty \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right). \tag{1.31}$$

We recall from §1.1 that the appropriate initial conditions for this problem are to specify $u(x,0) = f(x)$ and $(\partial u/\partial t)(x,0) = g(x)$. Setting $t = 0$ in (1.31), we find that

$$f(x) = \sum_1^\infty a_n \sin \frac{n\pi x}{l},$$

whereas if we differentiate (1.31) with respect to t (ignoring possible difficulties about differentiating an infinite series term by term) and then set $t = 0$, we get

$$g(x) = \sum_1^\infty \frac{n\pi c}{l} b_n \sin \frac{n\pi x}{l}.$$

Thus we are led once again to the problem of expanding f and g in a sine series of the form (1.22).

To sum up: in order to carry out the program of solving differential equations by separation of variables, there are two problems that have to be addressed. First, there are some technicalities connected with the convergence properties of infinite series; these are sometimes annoying but rarely are really serious. Second and more important, the following questions must be answered. *Can a given function on the interval* $(0,l)$ *be expanded in a sine series* (1.22) *or a cosine series* (1.25)? *Can a periodic function with period* 2π *be expanded in a series of the form* (1.27)? *If so, how?*

It is to these and related questions that the next chapter is devoted.

EXERCISES

1. Derive pairs of ordinary differential equations from the following partial differential equations by separation of variables, or show that it is not possible.
   a. $yu_{xx} + u_y = 0$.
   b. $x^2u_{xx} + xu_x + u_{yy} + u = 0$.
   c. $u_{xx} + u_{xy} + u_{yy} = 0$.
   d. $u_{xx} + u_{xy} + u_y = 0$.
2. Derive sets of three ordinary differential equations from the following partial differential equations by separation of variables.
   a. $yu_{xx} + xu_{yy} + xyu_{zz} = 0$.
   b. $x^2u_{xx} + xu_x + u_{yy} + x^2u_{zz} = 0$.
3. Use the results in the text to solve
$$u_{tt} = 9u_{xx}, \qquad u(0,t) = u(1,t) = 0,$$
$$u(x,0) = 2\sin \pi x - 3\sin 4\pi x, \qquad u_t(x,0) = 0 \qquad (0 < x < 1).$$
4. Use the results in the text to solve
$$u_t = \tfrac{1}{10}u_{xx}, \qquad u_x(0,t) = u_x(\pi,t) = 0,$$
$$u(x,0) = 3 - 4\cos 2x \qquad (0 < x < \pi).$$
   Determine a value of t_0 so that $|u(x,t) - 3| < 10^{-4}$ for $t > t_0$.
5. By separation of variables, derive the solutions $u_n(x,y) = \sin n\pi x \sinh n\pi y$ of
$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(1,y) = u(x,0) = 0$$
   that were discussed in Exercise 5a, §1.2.
6. By separation of variables, derive the family
$$u_{mn}^{\pm}(x,y,z) = \sin m\pi x \cos n\pi y \exp\left(\pm\sqrt{m^2+n^2}\,\pi z\right)$$
   of the problem
$$\nabla^2 u = 0, \qquad u(0,y,z) = u(1,y,z) = u_y(x,0,z) = u_y(x,1,z) = 0.$$
7. Use separation of variables to find an infinite family of independent solutions of
$$u_t = ku_{xx}, \qquad u(0,t) = 0, \qquad u_x(l,t) = 0,$$
   representing heat flow in a rod with one end held at temperature zero and the other end insulated.

CHAPTER 2
FOURIER SERIES

In Chapter 1 we derived three problems concerning the expansion of functions in terms of sines and cosines. The most fundamental of these is the expansion of periodic functions, which is of importance not only for boundary value problems but for the analysis of any sort of periodic phenomena, and which has provided either direct or indirect inspiration for many of the developments of modern mathematical analysis. Most of this chapter is devoted to the study of periodic functions. Once they are understood, the other two expansion problems of §1.3 can be solved without difficulty, as we shall see in §2.4.

In many respects it is simpler and neater to work with the complex exponential function $e^{i\theta}$ instead of the trigonometric functions $\cos\theta$ and $\sin\theta$. We recall that these functions are related by the formulas

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The advantages of cosine and sine are that they are real-valued and are, respectively, even and odd; the advantages of the exponential are that its differentiation formula $(e^{i\theta})' = ie^{i\theta}$ and addition formula $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$ are simpler than the corresponding formulas for cosine and sine. Accordingly, it is worthwhile to be able to translate one formulation into the other without much effort; we urge the readers who have not yet acquired this facility to spend a little time doing so. A more complete list of the properties of exponential and trigonometric functions of complex variables will be found in Appendix 2.

2.1 The Fourier series of a periodic function

Suppose that $f(\theta)$ is a function defined on the real line such that $f(\theta+2\pi) = f(\theta)$ for all θ. Such functions are said to be **periodic with period 2π**, or **2π-periodic** for short. We shall assume that f is Riemann integrable on every bounded interval; this will be the case if f is bounded and is continuous except perhaps at finitely many points in each bounded interval. (We shall consider various other hypotheses on f in subsequent sections.) Since we shall be using the complex exponential function, we shall allow f to be complex-valued rather than merely real-valued. This bit of extra generality causes no additional difficulties and indeed simplifies some things; moreover, in more advanced work it is often crucial to use complex functions.

We wish to know if f can be expanded in a series

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta). \tag{2.1}$$

Here $\frac{1}{2}a_0$ is the coefficient of the constant function $1 = \cos 0\theta$, and the factor of $\frac{1}{2}$ is incorporated in it for reasons of later convenience (see the remark following equation (2.6)). There is no b_0 because $\sin 0\theta = 0$.

In view of the formulas $\cos n\theta = (e^{in\theta} + e^{-in\theta})/2$ and $\sin n\theta = (e^{in\theta} - e^{-in\theta})/2i$, (2.1) can be rewritten as

$$f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{2.2}$$

where

$$c_0 = \tfrac{1}{2}a_0; \qquad c_n = \tfrac{1}{2}(a_n - ib_n) \text{ and } c_{-n} = \tfrac{1}{2}(a_n + ib_n) \text{ for } n = 1, 2, 3, \ldots \tag{2.3}$$

Alternatively, if we start out with (2.2), by using the formulas $e^{in\theta} = \cos n\theta + i \sin n\theta$, $\cos(-n)\theta = \cos n\theta$, and $\sin(-n)\theta = -\sin n\theta$, we can put it in the form (2.1) where

$$a_0 = 2c_0; \qquad a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}) \quad \text{for } n = 1, 2, 3, \ldots \tag{2.4}$$

In what follows we shall work primarily with (2.2), but we shall also show how to interpret the results in terms of (2.1).

As a first step towards analyzing general periodic functions in terms of trigonometric series, let us consider the following question. If we know to begin with that $f(\theta)$ has a series expansion of the form (2.2), how can the coefficients c_n be calculated in terms of f? The answer to this question is appealingly simple. Let us multiply both sides of (2.2) by $e^{-ik\theta}$ (k being an integer) and integrate from $-\pi$ to π. Taking on faith for the moment that it is permissible to integrate the series term by term, we obtain

$$\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta.$$

But

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta = \frac{1}{i(n-k)}e^{i(n-k)\theta}\Big|_{-\pi}^{\pi} = \frac{(-1)^{n-k} - (-1)^{n-k}}{i(n-k)} = 0 \qquad \text{if } n \neq k,$$

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta = \int_{-\pi}^{\pi} d\theta = 2\pi \qquad \text{if } n = k.$$

Hence the only term in the series that survives the integration is the term with $n = k$, and we obtain

$$\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta = 2\pi c_k.$$

In other words, relabeling the integer k as n, we have the desired formula for the coefficients c_n:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta. \tag{2.5}$$

It is now an easy matter to find the coefficients a_n and b_n for the series (2.1):

$$a_0 = 2c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta,$$

and for $n = 1, 2, 3, \ldots$,

$$a_n = c_n + c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(e^{-in\theta} + e^{in\theta})\,d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta,$$
$$b_n = i(c_n - c_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\theta)(e^{-in\theta} - e^{in\theta})\,d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta;$$

that is,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta \qquad (n \geq 0); \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta \qquad (n \geq 1). \end{aligned} \tag{2.6}$$

(Note that the formula for a_n here holds also for $n = 0$; this is the reason for the factor of $\frac{1}{2}$ in (2.1).)

To recapitulate: *if* f has a series expansion of the form (2.1) (or (2.2)), and *if* the series converges decently so that term-by-term integration is permissible, *then* the coefficients a_n and b_n [or c_n] are given by (2.6) [or (2.5)]. But now if f is *any* Riemann-integrable periodic function, the integrals in (2.5) and (2.6) make perfectly good sense, and we can use them to *define* the coefficients a_n, b_n, and c_n. We are now in a position to make a formal definition.

Definition. Suppose f is periodic with period 2π and integrable over $[-\pi, \pi]$. The numbers c_n defined by (2.5), or the numbers a_n and b_n defined by (2.6), are called the **Fourier coefficients** of f, and the corresponding series

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} \quad \text{or} \quad \tfrac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is called the **Fourier series** of f.

Instead of integrating from $-\pi$ to π in (2.5) and (2.6), one could equally well integrate over any interval of length 2π, for instance from 0 to 2π. The result will be the same since the integrands are all 2π-periodic. This is an instance of the following general fact, which is sufficiently useful to merit a special mention.

Lemma 2.1. *If F is periodic with period P, then $\int_a^{a+P} F(x)\,dx$ is independent of a.*

Proof: Let

$$g(a) = \int_a^{a+P} F(x)\,dx = \int_0^{a+P} F(x)\,dx - \int_0^a F(x)\,dx.$$

By the fundamental theorem of calculus, $g'(a) = F(a+P) - F(a)$, so by the periodicity of F, g' vanishes identically. Thus g is constant. ∎

Another useful observation in this context is that

$$\int_{-a}^{a} F(x)\,dx = \begin{cases} 2\int_0^a F(x)\,dx & \text{if } F \text{ is even,} \\ 0 & \text{if } F \text{ is odd.} \end{cases}$$

(Recall that F is **even** if $F(-x) = F(x)$ and **odd** if $F(-x) = -F(x)$.) Since $\cos n\theta$ is even and $\sin n\theta$ is odd, we have the following result.

Lemma 2.2. *With reference to the formulas (2.6),*

$$\text{if } f \text{ is even,}\qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\,d\theta \quad\text{and}\quad b_n = 0;$$

$$\text{if } f \text{ is odd,}\qquad a_n = 0 \quad\text{and}\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta.$$

Whether the Fourier series of a 2π-periodic function f is written in the trigonometric form (2.1) or the exponential form (2.2), the constant term in the series is

$$c_0 = \tfrac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta,$$

which is nothing but the average or mean value of f on the interval $[-\pi, \pi]$. By Lemma 2.1, it is also the mean value of f on *any* interval of length 2π. This fact is very useful, and it may be more easily remembered than the integral formula; accordingly, we display it as a lemma.

Lemma 2.3. *The constant term in the Fourier series of a 2π-periodic function f is the mean value of f on an interval of length 2π.*

The preceding discussion shows that if we wish to find a trigonometric series that converges to a given periodic function f, the Fourier series of f is the only reasonable candidate; but we do not yet know whether it always does the job. Before tackling this general question, let us compute a couple of examples.

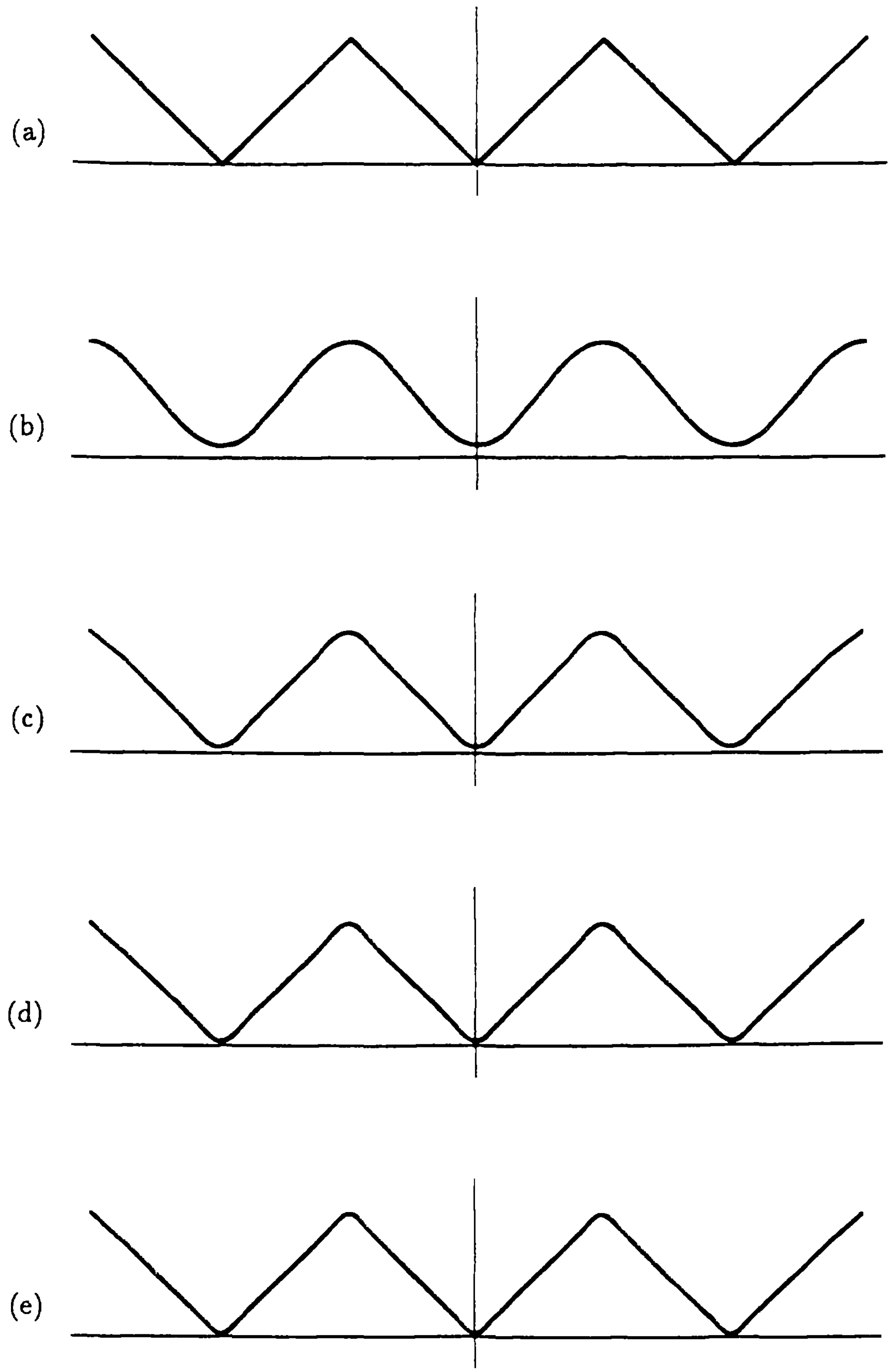


FIGURE 2.1. The triangle wave of Example 1 and some partial sums of its Fourier series: (a) the triangle wave, (b) S_1, (c) S_2, (d) S_3, and (e) S_4, where $S_K = \frac{1}{2}\pi - (4/\pi)\sum_1^K (2k-1)^{-2}\cos(2k-1)\theta$.

Example 1. Let f be the 2π-periodic function determined by the formula

$$f(\theta) = |\theta| \quad \text{for } -\pi \le \theta \le \pi;$$

that is, f is the triangle wave depicted in Figure 2.1(a). Since f is even, we can calculate the coefficients a_n and b_n by using Lemma 2.2. We have $b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta\,d\theta = \frac{2}{\pi}\int_0^\pi \theta\cos n\theta\,d\theta.$$

Thus, for $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^\pi \theta\,d\theta = \frac{1}{\pi}\theta^2\Big|_0^\pi = \pi,$$

and for $n > 0$,

$$a_n = \frac{2}{\pi}\frac{\theta\sin n\theta}{n}\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi \frac{\sin n\theta}{n}\,d\theta = \frac{2}{\pi}\frac{\cos n\theta}{n^2}\Big|_0^\pi = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2},$$

since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$. Now, $(-1)^n - 1$ equals -2 when n is odd and 0 when n is even. Therefore, the Fourier series of f is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\dots}\frac{1}{n^2}\cos n\theta = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\theta}{(2k-1)^2}. \tag{2.7}$$

The graphs of the first few partial sums of this series are shown in Figure 2.1(b–e). Evidently they provide good approximations to f: after only five terms (including the constant term), the graph of the partial sum is almost indistinguishable from the graph of f, except that the corners are a bit rounded. Moreover, we can easily see that the whole series converges absolutely, by comparison to the convergent series $\sum_1^\infty n^{-2}$.

Example 2. Let g be the 2π-periodic function determined by the formula

$$g(\theta) = \theta \quad \text{for } -\pi < \theta \le \pi.$$

In other words, g is the sawtooth wave depicted in Figure 2.2(a). We could use Lemma 2.2 to calculate a_n and b_n since g is odd, but for the sake of variety we shall use (2.5) to calculate c_n instead. For $n = 0$ we have

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta\,d\theta = 0,$$

and for $n \ne 0$ we integrate by parts to obtain

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta e^{-in\theta}\,d\theta = \frac{1}{2\pi}\frac{\theta e^{-in\theta}}{-in}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-in\theta}}{-in}\,d\theta \\ &= \frac{1}{2\pi}e^{-in\theta}\left(\frac{\theta}{-in} + \frac{1}{n^2}\right)\Big|_{-\pi}^{\pi} = \frac{(-1)^{n+1}}{in}, \end{aligned}$$

(a)

(b)

(c)

(d)

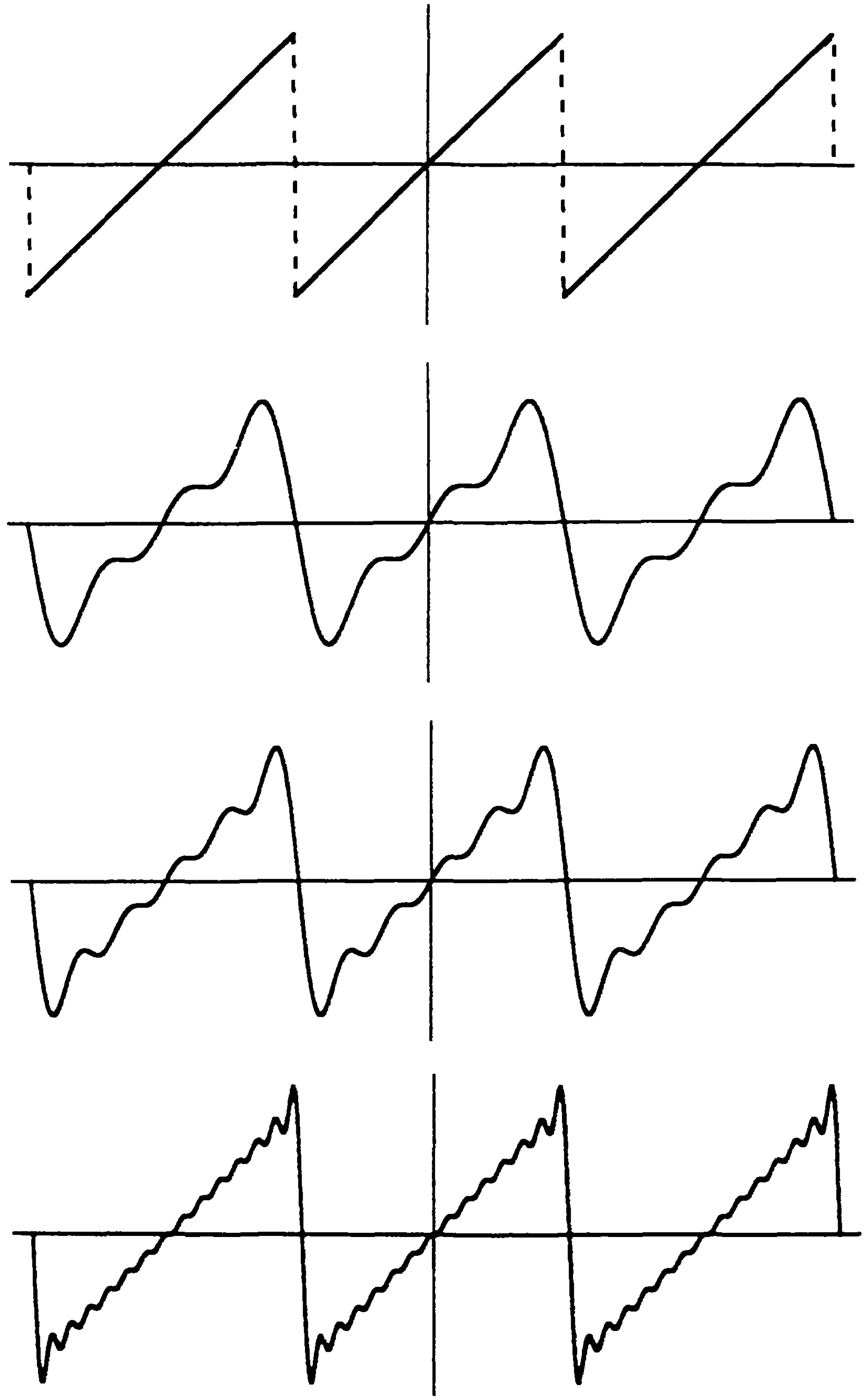


FIGURE 2.2. The sawtooth wave of Example 2 and some partial sums of its Fourier series: (a) the sawtooth wave, (b) S_3, (c) S_5, and (d) S_{14}, where $S_N = 2\sum_1^N (-1)^{n+1} n^{-1} \sin n\theta$.

since $e^{-in\pi} = (-1)^n$. Hence the Fourier series of g is

$$\sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{in\theta}.$$

Here n runs through all positive and negative integers. Since $(-1)^n = (-1)^{-n}$, the nth and $(-n)$th terms of this series can be combined to give

$$(-1)^{n+1}\left(\frac{e^{in\theta}}{in} + \frac{e^{-in\theta}}{-in}\right) = \frac{2(-1)^{n+1}}{n}\sin n\theta,$$

and thus the Fourier series of g is

$$2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\theta. \tag{2.8}$$

The graphs of some partial sums of this series are shown in Figure 2.2(b–d). One can see that these partial sums do approximate the original function g, but a comparison of Figures 2.1 and 2.2 shows that the quality of the approximation here is markedly inferior to that in Example 1. One must add many more terms to the series to get a comparably close fit to the original curve, particularly near the discontinuities. (See also Figure 2.8 in §2.6, showing the 40th partial sum of the Fourier series of the reversed sawtooth wave, for an even more dramatic demonstration of this fact.)

Analytically, the reason for this is that the terms in the series (2.7) tend to zero much more rapidly than the terms in the series (2.8). Precisely, if one disregards the even-order terms in (2.7) (which are all zero), the nth term in (2.7) is of the order of magnitude of $(2n-1)^{-2}$, whereas the nth term in (2.8) is of the order of magnitude of n^{-1}. Thus, the contributions of the high-order terms is much less in (2.7) than in (2.8). As we shall see in §2.3, this phenomenon is intimately related to the fact that the triangle wave is smoother than the sawtooth wave: the former is everywhere continuous, whereas the latter has jump discontinuities. The point is that the rougher a function is, the more difficult it is to approximate it with perfectly smooth functions like linear combinations of $\cos n\theta$ and $\sin n\theta$.

In fact, there sems to be some danger that the series (2.8) will not converge: the nth term has magnitude roughly n^{-1} in general, and $\sum_1^\infty n^{-1}$ diverges. On the other hand, at a given point θ some of the functions $\sin n\theta$ will be positive and others will be negative, so there may be some cancellation effects that will prevent divergence. This is in fact the case, as we shall prove in the next section. For the moment, we simply wish to impress on the reader that the convergence of Fourier series is not a simple matter.

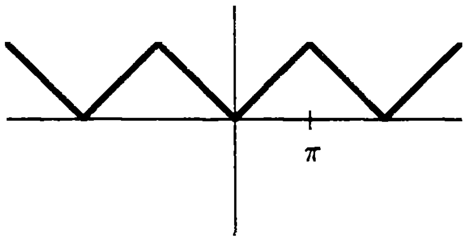
Table 1 gives a list of some elementary Fourier series. It includes all the examples we shall need later on. The fact that all the functions in this table really are the sums of their Fourier series (except perhaps at their points of discontinuity) follows from Theorem 2.1 in §2.2.

We conclude this section by deriving an estimate on the Fourier coefficients that will be needed to establish convergence results in the following sections.

TABLE 1. FOURIER SERIES

The functions f in this table are all understood to be 2π-periodic. The formula for $f(\theta)$ on either $(-\pi, \pi)$ or $(0, 2\pi)$ (except perhaps at its points of discontinuity) is given in the left column; the Fourier series of f is given in the right column; and the graph of f is sketched on the facing page.

1.	$f(\theta) = \theta \quad (-\pi < \theta < \pi)$	$2\sum_1^\infty \frac{(-1)^{n+1}}{n} \sin n\theta$
2.	$f(\theta) = \|\theta\| \quad (-\pi < \theta < \pi)$	$\frac{\pi}{2} - \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2}$
3.	$f(\theta) = \pi - \theta \quad (0 < \theta < 2\pi)$	$2\sum_1^\infty \frac{\sin n\theta}{n}$
4.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ \theta & (0 < \theta < \pi) \end{cases}$	$\frac{\pi}{4} - \frac{2}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} + \sum_1^\infty \frac{(-1)^{(n+1)}}{n} \sin n\theta$
5.	$f(\theta) = \sin^2 \theta$	$\frac{1}{2} - \frac{1}{2}\cos 2\theta$
6.	$f(\theta) = \begin{cases} -1 & (-\pi < \theta < 0) \\ 1 & (0 < \theta < \pi) \end{cases}$	$\frac{4}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$
7.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ 1 & (0 < \theta < \pi) \end{cases}$	$\frac{1}{2} + \frac{2}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$
8.	$f(\theta) = \|\sin\theta\|$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1}$
9.	$f(\theta) = \|\cos\theta\|$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{(-1)^n \cos 2n\theta}{4n^2-1}$
10.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ \sin\theta & (0 < \theta < \pi) \end{cases}$	$\frac{1}{\pi} - \frac{2}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1} + \frac{1}{2}\sin\theta$



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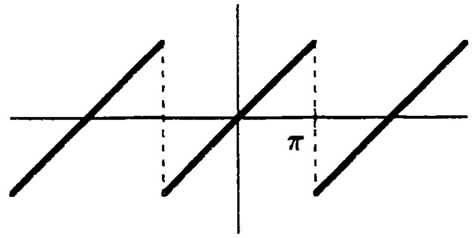

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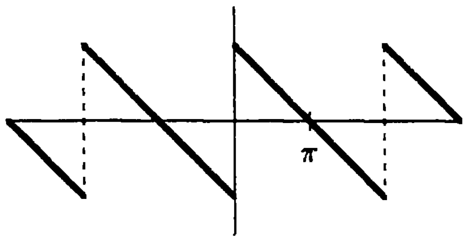

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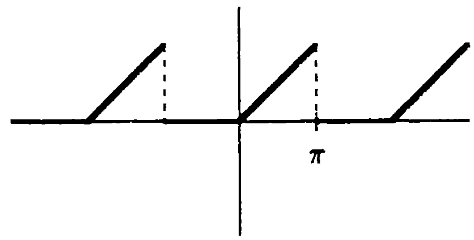

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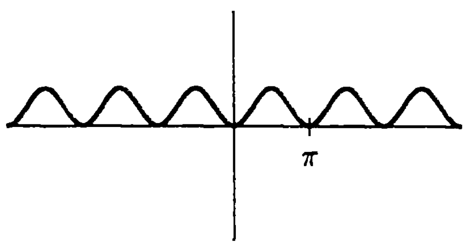

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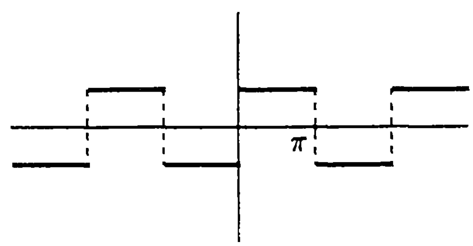

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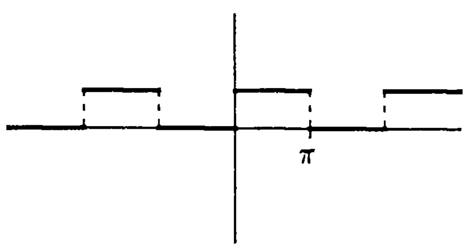

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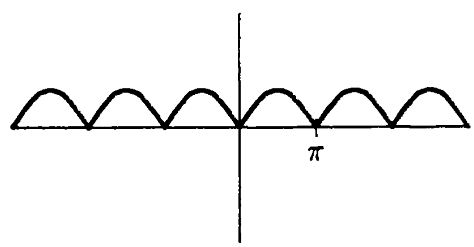

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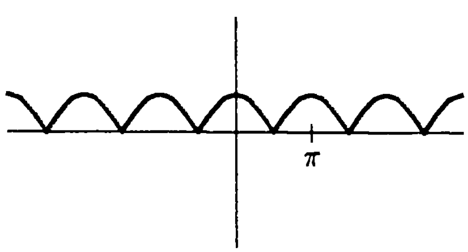

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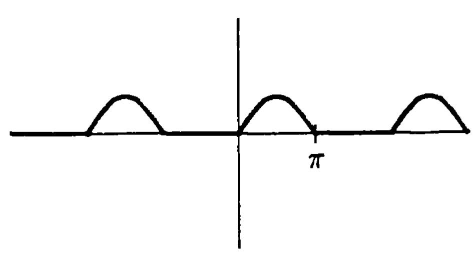

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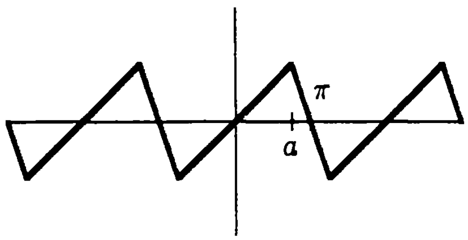
TABLE 1 (continued)

11.	$f(\theta)=\begin{cases}\theta & (-a<\theta<a)\\ a\frac{\pi-\theta}{\pi-a} & (a<\theta<\pi)\\ a\frac{\pi+\theta}{a-\pi} & (-\pi<\theta<-a)\end{cases}$	$\dfrac{2}{\pi-a}\sum_1^\infty \dfrac{\sin na}{n^2}\sin n\theta$
12.	$f(\theta)=\begin{cases}(2a)^{-1} & (\lvert\theta\rvert<a)\\ 0 & (a<\lvert\theta\rvert<\pi)\end{cases}$	$\dfrac{1}{2\pi}+\dfrac{1}{\pi}\sum_1^\infty \dfrac{\sin na}{na}\cos n\theta$
13.	$f(\theta)=\begin{cases}(2a)^{-1} & (\lvert\theta-\theta_0\rvert<a)\\ 0 & (a<\lvert\theta-\theta_0\rvert<\pi)\end{cases}$	$\dfrac{1}{2\pi}+\dfrac{1}{\pi}\sum_1^\infty \dfrac{\sin na}{na}(\cos n\theta_0\cos n\theta + \sin n\theta_0 \sin n\theta)$
14.	$f(\theta)=\begin{cases}1 & (-a<\theta<a)\\ -1 & (2a<\theta<4a)\\ 0 & \text{elsewhere in } (-\pi,\pi)\end{cases}$	$\sum_1^\infty \dfrac{\sin na}{n}\Big[(1-\cos 3na)\cos n\theta - \sin 3na \sin n\theta\Big]$
15.	$f(\theta)=\begin{cases}a^{-2}(a-\lvert\theta\rvert) & (\lvert\theta\rvert<a)\\ 0 & (a<\lvert\theta\rvert<\pi)\end{cases}$	$\dfrac{1}{2\pi}+\dfrac{2}{\pi}\sum_1^\infty \dfrac{1-\cos na}{n^2a^2}\cos n\theta$
16.	$f(\theta)=\theta^2 \quad (-\pi<\theta<\pi)$	$\dfrac{\pi^2}{3}+4\sum_1^\infty \dfrac{(-1)^n}{n^2}\cos n\theta$
17.	$f(\theta)=\theta(\pi-\lvert\theta\rvert) \quad (-\pi<\theta<\pi)$	$\dfrac{8}{\pi}\sum_1^\infty \dfrac{\sin(2n-1)\theta}{(2n-1)^3}$
18.	$f(\theta)=e^{b\theta} \quad (-\pi<\theta<\pi)$	$\dfrac{\sinh b\pi}{\pi}\sum_{-\infty}^\infty \dfrac{(-1)^n}{b-in}e^{in\theta}$
19.	$f(\theta)=e^{b\theta} \quad (0<\theta<2\pi)$	$\dfrac{e^{2\pi b}-1}{2\pi}\sum_{-\infty}^\infty \dfrac{e^{in\theta}}{b-in}$
20.	$f(\theta)=\sinh\theta \quad (-\pi<\theta<\pi)$	$\dfrac{2\sinh\pi}{\pi}\sum_1^\infty \dfrac{(-1)^{n+1}n}{n^2+1}\sin n\theta$



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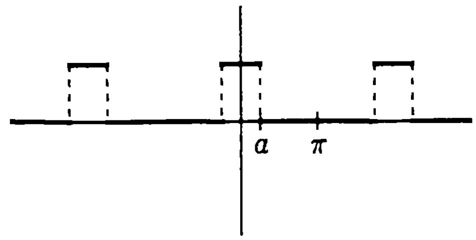

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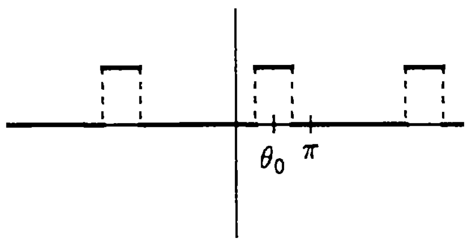

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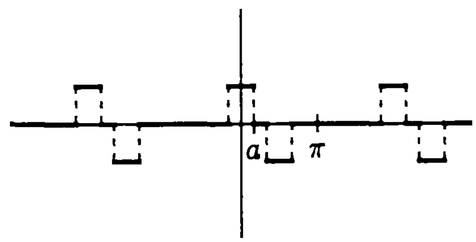

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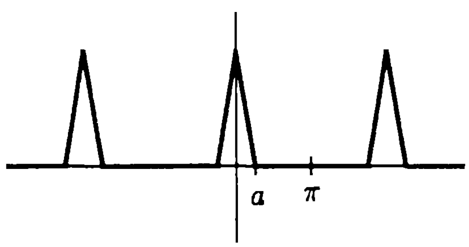

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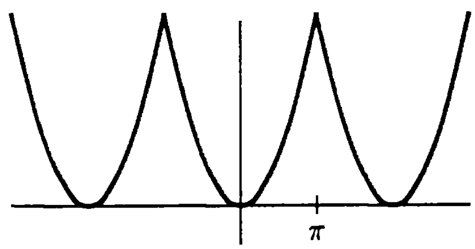

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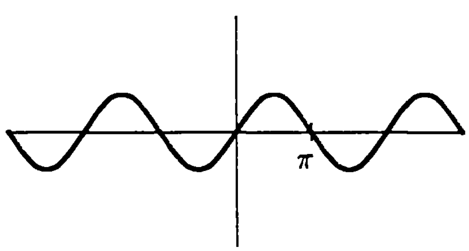

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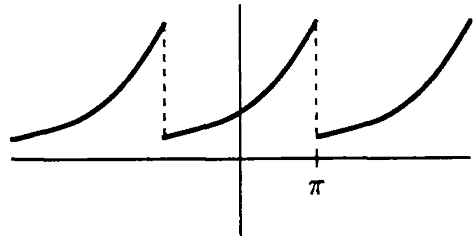

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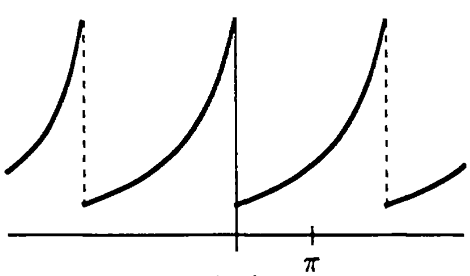

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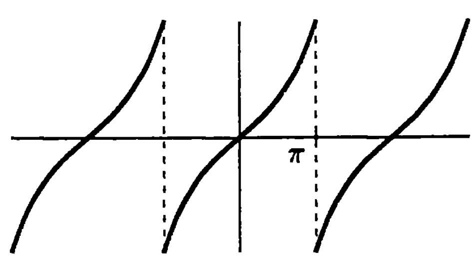


(20)

Bessel's Inequality. *If f is 2π-periodic and Riemann integrable on $[-\pi, \pi]$, and the Fourier coefficients c_n are defined by (2.5), then*

$$\sum_{-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.$$

Proof: Since $|z|^2 = z\overline{z}$ for any complex number z,

$$\begin{aligned}
&\left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 \\
&= \left(f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right) \left(\overline{f(\theta)} - \sum_{-N}^{N} \overline{c}_n e^{-in\theta} \right) \\
&= |f(\theta)|^2 - \sum_{-N}^{N} \left[c_n \overline{f(\theta)} e^{in\theta} + \overline{c}_n f(\theta) e^{-in\theta} \right] + \sum_{m,n=-N}^{N} c_m \overline{c}_n e^{i(m-n)\theta}.
\end{aligned}$$

Now divide both sides by 2π and integrate from $-\pi$ to π. Taking account of the formulas

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta = c_n, \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n, \end{cases}$$

we obtain

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} \left[c_n \overline{c}_n + \overline{c}_n c_n \right] + \sum_{-N}^{N} c_n \overline{c}_n \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} |c_n|^2.
\end{aligned}$$

But the integral on the left is certainly nonnegative, so

$$0 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} |c_n|^2.$$

Letting $N \to \infty$, we obtain the desired result. ∎

Bessel's inequality can also be stated in terms of the coefficients a_n and b_n defined by (2.6). Indeed, by equation (2.4), for $n \ge 1$ we have

$$\begin{aligned}
|a_n|^2 + |b_n|^2 &= a_n \overline{a}_n + b_n \overline{b}_n \\
&= (c_n + c_{-n})(\overline{c}_n + \overline{c}_{-n}) + i(c_n - c_{-n})(-i)(\overline{c}_n - \overline{c}_{-n}) \\
&= 2c_n \overline{c}_n + 2c_{-n} \overline{c}_{-n},
\end{aligned}$$

so that

$$|a_0|^2 = 4|c_0|^2, \qquad |a_n|^2 + |b_n|^2 = 2\left(|c_n|^2 + |c_{-n}|^2\right) \quad \text{for } n \geq 1.$$

Therefore,

$$\frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_1^\infty \left(|a_n|^2 + |b_n|^2\right) = \sum_{-\infty}^\infty |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.$$

It turns out, as we shall see later, that Bessel's inequality is actually an equality. For now, its main significance is simply the fact that the series $\sum |a_n|^2$, $\sum |b_n|^2$, and $\sum |c_n|^2$ are all convergent. As a consequence, we obtain the following result, which is a special case of a theorem known as the *Riemann-Lebesgue lemma.*

Corollary 2.1. *The Fourier coefficients a_n, b_n, and c_n all tend to zero as $n \to \infty$ (and as $n \to -\infty$ in the case of c_n).*

Proof: $|a_n|^2$, $|b_n|^2$, and $|c_n|^2$ are the nth terms of convergent series, so they tend to zero as $n \to \infty$; hence so do a_n, b_n, and c_n. ∎

EXERCISES

Verify the formulas of Table 1. That is, for $3 \leq n \leq 20$, Exercise n is to show that the series in the right column of entry n in Table 1 is the Fourier series of the function in the left column. (Entries 1 and 2 are Examples 1 and 2 in the text.) Some of these functions are related to each other, and you may be able to use this fact to avoid caclulating all the Fourier coefficients from scratch each time. Entries 3 and 4 can be derived from entries 1 and 2; entry 7 can be derived from entry 6; entries 9 and 10 can be derived from entry 8; entries 13 and 14 can be derived from entry 12; and entries 19 and 20 can be derived from entry 18.

2.2 A convergence theorem

Question: does the Fourier series of a periodic function f converge to f? The answer is certainly not obvious — for example, why should one be able to expand nonsmooth functions like the examples in §2.1 in a series whose individual terms $\cos nx$ and $\sin nx$ possess derivatives of all orders? Fourier's assertion that the answer is yes was initially greeted with a certain amount of disbelief. In fact, the answer *is* always yes provided that things are interpreted suitably, although the situation is somewhat more delicate than one might initially expect.

In this section we shall show that the Fourier series of f converges to f under certain reasonably general hypotheses on f; later, in §2.3, §2.6, §3.4, and §9.3, we shall present some variants of this result under other conditions on f. We first define the class of functions with which we shall be working.

Suppose $-\infty < a < b < \infty$. We say that a function f on the closed interval $[a, b]$ is **piecewise continuous** provided that

(i) f is continuous on $[a, b]$ except perhaps at finitely many points $x_1, \dots, x_k$;

(ii) at each of the points $x_1, \dots, x_k$, the left-hand and right-hand limits of f,

$$f(x_j-) = \lim_{h\to 0,\ h>0} f(x_j - h) \quad \text{and} \quad f(x_j+) = \lim_{h\to 0,\ h>0} f(x_j + h),$$

exist. (If the endpoint a (or b) is one of the exceptional points x_j, we require only the right-hand (or left-hand) limit to exist.)

That is, f is piecewise continuous on $[a, b]$ if f is continuous there except for finitely many finite jump discontinuities. (When we say that the limits $f(x_j\pm)$ exist, we mean in particular that they are finite: ∞ is not allowed as a value.) We denote the class of piecewise continuous functions on $[a, b]$ by $PC(a, b)$.

Next, we say that a function f on $[a, b]$ is **piecewise smooth** if f and its first derivative f' are both piecewise continuous on $[a, b]$, and we denote the class of piecewise smooth functions on $[a, b]$ by $PS(a, b)$. More precisely, $f \in PS(a, b)$ if and only if

(i) $f \in PC(a, b)$;

(ii) f' exists and is continuous on (a, b) except perhaps at finitely many points $x_1, \dots, x_K$ (which will include any points where f is discontinuous), and the one-sided limits $f'(x_j-)$ and $f'(x_j+)$ ($j = 1, \dots, K$), and also $f'(a+)$ and $f'(b-)$, exist.

In other words, f is piecewise smooth if its graph is a smooth curve except for finitely many jumps (where f is discontinuous) and corners (where f' is discontinuous). We do not allow infinite discontinuities (such as $f(x) = 1/x$ has at $x = 0$) or sharp cusps (where f' becomes infinite). See Figure 2.3.

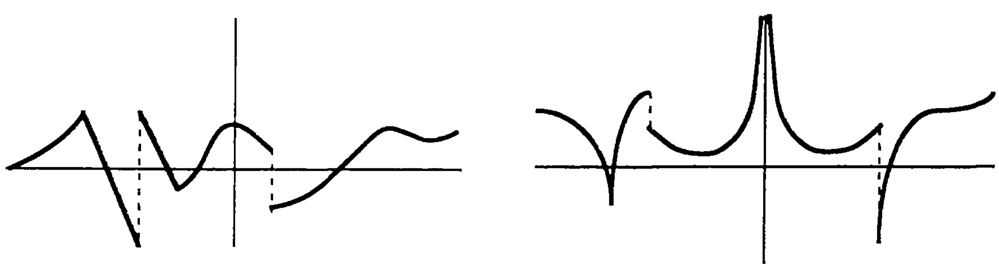


FIGURE 2.3. A piecewise smooth function (left) and a function that is not piecewise smooth (right).

One last bit of terminology: a function defined on the whole real line $\mathbf{R}$ is said to be **piecewise continuous** or **piecewise smooth** on $\mathbf{R}$ if it is so on every bounded interval $[a, b]$. (That is, f or f' may have infinitely many discontinuities on the whole line but only finitely many in any bounded interval.) We denote the spaces of piecewise continuous and piecewise smooth functions on $\mathbf{R}$ by $PC(\mathbf{R})$ and $PS(\mathbf{R})$.

We now return to consideration of the Fourier series of a 2π-periodic function $f(\theta)$. We recall that this is defined to be

$$\tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{2.9}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\cos n\psi\, d\psi, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\sin n\psi\, d\psi,$$
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)e^{-in\psi}\, d\psi. \tag{2.10}$$

(We have labeled the variable of integration in (2.10) as ψ simply for later convenience.)

What meaning is to be attached to this series? Of course, the sum of any infinite series is defined to be the limit of its partial sums. When we write the series (2.9) in trigonometric form, we agree always to group together the terms involving $\cos n\theta$ and $\sin n\theta$ as indicated above; correspondingly, when we write it in exponential form, we agree always to group together the terms involving $e^{in\theta}$ and $e^{-in\theta}$. (*This convention will always be in force.*) Thus we take the Nth partial sum of the series (2.9) to be the sum $S_N^f(\theta)$ defined by

$$S_N^f(\theta) = \tfrac{1}{2}a_0 + \sum_1^N (a_n \cos n\theta + b_n \sin n\theta) = \sum_{-N}^{N} c_n e^{in\theta}, \tag{2.11}$$

and our aim is to show that S_N^f converges to f as $N \to \infty$.

If we plug the definition (2.10) of c_n into (2.11), we see that

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\psi)e^{in(\theta-\psi)}d\psi = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\psi)e^{in(\psi-\theta)}d\psi.$$

The last equality results from replacing n by $-n$; this does not affect the sum since n ranges from $-N$ to N. If we now make the change of variable $\phi = \psi - \theta$ and use Lemma 2.1, we obtain

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi+\theta}^{\pi+\theta} f(\theta+\phi)e^{in\phi}d\phi = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\theta+\phi)e^{in\phi}d\phi.$$

In short,

$$S_N^f(\theta) = \int_{-\pi}^{\pi} f(\theta+\phi)D_N(\phi)\,d\phi, \quad \text{where } D_N(\phi) = \frac{1}{2\pi}\sum_{-N}^{N} e^{in\phi}. \tag{2.12}$$

The function $D_N(\phi)$ is called the Nth **Dirichlet kernel**. We can express D_N in a more computable form by recognizing that it is the sum of a finite geometric progression:

$$D_N(\phi) = \frac{1}{2\pi}e^{-iN\phi}(1 + e^{i\phi} + \cdots + e^{i2N\phi}) = \frac{1}{2\pi}e^{-iN\phi}\sum_{0}^{2N} e^{in\phi}.$$

Since $\sum_0^K r^n = (r^{K+1} - 1)/(r - 1)$ for any $r \neq 1$, for $\phi \neq 0$ we have

$$D_N(\phi) = \frac{1}{2\pi}e^{-iN\phi}\frac{e^{i(2N+1)\phi} - 1}{e^{i\phi} - 1} = \frac{1}{2\pi}\frac{e^{i(N+1)\phi} - e^{-iN\phi}}{e^{i\phi} - 1}. \tag{2.13}$$

Moreover, on multiplying top and bottom by $e^{-i\phi/2}$, we obtain

$$D_N(\phi) = \frac{1}{2\pi}\frac{\exp\left[i(N+\frac{1}{2})\phi\right] - \exp\left[-i(N+\frac{1}{2})\phi\right]}{\exp(i\frac{1}{2}\phi) - \exp(-i\frac{1}{2}\phi)} = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})\phi}{\sin\frac{1}{2}\phi}. \tag{2.14}$$

From this formula it is easy to sketch the graph of D_N: it is the rapidly oscillating sine wave $y = \sin(N + \frac{1}{2})\phi$ amplitude-modulated to fit inside the envelope $y = \pm(2\pi)^{-1}\csc\frac{1}{2}\phi$. See Figure 2.4.

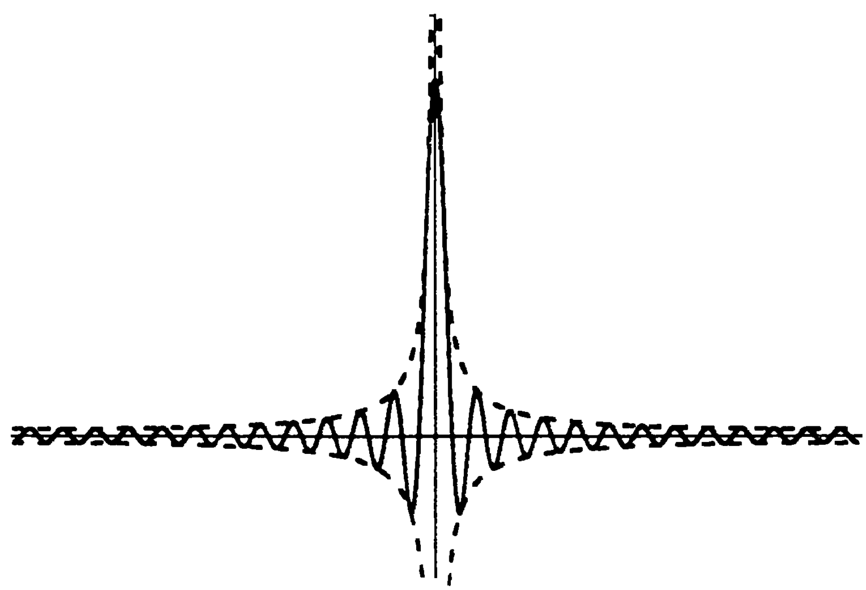


FIGURE 2.4. Graphs of the Dirichlet kernel $D_{25}(\phi)$ (solid) and its envelope $\pm(2\pi)^{-1}\csc\frac{1}{2}\phi$ (dashed) on the interval $-\pi < \phi < \pi$.

The pictorial intuition behind the fact that $S_N^f(\theta) \to f(\theta)$ is as follows: in the integral formula (2.12) for $S_N^f(\theta)$, the sharp central spike of $D_N(\phi)$ at $\phi = 0$ picks out the value $f(\theta)$, and the rapid oscillations of $D_N(\phi)$ away from $\phi = 0$ kill off most of the rest of the integral because of cancellations between positive and negative values. Before proceeding to the actual proof, however, we need one more fact about D_N.

Lemma 2.4. *For any N,*

$$\int_{-\pi}^{0} D_N(\theta)\, d\theta = \int_{0}^{\pi} D_N(\theta)\, d\theta = \frac{1}{2}.$$

Proof: From formula (2.12) we have

$$D_N(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{1}^{N} \cos n\theta,$$

so that

$$\int_{0}^{\pi} D_N(\theta)\, d\theta = \left[\frac{\theta}{2\pi} + \frac{1}{\pi}\sum_{1}^{N} \frac{\sin n\theta}{n}\right]_{0}^{\pi} = \frac{1}{2},$$

and likewise for the integral from $-\pi$ to 0. ∎

Here at last is our main convergence theorem. It says that the Fourier series of a function $f \in PS(\mathbf{R})$ converges pointwise to f, provided that we (re)define f at its points of discontinuity to be the average of its left- and right-hand limits.

Theorem 2.1. *If f is 2π-periodic and piecewise smooth on $\mathbf{R}$, and S_N^f is defined by (2.10) and (2.11), then*

$$\lim_{N\to\infty} S_N^f(\theta) = \tfrac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$

for every θ. In particular, $\lim_{N\to\infty} S_N^f(\theta) = f(\theta)$ for every θ at which f is continuous.

Proof: By Lemma 2.4, we have

$$\tfrac{1}{2}f(\theta-) = f(\theta-)\int_{-\pi}^{0} D_N(\phi)\, d\phi, \qquad \tfrac{1}{2}f(\theta+) = f(\theta+)\int_{0}^{\pi} D_N(\phi)\, d\phi,$$

and hence by equation (2.12),

$$\begin{aligned} &S_N^f(\theta) - \tfrac{1}{2}\left[f(\theta-) + f(\theta+)\right] \\ &= \int_{-\pi}^{0}\left[f(\theta+\phi) - f(\theta-)\right]D_N(\phi)\, d\phi + \int_{0}^{\pi}\left[f(\theta+\phi) - f(\theta+)\right]D_N(\phi)\, d\phi. \end{aligned}$$

We wish to show that for each fixed θ, this quantity approaches zero as $N \to \infty$. But by formula (2.13), we can write it as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)(e^{i(N+1)\phi} - e^{-iN\phi})\, d\phi \tag{2.15}$$

where $g(\phi)$ is defined to be

$$\frac{f(\theta+\phi)-f(\theta-)}{e^{i\phi}-1} \text{ for } -\pi<\phi<0, \qquad \frac{f(\theta+\phi)-f(\theta+)}{e^{i\phi}-1} \text{ for } 0<\phi<\pi.$$

g is a well-behaved function on $[-\pi,\pi]$, as smooth as f is, except near $\phi=0$ (where $e^{i\phi}-1$ vanishes). However, by l'Hôpital's rule,

$$\lim_{\phi\to 0+} g(\phi) = \lim_{\phi\to 0+} \frac{f(\theta+\phi)-f(\theta+)}{e^{i\phi}-1} = \lim_{\phi\to 0+} \frac{f'(\theta+\phi)}{ie^{i\varphi}} = \frac{f'(\theta+)}{i}.$$

Similarly, $g(\phi)$ approaches the finite limit $i^{-1}f'(\theta-)$ as ϕ approaches zero from the left. Hence g is actually piecewise continuous on $[-\pi,\pi]$, so by the corollary to Bessel's inequality in §2.1, its Fourier coefficients

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)e^{-in\phi}\,d\phi$$

tend to zero as $n\to\pm\infty$. But the expression (2.15) is nothing but $C_{-(N+1)}-C_N$, so it vanishes as $N\to\infty$; and this is what we needed to show. ∎

Let us see what this theorem says with regard to the two examples of the previous section. The function f of Example 1 is piecewise smooth and everywhere continuous, so the Fourier series of f converges to f at every point. Thus,

$$\frac{\pi}{2}-\frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos(2n-1)\theta}{(2n-1)^2} = |\theta| \quad \text{for } -\pi\le\theta\le\pi. \tag{2.16}$$

On the other hand, the function g of Example 2 is piecewise smooth and continuous except at the points $\theta=k\pi$ with k odd. At these discontinuities we have $g(k\pi-)=\pi$ and $g(k\pi+)=-\pi$, so $\frac{1}{2}[g(k\pi-)+g(k\pi+)]=0$. Thus the Fourier series of g converges to g at all points except $\theta=k\pi$ (k odd), where it converges to zero. Hence,

$$\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\theta = \frac{\theta}{2} \quad \text{for } -\pi<\theta<\pi. \tag{2.17}$$

In particular, if we take $\theta=0$ in (2.16), we obtain the formula

$$\sum_{1}^{\infty}\frac{1}{(2k-1)^2} = 1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots=\frac{\pi^2}{8}.$$

(As the reader may check, the same formula results from taking taking $\theta=\pi$.) Moreover, if we take $\theta=\frac{1}{2}\pi$ in (2.17) and use the fact that $\sin\frac{1}{2}n\pi$ is alternately 1 and -1 when n is odd and 0 when n is even, we find that

$$\sum_{1}^{\infty}\frac{(-1)^{k+1}}{2k-1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}.$$

These are two interesting instances where numerical series can be evaluated as special values of Fourier series. Others can be found in the exercises.

Theorem 2.1 says that the Fourier series of a 2π-periodic piecewise smooth function f converges to f everywhere, provided that f is (re)defined at each of its points of discontinuity to be the average of its left- and right-hand limits there. *Henceforth, when we speak of piecewise smooth functions, we shall assume that this adjustment has been made,* unless we explicitly state otherwise. This will obviate the need to single out the points of discontinuity for special attention. In particular, with this understanding, we have the following uniqueness theorem.

Corollary 2.2. *If f and g are 2π-periodic and piecewise smooth, and f and g have the same Fourier coefficients, then $f = g$.*

Proof: f and g are both the sum of the same Fourier series. ∎

EXERCISES

1. Which of the following functions are continuous, piecewise continuous, or piecewise smooth on $[-\pi, \pi]$?
   a. $f(\theta) = \csc\theta$. b. $f(\theta) = (\sin\theta)^{1/3}$. c. $f(\theta) = (\sin\theta)^{4/3}$.
   d. $f(\theta) = \cos\theta$ if $\theta > 0$, $f(\theta) = -\cos\theta$ if $\theta \le 0$.
   e. $f(\theta) = \sin\theta$ if $\theta > 0$, $f(\theta) = \sin 2\theta$ if $\theta \le 0$.
   f. $f(\theta) = (\sin\theta)^{1/5}$ if $\theta < \frac{1}{2}\pi$, $f(\theta) = \cos\theta$ if $\theta \ge \frac{1}{2}\pi$.
2. To what values do the series in entries 6, 7, 12, and 18 of Table 1, §2.1, converge at the points where their sums are discontinuous?

The Fourier series for a number of piecewise smooth functions are listed in Table 1 of §2.1, and Theorem 2.1 tells what the sums of these series are. By using this information and choosing suitable values of θ (usually 0, $\frac{1}{2}\pi$, or π), derive the following formulas for the sums of numerical series. (The relevant entries from Table 1 are indicated in parentheses.)

3. $\displaystyle\sum_1^\infty \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_1^\infty \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-2}{4}$ (8).
4. $\displaystyle\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ (16).
5. $\displaystyle\sum_1^\infty \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$ (17).
6. $\displaystyle\sum_1^\infty \frac{(-1)^n}{n^2+b^2} = \frac{\pi}{2b}\operatorname{csch} b\pi - \frac{1}{2b^2}$ (18 or 19).
7. $\displaystyle\sum_1^\infty \frac{1}{n^2+b^2} = \frac{\pi}{2b}\coth b\pi - \frac{1}{2b^2}$ (18 or 19; this is a bit tricky).

2.3 Derivatives, integrals, and uniform convergence

This section is devoted to an examination of the behavior of Fourier series in relation to the processes of calculus.

We shall be largely concerned here with periodic functions that are both continuous and piecewise smooth. Pictorially, the graph of such a function is a smooth curve except that it can have "corners" where the derivative jumps. The fundamental theorem of calculus,

$$f(b) - f(a) = \int_a^b f'(\theta)\,d\theta,$$

applies to functions f that are continuous and piecewise smooth, even though f' is undefined at the "corners." To see this it suffices to express the integral on the right as the sum of integrals over the subintervals of $[a, b]$ on which f is differentiable; the continuity of f guarantees that the endpoint evaluations at the intermediate subdivision points cancel out. For example, if f is differentiable except at the point $c \in (a, b)$, we have

$$\begin{aligned}\int_a^b f'(\theta)\,d\theta &= \int_a^c f'(\theta)\,d\theta + \int_c^b f'(\theta)d\theta \\ &= \big[f(c) - f(a)\big] + \big[f(b) - f(c)\big] = f(b) - f(a).\end{aligned}$$

This observation will be used implicitly in several of the following calculations, including the proof of Theorem 2.2.

Our first main result relates the Fourier coefficients of a function to those of its derivative. The fact that this relation is so simple is one of the main reasons underlying the utility of Fourier series.

Theorem 2.2. *Suppose f is 2π-periodic, continuous, and piecewise smooth. Let a_n, b_n, and c_n be the Fourier coefficients of f defined in (2.5) and (2.6), and let a_n', b_n', and c_n' be the corresponding Fourier coefficients of f'. Then*

$$a_n' = nb_n, \qquad b_n' = -na_n, \qquad c_n' = inc_n.$$

Proof: This is a simple matter of integration by parts. For example,

$$c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\,d\theta = \frac{1}{2\pi}f(\theta)e^{-in\theta}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(-ine^{-in\theta})\,d\theta.$$

The first term on the right vanishes because $f(-\pi) = f(\pi)$ and $e^{in\pi} = e^{-in\pi} = (-1)^n$, and the second term is inc_n. The argument for a_n' and b_n' is the same; we leave the details to the reader. ∎

Combining this result with the theorem of the previous section, we easily obtain the basic results on differentiation and integration of Fourier series.

Theorem 2.3. *Suppose f is 2π-periodic, continuous, and piecewise smooth, and suppose also that f' is piecewise smooth. If*

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} = \tfrac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is the Fourier series of $f(\theta)$, then $f'(\theta)$ is the sum of the derived series

$$\sum_{-\infty}^{\infty} inc_n e^{in\theta} = \sum_{1}^{\infty}(nb_n \cos n\theta - na_n \sin n\theta)$$

for all θ at which $f'(\theta)$ exists. At the exceptional points where f' has jumps, the series converges to $\frac{1}{2}\left[f'(\theta-) + f'(\theta+)\right]$.

Proof: Since f' is piecewise smooth, by Theorem 2.1 it is the sum of its Fourier series at every point (with appropriate modifications at the jumps). By Theorem 2.2, the coefficients of $e^{in\theta}$, $\cos n\theta$, and $\sin n\theta$ in this series are inc_n, nb_n, and $-na_n$. The result follows. ∎

In considering integration of Fourier series, one must keep in mind that the indefinite integral of a periodic function may not be periodic. For example, the constant function $f(\theta) = 1$ is periodic, but its antiderivative $F(\theta) = \theta$ is not. However, the integral of every term in a Fourier series is periodic except for the constant term, from which we see that a periodic function has a periodic integral precisely when the constant term in its Fourier series vanishes, i.e., when its mean value on $[-\pi, \pi]$ is zero. We therefore arrive at the following result.

Theorem 2.4. *Suppose f is 2π-periodic and piecewise continuous, with Fourier coefficients a_n, b_n, c_n, and let $F(\theta) = \int_0^\theta f(\phi)\,d\phi$. If $c_0\ (= \frac{1}{2}a_0) = 0$, then for all θ we have*

$$F(\theta) = C_0 + \sum_{n\neq 0} \frac{c_n}{in} e^{in\theta} = \tfrac{1}{2}A_0 + \sum_{1}^{\infty}\left(\frac{a_n}{n}\sin n\theta - \frac{b_n}{n}\cos n\theta\right) \tag{2.18}$$

where the constant term is the mean value of F on $[-\pi, \pi]$:

$$C_0 = \tfrac{1}{2}A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\,d\theta. \tag{2.19}$$

The series on the right of (2.18) is the series obtained by formally integrating the Fourier series of f term by term, whether the latter series actually converges or not. If $c_0 \neq 0$, the sum of the series on the right of (2.18) is $F(\theta) - c_0\theta$.

Proof: F is continuous and piecewise smooth, being the integral of a piecewise continuous function. Moreover, if $c_0 = 0$, F is 2π-periodic, for

$$F(\theta + 2\pi) - F(\theta) = \int_\theta^{\theta+2\pi} f(\phi)\,d\phi = \int_{-\pi}^{\pi} f(\phi)\,d\phi = 2\pi c_0 = 0.$$

Hence, by Theorem 2.1, $F(\theta)$ is the sum of its Fourier series at every θ. But by Theorem 2.2 applied to F, the Fourier coefficients A_n, B_n, and C_n of F are related to those of f by

$$A_n = -\frac{b_n}{n}, \qquad B_n = \frac{a_n}{n}, \qquad C_n = \frac{c_n}{in} \qquad (n \neq 0).$$

The formula (2.19) for the constant C_0 or $\frac{1}{2}A_0$ is just the usual formula for the zeroth Fourier coefficient of F. If $c_0 \neq 0$, these arguments can be applied to the function $f(\theta) - c_0$ rather than $f(\theta)$, yielding the final assertion. ∎

Example. Let f be the periodic function such that $f(\theta) = 1$ for $0 < \theta < \pi$ and $f(\theta) = -1$ for $-\pi < \theta < 0$, and let $F(\theta) = \int_0^\theta f(\phi)\,d\phi$. Clearly $F(\theta) = |\theta|$ for $|\theta| \leq \pi$. By entry 4 of Table 1, §2.1, the Fourier series of f is $(4/\pi)\sum_1^\infty (2n-1)^{-1}\sin(2n-1)\theta$, so by Theorem 2.4 we have

$$F(\theta) = C_0 - \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} \quad \text{where } C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|\,d\theta = \frac{\pi}{2}.$$

Thus we recover the result of entry 2 of Table 1.

Theorem 2.1 gave conditions under which the Fourier series of f converges pointwise to f. However, experience in working with infinite series teaches us that simple pointwise convergence of a series can be a tricky business, and that we are much better off if the convergence is absolute and uniform. We recall the definitions: suppose the series $\sum_1^\infty g_n(x)$ converges to $g(x)$ on a set S. The convergence is *absolute* if the series $\sum_1^\infty |g_n(x)|$ also converges for $x \in S$, and *uniform* if not only does the difference $g(x) - \sum_1^N g_n(x)$ tend to zero for each $x \in S$, but so does the maximum of this difference over the whole set S:

$$\sup_{x \in S}\left| g(x) - \sum_1^N g_n(x) \right| \to 0 \quad \text{as } N \to \infty.$$

The most useful criterion for guaranteeing absolute and uniform convergence is the **Weierstrass M-test**: if there is a sequence M_n of positive constants such that

$$|g_n(x)| \leq M_n \quad \text{for } x \in S, \quad \text{and} \quad \sum_1^\infty M_n < \infty,$$

then the series $\sum_1^\infty g_n(x)$ is absolutely and uniformly convergent.

In the case of Fourier series, we have the obvious estimates

$$|a_n \cos n\theta| \leq |a_n|, \qquad |b_n \sin n\theta| \leq |b_n|, \qquad |c_n e^{in\theta}| = |c_n|.$$

Hence the Weierstrass M-test will apply to a Fourier series in trigonometric form if $\sum_0^\infty |a_n| < \infty$ and $\sum_1^\infty |b_n| < \infty$, and to a Fourier series in exponential form if

$\sum_{-\infty}^{\infty} |c_n| < \infty$. Since it follows from the equations (2.3) and (2.4) relating a_n, b_n, and c_n that

$$|c_{\pm n}| \leq |a_n| + |b_n|, \qquad |a_n| \leq |c_n| + |c_{-n}|, \qquad |b_n| \leq |c_n| + |c_{-n}|,$$

the conditions $\sum_0^\infty |a_n| < \infty$ and $\sum_1^\infty |b_n| < \infty$ are completely equivalent to the condition $\sum_{-\infty}^{\infty} |c_n| < \infty$. We now present a sufficient (but not necessary) condition for them to hold.

Theorem 2.5. *If f is 2π-periodic, continuous, and piecewise smooth, then the Fourier series of f converges to f absolutely and uniformly on* $\mathbf{R}$.

Proof: By Theorem 2.1 and the remarks just made, it suffices to show that the series $\sum_{-\infty}^{\infty} |c_n|$ converges. Let c_n' denote the Fourier coefficients of f'. By Theorem 2.2 we know that $c_n = (in)^{-1}c_n'$ for $n \neq 0$, and by Bessel's inequality applied to f',

$$\sum_{-\infty}^{\infty} |c_n'|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(\theta)|^2 d\theta < \infty.$$

Hence, by the Cauchy-Schwarz inequality,

$$\sum_{-\infty}^{\infty} |c_n| = |c_0| + \sum_{n\neq 0} \left|\frac{c_n'}{n}\right| \leq |c_0| + \left(\sum_{n\neq 0} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n\neq 0} |c_n'|^2\right)^{1/2} < \infty,$$

since $\sum_{n\neq 0}(1/n^2) = 2\sum_1^\infty (1/n^2) < \infty$. (In case the reader needs reminding: the Cauchy-Schwarz inequality says that the dot product of two vectors is bounded by the product of their norms. It is valid in any number n of dimensions and also in the limit as $n \to \infty$. We shall discuss it in more detail in Chapter 3.) ∎

Let us return to Theorem 2.3. If f has many derivatives, Theorem 2.3 can be applied several times in succession to calculate the Fourier series of f', f'', f''', etc. Each time one takes a derivative, the magnitude of the Fourier coefficients c_n (or a_n and b_n) increases by a factor of $|n|$, which means that the derived series converges more slowly than the original series. Or, to put it another way, if the derived series converges at all, the original series must converge relatively rapidly. Thus there is a connection between the differentiability properties of a function and the rate of convergence of its Fourier series. Here is a precise result along these lines.

Theorem 2.6. *Suppose f is 2π-periodic. If f is of class $C^{(k-1)}$ and $f^{(k-1)}$ is piecewise smooth (thus $f^{(k)}$ exists except at finitely many points in each bounded interval and is piecewise continuous), then the Fourier coefficients of f satisfy*

$$\sum |n^k a_n|^2 < \infty, \qquad \sum |n^k b_n|^2 < \infty, \qquad \sum |n^k c_n|^2 < \infty.$$

In particular,

$$n^k a_n \to 0, \qquad n^k b_n \to 0, \qquad n^k c_n \to 0 \qquad \text{as } n \to \infty.$$

On the other hand, suppose the Fourier coefficients c_n $(n \neq 0)$ satisfy $|c_n| \leq C|n|^{-(k+\alpha)}$ (equivalently, $|a_n| \leq Cn^{-(k+\alpha)}$ and $|b_n| \leq Cn^{-(k+\alpha)}$) for some $C > 0$ and $\alpha > 1$. Then f is of class $C^{(k)}$.

Proof: For the first part, we apply Theorem 2.2 k times to conclude that the Fourier coefficients $c_n^{(k)}$ of $f^{(k)}$ are given by $c_n^{(k)} = (in)^k c_n$, and similarly for $a_n^{(k)}$ and $b_n^{(k)}$. The conclusions then follow from Bessel's inequality (applied to $f^{(k)}$) and its corollary. For the second part, we observe that since $\alpha > 1$,

$$\sum_{n\neq 0} |n^j c_n| \leq C \sum_{n\neq 0} |n|^{-(k-j+\alpha)} \leq 2C \sum_{n>0} n^{-\alpha} < \infty \quad \text{for } j \leq k.$$

Thus, by the Weierstrass M-test, the series $\sum_{-\infty}^{\infty} (in)^j c_n e^{in\theta}$ are absolutely and uniformly convergent for $j \leq k$. They therefore define continuous functions, which are the derivatives $f^{(j)}$ of $f(\theta) = \sum c_n e^{in\theta}$. ∎

The two halves of Theorem 2.6 are not perfect converses of each other; this is in the nature of things. (There is no simple "if and only if" theorem of this sort.) However, the moral is clear: the more derivatives a function has, the more rapidly its Fourier coefficients will tend to zero, and vice versa. In particular, f has derivatives of all orders precisely when its Fourier coefficients tend to zero more rapidly than any power of n (for example, $c_n = e^{-\epsilon|n|}$).

Another aspect of this phenomenon: the basic functions $e^{in\theta}$ or $\cos n\theta$ and $\sin n\theta$ are, of course, perfectly smooth individually, but they become more "jagged," that is, more highly oscillatory, as $n \to \infty$. In order to synthesize non-smooth functions from these smooth ingredients, then, the proper technique is to use relatively large amounts of the high-frequency (i.e., large-n) functions.

These points are worth remembering; they are among the basic lessons of Fourier analysis. The reader can see how they work by examining the entries Table 1 in §2.1. For instance, the sawtooth wave in entry 2 is piecewise smooth but not continuous; its Fourier coefficients are on the order of n^{-1}. The triangle wave in entry 1 is one step better — continuous and piecewise smooth, with a piecewise smooth derivative; its Fourier coefficients are on the order of n^{-2}. These examples are quite typical.

EXERCISES

1. Derive the result of entry 16 of Table 1, §2.1, by using equation (2.17) and Theorem 2.4.
2. Starting from entry 16 of Table 1 and using Theorem 2.4, show that

   a. $\theta^3 - \pi^2\theta = 12 \sum_1^{\infty} \dfrac{(-1)^n \sin n\theta}{n^3} \qquad (-\pi \leq \theta \leq \pi);$

   b. $\theta^4 - 2\pi^2\theta^2 = 48 \sum_1^{\infty} \dfrac{(-1)^{n+1} \cos n\theta}{n^4} - \dfrac{7\pi^4}{15} \qquad (-\pi \leq \theta \leq \pi);$

   c. $\sum_1^{\infty} \dfrac{1}{n^4} = \dfrac{\pi^4}{90}.$

3. Evaluate $\sum_1^{\infty} (2n-1)^{-4} \cos(2n-1)\theta$ by using entry 17 of Table 1.

4. By entry 8 of Table 1, we have

$$\sin\theta = \frac{2}{\pi} - \frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos 2n\theta}{4n^2-1} \qquad (0 \le \theta \le \pi), \tag{$*$}$$

and we also have

$$\cos\theta = \frac{d}{d\theta}\sin\theta = -\int_{\pi/2}^{\theta}\sin\phi\, d\phi.$$

Show that the series $(*)$ can be differentiated and integrated termwise to yield two apparently different expressions for $\cos\theta$ for $0 < \theta < \pi$, and reconcile these two expressions. (Hint: Equation (2.17) is useful.)

5. Let $f(\theta)$ be the periodic function such that $f(\theta) = e^\theta$ for $-\pi < \theta \le \pi$, and let $\sum_{-\infty}^{\infty} c_n e^{in\theta}$ be its Fourier series; thus $e^\theta = \sum c_n e^{in\theta}$ for $|\theta| < \pi$. If we formally differentiate this equation, we obtain $e^\theta = \sum inc_n e^{in\theta}$. But then $c_n = inc_n$, or $(1 - in)c_n = 0$, so $c_n = 0$ for all n. This is obviously wrong; where is the mistake?

6. The Fourier series in entries 11 and 12 of Table 1 are clearly related: the second is close to being the derivative of the first. Find the exact relationship (a) by examining the series and (b) by examining the functions that the series represent.

7. How smooth are the following functions? That is, how many derivatives can you guarantee them to have?

a. $f(\theta) = \displaystyle\sum_{-\infty}^{\infty}\frac{e^{in\theta}}{n^{13\,2} + 2n^6 - 1}$. b. $f(\theta) = \displaystyle\sum_{0}^{\infty}\frac{\cos n\theta}{2^n}$.

c. $f(\theta) = \displaystyle\sum_{0}^{\infty}\frac{\cos 2^n\theta}{2^n}$.

2.4 Fourier series on intervals

Fourier series give expansions of periodic functions on the line in terms of trigonometric functions. They can also be used to give expansions of functions defined on a finite interval in terms of trigonometric functions on that interval.

Suppose the interval in question is $[-\pi, \pi]$. (Other intervals can be transformed into this one by a linear change of variable; we shall discuss this point later.) Given a bounded, integrable function f on $[-\pi, \pi]$, we extend it to the whole real line by requiring it to be periodic of period 2π. Actually, to be completely consistent about this we should start out with f defined only on the half-open interval $(-\pi, \pi]$ or $[-\pi, \pi)$, or else (re)define f at the endpoints so that

$f(-\pi) = f(\pi)$. To be definite, we follow the first course of action; then the **periodic extension** of f to the whole line is given by

$$f(\theta + 2n\pi) = f(\theta) \quad \text{for all } \theta \in (-\pi, \pi] \text{ and all integers } n.$$

For instance, the periodic functions discussed in Examples 1 and 2 of §2.1 are the periodic extiensions of the functions $f(\theta) = |\theta|$ and $g(\theta) = \theta$ from $(-\pi, \pi]$ to the whole line.

If f is a piecewise smooth function on $(-\pi, \pi]$, we can expand its periodic extension in a Fourier series, and then by restricting the variable θ to $[-\pi, \pi]$, we obtain an expansion of the original function. All of what we have said in the previous sections applies to this situation, but there is one point that needs attention. If the original f is piecewise continuous or piecewise smooth on $[-\pi, \pi]$, then its periodic extension will be piecewise continuous or piecewise smooth on $\mathbf{R}$. However, even if f is perfectly smooth on $[-\pi, \pi]$, there will generally be discontinuities in the extended function or its derivatives at the points $(2n+1)\pi$, n an integer, where (so to speak) the copies of f are glued together. To be precise, suppose f is continuous on $[-\pi, \pi]$. Then the extension will be continuous at the points $(2n+1)\pi$ if and only if $f(-\pi) = f(\pi)$, and in this case the extension will have derivatives up to order k at $(2n+1)\pi$ if and only if $f^{(j)}(-\pi+) = f^{(j)}(\pi-)$ for $j \leq k$. (This is illustrated by the examples in §2.1: see Figures 2.1(a) and 2.2(a).) These phenomena must be kept in mind when one studies the relations between the smoothness properties of f and the size of its Fourier coefficients as in Theorem 2.6.

Two interesting variations can be made on this theme. Suppose now that we are interested in functions on the interval $[0, \pi]$ rather than $[-\pi, \pi]$. We can make such a function f into a 2π-periodic function, and hence obtain a Fourier expansions for it, by a twofold extension process: first we extend f in some simple way to the interval $[-\pi, \pi]$, then we extend the result periodically. There are two standard ways of performing the first step: we extend f to $[-\pi, \pi]$ by declaring it to be either even or odd. That is, we have the **even extension** f_{even} of f to $[-\pi, \pi]$ defined by

$$f_{\text{even}}(-\theta) = f(\theta) \text{ for } \theta \in [0, \pi]$$

and the **odd extension** f_{odd} of f to $[-\pi, \pi]$ defined by

$$f_{\text{odd}}(-\theta) = -f(\theta) \text{ for } \theta \in (0, \pi], \qquad f_{\text{odd}}(0) = 0.$$

(See Figure 2.5.) The advantage of using f_{even} or f_{odd} rather than any other extension is that the Fourier coefficients turn out very simply. Indeed, it follows from Lemma 2.2 of §2.1 that

$$\int_{-\pi}^{\pi} f_{\text{even}}(\theta) \cos n\theta \, d\theta = 2\int_0^{\pi} f(\theta) \cos n\theta \, d\theta, \qquad \int_{-\pi}^{\pi} f_{\text{even}}(\theta) \sin n\theta \, d\theta = 0,$$

whereas

$$\int_{-\pi}^{\pi} f_{\text{odd}}(\theta) \cos n\theta \, d\theta = 0, \qquad \int_{-\pi}^{\pi} f_{\text{odd}}(\theta) \sin n\theta \, d\theta = 2 \int_0^{\pi} f(\theta) \sin n\theta \, d\theta.$$

Thus the Fourier series of f_{even} involves only cosines and the Fourier series of f_{odd} involves only sines; moreover, the Fourier coefficients for these two cases can be computed in terms of the values of the original function f on $[0, \pi]$. We are thus led to the following definitions.

Definition. Suppose f is an integrable function on $[0, \pi]$. The series

$$\tfrac{1}{2}a_0 + \sum_1^{\infty} a_n \cos n\theta, \quad \text{where } a_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos n\theta \, d\theta,$$

is called the **Fourier cosine series** of f. The series

$$\sum_1^{\infty} b_n \sin n\theta, \quad \text{where } b_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta,$$

is called the **Fourier sine series** of f.

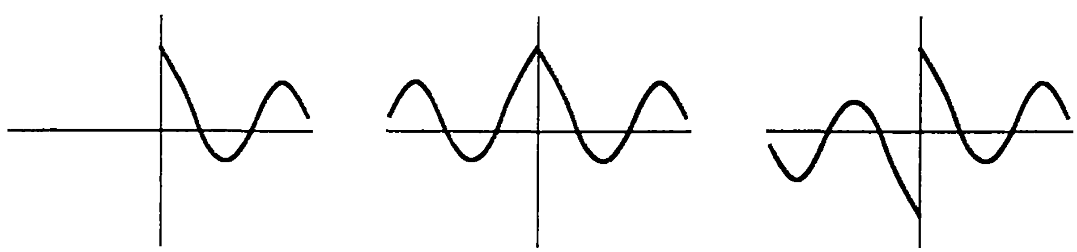


FIGURE 2.5. A function defined on $[0, \pi]$ (left), its even extension (middle), and its odd extension (right).

If f is piecewise continuous or piecewise smooth on $[0, \pi]$, its even periodic and odd periodic extensions will have the same properties on $\mathbf{R}$, but as before, one must watch for extra discontinuities at the points $n\pi$ (n an integer) where the pieces are joined together. If f is continuous on $[0, \pi]$, the even periodic extension will be continuous everywhere, but its derivative will have jumps at the points $2n\pi$ or $(2n+1)\pi$ unless $f'(0+) = 0$ or $f'(\pi-) = 0$, respectively. The odd periodic extension is less forgiving: it will have discontinuities at the points $2n\pi$ or $(2n+1)\pi$ unless $f(0) = 0$ or $f(\pi) = 0$, respectively. (As for higher derivatives: there are potential problems with the odd-order derivatives of the even periodic extension and with the even-order derivatives of the odd periodic extension at the points $n\pi$.)

Example 1. Consider the function $f(\theta) = \theta$ on $[0, \pi]$. Its even and odd periodic extensions are given on $(-\pi, \pi)$ by $f_{\text{even}}(\theta) = |\theta|$ and $f_{\text{odd}}(\theta) = \theta$; these are the functions whose Fourier series we worked out in §2.1. Hence,

$$\theta = 2\sum_{1}^{\infty} \frac{(-1)^{n+1}\sin n\theta}{n} = \frac{\pi}{2} - \frac{4}{\pi}\sum_{1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2} \qquad (0 < \theta < \pi).$$

Here f is perfectly smooth on $[0, \pi]$, but f_{odd} has discontinuities at the odd multiples of π. f_{even} is continuous everywhere, but its first derivative has discontinuities at all integer multiples of π. The reader may find other examples in Table 1.

At any rate, if we keep these remarks in mind and apply Theorem 2.1, we arrive at the following result.

Theorem 2.7. *Suppose f is piecewise smooth on $[0, \pi]$. The Fourier cosine series and the Fourier sine series of f converge to $\frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$ at every $\theta \in (0, \pi)$. In particular, they converge to $f(\theta)$ at every $\theta \in (0, \pi)$ where f is continuous. The Fourier cosine series of f converges to $f(0+)$ at $\theta = 0$ and to $f(\pi-)$ at $\theta = \pi$; the Fourier sine series of f converges to 0 at both these points.*

The results of the previous section on termwise differentiation and uniform convergence can be applied to these series, provided that one takes account of the behavior at the endpoints as indicated above.

Finally, we may wish to consider periodic functions whose period is something other than 2π, or functions defined on intervals other than $[-\pi, \pi]$ or $[0, \pi]$. These situations can be reduced to the ones we have already studied by making a simple change of variable.

For instance, suppose $f(x)$ is a periodic function with period $2l$. (The factor of 2 is merely for convenience.) We make the change of variables

$$x = \frac{l\theta}{\pi}, \qquad g(\theta) = f(x) = f\left(\frac{l\theta}{\pi}\right).$$

Then g is 2π-periodic, so if it is piecewise smooth we can expand it in a Fourier series:

$$g(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in\theta}\,d\theta.$$

If we now substitute $\theta = \pi x/l$ into these formulas, we obtain the $2l$-periodic Fourier series of the original function f:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}, \qquad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx. \tag{2.20}$$

The corresponding formula in terms of cosines and sines is

$$f(x) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right], \tag{2.21}$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx. \tag{2.22}$$

From this it follows that the Fourier cosine and sine expansions of a piecewise smooth function f on the interval $[0, l]$ are

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n\cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \tag{2.23}$$

and

$$f(x) = \sum_1^\infty b_n\sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \tag{2.24}$$

These formulas are probably worth memorizing; they are used very frequently. Another point worth remembering is that, just as in the case of Fourier series for periodic functions, *the constant term* $\frac{1}{2}a_0$ *in the Fourier cosine series of a function* f *on an interval is the mean value of* f *on that interval:* $\frac{1}{2}a_0 = l^{-1}\int_0^l f(x)\,dx$.

Example 2. Let us find the Fourier cosine and sine expansions of $f(x) = x$ on $[0, l]$. Having set $\theta = \pi x/l$, this amounts to finding the expansions of $g(\theta) = l\theta/\pi$ on $[0, \pi]$, which we have done above. Namely, for $0 < \theta < \pi$ we have

$$\frac{l\theta}{\pi} = \frac{2l}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin n\theta = \frac{l}{2} - \frac{4l}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2}\cos(2n-1)\theta,$$

so for $0 < x < l$,

$$x = \frac{2l}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l} = \frac{l}{2} - \frac{4l}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}.$$

Finally, what if we wish to use an interval of length l whose left endpoint is not 0, say $[a, a+l]$? Simply apply the preceding formulas to $g(x) = f(x+a)$; we leave it to the reader to write out the resulting formulas for $f(x)$.

EXERCISES

In Exercises 1–6, find both the Fourier cosine series and the Fourier sine series of the given function on the interval $[0, \pi]$. Try to use the results of Table 1, §2.1, rather than working from scratch. To what values do these series converge when $\theta = 0$ and $\theta = \pi$?

1. $f(\theta) = 1$.
2. $f(\theta) = \pi - \theta$.
3. $f(\theta) = \sin\theta$.
4. $f(\theta) = \cos\theta$.

5. $f(\theta) = \theta^2$. (For the sine series, use entries 1 and 17 of Table 1.)
6. $f(\theta) = \theta$ for $0 \le \theta \le \frac{1}{2}\pi$, $f(\theta) = \pi - \theta$ for $\frac{1}{2}\pi \le \theta \le \pi$. (For the sine series, use entry 11 of Table 1, and for the cosine series, entry 2.)

In Exercises 7–11, expand the function in a series of the indicated type. For example, "sine series on $[0, l]$" means a series of the form $\sum b_n \sin(n\pi x/l)$. Again, use previously derived results as much as possible.

7. $f(x) = 1$; sine series on $[0, 6\pi]$.
8. $f(x) = 1 - x$; cosine series on $[0, 1]$.
9. $f(x) = 1$ for $0 < x < 2$, $f(x) = -1$ for $2 < x < 4$; cosine series on $[0, 4]$.
10. $f(x) = lx - x^2$; sine series on $[0, l]$.
11. $f(x) = e^x$; series of the form $\sum_{-\infty}^{\infty} c_n e^{2\pi i n x}$ on $[0, 1]$.
12. Suppose f is a piecewise continuous function on $[0, \pi]$ such that $f(\theta) = f(\pi - \theta)$. (That is, the graph of f is symmetric about the line $\theta = \frac{1}{2}\pi$.) Let a_n and b_n be the Fourier cosine and sine coefficients of f. Show that $a_n = 0$ for n odd and $b_n = 0$ for n even.

2.5 Some applications

At this point we are ready to complete the solutions of the boundary value problems that were discussed in §1.3. The first of these problems was the one describing heat flow on an interval $[0, l]$, where the initial temperature is $f(x)$ and the endpoints are held at temperature zero,

$$u_t = k u_{xx}, \qquad u(x, 0) = f(x) \quad \text{for } x \in [0, l], \qquad u(0, t) = u(l, t) = 0 \quad \text{for } t > 0,$$

and we derived the following series as a candidate for a solution:

$$\begin{aligned} u(x,t) &= \sum_1^\infty b_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin\frac{n\pi x}{l}, \\ \text{where} \quad f(x) &= \sum_1^\infty b_n \sin\frac{n\pi x}{l}. \end{aligned} \tag{2.25}$$

The questions that we left open were: (1) Can the initial temperature f be expressed as such a sine series? (2) Does this formula for u actually define a solution of the heat equation with the given boundary conditions? We now know that the answer to the first question is *yes*, provided that f is piecewise smooth on $[0, l]$ (certainly a reasonable requirement from a physical point of view): we have merely to expand f in its Fourier sine series (2.24). Let us therefore address the second question.

The individual terms in the series for u solve the heat equation, by the way they were constructed. Moreover, when $t > 0$ the factor $\exp(-n^2\pi^2 kt/l)$ tends to zero very rapidly as $n \to \infty$, so that the series converges nicely. More precisely, since the coefficients b_n tend to zero as $n \to \infty$ and in particular are bounded by some constant C, for any positive ϵ we have

$$0 < \left| b_n \exp\left(\frac{-n^2\pi^2kt}{l^2}\right) \sin\frac{n\pi x}{l} \right| \le Ce^{-\delta n^2} \quad \text{for } t \ge \epsilon, \text{ where } \delta = \frac{\pi^2 k\epsilon}{l^2}.$$

The same sort of estimate also holds for the first t-derivative and the first two x-derivatives of the terms of the series for u, with an extra factor of n^2 thrown in. Since $\sum_1^\infty n^k e^{-\delta n^2}$ converges for any k, we see by the Weierstrass M-test that these derived series converge absolutely and uniformly in the region $0 \le x \le l$, $t \ge \epsilon$, and we deduce that termwise differentiation of the series is permissible. Conclusion: u is a solution of the heat equation.

As for the boundary conditions, it is evident that $u(0,t) = u(l,t) = 0$, since all the terms in the series for u vanish at $x = 0, l$, and $u(x,0) = f(x)$ by the choice of the coefficients b_n. However, as we pointed out in §1.1, we really want a bit more, namely, the continuity condition that $u(x,t)$ should tend to zero as $x \to 0, l$ and to $f(x)$ as $t \to 0$. The preceding discussion shows that the first of these requirements is always satisfied: for each $t > 0$, the series for $u(x,t)$ converges uniformly on $[0,l]$,so $u(x,t)$ is a continuous function of x. (In particular, as $x \to 0$ or $x \to l$, $u(x,t)$ approaches $u(0,t)$ or $u(l,t)$, which are zero.) Moreover, if f is continuous and piecewise smooth on $[0,l]$ and $f(0) = f(l) = 0$, then the odd periodic extension of f is continuous and piecewise smooth, so $\sum |b_n| < \infty$ by Theorem 2.5. The Weierstrass M-test then shows that the series for u converges uniformly on the whole region $0 \le x \le l$, $t \ge 0$, and hence that u is continuous there; in particular, $u(x,t) \to u(x,0) = f(x)$ as $t \to 0$.

If f has discontinuities or is nonzero at the endpoints, it is still true that $u(x,t) \to f(x)$ as $t \to 0$ provided that $0 < x < l$ and f is continuous at x, but the proof is more delicate. (See Walker [53], §4.7.) We shall not concern ourselves with such technical refinements, as we have already established the main point: under reasonable assumptions on the initial temperature f, the function u satisfies all the desired conditions.

One question we have not really settled is the uniqueness of the solution. That is, we have constructed *one* solution; is it the only one? The answer is *yes*. One can argue that any solution $u(x,t)$ must be expandable in a Fourier sine series in x for each t and then use the differential equation to show that the coefficients of this series must be the ones we found above. Alternatively, one can invoke some general uniqueness theorems for solutions of the heat equation; see John [33] or Folland [24]. Similar considerations apply to the other problems we solve later, and we shall not worry about uniqueness from now on except in situations where pitfalls actually exist.

Lest the reader become too complacent, however, let us briefly consider the problem of solving the heat equation for times $t < 0$ — that is, given the temperature distribution at time $t = 0$, to reconstruct the distribution at earlier times. If we take $t < 0$ in (2.25), the factors $e^{-n^2\pi^2kt/l^2}$ tend rapidly to *infinity* rather than zero as $n \to \infty$, with the result that the series for $u(x,t)$ will almost certainly *diverge* unless the coefficients b_n of the initial function f happen to decay extremely rapidly as $n \to \infty$. Thus (2.25), in general, does *not* give a solution to the heat equation when $t < 0$. This is not merely a failure of mathematical technique, however. The initial value problem for the time-reversed heat equation is simply not well posed, a reflection of the fundamental physical fact that the direction of time is irreversible in diffusion processes. One can mix hot water and cold water to get warm water, but one cannot then separate the warm water back into hot and cold components! More to the point, one cannot tell by examining the warm water which part was initially hot and which part was initially cold, or what their initial temperatures were.

Exactly the same considerations apply to the problem of heat flow on $[0, l]$ with insulated endpoints,

$$u_t = k u_{xx}, \qquad u(x,0) = f(x), \qquad u_x(0,t) = u_x(l,t) = 0,$$

whose solution is

$$u(x,t) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \cos\frac{n\pi x}{l},$$

where

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{l}.$$

The only difference is that now we expand f in its Fourier cosine series (2.22).

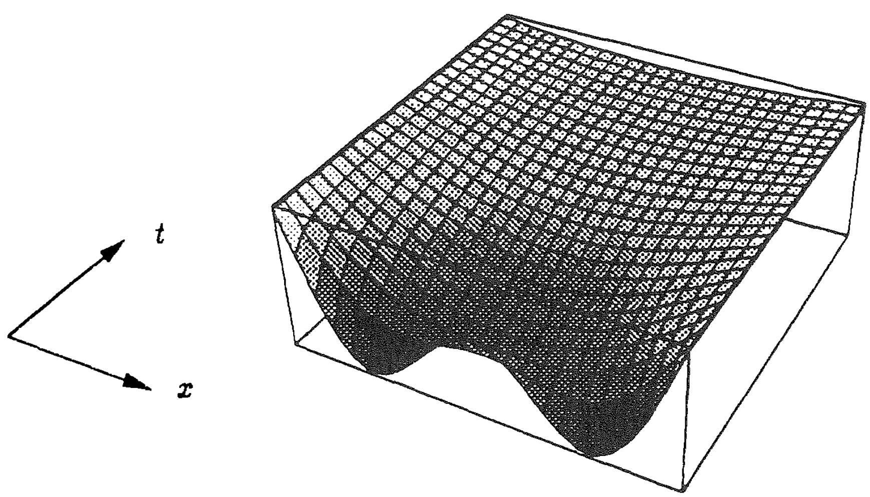


FIGURE 2.6. The solution (2.25) of the heat equation with $k = \frac{1}{4}$, $l = 1$, $b_1 = -\frac{1}{3}$, $b_2 = -\frac{1}{6}$, and $b_n = 0$ for $n > 2$, on the region $0 \le x \le 1$, $0 \le t \le 1$.

Let us pause a moment to see what these solutions tell us about the physics of the situation. In the limit as $t \to \infty$, the exponential factors all vanish, so the solution u approaches a constant — namely, 0 in the case where the endpoints are held at temperature 0 and $\frac{1}{2}a_0$ in the case of insulated endpoints. The first of these is easy to understand: the interval $[0, l]$ comes into thermal equilibrium with its surroundings. As for the second, if we recall that

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx,$$

we see that the limiting temperature $\frac{1}{2}a_0$ is simply the average value of the initial temperature. In other words, no heat enters or escapes, so the various parts of the interval simply come into thermal equilibrium with each other. Moreover, in both cases, the high-frequency terms (i.e., the terms with n large) damp out more quickly than the low-frequency terms: this expresses the fact that the diffusion of heat tends to quickly smooth out local variations in temperature. A simple illustration of these assertions can be found in Figure 2.6.

Now let us turn to the problem of the vibrating string:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u(0,t) = u(l,t) = 0.$$

According to the discussion in §1.3, we should expand f and g in their Fourier sine series,

$$f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}, \qquad g(x) = \sum_1^\infty B_n \sin\frac{n\pi x}{l}, \tag{2.26}$$

and then take

$$u(x,t) = \sum_1^\infty \sin\frac{n\pi x}{l}\left(b_n\cos\frac{n\pi ct}{l} + \frac{lB_n}{n\pi c}\sin\frac{n\pi ct}{l}\right). \tag{2.27}$$

Here the analysis is more delicate than for the heat equation, because there are no exponentially decreasing factors in this series to help the convergence. The series (2.27) for u is likely to converge about as well as the sine series for f and g, but if we differentiate it twice with respect to x or t in order to verify the wave equation, we introduce a factor of n^2; and this may well be enough to destroy the convergence.

We can avoid this difficulty by making sufficiently strong smoothness assumptions on f and g. For instance, let us suppose that f and g are of class $C^{(3)}$ and $C^{(2)}$, respectively, that f''' and g'' are piecewise smooth, and that f, g, f'', and g'' vanish at the endpoints 0 and l. These conditions guarantee that the odd periodic extensions of f and g will have the same smoothness properties (even at the points $n\pi$), and hence, by Theorem 2.6, that the coefficients b_n and B_n will satisfy

$$|b_n| \le Cn^{-4}, \qquad |B_n| \le Cn^{-3}.$$

Now the nth term in the series (2.27) will be dominated by n^{-4}, and if we differentiate it twice in either x or t it is still dominated by n^{-2}. Since $\sum_1^\infty n^{-2}$

converges, the M-test guarantees the absolute and uniform convergence of the twice-derived series, and we are in business.

This is not entirely satisfactory, however. It is physically reasonable to assume that f and g are continuous and perhaps piecewise smooth, but one may — and indeed should — have the feeling that the extra differentiability assumptions are annoyances that reflect a failure of technique rather than a real difficulty in the original problem.

We can obtain more insight into this problem by recalling the trigonometric identities

$$\sin a \cos b = \tfrac{1}{2}\Big[\sin(a+b)+\sin(a-b)\Big], \qquad \sin a \sin b = \tfrac{1}{2}\Big[\cos(a-b)-\cos(a+b)\Big],$$

by means of which the series (2.27) can be rewritten

$$\begin{aligned} u(x,t) =& \frac{1}{2}\sum_{1}^{\infty} b_n \sin\frac{n\pi}{l}(x+ct) + \frac{1}{2}\sum_{1}^{\infty} b_n \sin\frac{n\pi}{l}(x-ct) \\ &+ \frac{1}{2c}\sum_{1}^{\infty}\frac{lB_n}{n\pi}\cos\frac{n\pi}{l}(x-ct) - \frac{1}{2c}\sum_{1}^{\infty}\frac{lB_n}{n\pi}\cos\frac{n\pi}{l}(x+ct). \end{aligned}$$

The first two sums on the right are just the Fourier sine series for f, evaluated at $x \pm ct$, and the last two are (up to constant factors) just the Fourier sine series for g, integrated once and then evaluated at $x \pm ct$. To restate this: let us suppose that f and g are piecewise smooth, so that the expansions (2.26) are valid on the interval $(0, l)$. We use the formulas (2.26) to extend f and g from this interval to the whole line; that is, we extend f and g to $\mathbf{R}$ by requiring them to be odd and $2l$-periodic. We then have

$$u(x,t) = \frac{1}{2}\Big[f(x+ct)+f(x-ct)\Big] + \frac{1}{2c}\Big[G(x+ct)-G(x-ct)\Big], \qquad (2.28)$$

where G is any antiderivative of g.

From this closed formula it is perfectly plain that if f is twice differentiable and g is once differentiable, then u satisfies the wave equation, for

$$\frac{\partial^2}{\partial x^2}f(x \pm ct) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}f(x \pm ct) = f''(x \pm ct), \qquad (2.29)$$

and likewise for G. Even here the differentiability assumptions seem a bit artificial; one would like, for example, to allow f to be a function with corners in order to model plucked strings. Indeed, in some sense the first equation in (2.29) should be correct, simply as a formal consequence of the chain rule, even if f'' is ill-defined. The idea that is crying to be set free here is the notion of a "weak solution" of a differential equation, which enables one to consider functions u defined by (2.28) as solutions of the wave equation even when the requisite derivatives of f and g do not exist. We shall say more about this in §9.5.

Another point should be raised here. One does not have to go through Fourier series to produce the formula (2.28) for the solution of the vibrating string problem; an elementary derivation is sketched in Exercise 6 of §1.1. It is then fair to ask what good the complicated-looking formula (2.27) is when the simple (2.28) is readily available. There are two good answers. First, the trick in Exercise 6, §1.1, that quickly produces the general solution of the 1-dimensional wave equation does not work for other equations (including the higher-dimensional wave equation), whereas the Fourier method and its generalizations often do. Second, although (2.28) tells you what you see if you look at a vibrating string, (2.27) *tells you what you hear when you listen to it.* The ear, unlike the eye, has a built-in Fourier analyzer that resolves sound waves into their components of different frequencies, which are perceived as musical tones.* Typically, the first term in the series (2.27) is the largest one, so one hears the note with frequency $2\pi c/l$ colored by the "overtones" at the higher frequencies $2\pi nc/l$ with $n > 1$.

The difference in the convergence properties of the series solutions (2.25) and (2.27) of the heat and wave equations reflects a difference in the physics: diffusion processes such as heat flow tend to smooth out any irregularities in the initial data, whereas wave motion propagates singularities. Thus, the solution (2.25) of the heat equation becomes smoother as t increases, and this is reflected in the exponential decay of the high-frequency terms. (See the discussion of smoothness versus rates of convergence at the end of §2.3.) However, any sharp corners in the initial configuration of a vibrating string will not disappear but merely move back and forth, as is clear from (2.28); hence there is no improvement in the rate of convergence of the solution (2.27). (Compare Figures 2.6 and 2.7, which show solutions of the heat and wave equations with the same initial values up to a constant factor and the same boundary conditions; the initial variations damp out in the first case, but not in the second.)

We shall see other applications of Fourier expansions of functions on an interval in Chapter 4. Fourier expansions are also the natural tool for analyzing periodic functions on the line. In practice, there are two principal sources of such functions. The first is the angular variable in polar or cylindrical coordinates or the longitudinal angular variable in spherical coordinates; in this context periodicity is an immediate consequence of the geometry of the situation. The other is physical phenomena that vary periodically (or approximately periodically) in time, such as certain types of electrical signals, the length of a day, daily or seasonal variations in temperature, and so forth.

As an example, let us analyze the variations in temperature beneath the ground due to the daily and seasonal fluctuations of temperature at the surface of the earth. We shall concern ourselves only with the temperature near a particular spot on the surface, over distances of (say) at most 100 meters. We therefore neglect the fact that at great depths the earth is hotter than at the surface, and we assume that (i) the earth is of uniform composition; (ii) the temperature at the surface is a function $f(t)$ of time only, not of position; (iii) $f(t)$ is periodic

* Of course, this is an oversimplification.

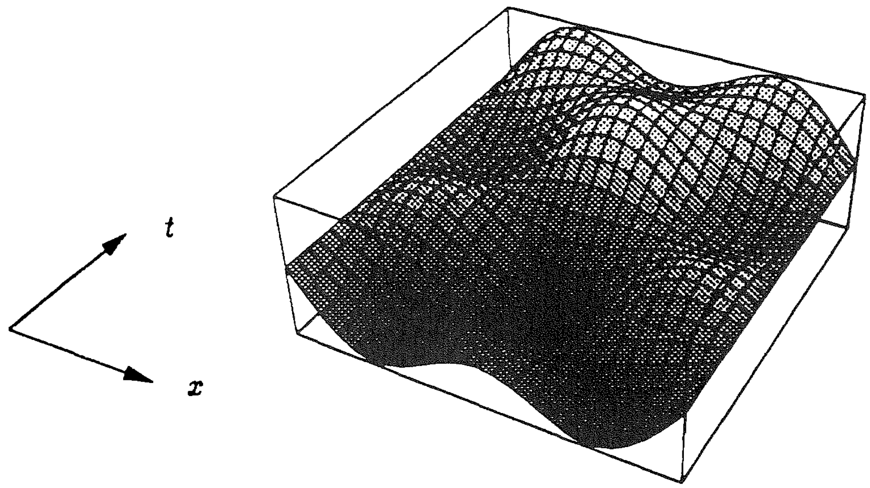


FIGURE 2.7. The solution (2.27) of the wave equation with $l = c = 1$, $b_1 = -0.2$, $b_2 = -0.1$, $b_n = 0$ for $n > 2$, and $B_n = 0$ for all n, on the region $0 \le x \le 1$, $0 \le t \le 1$.

of period 1 and so has a Fourier series

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n t}.$$

(We may take the unit of time to be 1 year, so that the dominant terms in the series will be $n = \pm 1$, corresponding to seasonal variations, and $n = \pm 365$, corresponding to daily variations. With a bit more accuracy, we could take the unit of time to be 4 years and the dominant terms to be $n = \pm 4$ and $n = \pm 1461$ $(= \pm 4 \times 365\frac{1}{4})$. Or, we could take an even longer period to account for long-term climatic changes.) The boundary value problem to be solved is therefore

$$u_t = k u_{xx} \quad \text{for } x > 0, \qquad u(0,t) = f(t).$$

Since f is periodic in t, we expect u to have the same property, so we look for solutions of the form

$$u(x,t) = \sum_{-\infty}^{\infty} C_n(x) e^{2\pi i n t}.$$

Taking on faith that this series can be differentiated termwise, we find that

$$u_t = \sum_{-\infty}^{\infty} (2\pi i n) C_n(x) e^{2\pi i n t}, \qquad u_{xx} = \sum_{-\infty}^{\infty} C_n''(x) e^{2\pi i n t}.$$

Hence, taking into account the initial condition, we have

$$C_n''(x) - 2\pi i n k^{-1} C_n(x) = 0, \qquad C_n(0) = c_n.$$

Since the square roots of $2in$ are $\pm(1+i)n^{1/2}$ if $n > 0$ and $\pm(1-i)|n|^{1/2}$ if $n < 0$, the general solution of this differential equation is

$$\begin{aligned} a\exp\left((1+i)\sqrt{\frac{\pi n}{k}}\,x\right) + b\exp\left(-(1+i)\sqrt{\frac{\pi n}{k}}\,x\right) &\quad \text{if} \quad n > 0, \\ a\exp\left((1-i)\sqrt{\frac{\pi |n|}{k}}\,x\right) + b\exp\left(-(1-i)\sqrt{\frac{\pi |n|}{k}}\,x\right) &\quad \text{if} \quad n < 0, \\ ax + b &\quad \text{if} \quad n = 0. \end{aligned}$$

In each case we must take $a = 0$ because of the physical requirement that the temperature should remain bounded as x increases. (In effect we are imposing a boundary condition at $x = \infty$ to supplement the one at $x = 0$.) The initial condition then implies that $b = c_n$. Hence, upon grouping together the nth and $(-n)$th terms, we obtain the solution

$$\begin{aligned} u(x,t) = c_0 + \sum_1^\infty \exp\left(-\sqrt{\frac{\pi n}{k}}\,x\right) \\ \times \left[c_n \exp\left(2\pi i n t - i\sqrt{\frac{\pi n}{k}}\,x\right) + c_{-n}\exp\left(-2\pi i n t + i\sqrt{\frac{\pi n}{k}}\,x\right)\right]. \end{aligned}$$

It is now easy to check that this function u really does solve the problem.

The main features to be noted here are the following. First, all of the nonconstant terms in u (the ones with $n \neq 0$) die out exponentially fast as x increases, and the high-frequency ones die out faster than the low-frequency ones. (In actual fact, the daily variations in temperature become negligible at a depth of a few centimeters, and the seasonal ones become negligible at a depth of a few meters, where the temperature remains essentially constant at the annual mean c_0.) Second, the temperature variations at depth x are out of phase with those at the surface by an amount proportional to x and $\sqrt{|n|}$, because the heat takes time to penetrate. For example, if the $n = 1$ term, representing the main seasonal variations, is the dominant one, at depth $x = \sqrt{\pi k}$ the temperature is warmer in winter and cooler in summer.

In considering the usefulness of Fourier series or any other sort of infinite series, one should not lose sight of the fact that the partial sums of the series provide approximations to the full sum, and that such approximations may be just what one needs to obtain a computationally manageable solution to a problem. The questions about smoothness and rates of convergence that we have discussed in some detail have a computational as well as a theoretical significance: rapidly converging series such as (2.25) yield accurate answers much more readily than slowly converging ones such as (2.27). An interesting discussion of rates of convergence of infinite series, and the implications for numerical calculations, can be found in Boas [7].

On the other hand, in many situations one knows the initial data only to a finite degree of accuracy. For example, one may be studying a physical quantity

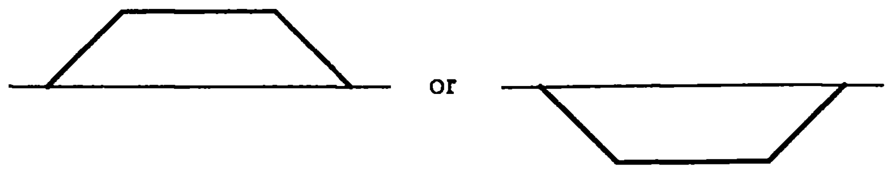
$f(t)$ that varies periodically with the time t, and one may know the values of f approximately from physical measurements. In this context the point of Fourier analysis is that it is usually appropriate to take a trigonometric polynomial of fairly low degree, whose coefficients are determined so as to fit the data well, as a mathematical model for f.

EXERCISES

1. A rod 100 cm long is insulated along its length and at both ends. Suppose that its initial temperature is $u(x, 0) = x$ (x in cm, u in °C, t in sec, $0 \leq x \leq 100$), and that its diffusivity coefficient k is 1.1 cm^2/sec (about right if the bar is made of copper).
   a. Find the temperature $u(x, t)$ for $t > 0$. (It is something of the form $50 + \sum_1^\infty a_n(t) \cos(n\pi x/100)$, and $a_n(t) = 0$ when n is even.)
   b. Show that the first three terms of the series (i.e., $50 + a_1(t)\cos(\pi x/100) + a_3(t)\cos(3\pi x/100)$) give the temperature accurately to within 1 unit when $t = 60$. Using this fact, find $u(0, 60)$, $u(10, 60)$, and $u(40, 60)$.

   $$\text{Hint:} \quad \sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}, \quad \text{so} \quad \sum_3^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - 1 - \frac{1}{9} \approx .123.$$

   c. Find a number $T > 0$ such that $u(x, t)$ is within 1 unit of its equilibrium value 50 for all x when $t > T$.
2. Redo Exercises 1a and 1c with $k = .01$ (a reasonable figure if the bar is made of ceramic). Now how many terms of the series are needed to get an accuracy of 1 unit when $t = 60$?
3. Consider again the copper rod of Exercise 1 ($k = 1.1$). Suppose that the rod is initially at temperature 100°C and that the ends are subsequently put into a bath of ice water (at 0°C).
   a. Assuming no heat loss along the length of the rod, find the temperature $u(x, t)$ at subsequent times.
   b. Use your answer to find $u(50, t)$ numerically when $t = 30$, 60, 300, 3600.
   c. Prove that your answers in (b) are correct to within 1 unit. (Hint: The series for $u(50, t)$ is alternating.)
4. Consider a vibrating string occupying the interval $0 \leq x \leq l$. Suppose the string is plucked in the middle in such a way that its initial displacement $u(x, 0)$ is $2mx/l$ for $0 \leq x \leq \frac{1}{2}l$ and $2m(l-x)/l$ for $\frac{1}{2}l \leq x \leq l$ (so the maximum displacement, at $x = \frac{1}{2}l$, is m), and its initial velocity $u_t(x, 0)$ is zero.
   a. Find the displacement $u(x, t)$ as a Fourier series.
   b. Describe $u(x, t)$ in the closed form (2.28) and show that at times $t > 0$, $u(x, t)$ (as a function of x) typically looks like the following figure:



5. Consider a vibrating string as in Exercise 4. Suppose the string is plucked at $x = a$ instead of $x = \frac{1}{2}l$, so the initial displacement is mx/a for $0 \le x \le a$ and $m(l-x)/(l-a)$ for $a \le x \le l$, and the initial velocity is zero.
   a. Find the displacement $u(x,t)$ as a Fourier series. (Entry 11 of Table 1, §2.1, will be helpful.)
   b. Convince yourself that the terms with large n contribute more to $u(x,t)$ when a becomes closer to l. (Musically: plucking near the end gives a tone with more higher harmonics.)
6. Suppose the string in Exercise 4 is initially struck in the middle so that its initial displacement is zero but its initial velocity $u_t(x,0)$ is 1 for $|x - \frac{1}{2}l| < \delta$ and 0 elsewhere. Find $u(x,t)$ for $t > 0$.
7. Suppose that the temperature at time t at a point on the surface of the earth is given by
$$u(0,t) = 10 - 7\cos 2\pi t - 5\cos 2\pi(365)t.$$
(Here u is measured in $°C$ and t is measured in years; the coefficients are roughly correct for Seattle, Washington.) Suppose that the diffusivity coefficient of the earth is $k = .003\ \mathrm{cm}^2/\mathrm{sec} \approx 9.46\ \mathrm{m}^2/\mathrm{yr}$.
   a. Find $u(x,t)$ for $x > 0$.
   b. At what depth x do the daily variations in temperature become less than 1 unit? What about the annual variations?

2.6 Further remarks on Fourier series

There is much more to be said about Fourier series than is contained in this chapter. Some good references for further information on both the theoretical aspects of the subject and its applications are the books of Dym-McKean [19], Körner [34], and Walker [53]. Also recommended is the article of Coppel [15] on the history of Fourier analysis and its influence on other branches of mathematics, and the articles by Zygmund, Hunt, and Ash in [2]. Finally, the serious student of Fourier analysis should become acquainted with the treatise of Zygmund [58], which gives an encyclopedic account of the subject.

We conclude this chapter with a brief discussion of a few other interesting aspects of Fourier series.

The transform point of view

Given a 2π-periodic function, its sequence $\{c_n\}$ of Fourier coefficients can be regarded as a function $\widehat{f}$ whose domain is the integers:

$$\widehat{f}(n) = c_n = \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta.$$

The mapping $f \to \widehat{f}$ is thus a *transform* that converts periodic functions on the line to functions on the integers. The inverse transform is the operation which assigns to a function $\phi(n)$ on the integers (that decays suitably as $n \to \infty$) the function $\sum_{-\infty}^{\infty} \phi(n)e^{in\theta}$. In principle all the information in f is also contained in its transform $\widehat{f}$, and vice versa, but the information may be encoded in a more convenient form on one side or the other. For example, Theorem 2.2 shows that the transform converts differentiation into a simple algebraic operation: $\widehat{f'}(n) = in\widehat{f}(n)$. We shall return to this point of view in Chapter 7.

Comparison with Taylor series

Perhaps the most well known and widely used type of infinite series expansion for functions is the Taylor series, and it is of interest to compare the features of Taylor series and Fourier series.

In order for a function $f(x)$ to have a Taylor expansion about a point x_0,

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad |x - x_0| < r,$$

f must have derivatives of all orders at x_0. If it does, the coefficients of the Taylor series are determined by these derivatives, and hence by the values of f in an arbitrarily small neighborhood of x_0. The rate at which these coefficients grow or decay as $n \to \infty$ is related to the radius of convergence of the series and hence to the distance from x_0 to the nearest singularity of f (in the complex plane). In general the partial sums of the Taylor series provide excellent approximations to f near x_0 but are often of little use when $|x - x_0|$ is large.

In contrast, a function f need have only minimal smoothness properties in order to have a convergent Fourier expansion

$$f(x) = \sum_{-\infty}^{\infty} \left((2l)^{-1} \textstyle\int_a^{a+2l} f(y)e^{-in\pi y/l}dy\right) e^{in\pi x/l}, \qquad x \in (a, a + 2l).$$

The coefficients of this series depend on the values of f over the entire interval $(a, a+2l)$. The rate at which they decay as $n \to \infty$ is related to the differentiability properties of f, or rather of its periodic extension. The partial sums of the Fourier series will converge to f only rather slowly if f is not very smooth, but they tend to provide good approximations over the whole interval $(a, a + 2l)$.

Thus Taylor series and Fourier series are of quite different natures: the first one is intimately connected with the local properties of f near x_0, whereas the second is related to global properties of f. There is a situation, however, in which the two can be seen as aspects of the same thing. Namely, suppose f is an analytic function of the complex variable z in some disc $|z - z_0| < R$. If we write $z - z_0$ in polar coordinates as $re^{i\theta}$, the Taylor series for f about z_0 turns into a Fourier series in θ for each fixed $r < R$:

$$\sum_0^\infty a_n(z - z_0)^n = \sum_0^\infty (a_n r^n)e^{in\theta}.$$

The formula (2.5) for the Fourier coefficients, in this case, is nothing but the Cauchy integral formula for the derivatives of f at z_0. This connection between Fourier analysis and complex function theory has many interesting consequences, which are discussed in more advanced books such as Dym-McKean [19] and Zygmund [58].

Convergence of Fourier series

The study of the convergence of Fourier series has a long and complex history. The convergence theorems we have presented in §§2.2–3 are sufficient for many purposes, but they do not give the whole picture. Here we briefly indicate a few other highlights of the story. In the first place, the hypotheses of our Theorem 2.1 can be weakened. The same conclusion is obtained if we assume only that f is of "bounded variation" on the interval $[-\pi, \pi]$, which means that it can be written as the difference of two nondecreasing functions on that interval. (It is not hard to show that piecewise smooth functions have this property.) On the other hand, it has been known since 1876 that there are continuous periodic functions whose Fourier series diverge at some points, and for almost a century it was an open question whether the Fourier series of a continuous function could be guaranteed to converge at *any* point. An affirmative answer was obtained only in 1966, with a deep theorem of L. Carleson to the effect that the Fourier series of any square-integrable function f must converge to f at "almost every" point, in a sense that we shall describe in §3.3. See the article by Hunt in [2].

One fundamental fact that has emerged over the years is that, in many situations, simple pointwise convergence of a series is not the appropriate thing to look at; and there are many other notions of convergence that may be used. For example, there is uniform convergence, which is stronger than pointwise convergence; there is also "pth power mean" convergence, according to which the series $\sum_1^\infty f_n$ converges to f on the interval $[a, b]$ if

$$\lim_{N\to\infty} \int_a^b \left| \sum_1^N f_n(x) - f(x) \right|^p dx = 0.$$

We shall say much more about the case $p = 2$ in the next chapter. There are also ways of summing divergent series that can be used to advantage; we shall now briefly discuss the simplest of these.

It is easy to see that if a sequence $\{a_n\}$ converges to a, then the average $k^{-1}\sum_1^k a_n$ of its first k terms also converges to a as $k \to \infty$, but these averages may converge when the original sequence does not. For example, the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

is divergent; but the average of its first k terms is $(k+1)/2k$ or $1/2$ according as k is odd or even, and this tends to $1/2$ as $k \to \infty$. Now, given an infinite series $\sum_0^\infty b_n$ with partial sums $s_N = \sum_0^N b_n$, the average of its first $k+1$ partial sums,

$$\frac{1}{k+1}(s_0 + s_1 + \cdots + s_k),$$

is called its kth **Cesàro mean**, and the series is said to be **Cesàro summable** to the number s if its Cesàro means (rather than just its partial sums) converge to s. We then have the following theorem, due to L. Fejér.

Theorem 2.8. *If f is 2π-periodic and piecewise continuous on* $\mathbf{R}$, *then the Fourier series of f is Cesàro summable to $\frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$ at every θ. Moreover, if f is everywhere continuous, the Cesàro means of the series converge to f uniformly.*

The proof of this theorem is similar in spirit to that of Theorem 2.1; it can be found, for example, in §2 of Körner [34] or §2.7 of Walker [53]. The significance of the theorem is twofold. First, it gives a way of recovering a piecewise continuous function f from its Fourier coefficients when the Fourier series fails to converge. Second, even when the Fourier series of f does converge, its Cesáro means tend to give better approximations to f than its partial sums: for example, they converge uniformly to f whenever f is continuous, whereas the partial sums converge uniformly only under stronger smoothness conditions (cf. Theorem 2.5).

The Gibbs phenomenon

Suppose f is a periodic function. If f has a discontinuity at x_0, the Fourier series of f cannot converge uniformly on any interval containing x_0, because the uniform limit of continuous functions is continuous. In fact, for the Fourier series of a piecewise smooth function f, the lack of uniformity manifests itself in a particularly dramatic way known as the **Gibbs phenomenon**: as one adds on more and more terms, the partial sums overshoot and undershoot f near the discontinuity and thus develop "spikes" that tend to zero in width but *not* in height. One can see this in Figure 2.8, which shows the fortieth partial sum of the Fourier series of the sawtooth wave function

$$f(\theta) = \pi - \theta \text{ for } 0 < \theta < 2\pi, \qquad f(\theta + 2n\pi) = f(\theta).$$

A precise statement and proof of the Gibbs phenomenon for this function is outlined in Exercise 1. It can be shown that the same behavior occurs at any discontinuity of any piecewise smooth function. See Körner [34] and Hewitt-Hewitt [28] for interesting discussions of the Gibbs phenomenon.

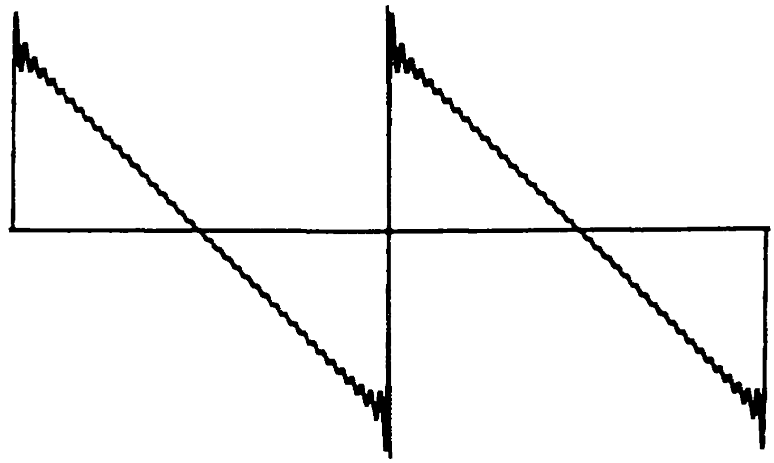


FIGURE 2.8. Graph of $2\sum_1^{40} n^{-1}\sin n\theta$, $-2\pi < \theta < 2\pi$ (an illustration of the Gibbs phenomenon).

EXERCISE

1. Recall from Table 1, §2.1, that $f(\theta) = 2\sum_1^{\infty} n^{-1}\sin n\theta$ is the 2π-periodic function that equals $\pi - \theta$ for $0 < \theta < 2\pi$. Let

$$g_N(\theta) = 2\sum_1^N \frac{\sin n\theta}{n} - (\pi - \theta),$$

so that $g(\theta)$ is the difference between $f(\theta)$ and its Nth partial sum for $0 < \theta < 2\pi$.

a. Show that $g_N'(\theta) = 2\pi D_N(\theta)$ where D_N is the Dirichlet kernel (2.10).

b. Using (2.12), show that the first critical point of $g_N(\theta)$ to the right of zero occurs at $\theta_N = \pi/(N+\frac{1}{2})$, and that

$$g_N(\theta_N) = \int_0^{\theta_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,d\theta - \pi.$$

c. Show that

$$\lim_{N\to\infty} g_N(\theta_N) = 2\int_0^{\pi} \frac{\sin\phi}{\phi}\,d\phi - \pi.$$

(Hint: Let $\phi = (N+\frac{1}{2})\theta$.) This limit is approximately equal to .562. Thus the difference between $f(\theta)$ and the Nth partial sum of its Fourier series develops a spike of height .562 (but of increasingly narrow width) just to the right of $\theta = 0$ as $N \to \infty$. (There is another such spike on the left.)

CHAPTER 3
ORTHOGONAL SETS OF FUNCTIONS

Fourier series are only one of a large class of interesting and useful infinite series expansions for functions that are based on so-called *orthogonal systems* or *orthogonal sets* of functions. This chapter is devoted to explaining the general conceptual framework for understanding such systems, and to showing how they arise from certain kinds of differential equations. Underlying these ideas is a profound analogy between the algebra of Fourier series and the algebra of n-dimensional vectors, which we now investigate.

3.1 Vectors and inner products

We recall some ideas from elementary 3-dimensional vector algebra and recast them in a more general form. We identify 3-dimensional vectors with ordered triples of real numbers; that is, we write

$$\mathbf{a} = (a_1, a_2, a_3) \qquad \text{rather than} \qquad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

The dot product or inner product of two vectors is then defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

and the norm or length of a vector is defined by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

We propose to generalize these ideas in two ways: by working in an arbitrary number k of dimensions, and by using complex numbers rather than real ones. This generalization is not just a mathematical fantasy. Although k-dimensional vectors do not have an immediate geometrical interpretation in physical space, they are still useful for dealing with problems involving k independent variables. For our purposes, the main motivation for the use of complex numbers is their connection with the exponentials $e^{i\theta}$; but it should be noted that the use of complex vectors is essential in quantum physics. However, in visualizing the ideas we shall be discussing, the reader should just think of real 3-dimensional vectors.

A **(complex)** k**-dimensional vector** is an ordered k-tuple of complex numbers:

$$\mathbf{a} = (a_1, a_2, \ldots, a_k).$$

The vector $\mathbf{a}$ is called **real** if its components a_j are all real numbers. Addition and scalar multiplication are defined just as in the 3-dimensional case, but now the scalars are allowed to be complex:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, \ldots, a_k + b_k),$$
$$c\mathbf{a} = (ca_1, \ldots, ca_k) \qquad (c \in \mathbf{C}).$$

We denote the zero vector $(0, 0, \ldots, 0)$ by $\mathbf{0}$, and we denote the space of all complex k-dimensional vectors by $\mathbf{C}^k$.

The **inner product** of two vectors is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1\overline{b}_1 + a_2\overline{b}_2 + \cdots + a_k\overline{b}_k, \tag{3.1}$$

and the **norm** of a vector is defined by

$$\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2} = \left(a_1\overline{a}_1 + \cdots + a_k\overline{a}_k\right)^{1/2} = \left(|a_1|^2 + \cdots + |a_k|^2\right)^{1/2}. \tag{3.2}$$

The reason for the complex conjugates in the definition of the inner product is to make the norm (3.2) positive, for we wish to interpret $\|\mathbf{a}\|$ as the magnitude or length of the vector $\mathbf{a}$. (Recall that the absolute value of a complex number $z = x+iy$ is $(x^2+y^2)^{1/2}$, and this is $(z\overline{z})^{1/2}$ rather than $(z^2)^{1/2}$.) Notice, however, that for real vectors, (3.1) and (3.2) become

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1b_1 + \cdots + a_kb_k, \qquad \|\mathbf{a}\| = (a_1^2 + \cdots + a_k^2)^{1/2},$$

the obvious generalization of the familiar 3-dimensional case.

A word about the notation: The inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ is often denoted by $\mathbf{a} \cdot \mathbf{b}$ or $(\mathbf{a}, \mathbf{b})$. Also, in the physics literature it is customary to switch the roles of $\mathbf{a}$ and $\mathbf{b}$, that is, to put the complex conjugates on the first variable rather than the second. This discrepancy is regrettable, but by now it is firmly entrenched in common usage.

The inner product (3.1) is clearly linear as a function of its first variable but *antilinear* or *conjugate linear* as a function of its second variable; that is, for any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and any complex numbers z, w,

$$\begin{aligned} \langle z\mathbf{a} + w\mathbf{b},\ \mathbf{c} \rangle &= z\langle \mathbf{a}, \mathbf{c} \rangle + w\langle \mathbf{b}, \mathbf{c} \rangle, \\ \langle \mathbf{a},\ z\mathbf{b} + w\mathbf{c} \rangle &= \overline{z}\langle \mathbf{a}, \mathbf{b} \rangle + \overline{w}\langle \mathbf{a}, \mathbf{c} \rangle \end{aligned} \tag{3.3}$$

Also, the inner product is *Hermitian symmetric,* which means that

$$\langle \mathbf{b}, \mathbf{a} \rangle = \overline{\langle \mathbf{a}, \mathbf{b} \rangle}, \tag{3.4}$$

and the norm satisfies the conditions

$$\|c\mathbf{a}\| = |c|\,\|\mathbf{a}\| \qquad (c \in \mathbf{C}), \tag{3.5}$$

$$\|\mathbf{a}\| > 0 \quad \text{for all } \mathbf{a} \neq \mathbf{0}. \tag{3.6}$$

Using these facts, we now derive some fundamental properties of inner products and norms.

Lemma 3.1. *For any* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbf{C}^k$,

$$\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2\,\mathrm{Re}\langle \mathbf{a}, \mathbf{b}\rangle + \|\mathbf{b}\|^2.$$

Proof: By (3.3), (3.4), and the definition of the norm,

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= \langle \mathbf{a}+\mathbf{b},\ \mathbf{a}+\mathbf{b}\rangle = \langle \mathbf{a}, \mathbf{a}\rangle + \langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{b}, \mathbf{a}\rangle + \langle \mathbf{b}, \mathbf{b}\rangle \\ &= \langle \mathbf{a}, \mathbf{a}\rangle + \langle \mathbf{a}, \mathbf{b}\rangle + \overline{\langle \mathbf{a}, \mathbf{b}\rangle} + \langle \mathbf{b}, \mathbf{b}\rangle = \|\mathbf{a}\|^2 + 2\,\mathrm{Re}\langle \mathbf{a}, \mathbf{b}\rangle + \|\mathbf{b}\|^2. \qquad \blacksquare\end{aligned}$$

The Cauchy-Schwarz Inequality. *For any* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbf{C}^k$,

$$\left|\langle \mathbf{a}, \mathbf{b}\rangle\right| \le \|\mathbf{a}\|\,\|\mathbf{b}\|. \tag{3.7}$$

Proof: We may assume that $\mathbf{b} \neq \mathbf{0}$, since otherwise both sides of (3.7) are 0. Also, neither $\left|\langle \mathbf{a}, \mathbf{b}\rangle\right|$ nor $\|\mathbf{a}\|\,\|\mathbf{b}\|$ is affected if we multiply $\mathbf{a}$ by a scalar of absolute value one, so we may replace $\mathbf{a}$ by $c\mathbf{a}$, with $|c| = 1$, so as to make $\langle \mathbf{a}, \mathbf{b}\rangle$ real. (That is, if $\langle \mathbf{a}, \mathbf{b}\rangle = re^{i\theta}$, we take $c = e^{-i\theta}$.) Assuming then that $\langle \mathbf{a}, \mathbf{b}\rangle$ is real, by Lemma 3.1 we see that for any real number t,

$$0 \le \|\mathbf{a}+t\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2t\langle \mathbf{a}, \mathbf{b}\rangle + t^2\|\mathbf{b}\|^2.$$

This last expression is a quadratic function of t, since $\|\mathbf{b}\| \neq 0$, and (by elementary calculus) it achieves its minimum value at $t = -\langle \mathbf{a}, \mathbf{b}\rangle/\|\mathbf{b}\|^2$. If we substitute this value for t, we obtain

$$0 \le \|\mathbf{a}\|^2 - 2\frac{\langle \mathbf{a}, \mathbf{b}\rangle^2}{\|\mathbf{b}\|^2} + \frac{\langle \mathbf{a}, \mathbf{b}\rangle^2}{\|\mathbf{b}\|^4}\|\mathbf{b}\|^2 = \|\mathbf{a}\|^2 - \frac{\langle \mathbf{a}, \mathbf{b}\rangle^2}{\|\mathbf{b}\|^2},$$

or

$$0 \le \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \langle \mathbf{a}, \mathbf{b}\rangle^2,$$

which, since $\langle \mathbf{a}, \mathbf{b}\rangle$ is assumed real, is equivalent to (3.7). ∎

The Triangle Inequality. *For any* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbf{C}^k$,

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|. \tag{3.8}$$

Proof: By Lemma 3.1, the Cauchy-Schwarz inequality, and the fact that $\mathrm{Re}\, z \le |z|$, we have

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + 2\,\mathrm{Re}\langle \mathbf{a}, \mathbf{b}\rangle + \|\mathbf{b}\|^2 \\ &\le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{b}\|^2 \\ &= \left(\|\mathbf{a}\| + \|\mathbf{b}\|\right)^2. \qquad \blacksquare\end{aligned}$$

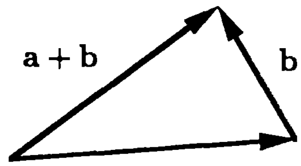


FIGURE 3.1. The sum of two vectors.

Geometrically, the triangle inequality just says that one side of a triangle can be no longer than the sum of the other two sides; see Figure 3.1. This picture is perfectly accurate, for the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}+\mathbf{b}$ always lie in the same plane no matter how many dimensions they live in.

We recall that two real 3-dimensional vectors are orthogonal or perpendicular to each other precisely when their inner product is zero. We shall take this as a definition in the general case: two complex k-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$. The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are called **mutually orthogonal** if $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 0$ for all $i \neq j$. With this terminology, we have a generalization of the classic theorem about the lengths of the sides of a right triangle:

The Pythagorean Theorem. *If* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *are mutually orthogonal, then*

$$\|\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_n\|^2 = \|\mathbf{a}_1\|^2 + \|\mathbf{a}_2\|^2 + \cdots + |\mathbf{a}_n\|^2. \tag{3.9}$$

Proof: We have

$$\|\mathbf{a}_1 + \cdots + \mathbf{a}_n\|^2 = \langle \mathbf{a}_1 + \cdots + \mathbf{a}_n,\, \mathbf{a}_1 + \cdots + \mathbf{a}_n \rangle.$$

If we multiply out the right side by (3.3), all the cross terms vanish because of the orthogonality condition, and we are left with

$$\langle \mathbf{a}_1, \mathbf{a}_1 \rangle + \cdots + \langle \mathbf{a}_n, \mathbf{a}_n \rangle = \|\mathbf{a}_1\|^2 + \cdots + \|\mathbf{a}_n\|^2. \qquad \blacksquare$$

Important Remark. The proofs of the Cauchy-Schwarz and triangle inequalities and the Pythagorean theorem depend only on the properties (3.3) and (3.4) of the inner product and the definition $\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2}$, not on the specific formula (3.1). They therefore remain valid for any other "inner product" that satisfies (3.3) and (3.4) and the "norm" associated to it.

Some more terminology: We say that a vector $\mathbf{u}$ is **normalized**, or is a **unit vector**, if $\|\mathbf{u}\| = 1$. Any nonzero vector $\mathbf{a}$ can be normalized by multiplying it by the reciprocal of its norm: If $\mathbf{u} = \|\mathbf{a}\|^{-1}\mathbf{a}$, then $\|\mathbf{u}\| = \|\mathbf{a}\|^{-1}\|\mathbf{a}\| = 1$. We shall call a collection $\{\mathbf{a}_1, \mathbf{a}_2, \ldots\}$ of vectors an **orthogonal set** if its elements are mutually orthogonal and nonzero, and an **orthonormal set** if its elements are mutually orthogonal and normalized. (See Figure 3.2.) Of course, any orthogonal set can be made into an orthonormal set by normalizing each of its elements. Thus, a set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots\}$ is orthonormal if and only if

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \delta_{ij}, \tag{3.10}$$

where δ_{ij} is the **Kronecker δ-symbol:**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.11}$$

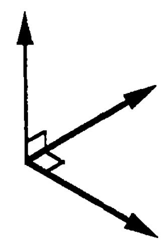


FIGURE 3.2. An orthonormal set of vectors.

The vectors in any orthogonal set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ are linearly independent; that is, the equation

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

can hold only when all the scalars c_j are zero. To see this, take the inner product of both sides with $\mathbf{a}_j$ $(1 \le j \le n)$; because of the orthogonality and the fact that $\mathbf{a}_j \neq \mathbf{0}$, the result is

$$c_j\langle \mathbf{a}_j, \mathbf{a}_j \rangle = c_j\|\mathbf{a}_j\|^2 = 0, \quad \text{hence} \quad c_j = 0.$$

It follows that *the number of vectors in any orthogonal set in $\mathbf{C}^k$ is at most k,* since $\mathbf{C}^k$ is k-dimensional.

An example of an orthonormal set of k vectors is given by the standard basis vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$, where

$$\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0) \qquad (1 \text{ in the } j\text{th position, } 0 \text{ elsewhere}).$$

For any $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbf{C}^k$, we clearly have

$$\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_k\mathbf{e}_k,$$

so $\mathbf{a}$ is expressed in a simple way as a linear combination of the $\mathbf{e}_j$'s. But sometimes it is more convenient to use other orthonormal sets that are adapted to a particular problem, and here too there is a simple way of expressing arbitrary vectors as linear combinations of the orthonormal vectors.

Indeed, suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal set in $\mathbf{C}^k$. If a vector $\mathbf{a} \in \mathbf{C}^k$ is expressed as a linear combination of the $\mathbf{u}_j$'s,

$$\mathbf{a} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k,$$

by taking the inner product of both sides with $\mathbf{u}_j$ and using (3.10) we find that the coefficients c_j are given by

$$c_j = \langle \mathbf{a}, \mathbf{u}_j \rangle \qquad (1 \le j \le k). \tag{3.12}$$

Conversely, if $\mathbf{a}$ is any vector in $\mathbf{C}^n$, we may define the constants $c_1, \ldots, c_k$ by (3.12) and form the linear combination

$$\widetilde{\mathbf{a}} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k.$$

Then the difference $\mathbf{b} = \mathbf{a} - \widetilde{\mathbf{a}}$ is orthogonal to all the u_j's:

$$\langle \mathbf{b}, \mathbf{u}_j \rangle = \langle \mathbf{a}, \mathbf{u}_j \rangle - \langle \widetilde{\mathbf{a}}, \mathbf{u}_j \rangle = c_j - c_j = 0.$$

But this means that $\mathbf{b} = \mathbf{0}$, for otherwise $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{b}\}$ would be an orthogonal set with $k+1$ elements, which is impossible. In other words, $\widetilde{\mathbf{a}} = \mathbf{a}$, and we have the following result.

Theorem 3.1. *Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal set of k vectors in $\mathbf{C}^k$. For any $\mathbf{a} \in \mathbf{C}^k$ we have*

$$\mathbf{a} = \langle \mathbf{a}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{a}, \mathbf{u}_k \rangle \mathbf{u}_k.$$

Moreover,

$$\|\mathbf{a}\|^2 = |\langle \mathbf{a}, \mathbf{u}_1 \rangle|^2 + \cdots + |\langle \mathbf{a}, \mathbf{u}_k \rangle|^2.$$

Proof: The first assertion has just been proved, and the second one follows from it by the Pythagorean theorem. ∎

EXERCISES

1. Show that $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right)$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^k$.
2. Suppose $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ is an orthogonal set in $\mathbf{C}^k$, not necessarily normalized. Use Theorem 3.1 to show that for any $\mathbf{a} \in \mathbf{C}^k$,
$$\mathbf{a} = \frac{\langle \mathbf{a}, \mathbf{y}_1 \rangle \mathbf{y}_1}{\|\mathbf{y}_1\|^2} + \cdots + \frac{\langle \mathbf{a}, \mathbf{y}_k \rangle \mathbf{y}_k}{\|\mathbf{y}_k\|^2}.$$
3. Let $\mathbf{y}_1 = (2, 3i, 5)$ and $\mathbf{y}_2 = (3i, 2, 0)$.
   a. Show that $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = 0$ and find a nonzero $\mathbf{y}_3$ that is orthogonal to both $\mathbf{y}_1$ and $\mathbf{y}_2$.
   b. What are the norms of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$?
   c. Use Theorem 3.1 or Exercise 2 to express the vectors $(1, 2, 3i)$ and $(0, 1, 0)$ as linear combinations of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$.

4. Let $\mathbf{u}_1 = \frac{1}{3}(1, 2i, -2i, 0)$, $\mathbf{u}_2 = \frac{1}{5}(2-4i, -2, i, 0)$, $\mathbf{u}_3 = \frac{1}{15}(4+2i, 5+8i, 4+10i, 0)$, and $\mathbf{u}_4 = (0, 0, 0, i)$.
   a. Show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_4\}$ is an orthonormal set in $\mathbf{C}^4$.
   b. Express the vectors $(1, 0, 0, 0)$ and $(2, 10-i, 10-9i, -3)$ as linear combinations of $\mathbf{u}_1, \ldots, \mathbf{u}_4$ by using Theorem 3.1.
5. Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an orthonormal set in $\mathbf{C}^k$ with $m < k$. Show that for any $\mathbf{a} \in \mathbf{C}^k$ there is a unique set of constants $\{c_1, \ldots, c_m\}$ such that $\mathbf{a} - \sum_1^m c_j \mathbf{u}_j$ is orthogonal to all the $\mathbf{u}_j$'s, and determine these constants explicitly. (Hint: Consider the proof of Theorem 3.1.)

The following problems deal with $k \times k$ complex matrices $T = (T_{ij})$. We recall that if $T = (T_{ij})$ and $S = (S_{ij})$ are $k \times k$ matrices, TS is the matrix whose (ij)th component is $\sum_l T_{il} S_{lj}$, and if $\mathbf{a} \in \mathbf{C}^k$, $T\mathbf{a}$ is the vector whose ith component is $\sum_j T_{ij} a_j$. The (Hermitian) **adjoint** of the matrix T is the matrix T^* obtained by interchanging rows and columns and taking complex conjugates, that is, $(T^*)_{ij} = \overline{T_{ji}}$.

6. Show that $\langle T\mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{a}, T^*\mathbf{b} \rangle$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^k$.
7. Show that if $T = T^*$, the "product" defined by $\langle \mathbf{a}, \mathbf{b} \rangle_T = \langle T\mathbf{a}, \mathbf{b} \rangle$ satisfies properties (3.3) and (3.4).
8. Let $\mathbf{t}_j = (T_{1j}, \ldots, T_{kj})$ be the vector that makes up the jth row of T. Show that the following properties of the matrix T are equivalent. (Hint: Show that the (ij)th component of T^*T is $\langle \mathbf{t}_j, \mathbf{t}_i \rangle$.)
   (i) $\{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ is an orthonormal basis for $\mathbf{C}^k$.
   (ii) T^*T is the identity matrix, i.e., $(T^*T)_{ij} = \delta_{ij}$.
   (iii) $\|T\mathbf{a}\| = \|\mathbf{a}\|$ for all $\mathbf{a} \in \mathbf{C}^k$.
9. Show that $|\langle \mathbf{a}, \mathbf{b} \rangle| = \|\mathbf{a}\|\,\|\mathbf{b}\|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are complex scalar multiples of one another, and that $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are positive scalar multiples of one another. (Examine the proofs of the Cauchy-Schwarz and triangle inequalities to see when equality holds.)

3.2 Functions and inner products

A vector $\mathbf{a} = (a_1, \ldots, a_k)$ in $\mathbf{C}^k$ can be regarded as a function on the set $\{1, \ldots, k\}$ that assigns to the integer j the jth component $\mathbf{a}(j) = a_j$, and with this notation we can write the inner product and norm as follows:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_1^k \mathbf{a}(j)\overline{\mathbf{b}(j)}, \qquad \|\mathbf{a}\| = \left(\sum_1^k |\mathbf{a}(j)|^2 \right)^{1/2}. \tag{3.13}$$

We now make a leap of imagination: Consider the space $PC(a, b)$ of piecewise continuous functions on the interval $[a, b]$, and think of functions $f \in PC(a, b)$ as infinite-dimensional vectors whose "components" are the values $f(x)$ as x ranges over the interval $[a, b]$. The operations of vector addition and scalar multiplication are just the usual addition of functions and multiplication of functions by constants. To define the inner product and the norm, we simply replace the sums in (3.13) by their continuous versions, i.e., integrals:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx, \qquad \|f\| = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}. \tag{3.14}$$

This inner product on functions evidently satisfies the linearity and symmetry properties (3.3) and (3.4), and it is related to the norm by the equation $\|f\| = \langle f, f\rangle^{1/2}$. *Hence the Cauchy-Schwarz inequality, the triangle inequality, and the Pythagorean theorem remain valid in this context, with the same proofs.* Explicitly, in terms of integrals, they say the following:

$$\left|\int_a^b f(x)\overline{g(x)}\,dx\right| \le \sqrt{\int_a^b |f(x)|^2\,dx}\sqrt{\int_a^b |g(x)|^2\,dx}, \tag{3.15}$$

$$\sqrt{\int_a^b |f(x)+g(x)|^2\,dx} \le \sqrt{\int_a^b |f(x)|^2\,dx} + \sqrt{\int_a^b |g(x)|^2\,dx}, \tag{3.16}$$

and

$$\begin{gathered}\int_a^b \left|\sum_1^n f_j(x)\right|^2 dx = \sum_1^n \int_a^b |f_j(x)|^2\,dx \\ \text{when} \quad \int_a^b f_i(x)\overline{f_j(x)}\,dx = 0 \quad \text{for } i \neq j.\end{gathered} \tag{3.17}$$

The homogeneity property (3.5) of the norm, i.e., $\|cf\| = |c|\,\|f\|$, is clearly valid in the present situation, but there is a slight problem with the positivity property (3.6). The integral of a function is not affected by altering the value of the function at a finite number of points, so if f is a function on $[a, b]$ that is zero except at a finite number of points, then $\|f\| = 0$ although f is not the zero function. For the class $PC(a, b)$ with which we are working, there are two ways out of this difficulty. One is to use the convention suggested by the Fourier convergence theorem, that is, to consider only functions $f \in PC(a, b)$ with the property that

$$f(x) = \tfrac{1}{2}\big[f(x-) + f(x+)\big] \quad \text{for all } x \in (a, b), \quad f(a) = f(a+), \quad f(b) = f(b-).$$

If $f \in PC(a, b)$ satisfies this condition and $f(x_0) \neq 0$, then $|f(x)| > 0$ on some interval containing x_0, and hence $\|f\| > 0$. (See Exercises 6 and 7.) The other is simply to agree to consider two functions as equal if they agree except at finitely many points. The reader can use whichever of these devices seems most comfortable; at any rate, we shall not worry any more about this matter.

The concepts of orthogonal and orthonormal sets of functions are defined just as for vectors in $\mathbf{C}^k$, and we can ask whether there is an analogue of Theorem 3.1. That is, given an orthonormal set $\{\phi_n\}$ in $PC(a, b)$, can we express an arbitrary $f \in PC(a,b)$ as $\sum \langle f, \phi_n \rangle \phi_n$? Here, for the first time, we have to confront the fact that the space $PC(a,b)$, unlike $\mathbf{C}^k$, is infinite-dimensional. This means, in particular, that we cannot tell whether the set $\{\phi_n\}$ contains "enough" functions to span the whole space just by counting how many functions are in it; after all, if one removes finitely many elements from an infinite set, there are still infinitely many left. It also means that the sum $\sum \langle f, \phi_n \rangle \phi_n$ will be an infinite series, so we have to worry about convergence. Hence there is some work to be done; but we can see that we are on the track of something very interesting by reconsidering the results of the previous chapter in the light of the ideas we have just developed.

Consider the functions

$$\phi_n(x) = (2\pi)^{-1/2} e^{inx}, \qquad n = 0, \pm 1, \pm 2, \dots$$

We regard these functions as elements of the space $PC(-\pi, \pi)$; we then have

$$\langle \phi_m, \phi_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} \overline{e^{inx}}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x}\, dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Thus $\{\phi_n\}_{-\infty}^{\infty}$ is an orthonormal set. Moreover, if the Fourier coefficients c_n of $f \in PC(-\pi, \pi)$ are defined as in Chapter 2, we have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{e^{inx}}\, dx = (2\pi)^{-1/2} \langle f, \phi_n \rangle,$$

and hence

$$\sum_{-\infty}^{\infty} c_n e^{inx} = \sum_{-\infty}^{\infty} \Big[(2\pi)^{-1/2} \langle f, \phi_n \rangle \Big] \Big[(2\pi)^{1/2} \phi_n(x) \Big] = \sum_{-\infty}^{\infty} \langle f, \phi_n \rangle \phi_n(x).$$

Thus, the Fourier series of f is just its expansion with respect to the orthonormal set $\{\phi_n\}$, as one would expect from the discussion in §3.1!

Let us try this again for Fourier cosine series on the interval $[0, \pi]$. From the trigonometric identity

$$\cos a \cos b = \tfrac{1}{2} \Big[\cos(a+b) + \cos(a-b) \Big]$$

and the fact that

$$\int_0^{\pi} \cos kx\, dx = \begin{cases} k^{-1} \sin kx \big|_0^{\pi} = 0 & \text{for } k \neq 0, \\ x \big|_0^{\pi} = \pi & \text{for } k = 0, \end{cases}$$

we see that for $m, n \geq 0$,

$$\begin{aligned} \int_0^{\pi} \cos mx \cos nx\, dx &= \frac{1}{2} \int_0^{\pi} \Big[\cos(m+n)x + \cos(m-n)x \Big] dx \\ &= \begin{cases} \pi & \text{if } m = n = 0, \\ \frac{1}{2}\pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n. \end{cases} \end{aligned}$$

That is, if we define

$$\psi_0(x) = (1/\pi)^{1/2}, \qquad \psi_n(x) = (2/\pi)^{1/2} \cos nx \quad \text{for } n > 0,$$

then $\{\psi_n\}_0^\infty$ is an orthonormal set in $PC(0,\pi)$. Moreover, if the Fourier cosine coefficients a_n of $f \in PC(0,\pi)$ are defined as before,

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \begin{cases} 2(1/\pi)^{1/2}\langle f, \psi_0\rangle & \text{for } n = 0, \\ (2/\pi)^{1/2}\langle f, \psi_n\rangle & \text{for } n > 0, \end{cases}$$

we have

$$\tfrac{1}{2}a_0 + \sum_0^\infty a_n \cos nx = \sum_0^\infty \langle f, \psi_n\rangle \psi_n(x).$$

The reader may verify that the trigonometric form of the Fourier series on $[-\pi,\pi]$ and the Fourier sine series on $[0,\pi]$ are also instances of expansions with respect to orthonormal sets.

Now, we have been a bit cavalier in this discussion. The reader will recall that we proved the validity of Fourier expansions only for piecewise smooth functions; for functions that are merely piecewise continuous there is no guarantee that the Fourier series will converge at any given point. What this means is that we need to take a closer look at questions of convergence in the context of the ideas from vector geometry that we are now using.

EXERCISES

1. Show that $\left\{(2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal set in $PC(0,l)$.
2. Show that $\left\{(2/l)^{1/2}\cos(n-\frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal set in $PC(0,l)$.
3. Show that $f_0(x) = 1$ and $f_1(x) = x$ are orthogonal on $[-1,1]$, and find constants a and b so that $f_2(x) = x^2 + ax + b$ is orthogonal to both f_0 and f_1 on $[-1,1]$. What are the normalizations of f_0, f_1, and f_2?
4. Suppose $\{\phi_n\}$ is an orthonormal set in $PC(0,l)$, and let ϕ_n^+ and ϕ_n^- be the even and odd extensions of ϕ_n to $[-l,l]$. Show that $\left\{2^{-1/2}\phi_n^+\right\} \cup \left\{2^{-1/2}\phi_n^-\right\}$ is an orthonormal set in $PC(-l,l)$. (Hint: First show that $\left\{2^{-1/2}\phi_n^+\right\}$ and $\left\{2^{-1/2}\phi_n^-\right\}$ are orthonormal, and then that $\langle \phi_n^+, \phi_m^-\rangle = 0$ for all m, n.)
5. Let $\{\phi_n : n \geq 0\}$ be an orthonormal set in $PC(-l,l)$ such that ϕ_n is even when n is even and ϕ_n is odd when n is odd. Show that $\{\sqrt{2}\,\phi_n : n \text{ even}\}$ and $\{\sqrt{2}\,\phi_n : n \text{ odd}\}$ are orthonormal sets in $PC(0,l)$.
6. Suppose $f \in PC(a,b)$ and $f(x) = \frac{1}{2}\big[f(x-) + f(x+)\big]$ for all $x \in (a,b)$. Show that if $f(x_0) \neq 0$ for some $x_0 \in (a,b)$, then $f(x) \neq 0$ for all x in some interval containing x_0. (x_0 may be an endpoint of the interval.)
7. Show that if $f \in PC(a,b)$, $f \geq 0$, and $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ except perhaps at finitely many points. (Hint: By redefining f at its discontinuities, you can make f satisfy the conditions of Exercise 6.)

3.3 Convergence and completeness

If we visualize a k-dimensional vector $\mathbf{a}$ as the point in k-space with coordinates $(a_1, \ldots, a_k)$ rather than as an arrow, then $\|\mathbf{a}-\mathbf{b}\|$ is just the distance between the points $\mathbf{a}$ and $\mathbf{b}$ as defined by Euclidean geometry. Accordingly, the natural notion of convergence for vectors is that $\mathbf{a}_n \to \mathbf{a}$ if and only if $\|\mathbf{a}_n - \mathbf{a}\| \to 0$. This suggests a new definition of convergence for functions. Namely, if $\{f_n\}$ is a sequence of functions in $PC(a,b)$, we say that $f_n \to f$ **in norm** if $\|f_n - f\| \to 0$, that is,

$$f_n \to f \text{ in norm} \quad \Longleftrightarrow \quad \int_a^b |f_n(x) - f(x)|^2 dx \to 0.$$

Convergence of f_n to f in norm thus means that the difference $f_n - f$ tends to zero in a suitable averaged sense over the interval $[a,b]$. It does not guarantee pointwise convergence, nor does pointwise convergence imply convergence in norm. For example, let $[a,b] = [0,1]$. If we define

$$f_n(x) = 1 \quad \text{for } 0 \le x \le 1/n, \quad f_n(x) = 0 \quad \text{elsewhere,}$$

then

$$\|f_n\|^2 = \int_0^1 |f_n(x)|^2 \, dx = \int_0^{1/n} dx = 1/n,$$

so $f_n \to 0$ in norm, but $f_n(0) = 1$ for all n, so f_n does not converge to zero pointwise. On the other hand, if

$$g_n(x) = n \quad \text{for } 0 < x < 1/n, \quad g_n(x) = 0 \quad \text{elsewhere,}$$

then $g_n \to 0$ pointwise (in fact, $g_n(0) = 0$ for all n, and for any $x > 0$, $g_n(x) = 0$ for $n > |x|^{-1}$), but

$$\|g_n\|^2 = \int_0^1 |g_n(x)|^2 \, dx = \int_0^{1/n} n^2 \, dx = n,$$

so $g_n \not\to 0$ in norm. However, we have the following simple and useful result.

Theorem 3.2. *If $f_n \to f$ uniformly on $[a,b]$ $(-\infty < a < b < \infty)$, then $f_n \to f$ in norm.*

Proof: Uniform convergence means that there is a sequence $\{M_n\}$ of constants such that $|f_n(x) - f(x)| \le M_n$ for all $x \in [a,b]$ and $M_n \to 0$. But then

$$\|f_n - f\|^2 = \int_a^b |f_n(x) - f(x)|^2 \, dx \le \int_a^b M_n^2 \, dx = (b-a)M_n^2,$$

so $\|f_n - f\|$ tends to zero along with M_n. ∎

It should be mentioned that the norm and inner product are themselves continuous with respect to convergence in norm; that is, if $f_n \to f$ in norm, then

$$\|f_n\| \to \|f\|, \qquad \langle f_n, g\rangle \to \langle f, g\rangle \quad \text{and} \quad \langle g, f_n\rangle \to \langle g, f\rangle \quad \text{for all } g.$$

The verification is left to the reader (Exercises 1 and 2).

$PC(a,b)$ fails in one crucial respect to be a good infinite-dimensional analogue of Euclidean space, namely, it is not *complete.* This means, intuitively, that there are sequences that look like they ought to converge in norm, but which fail to have a limit in the space $PC(a,b)$. The formal definition is as follows. A sequence $\{\mathbf{a}_n\}_1^\infty$ of vectors (or functions or numbers) is called a **Cauchy sequence** if $\|\mathbf{a}_m - \mathbf{a}_n\| \to 0$ as $m, n \to \infty$, that is, if the terms in the sequence get closer and closer to each other as one goes further out in the sequence. A space S of vectors (or functions or numbers) is called **complete** if every Cauchy sequence in S has a limit in S. The real and complex number systems are complete, and it follows easily that the vector spaces $\mathbf{C}^k$ are complete for any k. The set R of rational numbers is not: if $\{r_n\}$ is a sequence of rational numbers with an irrational limit, such as the sequence of decimal approximations to π, then $\{r_n\}$ is Cauchy but has no limit in R.

One can see that $PC(a,b)$ is not complete by the following simple example. Take $[a,b] = [0,1]$, and let

$$f_n(x) = x^{-1/4} \quad \text{for } x > 1/n, \qquad f_n(x) = 0 \quad \text{for } x \le 1/n.$$

If $m > n$, $f_m(x) - f_n(x)$ equals $x^{-1/4}$ when $m^{-1} < x \le n^{-1}$ and equals 0 otherwise, so

$$\|f_m - f_n\|^2 = \int_{1/m}^{1/n} x^{-1/2}\,dx = 2x^{1/2}\Big|_{1/m}^{1/n} = 2(n^{-1/2} - m^{-1/2}),$$

which tends to zero as $m, n \to \infty$. Thus the sequence $\{f_n\}$ is Cauchy; but clearly its limit, either pointwise or in norm, is the function

$$f(x) = x^{-1/4} \quad \text{for } x > 0, \qquad f(0) = 0, \tag{3.18}$$

and this function does not belong to $PC(0,1)$ because it becomes unbounded as $x \to 0$.

It is easy enough to enlarge the space $PC(a,b)$ to include functions such as (3.18) with one or more infinite singularities in the interval $[a,b]$: One simply allows improper (but absolutely convergent) integrals in the definition of the inner product and the norm. But even this is not enough. One can construct Cauchy sequences $\{f_n\}$ in which f_n acquires more and more singularities as n increases, in such a way that the limit function f is everywhere discontinuous — and in particular, not Riemann integrable on any interval.

Fortunately, there is a more sophisticated theory of integration, the *Lebesgue integral,* which allows one to handle such highly irregular functions. The Lebesgue theory does require a very weak regularity condition called *measurability*, but this technicality need not concern us. All functions that arise in practice are measurable, and *all functions mentioned in the remainder of this book are tacitly assumed to be measurable.* For our present purposes, we do not need to know anything about the construction or detailed properties of the Lebesgue integral; all we need is a couple of definitions and a couple of facts that we shall quote without proof. Rudin [47] and Dym-McKean [19] contain brief expositions of Lebesgue integration that include most of the results we shall use; more extensive accounts of the theory can be found, for example, in Folland [25] and Wheeden-Zygmund [56].

We denote by $L^2(a,b)$ the space of **square-integrable** functions on $[a,b]$, that is, the set of all functions on $[a,b]$ whose squares are absolutely Lebesgue-integrable over $[a,b]$:

$$L^2(a,b) = \left\{ f : \int_a^b |f(x)|^2 \, dx < \infty \right\}. \tag{3.19}$$

This space includes all functions for which the (possibly improper) Riemann integral $\int_a^b |f(x)|^2 dx$ converges, and one should think of it simply as the space of all functions f such that the region between the graph of $|f|^2$ and the x-axis has finite area. Since

$$st \leq \tfrac{1}{2}(s^2 + t^2)$$

(because $s^2 + t^2 - 2st = (s-t)^2 \geq 0$) for any real numbers s and t, we have

$$|f(x)\overline{g(x)}| \leq \tfrac{1}{2}\Big(|f(x)|^2 + |g(x)|^2\Big),$$

and thus if f and g are in $L^2(a,b)$, the integral

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)} \, dx$$

is absolutely convergent. Therefore, the definitions of the inner product and norm extend to the space $L^2(a,b)$, as do all their properties that we have discussed previously.

As in the space $PC(a,b)$, there is a slight problem with the positivity of the norm, as the condition $\int |f|^2 = 0$ does not imply that f vanishes identically but only that the $f = 0$ "almost everywhere." The precise interpretation of this phrase is as follows. A subset E of $\mathbf{R}$ is said to have **measure zero** if, for any $\epsilon > 0$, E can be covered by a sequence of open intervals whose total length is less than ϵ, that is, if there exist open intervals $I_1, I_2, \ldots$ of lengths $l_1, l_2, \ldots$ such that $E \subset \bigcup_1^\infty I_j$ and $\sum_1^\infty l_j < \epsilon$. (For example, any countable set has measure zero: If $E = \{x_1, x_2, \ldots\}$, let I_j be the interval of length $\epsilon/2^j$ centered at x_j.) A statement about real numbers that is true for all x except for those x in some set of measure zero is said to be true **almost everywhere**, or for **almost every** x.

It can be shown that if $f \in L^2(a,b)$, the norm of f is zero if and only if $f(x) = 0$ for almost every $x \in [a,b]$. Accordingly, we agree to regard two functions as equal if they are equal almost everywhere. This weakened notion of equality then validates the statement that $\|f\| = 0$ only when $f = 0$, and it turns out also to be appropriate in many other contexts. Moreover, if two *continuous* functions are equal almost everywhere then they are identically equal, so for continuous functions the ordinary notion of equality is entirely adequate.

The crucial properties of $L^2(a,b)$ that we shall need to state without proof are contained in the following theorem.

Theorem 3.3. (a) *$L^2(a,b)$ is complete with respect to convergence in norm.* (b) *For any $f \in L^2(a,b)$ there is a sequence f_n of continuous functions on $[a,b]$ such that $f_n \to f$ in norm. In fact, the functions f_n can be taken to be the restrictions to $[a,b]$ of functions on the line that possess derivatives of all orders at every point; moreover, the latter functions can be taken to be $(b-a)$-periodic or to vanish outside a bounded set.*

This theorem says that $L^2(a,b)$ is obtained by "filling in the holes" in the space $PC(a,b)$. The first assertion says that all the holes have been filled, and the second one says that nothing extra, beyond the completion of $PC(a,b)$, has been added in. For a proof, see Rudin [47], Theorems 11.38 and 11.42. We shall indicate how to prove the second assertion — that is, how to approximate arbitrary L^2 functions by smooth ones — in §7.1.

We are now ready to discuss the convergence of expansions with respect to orthonormal sets in $PC(a,b)$, or more generally in $L^2(a,b)$. The first step is to obtain the general form of Bessel's inequality, which is a straightforward generalization of the special case we proved in §2.1.

Bessel's Inequality. *If $\{\phi_n\}_1^\infty$ is an orthonormal set in $L^2(a,b)$ and $f \in L^2(a,b)$, then*

$$\sum_1^\infty |\langle f, \phi_n\rangle|^2 \le \|f\|^2. \tag{3.20}$$

Proof: Observe that

$$\Big\langle f, \langle f, \phi_n\rangle\phi_n \Big\rangle = \overline{\langle f, \phi_n\rangle}\langle f, \phi_n\rangle = |\langle f, \phi_n\rangle|^2$$

and that by the Pythagorean theorem,

$$\left\|\sum_1^N \langle f, \phi_n\rangle\phi_n\right\|^2 = \sum_1^N |\langle f, \phi_n\rangle|^2.$$

Hence, for any positive integer N, by Lemma 3.1,

$$\begin{aligned}
0 &\le \left\| f - \sum_1^N \langle f, \phi_n\rangle \phi_n \right\|^2 \\
&= \|f\|^2 - 2\,\mathrm{Re}\left\langle f, \sum_1^N \langle f, \phi_n\rangle \phi_n \right\rangle + \left\| \sum_1^N \langle f, \phi_n\rangle \phi_n \right\|^2 \\
&= \|f\|^2 - 2\sum_1^N |\langle f, \phi_n\rangle|^2 + \sum_1^N |\langle f, \phi_n\rangle|^2 \\
&= \|f\|^2 - \sum_1^N |\langle f, \phi_n\rangle|^2.
\end{aligned}$$

Letting $N \to \infty$, we obtain the desired result. ∎

We are now concerned with the following problem: given an orthonormal set $\{\phi_n\}_1^\infty$ in $L^2(a,b)$, is it true that

$$f = \sum_1^\infty \langle f, \phi_n\rangle \phi_n \tag{3.21}$$

for all $f \in L^2(a,b)$? First we assure ourselves that the series on the right actually makes sense.

Lemma 3.2. *If $f \in L^2(a,b)$ and $\{\phi_n\}$ is any orthonormal set in $L^2(a,b)$, then the series $\sum\langle f, \phi_n\rangle\phi_n$ converges in norm, and $\left\|\sum\langle f, \phi_n\rangle\phi_n\right\| \le \|f\|$.*

Proof: Bessel's inequality guarantees that the series $\sum |\langle f, \phi_n\rangle|^2$ converges, so by the Pythagorean theorem,

$$\left\| \sum_m^n \langle f, \phi_n\rangle \phi_n \right\|^2 = \sum_m^n |\langle f, \phi_n\rangle|^2 \to 0 \quad \text{as } m, n \to \infty.$$

Thus the partial sums of the series $\sum\langle f, \phi_n\rangle\phi_n$ form a Cauchy sequence, and since $L^2(a,b)$ is complete, the series converges. Finally, another application of the Pythagorean theorem and Bessel's inequality gives

$$\begin{aligned}
\left\| \sum_1^\infty \langle f, \phi_n\rangle \phi_n \right\|^2 = \lim_{N\to\infty} \left\| \sum_1^N \langle f, \phi_n\rangle \phi_n \right\|^2 &= \lim_{N\to\infty} \sum_1^N |\langle f, \phi_n\rangle|^2 \\
&= \sum_1^\infty |\langle f, \phi_n\rangle|^2 \le \|f\|^2.
\end{aligned}$$ ∎

Now, an obvious necessary condition for (3.21) to hold for arbitrary f is that the orthonormal set $\{\phi_n\}$ is as large as possible, that is, that there is no nonzero f which is orthogonal to all the ϕ_n's. (If $\langle f, \phi_n\rangle = 0$ for all n, then (3.21) implies that $f = 0$.) Moreover, if (3.21) holds and the Pythagorean theorem extends to infinite sums of orthogonal vectors, Bessel's inequality (3.20) should actually be an equality. With these thoughts in mind, we arrive at the main theorem.

Theorem 3.4. *Let $\{\phi_n\}_1^\infty$ be an orthonormal set in $L^2(a,b)$. The following conditions are equivalent:*
(a) If $\langle f, \phi_n\rangle = 0$ for all n, then $f = 0$.
(b) For every $f \in L^2(a,b)$ we have $f = \sum_1^\infty \langle f, \phi_n\rangle \phi_n$, where the series converges in norm.
(c) For every $f \in L^2(a,b)$, we have **Parseval's equation**:

$$\|f\|^2 = \sum_1^\infty |\langle f, \phi_n\rangle|^2. \tag{3.22}$$

Proof: We shall show that (a) implies (b), that (b) implies (c), and that (c) implies (a).

(a) implies (b): Given $f \in L^2(a,b)$, the series $\sum \langle f, \phi_n\rangle \phi_n$ converges in norm, by Lemma 3.2. We can see that its sum is f by showing that the difference $g = f - \sum \langle f, \phi_n\rangle \phi_n$ is zero. But

$$\langle g, \phi_m\rangle = \langle f, \phi_m\rangle - \sum_{n=1}^\infty \langle f, \phi_n\rangle \langle \phi_n, \phi_m\rangle = \langle f, \phi_m\rangle - \langle f, \phi_m\rangle = 0$$

for all m. Hence, if (a) holds, $g = 0$.

(b) implies (c): If $f = \sum \langle f, \phi_n\rangle \phi_n$, then by the Pythagorean theorem,

$$\|f\|^2 = \lim_{N\to\infty} \left\| \sum_1^N \langle f, \phi_n\rangle \phi_n \right\|^2 = \lim_{N\to\infty} \sum_1^N |\langle f, \phi_n\rangle|^2 = \sum_1^\infty |\langle f, \phi_n\rangle|^2.$$

(c) implies (a): If (c) holds and $\langle f, \phi_n\rangle = 0$ for all n then $\|f\| = 0$, and therefore $f = 0$. ∎

An orthonormal set that possesses the properties (a)–(c) of Theorem 3.4 is called a **complete orthonormal set** or an **orthonormal basis** for $L^2(a,b)$. This usage of the word *complete* is different from the one discussed earlier in this section, but it is obviously appropriate in the present context. If $\{\phi_n\}$ is an orthonormal basis of $L^2(a,b)$ and $f \in L^2(a,b)$, the numbers $\langle f, \phi_n\rangle$ are called the (generalized) **Fourier coefficients** of f with respect to $\{\phi_n\}$, and the series $\sum \langle f, \phi_n\rangle \phi_n$ is called the (generalized) **Fourier series** of f.

Often it is more convenient not to require the elements of a basis to be unit vectors. Accordingly, suppose $\{\psi_n\}$ is an orthogonal set (and recall that, according to our definition of orthogonal set, this entails $\psi_n \neq 0$ for all n). Let $\phi_n = \|\psi_n\|^{-1}\psi_n$; then $\{\phi_n\}$ is an orthonormal set. We say that $\{\psi_n\}$ is a **complete orthogonal set** or an **orthogonal basis** if $\{\phi_n\}$ is an orthonormal basis. In this case the expansion formula for $f \in L^2(a,b)$ and the Parseval equation take the form

$$f = \sum \frac{\langle f, \psi_n\rangle}{\|\psi_n\|^2}\psi_n \qquad \|f\|^2 = \sum \frac{|\langle f, \psi_n\rangle|^2}{\|\psi_n\|^2}. \tag{3.23}$$

Now, what about the orthonormal sets derived from Fourier series that we discussed in §3.2? We have not yet proved that they are complete, for we derived the expansion formula $f = \sum \langle f, \phi_n\rangle \phi_n$ only when f was piecewise smooth, not for an arbitrary $f \in L^2(a,b)$. But there is actually very little work left to do.

Theorem 3.5. *The sets*

$$\left\{e^{inx}\right\}_{n=-\infty}^{\infty} \quad \textit{and} \quad \left\{\cos nx\right\}_{n=0}^{\infty} \cup \left\{\sin nx\right\}_{n=1}^{\infty}$$

are orthogonal bases for $L^2(-\pi,\pi)$. *The sets*

$$\left\{\cos nx\right\}_{n=0}^{\infty} \quad \textit{and} \quad \left\{\sin nx\right\}_{n=1}^{\infty}$$

are orthogonal bases for $L^2(0,\pi)$.

Proof: First consider the functions $\psi_n(x) = e^{inx}$. Suppose $f \in L^2(-\pi,\pi)$ and ϵ is a (small) positive number; we wish to show that the Nth partial sum of the Fourier series of f approximates f in norm to within ϵ if N is sufficiently large. By part (b) of Theorem 3.3, we can find a 2π-periodic function $\widetilde{f}$, possessing derivatives of all orders, such that $\|f - \widetilde{f}\| < \epsilon/3$. Let $c_n = (2\pi)^{-1}\langle f, \psi_n\rangle$ and $\widetilde{c}_n = (2\pi)^{-1}\langle \widetilde{f}, \psi_n\rangle$ be the Fourier coefficients of f and $\widetilde{f}$. By Theorem 2.5 of §2.3, we know that the Fourier series $\sum \widetilde{c}_n \psi_n$ converges uniformly to $\widetilde{f}$; hence, by Theorem 3.3, it converges to $\widetilde{f}$ in norm. Thus, if we take N sufficiently large, we have

$$\left\|\widetilde{f} - \sum_{-N}^{N} \widetilde{c}_n \psi_n\right\| < \frac{\epsilon}{3}.$$

Moreover, by the Pythagorean theorem and Bessel's inequality,

$$\begin{aligned}\left\|\sum_{-N}^{N} \widetilde{c}_n \psi_n - \sum_{-N}^{N} c_n \psi_n\right\|^2 &\le \sum_{-N}^{N} |\widetilde{c}_n - c_n|^2 \\ &\le \sum_{-\infty}^{\infty} |\widetilde{c}_n - c_n|^2 \le \|\widetilde{f} - f\|^2 < \left(\frac{\epsilon}{3}\right)^2.\end{aligned}$$

Thus, if we write

$$f - \sum_{-N}^{N} c_n \psi_n = \left(f - \widetilde{f}\right) + \left(\widetilde{f} - \sum_{-N}^{N} \widetilde{c}_n \psi_n\right) + \left(\sum_{-N}^{N} \widetilde{c}_n \psi_n - \sum_{-N}^{N} c_n \psi_n\right)$$

and use the triangle inequality, we see that

$$\left\|f - \sum_{-N}^{N} c_n \psi_n\right\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves the completeness of the set $\{\psi_n\} = \{e^{inx}\}$ in $L^2(-\pi,\pi)$, and the completeness of $\{\cos nx\} \cup \{\sin nx\}$ is essentially a restatement of the same result. The completeness of $\{\cos nx\}$ and $\{\sin nx\}$ in $L^2(0,\pi)$ is an easy corollary. (Just consider the even or odd extension of $f \in L^2(0,\pi)$ to $[-\pi,\pi]$.) ∎

The normalizing constants for the functions in Theorem 3.5 are, of course, $\sqrt{1/2\pi}$ for e^{inx}, $\sqrt{1/\pi}$ for $\cos nx$ and $\sin nx$ on $[-\pi, \pi]$ (except for $n = 0$), and $\sqrt{2/\pi}$ for $\cos nx$ and $\sin nx$ on $[0, \pi]$ (except for $n = 0$). With this in mind, one easily sees that the Parseval equation takes the form

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi \sum_{-\infty}^{\infty} |c_n|^2 = \frac{\pi}{2}|a_0|^2 + \pi \sum_{1}^{\infty}\left(|a_n|^2 + |b_n|^2\right), \qquad f \in L^2(-\pi, \pi),$$

where a_n, b_n, and c_n are the Fourier coefficients of f as defined in §2.1, and

$$\int_0^{\pi} |f(x)|^2 \, dx = \frac{\pi}{4}|a_0|^2 + \frac{\pi}{2}\sum_{1}^{\infty} |a_n|^2 = \frac{\pi}{2}\sum_{1}^{\infty} |b_n|^2, \qquad f \in L^2(0, \pi),$$

where a_n and b_n are the Fourier cosine and sine coefficients of f as defined in §2.4. For example, if we consider the Fourier sine series of $f(x) = x$ on $[0, \pi]$ as derived in §2.1, we find that

$$\frac{\pi}{2}\sum_{1}^{\infty} \frac{4}{n^2} = \int_0^{\pi} x^2 \, dx = \frac{\pi^3}{3}, \quad \text{or} \quad \sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

a result which we derived by other means in Exercise 3, §2.3.

Let us sum up our theorems about the convergence of Fourier series. If f is a periodic function, then the Fourier series of f converges to f

(i) absolutely, uniformly, and in norm, if f is continuous and piecewise smooth;
(ii) pointwise and in norm, if f is piecewise smooth;
(iii) in norm, if $f \in L^2(a, b)$.

These results are sufficient for virtually all practical purposes. However, as we indicated in §2.6, there is more to be said on the subject. Here we shall just mention one more result that is a natural generalization of the theorems in this section. If $1 \le p < \infty$, we define $L^p(a, b)$ to be the space of Lebesgue-integrable functions f on $[a, b]$ such that

$$\int_a^b |f(x)|^p \, dx < \infty.$$

If $p > 1$, the Fourier series of any $f \in L^p(-\pi, \pi)$ converges to f in the "L^p norm," that is, if $\{c_n\}$ are the Fourier coefficients of f,

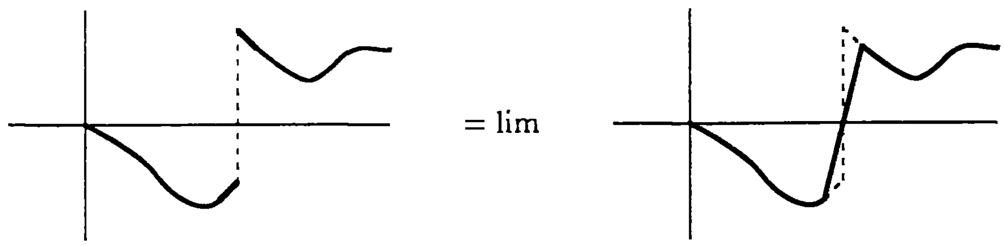
$$\int_a^b \left| \sum_{-N}^{N} c_n e^{inx} - f(x) \right|^p dx \to 0 \quad \text{as } N \to \infty.$$

However, this result is false for $p = 1$.

EXERCISES

1. Show that if $f_n \in L^2(a, b)$ and $f_n \to f$ in norm, then $\langle f_n, g\rangle \to \langle f, g\rangle$ for all $g \in L^2(a, b)$. (Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f, \, g\rangle$.)
2. Show that $\big|\, \|f\| - \|g\| \,\big| \le \|f - g\|$. (Use the triangle inequality; consider the cases $\|f\| \ge \|g\|$ and $\|f\| \le \|g\|$ separately.) Deduce that if $f_n \to f$ in norm then $\|f_n\| \to \|f\|$.

3. Show directly that any $f \in PC(a,b)$ is the limit in norm of a sequence of continuous functions on $[a,b]$, by the argument suggested by the following picture.



4. Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a,b)$. Suppose $c > 0$ and $d \in \mathbf{R}$, and let $\psi_n(x) = c^{1/2}\phi_n(cx+d)$. Show that $\{\psi_n\}$ is an orthonormal basis for $L^2(\frac{a-d}{c}, \frac{b-d}{c})$.
5. Finish the proof of Theorem 3.5. That is, from the completeness of $\{e^{inx}\}$ on $[-\pi,\pi]$, deduce the completeness of $\{\cos nx\} \cup \{\sin nx\}$ on $[-\pi,\pi]$ and the completeness of $\{\cos nx\}$ and $\{\sin nx\}$ on $[0,\pi]$.
6. Let $\phi_n(x) = (2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)$. In Exercise 1, §3.2, it was shown that $\{\phi_n\}_1^\infty$ is an orthonormal set in $L^2(0,l)$. Prove that it is actually a basis, via the following argument.
   a. Let $\psi_k(x) = l^{-1/2}\sin(k\pi x/2l)$. Show that $\{\psi_k\}_1^\infty$ is an orthonormal basis for $L^2(0,2l)$. (This follows from Theorem 3.5 and Exercise 4.)
   b. If $f \in L^2(0,l)$, extend f to $[0,2l]$ by making it symmetric about the line $x = l$, that is, define the extension $\tilde{f}$ by $\tilde{f}(x) = \tilde{f}(2l-x) = f(x)$ for $x \in [0,l]$. Show that $\langle \tilde{f}, \psi_{2n}\rangle = 0$ and $\langle \tilde{f}, \psi_{2n-1}\rangle = 2^{1/2}\langle f, \phi_n\rangle$.
   c. Conclude that if $\langle f, \phi_n\rangle = 0$ for all n, then $f = 0$.
7. Show that $\left\{(2/l)^{1/2}\cos(n-\frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal basis for $L^2(0,l)$. (The argument is similar to that in Exercise 6, but this time you should extend f to be skew-symmetric about $x = l$, that is, $\tilde{f}(2l-x) = -\tilde{f}(x) = -f(x)$ for $x \in [0,l]$.)
8. Find the expansions of the functions $f(x) = 1$ and $g(x) = x$ on $[0,l]$ with respect to the orthonormal bases in Exercises 6 and 7.
9. Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a,b)$. Show that for any $f, g \in L^2(a,b)$,
$$\langle f, g\rangle = \sum \langle f, \phi_n\rangle \overline{\langle g, \phi_n\rangle}.$$
(Note that the case $f = g$ is Parseval's equation.)
10. Evaluate the following series by applying Parseval's equation to certain of the Fourier expansions in Table 1 of §2.1.

a. $\displaystyle\sum_1^\infty \frac{1}{n^4}$ b. $\displaystyle\sum_1^\infty \frac{1}{(2n-1)^6}$ c. $\displaystyle\sum_1^\infty \frac{n^2}{(n^2+1)^2}$

d. $\displaystyle\sum_1^\infty \frac{\sin^2 na}{n^4}$ $(0 < a < \pi)$

11. Suppose f is of class $C^{(1)}$, 2π-periodic, and real-valued. Show that f' is orthogonal to f in $L^2(-\pi,\pi)$ in two ways: (a) by expanding f in a Fourier series and using Exercise 9 and (b) directly from the fact that $2ff' = (f^2)'$.

3.4 More about L^2 spaces; the dominated convergence theorem

In this section we continue the general discussion of L^2 spaces and introduce an extremely useful criterion for the integral of a limit to equal the limit of the integrals.

Other types of L^2 spaces

The results of the previous section concerning $L^2(a,b)$ can be generalized in various ways, and we shall need some of these generalizations later on.

First, one can replace the element dx of linear measure on $[a,b]$ by a weighted element of measure, $w(x)\,dx$. To be precise, suppose w is a continuous function on $[a,b]$ such that $w(x) > 0$ for all $x \in [a,b]$; we call such a w a **weight function** on $[a,b]$. We can then define the "weighted L^2 space" $L^2_w(a,b)$ to be the set of all (Lebesgue measurable) functions on $[a,b]$ such that

$$\int_a^b |f(x)|^2 w(x)\,dx < \infty,$$

and we define an inner product and norm on $L^2_w(a,b)$ by

$$\langle f,g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx, \qquad \|f\|_w = \left(\int_a^b |f(x)|^2 w(x)\,dx\right)^{1/2}.$$

This inner product and norm still satisfy the fundamental conditions (3.3)–(3.6), so the theorems of §3.1 apply in this situation. So do Theorems 3.2, 3.3, and 3.4. w could also be allowed to have some singularities, as long as $\int_a^b w(x)\,dx < \infty$, or to vanish at a few points. (If w vanishes on a whole subinterval of $[a,b]$, one loses the strict positivity of the norm.)

Second, one can replace the bounded interval $[a,b]$ with a half-line or the whole line, or by a region in the plane or in a higher-dimensional space. That is, let D be a region in $\mathbf{R}^k$. (A "region" can be anything reasonable: an open set, or the closure of an open set, or indeed any Lebesgue measurable set. It does not have to be bounded, and indeed may be the whole space.) We define $L^2(D)$ to be the set of all functions f such that

$$\int_D |f(\mathbf{x})|^2\,d\mathbf{x} < \infty,$$

and we define the inner product and norm on $L^2(D)$ by

$$\langle f,g\rangle = \int_D f(\mathbf{x})\overline{g(\mathbf{x})}\,d\mathbf{x}, \qquad \|f\| = \left(\int_D |f(\mathbf{x})|^2\,d\mathbf{x}\right)^{1/2}.$$

Here $\int_D$ is a k-tuple integral, and $d\mathbf{x}$ is the element of Euclidean measure in k-space (length when $k = 1$, area when $k = 2$, volume when $k = 3$, etc.). If one is working only with Riemann integrals, one has to worry a bit about improper integrals when D is unbounded, but this problem is not serious. (The Lebesgue theory handles integrals over unbounded regions rather more smoothly.) Again, this inner product and norm satisfy (3.3)–(3.6), so the results of §3.1 are available, as is Theorem 3.4. However, the analogue of Theorem 3.2 is *false* when D is unbounded (or more precisely, when D has infinite measure), and a glance at its proof should show why. (See Exercise 6.) We shall state a result shortly that can be used in its place.

Theorem 3.3 also needs to be reformulated; here is one good version of it.

Theorem 3.6. *$L^2(D)$ is complete. If $f \in L^2(D)$, there is a sequence $\{f_n\}$ that converges to f in norm, such that each f_n is continuous on D and vanishes outside some bounded set. The f_n's can be taken to be restrictions to D of functions defined on all of $\mathbf{R}^k$ that have derivatives of all orders and vanish outside bounded sets.*

One can also modify $L^2(D)$ by throwing in a weight function, as before.

As a matter of fact, all one needs to develop the ideas of §3.1 are the following ingredients:

(i) a vector space $\mathscr{H}$, that is, a collection of objects that can be added to each other and multiplied by complex numbers, such that the usual laws of vector addition and scalar multiplication hold;

(ii) an inner product $\langle u, v\rangle$ on $\mathscr{H}$ and associated norm $\|u\| = \langle u, u\rangle^{1/2}$ that satisfy (3.3)–(3.6).

If, in addition, the space $\mathscr{H}$ is *complete* with respect to convergence in norm, it is called a **Hilbert space**. In this case, Bessel's inequality and Theorem 3.4 also hold. This general setup includes, but is not limited to, the spaces $\mathbf{C}^k$, $L^2(a, b)$, $L^2_w(a, b)$, and $L^2(D)$ discussed above.

Another example of a Hilbert space is the space l^2 of square-summable sequences. That is, the elements of l^2 are sequences $\{c_n\}_1^\infty$ of complex numbers such that $\sum_1^\infty |c_n|^2 < \infty$, and the inner product and norm are defined by

$$\Big\langle \{c_n\}, \{d_n\} \Big\rangle = \sum_1^\infty c_n \overline{d}_n, \qquad \Big\|\{c_n\}\Big\| = \left(\sum_1^\infty |c_n|^2\right)^{1/2}.$$

We have encountered this space before without mentioning it explicitly. Indeed, suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(a, b)$. Then the mapping that takes an $f \in L^2(a, b)$ to its sequence of coefficients $\Big\{\langle f, \phi_n\rangle\Big\}$ sets up a one-to one correspondence between $L^2(a, b)$ and l^2 that is linear and (by Parseval's equation) norm-preserving. Such a mapping is called a **unitary operator**.

One further comment: We suggested thinking of functions $f \in L^2(a, b)$ as vectors whose components are the values $f(x)$, $x \in [a, b]$. The reader who knows about orders of infinity may be puzzled that there are uncountably many such "components," and yet the orthonormal bases we have displayed are countable sets. The explanation is that the elements of $L^2(a,b)$ are continuous functions or limits in norm of continuous functions, and the values of a continuous function are not completely independent of each other. For example, if f is continuous on $[a,b]$, then f is completely determined by its values at the rational points in $[a,b]$, of which there are only countably many.

The dominated convergence theorem

We now state one other result from the Lebesgue theory of integration that is of great utility even in the setting of Riemann integrable functions. It gives a general condition under which the integral of a limit is the limit of the integrals, and is an improvement on most of the theorems of this sort that one commonly encounters in calculus texts. We shall use it frequently throughout the rest of this book.

The Dominated Convergence Theorem. *Let D be a region in $\mathbf{R}^k$ $(k = 1,2,3,\ldots)$. Suppose g_n $(n = 1,2,3,\ldots)$, g, and ϕ are functions on D, such that*
(a) $\phi(\mathbf{x}) \geq 0$ and $\int_D \phi(\mathbf{x})\,d\mathbf{x} < \infty$,
(b) $|g_n(\mathbf{x})| \leq \phi(\mathbf{x})$ for all n and all $x \in D$,
(c) $g_n(\mathbf{x}) \to g(\mathbf{x})$ as $n \to \infty$ for all $\mathbf{x} \in D$.
Then $\int_D g_n(\mathbf{x})\,d\mathbf{x} \to \int_D g(\mathbf{x})\,d\mathbf{x}$.

The proof of this theorem is beyond the scope of this book (see Rudin [47], Folland [25], or Wheeden-Zygmund [56]), but the intuition behind it can be easily explained. If $g_n \to g$ pointwise, how can the relation $\int_D g_n \to \int_D g$ fail? Consider the following two examples, in which D is the real line:

$$f_n(x) = 1 \quad \text{for } n < x < n+1, \qquad f_n(x) = 0 \quad \text{otherwise.}$$
$$g_n(x) = n \quad \text{for } 0 < x < 1/n, \qquad g_n(x) = 0 \quad \text{otherwise.}$$

We have
$$\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} g_n(x)\,dx = 1 \quad \text{for all } n,$$

but $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) = 0$ for all x. The trouble is that as $n \to \infty$, the region under the graph of f_n moves out to infinity to the right, and the region under the graph of g_n moves out to infinity upwards, so in the limit there is nothing left. (See Figure 3.3.)

Now, the dominated convergence theorem essentially says that if this sort of bad behavior is eliminated, then the integral of the limit is the limit of the integrals. Hypothesis (a) says that the region under the graph of ϕ has finite area, and hypothesis (b) says that the graphs of $|g_n|$ are trapped inside this region, so they cannot leak out to infinity.

As a corollary, we obtain the following relation between pointwise convergence and convergence in norm.

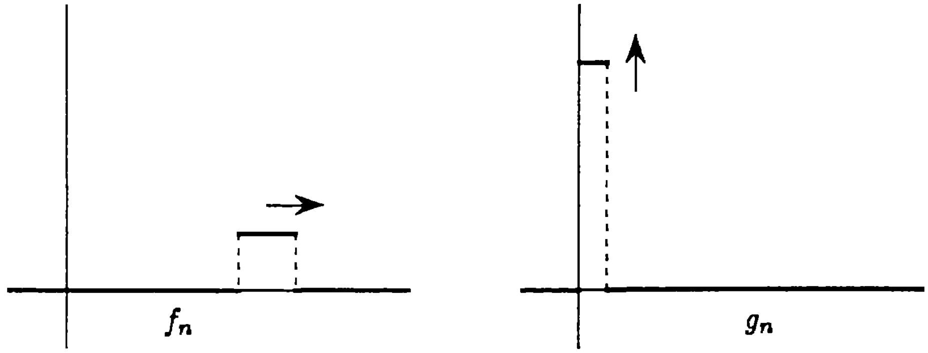


FIGURE 3.3. The examples f_n and g_n of sequences for which the integral of the limit is not the limit of the integral. The arrows indicate what happens as n increases.

Theorem 3.7. *Suppose $f_n \in L^2(D)$ for all n and $f_n \to f$ pointwise. If there exists $\psi \in L^2(D)$ such that $|f_n(\mathbf{x})| \le |\psi(\mathbf{x})|$ for all n and all $\mathbf{x} \in D$, then $f_n \to f$ in norm.*

Proof: We have $|f(\mathbf{x})| = \lim |f_n(\mathbf{x})| \le |\psi(\mathbf{x})|$, and hence

$$|f_n(\mathbf{x}) - f(\mathbf{x})|^2 \le \Big(|f_n(\mathbf{x})| + |f(\mathbf{x})|\Big)^2 \le |2\psi(\mathbf{x})|^2.$$

Therefore, we can apply the dominated convergence theorem, with $g_n = |f_n - f|^2$, $g = 0$, and $\phi = |2\psi|^2$, to conclude that

$$\|f_n - f\|^2 = \int_D |f_n(\mathbf{x}) - f(\mathbf{x})|^2 \, d\mathbf{x} \to 0. \qquad \blacksquare$$

Best approximations in L^2

If $\{\phi_n\}$ is an orthonormal basis for $L^2(D)$, where D is any interval in $\mathbf{R}$ or region in $\mathbf{R}^n$, we have $\sum\langle f, \phi_n\rangle\phi_n = f$ for all $f \in L^2(D)$. On the other hand, suppose $\{\phi_n\}$ is an orthonormal set in $L^2(D)$ that is not complete. If $f \in L^2(D)$, what significance can we attach to the series $\sum\langle f, \phi_n\rangle\phi_n$? We know that it converges by Lemma 3.2. In general its sum will not be f, but it is the unique *best approximation* to f in norm among all functions of the form $\sum c_n\phi_n$. (The latter sum converges in norm precisely when $\sum|c_n|^2 < \infty$, as the argument used to prove Lemma 3.2 shows.) We state this result as a theorem.

Theorem 3.8. *If $\{\phi_n\}$ is an orthonormal set in $L^2(D)$ and $f \in L^2(D)$, then*

$$\Big\|f - \sum\langle f, \phi_n\rangle\phi_n\Big\| \le \Big\|f - \sum c_n\phi_n\Big\|$$

for all choices of c_n with $\sum|c_n|^2 < \infty$. Equality holds only when $c_n = \langle f, \phi_n\rangle$ for all n.

Proof: We have

$$f - \sum c_n\phi_n = \Big(f - \sum\langle f, \phi_n\rangle\phi_n\Big) + \sum\Big(\langle f, \phi_n\rangle - c_n\Big)\phi_n.$$

Now, $f - \sum\langle f, \phi_n\rangle\phi_n$ is easily seen to be orthogonal to all ϕ_n; see the first part of the proof of Theorem 3.4. Hence, by the Pythagorean theorem (and a simple limiting argument, if there are infinitely many ϕ_n),

$$\Big\|f - \sum c_n\phi_n\Big\|^2 = \Big\|f - \sum\langle f, \phi_n\rangle\phi_n\Big\|^2 + \sum\Big|\langle f, \phi_n\rangle - c_n\Big|^2.$$

The last sum on the right is clearly nonnegative, and it is zero precisely when $c_n = \langle f, \phi_n\rangle$ for all n; this establishes the theorem. ∎

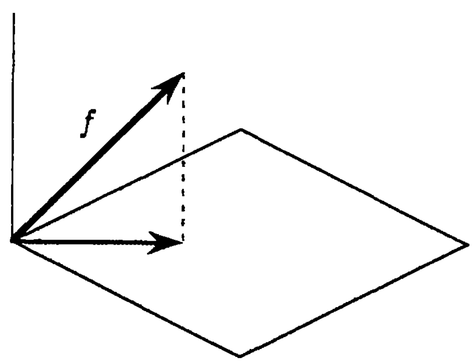


FIGURE 3.4. A vector f and its orthogonal projection onto a plane.

The pictorial intuition behind Theorem 3.8 is shown in Figure 3.4. The horizontal plane represents the space of functions (or vectors) of the form $\sum c_n\phi_n$; the sum $\sum\langle f, \phi_n\rangle\phi_n$ is the closest point to f in this plane, namely, the orthogonal projection of f onto the plane.

One situation in which Theorem 3.8 is particularly useful is when $\{\phi_n\}$ is simply a finite subset of an orthonormal basis.

Corollary 3.1. *Suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(D)$. If $f \in L^2(D)$, the partial sum $\sum_1^N\langle f, \phi_n\rangle\phi_n$ of the series $\sum_1^\infty\langle f, \phi_n\rangle\phi_n$ is the best approximation in norm to f among all linear combinations of $\phi_1, \ldots, \phi_N$.*

EXERCISES

1. Show that $\left\{e^{2\pi i(mx+ny)}\right\}_{m,n=-\infty}^{\infty}$ is an orthonormal set in $L^2(D)$ where D is any square whose sides have length one and are parallel to the coordinate axes.
2. Find constants a, b, A, B, C such that $f_0(x) = 1$, $f_1(x) = ax+b$, and $f_2(x) = Ax^2+Bx+C$ are an orthonormal set in $L^2_w(0,\infty)$ where $w(x) = e^{-x}$. (Hint: $\int_0^\infty x^n e^{-x}dx = n!$.)

3. Let D be the unit disc $\{x^2+y^2 \le 1\}$, and let $f_n(x,y) = (x+iy)^n$. Show that $\{f_n\}_0^\infty$ is an orthogonal set in $L^2(D)$, and compute $\|f_n\|$ for all n. (Hint: In polar coordinates, $x+iy = re^{i\theta}$ and $dx\,dy = r\,dr\,d\theta$.)
4. Suppose $\{\phi_n\}$ is an orthonormal set in $L^2_w(D)$. Show that $\{w^{1/2}\phi_n\}$ is an orthonormal set in $L^2(D)$ (with respect to the weight function 1).
5. Suppose $f : [a,b] \to [c,d]$ and $f'(x) > 0$ for $x \in [a,b]$. Show that if $\{\phi_n\}$ is an orthonormal basis for $L^2(c,d)$, then $\{\phi_n \circ f\}$ is an orthonormal basis for $L^2_w(a,b)$ where $w = f'$.
6. Find an example of a sequence $\{f_n\}$ in $L^2(0,\infty)$ such that $f_n \to 0$ uniformly but $f_n \not\to 0$ in norm.
7. What is the best approximation in norm to the function $f(x) = x$ on the interval $[0,\pi]$ among all functions of the form (a) $a_0 + a_1\cos x + a_2\cos 2x$, (b) $b_1 \sin x + b_2 \sin 2x$, (c) $a\cos x + b\sin x$?

3.5 Regular Sturm-Liouville problems

In §1.3 we arrived at the orthogonal bases $\{\cos nx\}_0^\infty$ and $\{\sin nx\}_1^\infty$ for $L^2(0,\pi)$ by solving the boundary value problems

$$u''(x) + \lambda^2 u(x) = 0, \qquad u'(0) = u'(\pi) = 0$$

and

$$u''(x) + \lambda^2 u(x) = 0, \qquad u(0) = u(\pi) = 0.$$

We derived the orthogonal basis $\{e^{inx}\}_{-\infty}^{\infty}$ for $L^2(-\pi,\pi)$ by considering periodic functions, but we could also have found it by solving the boundary value problem

$$u''(x) + \lambda^2 u(x) = 0, \qquad u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi).$$

In fact, there is a large class of boundary value problems on an interval $[a,b]$ that lead to orthogonal bases for $L^2(a,b)$. These problems are the subject of the present section.

First, a bit of conceptual background from finite-dimensional linear algebra. We recall that a linear transformation $T : \mathbf{C}^k \to \mathbf{C}^k$ is called *self-adjoint* or *Hermitian* if

$$\langle T\mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, T\mathbf{b}\rangle \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbf{C}^k.$$

(When T is described by a matrix (T_{ij}), this means that $T_{ji} = \overline{T_{ij}}$.) It is one of the basic results of linear algebra, known as the *spectral theorem* or the *principal axis theorem,* that whenever T is self-adjoint there is an orthonormal basis of $\mathbf{C}^k$ consisting of eigenvectors for T. What we are aiming for is an analogue of this theorem for differential operators acting on the space $L^2(a,b)$.

Suppose then that S and T are linear operators that are defined on certain subspaces $\mathscr{D}_S$ and $\mathscr{D}_T$ of $L^2(a,b)$ and map them into $L^2(a,b)$. We say that S and T are **adjoint** to each other (or that T is the adjoint of S, or vice versa) if

$$\langle S(f), g\rangle = \langle f, T(g)\rangle \quad \text{for all } f \in \mathscr{D}_S \text{ and } g \in \mathscr{D}_T.$$

S is called **self-adjoint** or **Hermitian** if

$$\langle S(f), g\rangle = \langle f, S(g)\rangle \quad \text{for all } f, g \in \mathscr{D}_S.$$

(These definitions will suffice for our purposes; in more advanced work one needs to be more careful about specifying the domains $\mathscr{D}_S$ and $\mathscr{D}_T$.)

Now suppose L is a second-order linear differential operator,

$$L(f) = rf'' + qf' + pf,$$

where r, q, and p are real functions of class $C^{(2)}$ on $[a,b]$. We shall assume that the leading coefficient r is nonvanishing on $[a,b]$, as the existence of "singular points" where $r = 0$ complicates the theory considerably. (Later we shall sometimes allow r to vanish at one or both endpoints.) For the time being, we take the domain of L to be the space of all twice continuously differentiable functions on $[a,b]$.

What is the adjoint of L? If we write out the integral defining $\langle L(f), g\rangle$, we can move the derivatives from f onto g by integration by parts, thus:

$$\int_a^b (rf'')\overline{g}\,dx = -\int_a^b f'(r\overline{g})'dx + rf'\overline{g}\Big|_a^b = \int_a^b f(r\overline{g})''dx + \Big[rf'\overline{g} - f(r\overline{g})'\Big]_a^b,$$

$$\int_a^b (qf')\overline{g}\,dx = -\int_a^b f(q\overline{g})'dx + qf\overline{g}\Big|_a^b.$$

We therefore have

$$\begin{aligned}\langle L(f), g\rangle &= \int_a^b (rf'' + qf' + pf)\overline{g}\,dx \\ &= \int_a^b f\Big[(r\overline{g})'' - (q\overline{g})' + p\overline{g}\Big]dx + \Big[rf'\overline{g} - f(r\overline{g})' + qf\overline{g}\Big]_a^b \qquad (3.24) \\ &= \langle f, L^*(g)\rangle + \Big[r(f'\overline{g} - f\overline{g}') + (q - r')f\overline{g}\Big]_a^b,\end{aligned}$$

where L^* is the **formal adjoint** of L defined by

$$L^*(g) = (rg)'' - (qg)' + pg = rg'' + (2r' - q)g' + (r'' - q' + p)g. \qquad (3.25)$$

(Here we have used the assumption that r, q, and p are real.) We say that L is **formally self-adjoint** if $L^* = L$. On comparing the coefficients of L^* with L, we see that this happens precisely when $2r' - q = q$ and $r'' - q' = 0$, that is, when $q = r'$. In this case, L has the form

$$L(f) = rf'' + r'f' + pf = (rf')' + pf, \qquad (3.26)$$

and moreover, the second boundary term at the end of (3.24) vanishes. We have therefore proved the following.

Lagrange's Indentity. *If L is formally self-adjoint,*

$$\langle L(f), g\rangle = \langle f, L(g)\rangle + \Big[r(f'\overline{g} - f\overline{g}')\Big]_a^b. \tag{3.27}$$

Evidently the discrepancy between formal and actual self-adjointness lies in the endpoint terms in (3.27). They can be eliminated by restricting L to a smaller domain, consisting of functions that satisfy suitable boundary conditions. More precisely, for a second-order operator L it is usually appropriate to impose two independent boundary conditions of the form

$$\begin{aligned} B_1(f) &= \alpha_1 f(a) + \alpha_1' f'(a) + \beta_1 f(b) + \beta_1' f'(b) = 0,\\ B_2(f) &= \alpha_2 f(a) + \alpha_2' f'(a) + \beta_2 f(b) + \beta_2' f'(b) = 0, \end{aligned} \tag{3.28}$$

where the α's and β's are constants. We say that the boundary conditions (3.28) are **self-adjoint** (relative to the operator L) if

$$\Big[r(f'\overline{g} - f\overline{g}')\Big]_a^b = 0 \quad \text{for all } f, g \text{ satisfying (3.28)}.$$

Almost all the boundary conditions that arise in practice are of the form

$$\begin{gathered} \alpha f(a) + \alpha' f'(a) = 0, \qquad \beta f(b) + \beta' f'(b) = 0\\ \Big(\alpha, \alpha', \beta, \beta' \in \mathbf{R}; \quad (\alpha, \alpha') \neq (0,0); \quad (\beta, \beta') \neq (0,0)\Big). \end{gathered} \tag{3.29}$$

Boundary conditions of the form (3.29) are called **separated**, since each one involves a condition at only one endpoint. Separated boundary conditions are always self-adjoint (relative to any operator L). In fact, if f and g both satisfy the boundary condition at a,

$$\alpha f(a) + \alpha' f'(a) = 0, \qquad \alpha g(a) + \alpha' g'(a) = 0, \tag{3.30}$$

then the expression $r(f'\overline{g} - f\overline{g}')$ vanishes at $x = a$; likewise at b. This is obvious when $\alpha' = 0$, in which case (3.30) becomes $f(a) = g(a) = 0$; on the other hand, if $\alpha' \neq 0$, we can rewrite (3.30) as

$$f'(a) = cf(a), \qquad g'(a) = cg(a) \qquad (c = -\alpha/\alpha'),$$

so that

$$r(a)[f'(a)\overline{g(a)} - f(a)\overline{g'(a)}] = cr(a)[f(a)\overline{g(a)} - f(a)\overline{g(a)}] = 0.$$

There is also one set of nonseparated boundary conditions that is commonly used, namely, the **periodic** boundary conditions

$$f(a) = f(b), \qquad f'(a) = f'(b). \tag{3.31}$$

These are self-adjoint relative to L provided that $r(a) = r(b)$, for then the endpoint evaluations at a and b in (3.27) cancel each other out.

Now we are ready to formulate the boundary value problems that lead to orthogonal bases for $L^2(a, b)$.

Definition. A **regular Sturm-Liouville problem** on the interval $[a, b]$ is specified by the following data:

(i) a formally self-adjoint differential operator L defined by $L(f) = (rf')' + pf$, where r, r', and p are real and continuous on $[a, b]$ and $r > 0$ on $[a, b]$;

(ii) a set of self-adjoint boundary conditions, $B_1(f) = 0$ and $B_2(f) = 0$, for the operator L;

(iii) a positive, continuous function w on $[a, b]$.

The object is *to find all solutions f of the boundary value problem*

$$\begin{aligned} &L(f) + \lambda w f = 0, \quad \text{i.e., } \left[r(x)f'(x)\right]' + p(x)f(x) + \lambda w(x)f(x) = 0, \\ &B_1(f) = B_2(f) = 0, \end{aligned} \tag{3.32}$$

where λ is an arbitrary constant.

(A comment on condition (i): We have assumed from the outset that r does not vanish on $[a, b]$, so either $r > 0$ or $r < 0$. If $r < 0$, we simply replace r, p, and λ by $-r$, $-p$, and $-\lambda$, which leaves (3.32) unchanged.)

For most values of λ, the only solution of (3.32) is the trivial one, $f(x) \equiv 0$. If (3.32) has nontrivial solutions, λ is called an **eigenvalue** for the Sturm-Liouville problem, and the corresponding nontrivial solutions are called **eigenfunctions**. (This usage of the term *eigenvalue* is somewhat specialized. λ is an eigenvalue in the usual sense of the word, not of the operator L but rather of the operator M defined by $M(f) = -w^{-1}L(f)$.) If f and g satisfy (3.32), then so does any linear combination $c_1 f + c_2 g$ (this is just the superposition principle at work), so the set of all eigenfunctions for a given eigenvalue λ, together with the zero function, is a linear space called the **eigenspace** for λ.

We summarize the elementary properties of eigenvalues and eigenfunctions in the following theorem, which displays the importance of eigenfunctions from the point of view of orthogonal sets. We recall that if $w > 0$ is a weight function on $[a, b]$, the weighted inner product $\langle f, g\rangle_w$ is given by

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx = \langle wf, g\rangle = \langle f, wg\rangle. \tag{3.33}$$

Theorem 3.9. *Let a regular Sturm-Liouville problem (3.32) be given.*

(a) All eigenvalues are real.

(b) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function w; that is, if f and g are eigenfunctions with eigenvalues λ and μ, $\lambda \neq \mu$, then

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx = 0.$$

(c) The eigenspace for any eigenvalue λ is at most 2-dimensional. If the boundary conditions are separated, it is always 1-dimensional.

Proof: (a) If λ is an eigenvalue, with eigenfunction f, then

$$\lambda\|f\|_w^2 = \langle \lambda w f, f\rangle = -\langle L(f), f\rangle = -\langle f, L(f)\rangle = \langle f, \lambda w f\rangle = \overline{\lambda}\langle f, wf\rangle = \overline{\lambda}\|f\|_w^2.$$

Here we have used (3.27) and (3.33) and the fact that f satisfies self-adjoint boundary conditions. Since $\|f\|_w^2 > 0$, we conclude that $\overline{\lambda} = \lambda$, that is, λ is real.

(b) Suppose $L(f) + \lambda w f = 0$ and $L(g) + \mu w g = 0$, where f and g are nonzero. We have just shown that λ and μ must be real, and by the same sort of argument,

$$\lambda\langle f, g\rangle_w = \langle \lambda w f, g\rangle = -\langle L(f), g\rangle = -\langle f, L(g)\rangle = \langle f, \mu w g\rangle = \mu\langle f, g\rangle_w.$$

Thus, if $\lambda \neq \mu$ we must have $\langle f, g\rangle_w = 0$.

(c) The fundamental existence theorem for ordinary differential equations (see Appendix 5) says that for any constants c_1 and c_2 there is a unique solution of $L(f) + \lambda w f = 0$ satisfying the initial conditions $f(a) = c_1$, $f'(a) = c_2$. That is, a solution is specified by two arbitrary constants c_1 and c_2, so the space of *all* solutions of $L(f) + \lambda w f = 0$ is 2-dimensional. Hence the space of solutions satisfying the given boundary conditions is *at most* 2-dimensional. Moreover, if the boundary conditions are separated, one of them has the form $\alpha f(a) + \alpha' f'(a) = 0$. This imposes the linear relation $\alpha c_1 + \alpha' c_2 = 0$ on the constants c_1 and c_2 and hence reduces the dimension of the solution space to one. (Of course the other boundary condition will usually reduce the dimension to zero; this is why there are nontrivial solutions only for certain special values of λ.) ∎

At this point it is not evident that a given Sturm-Liouville problem has any eigenfunctions at all. But, in fact, there are as many as anyone could wish for.

Theorem 3.10. *For every regular Sturm-Liouville problem*

$$(rf')' + pf + \lambda w f = 0, \qquad B_1(f) = B_2(f) = 0$$

on $[a, b]$, there is an an orthonormal basis $\{\phi_n\}_1^\infty$ of $L_w^2(a, b)$ consisting of eigenfunctions. If λ_n is the eigenvalue for ϕ_n, then $\lim_{n\to\infty} \lambda_n = +\infty$. Moreover, if f is of class $C^{(2)}$ on $[a, b]$ and satisfies the boundary conditions $B_1(f) = B_2(f) = 0$, then the series $\sum\langle f, \phi_n\rangle\phi_n$ converges uniformly to f.

In more detail, the content of Theorem 3.10 is as follows. By Theorem 3.9(c), for each eigenvalue λ there are either one or two independent eigenfunctions. In the latter case we can choose the two eigenfunctions to be orthogonal to each other with respect to the weight w. (If $\langle f_1, f_2\rangle_w \neq 0$, we can replace f_2 by $\widetilde{f}_2 = f_2 - cf_1$ where c is chosen to make $\langle f_1, \widetilde{f}_2\rangle = 0$.) If we put all these eigenfunctions together, by Theorem 3.9(b) we obtain an orthogonal set; and Theorem 3.10 says that this set is actually a basis. This implies, in particular, that the set of eigenvalues is countably infinite.

We shall take Theorem 3.10 on faith for the present, but we shall prove it in the case of separated boundary conditions in §10.3. A proof of the general case, as well as its generalization to higher-order differential equations, can be found in Naimark [40], Chapter II.

Example. Consider the problem

$$f'' + \lambda f = 0, \qquad f'(0) = \alpha f(0), \quad f'(l) = \beta f(l). \tag{3.34}$$

First let us dispose of the case $\lambda = 0$. The general solution of $f'' = 0$ is $f(x) = c_1 + c_2 x$. The boundary condition at 0 says that $c_2 = \alpha c_1$, and the boundary condition at l says that $c_2 = \beta(c_1 + c_2 l)$. The only solution of this pair of equations is $c_1 = c_2 = 0$ unless $\beta = \alpha/(1 + l\alpha)$, in which case we may take $c_1 = 1$ and $c_2 = \alpha$.

Now for $\lambda \neq 0$, let us set $\lambda = \nu^2$, where ν is positive real or positive imaginary according as $\lambda > 0$ or $\lambda < 0$. (By Theorem 3.9(a), we need only consider real λ.) The general solution of the differential equation $f'' + \lambda f = 0$ is

$$f(x) = c_1 \cos \nu x + c_2 \sin \nu x \qquad (\lambda = \nu^2).$$

Since $f(0) = c_1$ and $f'(0) = \nu c_2$, the boundary condition at 0 says that $c_2 = (\alpha/\nu)c_1$. Since a constant multiple of a solution is a solution, we may choose $c_1 = \nu$, $c_2 = \alpha$, so that

$$f(x) = \nu \cos \nu x + \alpha \sin \nu x. \tag{3.35}$$

Now the boundary condition at l says that

$$-\nu^2 \sin \nu l + \alpha\nu \cos \nu l = \beta(\nu \cos \nu l + \alpha \sin \nu l),$$

or

$$(\alpha - \beta)\nu \cos \nu l = (\alpha\beta + \nu^2) \sin \nu l,$$

or finally

$$\tan \nu l = \frac{(\alpha - \beta)\nu}{\alpha\beta + \nu^2}. \tag{3.36}$$

For the case of imaginary ν (i.e., $\lambda < 0$) we set $\nu = i\mu$ and use the fact that $\tan ix = i \tanh x$ to rewrite (3.36) as

$$\tanh \mu l = \frac{(\alpha - \beta)\mu}{\alpha\beta - \mu^2}. \tag{3.37}$$

In both cases we need only consider positive values of ν and μ, since the actual eigenvalue is ν^2 or $-\mu^2$.

If ν satisfies (3.36), then the function f defined by (3.35) is an eigenfunction for the problem (3.34). In general it is not normalized, but finding the normalization is a simple matter of calculus, and the equation (3.36) can often be used to simplify the result. As an illustration, let us work out the case $\beta = -\alpha$. (Other cases are considered in Exercises 5 and 6.) If f is given by (3.35), then

$$\begin{aligned}
\|f\|^2 &= \int_0^l (\nu^2 \cos^2 \nu x + 2\alpha\nu \sin \nu x \cos \nu x + \alpha^2 \sin^2 \nu x)\, dx \\
&= \Big[\tfrac{1}{2}\nu^2(x + \nu^{-1} \cos \nu x \sin \nu x) + \alpha \sin^2 \nu x + \tfrac{1}{2}\alpha^2(x - \nu^{-1} \cos \nu x \sin \nu x)\Big]_0^l \\
&= \tfrac{1}{2}(\nu^2 + \alpha^2)l + \frac{(\nu^2 - \alpha^2)}{2\nu} \cos \nu l \sin \nu l + \alpha \sin^2 \nu l.
\end{aligned}$$

But if $\beta = -\alpha$, (3.36) gives

$$\frac{(\nu^2 - \alpha^2)}{2\nu} = \frac{\alpha}{\tan \nu l} = \frac{\alpha \cos \nu l}{\sin \nu l},$$

so

$$\|f\|^2 = \tfrac{1}{2}(\nu^2 + \alpha^2)l + \alpha(\cos^2 \nu l + \sin^2 \nu l) = \tfrac{1}{2}(\nu^2 + \alpha^2)l + \alpha. \tag{3.38}$$

There is no way to describe the values of ν and μ that solve the transcendental equations (3.36) and (3.37) in closed form (except when $\alpha = \beta$), but it is easy to find them graphically. Namely, they are the values at which the curves $y = \tan \nu l$ and $y = (\alpha - \beta)\nu/(\alpha\beta + \nu^2)$ in the νy-plane, or $y = \tanh \mu l$ and $y = (\alpha - \beta)\mu/(\alpha\beta - \mu^2)$ in the μy-plane, intersect. The relative configuration of these curves depends on α and β; we shall display a couple of representative cases here and let the reader work out some others as exercises.

Case I. $\alpha = 1$, $\beta = -1$, $l = \pi$. Here the situation is as depicted in Figure 3.5. There is an infinite sequence of positive solutions to (3.36), say $\nu_1 < \nu_2 < \cdots$, and ν_n is approximately $n - 1$ when n is large. There are no positive solutions to (3.37). Hence, there is an infinite sequence of positive eigenvalues $\lambda_n = \nu_n^2$ for (3.34), with $\lambda_n \approx (n-1)^2$ for n large, and no negative eigenvalues. (Zero is not an eigenvalue since $-1 \neq 1/(1+\pi)$.) The (unnormalized) eigenfunctions are given by (3.35):

$$f_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x.$$

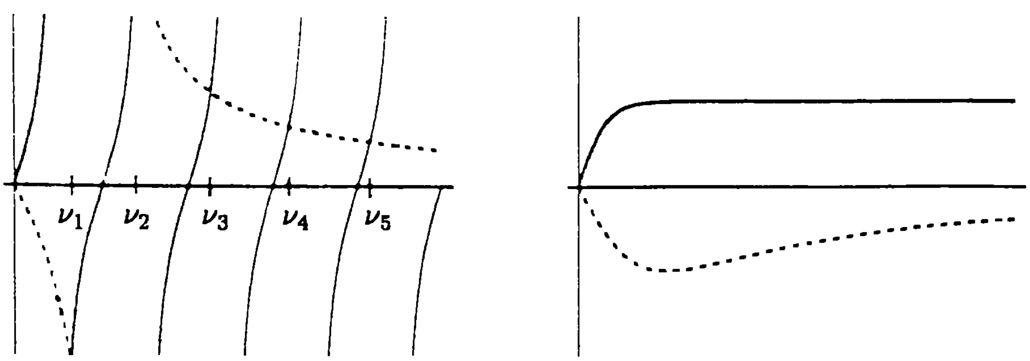


FIGURE 3.5. Left: the graphs of $\tan \pi\nu$ (solid) and $2\nu/(\nu^2 - 1)$ (dashed); the numbers ν_n are the values of ν at which the graphs intersect. Right: the graphs of $\tanh \pi\mu$ (solid) and $-2\mu/(\mu^2 + 1)$ (dashed).

Case II. $\alpha = 1$, $\beta = 4$, $l = \pi$. Here the situation is as depicted in Figure 3.6. Again there is an infinite sequence $\{\nu_n\}_1^\infty$ of positive solutions to (3.36), this time with $\nu_n \approx n$ for large n; and zero is not an eigenvalue of (3.34) since $4 \neq 1/(1+\pi)$. But now there is also one positive solution μ_0 to (3.37). Hence, there is an infinite sequence of positive eigenvalues $\lambda_n = \nu_n^2$ for (3.34) and one negative eigenvalue $\lambda_0 = -\mu_0^2$. The (unnormalized) eigenfunction for $\lambda_n = \nu_n^2$ is

$$f_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x,$$

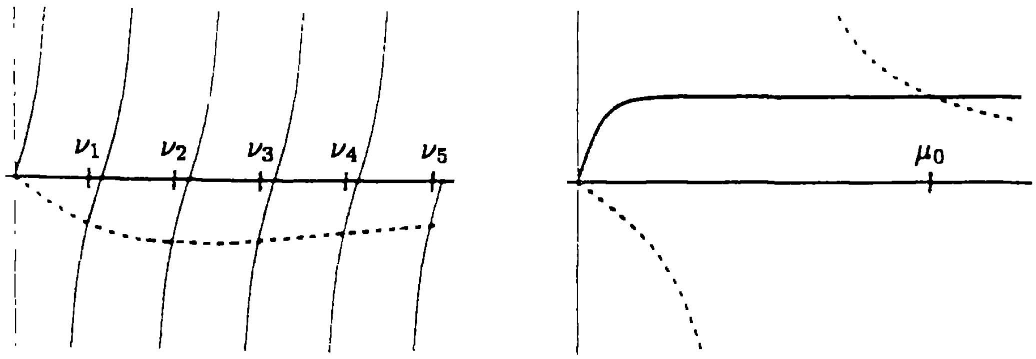


FIGURE 3.6. Left: the graphs of $\tan \pi\nu$ (solid) and $-3\nu/(\nu^2+4)$ (dashed). Right: the graphs of $\tanh \pi\mu$ (solid) and $3\mu/(\mu^2-4)$ (dashed). The numbers ν_n and μ_0 are the values of ν and μ at which the graphs intersect.

and the eigenfunction for $\lambda_0 = -\mu_0^2$ is

$$f_0(x) = \mu_0 \cosh \mu_0 x + \sinh \mu_0 x.$$

EXERCISES

1. Under what condition on the constants c and c' are the boundary conditions $f(b) = cf(a)$ and $f'(b) = c'f'(a)$ self-adjoint for the operator $L(f) = (rf')' + pf$ on $[a,b]$? (Assume as usual that r and p are real.)
2. Show that the problem (3.34) has no negative eigenvalues if $\alpha > 0 > \beta$ and exactly one negative eigenvalue if $\beta > \alpha > 0$ or $0 > \beta > \alpha$.
3. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f(0) = 0$, $f'(l) = 0$ on $[0,l]$. (Cf. Exercise 6, §3.3.)
4. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f'(0) = 0$, $f(l) = 0$ on $[0,l]$. (Cf. Exercise 7, §3.3.)
5. Find the normalized eigenfunctions for the problem (3.34) in the case $\alpha = 0$. (The answer is a bit different in the cases $\beta > 0$, $\beta = 0$, and $\beta < 0$.)
6. Find the normalized eigenfunctions for the problem (3.34) in the case $\beta = 0$. (Hint: The change of variable $x \to l - x$ essentially reduces this to Exercise 5.)
7. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f(0) = 0$, $f'(1) = -f(1)$.
8. The Sturm-Liouville theory can be generalized to higher-order equations. As an example, consider the operator $L(f) = f^{(4)}$ on the interval $[0,l]$.
   a. Prove the analogue of Lagrange's identity for L:
   $$\int_0^l \left[f^{(4)}(x)\overline{g}(x) - f(x)\overline{g}^{(4)}(x)\right]dx = \left[f'''\overline{g} - f\overline{g}''' + f'\overline{g}'' - f''\overline{g}'\right]_0^l. \quad (*)$$
   b. For the fourth-order equation $L(f) - \lambda f = 0$ one needs four boundary conditions involving f, f', f'', and f'''. Such a set of boundary

conditions is called self-adjoint for L if the right side of $(*)$ vanishes whenever f and g both satisfy the conditions. Show that one obtains a self-adjoint set of boundary conditions by imposing any of the following pairs of conditions at $x = 0$ and any one of them at $x = l$:

$$f = f' = 0, \qquad f = f'' = 0, \qquad f' = f''' = 0, \qquad f'' = f''' = 0.$$

c. Show that the eigenvalues for the equation $L(f) - \lambda f = 0$, subject to any self-adjoint set of boundary conditions, are all real, and that eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(0, l)$.

d. One can show that the analogue of Theorem 3.10 holds here, i.e., there is an orthonormal basis of eigenfunctions. For example, consider the boundary conditions $f(0) = f''(0) = 0$, $f(l) = f''(l) = 0$. Show that $f_n(x) = \sin(n\pi x/l)$ is an eigenfunction. What is its eigenvalue? Why can you guarantee immediately that there are no other independent eigenfunctions?

9. Suppose p, q, and r are real functions of class $C^{(2)}$ and that $r > 0$. The differential equation $rf'' + qf' + pf + \lambda f = 0$ can be written in the form $L(f) + \lambda w f = 0$ where w is an arbitrary positive function and $L(f) = wrf'' + wqf' + wpf$. Show that w can be always be chosen so that L is formally self-adjoint.

The following two problems use the fact that the general solution of the Euler equation

$$x^2 f''(x) + axf'(x) + bf(x) = 0 \qquad (x > 0)$$

is $c_1 x^{r_1} + c_2 x^{r_2}$ where r_1 and r_2 are the zeros of the polynomial $r(r-1) + ar + b$. (If the two zeros coincide, the general solution is $c_1 x^{r_1} + c_2 x^{r_1} \log x$.) In case r_1 and r_2 are complex, it is useful to recall that $x^{is} = e^{is \log x}$.

10. Find the eigenvalues and normalized eigenfunctions for the problem

$$(xf')' + \lambda x^{-1} f = 0, \qquad f(1) = f(b) = 0 \qquad (b > 1).$$

Expand the function $g(x) = 1$ in terms of these eigenfunctions. (Hint: in computing integrals, make the substitution $y = \log x$. Orthonormality here is with respect to the weight $w(x) = x^{-1}$.)

11. Find the eigenvalues and normalized eigenfunctions for the problem

$$(x^2 f')' + \lambda f = 0, \qquad f(1) = f(b) = 0 \qquad (b > 1).$$

12. Consider the Sturm-Liouville problem

$$(rf')' + pf + \lambda f = 0, \qquad f(a) = f(b) = 0. \qquad (**)$$

a. Show that if f satisfies $(**)$, then

$$\lambda \int_a^b |f|^2\, dx = \int_a^b r|f'|^2\, dx - \int_a^b p|f|^2\, dx.$$

(Hint: Use the fact that $\lambda f = -(rf')' - pf$ and integrate by parts.)

b. Deduce that if $p(x) \leq C$ for all x, then all the eigenvalues λ of (**) satisfy $\lambda \geq -C$.

c. Show that the conclusion of (b) still holds if the boundary conditions $f(a) = f(b) = 0$ are replaced by $f'(a) - \alpha f(a) = f'(b) - \beta f(b) = 0$ where $\alpha \leq 0$ and $\beta \geq 0$. (Hint: The analogue of part (a) in this situation is

$$\lambda \int_a^b |f|^2\,dx = \int_a^b r|f'|^2\,dx - \int_a^b p|f|^2\,dx + \beta r(b)|f(b)|^2 - \alpha r(a)|f(a)|^2.)$$

3.6 Singular Sturm-Liouville problems

In §3.5 we considered the differential equation

$$rf'' + r'f' + pf + \lambda wf = 0 \tag{3.39}$$

on a closed, bounded interval $[a, b]$, in which r, r', p, and w were assumed continuous on $[a, b]$ and r and w were assumed strictly positive on $[a, b]$. However, it often turns out in practice that one or more of these assumptions must be weakened, leading to the so-called **singular Sturm-Liouville problems.** Specifically, we allow the following modifications of the basic setup:

(i) The leading coefficient r may vanish at one or both endpoints of $[a, b]$. In addition, the weight w may vanish or tend to infinity at one or both endpoints, and the function $|p|$ may tend to infinity at one or both endpoints.

(ii) The interval $[a, b]$ may be unbounded, that is, $a = -\infty$ and/or $b = \infty$.

There is an extensive theory of these more general boundary value problems, but it is beyond the scope of this book. (Complete treatments can be found in Dunford-Schwartz [18] and Naimark [40]; see also Titchmarsh [52].) We shall merely sketch a few of the main features here, and we shall discuss specific examples in Chapters 5 and 6 and Sections 7.4 and 10.4.

The first problem is to decide what sort of boundary conditions to impose. Since we wish to use the machinery of inner products and orthogonality, we wish to use only solutions of (3.39) that are square-integrable. Now, in the regular case, all solutions of (3.39) are continuous on $[a, b]$ and hence belong to $L^2_w(a, b)$. However, under condition (i), the solutions to (3.39) may fail to be square-integrable because they blow up at one or both endpoints; whereas under condition (ii), solutions may fail to be square-integrable because they do not decay at infinity. Thus, we distinguish two cases concerning the behavior of solutions at each endpoint; to be definite, we consider the endpoint a.

Case I. All solutions of (3.39) belong to $L^2_w(a, c)$ for $a < c < b$. (It turns out that if this condition is satisfied for one value of λ, then it is satisfied for all values of λ.) In this case, we impose a boundary condition at a. In some cases it may be of the form $\alpha f(a) + \alpha' f'(a) = 0$, as before, but it may also be a condition on the limiting behavior of f and f' at a — for example, the condition that $f(x)$ should remain bounded as $x \to a$.

Case II. Not all solutions of (3.39) belong to $L^2_w(a, c)$. In this case we impose no boundary condition at a beyond the one that automatically comes with the problem, namely, that the solution should belong to $L^2_w(a, b)$.

In any event, we require the boundary conditions to be self-adjoint, i.e., if f and g satisfy the boundary conditions then the boundary term in Lagrange's identity should vanish. Precisely, since f and g may have singularities at a and b, or a and/or b may be infinite, this requirement should be formulated as

$$\lim_{\delta,\epsilon\to 0}\Big[r(f'\overline{g} - f\overline{g}')\Big]_{a+\delta}^{b-\epsilon} = 0. \tag{3.40}$$

(3.40) implies that

$$\langle L(f), g\rangle = \langle f, L(g)\rangle \quad \text{where} \quad L(f) = (rf')' + pf,$$

for any smooth functions f and g that satisfy the boundary conditions, and once this equation is established, the proof of Theorem 3.9 goes through without change. Therefore, the eigenvalues are all real and the eigenfunctions with distinct eigenvalues are orthogonal to each other.

However, the situation with Theorem 3.10 is different: in general, *there is no guarantee that there will be enough eigenfunctions to make an orthonormal basis.* Sometimes there are, sometimes there aren't. In the latter case, it is still possible to expand arbitrary functions in $L^2_w(a, b)$ in terms of solutions of the differential equation (3.39) that satisfy the given boundary conditions, but the expansion will involve an integral rather than (or in addition to) an infinite series.

For example, consider the differential equation

$$f'' + \lambda f = 0 \quad \text{on} \quad (-\infty, \infty).$$

The general solution is

$$c_1 \cos \nu x + c_2 \sin \nu x \quad \text{or} \quad c_1 e^{i\nu x} + c_2 e^{-i\nu x} \quad (\lambda = \nu^2).$$

None of these functions, for any value of λ, belongs to $L^2(-\infty, \infty)$, except for the trivial case $c_1 = c_2 = 0$. However, any $f \in L^2(-\infty, \infty)$ can be written as a "continuous superposition" (i.e., integral) of the functions $e^{i\nu x}$ as ν ranges over all real numbers, by means of the Fourier transform. This is the subject of Chapter 7.

CHAPTER 4
SOME BOUNDARY VALUE PROBLEMS

This chapter is devoted to the solution of various boundary value problems by the techniques we have developed so far, namely,

(i) separation of variables,
(ii) the superposition principle, and
(iii) expansion of functions in series of eigenfunctions.

This subject was begun in §2.5. All the major ideas we need are already in place, and it is just a question of learning how to combine them efficiently and developing a feeling for the connection between the mathematics and the physics. In the first section we discuss a few useful general techniques; the remainder of the chapter is largely devoted to working out a variety of examples.

Our methods generally lead to solutions in the form of infinite series. In this chapter we shall not worry much about technical questions of convergence, termwise differentiation, and such things. In some cases, one can verify that the series converge in a sufficiently strong sense to justify all the formal manipulations according to the principles of classical analysis; even when this is not the case, one can usually establish the validity of the solution by interpreting things properly — for example, by abandoning pointwise convergence in favor of norm convergence or the notion of weak convergence that we shall develop in Chapter 9. These issues were discussed in some detail in §2.5 for the boundary value problems solved there, and similar remarks apply to the problems considered in this chapter. At any rate, our concern here is with finding the solutions rather than with a rigorous justification of the calculations.

We shall also not worry about questions of uniqueness. That is, our methods will produce *one* solution to the boundary value problem, and we shall not try to prove rigorously that it is the *only* solution. In general, a problem that is properly posed from a physical point of view will indeed have a unique solution; or at least any non-uniqueness will be easily visible in the physical setup. (See John [33] or Folland [24] for further discussion of these matters.)

We shall point out here and there how Sturm-Liouville problems of a rather general sort turn up in applications. However, when we perform specific calculations, we must limit ourselves to the differential equations that we can solve explicitly — and at this point, this means mainly the equation $f'' + \lambda f = 0$ or its close relative $x^2 f'' + 2xf' + \lambda f = 0$ (discussed in §4.3). We shall solve some problems involving more complicated equations in Chapters 5 and 6.

4.1 Some useful techniques

We begin this chapter by discussing the sort of problems we shall be considering and assembling a bag of tricks for them. To put the discussion on a concrete level, let us think of the boundary value problems for the heat and wave equations that we solved in §2.5. In these problems we were solving a homogeneous linear differential equation $L(u) = 0$ (either the heat or the wave equation) for a function $u(x, t)$ on the region $a < x < b$, $t > 0$. We imposed some homogeneous linear boundary conditions on u at $x = a$ and $x = b$, and some linear initial conditions on u at $t = 0$. Let us write the boundary conditions as $B(u) = 0$ and the initial conditions as $I(u) = h(x)$, with the understanding that each of these single equations may stand for several equations grouped together. (For example, for the vibrating string problem, "$B(u) = 0$" stands for "$u(0, t) = 0$ and $u(l, t) = 0$," and "$I(u) = h(x)$" stands for "$u(x, 0) = h_1(x)$ and $u_t(x, 0) = h_2(x)$.") Thus the boundary value problem has the form

$$L(u) = 0, \qquad B(u) = 0, \qquad I(u) = h(x). \tag{4.1}$$

The technique for solving (4.1) was to use separation of variables to produce an infinite family of functions $u(x, t) = \sum c_n \phi_n(t) \psi_n(x)$ that satisfy $L(u) = 0$ and $B(u) = 0$, and then to choose the constants c_n appropriately to obtain $I(u) = h(x)$.

In the examples we considered in §2.5, the boundary conditions were such as to lead to Fourier sine or cosine series. In this chapter we shall consider other homogeneous boundary conditions. These will yield other Sturm-Liouville problems and hence lead to infinite series involving the eigenfunctions for these problems. The particular eigenfunctions will differ from problem to problem, but the method of solution is the same in all cases.

We shall also generalize (4.1) by considering inhomogeneous equations and inhomogeneous boundary conditions:

$$L(u) = F(x, t), \qquad B(u) = g(t), \qquad I(u) = h(x). \tag{4.2}$$

There are several techniques for reducing such problems to more manageable ones. We now discuss these techniques on the general level; specific examples will be found in subsequent sections. The reader may find it helpful to read this material in conjunction with the examples, rather than trying to absorb it completely before reading further.

Technique 1: Use the superposition principle to deal with inhomogeneous terms one at a time.

In problem (4.2) there are three inhomogeneous terms: F, g, and h. Suppose we can solve the three problems obtained by replacing all but one of these functions by zero:

$$L(u) = 0, \qquad B(u) = 0, \qquad I(u) = h(x), \tag{4.3}$$
$$L(u) = 0, \qquad B(u) = g(t), \qquad I(u) = 0, \tag{4.4}$$
$$L(u) = F(x,t), \qquad B(u) = 0, \qquad I(u) = 0. \tag{4.5}$$

If u_1, u_2, and u_3 are the solutions to (4.3), (4.4), and (4.5), respectively, then $u = u_1 + u_2 + u_3$ will be the solution of (4.2). In particular, (4.3) is just (4.1), which we already know how to deal with, so it suffices to solve (4.4) and (4.5).

We remark that this method can sometimes be used to break down the problem still further. For example, if we are working on the interval $a < x < b$, the boundary condition $B(u) = g(t)$ generally stands for two conditions, one at $x = a$ and one at $x = b$, say $B_a(u) = g_a(t)$ and $B_b(u) = g_b(t)$. If we can solve the (probably simpler) problems obtained by replacing one or the other of the functions g_a and g_b by zero, we can solve the original problem by adding the solutions to the two simpler problems.

Let us now turn to the inhomogeneous differential equation $L(u) = F(x,t)$. Suppose the homogeneous equation $L(u) = 0$ with homogeneous boundary conditions $B(u) = 0$ can be handled by separation of variables, leading to solutions $u(x,t) = \sum c_n\phi_n(x)\psi_n(t)$ where the ϕ_n's are the eigenfunctions for a Sturm-Liouville problem. Then the same sort of eigenfunction expansion can be used to produce solutions of the inhomogeneous equation $L(u) = F(x,t)$ subject to the same boundary conditions $B(u) = 0$. Namely, for each t we expand the function $F(x,t)$ in terms of the eigenfunctions $\phi_n(x)$,

$$F(x,t) = \sum c_n(t)\phi_n(x),$$

and we try to find a solution u in the form

$$u(x,t) = \sum \omega_n(t)\phi_n(x),$$

where the functions $\omega_n(t)$ are to be determined. If we plug these series into the differential equation $L(u) = F$, the result will be a sequence of *ordinary* differential equations for the unknown functions $\omega_n(t)$ in terms of the known functions $c_n(t)$. These equations can be solved subject to whatever initial conditions at $t = 0$ one may require. The resulting function $u(x,t)$ then satisfies the differential equation $L(u) = F$ and the desired initial conditions; it satisfies the boundary conditions $B(u) = 0$ because they are built into the eigenfunctions ϕ_n. In short, we have:

Technique 2: The Sturm-Liouville expansions used to solve $L(u) = 0$ *with homogeneous boundary conditions* $B(u) = 0$ *can also be used to solve the inhomogeneous equation* $L(u) = F(x,t)$ *with the same boundary conditions.*

Another useful device is available for solving (4.2) when the inhomogeneous terms F and g are independent of t:

$$L(u) = F(x), \qquad B(u) = c, \qquad I(u) = h(x). \tag{4.6}$$

(We have written c instead of g to remind ourselves that it is constant.) In this case, the differential equation $L(u) = F$ with boundary conditions $B(u) = c$ may admit **steady-state solutions**, that is, solutions that are independent of t. The superposition principle (Technique 1) can be used to break (4.6) down into the problem of finding a steady-state solution and solving the homogeneous equation with given initial conditions: If $u_0(x)$ and $v(x,t)$ satisfy

$$L(u_0) = F(x), \qquad B(u_0) = c, \tag{4.7}$$

$$L(v) = 0, \qquad B(v) = 0, \qquad I(v) = h(x) - u_0(x), \tag{4.8}$$

then $u(x,t) = u_0(x) + v(x,t)$ satisfies (4.6). (4.7) is relatively easy to solve because it is only an *ordinary* differential equation for u_0, and (4.8) is just (4.1) again (with different initial conditions). To summarize:

Technique 3: To solve an inhomogeneous problem with time-independent data, reduce to the homogeneous case by finding a steady-state solution.

Technique 3 is not infallible. Sometimes there is no steady-state solution; that is, the boundary conditions $B(u_0) = c$ are incompatible with the differential equation $L(u_0) = F(x)$ when u_0 is independent of t. (When this happens, there is usually a good physical reason for it.) We also observe that in the case of homogeneous boundary conditions $B(u) = 0$, Techniques 2 and 3 can both be used to solve the equation $L(u) = F(x)$. The solutions may differ in appearance (the first one involves a series expansion for F), but they are actually the same.

There remains the question of solving problems with inhomogeneous boundary conditions $B(u) = g(t)$ that are time-dependent. Often the most efficient tool for handling such problems is the Laplace transform; see §8.4. However, it is worth noting that the superposition principle can be used to trade off inhomogeneous boundary conditions for inhomogeneous equations. Namely, suppose we wish to solve

$$L(u) = 0, \qquad B(u) = g(t), \qquad I(u) = 0. \tag{4.9}$$

Let $w(x,t)$ be any smooth function that satisfies the boundary conditions $B(w) = g(t)$ and the initial conditions $I(w) = 0$; such functions are relatively easy to construct because no differential equation needs to be solved. But then u satisfies (4.9) if and only if $v = u - w$ satisfies

$$L(v) = F(x,t), \qquad B(v) = 0, \qquad I(v) = 0,$$

where $F(x,t) = -L(w)$. In this way, problem (4.4) can be reduced to problem (4.5), which we have already discussed.

The preceding discussion has been phrased in terms of time-dependent problems for ease of exposition, but the techniques we have presented apply equally well to problems not involving time, such as the Laplace equation in two space variables x and y, with one of these variables playing the role of t. Here there are no "initial conditions" as opposed to "boundary conditions" but rather boundary conditions pertaining to different parts of the boundary; and "steady-state solutions" are to be interpreted as solutions that depend on only one of the variables. But the same ideas still work.

4.2 One-dimensional heat flow

In §2.5 we solved the problem of finding the temperature $u(x,t)$ in a rod that is insulated along its length and occupies the interval $0 \leq x \leq l$, given that the ends of the rod are either (a) insulated or (b) held at temperature zero. (The reader may prefer to think instead of a slab occupying the region $0 \leq x \leq l$ of xyz-space, where conditions are such that variations in temperature in the yz-directions are insignificant. The mathematics is the same.) Here we play some more complicated variations on the same theme.

Newton's law of cooling

Consider the same rod as before, and suppose the ends of the rod are in contact with a medium at temperature zero; but now suppose that the boundary conditions are given by Newton's law of cooling: the temperature gradient across the ends is proportional to the temperature difference between the ends and the surrounding medium. That is, we have the boundary value problem

$$u_t = ku_{xx}, \qquad u_x(0,t) = \alpha u(0,t), \qquad u_x(l,t) = -\alpha u(l,t), \tag{4.10}$$

subject of course to an initial condition $u(x,0) = f(x)$. Here α is a positive constant; the fact that the coefficient is α at $x = 0$ and $-\alpha$ at $x = l$ expresses the fact that, if $u(x,t) > 0$, the temperature will be increasing as one crosses the boundary at $x = 0$ from left to right and decreasing as one crosses the boundary at $x = l$ from left to right (and vice versa if $u(x,t) < 0$). The cases of insulated boundary, or boundary held at temperature zero, are the limiting cases $\alpha \to 0$ and $\alpha \to \infty$ of this setup.

We apply separation of variables. As before, if we set $u(x,t) = X(x)T(t)$ in (4.10) and call the separation constant $-k\nu^2$, we obtain the differential equation $T' = -k\nu^2 T$ for T and the Sturm-Liouville problem

$$X'' + \nu^2 X = 0, \qquad X'(0) = \alpha X(0), \quad X'(l) = -\alpha X(l) \tag{4.11}$$

for X. We solved this problem in §3.5, and the analysis there shows the following:

(i) Zero is not an eigenvalue.

(ii) The positive eigenvalues are the numbers ν^2 such that ν satisfies

$$\tan \nu l = \frac{2\alpha\nu}{\nu^2 - \alpha^2},$$

and there is an infinite sequence $\nu_1 < \nu_2 < \cdots$ of such ν's. (See Figure 3.5 for the case $\alpha = 1$.) The normalized eigenfunction corresponding to the eigenvalue ν_n^2 is

$$\phi_n(x) = d_n^{-1}(\nu_n \cos \nu_n x + \alpha \sin \nu_n x)$$

where

$$d_n^2 = \tfrac{1}{2}(\nu_n^2 + \alpha^2)l + \alpha.$$

(iii) The negative eigenvalues are the numbers $-\mu^2$ such that μ satisfies

$$\tanh l\mu = \frac{-2\alpha\mu}{\mu^2 + \alpha^2},$$

but there are no solutions of this equation since the left and right sides always have opposite signs.

Now we can solve the boundary value problem (4.10) subject to the initial condition $u(x,0) = f(x)$. Namely, we expand f in terms of the functions ϕ_n, which we know to be an orthonormal basis for $L^2(0,l)$: $f = \sum\langle f, \phi_n\rangle\phi_n$. Then we solve the differential equation $T' = -k\nu_n^2 T$ with initial value $\langle f, \phi_n\rangle$, obtaining $T(t) = \langle f, \phi_n\rangle \exp(-k\nu_n^2 t)$. Finally, we put it all together, obtaining

$$\begin{aligned} u(x,t) &= \sum_1^\infty \langle f, \phi_n\rangle \exp(-k\nu_n^2 t)\phi_n(x) \\ &= \sum_1^\infty \frac{2c_n}{(\nu_n^2 + \alpha^2)l + 2\alpha} \exp(-k\nu_n^2 t)(\nu_n \cos \nu_n x + \alpha \sin \nu_n x) \end{aligned}$$

where

$$c_n = \int_0^l f(x)(\nu_n \cos \nu_n x + \alpha \sin \nu_n x)\, dx.$$

Since all the eigenvalues are positive, the solution approaches zero exponentially fast as $t \to \infty$: this is just what one would expect physically.

Suppose we replace (4.10) by

$$u_t = ku_{xx}, \qquad u_x(0,t) = u(0,t), \quad u_x(l,t) = 4u(l,t). \tag{4.12}$$

Here the left boundary condition is just as before with $\alpha = 1$, but the right boundary condition is physically unreasonable: It says that heat is being pumped into the rod at the right end when the temperature of the rod is already greater than that of the surroundings, and sucked out when the temperature of the rod is less. Nonetheless, we can still solve the mathematical problem and see what we get. We solved the relevant Sturm-Liouville problem in §3.5, and we found that in addition to the sequence $\{\nu_n^2\}_1^\infty$ of positive eigenvalues, with eigenfunctions $\{\phi_n\}_1^\infty$, there is one negative eigenvalue $-\mu_0^2$, with eigenfunction ϕ_0. The solution of (4.12) with initial data f is then

$$u(x,t) = \langle f, \phi_0\rangle \exp(k\mu_0^2 t)\phi_0(x) + \sum_1^\infty \langle f, \phi_n\rangle \exp(-k\nu_n^2 t)\phi_n(x).$$

Here the term involving the negative eigenvalue *grows* exponentially as $t \to \infty$, unless by some chance $\langle f, \phi_0\rangle = 0$. But this is to be expected: If the rod is initially hot, it keeps getting hotter because heat is being pumped in at the right end. So the mathematics still makes some physical sense even when the physics is unrealistic!

Inhomogeneous boundary conditions

So far we have always assumed that both ends of the rod or slab are exposed to the same outside temperature. But perhaps the rod goes through a wall (or the slab *is* a wall) between two rooms at different temperatures: The temperature on the left is zero, for instance, and the temperature on the right is $A \neq 0$. Then we should impose the boundary conditions

$$u(0,t) = 0, \qquad u(l,t) = A \tag{4.13}$$

or, for Newton's law of cooling,

$$u_x(0,t) = \alpha u(0,t), \qquad u_x(l,t) = -\alpha[u(l,t) - A]. \tag{4.14}$$

These are inhomogeneous boundary conditions that do not depend on time, so we apply Technique 3 of §4.1 to find a solution. That is, we begin by finding the steady-state solution $u_0(x)$ of the heat equation that satisfies (4.13) or (4.14). This is easy: For a function u_0 that does not depend on t, the heat equation simply becomes $u_0'' = 0$. The general solution of this equation is $u_0(x) = cx + d$, and we have merely to determine the coefficients c and d so that u_0 satisfies (4.13) or (4.14). For (4.13) the solution is

$$u_0(x) = (A/l)x,$$

and for (4.14) the solution is

$$u_0(x) = \frac{A}{2+\alpha l}(\alpha x + 1).$$

Now we can solve the heat equation with initial data $u(x,0) = f(x)$, subject to the boundary conditions (4.13) or (4.14) — let us say (4.13), to be definite. Namely, we set

$$u(x,t) = u_0(x) + v(x,t) = (A/l)x + v(x,t).$$

Then u satisfies

$$u_t = ku_{xx}, \qquad u(0,t) = 0, \quad u(l,t) = A, \quad u(x,0) = f(x)$$

if and only if v satisfies

$$v_t = kv_{xx}, \qquad v(0,t) = v(l,t) = 0, \qquad v(x,0) = f(x) - (A/l)x.$$

Thus we now have *homogeneous* boundary conditions for v, with slightly different initial data. As we know from §2.5, we can solve this problem by expanding $f(x) - (A/l)x$ in a Fourier sine series; and we have essentially computed the Fourier sine series for $(A/l)x$ in §2.1:

$$\frac{A}{l}x = \sum_1^\infty \frac{2A(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{l}.$$

The result is

$$v(x,t) = \sum_1^\infty \left(b_n - \frac{2A(-1)^{n+1}}{n\pi} \right) e^{-n^2\pi^2 kt/l^2} \sin \frac{n\pi x}{l},$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx,$$

and hence

$$\begin{aligned} u(x,t) &= \frac{A}{l}x + v(x,t) \\ &= \sum_1^\infty \frac{2A(-1)^{n+1}}{n\pi}(1 - e^{-n^2\pi^2 kt/l^2}) \sin \frac{n\pi x}{l} + \sum_1^\infty b_n e^{-n^2\pi^2 kt/l^2} \sin \frac{n\pi x}{l}. \end{aligned}$$

The first sum here represents the solution that starts at 0 at time $t = 0$ and rises to the steady state $(A/l)x$ because of the influx of heat from the right, whereas the second sum represents the transient effects of the initial temperature $f(x)$.

The inhomogeneous heat equation

Having considered inhomogeneous boundary conditions, we now consider the inhomogeneous differential equation $u_t = ku_{xx} + F(x,t)$. Here $F(x,t)$ models the effect of some mechanism that adds or subtracts heat from the rod — perhaps some heat sources along the length of the rod, or a chemical or nuclear reaction within the rod itself. (F is measured in degrees per unit time; it represents the rate at which heat is being produced.) To be definite, let us suppose that the rod is initially at temperature zero and is held at temperature zero at both ends; thus, we wish to solve

$$u_t = ku_{xx} + F(x,t), \qquad u(x,0) = 0, \qquad u(0,t) = u(l,t) = 0. \tag{4.15}$$

If the inhomogeneous term is independent of t, i.e., $F(x,t) = F(x)$, we can use the same device as in the previous example. That is, we first solve the steady-state problem

$$ku_0'' + F(x) = 0, \qquad u_0(0) = u_0(l) = 0,$$

which is easily accomplished by integrating $-F(x)/k$ twice and choosing the constants of integration appropriately. Then the substitution $u(x,t) = u_0(x) + v(x,t)$ turns (4.15) into

$$v_t = kv_{xx}, \qquad v(x,0) = -u_0(x), \qquad v(0,t) = v(l,t) = 0,$$

which we have already solved by means of Fourier sine series.

For the general case, we can use Technique 2 of §4.1. The eigenfunction expansion that solves the homogeneous case $F = 0$ is the Fourier sine series. Hence, we begin by expanding everything in sight in a Fourier sine series:

$$u(x,t) = \sum_1^\infty b_n(t)\sin\frac{n\pi x}{l}, \qquad F(x,t) = \sum_1^\infty \beta_n(t)\sin\frac{n\pi x}{l}. \tag{4.16}$$

Here the coefficients β_n are computed from the known function F in the usual way, and the coefficients b_n are to be determined. If we plug these series into the differential equation (4.15), we obtain

$$\sum_1^\infty b_n'(t)\sin\frac{n\pi x}{l} = \sum_1^\infty \left(-\frac{n^2\pi^2 k}{l^2}b_n(t) + \beta_n(t)\right)\sin\frac{n\pi x}{l},$$

and equating the coefficients of $\sin(n\pi x/l)$ on both sides yields

$$b_n'(t) + \frac{n^2\pi^2 k}{l^2}b_n(t) = \beta_n(t).$$

This is a first-order ordinary differential equation for b_n, and it is easily solved by multiplying through by the integrating factor $e^{n^2\pi^2 kt/l^2}$:

$$\frac{d}{dt}\left[b_n(t)\exp\left(\frac{n^2\pi^2 kt}{l^2}\right)\right] = \beta_n(t)\exp\left(\frac{n^2\pi^2 kt}{l^2}\right).$$

Integrating both sides and remembering that $b_n(0) = 0$ (from the initial condition in (4.15)), we find that

$$b_n(t) = \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right)\int_0^t \beta_n(s)\exp\left(\frac{n^2\pi^2 ks}{l^2}\right)ds, \tag{4.17}$$

and the solution u is obtained by substituting this formula for b_n into (4.16).

The sharp-eyed reader will have noticed that this line of argument needs some justification. We differentiated the series $u = \sum b_n(t)\sin(n\pi x/l)$ termwise with respect to t and x without really knowing what we were doing, since the coefficients b_n (and hence the convergence properties of the series) were as yet unknown. Only after we have found formula (4.17) and substituted it into (4.16) can we see that the function u thus defined really solves the problem. (It always does so in the weak sense discussed in §9.5. Moreover, if the function $F(x,t)$ is such that its Fourier sine coefficients β_n tend to zero reasonably rapidly as $n \to \infty$, the same will be true of the coefficients b_n of u in view of (4.17), and one can then show that u satisfies (4.15) in the ordinary pointwise sense.)

Example. Suppose the rod is radioactive and produces heat at a constant rate R. Thus the problem to be solved is

$$u_t = ku_{xx} + R, \qquad u(x,0) = 0, \qquad u(0,t) = u(l,t) = 0.$$

Employing Technique 3, we first solve

$$u_0''(x) = -R/k, \qquad u_0(0) = u_0(l) = 0,$$

to obtain

$$u_0(x) = (R/2k)x(l-x).$$

Next we solve

$$v_t = kv_{xx}, \qquad v(x,0) = -u_0(x), \qquad v(0,t) = v(l,t) = 0$$

by expanding u_0 in its Fourier sine series (cf. Exercise 10, §2.4)

$$\frac{R}{2k}x(l-x) = \frac{4l^2R}{\pi^3 k} \sum_{n=1,3,5,\dots} \frac{1}{n^3}\sin\frac{n\pi x}{l} \qquad (0 < x < l)$$

to obtain

$$v(x,t) = -\frac{4l^2R}{\pi^3 k} \sum_{n=1,3,5,\dots} \frac{1}{n^3}\exp\frac{-n^2\pi^2 kt}{l^2}\sin\frac{n\pi x}{l}$$

and hence

$$\begin{aligned} u(x,t) &= u_0(x) + v(x,t) \\ &= \frac{4l^2R}{\pi^3 k} \sum_{n=1,3,5,\dots} \frac{1}{n^3}\left(1 - \exp\frac{-n^2\pi^2 kt}{l^2}\right)\sin\frac{n\pi x}{l}. \end{aligned} \tag{4.18}$$

(See Figure 4.1.)

Employing Technique 2, we expand $u(x,t)$ and the constant function R in Fourier sine series (cf. Exercise 1, §2.4):

$$u(x,t) = \sum_1^\infty b_n(t)\sin\frac{n\pi x}{l}, \qquad R = \frac{4R}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n}\sin\frac{n\pi x}{l} \qquad (0 < x < l).$$

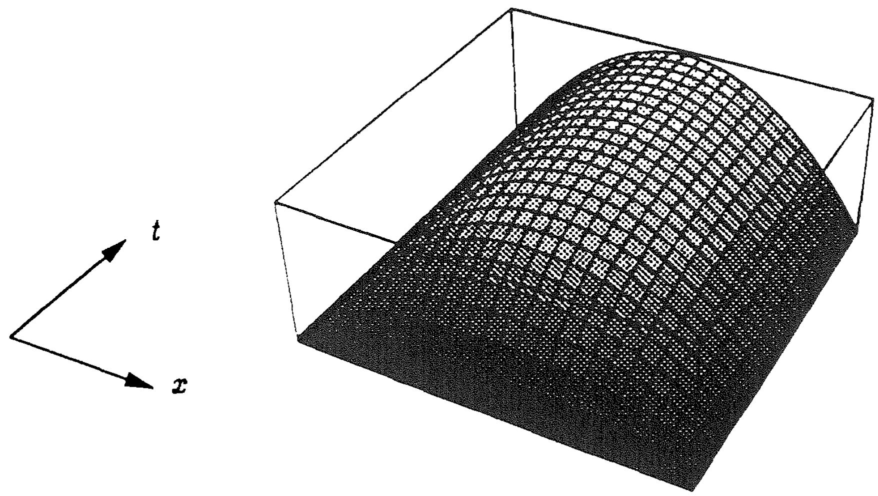


FIGURE 4.1. The temperature function $u(x,t)$ given by (4.18) with $k = 0.5$, $R = 1.3$, and $l = 1$, on the region $0 \le x \le 1$, $0 \le t \le 1$.

The differential equation for u then gives

$$b_n'(t) + \frac{n^2\pi^2 k}{l^2} b_n(t) = \begin{cases} 4R/n\pi, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

The solution to this equation with initial value 0 is

$$b_n(t) = \frac{4l^2 R}{n^3\pi^3 k}\left(1 - \exp\frac{-n^2\pi^2 kt}{l^2}\right)$$

for n odd and $b_n(t) = 0$ for n even, which again gives the solution (4.18).

Heat flow in nonuniform materials

One can also consider heat flow in rods or slabs of nonuniform composition, where the specific heat density σ and the thermal conductivity K vary from point to point. In this case the (homogeneous) heat equation becomes $\sigma(x)u_t = (K(x)u_x)_x$ (see Appendix 1). All of what we have done works in principle for this more general situation; the difference is that one must solve boundary value problems for the Sturm-Liouville equation $(K(x)f')' + \lambda\sigma(x)f = 0$ rather than $kf'' + \lambda f = 0$ (with k constant).

EXERCISES

All these problems concern heat flow in a rod on the interval $[0, l]$; in all except the last one, it is assumed that heat can enter or leave the rod only at the ends.

1. Suppose the end $x = 0$ is held at temperature zero while the end $x = l$ is insulated.
   a. Find a series expansion for the temperature $u(x,t)$ given the initial temperature $f(x)$.
   b. What is $u(x,t)$ when $f(x) \equiv 50$?

2. Repeat Exercise 1a, replacing the assumption $u(0,t) = 0$ by the assumption $u(0,t) = C \neq 0$.
3. Repeat Exercise 1a, replacing the assumption that $u_x(l,t) = 0$ by the assumption $u_x(l,t) = A$ (i.e., heat is being supplied at a constant rate at the right end).
4. Repeat Exercise 1a, assuming that the rod generates heat within itself at a constant rate R, so the heat equation is replaced by $u_t = ku_{xx} + R$.
5. Take $l = \pi$ and solve: $u_t = ku_{xx} + e^{-2t}\sin x$, $u(x,0) = u(0,t) = u(\pi,t) = 0$.
6. In the example of the radioactive rod, suppose that the reaction that produces the heat inside the rod dies out over time, so that the differential equation is $u_t = ku_{xx} + Re^{-ct}$. What is the solution?
7. Suppose that a rod is insulated at both ends, has initial temperature zero, and generates heat within itself at the constant rate R; thus, $u_t = ku_{xx} + R$ and $u(x,0) = u_x(0,t) = u_x(l,t) = 0$.
   a. Show that Technique 3 doesn't work here; that is, there is no steady-state solution of $u_t = ku_{xx} + R$, $u_x(0,t) = u_x(l,t) = 0$. Why is this to be expected physically?
   b. Solve the problem by Technique 2 (or by making a clever guess).
   c. Solve the problem with the constant R replaced by Re^{-ct}.
8. Solve: $u_t = ku_{xx}$, $u_x(0,t) = 0$, $u_x(l,t) + bu(l,t) = 0$ $(b > 0)$, $u(x,0) = 100$. (Cf. Exercise 5, §3.5. What is the physical interpretation?)
9. Let $k(x)$ be a smooth positive function on $[0,l]$. Solve the boundary value problem $u_t = (ku_x)_x + f(x,t)$, $u(0,t) = u(l,t) = u(x,0) = 0$, in terms of the eigenvalues $\{\lambda_n\}$ and the eigenfunctions $\{\phi_n\}$ of the Sturm-Liouville problem $(kf')' + \lambda f = 0$, $f(0) = f(l) = 0$.
10. We have always supposed that the rod is insulated along its length. Suppose instead that the surroundings are at temperature zero, and that heat transfer takes place at a rate proportional to the temperature difference (Newton's law). A reasonable model for this situation is the modified heat equation $u_t = ku_{xx} - hu$, where h is a positive constant.
    a. Show that u satisfies this equation if and only if $u(x,t) = e^{-ht}v(x,t)$ where v satisfies the ordinary heat equation. Show also how this result could be discovered by separation of variables.
    b. Suppose that both ends are insulated and that the initial temperature is $f(x) = x$. Solve for $u(x,t)$.
    c. Suppose instead that one end is held at temperature 0 and the other is held at temperature 100, and that the initial temperature is zero. Solve for $u(x,t)$. (Use Technique 3, and cf. entry 20 in Table 1, §2.1.)

4.3 One-dimensional wave motion

In §2.5 we analyzed the vibrating string problem,

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

by means of Fourier sine series. We now consider some related boundary value problems.

The inhomogeneous wave equation

We can add an inhomogeneous term to the vibrating string problem to represent an external force that affects the vibrations:

$$\begin{aligned} &u_{tt} = c^2 u_{xx} + F(x,t), \\ &u(0,t) = u(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned} \tag{4.19}$$

For example, if the string is an electrically charged wire, F could result from a surrounding electromagnetic field.

The techniques that we developed in §4.1 and used in §4.2 to solve the inhomogeneous heat equation work equally well here. If F is independent of t, one can first find a steady-state solution $u_0(x)$ by integrating F twice and then solve the homogeneous wave equation for $v = u - u_0$ with the initial displacement f replaced by $f - u_0$. Or, for the general case, one can expand $u(x,t)$ and $F(x,t)$ in Fourier sine series in x for each t,

$$u(x,t) = \sum_1^\infty b_n(t) \sin \frac{n\pi x}{l}, \qquad F(x,t) = \sum_1^\infty \beta_n(t) \sin \frac{n\pi x}{l},$$

yielding a sequence of ordinary differential equations for the Fourier coefficients of u in terms of those of F, namely,

$$b_n''(t) + \frac{n^2\pi^2c^2}{l^2} b_n(t) = \beta_n(t), \tag{4.20}$$

These equations can be solved by standard techniques such as variation of parameters or Laplace transforms (see §8.3 or Boyce-diPrima [10]); the solution with initial conditions $b_n(0) = b_n'(0) = 0$ is

$$b_n(t) = \frac{l}{n\pi c} \int_0^t \sin \frac{n\pi c(t-s)}{l} \beta_n(s)\, ds. \tag{4.21}$$

This formula leads to the solution of (4.19) with initial conditions $f = g = 0$. But then to solve (4.19) with arbitrary initial data f and g, by the superposition principle (Technique 1) one is reduced to solving the *homogeneous* wave equation with these initial data; and this we have already done. (As with the heat equation, these calculations show only that $u = \sum b_n(t) \sin(n\pi x/l)$, with $b_n(t)$ defined by (4.21), is a reasonable candidate for a solution; further arguments are needed for a rigorous establishment of the fact that it really works.)

Vibrations with free ends

Another boundary value problem of interest for the wave equation is obtained by requiring that the spatial derivative u_x rather than u itself should vanish at the endpoints:

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \\ &u_x(0,t) = u_x(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned} \tag{4.22}$$

If one thinks of a vibrating string, this represents a string whose ends are free to move on frictionless tracks along the lines $x = 0$ and $x = l$ in the xu-plane. The condition that $u_x = 0$ at the endpoints expresses the fact that there are no forces directed along the tracks to oppose the tension in the string. This may seem a rather artifical situation, but more natural interpretations of (4.22) are available. For one, (4.22) is a model for the longitudinal vibrations of an elastic rod or a spring that is free at both ends. ("Longitudinal" means that the vibrations involve displacements of the material along the x-axis by compression and extension of the rod or spring, rather than displacements perpendicular to the x-axis as in the vibrating string.) An even better interpretation of (4.22) is as the longitudinal vibrations of a column of air that is open at both ends, such as a flute or organ pipe. In the case of the flute, for example, musical notes are produced by vibrations of the air within the flute; these vibrations are largely confined to the interval between the hole where the moving air is introduced by the player and the first open finger-hole.

The mathematics of (4.22) is very similar to the vibrating string problem, except that one uses Fourier cosine series rather than sine series. Indeed, we can solve the problem by expanding everything in Fourier cosine series from the outset: If we substitute

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{l}, \qquad g(x) = \tfrac{1}{2}\alpha_0 + \sum_1^\infty \alpha_n \cos\frac{n\pi x}{l},$$
$$u(x,t) = \tfrac{1}{2}A_0(t) + \sum_1^\infty A_n(t)\cos\frac{n\pi x}{l},$$

into (4.22), we obtain the ordinary differential equations

$$A_n''(t) = -\frac{n^2\pi^2c^2}{l^2}A_n(t), \qquad A_n(0) = a_n, \quad A_n'(0) = \alpha_n.$$

The solution to this, for $n > 0$, is

$$A_n(t) = a_n \cos\frac{n\pi ct}{l} + \frac{l\alpha_n}{n\pi c}\sin\frac{n\pi ct}{l},$$

whereas for $n = 0$ it is

$$A_0(t) = a_0 + \alpha_0 t.$$

Hence, the solution u of (4.22) is given by

$$u(x,t) = \tfrac{1}{2}(a_0 + \alpha_0 t) + \sum_1^\infty \cos\frac{n\pi x}{l}\left(a_n \cos\frac{n\pi ct}{l} + \frac{l\alpha_n}{n\pi c}\sin\frac{n\pi ct}{l}\right). \tag{4.23}$$

(As usual, the formal differentiations used in arriving at this formula need to be justified after the fact. Alternatively, one could arrive at (4.23) by separation of variables.)

Here there is a bit of a surprise. The terms with $n > 0$ describe the vibratory motion of the string (or rod, or spring, or whatever; let us call it the "device"), and the term $\frac{1}{2}a_0$ is just a constant displacement, of no importance; but if $\alpha_0 \neq 0$, the term $\frac{1}{2}\alpha_0 t$ says that the device as a whole is moving with velocity $\frac{1}{2}\alpha_0$ — perpendicular to the x-axis in the case of a string, and along the x-axis in the other cases. Indeed, there is nothing in the setups we have described to prevent this, since the ends of the device are free to move. The constant $\frac{1}{2}\alpha_0 = l^{-1}\int_0^l g(x)\,dx$ is the average initial velocity of the device; and in the absence of any countervailing forces the device will continue to move with this velocity. If the device as a whole stays put, it simply means that $\alpha_0 = 0$.

Mixed boundary conditions

Another problem of interest is the one with mixed boundary conditions:

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \\ &u(0,t) = u_x(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned} \tag{4.24}$$

Here the left endpoint is fixed and the right endpoint is free. One can think of a string or elastic rod with one fixed end and one free end, or a column of air that is closed at one end and open at the other, such as a clarinet or a stopped organ pipe.

After separation of variables, the Sturm-Liouville problem to be solved in this case is

$$X'' + \lambda X = 0, \qquad X(0) = X'(l) = 0.$$

This was the subject of Exercise 3, §3.5, but we shall briefly derive the solution here. If we set $\lambda = \nu^2$, the general solution of the differential equation is a linear combination of $\sin\nu x$ and $\cos\nu x$. The condition $X(0) = 0$ implies that $X(x) = c\sin\nu x$, and the condition $X'(l) = 0$ then becomes $\cos\nu l = 0$. This means that νl must be a half-integer multiple of π, so the (unnormalized) eigenfunctions are

$$X_n(x) = \sin\frac{(n-\frac{1}{2})\pi x}{l} = \sin\frac{(2n-1)\pi x}{2l}, \qquad n = 1,2,3,\ldots$$

We leave to the reader to work out the details, but it should be pretty clear now that the solution $u(x,t)$ of (4.22) will have the form

$$\sum_1^\infty \sin\frac{(2n-1)\pi x}{2l}\left[a_n\cos\frac{(2n-1)\pi ct}{2l} + \frac{2l\alpha_n}{(2n-1)\pi c}\sin\frac{(2n-1)\pi ct}{2l}\right]. \tag{4.25}$$

There is an interesting difference between the frequency spectra of the waves (4.23) and (4.25). The nth term of the series in (4.23) represents a vibration with period $2l/nc$, or frequency $nc/2l$; so the allowable frequencies are the integer multiples of the "fundamental" frequency $c/2l$. The nth term in (4.25), on the other hand, has frequency $(2n-1)c/4l$; so the allowable frequencies are the *odd* integer multiples of the fundamental frequency $c/4l$. In particular, the fundamental frequency in the former case is twice as great as in the latter. In musical terms, this means that an air column open at only one end produces notes an octave lower than a column of the same length open at both ends, and that its odd harmonics are missing. (See Figure 4.2.)

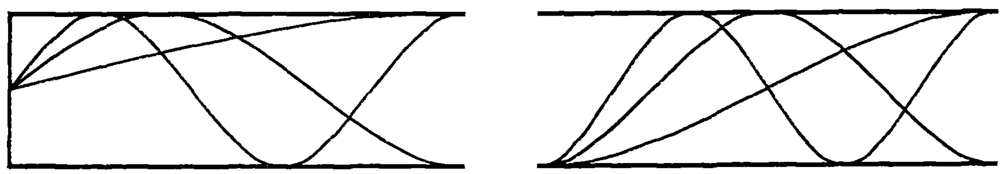


FIGURE 4.2. Profiles of the vibrations corresponding to the three lowest eigenvalues in a pipe open at one end (left) and a pipe open at both ends (right).

These remarks apply to clarinets but not to oboes, saxophones, or any of the brass instruments. Oboes and saxophones have conical bores (their interior diameter increases steadily from mouthpiece to bell) rather than the cylindrical bore of the clarinet (whose interior diameter is essentially constant). The effect of this, as we shall see in §5.6, is that the frequencies they produce are about the same as the frequencies of a cylindrical column of the same length that is *open at both ends.* In particular, they produce all integer multiples of the fundamental frequency. The physics of the brass instruments is considerably more complex, and we shall not discuss it here. For further information on the physics of musical instruments, we refer the reader to Hutchins [31] and Taylor [51].

Other problems in wave motion

A number of other variations on these themes are possible. For example, one can add an inhomogeneous term to the wave equation in (4.22) and (4.24), just as in (4.19), and the same techniques of solution are applicable. One can also consider inhomogeneous boundary conditions, such as

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = 0, \quad u(l,t) = h(t).$$

This might represent a string that is fixed at the left end and is being shaken at the right end, or an electromagnetic signal being sent down a wire from the end $x = l$. We shall solve this problem in §8.4 by using the Laplace transform; for the time being, we leave it to the reader to work out a special case in Exercise 7, using the trick mentioned at the end of §4.1.

One can also consider waves in nonuniform media — for example, a vibrating string whose linear mass density ρ varies from point to point. (Perhaps the string is thicker in some places than in others.) In this case the wave equation becomes

$$u_{tt} = T\rho(x)^{-1}u_{xx} \tag{4.26}$$

where T is the tension of the string (see Appendix 1). The Sturm-Liouville equation that results from separation of variables is then $f'' + \lambda\rho(x)f = 0$, which produces eigenfunctions that are orthogonal with respect to the weight $\rho(x)$. One can then solve the wave equation (4.26) by using these eigenfunctions in place of sines and cosines.

EXERCISES

1. Verify that the function $b_n(t)$ defined by (4.21) satisfies the differential equation (4.20) and the initial conditions $b_n(0) = b_n'(0) = 0$.
2. One end of an elastic bar of length l is held at $x = 0$, and the other end is stretched from its natural position $x = l$ to $x = (1 + b)l$. Thus, an arbitrary point x in the bar is moved to $(1+b)x$, so its displacement from equilibrium is bx. At time $t = 0$ the ends of the bar are released; thus, $u(x, 0) = bx$ and $u_t(x, 0) = 0$.
   a. Find the displacement $u(x, t)$ at times $t > 0$.
   b. Show that the velocity at the left end of the bar alternately takes the values bc and $-bc$ on time intervals of length l/c. (That is, $u_t(0, t) = bc$ for $2ml/c < t < (2m + 1)l/c$ and $u_t(0, t) = -bc$ for $(2m + 1)l/c < t < (2m + 2)l/c$, $m = 0, 1, 2, \ldots$. Hint: Entry 6 of Table 1, §2.1.)
3. Suppose a horizontally stretched string is heavy enough for the effects of gravity to be significant, so that the wave equation must be replaced by $u_{tt} = c^2u_{xx} - g$ where g is the acceleration of gravity. (The boundary conditions are still $u(0, t) = u(l, t) = 0$.)
   a. Find the steady-state solution $\phi(x)$.
   b. Suppose that initially $u(x, 0) = u_t(x, 0) = 0$. Find the solution $u(x, t)$ as a Fourier series, and show that
   $$u(x, t) = \phi(x) - \tfrac{1}{2}\Big[\Phi(x + ct) + \Phi(x - ct)\Big]$$
   where Φ is the odd $2l$-periodic extension of ϕ. (Cf. the discussion in §2.5.)
4. In problem (4.22) discussed in the text, assume that $\int_0^l g(x)\,dx = 0$ (average initial velocity is zero), and let $h(x) = \int_0^x g(\xi)\,d\xi$. Show that the solution (4.23) can be written as
   $$u(x, t) = \frac{1}{2}\Big[F(x + ct) + F(x - ct)\Big] + \frac{1}{2c}\Big[H(x + ct) - H(x - ct)\Big]$$
   where F and H are the even $2l$-periodic extensions of f and h. (Cf. the discussion in §2.5.)

5. Find the general solution of $u_{tt} = c^2u_{xx} - a^2u$, $u(0,t) = u(l,t) = 0$, with arbitrary initial conditions. This is a model for a string vibrating in an elastic medium; the term $-a^2u$ represents the force of reaction of the medium on the string. (Hint: The differential equation is homogeneous; start from scratch with separation of variables.)
6. In real-life vibrating strings, the vibrations damp out because the strings are not perfectly elastic. This situation can be modeled by the modified wave equation $u_{tt} = c^2u_{xx} - 2ku_t$; the term $-2ku_t$ represents the frictional forces that cause the damping. (The factor of 2 is purely for convenience.) Find the general solution, subject to the boundary conditions $u(0,t) = u(l,t) = 0$. Assume at first that $k < \pi c/l$. What happens if $k \geq \pi c/l$? (See the hint for Exercise 5.)
7. A string of length $l = \pi$ (for simplicity) is fixed at one end and attached to an oscillator at the other, so that $u(0,t) = 0$ and $u(\pi,t) = \sin kt$. If the string is initially at rest ($u(x,0) = u_t(x,0) = 0$), find $u(x,t)$. (Hints: (1) Let $u(x,t) = v(x,t) + (x/\pi)\sin kt$ and solve for v. (2) When $k \neq \alpha$ the general solution of $f'' + \alpha^2 f = \beta \sin kt$ is $c_1 \cos \alpha t + c_2 \sin \alpha t + (\beta \sin kt)/(\alpha^2 - k^2)$.) The typical case is when k/c is not an integer; if it is, the answer will have a different form due to resonance between the imposed oscillations and one of the natural frequencies of the string.
8. The total energy of a vibrating string at time t, up to a constant factor, is
$$E(t) = \int_0^l \left[u_t(x,t)^2 + c^2u_x(x,t)^2\right] dx.$$
(The first term is the kinetic energy and the second term is the potential energy. u is assumed to be real here.)
   a. If the string has fixed ends and $u(x,t)$ is written as a Fourier series as in equation (2.24), show that
   $$E(t) = \frac{\pi^2c^2}{2l}\sum_1^\infty (nb_n)^2 + \frac{l}{2}\sum_1^\infty B_n^2.$$
   (In particular, we have conservation of energy: $E(t)$ is independent of t. This also suggests that a natural physical requirement is that the series $\sum(nb_n)^2$ and $\sum B_n^2$ be convergent. This is the case if $u(x,0)$ is continuous and piecewise smooth and $u_t(x,0)$ is piecewise continuous. Why?)
   b. Derive a similar result for a vibrating string (or bar or air column) with free ends, with the same formula for $E(t)$.

4.4 The Dirichlet problem

The **Dirichlet problem** is to find a solution of Laplace's equation in a region D that assumes given values on the boundary ∂D of D:
$$\nabla^2 u = 0 \text{ in } D, \qquad u(x) = f(x) \quad \text{for } x \in \partial D.$$

This can be interpreted physically as finding the steady-state temperature in D when the temperature on ∂D is known, or as finding the electrostatic potential in the charge-free region D when the potential on ∂D is known. This problem can be studied in any number of dimensions; here we consider the 2-dimensional case for certain simple regions in which the method of separation of variables is effective. Some other boundary value problems for the equation $\nabla^2 u = 0$ are considered in the exercises.

The Dirichlet problem in a rectangle

The simplest situation is that of a rectangle. We take the sides of the rectangle to have length l and L, and we take the origin to be at the lower left corner. Thus,

$$D = [0, l] \times [0, L] = \Big\{(x, y) : 0 \le x \le l,\ 0 \le y \le L\Big\},$$

and the boundary value problem to be solved is

$$\begin{gathered} u_{xx} + u_{yy} = 0, \\ u(x, 0) = f_1(x), \quad u(x, L) = f_2(x), \quad u(0, y) = g_1(y), \quad u(l, y) = g_2(y). \end{gathered}$$

By the superposition principle (Technique 1) it will suffice to solve this problem in the special cases $g_1 = g_2 = 0$ and $f_1 = f_2 = 0$, as the solution in the general case is obtained by adding together the solutions for these two special cases. Moreover, the cases $g_1 = g_2 = 0$ and $f_1 = f_2 = 0$ are equivalent, just by interchanging the roles of x and y, so we work out only the first one:

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(0, y) = u(l, y) = 0, \quad u(x, 0) = f_1(x), \quad u(x, L) = f_2(x). \end{aligned} \tag{4.27}$$

We apply separation of variables. Neglecting the inhomogeneous boundary conditions for the moment, we search for solutions u that satisfy the homogeneous boundary conditions. Taking $u(x, y) = X(x)Y(y)$, we find from the differential equation that $X''Y + Y''X = 0$, or $Y''/Y = -X''/X$. Setting Y''/Y and $-X''/X$ equal to a constant ν^2, we obtain

$$\begin{aligned} &X'' + \nu^2 X = 0, \quad X(0) = X(l) = 0, \\ &Y'' - \nu^2 Y = 0. \end{aligned}$$

The Sturm-Liouville problem for X is a familiar one that we have seen many times before: The eigenvalues are $\nu^2 = (n\pi/l)^2$ where n is a positive integer, and the corresponding eigenfunctions are $\sin(n\pi x/l)$. In other words, we are working once again with Fourier sine series in x. (Readers who foresaw this outcome immediately upon looking at (4.27) are to be congratulated on their instincts.) As for Y, the general solution of the equation $Y'' - \nu^2 Y = 0$ with $\nu^2 = (n\pi/l)^2$

is a linear combination of $\cosh(n\pi y/l)$ and $\sinh(n\pi y/l)$, so we are looking at solutions u of the form

$$u(x,y) = \sum_1^\infty \sin\frac{n\pi x}{l}\left(\alpha_n \cosh\frac{n\pi y}{l} + \beta_n \sinh\frac{n\pi y}{l}\right), \tag{4.28}$$

and we must determine the coefficients α_n and β_n to get the right boundary conditions at $y = 0$ and $y = L$ ("initial" and "final" conditions, if you like). We expand the functions f_1 and f_2 in (4.27) in their Fourier sine series:

$$f_1(x) = \sum_1^\infty a_n \sin\frac{n\pi x}{l}, \qquad f_2(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}.$$

On setting $y = 0$ or $y = L$ in (4.28) and comparing coefficients, we find that

$$\alpha_n = a_n, \qquad \alpha_n \cosh\frac{n\pi L}{l} + \beta_n \sinh\frac{n\pi L}{l} = b_n,$$

or

$$\alpha_n = a_n, \qquad \beta_n = b_n \operatorname{csch}\frac{n\pi L}{l} - a_n \coth\frac{n\pi L}{l}.$$

The solution is obtained by substituting these formulas into (4.28). It can be expressed more symmetrically by taking $\sinh[n\pi(L-y)/l]$ and $\sinh(n\pi y/l)$ as a basis for solutions to $Y'' - (n\pi/l)^2 Y = 0$ instead of $\cosh(n\pi y/l)$ and $\sinh(n\pi y/l)$; the result is

$$u(x,y) = \sum_1^\infty \sin\frac{n\pi x}{l}\left(A_n \sinh\frac{n\pi(L-y)}{l} + B_n \sinh\frac{n\pi y}{l}\right),$$

$$A_n = a_n \operatorname{csch}\frac{n\pi L}{l}, \qquad B_n = b_n \operatorname{csch}\frac{n\pi L}{l}.$$

The Dirichlet problem in polar coordinates

We next solve the Dirichlet problem in a "polar-coordinate rectangle"

$$S = \left\{(r\cos\theta, r\sin\theta) : r_0 \le r \le r_1,\ \alpha \le \theta \le \beta\right\}.$$

(See Figure 4.3.) For this we need the formula for the Laplacian in polar coordinates:

$$\nabla^2 u = u_{xx} + u_{yy} = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}.$$

This formula is derived in Appendix 4.

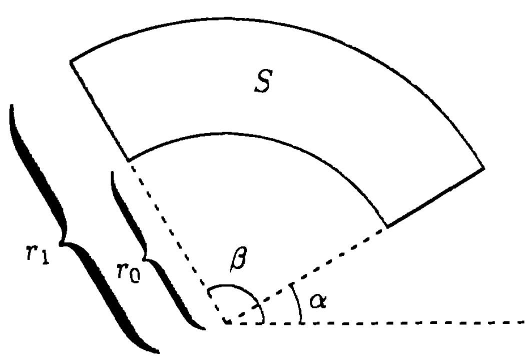


FIGURE 4.3. A "rectangular" region in polar coordinates.

In order to solve the Dirichlet problem on the region S, as in the rectangular case it will suffice to do the special cases when the solution is to vanish on the two radial pieces of the boundary or on the two circular pieces of the boundary. We shall work out the first of these cases here and leave the second one as Exercise 7. By rotating the coordinates suitably we may assume that the initial angle α is 0, so the problem we are to solve is

$$\begin{aligned} &u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 \quad \text{in } S, \\ &u(r,0) = u(r,\beta) = 0, \quad u(r_1,\theta) = f(\theta), \quad u(r_0,\theta) = g(\theta). \end{aligned} \tag{4.29}$$

As usual, we begin by looking for product solutions $u(r,\theta) = R(r)\Theta(\theta)$ that satisfy the homogeneous boundary conditions. For such a u, Laplace's equation becomes

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)},$$

so upon setting both these expressions equal to a constant ν^2 we obtain

$$\Theta''(\theta) + \nu^2\Theta(\theta) = 0, \qquad \Theta(0) = \Theta(\beta) = 0, \tag{4.30}$$

$$r^2R''(r) + rR'(r) - \nu^2R(r) = 0. \tag{4.31}$$

The Sturm-Liouville problem (4.30) is our old friend that leads to the eigenvalues $\nu^2 = (n\pi/\beta)^2$ and eigenfunctions $\sin(n\pi\theta/\beta)$. The equation (4.31) for R is a special case of the **Euler equation**

$$r^2f''(r) + arf'(r) + bf(r) = 0, \tag{4.32}$$

which is one of the few types of equations with variable coefficients that can be solved in an elementary way. Namely, just as one uses exponential functions to solve constant-coefficient equations, one uses power functions to solve the Euler equation. Substituting $f(r) = r^\lambda$ in (4.32) yields

$$\Big[\lambda(\lambda-1) + a\lambda + b\Big]r^\lambda = 0,$$

so if λ_1 and λ_2 are the roots of the quadratic polynomial $\lambda^2 + (a-1)\lambda + b$, the functions r^{λ_1} and r^{λ_2} satisfy (4.32). The general solution of (4.32) is then a linear combination of these two except when $\lambda_1 = \lambda_2$, in which case the general solution is a linear combination of r^{λ_1} and $r^{\lambda_1}\log r$.

In the case (4.31) with which we are concerned, we have $a = 1$ and $b = \nu^2 = (n\pi/\beta)^2$, so the quadratic polynomial becomes $\lambda^2 - (n\pi/\beta)^2$, whose roots are $\lambda = \pm n\pi/\beta$. Therefore, we have found the following sort of solutions to Laplace's equation in the region S:

$$u(r,\theta) = \sum_1^\infty \sin\frac{n\pi\theta}{\beta}\left(a_nr^{n\pi/\beta} + b_nr^{-n\pi/\beta}\right).$$

It remains only to choose a_n and b_n to satisfy the remaining boundary conditions in (4.29). But this is easy: If we expand f and g in their Fourier sine series,

$$f(\theta)=\sum_1^\infty c_n \sin\frac{n\pi\theta}{\beta}, \qquad g(\theta)=\sum_1^\infty d_n \sin\frac{n\pi\theta}{\beta},$$

we see that

$$a_n r_1^{n\pi/\beta}+b_n r_1^{-n\pi/\beta}=c_n, \qquad a_n r_0^{n\pi/\beta}+b_n r_0^{-n\pi/\beta}=d_n,$$

and it is a simple matter to solve these equations simultaneously for a_n and b_n.

In a similar way we can solve the Dirichlet problem in an annulus

$$A=\Big\{(r\cos\theta, r\sin\theta): r_0\le r\le r_1,\ \theta \text{ arbitrary}\Big\},$$

namely,

$$u_{rr}+r^{-1}u_r+r^{-2}u_{\theta\theta}=0 \quad \text{in } A, \qquad u(r_1,\theta)=f(\theta), \quad u(r_0,\theta)=g(\theta).$$

The boundary conditions at $\theta=0$ and $\theta=\beta$ are now replaced by the requirement that u be 2π-periodic in θ. Thus, instead of (4.30) we ask for periodic solutions of $\Theta''+\nu^2\Theta=0$; this forces ν to be an integer and gives the eigenfunctions $e^{\pm in\theta}$ or $\cos n\theta$ and $\sin n\theta$, with the result that

$$u(r,\theta)=(a_0+b_0\log r)+\sum_{n=\pm1,\pm2,..} e^{in\theta}(a_n r^n+b_n r^{-n}). \tag{4.33}$$

Now the coefficients a_n and b_n are found by expanding the periodic functions f and g in their full Fourier series rather than a Fourier sine series.

Finally, we can let the inner radius r_0 tend to zero and consider the Dirichlet problem on a disc

$$D=\Big\{(x,y): x^2+y^2\le r_1^2\Big\}=\Big\{(r\cos\theta,\ r\sin\theta): r\le r_1\Big\},$$

that is,

$$u_{rr}+r^{-1}u_r+r^{-2}u_{\theta\theta}=0 \quad \text{in } D, \qquad u(r_1,\theta)=f(\theta). \tag{4.34}$$

Here the inner boundary condition has disappeared, but there is still a condition to be satisfied at $r=0$. Functions of the form (4.33) will satisfy Laplace's equation in the punctured disc $\{0<r\le r_1\}$, but they will blow up at $r=0$ unless all the terms involving $\log r$ or negative powers of r vanish. In other words, we impose the "boundary condition" on the product solutions $u=R\Theta$ obtained from (4.30) and (4.31) that they should be continuous at $r=0$. The result is that (4.33) must be replaced by

$$u(r,\theta)=\sum_{-\infty}^{\infty} c_n r^{|n|}e^{in\theta},$$

and the condition $u(r_1, \theta) = f(\theta)$ means that the numbers $c_n r_1^{|n|}$ are the Fourier coefficients of f.

From this we can derive a useful formula for the solution to (4.34) as an integral rather than a series. To simplify the calculation a bit, we shall take $r_1 = 1$; the reader may verify that for the general case one merely replaces r by r/r_1 in the following formulas. We recall that the Fourier coefficients of f are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi}\, d\phi.$$

If we substitute this into the formula for u, we obtain

$$u(r, \theta) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi}\, d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) P(r, \theta - \phi)\, d\phi$$

where $P(r, \theta)$ is the **Poisson kernel**:

$$P(r, \psi) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\psi} = \sum_{0}^{\infty} r^n e^{in\psi} + \sum_{1}^{\infty} r^n e^{-in\psi}.$$

The series on the right are geometric series that converge nicely for $r < 1$. This fact justifies the interchange of integration and summation we have just performed, and it also allows one to sum the series in closed form:

$$\begin{aligned} P(r, \psi) &= \frac{1}{1 - re^{i\psi}} + \frac{re^{-i\psi}}{1 - re^{-i\psi}} = \frac{1 - r^2}{(1 - re^{i\psi})(1 - re^{-i\psi})} \\ &= \frac{1 - r^2}{1 + r^2 - 2r\cos\psi}. \end{aligned}$$

In short, we have the **Poisson integral formula** for the solution of (4.34) (with $r_1 = 1$):

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)} f(\phi)\, d\phi. \tag{4.35}$$

EXERCISES

Exercises 1–3 deal with the equation $\nabla^2 u = 0$ in the square

$$D = \left\{(x, y) : 0 \le x \le l,\ 0 \le y \le l\right\}.$$

1. Solve $\nabla^2 u = 0$ in D subject to the boundary conditions $u(x, 0) = u(0, y) = u(l, y) = 0$, $u(x, l) = x(l - x)$. (Cf. Exercise 10, §2.4.)
2. Find the steady-state temperature in D if the sides $x = 0$ and $x = l$ are insulated, the side $y = 0$ is held at temperature zero, and the side $y = l$ is held at temperature $u(x, l) = x$.

3. Consider the *Neumann problem*

$$\nabla^2 u = 0 \text{ in } D, \qquad u_x(0,y) = u_x(l,y) = u_y(x,0) = 0, \quad u_y(x,l) = f(x).$$

(Thus the normal derivative of u on the boundary is prescribed.) Use Fourier cosine series to find a solution, if possible. Show that a solution exists only if $\int_0^l f(x)\,dx = 0$, in which case it contains an arbitrary constant.

4. Find the steady-state temperature in the semi-infinite strip $0 \le x \le l$, $0 \le y < \infty$ if $u(0,y) = u(l,y) = 0$ and $u(x,0) = f(x)$. (Hint: On physical grounds, $u(x,y)$ must be bounded in the strip.)

Exercises 5–8 deal with the equation $\nabla^2 u = 0$ in polar coordinates.

5. Suppose the inner side of the annulus $\{(r,\theta) : r_0 \le r \le 1\}$ is insulated and the outer side is held at temperature $u(1,\theta) = f(\theta)$.
   a. Find the steady-state temperature.
   b. What is the solution if $f(\theta) = 1 + 2\sin\theta$?
6. Let D be the unit disc $\{(r,\theta) : 0 \le r \le 1\}$. Let $P(r,\theta)$ be the Poisson kernel, and let $u(r,\theta)$ be the solution of the Dirichlet problem $\nabla^2 u = 0$ in D, $u(1,\theta) = f(\theta)$.
   a. Show that the value of u at the origin is $(2\pi)^{-1}\int_{-\pi}^{\pi} f(\theta)\,d\theta$. (This is the *mean value theorem* for harmonic functions: the value of a harmonic function at the center of a circle is the average of its values on the circle.)
   b. Show that $P(r,\theta) > 0$ and that $\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 2\pi$ for all $r < 1$.
   c. Use part (b) to show that if $f(\theta) \le M$ for all θ, then $u(r,\theta) \le M$ for all θ and all $r < 1$. (This is the *maximum principle* for harmonic functions in a disc.)
7. Solve the following Dirichlet problem:

$$\nabla^2 u = 0 \quad \text{in } S = \left\{(r,\theta) : 0 < r_0 \le r \le 1,\ 0 \le \theta \le \beta\right\},$$
$$u(r_0,\theta) = u(1,\theta) = 0, \qquad u(r,0) = g(r), \quad u(r,\beta) = h(r).$$

   (Cf. Exercise 10, §3.5.)
8. Consider the Dirichlet problem on the limiting case

$$S_0 = \left\{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \beta\right\}$$

   of the region S in Exercise 7.
   a. Solve: $\nabla^2 u = 0$ in S_0, $u(r,0) = u(r,\beta) = 0$ for $r < 1$, $u(1,\theta) = f(\theta)$. (This is problem (4.29) in the limiting case $r_0 = 0$, and the method used to solve (4.29) can be adapted. Note that the piece of the boundary $r = r_0$ has collapsed to a point, at which f has already been prescribed to be zero.)
   b. Try to solve the limiting case of Exercise 7: $\nabla^2 u = 0$ in S_0, $u(1,\theta) = 0$, $u(r,0) = g(r)$, $u(r,\beta) = h(r)$. (You won't succeed with the present methods. Separation of variables leads to the problem $(rf')' + (\lambda/r)f = 0$, $f(1) = 0$, which has *no* eigenfunctions in $L^2_{1/r}(0,1)$. This is a singular Sturm-Liouville problem whose solution requires integrals rather than infinite series; see Exercise 9, §7.4.)

4.5 Multiple Fourier series and applications

We have seen how Sturm-Liouville problems give rise to orthonormal bases for $L^2(a,b)$, but we have not yet seen any examples of orthonormal bases for $L^2(D)$ where D is a region in $\mathbf{R}^n$ with $n > 1$. However, for rectangular regions — that is, regions that are products of intervals — there is a simple way of building orthonormal bases out of the one-dimensional ones. Specifically, we have the following theorem.

Theorem 4.1. *Suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(a,b)$ and $\{\psi_n\}_1^\infty$ is an orthonormal basis for $L^2(c,d)$. Let*

$$\chi_{mn}(x,y) = \phi_m(x)\psi_n(y).$$

Then $\{\chi_{mn}\}_{m,n=1}^\infty$ is an orthonormal basis for $L^2(D)$, where

$$D = [a,b] \times [c,d] = \Big\{(x,y) : a \le x \le b,\ c \le y \le d\Big\}.$$

Proof: Orthonormality is easy:

$$\begin{aligned}
\langle \chi_{mn}, \chi_{m'n'} \rangle &= \iint_D \chi_{mn}(x,y)\overline{\chi_{m'n'}(x,y)}\,dx\,dy \\
&= \int_c^d \int_a^b \phi_m(x)\psi_n(y)\overline{\phi_{m'}(x)\psi_{n'}(y)}\,dx\,dy \\
&= \left(\int_a^b \phi_m(x)\overline{\phi_{m'}(x)}\,dx\right)\left(\int_c^d \psi_n(y)\overline{\psi_{n'}(y)}\,dy\right) \\
&= \begin{cases} 1 & \text{if } m = m' \text{ and } n = n', \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

To prove completeness, we shall show that if $f \in L^2(D)$ and $\langle f, \chi_{mn}\rangle = 0$ for all m and n then $f = 0$. (The argument that follows is the truth but not quite the whole truth; we are glossing over some technical points about the workings of the Lebesgue integral. See Folland [25], §2.5, or Wheeden-Zygmund [56], Chapter 6.) Let

$$g_n(x) = \int_c^d f(x,y)\overline{\psi_n(y)}\,dy.$$

Then $g_n \in L^2(a,b)$, for by the Schwarz inequality and the fact that $\|\psi_n\| = 1$,

$$\begin{aligned}
\int_a^b |g_n(x)|^2 dx &\le \int_a^b \left(\int_c^d |f(x,y)|^2 dy\right)\left(\int_c^d |\psi_n(y)|^2 dy\right) dx \\
&= \int_a^b \int_c^d |f(x,y)|^2 dy\,dx < \infty.
\end{aligned}$$

Moreover,

$$\langle g_n, \phi_m \rangle = \int_a^b \int_c^d f(x,y)\overline{\psi_n(y)\phi_m(x)}\,dy\,dx = \langle f, \chi_{mn} \rangle = 0$$

for all m, so since $\{\phi_m\}$ is complete, we have $g_n(x) = 0$ for all n and (almost) all x. But $g_n(x) = \langle f(x,\cdot), \psi_n \rangle$, so the completeness of $\{\psi_n\}$ implies that $f(x,y) = 0$ for (almost) all x and y, that is, $f = 0$ as an element of $L^2(D)$. ∎

This theorem is valid (with essentially the same proof) in much greater generality than our statement of it. Here are four useful extensions of it:

(i) One can replace the intervals $[a,b]$ and $[c,d]$ by sets $A \subset \mathbf{R}^j$ and $B \subset \mathbf{R}^k$, in which case
$$D = A \times B = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbf{R}^{j+k} : \mathbf{x} \in A,\ \mathbf{y} \in B\Big\}.$$

(ii) One can introduce weight functions. If $\{\phi_n\}$ is an orthonormal basis for $L^2_u(a,b)$ and $\{\psi_n\}$ is an orthonormal basis for $L^2_v(c,d)$, then $\{\chi_{mn}\}$ is an orthonormal basis for $L^2_w(D)$, where $w(x,y) = u(x)v(y)$.

(iii) One can consider products with more than two factors. For example, suppose that in addition to the data in the theorem we have an orthonormal basis $\{\theta_n\}$ for $L^2(\alpha,\beta)$. Then the products $\phi_l(x)\psi_m(y)\theta_n(z)$ form an orthonormal basis for $L^2(D)$, where
$$D = \Big\{(x,y,z) : a \le x \le b,\ c \le y \le d,\ \alpha \le z \le \beta\Big\}.$$

(iv) One can start with an orthonormal basis $\{\psi_n\}_{n=1}^\infty$ for $L^2(c,d)$, and for *each* n a different orthonormal basis $\{\phi_{m,n}\}_{m=1}^\infty$ for $L^2(a,b)$. Then $\{\chi_{mn}\}_{m,n=1}^\infty$ is an orthonormal basis for $L^2(D)$, where $\chi_{mn}(x,y) = \phi_{m,n}(x)\psi_n(y)$. This situation will turn up in Chapters 5 and 6; in particular, see Theorem 5.4 of §5.5.

One more comment about the theorem should be made. The assertion that $\{\chi_{mn}\}$ is a basis should mean that if $f \in L^2(D)$ then $f = \sum \langle f, \chi_{mn} \rangle \chi_{mn}$, but one must assign a precise meaning to such a double infinite series. In fact, there is no problem. One arranges the terms $\langle f, \chi_{m,n} \rangle \chi_{mn}$ into a single sequence in any way one wishes, and the resulting ordinary infinite series always converges in norm to f.

With this bit of machinery in hand, we can find useful series expansions for functions of two or more variables. The most basic example is the multiple Fourier series for periodic functions. Suppose, to be specific, that we wish to study functions $f(x,y)$ that are 1-periodic in each variable: $f(x+1, y) = f(x,y)$ and $f(x, y+1) = f(x,y)$. (Functions of this sort arise, for example, in the theory of crystal lattices in solid-state physics.) Such doubly periodic functions are competely determined by their restrictions to the unit square

$$S = [0,1] \times [0,1] = \Big\{(x,y) : 0 \le x, y \le 1\Big\}.$$

We already know that $\{e^{2\pi inx}\}_{-\infty}^{\infty}$ is an orthonormal basis for $L^2(0,1)$, so it follows that

$$\left\{\chi_{mn}(x,y) = e^{2\pi i(mx+ny)} : -\infty < m,n < \infty\right\}$$

is an orthonormal basis for $L^2(S)$. Thus we can expand any doubly periodic f that is square-integrable on S in a double Fourier series:

$$f = \sum_{m,n=-\infty}^{\infty} c_{mn}\chi_{mn}, \qquad c_{mn} = \langle f, \chi_{mn}\rangle = \int_0^1\int_0^1 f(x,y)e^{-2\pi i(mx+ny)}\,dx\,dy,$$

where the series converges (at least) in norm. (The reader should be warned that the question of pointwise convergence of multiple Fourier series is even more delicate than in the one-dimensional case, but norm convergence works equally easily in any number of dimensions.)

Similarly, we can form multiple Fourier cosine or sine series, or combine other orthonormal bases arising from Sturm-Liouville problems, to construct bases for functions on rectangular regions; and this procedure can be used to solve boundary value problems in dimensions $n > 1$. We illustrate this with some examples.

Example 1. We analyze the vibrations of an elastic membrane stretched across a rectangular frame. That is, we study the following boundary value problem for the wave equation in two space dimensions:

$$u_{tt} = c^2(u_{xx} + u_{yy}) \text{ for } 0 < x < l,\ 0 < y < L,$$
$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y),$$
$$u(0,y,t) = u(l,y,t) = u(x,0,t) = u(x,L,t) = 0.$$

It is pretty clear that we shall want to use a double Fourier sine series to solve this problem, but let us see explicitly how separation of variables leads to this construction. Neglecting the initial conditions for the moment, we look for product solutions $X(x)Y(y)T(t)$ of the wave equation in the rectangle with zero boundary values. The wave equation for such functions is

$$XYT'' = c^2(X''YT + XY''T), \quad \text{or} \quad \frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

The quantities on either side of the last equation must equal some constant, which we shall call $-\nu^2$, so $T'' + \nu^2c^2T = 0$ and

$$\frac{X''}{X} = -\frac{Y''}{Y} - \nu^2.$$

Now the quantities on either side of *this* equation must equal another constant, which we call $-\mu^2$. Taking the boundary conditions into account, we therefore have

$$X'' + \mu^2X = 0, \qquad X(0) = X(l) = 0,$$
$$Y'' + (\nu^2 - \mu^2)Y = 0, \qquad Y(0) = Y(L) = 0,$$

and hence

$$\mu^2 = \left(\frac{m\pi}{l}\right)^2, \qquad X(x) = \sin\frac{m\pi x}{l}, \qquad (m = 1, 2, 3, \ldots),$$
$$\nu^2 - \mu^2 = \left(\frac{n\pi}{L}\right)^2, \qquad Y(y) = \sin\frac{n\pi y}{L}, \qquad (n = 1, 2, 3, \ldots).$$

Finally, since $T'' + \nu^2 c^2 T = 0$, we have

$$T(t) = a_\nu \cos\nu ct + b_\nu \sin\nu ct \quad \text{where } \nu^2 = \mu^2 + (\nu^2 - \mu^2) = \left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{L}\right)^2.$$

Thus we have the following solutions of the wave equation in the rectangle with zero boundary values:

$$u(x, y, t)$$
$$= \sum_{m,n=1}^{\infty} \sin\frac{m\pi x}{l}\sin\frac{n\pi y}{L}\left(a_{mn}\cos\pi ct\sqrt{\frac{m^2}{l^2} + \frac{n^2}{L^2}} + b_{mn}\sin\pi ct\sqrt{\frac{m^2}{l^2} + \frac{n^2}{L^2}}\right).$$

The coefficients a_{mn} and b_{mn} are determined by the initial conditions in the usual way: One expands f and g in their double Fourier sine series and matches coefficients with those of $u(x, y, 0)$ and $u_t(x, y, 0)$. Specifically,

$$u(x, y, 0) = f(x, y) = \sum_{m,n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{l}\sin\frac{n\pi y}{L},$$

so that

$$a_{mn} = \frac{4}{lL}\int_0^l \int_0^L f(x, y)\sin\frac{m\pi x}{l}\sin\frac{n\pi y}{L}\,dy\,dx.$$

Qualitatively, the interesting feature here is the set of allowable frequencies of vibration, namely,

$$\left\{\pi c\sqrt{(m/l)^2 + (n/L)^2} : m, n = 1, 2, 3, \ldots\right\}.$$

In contrast to the case of 1-dimensional vibrations, these are not integer multiples of a fundamental frequency. For example, if $l = L = \pi c$, the lowest frequencies are $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{13}$, $\sqrt{17}$, and so forth. For this reason a rectangular membrane does not usually produce a musical sound as it vibrates. (The more commonly encountered case of a circular membrane will be studied in Chapter 5.)

Example 2. We consider heat flow in a rectangular solid

$$D = \left\{(x, y, z) : 0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z \le l_3\right\},$$

where the top and bottom faces are held at temperature zero and the other four faces are insulated. Thus, if the initial temperature is $f(x,y,z)$, the problem to be solved is

$$u_t = k(u_{xx} + u_{yy} + u_{zz}), \qquad u(x,y,z,0) = f(x,y,z),$$
$$u(x,y,0,t) = u(x,y,l_3,t) = 0,$$
$$u_x(0,y,z,t) = u_x(l_1,y,z,t) = u_y(x,0,z,t) = u_y(x,l_2,z,t) = 0.$$

The process of separation of variables works here just as in the previous example, except that there is one more step (because there is one more variable), and the boundary conditions lead to Fourier cosine series in x and y rather than Fourier sine series. We leave it to the reader to work through the details; the upshot is that

$$u(x,y,z,t) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=1}^{\infty} \epsilon_{n_1n_2} a_{n_1n_2n_3} \exp\left[-\left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} + \frac{n_3^2}{l_3^2}\right)\pi^2 kt\right]$$
$$\times \cos\frac{n_1\pi x}{l_1}\cos\frac{n_2\pi y}{l_2}\sin\frac{n_3\pi z}{l_3}.$$

Here $\epsilon_{n_1n_2}$ equals 1 when n_1 and n_2 are both nonzero, $\frac{1}{2}$ when one of n_1 and n_2 is zero but not both, and $\frac{1}{4}$ when $n_1 = n_2 = 0$ (this is to account for the usual factor of $\frac{1}{2}$ in the constant term of a Fourier cosine series), and the coefficients $a_{n_1n_2n_3}$ are the Fourier coefficients of the initial temperature:

$$a_{n_1n_2n_3} = \frac{8}{l_1l_2l_3}\int_0^{l_1}\int_0^{l_2}\int_0^{l_3} f(x,y,z)\cos\frac{n_1\pi x}{l_1}\cos\frac{n_2\pi y}{l_2}\sin\frac{n_3\pi z}{l_3}\,dz\,dy\,dx.$$

Example 3. Suppose the rectangular box D of Example 2 is filled with a distribution of electric charge with density $\rho(x,y,z)$, and the faces of the box are grounded so that their electrostatic potential is zero. What is the potential inside the box? What we want is the solution of

$$u_{xx} + u_{yy} + u_{zz} = -4\pi\rho(x,y,z) \text{ in } D,$$
$$u(0,y,z) = u(l_1,y,z) = u(x,0,z) = u(x,l_2,z) = u(x,y,0) = u(x,y,l_3) = 0.$$

Here Technique 2 of §4.1 is effective. Namely, the zero boundary conditions suggest the use of Fourier sine series in each variable, so we expand u in such a series:

$$u(x,y,z) = \sum_{n_1,n_2,n_3=1}^{\infty} b_{n_1n_2n_3}\sin\frac{n_1\pi x}{l_1}\sin\frac{n_2\pi y}{l_2}\sin\frac{n_3\pi z}{l_3}.$$

Computing $\nabla^2 u$ by termwise differentiation, we find

$$\nabla^2 u(x,y,z)$$
$$= -\sum_{n_1,n_2,n_3=1}^{\infty}\left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} + \frac{n_3^2}{l_3^2}\right)\pi^2 b_{n_1n_2n_3}\sin\frac{n_1\pi x}{l_1}\sin\frac{n_2\pi y}{l_2}\sin\frac{n_3\pi z}{l_3}.$$

This must be the multiple sine series for $-4\pi\rho$, so we can solve immediately for the coefficients $b_{n_1 n_2 n_3}$:

$$b_{n_1 n_2 n_3} = \frac{32}{\pi l_1 l_2 l_3}\left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} + \frac{n_3^2}{l_3^2}\right)^{-1} \times \int_0^{l_1}\int_0^{l_2}\int_0^{l_3} \rho(x,y,z)\sin\frac{n_1\pi x}{l_1}\sin\frac{n_2\pi y}{l_2}\sin\frac{n_3\pi z}{l_3}\,dz\,dy\,dx.$$

This formal procedure for solving the problem can be justified easily if we impose conditions on ρ so that its Fourier coefficients tend rapidly to zero (e.g., ρ and its first few derivatives should vanish on the boundary of the box). It can also be justified by more sophisticated methods just under the condition that $\rho \in L^2(D)$.

EXERCISES

1. Show that if $v(x,t)$ and $w(y,t)$ are solutions of the 1-dimensional heat equation ($v_t = kv_{xx}$ and $w_t = kw_{yy}$), then $u(x,y,t) = v(x,t)w(y,t)$ satisfies the 2-dimensional heat equation. Can you generalize to 3 dimensions? Is the same result true for solutions of the wave equation?
2. Redo Example 1 in the text for the damped wave equation $u_{tt} + 2ku_t = c^2(u_{xx} + u_{yy})$. (Cf. Exercise 6, §4.2.)
3. Solve the wave equation (with general initial conditions) for a rectangular membrane if one pair of opposite edges is held fixed ($u(0,y,t) = u(l,y,t) = 0$) and the other pair is free ($u_y(x,0,t) = u_y(x,L,t) = 0$). How do the frequencies compare with those of Example 1?
4. Let D be the rectangular box of Example 2. Suppose the faces $z = 0$ and $z = l_3$ are insulated, and the other four faces are kept at temperature zero. Find the temperature $u(x,y,z,t)$ given that $u(x,y,z,0) = f(x,y)$. (Hint: Since f is independent of z and the z-faces are insulated, you can treat this as a 2-dimensional problem.)
5. In Example 3, suppose $l_1 = l_2 = l_3 = \pi$ and $\rho(x,y,z) = x$. What is the potential u?
6. Consider a cubic crystal lattice in which the charge density $\rho(x,y,z)$ is $2l$-periodic in each variable. (We suppose that the lattice extends infinitely in all directions; this is reasonable if its actual size is very large in comparison with the length scale being studied.) Use Fourier series to find a periodic solution of $\nabla^2 u = -4\pi\rho$, assuming that the net charge in any cube of side $2l$ is zero. (This assumption is generally valid in practice. Why is it needed mathematically?)

CHAPTER 5
BESSEL FUNCTIONS

The 2-dimensional wave equation in polar coordinates is

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}).$$

(See Appendix 4 for the calculation of the Laplace operator in polar coordinates.) If we apply separation of variables by taking $u = R(r)\Theta(\theta)T(t)$, the wave equation becomes

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta}.$$

Both sides must equal a constant, which we shall call $-\mu^2$. Setting the expression on the right equal to $-\mu^2$ and multiplying through by r^2, we obtain

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \mu^2r^2 = -\frac{\Theta''}{\Theta}.$$

Here both sides must equal another constant, which we call ν^2, so we arrive at the ordinary differential equations

$$T'' + c^2\mu^2T = 0 \quad \text{and} \quad \Theta'' + \nu^2\Theta = 0,$$

which are familiar enough, and

$$r^2R''(r) + rR'(r) + (\mu^2r^2 - \nu^2)R(r) = 0, \tag{5.1}$$

which is new. Equation (5.1) can be simplified a bit by the change of variable $x = \mu r$. (It is not yet clear whether we want μ to be real or imaginary, but at this point it doesn't matter; there is no harm in letting x be a complex variable.) That is, we substitute

$$R(r) = f(\mu r), \quad R'(r) = \mu f'(\mu r), \quad R''(r) = \mu^2 f''(\mu r), \quad \text{and} \quad r = x/\mu$$

into (5.1), obtaining

$$\left(\frac{x}{\mu}\right)^2 \mu^2 f''(x) + \frac{x}{\mu}\mu f'(x) + (x^2 - \nu^2)f(x) = 0,$$

or

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0. \tag{5.2}$$

This is **Bessel's equation of order** ν. It and its variants arise in many problems in physics and engineering, particularly where some sort of circular symmetry is involved. For this reason, its solutions are sometimes called **cylinder functions**, but we shall use the more common term **Bessel functions**. This chapter is an exposition of the basic properties of Bessel functions and some of their applications. Further information on Bessel functions can be found in Erdélyi et al. [21], Hochstadt [30], Lebedev [36], and especially the classic treatise of Watson [55].

5.1 Solutions of Bessel's equation

In this section we construct solutions of Bessel's equation (5.2) by means of power series. For the time being, the variable x and the parameter ν can be arbitrary complex numbers, although for most applications they will both be real and nonnegative. We shall make occasional references to the complex-variable properties of Bessel functions, but the reader who wishes to ignore them will not miss much as far as the material in this chapter is concerned.

At the outset, we note that (5.2) is unchanged if ν is replaced by $-\nu$, so we can take $\mathrm{Re}(\nu) \geq 0$ (in particular, $\nu \geq 0$ when ν is real) whenever it is convenient.

The differential equation (5.2) has a regular singular point at $x = 0$, so we expect to find solutions of the form

$$f(x) = \sum_0^\infty a_j x^{j+b} \qquad (a_0 \neq 0), \tag{5.3}$$

where the exponent b and the coefficients a_j are to be determined. If we substitute (5.3) into (5.2), we obtain

$$\sum_0^\infty a_j \Big[(j+b)(j+b-1)x^{j+b} + (j+b)x^{j+b} + x^{j+b+2} - \nu^2 x^{j+b}\Big] = 0. \tag{5.4}$$

We separate out the terms x^{j+b+2} and relabel the index of summation,

$$\sum_0^\infty a_j x^{j+b+2} = a_0 x^{2+b} + a_1 x^{3+b} + a_2 x^{4+b} + \cdots = \sum_2^\infty a_{j-2} x^{j+b},$$

thus transforming (5.4) into

$$\sum_0^\infty \Big[(j+b)^2 - \nu^2\Big] a_j x^{j+b} + \sum_2^\infty a_{j-2} x^{j+b} = 0.$$

Now, a power series can vanish identically only when all of its coefficients are zero, so we obtain the following sequence of equations:

$$\text{for } j = 0, \quad (b^2 - \nu^2)a_0 = 0, \tag{5.5}$$

$$\text{for } j = 1, \quad \left[(1+b)^2 - \nu^2\right]a_1 = 0, \tag{5.6}$$

$$\text{for } j \geq 2, \quad \left[(j+b)^2 - \nu^2\right]a_j + a_{j-2} = 0. \tag{5.7}$$

Since we assumed that $a_0 \neq 0$, equation (5.5) forces $b = \pm\nu$, and for the time being we take $b = \nu$. Then equation (5.6) becomes $(2\nu + 1)a_1 = 0$, so we must have $a_1 = 0$ except when $\nu = -\frac{1}{2}$; even when $\nu = -\frac{1}{2}$ it is consistent to take $a_1 = 0$, and we do so. Next, equation (5.7) with $b = \nu$ says that

$$a_j = -\frac{a_{j-2}}{(j+\nu)^2 - \nu^2} = -\frac{a_{j-2}}{j(j+2\nu)}. \tag{5.8}$$

From this recursion formula we can solve for all the even-numbered coefficients in terms of a_0:

$$a_2 = -\frac{a_0}{2(2+2\nu)}, \qquad a_4 = -\frac{a_2}{4(4+2\nu)} = \frac{a_0}{2\cdot 4(2+2\nu)(4+2\nu)},$$

and in general,

$$\begin{aligned} a_{2k} &= \frac{(-1)^k a_0}{2\cdot 4\cdots(2k)(2+2\nu)(4+2\nu)\cdots(2k+2\nu)} \\ &= \frac{(-1)^k a_0}{2^{2k}k!(1+\nu)(2+\nu)\cdots(k+\nu)}. \end{aligned} \tag{5.9}$$

In the same way, we obtain all of the odd-numbered coefficients in terms of a_1; but $a_1 = 0$, and hence

$$a_{2k+1} = 0.$$

The only time when this procedure runs into difficulties is when ν is a negative integer or half-integer. If $\nu = -n$, the numbers a_{2k} in (5.9) are ill-defined for $k \geq n$ because of a zero factor in the denominator, so in this case we do not obtain a solution. If $\nu = -n/2$ with n odd, the recursion formula (5.8) has a zero in the denominator for $j = n$, so our derivation of the string of equations $0 = a_1 = a_3 = \cdots$ breaks down at this point. However, if we rewrite (5.8) in the original form (5.7), then for $j = n$ it says that $0 \cdot a_n = a_{n-2}$. Since we already have $a_{n-2} = 0$, it is still consistent to take $a_n = 0$, and we do so. (We could take a_n to be something else; then we would get a different solution of the differential equation.)

In short, except when ν is a negative integer we have the solutions

$$f(x) = a_0 \sum_0^\infty \frac{(-1)^k x^{2k+\nu}}{2^{2k}k!(1+\nu)(2+\nu)\cdots(k+\nu)}$$

to Bessel's equation. It remains to pick the constant a_0, and the standard choice is

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

(See Appendix 3 for a discussion of the gamma function.) Since the functional equation $\Gamma(z+1) = z\Gamma(z)$ implies that

$$\Gamma(k+\nu+1) = (k+\nu)\cdots(1+\nu)\Gamma(\nu+1),$$

this choice of a_0 makes $f(x)$ equal to

$$J_\nu(x) = \sum_0^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu} \tag{5.10}$$

A simple application of the ratio test shows that this series is absolutely convergent for all $x \neq 0$ (and also for $x = 0$ when $\mathrm{Re}(\nu) > 0$ or $\nu = 0$). The function $J_\nu(x)$ thus defined is called the **Bessel function** (of the first kind) **of order** ν.

$J_\nu(x)$ is real when $x > 0$ and ν is real (the case we shall mainly be interested in). It tends to 0 as $x \to 0$ whenever $\mathrm{Re}(\nu) > 0$ and blows up as $x \to 0$ whenever $\mathrm{Re}(\nu) < 0$ and ν is not an integer. If we consider x as a complex variable, $J_\nu(x)$ is multivalued when ν is not an integer; to make a well-defined function we shall always take the principal branch of $(x/2)^\nu$ in (5.10). (That is, $(x/2)^\nu = e^{\nu \log(x/2)}$ where $-\pi < \mathrm{Im}\log(x/2) \leq \pi$.) However, $x^{-\nu}J_\nu(x)$ is an entire analytic function of x for any ν.

When ν is a nonnegative integer n, we can use the fact that $j! = \Gamma(j+1)$ to write

$$J_n(x) = \sum_0^\infty \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k+n} \qquad (n = 0, 1, 2, \ldots).$$

Also, it is to be observed that the definition (5.10) makes sense when ν is a negative integer, even though the formulas leading to it do not! Indeed, we recall that $1/\Gamma(z) = 0$ when $z = 0, -1, -2, \ldots$; hence if $\nu = -n$, we have $1/\Gamma(k+\nu+1) = 0$ for $k = 0, 1, \ldots, n-1$. Thus, the first n terms in the series (5.10) vanish, and by setting $k = j + n$ in (5.10) we find that

$$J_{-n}(x) = \sum_{k=n}^\infty \frac{(-1)^k}{k!(k-n)!} \left(\frac{x}{2}\right)^{2k-n} = \sum_{j=0}^\infty \frac{(-1)^{j+n}}{j!(j+n)!} \left(\frac{x}{2}\right)^{2j+n},$$

or

$$J_{-n}(x) = (-1)^n J_n(x). \tag{5.11}$$

(In effect, we have compensated for the zero in the denominators of (5.9) by taking the leading coefficient a_0 equal to 0.)

We arrived at (5.10) by taking $b = \nu$ in the recursion formula (5.7). It is easily checked that if we take $b = -\nu$ instead, we get the same results with ν replaced by $-\nu$ throughout; in other words, we end up with $J_{-\nu}(x)$. Thus, we have two solutions J_ν and $J_{-\nu}$ of Bessel's equation (5.2). When ν is not an integer they are clearly linearly independent, since

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(\nu+1)}, \qquad J_{-\nu}(x) \approx \frac{x^{-\nu}}{2^{-\nu}\Gamma(-\nu+1)} \quad \text{for } x \text{ near } 0.$$

In this case the general solution of (5.2) is a linear combination of J_ν and $J_{-\nu}$.

However, if ν is an integer then $J_{-\nu} = (-1)^\nu J_\nu$ by (5.11), so in this case we are still lacking a second independent solution. The standard way out of this difficulty is as follows. For ν not an integer, we define the **Weber function** or **Bessel function of the second kind** Y_ν by

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{5.12}$$

Y_ν is a linear combination of J_ν and $J_{-\nu}$, so it satisfies Bessel's equation (5.2). Also, since the coefficient of $J_{-\nu}(x)$ in (5.12) is nonzero, J_ν and Y_ν are linearly independent, and we may use them (rather than J_ν and $J_{-\nu}$) as a basis for solutions of (5.2). Now if we take ν to be an integer in (5.12), the expression on the right turns into the indeterminate 0/0, by (5.11) and the fact that $\cos n\pi = (-1)^n$. However, it can be shown that the limit as ν approaches an integer of $Y_\nu(x)$ exists and is finite for all $x \neq 0$, and that the function

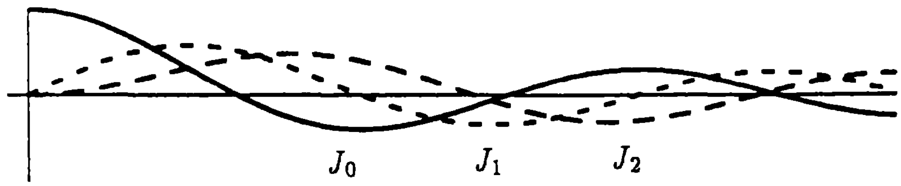
$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

thus defined is a solution of Bessel's equation. (See, for example, Lebedev [36], §5.4.) In fact, one can calculate $Y_n(x)$ from (5.12), (5.10), and l'Hôpital's rule. We shall not present the details since the explicit formula for Y_n will be of no particular use to us. (However, see Exercises 3–5.) The most important feature of Y_n is its asymptotic behavior as $x \to 0$, which for $n \geq 0$ is given by

$$Y_n(x) \approx -\frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n} \quad \text{as } x \to 0 \qquad (n = 1, 2, 3, \ldots),$$
$$Y_0(x) \approx \frac{2}{\pi}\log\frac{x}{2} \quad \text{as } x \to 0.$$

(It follows from (5.11) and (5.12) that $Y_{-n} = (-1)^n Y_n$, so the case $n < 0$ is also covered.) In particular, $Y_n(x)$ blows up as $x \to 0$, whereas $J_n(x)$ remains bounded; so Y_n and J_n are linearly independent and form a basis for all solutions of Bessel's equation of order n. See Figure 5.1.

One may wonder why Y_ν was chosen as the particular linear combination (5.12) of J_ν and $J_{-\nu}$. If the only object had been to find an expression that gives a second solution for integer values of ν, many other formulas would have worked equally well; indeed, several variants of Y_ν are found in the literature. (See Watson [55], §3.5.) The answer has to do with the behavior of $J_\nu(x)$ and $Y_\nu(x)$ as $x \to \infty$, a matter that will be discussed in §5.3.



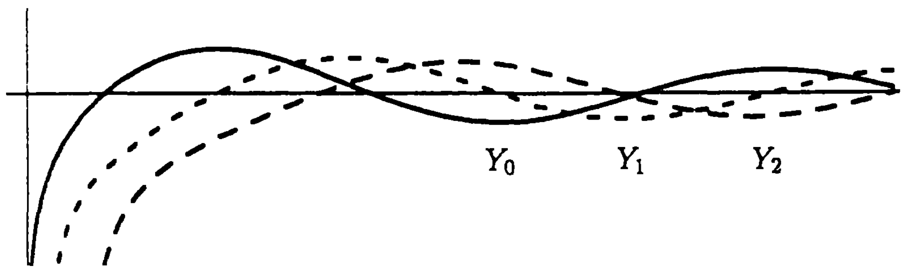


FIGURE 5.1. Graphs of some Bessel functions on the interval $0 \leq x \leq 10$. Top: J_0 (solid), J_1 (short dashes), and J_2 (long dashes). Bottom: Y_0 (solid), Y_1 (short dashes), and Y_2 (long dashes).

EXERCISES

1. Let f_1 and f_2 be solutions of Bessel's equation of order ν, and let W denote their Wronskian $f_1 f_2' - f_1' f_2$.
   a. Use Bessel's equation to show that $W'(x) = -W(x)/x$ and hence show that $W(x) = C/x$ for some constant C.
   b. Show that if $f_1 = J_\nu$ and $f_2 = J_{-\nu}$, then $W(x) = -2\sin \nu\pi/\pi x$. (Hint: Consider the limiting behavior of $J_\nu(x)$ and $J_{-\nu}(x)$ as $x \to 0$, and use the fact that $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin \nu\pi$.)
   c. Show that if $f_1 = J_\nu$ and $f_2 = Y_\nu$ then $W(x) = 2/\pi x$.
2. Deduce from (5.11) and (5.12) that $Y_{-n} = (-1)^n Y_n$ when n is an integer.
3. When $\nu > 0$ and ν is not an integer, $Y_\nu(x)$ is given by a power series whose lowest-order term is $c_\nu (x/2)^{-\nu}$. What is the constant c_ν? Show that if n is a positive integer, $\lim_{\nu\to n} c_\nu = -(n-1)!/\pi$. (Hint: $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin \nu\pi$.)
4. Show that when n is an integer,
$$Y_n(x) = \frac{1}{\pi}\left[\frac{\partial J_\nu}{\partial \nu} + (-1)^{n+1}\frac{\partial J_{-\nu}}{\partial \nu}\right]_{\nu=n}.$$
5. Let $\psi(z) = \Gamma'(z)/\Gamma(z)$. Use Exercise 4 to show that
$$Y_0(x) = \frac{2}{\pi} J_0(x) \log \frac{x}{2} + \frac{2}{\pi}\sum_0^\infty \frac{(-1)^{j+1}\psi(j+1)}{(j!)^2}\left(\frac{x}{2}\right)^{2j}.$$
(For the ambitious reader: Find the analogous formula for Y_n, $n > 0$. The answer can be found in Lebedev [36], §5.5.)

5.2 Bessel function identities

There is a vast assortment of formulas relating Bessel functions to one another and to various other special functions; some are algebraic relations, and others involve integrals or infinite series. In this section we discuss a few of the most elementary and useful of them.

To begin with, there is a nice set of algebraic identities relating J_ν and its derivative to the "adjacent" functions $J_{\nu-1}$ and $J_{\nu+1}$:

The Recurrence Formulas. *For all x and ν,*

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x), \tag{5.13}$$

$$\frac{d}{dx}\left[x^{\nu} J_\nu(x)\right] = x^{\nu} J_{\nu-1}(x), \tag{5.14}$$

$$x J_\nu'(x) - \nu J_\nu(x) = -x J_{\nu+1}(x), \tag{5.15}$$

$$x J_\nu'(x) + \nu J_\nu(x) = x J_{\nu-1}(x), \tag{5.16}$$

$$x J_{\nu-1}(x) + x J_{\nu+1}(x) = 2\nu J_\nu(x), \tag{5.17}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x). \tag{5.18}$$

Proof: To prove (5.13) we use the power series (5.10) for $J_\nu(x)$:

$$\begin{aligned}\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] &= \frac{d}{dx}\sum_0^\infty \frac{(-1)^k x^{2k}}{2^{2k+\nu} k!\Gamma(\nu+k+1)} = \sum_1^\infty \frac{(-1)^k (2k) x^{2k-1}}{2^{2k+\nu} k!\Gamma(\nu+k+1)} \\ &= \sum_1^\infty \frac{(-1)^k x^{2k-1}}{2^{2k+\nu-1}(k-1)!\Gamma(\nu+k+1)}.\end{aligned}$$

We relabel the index k in the last sum as $k+1$, thus obtaining

$$\sum_0^\infty \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+\nu+1} k!\Gamma(\nu+k+2)} = -x^{-\nu}\sum_0^\infty \frac{(-1)^k x^{2k+\nu+1}}{2^{2k+\nu+1} k!\Gamma(\nu+k+2)} = -x^{-\nu} J_{\nu+1}(x).$$

The proof of (5.14) is similar:

$$\begin{aligned}\frac{d}{dx}\left[x^{\nu} J_\nu(x)\right] &= \frac{d}{dx}\sum_0^\infty \frac{(-1)^k x^{2k+2\nu}}{2^{2k+\nu} k!\Gamma(k+\nu+1)} = \sum_0^\infty \frac{(-1)^k (2k+2\nu) x^{2k+2\nu-1}}{2^{2k+\nu} k!\Gamma(k+\nu+1)} \\ &= x^\nu \sum_0^\infty \frac{(-1)^k x^{2k+\nu-1}}{2^{2k+\nu-1} k!\Gamma(k+\nu)} = x^\nu J_{\nu-1}(x).\end{aligned}$$

Next, performing the indicated differentiations in (5.13) and (5.14), we obtain

$$(-\nu)x^{-\nu-1} J_\nu(x) + x^{-\nu} J_\nu'(x) = -x^{-\nu} J_{\nu+1}(x),$$
$$\nu x^{\nu-1} J_\nu(x) + x^{\nu} J_\nu'(x) = x^{\nu} J_{\nu-1}(x).$$

(5.15) and (5.16) are obtained by multiplying these equations through by $x^{\nu+1}$ and $x^{-\nu+1}$, respectively. Finally, (5.17) and (5.18) follow by subtracting and adding (5.15) and (5.16). ∎

As a first application of these formulas, we shall show that the Bessel functions of half-integer order can be expressed in terms of familiar elementary functions. To start with, consider $J_{-1/2}(x)$. Since

$$2^k k! = 2^k(1 \cdot 2 \cdot 3 \cdots k) = 2 \cdot 4 \cdot 6 \cdots (2k)$$

and

$$\begin{aligned} 2^k\Gamma(k + \tfrac{1}{2}) &= 2^k(k - \tfrac{1}{2})(k - \tfrac{3}{2}) \cdots (\tfrac{1}{2})\Gamma(\tfrac{1}{2}) \\ &= (2k - 1)(2k - 3) \cdots (1)\sqrt{\pi}, \end{aligned}$$

we have

$$J_{-1/2}(x) = \sum_0^\infty \frac{(-1)^k x^{2k-(1/2)}}{2^{-1/2}\left[2^k k!\right]\left[2^k\Gamma(k + \frac{1}{2})\right]} = \left(\frac{2}{\pi x}\right)^{1/2} \sum_0^\infty \frac{(-1)^k x^{2k}}{(2k)!}$$

But the last series is just the Taylor series of $\cos x$. This, together with a similar calculation for $J_{1/2}(x)$ (see Exercise 1), shows that

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x. \tag{5.19}$$

It now follows by repeated application of the recurrence formula (5.17) that *whenever* $\nu - \frac{1}{2}$ *is an integer,*

$$J_\nu(x) = x^{-1/2}\left[P_\nu(x)\cos x + Q_\nu(x)\sin x\right]$$

where P_ν *and* Q_ν *are rational functions.* For example, taking $\nu = \frac{1}{2}$ in (5.17), we find that

$$J_{3/2}(x) = x^{-1}J_{1/2}(x) - J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} - \cos x\right).$$

It is to be emphasized that the Bessel functions of half-integer order are the *only* ones that are elementary functions.

Our next group of results concerns the Bessel functions of integer order. We recall that if $\{a_n\}$ is a sequence of numbers, the *generating function* for a_n is the power series $\sum a_n z^n$.

The Generating Function for $J_n(x)$. *For all* x *and all* $z \neq 0$,

$$\sum_{-\infty}^{\infty} J_n(x) z^n = \exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right]. \tag{5.20}$$

Proof: We begin by observing that

$$\exp\frac{xz}{2} = \sum_0^\infty \frac{z^j}{j!}\left(\frac{x}{2}\right)^j, \qquad \exp\frac{-x}{2z} = \sum_0^\infty \frac{(-1)^k}{z^k k!}\left(\frac{x}{2}\right)^k.$$

Since these series are absolutely convergent, they can be multiplied together and the terms in the resulting double series summed in any order:

$$\exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right] = \sum_{j,k=0}^\infty \frac{(-1)^k z^{j-k}}{j!k!}\left(\frac{x}{2}\right)^{j+k}.$$

We sum this series by first adding up all the terms involving a given power z^n of z and then summing over n. That is, we set $j - k = n$ or $j = k + n$ and obtain (with the understanding that $1/(k+n)! = 1/\Gamma(k+n+1) = 0$ when $k + n < 0$)

$$\begin{aligned}\exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right] &= \sum_{n=-\infty}^\infty \left[\sum_{k=0}^\infty \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right] z^n \\ &= \sum_{-\infty}^\infty J_n(x) z^n.\end{aligned}$$

∎

In (5.20), z can be any nonzero complex number. In particular, we can take $z = e^{i\theta}$, in which case $\frac{1}{2}(z - z^{-1}) = i\sin\theta$, so that

$$e^{ix\sin\theta} = \sum_{-\infty}^\infty J_n(x)e^{in\theta}. \tag{5.21}$$

But the expression on the left is a 2π-periodic function of θ, and the expression on the right is visibly a Fourier series! Hence, the coefficients $J_n(x)$ in the series must be given by the usual formula for Fourier coefficients, namely,

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta - in\theta}\,d\theta. \tag{5.22}$$

Here is a new formula for J_n, quite different from its original definition as a power series and in some respects more useful. For instance, it shows immediately, what is not at all evident from the power series, that $|J_n(x)| \le 1$ for all real x:

$$|J_n(x)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |e^{ix\sin\theta - in\theta}|\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta = 1 \qquad (x \in \mathbf{R}).$$

In fact, the same is true of all the derivatives of $J_n(x)$, for differentiation of (5.22) yields

$$J_n^{(k)}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(i\sin\theta)^k e^{ix\sin\theta - in\theta}\,d\theta.$$

There are several other formulas for J_n that are equivalent to (5.22). We collect them in a theorem.

Bessel's Integral Formulas. *For any $x \in \mathbf{C}$ and any integer n,*

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta - in\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta. \tag{5.23}$$

Moreover,

$$\begin{aligned} J_n(x) &= \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\cos n\theta\,d\theta \quad \textit{if } n \textit{ is even;} \\ J_n(x) &= \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin\theta)\sin n\theta\,d\theta \quad \textit{if } n \textit{ is odd.} \end{aligned} \tag{5.24}$$

Proof: The change of variable $\theta \to -\theta$ in (5.22) leads to

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\theta + in\theta}\,d\theta.$$

Adding this equation to (5.22) and dividing by 2, we obtain

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta)\,d\theta.$$

Here the integrand is an even function of θ, so the integral from $-\pi$ to π is twice the integral from 0 to π; hence (5.23) follows.

Next, in (5.23) we make the change of variable $\theta \to \pi - \theta$ and use the fact that $\sin(\pi - \theta) = \sin\theta$ and $\cos(\phi - n\pi) = (-1)^n\cos\phi$ to obtain

$$(-1)^n J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta + n\theta)\,d\theta.$$

We now add or subtract this equation to (5.23), depending on whether n is even or odd, and use the identities

$$\begin{aligned} \cos(a-b) + \cos(a+b) &= 2\cos a\cos b, \\ \cos(a-b) - \cos(a+b) &= 2\sin a\sin b, \end{aligned}$$

with the result that

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\,d\theta \qquad (n \text{ even}), \\ J_n(x) &= \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta\,d\theta \qquad (n \text{ odd}). \end{aligned}$$

The integrands in these equations are invariant under the substitution $\theta \to \pi - \theta$, i.e., symmetric about $\theta = \pi/2$, so the integrals from 0 to π are twice the integrals from 0 to $\pi/2$. This proves (5.24). ∎

If one is interested in calculating $J_n(x)$ numerically, the power series (5.10) is effective for small values of x. However, Bessel's integrals (evaluated by a numerical integration scheme such as Simpson's rule) are much more efficient when x is reasonably large. Similar but more complicated formulas exist for $J_\nu(x)$ when ν is not an integer.

EXERCISES

1. Show that $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$.

Use the recurrence formulas to prove the identities in Exercises 2–7.

2. $J_{-3/2}(x) = -\sqrt{2/\pi x}\,(x^{-1}\cos x + \sin x)$
3. $x^2 J_\nu''(x) = (\nu^2 - \nu - x^2)J_\nu(x) + xJ_{\nu+1}(x)$
4. $\int_0^x sJ_0(s)\,ds = xJ_1(x)$ and $\int_0^x J_1(s)\,ds = 1 - J_0(x)$
5. $\int_0^x s^2 J_1(s)\,ds = 2xJ_1(x) - x^2 J_0(x)$
6. $\int_0^x J_3(s)\,ds = 1 - J_2(x) - 2x^{-1}J_1(x)$ (Hint: $J_3(s) = s^2 \cdot s^{-2}J_3(s)$.)
7. $(\nu+3)x^2 J_\nu(x) + 2(\nu+2)\left[x^2 - 2(\nu+1)(\nu+3)\right]J_{\nu+2}(x) + (\nu+1)x^2 J_{\nu+4}(x) = 0$ (Hint: Use (5.17) to express $J_{\nu+2}$ in terms of $J_{\nu+1}$ and $J_{\nu+3}$; then use (5.17) on the latter functions.)
8. Prove the reduction formula

$$\int_0^x s^n J_0(s)\,ds = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int_0^x s^{n-2}J_0(s)\,ds.$$

(Hint: Integrate by parts, using the facts that $(xJ_1)' = xJ_0$ and $J_0' = -J_1$.) Use this formula to show that

$$\int_0^x s^3 J_0(s)\,ds = (x^3 - 4x)J_1(x) + 2x^2 J_0(x),$$
$$\int_0^x s^5 J_0(s)\,ds = x(x^2 - 8)^2 J_1(x) + 4x^2(x^2 - 8)J_0(x).$$

Exercises 9–11 are applications of formulas (5.11) and (5.21).

9. Show that for all x,

$$J_0(x) + 2\sum_1^\infty J_{2n}(x) = 1, \qquad \sum_1^\infty (2n-1)J_{2n-1}(x) = \frac{x}{2}.$$

(Hint: To obtain the second formula, differentiate both sides of (5.21).)

10. Show that for each fixed x, $\lim_{n\to\infty} n^k J_n(x) = 0$ for all k. (Hint: Theorem 2.6, §2.3.)
11. Show that for all real x,

$$J_0(x)^2 + 2\sum_1^\infty J_n(x)^2 = 1.$$

(Hint: Parseval's equation.) Deduce that $|J_0(x)| \le 1$ and $|J_n(x)| \le 2^{-1/2}$ for $n > 0$.

12. Verify directly from the formula $J_0(x) = (2/\pi)\int_0^{\pi/2}\cos(x\sin\theta)\,d\theta$ that J_0 satisfies Bessel's equation of order zero.
13. Deduce from (5.24) that for $n = 0, 1, 2, \ldots$,

$$J_{2n}(x) = (-1)^n \frac{2}{\pi}\int_0^{\pi/2} \cos(x\cos\theta)\cos 2n\theta\,d\theta,$$

$$J_{2n+1}(x) = (-1)^n \frac{2}{\pi}\int_0^{\pi/2} \sin(x\cos\theta)\cos(2n-1)\theta\,d\theta.$$

14. (Poisson's integral for J_ν) Show that if $\mathrm{Re}(\nu) > -\frac{1}{2}$,

$$J_\nu(x) = \frac{x^\nu}{2^\nu \pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\nu-(1/2)}e^{ixt}\,dt.$$

(Hint: Write $e^{ixt} = \sum_0^\infty (ixt)^j/j!$ and integrate the series term by term. The resulting integrals can be evaluated in terms of the beta function; see Appendix 3.) Deduce that

$$J_\nu(x) = \frac{x^\nu}{2^\nu \pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_{-\pi/2}^{\pi/2} e^{ix\sin\theta}\cos^{2\nu}\theta\,d\theta.$$

5.3 Asymptotics and zeros of Bessel functions

The power series (5.10) that defines $J_\nu(x)$ readily yields precise information about $J_\nu(x)$ when x is near 0 but is of little value when x is large. The integral formulas of §5.2 are somewhat more helpful in this regard, but we still have only a rather vague idea of how $J_\nu(x)$ behaves as $x \to \infty$. One good reason to be concerned about this comes from the applications of Bessel functions to partial differential equations that were sketched in the introduction to this chapter and will be discussed more fully in §5.5. The solutions of these equations involve the functions $J_\nu(\mu x)$ where μ may be very large; and the boundary conditions generally turn into equations such as $J_\nu(\mu) = 0$ or $cJ_\nu(\mu) + \mu J_\nu'(\mu) = 0$. Hence, we are particularly interested in locating the zeros of functions such as $J_\nu(x)$ or $cJ_\nu(x) + xJ_\nu'(x)$.

In this section we shall assume throughout that ν is real and x is positive. The results can, however, be extended to complex ν and x with suitable modifications.

We can obtain a clue as to the behavior of Bessel functions for large x by the following device. Suppose $f(x)$ is a solution of Bessel's equation

$$x^2 f''(x) + xf'(x) + (x^2 - \nu^2)f(x) = 0.$$

Let us set $g(x) = x^{1/2}f(x)$, so that

$$f(x) = \frac{g(x)}{x^{1/2}}, \quad f'(x) = \frac{g'(x)}{x^{1/2}} - \frac{g(x)}{2x^{3/2}}, \quad f''(x) = \frac{g''(x)}{x^{1/2}} - \frac{g'(x)}{x^{3/2}} + \frac{3g(x)}{4x^{5/2}}.$$

If we substitute these formulas into Bessel's equation and multiply through by $x^{-3/2}$, it reduces to

$$g''(x) + g(x) + \frac{\frac{1}{4} - \nu^2}{x^2} g(x) = 0.$$

Now, when x is very large the coefficient $(\frac{1}{4} - \nu^2)/x^2$ is very small, so it is reasonable to expect solutions of this equation to behave for large x like solutions of $g''(x) + g(x) = 0$. But the latter are just the linear combinations of $\sin x$ and $\cos x$ or, equivalently, functions of the form $a\cos(x+b)$ or $a\sin(x+b)$. Hence, the solutions $f(x)$ of Bessel's equation should look like $ax^{-1/2}\cos(x+b)$ or $ax^{-1/2}\sin(x+b)$.

These intuitive ideas turn out to be completely correct, and there are various ways of justifying them rigorously. It is a somewhat more arduous task to identify the particular function $ax^{-1/2}\cos(x+b)$ that corresponds to a particular solution of Bessel's equation such as $J_\nu(x)$. Nonetheless, the answer is known, and here it is.

Theorem 5.1. *For each $\nu \in \mathbf{R}$ there is a constant C_ν such that if $x \geq 1$,*

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + E_\nu(x) \quad \textit{where} \quad |E_\nu(x)| \leq \frac{C_\nu}{x^{3/2}}. \tag{5.25}$$

Thus, $J_\nu(x) \approx ax^{-1/2}\cos(x+b)$ where $a = \sqrt{2/\pi}$ and $b = (2\nu - 1)/4\pi$, with an error term that tends to zero like $x^{-3/2}$, that is, one order faster than the $x^{-1/2}$ in the main term. It is important to note that the constant C_ν in the error estimate grows with $|\nu|$. (5.25) is a useful formula for $J_\nu(x)$ only when $x \gg |\nu|$.

The proof of this result requires some rather sophisticated techniques involving Laplace transforms and contour integrals. We shall give it in §8.6. The method discussed there actually gives much more precise information about the error terms $E_\nu(x)$, and it also gives results for nonreal x and ν, but the theorem as stated here is all we shall need. Further results on the asymptotics of Bessel functions can be found in Watson [55], Chapters VII and VIII.

At this point we can explain the significance of the second solution $Y_\nu(x)$ introduced in §5.1. Replacing ν by $-\nu$ in (5.25), we have

$$J_{-\nu}(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \qquad (x \gg 0),$$

and

$$\cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = \cos(\nu\pi)\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \sin(\nu\pi)\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

Hence, if ν is not an integer, by combining these formulas with (5.12) and (5.25) we find that

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{5.26}$$

for $x \gg 0$, where the error term is again bounded by some constant times $x^{-3/2}$. That is, if we think of J_ν as approximately a damped cosine wave, then Y_ν is the corresponding damped sine wave. One can show that the relation (5.26) continues to hold in the limiting cases when ν is an integer.

By combining (5.25) with the recurrence formula (5.16), we also obtain an asymptotic formula for J_ν'. Indeed, we have

$$J_\nu'(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x).$$

But by (5.25),

$$\begin{aligned} J_{\nu-1}(x) &= \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{(\nu-1)\pi}{2} - \frac{\pi}{4}\right) + E_{\nu-1}(x) \\ &= -\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + E_{\nu-1}(x), \end{aligned}$$

and

$$\left|\frac{\nu}{x} J_\nu(x)\right| \leq \left(\frac{2}{\pi}\right)^{1/2} \frac{|\nu|}{x^{3/2}} + \frac{|\nu| C_\nu}{x^{5/2}}.$$

For $x \geq 1$ we have $x^{-5/2} < x^{-3/2}$, and hence

$$J_\nu'(x) = -\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \widetilde{E}_\nu(x), \qquad |\widetilde{E}_\nu(x)| \leq \frac{C_\nu'}{x^{3/2}}. \tag{5.27}$$

Of course this is the result we would get by differentiating (5.25) *if* we knew that the derivative $E_\nu'(x)$ of the error term is also dominated by $x^{-3/2}$. But since the derivative of a small function need not be small, this is not automatic; the recurrence formula saves us the trouble of calculating E_ν'.

We now turn to the problem of describing the positive solutions of the equations

$$aJ_\nu(x) + bxJ_\nu'(x) = 0 \tag{5.28}$$

where $\nu \geq 0$, $a, b \in \mathbf{R}$, and $(a, b) \neq (0, 0)$. As we have indicated, these will be of importance in solving boundary value problems.

In the first place, the function $x^{-\nu}[aJ_\nu(x) + bJ_\nu'(x)]$ is an entire analytic function of the complex variable x, so its zeros are all isolated; that is, there are only finitely many zeros in any bounded region of the complex plane. It follows that the positive solutions of the equation (5.28) can be arranged in an increasing sequence,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

with $\lim \lambda_j = \infty$. The main features of interest in this sequence are (i) the location of the first few terms λ_1, λ_2, etc., and (ii) the asymptotic behavior of λ_k as $k \to \infty$. (There is a sort of "grey area" in between where not much precise information is known.) To investigate the second aspect, we must distinguish between the cases $b = 0$ and $b \neq 0$; that is, we consider separately the equations

$$J_\nu(x) = 0 \qquad \text{and} \qquad cJ_\nu(x) + xJ_\nu'(x) = 0 \quad (c = a/b).$$

First, the case $b = 0$. We can read off the asymptotics of the sequence λ_k of positive zeros of J_ν immediately from the preceding results on the asymptotics of the functions J_ν and J_ν'. Indeed, from (5.25) we know that $J_\nu(x)$ is approximately $x^{-1/2}\cos(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$ for large x, so its zeros should occur at approximately the same places as those of $\cos(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$, namely, $(j + \frac{1}{2}\nu + \frac{3}{4})\pi$ for large positive integers j. This can be made precise by the following lemma.

Lemma 5.1. *Suppose $f(x)$ is a differentiable real-valued function that satisfies*

$$|f(x) - \cos x| \leq \epsilon \quad \textit{and} \quad |f'(x) + \sin x| \leq \epsilon \quad \textit{for} \quad x \geq M\pi,$$

where $\epsilon \ll 1$. Then for all integers $m \geq M$, f has exactly one zero z_m in each interval $[m\pi, (m+1)\pi]$, and it satisfies $|z_m - (m + \frac{1}{2}\pi)| < 2\epsilon$.

Proof: We shall sketch the ideas and leave it to the reader to make the details precise. First, we have $\cos m\pi = (-1)^m$ and $\cos(m + \frac{1}{2})\pi = 0$. Since $|f(x) - \cos x| < \epsilon$, f has opposite signs at $m\pi$ and $(m+1)\pi$, so it must have at least one zero in between, and all such zeros must occur near $(m + \frac{1}{2})\pi$. But $\sin(m + \frac{1}{2}\pi) = (-1)^m$, so the condition $|f'(x) + \sin x| < \epsilon$ implies that $f'(x) \neq 0$ for x near $(m + \frac{1}{2})\pi$. Hence f is strictly increasing or decreasing near $(m + \frac{1}{2})\pi$, so it can have at most one zero there. ∎

We can apply Lemma 5.1 to the function

$$f(x) = \tilde{x}^{1/2} J_\nu(\tilde{x}), \qquad \tilde{x} = x + \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi.$$

Since $f'(x) = \tilde{x}^{1/2}J_\nu'(\tilde{x}) + \frac{1}{2}\tilde{x}^{-1/2}J_\nu(\tilde{x})$,the estimates (5.25) and (5.27) show that $f(x)$ and $f'(x)$ differ from $\cos x$ and $-\sin x$ by errors that are bounded by a constant times x^{-1} and so can be made as small as we please by taking x large enough. The conclusion is that for large M, the solutions of the equation $J_\nu(x) = 0$ such that $x > M\pi$ are approximately at the points $(m + \frac{1}{2}\nu + \frac{3}{4})\pi$ where m is an integer. Moreover, the approximation gets better the larger we take M.

A similar argument applies to the function

$$f(x) = c\tilde{x}^{-1/2}J_\nu(\tilde{x}) + \tilde{x}^{1/2}J_\nu'(\tilde{x}), \qquad \tilde{x} = x + \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi.$$

Indeed, the second term in $f(x)$ is approximately $\sin(x + \frac{1}{2}\pi) = \cos x$ by (5.27) (note that we defined $\tilde{x}$ so as to make this shift of $\frac{1}{2}\pi$!), whereas the first term is dominated by x^{-1} by (5.25). Moreover,

$$f'(x) = -\tfrac{1}{2}c\tilde{x}^{-3/2}J_\nu(\tilde{x}) + c\tilde{x}^{-1/2}J_\nu'(\tilde{x}) + \tfrac{1}{2}\tilde{x}^{-1/2}J_\nu'(\tilde{x}) + x^{1/2}J_\nu''(\tilde{x}).$$

The first three terms are all dominated by $\tilde{x}^{-1}$, and Bessel's equation says that

$$J_\nu''(\tilde{x}) = -\tilde{x}^{-1}J_\nu'(\tilde{x}) - (1 - \nu^2\tilde{x}^{-2})J_\nu(\tilde{x}),$$

so

$$f'(x) = \tilde{x}^{1/2} J_\nu(\tilde{x}) + (\text{terms of order } x^{-1}).$$

(5.25) then implies that $f'(x)$ is approximately $\cos(x + \frac{1}{2}\pi) = -\sin x$. Therefore, Lemma 5.1 can be applied to $f(x)$ to conclude that the function $cJ_\nu(\tilde{x}) + \tilde{x}J'_\nu(\tilde{x}) = x^{1/2}f(x)$ has zeros approximately at the points $(m + \frac{1}{2})\pi$ for large integers m. In other words, $cJ_\nu(x) + xJ'_\nu(x)$ has zeros approximately at $(m + \frac{1}{2}\nu + \frac{1}{4})\pi$ for large integers m.

There remains the question of locating the small positive zeros of $aJ_\nu(x) + bxJ'_\nu(x)$. We shall content ourselves with deriving a simple lower bound for the *smallest* positive zeros of these functions under the conditions $\nu \geq 0$ and $a, b \geq 0$. (The cases when $\nu < 0$, $a < 0$, or $b < 0$ arise only infrequently in applications.)

Lemma 5.2. *Suppose $\nu \geq 0$, $a, b \geq 0$, and $(a, b) \neq (0, 0)$. If ω_ν is the smallest positive zero of $aJ_\nu(x) + bxJ'_\nu(x)$, then $\omega_\nu > \nu$.*

Proof: The case $\nu = 0$ is trivial, so we assume $\nu > 0$. $J_\nu(x)$ and $J'_\nu(x)$ are clearly positive for small $x > 0$, since the leading terms of their power series (namely, $x^\nu/2^\nu\Gamma(\nu + 1)$ and $x^{\nu-1}/2^\nu\Gamma(\nu)$) are positive. Now, Bessel's equation can be written as

$$x\frac{d}{dx}\left[xJ'_\nu(x)\right] = (\nu^2 - x^2)J_\nu(x).$$

If the first zero ζ_ν of J_ν were $\leq \nu$, the expression on the right would be positive on the interval $(0, \zeta_\nu)$; hence $xJ'_\nu(x)$ would be increasing on this interval; hence J'_ν would be positive on this interval, which is impossible by Rolle's theorem. Therefore $\zeta_\nu > \nu$; the expression on the right *is* positive on the interval $(0, \nu)$; hence $xJ'_\nu(x)$ *is* increasing on $(0, \nu]$, so $J'_\nu > 0$ on $(0, \nu]$. We have now shown that $J_\nu(x)$ and $J'_\nu(x)$ are positive on $(0, \nu]$; but then so is $aJ_\nu(x) + bxJ'_\nu(x)$, so $\omega_\nu > \nu$. ∎

Lemma 5.2 will suffice for our purposes, but more precise estimates on these zeros are available. In particular, one can show that

$$\nu < \omega_\nu < \sqrt{2(\nu + 1)(\nu + 3)}. \tag{5.29}$$

(See Watson [55], §15.3.) Thus ω_ν is of the same order of magnitude as ν.

We sum up our results in a theorem.

Theorem 5.2. *Suppose $\nu \in \mathbf{R}$, $a, b \geq 0$, and $(a, b) \neq (0, 0)$. Let $\lambda_1, \lambda_2, \ldots$ be the positive zeros of $aJ_\nu(x) + bxJ'_\nu(x)$, arranged in increasing order. Then:*

(a) $\lambda_1 > \nu$.

(b) If $b = 0$, there is an integer $M = M(\nu)$ such that

$$\lambda_k \sim (k + M + \tfrac{1}{2}\nu + \tfrac{3}{4})\pi \quad \text{as} \quad k \to \infty.$$

(c) If $b > 0$, there is an integer $M = M(\nu, a/b)$ such that

$$\lambda_k \sim (k + M + \tfrac{1}{2}\nu + \tfrac{1}{4})\pi \quad \text{as} \quad k \to \infty.$$

Here "$\sim$" means that the difference between the quantities on the left and on the right tends to zero as $k \to \infty$.

EXERCISES

1. Fill in the details of the proof of Lemma 5.1.
2. In the text and exercises of §5.2, J_ν was computed explicitly for $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}$. Verify Theorem 5.1 in these cases.
3. Use Exercise 2 and the recurrence formulas to prove Theorem 5.1 when ν is a half-integer. (Proceed by induction on n, where $\nu = \pm n + \frac{1}{2}$.)
4. Let $\{\lambda_k\}$ be the positive zeros of J_ν ($\nu \in \mathbf{R}$). Show that $J_{\nu+1}(\lambda_k) \approx \pm\sqrt{2/\pi\lambda_k}$ for large k. (In view of Theorem 5.3(a) below, this is of interest in estimating the coefficients in Fourier-Bessel series.)
5. (The interlacing theorem) Suppose $\nu \in \mathbf{R}$. Prove that between every two positive zeros of J_ν there is a zero of $J_{\nu+1}$, and between every two positive zeros of $J_{\nu+1}$ there is a zero of J_ν. (Hint: Use Rolle's theorem and the recurrence formulas (5.13) and (5.14).)
6. Let $f(x) = x^{1/2}J_\nu(x)$. Then, as shown in the text, f satisfies $f'' + f = (\nu^2 - \frac{1}{4})x^{-2}f$.
   a. Use this differential equation to show that for $n = 1, 2, 3, \ldots$,
   $$\int_{2n\pi}^{(2n+1)\pi} (\tfrac{1}{4} - \nu^2)x^{-2}f(x)\sin x\,dx = -\Big[f\Big((2n+1)\pi\Big) + f(2n\pi)\Big].$$
   b. Suppose $-\frac{1}{2} < \nu < \frac{1}{2}$. Show that f must vanish somewhere in the interval $[2n\pi, (2n+1)\pi]$. (Hint: By comparing signs on the two sides of the equation in part (a), show that it is impossible for f to be everywhere positive or everywhere negative on this interval.) Note that Theorem 5.1 yields a sharper result when n is large, but this elementary argument is valid for all $n \geq 1$.
7. Exercise 6 shows that J_ν has infinitely many positive zeros when $-\frac{1}{2} < \nu < \frac{1}{2}$, and the same is obviously true of $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$. Use this fact together with Exercise 5 (but without invoking Theorem 5.1) to show that J_ν has infinitely many positive zeros for all real ν.
8. Let j_ν denote the smallest positive zero of J_ν. Show that $j_{\nu-1} < j_\nu$ for all $\nu \geq 1$. (Hint: Use formula (5.14) and Rolle's theorem.)

5.4 Orthogonal sets of Bessel functions

We recall that the differential equation (5.1) from which we derived Bessel's equation is
$$x^2 f''(x) + xf'(x) + (\mu^2 x^2 - \nu^2)f(x) = 0, \tag{5.30}$$
and that the solutions of this equation are the functions $f(x) = g(\mu x)$ where g satisfies Bessel's equation (i.e., equation (5.30) with $\mu = 1$). Upon dividing through by x, (5.30) can be rewritten as
$$xf''(x) + f'(x) - \frac{\nu^2}{x}f(x) + \mu^2 x f(x) = \Big[xf'(x)\Big]' - \frac{\nu^2}{x}f(x) + \mu^2 x f(x) = 0. \tag{5.31}$$

This is a Sturm-Liouville equation of the sort studied in §3.5, that is,

$$(rf')' + pf + \mu^2 wf = 0 \quad \text{where } r(x) = x,\ p(x) = -\frac{\nu^2}{x},\ w(x) = x.$$

If we consider this equation on an interval $[a, b]$ with $0 < a < b < \infty$ and impose suitable boundary conditions, say

$$\alpha f(a) + \alpha' f'(a) = 0, \qquad \beta f(b) + \beta' f'(b) = 0,$$

we obtain a regular Sturm-Liouville problem. The eigenfunctions will be of the form

$$f(x) = c_\mu J_\nu(\mu x) + d_\mu Y_\nu(\mu x) \tag{5.32}$$

where μ, c_μ, and d_μ must be chosen so that the boundary conditions hold. In this way we obtain an orthonormal basis of $L^2_w(a, b)$ consisting of functions of the form (5.32), where $w(x) = x$. In general the determination of the eigenvalues μ^2 and the coefficients c_μ and d_μ is a rather messy business, and we shall not pursue the matter further.

More important and more interesting, however, is to consider the equation (5.31) on an interval $[0, b]$ under the assumption $\nu \geq 0$. Here the Sturm-Liouville problem is singular, because the leading coefficient r vanishes at $x = 0$ and the coefficient p blows up there. As a result, it is inappropriate to impose the usual sort of boundary condition at $x = 0$ such as $\alpha f(0) + \alpha' f'(0) = 0$. Indeed, we know that the solutions are of the form (5.32), and such functions (and their derivatives) become infinite at $x = 0$ unless $d_\mu = 0$. Instead, the natural boundary condition at $x = 0$ is simply that the solution should be continuous there, i.e., that $d_\mu = 0$. We can still impose a boundary condition at $x = b$, so the Sturm-Liouville problem we propose to consider is

$$\begin{aligned} &xf''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0 \qquad (\nu \geq 0), \\ &f(0+) \text{ exists and is finite}, \qquad \beta f(b) + \beta' f'(b) = 0. \end{aligned} \tag{5.33}$$

The results of §3.5 concerning the reality of the eigenvalues and the orthogonality of the eigenfunctions for regular Sturm-Liouville problems are still valid in the present situation. What needs to be checked is that if f and g are eigenfunctions of (5.33), that is,

$$f(x) = J_\nu(\mu_j x), \qquad g(x) = J_\nu(\mu_k x),$$

then

$$\langle L(f), g\rangle = \langle f, L(g)\rangle \quad \text{where } L(f) = (xf')' - \frac{\nu^2}{x} f. \tag{5.34}$$

However, if we apply Lagrange's identity to this operator L on the interval $[\epsilon, b]$, we find that

$$\int_\epsilon^b \left[L(f)(x)\overline{g(x)} - f(x)\overline{L(g)(x)}\right] dx = \left[xf'(x)\overline{g(x)} - xf(x)\overline{g'(x)}\right]_\epsilon^b.$$

The boundary condition at $x = b$ causes the endpoint evaluation at $x = b$ to vanish. As for $x = \epsilon$: If $\nu > 0$, then $f(x)$, $g(x)$, $xf'(x)$, and $xg'(x)$ are asymptotic to constant multiples of x^ν as $x \to 0$, so

$$|\epsilon f'(\epsilon)\overline{g(\epsilon)} - \epsilon f(\epsilon)\overline{g'(\epsilon)}| \leq C\epsilon^{2\nu} \to 0 \quad \text{as } \epsilon \to 0.$$

If $\nu = 0$, then $f(0) = g(0) = 1$ and $f'(0) = g'(0) = 0$, so again the contribution at $x = \epsilon$ vanishes as $\epsilon \to 0$. In either case, we have verified (5.34).

Once this is known, the proof of Theorem 3.9 in §3.5 goes through to show that the eigenvalues of (5.33) are real, the eigenfunctions are orthogonal with respect to the weight function $w(x) = x$, and the eigenspaces are 1-dimensional.

Now, if $f(x) = J_\nu(\mu x)$, then $f'(x) = \mu J_\nu'(\mu x)$. Hence, the solutions of (5.33) are the functions $J_\nu(\mu x)$ such that

$$\beta J_\nu(\mu b) + \beta' \mu J_\nu'(\mu b) = 0.$$

It will now be convenient to set $\lambda = \mu b$, so that $\mu = \lambda / b$. We distinguish between $\beta' = 0$, in which case we have

$$J_\nu(\lambda) = 0, \tag{5.35}$$

and $\beta' \neq 0$, in which case we set $c = b\beta/\beta'$ and obtain

$$cJ_\nu(\lambda) + \lambda J_\nu'(\lambda) = 0. \tag{5.36}$$

Equations (5.35) and (5.36) are of the sort we analyzed in §5.3. In either case there is an infinite sequence $\{\lambda_k\}_1^\infty$ of positive solutions, and the corresponding eigenvalues of problem (5.33) are the numbers λ_k^2/b^2.

Thus we have identified the positive eigenvalues of the problem (5.33). There remains the question of zero or negative eigenvalues, concerning which we have the following result.

Lemma 5.3. *Zero is an eigenvalue of (5.33) if and only if $\beta/\beta' = -\nu/b$, in which case the eigenfunction is $f(x) = x^\nu$. If $\beta' = 0$ or if $\beta/\beta' \geq -\nu/b$, there are no negative eigenvalues.*

Proof: When $\mu = 0$, the differential equation in (5.33) becomes the Euler equation

$$x^2 f''(x) + xf'(x) - \nu^2 f(x) = 0,$$

which we analyzed in §4.3. The general solution is $c_1 x^\nu + c_2 x^{-\nu}$ if $\nu > 0$, or $c_1 + c_2 \log x$ if $\nu = 0$. The boundary condition at $x = 0$ forces $c_2 = 0$, and then the boundary condition at $x = b$ becomes $\beta b + \nu\beta' = 0$. This proves the first assertion.

To investigate negative eigenvalues, i.e., the case $\mu^2 < 0$ in (5.33), we set $\mu = i\kappa$. The general solution of (5.33) is then $c_1 J_\nu(i\kappa x) + c_2 Y_\nu(i\kappa x)$, and again the boundary condition at $x = 0$ forces $c_2 = 0$. (The behavior of $Y_\nu(x)$ that we described in §5.1 still holds when x is imaginary; in particular, $Y_\nu(x)$ blows up as $x \to 0$.) Hence the boundary condition at $x = b$ is still (5.35) or (5.36), with $\lambda = i\kappa b$, so we must investigate solutions of the equations

$$J_\nu(iy) = 0 \quad \text{or} \quad cJ_\nu(iy) + iyJ'_\nu(iy) = 0, \quad (c = b\beta/\beta', \ y > 0). \tag{5.37}$$

Now, from the defining formula (5.10) for J_ν we have

$$J_\nu(iy) = i^\nu I_\nu(y), \quad \text{where } I_\nu(y) = \sum_0^\infty \frac{1}{k!\Gamma(\nu + k + 1)} \left(\frac{y}{2}\right)^{\nu+2k}.$$

Moreover, the recurrence formula (5.15), which is valid for all complex x, shows that

$$\begin{aligned} iyJ'_\nu(iy) = \nu J_\nu(iy) - iyJ_{\nu+1}(iy) &= \nu i^\nu I_\nu(y) - i^{\nu+2} y I_{\nu+1}(y) \\ &= i^\nu \left[\nu I_\nu(y) + y I_{\nu+1}(y)\right], \end{aligned}$$

so (5.37) can be written as

$$I_\nu(y) = 0 \quad \text{or} \quad (c + \nu) I_\nu(y) + y I_{\nu+1}(y) = 0 \qquad (y > 0).$$

But it is obvious from the above definition of I_ν that $I_\nu(y) > 0$ (and likewise $I_{\nu+1}(y) > 0$) for all $y \neq 0$. Hence the first equation has no solutions, and neither does the second one when $c + \nu = (b\beta/\beta') + \nu \geq 0$. This completes the proof.

(A slightly more careful analysis would show that when $\beta/\beta' < -\nu/b$, there is exactly one negative eigenvalue; the situation is similar to the one analyzed in the examples of §3.5. However, this case arises only rarely in applications.) ∎

We have therefore constructed a whole family (depending on the parameters ν, β, and β') of orthogonal sets of functions on $[0, b]$, with respect to the weight $w(x) = x$, of the form $f_k(x) = J_\nu(\lambda_k x/b)$. In order to make this a useful tool for solving concrete problems, we need also to identify the norms of these functions, namely,

$$\|f_k\|_w^2 = \int_0^b |f_k(x)|^2 x \, dx.$$

Actually, all these eigenfunctions are real, so we can omit the absolute values here, and we have the following result.

Lemma 5.4. *If $\mu > 0$, $b > 0$, and $\nu \geq 0$,*

$$\int_0^b J_\nu(\mu x)^2 x \, dx = \frac{b^2}{2} J'_\nu(\mu b)^2 + \frac{\mu^2 b^2 - \nu^2}{2\mu^2} J_\nu(\mu b)^2. \tag{5.38}$$

Proof: Let $f(x) = J_\nu(\mu x)$. The differential equation satisfied by f is

$$x^2 f'' + x f' + (\mu^2 x^2 - \nu^2) f = 0, \quad \text{or} \quad x(xf')' = (\nu^2 - \mu^2 x^2) f.$$

If we multiply this through by $2f'$, it becomes

$$2(xf')'(xf') = (\nu^2 - \mu^2 x^2)(2f'f), \quad \text{or} \quad \left[(xf')^2\right]' = (\nu^2 - \mu^2 x^2)(f^2)'.$$

We integrate from 0 to b and use integration by parts on the right side:

$$(xf')^2\big|_0^b = (\nu^2 - \mu^2 x^2)f^2\big|_0^b + \mu^2 \int_0^b f(x)^2(2x)\,dx.$$

The endpoint evaluations at $x = 0$ all vanish. Indeed, this is obvious for $(xf')^2$ and $\mu^2x^2f^2$; and it is true for $\nu^2 f^2$ because either $\nu = 0$ or $\nu > 0$, and in the latter case $f(0) = J_\nu(0) = 0$. Therefore, we have

$$2\mu^2 \int_0^b f(x)^2 x\,dx = b^2 f'(b)^2 + (\mu^2b^2 - \nu^2)f(b)^2,$$

and since $f'(x) = \mu J_\nu'(\mu x)$, this proves (5.38). ∎

Equation (5.38) can be simplified in the cases where $\mu = \lambda/b$ and the boundary condition (5.35) or (5.36) is satisfied, namely,

$$\text{if } J_\nu(\lambda) = 0,\quad \int_0^b J_\nu\left(\frac{\lambda x}{b}\right)^2 x\,dx = \frac{b^2}{2}J_\nu'(\lambda)^2, \tag{5.39}$$

$$\text{if } cJ_\nu(\lambda) + \lambda J_\nu'(\lambda) = 0,\quad \int_0^b J_\nu\left(\frac{\lambda x}{b}\right)^2 x\,dx = \frac{b^2(\lambda^2 - \nu^2 + c^2)}{2\lambda^2}J_\nu(\lambda)^2. \tag{5.40}$$

It is customary to restate (5.39) by using the recurrence formula (5.15), which implies that $J_\nu'(\lambda) = -J_{\nu+1}(\lambda)$ whenever $J_\nu(\lambda) = 0$, thus:

$$\text{if } J_\nu(\lambda) = 0,\quad \int_0^b J_\nu\left(\frac{\lambda x}{b}\right)^2 x\,dx = \frac{b^2}{2}J_{\nu+1}(\lambda)^2. \tag{5.41}$$

We can now write down a number of orthonormal sets of Bessel functions on $[0, b]$ obtained from the Sturm-Liouville problems (5.33). One last, crucial question remains: *Are these sets orthonormal bases?* The theorems of §3.5 do not apply since these Sturm-Liouville problems are singular. Nonetheless, the answer is *yes*, and we shall explain a method for proving this in §10.4; a complete proof can be found in Watson [55], Chapter XVIII. We sum up the results in a theorem.

Theorem 5.3. *Suppose $\nu \ge 0$, $b > 0$, and $w(x) = x$.*

(a) Let $\{\lambda_k\}_1^\infty$ be the positive zeros of $J_\nu(x)$, and let $\phi_k(x) = J_\nu(\lambda_k x/b)$. Then $\{\phi_k\}_1^\infty$ is an orthogonal basis for $L_w^2(0, b)$, and

$$\|\phi_k\|_w^2 = \frac{b^2}{2}J_{\nu+1}(\lambda_k)^2.$$

(b) Suppose $c \ge -\nu$. Let $\{\tilde\lambda_k\}_1^\infty$ be the positive zeros of $cJ_\nu(x) + xJ_\nu'(x)$, and let $\psi_k(x) = J_\nu(\tilde\lambda_k x/b)$. If $c > -\nu$, then $\{\psi_k\}_1^\infty$ is an orthogonal basis for $L_w^2(0, b)$. If $c = -\nu$, then $\{\psi_k\}_0^\infty$ is an orthogonal basis for $L_w^2(0, b)$, where $\psi_0(x) = x^\nu$. Moreover,

$$\|\psi_k\|_w^2 = \frac{b^2(\tilde\lambda_k^2 - \nu^2 + c^2)}{2\tilde\lambda_k^2}J_\nu(\tilde\lambda_k)^2 \quad (k \ge 1), \qquad \|\psi_0\|_w^2 = \frac{b^{2\nu+2}}{2\nu+2}.$$

In practice, the constant c in part (b) is almost always nonnegative. Under this condition the Bessel functions $\{\psi_k\}_1^\infty$ form an orthogonal basis for $L^2_w(0,b)$ except in the single case $c = \nu = 0$, when one must add in the constant function $\psi_0(x) = 1$. The latter case is an important one, however, and one must not forget its exceptional character.

From Theorem 5.3 we know that any $f \in L^2_w(0,b)$ can be expanded in a **Fourier-Bessel series**

$$f = \sum c_k \phi_k, \qquad c_k = \frac{1}{\|\phi_k\|_w^2} \int_0^b f(x)\phi_k(x) x\, dx,$$
$$\text{or} \quad f = \sum d_k \psi_k, \qquad d_k = \frac{1}{\|\psi_k\|_w^2} \int_0^b f(x)\psi_k(x) x\, dx.$$

(The second of these expansions is also called a **Dini series.**) These series converge in the norm of $L^2_w(0,b)$, but under suitable conditions one can also prove pointwise or uniform convergence. In fact, except perhaps at the endpoints 0 and b, the behavior of these series is much like ordinary Fourier series. For example, if f is piecewise smooth on $[0,b]$ then $\sum c_j\phi_j(x)$ and $\sum d_j\psi_j(x)$ converge to $\frac{1}{2}\left[f(x-) + f(x+)\right]$ for all $x \in (0,b)$. Of course, $\phi_j(b) = 0$ for all j because of the boundary condition, and $\phi_j(x)$ and $\psi_j(x)$ vanish to order ν as $x \to 0$ for all j; thus one cannot expect the series to converge well near the endpoints unless f satisfies similar conditions. However, if f does satisfy such conditions and is suitably smooth, one can prove absolute and uniform convergence. See Watson [55], Chapter XVIII.

Example. Let $\{\lambda_k\}$ be the positive zeros of $J_0(x)$, and let $f(x) = 1$ for $0 \le x \le b$. According to Theorem 5.3(a), we have $f(x) = \sum_1^\infty c_k J_0(\lambda_k x/b)$ (the series converging at least in the norm of $L^2_w(0,b)$ with $w(x) = x$), where

$$c_k = \frac{2}{b^2 J_1(\lambda_k)^2} \int_0^b J_0\left(\frac{\lambda_k x}{b}\right) x\, dx.$$

Since $xJ_0(x)$ is the derivative of $xJ_1(x)$ by the recurrence formula (5.14), we make the substitution $x = bt/\lambda_k$ and obtain

$$c_k = \frac{2}{b^2 J_1(\lambda_k)^2} \frac{b^2}{\lambda_k^2} \int_0^{\lambda_k} J_0(t)\, t\, dt = \frac{2}{\lambda_k^2 J_1(\lambda_k^2)} \Big[tJ_1(t)\Big]_0^{\lambda_k} = \frac{2}{\lambda_k J_1(\lambda_k)}.$$

Other examples will be found in the exercises; the evaluations of the integrals are usually applications of the recurrence formulas.

EXERCISES

In Exercises 1–4, expand the given function on the interval $[0,b]$ in a Fourier-Bessel series $\sum c_k J_0(\lambda_k x/b)$ where $\{\lambda_k\}_1^\infty$ are the positive zeros of J_0.

1. $f(x) = x^2$. (Hint: Exercise 8, §5.2.)
2. $f(x) = b^2 - x^2$. (Hint: Exercise 1.)
3. $f(x) = x$. (Hint: Exercise 8, §5.2.)
4. $f(x) = 1$ for $0 \le x \le \frac{1}{2}b$, $f(x) = 0$ for $\frac{1}{2}b < x \le b$.
5. Expand $f(x) = 1$ on the interval $[0, b]$ in a series $\sum c_k J_0(\lambda_k x/b)$ where $\{\lambda_k\}$ are the positive zeros of $cJ_0(x) + xJ_0'(x)$, $c > 0$. What about the case $c = 0$? (Be careful!)
6. Expand $f(x) = x$ on the interval $[0, 1]$ in a series $\sum c_k J_1(\lambda_k x)$ where $\{\lambda_k\}$ are the positive zeros of J_1.
7. Expand $f(x) = x^\nu$ on the interval $[0, 1]$ in a series $\sum c_k J_\nu(\lambda_k x)$ where $\nu > 0$ and $\{\lambda_k\}$ are the positive zeros of J_ν'.
8. Let $f(x) = x$ for $0 \le x < 1$, $f(x) = 0$ for $1 \le x \le 2$. Expand $f(x)$ on the interval $[0, 2]$ in a series $\sum c_k J_1(\lambda_k x/2)$ where $\{\lambda_k\}$ are the positive zeros of J_1'.
9. Let $\{\lambda_k\}$ be the positive zeros of J_0, and let $\phi_k(x) = J_0(\lambda_k \sqrt{x/l})$. Show that $\{\phi_k\}$ is an orthogonal basis for $L^2(0, l)$ (with weight function 1). What is the norm of ϕ_k in $L^2(0, l)$?

5.5 Applications of Bessel functions

In the introduction to this chapter we showed that if one applies separation of variables to the two-dimensional wave equation in polar coordinates,

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}),$$

one obtains the ordinary differential equations

$$\begin{aligned} &T''(t) + c^2\mu^2 T(t) = 0, \qquad \Theta''(\theta) + \nu^2\Theta(\theta) = 0, \\ &r^2 R''(r) + rR'(r) + (\mu^2 r^2 - \nu^2)R(r) = 0. \end{aligned} \tag{5.42}$$

The heat equation works similarly, except that the equation for T is $T'(t) + k\mu^2 T(t) = 0$. Let us sketch the procedure for solving these equations.

Suppose we are interested in solving the partial differential equation in the disc of radius b about the origin, with some boundary conditions at $r = b$. In the first place, by the nature of polar coordinates, Θ must be 2π-periodic. Hence ν must be an integer n, which we can take to be nonnegative, and $\Theta(\theta) = c\cos n\theta + d\sin n\theta$. The equation for R is now the Bessel equation of order n. Since we want a solution of the partial differential equation in the whole disc including the origin, we forbid $R(r)$ to blow up at $r = 0$; hence $R(r)$ must be a constant multiple of $J_n(\mu r)$. Moreover, there will be a sequence of positive numbers μ_k for which $J_n(\mu_k r)$ satisfies the boundary conditions at $r = b$. Finally, we plug these numbers into the equation for T and solve it.

The problem of finding a solution with given initial conditions now reduces to the problem of expanding the initial data in a series involving the functions Θ and R derived above, that is, a series of the form

$$\sum_{n,k} (c_{nk} \cos n\theta + d_{nk} \sin n\theta) J_n(\mu_k r). \tag{5.43}$$

This is a *doubly infinite* series, involving the two indices n and k. However, we point out that if the θ-dependence of the initial data involves only finitely many of the functions $\cos n\theta$ and $\sin n\theta$, then only the terms in (5.43) involving those particular functions will be nonzero, so (5.43) will reduce to a finite sum of singly infinite series. For example, if the initial data are radial, i.e., independent of θ, then only the terms with $n = 0$ survive, and (5.43) reduces to $\sum a_k J_0(\mu_k r)$.

Similar remarks apply to boundary value problems in "polar-coordinate rectangles" bounded by circles $r = a$, $r = b$ and rays $\theta = \alpha$, $\theta = \beta$, except that the indices ν and eigenfunctions Θ will generally be different.

We now turn to some specific applications, in which these ideas will be explained more fully.

Vibrations of a circular membrane

Let us now solve the problem of the vibrations of a circular membrane fixed along its boundary, such as a drum, that occupies the disc of radius b centered at the origin. According to the preceding discussion, we need to solve (5.42) subject to the conditions that Θ should be 2π-periodic, that R should be continuous at $r = 0$, and that $R(b) = 0$, and we obtain

$$\Theta(\theta) = c_n \cos n\theta + d_n \sin n\theta,$$
$$R(r) = J_n\left(\frac{\lambda r}{b}\right) \quad \text{where } J_n(\lambda) = 0.$$

In particular $\mu = \lambda/b$ in (5.42), so

$$T(t) = a_1 \cos \frac{\lambda c t}{b} + a_2 \sin \frac{\lambda c t}{b},$$

where again λ satisfies $J_n(\lambda) = 0$.

Let $\lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}, \ldots$ be the positive zeros of $J_n(x)$. By Theorem 5.3 we know that $\{J_n(\lambda_{k,n} r/b)\}_{k=1}^{\infty}$ is an orthogonal basis for $L^2_w(0, b)$ where $w(r) = r$. Moreover, $\{\cos n\theta\}_0^\infty \cup \{\sin n\theta\}_1^\infty$ is an orthogonal basis for $L^2(-\pi, \pi)$. It follows that the products $J_n(\lambda_{k,n} r/b) \cos n\theta$ and $J_n(\lambda_{k,n} r/b) \sin n\theta$ will form an orthogonal set in $L^2_w(D)$, where

$$D = \Big\{(r, \theta) : 0 \le r \le b,\ -\pi \le \theta \le \pi\Big\}, \qquad w(r, \theta) = r.$$

But if we interpret r and θ as polar coordinates in the xy-plane, D is nothing but the disc of radius b about the origin, and the weighted measure $w(r, \theta)\, dr\, d\theta$ is Euclidean area measure:

$$w(r, \theta)\, dr\, d\theta = r\, dr\, d\theta = dx\, dy.$$

In fact, we have the following result.

Theorem 5.4. *Let $\lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}, \ldots$ be the positive zeros of $J_n(x)$. Then*

$$\left\{J_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos n\theta : n \geq 0,\ k \geq 1\right\} \cup \left\{J_n\left(\frac{\lambda_{k,n}r}{b}\right)\sin n\theta : n, k \geq 1\right\}$$

is an orthogonal basis for $L^2(D)$, where D is the disc of radius b about the origin.

Proof: This is not an instance of Theorem 4.1 of §4.4 because we are using a different basis for functions of r for each choice of the index n; nonetheless, the argument we used to prove that theorem also proves this one. That is, one checks orthogonality by evaluating the double integrals that define the inner products as iterated integrals. To prove completeness, suppose that $f \in L^2(D)$ is orthogonal to all the functions $J_n(\lambda_{k,n}r/b)\cos n\theta$ and $J_n(\lambda_{k,n}r/b)\sin n\theta$. Then the functions

$$g_n(r) = \int_{-\pi}^{\pi} f(r,\theta)\cos n\theta\,d\theta \quad \text{and} \quad h_n(r) = \int_{-\pi}^{\pi} f(r,\theta)\sin n\theta\,d\theta$$

are orthogonal to all the functions $J_n(\lambda_{k,n}r/b)$ $(k = 1, 2, \ldots)$ and hence are zero. But this says that for (almost) every r, $f(r,\theta)$ is orthogonal to all the functions $\cos n\theta$ and $\sin n\theta$; hence $f = 0$. ∎

Now we can solve the vibrating membrane problem with initial conditions. For simplicity, let us take them to be

$$u(r,\theta,0) = f(r,\theta), \qquad u_t(r,\theta,0) = 0.$$

The initial condition $u_t = 0$ means that we must drop the sine term in $T(t)$. Hence, taking u to be a general superposition of the solutions we have constructed,

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(c_{nk}\cos n\theta + d_{nk}\sin n\theta)J_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos\frac{\lambda_{k,n}ct}{b}. \tag{5.44}$$

To solve the problem we have merely to determine the coefficients c_{nk} and d_{nk} so that $u(r,\theta,0) = f(r,\theta)$, and this means expanding f in terms of the basis of Theorem 5.4. In view of the normalizations for the Bessel functions presented in Theorem 5.3, we have

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(c_{nk}\cos n\theta + d_{nk}\sin n\theta)J_n\left(\frac{\lambda_{k,n}r}{b}\right),$$

where

$$c_{0k} = \frac{1}{\pi b^2 J_1(\lambda_{k,0})^2}\int_{-\pi}^{\pi}\int_0^b f(r,\theta)J_0\left(\frac{\lambda_{k,0}r}{b}\right)r\,dr\,d\theta,$$

and for $n \geq 1$,

$$c_{nk} = \frac{2}{\pi b^2 J_{n+1}(\lambda_{k,n})^2}\int_{-\pi}^{\pi}\int_0^b f(r,\theta)J_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos(n\theta)r\,dr\,d\theta,$$

$$d_{nk} = \frac{2}{\pi b^2 J_{n+1}(\lambda_{k.n})^2}\int_{-\pi}^{\pi}\int_0^b f(r,\theta)J_n\left(\frac{\lambda_{k,n}r}{b}\right)\sin(n\theta)r\,dr\,d\theta.$$

Example 1. Suppose the initial displacement is $f(r,\theta) = b^2 - r^2$. Since f is independent of θ, so is u; hence the only nonzero terms in (5.44) are the ones with $n = 0$. That is,

$$u(r,t) = \sum_1^\infty c_k J_0\left(\frac{\lambda_k r}{b}\right) \cos\frac{\lambda_k ct}{b} \qquad (\lambda_k = \lambda_{k,0}),$$

where $\sum c_k J_0(\lambda_k r/b)$ is the Fourier-Bessel expansion of the function $b^2 - r^2$. Therefore, by Exercise 2 of §5.4,

$$u(r,t) = \sum_1^\infty \frac{8b^2}{\lambda_k^3 J_1(\lambda_k)} J_0\left(\frac{\lambda_k r}{b}\right) \cos\frac{\lambda_k ct}{b}.$$

The most interesting aspect of the solution (5.44) is the set of allowable frequencies, namely, the zeros of the Bessel functions,

$$\left\{\frac{\pi c\lambda_{k,n}}{b} : n \geq 0,\ k \geq 1\right\}.$$

An important feature of this set is that there are only finitely many frequencies less than a preassigned number M. Indeed, from Theorem 5.3 we know that all zeros of $J_n(\lambda)$ satisfy $\lambda > n$, i.e., $\lambda_{k,n} > n$ for all k and n. Consequently, if we are to have $\pi c\lambda_{k,n}/b < M$, we must have $n < bM/\pi c$. That is, for each n J_n has only finitely many zeros $\lambda_{k,n}$ satisfying $\pi c\lambda_{k,n}/b < M$, and if $n \geq bM/\pi c$ it has none at all; hence there are only finitely many altogether.

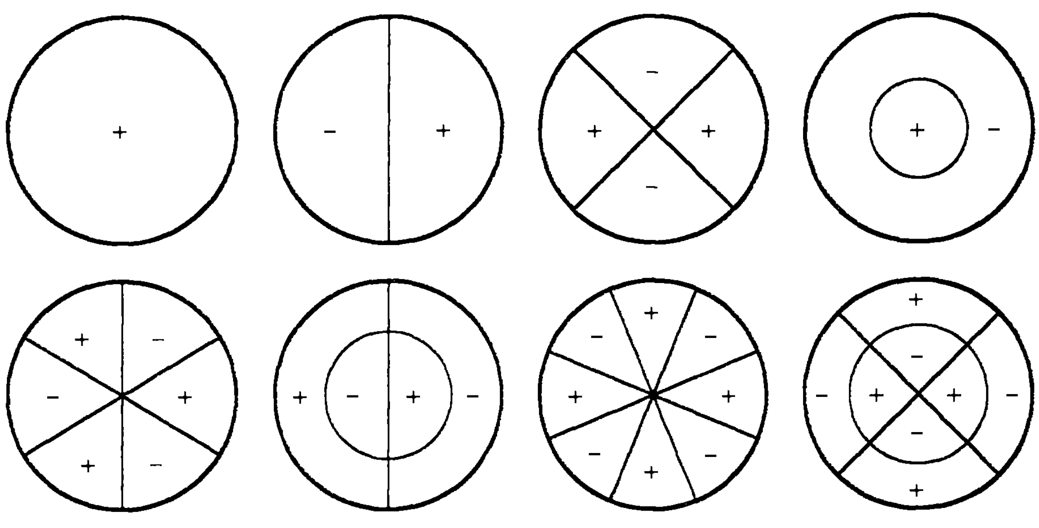


FIGURE 5.2. Diagrams of the eigenfunctions $J_n(\lambda_{k,n}r)\cos n\theta$ of the vibrating membrane for the eight smallest $\lambda_{k,n}$'s. The plus or minus signs indicate the regions where the eigenfunctions are positive or negative. Top: $\lambda_{1,0}$, $\lambda_{1,1}$, $\lambda_{1,2}$, and $\lambda_{2,0}$. Bottom: $\lambda_{1,3}$, $\lambda_{2,1}$, $\lambda_{1,4}$, and $\lambda_{2,2}$.

The smallest numbers $\lambda_{k,n}$ are as follows, correct to two decimal places.

$$\begin{array}{lllll} \lambda_{1,0} = 2.40, & \lambda_{1,1} = 3.83, & \lambda_{1,2} = 5.14, & \lambda_{2,0} = 5.52, & \lambda_{1,3} = 6.38, \\ \lambda_{2.1} = 7.02, & \lambda_{1,4} = 7.59, & \lambda_{2,2} = 8.42, & \lambda_{3,0} = 8.65, & \lambda_{1,5} = 8.77. \end{array}$$

The eigenfunctions corresponding to the first eight of these are drawn schematically in Figure 5.2. Observe that the frequencies are not integer multiples of a fundamental frequency, even approximately; hence drums have poorer tone quality than strings or wind instruments. In practice, drums that are designed to have a definite pitch possess structural features that make our simple mathematical model rather inaccurate. (For example, the vibrating membranes of Indian drums such as the tabla or mridangam are of nonuniform thickness.) It should also be mentioned that the pitch of a drum depends on how the drum is struck. If it is struck at the center, only the frequencies $\pi c\lambda_{k,0}/b$ (corresponding to the circularly symmetric vibrations with $n = 0$ in (5.44)) are significant. But if it is struck near the edge, as kettledrums normally are, the predominant frequency is likely to be $\pi c\lambda_{1,1}/b$. See Rossing [46] for a discussion of the physics of kettledrums.

The heat equation in polar coordinates

The heat equation in polar coordinates is

$$u_t = K(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}).$$

(We call the diffusivity coefficient K, since we are using k as an index for the zeros of Bessel functions.) We may imagine a solid body occupying some region in cylindrical coordinates,

$$R = \Big\{(r,\theta,z) : (r,\theta) \in D,\ z_1 \le z \le z_2\Big\}$$

(D being a region in the plane), in which the temperature, for one reason or another, is independent of z. Suppose D is the disc of radius b about the origin. If we impose the condition that $u = 0$ on the boundary of D, then exactly the same analysis as before leads to a solution that looks like (5.44) except that $\cos\lambda_{k,n}ct/b$ is replaced by $\exp[-\lambda_{k,n}^2 Kt/b^2]$. If instead we impose the "Newton's law of cooling" boundary condition $u_r + cu = 0$ ($c > 0$), the results are similar except that the numbers $\lambda_{k,n}$ should be the positive zeros of $bcJ_n(x) + xJ_n'(x)$.

Rather than work out these problems in detail, let us do a problem with some new features. Suppose D is the wedge-shaped region

$$D = \Big\{(r,\theta) : 0 \le r \le b,\ 0 \le \theta \le \alpha\Big\},$$

where $0 < \alpha < 2\pi$, and let us suppose that the boundary is insulated. This means that the normal derivative of u on the boundary must vanish, that is,

$$u_\theta(r,0) = u_\theta(r,\alpha) = u_r(b,\theta) = 0.$$

If we take $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, then, separation of variables leads to the following 1-dimensional problems:

$$r^2R''(r) + rR'(r) + (\mu^2r^2 - \nu^2)R(r) = 0, \qquad R'(b) = 0;$$
$$\Theta''(\theta) + \nu^2\Theta(\theta) = 0, \qquad \Theta'(0) = \Theta'(\alpha) = 0;$$
$$T'(t) + \mu^2KT(t) = 0.$$

(There is also an implied boundary condition at $r = 0$, namely, that R should not blow up there.) The differential equation for Θ together with the boundary condition at 0 imply that (up to a constant factor) $\Theta(\theta) = \cos\nu\theta$, and the boundary condition at α then forces $\nu = n\pi/\alpha$. In short, we obtain Fourier cosine series in θ, Bessel functions of order $n\pi/\alpha$ in r, and, of course, exponential functions in t. More precisely, let $\{\lambda_{k,n}\}$ now denote the positive zeros of $J'_{n\pi/\alpha}(x)$. Then $\{J_{n\pi/\alpha}(\lambda_{k,n}r/b)\}_{k=1}^{\infty}$ is a set of eigenfunctions for the Sturm-Liouville problem in r with eigenvalues $\mu^2 = (\lambda_{k,n}/b)^2$. Also, by Theorem 5.3 it is an orthogonal basis for $L^2_w(0,b)$ with $w(r) = r$, *except* in the case $n = 0$. For $n = 0$ one must augment this set by including the constant function 1 (corresponding to the eigenvalue $\mu = 0$) in order to make it complete; and constant functions are solutions of our boundary value problem. We therefore arrive at the following general solution:

$$u(r,\theta,t) = a_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{nk}J_{n\pi/\alpha}\left(\frac{\lambda_{k,n}r}{b}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right)\exp\left(-\frac{\lambda_{k,n}^2Kt}{b^2}\right).$$

If we wish to satisfy an initial condition $u(r,\theta,0) = f(r,\theta)$, we must have

$$f(r,\theta) = a_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{nk}J_{n\pi/\alpha}\left(\frac{\lambda_{k,n}r}{b}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right). \tag{5.45}$$

But the obvious analogue of Theorem 5.4 holds here, so such an expansion is possible. In fact, taking account of the normalizations in Theorem 5.3, we find that

$$a_{00} = \frac{2}{\alpha b^2}\int_0^b\int_0^\alpha f(r,\theta)r\,dr\,d\theta,$$
$$a_{0k} = \frac{2\lambda_{k,0}^2}{\alpha b^2(\lambda_{k,0})^2J_0(\lambda_{k,0})^2}\int_0^b\int_0^\alpha f(r,\theta)J_0\left(\frac{\lambda_{k,0}r}{b}\right)r\,dr\,d\theta \qquad (k \geq 1),$$

and for $n, k \geq 1$,

$$a_{nk} = \frac{4\lambda_{k,n}^2}{\alpha b^2\left[\lambda_{k,n}^2 - (n\pi/\alpha)^2\right]J_{n\pi/\alpha}(\lambda_{k,n})^2} \times \int_0^b\int_0^\alpha f(r,\theta)J_{n\pi/\alpha}\left(\frac{\lambda_{k,n}r}{b}\right)\cos\frac{n\pi\theta}{\alpha}r\,dr\,d\theta.$$

Example 2. Suppose that $\alpha = \frac{1}{2}\pi$ and the initial temperature is $f(r,\theta) = r^2 \cos 2\theta$ Then (5.45) becomes

$$r^2 \cos 2\theta = a_{00} + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} a_{nk} J_{2n}\left(\frac{\lambda_{k,n} r}{b}\right) \cos 2n\theta.$$

But the left side involves $\cos 2n\theta$ only for $n = 1$; so it is clear by inspection, or from the orthogonality relations for $\cos 2n\theta$ on $[0, \frac{1}{2}\pi]$, that only the terms with $n = 1$ on the right will be nonzero. Hence, after canceling the $\cos 2\theta$ on both sides, we are reduced to finding the coefficients $a_k = a_{1k}$ in the expansion

$$r^2 = \sum_{k=1}^{\infty} a_k J_2\left(\frac{\lambda_k r}{b}\right), \qquad \lambda_k = \lambda_{k,1}.$$

By Theorem 5.3, these are given by

$$a_k = \frac{2\lambda_k^2}{b^2(\lambda_k^2 - 4) J_2(\lambda_k)^2} \int_0^b r^3 J_2\left(\frac{\lambda_k r}{b}\right) dr,$$

and by the recurrence formula (5.14),

$$\int_0^b r^3 J_2\left(\frac{\lambda_k r}{b}\right) dr = \left(\frac{b}{\lambda_k}\right)^4 \int_0^{\lambda_k} x^3 J_2(x)\, dx = \left(\frac{b}{\lambda_k}\right)^4 \lambda_k^3 J_3(\lambda_k).$$

Combining these results, we obtain the solution:

$$u(r,\theta,t) = 2b^2 \cos 2\theta \sum_1^{\infty} \frac{\lambda_k J_3(\lambda_k)}{(\lambda_k^2 - 4) J_2(\lambda_k)^2} J_2\left(\frac{\lambda_k r}{b}\right) \exp\left(-\frac{\lambda_k^2 K t}{b^2}\right).$$

We can also solve the heat equation in an annulus $0 < a \le r \le b$ with boundary conditions at $r = a$ and $r = b$. Here the eigenfunctions in θ are linear combinations of $\cos n\theta$ and $\sin n\theta$ with n an integer, just as in the disc. But in the variable r we obtain a regular Sturm-Liouville problem on the interval $[a, b]$ involving the Bessel equation of order n, and the eigenfunctions will be linear combinations of $J_n(\lambda r/b)$ and $Y_n(\lambda r/b)$ chosen so as to satisfy the boundary conditions. See Exercise 8.

The Dirichlet problem in a cylinder

As a final application, let us consider the Dirichlet problem in the cylinder

$$D = \{(r,\theta,z) : 0 \le r \le b,\ 0 \le z \le l\}.$$

That is, we wish to solve

$$\begin{aligned} &u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \quad \text{in } D, \\ &u(r,\theta,0) = f(r,\theta), \qquad u(r,\theta,l) = g(r,\theta), \qquad u(b,\theta,z) = h(\theta,z). \end{aligned} \tag{5.46}$$

Here we shall work out the special case where $f = h = 0$ and g is independent of θ:

$$u(r,\theta,0) = 0, \qquad u(r,\theta,l) = g(r), \qquad u(b,\theta,z) = 0. \tag{5.47}$$

The generalization to arbitrary $g(r,\theta)$ is left as Exercise 5. The case $g = h = 0$ is entirely similar to this one, and the case $f = g = 0$ will be discussed in §5.6. Of course the general case (5.46) can be solved by superposing the solutions to these three special cases.

Since our boundary conditions are independent of θ, we expect the solution to be independent of θ too (but see Exercise 5), so we apply separation of variables to $u(r,z) = R(r)Z(z)$ and find that

$$r^2R''(r) + rR'(r) + \mu^2r^2R(r) = 0, \qquad R(b) = 0.$$
$$Z''(z) - \mu^2Z(z) = 0, \qquad Z(0) = 0.$$

The Sturm-Liouville problem for R has the eigenfunctions $J_0(\lambda_k r/b)$ where $\{\lambda_k\}_1^\infty$ are the positive zeros of J_0, with eigenvalues $\mu^2 = (\lambda_k/b)^2$. The corresponding solutions for Z are $\sinh(\lambda_k z/b)$. Hence, we obtain

$$u(r,z) = \sum_{k=1}^{\infty} a_k J_0\left(\frac{\lambda_k r}{b}\right) \sinh\frac{\lambda_k z}{b}.$$

To satisfy the boundary condition at $z = l$, we expand g in its Fourier-Bessel series,

$$g(r) = \sum_1^\infty c_k J_0\left(\frac{\lambda_k r}{b}\right),$$

and take

$$a_k = c_k \operatorname{csch}\frac{\lambda_k l}{b}.$$

Thus the Dirichlet problem with the special boundary conditions (5.47) is solved.

Example 3. If $g(r) \equiv 1$, the coefficient c_k was found at the end of §5.4 to be $2/\lambda_k J_1(\lambda_k)$. Therefore,

$$u(r,z) = 2\sum_1^\infty \frac{J_0(\lambda_k r/b)}{\lambda_k J_1(\lambda_k)} \frac{\sinh(\lambda_k z/b)}{\sinh(\lambda_k l/b)}.$$

EXERCISES

Exercises 1–4 deal with the heat equation in polar or cylindrical coordinates, in which we take the diffusivity coefficient k equal to 1.

1. A cylinder of radius b is initially at the constant temperature A. Find the temperatures in it at subsequent times if its ends are insulated and its circular surface obeys Newton's law of cooling, $u_r + cu = 0$ ($c > 0$).

2. A circular cylinder of radius ρ initially is at the constant temperature A. At time $t = 0$ it is tightly wrapped in a sheath of the same material of thickness δ, thus forming a cylinder of radius $\rho + \delta$. The sheath is initially at temperature B, and its outside surface is maintained at temperature B. If the ends of the new, enlarged cylinder are insulated, find the temperature inside at subsequent times.
3. A cylindrical core of radius 1 is removed from a block of material whose temperature increases linearly from left to right. (Thus, if the cylinder occupies the region $x^2 + y^2 \leq 1$, the initial temperature is $ax + b$ for some constants a and b.) Find the subsequent temperatures in the core if
   a. it is completely insulated;
   b. its ends are insulated and its circular surface is maintained at temperature zero.
4. A cylindrical uranium rod of radius 1 generates heat within itself at a constant rate a. Its ends are insulated and its circular surface is immersed in a cooling bath at temperature zero. (Thus, $u_t = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + a$ and $u(1,t) = 0$.)
   a. Find the steady-state temperature $v(r)$ in the rod. (Hint: By symmetry, the steady-state temperature is independent of θ. Since $v_{rr} + r^{-1}v_r = r^{-1}(rv_r)_r$, the steady-state equation can be solved by integrating twice.)
   b. Find the temperature in the rod if its initial temperature is zero. (Hint: Again, u is independent of θ. Let $u = v + w$ with v as in part (a) and solve for w. Exercise 2, §5.4, is helpful.)
5. Solve problem (5.46) for a general $g(r,\theta)$ when $f = h = 0$. Prove that if g is independent of θ, your solution reduces to the one in the text.
6. Find the steady-state temperature in the cylinder $0 \leq r \leq 1$, $0 \leq z \leq 1$ when the circular surface is insulated, the bottom is kept at temperature 0, and the top is kept at temperature $f(r)$.
7. Analyze the vibrations of an elastic solid cylinder occupying the region $0 \leq r \leq 1$, $0 \leq z \leq 1$ in cylindrical coordinates if its top and bottom are held fixed, its circular surface is free, and the initial velocity u_t is zero. That is, find the general solution of

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz}),$$
$$u(r,\theta,0,t) = u(r,\theta,1,t) = u_r(1,\theta,z,t) = u_t(r,\theta,z,0) = 0.$$

8. Show that the eigenvalues of the Sturm-Liouville problem

$$[xf'(x)]' - \nu^2 x^{-1} f(x) + \lambda^2 x f(x) = 0 \quad (0 < a < x < b), \qquad f(a) = f(b) = 0$$

are the numbers λ^2 such that $J_\nu(\lambda a)Y_\nu(\lambda b) = J_\nu(\lambda b)Y_\nu(\lambda a)$. What are the corresponding eigenfunctions?

5.6 Variants of Bessel functions

Bessel functions arise in very diverse ways in physics and engineering, and it is often more convenient to work with certain functions related to J_ν and Y_ν rather than with J_ν and Y_ν themselves. In this section we discuss some of these related functions and the differential equations from which they arise.

Hankel functions

We saw in §5.3 that $J_\nu(x)$ behaves like a damped cosine when x is large and positive, and $Y_\nu(x)$ behaves like the corresponding damped sine:

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi), \qquad Y_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \sin(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi).$$

As it is often better to use e^{ix} and e^{-ix} instead of $\cos x$ and $\sin x$, we are led to consider the linear combinations

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x), \qquad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x).$$

$H_\nu^{(1)}$ and $H_\nu^{(2)}$ are called the first and second **Hankel functions** or **Bessel functions of the third kind**. Their asymptotic behavior for large x is given by

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} \Big[\exp i(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\Big]\Big[1 + \boldsymbol{E}_1(x)\Big], \qquad |\boldsymbol{E}_1(x)| \le \frac{C_\nu}{x},$$
$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}} \Big[\exp i(-x + \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi)\Big]\Big[1 + \boldsymbol{E}_2(x)\Big], \qquad |\boldsymbol{E}_2(x)| \le \frac{C_\nu}{x}.$$

When stated in this form, with the error terms $\boldsymbol{E}_1(x)$ and $\boldsymbol{E}_2(x)$ multiplied by $e^{\pm ix}$, these formulas continue to hold for all *complex* $x = re^{i\theta}$ with $r \ge 1$ and $|\theta| < \pi$. (See Exercise 6, §8.6. There is a significant difference between $e^{ix} + \boldsymbol{E}_1(x)$ and $e^{ix}[1 + \boldsymbol{E}_1(x)]$ when x has an imaginary part, since then e^{ix} may be exponentially growing or decreasing.)

Modified Bessel functions

The **modified Bessel equation** is

$$x^2 f''(x) + xf'(x) - (x^2 + \nu^2)f(x) = 0. \tag{5.48}$$

It differs from the ordinary Bessel equation only in that x^2 is replaced by $-x^2$. In fact, it is the special case of the generalized Bessel equation

$$x^2 f''(x) + xf'(x) + (\mu^2 x^2 - \nu^2)f(x) = 0$$

in which $\mu = i$, so it can be reduced to Bessel's equation by the change of variable $x \to ix$. One solution of (5.48) is therefore $f(x) = J_\nu(ix)$, but it is more common use the constant multiple

$$I_\nu(x) = i^{-\nu} J_\nu(ix),$$

called the **modified Bessel function.** The reason is that, since $i^{\nu+2k} = i^\nu(-1)^k$,

$$I_\nu(x) = \sum_0^\infty \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k},$$

which has the obvious advantage of being real when x and ν are real.

For a second independent solution of (5.48), one could use $I_{-\nu}(x)$ when ν is not an integer, or $Y_\nu(ix)$ for arbitrary ν. However, the standard choice is the function

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}.$$

Just as with Y_ν, this formula is well-defined whenever ν is not an integer and can be evaluated by l'Hopital's rule when ν is an integer, and it defines a solution of (5.48), independent of I_ν, for all ν. See Lebedev [36], §5.7, or Watson [55], §3.7.

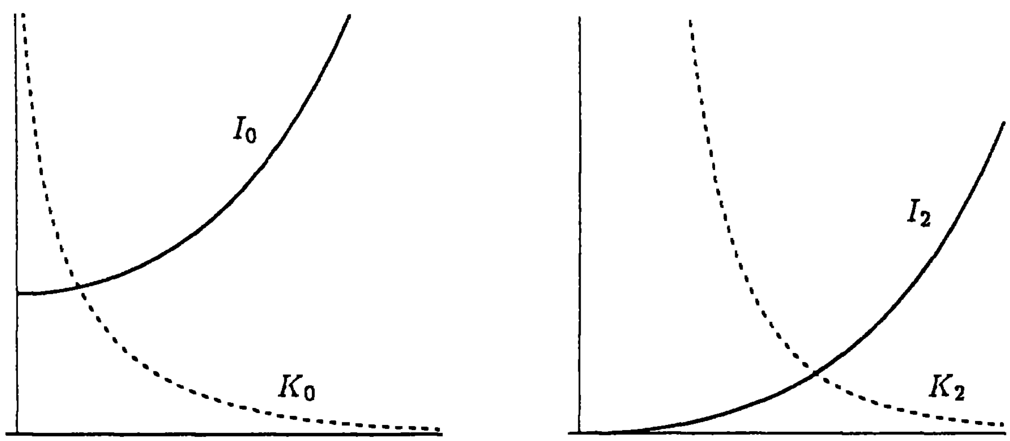


FIGURE 5.3. Graphs of some modified Bessel functions on the interval $0 \leq x \leq 3$. Left: I_0 (solid) and K_0 (dashed). Right: I_2 (solid) and K_2 (dashed).

The reason for choosing K_ν as the second independent solution of (5.48), like the reason for choosing Y_ν as the second solution of Bessel's equation, has to do with its asymptotic behavior for large x. Indeed, if we make the change of variable $f(x) = x^{-1/2}g(x)$ in (5.48), as we did with Bessel's equation in §5.3, we obtain

$$g''(x) - g(x) + \frac{\frac{1}{4} - \nu^2}{x^2} g(x) = 0.$$

When $x \gg 1$ we expect the last term in this equation to be negligibly small in comparison to the other two, so the solutions should look like the solutions of

$g'' - g = 0$, namely, $ae^x + be^{-x}$. Now, all the latter functions grow exponentially as $x \to +\infty$ except for the ones with $a = 0$, which decay exponentially; so we expect something similar to happen with solutions of (5.48). This is indeed the case, and K_ν is singled out as the only solution of (5.48) that tends to 0 rather than ∞ as $x \to +\infty$. More precisely, we have the following asymptotic formulas:

$$I_\nu(x) = \frac{1}{\sqrt{2\pi x}} e^x \Big[1 + E_1(x)\Big], \qquad K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \Big[1 + E_2(x)\Big] \qquad (x \geq 1),$$

where, as usual, $E_1(x)$ and $E_2(x)$ are bounded by a constant, depending on ν, times x^{-1}. See Exercise 7, §8.6.

The asymptotic behavior of the modified Bessel functions as $x \to 0$ is just like that of the ordinary Bessel functions, except for constant factors. That is, if $\nu > 0$, $I_\nu(x) \sim c_\nu x^\nu$ and $K_\nu(x) \sim c'_\nu x^{-\nu}$, whereas $I_0(x) \sim 1$ and $K_0(x) \sim c \log x$. Thus, when $\nu \geq 0$, I_ν and its constant multiples are the only solutions of (5.48) that do not blow up as $x \to 0$. See Figure 5.3 on the previous page.

We met I_ν briefly in §5.4 when we were showing that certain Sturm-Liouville problems had no negative eigenvalues. Let us now display a situation where I_ν enters the solution in a positive way, namely, the Dirichlet problem in a right circular cylinder. We solved part of this problem in §5.5, and we now deal with the remaining case:

$$\begin{gathered} u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \quad \text{for } 0 \leq r < b,\ 0 < z < l, \\ u(r,\theta,0) = u(r,\theta,l) = 0, \qquad u(b,\theta,z) = h(\theta,z). \end{gathered} \tag{5.49}$$

For simplicity we shall assume that h is independent of θ and leave the general case as Exercise 1. When the boundary conditions are independent of θ, the solution u will be too; so just as before, we try $u(r,z) = R(r)Z(z)$ and arrive at

$$\begin{gathered} r^2R''(r) + rR(r) + \mu^2r^2R(r) = 0, \\ Z''(z) - \mu^2 Z(z) = 0, \qquad Z(0) = Z(l) = 0. \end{gathered}$$

The boundary conditions on Z now force $-\mu^2 = (n\pi/l)^2$ and $Z(z) = \sin(n\pi z/l)$, so the equation for R becomes

$$r^2R''(r) + rR'(r) - (n\pi r/l)^2 R(r) = 0,$$

which reduces under that change of variable $x = n\pi r/l$ to the modified Bessel equation of order zero. Of course $R(r)$ cannot blow up as $r \to 0$, so $R(r) = I_0(n\pi r/l)$. Hence, by superposition we arrive at

$$u(r,z) = \sum_1^\infty a_n I_0\left(\frac{n\pi r}{l}\right) \sin\frac{n\pi z}{l}.$$

To satisfy the boundary condition $u(b,z) = h(z)$, we have merely to expand h in its Fourier sine series on $[0,l]$ and match up coefficients:

$$a_n = \left[I_0\left(\frac{n\pi b}{l}\right)\right]^{-1} \frac{2}{l}\int_0^l f(z)\sin\frac{n\pi z}{l}\,dz.$$

Equations reducible to Bessel's equation

A large number of ordinary differential equations can be transformed into Bessel's equation by appropriate changes of independent and dependent variables. Here is one class of such equations.

Theorem 5.5. *Consider the equation*

$$x^p f''(x) + px^{p-1} f'(x) + (ax^q + bx^{p-2})f(x) = 0, \tag{5.50}$$

where $(1-p)^2 - 4b \geq 0$ *and* $q - p + 2 > 0$. *Let*

$$\alpha = \frac{1-p}{2}, \qquad \beta = \frac{q-p+2}{2}, \qquad \lambda = \frac{2\sqrt{|a|}}{q-p+2}, \qquad \nu = \frac{\sqrt{(1-p)^2 - 4b}}{q-p+2}.$$

If $a > 0$, *the general solution of (5.50) is*

$$f(x) = x^\alpha \Big[c_1 J_\nu(\lambda x^\beta) + c_2 Y_\nu(\lambda x^\beta) \Big],$$

whereas if $a < 0$, *the general solution of (5.50) is*

$$f(x) = x^\alpha \Big[c_1 I_\nu(\lambda x^\beta) + c_2 K_\nu(\lambda x^\beta) \Big].$$

The proof, simple in principle but tedious to write out, consists merely of substituting $f(x) = x^\alpha g(x)$ and $x = (y/\lambda)^{1/\beta}$ into (5.50) and using the chain rule to reduce the resulting equation to the Bessel equation or modified Bessel equation relating g and y. Instead of presenting the messy details, let us look at some examples.

Example 1. If $p = q = a = 1$ and $b = -\nu^2$, (5.50) is just Bessel's equation of order ν (after multiplying through by x).

Example 2. If $p = q = b = 0$ and $a = 1$, (5.50) is the equation $f'' + f = 0$, whose solutions are linear combinations of $\cos x$ and $\sin x$. The solutions given by Theorem 5.5 are linear combinations of $x^{1/2} J_{1/2}(x)$ and $x^{1/2} Y_{1/2}(x)$; by (5.19) and the fact that $Y_{1/2} = -J_{-1/2}$, these are just what they should be.

Example 3. If $p = b = 0$ and $a = -1$, and $q = 1$, (5.50) becomes the **Airy equation** $f''(x) - xf(x) = 0$, which arises in the study of diffraction and related phenomena in optics. In Theorem 5.5 we have $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\lambda = \frac{2}{3}$, and $\nu = \frac{1}{3}$, so the solutions are

$$f(x) = x^{1/2} \Big[c_1 I_{1/3}(\tfrac{2}{3} x^{3/2}) + c_2 K_{1/3}(\tfrac{2}{3} x^{3/2}) \Big]. \tag{5.51}$$

On the other hand, it is not hard to solve the Airy equation directly by assuming the solution to be of the form $f(x) = \sum a_n x^n$ and determining the coefficients a_n. We leave it as an exercise for the reader to do this and compare the results with the formula (5.51).

Example 4. As a final example, let us consider spherically symmetric waves in three dimensions, that is, solutions of the wave equation $u_{tt} = c^2\nabla^2 u$ of the form

$$u(x,y,z,t) = v(r,t), \quad \text{where } r = \sqrt{x^2+y^2+z^2}.$$

From the formula for the Laplacian in spherical coordinates (see Appendix 4), we have $\nabla^2 u = v_{rr} + 2r^{-1}v_r$, so the wave equation becomes

$$v_{tt} = c^2(v_{rr} + 2r^{-1}v_r). \tag{5.52}$$

Separation of variables, with $v(r,t) = R(r)T(t)$, leads to the equations

$$T''(t) + c^2\lambda^2 T(t) = 0, \qquad R''(r) + 2r^{-1}R'(r) + \lambda^2 R(r) = 0.$$

After being multiplied through by r^2, the equation for R is of the form (5.50) with $p = q = 2$, $a = \lambda^2$, and $b = 0$. Hence, by Theorem 5.5, the solutions are

$$R(r) = c_1 r^{-1/2} J_{1/2}(\lambda r) + c_2 r^{-1/2} Y_{1/2}(\lambda r) = \frac{a_1 \sin\lambda r + a_2 \cos\lambda r}{r}.$$

We therefore obtain as solutions of the spherical wave equation (5.52) the functions

$$v(r,t) = \frac{a_1 \sin\lambda r + a_2\cos\lambda r}{r}(b_1 \cos\lambda ct + b_2 \sin\lambda ct) \tag{5.53}$$

and their superpositions obtained from taking different values of λ. Of course, if we want solutions that are nonsingular at the origin we must take $a_2 = 0$. This result can be generalized to arbitrary waves in spherical coordinates (without special symmetry properties), and the answer turns out to involve the Bessel functions J_ν where ν is an arbitrary half-integer. We shall work this out in Chapter 6.

We can now construct a model for vibrations of air in a conical pipe such as an oboe or saxophone. We place the vertex of the cone at the origin and take the length of the pipe, measured along the slanting side, to be l. We also assume that the vibrations depend only on the distance from the vertex, so that equation (5.52) applies. Here the simplest interpretation of v is as the so-called *condensation,* which is the change in the air pressure p relative to the ambient pressure p_0: $v = (p - p_0)/p_0$. (In analyzing the vibrations of a cylindrical pipe by the 1-dimensional wave equation in §4.2, we took the solution u to represent the displacement of the air. At least for small vibrations, the condensation in this situation is given by $v = -u_x$. If u satisfies the wave equation then so does u_x; so these descriptions are equivalent, although the boundary conditions look different. In higher dimensions the displacement is a vector quantity and so is more complicated to study. See Ingard [32] or Taylor [51].) The pressure must be equal to the ambient pressure at the open end of the pipe, so $v(l,t) = 0$, and it must be finite at the vertex. Therefore, $a_2 = 0$ and $\lambda = n\pi/l$ in (5.53), where n is a positive integer, and we obtain the general solution

$$v(r,t) = \sum_1^\infty \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\frac{1}{r}\sin\frac{n\pi r}{l}. \tag{5.54}$$

Initial conditions $v(r,0) = f(r)$ and $v_t(r,0) = g(r)$ can be satisfied by expanding $rf(r)$ and $rg(r)$ in their Fourier sine series on $[0,l]$; see Exercise 3. Note that the allowable frequencies are the integer multiples of the fundamental frequency $l/2c$, just as in an open cylindrical pipe.

EXERCISES

1. Solve (5.49) for a general $h(\theta, z)$.
2. Find the general solution of the Airy equation $f''(x) - xf(x) = 0$ by assuming that $f(x) = \sum_0^\infty a_n x^n$ and determining the coefficients a_n. Since $x = 0$ is a regular point of the differential equation, a_0 and a_1 can be chosen arbitrarily; show that the solution f_1 with $a_0 = 1$, $a_1 = 0$ and the solution f_2 with $a_0 = 0$, $a_1 = 1$ are given by

$$f_1(x) = \Gamma(\tfrac{2}{3})3^{-1/3}x^{1/2}I_{-1/3}(\tfrac{2}{3}x^{3/2}), \qquad f_2(x) = \Gamma(\tfrac{4}{3})3^{1/3}x^{1/2}I_{1/3}(\tfrac{2}{3}x^{3/2}).$$

3. Determine the coefficients a_n and b_n in (5.54) if $v(r,0) = 0$ and $v_t(r,0) = l-r$.
4. A flexible cable hangs from a hook. (Assume the cable is on the z-axis, with bottom at $z = 0$ and top at $z = l$.) Since the tension at the point z on the cable is proportional to the weight of the portion of the cable below z, i.e., proportional to z, the appropriate wave equation to describe oscillations of the cable is $u_{tt} = c^2(zu_z)_z$ where u is the displacement. Since the top of the cable is fixed, $u(l,t) = 0$; and obviously the displacement $u(0,t)$ at the bottom must be finite. Find the general solution of this boundary value problem. (Hint: Use Theorem 5.5 and Exercise 9, §5.4.)

CHAPTER 6
ORTHOGONAL POLYNOMIALS

Some of the most useful orthogonal bases for L^2 spaces consist of polynomial functions. This chapter is a brief introduction to the most important of these orthogonal systems of polynomials; the last section also contains a discussion of some other interesting orthogonal bases.

6.1 Introduction

Let (a, b) be any open interval in $\mathbf{R}$, finite or infinite, and let $w(x)$ be a positive function on (a, b) such that the integrals $\int_a^b x^n w(x)\,dx$ $(n = 0, 1, 2, \ldots)$ are all absolutely convergent. Then there is a unique sequence $\{p_n\}_0^\infty$ of polynomials of the form

$$p_0(x) = 1, \qquad p_1(x) = x + a_0,$$
$$p_2(x) = x^2 + b_1 x + b_0, \qquad p_3(x) = x^3 + c_2 x^2 + c_1 x + c_0, \ldots$$

which is orthogonal with respect to the weight function w on (a, b). Indeed, the constant a_0 is fixed by the requirement that p_1 should be orthogonal to p_0:

$$0 = \langle p_1, p_0 \rangle_w = \int_a^b (x + a_0) w(x)\,dx \implies a_0 = -\frac{\int_a^b x w(x)\,dx}{\int_a^b w(x)\,dx}.$$

Once a_0, and hence p_1, is known, the orthogonality conditions $\langle p_2, p_1 \rangle_w = 0$ and $\langle p_2, p_0 \rangle_w = 0$ give two linear equations that can be solved for the two constants b_0 and b_1 in p_2. Once these have been found, the three equations $\langle p_3, p_2 \rangle_w = \langle p_3, p_1 \rangle_w = \langle p_3, p_0 \rangle_w = 0$ can be solved for the constants c_0, c_1, c_2 in p_3. Continuing in this way, we see that the coefficients of all the polynomials p_n are determined by the orthogonality conditions. (It is not difficult to show by induction that the systems of linear equations determining the coefficients all have unique solutions, and to construct a recursive formula for the solutions.)

In short, associated to each weight function w on an interval (a, b) as above, there is a unique sequence $\{p_n\}_0^\infty$ of polynomials determined by the requirements that

(i) p_n is a polynomial of degree n,
(ii) $\langle p_n, p_m\rangle_w = 0$ for all $n \neq m$,
(iii) the coefficient of x^n in p_n is 1.

If we keep conditions (i) and (ii) but drop condition (iii), we have the freedom to multiply each p_n by an arbitrary nonzero constant c_n, and c_n can be chosen to make p_n satisfy another auxiliary condition in place of (iii). For example, it can be chosen so as to make $\|p_n\|_w = 1$ or to fix the value of p_n at some point.

Before proceeding, let us point out one simple fact that will be used repeatedly in this chapter.

Lemma 6.1. *Suppose $\{p_n\}_0^\infty$ is a sequence of polynomials such that p_n is of (exact) degree n for all n. Then every polynomial of degree k ($k = 0, 1, 2, \ldots$) is a linear combination of $p_0, \ldots, p_k$.*

Proof: If f is a polynomial of degree k, choose the constant c_k so that f and $c_k p_k$ have the same coefficient of x^k. Then $f - p_k c_k$ is a polynomial of degree $k-1$, so we can choose c_{k-1} so that $f - c_k p_k$ and $c_{k-1}p_{k-1}$ have the same coefficient of x^{k-1}. Then $f - c_k p_k - c_{k-1}p_{k-1}$ is a polynomial of degree $k-2$, and we can proceed inductively to choose $c_{k-2}, \ldots, c_0$ so that $f - \sum_0^k c_n p_n = 0$. ∎

The classical orthogonal polynomials we shall be studying in this chapter are eigenfunctions for certain singular Sturm-Liouville problems. We could proceed as we did in the case of Bessel functions, by first writing down the differential equation to be solved, finding its complete solution by the method of power series, and then singling out the polynomial solutions for special attention. However, since our aim here is to develop the basic properties of these polynomials as quickly and cleanly as possible, we have chosen to relegate these calculations to the exercises and to adopt a more direct approach. In each of the classical systems the polynomials p_n can be defined by a formula of the form

$$p_n(x) = \frac{C_n}{w(x)}\frac{d^n}{dx^n}\left[w(x)P(x)^n\right] \tag{6.1}$$

where C_n is a constant, $w(x)$ is the weight function with respect to which the p_n's are orthogonal, and $P(x)$ is a certain fixed polynomial. These formulas are known as **Rodrigues formulas** (the original formula of Rodrigues being the one pertaining to Legendre polynomials). From (6.1) it is easy to prove the orthogonality relations for the p_n's, to derive the differential equation that they satisfy, and to find their normalization constants. (The constants C_n in (6.1) are firmly fixed by tradition in each case, and they usually are not chosen to make $\|p_n\|_w = 1$. Hence, in order to expand general functions in terms of the basis $\{p_n\}$, it is necessary to know $\|p_n\|_w$.)

Since the Sturm-Liouville problems leading to the classical orthogonal polynomials are all singular, the general Sturm-Liouville theory does not guarantee that these orthogonal systems are complete. However, they *are* complete, and we shall establish this by invoking some theorems from Chapter 7. (The results in

Chapter 7 do not depend on the material in Chapter 6, so the reader is free to skip ahead and read them at any time.) We shall also derive generating functions for these polynomials and sketch some of their applications. Much more can be said, but for a more complete discussion we refer the reader to the books of Erdélyi et al. [21], Hochstadt [30], Lebedev [36], Rainville [44], and Szegö [50].

EXERCISE

1. Let $\{p_n\}_0^\infty$ be an orthogonal set in $L^2_w(a,b)$, where p_n is a polynomial of degree n.
   a. Fix a value of n. Let $x_1, x_2, \dots x_k$ be the points in (a,b) where p_n changes sign, i.e., where its graph crosses the x-axis, and let $q(x) = \prod_1^k (x - x_j)$. Show that $p_n q$ never changes sign on (a,b) and hence that $\langle p_n, q\rangle_w \neq 0$.
   b. Show that the number k of sign changes in part (a) is at least n. (Hint: If $k < n$ then $\langle p_n, q\rangle_w = 0$. Why?)
   c. Conclude that p_n has exactly n distinct zeros, all of which lie in (a,b). (Geometrically, this indicates that p_n becomes more and more oscillatory on (a,b) as $n \to \infty$, rather like $\sin nx$.)

6.2 Legendre polynomials

The nth **Legendre polynomial**, denoted by P_n, is defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{6.2}$$

The function $(x^2-1)^n$ is a polynomial of degree $2n$ with leading term x^{2n}, so P_n is a polynomial of degree n. For the first few values of n we have

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2-1),$$
$$P_3(x) = \tfrac{1}{2}(5x^3-3x), \qquad P_4(x) = \tfrac{1}{8}(35x^4-30x^2+3).$$

See Figure 6.1. The coefficients of P_n can be calculated by using the binomial theorem (Exercise 1), but all we shall need is the leading one:

$$\begin{aligned} P_n(x) &= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^{2n}+\cdots) = \frac{1}{2^n n!}\Big[(2n)(2n-1)\cdots(n+1)x^n+\cdots\Big] \\ &= \frac{(2n)!}{2^n(n!)^2}x^n+\cdots, \end{aligned} \tag{6.3}$$

where the dots denote terms of lower degree.

We begin by establishing the orthogonality properties of the Legendre polynomials. In what follows we shall be working in the space $L^2(-1,1)$ (with weight function $w(x)\equiv 1$), and $\langle\ ,\ \rangle$ will denote the inner product in this space.

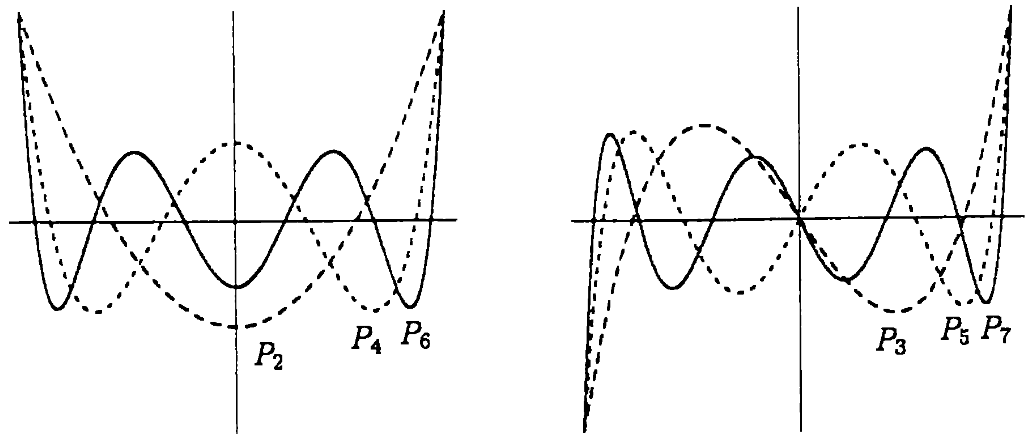


FIGURE 6.1. Graphs of some Legendre polynomials on the interval $-1 \leq x \leq 1$. Left: P_2 (long dashes), P_4 (short dashes), and P_6 (solid). Right: P_3 (long dashes), P_5 (short dashes), and P_7 (solid).

Theorem 6.1. *The Legendre polynomials $\{P_n\}_0^\infty$ are orthogonal in $L^2(-1,1)$, and*

$$\|P_n\|^2 = \frac{2}{2n+1}. \tag{6.4}$$

Proof: The key observation is that if f is any function of class $C^{(n)}$ on $[-1,1]$, we have

$$2^n n! \langle f, P_n \rangle = \int_{-1}^{1} f(x) \frac{d^n}{dx^n}(x^2-1)^n \, dx = (-1)^n \int_{-1}^{1} f^{(n)}(x)(x^2-1)^n \, dx. \tag{6.5}$$

The second of these equations follows by an n-fold integration by parts; the endpoint terms are all zero because the function $(x^2-1)^n = (x-1)^n(x+1)^n$ vanishes to nth order at $x = \pm 1$ and hence its first $n-1$ derivatives all vanish at $x = \pm 1$. If f is a polynomial of degree less than n then $f^{(n)} \equiv 0$, so $\langle f, P_n \rangle = 0$. In particular, this is true of $P_0, \ldots, P_{n-1}$, so $\langle P_m, P_n \rangle = 0$ for $m < n$. By the same reasoning with m and n interchanged, we also have $\langle P_m, P_n \rangle = 0$ for $m > n$, so the P_n's are mutually orthogonal.

On the other hand, if we take $f = P_n$, by (6.3) we have

$$f^{(n)}(x) \equiv \frac{(2n)!}{2^n n!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{2 \cdot 4 \cdots (2n)} = 1 \cdot 3 \cdot 5 \cdots (2n-1),$$

so by (6.5),

$$\|P_n\|^2 = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} \int_{-1}^{1} (1-x^2)^n \, dx.$$

But by the substitution $x = \sqrt{y}$ and the formula for the beta integral (Appendix 3),

$$\int_{-1}^{1}(1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx = \int_0^1 (1-y)^n y^{-1/2} dy = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}$$

$$= \frac{n!\sqrt{\pi}}{\Gamma(n+\frac{3}{2})} = \frac{n!}{(\frac{1}{2})(\frac{3}{2})\cdots(n+\frac{1}{2})} = \frac{2^{n+1}n!}{1\cdot 3\cdot 5\cdots(2n+1)},$$

which proves (6.4). ∎

We next derive the differential equation satisfied by the Legendre polynomials.

Theorem 6.2. *For all $n \geq 0$ we have*

$$\Big[(1-x^2)P_n'(x)\Big]' + n(n+1)P_n(x) = 0. \tag{6.6}$$

Proof: Let $g(x) = \Big[(1-x^2)P_n'(x)\Big]'$. Since P_n' is a polynomial of degree $n-1$, x^2P_n' is of degree $n+1$, and hence g is of degree n. In fact, by (6.3), its leading term is

$$\frac{(2n)!}{2^n(n!)^2}\frac{d}{dx}\Big[(-x^2)(nx^{n-1})\Big] = -n(n+1)\frac{(2n)!}{2^n(n!)^2}x^n.$$

Thus, in view of (6.3), $g + n(n+1)P_n$ is a polynomial of degree $n-1$, so by Lemma 6.1 it is a linear combination of $P_0, \dots, P_{n-1}$:

$$g(x) + n(n+1)P_n(x) = \Big[(1-x^2)P_n'(x)\Big]' + n(n+1)P_n(x) = \sum_0^{n-1} c_j P_j(x).$$

By orthogonality, the coefficients c_j can be calculated in terms of inner products:

$$c_j = \frac{\langle g + n(n+1)P_n, P_j\rangle}{\|P_j\|^2} = \frac{\langle g, P_j\rangle + n(n+1)\langle P_n, P_j\rangle}{\|P_j\|^2}.$$

Now $\langle P_n, P_j\rangle = 0$ for $j < n$, and

$$\langle g, P_j\rangle = \int_{-1}^{1}\Big[(1-x^2)P_n'(x)\Big]' P_j(x)\,dx.$$

After two integrations by parts, in which the boundary terms vanish since $x^2 - 1 = 0$ when $x = \pm 1$,

$$\langle g, P_j\rangle = \int_{-1}^{1} P_n(x)\Big[(1-x^2)P_j'(x)\Big]' dx.$$

But $\Big[(1-x^2)P_j'(x)\Big]'$ is a polynomial of degree j, hence is a linear combination of $P_0, \dots, P_j$, hence is orthogonal to P_n. Therefore, $c_j = 0$ for all $j < n$, and we are done. ∎

Theorem 6.2 says that the Legendre polynomials are eigenfunctions for the **Legendre equation**

$$\left[(1-x^2)y'\right]' + \lambda y = 0, \tag{6.7}$$

the eigenvalue for P_n being $n(n+1)$. This is an equation of Sturm-Liouville type on the interval $(-1, 1)$, but it is singular since the leading coefficient $1 - x^2$ vanishes at both endpoints. To arrive at the appropriate boundary conditions to define a Sturm-Liouville problem, one must examine the behavior of the solutions of (6.7) near $x = \pm 1$.

Briefly, the situation is as follows. The points $x = \pm 1$ are regular singular points for equation (6.7), and it is easily verified that the characteristic exponents at each of these singular points are both zero. Hence, for any λ, equation (6.7) will have one nontrivial solution that is analytic at $x = 1$, whereas any second independent solution will have a logarithmic singularity there; the same is true at $x = -1$. We may therefore impose boundary conditions on (6.7) by requiring that the solutions have no singularity at $x = \pm 1$, a requirement that can be phrased as follows:

$$\lim_{x \to 1} y(x) \text{ and } \lim_{x \to -1} y(x) \text{ exist.} \tag{6.8}$$

The Legendre polynomials are then eigenfunctions for the Sturm-Liouville problem defined by (6.7) and (6.8). We shall now establish the completeness of the Legendre polynomials, which implies in particular that there are no other eigenfunctions for (6.7) and (6.8).

Theorem 6.3. *$\{P_n\}_0^\infty$ is an orthogonal basis for $L^2(-1, 1)$.*

Proof: Suppose $f \in L^2(-1, 1)$ is orthogonal to all the P_n's, and hence (by Lemma 6.1) orthogonal to every polynomial. Given a small positive number ϵ, there is a continuous function g on $[-1, 1]$ such that $\|f - g\| < \frac{1}{2}\epsilon$ (Theorem 3.3, §3.3). By the Weierstrass approximation theorem, which we shall prove in §7.1, there is a polynomial P such that $|P(x) - g(x)| < \frac{1}{4}\epsilon$ for all $x \in [-1, 1]$, and hence such that

$$\|P - g\| = \left(\int_{-1}^{1} |P(x) - g(x)|^2 dx\right)^{1/2} < \tfrac{1}{4}\epsilon\sqrt{2} < \tfrac{1}{2}\epsilon.$$

But then

$$\|f\|^2 = \langle f, f\rangle = \langle f - g, f\rangle + \langle g - P, f\rangle + \langle P, f\rangle,$$

and since $\langle P, f\rangle = 0$ by hypothesis, the Cauchy-Schwarz inequality yields

$$\|f\|^2 \le \|f - g\| \, \|f\| + \|g - P\| \, \|f\| < \epsilon \|f\|,$$

so that $\|f\| < \epsilon$. Since ϵ is arbitrary, $f = 0$. ∎

In view of Theorem 6.1, the expansion of a function $f \in L^2(-1,1)$ in terms of Legendre polynomials is given by

$$f = \sum_0^\infty c_n P_n, \quad \text{where } c_n = \frac{2n+1}{2}\langle f, P_n\rangle = \frac{2n+1}{2}\int_{-1}^1 f(x)P_n(x)\,dx. \tag{6.9}$$

The series $\sum c_n P_n$ converges in norm; it can also be shown to converge pointwise provided that f is piecewise smooth, just as in the case of Fourier series.

A related type of expansion is sometimes useful. We observe that $(x^2-1)^n$ is an even function of x, so that its nth derivative $2^n n! P_n(x)$ is even or odd according as n is even or odd. Therefore, just as we passed from Fourier series on $[-\pi, \pi]$ to Fourier cosine and sine series on $[0, \pi]$, we can pass from series of Legendre polynomials on $[-1, 1]$ to series of even or odd Legendre polynomials on $[0, 1]$. The result is the following.

Theorem 6.4. *$\{P_{2n}\}_{n=0}^\infty$ and $\{P_{2n+1}\}_{n=0}^\infty$ are orthogonal bases for $L^2(0,1)$. The norm of P_k in $L^2(0,1)$ is $(2k+1)^{-1/2}$.*

The details of the proof are left to the reader (Exercise 11). The functions P_{2n} and P_{2n+1} are the eigenfunctions of the Sturm-Liouville problems on $(0, 1)$ defined by the Legendre equation (6.7) and the boundary conditions

$$\begin{aligned} &\lim_{x\to 1} y(x) \text{ exists}, \quad y'(0) = 0 \quad (\text{for } P_{2n}),\\ &\lim_{x\to 1} y(x) \text{ exists}, \quad y(0) = 0 \quad (\text{for } P_{2n+1}). \end{aligned}$$

The following identity gives the generating function for the Legendre polynomials. We shall derive it by means of contour integrals. Another approach (see Rainville [44] and Walker [53]) is to take this identity as a *definition* of the Legendre polynomials and develop the theory from there; in fact, this is what Legendre did originally.

Theorem 6.5. *For $-1 \le x \le 1$ and $|z| < 1$ we have*

$$\sum_0^\infty P_n(x) z^n = (1 - 2xz + z^2)^{-1/2}. \tag{6.10}$$

(Here z may be complex, and the principal branch of the square root function is used on the right.)

Proof: Given $x \in [-1, 1]$, let γ denote the circle of radius 1 about x in the complex plane. Applying the Cauchy formula for derivatives (Appendix 2) to the formula (6.2), we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2\pi i}\int_\gamma \frac{(\zeta^2-1)^n}{2^n(\zeta - x)^{n+1}}\,d\zeta.$$

Thus, if $|z|$ is small enough so that the geometric series $\sum\left[z(\zeta^2-1)/2(\zeta-x)\right]^n$ converges uniformly for $\zeta \in \gamma$ ($|z| < \frac{2}{5}$ is good enough),

$$\begin{aligned}
\sum_0^\infty P_n(x)z^n &= \frac{1}{2\pi i}\int_\gamma \sum_0^\infty \left(\frac{z}{2}\right)^n \frac{(\zeta^2-1)^n}{(\zeta-x)^{n+1}}\,d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta-x}\left[1-\frac{z(\zeta^2-1)}{2(\zeta-x)}\right]^{-1} d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \frac{2\,d\zeta}{z-2x+2\zeta-z\zeta^2}.
\end{aligned}$$

The zeros of $z-2x+2\zeta-z\zeta^2$, as a function of ζ, occur at

$$\zeta_1 = \frac{1-\sqrt{1-2xz+z^2}}{z} \quad \text{and} \quad \zeta_2 = \frac{1+\sqrt{1-2xz+z^2}}{z}.$$

When $|z|$ is small, $\sqrt{1-2xz+z^2}$ is approximately $1-xz$ (by the tangent line approximation), so ζ_1 is close to x while ζ_2 is very large. In particular, ζ_1 is inside the circle γ and ζ_2 is outside, so a simple calculation with the residue theorem gives

$$\sum_0^\infty P_n(x)z^n = \operatorname{Res}_{\zeta=\zeta_1} \frac{2}{z-2x+2\zeta-z\zeta^2} = (1-2xz+z^2)^{-1/2}.$$

Thus (6.10) is proved assuming that $|z|$ is sufficiently small. But then the series on the left of (6.10) is the Taylor series of the analytic function on the right, and its radius of convergence is the distance from the origin to the singularities of the latter function at $z = x \pm i\sqrt{1-x^2}$, namely, 1. The formula is therefore valid for all z such that $|z| < 1$. ∎

Corollary 6.1. *For all n we have $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.*

Proof: On setting $x = \pm 1$ in (6.10) we have

$$\sum_0^\infty P_n(1)z^n = \frac{1}{1-z}, \qquad \sum_0^\infty P_n(-1)z^n = \frac{1}{1+z}.$$

But the Taylor series of the functions $(1-z)^{-1}$ and $(1+z)^{-1}$ are just the geometric series $\sum z^n$ and $\sum(-1)^n z^n$. The result follows by comparing coefficients of z^n. ∎

Formula (6.10) has an interesting physical interpretation. If a charge (or mass) is located at the point $\mathbf{a}$ in $\mathbf{R}^3$, the induced electrostatic (or gravitational) potential at the point $\mathbf{x}$ is, up to a constant multiple, $|\mathbf{x}-\mathbf{a}|^{-1}$. Suppose that $\mathbf{a}$ is at a unit distance from the origin; let $r = |\mathbf{x}|$, and let θ be the angle between the vectors $\mathbf{x}$ and $\mathbf{a}$. Then by the geometric interpretation of the dot product,

$$|\mathbf{x}-\mathbf{a}|^{-1} = \left[(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})\right]^{-1/2} = (r^2 - 2r\cos\theta + 1)^{-1/2}.$$

Therefore, by (6.10), if $r < 1$ we have

$$|\mathbf{x} - \mathbf{a}|^{-1} = \sum_0^\infty P_n(\cos\theta)r^n.$$

That is, the Legendre polynomials give the expansion of the potential about the origin in powers of $r = |\mathbf{x}|$. Other applications of Legendre polynomials will be given in §6.2.

We conclude this section with a formula relating the Legendre polynomials and their derivatives.

Theorem 6.6. *For all $n \geq 1$ we have*

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$$

Proof: The second derivative of $(x^2 - 1)^{n+1}$ is

$$\frac{d}{dx}\Big[2(n+1)x(x^2-1)^n\Big] = 2(n+1)\Big[(x^2-1)^n + 2nx^2(x^2-1)^{n-1}\Big],$$

and by writing $x^2 = (x^2 - 1) + 1$ on the right we see that

$$\frac{d^2}{dx^2}(x^2-1)^{n+1} = 2(n+1)(2n+1)(x^2-1)^n + 4(n+1)n(x^2-1)^{n-1}.$$

Therefore, by formula (6.2),

$$\begin{aligned} P_{n+1}(x) &= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n-1}}{dx^{n-1}}\frac{d^2}{dx^2}(x^2-1)^{n+1} \\ &= \frac{(2n+1)}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n + \frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1} \\ &= \frac{(2n+1)}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n + P_{n-1}(x). \end{aligned}$$

Differentiating both sides and applying formula (6.2) once more, we are done. ∎

Theorem 6.6 can also be stated as

$$\int P_n(x)\,dx = \frac{1}{2n+1}\Big[P_{n+1}(x) - P_{n-1}(x)\Big] + C \qquad (n \geq 1), \tag{6.11}$$

a useful integration formula. (This formula holds also for $n = 0$ if we set $P_{-1}(x) = 0$.)

EXERCISES

1. Show that
$$P_n(x) = \frac{1}{2^n} \sum_{j \le n/2} \frac{(-1)^j (2n-2j)!}{j!(n-j)!(n-2j)!} x^{n-2j}.$$
2. Deduce from Exercise 1 that
$$P_{2k-1}(0) = 0, \qquad P_{2k}(0) = \frac{(-1)^k (2k)!}{2^{2k}(k!)^2}.$$
3. Find the general solution of the Legendre equation
$$\left[(1-x^2)y'\right]' + \lambda y = 0 \qquad (-1 < x < 1)$$
where λ is an arbitrary complex number. To do this, rewrite the equation in the form
$$(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$$
where ν is again an arbitrary complex number, set $y = \sum_0^\infty a_n x^n$, and determine the coefficients a_n recursively in terms of a_0 and a_1. Use the ratio test to verify that the resulting series converge on $(-1, 1)$.
4. With reference to Exercise 3, show that the Legendre equation has a polynomial solution precisely when ν is an integer, and that this solution is a constant multiple of P_ν if $\nu \ge 0$ or $P_{-\nu-1}$ if $\nu < 0$.
5. Show that the generating function $F(x, z) = (1 - 2xz + z^2)^{-1/2}$ of Theorem 6.5 satisfies $(1 - 2xz + z^2)(\partial F/\partial z) = (x - z)F$, and deduce the recursion formula
$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$
6. Expand x^2, x^3, and x^4 in series of Legendre polynomials. (Hint: No calculus is needed. Cf. the proof of Lemma 6.1.)
7. Let $f(x) = 1$ for $0 < x < 1$ and $f(x) = -1$ for $-1 < x < 0$. Expand f is a series of Legendre polynomials. (Hint: Use equation (6.11) and Exercise 2.)
8. Let $f(x) = x$ for $0 < x < 1$ and $f(x) = 0$ for $-1 < x < 0$. Expand f in a series of Legendre polynomials. (Hint: Use equation (6.11); you can leave the answer in terms of the numbers $P_n(0)$ or evaluate the latter by Exercise 2.)
9. As in the proof of Theorem 6.5, write
$$P_n(x) = \frac{1}{2\pi i} \int_\gamma \frac{(\zeta^2 - 1)^n}{2^n (\zeta - x)^{n+1}} \, d\zeta.$$
For $-1 < x < 1$, take γ to be the circle of radius $\sqrt{1 - x^2}$ about x, and deduce *Laplace's integral formula*
$$P_n(x) = \frac{1}{\pi} \int_0^\pi \left(x + i\sqrt{1 - x^2} \cos\theta\right)^n d\theta.$$
10. Deduce from Exercise 9 that $|P_n(x)| \le 1$ for $-1 \le x \le 1$.
11. Deduce Theorem 6.4 from Theorem 6.3.

6.3 Spherical coordinates and Legendre functions

In this section we shall solve some boundary value problems involving the Laplacian in spherical coordinates. We recall (see Appendix 4) that the spherical coordinates of a point $\mathbf{x} = (x, y, z) \in \mathbf{R}^3$ are given by

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi,$$

and that the Laplacian in spherical coordinates is given by

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin\phi}(u_\phi\sin\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}. \tag{6.12}$$

To begin with, we consider the Dirichlet problem for the unit ball in $\mathbf{R}^3$:

$$\nabla^2 u(r,\theta,\phi) = 0 \quad \text{for } r < 1, \qquad u(1,\theta,\phi) = f(\theta,\phi). \tag{6.13}$$

Applying the method of separation of variables, we look for solutions of $\nabla^2 u = 0$ in the form $u = R(r)\Theta(\theta)\Phi(\phi)$. Substituting this expression into the equation $\nabla^2 u = 0$ and rearranging the terms, we obtain

$$r^2\sin^2\phi\left[\frac{R''}{R} + \frac{2R'}{rR}\right] + \sin\phi\frac{(\Phi'\sin\phi)'}{\Phi} = -\frac{\Theta''}{\Theta}. \tag{6.14}$$

Both sides must equal a constant m^2. Thus $\Theta'' + m^2\Theta = 0$ and hence

$$\Theta(\theta) = ae^{im\theta} + be^{-im\theta}.$$

Since θ represents the longitude in spherical coordinates, Θ must be 2π-periodic; hence m must be an integer, which we may take to be nonnegative.

We now set the left side of (6.14) equal to m^2 and separate r and ϕ:

$$\frac{r^2R'' + 2rR'}{R} = \frac{m^2}{\sin^2\phi} - \frac{(\Phi'\sin\phi)'}{\Phi\sin\phi}.$$

Here both sides must equal a constant λ, and the equations for Φ and R can be written

$$\frac{(\Phi'\sin\phi)'}{\sin\phi} - \frac{m^2\Phi}{\sin^2\phi} + \lambda\Phi = 0, \tag{6.15}$$

$$r^2R'' + 2rR' - \lambda R = 0. \tag{6.16}$$

Equation (6.15) can be transformed into a close relative of the Legendre equation (6.7) by the substitution $s = \cos\phi$. (Recall that ϕ, the co-latitude, ranges over the interval $[0, \pi]$. The transformation $\phi \to s = \cos\phi$ is a one-to-one correspondence between $[0, \pi]$ and $[-1, 1]$.) Indeed, if q is a quantity depending on s and hence on ϕ, we have

$$\frac{dq}{d\phi} = \frac{dq}{ds}\frac{ds}{d\phi} = -\sin\phi\frac{dq}{ds}, \quad \text{or} \quad \frac{1}{\sin\phi}\frac{dq}{d\phi} = -\frac{dq}{ds}.$$

Hence, if we set

$$s = \cos\phi, \qquad S(s) = S(\cos\phi) = \Phi(\phi)$$

and note that $\sin^2\phi = 1 - s^2$, we have

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) = \frac{d}{ds}\left((1-s^2)\frac{dS}{ds}\right).$$

Therefore, $\Phi(\phi)$ satisfies (6.15) if and only if $S(s) = \Phi(\arccos s)$ satisfies

$$\left[(1-s^2)S'\right]' - \frac{m^2 S}{1-s^2} + \lambda S = 0. \tag{6.17}$$

When $m = 0$ this is just the Legendre equation (6.7). In general, (6.17) is called the **associated Legendre equation** of order m.

When m is a positive integer, as it is in our case, it is easy to find solutions of (6.17) in terms of the solutions of the ordinary Legendre equation

$$\left[(1-s^2)w'\right]' + \lambda w = 0. \tag{6.18}$$

Indeed, let w be a solution of (6.18). If we apply the product rule for $(m+1)$th order derivatives,

$$(fg)^{(m+1)} = \sum_0^{m+1} \frac{(m+1)!}{k!(m+1-k)!} f^{(k)} g^{(m+1-k)}$$

to $f(s) = 1 - s^2$ and $g(s) = w'(s)$, we obtain

$$\left[(1-s^2)w'\right]^{(m+1)} = (1-s^2)w^{(m+2)} - 2(m+1)sw^{(m+1)} - m(m+1)w^{(m)},$$

so by differentiating (6.18) m times we obtain

$$(1-s^2)w^{(m+2)} - 2(m+1)sw^{(m+1)} - m(m+1)w^{(m)} + \lambda w^{(m)} = 0. \tag{6.19}$$

Now let

$$S = (1-s^2)^{m/2} w^{(m)}.$$

We have

$$(1-s^2)S' = -ms(1-s^2)^{m/2} w^{(m)} + (1-s^2)^{(m/2)+1} w^{(m+1)},$$

and hence, after a straightforward calculation,

$$\Big[(1-s^2)S'\Big]' = (1-s^2)^{m/2} \times \left[(1-s^2)w^{(m+2)} - 2(m+1)sw^{(m+1)} + \frac{m^2 w^{(m)}}{1-s^2} - m(m+1)w^{(m)}\right].$$

But in view of (6.19), this means that

$$\Big[(1-s^2)S'\Big]' = \frac{m^2 S}{1-s^2} - \lambda S.$$

In other words, if w satisfies (6.18), then $S = (1-s^2)^{m/2}w^{(m)}$ satisfies (6.17).

In particular, if we take $\lambda = n(n+1)$ and take w to be the Legendre polynomial P_n, we obtain the **associated Legendre function** P_n^m:

$$P_n^m(s) = (1-s^2)^{m/2}\frac{d^m P_n(s)}{ds^m} = \frac{(1-s^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{ds^{n+m}}(s^2-1)^n. \tag{6.20}$$

(*Note:* Some authors insert an extra factor of $(-1)^m$ into the definition of $P_n^m(s)$.) We observe that $P_n^m(s) \equiv 0$ when $m > n$, since P_n is a polynomial of degree n, so P_n^m is of interest only for $n \geq m$. But for $m = 1, 2, 3, \ldots$ and $n \geq m$, P_n^m is a solution of the boundary value problem

$$\begin{gathered}\Big[(1-s^2)y'\Big]' + \frac{m^2 y}{1-x^2} + n(n+1)y = 0, \\ y(-1) = y(1) = 0.\end{gathered} \tag{6.21}$$

Theorem 6.7. *For each positive integer m, $\{P_n^m\}_{n=m}^\infty$ is an orthogonal basis for $L^2(-1,1)$, and*

$$\|P_n^m\|^2 = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}.$$

Proof: The orthogonality of P_n^m and $P_{n'}^m$ for $n \neq n'$ follows by the usual integration by parts from the fact that P_n^m satisfies (6.21); cf. the proof of Theorem 3.9 in §3.5. Also, from (6.20) we see that

$$P_{m+k}^m(s) = (1-s^2)^{m/2} q_k(s)$$

where q_k is a polynomial of degree k (we have set $n = m+k$ since $n \geq m$), and

$$\langle P_{m+k}^m, P_{m+k'}^m\rangle = \int_{-1}^1 q_k(s)q_{k'}(s)(1-s^2)^m\, ds.$$

That is, the polynomials q_k are orthogonal with respect to the weight function $w(s) = (1-s^2)^m$ on $(-1,1)$. The completeness of the set $\{q_k\}_0^\infty$ in $L_w^2(-1,1)$

follows from the Weierstrass approximation theorem, just as in the proof of Theorem 6.3. But then if $f \in L^2(-1,1)$ is orthogonal to all the P_{m+k}^m in $L^2(-1,1)$, the function $g(s) = (1-s^2)^{-m/2} f(s)$ will be orthogonal to all the q_k in $L_w^2(-1,1)$:

$$0 = \int_{-1}^{1} f(s) P_{m+k}^m(s)\, ds = \int_{-1}^{1} g(s) q_k(s)(1-s^2)^m\, ds.$$

It follows that $g = 0$ and hence $f = 0$, so the set $\{P_{m+k}^m\}_{k=0}^{\infty}$ is complete.

Finally, we compute the norm of P_n^m. To simplify the notation we fix n and set

$$y_m = P_n^m(s) \quad (m = 1, 2, \ldots, n), \qquad y_0 = P_n(s).$$

First, from (6.20) we have

$$\begin{aligned} y'_{m-1} &= \frac{d}{ds}\left[(1-s^2)^{(m-1)/2} \frac{d^{m-1} P_n(s)}{ds^{m-1}}\right] \\ &= -(m-1)s(1-s^2)^{(m-3)/2} \frac{d^{m-1} P_n(s)}{ds^{m-1}} + (1-s^2)^{(m-1)/2} \frac{d^m P_n(s)}{ds^m} \\ &= -(m-1)s(1-s^2)^{-1} y_{m-1} + (1-s^2)^{-1/2} y_m. \end{aligned}$$

In other words,

$$y_m = \sqrt{1-s^2}\, y'_{m-1} + \frac{(m-1)s}{\sqrt{1-s^2}} y_{m-1}.$$

Square both sides and integrate from -1 to 1:

$$\|y_m\|^2 = \int_{-1}^{1} \left[(1-s^2)(y'_{m-1})^2 + 2(m-1)s y_{m-1} y'_{m-1} + \frac{(m-1)^2 s^2}{1-s^2}(y_{m-1})^2\right] ds. \tag{6.22}$$

Now integrate the first two terms on the right by parts. For the first one, by (6.21) (with m replaced by $m-1$) we obtain

$$\begin{aligned} \int_{-1}^{1} (1-s^2)(y'_{m-1})^2\, ds &= -\int_{-1}^{1} y_{m-1}\left[(1-s^2) y'_{m-1}\right]' ds \\ &= \int_{-1}^{1} \left[n(n+1) - \frac{(m-1)^2}{1-s^2}\right] (y_{m-1})^2\, ds, \end{aligned}$$

whereas for the second one, we have

$$\begin{aligned} \int_{-1}^{1} s y_{m-1} y'_{m-1}\, ds &= -\int_{-1}^{1} y_{m-1} [s y_{m-1}]'\, ds \\ &= -\int_{-1}^{1} s y_{m-1} y'_{m-1}\, ds - \int_{-1}^{1} (y_{m-1})^2\, ds, \end{aligned}$$

which implies that

$$2\int_{-1}^{1} s y_{m-1} y'_{m-1}\, ds = -\int_{-1}^{1} (y_{m-1})^2\, ds.$$

Substituting these results into (6.22), we find that

$$\|y_m\|^2 = [n(n+1) - m(m-1)]\int_{-1}^{1}(y_{m-1})^2\, ds = (n+m)(n-m+1)\|y_{m-1}\|^2.$$

It therefore follows by induction that

$$\|y_m\|^2 = (n+m)\cdots(n+2)(n+1)(n-m+1)\cdots(n-1)n\|y_0\|^2,$$

or, in other words,

$$\|P_n^m\|^2 = \frac{(n+m)!}{(n-m)!}\|P_n\|^2 = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1},$$

where we have used Theorem 6.1 for the last equation. ∎

We now return to the Dirichlet problem (6.13). What we have found so far is that in the separated solution $u = R(r)\Theta(\theta)\Phi(\phi)$ of Laplace's equation, $\Theta(\theta)$ has the form $ae^{im\theta} + be^{-im\theta}$ with m a nonnegative integer, and $\Phi(\phi)$ has the form $y(\cos\phi)$ where y is a solution of the associated Legendre equation (6.17). Moreover, since we wish u to be a continuous function on the unit ball, $y(\pm 1)$ must be finite, and when $m > 0$, $y(\pm 1)$ must actually be zero. The reason is that the longitude θ is not well-defined along the z-axis, where $\cos\phi = \pm 1$, so the function $y(\cos\phi)(ae^{im\theta} + be^{-im\theta})$ will be discontinuous there unless $y(\pm 1) = 0$. The Legendre polynomials P_n (for $m = 0$) and the associated Legendre functions P_n^m (for $m > 0$) are solutions of these boundary value problems, and since they form complete orthogonal sets in $L^2(-1, 1)$, there are no other independent solutions.

In particular, the eigenvalue λ in the Legendre equation must be of the form $n(n+1)$ where n is a nonnegative integer, so the equation (6.16) for R becomes

$$r^2R'' + 2rR' - n(n+1)R = 0. \tag{6.23}$$

This is an Euler equation, and its general solution is

$$R(r) = ar^n + br^{-n-1}.$$

Since we want the solution to be continuous at the origin, we must take $b = 0$.

In short, we have found the following family of solutions of Laplace's equation:

$$u_{mn}(r,\theta,\phi) = r^n e^{im\theta} P_n^{|m|}(\cos\phi) \qquad (n = 0, 1, 2, \ldots;\ |m| \le n).$$

Here it is understood that $P_n^0 = P_n$, and we are using the letter m in a slightly different way than we did before in order to list $e^{im\theta}$ and $e^{-im\theta}$ separately. We therefore hope to solve our original problem (6.13) by taking a superposition of these solutions:

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} c_{mn} r^n e^{im\theta} P_n^{|m|}(\cos\phi), \tag{6.24}$$

for which the boundary condition $u(1,\theta,\phi) = f(\theta,\phi)$ becomes

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} c_{mn} e^{im\theta} P_n^{|m|}(\cos\phi). \tag{6.25}$$

Now, $\{e^{im\theta}\}_{m=-\infty}^{\infty}$ is a complete orthogonal set on $(-\pi,\pi)$, and since the substitution $s = \cos\phi$ gives

$$\int_{-1}^{1} F(s)\,ds = \int_0^{\pi} F(\cos\phi)\sin\phi\,d\phi$$

for any F, Theorems 6.1 and 6.7 show that for each m, $\{P_n^{|m|}(\cos\phi)\}_{n=|m|}^{\infty}$ is a complete orthogonal set on $(0,\pi)$ with respect to the weight function $\sin\phi$. It follows that the functions

$$Y_{mn}(\theta,\phi) = e^{im\theta} P_n^{|m|}(\phi) \qquad (n = 0,1,2,\ldots;\ |m| \le n),$$

considered as functions on the unit sphere S in $\mathbf{R}^3$, form an orthogonal basis of $L^2(S)$ with respect to the surface measure $d\sigma(\theta,\phi) = \sin\phi\,d\theta\,d\phi$. Moreover, the normalization constants can be read off from Theorems 6.1 and 6.7:

$$\|Y_{mn}\|^2 = 2\pi\frac{(n+|m|)!}{(n-|m|)!}\frac{2}{2n+1} = \frac{4\pi}{2n+1}\frac{(n+|m|)!}{(n-|m|)!}.$$

The functions Y_{mn} are called **spherical harmonics.**

The series (6.25) is just the expansion of f with respect to the basis of spherical harmonics, so the coefficients c_{mn} in (6.25) are given by

$$\begin{aligned} c_{mn} &= \frac{\langle f, Y_{mn}\rangle}{\|Y_{mn}\|^2} \\ &= \frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}\int_0^{\pi}\int_{-\pi}^{\pi} f(\theta,\phi)e^{-im\theta}P_n^{|m|}(\phi)\sin\phi\,d\theta\,d\phi. \end{aligned} \tag{6.26}$$

We have therefore proved the following result.

Theorem 6.8. *The solution of the Dirichlet problem (6.13) is the series (6.24) in which the coefficients c_{mn} are given by (6.26).*

This is not the end of the story, however. There are two additional important facts about the solution (6.24) that should be pointed out. The first, a significant feature that is obscured by the use of spherical coordinates, is that each term of the series in (6.24) is a homogeneous polynomial in the Cartesian coordinates (x,y,z). The second is that the infinite series in (6.24) can be re-expressed as an integral that is in some respects more useful; it is the 3-dimensional analogue of the Poisson integral formula that we presented in §4.4. We now discuss these two facts in the form of theorems.

Theorem 6.9. *For each m and n, the (m,n)th term in (6.24) is a homogeneous polynomial of degree n in the Cartesian coordinates (x,y,z).*

Proof: First consider the case $m \geq 0$. As in the proof of Theorem 6.7, we observe that $P_n^m(s) = (1-s^2)^{m/2} q_{n-m}(s)$ where q_{n-m} is a polynomial of degree $n-m$ that is even or odd according as $n-m$ is even or odd. Thus we can write $q_{n-m}(s) = \sum_{2j \leq n-m} a_j s^{n-m-2j}$, and hence

$$P_n^m(\cos\phi) = \sin^m\phi \sum_{2j\leq n-m} a_j \cos^{n-m-2j}\phi.$$

Therefore,

$$\begin{aligned} r^n e^{im\theta} P_n^m(\cos\phi) &= [re^{i\theta}\sin\phi]^m \sum_{2j\leq n-m} a_j r^{2j}(r\cos\phi)^{n-m-2j} \\ &= (x+iy)^m \sum_{2j\leq n-m} a_j (x^2+y^2+z^2)^j z^{n-m-2j}, \end{aligned}$$

which is a homogeneous polynomial of degree n. The same calculation shows that for $m < 0$,

$$r^n e^{im\theta} P_n^{|m|}(\cos\phi) = (x-iy)^{|m|} \sum_{2j\leq n-|m|} a_j (x^2+y^2+z^2)^j z^{n-|m|-2j}. \qquad \blacksquare$$

Theorem 6.9 implies that the series (6.24), when rewritten in Cartesian coordinates, is just the Taylor series of the solution u about the origin. It also implies that the spherical harmonics Y_{mn} are the restrictions of homogeneous harmonic polynomials to the unit sphere. The theory of spherical harmonics can also be developed from the beginning from this point of view; see Folland [24], Stein-Weiss [49], or Walker [53].

Theorem 6.10. *If f is a continuous function on the unit sphere $|\mathbf{x}| = 1$, the solution of the Dirichlet problem*

$$\nabla^2 u(\mathbf{x}) = 0 \text{ for } |\mathbf{x}| < 1, \qquad u(\mathbf{x}) = f(\mathbf{x}) \text{ for } |\mathbf{x}| = 1$$

is given by

$$u(\mathbf{x}) = \frac{1}{4\pi} \iint_{|\mathbf{y}|=1} \frac{1-|\mathbf{x}|^2}{(1-2|\mathbf{x}|\cos\alpha + |\mathbf{x}|^2)^{3/2}} f(\mathbf{y})\, d\sigma(\mathbf{y}), \tag{6.27}$$

where α is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$ and σ is the surface measure on the unit sphere.

Proof: First suppose $\mathbf{x}$ is on the positive z-axis, so that the spherical coordinates of $\mathbf{x}$ are $(r, *, 0)$ (the θ coordinate is undefined along the z-axis). Since $P_n^{|m|}(1) = 0$ for $m \neq 0$ (obvious from the definition) and $P_n^0(1) = P_n(1) = 1$ (Corollary 6.1), by Theorem 6.8 we have

$$u(r, *, 0) = \sum_{n=0}^{\infty} c_n r^n, \qquad c_n = \frac{2n+1}{4\pi} \int_0^{\pi} \int_{-\pi}^{\pi} f(\theta, \phi) P_n(\cos\phi) \sin\phi \, d\theta \, d\phi.$$

In other words,

$$u(r, *, 0) = \frac{1}{4\pi} \int_0^{\pi} \int_{-\pi}^{\pi} \left[\sum_0^{\infty} (2n+1) P_n(\cos\phi) r^n \right] f(\theta, \phi) \sin\phi \, d\theta \, d\phi \tag{6.28}$$

But since

$$(2n+1)r^n = \left(2r\frac{d}{dr} + 1 \right) r^n,$$

by Theorem 6.5 we have

$$\begin{aligned} \sum_0^{\infty} (2n+1) P_n(\cos\phi) r^n &= \left(2r\frac{d}{dr} + 1 \right) \frac{1}{(1 - 2r\cos\phi + r^2)^{1/2}} \\ &= \frac{1 - r^2}{(1 - 2r\cos\phi + r^2)^{3/2}}. \end{aligned}$$

If we substitute this into (6.28) and write $\mathbf{y}$ for the point whose spherical coordinates are $(1, \theta, \phi)$, we obtain (6.27) for the special case that $\mathbf{x}$ is on the positive z-axis. But (6.27) is expressed in a way that is independent of the particular Cartesian coordinate system used. Thus, given any vector $\mathbf{x}$, we can choose the positive z-axis to be in the same direction as $\mathbf{x}$, and the same reasoning then shows that (6.27) is valid at $\mathbf{x}$. ∎

A number of other boundary value problems involving the Laplacian in spherical coordinates can be solved by modifying the calculations leading up to Theorem 6.8. Some of these problems are examined in Exercises 3–6. We shall conclude this discussion by showing how to handle problems that lead to the equation $\nabla^2 u = -\mu^2 u$ rather than $\nabla^2 u = 0$.

Suppose, for example, that we wish to calculate the temperature $u(\mathbf{x}, t)$ in a solid ball of radius 1 given that the initial temperature is $f(\mathbf{x})$ and the surface of the ball is held at temperature zero:

$$u_t = k\nabla^2 u \text{ for } |\mathbf{x}| < 1, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}), \qquad u(\mathbf{x}, t) = 0 \text{ for } |\mathbf{x}| = 1. \tag{6.29}$$

We first separate out the t dependence by taking $u = X(\mathbf{x})T(t)$, which gives

$$\frac{T'}{kT} = \frac{\nabla^2 X}{X} = -\mu^2,$$

so that $T(t) = Ce^{-\mu^2 kt}$ and

$$\nabla^2 X = -\mu^2 X \quad \text{for } |\mathbf{x}| < 1, \qquad X(\mathbf{x}) = 0 \quad \text{for } |\mathbf{x}| = 1. \tag{6.30}$$

We now express $\mathbf{x}$ in spherical coordinates and proceed just as we did for the Dirichlet problem (6.13). The reader may verify without difficulty that if we take $X = R(r)\Theta(\theta)\Phi(\phi)$ in (6.30), we obtain exactly the same equations for Θ and Φ as we did before, so that $\Theta(\theta) = e^{im\theta}$ and $\Phi(\phi) = P_n^{|m|}(\cos\phi)$ where m and n are integers with $|m| \leq n$. The only change is that instead of the Euler equation (6.23) for R, we obtain

$$r^2 R'' + 2rR' + [\mu^2 r^2 - n(n+1)]R = 0. \tag{6.31}$$

This is almost, but not quite, a Bessel equation. In fact, if we set

$$R(r) = r^{-1/2} g(r),$$

a simple calculation shows that (6.31) turns into

$$r^2 g''(r) + rg'(r) + [\mu^2 r^2 - (n + \tfrac{1}{2})^2] g(r) = 0.$$

As we observed at the beginning of Chapter 5, the change of variable $r \to r/\mu$ transforms this into Bessel's equation of order $n + \frac{1}{2}$. Therefore, the solutions of (6.31) that are finite at $r = 0$ are constant multiples of

$$R(r) = r^{-1/2} J_{n+(1/2)}(\mu r).$$

(Recall that the power series expansion of $J_{n+(1/2)}(\mu r)$ about $r = 0$ involves the powers $r^{n+(1/2)+2j}$ with $j \geq 0$, so the expansion of R involves the powers r^{n+2j}. These exponents are positive integers, so R is analytic at the origin.) The boundary condition in (6.30) becomes $R(1) = 0$, so μ must be one of the positive zeros of $J_{n+(1/2)}$. Denoting these zeros by $\mu_1^n, \mu_2^n, \ldots$, we arrive at the following family of solutions of the heat equation that vanish on the unit sphere:

$$u(r, \theta, \phi, t) = \sum_{l,m,n} c_{lmn} r^{-1/2} J_{n+(1/2)}(\mu_l^n r) e^{im\theta} P_n^{|m|}(\cos\phi) e^{-(\mu_l^n)^2 t}. \tag{6.32}$$

Since the spherical harmonics $e^{im\theta} P_n^{|m|}(\cos\phi)$ form a complete orthogonal set with respect to the surface measure $\sin\phi \, d\theta \, d\phi$ on the unit sphere, and the Bessel functions $J_{n+(1/2)}(\mu_l^n r)$ form (for each fixed n) a complete orthogonal set with respect to the measure $r\,dr$ on $(0, 1)$, it follows that the functions

$$F_{lmn}(r, \theta, \phi) = r^{-1/2} J_{n+(1/2)}(\mu_l^n r) e^{im\theta} P_n^{|m|}(\cos\phi)$$

in (6.32) form a complete orthogonal set with respect to the volume measure

$$dV(r, \theta, \phi) = r^2 \sin\phi \, dr \, d\theta \, d\phi$$

on the unit ball. The normalizations are given by Theorem 6.7 and Theorem 5.3:

$$\|F_{lmn}\|^2 = \frac{2\pi(n+|m|)!J_{n+(3/2)}(\mu_l^n)^2}{(2n+1)(n-|m|)!}$$

To solve problem (6.29) we merely have to expand f with respect to this basis and plug the resulting coefficients c_{kmn} into (6.32).

The Legendre polynomials P_n and associated Legendre functions P_n^m are special solutions of the Legendre equations (6.18) and (6.17). For other applications — for example, the solution of the Dirichlet problem in a conical region — it is important to study the general solutions of these equations and to allow the parameters λ and m to be arbitrary complex numbers. These solutions go under the general name of **Legendre functions**. Accounts of the theory of Legendre functions can be found in Erdélyi et al. [21], Hochstadt [30], and Lebedev [36].

There are several other coordinate systems in $\mathbf{R}^3$ in which the technique of separation of variables can be applied to the Laplace operator, including the so-called *spheroidal, toroidal,* and *bipolar* coordinates. Separation of variables in these coordinates can be used to solve, for example, the Dirichlet problem in the interior of an ellipsoid of revolution, the interior of a torus, or the region between two intersecting spheres; the solutions all involve the Legendre functions. For a detailed account of these matters, we refer the reader to Lebedev [36].

EXERCISES

1. Solve the following Dirichlet problem: $\nabla^2 u(r,\theta,\phi) = 0$ for $r < 1$, $u(1,\theta,\phi) = \cos\phi$ for $0 \le \phi \le \frac{1}{2}\pi$, $u(1,\theta,\phi) = 0$ for $\frac{1}{2}\pi \le \phi \le \pi$. (Hint: Exercise 8, §6.2.)
2. Solve the following Dirichlet problem: $\nabla^2 u(r,\theta,\phi) = 0$ for $r < 1$, $u(1,\theta,\phi) = \cos^2\theta \sin^2\phi$. Express the answer both in spherical coordinates and in Cartesian coordinates.
3. Solve the Dirichlet problem for the *exterior* of a sphere: $\nabla^2 u(r,\theta,\phi) = 0$ for $r > 1$, $u(1,\theta,\phi) = f(\theta,\phi)$, and $u(r,\theta,\phi) \to 0$ as $r \to \infty$.
4. Solve the following Dirichlet problem in the upper hemisphere $r < 1$, $\phi < \frac{1}{2}\pi$: $\nabla^2 u(r,\theta,\phi) = 0$ for $r < 1$ and $\phi < \frac{1}{2}\pi$, $u(1,\theta,\phi) = f(\phi)$ for $\phi < \frac{1}{2}\pi$, $u(r,\theta,\frac{1}{2}\pi) = 0$. (Hint: Theorem 6.4.) What is the answer, explicitly, when $f(\phi) \equiv 1$? (Use Exercise 7, §6.2.)
5. Suppose the base of the hemispherical solid $r < 1$, $\phi < \frac{1}{2}\pi$ is insulated while its spherical surface $r = 1$ is held at a steady temperature $f(\phi)$. Find the steady-state temperature in the solid. (Hint: Theorem 6.4.)
6. Solve the Dirichlet problem in a spherical shell $a < r < b$: $\nabla^2 u(r,\theta,\phi) = 0$ for $a < r < b$, $u(a,\theta,\phi) = f(\theta,\phi)$, $u(b,\theta,\phi) = g(\theta,\phi)$. (Hint: Do the cases $f = 0$ and $g = 0$ separately; then use superposition.)
7. Solve the wave equation for the vibrations in a spherical cavity when the boundary is held fixed: $u_{tt} = c^u\nabla^2 u$ for $r < 1$, $u(1,\theta,\phi,t) = 0$. (Find the general solution for arbitrary initial conditions.)

8. Solve the initial value problem $u_t = k\Big[(1-x^2)u_x\Big]_x$ for $-1 < x < 1$, $u(x,0) = f(x)$. (This could be a model for the diffusion of a liquid through a gel whose diffusivity at position x is proportional to $1 - x^2$.)

6.4 Hermite polynomials

The nth **Hermite polynomial** $H_n(x)$ is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{6.33}$$

Simple calculations show that

$$\begin{aligned} &H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2, \\ &H_3(x) = 8x^3 - 12x, \qquad H_4(x) = 16x^4 - 48x^2 + 12. \end{aligned}$$

In general, we have

$$\begin{aligned} e^{-x^2} H_n(x) = (-1)^n \frac{d^n}{dx^n} e^{-x^2} &= -\frac{d}{dx}[e^{-x^2} H_{n-1}(x)] \\ &= e^{-x^2}[2xH_{n-1}(x) - H'_{n-1}(x)], \end{aligned}$$

or

$$H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x), \tag{6.34}$$

which allows one to compute H_n by induction on n (see Exercise 1). In particular, it follows from this formula that the leading term of $H_n(x)$ is $2x$ times the leading term of $H_{n-1}(x)$, and hence that H_n is a polynomial of degree n whose leading term is $(2x)^n$. Moreover, since e^{-x^2} is an even function, H_n is even or odd according as n is even or odd.

We now investigate the orthogonality properties of the Hermite polynomials. We shall be working with the spaces $L^2(\mathbf{R})$ and $L^2_w(\mathbf{R})$ where

$$w(x) = e^{-x^2}.$$

The symbol w will always have this meaning throughout this section. For future reference we note that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx = \int_0^{\infty} y^{-1/2}e^{-y}\,dy = \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Theorem 6.11. *The Hermite polynomials $\{H_n\}_0^\infty$ are orthogonal on $\mathbf{R}$ with respect to the weight function $w(x) = e^{-x^2}$, and*

$$\|H_n\|_w^2 = 2^n n! \sqrt{\pi}.$$

Proof: If f is any polynomial we have

$$\langle f, H_n\rangle_w = \int_{-\infty}^{\infty} f(x)H_n(x)e^{-x^2}dx = (-1)^n \int_{-\infty}^{\infty} f(x)\frac{d^n}{dx^n}e^{-x^2}dx$$
$$= \int_{-\infty}^{\infty} f^{(n)}(x)e^{-x^2}dx.$$

For the last equation we have integrated by parts n times; the boundary terms vanish because $P(x)e^{-x^2} \to 0$ as $x \to \pm\infty$ for any polynomial P. If f is a polynomial of degree less than n, and in particular if $f = H_m$ with $m < n$, then $f^{(n)} \equiv 0$ and hence $\langle f, H_n\rangle_w = 0$. This proves the orthogonality of the Hermite polynomials. On the other hand, if $f = H_n$ we have $f(x) = (2x)^n + \cdots$ and hence $f^{(n)} \equiv 2^n n!$, so

$$\|H_n\|_w^2 = 2^n n! \int_{-\infty}^{\infty} e^{-x^2}dx = 2^n n!\sqrt{\pi}. \qquad \blacksquare$$

We next establish the completeness of the orthogonal set $\{H_n\}_0^\infty$ in $L^2_w(\mathbf{R})$. Actually we shall prove a slightly stronger statement, for use in the next section.

Theorem 6.12. *Suppose f is a function on $\mathbf{R}$ such that $|f(x)|e^{|tx|}e^{-x^2}$ is integrable on $\mathbf{R}$ for all $t \in \mathbf{R}$. If*

$$\int_{-\infty}^{\infty} f(x)P(x)e^{-x^2}dx = 0 \quad \textit{for all polynomials } P,$$

then $f = 0$ (almost everywhere).

Proof: Since $e^{itx} = \sum_0^\infty (itx)^n/n!$ and

$$\left|\sum_0^N \frac{(itx)^n}{n!}\right| \le \sum_0^\infty \frac{|tx|^n}{n!} = e^{|tx|} \quad \text{for all } N \ge 0,$$

the dominated convergence theorem (applied to the partial sums of the series) implies that

$$\int_{-\infty}^{\infty} e^{itx}f(x)e^{-x^2}dx = \sum_0^\infty \frac{(it)^n}{n!}\int_{-\infty}^{\infty} x^n f(x)e^{-x^2}dx.$$

The hypothesis on f implies that all the integrals on the right vanish. By the Fourier inversion theorem, which we shall prove in §7.2, it follows that $f(x)e^{-x^2} = 0$, and hence $f(x) = 0$, almost everywhere. $\blacksquare$

Corollary 6.2. *The set $\{H_n\}_0^\infty$ is an orthogonal basis for $L^2_w(\mathbf{R})$.*

Proof: If $f \in L^2_w(\mathbf{R})$ and $\langle f, H_n\rangle_w = 0$ for all n, then $\langle f, P\rangle_w = 0$ for all polynomials P by Lemma 6.1, §6.1, and

$$\int_{-\infty}^{\infty} |f(x)|e^{|tx|}e^{-x^2}dx \le \left(\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2}dx\right)^{1/2}\left(\int_{-\infty}^{\infty} e^{2|tx|}e^{-x^2}dx\right)^{1/2} < \infty$$

by the Cauchy-Schwarz inequality. It follows from Theorem 6.12 that $f = 0$ in $L^2_w(\mathbf{R})$. $\blacksquare$

Our next step in investigating the Hermite functions is to derive their generating function.

Theorem 6.13. *For any $x \in \mathbf{R}$ and $z \in \mathbf{C}$ we have*

$$\sum_0^\infty H_n(x)\frac{z^n}{n!} = e^{2xz-z^2}. \tag{6.35}$$

Proof: This formula can be proved by the same method as Theorem 6.5 (see Exercise 7), but we shall adopt a different approach here. We begin by observing that if $u = x - z$ (where x is fixed) we have $d/du = -d/dz$, and hence

$$\frac{d^n}{dz^n}e^{-(x-z)^2}\Big|_{z=0} = (-1)^n\frac{d^n}{du^n}e^{-u^2}\Big|_{u=x} = e^{-u^2}H_n(u)\Big|_{u=x} = e^{-x^2}H_n(x).$$

Therefore, by Taylor's formula,

$$e^{-(x-z)^2} = \sum_0^\infty e^{-x^2}H_n(x)\frac{z^n}{n!}.$$

Multiplying through by e^{x^2}, we obtain (6.35). ∎

Differentiation of (6.35) with respect to x yields

$$\sum_0^\infty H_n'(x)\frac{z^n}{n!} = 2ze^{-2xz-z^2} = 2\sum_0^\infty H_n(x)\frac{z^{n+1}}{n!} = 2\sum_1^\infty H_{n-1}(x)\frac{z^n}{(n-1)!},$$

where for the last equation we have made the substitution $n \to n-1$. If we equate coefficients of z^n on the left and right, we find that

$$H_0' = 0, \qquad H_n' = 2nH_{n-1} \quad \text{for } n > 0. \tag{6.36}$$

Combining (6.36) with (6.34) yields the recursion formula

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \tag{6.37}$$

and also the differential equation

$$H_n(x) = \frac{x}{n}H_n'(x) - \frac{1}{2n}H_n''(x),$$

or

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

This equation can be written in Sturm-Liouville form by multiplying through by e^{-x^2}:

$$\left[e^{-x^2}H_n'(x)\right]' + 2ne^{-x^2}H_n(x) = 0.$$

In short, the Hermite polynomials are the eigenfunctions for the singular Sturm-Liouville problem

$$\left[e^{-x^2}y'\right]' + \lambda e^{-x^2}y = 0, \qquad -\infty < x < \infty, \tag{6.38}$$

the only "boundary condition" being that the solutions are required to be in $L^2_w(\mathbf{R})$.

For many purposes it is preferable to replace the Hermite polynomials by the **Hermite functions** h_n defined by

$$h_n(x) = e^{-x^2/2}H_n(x).$$

See Figure 6.2. We summarize their properties in a theorem.

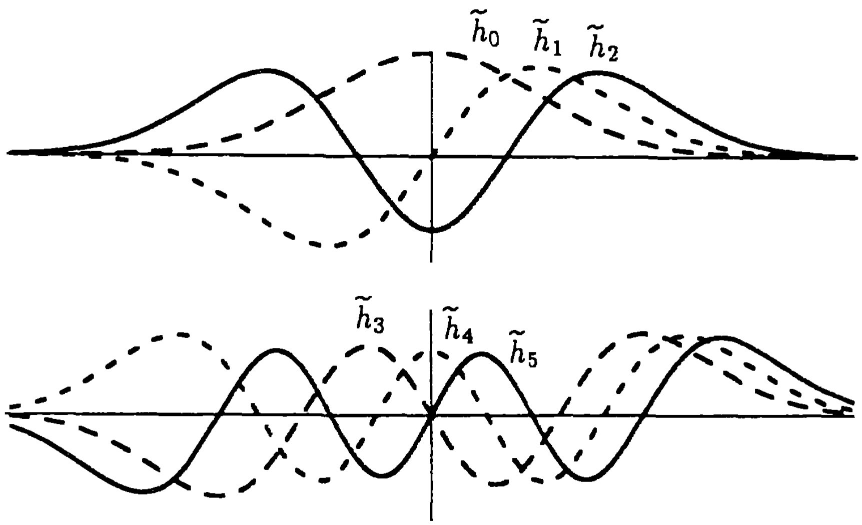


FIGURE 6.2. Graphs of some normalized Hermite functions $\widetilde{h}_n = h_n/\|h_n\| = h_n/\sqrt{2^n\pi^{1/2}n!}$ on the interval $-4 \le x \le 4$. Top: $\widetilde{h}_0$ (long dashes), $\widetilde{h}_1$ (short dashes), and $\widetilde{h}_2$ (solid). Bottom: $\widetilde{h}_3$ (long dashes), $\widetilde{h}_4$ (short dashes), and $\widetilde{h}_5$ (solid).

Theorem 6.14. *The Hermite functions* $\{h_n\}_0^\infty$ *are an orthogonal basis for* $L^2(\mathbf{R})$ *(with weight function* 1*). They satisfy*

$$xh_n(x) + h_n'(x) = 2nh_{n-1}(x), \tag{6.39}$$

$$xh_n(x) - h_n'(x) = h_{n+1}(x), \tag{6.40}$$

$$h_n''(x) - x^2h_n(x) + (2n+1)h_n(x) = 0. \tag{6.41}$$

Proof: The orthogonality of the h_n's follows from Theorem 6.11, since

$$\langle h_n, h_m\rangle = \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \langle H_n, H_m\rangle_w.$$

Similarly, their completeness follows from Theorem 6.12. If we write $H_n(x) = e^{x^2/2}h_n(x)$ in (6.36), we have

$$2ne^{x^2/2}h_{n-1}(x) = [e^{x^2/2}h_n(x)]' = e^{x^2/2}[xh_n(x) + h_n'(x)],$$

which is (6.39). In view of this result, (6.37) (with n replaced by $n+1$) becomes

$$h_{n+1}(x) = 2xh_n(x) - 2nh_{n-1}(x) = 2xh_n(x) - [xh_n(x) + h_n'(x)] = xh_n(x) - h_n'(x),$$

which is (6.40). Finally, if we combine (6.40) (with n replaced by $n-1$) and (6.39), we obtain

$$\begin{aligned}2nh_n(x) &= 2n[xh_{n-1}(x) - h_{n-1}'(x)] = x[xh_n(x) + h_n'(x)] - [xh_n(x) + h_n'(x)]' \\ &= x^2h_n(x) - h_n''(x) - h_n(x),\end{aligned}$$

which is (6.41). ∎

Equation (6.41) shows that the Hermite functions are the L^2 eigenfunctions for the Sturm-Liouville equation

$$y'' - x^2 y + \lambda y = 0. \tag{6.42}$$

(6.42) and (6.38) are both referred to in the literature as the **Hermite equation.**

The Hermite equation (6.42) arises in the study of the classical boundary value problems in parabolic regions, through the use of **parabolic coordinates**. These are coordinates (s,t) in the plane related to the Cartesian coordinates (x,y) by

$$x = s^2 - t^2, \qquad y = 2st \qquad (-\infty < s < \infty,\ t \geq 0).$$

The curves $s = c$ and $t = c$ (c constant) are the parabolas $x = c^2 - (y/2c)^2$ and $x = (y/2c)^2 - c^2$ opening to the left or the right, with focus at the origin. See Figure 6.3.

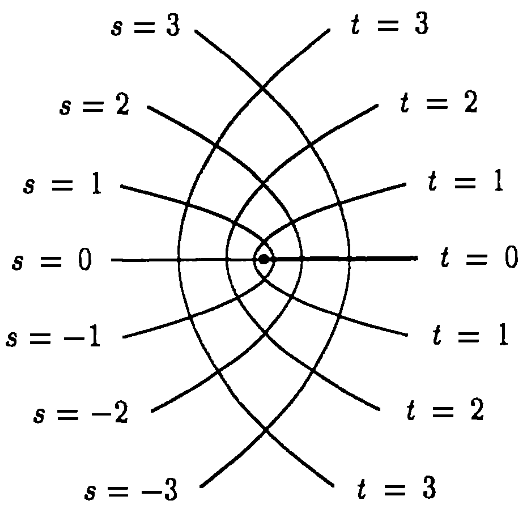


FIGURE 6.3. The parabolic coordinate system. This system is singular along the ray $t = 0$ (indicated by a heavy line), where the coordinates $(s,0)$ and $(-s,0)$ define the same point.

Let us consider the Laplace's equation in $\mathbf{R}^3$, in which we convert to parabolic coordinates in the xy-plane. A routine calculation with the chain rule shows that

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = \frac{1}{4(s^2 + t^2)}(u_{ss} + u_{tt}) + u_{zz}.$$

As the reader may readily verify, if we substitute $u = S(s)T(t)Z(z)$ into the equation $\nabla^2 u = 0$ and separate out Z first, we obtain the ordinary differential equations

$$\begin{aligned} Z''(z) + \mu^2 Z(z) &= 0, \\ S''(s) - 4\mu^2 s^2 S(s) + \lambda S(s) &= 0, \\ T''(t) - 4\mu^2 t^2 T(x) - \lambda T(t) &= 0. \end{aligned}$$

Moreover, the substitutions $S(s) = f(\sqrt{2\mu}s)$ and $T(t) = g(i\sqrt{2\mu}\,t)$ convert the equations for S and T into the Hermite equation (6.42); more precisely, with $\sigma = \sqrt{2\mu}s$ and $\tau = i\sqrt{2\mu}\,t$ we have

$$f''(\sigma) - \sigma^2 f(\sigma) + \frac{\lambda}{2\mu}f(\sigma) = g''(\tau) - \tau^2 g(\tau) + \frac{\lambda}{2\mu}g(\tau) = 0.$$

Hence, by taking $\lambda = 2\mu(2n+1)$ we obtain the solutions

$$u = e^{\pm i\mu z}h_n(\sqrt{2\mu}\,s)h_n(i\sqrt{2\mu}\,t) = e^{\pm i\mu z}e^{2\mu(t^2-s^2)}H_n(\sqrt{2\mu}\,s)H_n(i\sqrt{2\mu}\,t).$$

Of course, a complete analysis of the Laplacian in parabolic coordinates requires a study of all solutions of the Hermite equation (6.42) for arbitrary values of λ; these solutions are known as **parabolic cylinder functions**. This analysis is beyond the scope of this book, and we refer the reader to Erdélyi et al. [21] and Lebedev [36]; however, see Exercise 9.

The Hermite functions are also of importance in quantum mechanics, as they are the wave functions for the stationary states of the quantum harmonic oscillator. To be more precise, the wave functions for the stationary states of a quantum particle moving along a line in a potential $V(x)$ are the L^2 solutions of the equation

$$\frac{\hbar^2}{2m}u''(x) - V(x)u(x) + Eu(x) = 0,$$

where $\hbar$ is Planck's constant, m is the mass of the particle, and the eigenvalue E is the energy level. For a harmonic oscillator the potential is $V(x) = ax^2$ $(a > 0)$, so the substitutions $u(x) = f\Big([2am/\hbar^2]^{1/4}x\Big)$, $\xi = [2am/\hbar^2]^{1/4}x$ turn this equation into the Hermite equation

$$f''(\xi) - \xi^2 f(\xi) + \frac{\lambda}{\hbar}\sqrt{\frac{2m}{a}} = 0.$$

Thus the stationary wave functions are the Hermite functions $h_n\Big([2am/\hbar^2]^{1/4}x\Big)$, and the corresponding energy levels are $(2n+1)\hbar\sqrt{a/2m}$.

EXERCISES

1. Show by induction on n that

$$H_n(x) = n!\sum_{j\le n/2}\frac{(-1)^j(2x)^{n-2j}}{j!(n-2j)!}.$$

2. Find the general solution of the Hermite equation $y'' - 2xy' + \lambda y = 0$, where λ is an arbitrary complex number, by taking $y = \sum_0^\infty a_n x^n$ and solving for a_n in terms of a_0 and a_1. Show that the resulting series converge for all x.

3. Show that the Hermite equation in Exercise 2 has a polynomial solution of degree n precisely when $\lambda = 2n$, and that this solution is (a constant multiple of) H_n.
4. Expand the function $f(x) = x^{2m}$ (m a positive integer) in a series of Hermite polynomials. (Hint: Apply the formula used in the proof of Theorem 6.11.)
5. Expand the function $f(x) = e^{ax}$ in a series of Hermite polynomials. (Hint: Either proceed as in Exercise 4 or use Theorem 6.13.)
6. Let $f(x) = 1$ for $x > 0$, $f(x) = 0$ for $x < 0$. Expand f in a series of Hermite polynomials. (Hint: $e^{-x^2}H_n = -[e^{-x^2}H_{n-1}]'$. Use Exercise 1 to evaluate $H_n(0)$.)
7. Prove formula (6.35) by the method used to prove Theorem 6.5 — namely, plug definition (6.33) into the series $\sum H_n(x)z^n/n!$, apply the Cauchy integral formula for derivatives, sum the resulting geometric series, and finally apply the residue theorem.
8. Show that if $\phi_n(x) = h_n(ax)$ where $a > 0$, then $\{\phi_n\}_0^\infty$ is an orthogonal basis for $L^2(\mathbf{R})$, and that $\|\phi_n\|^2 = a^{-1}2^n n!\sqrt{\pi}$.
9. Let (s, t, z) denote parabolic coordinates in $\mathbf{R}^3$ as in the text. Consider the following Dirichlet problem in the parabolic slab $0 \le t \le 1$, $0 \le z \le 1$:

$$\nabla^2 u(s,t,z) = 0 \quad \text{for } t < 1,\ 0 < z < 1;$$
$$u(s,t,0) = u(s,t,1) = 0, \quad u(s,1,z) = f(s,z).$$

Assume that $\int_0^1 \int_{-\infty}^{\infty} |f(s,z)|^2 ds\,dz < \infty$ and find a solution in terms of Hermite functions.

6.5 Laguerre polynomials

Let α be a real number such that $\alpha > -1$. The nth **Laguerre polynomial** L_n^α corresponding to the parameter α is defined by

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x}). \tag{6.43}$$

(This formula makes perfectly good sense for any complex number α. However, it defines a polynomial of degree n only when α is not a negative integer, and these polynomials satisfy orthogonality relations only when $\alpha > -1$.) Some authors reserve the term *Laguerre polynomial* for the case $\alpha = 0$ and call the L_n^α's for $\alpha \neq 0$ *generalized Laguerre polynomials.*

By the product formula for nth derivatives, we have

$$\begin{aligned} L_n^\alpha(x) &= x^{-\alpha}e^x \sum_{k=0}^n \frac{1}{k!(n-k)!}\frac{d^k e^{-x}}{dx^k}\frac{d^{n-k}x^{\alpha+n}}{dx^{n-k}} \\ &= \sum_{k=0}^n \frac{(n+\alpha)(n-1+\alpha)\cdots(k+1-\alpha)}{k!(n-k)!}(-x)^k. \end{aligned} \tag{6.44}$$

Thus L_n^α is indeed a polynomial of degree n, and its leading term is $(-1)^n x^n/n!$. We shall now develop the basic properties of the polynomials L_n^α. The techniques will be very similar to those used for Legendre polynomials in §6.2, so we shall be brief.

Theorem 6.15. *The Laguerre polynomials* $\{L_n^\alpha\}_{n=0}^\infty$ *are a complete orthogonal set on* $(0,\infty)$ *with respect to the weight function*

$$w(x) = x^\alpha e^{-x},$$

and their norms are given by

$$\|L_n^\alpha\|_w^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

Proof: If f is any polynomial, an n-fold integration by parts shows that

$$\begin{aligned}\int_0^\infty f(x)L_n^\alpha(x)x^\alpha e^{-x}\,dx &= \frac{1}{n!}\int_0^\infty f(x)\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x})\,dx \\ &= \frac{(-1)^n}{n!}\int_0^\infty f^{(n)}(x)x^{\alpha+n}e^{-x}\,dx.\end{aligned}$$

If f is of degree less than n, in particular if $f = L_m^\alpha$ with $m < n$, then $f^{(n)} \equiv 0$; this proves that $\langle L_n^\alpha, L_m^\alpha\rangle_w = 0$ for $n \neq m$. On the other hand, if $f = L_n^\alpha$ then $f^{(n)} \equiv (-1)^n$ by (6.44), so

$$\|L_n^\alpha\|_w^2 = \frac{1}{n!}\int_0^\infty x^{\alpha+n}e^{-x}\,dx = \frac{\Gamma(n+\alpha+1)}{n!}.$$

To prove completeness, we assume that $g \in L_w^2(0,\infty)$ satisfies $\langle g, L_n^\alpha\rangle = 0$ for all n and show that $g = 0$. To do this, we transfer the problem from $(0,\infty)$ to $(-\infty,\infty)$ by using the formula

$$\int_0^\infty F(x)\,dx = \int_0^\infty F(y^2)\,2y\,dy = \int_{-\infty}^\infty F(y^2)|y|\,dy, \tag{6.45}$$

valid for any integrable function F on $(0,\infty)$. To begin with, since every polynomial, and in particular every monomial x^n, is a linear combination of L_n^α's (by Lemma 6.1, §6.1), the conditions $\langle g, L_n^\alpha\rangle_w = 0$ together with (6.45) imply that

$$0 = \int_0^\infty g(x)x^n x^\alpha e^{-x}\,dx = \int_{-\infty}^\infty g(y^2)|y|^{2\alpha+1}y^{2n}e^{-y^2}\,dy$$

for all n. But also

$$0 = \int_{-\infty}^\infty g(y^2)|y|^{2\alpha+1}y^{2n+1}e^{-y^2}\,dy$$

for all n, simply because the integrand is an odd function. Therefore,

$$0 = \int_{-\infty}^{\infty} g(y^2)|y|^{2\alpha+1} P(y) e^{-y^2}\, dy$$

for every polynomial P. But then the function $f(y) = g(y^2)|y|^{2\alpha+1}$ satisfies the hypotheses of Theorem 6.12, for by the Cauchy-Schwarz inequality and (6.45),

$$\begin{aligned}
&\int_{-\infty}^{\infty} |g(y^2)||y|^{2\alpha+1} e^{|ty|} e^{-y^2}\, dy \\
&\quad\le \left(\int_{-\infty}^{\infty} |g(y^2)|^2 |y|^{2\alpha+1} e^{-y^2}\, dy\right)^{1/2} \left(\int_{-\infty}^{\infty} |y|^{2\alpha+1} e^{2|ty|} e^{-y^2}\, dy\right)^{1/2} \\
&\quad= \|g\|_w \left(\int_{-\infty}^{\infty} |y|^{2\alpha+1} e^{2|ty|} e^{-y^2}\, dy\right)^{1/2} < \infty.
\end{aligned}$$

Therefore, by Theorem 6.12, $g = 0$. ∎

Remark. The assumption $\alpha > -1$ is necessary in Theorem 6.15. If $\alpha \le -1$ then the function $w(x) = x^\alpha e^{-x}$ is not integrable at the origin, so the integrals defining $\langle L_n^\alpha, L_k^\alpha \rangle_w$ and $\|L_n^\alpha\|_w^2$ all diverge.

We next show that the Laguerre polynomials satisfy the **Laguerre equation**

$$[x^{\alpha+1} e^{-x} y']' + n x^\alpha e^{-x} y = 0, \tag{6.46}$$

which can also be written in the form

$$xy'' + (\alpha + 1 - x)y' + ny = 0 \tag{6.47}$$

in view of the fact that

$$[x^{\alpha+1} e^{-x} y']' = x^{\alpha+1} e^{-x} y'' + (\alpha + 1 - x) x^\alpha e^{-x} y'. \tag{6.48}$$

Theorem 6.16. *The Laguerre polynomial L_n^α satisfies equation (6.46).*

Proof: Let $y_n = L_n^\alpha$. By (6.48),

$$[x^{\alpha+1} e^{-x} y_n']' = x^\alpha e^{-x} \Big[-x y_n' + x y_n'' + (\alpha + 1) y_n'\Big].$$

The expression in square brackets on the right is a polynomial of degree n whose leading term is the leading term of $-xy_n'$. By (6.44), this is the same as the leading term of $-ny_n$, namely, $(-1)^{n-1} x^n/(n-1)!$. In other words,

$$[x^{\alpha+1} e^{-x} y_n']' = x^\alpha e^{-x}(-ny_n + P) \tag{6.49}$$

where P is a polynomial of degree less than n. By Lemma 6.1, §6.1, P must be a linear combination of the Laguerre polynomials $y_k = L_k^\alpha$ with $k < n$. We shall show that P is orthogonal to all these polynomials with respect to the weight $w(x) = x^\alpha e^{-x}$, from which it follows that $P = 0$ and hence that y_n satisfies (6.46).

Indeed, by (6.49),

$$\int_0^\infty P(x)y_k(x)x^\alpha e^{-x}\,dx$$
$$= \int_0^\infty n y_n(x)y_k(x)x^\alpha e^{-x}\,dx + \int_0^\infty [x^{\alpha+1}e^{-x}y_n'(x)]'y_k(x)\,dx.$$

The first term on the right vanishes since $\langle y_n, y_k\rangle_w = 0$, and after two integrations by parts and another use of (6.48), the second term becomes

$$\int_0^\infty y_n(x)[x^{\alpha+1}e^{-x}y_k'(x)]'\,dx = \int_0^\infty y_n(x)Q(x)x^\alpha e^{-x}\,dx$$

where Q is a polynomial of degree k. But y_n is orthogonal to all such polynomials, so the integral vanishes. ∎

Theorem 6.16 implies that the Laguerre polynomials L_n^α are the eigenfunctions for a Sturm-Liouville problem on the interval $(0,\infty)$ associated to the differential equation

$$[x^{\alpha+1}e^{-x}y']' + \lambda x^\alpha e^{-x}y = 0. \tag{6.50}$$

This problem is singular both because $(0,\infty)$ is an infinite interval and because (6.50) has a regular singular point at $x = 0$. The boundary conditions for this problem are as follows. At infinity the only condition is that the solution should be square-integrable with respect to the weight $w(x) = x^\alpha e^{-x}$. On the other hand, from the theory of regular singular points, one knows that (6.50) has one solution that is analytic at $x = 0$ and another that is asymptotic to $x^{-\alpha}$ (or $\log x$ when $\alpha = 0$) as $x \to 0$. For $\alpha \geq 0$ the analytic solution is singled out by requiring it to remain finite as $x \to 0$, whereas for $-1 < \alpha < 0$ it is singled out by requiring that its first derivative remain finite as $x \to 0$.

We now derive the generating function for the Laguerre polynomials.

Theorem 6.17. *For $x > 0$ and $|z| < 1$,*

$$\sum_0^\infty L_n^\alpha(x)z^n = \frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1}} \tag{6.51}$$

Proof: The idea is the same as the proof of Theorem 6.5. Namely, if $x > 0$, let γ denote a circle in the right half-plane centered at x. We successively apply the defninion (6.43), the Cauchy integral formula for derivatives, the formula for the sum of a geometric series, the substitution $\sigma = (1-z)\zeta$ (which transforms γ into another circle γ'), and the Cauchy integral formula once again to obtain

$$\begin{aligned}
\sum_0^\infty L_n^\alpha(x)z^n &= \sum_0^\infty \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x})z^n \\
&= \frac{x^{-\alpha}e^x}{2\pi i}\sum_0^\infty z^n \int_\gamma \frac{\zeta^{\alpha+n}e^{-\zeta}}{(\zeta-x)^{n+1}}\,d\zeta \\
&= \frac{x^{-\alpha}e^x}{2\pi i}\int_\gamma \frac{\zeta^\alpha e^{-\zeta}}{\zeta-x}\sum_0^\infty\left(\frac{\zeta z}{\zeta-x}\right)^n d\zeta \\
&= \frac{x^{-\alpha}e^x}{2\pi i}\int_\gamma \frac{\zeta^\alpha e^{-\zeta}}{\zeta(1-z)-x}\,d\zeta \\
&= \frac{x^{-\alpha}e^x}{2\pi i(1-z)^{\alpha+1}}\int_{\gamma'} \frac{\sigma^\alpha e^{-\sigma/(1-z)}}{\sigma-x}\,d\sigma \\
&= \frac{x^{-\alpha}e^x}{(1-z)^{\alpha+1}}x^\alpha e^{-x/(1-z)}.
\end{aligned}$$

These formal calculations are valid for $|z|$ sufficiently small and prove (6.51) for such z; but then (6.51) is valid for $|z| < 1$ since the right side is analytic there. We leave the details of the justification to the reader. ∎

Perhaps the most striking application of Laguerre polynomials is in the quantum-mechanical analysis of the hydrogen atom. We shall sketch the ideas very briefly and refer the reader to Landau-Lifshitz [35] (whose notation for Laguerre polynomials, however, differs from ours) for a complete treatment.

Consider a system consisting of an electron and a proton. Since the proton is about 2,000 times more massive than the electron, we shall neglect its motion and consider it to be fixed at the origin. The electron is then moving in an electrostatic force field with potential $-\epsilon^2/r$ where ϵ is the charge of the proton and r is the distance from the origin. According to quantum mechanics, if the electron is in a stationary state at the energy level $E \in \mathbf{R}$, its wave function u is a function in $L^2(\mathbf{R}^3)$ satisfying the equation

$$\frac{\hbar^2}{2m}\nabla^2 u + \frac{\epsilon^2}{r}u + Eu = 0, \tag{6.52}$$

where $\hbar$ is Planck's constant and m is the mass of the electron. By an appropriate choice of units we may, and shall, assume that $\hbar = m = \epsilon = 1$.

We apply the method of separation of variables to solve (6.52), using spherical coordinates and taking $u = R(r)\Theta(\theta)\Phi(\phi)$. By the same calculations as in §6.3 we find that $\Theta(\theta) = e^{im\theta}$ and $\Phi(\phi) = P_n^{|m|}(\cos\phi)$ where m and n are integers with $n \geq |m|$, and R satisfies

$$r^2R'' + 2rR' + [2Er^2 + 2r - n(n+1)]R = 0. \tag{6.53}$$

We are primarily interested in the states where the energy level E is negative, that is, where the electron and proton are bound together in an atom. Assuming $E < 0$, then, we make the substitutions

$$\nu = (-2E)^{-1/2}, \qquad s = 2\nu^{-1}r, \qquad R(r) = S(2\nu^{-1}r) = S(s),$$

which (by a routine calculation) turn (6.53) into

$$s^2S'' + 2sS' + [\nu s - \tfrac{1}{4}s^2 - n(n+1)]S = 0. \tag{6.54}$$

Finally, we set $S = s^n e^{-s/2}\Sigma$ in (6.54), and after some more computation we obtain

$$s\Sigma'' + (2n + 2 - s)\Sigma' + (\nu - n - 1)\Sigma = 0.$$

This is the Laguerre equation (6.47) with $\alpha = 2n + 1$ and n replaced by $\nu - n - 1$. The only solutions of this equation that lead to solutions $u = R\Theta\Phi$ of (6.52) that are in $L^2(\mathbf{R}^3)$ are the Laguerre polynomials. Hence ν must be an integer $\geq n + 1$, and after reversing these substitutions we end up with the solution

$$R_{n\nu}(r) = (2\nu^{-1}r)^n e^{-r/\nu} L_{\nu-n-1}^{2n+1}(2\nu^{-1}r) \tag{6.55}$$

of (6.53). The eigenfunctions for the original problem (6.52) are

$$u_{mn\nu} = R_{n\nu}(r)e^{im\theta}P_n^{|m|}(\cos\phi) \qquad (|m| \leq n < \nu), \tag{6.56}$$

and the eigenvalue E of $u_{mn\nu}$ is $-\frac{1}{2}\nu^{-2}$.

We conclude with two important points concerning the physical interpretation of these results. First, when an electron jumps from one energy level $-\frac{1}{2}\nu^{-2}$ to a lower one $-\frac{1}{2}\mu^{-2}$, it emits a photon of frequency $(h/2)(\mu^{-2} - \nu^{-2})$. The fact that these frequencies are (up to a constant factor) differences of reciprocal squares of integers was known experimentally before the invention of quantum mechanics, and it provided one of the decisive early confirmations of the quantum theory.

Second, for each eigenvalue $-\frac{1}{2}\nu^{-2}$ with $\nu > 1$ there are several different eigenfunctions in the list (6.56). In fact, there are ν allowable values of n (namely, $0, \ldots, \nu - 1$), and for each such n there are $2n + 1$ allowable values of m (namely, $-n, \ldots, n$). Hence for each ν there are

$$\sum_{n=0}^{\nu-1}(2n+1) = \nu^2$$

independent eigenfunctions. Actually, one must also take into account the spin of the electron, which has two eigenstates ("up"and "down"); this effectively doubles the number of independent eigenfunctions at each energy level. The collections of eigenfunctions at the various energy levels constitute the "electron shells" that form the basis for the periodic table of elements.

One final remark: The eigenfunctions (6.55) for the Sturm-Liouville problem (6.53) do *not* form a complete orthogonal set. Problem (6.53) has both a discrete and a continuous spectrum; this means that the expansion of a general function in terms of the eigenfunctions of (6.53) involves not only a sum over the eigenfunctions with eigenvalues $-\frac{1}{2}\nu^{-2}$ that we found above, but also an integral over a collection of (non-L^2) eigenfunctions corresponding to eigenvalues $E \geq 0$. Physically this means that an electron-proton system has not only bound states but also unbound states in which the electron has enough energy to escape from the electrostatic potential well. An analysis of the unbound states can be found in Landau-Lifshitz [35].

EXERCISES

1. Consider the Laguerre equation $xy'' + (\alpha + 1 - x)y' + \lambda y = 0$ where λ and α are arbitrary complex numbers.
   a. Assuming that α is not a negative integer, find a solution in the form $y = \sum_0^\infty a_n x^n$ with $a_0 = 1$, and show that this solution is a constant multiple of L_n^α when λ is a nonnegative integer n.
   b. Assuming that α is not a positive integer, find a solution in the form $y = \sum_0^\infty b_n x^{n-\alpha}$ with $b_0 = 1$.
2. By differentiating formula (6.51) with respect to z, show that
$$(1 - z^2)\frac{\partial}{\partial z}\sum L_n^\alpha(x)z^n = \left[x + (1+\alpha)(z-1)\right]\sum L_n^\alpha(x)z^n,$$
and hence derive the recursion formula
$$(n+1)L_{n+1}^\alpha(x) + (x - \alpha - 2n - 1)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0.$$
3. By differentiating formula (6.51) with respect to x as in Exercise 2, show that
$$(L_n^\alpha)'(x) - (L_{n-1}^\alpha)'(x) + L_{n-1}^\alpha(x) = 0.$$
4. Use formula (6.44) and Exercise 1, §6.4, to show that
$$L_n^{-1/2}(x) = \frac{(-1)^n}{2^{2n}n!}H_{2n}(\sqrt{x}), \qquad L_n^{1/2}(x) = \frac{(-1)^n}{2^{2n+1}n!}\frac{H_{2n+1}(\sqrt{x})}{\sqrt{x}}.$$
5. Expand the function $f(x) = x^\nu$ ($\nu \geq 0$) in a series of Laguerre polynomials. (Hint: To compute $\langle f, L_n^\alpha\rangle_w$, use formula (6.43) and integrate by parts n times.)
6. Expand the function $f(x) = e^{-bx}$ ($b > 0$) in a series of Laguerre polynomials. (Hint: Either proceed as in Exercise 5 or use Theorem 6.17.)

6.6 Other orthogonal bases

In this section we give a brief introduction to the other classical orthogonal sets of polynomials and to a few other orthonormal bases for L^2 spaces, not connected with differential equations, that have proved to be of importance.

Chebyshev polynomials*

The nth **Chebyshev polynomial** T_n is defined by the formula

$$T_n(\cos\theta) = \cos n\theta. \tag{6.57}$$

* The number of ways of transliterating the Russian name Chebyshev is almost infinite: Tchebyshev, Tchebichef, Tschebyschev, Čebyšev, etc.

Explicitly, we have

$$\cos n\theta = \operatorname{Re} e^{in\theta} = \operatorname{Re}(\cos\theta + i\sin\theta)^n = \operatorname{Re}\sum_{j=0}^{n} \frac{n!}{j!(n-j)!}(\cos\theta)^{n-j}(i\sin\theta)^j.$$

The real terms in the sum are those with j even, say $j = 2k$, and $(i\sin\theta)^{2k} = (\cos^2\theta - 1)^k$, so

$$\cos n\theta = \sum_{k\le n/2} \frac{n!}{(2k)!(n-2k)!}\cos^{n-2k}\theta\,(\cos^2\theta - 1)^k.$$

Therefore,

$$T_n(x) = \sum_{k\le n/2} \frac{n!}{(2k)!(n-2k)!}x^{n-2k}(x^2-1)^k.$$

Since $\{\cos n\theta\}_0^\infty$ is an orthogonal basis for $L^2(0,\pi)$, the substitution $\theta = \arccos x$ shows that $\{T_n\}_0^\infty$ is an orthogonal basis for $L^2_w(-1,1)$ where $w(x) = (1-x^2)^{-1/2}$. Indeed, if $m \ne n$,

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{(1-x^2)^{1/2}}dx = \int_0^\pi T_n(\cos\theta)T_m(\cos\theta)\,d\theta = \int_0^\pi \cos n\theta\cos m\theta\,d\theta = 0,$$

which gives the orthogonality. Likewise, if f is orthogonal to all T_n,

$$0 = \int_{-1}^{1} \frac{f(x)T_n(x)}{(1-x^2)^{1/2}}dx = \int_0^\pi f(\cos\theta)\cos n\theta\,d\theta,$$

whence $f = 0$; this gives the completeness. The same substitution shows that the differential equation $y'' + n^2y = 0$ for $\cos n\theta$ turns into the **Chebyshev equation**

$$(1-x^2)y'' - xy' - n^2y = 0, \qquad \text{or} \qquad \left[(1-x^2)^{1/2}y'\right]' + n^2(1-x^2)^{-1/2}y = 0,$$

satisfied by T_n.

The generating function for the Chebyshev polynomials is given by

$$1 + 2\sum_1^\infty T_n(x)z^n = \frac{1-z^2}{1-2xz+z^2}.$$

This formula is easily proved by substituting $x = \cos\theta$, writing

$$1 + 2\sum_1^\infty T_n(\cos\theta)z^n = 1 + 2\sum_1^\infty(\cos n\theta)z^n = \sum_{-\infty}^\infty e^{in\theta}z^n,$$

and summing the geometric series. We actually performed this calculation in §4.4, where this generating function (with $x = \cos\theta$ and $z = r$) turned out to be the Poisson kernel.

Chebyshev polynomials are of great importance in the theory of polynomial interpolation and approximation. We refer the reader to Rivlin [45] for a comprehensive account; see also Körner [34], §§43–45.

Jacobi polynomials

Let α and β be real numbers greater than -1. The nth **Jacobi polynomial** $P_n^{(\alpha,\beta)}$ associated to the parameters α and β is defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Big]. \qquad (6.58)$$

When $\alpha = \beta = 0$, $P_n^{(\alpha,\beta)}$ is the Legendre polynomial P_n. The techniques we used in §6.2 to investigate the Legendre polynomials can be generalized to yield analogous results for the Jacobi polynomials:

(i) For each α and β, $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ is an orthogonal basis for $L_w^2(-1,1)$ where $w(x) = (1-x)^\alpha(1+x)^\beta$, and

$$\|P_n^{(\alpha,\beta)}\|_w = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

(ii) $P_n^{(\alpha,\beta)}$ satisfies the **Jacobi equation**

$$(1-x^2)y'' + \Big[\beta-\alpha-(\alpha+\beta+2)x\Big]y' + n(n+\alpha+\beta+1)y = 0.$$

(iii) The generating function for the Jacobi polynomials $P_n^{(\alpha,\beta)}$ is

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)z^n = \frac{2^{\alpha+\beta}}{W(1-z+W)^\alpha(1+z+W)^\beta}, \qquad W = \sqrt{1-2xz+z^2}.$$

For more details, see Erdélyi et al. [21], Hochstadt [30], and Szegö [50].

We have observed that Legendre polynomials are the special case of Jacobi polynomials with $\alpha = \beta = 0$. Chebyshev polynomials are also essentially a special case of Jacobi polynomials, with $\alpha = \beta = -\frac{1}{2}$. Indeed, from the fact that $\{P_n^{(-1/2,-1/2)}\}$ and $\{T_n\}$ are both orthogonal bases for $L_w^2(-1,1)$ where $w(x) = (1-x^2)^{-1/2}$, or from the fact that the Jacobi differential equation reduces to the Chebyshev equation when $\alpha = \beta = -\frac{1}{2}$, it follows that T_n must be a constant multiple of $P_n^{(-1/2,-1/2)}$. In fact, it turns out that

$$T_n = \frac{2^{2n}(n!)^2}{(2n)!}P_n^{(-1/2,-1/2)}.$$

In the cases $\alpha = \beta > -\frac{1}{2}$, the Jacobi polynomials are sometimes given a different normalization and called **Gegenbauer polynomials** or **ultraspherical polynomials**. Precisely, the nth Gegenbauer polynomial C_n^λ associated to the parameter $\lambda > 0$ is defined by

$$C_n^\lambda(x) = \frac{\Gamma(2\lambda+n)\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)\Gamma(\lambda+n+\frac{1}{2})}P_n^{(\lambda-(1/2),\lambda-(1/2))}(x).$$

The reason for the new normalization is that the Gegenbauer polynomials have the simple generating function

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x) z^n = (1 - 2xz + z^2)^{-\lambda}.$$

The Jacobi polynomials $P_n^{((k-3)/2,(k-3)/2)}$, or equivalently the Gegenbauer polynomials $C_n^{(k-2)/2}$, play the same role in the theory of spherical harmonics in $\mathbf{R}^k$ as the Legendre polynomials do in $\mathbf{R}^3$; see Erdélyi et al. [21] and Stein-Weiss [49]. For some of the deeper properties and uses of Jacobi polynomials, see Askey [3].

Haar and Walsh functions

There are two interesting orthonormal bases for $L^2(0,1)$ consisting of step functions. The first one is the system of **Haar functions**

$$\{h_{(0)}\} \cup \{h_{jn} : j \geq 0,\ 0 \leq n < 2^j\}$$

constructed as follows:

$$h_{(0)}(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and for $j \geq 0$ and $0 \leq n < 2^j$,

$$h_{jn}(x) = \begin{cases} 2^{j/2} & \text{if } 2^{-j}n < x < 2^{-j}(n + \frac{1}{2}), \\ -2^{j/2} & \text{if } 2^{-j}(n + \frac{1}{2}) < x < 2^{-j}(n+1), \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 6.4.

It is customary to parametrize the Haar functions by a single index m rather than the two indices j and n by defining

$$h_{(m)} = h_{jn} \text{ for } m = 2^j + n.$$

However, the use of two indices makes the geometry clearer; namely, j indicates the length of the interval on which h_{jn} is nonzero (to wit, 2^{-j}), whereas n indicates the position of that interval within $[0,1]$.

It is an easy exercise to see that the Haar functions are orthonormal. Indeed, the product $h_{jn}(x)h_{j'n'}(x)$ vanishes identically if $j = j'$ and $n \neq n'$, whereas if $j > j'$ it either vanishes identically or equals $\pm 2^{j'/2} h_{jn}(x)$. This is obvious if you think about the graphs of the h_{jn}'s for a minute; and it is equally obvious that $\int_0^1 h_{jn}(x)\,dx = 0$. Thus the h_{jn}'s are orthogonal to one another; similarly, $h_{jn}(x)h_{(0)}(x) = h_{jn}(x)$, so h_{jn} is orthogonal to $h_{(0)}$. Moreover,

$$\|h_{jn}\|^2 = \int_0^1 h_{jn}(x)^2\,dx = \int_{2^{-j}n}^{2^{-j}(n+1)} 2^j\,dx = 1.$$

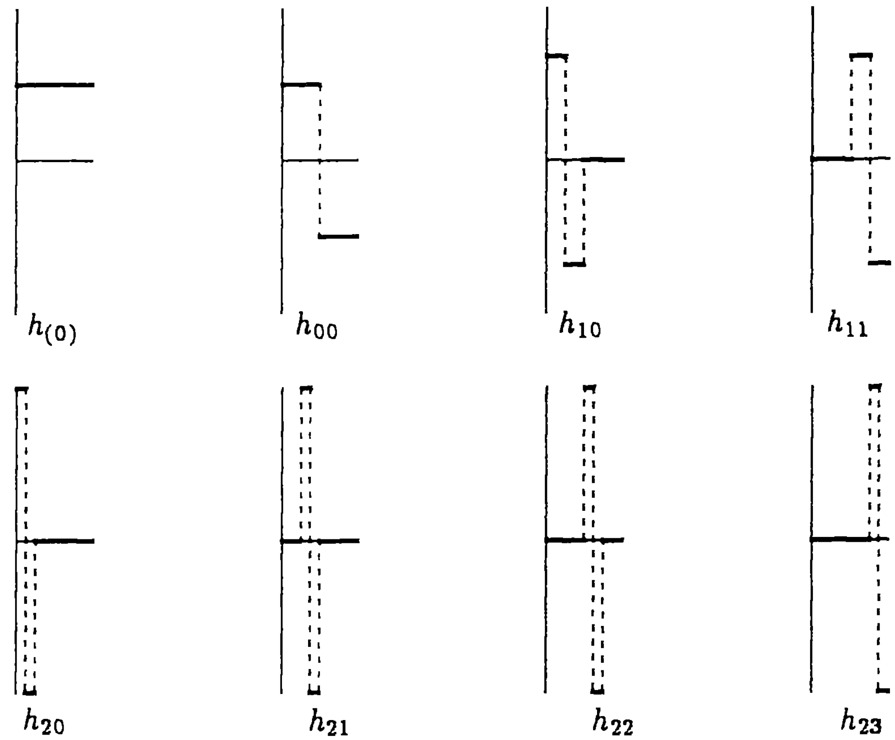


FIGURE 6.4. Graphs of the first eight Haar functions.

It is also easy to see that the Haar functions are complete in $L^2(0,1)$. The key observation is that the space of linear combinations of $h_{(0)}$ and the h_{jn}'s with $j < J$ equals the space of functions on $[0,1]$ that are constant on each interval $\left(2^{-J}k, 2^{-J}(k+1)\right)$ $(0 \le k < 2^J)$. (The former space is evidently contained in the latter one, and they both have dimension 2^J, so they coincide.) It follows that the set of all finite linear combinations of the Haar functions is the space of all step functions on $[0,1]$ whose discontinuities occur among the dyadic rational numbers $2^{-j}k$ $(j, k \ge 0)$, and this space is dense in $L^2(0,1)$.

In short, the Haar functions form an orthonormal basis for $L^2(0,1)$.

To construct our second orthonormal basis consisting of step functions, we begin with the **Rademacher functions** $r_n(x)$. For $n \ge 0$, one divides the interval $[0,1]$ into 2^n equal subintervals; $r_n(x)$ is the function which alternately takes the values $+1$ and -1 on these subintervals, beginning with $+1$ on the first subinterval. In other words, $r_n(x) = (-1)^{d_n(x)}$ where $d_n(x)$ is the nth digit in the binary decimal expansion of x. See Figure 6.5.

A **Walsh function** is a finite product of Rademacher functions. More precisely, if n is a nonnegative integer, let $b_k, \ldots, b_1$ be the digits in the binary decimal for n (i.e., $n = b_k \cdots b_1$ in base 2); then the nth Walsh function $w_n(x)$ is defined to be

$$w_n(x) = r_1(x)^{b_1} \cdots r_k(x)^{b_k}.$$

See Figure 6.6.

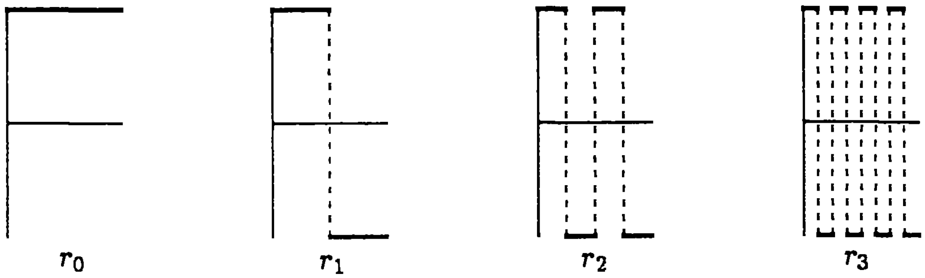


FIGURE 6.5. Graphs of the first four Rademacher functions.

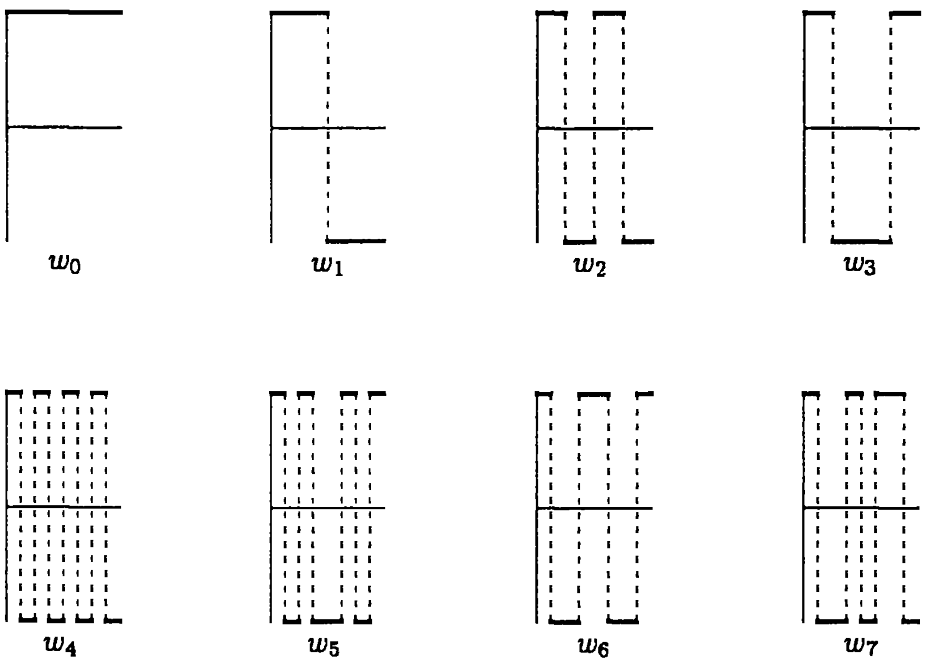


FIGURE 6.6. Graphs of the first eight Walsh functions.

The set $\{w_n\}_0^\infty$ of Walsh functions is an orthonormal basis for $L^2(0, 1)$. Indeed, since the product of two Walsh functions is again a Walsh function, the orthogonality follows from the fact that $\int_0^1 w_n(x)\,dx = 0$ for $n > 0$; this, in turn, is true because the total length of the intervals on which $w_n(x) = 1$, and of the intervals on which $w_n(x) = -1$, is $\frac{1}{2}$. Also, $w_n(x)^2 \equiv 1$ (except at a finite number of points), so clearly $\|w_n\|^2 = 1$. The completeness follows by the same argument as for the Haar functions.

The property of the Haar functions that was emphasized by Haar in the 1910 paper where he introduced them is the fact that the expansion of any continuous function f on $[0, 1]$ in a series of Haar functions converges uniformly to f — a feature that is conspicuously false for Fourier series and other orthogonal series arising from Sturm-Liouville problems. Walsh subsequently introduced his functions w_n in 1923 as an orthonormal set of step functions that qualitatively resemble the trigonometric functions more than the Haar functions, in that they live on the whole interval $[0, 1]$ rather than on small subintervals and become more and more oscillatory as n increases. Haar and Walsh functions have since been found to be interesting for various other theoretical reasons. They have also found a lot of practical applications due to their simplicity from the point of view of numerical calculations. In particular, the fact the the Walsh functions assume only the two values ± 1 makes them particularly handy to use with digital processing equipment. An account of the applications of Haar and Walsh functions in signal and image processing and related fields can be found in Beauchamp [4].

Wavelets

The Haar functions h_{jn} are generated from a single function by dilations and translations. Indeed, if

$$\chi(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$h_{jn}(x) = 2^{j/2}\chi(2^j x - n). \tag{6.59}$$

Since we were interested in functions on $[0, 1]$, we assumed that $0 \leq n < 2^j$, but (6.59) makes sense for any integers j and n, and a modification of the arguments we gave above shows that $\{h_{jn}\}_{j,n=-\infty}^{\infty}$ is an orthonormal basis for $L^2(\mathbf{R})$. Like the Haar basis for $L^2(0, 1)$, this basis has both good and bad features. One of its main advantages is that it is "localized": The term $\langle f, h_{jn}\rangle h_{jn}$ in the expansion of a function f affects, and is affected by, the behavior of f *only* in the interval where $h_{jn} \neq 0$, and as $j \to +\infty$ this interval becomes smaller and smaller. Hence, in order to study the behavior of f in a small interval, one needs to look only at the terms in the series $\sum\langle f, h_{jn}\rangle h_{jn}$ such that h_{jn} "lives" on that interval. On the other hand, there is a disadvantage: When f is a smooth function the expansion $\sum\langle f, h_{jn}\rangle h_{jn}$ is only slowly convergent, and of course the partial sums are not smooth functions but step functions.

One of the exciting discoveries of recent years (1986–88, to be precise) is that this defect can be remedied while still preserving the localization property. In fact, we have the following result.

Theorem 6.18. *For any positive integer k there exist functions ψ of class $C^{(k)}$ on $\mathbf{R}$ that vanish outside a finite interval, such that the functions*

$$\psi_{jn}(x) = 2^{j/2}\psi(2^j x - n) \qquad (j, n = 0, \pm 1, \pm 2, \pm 3, \dots)$$

constitute an orthonormal basis for $L^2(\mathbf{R})$.

The functions ψ_{jn} in this theorem are called **wavelets**; the basic function ψ is called the **mother wavelet.** The mother wavelets are *not* given by any simple formula but rather by a computationally effective recursive algorithm. Other constructions, involving spline (piecewise polynomial) functions or Fourier integrals, lead to variants of this theorem in which the mother wavelets $\psi(x)$ do not vanish outside a finite interval but do decay rapidly as $x \to \pm\infty$. (Theorem 6.18, as stated, is due to I. Daubechies; the variants just mentioned, which came a little earlier, are due to G. Battle, P. G. Lemarié, and Y. Meyer.)

Wavelet expansions share with Fourier series the property of being rapidly convergent when the function in question is smooth, but since wavelets (unlike trigonometric functions) are localized, one can use them to study *local* smoothness properties of functions. In fact, there is a close relationship between the smoothness properties of f near a point x_0 and the decay properties of the coefficients $\langle f, \psi_{jn} \rangle$ as $j \to +\infty$ for those j, n such that $\psi_{jn}(x_0) \neq 0$. (Of course this relationship holds only for properties involving only derivatives of order $\leq k$, k being the order of smoothness of the wavelets themselves.)

From a practical point of view, this has the following consequence. To be definite, let us consider a function $f \in L^2(\mathbf{R})$ that vanishes outside an interval $[-l, l]$. We can expand f in a Fourier series $\sum c_n e^{\pi i n x/l}$ or a wavelet series $\sum \langle f, \psi_{jn} \rangle \psi_{jn}$. If f is everywhere smooth, these two representations of f are comparably efficient, that is, one has to take about the same number of terms in both cases to approximate f to a given accuracy. However, suppose f is smooth except for a small number of singularities such as jump discontinuities. The presence of even one singularity ruins the rapid convergence of the whole Fourier series, but the presence of a singularity at x_0 has little effect on the terms in the wavelet series except for the ones with $\psi_{jn}(x_0) \neq 0$. Hence, for functions with a small number of singularities the wavelet series is a much more efficient representation than the Fourier series. This makes wavelet series (and their higher-dimensional analogues) particularly useful in problems in signal and image processing having to do with edge detection and related phenomena.

The subject of wavelets and their applications (both in engineering and in pure mathematics) underwent an explosive development in the late 1980s. A more detailed discussion of these matters is beyond the scope of this book; we refer the reader to Daubechies [16], Mallat [38], and the articles by Daubechies and Meyer in [13].

CHAPTER 7
THE FOURIER TRANSFORM

Up to this point, the main theme of this book has been the theory and application of infinite series expansions involving various orthonormal sets of functions. We now turn to the study of integral transforms, a different but related collection of techniques for analyzing functions and solving differential equations. We begin with the Fourier transform, which provides a way of expanding functions on the whole real line $\mathbf{R} = (-\infty, \infty)$ as (continuous) superpositions of the basic oscillatory functions $e^{i\xi x}$ ($\xi \in \mathbf{R}$) in much the same way that Fourier series are used to expand functions on a finite interval. To provide some motivation, let us perform a few formal calculations.

Suppose that f is a function on $\mathbf{R}$. For any $l > 0$ we can expand f on the interval $[-l, l]$ in a Fourier series, and we wish to see what happens to this expansion as we let $l \to \infty$. To this end, we write the Fourier expansion as follows: For $x \in [-l, l]$,

$$f(x) = \frac{1}{2l} \sum_{-\infty}^{\infty} c_{n,l} e^{i\pi n x/l}, \qquad c_{n,l} = \int_{-l}^{l} f(y) e^{-i\pi n y/l} \, dy.$$

Let $\Delta\xi = \pi/l$ and $\xi_n = n\Delta\xi = n\pi/l$; then these formulas become

$$f(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} c_{n,l} e^{i\xi_n x} \Delta\xi, \qquad c_{n,l} = \int_{-l}^{l} f(y) e^{-i\xi_n y} \, dy.$$

Let us suppose that $f(x)$ vanishes rapidly as $x \to \pm\infty$; then $c_{n,l}$ will not change much if we extend the region of integration from $[-l, l]$ to $(-\infty, \infty)$:

$$c_{n,l} \approx \int_{-\infty}^{\infty} f(y) e^{-i\xi_n y} \, dy.$$

This last integral is a function only of ξ_n, which we call $\widehat{f}(\xi_n)$, and we now have

$$f(x) \approx \frac{1}{2\pi} \sum_{-\infty}^{\infty} \widehat{f}(\xi_n) e^{i\xi_n x} \Delta\xi \qquad (|x| < l).$$

This looks very much like a Riemann sum. If we now let $l \to \infty$, so that $\Delta\xi \to 0$, the $\approx$ should become $=$ and the sum should turn into an integral, thus:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{i\xi x}\,d\xi, \quad \text{where } \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx. \tag{7.1}$$

These limiting calculations are utterly nonrigorous as they stand; nonetheless, the final result is correct under suitable conditions on f, as we shall prove in due course. The function $\widehat{f}$ is called the **Fourier transform** of f, and (7.1) is the **Fourier inversion theorem.**

Before proceeding, we establish a couple of notational conventions. We shall be dealing with functions defined on the real line, and most of our integrals will be definite integrals over the whole line. Accordingly, we shall agree that an integral sign with no explicit limits means the integral over $\mathbf{R}$ (and not an indefinite integral):

$$\int f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Moreover, L^2 will mean $L^2(\mathbf{R})$, the space of square-integrable functions on $\mathbf{R}$.

We also introduce the space $L^1 = L^1(\mathbf{R})$ of **(absolutely) integrable** functions on $\mathbf{R}$:

$$L^1 = \left\{f : \int |f(x)|\,dx < \infty\right\}.$$

(Here, as with L^2, the integral should be understood in the Lebesgue sense, but this technical point will not be of any great concern to us.) We remark that L^1 is not a subset of L^2, nor is L^2 a subset of L^1. The singularities of a function in L^1 (that is, places where the values of the function tend to ∞) can be somewhat worse than those of a function in L^2, since squaring a large number makes it larger; on the other hand, functions in L^2 need not decay as rapidly at infinity as those in L^1, since squaring a small number makes it smaller. For example, let

$$f(x) = \begin{cases} x^{-2/3} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$g(x) = \begin{cases} x^{-2/3} & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is in L^1 but not in L^2, whereas g is in L^2 but not in L^1. (The easy verification is left to the reader.) However, we have the following useful facts:

(i) If $f \in L^1$ and f is bounded, then $f \in L^2$. Indeed,

$$|f| \le M \implies |f|^2 \le M|f| \implies \int |f(x)|^2 dx \le M \int |f(x)|\,dx < \infty.$$

(ii) If $f \in L^2$ and f vanishes outside a finite interval $[a, b]$, then $f \in L^1$. This follows from the Cauchy-Schwarz inequality:

$$\int |f(x)|\,dx = \int_a^b 1 \cdot |f(x)|\,dx \le (b-a)^{1/2}\left(\int_a^b |f(x)|^2 dx\right)^{1/2} < \infty.$$

7.1 Convolutions

Before studying the Fourier transform, we need to introduce the convolution product of two functions, a device that will be very useful both as a theoretical tool and in applications. This idea may seem a bit mysterious to the reader who has never seen it before; we shall make a few comments on its meaning following Theorem 7.2, but a fuller appreciation of its significance can best be achieved by seeing how it arises throughout the course of this chapter.

If f and g are functions on $\mathbf{R}$, their **convolution** is the function $f * g$ defined by

$$f * g(x) = \int f(x-y)g(y)\,dy, \tag{7.2}$$

provided that the integral exists. Various conditions can be imposed on f and g to ensure that the integral will be absolutely convergent for all $x \in \mathbf{R}$, for example:

(i) If $f \in L^1$ and g is bounded (say $|g| \leq M$), then

$$\int |f(x-y)g(y)|\,dy \leq M \int |f(x-y)|\,dy = M \int |f(y)|dy < \infty.$$

(ii) If f is bounded (say $|f| \leq M$) and $g \in L^1$, then

$$\int |f(x-y)g(y)|\,dy \leq M \int |g(y)|\,dy < \infty.$$

(iii) If f and g are both in L^2, then by the Cauchy-Schwarz inequality,

$$\int |f(x-y)g(y)|\,dy \leq \sqrt{\int |f(x-y)|^2 dy}\sqrt{\int |g(y)|^2 dy} = \|f\|\,\|g\| < \infty.$$

(iv) If f is piecewise continuous and g is bounded and vanishes outside a finite interval $[a, b]$, then $f * g(x)$ exists for all x, since the function $y \to f(x-y)$ is bounded on $[a, b]$ for any x.

(v) It can be shown that if f and g are both in L^1, then $f * g(x)$ exists for "almost every" x, i.e., for all x except for some set having Lebesgue measure zero; moreover, $f * g \in L^1$. See Folland [25], §8.1, or Wheeden-Zygmund [56], §9.1.

This list can be extended. In what follows we assume implicitly that the functions we mention satisfy appropriate conditions so that all integrals in question are absolutely convergent. The reader may supply specific hypotheses at will; it would often be quite tedious to list all possible ones.

We now investigate the basic algebraic and analytic properties of convolutions.

Theorem 7.1. *Convolution obeys the same algebraic laws as ordinary multiplication:*

(i) $f * (ag + bh) = a(f * g) + b(f * h)$ *for any constants* a, b*;*

(ii) $f * g = g * f$*;*

(iii) $f * (g * h) = (f * g) * h$.

Proof: (i) is obvious since integration is a linear operation. For (ii), make the change of variable $z = x - y$:

$$f * g(x) = \int f(x-y)g(y)\,dy = \int f(z)g(x-z)\,dz = g * f(x).$$

For (iii), use (ii) and interchange the order of integration:

$$(f * g) * h(x) = \int f * g(x-y)h(y)\,dy = \iint f(z)g(x-y-z)h(y)\,dz\,dy$$

$$= \iint f(z)g(x-z-y)h(y)\,dy\,dz = \int f(z)g * h(x-z)\,dz = f * (g * h)(x). \quad \blacksquare$$

Theorem 7.2. *Suppose that f is differentiable and the convolutions $f * g$ and $f' * g$ are well-defined. Then $f * g$ is differentiable and $(f * g)' = f' * g$. Likewise, if g is differentiable, then $(f * g)' = f * g'$.*

Proof: Just differentiate under the integral sign:

$$(f * g)'(x) = \frac{d}{dx} \int f(x-y)g(y)\,dy = \int f'(x-y)g(y)\,dy = f' * g(x).$$

Since $f * g = g * f$, the same argument works with f and g interchanged. $\blacksquare$

We emphasize that in Theorem 7.2 one can throw the derivative in $(f * g)'$ onto either factor. Thus $f * g$ is at least as smooth as either f or g, even when the other factor has no smoothness properties.

Let us pause to make a few remarks that may shed some light on the meaning of convolutions. In the first place, let us think of the convolution integral as a limit of Riemann sums,

$$\int f(x-y)g(y)\,dy \approx \sum f(x-y_j)g(y_j)\Delta y_j.$$

The function $f_j(x) = f(x - y_j)$ is the function f translated along the x-axis by the amount y_j, so the sum on the right is a linear combination of translates of f with coefficients $g(y_j)\Delta y_j$. We can therefore think of $f * g$ as a continuous superposition of translates of f; and since $f * g = g * f$, it is also a continuous superposition of translates of g.

Second, convolutions may be interpreted as "moving weighted averages." We recall that the average value of a function f on the interval $[a, b]$ is defined to be $(b-a)^{-1} \int_a^b f(y)\,dy$. More generally, the weighted average of f on $[a, b]$ with respect to a nonnegative weight function w is

$$\frac{\int_a^b f(y)w(y)\,dy}{\int_a^b w(y)\,dy}.$$

Suppose now that g is nonnegative and $\int g(y)\,dy = 1$. If we write $f * g(x)$ as $\int f(y)g(x-y)\,dy$, we see that $f * g(x)$ is the weighted average of f (on the whole line) with respect to the weight function $w(y) = g(x - y)$. If $g(x) = 0$ for $|x| > a$ then $g(x - y) = 0$ for $|x - y| > a$, so $f * g(x)$ is a weighted average of f on the interval $[x - a, x + a]$. In particular, if

$$g(x) = \begin{cases} (2a)^{-1} & \text{if } -a < x < a, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$f * g(x) = \frac{1}{2a} \int_{x-a}^{x+a} f(y)\, dy,$$

which is the (ordinary) average of f on the interval $[x - a, x + a]$.

One respect in which convolution does not resemble ordinary multiplication is that whereas $f \cdot 1 = f$ for all f, there is no function g such that $f * g = f$ for all f. (The Dirac "δ-function" does the job, but it is not a genuine function; we shall discuss it in Chapter 9.) However, we can easily find sequences $\{g_n\}$ such that $f * g_n$ converges to f as $n \to \infty$. The intuition is provided by the remarks of the preceding paragraph: If $g(x)$ vanishes (or at least is negligibly small) outside an interval $|x| < a$, then $f * g(x)$ will be a weighted average of the values of f on the interval $[x - a, x + a]$, and if a is very small this should be approximately $f(x)$.

To be precise, suppose $g \in L^1$, and for $\epsilon > 0$ let

$$g_\epsilon(x) = \frac{1}{\epsilon} g\left(\frac{x}{\epsilon}\right). \tag{7.3}$$

That is, g_ϵ is obtained from g by compressing the graph in the x-direction by a factor of ϵ and simultaneously stretching it in the y direction by a factor of $1/\epsilon$. (We are thinking of the case $\epsilon < 1$; if $\epsilon > 1$ the words *compressing* and *stretching* should be interchanged. See Figure 7.1.) As $\epsilon \to 0$ the graph of g_ϵ becomes a sharp spike at $x = 0$, but the area under the graph remains constant:

$$\int g_\epsilon(x)\, dx = \int g\left(\frac{x}{\epsilon}\right) d\left(\frac{x}{\epsilon}\right) = \int g(y)\, dy.$$

More generally, the substitution $x = \epsilon y$ yields

$$\int_a^b g_\epsilon(x)\, dx = \int_{a/\epsilon}^{b/\epsilon} g(y)\, dy. \tag{7.4}$$

With this in mind, we can state a precise theorem.

Theorem 7.3. *Let g be an L^1 function such that $\int_{-\infty}^{\infty} g(y)\, dy = 1$, and let $\alpha = \int_{-\infty}^{0} g(y)\, dy$ and $\beta = \int_0^\infty g(y)\, dy$. (Note that $\alpha + \beta = 1$ and that $\alpha = \beta = \frac{1}{2}$ if g is even.) Suppose that f is piecewise continuous on $\mathbf{R}$, and suppose either that f is bounded or that g vanishes outside a finite interval so that $f * g(x)$ is well-defined for all x. If g_ϵ is defined by (7.3), then*

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \textit{for all } x.$$

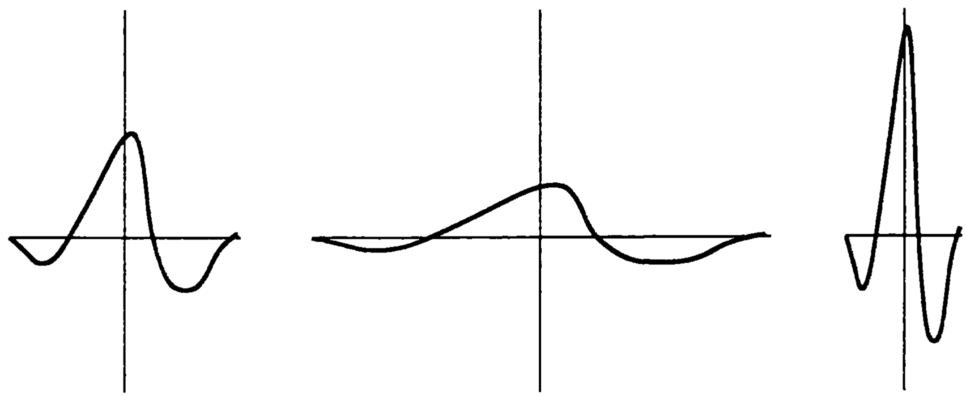


FIGURE 7.1. A function $g(x)$ (left) and its dilates $g_2(x) = \frac{1}{2}g(\frac{1}{2}x)$ (middle) and $g_{1/2}(x) = 2g(2x)$ (right).

In particular, if f is continuous at x, then

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = f(x). \tag{7.5}$$

Moreover, if f is continuous at every point in the bounded interval $[a, b]$, the convergence in (7.5) is uniform on $[a, b]$.

Proof: We have

$$f * g_\epsilon(x) - \alpha f(x+) - \beta f(x-) = \int_{-\infty}^{0} \Big[f(x-y) - f(x+)\Big] g_\epsilon(y)\,dy \\ + \int_0^\infty \Big[f(x-y) - f(x-)\Big] g_\epsilon(y)\,dy,$$

so we wish to show that both integrals on the right can be made arbitrarily small by taking ϵ sufficiently small. The argument is the same for both of them, so we consider only the second one. Given $\delta > 0$, we can choose $c > 0$ small enough so that $|f(x-y) - f(x-)| < \delta$ when $0 < y < c$, and we break up the integral as $\int_0^c + \int_c^\infty$. By (7.4),

$$\left|\int_0^c \Big[f(x-y) - f(x-)\Big] g_\epsilon(y)\,dy\right| \le \delta \int_0^c |g_\epsilon(y)|\,dy = \delta \int_0^{c/\epsilon} |g(y)|\,dy \\ \le \delta \int_0^\infty |g(y)|\,dy,$$

and we can make this as small as we wish by choosing δ suitably. To estimate the integral from c to ∞, we use the assumption that either f is bounded (say $|f| \le M$) or g vanishes outside a finite interval (say $g(x) = 0$ for $|x| > R$). In the first case, by (7.4),

$$\left|\int_c^\infty \Big[f(x-y) - f(x-)\Big] g_\epsilon(y)\,dy\right| \le 2M \int_c^\infty |g_\epsilon(y)|\,dy = 2M \int_{c/\epsilon}^\infty |g(y)|\,dy,$$

which tends to zero along with ϵ. In the second case, $g_\epsilon(x) = 0$ for $|x| > \epsilon R$, and in particular $g_\epsilon(x) = 0$ for $x > c$ if $\epsilon < c/R$, so the integral from c to ∞ actually vanishes for ϵ small.

Finally, if f is continuous on $[a, b]$, then f is uniformly continuous there, so the choice of c in the preceding argument can be made independent of x for $x \in [a, b]$. It follows easily that the convergence of $f * g_\epsilon(x)$ to $f(x)$ is uniform on $[a, b]$. ∎

There are several variants of Theorem 7.3, which say that $f * g_\epsilon \to f$ in some sense or other as $\epsilon \to 0$ under suitable hypotheses on f and g. We shall content ourselves with stating a result for norm convergence of L^2 functions.

Theorem 7.4. *Suppose $g \in L^1$ is bounded and satisfies $\int g(y)\,dy = 1$. If $f \in L^2$, then $f * g(x)$ is well-defined for all x, and if g_ϵ is defined as in (7.3), $f * g_\epsilon$ converges to f in norm as $\epsilon \to 0$.*

The proof of this result is not really difficult, but it involves some approximation arguments that are a bit beyond the level of the present discussion. See Folland [25], Theorem 8.14, or Wheeden-Zygmund [56], Theorem 9.6.

The family $\{g_\epsilon\}$ in Theorems 7.3 and 7.4 is called an **approximate identity,** since the operation of convolution with g_ϵ tends to the identity operator as $\epsilon \to 0$. See Figure 7.2.

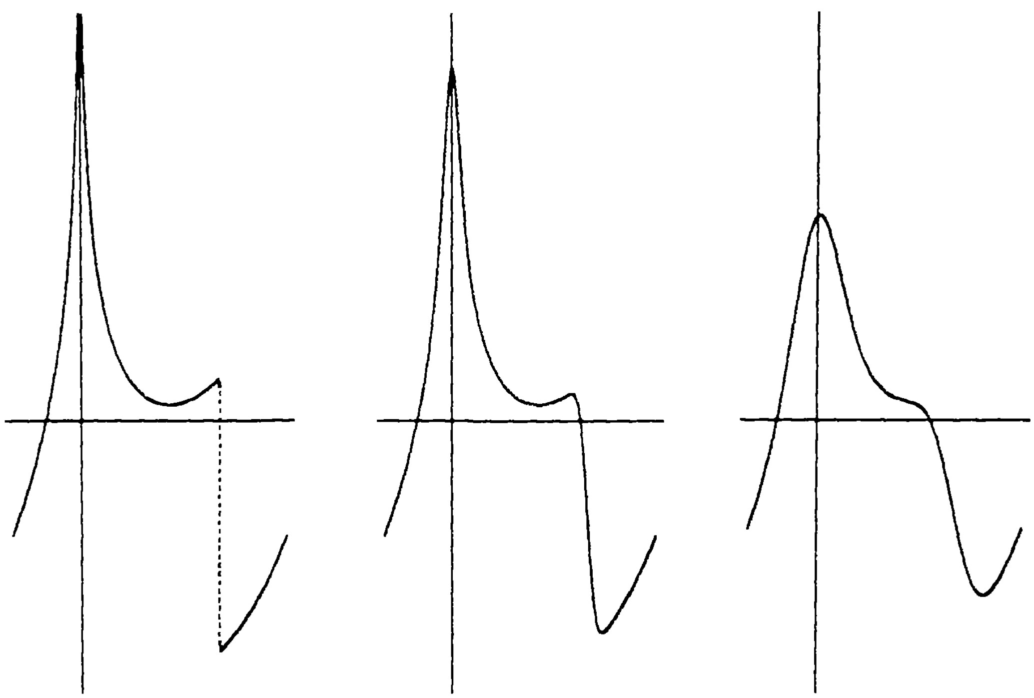


FIGURE 7.2. A function f with an infinite singularity and a jump discontinuity (left), $f * G_{0.1}$ (middle), and $f * G_{0.3}$ (right), where G is the Gaussian (7.6).

One of the functions g that is most often used in this context is the **Gaussian**

$$G(y) = \pi^{-1/2} e^{-y^2}. \tag{7.6}$$

It satisfies $\int G(y)\,dy = 1$ because

$$\int_{-\infty}^{\infty} e^{-y^2} dy = 2\int_0^{\infty} e^{-y^2} dy = \int_0^{\infty} e^{-t} t^{-1/2} dt = \Gamma(\tfrac{1}{2}) = \pi^{1/2}. \tag{7.7}$$

G is even, so that when it is used as the g in Theorem 7.3 we have $\alpha = \beta = \frac{1}{2}$. G and its dilated versions G_ϵ have the property that all their derivatives are bounded integrable functions. Indeed, it is easily established by induction that $G^{(k)}(y) = P_k(y)e^{-y^2}$ where P_k is a polynomial of degree k, and it follows that $|G^{(k)}(y)| \le C_k e^{-|y|}$, with similar estimates (involving some powers of ϵ) for G_ϵ. Hence we can apply Theorems 7.3 and 7.4: If f is (say) bounded and piecewise continuous, then $f * G_\epsilon$ is of class $C^{(\infty)}$, and it approximates f when ϵ is small. These convolutions may be regarded as "smeared out" or "smoothed out" versions of f. What we have developed here is a method of approximating general functions by smooth ones, a useful technical tool in many situations. In particular, it yields a proof of the following fundamental result.

The Weierstrass Approximation Theorem. *If f is a continuous function on $[a,b]$ $(-\infty < a < b < \infty)$, then f is the uniform limit of polynomials on $[a,b]$. That is, for any $\delta > 0$ there is a polynomial P such that*

$$\sup_{a \le x \le b} |f(x) - P(x)| < \delta.$$

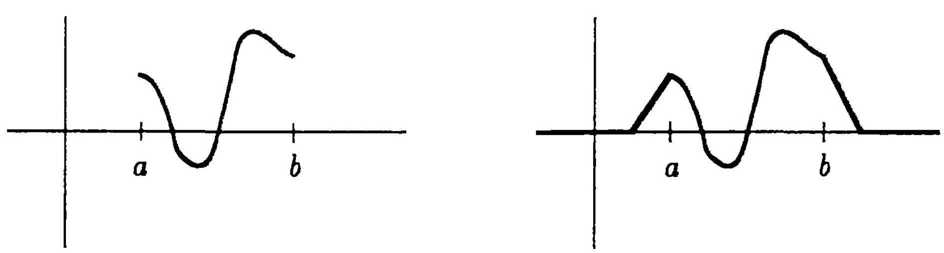


FIGURE 7.3. A continuous function f on $[a,b]$ (left) and a continuous extension of f to $\mathbf{R}$ (right).

Proof: Extend f to be a continuous function on the whole real line that vanishes outside the interval $[a-1, b+1]$; see Figure 7.3. By Theorem 7.3, $f * G_\epsilon \to f$ uniformly on $[a,b]$ where G is given by (7.6). Thus, given $\delta > 0$, if ϵ is sufficiently small we have

$$\sup_{a \le x \le b} \left| f(x) - \frac{1}{\epsilon\sqrt{\pi}} \int_{a-1}^{b+1} e^{-(x-y)^2/\epsilon^2} f(y)\,dy \right| < \frac{\delta}{2}.$$

As x ranges over $[a,b]$ and y ranges over $[a-1, b+1]$, $(x-y)/\epsilon$ ranges over the bounded set $[c,d]$ where $c = (a-b-1)/\epsilon$ and $d = (b-a+1)/\epsilon$, and the Taylor series $\sum_0^\infty (-1)^n t^{2n}/n!$ for e^{-t^2} converges uniformly on this set. It follows easily that we can replace $e^{-(x-y)^2/\epsilon^2}$ in the above integral by a suitable Taylor polynomial without changing the integral by more than $\frac{1}{2}\delta$. In other words, if N is sufficiently large,

$$\sup_{a\le x\le b} |f(x) - P(x)| < \delta$$

where

$$P(x) = \frac{1}{\epsilon\sqrt{\pi}} \sum_0^N \int_{a-1}^{b+1} \frac{(-1)^n (x-y)^{2n}}{\epsilon^{2n} n!} f(y)\, dy.$$

But $P(x)$ is a polynomial of degree $2N$, as one can see by expanding $(x-y)^{2n}$ by the binomial theorem:

$$P(x) = \sum_{n=0}^{N} \sum_{k=0}^{2n} c_{kn} x^k, \quad c_{kn} = \frac{(-1)^{k-n}(2n)!}{\epsilon^{2n+1} n! k! (2n-k)! \sqrt{\pi}} \int_{a-1}^{b+1} y^{2n-k} f(y)\, dy. \quad \blacksquare$$

The Gaussian is not the only commonly used approximate identity. Another one is given by

$$H(y) = \frac{1}{\pi(1+y^2)},$$

which, as we shall see, arises in the solution of the Dirichlet problem for a half-plane. It shares with G the properties of being even and having derivatives of all orders that are bounded integrable functions, so it also provides smooth approximations to general bounded functions. Another approximate identity with these properties, and an extra one that makes it particularly useful in some situations, is given by

$$K(y) = \begin{cases} C^{-1} e^{-1/(1-y^2)} & \text{for } |y| < 1, \\ 0 & \text{for } |y| \ge 1, \end{cases} \qquad C = \int_{-1}^{1} e^{-1/(1-y^2)} dy. \tag{7.8}$$

K possesses derivatives of all orders, even at $y = \pm 1$ (because $e^{-1/(1-y^2)}$ vanishes to infinite order as y approaches 1 from the left or -1 from the right), and it vanishes outside the bounded set $|y| \le 1$. Hence the convolutions $f * K_\epsilon$ are well-defined for any piecewise continuous f, bounded or not, and they provide smooth approximations to all such f. Some other applications of K are given in Exercises 7 and 8.

EXERCISES

1. Which of the following functions are in L^1? in L^2?

   a. $\dfrac{\sin x}{|x|^{3/2}}$ b. $(1+x^2)^{-1/2}$ c. $\dfrac{1}{x^2-1}$ d. $\dfrac{1-\cos x}{x^2}$

2. Let $f(x) = |x|^{-p}$ where $\frac{1}{2} < p < 1$. Show that f is in neither L^1 nor L^2, but that f can be expressed as the sum of an L^1 function and an L^2 function.
3. Let $f(x) = 1$ if $-1 < x < 1$, $f(x) = 0$ otherwise.
   a. Compute $f * f$ and $f * f * f$.
   b. Let $f_\epsilon(x) = \epsilon^{-1} f(\epsilon^{-1}x)$ as in (7.3) and let $g(x) = x^3 - x$. Compute $f_\epsilon * g$ and check directly that $f_\epsilon * g \to 2g$ as $\epsilon \to 0$. (Note that $2 = \int f(x)\,dx$.)
4. Let $f(x) = e^{-x^2}$ and $g(x) = e^{-2x^2}$. Compute $f * g$. (Hint: Complete the square in the exponent and use the fact that $\int e^{-x^2} dx = \sqrt{\pi}$.)
5. For $t > 0$, let $f_t(x) = (4\pi t)^{-1/2} e^{-x^2/4t}$. Show that $f_t * f_s = f_{t+s}$. (Hint: First do Exercise 4 as a warmup.)
6. For $t > 0$, let $f_t(x) = x^{t-1}/\Gamma(t)$ for $x > 0$ and $f_t(x) = 0$ for $x \le 0$. Show that $f_t * f_s = f_{t+s}$. (Hint: The integral defining f_{t+s} can be reduced to the integral for the beta function.)
7. Show that for any $\delta > 0$ there is a function ϕ on $\mathbf{R}$ with the following properties: (i) ϕ is of class $C^{(\infty)}$, (ii) $0 \le \phi(x) \le 1$ for all x, (iii) $\phi(x) = 1$ when $0 \le x \le 1$, (iv) $\phi(x) = 0$ when $x < -\delta$ or $x > 1 + \delta$. (Hint: Define f by $f(x) = 1$ if $-\frac{1}{2}\delta \le x \le 1 + \frac{1}{2}\delta$, $f(x) = 0$ otherwise. Show that $f * K_\epsilon$ does the job if K is as in (7.8) and $\epsilon < \frac{1}{2}\delta$.)
8. Show that for any $f \in L^2$ and any $\delta > 0$, there is a function g such that (i) g is of class $C^{(\infty)}$, (ii) g vanishes outside a finite interval, and (iii) $\|f - g\| < \delta$. Proceed by the following steps.
   a. Let $F(x) = f(x)$ if $|x| < N$, $F(x) = 0$ otherwise. Show that $\|F - f\| < \frac{1}{2}\delta$ if N is sufficiently large.
   b. Show that $g = F * K_\epsilon$ does the job if K is as in (7.8) and ϵ is sufficiently small.

7.2 The Fourier transform

If f is an integrable function on $\mathbf{R}$, its **Fourier transform** is the function $\widehat{f}$ on $\mathbf{R}$ defined by

$$\widehat{f}(\xi) = \int e^{-i\xi x} f(x)\,dx.$$

We shall sometimes write $f^\frown$ instead of $\widehat{f}$, particularly when the label f is replaced by a more complicated expression. We shall also occasionally write

$$\mathscr{F}\Big[f(x)\Big] = \widehat{f}(\xi)$$

for the Fourier transform of f. (This involves an ungrammatical use of the symbols x and ξ but is sometimes the clearest way of expressing things.)

Since $e^{-i\xi x}$ has absolute value 1, the integral converges absolutely for all ξ and defines a bounded function of ξ:

$$|\widehat{f}(\xi)| \le \int |f(x)|\,dx. \tag{7.9}$$

Moreover, since $|e^{-i\eta x}f(x) - e^{-i\xi x}f(x)| \le 2|f(x)|$, the dominated convergence theorem implies that $\widehat{f}(\eta) - \widehat{f}(\xi) \to 0$ when $\eta \to \xi$, that is, $\widehat{f}$ is continuous.

The following theorem summarizes some of the other basic properties of the Fourier transform.

Theorem 7.5. *Suppose $f \in L^1$.*
(a) For any $a \in \mathbf{R}$,

$$\mathscr{F}\Big[f(x-a)\Big] = e^{-ia\xi}\widehat{f}(\xi) \quad \text{and} \quad \mathscr{F}\Big[e^{iax}f(x)\Big] = \widehat{f}(\xi - a).$$

(b) If $\delta > 0$ and $f_\delta(x) = \delta^{-1}f(x/\delta)$ as in (7.3), then

$$[f_\delta]^\frown(\xi) = \widehat{f}(\delta\xi) \quad \text{and} \quad \mathscr{F}\Big[f(\delta x)\Big] = [\widehat{f}]_\delta(\xi).$$

(c) If f is continuous and piecewise smooth and $f' \in L^1$, then

$$[f']^\frown(\xi) = i\xi\widehat{f}(\xi).$$

On the other hand, if $xf(x)$ is integrable, then

$$\mathscr{F}\Big[xf(x)\Big] = i[\widehat{f}]'(\xi).$$

(d) If also $g \in L^1$, then

$$(f * g)^\frown = \widehat{f}\widehat{g}.$$

Proof: For the first equation of (a), we have

$$\mathscr{F}\Big[f(x-a)\Big] = \int e^{-i\xi x}f(x-a)\,dx = \int e^{-i\xi x - i\xi a}f(x)\,dx = e^{-ia\xi}\widehat{f}(\xi).$$

The other equations of (a) and (b) are equally easy to prove; we leave them as exercises for the reader. As for (c), observe that since $f' \in L^1$, the limit

$$\lim_{x\to+\infty} f(x) = f(0) + \int_0^\infty f'(x)\,dx$$

exists, and since $f \in L^1$ this limit must be zero. Likewise, $\lim_{x\to-\infty} f(x) = 0$. Hence we can integrate by parts, and the boundary terms vanish:

$$[f']^\frown(\xi) = \int e^{-i\xi x}f'(x)\,dx = -\int(-i\xi)e^{-i\xi x}f(x)\,dx = i\xi\widehat{f}(\xi).$$

On the other hand, if $xf(x)$ is integrable, since $xe^{-i\xi x} = i(d/d\xi)e^{-i\xi}$ we have

$$\mathscr{F}\Big[xf(x)\Big] = \int e^{-i\xi x}xf(x)\,dx = i\frac{d}{d\xi}\int e^{-i\xi x}f(x)\,dx = i[\widehat{f}]'(\xi).$$

Finally, for (d),

$$
\begin{aligned}
(f*g)\widehat{\ }(\xi) &= \iint e^{-i\xi x} f(x-y)g(y)\,dy\,dx \\
&= \iint e^{-i\xi(x-y)} f(x-y)e^{-i\xi y}g(y)\,dx\,dy \\
&= \iint e^{-i\xi z} f(z)e^{-i\xi y}g(y)\,dz\,dy \qquad (z=x-y) \\
&= \widehat{f}(\xi)\widehat{g}(\xi).
\end{aligned}
$$

∎

Parts (a), (b), and (c) exhibit a remarkable set of correspondences between functions and their Fourier transforms. In essence: Translating a function corresponds to multiplying its Fourier transform by an exponential and vice versa; dilating a function by the factor δ corresponds to dilating its Fourier transform by the factor $1/\delta$ and vice versa; differentiating a function corresponds to multiplying its Fourier transform by the coordinate variable and vice versa. (Of course, this formulation is a bit imprecise; there are factors of -1, i, and δ to be sorted out.) This symmetry between f and $\widehat{f}$ extends also to part (d): It will follow from (d) and the Fourier inversion formula below that

$$\widehat{f}*\widehat{g} = 2\pi(fg)\widehat{\ }. \tag{7.10}$$

Before developing the theory further, let us compute three basic examples of Fourier transforms.

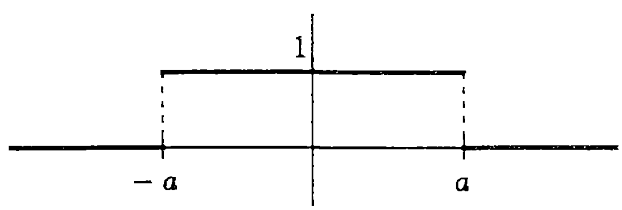


FIGURE 7.4. Graph of the function χ_a.

Example 1. Let χ_a be the function depicted in Figure 7.4:

$$\chi_a(x) = \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\widehat{\chi}_a(\xi) = \int_{-a}^{a} e^{-i\xi x}\,dx = \frac{e^{-ia\xi}-e^{ia\xi}}{-i\xi} = 2\frac{\sin a\xi}{\xi}. \tag{7.11}$$

Example 2. Let $f(x) = e^{-ax^2/2}$ where $a>0$. We observe that f satisfies the differential equation $f'(x) + axf(x) = 0$. If we apply the Fourier transform to this equation, by Theorem 7.5(c) we obtain $i\xi\widehat{f}(\xi) + ia[\widehat{f}]'(\xi) = 0$, or $[\widehat{f}]'(\xi) + a^{-1}\xi\widehat{f}(\xi) = 0$. This differential equation for $\widehat{f}$ is easily solved:

$$\frac{[\widehat{f}]'(\xi)}{\widehat{f}(\xi)} = -\frac{\xi}{a} \implies \log\widehat{f}(\xi) = -\frac{\xi^2}{2a} + \log C \implies \widehat{f}(\xi) = Ce^{-\xi^2/2a}.$$

To evaluate the constant C, we set $\xi = 0$ and use (7.7):

$$C = \widehat{f}(0) = \int f(x)\,dx = \int e^{-ax^2/2}\,dx = \sqrt{\frac{2}{a}}\int e^{-y^2}\,dy = \sqrt{\frac{2\pi}{a}}.$$

Therefore,

$$\mathscr{F}\left[e^{-ax^2/2}\right] = \sqrt{\frac{2\pi}{a}}e^{-\xi^2/2a}. \tag{7.12}$$

(A neat derivation of this result using contour integrals is sketched in Exercise 1.)

Example 3. Let $f(x) = (x^2 + a^2)^{-1}$ where $a > 0$. We shall calculate $\widehat{f}$ here by contour integration; another derivation that uses the Fourier inversion formula but no complex variable theory is sketched in Exercise 2. If $\xi < 0$, $e^{-i\xi z}$ is a bounded analytic function of z in the upper half-plane, so by applying the residue theorem on the contour in Figure 7.5 and letting $N \to \infty$ we obtain

$$\widehat{f}(\xi) = \int \frac{e^{-i\xi x}}{x^2 + a^2}\,dx = 2\pi i \operatorname{Res}_{z=i}\frac{e^{-i\xi z}}{(z^2 + a^2)} = 2\pi i\frac{e^{a\xi}}{2ia} = \frac{\pi}{a}e^{a\xi} \qquad (\xi < 0).$$

Similarly, if $\xi > 0$ we can integrate around the lower half-plane to obtain $\widehat{f}(\xi) = (\pi/a)e^{-a\xi}$. Of course, for $\xi = 0$, $\widehat{f}(0) = \int f(x)\,dx = \pi/a$. Conclusion:

$$\mathscr{F}\left[(x^2 + a^2)^{-1}\right] = \frac{\pi}{a}e^{-a|\xi|}. \tag{7.13}$$

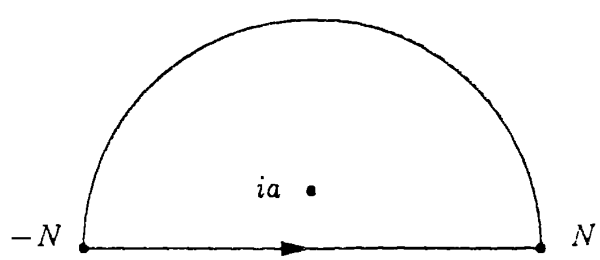


FIGURE 7.5. The contour for Example 3.

The equation $[f']\widehat{\ }(\xi) = i\xi \widehat{f}(\xi)$ of Theorem 7.5(c) is the analogue of Theorem 2.2 in §2.3 for Fourier series. The moral here, as in the theory of Fourier series, is that the smoother f is, the faster $\widehat{f}$ decays at infinity, and vice versa. The examples worked out above illustrate this principle. The function of Example 1 vanishes outside a finite interval but is not continuous; its Fourier transform is analytic but decays only like $1/\xi$ at infinity. The function of Example 3 is smooth but decays slowly; its Fourier transform decays rapidly but is not differentiable at $\xi = 0$. The function of Example 2 has both smoothness and decay, and is essentially its own Fourier transform.

One other basic property of Fourier transforms of L^1 functions should be mentioned here. We observed earlier that if $f \in L^1$, then $\widehat{f}$ is a bounded, continuous function on $\mathbf{R}$; but something more is true.

The Riemann-Lebesgue Lemma. *If $f \in L^1$, then $\widehat{f}(\xi) \to 0$ as $\xi \to \pm\infty$.*

Proof: First suppose that f is a step function, that is, $f(x) = \sum_1^k c_j \phi_j(x)$ where each ϕ_j is a function that equals 1 on some bounded interval $|x - x_j| < a_j$ and equals 0 elsewhere. By (7.11) and Theorem 7.5(a), $\widehat{\phi}_j(\xi) = 2\xi^{-1} e^{-ix_j\xi} \sin a_j\xi$, which vanishes at infinity. Hence, so does $\widehat{f}$.

For the general case, if $f \in L^1$ one can find a sequence $\{f_n\}$ of step functions such that $\int |f_n(x) - f(x)|\, dx \to 0$. (When f is Riemann integrable, this assertion is essentially a restatement of the fact that the integral of f is the limit of Riemann sums. It is true also for Lebesgue integrable functions, but the proof naturally requires some results from Lebesgue integration theory. See Folland [25], Theorems 2.26 and 2.41.) But then by (7.9),

$$\sup_{\xi} |\widehat{f}_n(\xi) - \widehat{f}(\xi)| \le \int |f_n(x) - f(x)|\, dx \to 0 \quad \text{as } n \to \infty,$$

that is, $\widehat{f}_n \to \widehat{f}$ uniformly. Since each $\widehat{f}_n$ vanishes at infinity, it follows easily that $\widehat{f}$ does too. ∎

The Fourier inversion theorem

We now turn to the Fourier inversion formula, that is, the procedure for recovering f from $\widehat{f}$. The heuristic arguments in the introduction to this chapter led us to the formula

$$f(x) = \frac{1}{2\pi} \int e^{i\xi x} \widehat{f}(\xi)\, d\xi. \tag{7.14}$$

(Note that this is the same as the formula that gives $\widehat{f}$ in terms of f, except for the plus sign in the exponent and the factor of 2π. This accounts for the symmetry between f and $\widehat{f}$ in Theorem 7.5.) Our task is to investigate the validity of (7.14). Like the question of whether the Fourier series of a periodic function f converges to f, this is not entirely straightforward.

The first difficulty is that $\widehat{f}$ may not be in L^1, as (7.11) shows, and in this case the integral in (7.14) is not absolutely convergent. Even if it is, one cannot establish (7.14) simply by substituting in the defining formula for $\widehat{f}$,

$$\int e^{i\xi x}\widehat{f}(\xi)\,d\xi = \iint e^{i\xi(x-y)}f(y)\,dy\,d\xi,$$

and interchanging the order of integration, because the integral $\int e^{i\xi(x-y)}d\xi$ is divergent. The simplest remedy for both these problems is to multiply $\widehat{f}$ by a "cutoff function" to make the integrals converge and then to pass to the limit as the cutoff is removed.

One convenient cutoff function is $e^{-\epsilon^2\xi^2/2}$: For any fixed $\epsilon > 0$ it decreases rapidly as $\xi \to \pm\infty$, and to remove it we simply let $\epsilon \to 0$. Accordingly, instead of (7.14), for $f \in L^1$ we consider

$$\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi = \frac{1}{2\pi}\iint e^{i\xi(x-y)}e^{-\epsilon^2\xi^2/2}f(y)\,dy\,d\xi.$$

Now the double integral is absolutely convergent and it is permissible to interchange the order of integration. The ξ-integral is evaluated by (7.12):

$$\int e^{i\xi(x-y)}e^{-\epsilon^2\xi^2/2}d\xi = \mathscr{F}\left[e^{-\epsilon^2\xi^2/2}\right](y-x) = \frac{\sqrt{2\pi}}{\epsilon}e^{-(x-y)^2/2\epsilon^2}.$$

In other words,

$$\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi = \frac{1}{\epsilon\sqrt{2\pi}}\int f(y)e^{-(x-y)^2/2\epsilon^2}dy = f * \phi_\epsilon(x)$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad \phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right) = \frac{1}{\epsilon\sqrt{2\pi}}e^{-x^2/2\epsilon^2}.$$

But this is precisely the situation of Theorem 7.3 and example (7.6) in §7.1 (with ϵ replaced by $\epsilon\sqrt{2}$), and we conclude that if f is piecewise continuous,

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi = \tfrac{1}{2}\left[f(x-)+f(x+)\right]$$

for all x. We have arrived at our main result.

The Fourier Inversion Theorem. *Suppose f is integrable and piecewise continuous on $\mathbf{R}$, defined at its points of discontinuity so as to satisfy $f(x) = \frac{1}{2}\left[f(x-)+f(x+)\right]$ for all x. Then*

$$f(x) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi, \qquad x \in \mathbf{R}. \tag{7.15}$$

Moreover, if $\widehat{f} \in L^1$, then f is continuous and

$$f(x) = \frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)\,d\xi, \qquad x \in \mathbf{R}. \tag{7.16}$$

Proof: The only thing left to prove is the last assertion. But

$$\left|e^{i\xi x}e^{-\epsilon^2|\xi|^2/2}\widehat{f}(\xi)\right| \le \left|\widehat{f}(\xi)\right|,$$

so if $\widehat{f} \in L^1$, we can apply the dominated convergence theorem to evaluate the limit in (7.15):

$$\frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)\,d\xi = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi = \tfrac{1}{2}\Big[f(x-)+f(x+)\Big].$$

The integral on the left is $1/2\pi$ times the Fourier transform of $\widehat{f}$ evaluated at $-x$, and we have already pointed out that Fourier transforms of integrable functions are continuous. Hence f is continuous and $\frac{1}{2}\Big[f(x-)+f(x+)\Big] = f(x)$. ∎

The inversion formula (7.16), or its variant (7.15), expresses a general function f as a continuous superposition of the exponential functions $e^{i\xi x}$. In this way it provides an analogue for nonperiodic functions of the Fourier series expansion of periodic functions.

Corollary 7.1. *If $\widehat{f} = \widehat{g}$, then $f = g$.*

Proof: If $\widehat{f} = \widehat{g}$, then $(f-g)\widehat{\ } = 0$, so $f - g = 0$ by (7.16). ∎

If ϕ is the Fourier transform of $f \in L^1$, we say that f is the **inverse Fourier transform** of ϕ and write $f = \mathscr{F}^{-1}\phi$. The operation $\mathscr{F}^{-1}$ is well-defined by Corollary 7.1.

Remark. Functions f such that f and $\widehat{f}$ are both in L^1 exist in great abundance; one needs only a little smoothness of f to ensure the necessary decay of $\widehat{f}$ at infinity. For example, if f is twice differentiable and f' and f'' are also integrable, then $(f'')\widehat{\ }(\xi) = -\xi^2\widehat{f}(\xi)$ is bounded, so $|\widehat{f}(\xi)| \le C/(1+\xi^2)$, whence $\widehat{f} \in L^1$. (See also Exercise 7.) Such functions have the property that f and $\widehat{f}$ are bounded and continuous as well as integrable, and hence f and $\widehat{f}$ are also in L^2.

A number of variations on the Fourier inversion theorem are possible. For one thing, a version of (7.15) is true for functions $f \in L^1$ that are not piecewise continuous; namely, if $f \in L^1$, we have

$$f(x) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi$$

for "almost every" $x \in \mathbf{R}$, in the sense of Lebesgue measure. For another, one can replace the cutoff function $e^{-\epsilon^2\xi^2/2}$ in (7.15) by any of a large number of other functions with similar properties. (See Folland [25], Theorem 8.31; also Exercise 5.)

On the more naive level, one can ask whether the integral in (7.14) can be interpreted simply as a (principal value) improper integral, that is, whether

$$f(x) = \lim_{r\to\infty}\frac{1}{2\pi}\int_{-r}^{r} e^{i\xi x}\widehat{f}(\xi)\,d\xi.$$

This amounts to using the cutoff function that equals 1 on $[-r, r]$ and 0 elsewhere, and letting $r \to \infty$; it is the obvious analogue of evaluating a Fourier series as the limit of its symmetric partial sums as we did in §2.2. Just as in that case, piecewise continuity of f does not suffice, but piecewise smoothness does.

Theorem 7.6. *If f is integrable and piecewise smooth on* $\mathbf{R}$, *then*

$$\lim_{r\to\infty} \frac{1}{2\pi}\int_{-r}^{r} e^{i\xi x}\widehat{f}(\xi)\,d\xi = \tfrac{1}{2}\Big[f(x-)+f(x+)\Big] \tag{7.17}$$

for every $x \in \mathbf{R}$.

Proof: We have

$$\int_{-r}^{r} e^{i\xi(x-y)}d\xi = \frac{e^{ir(x-y)} - e^{-ir(x-y)}}{i(x-y)} = \frac{2\sin r(x-y)}{x-y},$$

so

$$\begin{aligned}\frac{1}{2\pi}\int_{-r}^{r} e^{i\xi x}\widehat{f}(\xi)\,d\xi &= \frac{1}{2\pi}\int_{-r}^{r}\!\int e^{i\xi(x-y)}f(y)\,dy\,d\xi \\ &= \frac{1}{\pi}\int \frac{\sin r(x-y)}{x-y}f(y)\,dy = \frac{1}{\pi}\int \frac{\sin ry}{y}f(x-y)\,dy.\end{aligned}$$

This has the form $f * g_\epsilon(x)$ where $\epsilon = 1/r$ and $g(x) = (\sin x)/x$, just as in the arguments leading to (7.15). The trouble is that $(\sin x)/x$ is not in L^1, so Theorem 7.3 is not applicable. (See Exercise 6.) However, it is well known that

$$\int_0^\infty \frac{\sin ry}{y}\,dy = \int_{-\infty}^0 \frac{\sin ry}{y}\,dy = \frac{\pi}{2},$$

where the integrals are conditionally convergent. (See Boas [8], §9D.) Hence, we can write

$$\begin{aligned}&\frac{1}{2\pi}\int_{-r}^{r} e^{i\xi x}\widehat{f}(\xi)\,d\xi - \tfrac{1}{2}\Big[f(x-)+f(x+)\Big] \\ &\quad= \frac{1}{\pi}\int_{-\infty}^{0}\frac{\sin ry}{y}\Big[f(x-y)-f(x+)\Big]\,dy + \frac{1}{\pi}\int_0^\infty \frac{\sin ry}{y}\Big[f(x-y)-f(x-)\Big]\,dy,\end{aligned}$$

and it suffices to show that both integrals on the right tend to zero as $r \to \infty$.

Consider the integral over $(0,\infty)$; the argument for the other one is much the same. For any $K > 0$ we can write it as

$$\int_0^K \frac{\sin ry}{y}\Big[f(x-y)-f(x-)\Big] + \int_K^\infty \frac{\sin ry}{y}\Big[f(x-y)-f(x-)\Big]\,dy. \tag{7.18}$$

If $K \geq 1$ we have

$$\left|\int_K^\infty \frac{\sin ry}{y}f(x-y)\,dy\right| \leq \int_K^\infty |f(x-y)|\,dy$$

and

$$\int_K^\infty \frac{\sin ry}{y} f(x-)\,dy = f(x-)\int_{rK}^\infty \frac{\sin z}{z}\,dz.$$

These are the tail ends of convergent integrals, so they tend to zero as $K \to \infty$, no matter what r is (as long as, say, $r \geq 1$). Hence we can make the integral over (K, ∞) in (7.18) as small as we wish by taking K sufficiently large.

On the other hand, the integral over $(0, K)$ in (7.18) equals

$$\int \frac{e^{iry} - e^{-iry}}{2i} g(y)\,dy = \frac{1}{2i}\left[\widehat{g}(r) - \widehat{g}(-r)\right]$$

where

$$g(y) = \frac{f(x-y) - f(x-)}{y} \text{ if } 0 < y < K, \qquad g(y) = 0 \text{ otherwise.}$$

Since f is piecewise smooth, g is piecewise smooth except perhaps near $y = 0$, and $g(y)$ approaches the finite limit $f'(x-)$ as y decreases to 0. Hence g is bounded on $[0, K]$ and thus is integrable on $\mathbf{R}$. But then $\widehat{g}(r)$ and $\widehat{g}(-r)$ tend to zero as $r \to \infty$ by the Riemann-Lebesgue lemma, so we are done. ∎

The Fourier transform on L^2

We have developed the Fourier transform in the setting of the space L^1, but our experience with Fourier series suggests that the space L^2 should also play a significant role. This is indeed the case. There is an initial difficulty to be overcome, in that the integral $\int e^{-i\xi x} f(x)\,dx$ may not converge if f is in L^2 but not in L^1, but there is a way around this problem. The key observation is that the analogue of Parseval's formula holds for the Fourier transform. Namely, suppose that f and g are L^1 functions such that $\widehat{f}$ and $\widehat{g}$ are in L^1. Then f, g, $\widehat{f}$, and $\widehat{g}$ are also in L^2 (cf. the remark following the Fourier inversion theorem), and by (7.16) we have

$$\begin{aligned} 2\pi\langle f, g\rangle = 2\pi \int f(x)\overline{g(x)}\,dx &= \iint f(x)\overline{e^{i\xi x}\widehat{g}(\xi)}\,d\xi\,dx \\ &= \iint f(x)e^{-i\xi x}\overline{\widehat{g}(\xi)}\,dx\,d\xi = \int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi = \langle \widehat{f}, \widehat{g}\rangle. \end{aligned}$$

In other words, the Fourier transform preserves inner products up to a factor of 2π. In particular, taking $g = f$, we obtain

$$\|\widehat{f}\|^2 = 2\pi\|f\|^2,$$

which is the "Parseval formula" for the Fourier transform.

Now, if f is an arbitrary L^2 function, we can find a sequence $\{f_n\}$ such that f_n and $\widehat{f}_n$ are in L^1 and $f_n \to f$ in the L^2 norm. (This follows from Theorem

2.7 of §2.4, which we stated without any proof, but we now have the machinery to construct such sequences explicitly; see Exercise 8, §7.1.) Then

$$\|\widehat{f}_n - \widehat{f}_m\|^2 = 2\pi\|f_n - f_m\|^2 \to 0 \quad \text{as} \quad m, n \to \infty,$$

so $\{\widehat{f}_n\}$ is a Cauchy sequence in L^2. Since L^2 is complete, it has a limit, which is easily seen to depend only on f and not on the approximating sequence $\{f_n\}$. We *define* this limit to be $\widehat{f}$. In this way, the domain of the Fourier transform is extended to include all of L^2, and a simple limiting argument shows that this extended Fourier transform still preserves the norm and inner product up to a factor of 2π, and that it still satisfies the properties of Theorem 7.5. In short, we have the following result.

The Plancherel Theorem. *The Fourier transform, defined originally on $L^1 \cap L^2$, extends uniquely to a map from L^2 to itself that satisfies*

$$\langle \widehat{f}, \widehat{g} \rangle = 2\pi\langle f, g\rangle \quad \text{and} \quad \|\widehat{f}\|^2 = 2\pi\|f\|^2 \quad \text{for all } f, g \in L^2.$$

Moreover, the formulas of Theorem 7.5 still hold for L^2 functions.

If f is in L^2 but not in L^1, the integral $\int f(x)e^{-i\xi x}\,dx$ defining $\widehat{f}$ may not converge pointwise, but it may be interpreted by a limiting process like the one we used in the inversion formula (7.15). That is, if $f \in L^2$, as $\epsilon \to 0$ the functions g^ϵ defined by

$$g^\epsilon(\xi) = \int e^{-i\xi x} e^{-\epsilon^2 x^2/2} f(x)\,dx$$

converge in the L^2 norm, and pointwise almost everywhere, to $\widehat{f}$. Likewise, the functions f^ϵ defined by

$$f^\epsilon(x) = \frac{1}{2\pi}\int e^{i\xi x} e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)\,d\xi$$

converge in the L^2 norm, and pointwise almost everywhere, to f.

The Fourier inversion theorem is also a useful device for computing Fourier transforms. Indeed, upon setting $\phi = \widehat{f}$ the inversion formula (7.16) can be restated as

$$\phi = \widehat{f} \quad \Longrightarrow \quad f(x) = (2\pi)^{-1}\widehat{\phi}(-x).$$

(The original formula (7.16) is valid when ϕ and $\widehat{\phi}$ are in L^1, but in the present form it continues to hold for any $\phi \in L^2$.) But this means that if ϕ is the Fourier transform of a known function f, we can immediately write down the Fourier transform of ϕ by setting $\xi = -x$:

$$\phi = \widehat{f} \quad \Longrightarrow \quad \widehat{\phi}(\xi) = 2\pi f(-\xi).$$

For example, from formula (7.11) we have

$$\mathscr{F}\left[\frac{\sin ax}{x}\right] = \begin{cases} \pi & \text{if } |\xi| < a, \\ 0 & \text{otherwise.} \end{cases}$$

TABLE 2. SOME BASIC FOURIER TRANSFORMS

Functions are listed on the left, their Fourier transforms on the right. a and c denote constants with $a > 0$ and $c \in \mathbf{R}$.

1.	$f(x)$	$\widehat{f}(\xi)$
2.	$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
3.	$e^{icx}f(x)$	$\widehat{f}(\xi - c)$
4.	$f(ax)$	$a^{-1}\widehat{f}(a^{-1}\xi)$
5.	$f'(x)$	$i\xi\widehat{f}(\xi)$
6.	$xf(x)$	$i(\widehat{f})'(\xi)$
7.	$(f * g)(x)$	$\widehat{f}(\xi)\widehat{g}(\xi)$
8.	$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f} * \widehat{g})(\xi)$
9.	$e^{-ax^2/2}$	$\sqrt{2\pi/a}\, e^{-\xi^2/2a}$
10.	$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\lvert\xi\rvert}$
11.	$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
12.	$\chi_a(x) = \begin{cases} 1 & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$2\xi^{-1}\sin a\xi$
13.	$x^{-1}\sin ax$	$\pi\chi_a(\xi) = \begin{cases} \pi & (\lvert\xi\rvert < a) \\ 0 & (\lvert\xi\rvert > a) \end{cases}$

Here $x^{-1}\sin ax$ is a function that is in L^2 but not in L^1; the calculation of its Fourier transform directly from the definition is a somewhat tricky business.

Table 2 contains a brief list of basic Fourier transform formulas that we have derived in this section. All of them will be used repeatedly in what follows. Much more extensive tables of Fourier transforms are available — for example, Erdélyi et al. [22]. (Most of the entries in [22] are in the form of Fourier sine or cosine transforms; see §7.4, especially the concluding remarks.)

One final remark: The definition of the Fourier transform that we have adopted here is not universally accepted. Two other frequently used definitions are

$$\widetilde{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-i\xi x} f(x)\,dx, \qquad \check{f}(\xi) = \int e^{-2\pi i\xi x} f(x)\,dx,$$

for which the inversion formulas are

$$f(x) = \frac{1}{\sqrt{2\pi}}\int e^{i\xi x}\widetilde{f}(\xi)\,d\xi, \qquad f(x) = \int e^{2\pi i\xi x}\check{f}(\xi)\,d\xi.$$

Some people also omit the minus sign in the exponent in the formulas defining $\widehat{f}$, $\widetilde{f}$, and $\check{f}$; it then reappears in the exponent in the inversion formula. $\widetilde{f}$ has the advantage of getting rid of the 2π in the Plancherel theorem, $\|\widetilde{f}\|^2 = \|f\|^2$, but the disadvantage of introducing one in the convolution formula, $(f * g)\widetilde{} = \sqrt{2\pi}\widetilde{f}\widetilde{g}$. $\check{f}$ obviates the 2π's in both these formulas, $\|\check{f}\|^2 = \|f\|^2$ and $(f * g)\check{} = \check{f}\check{g}$, but introduces them in the formula for derivatives, $(f')\check{}(\xi) = 2\pi i\xi\check{f}(\xi)$. In short, one can choose to put the 2π's where one finds them least annoying, but one cannot get rid of them entirely.

EXERCISES

1. If $f(x) = e^{-ax^2/2}$ with $a > 0$, then $\widehat{f}(\xi) = \int e^{-i\xi x - ax^2/2}dx$. Derive formula (7.12) by completing the square in the exponent, using Cauchy's theorem to shift the path of integration from the real axis ($\operatorname{Im} x = 0$) to the horizontal line $\operatorname{Im} x = -\xi/a$, and finally using (7.7).
2. Show directly that $\mathscr{F}\left[e^{-a|x|}\right] = 2a(\xi^2 + a^2)^{-1}$ and hence derive (7.13) from the Fourier inversion formula.
3. Complete the proof of Theorem 7.5(a, b).
4. Let f be as in Exercise 3, §7.1. Compute $\widehat{f}$ and $(f * f)\widehat{}$ from the formulas in that exercise and verify that $(f * f)\widehat{} = (\widehat{f})^2$.
5. Suppose $g \in L^1$, $\int g(x)\,dx = 1$, and $\widehat{g} \in L^1$.
   a. Show that $\widehat{g}(\delta\xi) \to 1$ as $\delta \to 0$ for all $\xi \in \mathbf{R}$.
   b. Show that for any continuous $f \in L^1$,
   $$\lim_{\delta\to 0} \frac{1}{2\pi} \int e^{i\xi x}\widehat{g}(\delta\xi)\widehat{f}(\xi)\,d\xi = f(x)$$
   for all x. What if f is only piecewise continuous? (Mimic the argument leading to (7.15), using the Fourier inversion theorem for g.)
6. Show that $\int_0^\infty x^{-1}|\sin x|\,dx = \infty$. (Hint: Show that $\int_{(n-1)\pi}^{n\pi} x^{-1}|\sin x|\,dx > 2/n$.)
7. Suppose that f is continuous and piecewise smooth, $f \in L^2$, and $f' \in L^2$. Show that $\widehat{f} \in L^1$. (Hint: First show that $\int(1 + \xi^2)|\widehat{f}(\xi)|^2 d\xi$ is finite; then use the Cauchy-Schwarz inequality as in the proof of Theorem 2.3, §2.3.)
8. Given $a > 0$, let $f(x) = e^{-x}x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\widehat{f}(\xi) = \Gamma(a)(1 + i\xi)^{-a}$.
9. Use the residue theorem to show that
   $$\mathscr{F}\left[\frac{1}{x^4 + 1}\right] = \frac{\pi}{\sqrt{2}}e^{-|\xi|/\sqrt{2}}\left(\cos\frac{\xi}{\sqrt{2}} + \sin\frac{|\xi|}{\sqrt{2}}\right).$$
10. Let $f(x) = (\sinh ax)/(\sinh \pi x)$ where $0 < a < \pi$.
    a. Use the residue theorem to show that
    $$\widehat{f}(\xi) = 2i\sum_1^\infty (-1)^n e^{-n|\xi|} \sinh ina.$$
    b. Use the fact that $2\sinh ina = e^{ina} - e^{-ina}$ and sum the geometric series to show that
    $$\widehat{f}(\xi) = \frac{\sin a}{\cosh\xi + \cos a}.$$

11. Given $\nu > -\frac{1}{2}$, let $f(x) = (1 - x^2)^{\nu-(1/2)}$ if $|x| < 1$, $f(x) = 0$ if $|x| > 1$. Show that $\widehat{f}(\xi) = 2^\nu \pi^{1/2}\Gamma(\nu + \frac{1}{2})\xi^{-\nu} J_\nu(\xi)$. (Cf. Exercise 14, §5.2.)
12. For $a > 0$, let $f_a(x) = a/\left[\pi(x^2 + a^2)\right]$ and $g_a(x) = (\sin ax)/\pi x$. Use the Fourier transform to show that:
    a. $f_a * f_b = f_{a+b}$, b. $g_a * g_b = g_{\min(a,b)}$.
13. Use the Plancherel theorem to prove the indicated formulas. In all of them, a and b denote positive numbers.
    a. $\displaystyle\int \frac{\sin(at)\sin(bt)}{t^2}\,dt = \pi \min(a,b)$. (Cf. formula (7.11).)
    b. $\displaystyle\int \frac{t^2}{(t^2 + a^2)(t^2 + b^2)}\,dt = \frac{\pi}{a+b}$. (Cf. formula (7.13).)
    c. $\displaystyle\int (1 + it)^{-a}(1 - it)^{-b}\,dt = \frac{2^{2-a-b}\pi\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}$. (Here $a, b > \frac{1}{2}$; use Exercise 8.)
14. Let h_n be the nth Hermite function as defined in (6.33). Show that $\widehat{h}_n(\xi) = \sqrt{2\pi}\,(-i)^n h_n(\xi)$. (Hint: Use induction on n. For h_0 the result is true by formula (7.12). Assuming the result for h_n, prove it for h_{n+1} by applying the Fourier transform to equation (6.40).) This shows that the Hermite functions are an orthogonal basis of eigenfunctions for the Fourier transform.
15. Let $l_n(x) = e^{-x/2}L_n^0(x)$ for $x > 0$, $l_n(x) = 0$ for $x < 0$, where L_n^0 denotes the Laguerre polynomial defined by (6.43), and let $\phi_n(\xi) = (2\pi)^{-1/2}\widehat{l}_n(\xi)$.
    a. Show that
    $$\phi_n(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{(2i\xi - 1)^n}{(2i\xi + 1)^{n+1}}.$$
    (Hint: Plug the definition of L_n^0 into the formula defining $\widehat{l}_n$ and integrate by parts n times.)
    b. Deduce from part (a) and Theorem 6.15 that $\{\phi_n\}_0^\infty$ is an orthonormal basis for the space $\{f \in L^2 : \mathcal{F}^{-1}f(x) = 0 \text{ for } x < 0\}$.

7.3 Some applications

The Fourier transform is a useful tool for analyzing a great variety of problems in mathematics and the physical sciences. Underlying most of these applications is the following fundamental fact.

Suppose L is a linear operator on functions on $\mathbf{R}$ that commutes with translations; that is, if $L[f(x)] = g(x)$ then $L[f(x+s)] = g(x+s)$ for any $s \in \mathbf{R}$. Then any exponential function e^{ax} ($a \in \mathbf{C}$) that belongs to the domain of L is an eigenfunction of L.

The proof of this is very simple: let $f(x) = e^{ax}$ and $g = L[f]$. Then for any $s \in \mathbf{R}$,

$$g(x+s) = L[e^{a(x+s)}] = L[e^{as}e^{ax}] = e^{as}L[e^{ax}] = e^{as}g(x).$$

Setting $x = 0$, we find that $g(s) = g(0)e^{as}$ for all $s \in \mathbf{R}$; in other words, $g = Cf$ where $C = g(0)$. Thus $L[f] = Cf$.

Suppose in particular that the domain of L includes all the imaginary exponentials $e^{i\xi x}$, and let $h(\xi)$ be the eigenvalue for $e^{i\xi x}$; thus $L[e^{i\xi x}] = h(\xi)e^{i\xi x}$. If L satisfies some very mild continuity conditions, one can read off the action of L on a more or less arbitrary function f from the Fourier inversion formula. Indeed, that formula expresses f as a continuous superposition of the exponentials $e^{i\xi x}$,

$$f(x) = \frac{1}{2\pi}\int \widehat{f}(\xi)e^{i\xi x}\,d\xi,$$

and so

$$L[f](x) = \frac{1}{2\pi}\int \widehat{f}(\xi)L[e^{i\xi x}]\,d\xi = \frac{1}{2\pi}\int \widehat{f}(\xi)h(\xi)e^{i\xi x}\,d\xi.$$

(The continuity conditions on L are used here to justify treating the integral as if it were a finite sum.) Thus, in terms of the Fourier transform $\widehat{f}$, the action of L reduces to the simple algebraic operation of multiplication by the function h, $(L[f])\widehat{\ } = h\widehat{f}$, so passage from f to $\widehat{f}$ may simplify the analysis of L immensely. Alternatively, if h is the Fourier transform of a function H, we can express L as a convolution: $Lf = f * H$.

In order for these calculations to work according to the theory we have developed so far, f and H must be L^1 or L^2 functions. However, as we shall see in §9.4, it is possible to extend the domain of the Fourier transform to include much more general sorts of functions, and the present discussion then extends to the more general situation. In any case, at this point we are only giving an informal presentation of the ideas rather than precise results.

Perhaps the single most important class of operators L to which this analysis applies is the class of linear differential operators with constant coefficients: $L[f] = \sum_0^k c_j f^{(j)}$. Here, of course, we have $L[e^{i\xi x}] = \sum_0^k c_j (i\xi)^j e^{i\xi x}$; hence for any f, $(L[f])\widehat{\ }(\xi) = \sum_0^k c_j (i\xi)^j \widehat{f}(\xi)$, as we already know from Theorem 7.5(c). This fact is the basis for the use of the Fourier transform in solving differential equations, as we shall now demonstrate.

Partial differential equations

In Chapters 2 and 3 we saw how to solve certain boundary value problems for the heat, wave, and Laplace equations by means of Fourier series. We now use the Fourier transform to solve analogous problems on unbounded regions. The crux of the matter is Theorem 7.5(c), which says that the Fourier transform converts differentiation into a simple algebraic operation. By utilizing this fact we can reduce partial differential equations to easily solvable ordinary differential equations.

To begin with, consider heat flow in an infinitely long rod, given the initial temperature $f(x)$:

$$u_t = ku_{xx} \quad (-\infty < x < \infty), \qquad u(x,0) = f(x). \tag{7.19}$$

There are no boundary conditions for $t > 0$ because there is no boundary, but we shall assume (for the time being) that $u(x,t)$ and $f(x)$ vanish sufficiently rapidly as $x \to \pm\infty$ to be integrable over the whole line. Then we can apply the Fourier transform in x to convert (7.19) into

$$\frac{\partial \widehat{u}}{\partial t}(\xi,t) = -k\xi^2\widehat{u}(\xi,t), \qquad \widehat{u}(\xi,0) = \widehat{f}(\xi).$$

For each fixed ξ, this is a simple ordinary differential equation in t with an initial condition; its solution is

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)e^{-k\xi^2 t}.$$

It remains to invert the Fourier transform, which can be done in either of two ways. The first is to apply the Fourier inversion theorem to obtain the **Fourier integral formula** for u:

$$u(x,t) = \frac{1}{2\pi}\int \widehat{f}(\xi)e^{-k\xi^2 t}e^{i\xi x}\,d\xi.$$

The second is to use Theorem 7.5(d) to obtain a formula for u as a convolution; this has the advantage of expressing u in terms of f rather than $\widehat{f}$. Namely, by formula (7.12) with $a = 1/2kt$, we see that the inverse Fourier transform of $e^{-k\xi^2 t}$ is

$$K_t(x) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}.$$

Therefore,

$$u(x,t) = f * K_t(x) = \frac{1}{\sqrt{4\pi kt}}\int f(y)e^{-(x-y)^2/4kt}\,dy. \tag{7.20}$$

Once we have this formula in hand, we can verify directly that it works. It is a simple exercise (Exercise 1 of §1.1) to check that $K_t(x)$ satisfies the heat equation, from which it follows by differentiating under the integral that $u(x,t)$ does also; and that $u(x,t) \to f(x)$ as $t \to 0$ (assuming, say, that f is continuous) follows from Theorem 7.3. Moreover, the hypothesis that $f \in L^1$ can be relaxed considerably. Since $K_t(x)$ decays very rapidly as $x \to \pm\infty$, the integral in (7.20) will converge as long as $f(x)$ grows less rapidly at infinity than any function $e^{\epsilon x^2}$ ($\epsilon > 0$), and an easy extension of the arguments just sketched shows that u still satisfies (7.19) in this case.

The physical interpretation of (7.20) is as follows. Imagine that the whole infinite rod starts out at temperature zero, and at time $t = 0$ a unit quantity of heat is injected at the origin. As t increases this heat spreads out along the rod, producing the temperature distribution $K_t(x)$. If, instead, the heat is injected at the point y, the resulting temperature distribution is $K_t(x-y)$. Now, in the problem (7.19), at time $t = 0$ there is an amount $f(y)\,dy$ of heat at the point y, which spreads out to give $K_t(x-y)f(y)\,dy$ at time $t > 0$. By the superposition principle, these temperatures can be added up to form (7.20).

Is (7.20) the *only* solution of (7.19)? Alas, the answer is *no*, for there exist nonzero solutions $v(x,t)$ of the heat equation with $v(x,0) = 0$. (The construction is rather complicated; see John [33] or Körner [34].) However, such functions $v(x,t)$ grow very rapidly as $x \to \pm\infty$, so they can be dismissed as physically unrealistic. What *is* true is that if the initial temperature $f(x)$ is bounded, then (7.20) is the only *bounded* solution of (7.19).

Let us now turn to the Dirichlet problem for a half-plane:

$$u_{xx} + u_{yy} = 0 \quad (x \in \mathbf{R},\ y > 0), \qquad u(x,0) = f(x). \tag{7.21}$$

Here again we must impose a boundedness condition to obtain uniqueness, and the reason is simple: If $u(x,y)$ satisfies (7.21), then so does $u(x,y) + y$. We therefore assume that f is bounded and (for the moment) integrable, and we seek a bounded solution of (7.21).

As with the heat equation, we begin by taking the Fourier transform in x:

$$-\xi^2\widehat{u}(\xi,y) + \widehat{u}(\xi,y)_{yy} = 0, \qquad \widehat{u}(\xi,0) = \widehat{f}(\xi).$$

This is an ordinary differential equation in y, and its general solution is

$$\widehat{u}(\xi,y) = C_1(\xi)e^{|\xi|y} + C_2(\xi)e^{-|\xi|y}, \qquad C_1(\xi) + C_2(\xi) = \widehat{f}(\xi).$$

Because of the boundedness requirement, we must reject the solution $e^{|\xi|y}$, so we take $C_1 = 0$ and $C_2 = \widehat{f}$. Hence, by Theorem 7.5(d), $u(x,y) = f * P_y(x)$ where $\widehat{P}_y(\xi) = e^{-|\xi|y}$, so by (7.13) (with y in place of a),

$$P_y(x) = \frac{y}{\pi(x^2 + y^2)}.$$

Thus,

$$u(x,y) = f * P_y(x) = \int \frac{yf(x-t)}{\pi(t^2 + y^2)}\,dt. \tag{7.22}$$

This is the **Poisson integral formula** for the solution of (7.21), and $P_y(x)$ is called the **Poisson kernel**. Since $P_y \in L^1$, (7.22) makes sense for any bounded f and defines a bounded function u:

$$|f| \le M \quad \Longrightarrow \quad |u(x,y)| \le M \int \frac{y}{\pi(t^2 + y^2)}\,dt = M.$$

One can check directly that it satisfies Laplace's equation; and if f is (say) continuous, we have $u(x,y) \to f(x)$ as $y \to 0$ by Theorem 7.3.

The 1-dimensional wave equation can be solved by the same technique, leading to the solution found in Exercise 6, §1.1. We leave the details to the reader (Exercise 3).

Signal analysis

Let $f(t)$ represent the amplitude of a signal, perhaps a sound wave or an electromagnetic wave, at time t. The Fourier representation

$$f(t) = \frac{1}{2\pi}\int \widehat{f}(\omega)e^{i\omega t}\,d\omega, \qquad \widehat{f}(\omega) = \int f(t)e^{-i\omega t}\,dt$$

exhibits f as a continuous superposition of the simple periodic waves $e^{i\omega t}$ as ω ranges over all possible frequencies. This representation is absolutely basic in the analysis of signals in electrical engineering and information theory. Whole books can be, and have been, written on this subject; here we shall just present a couple of basic results to give the flavor of the ideas. For more extensive treatments we refer the reader to Bracewell [11], Dym-McKean [19], Papoulis [42], and Taylor [51].

In the first place, the power of a signal $f(t)$ is proportional to the square of the amplitude, $|f(t)|^2$, so the total energy of the signal is proportional to $\int |f(t)|^2\,dt$. Hence, the condition that the total energy be finite is just that $f \in L^2$.

Second, electrical systems can be mathematically modeled as operators L that transform an input signal f into an output signal $L[f]$. Many (but of course not all) such systems have a linear response, which means that the operator L is linear. Also, their action is generally unaffected by the passage of time (a given input signal produces the same response whether it was fed in yesterday or today), which means that L commutes with time translations. In this case, the general principles enunciated at the beginning of this section apply, and we see that L is described by

$$(L[f])\widehat{\ } = h\widehat{f}, \quad \text{or} \quad L[f] = f * H,$$

where h is a certain complex-valued function and H is its inverse Fourier transform. h is called the **system function** and H is called the **impulse response**. ($H(t)$ is the output when the input is the Dirac δ-function; see Chapter 9.) If we write $h(\omega)$ in polar form as $h(\omega) = A(\omega)e^{i\theta(\omega)}$, $A(\omega)$ and $\theta(\omega)$ represent the amplitude and phase modulation due to the operator L at frequency ω.

Physical devices that work in real time must obey the law of causality, which means that if the input signal $f(t)$ is zero for $t < t_0$, then the output $g(t) = L[f](t)$ must also be zero for $t < t_0$. In other words,

$$f(t) = 0 \text{ for } t < t_0 \quad \Longrightarrow \quad f * H(t) = \int_{t_0}^{\infty} f(s)H(t-s)\,ds = 0 \text{ for } t < t_0.$$

The only way this can hold for all inputs f is to have $H(t-s) = 0$ when $t < t_0$ and $s > t_0$, and for this to be true for all t_0 we must have $H(t-s) = 0$ when $t - s < 0$, i.e., $H(t) = 0$ for $t < 0$.

This condition on H places rather severe restrictions on the system function h, and it often implies that certain desirable characteristics of an electrical system can be achieved only approximately. For example, one often wishes to filter out all frequencies outside some finite interval — say, outside the interval $[-\Omega, \Omega]$.

A device to accomplish this is called a *band-pass filter*, and the system function for an ideal band-pass filter would be the function $h(\omega)$ that equals 1 if $|\omega| \leq \Omega$ and equals 0 otherwise. But by a slight modification of formula (7.11), the corresponding impulse response $H(t)$ would be $H(t) = (\sin \Omega t)/2\pi t$, which does not obey the causality principle. It is, of course, the business of engineers to figure out ways to circumvent such difficulties!

Because of their importance in engineering as well as pure analysis, L^2 functions whose inverse Fourier transforms vanish on a half-line have been studied extensively. They are known as **Hardy functions**, and the space of Hardy functions is denoted by H^2. See Exercise 15 of §7.2 for the construction of a useful orthonormal basis for H^2, and Dym-McKean [19] for the physical interpretation of this basis and further information on H^2.

Let us now turn to a basic theorem of signal analysis that involves a neat interplay of Fourier transforms and Fourier series. Suppose f represents a physical signal that we are allowed to investigate by measuring its values at some sequence of times $t_1 < t_2 < \dots$. How much information can we gain this way? Of course, for an arbitrary function $f(t)$, knowing a discrete set of values $f(t_1), f(t_2), \dots$ tells us essentially nothing about the values of f at other points. However, if f is known to involve only certain frequencies, we can say quite a bit. To be precise, the signal f is called **band-limited** if it involves only frequencies smaller than some constant Ω, that is, if $\widehat{f}$ vanishes outside the finite interval $[-\Omega, \Omega]$. In this case, since $e^{i\omega t}$ does not change much on any interval of length $\Delta t \ll \omega^{-1}$, one has the intuitive feeling that $f(t + \Delta t)$ cannot differ much from $f(t)$ when $\Delta t \ll \Omega^{-1}$; hence one should pretty well know f once one knows the values $f(t_j)$ at a sequence $\{t_j\}$ of points with $t_{j+1} - t_j \approx \Omega^{-1}$.

This bit of folk wisdom can be made into an elegant and precise theorem by combining the techniques of Fourier series and Fourier integrals. Namely, suppose that $f \in L^2$ and f is band-limited. Then $\widehat{f} \in L^2$ and $\widehat{f}$ vanishes outside a finite interval, so $\widehat{f} \in L^1$. It follows that the Fourier inversion formula holds in the form (7.16). With this in mind, we have the following result.

The Sampling Theorem. *Suppose $f \in L^2$ and $\widehat{f}(\omega) = 0$ for $|\omega| \geq \Omega$. Then f is completely determined by its values at the points $t_n = n\pi/\Omega$, $n = 0, \pm 1, \pm 2, \dots$. In fact,*

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}. \tag{7.23}$$

Proof: Let us expand $\widehat{f}$ in a Fourier *series* on the interval $[-\Omega, \Omega]$, writing $-n$ in place of n for reasons of later convenience:

$$\widehat{f}(\omega) = \sum_{-\infty}^{\infty} c_{-n} e^{-in\pi\omega/\Omega} \qquad (|\omega| \leq \Omega).$$

The Fourier coefficients c_{-n} are given by

$$c_{-n} = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \widehat{f}(\omega) e^{in\pi\omega/\Omega} d\omega = \frac{1}{2\Omega}\int_{-\infty}^{\infty} \widehat{f}(\omega) e^{in\pi\omega/\Omega} d\omega = \frac{\pi}{\Omega} f\left(\frac{n\pi}{\Omega}\right).$$

Here we have used the fact that $\widehat{f}(\omega) = 0$ for $|\omega| > \Omega$ and the Fourier inversion formula (7.16). Using these two ingredients again, we obtain

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \widehat{f}(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)e^{-in\pi\omega/\Omega}e^{i\omega t}\,d\omega$$
$$= \frac{1}{2\Omega}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)\frac{e^{i(\Omega t - n\pi)\omega/\Omega}}{i(\Omega t - n\pi)/\Omega}\bigg|_{-\Omega}^{\Omega} = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}.$$

(Termwise integration of the sum is permissible because the Fourier series of $\widehat{f}$ converges in $L^2(-\Omega, \Omega)$, and we are essentially taking the inner product of this series with $e^{i\omega t}$.) ∎

There is a dual formulation of this theorem for frequency sampling of time-limited functions. That is, suppose $f(t)$ vanishes for $|t| > L$. Then $\widehat{f}$ is determined by its values at the points $\omega_n = n\pi/L$ by the same formula (7.23) (with f replaced by $\widehat{f}$). The proof is essentially the same, because of the symmetry between f and $\widehat{f}$. The sampling theorem can also be modified to deal with signals whose Fourier transform vanishes outside an interval $[a, b]$ that is not centered at 0; see Exercise 7.

It is worth noting that the functions

$$s_n(t) = \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

form an orthogonal basis for the space of L^2 functions whose Fourier transforms vanish outside $[-\Omega, \Omega]$, and that the sampling formula (7.23) is merely the expansion of f with respect to this basis. Indeed, the calculations in the proof of the sampling theorem show that s_n is the inverse Fourier transform of the function

$$\widehat{s}_n(\omega) = \begin{cases} (\pi/\Omega)e^{-in\pi\omega/\Omega} & \text{if } |\omega| < \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

The assertion therefore follows from the Plancherel theorem and the fact that the functions $e^{-in\pi\omega/\Omega}$ constitute an orthogonal basis for $L^2(-\Omega, \Omega)$. Further discussion of the expansion (7.23) and related topics can be found in Higgins [29].

From a practical point of view, the expansion (7.23) has the disadvantage that it generally does not converge very rapidly, because the function $(\sin x)/x$ decays slowly as $x \to \infty$. A more rapidly convergent expansion for a function f can be obtained by *oversampling*, that is, by replacing the sequence of points $n\pi/\Omega$ at which f is sampled by a more closely spaced sequence $n\pi/\lambda\Omega$ ($\lambda > 1$). If this is done, one can replace $(\sin x)/x$ by a function that vanishes like x^{-2} as $x \to \infty$. The precise result is worked out in Exercise 8.

Heisenberg's inequality

It is impossible for a signal to be both band-limited and time-limited; that is, it is impossible for f and $\widehat{f}$ both to vanish outside a finite interval unless f is identically zero. Indeed, if $f \in L^2$ (say) and $\widehat{f}(\omega) = 0$ for $|\omega| > \Omega$, the integral

$$F(z) = \frac{1}{2\pi} \int e^{i\omega z} \widehat{f}(\omega)\, d\omega$$

makes sense for any complex number z; moreover, we can differentiate under the integral to see that $F(z)$ is analytic. Thus, f is the restriction to the real axis of the entire analytic function F, and in particular, f cannot vanish except at isolated points unless it vanishes identically. In exactly the same way, if $f \neq 0$ vanishes outside a finite interval then $\widehat{f}$ has only isolated zeros.

These facts are aspects of a general principle that says that f and $\widehat{f}$ cannot both be highly localized. That is, if f vanishes (or is very small) outside some small interval, then $\widehat{f}$ has to be quite "spread out," and vice versa. Another piece of supporting evidence for this idea is Theorem 7.5(b), which says in essence that composing f with a compression or expansion corresponds to composing $\widehat{f}$ with an expansion or compression, respectively. To obtain a precise quantitative result along these lines, we introduce the notion of the **dispersion** of f about the point a,

$$\Delta_a f = \int (x-a)^2 |f(x)|^2\, dx \Big/ \int |f(x)|^2\, dx.$$

$\Delta_a f$ is a measure of how much f fails to be concentrated near a. If f "lives near a," that is, if f is very small outside a small neighborhood of a, then the factor of $(x-a)^2$ will make the numerator of $\Delta_a f$ small in comparison to the denominator, whereas if f "lives far away from a," the same factor will make the numerator large in comparison to the denominator. The following theorem therefore says that f and $\widehat{f}$ cannot both be concentrated near single points.

Heisenberg's Inequality. *For any $f \in L^2$,*

$$(\Delta_a f)(\Delta_\alpha \widehat{f}) \geq \frac{1}{4} \quad \textit{for all } a, \alpha \in \mathbf{R}. \tag{7.24}$$

Proof: For technical convenience we shall assume that f is continuous and piecewise smooth, and that the functions $xf(x)$ and $f'(x)$ are in L^2. (The smoothness assumption can be removed by an additional limiting argument; see Dym-McKean [19]. If $xf(x)$ is not in L^2 then $\Delta_a f = \infty$, whereas if $f'(x)$ is not in L^2 then $\Delta_\alpha \widehat{f} = \infty$, as the calculations below will show; in either case, (7.24) is trivially true.) Let us first consider the case $a = \alpha = 0$. By integration by parts, we have

$$\int_A^B x\overline{f(x)} f'(x)\, dx = x|f(x)|^2 \Big|_A^B - \int_A^B \left(|f(x)|^2 + xf(x)\overline{f'(x)} \right) dx,$$

or

$$\int_A^B |f(x)|^2 dx = -2\,\mathrm{Re}\int_A^B \overline{xf(x)}f'(x)\,dx + x|f(x)|^2\Big|_A^B$$
$$= -2\,\mathrm{Re}\int_A^B \overline{g(x)}f'(x)\,dx + x|f(x)|^2\Big|_A^B.$$

Since f, g, and f' are in L^2, the limits of the integrals in this equation as $A \to -\infty$ and $B \to \infty$ exist. Hence, so do the limits of $A|f(A)|^2$ and $B|f(B)|^2$, and these limits must be zero. (Otherwise, $|f(x)| \sim |x|^{-1/2}$ for large x, and f would not be in L^2.) We can therefore let $A \to -\infty$ and $B \to \infty$ to obtain

$$\int |f(x)|^2 dx = -2\,\mathrm{Re}\int \overline{xf(x)}f'(x)\,dx.$$

By the Cauchy-Schwarz inequality, then,

$$\left(\int |f(x)|^2 dx\right)^2 \le 4\left(\int x^2|f(x)|^2 dx\right)\left(\int |f'(x)|^2 dx\right). \tag{7.25}$$

But by the Plancherel theorem, $\int |f|^2 = (2\pi)^{-1}\int |\widehat{f}|^2$ and

$$\int |f'(x)|^2 dx = \frac{1}{2\pi}\int |[f']\widehat{\ }(\xi)|^2\,d\xi = \frac{1}{2\pi}\int \xi^2|\widehat{f}(\xi)|^2\,d\xi.$$

Hence (7.25) can be rewritten as

$$\left(\int |f(x)|^2\,dx\right)\left(\int |\widehat{f}(\xi)|^2\,d\xi\right) \le 4\left(\int x^2|f(x)|^2\,dx\right)\left(\int \xi^2|\widehat{f}(\xi)|^2\,d\xi\right),$$

which is (7.24) with $a = \alpha = 0$.

The general case is easily reduced to this one by a change of variable. Namely, given a and α, let

$$F(x) = e^{-i\alpha x}f(x+a).$$

It is easily verified that F satisfies the hypotheses of the theorem and that $\Delta_a f = \Delta_0 F$ and $\Delta_\alpha \widehat{f} = \Delta_0 \widehat{f}$. (See Exercise 9.) We can therefore apply the preceding argument to F to conclude that

$$(\Delta_a f)(\Delta_\alpha \widehat{f}) = (\Delta_0 F)(\Delta_0 \widehat{f}) \ge \frac{1}{4}. \qquad \blacksquare$$

Quantum mechanics

The Fourier transform is an essential tool for the quantum-mechanical description of nature. It would take us too far afield to explain the physics here; but for those who have the necessary physical background, we present a brief discussion of the mathematical formalism. For more details, see Messiah [39] or Landau-Lifshitz [35].

In quantum mechanics, a particle such as an electron that moves along the x-axis is described by a "wave function" $f(x)$, which is a complex-valued L^2 function such that $\|f\| = 1$. $|f(x)|^2$ is interpreted as the probability density that the particle will be found at position x; that is, $\int_a^b |f(x)|^2\,dx$ is the probability that the particle will be found in the interval $[a, b]$. (The condition $\|f\| = 1$ guarantees that the total probability is 1.)

The Fourier transform of the wave function f essentially gives the probability density for the momentum of the particle. More precisely, we define a modified Fourier transform $\widetilde{f}$ by

$$\widetilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\widehat{f}\left(\frac{p}{\hbar}\right) = \frac{1}{\sqrt{2\pi\hbar}}\int f(x)e^{-ixp/\hbar}\,dx,$$

where $\hbar$ is Planck's constant. Then the Plancherel theorem implies that

$$\int |\widetilde{f}(p)|^2\,dp = \frac{1}{2\pi\hbar}\int |\widehat{f}(\hbar^{-1}p)|^2\,dp = \frac{1}{2\pi}\int |\widehat{f}(\xi)|^2\,d\xi = \int |f(x)|^2\,dx = 1.$$

Thus $|\widetilde{f}(p)|^2$ can be interpreted as a probability density, and it is the probability density for momentum.

Similar considerations apply to particles moving in 3-space. One merely has to use the 3-dimensional version of the Fourier transform; see §7.5.

Heisenberg's inequality is a precise formulation of the position-momentum uncertainty principle. The numbers $\Delta_a f$ and $\Delta_\alpha \widetilde{f}$ are measures of how much the probability distributions $|f|^2$ and $|\widetilde{f}|^2$ are spread out away from the points a and α. (If we take a and α to be the mean values of these distributions, $\Delta_a f$ and $\Delta_\alpha \widetilde{f}$ are their variances.) A simple change of variable shows that $\Delta_\alpha \widetilde{f} = \hbar^2 \Delta_{\alpha/\hbar} \widehat{f}$, so Heisenberg's inequality says that

$$(\Delta_a f)(\Delta_\alpha \widetilde{f}) \geq \hbar^2/4.$$

The uncertainty principle is often cited as one of the mysteries of quantum mechanics, but the inverse relationship between the spatial or temporal localization of a wave and the localization of its frequency spectrum is a general phenomenon that pertains to waves of any sort. What is strange about the quantum world is that particles behave in some respects like waves.

Other applications

The Fourier transform is a ubiquitous tool in many fields of science as well as in pure mathematics. For discussions of some of its other applications we refer the reader to Bracewell [11], Dym-McKean [19], Körner [34], Papoulis [42], and Walker [53], [54].

EXERCISES

1. Use the Fourier transform to find a solution of the ordinary differential equation $u'' - u + 2g(x) = 0$ where $g \in L^1$. (The solution obtained this way is the one that vanishes at $\pm\infty$. What is the general solution?)

2. Use the Fourier transform to derive the solution

$$u(x,t) = f * K_t(x) + \int_{-\infty}^{\infty} \int_0^t G(y,s) K_{t-s}(x-y)\, dy\, ds$$

of the inhomogeneous heat equation $u_t = ku_{xx} + G(x,t)$ with initial condition $u(x,0) = f(x)$, where K_t is as in (7.20). (Observe that if we set $K_t(x) = 0$ and $G(x,t) = 0$ for $t \le 0$, then the second term of u is the convolution of G with K in the x *and* t variables.)

3. Consider the wave equation $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.
   a. Assuming that all the Fourier transforms in question exist, show that

$$\widehat{u}(\xi,t) = \widehat{f}(\xi) \cos ct\xi + \widehat{g}(\xi)(c\xi)^{-1} \sin ct\xi.$$

   b. Invert the Fourier transform to obtain d'Alembert's formula for u (Exercise 6, §1.1). (Hint: For the first term, write $\cos ct\xi = \frac{1}{2}(e^{ict\xi} + e^{-ict\xi})$ and use Theorem 7.5(a); for the second one, cf. formula (7.11).)

4. Solve the Dirichlet problem in an infinite strip: $u_{xx} + u_{yy} = 0$ for $x \in \mathbf{R}$ and $0 < y < b$, $u(x,0) = f(x)$, $u(x,b) = g(x)$. (Hint: First do the case $f = 0$. The case $g = 0$ reduces to this one by the substitution $y \to b - y$, and the general case is obtained by superposition. Exercise 10, §7.2 is useful.)

5. Let S be the infinite cylinder of radius a, given in cylindrical coordinates (r, θ, z) by the equation $r = a$. Find the electrostatic potential u inside S if the portion of S with $|z| < l$ is held at potential 1 and the rest of S is held at potential 0. (u is clearly independent of θ, so the problem to be solved is $u_{rr} + r^{-1}u_r + u_{zz} = 0$ inside S, $u(a,z) = 1$ if $|z| < l$ and $u(a,z) = 0$ otherwise. Use the Fourier transform in z, and express the answer as a Fourier integral.)

6. Suppose $f \in L^2$ represents a signal. Show that the best approximation to f in the L^2 norm among all signals that are band-limited to the interval $[-\Omega, \Omega]$ is $g_0(t) = (2\pi)^{-1} \int_{-\Omega}^{\Omega} \widehat{f}(\omega) e^{i\omega t}\, d\omega$. That is, show that $\|g_0 - f\| \le \|g - f\|$ for all g such that $\widehat{g}(\omega) = 0$ for $|\omega| > \Omega$. (Use the Plancherel theorem, and cf. Theorem 3.8 of §3.4.)

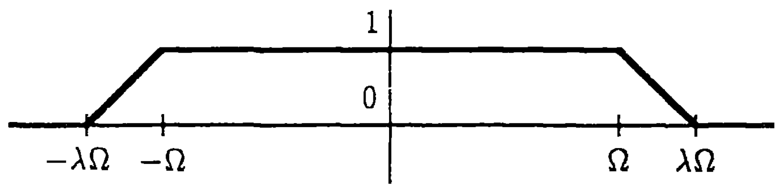
7. State and prove a version of the sampling theorem for signals whose Fourier transforms vanish outside an interval $[a,b]$. (A simple-minded answer to this problem is the following: If, say, $0 < a < b$, then $[a,b] \subset [-b,b]$, and one can apply the sampling theorem with $\Omega = b$. But this gives a formula for f in terms of its values at the points $n\pi/b$, whereas the optimal theorem involves the more widely spaced points $2n\pi/(b-a)$. Hint: If $\widehat{f}$ vanishes outside $[a,b]$, consider $g(t) = e^{-i(b-a)t/2} f(t)$.)

8. Suppose $f \in L^2(\mathbf{R})$, $\widehat{f}(\omega) = 0$ for $|\omega| > \Omega$, and $\lambda > 1$.
   a. As in the proof of the sampling theorem, show that

$$\widehat{f}(\omega) = \frac{\pi}{\lambda\Omega} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\lambda\Omega}\right) e^{-in\pi\omega/\lambda\Omega} \quad \text{for} \quad |\omega| \le \lambda\Omega.$$

b. Let $\widehat{g}_\lambda$ be the piecewise linear function sketched below. Show that the inverse Fourier transform of $\widehat{g}_\lambda$ is

$$g_\lambda(t) = \frac{\cos\Omega t - \cos\lambda\Omega t}{\pi(\lambda-1)\Omega t^2}.$$



c. Observe that $\widehat{f} = \widehat{g}_\lambda \widehat{f}$. By substituting the expansion in part (a) into the Fourier inversion formula, show that

$$f(t) = \frac{1}{2\pi}\int_{-\lambda\Omega}^{\lambda\Omega} \widehat{f}(\omega)\widehat{g}_\lambda(\omega)e^{i\omega t}\,d\omega = \frac{\pi}{\lambda\Omega}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\lambda\Omega}\right) g_\lambda\left(t - \frac{n\pi}{\lambda\Omega}\right).$$

This gives a sampling formula for f in which the basic functions $g_\lambda(t)$ decay like t^{-2} at infinity.

9. Suppose that f satisfies the hypotheses of Heisenberg's inequality, and let $F(x) = e^{-i\alpha x}f(x+a)$.
   a. Show that $\Delta_a f = \Delta_0 F$.
   b. Show that $\widehat{F}(\xi) = e^{ia(\xi+\alpha)}\widehat{f}(\xi+\alpha)$ and thence that $\Delta_\alpha \widehat{f} = \Delta_0 \widehat{F}$.
10. Show that Heisenberg's inequality $(\Delta_0 f)(\Delta_0 \widehat{f}) \geq \frac{1}{4}$ is an equality if and only if $f' + cf = 0$ where c is a real constant, and hence show that the functions that minimize the uncertainty product $(\Delta_0 f)(\Delta_0 \widehat{f})$ are precisely those of the form $f(x) = Ce^{-cx^2/2}$ for some $c > 0$. (Hint: Examine the proof of Heisenberg's inequality and recall that the Cauchy-Schwarz inequality $|\int fg| \leq \|f\|\,\|g\|$ is an equality if and only if f and g are scalar multiples of one another.) What are the minimizing functions for the uncertainty product $(\Delta_a f)(\Delta_\alpha \widehat{f})$ for general a, α? (Cf. Exercise 9.)

7.4 Fourier transforms and Sturm-Liouville problems

In §7.3 we solved some boundary value problems by applying the Fourier transform. The same results could have been obtained from a slightly different point of view, starting with separation of variables. For example, for functions $u(x,t) = X(x)T(t)$ the heat equation $u_t = ku_{xx}$ separates into the ordinary differential equations $T' = -\xi^2 kT$ and $X'' + \xi^2 X = 0$ where ξ^2 is the separation constant. Solution of these equations leads to the products $u(x,t) = e^{-\xi^2 kt}e^{i\xi x}$ and hence to their continuous superpositions

$$u(x,t) = \int c(\xi)e^{-\xi^2 kt}e^{i\xi x}\,d\xi.$$

If the initial condition is $u(x,0) = f(x)$, one sees from the Fourier inversion formula that $c(\xi) = (2\pi)^{-1}\widehat{f}(\xi)$, which leads to the solution (7.20).

What is at issue here is the singular Sturm-Liouville problem

$$X''(x) + \xi^2 X(x) = 0, \qquad -\infty < x < \infty.$$

The general solution of this equation is $c_1 e^{i\xi x} + c_2 e^{-i\xi x}$ for $\xi \neq 0$ or $c_1 + c_2 x$ for $\xi = 0$. None of these functions is in $L^2(\mathbf{R})$ except for the trivial case $c_1 = c_2 = 0$, so there is no possibility of finding an orthonormal basis of eigenfunctions. Instead, the expansion of an arbitrary $f \in L^2(\mathbf{R})$ in terms of these eigenfunctions is given by the Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{i\xi x}\,d\xi = \frac{1}{2\pi}\int_0^{\infty}\left[\widehat{f}(\xi)e^{i\xi x} + \widehat{f}(-\xi)e^{-i\xi x}\right]d\xi \tag{7.26}$$

(with the integral suitably interpreted).

The reader may wonder what justification we have for restricting attention to real values of ξ in this situation. The practical answer is that (7.26) works, so no nonreal values of ξ are needed. A rather vague but more satisfying reason, which applies also to other problems of this sort, is the following. When $\operatorname{Im}\xi \neq 0$, $e^{i\xi x}$ blows up exponentially as $x \to \infty$ or $x \to -\infty$, so it fails so miserably to be in L^2 that it cannot be of any pertinence to an L^2 problem. But when ξ is real, $e^{i\xi x}$ is close enough to being in L^2 that it can contribute to an eigenfunction expansion by an infinitesimal amount, as in (7.26).

Let us now consider two singular Sturm-Liouville problems pertaining to functions on the half-line $[0,\infty)$:

$$X''(x) + \xi^2 X(x) = 0 \quad (0 < x < \infty), \qquad X'(0) = 0; \tag{7.27a}$$

$$X''(x) + \xi^2 X(x) = 0 \quad (0 < x < \infty), \qquad X(0) = 0. \tag{7.27b}$$

In (7.27a) the solutions that satisfy the boundary condition are multiples of $\cos\xi x$, whereas in (7.27b) they are multiples of $\sin\xi x$. Again, none of these functions are in $L^2(0,\infty)$, so there is no orthonormal basis of eigenfunctions. Instead, we can hope to find expansion formulas similar to (7.26), namely,

$$f(x) = \int_0^{\infty} a(\xi)\cos\xi x\,d\xi, \qquad f(x) = \int_0^{\infty} b(\xi)\sin\xi x\,d\xi$$

for $f \in L^2(0,\infty)$. In fact, such formulas can easily be derived from the Fourier transform by the same device by which we obtained Fourier sine and cosine series on $[0,\pi]$ from Fourier series on $[-\pi,\pi]$, namely, consideration of the even and odd extensions of f to $\mathbf{R}$.

Indeed, if $f \in L^1(\mathbf{R})$ and f is even, then

$$\widehat{f}(\xi) = \int f(x)(\cos\xi x - i\sin\xi x)\,dx = \int f(x)\cos\xi x\,dx = 2\int_0^{\infty} f(x)\cos\xi x\,dx.$$

From this it is clear that $\widehat{f}$ is also even, so the inversion formula (suitably interpreted as a limit as in Theorem 7.6) becomes

$$f(x) = \frac{1}{2\pi}\int \widehat{f}(\xi)(\cos\xi x + i\sin\xi x)\,d\xi = \frac{1}{\pi}\int_0^\infty \widehat{f}(\xi)\cos\xi x\,d\xi.$$

In the same way, we see that if f is odd, then so is $\widehat{f}$, and

$$\widehat{f}(\xi) = -2i\int_0^\infty f(x)\sin\xi x\,dx, \qquad f(x) = \frac{i}{\pi}\int_0^\infty \widehat{f}(\xi)\sin\xi x\,d\xi.$$

These formulas involve only the values of f and $\widehat{f}$ on $[0,\infty)$, so we can use them on functions that are initially defined only on $[0,\infty)$. This suggests the following definitions.

Suppose now that $f \in L^1(0,\infty)$. We define the **Fourier cosine transform** and **Fourier sine transform** of f to be the functions $\mathscr{F}_c[f]$ and $\mathscr{F}_s[f]$ on $[0,\infty)$ defined by

$$\mathscr{F}_c[f](\xi) = \int_0^\infty f(x)\cos\xi x\,dx \quad\text{and}\quad \mathscr{F}_s[f](\xi) = \int_0^\infty f(x)\sin\xi x\,dx.$$

Thus, if f_{even} and f_{odd} are the even and odd extensions of f to $\mathbf{R}$, $\mathscr{F}_c[f]$ and $\mathscr{F}_s[f]$ are the restrictions to $[0,\infty)$ of $\frac{1}{2}\widehat{f}_{\text{even}}$ and $\frac{1}{2}i\widehat{f}_{\text{odd}}$. The inversion formulas therefore become

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathscr{F}_c[f](\xi)\cos\xi x\,d\xi = \frac{2}{\pi}\int_0^\infty \mathscr{F}_s[f](\xi)\sin\xi x\,d\xi, \tag{7.28}$$

giving the desired expansions of f in terms of cosines and sines. Here, of course, the integrals must be interpreted suitably. For example, if f is piecewise continuous we have

$$\lim_{\epsilon\to 0}\frac{2}{\pi}\int_0^\infty e^{-\epsilon^2\xi^2/2}\mathscr{F}_c[f](\xi)\cos\xi x\,d\xi = \tfrac{1}{2}\big[f(x-) + f(x+)\big].$$

The Parseval formula for $\mathscr{F}_c$ is obtained as follows:

$$\begin{aligned}\int_0^\infty |\mathscr{F}_c[f](\xi)|^2\,d\xi &= \frac{1}{4}\int_0^\infty |\widehat{f}_{\text{even}}(\xi)|^2\,d\xi = \frac{1}{8}\int_{-\infty}^\infty |\widehat{f}_{\text{even}}(\xi)|^2\,d\xi \\ &= \frac{\pi}{4}\int_{-\infty}^\infty |f_{\text{even}}(x)|^2\,dx = \frac{\pi}{2}\int_0^\infty |f(x)|^2\,dx,\end{aligned}$$

and similarly for $\mathscr{F}_s$. From this one obtains the analogue of the Plancherel theorem: *$\mathscr{F}_c$ and $\mathscr{F}_s$ extend to maps from $L^2(0,\infty)$ onto itself that satisfy*

$$\|\mathscr{F}_c[f]\|^2 = \|\mathscr{F}_s[f]\|^2 = \frac{\pi}{2}\|f\|^2.$$

This fact, in conjunction with (7.28), gives the eigenfunction expansions for $L^2(0,\infty)$ associated to the Sturm-Liouville problems (7.27).

$\mathscr{F}_c$ and $\mathscr{F}_s$ have operational properties similar to the ones for the ordinary Fourier transform given in Theorem 7.5, but they are not quite as simple. For example, here is how $\mathscr{F}_c$ interacts with convolutions. Suppose f and g are (say) bounded and integrable on $[0,\infty)$, and let F and G be their even extensions to $\mathbf{R}$. It is easily verified that $F * G$ is also even, so the convolution formula $(F * G)\widehat{} = \widehat{f}\widehat{G}$ turns into $\mathscr{F}_c[h] = \mathscr{F}_c[f] \cdot \mathscr{F}_c[g]$ where $2h$ is the restriction of $F * G$ to $[0,\infty)$. (The factor of 2 is there because $\mathscr{F}_c[f]$ is the restriction of $\frac{1}{2}\widehat{f}$, rather than $\widehat{f}$, to $[0,\infty)$.) We can evaluate h directly in terms of f and g, as follows: We have

$$\begin{aligned} F * G(x) &= \int_{-\infty}^{0} F(y)G(x-y)\,dy + \int_{0}^{\infty} F(y)G(x-y)\,dy \\ &= \int_{0}^{\infty} F(y)G(x+y)\,dy + \int_{0}^{\infty} F(y)G(|x-y|)\,dy, \end{aligned}$$

where we have substituted $-y$ for y in the first integral and used the evenness of F and G. When $x \geq 0$, the arguments y, $x+y$, and $|x-y|$ are all positive, so in this last expression we can replace F and G by f and g. In short, we have

$$\mathscr{F}_c[f] \cdot \mathscr{F}_c[g] = \mathscr{F}_c[h], \qquad h(x) = \frac{1}{2}\int_{0}^{\infty} f(y)\Big[g(x+y) + g(|x-y|)\Big]\,dy. \quad (7.29)$$

Similar formulas for $\mathscr{F}_c[f] \cdot \mathscr{F}_s[g]$ and $\mathscr{F}_s[f] \cdot \mathscr{F}_s[g]$ exist; see Exercise 2. Also, see Exercise 3 for the interaction of $\mathscr{F}_c$ and $\mathscr{F}_s$ with derivatives.

Example. Consider heat flow in a semi-infinite rod insulated along its length and at the end:

$$u_t = ku_{xx} \quad \text{for } x, t > 0, \qquad u_x(0,t) = 0, \qquad u(x,0) = f(x).$$

Separation of variables in the differential equation together with the boundary condition $u_x(0,t) = 0$ leads to the product solutions $e^{-\xi^2 kt}\cos\xi x$ and hence to their superpositions

$$u(x,t) = \int_{0}^{\infty} c(\xi)e^{-\xi^2 kt}\cos\xi x\,d\xi.$$

Setting $t = 0$ and applying (7.28), we see that $c(\xi) = (2/\pi)\mathscr{F}_c[f](\xi)$, or

$$u(x,t) = \frac{2}{\pi}\int_{0}^{\infty} \mathscr{F}_c[f](\xi)e^{\xi^2 kt}\cos\xi x\,d\xi.$$

This is the **Fourier integral** formula for the solution. A formula that gives the solution directly in terms of f instead of $\mathscr{F}_c[f]$ may be derived from it as follows. By a simple modification of (7.12),

$$e^{-\nu^2 kt} = \mathscr{F}_c[g_t](\nu) \quad \text{where } g_t(x) = \frac{1}{\sqrt{\pi kt}}e^{-x^2/4kt},$$

so by (7.29),

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_0^\infty f(y)\Big[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\Big]\,dy.$$

Of course this is nothing but the solution $u(x,t) = F * K_t(x)$ derived in §7.3, where F is the even extension of f to $\mathbf{R}$. The reader should take a minute or two to figure out why the latter construction gives the right answer.

We have shown how to derive Fourier cosine and sine transforms from the ordinary Fourier transform; one can also go the other way. Indeed, any function f on $\mathbf{R}$ is the sum of an even function and an odd function:

$$f = f_0 + f_1 \quad \text{where } f_0(x) = \frac{f(x)+f(-x)}{2},\ f_1(x) = \frac{f(x)-f(-x)}{2}.$$

Since the even and odd parts of $e^{-i\xi x}$ are $\cos\xi x$ and $-i\sin\xi x$, we then have

$$\widehat{f}(\xi) = \int_{-\infty}^\infty f_0(x)\cos\xi x\,dx - i\int_\infty^\infty f_1(x)\sin\xi x\,dx = 2\mathscr{F}_c[f_0](\xi) - 2i\mathscr{F}_s[f_1](\xi).$$

This observation is helpful for using tables of Fourier transforms such as Erdélyi et al. [22], where most of the entries are given in terms of cosine and sine transforms.

EXERCISES

1. Compute the following transforms, where $k > 0$.

   a. $\mathscr{F}_s[e^{-kx}]$ b. $\mathscr{F}_c[e^{-kx}]$ c. $\mathscr{F}_c\Big[(1+x)e^{-x}\Big]$ d. $\mathscr{F}_s[xe^{-x}]$

2. Let f and g be in $L^1(0,\infty)$. Show that $\mathscr{F}_s[f]\mathscr{F}_c[g] = \mathscr{F}_s[h]$ where

   $$h(x) = \int_0^\infty f(y)\frac{g(|x-y|) - g(x+y)}{2}\,dy,$$

   and that $\mathscr{F}_s[f]\mathscr{F}_s[g] = \mathscr{F}_c[H]$ where

   $$H(x) = \int_0^\infty f(y)\frac{\operatorname{sgn}(x-y)g(|x-y|) - g(x+y)}{2}\,dy,$$

   with $\operatorname{sgn}(t) = 1$ if $t > 0$ and $\operatorname{sgn}(t) = -1$ if $t < 0$.
3. Suppose that f is continuous and piecewise smooth and that f and f' are in $L^1(0,\infty)$. Show that

   $$\mathscr{F}_c[f'](\xi) = \xi\mathscr{F}_s[f](\xi) - f(0), \qquad \mathscr{F}_s[f'](\xi) = -\xi\mathscr{F}_c[f](\xi).$$

4. Solve the heat equation $u_t = ku_{xx}$ on the half-line $x > 0$ with boundary conditions $u(x,0) = f(x)$ and $u(0,t) = 0$. (Exercise 2 is useful.) Then solve

the inhomogeneous equation $u_t = ku_{xx} + G(x,t)$ with the same boundary conditions. (Cf. Exercise 1, §7.3.)

5. Solve the Dirichlet problem in the first quadrant: $u_{xx} + u_{yy} = 0$ for $x, y > 0$, $u(x,0) = f(x)$, $u(0,y) = g(y)$. (Hint: Consider the special cases $f = 0$ and $g = 0$. Use Exercise 2.)
6. Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $x > 0$, $0 < y < 1$ with boundary conditions $u_x(0,y) = 0$, $u_y(x,0) = 0$, $u(x,1) = e^{-x}$. Express the answer as a Fourier integral. (Use Exercise 1.)
7. Do Exercise 6 with the first two boundary conditions replaced by $u(0,y) = 0$ and $u(x,0) = 0$.
8. Find the steady-state temperature in a plate occupying the semi-infinite strip $x > 0$, $0 < y < 1$ if the edges $y = 0$ and $x = 0$ are insulated, the edge $y = 1$ is maintained at temperature 1 for $x < c$ and at temperature zero for $x > c$, and the faces of the plate lose heat to the surroundings according to Newton's law with proportionality constant h. That is, solve

$$\begin{gathered} u_{xx} + u_{yy} - hu = 0, \quad u_x(0,y) = u_y(x,0) = 0, \\ u(x,1) = 1 \text{ if } x < c, \quad u(x,1) = 0 \text{ if } x \geq c. \end{gathered}$$

Express the answer as a Fourier integral.

9. Consider the singular Sturm-Liouville problem

$$\left(rf'(r)\right)' + \lambda r^{-1} f(r) = 0 \quad \text{for } 0 < r < 1, \qquad f(1) = 0. \tag{$*$}$$

   a. Show that the substitution $r = e^{-x}$, $g(x) = f(e^{-x})$ converts $(*)$ into the problem $g'' + \lambda g = 0$ for $0 < x < \infty$, $g(0) = 0$. Put this together with the Fourier sine transform to derive the "eigenfunction expansion" of a function $f \in L^2_{1/r}(0,1)$ associated to $(*)$.
   b. Use the result of (a) to solve Exercise 8b in §4.4:

$$\begin{gathered} \nabla^2 u = 0 \quad \text{in } \left\{(r,\theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \beta\right\}, \\ u(r,0) = f(r), \qquad u(r,\beta) = g(r), \qquad u(1,\theta) = 0. \end{gathered}$$

7.5 Multivariable convolutions and Fourier transforms

In this section we consider functions of n real variables, that is, functions on the space $\mathbf{R}^n$ of n-tuples of real numbers. The notation for points in $\mathbf{R}^n$ will be $\mathbf{x} = (x_1, \ldots, x_n)$. We denote by $\mathbf{x} \cdot \mathbf{y}$ and $|\mathbf{x}|$ the usual dot product and norm on $\mathbf{R}^n$:

$$\begin{gathered} \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\ |\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}. \end{gathered}$$

Also, an integral sign with no limits will denote integration over all of n-space:

$$\int f(\mathbf{x})\,d\mathbf{x} \quad \text{means} \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n.$$

Most of the ideas of §§7.1–2 can be generalized in a straightforward way to functions of several variables.* We proceed to sketch the extensions of our previous results to functions on $\mathbf{R}^n$, providing details only where new ideas are required.

Convolutions are defined just as before:

$$f * g(\mathbf{x}) = \int f(\mathbf{y}) g(\mathbf{x} - \mathbf{y})\,d\mathbf{y}.$$

The basic algebraic properties of convolution stated in Theorem 7.1 and the differentiation property of Theorem 7.2 (with ordinary derivatives replaced by partial derivatives) remain valid in the n-variable case, with the same proofs:

$$f * (ag + bh) = a(f * g) + b(f * h), \qquad f * g = g * f, \qquad (f * g) * h = f * (g * h),$$
$$\partial_j (f * g) = (\partial_j f) * g = f * (\partial_j g) \qquad (\partial_j = \partial / \partial x_j).$$

Since

$$d(r\mathbf{x}) = (r\,dx_1) \cdots (r\,dx_n) = r^n d\mathbf{x} \qquad (r > 0),$$

the appropriate analogue of the dilation formula (7.3) is

$$g_\epsilon(\mathbf{x}) = \epsilon^{-n} g(\epsilon^{-1}\mathbf{x}). \tag{7.30}$$

The factor ϵ^{-n} is the right one to ensure that the integral of g_ϵ is independent of ϵ:

$$\int g_\epsilon(\mathbf{x})\,d\mathbf{x} = \int g(\epsilon^{-1}\mathbf{x})\,d(\epsilon^{-1}\mathbf{x}) = \int g(\mathbf{y})\,d\mathbf{y}.$$

The notion of one-sided limits does not make sense for functions of several variables, but we still have the following analogue of Theorems 7.3 and 7.4.

Theorem 7.7. *Suppose $g \in L^1$ and $\int g(\mathbf{x})\,d\mathbf{x} = 1$, and let g_ϵ be defined by (7.30).*

(a) Suppose that either f is bounded or g vanishes outside a bounded set, so that $f * g$ is well-defined. If f is continuous at $\mathbf{x}$, then

$$\lim_{\epsilon \to 0} f * g_\epsilon(\mathbf{x}) = f(\mathbf{x}).$$

If f is continuous on a closed, bounded set D, the convergence is uniform on D.

(b) If $f \in L^2$, then

$$\lim_{\delta \to 0} \|f * g_\epsilon - f\| = 0.$$

---

* An exception The notions of one-sided limits, piecewise continuity, and piecewise smoothness have no obvious counterparts for functions of more than one variable. We shall not concern ourselves with minimal smoothness hypotheses for functions of several variables, which require a more sophisticated theory

Proof: The proof of part (a) is essentially the same as the proof of Theorem 7.3: We write

$$f * g_\epsilon(\mathbf{x}) - f(\mathbf{x}) = \int \Big[f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})\Big] g_\epsilon(\mathbf{y})\, d\mathbf{y}$$

and estimate the integral over the regions $|\mathbf{y}| < c$ and $|\mathbf{y}| > c$ separately. The integral over $|\mathbf{y}| < c$ is small for c near 0 because $f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})$ is small there and the integral of $|g_\epsilon|$ is bounded by a constant independent of c; the integral over $|y| > c$ tends to zero along with ϵ because g_ϵ concentrates at zero as $\epsilon \to 0$. We leave the details to the reader. (The proof of part (b) is also the same as the proof of Theorem 7.4 — which we omitted.) ∎

The Fourier transform of an integrable function f on $\mathbf{R}^n$ is defined by

$$\widehat{f}(\boldsymbol{\xi}) = \mathscr{F}\Big[f(\mathbf{x})\Big] = \int e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x})\, d\mathbf{x}.$$

The estimate $|\widehat{f}(\boldsymbol{\xi})| \le \int |f(\mathbf{x})|\, d\mathbf{x}$ and the fact that $\widehat{f}(\boldsymbol{\xi})$ is continuous and tends to zero as $|\boldsymbol{\xi}| \to \infty$ are still valid. The basic transformational properties of the n-dimensional Fourier transform are just as in Theorem 7.5, with one new feature.

Theorem 7.8. *Suppose $f \in L^1(\mathbf{R}^n)$.*
(a) For any $\mathbf{a} \in \mathbf{R}^n$,

$$\mathscr{F}\Big[f(\mathbf{x}-\mathbf{a})\Big] = e^{-i\mathbf{a}\cdot\boldsymbol{\xi}}\widehat{f}(\boldsymbol{\xi}) \quad \textit{and} \quad \mathscr{F}\Big[e^{i\mathbf{a}\cdot\mathbf{x}} f(\mathbf{x})\Big] = \widehat{f}(\boldsymbol{\xi}-\mathbf{a}).$$

(b) If $\delta > 0$ and $f_\delta(\mathbf{x}) = \delta^{-n} f(\delta^{-1}\mathbf{x})$, then

$$[f_\delta]\widehat{\ }(\boldsymbol{\xi}) = \widehat{f}(\delta\boldsymbol{\xi}) \quad \textit{and} \quad \mathscr{F}\Big[f(\delta\mathbf{x})\Big] = (\widehat{f})_\delta(\boldsymbol{\xi}).$$

(c) If $\partial f/\partial x_j$ exists and is in L^1, then

$$[\partial f/\partial x_j]\widehat{\ }(\boldsymbol{\xi}) = i\xi_j \widehat{f}(\boldsymbol{\xi}),$$

whereas if $x_j f(\mathbf{x})$ is integrable, then

$$\mathscr{F}\Big[x_j f(\mathbf{x})\Big] = i\partial\widehat{f}/\partial\xi_j.$$

(d) If $g \in L^1$ and $f * g \in L^1$, then

$$(f * g)\widehat{\ } = \widehat{f}\widehat{g}.$$

(e) The Fourier transform commutes with rotations: If R is a rotation of $\mathbf{R}^n$, then

$$\mathscr{F}\Big[f(R\mathbf{x})\Big] = \widehat{f}(R\boldsymbol{\xi}).$$

Proof: The proof of (a)–(d) is just as in the one-variable case. As for (e), what we need is the fact that rotations preserve dot products and volumes, that is, $\boldsymbol{\xi} \cdot \mathbf{x} = R\boldsymbol{\xi} \cdot R\mathbf{x}$ and $d(R\mathbf{x}) = d\mathbf{x}$:

$$\begin{aligned}\mathscr{F}\left[f(R\mathbf{x})\right] &= \int e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} f(R\mathbf{x})\, d\mathbf{x} = \int e^{-iR\boldsymbol{\xi}\cdot R\mathbf{x}} f(R\mathbf{x})\, d(R\mathbf{x}) \\ &= \int e^{-iR\boldsymbol{\xi}\cdot\mathbf{y}} f(\mathbf{y})\, d\mathbf{y} = \widehat{f}(R\boldsymbol{\xi}).\end{aligned}$$

∎

Another useful and elementary fact is the following. Suppose f is a product of functions of the individual variables:

$$f(\mathbf{x}) = f_1(x_1)f_2(x_2)\cdots f_n(x_n).$$

Then the Fourier transform of f is the corresponding product of one-dimensional Fourier transforms:

$$\widehat{f}(\boldsymbol{\xi}) = \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\cdots\widehat{f}_n(\xi_n).$$

This is so because $e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} = (e^{-i\xi_1x_1})\cdots(e^{-i\xi_nx_n})$, so the n-dimensional integral defining $\widehat{f}$ decomposes into a product of one-dimensional integrals. An important example is the Gaussian $f(\mathbf{x}) = e^{-a|\mathbf{x}|^2/2}$ $(a > 0)$, which is the product of the functions $f_j(x_j) = e^{-ax_j^2/2}$. In view of formula (7.12), we have

$$\mathscr{F}\left[e^{-a|\mathbf{x}|^2/2}\right] = \left(\frac{2\pi}{a}\right)^{n/2} e^{-|\boldsymbol{\xi}|^2/2a} \qquad (a > 0). \tag{7.31}$$

With this fact in hand, the arguments leading up to the formulas (7.15) and (7.16) give the following n-dimensional Fourier inversion formula:

The Fourier Inversion Theorem. *If f is integrable and continuous on $\mathbf{R}^n$, then*

$$f(\mathbf{x}) = \lim_{\epsilon\to 0} \frac{1}{(2\pi)^n} \int e^{i\boldsymbol{\xi}\cdot\mathbf{x}} e^{-\epsilon^2|\boldsymbol{\xi}|^2/2} \widehat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \qquad \mathbf{x} \in \mathbf{R}^n. \tag{7.32}$$

If also $\widehat{f}$ is integrable, then

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int e^{i\boldsymbol{\xi}\cdot\mathbf{x}} \widehat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{7.33}$$

As in the one-dimensional case, if f is merely in L^1, formula (7.32) remains valid for almost every x, and the cutoff function $e^{-\epsilon^2\xi^2/2}$ can be replaced by a number of others. However, The n-dimensional analogues of Theorem 7.6, in which the Fourier inversion integral is interpreted as the pointwise limit of integrals over a family of bounded regions Ω_r that expand to fill out $\mathbf{R}^n$ as $r \to \infty$, are rather delicate and not terribly useful, and we shall not attempt to discuss them here.

The Plancherel theorem also remains true in the n-dimensional setting, with the same proof, except that the factor of 2π must be replaced by $(2\pi)^n$:

$$\langle \widehat{f}, \widehat{g}\rangle = (2\pi)^n \langle f, g\rangle, \qquad \|\widehat{f}\|^2 = (2\pi)^n \|f\|^2.$$

As before, the Fourier transform and inverse Fourier transform of L^2 functions can be calculated by limiting processes like (7.32).

Applications

The general discussion of linear operators that commute with translations at the beginning of §7.3 is valid in any number of dimensions. In particular, the Fourier transform converts any constant-coefficient differential operator into multiplication by a polynomial. For example, if

$$L[u] = au + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + \sum_{j,k=1}^{n} c_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k}$$

then

$$\big(L[u]\big)\widehat{\ }(\boldsymbol{\xi}) = P(\boldsymbol{\xi})\widehat{u}(\boldsymbol{\xi}) \quad \text{where } P(\boldsymbol{\xi}) = a + i\sum_{j=1}^{n} b_j \xi_j - \sum_{j\kappa=1}^{n} c_{jk}\xi_j\xi_k.$$

Formally, then, we can solve the inhomogeneous equation $L[u] = f$ on $\mathbf{R}^n$ by taking $\widehat{u} = \widehat{f}/P$. If $P(\boldsymbol{\xi})$ is never zero, this leads to a Fourier integral formula for u,

$$u(\mathbf{x}) = \frac{1}{(2\pi)^n} \int \frac{\widehat{f}(\boldsymbol{\xi})}{P(\boldsymbol{\xi})} e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\, d\boldsymbol{\xi}, \tag{7.34}$$

or to the convolution formula $u = f * K$ where K is the inverse Fourier transform of $1/P$. If P has zeros, the situation is technically more complicated since one has to worry about the integrability of $\widehat{f}/P$, but the same ideas work in principle. We shall discuss the examples of the Laplace, heat, and wave operators in §10.2; see also Folland [24], §1F. (Note, in any event, that (7.34) is not the only solution of the equation $L[u] = f$. In general there will be many others, but most or all of them grow rapidly at infinity so that they cannot be represented in terms of Fourier transforms.)

We can solve the initial value problem for the heat equation in $\mathbf{R}^n$,

$$u_t = k\nabla^2 u, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}),$$

by taking the Fourier transform in $\mathbf{x}$ just as before. In view of (7.31), the result is

$$u(\mathbf{x}, t) = f * K_t(\mathbf{x}), \qquad K_t(\mathbf{x}) = (4\pi k t)^{-n/2} e^{-|\mathbf{x}|^2/4kt}, \tag{7.35}$$

which has the same physical interpretation as before. Similarly, to solve the Dirichlet problem in a half-space

$$H = \Big\{(x_1, \dots, x_{n+1}) : x_{n+1} > 0\Big\}$$

in $\mathbf{R}^{n+1}$, we adopt the notation $y = x_{n+1}$, $\mathbf{x} = (x_1, \dots, x_n)$, so that the problem is

$$\nabla^2 u = \nabla_{\mathbf{x}}^2 u + u_{yy} = 0 \text{ in } H, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}).$$

Taking the Fourier transform in $\mathbf{x}$ and imposing the requirement of boundedness, we obtain $\widehat{u}(\boldsymbol{\xi}, t) = \widehat{f}(\boldsymbol{\xi})e^{-|\boldsymbol{\xi}|y}$ as before, and hence $u(\mathbf{x}, y) = f * P_y(\mathbf{x})$ where $\widehat{P}_y(\boldsymbol{\xi}) = e^{-|\boldsymbol{\xi}|y}$. The calculation of the inverse Fourier transform of $e^{-|\boldsymbol{\xi}|y}$ is trickier than in the 1-dimensional case, but it is still feasible; we leave it to the reader as Exercise 2. The result is the n-dimensional Poisson integral formula:

$$u(\mathbf{x}, y) = f * P_y(\mathbf{x}), \qquad P_y(\mathbf{x}) = \frac{\Gamma\big((n+1)/2\big)}{\pi^{(n+1)/2}} \frac{y}{\big(|\mathbf{x}|^2 + y^2\big)^{(n+1)/2}}.$$

If we try to solve the initial value problem for the wave equation

$$u_{tt} = c^2\nabla^2 u, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}), \qquad u_t(\mathbf{x}, 0) = g(\mathbf{x})$$

by taking the Fourier transform in $\mathbf{x}$, we obtain the ordinary differential equation

$$\widehat{u}_{tt}(\boldsymbol{\xi}, t) = -c^2|\boldsymbol{\xi}|^2\widehat{u}(\boldsymbol{\xi}, t), \qquad \widehat{u}(\boldsymbol{\xi}, 0) = \widehat{f}(\boldsymbol{\xi}), \qquad \widehat{u}_t(\boldsymbol{\xi}, 0) = \widehat{g}(\boldsymbol{\xi}),$$

the solution to which is

$$\widehat{u}(\boldsymbol{\xi}, t) = \widehat{f}(\boldsymbol{\xi}) \cos ct|\boldsymbol{\xi}| + \widehat{g}(\boldsymbol{\xi}) \frac{\sin ct|\boldsymbol{\xi}|t}{c|\boldsymbol{\xi}|}.$$

Here the inversion of the Fourier transform requires the theory of generalized functions, which we shall discuss in Chapter 9. In §9.5 we shall obtain the formula for u when $n = 2$ and $n = 3$, the most important cases for physics. (The formula gets progressively more complicated as n increases.)

The n-dimensional version of Heisenberg's inequality is as follows. We define the dispersion of f about the point $\mathbf{a} \in \mathbf{R}^n$ to be the vector $\Delta_{\mathbf{a}} f$ whose jth component is

$$(\Delta_{\mathbf{a}} f)_j = \int (x_j - a_j)^2 |f(\mathbf{x})|^2 d\mathbf{x} \Big/ \int |f(\mathbf{x})|^2\, d\mathbf{x}.$$

Then Heisenberg's inequality says that $(\Delta_{\mathbf{a}} f)_j(\Delta_{\alpha}\widehat{f})_j \geq \frac{1}{4}$ for each j and each $\mathbf{a}, \alpha \in \mathbf{R}^n$. The proof is exactly the same as in the 1-dimensional case. Quantum-mechanically, this means that there is an uncertainty inequality for each of the n components of the position and momentum vectors.

Fourier transforms of radial functions

The fact that the Fourier transform commutes with rotations (Theorem 7.8(e)) has the following interesting consequence. A function f on $\mathbf{R}^n$ is called **radial** if $f(R\mathbf{x}) = f(\mathbf{x})$ for all rotations R, that is, if $f(\mathbf{x})$ depends only on $|\mathbf{x}|$. Theorem 7.8(e) implies that if f is radial — say, $f(\mathbf{x}) = f_0(|\mathbf{x}|)$ — then so is $\widehat{f}$ — say, $\widehat{f}(\boldsymbol{\xi}) = \widetilde{f}_0(|\boldsymbol{\xi}|)$. In this case the integral formula relating f and $\widehat{f}$ can be rewritten in polar coordinates to yield $\widetilde{f}_0$ directly in terms of f_0. The two-dimensional case works as follows: With $\mathbf{x} = (r\cos\theta, r\sin\theta)$ and $\boldsymbol{\xi} = (\rho\cos\phi, \rho\sin\phi)$, we have $\mathbf{x}\cdot\boldsymbol{\xi} = r\rho\cos(\theta-\phi)$, and hence

$$\widehat{f}(\boldsymbol{\xi}) = \int f(\mathbf{x})e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\,d\mathbf{x} = \int_0^\infty \int_0^{2\pi} f_0(r)e^{-ir\rho\cos(\theta-\phi)}r\,d\theta\,dr.$$

By the substitution $\theta \to \theta + \phi + \frac{1}{2}\pi$ and Bessel's integral formula (5.23) we see that the θ-integral is independent of ϕ and equals

$$\int_0^{2\pi} e^{ir\rho\sin\theta}\,d\theta = 2\pi J_0(r\rho).$$

In other words,

$$\widetilde{f}_0(\rho) = 2\pi\int_0^\infty f_0(r)J_0(r\rho)r\,dr \qquad (n=2). \tag{7.36}$$

The integral on the right (without the factor of 2π) is called the **Hankel transform of order zero** of f_0. The Hankel transform and its relatives are explored in Exercises 7–9. The 3-dimensional version of (7.36) is simpler:

$$\widetilde{f}_0(\rho) = \frac{4\pi}{\rho}\int_0^\infty f_0(r)r\sin r\rho\,dr \qquad (n=3). \tag{7.37}$$

This can be proved as an elementary exercise in spherical coordinates (Exercise 3). The formula for general n turns out to be

$$\widetilde{f}_0(\rho) = (2\pi)^{n/2}\int_0^\infty f_0(r)(r\rho)^{1-(n/2)}J_{(n/2)-1}(r\rho)r^{n-1}\,dr. \tag{7.38}$$

We leave the verification of this as an exercise for the ambitious reader (Exercise 4). Observe that when $n=3$, (7.38) agrees with (7.37) since $J_{1/2}(x) = (2/\pi x)^{1/2}\sin x$. A complete discussion of the Fourier transform in n-dimensional spherical coordinates can be found in Stein-Weiss [49], Chapter 4.

EXERCISES

1. Find a solution of the following differential equations in $\mathbf{R}^2$ in terms of Fourier integrals.
   a. $u_{xx} + 2u_{yy} + 3u_x - 4u = f$.
   b. $u_{xxxx} - u_{yy} + 2u = f$.
2. The purpose of this exercise is to show that for $\mathbf{x}\in\mathbf{R}^n$ and $y>0$,

$$\frac{1}{(2\pi)^n}\int e^{i\boldsymbol{\xi}\cdot\mathbf{x}}e^{-y|\boldsymbol{\xi}|}d\boldsymbol{\xi} = \frac{\Gamma\big((n+1)/2\big)}{\pi^{(n+1)/2}}\frac{y}{\big(|\mathbf{x}|^2+y^2\big)^{(n+1)/2}}. \tag{$*$}$$

a. Combining the formula $(t^2+1)^{-1} = \int_0^\infty e^{-(t^2+1)s}ds$ with (7.13), we have

$$e^{-b} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-ibt}}{t^2+1}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{-ibt}e^{-t^2 s}e^{-s}\,ds\,dt \qquad (b>0).$$

Reverse the order of integration and apply (7.12) to obtain

$$e^{-b} = \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}}e^{-b^2/4s}\,ds \qquad (b>0).$$

b. Now set $b = y|\boldsymbol{\xi}|$ and plug into the left side of (*), obtaining

$$\frac{1}{(2\pi)^n}\int e^{i\boldsymbol{\xi}\cdot\mathbf{x}}e^{-y|\boldsymbol{\xi}|}\,d\boldsymbol{\xi} = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_0^\infty e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\frac{e^{-s}}{\sqrt{\pi s}}e^{-y^2|\boldsymbol{\xi}|^2/4s}\,ds\,d\boldsymbol{\xi}.$$

Reverse the order of integration and apply (7.31) to obtain the right side of (*).

3. Prove (7.37). (Hint: $\widetilde{f}_0(\rho) = \widehat{f}(\boldsymbol{\xi})$ where $\boldsymbol{\xi} = (0,0,\rho)$. Recall that the volume element in spherical coordinates (r,θ,ϕ) is $r^2\sin\phi\,dr\,d\theta\,d\phi$ where ϕ is the angle from the z-axis.)
4. (For those with experience in n-dimensional integration) Prove formula (7.38). (Hint: Reduce the problem to showing that

$$\int_S e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\,d\sigma(\mathbf{x}) = (2\pi)^{n/2}\rho^{1-(n/2)}J_{(n/2)-1}(\rho)$$

where $\boldsymbol{\xi} = (\rho,0,\dots,0)$, S is the unit sphere in $\mathbf{R}^n$, and $d\sigma$ is the element of surface measure on S. Then show that this formula follows from Exercise 14, §5.2.)
5. Use (7.37) to find the Fourier transform of $|\mathbf{x}|^{-1}e^{-|\mathbf{x}|}$ on $\mathbf{R}^3$. Then use this result to find a solution of the equation $u - \nabla^2 u = f$ on $\mathbf{R}^3$ in the form $u = f * K$. (Assume at first that $f \in L^1(\mathbf{R}^3)$ so that the Fourier transform can be applied, but then show that the solution exists and is a bounded function whenever f is a bounded function.)
6. On $\mathbf{R}^n$, let $f(\mathbf{x}) = 1$ if $|\mathbf{x}| < 1$, $f(\mathbf{x}) = 0$ otherwise. If $\boldsymbol{\xi} \in \mathbf{R}^n$ and $|\boldsymbol{\xi}| = \rho$, show that:
   a. if $n = 2$, $\widehat{f}(\boldsymbol{\xi}) = 2\pi\rho^{-1}J_1(\rho)$;
   b. if $n = 3$, $\widehat{f}(\boldsymbol{\xi}) = 4\pi\rho^{-3}(\sin\rho - \rho\cos\rho)$;
   c. for general n, $\widehat{f}(\boldsymbol{\xi}) = (2\pi)^{n/2}\rho^{-n/2}J_{n/2}(\rho)$.
7. Let us identify $\mathbf{R}^2$ with $\mathbf{C}$ and write points in $\mathbf{R}^2$ in polar coordinates as $re^{i\theta}$. Suppose $f \in L^1(\mathbf{R}^2)$ is of the form $f(re^{i\theta}) = f_0(r)e^{ik\theta}$ for some integer k. Show that $\widehat{f}(\rho e^{i\phi}) = 2\pi i^k g_0(\rho)e^{ik\phi}$ where

$$g_0(\rho) = (\mathscr{H}_k f_0)(\rho) = \int_0^\infty f_0(r)J_k(\rho r)r\,dr$$

is the **Hankel transform of order** k of f_0.

8. Deduce from the Fourier inversion formula and the Plancherel theorem on $\mathbf{R}^2$ that the Hankel transform in Exercise 7 is self-inverse and preserves the norm on $L_r^2(0,\infty)$:

$$\mathscr{H}_k(\mathscr{H}_k f) = f, \qquad \int_0^\infty |f(r)|^2 r\,dr = \int_0^\infty |\mathscr{H}_k f(r)|^2 r\,dr.$$

9. Suppose that $f(r)$ is of class $C^{(2)}$ on $[0,\infty)$ and that $rf(r)$, $rf'(r)$, and $rf''(r)$ are integrable on $[0,\infty)$ and tend to zero as $r \to \infty$. Show that

$$\mathscr{H}_k\Big[f''(r) + r^{-1}f'(r) - (k/r)^2 f(r)\Big] = -\rho^2 \mathscr{H}_k f(\rho),$$

where $\mathscr{H}_k$ is the Hankel transform of Exercise 7.

10. Consider the heat equation in $\mathbf{R}^n$ in which the initial values are assumed radial:

$$u_t = k\nabla^2 u, \qquad u(\mathbf{x}, t) = f(|\mathbf{x}|).$$

   a. Show that the solution u is radial, so that $u(\mathbf{x},t) = u_0(|\mathbf{x}|, t)$. (This is easily seen by taking the Fourier transform.)
   b. Show that for $n = 2$ or $n = 3$ we have

$$u_0(r,t) = \frac{e^{-r^2/4kt}}{2kt}\int_0^\infty f(s)e^{-s^2/4kt} I_0\left(\frac{rs}{2kt}\right) s\,ds \qquad (n = 2),$$

$$u_0(r,t) = \frac{e^{-r^2/4kt}}{r\sqrt{\pi kt}}\int_0^\infty f(s)e^{-s^2/4kt} \sinh\left(\frac{rs}{2kt}\right) s\,ds \qquad (n = 3),$$

   where I_0 is the modified Bessel function of order zero. (Hint: Take $\mathbf{x} = (0,r)$ $(n = 2)$ or $\mathbf{x} = (0,0,r)$ $(n = 3)$ and write the convolution (7.35) for $u(\mathbf{x},t)$ in polar or spherical coordinates.)

11. Suppose T is an invertible linear transformation of $\mathbf{R}^n$. Show that $(f \circ T)\widehat{\ } = |\det T|^{-1}\widehat{f} \circ (T^{-1})^*$, where $(T^{-1})^*$ is the inverse transpose of T.

7.6 Transforms related to the Fourier transform

The concept of the Fourier transform can be modified to fit into a number of other situations. One of these is the expansion of a periodic function in a Fourier series, as we mentioned in §2.6. Here we briefly discuss two other important transforms of Fourier type and then sketch the big picture that encompasses all these ideas.

The discrete Fourier transform

The discrete Fourier transform is a linear mapping that operates on complex N-dimensional vectors in much the same way that the Fourier transform operates on functions on $\mathbf{R}$. To motivate it, we consider the problem of numerical approximation of Fourier transforms.

In order to use the Fourier transform

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx$$

in machine computations, we must approximate it by something that involves only a finite number of algebraic calculations performed on a finite set of data. We accomplish this in several steps.

First, we replace the integral over $(-\infty, \infty)$ by the integral over a finite interval. In other words, we assume that f vanishes outside some bounded interval $[a, a + \Omega]$. It will be convenient to assume that $a = 0$, which can always be achieved by replacing $f(x)$ by $f(x - a)$.

Next, we shall try to calculate $\widehat{f}(\xi)$, not at every $\xi \in \mathbf{R}$ but only at a finite sequence of points contained in some bounded interval $[-C, C]$. (The choice of C may perhaps be determined by a knowledge of the rate at which $\widehat{f}(\xi)$ decays as $\xi \to \infty$.) Which sequence shall we choose? Since f vanishes outside an interval of length Ω, the sampling theorem (or rather its analogue with f and $\widehat{f}$ interchanged; cf. also Exercise 7 of §7.3) indicates that $\widehat{f}$ can be completely reconstructed from its values at the points $2\pi m/\Omega$ with m an integer. Hence, we shall content ourselves with calculating

$$\widehat{f}\left(\frac{2\pi m}{\Omega}\right) = \int_0^{\Omega} e^{-2\pi i m x/\Omega} f(x)\, dx, \qquad |m| \leq \frac{C\Omega}{2\pi}.$$

Finally, we replace the integral on the right by the left-endpoint Riemann sum obtained by subdividing the interval $[0, \Omega]$ into N equal subintervals with endpoints $n\Omega/N$ $(n = 0, \dots, N)$:

$$\widehat{f}\left(\frac{2\pi m}{\Omega}\right) \approx \sum_{n=0}^{N-1} e^{-2\pi i m n/N} f\left(\frac{n\Omega}{N}\right) \frac{\Omega}{N}.$$

(For this to be a good approximation, the exponentials $e^{-2\pi i m x/\Omega}$ must be essentially constant over each subinterval of length Ω/N, and this entails $|m| \ll N$. Hence, we should choose $N \gg C\Omega/2\pi$.)

In short, our approximation procedure leads to the following:

$$\text{if } f\left(\frac{n\Omega}{N}\right) = a_n, \text{ then } \widehat{f}\left(\frac{2\pi m}{\Omega}\right) \approx \frac{\Omega}{N}\widehat{a}_m \text{ for } |m| \ll N, \text{ where}$$

$$\widehat{a}_m = \sum_{n=0}^{N-1} e^{-2\pi i m n/N} a_n.$$

Now, since $e^{-2\pi i m} = 1$, the sequence $\{\widehat{a}_m\}$ is periodic with period N: $\widehat{a}_{m+N} = \widehat{a}_n$. Hence, the information in it is completely contained in the finite sequence $\widehat{a}_0, \dots, \widehat{a}_{N-1}$. We have therefore arrived at a mapping that transforms a vector

$\mathbf{a} = (a_0, \ldots, a_{N-1}) \in \mathbf{C}^N$ into another vector $\hat{\mathbf{a}} = (\hat{a}_0, \ldots, \hat{a}_{N-1}) \in \mathbf{C}^N$. This is the discrete Fourier transform.

Precisely, the N-point **discrete Fourier transform** is the linear map $\mathscr{F}_N : \mathbf{C}^N \to \mathbf{C}^N$ defined by

$$\mathscr{F}_N \mathbf{a} = \hat{\mathbf{a}}, \qquad \hat{a}_m = \sum_{m=0}^{N-1} e^{-2\pi imn/N} a_n \quad (0 \le m < N). \tag{7.39}$$

The discrete Fourier transform shares a number of nice properties with the ordinary Fourier transform. One is that it converts discrete convolution into ordinary multiplication:

$$\mathscr{F}_N(\mathbf{a} * \mathbf{b}) = (\hat{a}_0 \hat{b}_0, \ldots, \hat{a}_{N-1} \hat{b}_{N-1}), \tag{7.40}$$

where the discrete convolution $\mathbf{a} * \mathbf{b}$ is defined by

$$(\mathbf{a} * \mathbf{b})_n = \sum_{k=0}^{N-1} a_k b_{[n-k]}, \qquad [n-k] = \begin{cases} n-k & \text{if } n \ge k, \\ n-k+N & \text{if } n > k. \end{cases}$$

(Alternatively, $(\mathbf{a} * \mathbf{b})_n = \sum_{k=0}^{N-1} a_k b_{n-k}$ where n and k are considered as integers modulo N.) The proof of (7.40) is an easy exercise that we leave to the reader. Another important point in which the discrete Fourier transform resembles the Fourier transform is in its inversion formula, which we now derive.

Lemma 7.1. *For $m = 0, \ldots, N-1$ let*

$$\mathbf{e}_m = (1, e^{2\pi im/N}, e^{2\pi i2m/N}, \ldots, e^{2\pi i(N-1)m/N}).$$

Then $\{\mathbf{e}_m\}_{m=0}^{N-1}$ is an orthogonal basis for $\mathbf{C}^N$, and $\|\mathbf{e}_m\|^2 = N$ for all m.

Proof: Since the components of $\mathbf{e}_m$ all have absolute value 1, we have

$$\|\mathbf{e}_m\|^2 = 1 + 1 + 1 + \cdots + 1 = N.$$

On the other hand, if $l \ne m$ then $\langle \mathbf{e}_l, \mathbf{e}_m \rangle$ can be calculated as the sum of a geometric series:

$$\langle \mathbf{e}_l, \mathbf{e}_m \rangle = \sum_{n=0}^{N-1} e^{2\pi i(l-m)n/N} = \frac{1 - e^{2\pi i(l-m)}}{1 - e^{2\pi i(l-m)/N}} = 0$$

since $e^{2\pi i(l-m)} = 1$. ∎

According to this lemma, for any $\mathbf{a} \in \mathbf{C}^N$ we have

$$\mathbf{a} = \frac{1}{N} \sum_{m=0}^{N-1} \langle \mathbf{a}, \mathbf{e}_m \rangle \mathbf{e}_m. \tag{7.41}$$

But the inner products $\langle \mathbf{a}, \mathbf{e}_m \rangle$ are just the components of $\widehat{\mathbf{a}}$:

$$\langle \mathbf{a}, \mathbf{e}_m \rangle = \sum_{n=0}^{N-1} a_n e^{-2\pi i mn/N} = \widehat{a}_m,$$

so (7.41) becomes $\mathbf{a} = N^{-1} \sum_{m=0}^{N-1} \widehat{a}_m \mathbf{e}_m$, or in other words,

$$a_n = \frac{1}{N} \sum_{m=0}^{N-1} e^{2\pi i mn/N} \widehat{a}_m.$$

This is the inversion formula for the discrete Fourier transform.

As we have already indicated, the discrete Fourier transform is commonly used to provide a numerical approximation to the ordinary Fourier transform. From a computational point of view, however, the discrete Fourier transform has a potentially unpleasant feature. Namely, let us define an "elementary operation" to be a multiplication of two complex numbers followed by the addition of two complex numbers. It is clear from the definition (7.39) that the calculation of each $\widehat{a}_m$ requires N elementary operations, and there are N $\widehat{a}_m$'s; hence the calculation of $\widehat{\mathbf{a}}$ requires a total of N^2 elementary operations. When N is large, as it often must be for high-quality numerical work, N^2 is enormous, so the discrete Fourier transform may become computationally unmanageable.

When N is prime, not much can be done about this. However, when N is composite, the calculations can be substantially reduced by arranging them more efficiently. Suppose $N = N_1 N_2$, and let us write the indices m and n in (7.39), respectively, as multiples of N_1 and N_2 plus remainders:

$$m = m' N_1 + m'', \qquad n = n' N_2 + n'',$$

where m'' and n' range from 0 to $N_1 - 1$ and m' and n'' range from 0 to $N_2 - 1$. Then

$$e^{-2\pi i mn/N} = e^{-2\pi i [(m''n'/N_1) + (m'n''/N_2) + (m''n''/N)]},$$

so

$$\widehat{a}_m = \sum_{n''=0}^{N_2 - 1} C(m'', n'') e^{-2\pi i [(m'n''/N_2) + (m''n''/N)]}$$

where

$$C(m'', n'') = \sum_{n'=0}^{N_1 - 1} e^{-2\pi i m''n'/N_1} a_{n'N_2 + n''}.$$

The point is that the same sum $C(m'', n'')$ occurs in the calculation of a_m for N_2 different values of m (namely, m'', $N_1 + m''$, $2N_1 + m''$, etc.), so one can save time by just calculating it once and using it over and over. In fact, the calculation of each $C(m'', n'')$ requires N_1 elementary operations, and there are $N_1 N_2 = N$ different $C(m'', n'')$'s, so NN_1 elementary operations are needed to calculate them all. Once this is done, N_2 elementary operations are required to calculate each $\widehat{a}_m$, and there are N of those. The total is therefore $N(N_1 + N_2)$ elementary operations, as opposed to $N^2 = N(N_1 N_2)$ by the original method. Since $N_1 + N_2$ is much less than $N_1 N_2$ when either of N_1 or N_2 is large, this represents a substantial improvement.

When N_1 or N_2 can be factored further, this process can be repeated to increase the efficiency still more. The end result is that if $N = N_1 N_2 \cdots N_k$, the number of elementary operations can be reduced from N^2 to $N(N_1 + \cdots + N_k)$. In particular, if N is a power of 2, say $N = 2^k$, the reduction is from $2^{2k} = N^2$ to $k2^{k+1} = 2N \log_2 N$; this is the case most frequently used in practice. The resulting algorithm for calculating discrete Fourier transforms is known as the **fast Fourier transform**.

The fast Fourier transform, or variants of it, has been used by numerical analysts as far back as Gauss in 1805 (see Heideman et al. [27] for an interesting account of its history), but it did not become widely known until it was set forth by Cooley and Tukey [14] in 1965. Since then it has effected a marriage of Fourier analysis and numerical analysis whose impact on scientific computing has been nothing short of revolutionary. The paper of Cooley and Tukey [14] is still hard to beat as an introduction to the fast Fourier transform; for more extensive treatments and some applications, see the books of Brigham [12] and Walker [54].

The discrete Fourier transform is not merely a computational device. It also has important applications in pure mathematics, particularly in number theory. See Dym-McKean [19] and Körner [34].

The Mellin Transform

The **Mellin transform** $\mathscr{M}$ is a variant of the Fourier transform that pertains to functions on the interval $(0, \infty)$ rather than the whole line $\mathbf{R}$. It is defined by

$$\mathscr{M} f(\xi) = \int_0^\infty y^{-i\xi} f(y) \frac{dy}{y}.$$

If we make the substitution $x = \log y$ in this integral, we find that

$$\mathscr{M} f(\xi) = \int_{-\infty}^\infty e^{-ix\xi} f(e^x)\, dx.$$

In other words, $\mathscr{M} f$ is simply the Fourier transform of $f \circ \exp$, so that properties of the Mellin transform can easily be derived from the corresponding properties of the Fourier transform. Here are two examples. First, the inversion formula for the Mellin transform is

$$f(y) = f(e^x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ix\xi} \mathscr{M} f(\xi)\, d\xi = \frac{1}{2\pi} \int_{-\infty}^\infty y^{i\xi} \mathscr{M} f(\xi)\, d\xi.$$

Second, the fact that Fourier transforms convert convolution into multiplication becomes

$$\mathscr{M}(f \times g) = (\mathscr{M} f)(\mathscr{M} g),$$

where the "multiplicative convolution" $f \times g$ is defined by

$$f \times g(y) = \int_0^\infty f(z) g\left(\frac{y}{z}\right) \frac{dz}{z}.$$

Fourier Transforms on Groups

By now we have encountered a number of "Fourier-type" transforms:

i. The map that takes a 2π-periodic function f to its sequence $\{c_n\}_{-\infty}^{\infty}$ of Fourier coefficients,

$$f \longrightarrow c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx,$$

and the inverse map that takes a sequence $\{c_n\}_{-\infty}^{\infty}$ to the corresponding Fourier series:

$$\{c_n\}_{-\infty}^{\infty} \longrightarrow \sum_{-\infty}^{\infty} c_n e^{inx}.$$

ii. The Fourier transform on $\mathbf{R}$,

$$\mathscr{F} f(\xi) = \int_{\mathbf{R}} e^{-ix\xi} f(x)\, dx,$$

and its inverse

$$\mathscr{F}^{-1} g(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{ix\xi} g(\xi)\, d\xi.$$

iii. The discrete Fourier transform

$$(\mathscr{F}_N \mathbf{a})_m = \sum_{n=0}^{N-1} e^{-2\pi imn/N} a_n,$$

and its inverse

$$(\mathscr{F}_N^{-1} \mathbf{b})_n = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi imn/N} b_m.$$

iv. The Mellin transform

$$\mathscr{M} f(\xi) = \int_0^\infty y^{-i\xi} f(y) \frac{dy}{y},$$

and its inverse

$$\mathscr{M}^{-1} g(x) = \frac{1}{2\pi} \int_{\mathbf{R}} y^{i\xi} g(\xi)\, d\xi.$$

In each of these transform pairs, the spaces on which the functions are defined have the mathematical structure of a *group.* The direct transform maps functions on a certain group G to functions on another group $\widehat{G}$, and its inverse maps functions on $\widehat{G}$ back to functions on G. In (i) G is the additive group $\mathbf{R}/2\pi\mathbf{Z}$ of real numbers modulo 2π and $\widehat{G}$ is the additive group $\mathbf{Z}$ of integers; in (ii) both G and $\widehat{G}$ are the additive group $\mathbf{R}$; in (iii) both G and $\widehat{G}$ are the additive group $\mathbf{Z}/N\mathbf{Z}$ of integers modulo N; and in (iv) G is the multiplicative group of positive real numbers, whereas $\widehat{G}$ is the additive group $\mathbf{R}$. Moreover, in each of these cases the transform T and its inverse T^{-1} have the form

$$Tf(v) = C\int_G \overline{E(u,v)}f(u)\,du, \qquad T^{-1}g(u) = C'\int_{\widehat{G}} E(u,v)g(v)\,dv. \tag{7.42}$$

(The integral should be replaced by a sum in the cases where G or $\widehat{G}$ is $\mathbf{Z}$ or $\mathbf{Z}/N\mathbf{Z}$, and du should be replaced by du/u in case (iv).) Here C and C' are constants, and $E(u,v)$ is a map from $G\times\widehat{G}$ to the multiplicative group of complex numbers of absolute value 1 that is a group homomorphism in each variable: $E(x,n) = e^{inx}$ in case (i), $E(x,\xi) = e^{ix\xi}$ in case (ii), $E(n,m) = e^{2\pi imn/N}$ in case (iii), and $E(y,\xi) = y^{i\xi}$ in case (iv).

In fact, the Walsh expansions discussed in §6.6 also fit into this setup. There G is the group whose elements are sequences $\{a_n\}_1^\infty$ where each a_n is 0 or 1, and the group operation is addition modulo 2 in each component; G can be identified as a set with the unit interval $[0,1]$ by identifying the sequence $\{a_n\}$ with the digits in the binary decimal expansion of a number in $[0,1]$. $\widehat{G}$ is just the set $\{w_n\}_0^\infty$ of Walsh functions themselves, which form a group under multiplication. The transform taking a function f on the unit interval $[0,1]$ to its sequence $\{c_n\}_0^\infty$ of Walsh coefficients, and the inverse transform that combines these coefficients into a Walsh series,

$$c_n = \int_0^1 f(x)w_n(x)\,dx, \qquad f(x) = \sum_0^\infty c_n w_n(x),$$

are of the form (7.42) when we identify $[0,1]$ with the group G; the pairing E is given by $E(x,w_n) = w_n(x)$.

There is a general theory that encompasses all these examples. Its development requires ideas from topology and measure theory that are beyond the scope of this book, but we shall attempt a brief statement of the main idea. Let G be any locally compact Abelian group, that is, an Abelian (commutative) group with a topological structure such that the group operations are continuous and every point has a compact neighborhood. Then there exist (a) another locally compact Abelian group $\widehat{G}$, called the *dual group*, (b) translation-invariant volume elements du on G and dv on $\widehat{G}$, and (c) a map $E(u,v)$ from $G\times\widehat{G}$ to the group $\{z\in\mathbf{C} : |z| = 1\}$ which is a homomorphism in each variable, such that (7.42) holds. The functions $e_v(u) = E(u,v)$ can be thought of as basic building blocks for synthesizing general functions on G via (7.42) just as the exponential functions e^{inx} are the basic building blocks for synthesizing periodic functions on the real line. This theory can also be extended to non-Abelian locally compact groups, but the functions $E(u,v)$ must be replaced by more complicated functions coming from the linear representations of the group.

An introduction to these general theories of Fourier analysis may be found in Dym-McKean [19], Gross [26], and the articles of Graham, Weiss, and Sally in Ash [2].

CHAPTER 8
THE LAPLACE TRANSFORM

Suppose f is an integrable function on $\mathbf{R}$ such that $f(t) = 0$ for $t < 0$. In the integral defining the Fourier transform of f,

$$\widehat{f}(\omega) = \int_0^\infty f(t)e^{-i\omega t}\,dt, \tag{8.1}$$

we can take ω to be a *complex* number. Indeed, if $\omega = \alpha + i\beta$ with α and β real, we have $|e^{-i\omega t}| = e^{\beta t}$, so the integral (8.1) converges not just for real ω but for all ω with negative imaginary part, and it defines an analytic function on this domain. Moreover, if we restrict ω to the half-plane $\operatorname{Im}\omega < -a$, we can weaken the requirement that f be integrable: The integral (8.1) will converge for all functions $f(t)$ that grow no faster than e^{at}. In this situation it is customary to make the change of variable $z = i\omega$ and to define the **Laplace transform** of f to be

$$\mathscr{L}f(z) = \widehat{f}(-iz) = \int_0^\infty f(t)e^{-zt}\,dt.$$

The Laplace transform is really a special case of the Fourier transform, pertaining to functions that live on the positive half-line, but it is more "user-friendly" than the general Fourier transform, for two reasons. First, because of the exponential decrease of the function e^{-zt} for $\operatorname{Re} z > 0$, $t > 0$, the problems about convergence of the integral are much less delicate. Second, because Laplace transforms are analytic functions of z, one has all the machinery of complex variable theory to help study them. In particular, as we shall see, the Fourier inversion formula leads to an inversion formula for the Laplace transform in terms of a contour integral, and the techniques of residues, deformation of contours, etc., are at our disposal.

For a comprehensive treatment of the Laplace transform we refer the reader to Doetsch [17]; see also Bellman-Roth [5] for a different slant on the Laplace transform and its uses.

8.1 The Laplace transform

Let us begin with some precise definitions. We shall denote by $\mathscr{E}$ the class of functions f on the positive half-line $[0, \infty)$ that are piecewise continuous and satisfy an estimate

$$|f(t)| \le Ce^{at} \quad \text{for some } C \ge 0,\ a \in \mathbf{R}. \tag{8.2}$$

Occasionally, negative values of the variable t will intervene, and we adopt the convention that *elements of $\mathscr{E}$ are always extended to be zero on the negative half-line.* In other words, we regard the elements of $\mathscr{E}$ *either* as functions on $[0, \infty)$ *or* as functions on $\mathbf{R}$ that satisfy $f(t) = 0$ for $t < 0$.

In this connection, it will be useful to introduce the **Heaviside step function** $H(t)$ defined by

$$H(t) = \begin{cases} 1 & \text{if } t \ge 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{8.3}$$

Any possible confusion about whether or not a function $f(t)$ vanishes for $t < 0$ can be removed by multiplying $f(t)$ by $H(t)$. Thus, for example, we may wish to consider the function $f \in \mathscr{E}$ given by $f(t) = \cos t$ for $t \ge 0$. According to the convention in the preceding paragraph, the extension of f to $\mathbf{R}$ is given by $f(t) = H(t)\cos t$.

If $f \in \mathscr{E}$, its **Laplace transform** $\mathscr{L}f$ is the function of the complex variable z defined for $\operatorname{Re} z \gg 0$ by

$$\mathscr{L}f(z) = \int_0^\infty f(t)e^{-zt}\,dt. \tag{8.4}$$

More precisely, if $|f(t)| \le Ce^{at}$ and $\operatorname{Re} z = a + r$ for some $r > 0$, then

$$\int_0^\infty |f(t)e^{-zt}|\,dt \le \int_0^\infty Ce^{-rt}\,dt < \infty.$$

Thus the integral in (8.4) converges absolutely for $\operatorname{Re} z > a$, and the convergence is uniform in the half-plane $\operatorname{Re} z \ge a + \epsilon$ for any $\epsilon > 0$. The same is true of the derived integral $\int_0^\infty f(t)(-t)e^{-zt}\,dt$, so differentiation under the integral sign is legitimate, and we conclude that $\mathscr{L}f$ is an analytic function of z in the half-plane $\operatorname{Re} z > a$. It may happen that $\mathscr{L}f$ can be extended analytically to a larger (connected) domain in the complex plane; such an extension is unique if it exists, and we shall also denote the extended function by $\mathscr{L}f$.

Remark. In developing the theory of the Laplace transform, we could enlarge the class $\mathscr{E}$ somewhat by allowing functions that have integrable singularities; what one really needs is for $f(t)e^{-zt}$ to be in $L^1(0, \infty)$ for $\operatorname{Re} z$ sufficiently large. However, the only functions not in $\mathscr{E}$ that we shall wish to consider are those of the form $f(t) = t^{-\alpha} + g(t)$ where $0 < \alpha < 1$ and $g \in \mathscr{E}$. We shall calculate the Laplace transform of $t^{-\alpha}$ shortly (formula (8.6)); beyond this, we shall not bother to extend the theory beyond the class $\mathscr{E}$.

Besides being analytic, Laplace transforms of functions in $\mathscr{E}$ share one other basic property.

Lemma 8.1. *Suppose $f \in \mathscr{E}$ and $|f(t)| \le Ce^{at}$. Then*
(a) $\mathscr{L}f(x+iy) \to 0$ as $|y| \to \infty$ for each fixed $x > a$;
(b) $\mathscr{L}f(x+iy) \to 0$ as $x \to +\infty$, with y arbitrary.

Proof: (a) follows from the Riemann-Lebesgue lemma, since $\mathscr{L}f(x+iy) = \widehat{g}(y)$ where $g(t) = e^{-xt}f(t)$. As for (b), for $x > a+1$ we have $|f(t)e^{-(x+iy)t}| \le Ce^{-t}$, and e^{-t} is integrable on $[0,\infty)$. We can therefore apply the dominated convergence theorem to see that for any sequence $\{z_n\}$ such that $x_n = \operatorname{Re} z_n \to \infty$,

$$\lim \mathscr{L}f(z_n) = \int_0^\infty \lim f(t)e^{-z_n t}\,dt = \int_0^\infty 0\,dt = 0. \qquad \blacksquare$$

In working with Laplace transforms of several functions at once, the following point arises. We would like, for example, to say that the Laplace transform is linear:

$$\mathscr{L}(c_1 f + c_2 g) = c_1 \mathscr{L}f + c_2 \mathscr{L}g.$$

The trouble is that if f and g satisfy estimates of the form (8.2) with *different* values of a, $\mathscr{L}f$ and $\mathscr{L}g$ will not have the same domain. (If $|f(t)| \le Ce^{at}$ and $|g(t)| \le C'e^{a't}$, then $\mathscr{L}f$ and $\mathscr{L}g$ are defined for $\operatorname{Re} z > a$ and $\operatorname{Re} z > a'$, respectively.) However, the intersection of their domains will always include a half-plane of the form $\operatorname{Re} z > A$ (namely, $A = \max(a, a')$), and when we say "$\mathscr{L}(c_1 f + c_2 g) = c_2 \mathscr{L}f + c_2 \mathscr{L}g$" we mean that the equation holds on this half-plane. Of course, if $\mathscr{L}f$ and $\mathscr{L}g$ can be continued analytically to a larger common domain, this equation will still hold in the larger domain, so the question of domains will never be really troublesome. Similar remarks hold for other equations relating Laplace transforms of different functions.

Operational properties

As one would expect, Laplace transforms enjoy a set of operational properties similar to those of the Fourier transform. In adapting the latter results to the Laplace transform, however, we must take account of a couple of points arising from the fact that we are now dealing with functions that live only on the positive half-line and are zero on the negative half-line. This will affect the differentiation formulas, since there will probably be some discontinuities in f and/or its derivatives when it is brought suddenly to zero at $t = 0$ even if $f(t)$ is perfectly smooth on $[0,\infty)$, and it also affects the translation formulas.

Moreover, the convolution integral for functions that vanish for $t < 0$ reduces to an integral over finite intervals. Indeed, suppose that $f(t)$ and $g(t)$ both vanish when $t < 0$. Then $f(t-s) = 0$ for $s > t$ and $g(s) = 0$ for $s < 0$, so

$$f * g(t) = \begin{cases} \int_0^t f(t-s)g(s)\,ds & \text{if } t \ge 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Since the integration is over the finite interval $[0, t]$, there is no problem with behavior at infinity: it suffices for f and g to be, say, piecewise continuous, no matter how rapidly they may grow at infinity, in order for $f * g$ to be well-defined. In particular, $f * g$ is well-defined for $f, g \in \mathscr{E}$, and in that case $f * g$ also belongs to $\mathscr{E}$. (See Exercise 17.) The formula

$$f * g(t) = \int_0^t f(t-s)g(s)\,ds \qquad (t \geq 0)$$

may be taken as a definition of convolution for functions f and g that are defined only for $t \geq 0$; it coincides with our old definition if we agree to extend such functions to be zero for $t < 0$.

With these things in mind, we state the properties of Laplace transforms analogous to the properties of Fourier transforms in Theorem 7.5.

Theorem 8.1. *Suppose $f \in \mathscr{E}$.*
(a) For any $a > 0$ and $c \in \mathbf{C}$,

$$\mathscr{L}\Big[H(t-a)f(t-a)\Big] = e^{-az}\mathscr{L}f(z) \quad \text{and} \quad \mathscr{L}\Big[e^{ct}f(t)\Big] = \mathscr{L}f(z-c).$$

(b) If $a > 0$,

$$\mathscr{L}\Big[f(at)\Big] = a^{-1}\mathscr{L}f(a^{-1}z).$$

(c) If f is continuous and piecewise smooth on $[0, \infty)$ and $f' \in \mathscr{E}$, then

$$\mathscr{L}\Big[f'(t)\Big] = z\mathscr{L}f(z) - f(0).$$

(d) $\mathscr{L}\Big[\int_0^t f(s)\,ds\Big] = z^{-1}\mathscr{L}f(z)$.
(e) $\mathscr{L}\Big[tf(t)\Big] = -(\mathscr{L}f)'(z)$.
(f) If $t^{-1}f(t) \in \mathscr{E}$, then

$$\mathscr{L}\Big[t^{-1}f(t)\Big] = \int_z^\infty \mathscr{L}f(w)\,dw.$$

Here $\int_z^\infty$ denotes integration over any contour in the w-plane starting at z along which $\operatorname{Im} w$ stays bounded and $\operatorname{Re} w \to \infty$.
(g) If $g \in \mathscr{E}$ then $\mathscr{L}(f * g) = (\mathscr{L}f)(\mathscr{L}g)$.

Proof: The verification of these formulas is entirely similar to the proof of Theorem 7.5. (a) and (b) are proved by a change of variable in the integral defining $\mathscr{L}f$. For (c), integrate by parts: If a is sufficiently large so that $|f(t)| \leq Ce^{at}$ and $|f'(t)| \leq Ce^{at}$, then for $\operatorname{Re} z > a$ we have

$$\begin{aligned}\mathscr{L}[f'](z) = \int_0^\infty f'(t)e^{-zt}\,dt &= f(z)e^{-zt}\Big|_0^\infty - \int_0^\infty f(t)(-z)e^{-zt}\,dt \\ &= -f(0) + z\mathscr{L}f(z).\end{aligned}$$

(d) results from applying (c) to $\phi(t) = \int_0^t f(s)\,ds$: We have $\phi(0) = 0$ and $\phi' = f$, so $\mathscr{L}f(z) = z\mathscr{L}\phi(z)$. (e) is just a matter of differentiating under the integral. For (f), let $g(t) = t^{-1}f(t)$: By (e), $\mathscr{L}f = -(\mathscr{L}g)'$, and by Lemma 8.1, $\mathscr{L}g(z) \to 0$ as $\operatorname{Re} z \to \infty$. These two conditions determine $\mathscr{L}g$ uniquely as $\int_z^\infty \mathscr{L}f(w)\,dw$. (The precise contour of integration is immaterial, by Cauchy's theorem.) Finally, (g) is obtained by writing $\mathscr{L}(f * g)$ as a double integral and reversing the order of integration. We leave the details to the reader (Exercise 18). ∎

If f possesses derivatives of higher order in $\mathscr{E}$, the formula of Theorem 8.1(c) can be iterated; for example,

$$\mathscr{L}[f''](z) = z\mathscr{L}[f'](z) - f'(0) = z^2\mathscr{L}f(z) - zf(0) - f'(0).$$

A bit of confusion may arise here, as our convention that $f(t) = 0$ for $t < 0$ usually destroys any differentiability of f at $t = 0$. Rather, $f'(0)$ should be interpreted as the right-hand derivative of f at 0 or as the limit $f'(0+)$. In the same way, if f is k times differentiable on $[0, \infty)$ (including right-hand derivatives at 0) and $f, f', \dots, f^{(k)} \in \mathscr{E}$, then

$$\mathscr{L}[f^{(k)}](z) = z^k\mathscr{L}f(z) - z^{k-1}f(0+) - z^{k-2}f'(0+) - \cdots - f^{(k-1)}(0+). \quad (8.5)$$

Computation of Laplace transforms

Let us now compute some Laplace transforms. As a first example, consider $f(t) = t^\nu$ where $\nu > -1$. f is in $\mathscr{E}$ if $\nu \geq 0$; in fact, f satisfies estimates $|f(t)| \leq C_a e^{at}$ for any $a > 0$ (although the constant C_a blows up as $a \to 0$), so $\mathscr{L}f$ is defined for $\operatorname{Re} z > 0$. This conclusion also holds for $-1 < \nu < 0$: The singularity of f at $t = 0$ is integrable, so $e^{-zt}f(t)$ is integrable on $[0, \infty)$ whenever $\operatorname{Re} z > 0$. If z is real and positive, we can make the substitution $u = zt$ to obtain

$$\mathscr{L}f(z) = \int_0^\infty t^\nu e^{-zt}dt = \frac{1}{z^{\nu+1}}\int_0^\infty u^\nu e^{-u}\,du = \frac{\Gamma(\nu+1)}{z^{\nu+1}}.$$

This formula continues to hold in the whole half-plane $\operatorname{Re} z > 0$, for either of two reasons: (i) because two analytic functions in the half-plane $\operatorname{Re} z > 0$ that agree on the real axis must agree everywhere, or (ii) by the same substitution $u = zt$ as above, although for z nonreal one must use Cauchy's theorem to shift the contour from $\arg t = 0$ to $\arg(zt) = 0$. In short, we have

$$\mathscr{L}[t^\nu] = \Gamma(\nu+1)z^{-\nu-1} \qquad (\nu > -1). \quad (8.6)$$

(This formula also holds for complex ν with $\operatorname{Re}\nu > -1$.) (8.6) can be generalized by applying Theorem 8.1(a): For any $\alpha \in \mathbf{C}$,

$$\mathscr{L}[t^\nu e^{at}] = \Gamma(\nu+1)(z-a)^{-\nu-1}. \quad (8.7)$$

Of particular interest is the case where ν is a nonnegative integer n. Let us define an **exponential polynomial** to be a linear combination of functions of the form $t^n e^{at}$ with $n = 0, 1, 2, \dots$ and $a \in \mathbf{C}$. Moreover, we recall that a rational function $F(z)$ is called **proper** if, when F is written as the quotient of two polynomials, the numerator has smaller degree than the denominator. Equivalently, a rational function F is proper if and only if $F(z) \to 0$ as $|z| \to \infty$.

Theorem 8.2. *f is an exponential polynomial if and only if $\mathcal{L} f$ is a proper rational function.*

Proof: If f is an exponential polynomial, then $\mathcal{L} f$ is a linear combination of terms of the form $(z-a)^{-n-1}$ and hence is a rational function that tends to zero as $|z| \to \infty$. Conversely, if $F(z)$ is a proper rational function, F can be decomposed by partial fractions into a linear combination of functions of the form $(z-a)^{-n-1}$ with $n \geq 0$, so F is the Laplace transform of an exponential polynomial. (Recall that we are working over the complex numbers, so the denominator can be completely factored into linear factors; there are no irreducible quadratics.) ∎

The Laplace transforms of the trig functions $\cos at$ and $\sin at$ can be computed directly from the integral (8.2), or by expressing them in terms of $e^{\pm iat}$ and using (8.7) (with $\nu = 0$), or by using formula (8.5). The latter method works as follows: Setting $\mathcal{L}[\cos at] = F(z)$, we have

$$-a^2 F(z) = \mathcal{L}\left[-a^2 \cos at\right] = \mathcal{L}\left[(\cos at)''\right] = z^2 F(z) - z,$$

since $\cos 0 = 1$ and $(\cos)'(0) = 0$. Likewise, if $\mathcal{L}[\sin at] = G(z)$ then $-a^2 G(z) = z^2 G(z) - a$. Solving these equations for F and G, we find that

$$\mathcal{L}[\cos at] = \frac{z}{z^2 + a^2}, \qquad \mathcal{L}[\sin at] = \frac{a}{z^2 + a^2}.$$

Once these results are known, one can compute other Laplace transforms by using Theorem 8.1. For example, by Theorem 8.1(f),

$$\mathcal{L}\left[t^{-1} \sin t\right] = \int_z^\infty \frac{dw}{w^2+1} = \arctan w \Big|_z^\infty = \frac{\pi}{2} - \arctan z = \arctan \frac{1}{z},$$

and then by Theorem 8.1(d),

$$\mathcal{L}\left[\int_0^t \frac{\sin s}{s}\, ds\right] = \frac{1}{z} \arctan \frac{1}{z}.$$

The function on the left side of this last formula is known as the **sine integral** function and is denoted by $\mathrm{Si}(t)$:

$$\mathrm{Si}(t) = \int_0^t \frac{\sin s}{s}\, ds.$$

Some other related functions that will turn up in applications are the **exponential integral, error function**, and **complementary error function**:

$$E_1(t) = \int_t^\infty \frac{e^{-s}}{s}\, ds$$

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds, \qquad \mathrm{erfc}(t) = 1 - \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-s^2} ds.$$

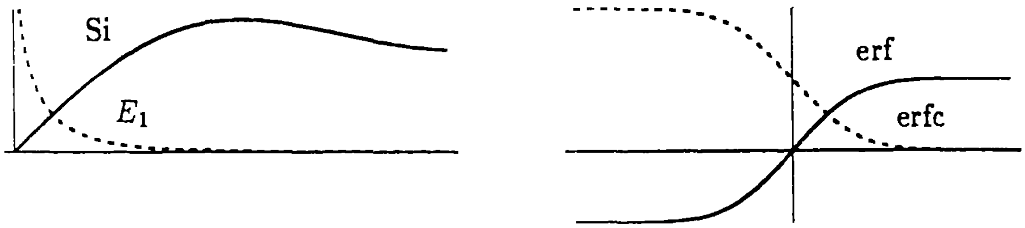


FIGURE 8.1. Left: the functions $Si(t)$ (solid) and $E_1(t)$ (dashed) on the interval $0 \leq t \leq 6$. Right: the functions $\operatorname{erf}(t)$ (solid) and $\operatorname{erfc}(t)$ (dashed) on the interval $-3 \leq t \leq 3$.

See Figure 8.1. (There is another variant of the exponential integral, $\operatorname{Ei}(t) = \int_{-\infty}^{t} s^{-1} e^s \, ds$, but this is more appropriate for negative values of t.)

Equation (8.6) suggests a method of computing Laplace transforms of functions that are represented by power series on $[0, \infty)$. Namely, suppose $f \in \mathscr{E}$ and $f(t) = \sum_0^\infty a_n t^n$ for $t \geq 0$. If it is possible to treat the series as a finite sum, we should have

$$\mathscr{L} f = \sum_0^\infty a_n \mathscr{L}[t^n] = \sum_0^\infty a_n \frac{n!}{z^{n+1}}.$$

One difficulty immediately presents itself: Because of the factors of $n!$, the series on the right may not converge. For example, if $f(t) = e^{-t^2} = \sum_0^\infty (-1)^n z^{2n}/n!$, the transformed series is $\sum_0^\infty (-1)^n (2n)!/n! z^{2n+1}$, and it is easily verified by the ratio test that this series diverges for every z. However, it turns out that there are no other difficulties: If the transformed series converges at all, its sum is indeed $\mathscr{L} f$. More precisely, we have the following theorem, in which we can also allow fractional powers in the series for f.

Theorem 8.3. *Suppose $f(t) = \sum_0^\infty a_n t^{n+\alpha}$ for $t \geq 0$, where $0 \leq \alpha < 1$. If the series*

$$\sum_0^\infty |a_n| \Gamma(n + \alpha + 1) b^{-n} \tag{8.8}$$

converges for some $b > 0$, then the series and the integral

$$F(z) = \sum_0^\infty a_n \Gamma(n + \alpha + 1) z^{-n-\alpha-1} \quad \text{and} \quad \mathscr{L} f(z) = \int_0^\infty f(t) e^{-zt} \, dt$$

both converge for $\operatorname{Re} z > b$, *and they are equal there.*

Proof: The series $F(z)$ converges for $|z| > b$, and in particular for $\operatorname{Re} z > b$, by comparison to the convergent series (8.8). Moreover, since (8.8) converges, its nth term must tend to zero as $n \to \infty$, and it follows that

$$|a_n| \leq \frac{Cb^n}{\Gamma(n + \alpha + 1)} \leq \frac{Cb^n}{\Gamma(n + 1)} = \frac{Cb^n}{n!}.$$

Therefore,

$$|f(t)| \le \sum_0^\infty |a_n| t^{n+\alpha} \le C t^\alpha \sum_0^\infty \frac{(bt)^n}{n!} = C t^\alpha e^{bt},$$

which implies that the integral defining $\mathscr{L} f$ converges for $\operatorname{Re} z > b$. Moreover, by the same estimate, if $\operatorname{Re} z = b + \delta > b$,

$$\left| \sum_0^N a_n t^{n+\alpha} e^{-zt} \right| \le e^{-(b+\delta)t} \sum_0^N |a_n| t^{n+\alpha} \le e^{-(b+\delta)t} \sum_0^\infty |a_n| t^{n+\alpha} \le C t^\alpha e^{-\delta t}$$

for all N. $t^\alpha e^{-\delta t}$ is integrable on $[0, \infty)$, so we can apply the dominated convergence theorem to conclude that

$$\mathscr{L} f(z) = \lim_{N \to \infty} \int_0^\infty \sum_0^N a_n t^{n+\alpha} e^{-zt}\, dt = \lim_{N \to \infty} \sum_0^N \frac{a_n \Gamma(n+\alpha+1)}{z^{n+\alpha+1}} = F(z). \quad \blacksquare$$

Example. Let us compute the Laplace transform of the Bessel function

$$J_0(t) = \sum_0^\infty \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Formally transforming the series termwise, we obtain

$$\mathscr{L} J_0(z) = \sum_0^\infty \frac{(-1)^n (2n)!}{2^{2n}(n!)^2 z^{2n+1}}.$$

An application of the ratio test shows that the series on the right converges for $|z| > 1$, so by Theorem 8.3 this formula is correct for $\operatorname{Re} z > 1$.

With a little more work we can find the sum of the series. We have

$$\frac{(2n)!}{2^n n!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{2 \cdot 4 \cdot 6 \cdots (2n)} = 1 \cdot 3 \cdot 5 \cdots (2n-1),$$

so

$$\frac{(-1)^n (2n)!}{2^{2n}(n!)^2} = \frac{(-1) \cdot (-3) \cdots (-2n+1)}{2^n n!} = \frac{(-1)}{2} \frac{(-3)}{2} \frac{(-2n+1)}{2} \frac{1}{n!}.$$

Now recall the binomial series:

$$(1+w)^\alpha = \sum_0^\infty \alpha(\alpha-1)\cdots(\alpha-n+1) \frac{w^n}{n!} \qquad (|w| < 1).$$

If we take $\alpha = -\frac{1}{2}$ and compare with the preceding formulas, we see that for $|z| > 1$,

$$\sum_0^\infty \frac{(-1)^n (2n)!}{2^{2n}(n!)^2 z^{2n+1}} = \frac{1}{z} \left(1 + \frac{1}{z^2}\right)^{-1/2} = (z^2+1)^{-1/2}.$$

Thus,

$$\mathscr{L} J_0(z) = (z^2+1)^{-1/2}, \tag{8.9}$$

where the branch of the square root is the one that is positive for $z > 0$. We have proved this formula for $\operatorname{Re} z > 1$. However, since J_0 is a bounded function on $[0, \infty)$, the integral defining $\mathscr{L} J_0$ converges for $\operatorname{Re} z > 0$, and $(z^2+1)^{-1/2}$ is an analytic function in this half-plane (its singularities are at $z = \pm i$). Hence, by analytic continuation, (8.9) is valid for $\operatorname{Re} z > 0$.

TABLE 3. SOME BASIC LAPLACE TRANSFORMS

Functions are listed on the left; their Laplace transforms are on the right. a and c denote constants with $a > 0$ and $c \in \mathbf{C}$.

1.	$f(t)$	$F(z) = \mathscr{L}f(z)$
2.	$H(t-a)f(t-a)$	$e^{-az}F(z)$
3.	$e^{ct}f(t)$	$F(z-c)$
4.	$f(at)$	$a^{-1}F(a^{-1}z)$
5.	$f'(t)$	$zF(z) - f(0)$
6.	$f^{(k)}(t)$	$z^k F(z) - \sum_0^{k-1} z^{k-1-j} f^{(j)}(0)$
7.	$\int_0^t f(s)\,ds$	$z^{-1}F(z)$
8.	$tf(t)$	$-F'(z)$
9.	$t^{-1}f(t)$	$\int_z^\infty F(w)\,dw$
10.	$f * g$	FG
11.	$t^\nu e^{ct}$ $\quad(\mathrm{Re}\,\nu > -1)$	$\Gamma(\nu+1)/(z-c)^{\nu+1}$
12.	$t^n e^{ct}$ $\quad(n = 0, 1, 2, \ldots)$	$n!/(z-c)^{n+1}$
13.	$(t+a)^{-1}$	$e^{az}E_1(az)$
14.	$\sin ct$	$c/(z^2+c^2)$
15.	$\cos ct$	$z/(z^2+c^2)$
16.	$\sinh ct$	$c/(z^2-c^2)$
17.	$\cosh ct$	$z/(z^2-c^2)$
18.	$\sin\sqrt{at}$	$\sqrt{\pi a/4z^3}\, e^{-a/4z}$
19.	$t^{-1}\sin\sqrt{at}$	$\pi\,\mathrm{erf}(\sqrt{a/4z})$
20.	$e^{-a^2t^2}$	$(\sqrt{\pi}/2a)e^{z^2/4a^2}\,\mathrm{erfc}(z/2a)$
21.	$\mathrm{erf}\, at$	$z^{-1}e^{z^2/4a^2}\,\mathrm{erfc}(z/2a)$
22.	$\mathrm{erf}\sqrt{t}$	$1/z\sqrt{z+1}$
23.	$e^t\,\mathrm{erf}\sqrt{t}$	$1/(z-1)\sqrt{z}$
24.	$\mathrm{erfc}(a/2\sqrt{t})$	$z^{-1}e^{-a\sqrt{z}}$
25.	$t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}\, e^{a/4z}\,\mathrm{erfc}(\sqrt{a/4z})$
26.	$t^{-1/2}e^{-a^2/4t}$	$\sqrt{\pi/z}\, e^{-a\sqrt{z}}$
27.	$t^{-3/2}e^{-a^2/4t}$	$2a^{-1}\sqrt{\pi}\, e^{-a\sqrt{z}}$
28.	$t^\nu J_\nu(t)$ $\quad(\nu > -\frac{1}{2})$	$2^\nu \pi^{-1/2}\Gamma(\nu+\frac{1}{2})(z^2+1)^{-\nu-(1/2)}$
29.	$J_0(\sqrt{t})$	$z^{-1}e^{-1/4z}$

Table 3 gives a brief list of basic Laplace transform formulas. The derivations of all the entries in it are contained in the text or exercises of §§8.1–2. Much more extensive tables of Laplace transforms and inverse Laplace transforms are available; see Erdélyi et al. [22] and Oberhettinger-Badii [41].

EXERCISES

In Exercises 1–7, compute the Laplace transform of the given functions. Assume throughout that $a > 0$. When two functions are in the same problem, the transform of one can be easily obtained from the transform of the other by means of Theorem 8.1.

1. a. $\sinh at$ b. $\cosh at$
2. $\cos^2 t$ (Hint: double angle formula.)
3. a. $e^{-a^2t^2}$ b. $\operatorname{erf} at$
4. a. $(t+a)^{-1}$ b. $(t+a)^{-2}$
5. $f(t) = t$ for $0 \le t \le 1$, $= e^{1-t}$ for $t > 1$ (Do this two ways: by computing the integral defining $\mathscr{L}f$ directly, and by writing $f(t) = t[H(t) - H(t-1)] + H(t-1)e^{1-t}$ and using Theorem 8.1(a).)
6. $t^{-1/2}e^{-\sqrt{at}}$
7. $t^{\alpha}L_n^{\alpha}(t)$ where L_n^{α} is the Laguerre polynomial of §6.5
8. Use partial fractions to find the exponential polynomial whose Laplace transform is:

   a. $\dfrac{2(z+1)}{z^2+2z}$ b. $\dfrac{4}{z(z+2)^2}$ c. $\dfrac{1}{z(z^2+1)}$

9. Let $f(t) = e^{at}$ and $g(t) = e^{bt}$. Compute $f * g$ directly and by using the Laplace transform.
10. Compute $f * g$ by using the Laplace transform:
    a. $f(t) = g(t) = J_0(t)$
    b. $f(t) = t^{a-1}$, $g(t) = t^{b-1}$ $(a, b > 0)$
    c. $f(t) = \sin t$, $g(t) = \sin 2t$.
11. Suppose f is a-periodic, that is, $f(t+a) = f(t)$ for $t > 0$. Show that

$$\mathscr{L}f(z) = \frac{F(z)}{1 - e^{-az}} \quad \text{where} \quad F(z) = \int_0^a f(t)e^{-zt}dt.$$

    (One way is to use the fact that $\int_0^\infty = \sum_0^\infty \int_{na}^{(n+1)a}$. Another is to compare the Laplace transforms of $f(t)$ and $H(t-a)f(t)$.)
12. Use the result of Exercise 9 to compute the Laplace transforms of the 2π-periodic functions that are given on the interval $(0, 2\pi)$ as follows:
    a. $f(t) = t$ for $0 < t < \pi$, $= t - 2\pi$ for $\pi < t < 2\pi$ (the sawtooth wave)
    b. $f(t) = 1$ for $0 < t < \pi$, $= -1$ for $\pi < t < 2\pi$ (the square wave)
    c. $f(t) = t$ for $0 < t < \pi$, $= 2\pi - t$ for $\pi < t < 2\pi$ (the triangle wave)

13. Let $f(t) = e^t$ and $g(t) = (\pi t)^{-1/2}$. Show that $f * g(t) = e^t \operatorname{erf}\sqrt{t}$. Then use this fact to evaluate $\mathscr{L}[e^t \operatorname{erf}\sqrt{t}]$ and $\mathscr{L}[\operatorname{erf}\sqrt{t}]$.

In Exercises 14–16, compute the Laplace transforms of the given functions by using Theorem 8.3, perhaps in conjunction with Theorem 8.1. Assume that $a > 0$. You may also need the duplication formula $\Gamma(2z)\sqrt{\pi} = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$.

14. a. $\sin\sqrt{at}$ b. $t^{-1}\sin\sqrt{at}$ c. $t^{-1/2}\cos\sqrt{at}$
15. $J_0(\sqrt{at})$
16. $t^\nu J_\nu(t)$ $(\nu > -\frac{1}{2})$
17. Prove that if $f \in \mathscr{E}$ and $g \in \mathscr{E}$, then $f * g \in \mathscr{E}$. More precisely, show that:
    a. If f and g are piecewise continuous, then $f * g$ is continuous.
    b. If $|f(t)| \le C_1 e^{at}$ and $|g(t)| \le C_2 e^{bt}$, then $|f * g(t)| \le C_3 e^{ct}$ where $c = \max(a, b)$ if $a \ne b$ and $c = a + 1$ (or indeed $a + \epsilon$) if $a = b$.
18. Fill in the details of the proof of Theorem 8.1.

8.2 The inversion formula

In order to make the Laplace transform an effective tool, one needs to know how to recover f from $\mathscr{L}f$. This can be accomplished by a suitable adaptation of the Fourier inversion formula.

Specifically, suppose that $f \in \mathscr{E}$ and $|f(t)| \le Ce^{at}$. Choose some number $b > a$ and set $g(t) = e^{-bt}f(t)$. Then g is integrable on $[0, \infty)$, so we can take its Fourier transform (recalling that $g(t) = 0$ for $t < 0$):

$$\widehat{g}(\omega) = \int_0^\infty e^{-bt} f(t) e^{-i\omega t}\,dt = \mathscr{L}f(b + i\omega).$$

Assuming that f (and hence g) is piecewise smooth and is (re)defined to be the average of its right and left limits at any points of discontinuity, we can apply Theorem 7.6 of §7.2 to obtain

$$e^{-bt}f(t) = g(t) = \lim_{r\to\infty} \frac{1}{2\pi}\int_{-r}^{r} \widehat{g}(\omega) e^{i\omega t}\,d\omega = \lim_{r\to\infty} \frac{1}{2\pi}\int_{-r}^{r} \mathscr{L}f(b + i\omega) e^{i\omega t}\,d\omega.$$

Let us set $z = b + i\omega$; as ω ranges over the interval $[-r, r]$, z ranges over the vertical segment from $b - ir$ to $b + ir$, and we denote the integral over this segment by $\int_{b-ir}^{b+ir}$. Also, $dz = i\,d\omega$, so

$$e^{-bt}f(t) = \lim_{r\to\infty} \frac{1}{2\pi i}\int_{b-ir}^{b+ir} \mathscr{L}f(z) e^{(z-b)t}\,dz.$$

Finally, we multiply through by e^{bt} and obtain

$$f(t) = \lim_{r\to\infty} \frac{1}{2\pi i}\int_{b-ir}^{b+ir} \mathscr{L}f(z) e^{zt}\,dz.$$

This is the desired inversion formula. Notice, in particular, that it does not depend on the choice of b, as long as b is large enough so that the line $\operatorname{Re} z = b$ lies in the half-plane where $\mathscr{L}f$ is analytic.* We sum up the results in a theorem.

* This could also be proved by invoking Cauchy's theorem.

Theorem 8.4. *Suppose that $f \in \mathscr{E}$ — say $|f(t)| \le Ce^{at}$ — and that f is piecewise smooth on $[0, \infty)$. If $b > a$,*

$$\tfrac{1}{2}\Big[f(t-) + f(t+)\Big] = \lim_{r \to \infty} \frac{1}{2\pi i} \int_{b-ir}^{b+ir} \mathscr{L} f(z) e^{zt}\, dz \tag{8.10}$$

for all $t \in \mathbf{R}$. (In particular, the expression on the right equals $\frac{1}{2} f(0+)$ when $t = 0$ and 0 when $t < 0$.)

Corollary 8.1. *If f and g are in $\mathscr{E}$ and $\mathscr{L} f = \mathscr{L} g$, then $f = g$. (More precisely, $f(t) = g(t)$ at all t where both f and g are continuous.)*

Proof: After suitable modification at points of discontinuity, f and g are both given by the integral (8.10). ∎

By Corollary 8.1, a function $f \in \mathscr{E}$ is uniquely determined (up to modifications at discontinuities) by its Laplace transform F; we shall say that f is the **inverse Laplace transform** of F and write $f = \mathscr{L}^{-1} F$:

$$f = \mathscr{L}^{-1} F \quad \Longleftrightarrow \quad F = \mathscr{L} f.$$

In practice, one is often given an analytic function $F(z)$ that one hopes to be the Laplace transform of some function $f(t)$, and one tries to compute f by substituting F for $\mathscr{L} f$ in the inversion formula (8.10). This procedure is a bit risky if one does not know in advance that F really is a Laplace transform, as the example in Exercise 10 shows. However, it usually works, and the following theorem gives some conditions, sufficiently general to include all the examples we shall discuss, which guarantee that it does. The proof is a bit technical, but we present it in some detail since there seems to be no convenient reference for it. (See also Doetsch [17] for some other results along these lines.)

Theorem 8.5. *Let $F(z)$ be an analytic function in the half-plane $\operatorname{Re} z > a$, and for $b > a$, $r > 0$, and $t \in \mathbf{R}$ let*

$$f_{r,b}(t) = \frac{1}{2\pi i} \int_{b-ir}^{b+ir} F(z) e^{zt}\, dz.$$

Suppose that $F(z)$ satisfies an estimate

$$|F(z)| \le C\Big(1 + |z|\Big)^{-\alpha} \qquad (\alpha > \tfrac{1}{2}) \tag{8.11}$$

for $\operatorname{Re} z > a$, and that for some $b > a$, $f_{r,b}(t)$ converges pointwise as $r \to \infty$ to a function $f(t)$ in the class $\mathscr{E}$. Then $\lim_{r \to \infty} f_{r,b}(t) = f(t)$ for all $b > a$, and $F = \mathscr{L} f$.

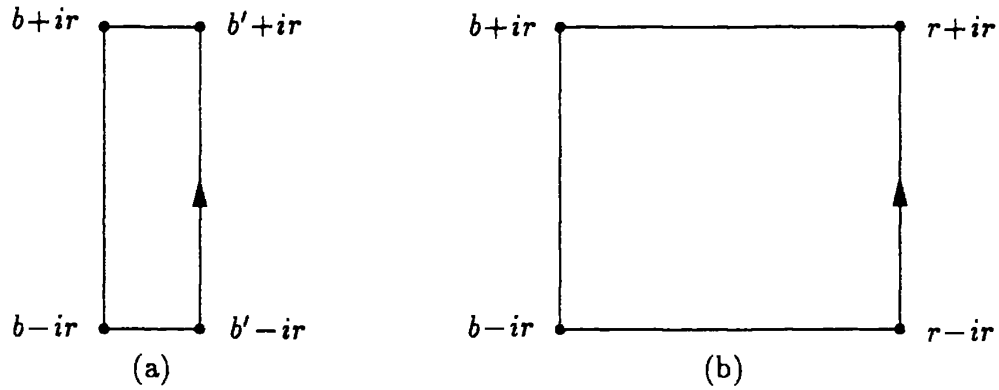


FIGURE 8.2. Contours for the proof of Theorem 8.5.

Proof: The fact that $\lim_{r\to\infty} f_{r,b}(t)$ is independent of b follows by applying Cauchy's theorem to the rectangle in Figure 8.2(a); the estimate (8.11) guarantees that the integrals over the top and bottom sides tend to zero as $r \to \infty$. Moreover, if $t < 0$, the estimate (8.11) together with the exponential decay of e^{zt} as $\operatorname{Re} z \to +\infty$ shows that the integral of $F(z)e^{zt}$ over the top, bottom, and right sides of the rectangle in Figure 8.2(b) tend to zero as $r \to \infty$, and it follows that $f(t) = 0$ for $t < 0$.

Now let $g_b(\omega) = F(b + i\omega)$, and let $g_{r,b}(\omega) = g_b(\omega)$ if $\omega \in [-r, r]$ and $g_{r,b}(\omega) = 0$ otherwise. Estimate (8.11) implies that g_b and $g_{r,b}$ are in $L^2(\mathbf{R})$ and that $g_{r,b} \to g_b$ in norm as $r \to \infty$. By the Plancherel theorem, $\widehat{g}_{r,b} \to \widehat{g}_b$ in norm as $r \to \infty$. It then follows from a standard lemma of measure theory that there is a sequence $r_j \to \infty$ so that $\widehat{g}_{r_j,b} \to \widehat{g}_b$ almost everywhere. (See, for example, Folland [25], §2.4, where the essentially identical argument for L^1 is given.) But

$$f_{r,b}(t) = \frac{1}{2\pi}\int_{-r}^{r} F(b+i\omega)e^{(b+i\omega)t}\,d\omega = \frac{e^{bt}}{2\pi}\int_{-\infty}^{\infty} g_{r,b}(\omega)e^{i\omega t}\,d\omega = \frac{e^{bt}}{2\pi}\widehat{g}_{r,b}(-t).$$

Hence,

$$2\pi e^{-bt} f(t) = \lim_{j\to\infty} 2\pi e^{-bt} f_{r_j,b}(t) = \widehat{g}_b(-t).$$

Since $f \in \mathscr{E}$ by assumption, we can take b so large that $e^{-bt}f(t)$ is integrable on $(0,\infty)$; then by the Fourier inversion theorem (with the substitution $t \to -t$) and the fact that $f(t) = 0$ for $t < 0$, we have

$$\begin{aligned} F(b+i\omega) = g_b(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}_b(-t)e^{-i\omega t}\,dt \\ &= \int_0^{\infty} e^{-bt} f(t)e^{-i\omega t}\,dt = \mathscr{L}f(b+i\omega). \end{aligned}$$

That is, $F = \mathscr{L}f$. ∎

Theorems 8.4 and 8.5 are both aspects of the Laplace inversion theorem. Their power lies in the fact that the methods of complex analysis can be brought to bear to evaluate the integral (8.10) or at least to extract significant information about it. This general statement can best be substantiated by considering some examples.

Example 1. As a first class of examples, consider the proper rational functions of z. We saw in Theorem 8.2 that the inverse Laplace transforms of such functions F are exponential polynomials, and $\mathscr{L}^{-1}F$ can be evaluated by decomposing F into partial fractions and using the formula $\mathscr{L}^{-1}\left[(z-a)^{-n-1}\right] = e^{at}t^n/n!$. However, rather than using partial fractions, one can use the residue theorem. Namely, we choose b large enough so that $F(z)$ is analytic for $\operatorname{Re} z \geq b$ (i.e., so that all the poles of F have real part less than b), and we choose r large enough so that all the poles of F are contained inside the rectangular contour γ_r shown in Figure 8.3(a). Then

$$\frac{1}{2\pi i}\int_{\gamma_r} F(z)e^{zt}\,dz = \sum_{\text{all poles of } F} \operatorname{Res} F(z)e^{zt}.$$

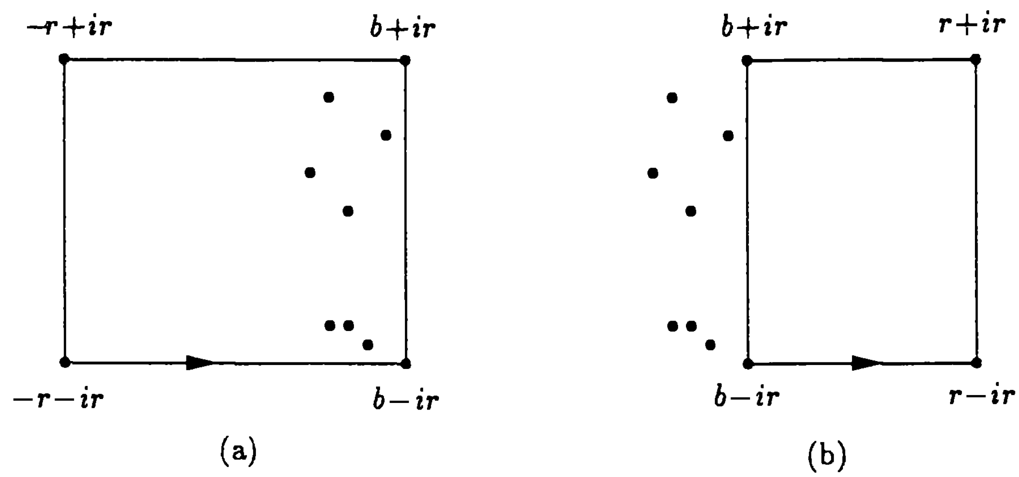


FIGURE 8.3. Contours for Example 1. The poles of the function F are indicated by dots.

We claim that the integrals over the top, bottom, and left sides of γ_r vanish as $r \to \infty$ when $t > 0$. Indeed,

$$\left|\int_{\text{top}} F(z)e^{zt}\,dz\right| \leq \max_{x\leq b}|F(x+ir)|\int_{-\infty}^{b} e^{xt}\,dx = \frac{e^{bt}}{t}\max_{x\leq b}|F(x+ir)|,$$

which tends to zero as $r \to \infty$ by the assumption on F. Similarly for the integral over the bottom; the integral over the left side is even better since $|e^{zt}| = e^{-rt}$ there. In view of Theorem 8.4, then, we have

$$\mathscr{L}^{-1}F(t) = \sum_{\text{all poles of } F} \operatorname{Res} F(z)e^{zt} \qquad (t > 0). \tag{8.12}$$

This is an effective formula for calculating the inverse Laplace transform of a proper rational function.

For $t < 0$, of course, e^{zt} is small when $\operatorname{Re} z$ is large positive rather than large negative, so by the same argument as before, one can close up the contour in the right half-plane as in Figure 8.3(b) and conclude by Cauchy's theorem that $\mathscr{L}^{-1}F(t) = 0$.

Example 2. The same technique can be applied to meromorphic functions with infinitely many poles, provided that a suitable way can be found to close up the contour of integration. As an example, let us find the inverse Laplace transform of

$$F(z) = \frac{1}{z^2} - \frac{\pi}{z\sinh \pi z}.$$

F is analytic in the whole complex plane except for poles at the points $z = \pm in$ for $n = 1, 2, 3, \ldots$. The singularity at $z = 0$ is removable, for as $z \to 0$,

$$F(z) = \frac{\sinh \pi z - \pi z}{z^2 \sinh \pi z} = \frac{\left[(\pi z) + (\pi^3 z^3/3!) + \cdots\right] - \pi z}{z^2[\pi z + \cdots]} = \frac{(\pi^3/3!) + \cdots}{\pi + \cdots} \to \frac{\pi^2}{3!}.$$

We shall consider the integral of $F(z)e^{zt}$ over the rectangular contour γ_r as in Figure 8.3(a), but we shall choose r carefully so as to avoid the poles of F, namely, $r = N + \frac{1}{2}$ where N is a positive integer. Since

$$\sinh \pi\left[x \pm i(N + \tfrac{1}{2})\right] = \pm(-1)^N i \cosh \pi x,$$

we have

$$\left|F\left(x \pm i(N + \tfrac{1}{2})\right)\right| \leq \frac{C_1}{N^2} + \frac{C_2}{N \cosh x} \leq \frac{C}{N}.$$

Hence, the same estimates as before show that the integral of $F(z)e^{zt}$ over the top, bottom, and left sides of γ_r ($r = N + \frac{1}{2}$) tends to zero as $N \to \infty$, whereas the integral over the whole contour is $2\pi i$ times the sum of the residues inside. Moreover,

$$\operatorname{Res}_{in} F(z)e^{zt} = \lim_{z \to in} \frac{e^{zt}}{z} \cdot \frac{-\pi(z - in)}{\sinh \pi z} = \frac{e^{int}}{in} \cdot \frac{-1}{\cosh \pi in} = \frac{(-1)^{n+1} e^{int}}{in},$$

since $\cosh \pi in = \cos \pi n = (-1)^n$. Therefore, letting $N \to \infty$, we obtain

$$\begin{aligned}\mathscr{L}^{-1}F(t) &= \lim_{N \to \infty} \sum_1^N \left(\operatorname{Res}_{in} + \operatorname{Res}_{-in}\right) F(z)e^{zt} \\ &= \lim_{N \to \infty} \sum_1^N (-1)^{n+1} \left(\frac{e^{int}}{in} + \frac{e^{-int}}{-in}\right) = 2\sum_1^\infty \frac{(-1)^{n+1}}{n} \sin nt.\end{aligned}$$

This we recognize as the Fourier series of the sawtooth wave function, discussed in §2.1; and it is easy to check directly that the Laplace transform of this function is indeed F (cf. Exercise 12a, §8.1). Thus the Laplace inversion formula provides a new way of deriving Fourier series! (See also Exercises 4 and 9.)

Example 3. If the function $F(z)$ has branch points, the preceding ideas do not work; however, one can often gain information by making suitable branch cuts and deforming the contour of integration onto the cut. To illustrate this technique, consider the following example. Let $\sqrt{z}$ denote the principal branch of the square root function — that is, let the branch cut be made along the negative real axis $(-\infty, 0]$, and for $z \notin (-\infty, 0]$ let $\sqrt{z}$ be the square root of z with positive real part — and let

$$F(z) = \frac{e^{-a\sqrt{z}}}{\sqrt{z}} \qquad (a > 0).$$

F is analytic in the cut plane $\mathbf{C}\backslash(-\infty, 0]$; it is bounded except near the origin and tends to zero faster than any power of $|z|$ as $z \to \infty$ in any sector $|\arg z| \le \pi - \epsilon$. (The latter is true since $\operatorname{Re}\sqrt{z} \to \infty$ as $z \to \infty$ in this region.)

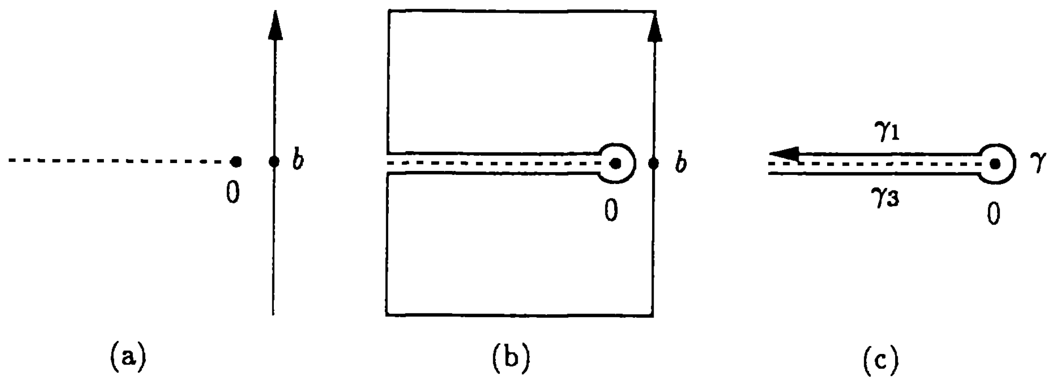


FIGURE 8.4. Contours for Example 3: (a) the initial contour, (b) transition, (c) the final contour.

To compute $\mathscr{L}^{-1}F$, we apply Theorem 8.5 and deform the contour $\operatorname{Re} z = b$ $(b > 0)$ in Figure 8.4(a) so that it wraps around the branch cut, as in Figure 8.4(c). This procedure may be justified by connecting the old and new contours with rectangular arcs, as in Figure 8.4(b). The integral of $F(z)e^{zt}$ over the whole contour in Figure 8.4(b) is zero by Cauchy's theorem; and since $|F(z)| \le 1/|z|^{1/2}$, the same estimates as in the previous examples show that the integral of $F(z)e^{zt}$ over the top, bottom, and left sides of the rectangle vanish as $r \to \infty$. Hence, in the limit we are left with the contour in Figure 8.4(c), where the rays γ_1 and γ_3 are to be envisioned as the top and bottom edges of the branch cut and γ_2 is a circle of radius ϵ about the origin, with ϵ ready to vanish on demand.

The integral over γ_2 tends to zero along with the radius ϵ, for

$$\left|\int_{\gamma_2} F(z)e^{zt}\,dz\right| \le \left(\max_{\gamma_2}|F(z)e^{zt}|\right)\left(\text{length of } \gamma_2\right) \le \frac{e^{\epsilon t}}{\sqrt{\epsilon}}2\pi\epsilon = 2\pi e^{\epsilon t}\sqrt{\epsilon}.$$

Now, with $\epsilon = 0$, we parametrize γ_1 by $z = -x$, $0 < x < \infty$, and note that $\sqrt{z} = i\sqrt{x}$ and $dz = -dx$, so that

$$\frac{1}{2\pi i}\int_{\gamma_1} F(z)e^{zt}\,dz = -\frac{1}{2\pi i}\int_0^\infty \frac{e^{-ia\sqrt{x}}}{i\sqrt{x}}e^{-xt}\,dx.$$

The integral over γ_3 is similar except that the orientation is reversed and $\sqrt{z} = -i\sqrt{x}$:

$$\frac{1}{2\pi i}\int_{\gamma_3} F(z)e^{zt}\,dz = \frac{1}{2\pi i}\int_0^\infty \frac{e^{ia\sqrt{x}}}{-i\sqrt{x}}e^{-xt}\,dx.$$

Hence, adding things up, we find that

$$\mathscr{L}^{-1}F(t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} F(z)e^{zt}\,dz = \frac{1}{\pi}\int_0^\infty \frac{\cos a\sqrt{x}}{\sqrt{x}}e^{-xt}\,dx.$$

With the substitution $y = \sqrt{x}$, this becomes

$$\mathscr{L}^{-1}F(t) = \frac{2}{\pi}\int_0^\infty e^{-y^2t}\cos ay\,dy.$$

But this is a Fourier cosine transform that we have already evaluated in Chapter 7: Since e^{-y^2t} is an even function of y,

$$\frac{2}{\pi}\int_0^\infty e^{-y^2t}\cos ay\,dy = \frac{1}{\pi}\int_{-\infty}^\infty e^{-y^2t}e^{iay}\,dy = \frac{1}{\sqrt{\pi t}}e^{-a^2/4t}.$$

Thus we have obtained the result

$$\mathscr{L}^{-1}\left[\frac{e^{-a\sqrt{z}}}{\sqrt{z}}\right] = \frac{1}{\sqrt{\pi t}}e^{-a^2/4t}, \quad \text{or} \quad \mathscr{L}\left[\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}\right] = \frac{e^{-a\sqrt{z}}}{\sqrt{z}}. \tag{8.13}$$

Here we have actually obtained a new Laplace transform formula. The Laplace transform of $(\pi t)^{-1/2}e^{-a^2/4t}$ is not easy to evaluate directly; it can be done by a tricky calculation, but one would be hard put to discover that calculation without knowing the answer in the first place.

EXERCISES

In Exercises 1–5, use the residue theorem to evaluate the inverse Laplace transforms of the given functions.

1. The functions in Exercise 6, §8.1.
2. $\dfrac{3z^2 + 12z + 8}{(z+2)^2(z+4)}$
3. $\dfrac{4z^2 + z + 15}{z(z^2 - 2z + 5)}$
4. $z^{-2}\tanh(\pi z/2)$ (Cf. Exercise 12c, §8.1)
5. $[z\cosh\sqrt{z}\,]^{-1}$ (Note: Despite the square root, $\cosh\sqrt{z} = \sum_0^\infty z^n/(2n)!$ is an entire function of z.)
6. Find the inverse Laplace transforms of $e^{-a\sqrt{z}}$ and $z^{-1}e^{-a\sqrt{z}}$ by using formula (8.13) and Theorem 8.1.

7. Find the inverse Laplace transform of $F(z) = z^{-1}\log(1+z)$ by the technique used to derive (8.13). (Note: $\mathscr{L}^{-1}F$ is not in $\mathscr{E}$ because it has a logarithmic singularity at $t = 0$; however, it is the derivative of a function in $\mathscr{E}$.)
8. Find the inverse Laplace transform of $z^{-1}e^{-a\sqrt{z}}$ by the technique used to derive (8.13). (Hint: In contrast to the situation in the text, the integral over the circle γ_2 in Figure 8.4c does not tend to zero. Show that

$$\mathscr{L}^{-1}[z^{-1}e^{-a\sqrt{z}}] = 1 - \frac{1}{\pi}\int_0^\infty \frac{e^{-xt}\sin a\sqrt{x}}{x}\,dx$$

and use Exercise 14b, §8.1.)
9. Suppose f is 2π-periodic: $f(t+2\pi) = f(t)$ for $t > 0$. Show that if one calculates the Laplace transform $F = \mathscr{L}f$ according to Exercise 11, §8.1, and then calculates $\mathscr{L}^{-1}F$ by the method used in Example 2, the result is always the Fourier series expansion of f.
10. Let $F(z) = e^{z^2}$ and $f(t) = (1/2\sqrt{\pi})e^{-t^2/4}$. Show that for any real b,

$$f(t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} F(z)e^{zt}\,dz,$$

but F is *not* the Laplace transform of f. (Cf. Exercise 3, §8.1, and formula (7.12). In fact, F cannot be the Laplace transform of any function in $\mathscr{E}$ because $F(z) \not\to 0$ as $\operatorname{Re} z \to \infty$. This example shows that one must take care in applying the inversion formula to a function F when one does not know in advance that F is the Laplace transform of something.)

8.3 Applications: Ordinary differential equations

The Laplace transform is a very efficient way of solving ordinary differential equations with constant coefficients. Of course, such equations can also be solved by more elementary methods, but the Laplace transform has the advantage of handling inhomogeneous equations and incorporating initial conditions with a minimum of fuss. Moreover, one can often read off important properties of the solution from its Laplace transform without computing the inverse transform explicitly.

Let us therefore consider a kth-order equation with initial conditions:

$$\begin{aligned} &u^{(k)} + a_{k-1}u^{(k-1)} + \cdots + a_1u' + a_0u = f, \\ &u(0) = c_0,\ u'(0) = c_1, \dots,\ u^{(k-1)}(0) = c_{k-1}. \end{aligned} \tag{8.14}$$

Here $a_0, \dots, a_{k-1}, c_0, \dots, c_{k-1}$ are constants, and we assume that $f \in \mathscr{E}$. Hoping that the solution u will also be in $\mathscr{E}$ (which in fact it is), we apply the Laplace transform to (8.14). In view of formula (8.5), we obtain

$$\begin{aligned} &z^k\mathscr{L}u - (z^{k-1}c_0 + z^{k-2}c_1 + \cdots + c_{k-1}) + a_{k-1}z^{k-1}\mathscr{L}u \\ &- a_{k-1}(z^{k-2}c_0 + \cdots + c_{k-2}) \cdots + a_1z\mathscr{L}u - a_1c_0 + a_0\mathscr{L}u = \mathscr{L}f. \end{aligned}$$

In other words, we have

$$PU = F + Q$$

where

$$U = \mathscr{L}u, \qquad F = \mathscr{L}f, \qquad P(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0,$$
$$Q(z) = c_0 z^{k-1} + (c_1 + a_{k-1}c_0)z^{k-2} + \cdots + (c_{k-1} + a_{k-1}c_{k-2} + \cdots + a_2 c_1 + a_1 c_0).$$

The essential features here are the following. In the original problem (8.14) there are three ingredients: the differential operator $Lu = u^{(k)} + \cdots + a_0 u$, the inhomogeneous term f, and the initial data c_j. On the Laplace transform side, these ingredients turn into the polynomials P and Q and the function F. P depends only on the operator L; F depends only on f, and Q depends on L and the initial data but not on f.

It is now easy to obtain the solution. In fact, we have

$$U = FG + R \quad \text{where} \quad G = \frac{1}{P} \text{ and } R = \frac{Q}{P}.$$

G and R are both proper rational functions, so they are Laplace transforms of exponential polynomials g and r; as we saw in §§8.1–2, g and r can be calculated by using the inversion formula and the residue theorem or by decomposing G and R into partial fractions. Therefore, in view of Theorem 8.1(g), the solution u is given by

$$u = f * g + r.$$

The term $f * g$ represents the solution to (8.14) in which the initial data c_j are all replaced by zero, whereas the term r represents the solution to (8.14) in which f is replaced by zero. In other words, if we think of (8.14) as a model for a physical system influenced by an external force f, $f * g$ is the response of the system to f while r is the response of the system to the initial conditions.

In most real-life situations, the motion of the system will damp out in time if no external forces are applied; that is, the solutions of (8.14) with $f = 0$ (namely, the functions r above) will decay exponentially as $t \to \infty$. In this case the solutions r are called **transients**. It is easy to determine when this happens in terms of the Laplace transform. Indeed, the Laplace transform of r is the rational function Q/P. The partial fraction decomposition of Q/P is a linear combination of terms of the form $(z - z_j)^{-n_j}$ where the z_j's are the zeros of P, and $\mathscr{L}^{-1}\left[(z - z_j)^{-n_j}\right] = t^{n_j - 1}e^{z_j t}$. Hence, *the solutions of* (8.14) *with* $f = 0$ *tend to zero as* $t \to \infty$ *if and only if the zeros of* P *all have negative real part.*

Example 1. Consider the equation

$$u'' + 2u' + 5u = f, \qquad u(0) = c_0, \qquad u'(0) = c_1. \tag{8.15}$$

u can be interpreted, for instance, as the displacement of a unit mass attached to a spring: u'' is the acceleration, $5u$ is the restoring force of the spring, $2u'$

represents a frictional force proportional to the velocity; and f is an external force acting on the mass. With $U = \mathscr{L}u$ and $F = \mathscr{L}f$ as before, we have

$$z^2U(z) - (c_0z + c_1) + 2zU(z) - 2c_0 + 5U(z) = F(z),$$

or

$$U(z) = \frac{F(z)}{z^2+2z+5} + \frac{c_0z + c_1 + 2c_0}{z^2+2z+5}.$$

The zeros of $z^2 + 2z + 5$ are at $z = -1 \pm 2i$, so by formula (8.12) we have

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{z^2+2z+5}\right] &= \operatorname{Res}_{-1+2i}\frac{e^{zt}}{z^2+2z+5} + \operatorname{Res}_{-1-2i}\frac{e^{zt}}{z^2+2z+5}\\ &= \frac{e^{(-1+2i)t}}{4i} + \frac{e^{(-1-2i)t}}{-4i} = \tfrac{1}{2}e^{-t}\sin 2t,\end{aligned}$$

and likewise

$$\mathscr{L}^{-1}\left[\frac{c_0z + c_1 + 2c_0}{z^2+2z+5}\right] = c_0e^{-t}\cos 2t + \tfrac{1}{2}(c_1 + c_0)e^{-t}\sin 2t.$$

Hence,

$$u(t) = \tfrac{1}{2}\int_0^t e^{s-t}\sin 2(t-s)\, f(s)\, ds + c_0e^{-t}\cos 2t + \tfrac{1}{2}(c_1 + c_0)e^{-t}\sin 2t. \quad (8.16)$$

For a particular f, it is a matter of choice whether one computes u from the convolution integral in (8.16) or by inverting its Laplace transform directly. For instance, suppose the system is initially at rest, is subjected to a unit force between time 0 and time 1, and then is allowed to move unhindered. That is, suppose that in (8.15) we have

$$c_0 = c_1 = 0, \qquad f(t) = H(t) - H(t-1) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ 0 & \text{if } t > 1, \end{cases}$$

where H is the Heaviside function (8.3). Then

$$u(t) = \tfrac{1}{2}\int_0^{\min(1,t)} e^{s-t}\sin(t-s)\, ds,$$

which can be evaluated by some (slightly tedious) calculus. Alternatively, we have $F(z) = \mathscr{L}f(z) = (1 - e^{-z})/z$, so

$$U(z) = \frac{1}{z(z^2+2z+5)} - \frac{e^{-z}}{z(z^2+2z+5)}. \quad (8.17)$$

By formula (8.12), one easily obtains

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{z(z^2+2z+5)}\right] &= \frac{1}{5} + \frac{e^{(-1+2i)t}}{-8-4i} + \frac{e^{(-1-2i)t}}{-8+4i}\\ &= \frac{1}{5} - \frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t \equiv v(t),\end{aligned}$$

and then by Theorem 8.1(a),

$$u(t) = v(t) - H(t-1)v(t-1).$$

Warning. One must take care if one tries to compute the inverse Laplace transform of (8.17) directly from the inversion formula:

$$u(t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{e^{zt}\,dz}{z(z^2+2z+5)} - \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{e^{zt}e^{-z}\,dz}{z(z^2+2z+5)} \qquad (b > 0).$$

If $t > 1$, one can close up the contour in the left half-plane and obtain both integrals as the sum of the residues at $z = 0$ and $z = -1 \pm 2i$, as in (8.12). However, if $0 < t < 1$, the exponential $e^{(t-1)z}$ in the second integral tends to zero in the *right* half-plane. Hence the contour for the second integral must be closed up to the right; there are no poles in that region, so the integral is zero. Hence one obtains the solution $u(t) = v(t)$ for $t < 1$ and $u(t) = v(t) - v(t-1)$ for $t > 1$, as above.

The Laplace transform works equally well for systems of simultaneous ordinary differential equations. Consider, for example, the first-order system

$$\mathbf{u}' = A\mathbf{u} + \mathbf{f}, \qquad \mathbf{u}(0) = \mathbf{c}.$$

Here $\mathbf{u} = (u_1, \ldots, u_k)$ and $\mathbf{f} = (f_1, \ldots, f_k)$ are (column) vectors of functions, $A = (A_{ij})$ is a $k \times k$ matrix of constants, and $\mathbf{c}$ is a constant vector. Assuming that the components of $\mathbf{f}$ are in $\mathscr{E}$, from which it will follow that the same is true of the components of $\mathbf{u}$, we apply the Laplace transform componentwise to obtain

$$z\mathbf{U}(z) - \mathbf{c} = A\mathbf{U}(z) + \mathbf{F}(z), \quad \text{or} \quad (zI - A)\mathbf{U}(z) = \mathbf{F}(z) + \mathbf{c},$$

where $\mathbf{U} = \mathscr{L}\mathbf{u}$, $\mathbf{F} = \mathscr{L}\mathbf{f}$, and I is the identity matrix.

To solve for $\mathbf{U}$, we must invert the matrix $(zI - A)$. Cramer's rule (the formula for inverting a matrix in terms of determinants) shows that the entries of $(zI - A)^{-1}$ are proper rational functions of z whose denominators are the characteristic polynomial $\det(zI - A)$ (and whose poles are therefore the eigenvalues of A). Hence, we can form the matrix

$$\Phi = (\phi_{ij}) = \mathscr{L}^{-1}\left[(zI - A)^{-1}\right]$$

whose entries are the inverse Laplace transforms of the entries of $(zI - A)^{-1}$; these entries will be certain exponential polynomials. Now, since

$$\mathbf{U} = (zI - A)^{-1}\mathbf{F} + (zI - A)^{-1}\mathbf{c},$$

we have

$$\mathbf{u} = \Phi * \mathbf{f} + \Phi\mathbf{c},$$

where $\Phi * \mathbf{f}$ is the vector whose ith component is $\sum_j \phi_{ij} * f_j$ and $\Phi\mathbf{c}$ is the vector whose ith component is $\sum_j \phi_{ij} c_j$. This formula has the same interpretation as in the scalar case: $\Phi * \mathbf{f}$ is the response of the system to $\mathbf{f}$ and $\Phi\mathbf{c}$ is the response to the initial conditions.

Example 2. Consider the system

$$\begin{aligned} u_1' &= -8u_1 - 9u_2 + f_1, \quad & u_1(0) &= c_1, \\ u_2' &= 4u_1 + 4u_2 + f_2, \quad & u_2(0) &= c_2, \end{aligned}$$

Here we have

$$A = \begin{pmatrix} -8 & -9 \\ 4 & 4 \end{pmatrix}, \qquad zI - A = \begin{pmatrix} z+8 & 9 \\ -4 & z-4 \end{pmatrix},$$

so by an easy calculation, $\det(zI - A) = (z+2)^2$ and

$$(zI - A)^{-1} = \begin{pmatrix} \frac{z-4}{(z+2)^2} & \frac{-9}{(z+2)^2} \\ \frac{4}{(z+2)^2} & \frac{z+8}{(z+2)^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{z+2} - \frac{6}{(z+2)^2} & \frac{-9}{(z+2)^2} \\ \frac{4}{(z+2)^2} & \frac{1}{z+2} + \frac{6}{(z+2)^2} \end{pmatrix}.$$

By formula (8.7), the inverse Laplace transform of this is

$$\Phi(t) = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{pmatrix} = \begin{pmatrix} e^{-2t} - 6te^{-2t} & -9te^{-2t} \\ 4te^{-2t} & e^{-2t} + 6te^{-2t} \end{pmatrix}.$$

Hence, the solution of the homogeneous equation $\mathbf{u}' = A\mathbf{u}$ with initial condition $\mathbf{u}(0) = \mathbf{c}$ is

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} - (6c_1 + 9c_2)te^{-2t} \\ c_2 e^{-2t} + (4c_1 + 6c_2)te^{-2t} \end{pmatrix},$$

and the solution of the inhomogeneous equation $\mathbf{u}' = A\mathbf{u} + \mathbf{f}$ with $\mathbf{u}(0) = \mathbf{0}$ is

$$\mathbf{u} = \begin{pmatrix} \phi_{11} * f_1 + \phi_{12} * f_2 \\ \phi_{21} * f_1 + \phi_{22} * f_2 \end{pmatrix}.$$

For example, let us take $f_1(t) = e^{-t}$ and $f_2(t) = 0$; then

$$u_1(t) = (e^{-2t} - 6te^{-2t}) * e^{-t} = \int_0^t (e^{-2s} - 6se^{-2s})e^{s-t}\,ds = (6t+5)e^{-2t} - 5e^{-t},$$

$$u_2(t) = 4te^{-2t} * e^{-t} = \int_0^t 4se^{-2s}e^{s-t}\,ds = 4e^{-t} - 4(t+1)e^{-2t}.$$

The Laplace transform can sometimes be useful in solving differential equations with nonconstant coefficients. However, one potential difficulty must be kept in mind: Some of the solutions to the equation may not belong to the class $\mathscr{E}$ because they either blow up at some point t_0 or grow faster than exponentially as $t \to \infty$. In this case, the Laplace transform method will *not* detect such solutions.

Example 3. Consider Bessel's equation of order zero:

$$tu''(t) + u'(t) + tu(t) = 0.$$

If u satisfies this equation *and* u, u', and u'' are in $\mathscr{E}$, we can apply the Laplace transform, setting $U = \mathscr{L}u$, and use Theorem 8.1 to obtain

$$-\frac{d}{dz}\left[z^2U(z) - zu(0) - u'(0)\right] + \left[zU(z) - u(0)\right] - \frac{d}{dz}U(z) = 0,$$

or

$$(z^2 + 1)U'(z) + zU(z) = 0.$$

This gives $U'/U = -z/(z^2 + 1)$, whence $\log U = -\frac{1}{2}\log(z^2 + 1) + \log C$, or

$$U(z) = C(z^2 + 1)^{-1/2}.$$

We recognize this as the Laplace transform of CJ_0 that we calculated in §8.1. However, the second independent solution Y_0 of Bessel's equation has completely disappeared. The reason is that Y_0 is singular at the origin. Y_0 itself blows up only logarithmically as $t \to 0$, which is not bad enough to cause serious problems; but Y_0' and Y_0'' blow up like t^{-1} and t^{-2}, respectively. This means that they do not have Laplace transforms, as the integrals $\int_0^\infty Y_0'(t)e^{-zt}\,dt$ and $\int_0^\infty Y_0''(t)e^{-zt}\,dt$ diverge; therefore, the preceding calculations cannot be applied to them.

EXERCISES

In Exercises 1–8, solve the given initial value problem by using the Laplace transform.

1. $u'' + 4u = \sin \omega t$, $u(0) = u'(0) = 0 \quad (\omega > 0)$
2. $u'' + 4u' + 4u = f(t)$, $u(0) = c_0$, $u'(0) = c_1$
3. $u'' + 2u' + 2u = H(t - \pi) - H(t - 2\pi)$, $u(0) = 0$, $u'(0) = 1$
4. $u''' - u' = f(t)$, $u(0) = 1$, $u'(0) = -1$, $u''(0) = 0$
5. $u^{(4)} - u = t$, $u(0) = u'(0) = u''(0) = u'''(0) = 0$
6. $\begin{aligned} u_1' &= 3u_1 - 2u_2, & u_1(0) &= c_1 \\ u_2' &= 4u_1 - u_2, & u_2(0) &= c_2 \end{aligned}$
7. $\begin{aligned} u_1' &= u_1 - 2u_2 + 2e^{4t}, & u_1(0) &= 3 \\ u_2' &= -3u_1 + 2u_2 - 3e^{4t}, & u_2(0) &= 8 \end{aligned}$
8. $\begin{aligned} u_1' - u_2' + u_1 - -u_1u_2 &= 2t + 1, & u_1(0) &= u_2(0) = 1 \\ u_1' + u_2'' &= t, & u_2'(0) &= -1 \end{aligned}$
9. Consider the differential equation $t^2u'' - 2u = 2t$ $(t > 0)$.
   a. Use the Laplace transform to find a family of solutions containing one arbitrary constant.
   b. What is the general solution of this equation, and why does the procedure in part (a) not yield it? (Hint: The equation $t^2u'' - 2u = 0$ is of Euler type.)

10. Solve the equation $tu'' - (1+t)u' + u = 0$ by Laplace transforms. (For this equation the Laplace transform does yield the general solution.)
11. A radioactive isotope I_1 decays into another isotope I_2. I_2 decays, in turn, into I_3, and I_3 decays into the stable isotope I_4. Let $N_j(t)$ be the amount of isotope I_j at time t, and let c_1, c_2, and c_3 be the decay rates. Then,

$$N_1' = -c_1N_1, \quad N_2' = c_1N_1 - c_2N_2, \quad N_3' = c_2N_2 - c_3N_3, \quad N_4' = c_3N_3.$$

Suppose one starts at $t = 0$ with an amount A of the pure isotope I_1. How much I_4 is there at time t?

8.4 Applications: Partial differential equations

The Laplace transform is a useful tool for solving a wide variety of boundary value problems. In this section we present a small but fairly diverse selection of applications. The reader may observe that the initial stages of the calculations in these problems, leading to a formula for the Laplace transform of the solution, are similar to one another; but the final forms of the solutions are quite different. This illustrates the flexibility and power of the Laplace transform technique.

Let us consider the **telegraph equation**

$$u_{xx} = \alpha u_{tt} + \beta u_t + \gamma u \qquad (\alpha, \beta, \gamma \geq 0,\ \alpha\beta \neq 0). \tag{8.18}$$

This includes as special cases the wave equation ($\alpha = c^{-2}$, $\beta = \gamma = 0$) and the heat equation ($\beta = k^{-1}$, $\alpha = \gamma = 0$). (8.18) may be interpreted as the equation for transmission of an electromagnetic signal along a cable, where the constants α, β, γ are related to the resistance, inductance, capacitance, and leakage conductance of the cable. (See Exercise 7, §1.1, and also Körner [34], §§65–66.) We shall study (8.18) for $t > 0$ and x in either an interval $[0, l]$ or a half-line $[0, \infty)$, the latter being a useful model for a very long cable. We shall be particularly interested in solving (8.18) subject to an inhomogeneous boundary condition $u(0, t) = f(t)$. One should think of $f(t)$ as the signal that is sent from the end $x = 0$ of the cable; in the case $\alpha = 0$ one may think of a time-varying temperature that is imposed on the end of a rod.

In addition to the boundary condition $u(0, t) = f(t)$ we need some initial conditions:

$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = \psi(x).$$

(If $\alpha = 0$, so that (8.18) is first-order in t, the second condition should be omitted.) However, by the superposition principle, this general problem can be reduced to the two special cases $f = 0$ and $\phi = \psi = 0$. Although the Laplace transform can be used for both of these cases, we shall concentrate on the second one. The first one can also be handled by Fourier sine series (in the case of a finite interval) or the Fourier sine transform (in the case of a half-line); but see Exercises 6–8.

We first consider the case of a half-line, where our problem is

$$u_{xx} = \alpha u_{tt} + \beta u_t + \gamma u, \qquad u(0,t) = f(t), \qquad u(x,0) = u_t(x,0) = 0. \tag{8.19}$$

(There is also an implicit assumption that $u(x,t)$ remains bounded as $x \to \infty$.) We wish to apply the Laplace transform with respect to t, so we assume that $f \in \mathscr{E}$ and search for a solution u that is in $\mathscr{E}$ as a function of t along with its derivatives u_x, u_{xx}, u_t, and u_{tt}. In this case we have

$$\mathscr{L}[u_{xx}] = \int_0^\infty \frac{\partial^2 u}{\partial x^2}(x,t)e^{-zt}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,t)e^{-zt}\,dt = [\mathscr{L}u]_{xx},$$

so if we set $F = \mathscr{L}f$ and $U = \mathscr{L}u$, the Laplace transform of (8.19) is

$$U_{xx} = (\alpha z^2 + \beta z + \gamma)U, \qquad U(0,z) = F(z).$$

Here we have an ordinary differential equation for U as a function of x, with z playing the role of an extra parameter. The general solution is

$$U = a_1 e^{xq} + a_2 e^{-xq}, \qquad a_1 + a_2 = F(z), \qquad q = \sqrt{\alpha z^2 + \beta z + \gamma},$$

where we take the principal branch of the square root (the one lying in the right half-plane). Since $x > 0$, to have something that is bounded and analytic in some half-plane $\operatorname{Re} z > a$ so as to invert the Laplace transform, we must reject the solution e^{xq} and take

$$U(x,z) = F(z)e^{-xq}, \qquad q = \sqrt{\alpha z^2 + \beta z + \gamma}.$$

It remains to invert the Laplace transform. In general this results in a rather complicated expression involving Bessel functions; we shall restrict our attention to some important special cases that turn out more simply.

The first such case is that of the wave equation, namely, $\alpha = c^{-2}$, $\beta = \gamma = 0$. Here we have

$$U(x,z) = e^{-xz/c}F(z),$$

whose inverse Laplace transform can be read off immediately from Theorem 8.1(a):

$$u(x,t) = H(t - c^{-1}x)f(t - c^{-1}x).$$

This comes as no surprise: the signal $f(t)$ simply moves down the line, reaching position x after a time delay of $c^{-1}x$.

Almost equally easy is the case in which $\beta^2 = 4\alpha\gamma$, so that $\alpha z^2 + \beta z + \gamma$ is a perfect square. In this case, let us write $\alpha = c^{-2}$ (as before) and $b = \beta/2\alpha$; then $\alpha z^2 + \beta z + \gamma = c^{-2}(z+b)^2$, so

$$U(x,z) = e^{-x(z+b)/c}F(z).$$

Again the inverse Laplace transform can be read off from Theorem 8.1:

$$u(x,t) = e^{-bx/c} H(t - c^{-1}x) f(t - c^{-1}x).$$

This represents the important situation in which the signal gets weaker by the factor $e^{-bx/c}$ as it progresses down the line but is otherwise undistorted.

Next we have the case of the heat equation, that is, $\alpha = \gamma = 0$, $\beta = k^{-1}$, in which

$$U(x,z) = e^{-x\sqrt{z/k}} F(z).$$

Here we can express the solution u as a convolution by computing the inverse Laplace transform of $e^{-x\sqrt{z/k}}$. In §8.2 we showed that

$$\mathscr{L}\left[\frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}\right] = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

It follows from Theorem 8.1(f) that

$$\mathscr{L}\left[\frac{x}{\sqrt{4\pi t^3}} e^{-x^2/4t}\right] = \int_z^\infty \frac{xe^{-x\sqrt{w}}}{2\sqrt{w}}\,dw = e^{-x\sqrt{z}}$$

and then from Theorem 8.1(b) that

$$\mathscr{L}\left[\frac{x}{\sqrt{4\pi k t^3}} e^{-x^2/4kt}\right] = e^{-x\sqrt{z/k}}.$$

The solution u is therefore the convolution of f (with respect to t) with the function $x(4\pi k t^3)^{-1/2} e^{-x^2/4kt}$:

$$u(x,t) = \frac{x}{\sqrt{4\pi k}} \int_0^t f(t-s) s^{-3/2} e^{-x^2/4ks}\,ds.$$

The case $\alpha = 0$, $\beta > 0$, $\gamma > 0$ is similar; see Exercise 1.

Let us now look at the transmission problem on a finite interval $[0, l]$, with zero boundary conditions at the right end. We shall restrict attention to the case of the wave equation:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = u_t(x,0) = 0, \qquad u(0,t) = f(t), \qquad u(l,t) = 0.$$

Taking the Laplace transform, we obtain the ordinary differential equation

$$z^2 U(x,z) = c^2 U_{xx}(x,z), \qquad U(0,z) = F(z), \qquad U(l,z) = 0,$$

whose solution is

$$U(x,z) = F(z) \frac{\sinh[(l-x)z/c]}{\sinh(lz/c)}. \tag{8.20}$$

At this point there are a couple of options for inverting the Laplace transform. Perhaps the most illuminating method is as follows: We multiply and divide by $e^{-lz/c}$ to obtain

$$\frac{\sinh[(l-x)z/c]}{\sinh(lz/c)} = \frac{e^{(l-x)z/c} - e^{-(l-x)z/c}}{e^{lz/c} - e^{-lz/c}} = \frac{e^{-zx/c} - e^{z(x-2l)/c}}{1 - e^{-2lz/c}}$$

and then expand the denominator in a geometric series:

$$\begin{aligned}\frac{\sinh[(l-x)z/c]}{\sinh(lz/c)} &= (e^{-xz/c} - e^{-(2l-x)z/c})\sum_0^\infty e^{-2nlz/c} \\ &= e^{-xz/c} - e^{-(2l-x)z/c} + e^{-(2l+x)z/c} - e^{-(4l-x)z/c} + e^{-(4l+x)z/c} - \cdots.\end{aligned}$$

After multiplying this by $F(z)$, we can invert the Laplace transform simply by using Theorem 8.1(a):

$$\begin{aligned}u(x,t) = &H\left(t-\frac{x}{c}\right)f\left(t-\frac{x}{c}\right) - H\left(t-\frac{2l-x}{c}\right)f\left(t-\frac{2l-x}{c}\right) \\ &+ H\left(t-\frac{2l+x}{c}\right)f\left(t-\frac{2l+x}{c}\right) - \cdots.\end{aligned} \tag{8.21}$$

Observe that for any fixed t, only finitely many terms of this series are nonzero, namely, the first n terms when $t < nl/c$. The physical interpretation is immediate. The first term represents the wave traveling down the line from the source $x = 0$. When it reaches the end $x = l$, it reflects and returns upside down; this is the second term, with its minus sign. When it gets back to $x = 0$ it reflects again and goes forward in its initial form; this is the third term. And so forth.

Another way one might think of obtaining u from (8.20) is to write u as the convolution of f with the inverse Laplace transform of

$$G(z,x) = \frac{\sinh[(l-x)z/c]}{\sinh(lz/c)}.$$

The trouble with this is that G is not the Laplace transform of a function: Although it decays exponentially as $|\operatorname{Re} z| \to \infty$ for $0 < x < l$, it does not vanish as $|\operatorname{Im} z| \to \infty$. (It is the Laplace transform of a generalized function, but that's another story.) However, if $F = \mathscr{L}f$ is a meromorphic function with suitable decay at infinity, one may be able to invert the Laplace transform of $F(z)G(x,z)$ directly by using the calculus of residues as we illustrated in §8.2. For example, suppose that the signal f is a simple sine wave:

$$f(t) = \sin t.$$

Then $F(z) = (z^2+1)^{-1}$, and so

$$u(x,t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{zt}F(z)G(x,z)\,dz = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\sinh[(l-x)z/c]}{(z^2+1)\sinh(lz/c)}\,dz.$$

The integrand is a meromorphic function of z with poles at $z = \pm i$ and $z = \pm in\pi l/c$ ($n = 1, 2, 3, \ldots$; the singularity at $z = 0$ is removable). The extra factor of $z^2 + 1$ is enough to guarantee that the contour of integration can be closed up in the left half-plane, so that $u(x,t)$ is the sum of the residues of $e^{zt}F(z)G(x,z)$ at all its poles.

We shall leave the details of the this calculation to the reader (Exercise 10); the results are as follows. When $l/\pi c$ is not an integer (the typical case), the poles are all simple. After adding together the residues at i and $-i$ and those at $in\pi l/c$ and $-in\pi l/c$, one finds that

$$u(x,t) = \sin t \frac{\sin[(l-x)/c]}{\sin(l/c)} + \sum_{1}^{\infty} \frac{2lc}{l^2 - n^2\pi^2c^2} \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}. \tag{8.22}$$

If $l/\pi c$ is an integer N, the frequency of the signal $f(t)$ coincides with one of the natural frequencies of the transmission line, and a resonance phenomenon occurs. Mathematically, this is reflected in the fact that the poles of $F(z)$ coincide with two of the poles of $G(x,z)$, resulting in a pair of double poles at $\pm i = \pm iN\pi c/l$. These two poles must be treated separately, and the result is

$$\begin{aligned} u(x,t) = \sum_{n \neq N} \frac{2lc}{l^2 - n^2\pi^2c^2} \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} + \left(1 - \frac{x}{l}\right) \sin t \cos\frac{N\pi x}{l} \\ + \frac{1}{2N\pi} \sin t \sin\frac{N\pi x}{l} - \frac{1}{N\pi} t \cos t \sin\frac{N\pi x}{l}. \end{aligned} \tag{8.23}$$

The most important term in (8.23) for large t is the last one, which becomes unbounded as $t \to \infty$: this is the resonance effect. It is interesting to note that the sum of the other terms in (8.23), as well as the whole sum in (8.22), remains bounded as $t \to \infty$, for the series can be bounded uniformly in x and t by a multiple of $\sum n^{-2}$. Even though the line is (so to speak) becoming more and more crowded with multiply reflected signals as t increases, as formula (8.21) shows, they tend to cancel one another out unless resonance occurs.

EXERCISES

1. Solve: $u_t = ku_{xx} - au$ for $x > 0$, $u(x,0) = 0$, $u(0,t) = f(t)$.
2. Find the temperature in a semi-infinite rod (the half-line $x > 0$) if its initial temperature is 0 and the end $x = 0$ is held at temperature 1 for $0 < t < 1$ and thereafter held at temperature 0.
3. Consider heat flow in a semi-infinite rod when heat is supplied to the end at a constant rate c:

$$u_t = ku_{xx} \text{ for } x > 0, \quad u(x,0) = 0, \quad u_x(0,t) = -c.$$

   a. With the aid of formula (8.13), show that

$$u(x,t) = c\sqrt{\frac{k}{\pi}} \int_0^t s^{-1/2} e^{-x^2/4ks}\, ds.$$

b. By substituting $\sigma = x/\sqrt{4ks}$ and then integrating by parts, show that

$$u(x,t) = c\sqrt{\frac{4kt}{\pi}}e^{-x^2/4kt} - cx\,\mathrm{erfc}\,\frac{x}{\sqrt{4kt}}.$$

4. A semi-infinite rod is initially at temperature 1. Its end is in contact with a medium at temperature zero and loses heat according to Newton's law of cooling:

$$u_t = ku_{xx} \text{ for } x > 0, \quad u(x,0) = 1, \quad u_x(0,t) = cu(0,t).$$

a. Show that

$$\mathscr{L}u(x,z) = \frac{1}{z} - \frac{c\sqrt{k}}{z(c\sqrt{k}+\sqrt{z})}e^{-x\sqrt{z/k}}.$$

b. Using entry 23 of Table 3, show that the temperature at the end is given by $u(0,t) = e^{c^2kt}\,\mathrm{erfc}(c\sqrt{kt})$. (Hint: Multiply and divide by $\sqrt{z} - c\sqrt{k}$.)

5. Consider heat flow in a rod of length l with initial temperature zero when one end is held at temperature zero and the other end at a variable temperature $f(t)$:

$$u_t = ku_{xx}, \quad u(x,0) = 0, \quad u(0,t) = 0, \quad u(l,t) = f(t).$$

Let $v(x,t)$ be the solution to this problem in the special case $f(t) \equiv 1$. (The explicit formula for v is given in Exercise 7, but it is not needed here.) Use the Laplace transform to obtain *Duhamel's formula* for u in terms of v:

$$u(x,t) = \frac{\partial}{\partial t}\int_0^t f(s)v(x,\,t-s)\,ds.$$

6. Consider heat flow in an infinite rod: $u_t = ku_{xx}$ for $x \in \mathbf{R}$, $u(x,0) = f(x)$.

a. Obtain the differential equation $kU_{xx} = zU - f(x)$ for $U = \mathscr{L}u$, and show that the only solution of this equation that tends to zero as $\mathrm{Re}\,z \to +\infty$ is

$$U(x,z) = \frac{1}{\sqrt{4kz}}\left[\int_{-\infty}^{x} e^{(y-x)\sqrt{z/k}}f(y)\,dy + \int_x^\infty e^{(x-y)\sqrt{z/k}}f(y)\,dy\right].$$

(Cf. Exercise 1, §7.3.)

b. Invert the Laplace transform by using formula (8.13) to obtain the convolution formula for u that was derived in §7.3.

7. Consider heat flow in a rod of length l:

$$u_t = ku_{xx}, \quad u(x,0) = 0, \quad u(0,t) = 0, \quad u(l,t) = A.$$

a. Show that $\mathscr{L}u(x,z) = [A\sinh x\sqrt{z/k}\,]/[z\sinh l\sqrt{z/k}\,]$.

b. By the technique used to obtain (8.21), show that

$$u(x,t) = A\sum_0^\infty \left(\operatorname{erf}\frac{(2n+1)l+x}{\sqrt{4kt}} - \operatorname{erf}\frac{(2n+1)l-x}{\sqrt{4kt}}\right).$$

(Compare this with the Fourier expansion of u obtained in §4.2:

$$u(x,t) = \frac{Ax}{l} - 2A\sum_1^\infty \frac{(-1)^{n+1}}{n\pi} e^{-n^2\pi^2kt/l^2} \sin\frac{n\pi x}{l}.$$

The former series converges very rapidly when t is small, whereas the latter one converges very rapidly when t is large. Hence they are both computationally useful.)

8. In this exercise we solve the vibrating string problem

$$u_{tt} = c^2 u_{xx}, \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0$$

by the Laplace transform.

a. Show that $U = \mathscr{L}u$ satisfies $c^2 U_{xx} - z^2 U = -zf(x)$, $U(0,z) = U(l,z) = 0$.

b. Show that $U(x,z) = G(x,z)/\sinh(lz/c)$ where $G(x,z)$ equals

$$\frac{1}{c}\sinh\frac{(l-x)z}{c}\int_0^x f(y)\sinh\frac{yz}{c}\,dy + \frac{1}{c}\sinh\frac{xz}{c}\int_x^l f(y)\sinh\frac{(l-y)z}{c}\,dy.$$

(Hint: Use variation of parameters, taking $\sinh xz/c$ and $\sinh(l-x)z/c$ as a basis for the solutions of the homogeneous equation $c^2 v_{xx} - z^2 v = 0$.)

c. Show that

$$G(x, in\pi c/l) = \frac{(-1)^n l}{2c} b_n \sin\frac{n\pi x}{l} \quad \text{where} \quad b_n = \frac{2}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy.$$

d. Observe that $G(x,z)$ is an entire analytic function of z, so $U(x,z)$ is analytic except for poles at the points $in\pi c/l$. Thence apply the Laplace inverson formula to obtain the Fourier series for u that was derived in §2.5.

9. Consider the boundary value problem

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = u_t(x,0) = 0, \quad u(0,t) = 0, \quad u_x(l,t) = f(t).$$

(This represents vibrations in a string or air column when the left end is fixed and the right end is subjected to a force proportional to $f(t)$.) Let $U = \mathscr{L}u$ and $F = .\mathscr{L}f$.

a. Show that

$$U(x,z) = F(z)\frac{c\sinh(xz/c)}{z\cosh(lz/c)}.$$

b. By the method used to derive (8.21), show that

$$u(x,t) = c\sum_0^\infty (-1)^n\Big[g\big(t-\phi_n(x)\big) - g\big(t-\phi_n(-x)\big)\Big] \quad \text{where}$$

$$g(\tau) = \int_0^\tau f(t)\,dt, \qquad \phi_n(x) = \frac{(2n+1)l-x}{c}.$$

c. Show that $u(l,t) = c(f*S)(t)$ where S is the square wave with period $2l$ (cf. Exercise 12b, §8.1). In particular, if f is constant, show that $u(l,t)$ is a triangle wave (cf. Exercise 12c, §8.1).

10. Complete the derivation of (8.22) and (8.23).

8.5 Applications: Integral equations

A number of problems in pure and applied mathematics lead to equations of the form

$$\int_0^t \kappa(t,s)u(s)\,ds = f(t) \quad \text{or} \quad u(t) + \int_0^t \kappa(t,s)u(s)\,ds = f(t). \tag{8.24}$$

Here t ranges over $(0,\infty)$, f and κ are given functions, and the equation is to be solved for u. (Generally κ is considered as "fixed," and one is interested in the relationship between f and u.) These equations are known as **Volterra integral equations** of the first and second kind, respectively. There are several techniques for studying such equations; here we are concerned with the use of the Laplace transform in the situation where the function $\kappa(t,s)$ depends only on the difference $t-s$:

$$\kappa(t,s) = k(t-s).$$

In this case the integrals in (8.24) are convolutions, and the integral equations become

$$k*u = f \quad \text{or} \quad u + k*u = f.$$

Assuming that u, f, and k possess Laplace transforms U, F, and K, these equations are equivalent to

$$KU = F \quad \text{or} \quad (1+K)U = F,$$

so that

$$U = \frac{F}{K} \quad \text{or} \quad U = \frac{F}{1+K}.$$

It now remains to invert the Laplace transform to obtain the solution u.

In the first case, $U = F/K$, this may not be possible. $K(z)$ is analytic in some half-plane $\operatorname{Re} z > a$ and tends to zero as $z \to \infty$ there, so $1/K$ blows up as $z \to \infty$. In particular, $1/K$ is not the Laplace transform of any function, and neither is F/K unless F has suitable decay properties at infinity. (Worse things can happen: K may have zeros with arbitrarily large real part, in which case F/K will not be analytic in any right half-plane unless F has zeros at the same places as K.) The intuitive reason for this difficulty is that the operation of convolution with k tends to make functions smoother, so the equation $u * k = f$ has no solution unless f is already smooth.

In the second case, $U = F/(1+K)$, this problem does not arise. Since $K(z)$ vanishes as $z \to \infty$, $1 + K(z)$ is approximately 1 for large z, so applying the inversion formula to $F/(1+K)$ is not too different from applying it to F. In fact,

$$\frac{F}{1+K} = F + K'F \quad \text{where } K' = \frac{-K}{1+K}.$$

Assuming that K' is the Laplace transform of a function k' (which it generally is), we obtain

$$u = f + k' * f.$$

Thus, the solution u is given in terms of f by the same sort of integral equation as the original one that gave f in terms of u.

Let us compute a few examples.

Example 1. Let $k(t) = \lambda e^{-t}$, where λ is a constant. To solve

$$u(t) + \lambda \int_0^t e^{s-t} u(s)\, ds = f(t),$$

we apply the Laplace transform to obtain

$$U(z) + \frac{\lambda U(z)}{z+1} = F(z).$$

Hence

$$U(z) = \frac{z+1}{z+\lambda+1} F(z) = F(z) - \frac{\lambda}{z+\lambda+1} F(z).$$

Inversion of the Laplace transform now yields

$$u(t) = f(t) - \lambda \int_0^t e^{(\lambda+1)(s-t)} f(s)\, ds.$$

Example 2. Let $k(t) = t$. Applying the Laplace transform to

$$u(t) + \int_0^t (t-s) u(s)\, ds = f(t)$$

yields

$$U(z) + z^{-2}U(z) = F(z), \quad \text{or} \quad U(z) = \frac{z^2 F(z)}{z^2+1} = F(z) - \frac{F(z)}{z^2+1}.$$

Therefore,

$$u(t) = f(t) - \int_0^t \sin(t-s)\, f(s)\, ds.$$

For particular functions f it may be easier to find the inverse Laplace transform of $z^2F(z)/(z^2+1)$ directly instead of using the convolution integral. For example, if $f(t) = t$, then $F(z) = 1/z^2$, so $U(z) = 1/(z^2+1)$ and $u(t) = \sin t$.

Example 3. Consider the Volterra equation of the first kind

$$\int_0^t (t-s)^4 u(s)\, ds = f(t).$$

The Laplace transform of this equation is

$$\frac{4!}{z^5}U(z) = F(z), \quad \text{or} \quad U(z) = \frac{z^5 F(z)}{24}.$$

Now, by formula (8.5),

$$z^5 F(z) = \mathscr{L}\big[f^{(5)}\big](z) + z^4 f(0) + z^3 f'(0) + z^2 f''(0) + z f^{(3)}(0) + f^{(4)}(0).$$

Hence, *if* f is five times differentiable, *if* $f^{(j)} \in \mathscr{E}$ for $j \leq 5$, and *if* $f^{(j)}(0) = 0$ for $j \leq 4$, *then* the solution is $u(t) = f^{(5)}(t)/24$. But if f is not sufficiently smooth, or if it does not vanish to fourth order at $t = 0$, then $z^5F(z)$ is not the Laplace transform of any function, and the original equation can be solved only in terms of generalized functions.

The integral equation $u + k * u = f$ is often called the **renewal equation** because it occurs in the study of renewal processes such as changes in population because of births and deaths. Suppose, for example, that a community contains N_0 people at time $t = 0$, and we wish to determine the population $N(t)$ at times $t > 0$. This can be done in terms of the following empirically determined pieces of information:

(i) For all $T > 0$, the probability $p(T)$ that a person will live for at least T units of time after birth.

(ii) The age distribution $a(\tau)$ of the initial population. That is, for $\Delta\tau$ small, $a(\tau)\Delta\tau$ is the proportion of the initial population whose age is between τ and $\tau + \Delta\tau$.

(iii) The birth rate R, which we shall assume to be constant. That is, in a time interval Δt, on the average each person will produce $R\Delta t$ offspring.

Now, at time t the population will consist of people from the original group and people who have been born in the meantime. In the initial population, for each $\tau > 0$ there are $N_0 a(\tau)\,d\tau$ people of age τ, and of these, $N_0 p(\tau + t)a(\tau)\,d\tau$ will survive to time t (when their age will be $\tau + t$). Adding all these up, we find that the number of survivors of the initial population is

$$\phi(t) = N_0 \int_0^\infty p(\tau + t)a(\tau)\,d\tau.$$

This is not quite a convolution, but it is a function that can be calculated from the known functions p and a. Also, at each time s with $0 < s < t$ there will be $RN(s)\,ds$ people being born, and of these, $Rp(t-s)N(s)\,ds$ will survive to time t (when their age will be $t - s$). Adding all these up, we find that the added population due to births is

$$R \int_0^t p(t-s)N(s)\,ds.$$

Therefore,

$$N(t) = \phi(t) + R \int_0^t p(t-s)N(s)\,ds.$$

This is a Volterra equation that can be solved for $N(t)$ in terms of ϕ and p.

A number of other processes can be modeled by the same equation. For instance, instead of people dying and being born, one can think of mechanical equipment wearing out and being replaced. See Bellman-Roth [5].

We conclude this section with one of the oldest examples (due to Niels Abel, around 1825) of a problem in analytical mechanics that leads to an integral equation. Think of the y-axis as being vertical, and think of particles moving in the xy-plane under the influence of gravity. Consider a curve C rising to the right from the origin, such as the solid curve in Figure 8.5, and consider a particle constrained to move along C (a skier on a ski jump, perhaps). Suppose the particle starts at rest at the point P on C and falls along C under the influence of gravity (with no other forces present). If we know the equation of C, it is a fairly simple matter to calculate the time for the particle to reach the origin. Abel's problem is the inverse of this one: *Determine the curve C so that the time for a particle to reach the origin from the point P at height h on C is a given function $f(h)$.*

To analyze Abel's problem, let σ denote the arc length of the curve C, measured from the origin. We think of σ as a function of the y-coordinate along C, $\sigma = \sigma(y)$. The kinetic energy of the particle at height y is equal to the change in potential energy from height h to height y,

$$\frac{1}{2}m\left(\frac{d\sigma}{dt}\right)^2 = mg(h-y),$$

where m is the mass of the particle and g is the acceleration due to gravity. Hence,

$$d\sigma = -\sqrt{2g(h-y)}\,dt.$$

(The minus sign is there because the particle is moving towards the origin, so that σ is a decreasing function of t.) On the other hand, $d\sigma = \sigma'(y)\,dy$, so

$$dt = -\frac{\sigma'(y)\,dy}{\sqrt{2g(h-y)}}.$$

If we integrate from $y = h$ to $y = 0$, on the left we obtain the time $f(h)$ for the particle to reach the origin, so

$$f(h) = -\int_h^0 \frac{\sigma'(y)\,dy}{\sqrt{2g(h-y)}} = \int_0^h \frac{\sigma'(y)\,dy}{\sqrt{2g(h-y)}}. \tag{8.25}$$

This is Abel's integral equation for the arc-length function σ. It is a Volterra equation of the first kind.

FIGURE 8.5. A cycloid; the solid portion of the curve is a tautochrone.

Let us solve Abel's equation in the special case when $f(h)$ is a constant T — that is, when the particle takes the same time T to reach the origin no matter where it starts. (The curve C that has this property is called the **tautochrone**. The idea is that C should be nearly flat near the origin but steep farther away, so that a particle that starts near the origin will travel slowly, whereas a particle that starts far away will pick up speed quickly.) We apply the Laplace transform to (8.25), with $f(h) \equiv T$, using the fact that $\mathscr{L}[t^{-1/2}] = \Gamma(\frac{1}{2})z^{-1/2} = \sqrt{\pi/z}$. Setting $S = \mathscr{L}\sigma'$, we obtain

$$\frac{T}{z} = S(z)\sqrt{\frac{\pi}{2gz}}, \quad \text{or} \quad S(z) = \frac{T\sqrt{2g}}{\sqrt{\pi}}\frac{1}{\sqrt{z}},$$

and hence

$$\sigma'(y) = \frac{T\sqrt{2g}}{\pi}\frac{1}{\sqrt{y}}.$$

We can use this formula to find an equation for the curve C. Recalling the standard formula for arc length, we have

$$1 + \left(\frac{dx}{dy}\right)^2 = \left[\sigma'(y)\right]^2 = \frac{A}{y}, \qquad A = \frac{2gT^2}{\pi^2},$$

or

$$dx = \sqrt{\frac{A-y}{y}}\,dy.$$

This equation can be integrated by the substitution $y = \frac{1}{2}A(1-\cos\theta)$, which gives

$$\frac{A-y}{y} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{(1+\cos\theta)^2}{\sin^2\theta}, \qquad dy = \tfrac{1}{2}A\sin\theta\,d\theta,$$

so that

$$dx = \tfrac{1}{2}A(1+\cos\theta)d\theta.$$

Integrating this and taking account of the fact that $x = 0$ when $y = 0$ (i.e., when $\theta = 0$), we obtain the parametric equations for C:

$$x = \tfrac{1}{2}A(\theta + \sin\theta), \qquad y = \tfrac{1}{2}A(1-\cos\theta), \qquad A = \frac{2gT^2}{\pi^2}.$$

These are the equations for a cycloid with a minimum at the origin and height A; see Figure 8.5. This cycloid extends indefinitely to the left and the right as indicated by the dashed curve in the figure, but the portion of it that solves the tautochrone problem as stated above is the half-arch between the origin and $(\frac{1}{2}A\pi, A)$, corresponding to the parameter interval $0 \le \theta \le \pi$ and shown as a solid curve in the figure. (All the other half-arches of the cycloid are also tautochrones.)

EXERCISES

In Exercises 1–4, solve the given integral equation for u.

1. $u(t) - a^2 \int_0^t (t-s)u(s)\,ds = t^2 \quad (a > 0)$
2. $u(t) - \frac{1}{6}\int_0^t (t-s)^3 u(s)\,ds = f(t)$
3. $u(t) + 2\int_0^t u(t-s)\cos as\,ds = \sin at \quad (a > 0)$
4. $\int_0^t u(s)u(t-s)\,ds = t^5 e^{-3t}$
5. Suppose $k \in \mathscr{E}$ and $\mathscr{L}k = 1/\sqrt{P}$ where P is a polynomial of degree n. Suppose also that f and its first n derivatives are in $\mathscr{E}$ and that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Show that the equation $u * k = f$ has the solution $u = k * [P(D)f]$ where

$$P(z) = a_n z^n + \cdots + a_1 z + a_0, \qquad P(D)f = a_n f^{(n)} + \cdots + a_1 f' + a_0 f.$$

6. Use Exercise 5 to solve $\int_0^t u(t-s)\operatorname{erf}\sqrt{s}\,ds = f(t)$, where $f(0) = f'(0) = f''(0) = 0$. (Cf. entry 22 of Table 3.)
7. A company buys N_0 new light bulbs at time $t = 0$ and thereafter buys new bulbs at the rate $r(t)$. Suppose the probability that a light bulb will last T

units of time after purchase is $p(T)$. Let $N(t)$ be the number of light bulbs in use at time t. Show that

$$N(t) = N_0 p(t) + \int_0^t r(s)p(t-s)\,ds.$$

8. In Exercise 7, suppose that $p(T) = e^{-cT}$. What should the replacement rate $r(t)$ be if the number of light bulbs needed for use at time t is (a) $N(t) \equiv N_0$ or (b) $N(t) = N_0(2 - e^{-t})$?
9. Show that there exists a curve C as in Figure 8.5 such that the time to fall to the origin along C from height h is proportional to h^α, if and only if $-\frac{1}{2} < \alpha \le \frac{1}{2}$. What are the solution curves in the case $\alpha = \frac{1}{2}$? (Hint: The restriction on α appears only after Abel's equation is solved for σ'.)

8.6 Asymptotics of Laplace transforms

In this section we explore an important and useful principle, namely, that *the behavior of $\mathscr{L}f(z)$ as $z \to \infty$ is intimately connected with the behavior of $f(t)$ as $t \to 0$.*

To avoid confusion, we need to make a few prefatory remarks. We shall be looking at functions $f(t)$ on the half-line $[0, \infty)$ and considering their Taylor expansions about $t = 0$. This is in conflict with the convention we adopted earlier that $f(t)$ is taken to be zero for $t < 0$, for functions that vanish for $t < 0$ (or their derivatives) will usually have a discontinuity at $t = 0$, so they will not possess a Taylor expansion. Rather, when we speak of a function f being of class $C^{(k)}$ near 0, we mean that it is the restriction to $[0, \infty)$ of a function $\tilde{f}$ on $\mathbf{R}$ that is of class $C^{(k)}$ on some interval centered at 0, and $f^{(k)}(0)$ is by definition $\tilde{f}^{(k)}(0)$. (The same point arose in connection with formula (8.5).)

The crucial point is that if $f(t)$ vanishes to high order as $t \to 0$, then $\mathscr{L}f(z)$ vanishes to high order as $\operatorname{Re} z \to \infty$. More precisely, suppose $f \in \mathscr{E}$ and $|f(t)| \le c_0 t^b$ for $t \le 1$, where c_0 and b are positive numbers. Since f grows at most exponentially for large t, it then satisfies an estimate of the form

$$|f(t)| \le ct^b e^{at} \quad \text{for all } t > 0 \qquad (a, b, c \ge 0). \tag{8.26}$$

$\mathscr{L}f(z)$ is then defined for $\operatorname{Re} z > a$, and it satisfies the following estimate.

Lemma 8.2. *If $f \in \mathscr{E}$ and f satisfies (8.26), then for any $\epsilon > 0$ there is a constant C_ϵ such that*

$$|\mathscr{L}f(z)| \le C_\epsilon |z|^{-b-1} \quad \textit{when } \operatorname{Re} z \ge a+1 \textit{ and } |\arg z| \le \tfrac{1}{2}\pi - \epsilon. \tag{8.27}$$

Proof: If $z = x + iy$ with $x > a$, we have

$$|\mathscr{L}f(z)| \le \int_0^\infty |f(t)e^{-zt}|\,dt \le c\int_0^\infty t^b e^{-(x-a)t}\,dt = c\Gamma(b+1)(x-a)^{-b-1}$$

(by the substitution $t = u/(x-a)$). Now, if $\operatorname{Re} z = x \ge a + 1$ and $|\arg z| = |\arctan(y/x)| \le \frac{1}{2}\pi - \epsilon$, then

$$x - a = \frac{x-a}{x}x \ge \frac{1}{a+1}x \ge \frac{1}{a+1}(\sin\epsilon)(x^2+y^2)^{1/2} = \frac{1}{a+1}(\sin\epsilon)|z|,$$

so

$$|\mathscr{L}f(z)| \le C_\epsilon|z|^{-b-1}, \qquad C_\epsilon = c\Gamma(b+1)(a+1)^{b+1}(\csc\epsilon)^{b+1}. \qquad \blacksquare$$

Now suppose that $g \in \mathscr{E}$ and and g is of class $C^{(k+1)}$ near $t = 0$. Taylor's theorem says that g is the sum of a polynomial of degree k and a remainder term that vanishes to order $k+1$ at $t = 0$:

$$\begin{aligned} g(t) &= \sum_0^k c_j t^j + r_k(t), \\ c_j &= \frac{g^{(j)}(0)}{j!}, \qquad |r_k(t)| \le ct^{k+1} \quad \text{for } t \text{ near } 0. \end{aligned} \tag{8.28}$$

(The estimate for r_k can be deduced from any of the standard formulas for the remainder term in Taylor's theorem. For example, Lagrange's formula says that for some $\tau \in (0, t)$,

$$r_k(t) = \frac{g^{(k+1)}(\tau)}{(k+1)!}t^{k+1},$$

and for τ sufficiently near 0, $g^{(k+1)}(\tau)/(k+1)!$ is bounded by some constant c.) Since g satisfies an estimate $|f(t)| \le ce^{at}$, where we may assume $a > 0$, and since the Taylor polynomial also satisfies such an estimate, so does the remainder r_k. Therefore, r_k satisfies (8.26) with $b = k + 1$. Hence, applying the Laplace transform to (8.28) and using (8.6), we obtain

$$\mathscr{L}g(z) = \sum_0^k j!c_j z^{-(j+1)} + \mathscr{L}r_k(z),$$

where $\mathscr{L}r_k(z)$ satisfies the estimate (8.27) with $b = k + 1$.

More generally, suppose that $f \in \mathscr{E}$ has the form $f(t) = t^\alpha g(t^\beta)$ where $\alpha > -1$, $\beta > 0$, and g is of class $C^{(k+1)}$ near $t = 0$. (We could even allow α to be complex here, with $\operatorname{Re}\alpha > -1$, but we shall stick with the real case for simplicity.) Substituting the expansion (8.28) into the formula for $f(t)$, we obtain

$$f(t) = \sum_0^k c_j t^{\alpha+j\beta} + t^\alpha r_k(t_\beta), \qquad |t^\alpha r_k(t^\beta)| \le ct^{\alpha+(k+1)\beta} \quad \text{for } t \text{ near } 0.$$

Then the same reasoning as before gives

$$\mathscr{L}f(z) = \sum_0^k \Gamma(\alpha + j\beta + 1)c_j z^{-\alpha - j\beta - 1} + R_k(z),$$

where R_k satisfies the estimate (8.27) with $b = \alpha + (k+1)\beta + 1$.

In brief, if f can be expanded in powers of t (including fractional ones) about $t = 0$, the leading terms of the expansion give the leading terms of the expansion of $\mathscr{L}f(z)$ about $z = +\infty$. The formula for such an expansion and the relevant estimates for the remainder term are connected to the derivatives of f (or of a related function g) at $t = 0$ via Taylor's theorem, but this is largely beside the point as far as the calculation of Laplace transforms is concerned. The essential features of the results derived above are summed up in the following theorem.

Watson's Lemma. *Suppose that $f \in \mathscr{E}$ and that for some $\alpha > -1$ and $\beta > 0$ we have*

$$f(t) = \sum_0^k c_j t^{\alpha + j\beta} + r_k(t) \quad \textit{where } |r_k(t)| \le ct^{\alpha + (k+1)\beta} \textit{ for } t \textit{ near } 0.$$

Then for any $\epsilon > 0$ there are positive constants $C_{k,\epsilon}$ and A such that

$$\begin{gathered}\mathscr{L}f(z) = \sum_0^k \frac{\Gamma(\alpha + j\beta + 1)c_j}{z^{\alpha + j\beta + 1}} + R_k(z) \quad \textit{where} \\ |\widetilde{R}_k(z)| \le \widetilde{C}_{k,\epsilon}|z|^{-[\alpha + (k+1)\beta + 1]} \textit{ when } \operatorname{Re} z \ge A \textit{ and } |\arg z| \le \tfrac{1}{2}\pi - \epsilon.\end{gathered} \tag{8.29}$$

In particular, if f is of class $C^{(k+1)}$ near $t = 0$, then

$$\begin{gathered}\mathscr{L}f(z) = \sum_0^k \frac{f^{(j)}(0)}{z^{j+1}} + R_k(z) \quad \textit{where} \\ |R_k(z)| \le C_{k,\epsilon}|z|^{-(k+2)} \textit{ when } \operatorname{Re} z \ge A \textit{ and } |\arg z| \le \tfrac{1}{2}\pi - \epsilon.\end{gathered} \tag{8.30}$$

The significance of Watson's lemma is that, under the stated conditions on f, $\mathscr{L}f(z)$ can be written as a simple sum of powers of z plus an error term that is of smaller order of magnitude than any of the terms in the sum as $\operatorname{Re} z \to \infty$. This often provides an effective means of calculating $\mathscr{L}f(z)$ when $\operatorname{Re} z \gg 0$. Moreover, the constant $C_{k,\epsilon}$ that bounds the error term can be estimated by the calculation in the proof of Lemma 8.2.

One important point needs to be explained here. Suppose f has derivatives of all orders at $t = 0$. Then one is tempted to let $k \to \infty$ in (8.30) to obtain the series $\sum_0^\infty f^{(j)}(0)/z^{j+1}$. However, *this series may be divergent for every z*. The trouble is that the constants $C_{k,\epsilon}$ in the error estimate may grow rapidly as k increases, so that for any fixed value of z, the error term may actually get larger instead of smaller as more and more terms are added. This divergence, however, does not prevent the partial sums of the series from being useful approximations to $\mathscr{L}f(z)$ for $\operatorname{Re} z$ large.

This rather paradoxical phenomenon is commonly expressed by saying that the series $\sum_0^\infty f^{(j)}(0)/z^{j+1}$ is an **asymptotic expansion** of $\mathscr{L}f(z)$ and writing

$$\mathscr{L}f(z) \sim \sum_0^\infty \frac{f^{(j)}(0)}{z^{j+1}} \quad \text{as } |z| \to \infty, \ |\arg z| \le \tfrac{1}{2}\pi - \epsilon. \tag{8.31}$$

The meaning of (8.31) is precisely that (8.30) holds for all k. The difference between a convergent power series and an asymptotic expansion,

$$F(z) = \sum a_j z^{-j} \quad \text{versus} \quad F(z) \sim \sum a_j z^{-j},$$

is this: In both cases, the partial sums $\sum_0^k a_j z^{-j}$ provide approximations to $F(z)$ in some sense; but in the first case, one can fix z and let $k \to \infty$ to obtain $F(z)$ exactly, whereas in the second case, one must fix k to obtain an approximation that gets better and better as $z \to \infty$. (Similar remarks apply to the more general asymptotic series obtained by letting $k \to \infty$ in (8.29).)

Even if the series $\sum f^{(j)}(0) z^{-(j+1)}$ does converge, its sum may not be $\mathscr{L}f(z)$. All that is guaranteed is that it will equal $\mathscr{L}f(z)$ plus an error term that tends to zero more rapidly than any power of z as $\operatorname{Re} z \to \infty$. In this connection one should compare Watson's lemma with Theorem 8.3 of §8.1. That theorem asserts that if f is the sum of its Taylor series on the whole positive real line, and if the series $\sum f^{(j)}(0)z^{-(j+1)}$ converges, then its sum is actually $\mathscr{L}f(z)$. (Thus, $\mathscr{L}f(z)$ is analytic in the exterior of some circle $|z| = A$, and $\sum f^{(j)}(0)z^{-(j+1)}$ is its Laurent series there.) In Watson's lemma one assumes less about f — the Taylor series of f might not converge everywhere, and its sum need not be f itself (i.e., f need not be analytic) — and one gets a weaker conclusion about $\mathscr{L}f$.

A couple of examples may help to elucidate this situation.

Example 1. Let $f(t) = 1/(1+t)$. For $z > 0$ we can make the substitution $t = (u/z) - 1$ in the integral defining $\mathscr{L}f(z)$, obtaining

$$\mathscr{L}f(z) = \int_0^\infty \frac{e^{-tz}}{1+t} dt = \int_z^\infty \frac{e^{z-u}}{u} du = e^z E_1(z),$$

where $E_1(z) = \int_z^\infty u^{-1}e^{-u} du$ is the exponential integral function mentioned in §8.1. (This result remains valid for $\operatorname{Re} z > 0$ by analytic continuation.) E_1 is a non-elementary function that turns up in many applications. It has been extensively tabulated (and, more recently, built into various pieces of computer software); but one can obtain $E_1(z)$ only for a finite range of values of z in this way. Hence, a simple approximation to $E_1(z)$ that is valid for large z is of considerable utility.

The Taylor series of f about $t = 0$ is the geometric series $\sum_0^\infty (-1)^n t^n$, so Watson's lemma gives the asymptotic expansion of E_1:

$$E_1(z) \sim e^{-z} \sum_0^\infty \frac{(-1)^n n!}{z^{n+1}} = e^{-z}(z^{-1} - z^{-2} + 2z^{-3} - 6z^{-4} + \cdots). \tag{8.32}$$

It is easily seen by the ratio test that the series on the right diverges for all z. However, the content of (8.32) is that the partial sums of this series give good approximations to $E_1(z)$ when $\operatorname{Re} z$ is large. From a practical point of view, it is usually sufficient just to take the first term:

$$E_1(z) \approx z^{-1}e^{-z}.$$

How good an approximation is this? We can estimate the error by the argument used to prove Lemma 8.2. Indeed, $1/(1+t) = 1 + r_0(t)$ where

$$|r_0(t)| = \left|\frac{1}{1+t} - 1\right| = \left|\frac{-t}{1+t}\right| \le t.$$

Hence,

$$|\mathscr{L} r_0(x+iy)| \le \int_0^\infty te^{-xt}\,dt = \frac{1}{x^2}\int_0^\infty ue^{-u}\,du = \frac{1}{x^2}.$$

Hence, the approximation $e^z E_1(z) \approx z^{-1}$ gives accuracy to four decimal places if $\operatorname{Re} z > 100$. If a finer approximation is needed, one can add on the next few terms. This will give increased accuracy if $\operatorname{Re} z$ is sufficiently large, but the more terms one adds on, the larger $\operatorname{Re} z$ must be for the extra accuracy to take effect.

Example 2. Let $f(t) = H(t-1)$, so that $\mathscr{L}f(z) = z^{-1}e^{-z}$. Here $f(t)$ vanishes identically for $t < 1$, so all the terms in the series $\sum f^{(j)}(0)z^{-(j+1)}$ are zero. The conclusion to be drawn from Watson's lemma is that $\mathscr{L}f(z)$ tends to zero more rapidly than any power of $|z|$ in the sector $|\arg z| \le \frac{1}{2}\pi - \epsilon$, which is indeed the case since $|\mathscr{L}f(z)| = |z|^{-1}e^{-\operatorname{Re} z}$. This example also shows that one cannot take $\epsilon = 0$, for if $\operatorname{Im} z \to \infty$ while $\operatorname{Re} z$ remains fixed, $z^{-1}e^{-z}$ tends to zero no more rapidly than z^{-1}.

We now show how to use Watson's lemma to derive the asymptotic formulas for Bessel functions stated in §5.3. The main point here is that the Hankel functions

$$H_\nu^{(1)} = J_\nu + iY_\nu = i\frac{e^{-i\nu\pi}J_\nu - J_{-\nu}}{\sin \nu\pi} \quad \text{and} \quad H_\nu^{(2)} = J_\nu - iY_\nu = -i\frac{e^{i\nu\pi}J_\nu - J_{-\nu}}{\sin \nu\pi}$$

can be represented as Laplace transforms. The proof of this fact is rather complicated, but we recommend it to the reader as a virtuoso display of contour integral techniques. See Watson [55], Chapter VI, for further information on integral representations of Bessel functions.

Theorem 8.6. *For* $\operatorname{Re}\nu > -\frac{1}{2}$ *and* $\operatorname{Re} w > 0$,

$$H_\nu^{(1)}(w) = \frac{2}{i\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{w}{2}\right)^\nu e^{iw}\int_0^\infty (t^2-2it)^{\nu-(1/2)}e^{-wt}dt, \tag{8.33}$$

$$H_\nu^{(2)}(w) = \frac{2i}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{w}{2}\right)^\nu e^{-iw}\int_0^\infty (t^2+2it)^{\nu-(1/2)}e^{-wt}dt. \tag{8.34}$$

Proof: The starting point is two formulas that we have asked the reader to verify as Exercise 16 of §8.1 and Exercise 14 of §5.2, namely,

$$\mathscr{L}\left[t^\nu J_\nu(t)\right] = \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}(z^2+1)^{-\nu-(1/2)} \qquad (\operatorname{Re}\nu > -\tfrac{1}{2}) \tag{8.35}$$

and

$$J_\nu(w) = \frac{w^\nu}{2^\nu\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\int_{-1}^1 e^{irw}(1-r^2)^{\nu-(1/2)}dr \qquad (\operatorname{Re}\nu > -\tfrac{1}{2}). \tag{8.36}$$

By the substitution $r = iz$, formula (8.36) can be restated as

$$\int_{-i}^{i} e^{-zw}(z^2+1)^{\nu-(1/2)}dz = \frac{i2^\nu\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}{w^\nu}J_\nu(w) \qquad (\operatorname{Re}\nu > -\tfrac{1}{2}). \tag{8.37}$$

We apply the Laplace inversion formula to (8.35), obtaining

$$w^\nu J_\nu(w) = \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{zw}(z^2+1)^{-\nu-(1/2)}dz \qquad (b>0) \tag{8.38}$$

for $\operatorname{Re}\nu > -\frac{1}{2}$ and $w > 0$. The function $(z^2+1)^{-\nu-(1/2)}$ has branch points at $z = \pm i$; we are using the branch of it in the right half-plane specified by the stipulation that $\arg(z^2+1) = 0$ when z is real and positive. This branch extends to the complement of the shaded region in Figure 8.6(a), and by deforming the contour in (8.38) as in Example 3 of §8.2, we obtain

$$J_\nu(w) = \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{w^\nu\sqrt{\pi}}\frac{1}{2\pi i}\int_{\gamma_1} e^{zw}(z^2+1)^{-\nu-(1/2)}dz, \tag{8.39}$$

where γ_1 is the contour in Figure 8.6(a).

The integral in (8.39) has the advantage over the one in (8.38) that it converges for all $\nu \in \mathbf{C}$ whenever $\operatorname{Re} w > 0$, because the factor e^{zw} now provides exponential decay. Moreover, the integral in (8.39) vanishes when $\nu = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$ by Cauchy's theorem, since then there are no branch cuts; this cancels the pole of $\Gamma(\nu+\frac{1}{2})$ at these points. Thus the left and right sides of (8.39) are analytic functions of ν and w for $\nu \in \mathbf{C}$ and $\operatorname{Re} w > 0$, and since they agree for $\operatorname{Re}\nu > -\frac{1}{2}$ and $w > 0$, they agree everywhere. In other words, (8.39) is

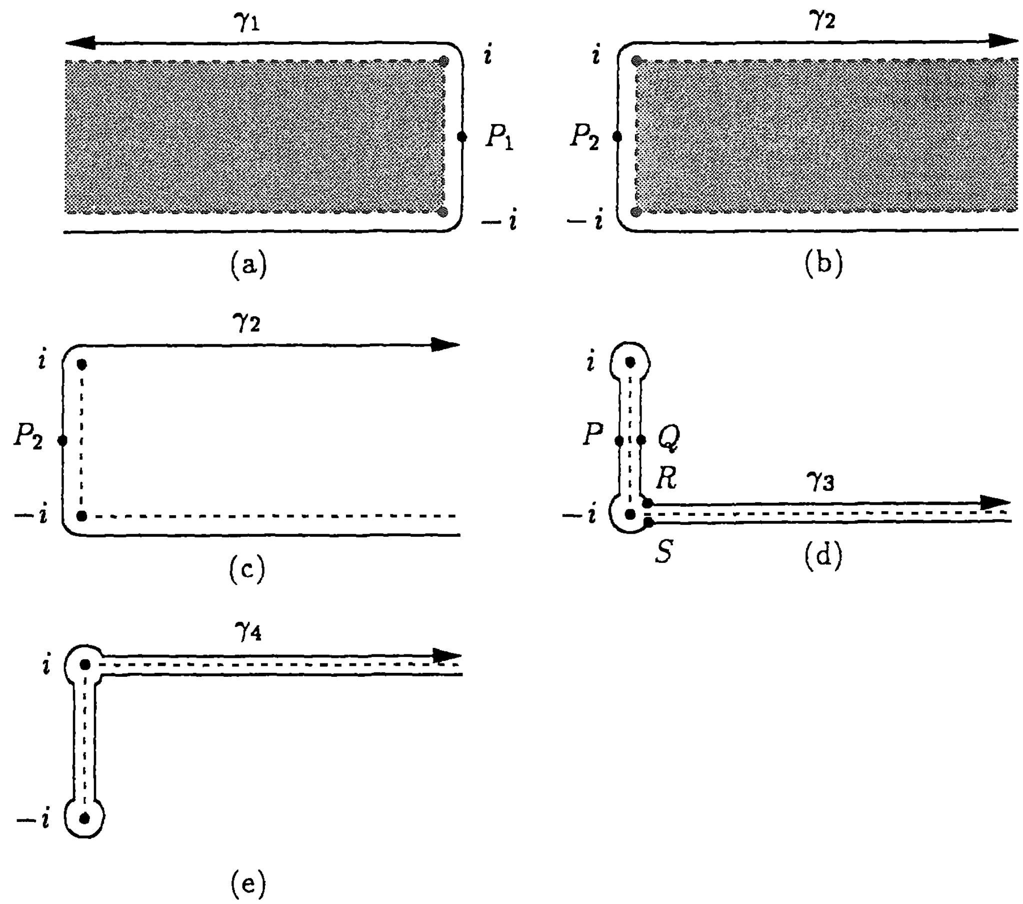


FIGURE 8.6. Contours for the proof of Theorem 8.6.

valid for $\operatorname{Re} w > 0$ and all $\nu \in \mathbf{C}$ (except for $\nu = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$, where the right side has a removable singularity); the branch of $(z^2+1)^{-\nu-(1/2)}$ is specified by the requirement that $\arg(z^2+1) = 0$ at P_1.

We make the substitution $z \to -z$ in (8.39):

$$\int_{\gamma_1} e^{zw}(z^2+1)^{-\nu-(1/2)} dz = \int_{\gamma_2} e^{-zw}(z^2+1)^{-\nu-(1/2)}\, dz,$$

where γ_2 is the contour in Figure 8.6(b) and $\arg(z^2+1) = 0$ at P_2. Further, we replace ν by $-\nu$, obtaining

$$J_{-\nu}(w) = \frac{w^\nu \Gamma(-\nu+\frac{1}{2})}{2^\nu \sqrt{\pi}} \frac{1}{2\pi i} \int_{\gamma_2} e^{-zw}(z^2+1)^{\nu-(1/2)}\, dz \qquad (\operatorname{Re} w > 0). \quad (8.40)$$

Next, we manipulate (8.40) by deforming the contour still further. Namely, we make the branch cuts for $(z^2+1)^{\nu-(1/2)}$ along the segment $[-i, i]$ and the ray $[-i, \infty - i]$ as in Figure 8.6(c). The complement of these cuts is a simply connected region, and the branch of $(z^2+1)^{\nu-(1/2)}$ there is specified by the requirement that

$\arg(z^2+1)=0$ at P_2. The contour γ_2 can then be deformed into the contour γ_3 in Figure 8.6(d). $\arg(z^2+1)$ is equal to 0 at P and hence is equal to -2π at Q, $-5\pi/2$ at R, and $3\pi/2$ at S.

We now make the assumption that $\operatorname{Re}\nu > -\frac{1}{2}$. In this case the singularities of $(z^2+1)^{\nu-(1/2)}$ at $\pm i$ are integrable, so we can squeeze the contour γ_3 onto the branch cuts. Taking account of the behavior of $\arg(z^2+1)$ on the different sides of the branch cuts, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_3} & e^{-zw}(z^2+1)^{\nu-(1/2)}\,dz \\ &= \frac{1-e^{-2\pi i[\nu-(1/2)]}}{2\pi i}\int_{-i}^{i} e^{-zw}(z^2+1)^{\nu-(1/2)}\,dz \\ &\quad + \frac{e^{-2\pi i[\nu-(1/2)]}-e^{2\pi i[\nu-(1/2)]}}{2\pi i}\int_{-i}^{\infty-i} e^{-zw}(z^2+1)^{\nu-(1/2)}\,dz,\end{aligned}\tag{8.41}$$

where the branch of the power function in these last integrals is specified by the condition $|\arg(z^2+1)|<\pi$. In view of (8.37), first term on the right of (8.41) equals

$$\begin{aligned}-\frac{ie^{-i\nu\pi}\sin\pi(-\nu+\frac{1}{2})}{\pi}&\int_{-i}^{i} e^{-zw}(z^2+1)^{\nu-(1/2)}dz \\ &= e^{-i\nu\pi}\frac{2^\nu\sin\pi(-\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{w^\nu\sqrt{\pi}}J_\nu(z),\end{aligned}$$

whereas the second term equals

$$\frac{\sin 2\pi(-\nu+\frac{1}{2})}{\pi}\int_{-i}^{\infty-i} e^{-zw}(z^2+1)^{\nu-(1/2)}\,dz.$$

If we substitute these results into (8.41) and thence into (8.40), and use the facts that

$$\Gamma(\nu+\tfrac{1}{2})\Gamma(-\nu+\tfrac{1}{2}) = \frac{\pi}{\sin\pi(-\nu+\frac{1}{2})},$$

$$\sin 2\pi(-\nu+\tfrac{1}{2}) = 2\sin\pi(-\nu+\tfrac{1}{2})\cos\pi(-\nu+\tfrac{1}{2}) = 2\sin\pi(-\nu+\tfrac{1}{2})\sin\nu\pi,$$

we obtain

$$J_{-\nu}(w) = e^{-i\nu\pi}J_\nu(w) + \frac{2w^\nu\sin\nu\pi}{2^\nu\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\int_{-i}^{\infty-i} e^{-zw}(z^2+1)^{\nu-(1/2)}\,dz,$$

from which it follows immediately from the definition of $H_\nu^{(1)}$ that

$$H_\nu^{(1)}(w) = \frac{2}{i\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{w}{2}\right)^\nu\int_{-i}^{\infty-i} e^{-zw}(z^2+1)^{\nu-(1/2)}\,dz.$$

But this is nothing but formula (8.33), as one sees by making the substitution $z=t-i$. Formula (8.34) is proved in exactly the same way by making the branch cuts for $(z^2+1)^{\nu-(1/2)}$ along $[-i,i]$ and $[i,\infty+i)$ and deforming the contour γ_2 in (8.40) onto the contour γ_4 in Figure 8.6(e). ∎

Now we can obtain the asymptotic formulas for the Bessel functions. For simplicity we shall assume that ν is real, although this is not actually necessary. For $\nu > -\frac{1}{2}$ we have

$$\begin{aligned}(t^2 \pm 2it)^{\nu-(1/2)} &= (\pm 2it)^{\nu-(1/2)}\left(1 \pm \frac{t}{2i}\right)^{\nu-(1/2)} \\ &= 2^{\nu-(1/2)} \exp\Big[(\pm i)(\tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi)\Big] t^{\nu-\frac{1}{2}} + r(t)\end{aligned} \tag{8.42}$$

where $|r(t)| \le Ct^{\nu+(1/2)}$, so Watson's lemma gives

$$\mathscr{L}\Big[(t^2 \pm 2it)^{\nu-(1/2)}\Big](w) = 2^{\nu-(1/2)} \exp\Big[(\pm i)(\tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi)\Big]\Gamma(\nu + \tfrac{1}{2})w^{-\nu-\frac{1}{2}} + R(w),$$

where $|R(w)| \le C|w|^{-\nu-(3/2)}$ as $\operatorname{Re} w \to \infty$. Substituting this result into (8.33) and (8.34), we obtain

$$H_\nu^{(1)}(w) = \exp\Big[i(w - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\Big]\left(\sqrt{\frac{2}{\pi w}} + e_1(w)\right), \tag{8.43}$$

$$H_\nu^{(2)}(w) = \exp\Big[-i(w - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\Big]\left(\sqrt{\frac{2}{\pi w}} + e_2(w)\right), \tag{8.44}$$

where $e_1(w)$ and $e_2(w)$ are dominated by $|w|^{-3/2}$ as $\operatorname{Re} w \to \infty$.

Formulas (8.43) and (8.44) were derived under the assumption that $\nu > -\frac{1}{2}$, but they are actually valid for all ν. Indeed, we have

$$H_{-\nu}^{(1)} = i\frac{e^{i\nu\pi}J_{-\nu} - J_\nu}{\sin(-\nu\pi)} = ie^{i\nu\pi}\frac{e^{-i\nu\pi}J_\nu - J_{-\nu}}{\sin\nu\pi} = e^{i\nu\pi}H_\nu^{(1)},$$

and likewise $H_{-\nu}^2 = e^{-i\nu\pi}H_\nu^{(2)}$. Hence, the effect of replacing ν by $-\nu$ in (8.43) or (8.44) is merely to multiply both sides by $e^{i\nu\pi}$ or $e^{-i\nu\pi}$.

Let us now take w to be real and positive, and write x instead of w to emphasize this restriction. Then the exponentials $e^{\pm ix}$ have absolute value 1, so (8.43) and (8.44) can be rewritten as

$$H_\nu^{(1)}(x) = \exp\Big[i(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\Big]\sqrt{\frac{2}{\pi x}} + E_1(x),$$
$$H_\nu^{(2)}(x) = \exp\Big[-i(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\Big]\sqrt{\frac{2}{\pi x}} + E_2(x),$$

where $E_1(x)$ and $E_2(x)$ are dominated by $x^{-3/2}$ as $x \to \infty$. These formulas at last yield Theorem 5.1 of §5.3:

$$J_\nu(x) = \frac{H_\nu^{(1)}(x) + H_\nu^{(2)}(x)}{2} = \sqrt{\frac{2}{\pi x}}\cos(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + E(x)$$

where $E(x) = \frac{1}{2}[E_1(x) + E_2(x)]$ is dominated by $x^{-3/2}$ as $x \to \infty$. They also yield

$$Y_\nu(x) = \frac{H_\nu^{(1)}(x) - H_\nu^{(2)}(x)}{2i} = \sqrt{\frac{2}{\pi x}}\sin(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + \frac{E_1(x) - E_2(x)}{2i},$$

which establishes formula (5.26) even in the case when ν is an integer.

Of course (8.43) and (8.44) give just the leading terms of the asymptotic expansions of $H_\nu^{(1)}(w)$ and $H_\nu^{(2)}(w)$. The complete expansions may be obtained by expanding the term $[1\pm(t/2i)]^{\nu-(1/2)}$ in (8.42) in its Taylor series and applying Watson's lemma. These expansions can be shown to be valid not just in the sector $|\arg w| < \frac{1}{2}\pi - \epsilon$ but in the larger sector $|\arg w| < \pi - \epsilon$; they therefore yield the asymptotic formulas for the modified Bessel functions I_ν and K_ν by taking w to be purely imaginary; see Exercises 5–7. Further results about the asymptotics of Bessel functions can be found in Watson [55], Chapters VII and VIII.

EXERCISES

1. From the result of Exercise 3, §8.1, find an asymptotic expansion for $e^{z^2}\operatorname{erfc} z$ in the sector $|\arg z| \le \frac{1}{2}\pi - \epsilon$.
2. Use the result of Exercise 1 to do the following:
   a. Evaluate $\lim_{x\to+\infty}[xe^{4x^2}\operatorname{erfc}(2x)]$.
   b. Show that $u(0,t) \approx 1/c\sqrt{\pi k t}$ for large t, where $u(x,t)$ is the solution of Exercise 4, §8.4.
3. The *incomplete gamma function* is defined by

$$\Gamma(a,z) = \int_z^\infty e^{-w}w^{a-1}\,dw$$

   for $a \in \mathbf{C}$ and $|\arg z| < \pi$. (The path of integration is a ray parallel to the positive real axis.) Show that for fixed a,

$$\Gamma(a,z) = z^a e^{-z}\mathscr{L}\Big[(1+t)^{a-1}\Big],$$

   and deduce an asymptotic formula for $\Gamma(a,z)$ in the region $|\arg z| \le \frac{1}{2}\pi - \epsilon$.
4. Find the asymptotic expansion of $\mathscr{L}\Big[(1+t^2)^{-1}\Big]$ in the sector $|\arg z| \le \frac{1}{2}\pi - \epsilon$.
5. Derive the complete asymptotic expansions of $H_\nu^{(1)}(w)$ and $H_\nu^{(2)}(w)$ in the sector $|\arg w| < \frac{1}{2}\pi - \epsilon$ from Theorem 8.6.
6. Suppose $\nu > -\frac{1}{2}$. The function $f(z) = (z^2 - 2iz)^{\nu-(1/2)}$ is analytic in the right half plane when we take the principal branch of the power function. For $|\alpha| < \frac{1}{2}\pi$ and $t > 0$, let $f_\alpha(t) = f(e^{i\alpha}t)$.
   a. Show that $\mathscr{L}f_\alpha(w)$ is analytic in the half-plane $\operatorname{Re} w > 0$.
   b. By Cauchy's theorem, show that the formula $\mathscr{L}f_0(w) = e^{i\alpha}\mathscr{L}f_\alpha(e^{i\alpha}w)$ holds in the intersection of the half-planes $\operatorname{Re} w > 0$ and $\operatorname{Re}(e^{i\alpha}w) > 0$,

and hence that it gives the analytic continuation of $\mathscr{L}f_0(w)$ to the half-plane $\mathrm{Re}(e^{i\alpha}w) > 0$. As α varies from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, this yields the analytic continuation of $\mathscr{L}f_0(w)$ to the slit plane $|\arg w| < \pi$.

c. Let $\sum a_k w^{-\nu-k-(1/2)}$ be the asymptotic expansion of $\mathscr{L}f_0(w)$ for $|\arg w| \leq \frac{1}{2}\pi - \epsilon$. By applying Watson's lemma to f_α ($|\alpha| < \frac{1}{2}\pi$) and using part (b), show that the expansion $\mathscr{L}f_0(w) \sim \sum a_k w^{-\nu-k-(1/2)}$ is actually valid in the larger sector $|\arg w| \leq \pi - \epsilon$.

d. The results of parts (a)–(c) still hold if $z^2 - 2iz$ is replaced by $z^2 + 2iz$. Conclude that the expansions of $H_\nu^{(1)}$ and $H_\nu^{(2)}$ in Exercise 5 are valid for $|\arg w| \leq \pi - \epsilon$.

7. Use Exercise 6 to show that the modified Bessel functions I_ν and K_ν satisfy

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 + E_1(x)\right], \qquad K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\left[1 + E_2(x)\right]$$

for $x \geq 1$, where $E_1(x)$ and $E_2(x)$ are bounded by a constant (depending on ν) times x^{-1}.

CHAPTER 9
GENERALIZED FUNCTIONS

The concept of a function has undergone a profound transformation over the past couple of centuries. In the early days of mathematical analysis, people thought of functions as being defined by specific formulas involving algebraic operations and perhaps some limiting processes. The modern notion of a function as a more or less arbitrary rule that assigns to each number x in some domain another number $f(x)$ evolved during the nineteenth century — a development that was largely influenced by the new science of Fourier analysis, where functions of a quite general sort arise naturally. By the end of that century, a few visionary souls began to realize that even this broad concept of a function was in some ways too confining and that it would be useful to consider objects that are "like functions but more singular than functions."

The most widely known example of such a generalized function is the so-called Dirac delta function $\delta(x)$, which is supposed to have the properties that $\delta(x) = 0$ for all $x \neq 0$ but $\int_{-a}^{a} \delta(x)\,dx = 1$ for any $a > 0$. One can think of $c\delta(x)$ as representing the charge density of a particle of charge c on the x-axis that occupies only the single point $x = 0$: There is no charge except at the origin, but the total charge is c. This may be an idealization (although it seems to be a highly accurate characterization of an electron), but it is a very useful one.

Within the traditional realm of functions, the Dirac function does not make sense: The integral of a function $\delta(x)$ that vanishes except at $x = 0$ is zero no matter what value is assigned to $\delta(0)$. Even if one tries to set $\delta(0) = \infty$ and modify the notion of integral so that this infinite value will have a real effect, one is at a loss to distinguish between "∞" and "2∞" in such a way that the integral of δ is 1 while the integral of 2δ is 2.

Engineers and physicists began using generalized functions in a nonrigorous way in the early part of this century, and they usually managed to produce the right answer in the end — a fact which caused much gnashing of teeth among professional mathematicians. It was not until the late 1940s that a way was found to put these ideas on a firm footing. In fact, there are several approaches to the subject of generalized functions. The one we shall follow is the one most widely used, Laurent Schwartz's theory of distributions. It will take some work to master the conceptual foundations of this theory, but the effort will be repaid by the power of the new tools at our disposal and the ease with which they can be used.

A more extensive and detailed treatment of the material in this chapter can be found in Zemanian [57], and an alternative approach to the theory of distributions in Lighthill [37]. There is also a different but related way of developing generalized functions via the operational calculus of Mikusiński; see Erdélyi [20].

In this chapter we shall be working with complex-valued functions of n real variables. We shall continue to use the convention we adopted in Chapter 7 that an integral sign with no limits attached denotes integration over all of n-space:

$$\int f(\mathbf{x})\,d\mathbf{x} = \int_{\mathbf{R}^n} f(\mathbf{x})\,d\mathbf{x}; \qquad \text{for } n=1, \quad \int f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

9.1 Distributions

Before proceeding to the main business at hand, we need to introduce a little terminology. First, a bit of shorthand notation for partial derivatives in n variables: If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers, we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Next, if f is a function on $\mathbf{R}^n$, the **support** of f is the closure of the set of all points $\mathbf{x}$ such that $f(\mathbf{x}) \neq 0$, in other words, the smallest closed set outside of which f vanishes identically. We denote the support of f by $\mathrm{supp}(f)$, and if $\mathrm{supp}(f) \subset E$ we say that f is **supported** in E. (Most of the functions one encounters in elementary mathematics vanish only at isolated points, so their support is all of $\mathbf{R}^n$. However, functions with smaller support turn up frequently in more advanced topics; for example, the theory of Laplace transforms pertains to functions supported in the half-line $[0, \infty)$.)

We shall denote by $C_0^{(\infty)}(\mathbf{R}^n)$, or just $C_0^{(\infty)}$ if the dimension n is understood, the space of functions on $\mathbf{R}^n$ whose (partial) derivatives of all orders exist and are continuous on $\mathbf{R}^n$ and whose support is a bounded subset of $\mathbf{R}^n$. We call the elements of $C_0^{(\infty)}$ **test functions**. We have encountered test functions already in Chapter 7, where we wrote down the example

$$\psi_0(x) = \begin{cases} e^{-1/(1-x^2)} & \text{if } |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

for the case $n = 1$. To make a similar example in several variables, simply apply this function to each variable and multiply the results together:

$$\Psi_0(\mathbf{x}) = \psi_0(x_1)\psi_0(x_2)\cdots\psi_0(x_n).$$

The partial derivatives of Ψ_0 are just products of the ordinary derivatives of $\psi_0(x_j)$, and

$$\operatorname{supp}(\Psi_0) = \left\{\mathbf{x} : |x_j| \le 1,\ 1 \le j \le n\right\}.$$

It is in the nature of test functions that they are usually not expressible by simple elementary formulas. Nonetheless, they exist in great abundance — great enough to provide close approximations to quite general functions. Indeed, it follows from Theorem 7.7 of §7.5 that for any continuous function f of bounded support there is a sequence of test functions that converges to f uniformly, and for any $g \in L^2(\mathbf{R}^n)$ there is a sequence of test functions that converges to g in norm.

Now we come to the key idea that will enable us to generalize the concept of function. A continuous function f on $\mathbf{R}^n$ is traditionally specified by giving its values $f(\mathbf{x})$ at all points $\mathbf{x} \in \mathbf{R}^n$, but f can equally well be specified by giving the values of the integrals $\int f(\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y}$ as ϕ ranges over all test functions. Indeed, let us fix a particular $\psi \in C_0^{(\infty)}$ such that $\int \psi(\mathbf{y})\,d\mathbf{y} = 1$ (for example, we could take $\psi = c\Psi_0$ where Ψ_0 is as above and $1/c = \int \Psi_0(\mathbf{y})\,d\mathbf{y}$), and let us set

$$\psi_{\mathbf{x},\epsilon}(\mathbf{y}) = \epsilon^{-n}\psi\left(\epsilon^{-1}(\mathbf{x}-\mathbf{y})\right) \qquad (\mathbf{x} \in \mathbf{R}^n,\ \epsilon > 0).$$

Now, if we know $\int f(\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y}$ for all test functions ϕ, then in particular we know $\int f(\mathbf{y})\psi_{\mathbf{x},\epsilon}(\mathbf{y})\,d\mathbf{y}$ for all $\mathbf{x}$ and ϵ. But by Theorem 7.7,

$$\lim_{\epsilon\to 0}\int f(\mathbf{y})\psi_{\mathbf{x},\epsilon}(\mathbf{y})\,d\mathbf{y} = \lim_{\epsilon\to 0}\epsilon^{-n}\int f(\mathbf{y})\psi\left(\epsilon^{-1}(\mathbf{x}-\mathbf{y})\right)d\mathbf{y} = f(\mathbf{x}), \tag{9.1}$$

and we therefore know $f(\mathbf{x})$ for all $\mathbf{x}$.

The intuition behind this is the following. ψ is supported in some ball $\{\mathbf{y} : |\mathbf{y}| < R\}$, so $\psi_{\mathbf{x},\epsilon}$ is supported in the ball $B_{\mathbf{x},\epsilon} = \{\mathbf{y} : |\mathbf{x}-\mathbf{y}| < R\epsilon\}$, which is a small neighborhood of $\mathbf{x}$ when ϵ is small. As we explained in §7.1, $\int f(\mathbf{y})\psi_{\mathbf{x},\epsilon}(\mathbf{y})\,d\mathbf{y}$ is a weighted average of the values of f in this neighborhood, and these values are close to $f(\mathbf{x})$ when ϵ is small since f is continuous. In other words, we can think of $\int f(\mathbf{y})\psi_{\mathbf{x},\epsilon}(\mathbf{y})\,d\mathbf{y}$ as a "smeared-out" version of $f(\mathbf{x})$, and $f(\mathbf{x})$ itself can be recovered as a limit of these smeared-out values. Let us introduce a notation that will suggest this idea as well as saving space:

$$f[\phi] = \int f(\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y} \qquad (\phi \in C_0^{(\infty)}).$$

The same considerations apply to functions on $\mathbf{R}^1$ that are only piecewise continuous, except that (9.1) does not hold at the points of discontinuity. (If f has a jump discontinuity at x, the integral $f[\psi_{x,\epsilon}]$ may converge to any value as $\epsilon \to 0$, depending on what ψ is. For example, if ψ is supported in $[0,\infty)$ the limit will be $f(x-)$, whereas if ψ is even the limit will be $\frac{1}{2}[f(x-)+f(x+)]$. See Theorem 7.3, §7.1.) More generally, suppose f is any function on $\mathbf{R}^n$ such that

$$\int_K |f(\mathbf{y})|\,d\mathbf{y} < \infty \text{ for any bounded set } K \subset \mathbf{R}^n.$$

Such functions are called **locally integrable.** Then the integrals $f[\phi]$ are absolutely convergent for all test functions ϕ; and it can be shown that (9.1) holds, perhaps not at every single $\mathbf{x}$, but at least for "almost every" $\mathbf{x}$ in the sense of Lebesgue. (See Folland [25], Theorem 8.15, or Wheeden-Zygmund [56], Theorem 9.13.)

In short, if f is a continuous function on $\mathbf{R}^n$, then f can be specified either by giving the values $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^n$, or the smeared-out values $f[\phi]$, $\phi \in C_0^{(\infty)}$. The same is true of locally integrable functions with discontinuities, provided that two functions that agree "almost everywhere" are regarded as identical.

The dependence of $f[\phi]$ on ϕ is simpler than the dependence of $f(\mathbf{x})$ on $\mathbf{x}$ in two important respects.

(i) It is linear: $f[c_1\phi_1+c_2\phi_2] = c_1 f[\phi_1]+c_2 f[\phi_2]$ for any $\phi_1, \phi_2 \in C_0^{(\infty)}$ and any complex constants c_1, c_2. This is a consequence of the fact that integration is a linear operation, and it is expressed by saying that the correspondence $\phi \to f[\phi]$ is a **linear functional** on the space $C_0^{(\infty)}$.

(ii) It is continuous, in the sense that if $\{\phi_k\}$ is a sequence of test functions that are all supported in a fixed bounded set K and $\phi_k \to 0$ uniformly, then $f[\phi_k] \to 0$. This is obvious:

$$\left| \int f(\mathbf{y})\phi_k(\mathbf{y})\,d\mathbf{y} \right| \le \sup_{\mathbf{y}} |\phi_k(\mathbf{y})| \int_K |f(\mathbf{y})|\,d\mathbf{y} \to 0 \int_K |f(\mathbf{y})|\,d\mathbf{y} = 0.$$

Now, the point is that there are *other* linear functionals on $C_0^{(\infty)}$ that are *not* given by integration against a function f, especially if we relax the continuity condition (ii), and these functionals will be our "generalized functions," or "distributions" as they are commonly called. We proceed to a formal definition.

Definition. A **distribution** is a mapping $F : C_0^{(\infty)} \to \mathbf{C}$ that satisfies the following conditions:

(i) Linearity: $F[c_1\phi_1 + c_2\phi_2] = c_1 F[\phi_1] + c_2 F[\phi_2]$ for all $\phi_1, \phi_2 \in C_0^{(\infty)}$ and all $c_1, c_2 \in \mathbf{C}$.

(ii) Continuity: Suppose $\{\phi_k\}$ is a sequence in $C_0^{(\infty)}$ such that $\operatorname{supp}(\phi_k)$ is contained in a fixed bounded set D for all k, and suppose that the functions ϕ_k and all their derivatives $\partial^\alpha\phi_k$ converge uniformly to zero as $k \to \infty$. Then $F[\phi_k] \to 0$. We shall adopt the traditional notation $\mathscr{D}'(\mathbf{R}^n)$, or $\mathscr{D}'$ for short, for the space of all distributions on $\mathbf{R}^n$. (The prime is an indication that $\mathscr{D}'$ is a space of linear functionals.)

Remark: The continuity requirement (ii) is an extremely weak one. It is used occasionally in deriving properties of distributions, but verifying that it is satisfied is almost never an important issue. The fact is that *any* linear functional $F[\phi]$ that is defined in any reasonably explicit way and makes sense for all $\phi \in C_0^{(\infty)}$ will automatically satisfy (ii). In what follows, when we define distributions we shall usually omit mentioning the condition (ii) and leave its verification as an easy exercise for the reader.

One should understand the definition of distribution intuitively as follows. A distribution F is like a function, but it may be too singular for the pointwise values $F(\mathbf{x})$ to make sense; only the smeared-out values $F[\phi]$ are well-defined. Nevertheless, it will often be convenient to use the notation $F(\mathbf{x})$, and in particular to write

$$F[\phi] = \int F(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x},$$

as if F were a function. This is a useful fiction, like the fiction that a derivative dy/dx is a quotient of infinitely small quantities dy and dx.

Let us look at a few examples.

Example 1. As the preceding discussion shows, every locally integrable function can be thought of as a distribution.

Example 2. The simplest example of a distribution that is not a function is the **Dirac delta function** δ, which is defined by

$$\delta[\phi] = \phi(\mathbf{0}).$$

If we think of $\delta[\phi]$ as being formally an integral $\int \delta(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x}$, this agrees with the idea of δ as a function that is 0 for all $\mathbf{x} \neq \mathbf{0}$ but with integral equal to 1, for then $\delta(\mathbf{x})\phi(\mathbf{x}) = \phi(\mathbf{0})\delta(\mathbf{x})$ will be 0 for $\mathbf{x} \neq \mathbf{0}$ and have integral equal to $\phi(\mathbf{0})$.

Example 3. Let C be a smooth curve in $\mathbf{R}^n$, and let $d\sigma$ denote the element of arc length on σ. We can then define a distribution F on $\mathbf{R}^n$ by

$$F[\phi] = \int_C \phi(\mathbf{x})\,d\sigma(\mathbf{x}).$$

In other words, if C is parametrized by $\mathbf{x} = \mathbf{x}(t)$,

$$F[\phi] = \int_{\mathbf{R}} \phi\big(\mathbf{x}(t)\big)|\mathbf{x}'(t)|\,dt.$$

This distribution is a sort of "delta function along C." That is, if one wishes to visualize F as a sort of function, one should think of a function $F(\mathbf{x})$ that is 0 when $\mathbf{x}$ is not on C and infinite when $\mathbf{x}$ is on C, in such a way that integral $\int F(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x}$ picks out the values of ϕ on C and then integrates them over C.

For instance, suppose C is the unit circle in the plane $\mathbf{R}^2$. In polar coordinates (r,θ), the distribution F is precisely $\delta(r-1)$; that is,

$$F[\phi] = \int_0^{2\pi} \phi(1,\theta)\,d\theta = \int_0^{2\pi}\int_0^{\infty} \phi(r,\theta)\delta(r-1)r\,dr\,d\theta.$$

Example 4. Let S be a smooth surface in $\mathbf{R}^3$, and let dS be the element of surface area on S. Just as in Example 3, we can define a distribution F on $\mathbf{R}^3$ by

$$F[\phi] = \int_S \phi(\mathbf{x})\,dS(\mathbf{x}),$$

and F can be thought of as a "delta function along S." More generally, if M is any smooth k-dimensional submanifold of $\mathbf{R}^n$ $(0 < k < n)$, the natural k-dimensional measure on S defines a distribution on $\mathbf{R}^n$.

It does not make sense to say that two distributions F and G are equal at a single point, but it does make sense to say that they are equal on an open set; namely, if $U \subset \mathbf{R}^n$ is open,

$$F = G \text{ on } U \quad \Longleftrightarrow \quad F[\phi] = G[\phi] \text{ for all } \phi \text{ with } \operatorname{supp}(\phi) \subset U.$$

(If F and G are locally integrable functions, this notion of equality on U really means that $F(\mathbf{x}) = G(\mathbf{x})$ for almost every $\mathbf{x} \in U$.) For example, if f is any continuous function on $\mathbf{R}^n$, $f + \delta = f$ on $\{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \neq \mathbf{0}\}$, because $\delta[\phi] = 0$ for all test functions ϕ whose support does not contain $\mathbf{0}$. The **support** of a distribution F can then be defined as the smallest closed set K such that $F = 0$ on the complement of K. Thus, the support of the distribution defined by arc length on a curve C in Example 3 is the closure of C.

Operations on distributions

The next order of business is to establish the following fundamental fact: *It is possible to extend the operation of differentiation from functions to distributions in such a way that every distribution possesses derivatives of all orders that are also distributions.* To explain the idea, let us first consider functions of one variable. If F is a continuously differentiable function on $\mathbf{R}$, we can calculate the smeared-out values of its derivative F' in terms of the smeared-out values of F by integration by parts. Indeed, if $\phi \in C_0^{(\infty)}(\mathbf{R})$,

$$F'[\phi] = \int F'(x)\phi(x)\,dx = F(x)\phi(x)\Big|_{-\infty}^{\infty} - \int F(x)\phi'(x)\,dx.$$

But $\phi(x) = 0$ when $|x|$ is large, so

$$F'[\phi] = -\int F(x)\phi'(x)\,dx = -F[\phi'].$$

Now, ϕ' is again a test function, so the expression $F[\phi']$ makes sense not just when F is a differentiable function but when F is an arbitrary distribution. We can therefore take this equation as a *definition* of $F'[\phi]$ when F is a distribution. That is, for any distribution F on $\mathbf{R}$ we define the distribution F' by

$$F'[\phi] = -F[\phi'] \qquad (F \in \mathscr{D}'(\mathbf{R}),\ \phi \in C_0^{(\infty)}(\mathbf{R})).$$

This operation can be repeated any number of times, yielding

$$F^{(k)}[\phi] = (-1)^k F[\phi^{(k)}] \qquad (F \in \mathscr{D}'(\mathbf{R}),\ \phi \in C_0^{(\infty)}(\mathbf{R})).$$

Moreover, the same calculation also works for partial derivatives of functions of several variables, enabling us to define the partial derivatives of all orders of a distribution on $\mathbf{R}^n$ (with the notation introduced at the beginning of the section):

$$(\partial^\alpha F)[\phi] = (-1)^{|\alpha|} F[\partial^\alpha \phi] \qquad (F \in \mathscr{D}'(\mathbf{R}^n),\ \phi \in C_0^{(\infty)}(\mathbf{R}^n)). \tag{9.2}$$

This freedom in taking derivatives is one of the main reasons why distributions are nice to work with, and it is also one of the best sources of interesting examples. For instance, we have the derivatives of the delta function:

$$\partial^\alpha \delta[\phi] = (-1)^{|\alpha|}\delta[\partial^\alpha \phi] = (-1)^{|\alpha|}\partial^\alpha \phi(\mathbf{0}).$$

We can also obtain distributions as derivatives of functions that are not differentiable in the ordinary sense. For instance, let $n = 1$ and consider the Heaviside function $H(x)$ that equals 0 for $x < 0$ and equals 1 for $x \geq 0$. If ϕ is any test function on $\mathbf{R}$, we have

$$H'[\phi] = -H[\phi'] = -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx = -\int_0^\infty \phi'(x)\,dx = \phi(0),$$

since $\phi(\infty) = 0$. In other words, $H' = \delta$. This accords with the intuition that $H'(x) = 0$ when $x \neq 0$ while $H'(0)$ is infinite in such a way as to make the jump $H(0+) - H(0-)$ equal to 1.

More generally, suppose f is a piecewise smooth function on $\mathbf{R}$ that is differentiable at all $x \neq 0$ but has a jump discontinuity at 0. What is the relation between the distribution derivative of f and its ordinary pointwise derivative, which exists for all $x \neq 0$? To avoid confusion, let us denote the distribution derivative by f' and the pointwise derivative by $f^{(1)}$. We then have

$$\begin{aligned}
f'[\phi] &= -\int f(x)\phi'(x)\,dx = -\int_{-\infty}^{0} f(x)\phi'(x)\,dx - \int_0^\infty f(x)\phi'(x)\,dx \\
&= -f(x)\phi(x)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} f^{(1)}(x)\phi(x)\,dx - f(x)\phi(x)\Big|_0^\infty + \int_0^\infty f^{(1)}(x)\phi(x)\,dx \\
&= -f(0-)\phi(0) + f(0+)\phi(0) + \int_{-\infty}^{\infty} f^{(1)}(x)\phi(x)\,dx,
\end{aligned}$$

that is,

$$f' = \big[f(0+) - f(0-)\big]\delta + f^{(1)}.$$

In other words, the distribution derivative f' equals the ordinary derivative on the set $\{x : x \neq 0\}$, but it has an extra multiple of the delta function thrown in to account for the jump of f at 0.

Similar formulas apply to all piecewise smooth functions. Indeed, a slight elaboration of the preceding argument establishes the following result.

Theorem 9.1. *Suppose f is piecewise smooth on $\mathbf{R}$ with discontinuities at $x_1, x_2, \ldots$. Let $f^{(1)}$ denote the pointwise derivative of f, which exists and is continuous except at the x_j's and perhaps some points where it has jump discontinuities, and let f' denote the distribution derivative of f. Then for any test function ϕ,*

$$f'[\phi] = \int \phi(x) f^{(1)}(x)\,dx + \sum \big[f(x_j+) - f(x_j-)\big]\phi(x_j).$$

In other words,

$$f'(x) = f^{(1)}(x) + \sum \big[f(x_j+) - f(x_j-)\big]\delta(x - x_j). \tag{9.3}$$

We leave the verification of Theorem 9.1 to the reader as Exercise 2. Observe that only the discontinuities of f, not those of f', show up here. It is easy to see that if f is piecewise smooth and continuous, then $\int f^{(1)}\phi = -\int f\phi'$ for all test functions ϕ even if $f^{(1)}$ has jumps — simply integrate by parts over each interval where $f^{(1)}$ is continuous and add up the results — so in this case the distribution derivative f' coincides with $f^{(1)}$.

In connection with partial derivatives of distributions in several variables, the following fact is worth pointing out. One is taught in advanced calculus that if the second partial derivatives of f exist and are continuous, then the mixed partials

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) \quad \text{and} \quad \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

are equal, but there are examples where these derivatives exist but are discontinuous and unequal. In the context of distributions, the latter pathology never arises: *Equality of mixed partials always holds for distribution derivatives.* This is because equality of mixed partials always holds for test functions (whose derivatives are all continuous), so

$$\frac{\partial^2 F}{\partial x_i \partial x_j}[\phi] = F\left[\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right] = F\left[\frac{\partial^2 \phi}{\partial x_j \partial x_i}\right] = \frac{\partial^2 F}{\partial x_j \partial x_i}[\phi].$$

This does not contradict the calculus books, for in the nasty example always presented there the derivatives $\partial^2 f/\partial x_i \partial x_j$ and $\partial^2 f/\partial x_j \partial x_i$ are equal except at one point, and hence they are indistinguishable as distributions.

A number of other basic operations can be extended from functions to distributions in much the same way as differentiation. One such operation is *translation:* If $F(\mathbf{x})$ is a function on $\mathbf{R}^n$ and $\mathbf{y} \in \mathbf{R}^n$, we can translate $F(\mathbf{x})$ by $\mathbf{y}$ to form the function $F_{\mathbf{y}}(\mathbf{x}) = F(\mathbf{x}-\mathbf{y})$. In terms of the smeared-out values $F[\phi]$, we have

$$\int F_{\mathbf{y}}(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} = \int F(\mathbf{x}-\mathbf{y})\phi(\mathbf{x})\,d\mathbf{x} = \int F(\mathbf{x})\phi(\mathbf{x}+\mathbf{y})\,d\mathbf{x} = \int F(\mathbf{x})\phi_{-\mathbf{y}}(\mathbf{x})\,d\mathbf{x},$$

by the substitution $\mathbf{x} \to \mathbf{x}+\mathbf{y}$; in other words, $F_{\mathbf{y}}[\phi] = F[\phi_{-\mathbf{y}}]$. We can now use this equation to *define* the translate $F_{\mathbf{y}}$ of a distribution F by a vector $\mathbf{y}$:

$$F_{\mathbf{y}}[\phi] = F[\phi_{-\mathbf{y}}], \quad \text{or} \quad \int F(\mathbf{x}-\mathbf{y})\phi(\mathbf{x})\,d\mathbf{x} = \int F(\mathbf{x})\phi(\mathbf{x}+\mathbf{y})\,d\mathbf{x}. \tag{9.4}$$

(This is one of the situations where it is notationally convenient to pretend that distributions are functions. The second formula in (9.4) is only a symbolic way of writing the first one, but it is surely easier to remember.) For example, the translates of the delta function are given by

$$\int \delta(\mathbf{x}-\mathbf{y})\phi(\mathbf{x})\,d\mathbf{x} = \int \delta(\mathbf{x})\phi(\mathbf{x}+\mathbf{y})\,d\mathbf{x} = \phi(\mathbf{y}).$$

Another important operation is *dilation:* If $F(\mathbf{x})$ is a function on $\mathbb{R}^n$ and a is a nonzero real number, we can form the new function $F(a\mathbf{x})$. When $n = 1$, by the substitution $y = ax$ we have

$$\int_{-\infty}^{\infty} F(ax)\phi(x)\,dx = |a|^{-1}\int_{-\infty}^{\infty} F(y)\phi(a^{-1}y)\,dy.$$

(If a is negative, the substitution $y = ax$ reverses the limits $+\infty$ and $-\infty$; putting them back in the right order changes the factor a^{-1} to $|a|^{-1}$.) For general n, the same calculation can be performed in each variable, with the result that

$$\int F(a\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} = |a|^{-n}\int F(\mathbf{x})\phi(a^{-1}\mathbf{x})\,d\mathbf{x}. \tag{9.5}$$

If $\phi(\mathbf{x})$ is a test function, then so is $\phi(a^{-1}\mathbf{x})$, hence this equation can be taken as the definition of the dilation of an arbitrary distribution F by the factor a. That is, if we introduce the notation $f^{[a]}(\mathbf{x})$ for $f(a\mathbf{x})$, the dilation $F^{[a]}$ of a distribution F is defined by

$$F^{[a]}[\phi] = |a|^{-n}F[\phi^{[1/a]}].$$

As in the case of translations, this formula gives the precise definition of $F^{[a]}$ when F is a distribution, but the idea is more clearly expressed in the symbolic integral formula (9.5).

For example, let $F = \delta$. We have

$$\begin{aligned}\int \delta(a\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} &= |a|^{-n}\int \delta(\mathbf{x})\phi(a^{-1}\mathbf{x})\,d\mathbf{x} = |a|^{-n}\phi(a^{-1}0)\\ &= |a|^{-n}\phi(0) = |a|^{-n}\int \delta(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x}.\end{aligned}$$

In other words, $\delta(a\mathbf{x}) = |a|^{-n}\delta(\mathbf{x})$. This is a nice illustration of the care that must be taken in manipulating distributions. If one merely thinks of $\delta(\mathbf{x})$ as a function that is zero for $\mathbf{x} \neq 0$ and ∞ for $\mathbf{x} = 0$, then $\delta(a\mathbf{x})$ is a function with exactly the same properties, and there is no way to distinguish between them. Nonetheless, they differ by a factor of $|a|^{-n}$ when they are integrated against test functions!

The last operation we shall consider here is *multiplication by smooth functions*. Suppose g is an infinitely differentiable function on $\mathbb{R}^n$. Then $g\phi$ is a test function whenever ϕ is, and for any continuous function F we have

$$\int \Big[g(\mathbf{x})F(\mathbf{x})\Big]\phi(\mathbf{x})\,d\mathbf{x} = \int F(\mathbf{x})\Big[g(\mathbf{x})\phi(\mathbf{x})\Big]\,d\mathbf{x}.$$

Now, if F is a distribution, this equation gives the definition of the distribution gF. In other words,

$$(gF)[\phi] = F[g\phi] \tag{9.6}$$

This seemingly innocuous definition can contain some surprises. As an example, let $n = 1$, $g(x) = x$, and take F to be either the delta function or its derivative. Both $\delta(x)$ and $\delta'(x)$ vanish for $x \neq 0$, so one might guess that $x\delta(x)$ and $x\delta'(x)$ should be identically zero. The first of these guesses is correct, but the second is not. One must go back to the definition to see what happens:

$$\int \Big[x\delta(x)\Big]\phi(x)\,dx = \int \delta(x)\Big[x\phi(x)\Big]\,dx = \Big[x\phi(x)\Big]_{x=0} = 0,$$

so $x\delta(x) = 0$, but

$$\begin{aligned}\int \Big[x\delta'(x)\Big]\phi(x)\,dx &= \int \delta'(x)\Big[x\phi(x)\Big]\,dx = -\int \delta(x)\Big[x\phi(x)\Big]'\,dx \\ &= -\Big[\phi(x) + x\phi'(x)\Big]_{x=0} = -\phi(0) = -\int \delta(x)\phi(x)\,dx.\end{aligned}$$

In other words, $x\delta'(x) = -\delta(x)$! The reader should work out some more examples of this sort to become comfortable with the ideas; see Exercises 5–7.

We conclude this section with one final example to illustrate how theorems of analysis can be expressed in the language of distributions. For simplicity we take $n = 3$. If f is a differentiable function on $\mathbf{R}^3$, the smeared-out values of its gradient ∇f can be defined by integration against a vector-valued (i.e., $\mathbf{R}^3$-valued) function ϕ via the dot product:

$$\int (\nabla f)\cdot\phi = \sum_1^3 \int \frac{\partial f}{\partial x_j}\phi_j = -\sum_1^3 \int f\frac{\partial \phi_j}{\partial x_j} = -\int f(\nabla\cdot\phi).$$

Now, suppose R is a region in $\mathbf{R}^3$ that is bounded by a smooth, closed surface S, and let χ be the function defined by $\chi(\mathbf{x}) = 1$ for $\mathbf{x} \in R$ and $\chi(\mathbf{x}) = 0$ for $\mathbf{x} \notin R$. Then the distribution gradient of χ is defined according to the above formula by

$$(\nabla\chi)[\phi] = -\int_R (\nabla\cdot\phi)(\mathbf{x})\,d\mathbf{x}.$$

On the other hand, from a geometric point of view $\nabla f(\mathbf{x})$ is a vector that points in the direction of steepest increase of f at $\mathbf{x}$, with magnitude equal to the rate of increase in that direction. In the case of our function χ, the only place where the values of χ change is along the surface S, and there the "direction of steepest increase" is the direction pointing directly into R, that is, the inward normal. Moreover, the "rate of increase" in this direction is infinite in such a way as to give a jump from 0 to 1 as S is crossed. In short, it seems a reasonable guess that $\nabla f = -\mathbf{n}\,dS$ where $\mathbf{n}$ is the unit outward normal to S and dS is surface measure on S, i.e., the "delta function along S" in the sense of Example 4. In other words, for any vector-valued test function ϕ we should have

$$\int_S \mathbf{n}\cdot\phi(\mathbf{x})\,dS(\mathbf{x}) = -(\nabla\chi)[\phi] = \int_R (\nabla\cdot\phi)(\mathbf{x})\,d\mathbf{x}.$$

This is precisely the divergence theorem. Analogous results hold in any number of dimensions. (For $n = 2$, see Exercise 8.)

EXERCISES

1. If f is a function on $\mathbf{R}^n$ and $a > 0$, let $f^{[a]}(\mathbf{x}) = f(a\mathbf{x})$. A function f is called **homogeneous of degree** λ if $f^{[a]}(\mathbf{x}) = a^{\lambda} f(\mathbf{x})$, and a distribution F is called **homogeneous of degree** λ if $F[\phi^{[a]}] = a^{-n-\lambda} F[\phi]$ for all $\phi \in C_0^{(\infty)}(\mathbf{R}^n)$.
   a. Show that these definitions coincide for locally integrable functions.
   b. Show that if F is homogeneous of degree λ, then $\partial^\alpha F$ is homogeneous of degree $\lambda - |\alpha|$.
   c. Show that $\partial^\alpha \delta$ is homogeneous of degree $-n - |\alpha|$.
2. Prove Theorem 9.1.
3. Show that the product rule holds for derivatives of products of $C^{(\infty)}$ functions and distributions. Specifically, suppose $g \in C^{(\infty)}(\mathbf{R})$ and $F \in \mathscr{D}'(\mathbf{R})$. Show that $(gF)' = g'F + gF'$ in the sense of definitions (9.2) and (9.6). (In $\mathbf{R}^n$, the same result holds for partial derivatives.)
4. Let $f(x) = x^2$ for $x < 1$, $f(x) = x^2 + 2x$ for $1 \le x < 2$, and $f(x) = 2x$ for $x \ge 2$. Compute the distribution derivative f' (a) by using Theorem 9.1, (b) by writing $f(x) = x^2 H(2 - x) + 2xH(x - 1)$ where H is the Heaviside function, and using Exercise 3.
5. In dimension $n = 1$, show that

$$x^j \delta^{(k)}(x) = \begin{cases} (-1)^j k(k-1)\cdots(k-j+1)\delta^{(k-j)}(x) & \text{if } j \le k, \\ 0 & \text{if } j > k. \end{cases}$$

   (Hint: $(fg)^{(k)} = \sum_{j=0}^{k} f^{(j)} g^{(k-j)} k!/j!(k-j)!$.)
6. In dimension $n = 1$, show that if $g \in C^{(\infty)}$, then

$$g(x)\delta^{(k)}(x) = \sum_{0}^{k} (-1)^j \frac{k!}{j!(k-j)!} g^{(j)}(0)\delta^{(k-j)}(x).$$

   (This generalizes Exercise 5.)
7. In $\mathbf{R}^n$, show that $|\mathbf{x}|^2 \nabla^2 \delta(\mathbf{x}) = 2n\delta(\mathbf{x})$.
8. In dimension $n = 2$, show that the following assertion is equivalent to Green's theorem. Let D be a region in $\mathbf{R}^2$ bounded by a smooth closed curve C. Let $\mathbf{n}$ be the unit outward normal to D on C and σ the arc length measure on C, and define $f(\mathbf{x}) = 1$ for $\mathbf{x} \in D$, $f(\mathbf{x}) = 0$ for $\mathbf{x} \notin D$. Then $\nabla f = -\mathbf{n}\sigma$.
9. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be an invertible linear transformation. If F is a distribution on $\mathbf{R}^n$, define the composition $F \circ T$ by $(F \circ T)[\phi] = |\det T|^{-1} F[\phi \circ T^{-1}]$. Show that this definition coincides with the usual definition of $F \circ T$ when F is a function. Can you generalize this to a nonlinear but continuously differentiable invertible transformation T?

9.2 Convergence, convolution, and approximation

There is a natural notion of convergence for distributions, an analogue of pointwise convergence for functions, called *weak convergence*; namely, we say that a sequence $\{F_k\}$ of distributions **converges weakly** to a distribution F if

$$F_k[\phi] \to F[\phi] \text{ for all test functions } \phi.$$

Similarly, if $\{F_t\}$ is a family of distributions depending on a real parameter t, we say that $F_t \to F$ weakly as $t \to t_0$ if $F_t[\phi] \to F[\phi]$ as $t \to t_0$ for all ϕ.

This notion of convergence can be applied, in particular, when the F_k's and F are all locally integrable functions. In this context, the appropriateness of the term "weak convergence" comes from the fact that if F_k converges to F in almost any reasonable sense (with the unfortunate exception of simple pointwise convergence), then F_k converges to F weakly. For instance, if F_k and F are locally integrable functions, then $F_k \to F$ weakly if any one of the following conditions is satisfied:

(i) $F_k \to F$ pointwise, and there is a fixed locally integrable function G such that $|F_k(\mathbf{x})| \le G(\mathbf{x})$ for all k and $\mathbf{x}$.

(ii) $F_k \to F$ uniformly on every bounded subset of $\mathbf{R}^k$.

(iii) $F_k \to F$ in the L^2 norm on every bounded subset of $\mathbf{R}^n$; that is, if $K \subset \mathbf{R}^n$ is bounded, $\int_K |F_k - F|^2 \to 0$.

The first two of these assertions are easy consequences of the dominated convergence theorem (Exercise 1), and the third follows from the Cauchy-Schwarz inequality: If $K = \text{supp}(\phi)$,

$$\left|F_k[\phi] - F[\phi]\right|^2 = \left|\int_K (F_k - F)\phi\right|^2 \le \int_K |F_k - F|^2 \int_K |\phi|^2 \to 0.$$

The main interest of weak convergence, however, derives from the fact that the weak limit of a sequence of functions may be no longer a function but a more singular distribution. The most basic example is the following.

Theorem 9.2. *Suppose f is an integrable function on $\mathbf{R}^n$ such that $\int f(\mathbf{x})\,d\mathbf{x} = 1$, and let $f_\epsilon(\mathbf{x}) = \epsilon^{-n} f(\epsilon^{-1}\mathbf{x})$ for $\epsilon > 0$. Then $f_\epsilon \to \delta$ weakly as $\epsilon \to 0$.*

Proof: Let $\widetilde{f}(\mathbf{x}) = f(-\mathbf{x})$ and $\widetilde{f}_\epsilon(\mathbf{x}) = \epsilon^{-n} f(-\epsilon^{-1}\mathbf{x})$. $\widetilde{f}$ is also integrable and satisfies $\int \widetilde{f} = 1$, so Theorem 7.7 of §7.5 asserts that $\widetilde{f}_\epsilon * \phi(\mathbf{x}) \to \phi(\mathbf{x})$ for all $\mathbf{x}$ as $\epsilon \to 0$ when ϕ is any bounded continuous function. In particular, this is true when ϕ is a test function and $\mathbf{x} = \mathbf{0}$, that is,

$$\widetilde{f}_\epsilon * \phi(\mathbf{0}) = \int f_\epsilon(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} \to \phi(\mathbf{0}) = \int \delta(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x}.$$

But this says precisely that $f_\epsilon \to \delta$ weakly. ∎

The geometric interpretation of this theorem is clear: As $\epsilon \to 0$, the graph of f_ϵ becomes a sharp spike near the origin that gets both taller and narrower as $\epsilon \to 0$ in such a way that the integral $\int f_\epsilon$ remains equal to 1; in the limit one gets the intuitive picture of δ as a spike of infinite height and zero width, with integral equal to 1. See Figure 9.1(a).

As we pointed out in §9.1, one of the main reasons for the utility of distributions is the fact that one can differentiate much more fearlessly than with functions. In this connection, the following result is of sufficient importance to be stated as a theorem, although its proof is a triviality.

Theorem 9.3. *Differentiation is continuous with respect to weak convergence; that is, if $F_k \to F$ weakly, then $\partial^\alpha F_k \to \partial^\alpha F$ weakly for any α.*

Proof: For any test function ϕ,

$$(\partial^\alpha F_k)[\phi] = (-1)^{|\alpha|} F_k[\partial^\alpha \phi] \to (-1)^{|\alpha|} F[\partial^\alpha \phi] = (\partial^\alpha F)[\phi]. \qquad \blacksquare$$

Theorem 9.3 is in marked constrast with the usual situation for functions: If $f_k \to f$ pointwise or uniformly, say, there is no guarantee that the derivatives of f_k will converge to the derivatives of f (or anything else) pointwise or uniformly. The following example may be illuminating. Take $n = 1$, and let

$$f_k(x) = k^{-1} \cos kx.$$

Then $|f_k(x)| \le k^{-1}$, so $f_k \to 0$ uniformly, and this also implies that $f_k \to 0$ weakly. Now, $f_k'(x) = -\sin kx$, so $\lim_{k\to\infty} f_k'(x)$ does not exist for *any* x except the integer multiples of π. Nonetheless, $f_k' \to 0$ weakly, that is,

$$\lim_{k\to\infty} \int \phi(x) \sin kx \, dx = 0 \text{ for any test function } \phi.$$

This also follows from the Riemann-Lebesgue lemma, since

$$\int \phi(x) \sin kx \, dx = \frac{\widehat{\phi}(k) - \widehat{\phi}(-k)}{2i}.$$

Intuitively, the idea is that f_k' is very rapidly oscillatory when k is large, and its average value is zero, so its smeared-out values $f_k'[\phi]$ are close to zero.

If f_ϵ is as in Theorem 9.2, then $\partial^\alpha f_\epsilon \to \partial^\alpha \delta$ as $\epsilon \to 0$, for any derivative ∂^α. By choosing f to be something simple and graphing the derivatives $\partial^\alpha f_\epsilon$ for ϵ small, one can obtain some pictorial intuition about the derivatives of the delta function. Figure 9.1(b) depicts f_ϵ' in the case $n = 1$, $f(x) = \pi^{-1/2} e^{-x^2}$. One can see that f_ϵ' has a positive spike to the left of the origin and a negative spike to the right. The height of these spikes is proportional to ϵ^{-2} and the area between them and the x-axis is proportional to ϵ^{-1}, in contrast to f_ϵ where the height is proportional to ϵ and the area is constant. In fact, by some simple calculations that we leave to the reader, the area of these spikes is $1/\epsilon\sqrt{\pi}$ and

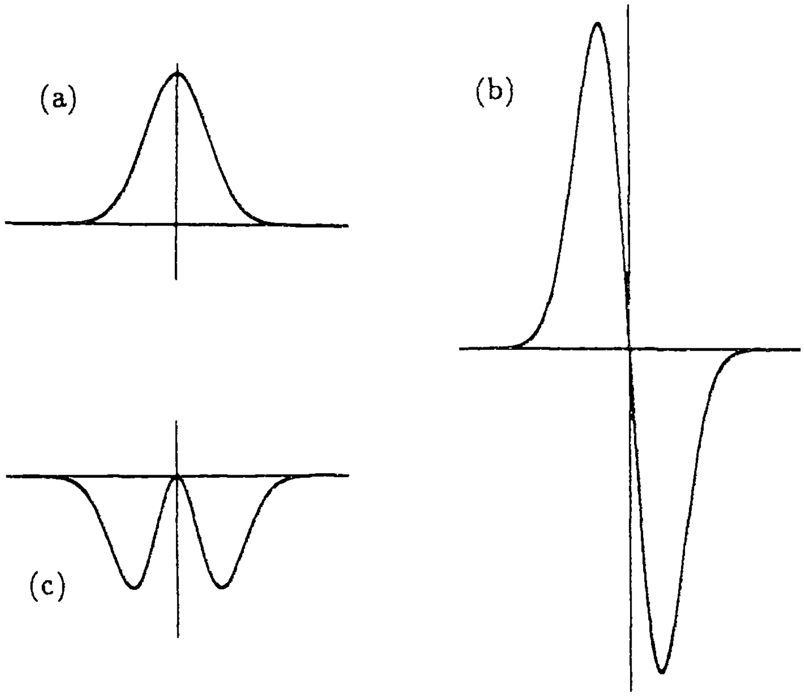


FIGURE 9.1. The functions (a) $f_\epsilon(x)$, (b) $f_\epsilon'(x)$, and (c) $xf_\epsilon'(x)$, where $f(x) = \pi^{-1/2}e^{-x^2}$ and $\epsilon = 0.4$.

the x-coordinates of their centers of mass are $\pm\frac{1}{2}\epsilon\sqrt{\pi}$. Hence, one can think of these spikes as being roughly like delta functions centered at $\pm\frac{1}{2}\epsilon\sqrt{\pi}$ and scaled by the factor $1/\epsilon\sqrt{\pi}$, so the effect of integrating a test function ϕ against f_ϵ' is roughly

$$\frac{1}{\epsilon\sqrt{\pi}}\int\left[\delta(x+\tfrac{1}{2}\epsilon\sqrt{\pi}) - \delta(x-\tfrac{1}{2}\epsilon\sqrt{\pi})\right]\phi(x)\,dx = \frac{\phi(-\frac{1}{2}\epsilon\sqrt{\pi}) - \phi(\frac{1}{2}\epsilon\sqrt{\pi})}{\epsilon\sqrt{\pi}},$$

which converges as $\epsilon \to 0$ to $-\phi'(0) = \delta'[\phi]$.

These approximations can also shed light on mysterious results like the identity $x\delta'(x) = -\delta(x)$ derived in §9.1. With $f(x) = \pi^{-1/2}e^{-x^2}$ as above, we have

$$xf_\epsilon'(x) = -\frac{2x^2}{\epsilon^3\sqrt{\pi}}e^{-x^2/\epsilon^2} = g_\epsilon(x) \quad \text{where } g(x) = -\frac{2x^2}{\sqrt{\pi}}e^{-x^2}.$$

A straightforward integration by parts shows that $\int g(x)\,dx = -\int f(x)\,dx = -1$, so by Theorem 9.2, $g_\epsilon \to -\delta$ weakly as $\epsilon \to 0$. Figure 9.1(c) gives the picture: $xf_\epsilon'(x)$ has a pair of negative spikes near the origin, and the area between the spikes and the x-axis is 1. As $\epsilon \to 0$, this approaches the negative of the delta function.

By now it should be clear that if one can represent a distribution as a weak limit of smooth functions, one can obtain insights into its structure that may be hard to obtain otherwise. The question therefore arises as to which distributions can be so represented, and fortunately the answer is: *all of them.* The idea, just as in Theorem 9.2, is to smooth a given distribution out by convolving it with a smooth approximate identity.

To make this work, we need to add one more thing to our list of operations that can be performed on distributions: *convolution with test functions.* If F is a locally integrable function and ϕ is a test function, we recall that the convolution $F * \phi$ is defined by

$$F * \phi(\mathbf{x}) = \int F(\mathbf{y})\phi(\mathbf{x} - \mathbf{y})\, d\mathbf{y}. \tag{9.7}$$

But the function $\widetilde{\phi}_{\mathbf{x}}(\mathbf{y}) = \phi(\mathbf{x} - \mathbf{y})$ is a test function for each $\mathbf{x}$, so the integral on the right makes sense when F is an arbitrary distribution if it is interpreted as $F[\widetilde{\phi}_{\mathbf{x}}]$. Hence we can take (9.7) as a definition of $F * \phi$ for any $F \in \mathscr{D}'$ and $\phi \in C_0^{(\infty)}$. $F * \phi$ is a genuine function, defined pointwise by (9.7), and in fact it possesses derivatives of all orders in the traditional sense.

Theorem 9.4. *If F is a distribution and ϕ is a test function, then $F * \phi$ is a function of class $C^{(\infty)}$, and $\partial^\alpha(F * \phi) = F * \partial^\alpha \phi$.*

Proof: For simplicity, suppose for the moment that $n = 1$. The support of ϕ is contained in $\{y : |y| \leq r\}$ for some $r > 0$, so if $|x| \leq C$ and $|h| \leq 1$ the functions $y \to \phi(x + h - y)$ are all supported in $\{y : |y| \leq R\}$ where $R = r + C + 1$. Moreover, ϕ and all its derivatives are uniformly continuous. It follows that the functions $\psi_{x,h}$ and $\chi_{x,h}$ defined by

$$\psi_{x,h}(y) = \phi(x+h-y) - \phi(x-y), \quad \chi_{x,h}(y) = \frac{\phi(x+h-y) - \phi(x-y)}{h} - \phi'(x-y)$$

and all their derivatives are supported in $\{y : |y| \leq R\}$ and converge uniformly to zero as $h \to 0$. Therefore, by the continuity condition in the definition of distribution,

$$(F * \phi)(x + h) - (F * \phi)(x) = F[\psi_{x,h}] \to 0 \text{ as } h \to 0,$$

which shows that $F * \phi$ is continuous, and

$$\frac{(F * \phi)(x + h) - (F * \phi)(x)}{h} - (F * \phi')(x) = F[\chi_{x,h}] \to 0 \text{ as } h \to 0,$$

which shows that $F * \phi$ is differentiable (in the classical sense) and its derivative is $F * \phi'$. The same argument can now be applied to $F * \phi'$ to obtain $(F * \phi)'' = F * \phi''$, and so forth; by induction, $(F * \phi)^{(k)} = F * \phi^{(k)}$ for all k. For functions of several variables, the same argument can be applied in each variable to show that the partial derivatives of $F * \phi$ of all orders exist and are continuous. ∎

We need one more preliminary result before coming to the theorem about approximating distributions by smooth functions.

Lemma 9.1. *Suppose F is a distribution and ϕ and ψ are test functions. Then $(F * \phi)[\psi] = F[\widetilde{\phi} * \psi]$ where $\widetilde{\phi}(\mathbf{x}) = \phi(-\mathbf{x})$.*

Proof: If F is a locally integrable function, the assertion is that

$$\int (F * \phi)(\mathbf{x})\psi(\mathbf{x})\,d\mathbf{x} = \int F(\mathbf{y})(\widetilde{\phi} * \psi)(\mathbf{y})\,d\mathbf{y},$$

which is true since both of these integrals are equal to the double integral

$$\iint F(\mathbf{y})\phi(\mathbf{x}-\mathbf{y})\psi(\mathbf{x})\,d\mathbf{x}\,d\mathbf{y}.$$

The idea is the same when F is a distribution, but a bit more needs to be said in that case since the integrals are only symbolic. The trick is to think of the integral defining $\widetilde{\phi} * \psi$ as a limit of Riemann sums:

$$\int \phi(\mathbf{x}-\mathbf{y})\psi(\mathbf{x})\,d\mathbf{x} = \lim \sum \phi(\mathbf{x}_j - \mathbf{y})\psi(\mathbf{x}_j)\Delta\mathbf{x}_j.$$

(To be precise, one can fix a cube that contains $\operatorname{supp}(\psi)$ and subdivide it into small subcubes; $\mathbf{x}_j$ is the center of the jth subcube and $\Delta\mathbf{x}_j$ is its volume, and the limit is taken over finer and finer subdivisions.) Since $\phi \in C_0^{(\infty)}$, it is not hard to verify that the sums on the right and their derivatives converge uniformly to $\widetilde{\phi} * \psi$ and its derivatives, and they are all supported in the bounded set $\{\mathbf{x}+\mathbf{y} : \mathbf{x} \in \operatorname{supp}(\phi),\ \mathbf{y} \in \operatorname{supp}(\psi)\}$. Hence, by the continuity condition in the definition of distribution,

$$\begin{aligned} F[\widetilde{\phi} * \psi] &= \lim F\left[\sum \phi(\mathbf{x}_j - \mathbf{y})\psi(\mathbf{x}_j)\Delta\mathbf{x}_j\right] = \lim \sum F[\phi(\mathbf{x}_j - \mathbf{y})]\psi(\mathbf{x}_j)\Delta\mathbf{x}_j \\ &= \lim \sum (F * \phi)(\mathbf{x}_j)\psi(\mathbf{x}_j)\Delta\mathbf{x}_j = \int (F * \phi)(\mathbf{x})\psi(\mathbf{x})\,d\mathbf{x} = (F * \phi)[\psi]. \quad \blacksquare \end{aligned}$$

Theorem 9.5. *Let ϕ be a test function on $\mathbf{R}^n$ with $\int \phi(\mathbf{x})\,d\mathbf{x} = 1$, and let $\phi_\epsilon(\mathbf{x}) = \epsilon^{-n}\phi(\epsilon^{-1}\mathbf{x})$. Then for any distribution F, $F * \phi_\epsilon \to F$ weakly as $\epsilon \to 0$.*

Proof: Let $\widetilde{\phi}(\mathbf{x}) = \phi(-\mathbf{x})$ as in Lemma 9.1; then $\widetilde{\phi}$ is again a test function whose integral equals 1. Hence, for any test function ψ,

$$\widetilde{\phi}_\epsilon * \psi \to \psi \quad \text{and} \quad \partial^\alpha(\widetilde{\phi}_\epsilon * \psi) = \widetilde{\phi}_\epsilon * \partial^\alpha\psi \to \partial^\alpha\psi$$

uniformly as $\epsilon \to 0$. Moreover, for $\epsilon \le 1$ the functions $\widetilde{\phi}_\epsilon * \psi$ are all supported in a common compact set. (Namely, if ϕ is supported in $\{\mathbf{x} : |\mathbf{x}| \le a\}$ and ψ is supported in $\{\mathbf{x} : |\mathbf{x}| \le b\}$, then $\widetilde{\phi}_\epsilon * \psi$ is supported in $\{\mathbf{x} : |\mathbf{x}| \le a+b\}$.) It then follows from Lemma 9.1 that

$$(F * \phi_\epsilon)[\psi] = F[\widetilde{\phi}_\epsilon * \psi] \to F[\psi],$$

that is, $F * \phi_\epsilon \to F$ weakly. $\blacksquare$

Theorems 9.4 and 9.5 together show that every distribution is the weak limit of a sequence of infinitely differentiable functions. (It is even the weak limit of a sequence of test functions; see Exercises 4–5.) Thus, although from one point of view distributions are much more general than functions, from another one they are not far from being functions. In fact, one can adopt the notion of weak limits of functions as a definition of distributions and develop the theory from there; see Lighthill [37].

It should be pointed out that Theorem 9.5 is not the only way of producing smooth approximations to a distribution F. (For instance, as Theorem 9.2 shows, when $F = \delta$ one does not need ϕ to be a test function.) In particular cases one can often produce approximating sequences directly without invoking a general theorem.

One can use the formula of Lemma 9.1 to define the convolution of two distributions *provided that at least one of them has bounded support.* Namely, if F and G are distributions and G has bounded support, one defines

$$(F * G)[\phi] = F[\widetilde{G} * \phi], \qquad \widetilde{G}(\mathbf{x}) = G(-\mathbf{x}). \tag{9.8}$$

Here $\widetilde{G} * \phi$ is defined by (9.7). It is infinitely differentiable by Theorem 9.4 and has bounded support since both $\widetilde{G}$ and ϕ do; hence it is a test function and F can be applied to it. For example, take $G = \delta$: then $\widetilde{G} = G$ and

$$(F * \delta)[\phi] = f[\delta * \phi] = F[\phi].$$

That is, $F * \delta = F$ for any F, so δ is the identity element for the convolution product.

If F and G are distributions on $\mathbf{R}^n$, one can define $F(\mathbf{x})G(\mathbf{y})$ as a distribution on $\mathbf{R}^{2n}$ (the so-called *tensor product* of F and G) in the obvious way: Its action on the test function $\phi \in C_0^{(\infty)}(\mathbf{R}^{2n})$ is

$$\iint F(\mathbf{x})G(\mathbf{y})\phi(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}.$$

More precisely, for each fixed $\mathbf{y}$, $\phi(\mathbf{x},\mathbf{y})$ is a test function of $\mathbf{x}$, and we can apply F to it. The resulting function $F[\phi(\cdot,\mathbf{y})]$ is easily seen to be a test function of $\mathbf{y}$, and we can apply G to it. We could also perform these operations in the opposite order; the answer is the same. For example, it is easily verified that $\delta(\mathbf{x})\delta(\mathbf{y}) = \delta(\mathbf{x},\mathbf{y})$, $\delta(\mathbf{x},\mathbf{y})$ being the delta function on $\mathbf{R}^{2n}$.

However, in general it does *not* make sense to try to define the product $F(\mathbf{x})G(\mathbf{x})$ as a distribution on $\mathbf{R}^n$. For example, there is no reasonable way to define $\delta^2(\mathbf{x}) = \delta(\mathbf{x}) \cdot \delta(\mathbf{x})$. (See Exercise 2.)

EXERCISES

1. Suppose F_k $(k = 1,2,\ldots)$ and F are locally integrable functions on $\mathbf{R}^n$ and $F_k \to F$ pointwise. Show that $F_k \to F$ weakly if there is a locally integrable G such that $|F_k(\mathbf{x})| \le G(\mathbf{x})$ for all k and $\mathbf{x}$, or if $F_k \to F$ uniformly on every bounded subset of $\mathbf{R}^n$.

2. On **R**, let $F_k(x) = k$ if $k^{-1} < x < 2k^{-1}$, $F_k(x) = 0$ otherwise. Show that:
   a. $F_k \to 0$ pointwise, but $F_k \to \delta$ weakly;
   b. $F_k^2 \to 0$ pointwise, but F_k^2 diverges weakly.
3. Construct a test function χ on $\mathbf{R}^n$ such that $\chi(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq 1$ and $\chi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq 3$. (Hint: Let $f(\mathbf{x}) = 1$ if $|\mathbf{x}| < 2$, $f(\mathbf{x}) = 0$ otherwise, and pick $\psi \in C_0^{(\infty)}$ supported in $\{\mathbf{x} : |\mathbf{x}| < 1\}$ such that $\int \psi = 1$. Show that $\chi = f * \psi$ does the job.)
4. Let χ be as in Exercise 3. If g is locally integrable and $\epsilon > 0$, let $g^\epsilon(\mathbf{x}) = \chi(\epsilon\mathbf{x})g(\mathbf{x})$. Show that $g^\epsilon \to g$ weakly as $\epsilon \to 0$.
5. Show that any distribution F is the weak limit of a sequence of test functions. (Hint: Let ϕ_ϵ be as in Theorem 9.5, and let χ be as in Exercise 3. Consider $\chi(\epsilon\mathbf{x})(F * \phi_\epsilon)(\mathbf{x})$.)
6. Show that if $\{\mathbf{x}_k\}$ is any sequence of points in $\mathbf{R}^n$ with $|\mathbf{x}_k| \to \infty$, then $\delta(\mathbf{x} - \mathbf{x}_k) \to 0$ weakly.
7. In dimension $n = 1$, show that $h^{-1}\left[\delta(x+h) - \delta(x)\right]$ converges weakly to $\delta'(x)$ as $h \to 0$.
8. In dimension $n = 1$, show that $x^\epsilon H(x) \to H(x)$ weakly as $\epsilon \to 0$ where H is the Heaviside function. Then show that $\epsilon x^{\epsilon-1}H(x) \to \delta(x)$ weakly as $\epsilon \to 0$.
9. In dimension $n = 1$, show that $k^N \sin kx \to 0$ weakly as $k \to \infty$, for any N.
10. In dimension $n = 2$, let σ_r be the arc length measure on the circle of radius r about the origin. Show that $(2\pi r)^{-1}\sigma_r \to \delta$ weakly as $r \to 0$. What is the analogue in dimension $n = 3$?
11. Suppose F is a distribution and ϕ is a test function on $\mathbf{R}^n$. Show that $(\partial^\alpha F) * \phi = F * (\partial^\alpha \phi) = \partial^\alpha (F * \phi)$ for any α.

9.3 More examples: Periodic distributions and finite parts

Throughout this section we shall be working in dimension one, i.e., with functions and distributions of one real variable. (The ideas, however, can be extended to functions of several variables.)

Periodic distributions

A distribution F on **R** is called **periodic** with period P if $F(x+P) = F(x)$ in the sense of equation (9.4):

$$F\left[\phi(x-P)\right] = F\left[\phi(x)\right] \text{ for all test functions } \phi.$$

We shall consider the case $P = 2\pi$; the general case is easily reduced to this one by the change of variable $x \to 2\pi x/P$.

Let $\{c_k\}_{-\infty}^{\infty}$ be a sequence of constants such that for some positive constants C and N we have

$$|c_k| \leq C|k|^N \quad \text{for } k \neq 0, \tag{9.9}$$

and let us form the Fourier series

$$\sum_{-\infty}^{\infty} c_k e^{ikx}. \tag{9.10}$$

Of course this series cannot converge pointwise unless $c_k \to 0$ as $k \to \pm\infty$, but we claim that it converges weakly to a periodic distribution. Indeed, let us consider the series obtained by omitting the constant term from (9.10) and formally integrating $N+2$ times:

$$f(x) = \sum_{k \neq 0} (ik)^{-N-2} c_k e^{ikx}. \tag{9.11}$$

By the assumption (9.9), the coefficients $b_k = (ik)^{-N-2} c_k$ satisfy $|b_k| \leq C|k|^{-2}$; hence the series (9.11) converges absolutely and uniformly, and its sum f is a continuous periodic function. But since (9.11) converges uniformly, it also converges weakly, i.e., the partial sums of the series converge weakly to f. By Theorem 9.2, then, for any positive integer j the j-times differentiated series converges weakly to the jth distribution derivative of f. In particular, for $j = N+2$, we obtain

$$\sum_{-\infty}^{\infty} c_k e^{ikx} = c_0 + \operatorname*{weak\,lim}_{K \to \infty} \sum_{1 \leq |k| \leq K} c_k e^{ikx} = c_0 + f^{(N+2)}(x);$$

that is, the series (9.10) converges weakly to the periodic distribution $f^{(N+2)} + c_0$.

Conversely, suppose F is a periodic distribution obtained by differentiating a continuous periodic function f some number of times, say $F = f^{(m)}$. We can assume that f is actually continuously differentiable (if not, just integrate it once after subtracting off a constant so that it has mean value zero; see Theorem 2.5, §2.3). By Theorem 2.3 of §2.3, the Fourier series of f converges absolutely and uniformly, and hence weakly, to f:

$$f(x) = \sum C_k e^{ikx}.$$

But then by Theorem 9.3, the m-times differentiated series converges weakly to $F = f^{(m)}$:

$$F = \sum_{-\infty}^{\infty} (ik)^m C_k e^{ikx}.$$

Moreover, the sequence $\{C_k\}$ is bounded (in fact $C_k \to 0$ as $k \to \pm\infty$), so the coefficients $c_k = (ik)^m C_k$ satisfy (9.9) with $N = m$.

One more thing is needed to give a complete picture: the fact that *every periodic distibution can be written in the form $F = c_0 + f^{(m)}$ where c_0 is a constant, f is a continuous periodic function, and m is a positive integer.* The proof is not really deep, but it involves a rather technical use of the continuity condition in the definition of distribution, and we shall omit it. Taking this fact for granted, then, we can sum up our results in the following theorem, a full proof of which can be found in Zemanian [57].

Theorem 9.6. *If F is any periodic distribution, then F can be expanded in a weakly convergent Fourier series,*

$$F(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}, \quad \text{that is,} \quad F[\phi] = \sum_{-\infty}^{\infty} c_k \int \phi(x) e^{ikx} dx \text{ for } \phi \in C_0^{(\infty)},$$

and the coefficients c_k satisfy

$$c_k \leq C\Big(1 + |k|\Big)^N$$

for some $C, N \geq 0$. Conversely, if $\{c_k\}$ is any sequence satisfying this estimate, the series $\sum_{-\infty}^{\infty} c_k e^{ikx}$ converges weakly to a periodic distribution.

If F is a continuous periodic function, its Fourier coefficients are given by the familiar formula

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) e^{-ikx}\, dx. \tag{9.12}$$

What if F is merely a distribution? Unfortunately, in that case formula (9.12) does not make sense as it stands. The integral of a distribution over a finite interval such as $[-\pi, \pi]$ is generally not well-defined because of possible singular behavior of the distribution at the endpoints. However, there are ways to finesse this difficulty, one of which is indicated in Exercise 1: Formula (9.12) is essentially correct. In practice, one usually represents a distribution F as some mth-order derivative of a piecewise smooth function f as above, and computes the Fourier series for F by differentiating the Fourier series for f.

Perhaps the most fundamental example of a periodic distribution that is not a function is the *periodic delta function*

$$\delta_{\text{per}}(x) = \sum_{-\infty}^{\infty} \delta(x - 2k\pi), \tag{9.13}$$

the sum of the delta functions at all the integer multiples of 2π. δ_{per} does not have mean value 0 — its integral over any interval of length 2π is 1 — so it is not the derivative of any periodic function. But $\delta_{\text{per}} - (2\pi)^{-1}$ has mean value 0, and it is the derivative of the sawtooth wave

$$f(x) = \frac{1}{2\pi}(\pi - x) \quad \text{for } 0 < x < 2\pi, \qquad f(x + 2k\pi) = f(x).$$

This is clear from Theorem 9.1 and Figure 9.2(a): f has slope $-1/2\pi$ except at the points $2k\pi$, where the jump from $-\frac{1}{2}$ to $\frac{1}{2}$ produces a delta function. The Fourier series of f is easily computed to be $\sum_1^{\infty}(\sin kx)/\pi k$ (see Table 1, §2.1), and hence

$$\delta_{\text{per}}(x) = \frac{1}{2\pi} + f'(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{1}^{\infty} \cos kx = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{ikx}. \tag{9.14}$$

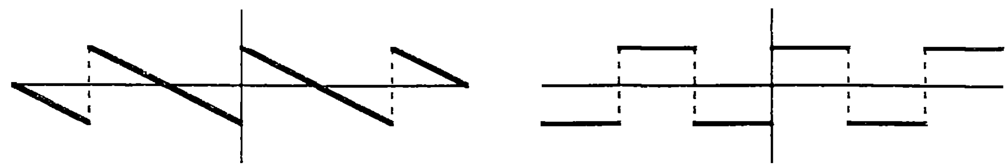


FIGURE 9.2. (a) The sawtooth wave f; (b) the square wave g.

In other words, δ_{per} is the distribution whose Fourier coefficients c_k are all equal to $1/2\pi$. This is also the result one gets by formally using the standard formula for Fourier coefficients:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta_{\text{per}}(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int \delta(x)e^{-ikx}\,dx = \frac{1}{2\pi}.$$

As another example, let us consider the square wave

$$g(x) = -1 \text{ for } -\pi < x < 0, \quad g(x) = 1 \text{ for } 0 < x < \pi, \qquad g(x + 2n\pi) = g(x),$$

whose Fourier series is

$$g(x) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin kx}{k}. \tag{9.15}$$

What is g'? On the one hand, by differentiating (9.15) we obtain

$$g'(x) = \frac{4}{\pi}\sum_{k \text{ odd}} \cos kx. \tag{9.16}$$

On the other hand, from Figure 9.2(b) it is clear that g' vanishes except at multiples of π; at the even multiples there is 2 times a delta function, and at the odd multiples there is -2 times a delta function — that is,

$$g'(x) = 2\sum_{-\infty}^{\infty}\delta(x - 2k\pi) - 2\sum_{-\infty}^{\infty}\delta(x-(2k+1)\pi) = 2\delta_{\text{per}}(x) - 2\delta_{\text{per}}(x-\pi).$$

Hence, by (9.14),

$$g'(x) = \frac{2}{\pi}\sum_{1}^{\infty}\cos kx - \frac{2}{\pi}\sum_{1}^{\infty}\cos k(x-\pi) = \frac{2}{\pi}\sum_{1}^{\infty}\cos kx - \frac{2}{\pi}\sum_{1}^{\infty}(-1)^k\cos kx,$$

which agrees with (9.16).

Finite parts of divergent integrals

The only functions that define distributions by the rule $F[\phi] = \int F(x)\phi(x)\,dx$ are the locally integrable ones. However, certain kinds of functions with nonintegrable singularities can also be made into distributions. Here we shall consider the basic examples arising from negative powers of the variable x.

The simplest example is the function $f(x) = x^{-1}$. If ϕ is a test function, the integral $\int x^{-1}\phi(x)\,dx$ is not absolutely convergent unless $\phi(0) = 0$, but it has an obvious interpretation as a so-called *principal value integral*:

$$P.V. \int \frac{\phi(x)}{x}\,dx = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\phi(x)}{x}\,dx.$$

The limit on the right exists because of cancellation between the negative values of x^{-1} on the left and the positive values on the right. More precisely, since x^{-1} is an odd function, whenever $0 < \epsilon < a$ we have

$$\int_{\epsilon<|x|<a} \frac{1}{x}\,dx = \int_{-a}^{-\epsilon} \frac{1}{x}\,dx + \int_{\epsilon}^{a} \frac{1}{x}\,dx = 0,$$

so

$$\int_{|x|>\epsilon} \frac{\phi(x)}{x}\,dx = \int_{\epsilon<|x|<a} \frac{\phi(x)-\phi(0)}{x}\,dx + \int_{|x|>a} \frac{\phi(x)}{x}\,dx.$$

Now, $[\phi(x) - \phi(0)]/x$ approaches the finite limit $\phi'(0)$ as $x \to 0$, so it is bounded on $[-a, a]$. Hence, we can let $\epsilon \to 0$ and define a distribution X^{-1} by the formula

$$X^{-1}[\phi] = P.V. \int \frac{\phi(x)}{x}\,dx = \int_{|x|<a} \frac{\phi(x)-\phi(0)}{x}\,dx + \int_{|x|>a} \frac{\phi(x)}{x}\,dx, \tag{9.17}$$

where the integrals on the right are absolutely convergent. Moreover, since the result is independent of a, we can let $a \to \infty$ to write

$$X^{-1}[\phi] = P.V. \int \frac{\phi(x)-\phi(0)}{x}\,dx.$$

Here the letters $P.V.$ mean that the integral over $\mathbf{R}$ is interpreted as the limit of the integrals over the symmetric intervals $[-a, a]$. X^{-1} agrees with $1/x$ on the complement of the origin, that is, if $0 \notin \operatorname{supp}(\phi)$ then $X^{-1}[\phi] = \int [\phi(x)/x]\,dx$, the integral here being absolutely convergent. We also have, from the definition of the principal value integral, that X^{-1} is the weak limit of the locally integrable functions F_ϵ as $\epsilon \to 0$, where

$$F_\epsilon(x) = \begin{cases} 1/x & \text{if } |x| > \epsilon, \\ 0 & \text{if } |x| \le \epsilon. \end{cases} \tag{9.18}$$

(For a smooth approximation to X^{-1}, see Exercise 4.)

Now let us consider a higher negative integer power of x, say $f(x) = x^{-k}$ where $k \geq 2$. Here the problems of making f into a distribution are a bit more serious. If k is even, there is no cancellation of positive and negative values to help make an integral $\int x^{-k}\phi(x)\,dx$ converge, and even if k is odd, the principal value of this integral usually does not exist; see Exercise 7. However, another method is at our disposal. In the sense of first-year calculus we have

$$x^{-k} = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}x^{-1},$$

and we have merely to reinterpret this equation in the sense of distributions; namely, if X^{-1} is the distribution defined by (9.17), we set

$$X^{-k} = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}X^{-1}.$$

Here the derivatives are to be interpreted in the sense of formula (9.2), that is,

$$\begin{aligned} X^{-k}[\phi] &= \frac{1}{(k-1)!}X^{-1}[\phi^{(k-1)}] \\ &= \frac{1}{(k-1)!}\,P.V.\int \frac{\phi^{(k-1)}(x) - \phi^{(k-1)}(0)}{x}\,dx. \end{aligned} \tag{9.19}$$

It is sometimes preferable to have a formula for $X^{-k}[\phi]$ that more closely resembles $\int x^{-k}\phi(x)\,dx$. If ϕ and its first $k-1$ derivatives vanish at $x = 0$, then $|\phi(x)| \leq C|x|^k$ for x near 0 by Taylor's theorem; one can then simply integrate (9.19) by parts $k-1$ times to obtain $\int x^{-k}\phi(x)\,dx$, the latter integral being absolutely convergent. The same idea will work in general if we choose the constants of integration cleverly at each stage, by taking the antiderivative of $\phi^{(j+1)}(x)$ to be not $\phi^{(j)}(x)$ but $\phi^{(j)}(x) - \phi^{(j)}(0)$. Indeed, if we set

$$\psi(x) = \phi^{(k-2)}(x) - \phi^{(k-2)}(0) - x\phi^{(k-1)}(0),$$

we have $\psi'(x) = \phi^{(k-1)}(x) - \phi^{(k-1)}(0)$ and hence

$$\begin{aligned} P.V.\int \frac{\phi^{(k-1)}(x) - \phi^{(k-1)}(0)}{x}\,dx &= \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\left[\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right]\frac{\phi^{(k-1)}(x) - \phi^{(k-1)}(0)}{x}\,dx \\ &= \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\left[\frac{\psi(x)}{x}\bigg|_{-R}^{-\epsilon} + \frac{\psi(x)}{x}\bigg|_{\epsilon}^{R}\right] + P.V.\int \frac{\psi(x)}{x^2}\,dx. \end{aligned}$$

Since $\psi(x)$ vanishes to second order at $x = 0$ by Taylor's theorem, there is no singularity in the last integral at $x = 0$, and also $\psi(x)/x \to 0$ as $x \to 0$. Hence the endpoint terms at $\pm\epsilon$ vanish as $\epsilon \to 0$, and since $\psi(x)/x \to \phi^{(k-1)}(0)$ as $x \to \pm\infty$, the endpoint terms at $\pm R$ cancel out as $R \to \infty$. In short,

$$P.V.\int \frac{\phi^{(k-1)}(x)-\phi^{(k-1)}(0)}{x}\,dx = P.V.\int \frac{\phi^{(k-2)}(x)-\phi^{(k-2)}(0)-x\phi^{(k-1)}(0)}{x^2}\,dx.$$

We now repeat the process, taking the antiderivative of $\phi^{(k-2)}(x) - \phi^{(k-2)}(0) - x\phi^{(k-1)}(0)$ to be $\phi^{(k-3)}(x) - \phi^{(k-3)}(0) - x\phi^{(k-2)}(0) - \frac{1}{2}x^2\phi^{(k-1)}(0)$, and obtain

$$P.V. \int \frac{\phi^{(k-1)}(x) - \phi^{(k-1)}(0)}{x}\,dx$$
$$= 2\,P.V. \int \frac{\phi^{(k-3)}(x) - \phi^{(k-3)}(0) - x\phi^{(k-2)}(0) - \frac{1}{2}x^2\phi^{(k-1)}(0)}{x^3}\,dx.$$

After $k-1$ such steps and division by $(k-1)!$, we obtain the desired result:

$$X^{-k}[\phi] = P.V. \int \frac{1}{x^k}\left[\phi(x) - \sum_0^{k-1} \frac{\phi^{(j)}(0)}{j!}x^j\right]dx. \tag{9.20}$$

Again, the singularity at the origin has disappeared because the remainder term in Taylor's formula is dominated by $|x|^k$. Each of the terms $x^{-k}\phi(x)$ and $x^{j-k}\phi^{(j)}(0)/j!$ is absolutely integrable at infinity except for $j = k-1$, and the latter one is integrable in the principal-value sense. The convergent integral (9.20) is called the finite part of the divergent integral $\int x^{-k}\phi(x)\,dx$ and is often written as such:

$$X^{-k}[\phi] = F.P. \int \frac{\phi(x)}{x^k}\,dx.$$

Formally, one can rewrite (9.20) as

$$X^{-k}[\phi] = \int x^{-k}\phi(x)\,dx - \sum_0^{k-1} \frac{\phi^{(j)}(0)}{j!}\int x^{j-k}\,dx,$$

or

$$X^{-k}(x) = x^{-k} - \sum_0^{k-1} \frac{\delta^{(j)}(x)}{j!}\int x^{j-k}\,dx.$$

Of course, all the integrals on the right are divergent; the intuitive idea is that we have converted the nonintegrable function x^{-k} into a good distribution by subtracting off some infinite multiples of the delta function and its derivatives. This point can also be made by considering the approximations F_ϵ to X^{-1} defined in (9.18). For example, the distribution derivative of $F_\epsilon(x)$ is the sum of (i) the function that equals $-x^{-2}$ for $|x| > \epsilon$ and 0 for $|x| < \epsilon$ and (ii) the delta functions coming from the jumps, namely, $\epsilon^{-1}\delta(x+\epsilon) + \epsilon^{-1}\delta(x-\epsilon)$. Thus, $X^{-2} = -\lim_{\epsilon\to 0} F_\epsilon'$ is formally x^{-2} minus an infinite multiple of $\delta(x)$. (See also Exercise 4.)

Similar but more complicated instances of systematically "subtracting off infinities" from divergent integrals to obtain finite answers occur with some frequency in quantum field theory, where they play an essential role in perturbative renormalization theory. See, for example, Bogoliubov-Shirkov [9].

One can play a similar game with nonintegral powers. To avoid worrying about fractional powers of negative numbers, let us consider the functions

$$\chi_+^{-\lambda}(x) = \begin{cases} x^{-\lambda} & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases} \qquad (\lambda < 1).$$

$\chi_+^{-\lambda}$ is a continuous function if $\lambda < 0$, and for $0 \le \lambda < 1$ its singularity at the origin is integrable; hence it defines a distribution in the obvious way. If $\lambda \ge 1$ then $\int_0^a x^{-\lambda}\,dx = \infty$ for any $a > 0$, but if λ is not an integer we can still make $x^{-\lambda}$ into a distribution by the same procedure as above. Namely, we define

$$\chi_+^{-\lambda} = \frac{(-1)^k}{(\lambda-1)(\lambda-2)\cdots(\lambda-k)}\frac{d^k}{dx^k}\chi_+^{k-\lambda} \quad \text{for} \quad k < \lambda < k+1, \tag{9.21}$$

that is,

$$\chi_+^{-\lambda}[\phi] = \frac{1}{(\lambda-1)(\lambda-2)\cdots(\lambda-k)}\int_0^\infty x^{k-\lambda}\phi^{(k)}(x)\,dx. \tag{9.22}$$

$\chi_+^{-\lambda}[\phi]$ is again called the **finite part** of the divergent integral $\int_0^\infty x^{-\lambda}\phi(x)\,dx$. One can obtain a formula similar to (9.20) that expresses $\chi_+^{-\lambda}[\phi]$ as the integral of $x^{-\lambda}$ against a modified form of ϕ obtained by subtraction of a Taylor polynomial, but it is not quite as simple as (9.20). See Exercise 9.

This procedure does not work when λ is an integer. One can define χ_+^{-k} by taking distribution derivatives of the function $f(x) = \log x$ for $x > 0$, $f(x) = 0$ for $x \le 0$; but the behavior of the distributions thus defined is not what one would expect in some respects. Most strikingly, χ_+^{-k} is *not* the weak limit of $\chi_+^{-\lambda}$ as $\lambda \to k$. In fact, $\lim_{\lambda\to k}\chi_+^{-\lambda}$ does not exist! What actually happens is this: One can allow the exponent λ to be a complex number, and then $\chi_+^{-\lambda}$ becomes a meromorphic distribution-valued function of $\lambda \in \mathbf{C}$ with simple poles at the negative integers, and the residue at $\lambda = -k$ is a multiple of $\delta^{(k-1)}$. See Exercise 10.

In the formulas (9.19), (9.20), and (9.22), it is not really necessary for ϕ to belong to $C_0^{(\infty)}$. It suffices for ϕ and its derivatives up to order k to vanish at infinity rapidly enough to guarantee the absolute convergence of all the relevant integrals at infinity: in this more general situation these formulas are still said to define the finite part of the integral $\int_{-\infty}^\infty x^{-k}\phi(x)\,dx$ or $\int_0^\infty x^{-\lambda}\phi(x)\,dx$. Moreover, one can define the finite part of the integral $\int_0^a x^{-\lambda}\phi(x)\,dx$, where $0 < a < \infty$, in an entirely similar way; see Exercise 13.

EXERCISES

1. Let

$$\chi(x) = \int_{|x|}^{2\pi} e^{-1/t(1-t)}\,dt \Big/ \int_0^{2\pi} e^{-1/t(1-t)}\,dt \quad \text{for } |x| < 2\pi,$$
$$= 0 \quad \text{for } |x| \ge 2\pi$$

a. Show that $\chi \in C_0^{(\infty)}$, $\operatorname{supp}(\chi) = [-2\pi, 2\pi]$, $\chi(x) + \chi(x - 2\pi) = 1$ for $0 \le x \le 2\pi$, and $\chi(x) + \chi(x + 2\pi) = 1$ for $-2\pi \le x \le 0$. (The last two statements are equivalent; to prove them, substitute $s = 1 - t$ in the integral defining χ.)

b. Show that $\int_{-\infty}^{\infty} f(x)\chi(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx$ if f is continuous and 2π-periodic. (Thus, if F is a 2π-periodic distribution, one can take $F[\chi]$ as a definition of $\int_{-\pi}^{\pi} F(x)\,dx$.)

c. Show that if F is a 2π-periodic distribution with Fourier series $\sum c_k e^{ikx}$ then $c_k = (1/2\pi)F\left[\chi(x)e^{-ikx}\right]$.

2. Let $f(x) = |\sin x|$, and recall from Table 1, §2.1, that

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{1}^{\infty} \frac{\cos 2kx}{4k^2 - 1}.$$

a. Show directly from the formula $f(x) = |\sin x|$ that $f''(x) = -|\sin x| + 2\delta_{\text{per}}(x) + 2\delta_{\text{per}}(x - \pi)$.

b. Show that the twice-differentiated Fourier series of f is the sum of the Fourier series of $-f(x)$, $2\delta_{\text{per}}(x)$, and $2\delta_{\text{per}}(x - \pi)$.

3. Experiment with differentiating some of the other Fourier series in Table 1, §2.1, in the manner of Exercise 2.

4. Let $g_\epsilon(x) = x/(x^2 + \epsilon^2)$ $(\epsilon > 0)$.

a. Show that $g_\epsilon \to X^{-1}$ weakly as $\epsilon \to 0$. (Hint: Show that

$$g_\epsilon[\phi] = \int_{|x|<a} x\frac{\phi(x) - \phi(0)}{x^2 + \epsilon^2}\,dx + \int_{|x|\ge a} \frac{x\phi(x)}{x^2 + \epsilon^2}\,dx$$

and use the dominated convergence theorem to obtain (9.17) as $\epsilon \to 0$.)

b. Sketch the graphs of g_ϵ and $-g_\epsilon'$. You should be able to see that for ϵ near 0, $-g_\epsilon'(x)$ is approximately x^{-2} for $|x| > 2\epsilon$, but that there is a sharp negative spike in the region $|x| < \epsilon\sqrt{3}$ that (roughly speaking) approaches $-\infty\delta(x)$ as $\epsilon \to 0$. Cf. the discussion following (9.20).

5. Let $f_a(x) = (x - ia)^{-1}$. Show that as a approaches 0 from the right, $f_a \to X^{-1} + \pi i\delta$ weakly, whereas as a approaches 0 from the left, $f_a \to X^{-1} - \pi i\delta$ weakly. (Hint: $(x - ia)^{-1} = (x + ia)/(x^2 + a^2)$; use Exercise 4a.)

6. Show that X^{-1} is the distribution derivative of the locally integrable function $f(x) = \log|x|$.

7. Suppose ϕ is a test function on $\mathbb{R}$. Show that $\int_{|x|>\epsilon} x^{-3}\phi(x)\,dx$ has a finite limit as $\epsilon \to 0$ if and only if $\phi'(0) = 0$. (Hint: Write $\phi(x) = \phi(0) + x\phi'(0) + \frac{1}{2}x^2\phi''(0) + R(x)$ according to Taylor's theorem.)

8. Show that (9.22) holds not just for $k < \lambda < k + 1$ but for all $\lambda < k + 1$ except $\lambda = 1, \cdots, k$. (Hint: First show that (9.22) holds if $\lambda < 1$ and k is any positive integer. For this range of λ, $X_+^{-\lambda}$ is an ordinary function, and you can use elementary calculus.)

9. Show that for $\lambda < k + 1$,

$$X_+^{-\lambda}[\phi] = \int_0^1 x^{-\lambda}\left[\phi(x) - \sum_0^{k-1} \frac{\phi^{(j)}(0)}{j!}x^j\right] dx + \int_1^\infty x^{-\lambda}\phi(x)\,dx + \sum_0^{k-1} \frac{\phi^{(j)}(0)}{j!(j+1-\lambda)}.$$

(Perhaps the best way is to allow λ to be complex. Show by elementary calculus that the quantity on the right equals $\int_0^\infty x^{-\lambda}\phi(x)\,dx$ when $\operatorname{Re}\lambda < 1$, and that this quantity and $X_+^{-\lambda}[\phi]$ (as defined in (9.22)) are both analytic functions on the half-plane $\operatorname{Re}\lambda < k+1$ except at $\lambda = 1, \ldots, k$. The result follows.)

10. Show (a) by using Exercise 8 and (b) by using Exercise 9 that if k is a positive integer, $(\lambda - k)X_+^{-\lambda}$ converges weakly as $\lambda \to k$ to $(-1)^k\delta^{(k-1)}/(k-1)!$. (Corollary: The weak limit as $\lambda \to k$ of $X_+^{-\lambda}$ does not exist.)
11. Show that $X_+^{-\lambda}$ is homogeneous of degree $-\lambda$ in the sense of Exercise 1, §9.1. Show also that X^{-k} is homogeneous of degree $-k$.
12. Let $L_+(x) = \log x$ if $x > 0$, $L_+(x) = 0$ if $x \le 0$. For $n = 1, 2, 3, \ldots$, define X_+^{-k} to be $(-1)^{k-1}/(k-1)!$ times the kth distribution derivative of L_+. Show that X_+^{-k} is not homogeneous of degree $-k$, but rather (with the notation of Exercise 1, §9.1)

$$X_+^{-k}[\phi^a] = a^{-1+k}X_+^{-k}[\phi] - a^{-1+k}\log a\,\phi^{(k-1)}(0).$$

13. Show that if $\lambda < 1$ and ϕ is $C^{(\infty)}$ on $[0, a]$,

$$\int_0^a x^{-\lambda}\phi(x)\,dx = \frac{\int_0^a x^{k-\lambda}\phi^{(k)}(x)\,dx}{(\lambda-1)\cdots(\lambda-k)} - \sum_{j=1}^{k} \frac{a^{j-\lambda}\phi^{(j-1)}(a)}{(\lambda-1)\cdots(\lambda-j)}$$

for any positive integer k. Show moreover that the expression on the right makes sense for any noninteger $\lambda < k+1$. (The expression on the right is taken as the definition of the finite part of $\int_0^a x^{-\lambda}\phi(x)\,dx$ for $k < \lambda < k+1$.)
14. Use formulas (9.19) and (9.22), together with the remarks at the end of the section, to derive the following formulas concerning finite parts of divergent integrals. (Note, in parts (b) and (c), that the answer may be negative even when the integrand is positive!)
    a. $F.P. \int_{-\infty}^{\infty} x^{-4}e^{-x^2}dx = \frac{4}{3}\sqrt{\pi}$.
    b. $F.P. \int_{-\infty}^{\infty} \frac{dx}{x^2(x^2+1)} = -\pi$.
    c. $F.P. \int_0^\infty x^{\mu-1}e^{-x}dx = \Gamma(\mu)$ for $\mu < 0$, μ not an integer. (Of course, this integral *defines* $\Gamma(\mu)$ for $\mu > 0$.)

9.4 Tempered distributions and Fourier transforms

In Chapter 7 we developed the theory of the Fourier transform for functions that are integrable or square-integrable on $\mathbf{R}^n$; we now wish to extend it to more general sorts of distributions, including functions that do not decay at infinity.

If f and g are integrable functions, by reversing the order of integration we have

$$\int \widehat{f}(\mathbf{y})g(\mathbf{y})\,d\mathbf{y} = \iint f(\mathbf{x})g(\mathbf{y})e^{-i\mathbf{y}\cdot\mathbf{x}}\,d\mathbf{x}\,d\mathbf{y} = \int f(\mathbf{x})\widehat{g}(\mathbf{x})\,d\mathbf{x}.$$

That is, in our distributional notation, $\widehat{f}[g] = f[\widehat{g}]$. This suggests that we define the Fourier transform $\widehat{f}$ of a distribution F by $\widehat{F}[\phi] = F[\widehat{\phi}]$, but there is a problem: When ϕ is a test function, $\widehat{\phi}$ is *not* a test function (unless $\phi = 0$), so $F[\widehat{\phi}]$ may not make sense. Indeed, since ϕ has bounded support, the integral defining $\widehat{\phi}(\boldsymbol{\xi})$ converges for all *complex* vectors $\boldsymbol{\xi}$, and $\widehat{\phi}(\boldsymbol{\xi})$ is an entire analytic function of each variable $\xi_1, \cdots, \xi_n$. In particular, for each fixed $\xi_2, \cdots, \xi_n$, $\widehat{\phi}(\boldsymbol{\xi})$ can vanish only for isolated values of ξ_1, so it cannot have bounded support.

One possible way to proceed is to define the Fourier transform of a distribution to be a linear functional on a suitable space of analytic functions that includes the Fourier transforms of test functions (a so-called *analytic functional* or *ultradistribution*); see Zemanian [57]. The disadvantage of this, beyond the necessity of developing a theory of analytic functionals, is that distributions and their Fourier transforms are no longer the same sort of object, so one loses much of the useful and elegant symmetry between F and $\widehat{f}$.

For most purposes, a better solution to this problem is to enlarge the class of test functions to make it invariant under Fourier transformation and, correspondingly, to restrict the class of allowable distributions. The nicest way of doing this was discovered by Laurent Schwartz. We define a **Schwartz function** to be a function ϕ of class $C^{(\infty)}$ such that ϕ and all its derivatives $\partial^\alpha\phi$ vanish at infinity more rapidly than any power of $|\mathbf{x}|$, or equivalently, such that $P(\mathbf{x})\partial^\alpha\phi(\mathbf{x})$ is bounded on $\mathbf{R}^n$ for any α and any polynomial P. We call the set of all Schwartz functions on $\mathbf{R}^n$ the **Schwartz class** and denote it by $\mathscr{S}(\mathbf{R}^n)$, or just $\mathscr{S}$.

Every polynomial is a sum of monomials, for which we introduce a notation analogous to our notation for partial derivatives:

$$\mathbf{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}.$$

The definition of the Schwartz class can then be expressed as follows:

$$\phi \in \mathscr{S} \iff \sup_{\mathbf{x}} |\mathbf{x}^\beta \partial^\alpha \phi(\mathbf{x})| < \infty \text{ for all } \alpha, \beta. \tag{9.23}$$

The Schwartz class includes all test functions, as well as functions like $e^{-|\mathbf{x}|^2}$, or more generally $P(\mathbf{x})e^{-|\mathbf{x}|^2}$ where P is any polynomial. Another example is given by

$$\phi(\mathbf{x}) = \Big[1 - \chi(\mathbf{x})\Big]\exp(-|\mathbf{x}|^{1/2}), \tag{9.24}$$

where χ is a test function such that $\chi(\mathbf{x}) = 1$ for $\mathbf{x}$ near $\mathbf{0}$. (See Exercise 3 of §9.2 for the construction of such a function. Its purpose here is to kill off the non-differentiability of $\exp(-|\mathbf{x}|^{1/2})$ at the origin.) This function and all its derivatives decay at infinity more slowly than exponentially but still faster than any power of $|\mathbf{x}|$.

As we pointed out in Chapter 7, the smoothness of a function on $\mathbf{R}^n$ tends to be reflected in the rapid decay of its Fourier transform and vice versa. Since Schwartz functions have both smoothness and rapid decay, the same should be true of their Fourier transforms. Indeed, we have the following result.

Theorem 9.7. *If $\phi \in \mathcal{S}$, then $\widehat{\phi} \in \mathcal{S}$.*

Proof: If $\phi \in \mathcal{S}$, then $\partial^\alpha \phi$ remains bounded when multiplied by any polynomial. In particular, for any α and β,

$$(1 + x_1^2)(1 + x_2^2) \cdots (1 + x_n^2)\mathbf{x}^\beta \partial^\alpha \phi(\mathbf{x})$$

is bounded, that is,

$$|\mathbf{x}^\beta \partial^\alpha \phi(\mathbf{x})| \leq \frac{C_{\alpha,\beta}}{(1 + x_1^2)(1 + x_2^2) \cdots (1 + x_n^2)}.$$

The function on the right is clearly integrable on $\mathbf{R}^n$ (its integral equals $C_{\alpha,\beta}\pi^n$), and therefore so is $\mathbf{x}^\beta \partial^\alpha \phi(\mathbf{x})$.

Next, the integrability of $\mathbf{x}^\beta \partial^\alpha \phi(\mathbf{x})$ for all α, β implies the integrability of $\partial^\alpha \left[\mathbf{x}^\beta \phi(\mathbf{x})\right]$ for all α, β. Indeed, by the product rule for derivatives, $\partial^\beta \left[\mathbf{x}^\alpha \phi(\mathbf{x})\right]$ is a linear combination of terms of the form $\mathbf{x}^{\beta'} \partial^{\alpha'} \phi(\mathbf{x})$ with $|\beta'| \leq |\beta|$ and $|\alpha'| \leq |\alpha|$, and the latter functions are integrable.

This being the case, for any α, β the Fourier transform of $\partial^\beta \left[\mathbf{x}^\alpha \phi(\mathbf{x})\right]$ is a bounded function. But by Theorem 7.8 of §7.5,

$$\left\{\partial^\beta \left[\mathbf{x}^\alpha \phi(\mathbf{x})\right]\right\}^\wedge = i^{|\alpha|+|\beta|} \boldsymbol{\xi}^\beta \partial^\alpha \widehat{\phi}(\boldsymbol{\xi}).$$

Hence $\widehat{\phi}$ satisfies (9.23), so $\widehat{\phi} \in \mathcal{S}$. ∎

Remark. The Fourier inversion formula says that $\phi(\mathbf{x}) = (2\pi)^{-n}(\widehat{\phi})^\wedge(-\mathbf{x})$. Hence, it follows from Theorem 9.7 that $\phi \in \mathcal{S}$ *if and only if* $\widehat{\phi} \in \mathcal{S}$.

The Schwartz class is preserved not only by the Fourier transform but by a number of other basic operations. For example, it is obvious that if $\phi \in \mathcal{S}$, then so are $\partial^\alpha \phi$ and $\mathbf{x}^\alpha \phi$ for any α, and so is $\phi(\mathbf{x} - \mathbf{y})$ for any $\mathbf{y}$. It follows easily from the product rule for derivatives that the product of two Schwartz functions is a Schwartz function. In view of Theorem 9.7, the fact that $(\phi * \psi)^\wedge = \widehat{\phi}\widehat{\psi}$, and the Fourier inversion formula, this also implies that the convolution of two Schwartz functions is a Schwartz function.

The distributions with which one can do Fourier analysis are those that are defined not only on test functions but also on Schwartz functions. To be precise, we make the following definition.

Definition. A **tempered distribution** is a mapping $F : \mathscr{S} \to \mathbf{C}$ that satisfies the following conditions:

(i) $F[c_1\phi_1 + c_2\phi_2] = c_1 F[\phi_1] + c_2 F[\phi_2]$ for all $\phi_1, \phi_2 \in \mathscr{S}$ and all $c_1, c_2 \in \mathbf{C}$.

(ii) There exist a positive integer N and a constant $C \geq 0$ such that for all $\phi \in \mathscr{S}$,

$$\left|F[\phi]\right| \leq C \sum_{|\alpha|+|\beta|\leq N} \sup_{\mathbf{x}} |\mathbf{x}^\beta \partial^\alpha \phi(\mathbf{x})|. \tag{9.25}$$

The standard notation for the set of all tempered distributions on $\mathbf{R}^n$ is $\mathscr{S}'(\mathbf{R}^n)$, or $\mathscr{S}'$ for short.

Remark. The estimate (ii) looks rather different from the continuity condition in the definition of distribution in §9.1. However, it clearly implies (and is actually equivalent to) the following condition, which is analogous to the one in §9.1:

(ii′) Suppose $\{\phi_k\}$ is a sequence in $\mathscr{S}$ such that $\mathbf{x}^\beta \partial^\alpha \phi_k(\mathbf{x})$ converges uniformly to zero on $\mathbf{R}^n$ as $k \to \infty$, for all α and β. Then $F[\phi_k] \to 0$.

It is easy to verify (see Exercise 1) that if $\phi \in \mathscr{S}$, there is a sequence $\{\phi_k\} \subset C_0^{(\infty)}$ that converges to ϕ in the sense of (ii′): $\mathbf{x}^\beta \partial^\alpha [\phi_n(\mathbf{x}) - \phi(\mathbf{x})] \to 0$ uniformly for all α, β. From this we draw two conclusions. First, if F and G are tempered distributions and $F[\phi] = G[\phi]$ for all $\phi \in C_0^{(\infty)}$, then $F[\phi] = G[\phi]$ for all $\phi \in \mathscr{S}$. (If $\phi \in \mathscr{S}$, choose a sequence $\{\phi_k\} \subset C_0^{(\infty)}$ as above; then $F[\phi] = \lim F[\phi_k] = \lim G[\phi_k] = G[\phi]$.) Second, if F is a distribution that satisfies the estimate (9.25) for all $\phi \in C_0^{(\infty)}$, then F can be extended in a unique way to $\mathscr{S}$ to make a tempered distribution. (If $\phi \in \mathscr{S}$, again choose $\{\phi_k\} \subset C_0^{(\infty)}$ converging to ϕ: Then $\mathbf{x}^\beta \partial^\alpha [\phi_j(\mathbf{x}) - \phi_k(\mathbf{x})] \to 0$ uniformly as $j, k \to \infty$, so $F[\phi_j - \phi_k] \to 0$; hence $\lim_{k\to\infty} F(\phi_k)$ exists, and we define $F[\phi]$ to be this limit.) The upshot is that *the set of tempered distributions can be identified with the set of distributions that satisfy the estimate (9.25) for some N.*

What (9.25) means, roughly, is that F is of "finite order" (so to speak, F is only finitely many derivatives away from being a genuine function) and that F grows at most polynomially at infinity. If F is actually a locally integrable function, F is tempered whenever

$$\int \left(1 + |\mathbf{x}|^2\right)^{-N} |F(\mathbf{x})|\, d\mathbf{x} < \infty \tag{9.26}$$

for some N, as this condition implies the absolute convergence of $\int F(\mathbf{x})\phi(\mathbf{x})\, d\mathbf{x}$ whenever ϕ vanishes at infinity faster than any power of $|\mathbf{x}|$. All polynomials and all L^1 and L^2 functions are tempered, as is $\left(1 + |\mathbf{x}|^2\right)^\alpha$ for any α, but $e^{|\mathbf{x}|}$ is not. (One can easily see that $\int e^{|\mathbf{x}|}\phi(\mathbf{x})\, d\mathbf{x}$ diverges when ϕ is given by (9.24).) Also, the periodic delta function $\sum_{k=-\infty}^{\infty} \delta(x - 2\pi k)$ on $\mathbf{R}$ is tempered (we shall discuss this below), but $\sum_{k=1}^{\infty} \delta^{(k)}(x - 2\pi k)$ is not, because it involves derivatives of arbitrarily high order.

In §9.1 we discussed several basic operations on functions that can be extended to distributions: differentiation, translation, dilation, and multiplication by smooth functions. The first three of these operations map tempered distributions to tempered distributions, simply because they map the Schwartz class to itself. For instance, if $F \in \mathscr{S}$, then $\partial^\alpha F$ is given by $(\partial^\alpha F)[\phi] = (-1)^{|\alpha|}F[\partial^\alpha \phi]$. If ϕ is a Schwartz function, then so is $\partial^\alpha \phi$, so this formula defines $\partial^\alpha F$ as a tempered distribution. The same is true for translations and dilations. As for multiplication by smooth functions, if F is a tempered distribution and g is a $C^{(\infty)}$ function such that g and all its derivatives grow at most polynomially at infinity (i.e., $|\partial^\alpha g(\mathbf{x})| \le C_\alpha\left(1 + |\mathbf{x}|^{N_\alpha}\right)$ for all α), then gF is a tempered distribution; see Exercise 2. Examples of functions satisfying this condition include all polynomials and the imaginary exponentials $g(\mathbf{x}) = e^{i\mathbf{a}\cdot\mathbf{x}}$. (If g does not satisfy this condition, gF may grow too rapidly at infinity to be tempered.)

In §9.2 we showed how to define the convolution of a distribution and a test function. Exactly the same process gives the convolution of a tempered distribution with a Schwartz function; namely, if $F \in \mathscr{S}'$ and $\phi \in \mathscr{S}$, $F * \phi(\mathbf{x}) = F[\widetilde{\phi}_{\mathbf{x}}]$ where $\widetilde{\phi}_{\mathbf{x}}(\mathbf{y}) = \phi(\mathbf{x} - \mathbf{y})$. A slight modification of the proof of Theorem 9.4 shows that this is a $C^{(\infty)}$ function. Moreover, $F * \phi$ and all its derivatives have at most polynomial growth at infinity (see Exercise 3), so $F * \phi$ can also be regarded as a tempered distribution. With this understanding, the obvious analogue of Lemma 9.1 holds:

$$(F * \phi)[\psi] = F[\widetilde{\phi} * \psi], \quad \text{where } \widetilde{\phi}(\mathbf{x}) = \phi(-\mathbf{x}). \tag{9.27}$$

To this list of operations we can now add the Fourier transform; namely, if F is a tempered distribution, we define $\widehat{f}$, another tempered distribution, by

$$\widehat{f}[\phi] = F[\widehat{\phi}], \qquad \phi \in \mathscr{S}. \tag{9.28}$$

This is well-defined by Theorem 9.7. The basic operational properties of the Fourier transform given in Theorem 7.8, §7.5, all remain valid for tempered distributions F. To wit,

$$\mathscr{F}\Big[F(\mathbf{x} - \mathbf{a})\Big] = e^{-i\mathbf{a}\cdot\boldsymbol{\xi}}\widehat{f}(\boldsymbol{\xi}), \quad \mathscr{F}\Big[e^{i\mathbf{a}\cdot\mathbf{x}}F(\mathbf{x})\Big] = \widehat{f}(\boldsymbol{\xi} - \mathbf{a}), \tag{9.29}$$

$$\mathscr{F}\Big[F(t\mathbf{x})\Big] = |t|^{-n}\widehat{f}(t^{-1}\boldsymbol{\xi}), \tag{9.30}$$

$$\mathscr{F}\Big[\partial^\alpha F(\mathbf{x})\Big] = i^{|\alpha|}\boldsymbol{\xi}^\alpha \widehat{f}(\boldsymbol{\xi}), \quad \mathscr{F}\Big[\mathbf{x}^\alpha F(\mathbf{x})\Big] = i^{|\alpha|}\partial^\alpha \widehat{f}(\boldsymbol{\xi}), \tag{9.31}$$

and if g is a Schwartz function,

$$(F * g)\widehat{\ } = \widehat{f}\widehat{g}. \tag{9.32}$$

The verification of these formulas is a simple matter of plugging in the definitions and using the fact that the formulas are true for Schwartz functions. For example,

$$\begin{aligned}\Big(\mathscr{F}[\partial^\alpha F]\Big)[\phi] &= (\partial^\alpha F)[\widehat{\phi}] = (-1)^{|\alpha|}F[\partial^\alpha \widehat{\phi}] = (-1)^{|\alpha|}i^{-|\alpha|}F[(\mathbf{x}^\alpha \phi)\widehat{\ }] \\ &= i^{|\alpha|}\widehat{f}[\mathbf{x}^\alpha \phi] = i^{|\alpha|}(\mathbf{x}^\alpha \widehat{f})[\phi].\end{aligned}$$

The formula (9.32) is a bit trickier; perhaps the easiest way is to use (9.27) and the easily verified fact that $\mathscr{F}^{-1}(\widetilde{\phi} * \widehat{\psi}) = \widehat{\phi}\psi$:

$$(F * \phi)\widehat{\ }[\psi] = (F * \phi)[\widehat{\psi}] = F[\widetilde{\phi} * \widehat{\psi}] = F[(\widehat{\phi}\psi)\widehat{\ }] = \widehat{f}[\widehat{\phi}\psi] = (\widehat{\phi}\widehat{f})[\psi].$$

We also have the Fourier inversion theorem. For Schwartz functions, the theorem can be stated as

$$\mathscr{F}^{-1}\phi(\mathbf{x}) = \frac{1}{(2\pi)^n}\int e^{i\mathbf{x}\cdot\xi}\phi(\xi)\,d\xi = \frac{1}{(2\pi)^n}\widehat{\phi}(-\mathbf{x}),$$

and in this form it holds also for tempered distributions:

$$\mathscr{F}^{-1}F(\mathbf{x}) = (2\pi)^{-n}\widehat{f}(-\mathbf{x}). \tag{9.33}$$

(Here, of course, $F(-\mathbf{x})$ denotes the distribution $\widetilde{f}$ defined by $\widetilde{F}[\phi] = F[\widetilde{\phi}]$, $\widetilde{\phi}$ being defined by (9.27).) Indeed, if we set $G(\mathbf{x}) = \widehat{f}(-\mathbf{x})$ and $\psi(\mathbf{x}) = \widehat{\phi}(-\mathbf{x})$, we have

$$\widehat{G}[\phi] = G[\widehat{\phi}] = \widehat{f}[\psi] = F[\widehat{\psi}] = (2\pi)^n F[\phi],$$

so that $F = (2\pi)^{-n}\widehat{G}$, or $\mathscr{F}^{-1}F = (2\pi)^{-n}G$.

Finally, the natural notion of convergence for tempered distributions is that

$$F_k \to F \iff F_k[\phi] \to F[\phi] \quad \text{for all } \phi \in \mathscr{S}.$$

We shall call this **temperate convergence**. It is very much like weak convergence but is a bit more stringent, since the condition $F_k[\phi] \to F[\phi]$ must hold for a larger class of ϕ's.

If F_k and F are functions, a sufficient condition to have $F_k \to F$ temperately is that $F_k \to F$ pointwise and $|F_k(\mathbf{x})| \le G(\mathbf{x})$ for all k and $\mathbf{x}$ where G satisfies a condition of the form (9.26) — for example, $|F_k(\mathbf{x})| \le C(1 + |\mathbf{x}|)^N$ for some N. This follows easily from the dominated convergence theorem.

Theorem 9.8. *If $F_k \to F$ temperately, then $\partial^\alpha F_k \to \partial^\alpha F$ temperately and $\widehat{f}_k \to \widehat{f}$ temperately.*

Proof: This is the same one-line argument that proved Theorem 9.3:

$$\widehat{f}_k[\phi] = F_k[\widehat{\phi}] \to F[\widehat{\phi}] = \widehat{f}[\phi]$$

for all $\phi \in \mathscr{S}$, and likewise for $\partial^\alpha F_k$. ∎

Theorem 9.8 is useful for computing specific Fourier transforms: if one can express a distribution F as the temperate limit of a sequence of integrable functions F_k, one can compute $\widehat{f}_k$ in the old-fashioned way and then pass to the limit.

It is high time we computed some examples. Others will be found in the exercises and in Sections 9.5 and 10.2.

Example 1. Let us start with the delta function:

$$\widehat{\delta}[\phi] = \delta[\widehat{\phi}] = \widehat{\phi}(\mathbf{0}) = \int \phi(\mathbf{x})\,d\mathbf{x}.$$

The integral on the right is $F[\phi]$ where F is the constant function 1, so

$$\widehat{\delta} \equiv 1.$$

We can generalize this by using formulas (9.29) and (9.31):

$$\left[\partial^\alpha \delta(\mathbf{x}-\mathbf{a})\right]^\frown(\boldsymbol{\xi}) = i^{|\alpha|}\boldsymbol{\xi}^\alpha e^{-i\mathbf{a}\cdot\boldsymbol{\xi}}\widehat{\delta}(\boldsymbol{\xi}) = i^{|\alpha|}\boldsymbol{\xi}^\alpha e^{-i\mathbf{a}\cdot\boldsymbol{\xi}}. \tag{9.34}$$

Then, by (9.33) and the fact that $\partial^\alpha\delta(-\mathbf{x}) = (-1)^{|\alpha|}\partial^\alpha\delta(\mathbf{x})$, we also have

$$\left[\mathbf{x}^\alpha e^{-i\mathbf{a}\cdot\mathbf{x}}\right]^\frown(\boldsymbol{\xi}) = (2\pi)^n i^{|\alpha|}\partial^\alpha\delta(\boldsymbol{\xi}+\mathbf{a}). \tag{9.35}$$

In particular, taking $\mathbf{a} = \mathbf{0}$, we see that, up to constant factors, the Fourier transforms of the derivatives of the delta function are the monomials $\boldsymbol{\xi}^\alpha$ and vice versa. Thus, the Fourier transforms of the linear combinations of δ and its derivatives are precisely the polynomials and vice versa.

Incidentally, the fact that δ is the inverse Fourier transform of the constant function 1 can be written *formally* as

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^n}\int e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\,d\boldsymbol{\xi}.$$

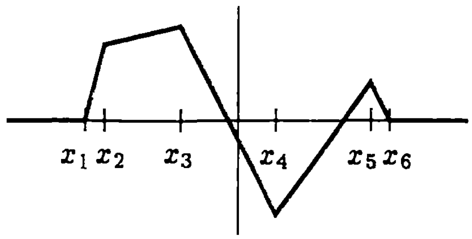
This bit of apparent nonsense is actually a convenient shorthand for the Fourier inversion formula; namely, if we replace $\mathbf{x}$ by $\mathbf{x}-\mathbf{y}$, integrate both sides against a test function ϕ, and reverse the order of integration on the right, we obtain

$$\phi(\mathbf{x}) = \int \phi(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\,d\mathbf{y} = \frac{1}{(2\pi)^n}\iint \phi(y)e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}}\,d\mathbf{y}\,d\boldsymbol{\xi} = \frac{1}{(2\pi)^n}\int \widehat{\phi}(\boldsymbol{\xi})e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\,d\boldsymbol{\xi},$$

which is indeed the Fourier inversion formula.

Example 2. Formula (9.34) gives a painless way of computing the Fourier transforms of piecewise polynomial functions. We shall illustrate the technique on the case of continuous, piecewise linear functions. Suppose $x_1 < x_2 < \cdots < x_k$, and suppose f is a continuous function that vanishes outside $[x_1, x_k]$ and is linear on each interval $[x_j, x_{j+1}]$ for $1 \le j < k$, as in Figure 9.3(a). Then f' is a piecewise constant function, so by Theorem 9.1, f'' is a linear combination of delta functions; namely,

$$f''(x) = \sum_1^k m_j\delta(x-x_j)$$



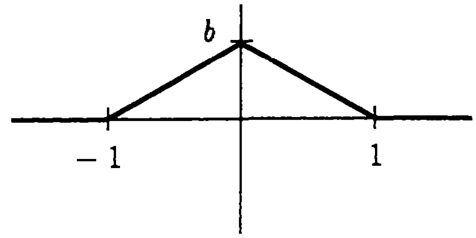


FIGURE 9.3. The functions of Example 2.

where m_j is the difference in the slopes of f to the right and to the left of x_j. But then by (9.34), $(f'')\widehat{\ }(\xi) = \sum_1^k m_j e^{-ix_j\xi}$, so by (9.31),

$$\widehat{f}(\xi) = -\frac{1}{\xi^2}\sum_1^k m_j e^{-ix_j\xi}.$$

Despite the factor of $1/\xi^2$, there is no singularity at $\xi = 0$ because f is in L^1 and hence $\widehat{f}$ is continuous: The sum on the right is guaranteed to vanish to second order at $\xi = 0$.

For instance, if f is the "triangle function" in Figure 9.3(b), the slope of f changes from 0 to b at $x_1 = -1$, from b to $-b$ at $x_2 = 0$, and from $-b$ to 0 at $x_3 = 1$. Hence

$$\widehat{f}(\xi) = -\frac{1}{\xi^2}(be^{i\xi} - 2b + be^{-i\xi}) = \frac{2b}{\xi^2}(1 - \cos\xi).$$

Example 3. We showed in §7.5 that the Fourier transform of a radial function $f(\mathbf{x}) = f_0(|\mathbf{x}|)$ is another radial function, $\widehat{f}(\boldsymbol{\xi}) = \widetilde{f}_0(|\boldsymbol{\xi}|)$, where in dimensions 2 and 3, $\widetilde{f}_0(\rho)$ is given in terms of $f_0(r)$ by

$$2\pi\int_0^\infty f_0(r)J_0(r\rho)\,dr \quad (n = 2), \qquad \frac{4\pi}{\rho}\int_0^\infty f_0(r)r\sin r\rho\,dr \quad (n = 3).$$

If we choose a sequence of radial f's so that the corresponding f_0's converge temperately to $\delta(r-1)$, we obtain the Fourier transforms of the "delta functions" on the unit circle in $\mathbf{R}^2$ and the unit sphere in $\mathbf{R}^3$, that is, arc length σ on the unit circle and area measure S on the unit sphere:

$$\widehat{\sigma}(\boldsymbol{\xi}) = 2\pi J_0(|\boldsymbol{\xi}|), \qquad \widehat{S}(\boldsymbol{\xi}) = 4\pi\frac{\sin|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}. \tag{9.36}$$

Remark. One could also arrive at (9.36) by applying the definition of the Fourier transform in a more straightforward way, namely,

$$\widehat{\sigma}(\boldsymbol{\xi}) = \int_{|\mathbf{x}|=1} e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\,d\sigma(\mathbf{x}) = \int_{-\pi}^{\pi} e^{-i\xi_1\cos\theta - i\xi_2\sin\theta}\,d\theta$$

and similarly for $\widehat{S}$. Indeed, this is really nothing but the calculation that produced the formula for Fourier transforms of radial functions in §7.5.

Example 4. In dimension $n = 1$, let

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases} \qquad f_\epsilon(x) = \begin{cases} -e^{\epsilon x} & \text{if } x < 0, \\ e^{-\epsilon x} & \text{if } x > 0. \end{cases}$$

For $\epsilon > 0$, f_ϵ is an integrable function whose Fourier transform is easily calculated:

$$\widehat{f}_\epsilon(\xi) = -\int_{-\infty}^{0} e^{(\epsilon - i\xi)x}\,dx + \int_0^\infty e^{-(\epsilon + i\xi)x}\,dx = \frac{-1}{\epsilon - i\xi} + \frac{1}{\epsilon + i\xi} = \frac{-2i\xi}{\epsilon^2 + \xi^2}.$$

Clearly, $f_\epsilon \to \operatorname{sgn}$ temperately as $\epsilon \to 0$, so in view of Exercise 4 of §9.3 and Theorem 9.8, we have

$$(\operatorname{sgn})\widehat{\ } = -2iX^{-1} \quad \text{where } X^{-1}[\phi] = P.V. \int x^{-1}\phi(x)\,dx.$$

The Fourier inversion theorem then also gives

$$(X^{-1})\widehat{\ }(\xi) = -i\pi \operatorname{sgn}(\xi).$$

Example 5. Every continuous periodic function on $\mathbf{R}$ is bounded on $\mathbf{R}$ and hence tempered. Since repeated differentiation of continuous periodic functions yields all periodic distributions on $\mathbf{R}$, periodic distributions are all tempered, and their Fourier series (as in Theorem 9.6) converge temperately. Hence, by Theorem 9.8 and formula (9.35),

$$F(x) = \sum_{-\infty}^{\infty} c_k e^{ikx} \quad \Longrightarrow \quad \widehat{F}(\xi) = 2\pi \sum_{-\infty}^{\infty} c_k \delta(\xi - k).$$

In particular, taking F to be the periodic delta function δ_{per}, for which $c_k = (2\pi)^{-1}$ for all k (cf. (9.14)), we obtain

$$\delta_{\text{per}}(x) = \sum_{-\infty}^{\infty} \delta(x - 2\pi k), \qquad \widehat{\delta}_{\text{per}}(\xi) = \sum_{-\infty}^{\infty} \delta(\xi - k). \tag{9.37}$$

Thus, δ_{per} is its own Fourier transform, except for some adjustment of 2π's! (In fact, since $\delta(ax) = a^{-1}\delta(x)$, we have $\delta(\xi - k) = 2\pi\delta(2\pi\xi - 2\pi k)$; hence (9.37) can be rewritten as $\widehat{\delta}_{\text{per}}(\xi) = 2\pi\delta_{\text{per}}(2\pi\xi)$.) If we now go back to the definition of the distribution Fourier transform, $\widehat{f}[\phi] = F[\widehat{\phi}]$, and take $F = \delta_{\text{per}}$, we obtain the following theorem.

The Poisson Summation Formula. *If $\phi \in \mathscr{S}(\mathbf{R})$, then*

$$\sum_{-\infty}^{\infty} \phi(k) = \sum_{-\infty}^{\infty} \widehat{\phi}(2\pi k). \tag{9.38}$$

A slightly more elaborate argument will show that (9.38) is valid not only for $\phi \in \mathscr{S}$ but for any ϕ such that $|\phi(x)| \le C|x|^{-1-\epsilon}$ and $|\widehat{\phi}(\xi)| \le C|\xi|^{-1-\epsilon}$ for some $\epsilon > 0$, a condition which ensures the absolute convergence of both series in (9.38). See Dym-McKean [19] or Folland [25].

We conclude this section with a few remarks about Laplace transforms of distributions. The Laplace transform can be applied to any distribution F on $\mathbf{R}$ that (i) is supported in the half-line $[0, \infty)$ and (ii) is of at most exponential growth, in the sense that the distribution $F^a(t) = e^{-at}F(t)$ is tempered for some $a \ge 0$. (This class of distributions includes all tempered distributions supported in $[0, \infty)$ as well as all functions of the class $\mathscr{E}$ of Chapter 8.) Indeed, if F is such a distribution, for $\operatorname{Re} z > a$ the integral

$$\mathscr{L}F(z) = \int_0^\infty e^{-zt}F(t)\,dt = \int \left[e^{-at}F(t)\right]e^{-(z-a)t}\,dt$$

can be interpreted as $F^a[E_{z-a}]$ where $E_{z-a}(t)$ is any Schwartz function that agrees with $e^{-(z-a)t}$ on $[0, \infty)$. (The function $e^{-(z-a)t}$ itself is not Schwartz, since it blows up as $t \to -\infty$, but this is irrelevant since F^a is supported in $[0, \infty)$.) This formula then defines the Laplace transform $\mathscr{L}F(z)$ as an analytic function in the half-plane $\operatorname{Re} z > a$.

For instance, the Laplace transforms of the delta function and its derivatives are given by

$$\mathscr{L}[\delta^{(k)}](z) = \int \delta^{(k)}(t)e^{-zt}\,dt = (-1)^k \frac{d^k}{dt^k}e^{-zt}\bigg|_{t=0} = z^k. \tag{9.39}$$

Most of the basic operational properties of the Laplace transform can be generalized in a straightforward way to distributions. The formula for the Laplace transform of a derivative is even simpler in this setup than it was before:

$$\mathscr{L}[F'](z) = z\mathscr{L}F(z). \tag{9.40}$$

The discrepancy between (9.40) and the formula $\mathscr{L}[f'](z) = z\mathscr{L}f(z) - f(0)$ derived in §8.1 is merely a reflection of the discrepancy between the pointwise and distributional derivatives of a piecewise smooth function. Indeed, if $f(t)$ is continuous and piecewise smooth on $[0, \infty)$ and $f(t) = 0$ for $t < 0$, according to Theorem 9.1 the distribution derivative f' and the pointwise derivative $f^{(1)}$ are related by $f' = f^{(1)} + f(0)\delta$. Therefore, by (9.39) and (9.40),

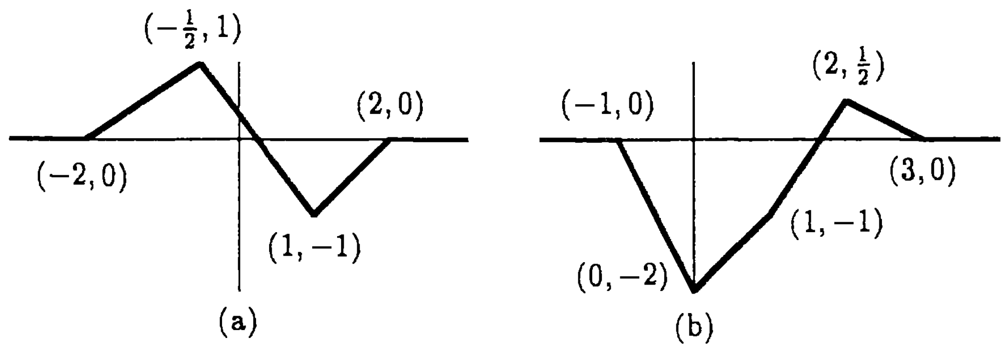
$$\mathscr{L}[f^{(1)}](z) = \mathscr{L}[f'](z) - f(0)\mathscr{L}\delta(z) = z\mathscr{L}f(z) - f(0).$$

In the same way, in the formula (8.5) for the Laplace transform of $f^{(k)}$, the terms involving the initial values $f^{(j)}(0+)$ $(j < k)$ are a reflection of the derivatives of the delta function that enter into the distributional kth derivative of f because of the fact that $f(t) = 0$ for $t < 0$.

EXERCISES

1. Suppose F is a tempered distribution on $\mathbf{R}^n$ that is homogeneous of degree a in the sense of Exercise 1, §9.1. Show that $\widehat{f}$ is homogeneous of degree $-n-a$.
2. Let χ be a test function on $\mathbf{R}^n$ such that $\chi(\mathbf{x}) = 1$ for $|\mathbf{x}| \le 1$. (See Exercise 3, §9.2.) Given $\phi \in \mathscr{S}$, let $\phi_\epsilon(\mathbf{x}) = \chi(\epsilon\mathbf{x})\phi(\mathbf{x})$. Show that $\mathbf{x}^\beta\partial^\alpha\phi_\epsilon(\mathbf{x}) \to \mathbf{x}^\beta\partial^\alpha\phi(\mathbf{x})$ uniformly on $\mathbf{R}^n$ for all α and β.
3. Suppose g is a $C^{(\infty)}$ function on $\mathbf{R}^n$ such that $|\partial^\alpha g(\mathbf{x})| \le C_\alpha(1+|\mathbf{x}|)^{N_\alpha}$ for all α.
   a. Show that $g\phi \in \mathscr{S}$ whenever $\phi \in \mathscr{S}$.
   b. Show that if F is any tempered distribution, gF (defined by (9.6)) is also tempered.
4. Suppose F is a tempered distribution and ϕ is a Schwartz function. We wish to show that for some $N \ge 0$, $|\partial^\alpha(F*\phi)(\mathbf{x})| \le C_\alpha(1+|\mathbf{x}|)^N$ for all α. Do this by proving the following estimates:
   a. For any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and any $K \ge 0$, $|\mathbf{y}|^K \le (1+|\mathbf{x}-\mathbf{y}|)^K(1+|\mathbf{x}|)^K$.
   b. $\sup_{\mathbf{y}} |\mathbf{y}^\beta\partial^\alpha\phi(\mathbf{x}-\mathbf{y})| \le C_{\alpha\beta}(1+|\mathbf{x}|)^{|\beta|}$.
   c. $|(F*\phi)(\mathbf{x})| \le C(1+|\mathbf{x}|)^N$ for some $N \ge 0$. (Use (9.25).)
   d. Finally, repeat this argument with ϕ replaced by $\partial^\alpha\phi$ to obtain the estimate for $\partial^\alpha(F*\phi)$.
5. In §7.5 we showed that $\mathscr{F}[e^{-a|\mathbf{x}|^2}] = (\pi/a)^{n/2}e^{-|\xi|^2/4a}$ for $a > 0$.
   a. Show that this formula remains valid for all complex a such that $\operatorname{Re} a > 0$. (These are the values of a for which $e^{-a|\mathbf{x}|^2}$ is an integrable function on $\mathbf{R}^n$.) (Hint: Both sides of the formula are analytic functions of a in the half-plane $\operatorname{Re} a > 0$.)
   b. Show that if $b \in \mathbf{R}$ and $b \ne 0$, $e^{-a|\mathbf{x}|^2}$ converges temperately to $e^{-ib|\mathbf{x}|^2}$ as a approaches ib from the right half-plane. Deduce that
   $$\mathscr{F}\left[e^{-ib|\mathbf{x}|^2}\right] = e^{-in\pi(\operatorname{sgn} b)/4}\left(\frac{\pi}{|b|}\right)^{n/2} e^{i|\xi|^2/4b},$$
   where $\operatorname{sgn} b$ is as in Example 4.
   c. Show that the solution of the free-particle Schrödinger equation
   $$\frac{1}{i}\frac{\partial u}{\partial t} = \nabla^2 u, \qquad u(\mathbf{x},0) = f(\mathbf{x}) \qquad (f \in \mathscr{S})$$
   for $t > 0$ is
   $$u(\mathbf{x},t) = \frac{e^{-in\pi/4}}{(4\pi t)^{n/2}}\int f(\mathbf{x}-\mathbf{y})e^{i|\mathbf{y}|^2/4t}\,d\mathbf{y}.$$

6. Use the method of Example 2 to compute the Fourier transforms of the piecewise linear functions pictured below.



7. Let $f(x) = 0$ for $x < -2$, $f(x) = (x+2)^2$ for $-2 \le x \le -1$, $f(x) = 2x+3$ for $-1 \le x \le 0$, $f(x) = 3(x-1)^2$ for $0 \le x \le 1$, and $f(x) = 0$ for $x \ge 1$. Compute $\widehat{f}(\xi)$; use the method of Example 2, but work with f''' rather than f''. (Note that f''' involves both the delta function and its derivative.)

The following problems deal with distributions on the real line. H is the Heaviside function, X^{-k} and $X_+^{-\lambda}$ are the distributions defined in §9.3, and $\operatorname{sgn} x$ is as in Example 4.

8. For $a \in \mathbf{C}$ and $x \in \mathbf{R}$, let $f_a(x) = 1/(x-a)$. (If $a \notin \mathbf{R}$, f is a smooth function on $\mathbf{R}$. If $a \in \mathbf{R}$, f_a is to be interpreted as a distribution defined by a principal-value integral; in other words, $f_a(x) = X^{-1}(x-a)$.) Show that
$$\widehat{f}_a(\xi) = \begin{cases} 2\pi i e^{-ia\xi} H(-\xi) & \text{if } \operatorname{Im} a > 0, \\ -2\pi i e^{-ia\xi} H(\xi) & \text{if } \operatorname{Im} a < 0, \\ -\pi i e^{-ia\xi} \operatorname{sgn}(\xi) & \text{if } \operatorname{Im} a = 0. \end{cases}$$
(Hint: If a is real, use Example 4 and (9.29). If $\operatorname{Im} a \neq 0$, it is probably easiest to show that f_a is the inverse Fourier transform of the integrable function on the right.)
9. Show that $(X^{-k})\widehat{\ }(\xi) = (-i)^k \pi (\operatorname{sgn}\xi)\xi^{k-1}/(k-1)!$ for any positive integer k. (Use Example 4 and (9.31).)
10. Show that $\widehat{H} = \pi\delta - iX^{-1}$. (Use either the result or the method of Example 4.)
11. Show that $(X_+^k)\widehat{\ } = \pi i^k \delta^{(k)} + (-i)^{k+1} k! X^{-k-1}$. (Use Exercise 10 and (9.31).)
12. Show that $(X_+^{-\lambda})\widehat{\ }(\xi) = e^{i\pi(\lambda-1)(\operatorname{sgn}\xi)/2}\Gamma(1-\lambda)|\xi|^{\lambda-1}$ if $\lambda > 0$ and λ is not an integer. (First do the case $0 < \lambda < 1$, for which $X_+^{-\lambda}$ is a function, by considering $X_+^{-\lambda}$ as the limit as $\epsilon \to 0$ of the integrable function $e^{-\epsilon x}x^{-\lambda}$. Then, for $k < \lambda < k+1$, use (9.21) and (9.31).)
13. Show that
$$(X_+^{\mu})\widehat{\ }(\xi) = e^{-i\pi(\mu+1)/2}\Gamma(1+\mu)X_+^{-\mu-1}(\xi) + e^{i\pi(\mu+1)/2}\Gamma(1+\mu)X_+^{-\mu-1}(-\xi)$$
whenever $\mu > -1$ and μ is not an integer. (If $-1 < \mu < 0$, this is a restatement of Exercise 12 with $\lambda = -\mu$. For $k < \mu < k+1$, use (9.31). For the case where μ is an integer, see Exercise 11.)

14. Use the Poisson summation formula (and the remark following it) to derive the following formulas for $a > 0$.
    a. $\displaystyle\sum_{-\infty}^{\infty} e^{-a^2k^2} = \frac{\sqrt{\pi}}{a}\sum_{-\infty}^{\infty} e^{-k^2\pi^2/a^2}$.
    b. $\displaystyle\sum_{-\infty}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi}{a}\coth \pi a$, and $\displaystyle\sum_{1}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi}{2a}\coth \pi a - \frac{1}{2a^2}$.
15. Prove the following generalization of the Poisson summation formula:

$$\sum_{\infty}^{\infty} \phi(x+k) = \sum_{-\infty}^{\infty} \widehat{\phi}(2\pi k)e^{2\pi i k x}.$$

(Hint: Apply the Poisson formula to the function $\psi(y) = \phi(x+y)$.)

9.5 Weak solutions of differential equations

Let L be a linear partial differential operator, say $L(u) = \sum_{|\alpha|\le k} a_\alpha \partial^\alpha u$, whose coefficients $a_\alpha = a_\alpha(\mathbf{x})$ are $C^{(\infty)}$ functions. Classically, when one considers the differential equation $L(u) = f$, one envisages f as a continuous function and u as a function such that the derivatives $\partial^\alpha u$ that occur in $L(u)$ all exist in the sense of elementary calculus, and the equation $L(u) = f$ is to hold pointwise. However, we can now broaden the scope of the subject: f and u can be arbitrary distributions, and the equation $L(u) = f$ should hold in the sense of distributions, i.e., for any test function ϕ, $\big(L(u)\big)[\phi] = f[\phi]$. More generally, we say that the equation $L(u) = f$ holds on an open set $V \subset \mathbf{R}^n$ if this relation is valid for any test function ϕ that is supported in V, that is,

$$\sum_{|\alpha|\le k} (-1)^{|\alpha|} u\Big[\partial^\alpha(a_\alpha \phi)\Big] = f[\phi] \qquad (\phi \in C_0^{(\infty)},\ \operatorname{supp}(\phi) \subset V). \tag{9.41}$$

Of course, one should ask what is gained by this added generality, as one is not often called on in practice to solve $L(u) = f$ when f is an arbitrary distribution. The special case where f is a delta function, $f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0)$, is extremely important; it is the subject of Chapter 10, and its significance will be explained there. In this section, however, we are concerned with the case where f is a smooth (or at least continuous) function on some open set V. That is, we have a *classical* differential equation, but we look for functions or distributions u that satisfy the equation in the sense (9.41) rather than pointwise. These are called **weak solutions**.

Series solutions as weak solutions

Weak solutions are often easier to obtain than classical solutions. For instance, they are the natural outcome of the techniques of separation of variables and orthogonal expansions discussed in the earlier chapters of this book. These techniques typically yield a sequence $\{u_k\}$ of special classical solutions to the differential equation $L(u) = 0$ on a region V, and one then attempts to form a larger class of solutions by taking infinite linear combinations $\sum c_k u_k$. Such infinite series will indeed be solutions *if* they converge and *if* one can justify interchange of differentiation and summation so that $L\left(\sum c_k u_k\right) = \sum c_k L(u_k)$. In the classical regime these convergence questions are sometimes rather painful, but if one is content with weak solutions the problems usually disappear. It suffices for the series $\sum c_k u_k$ to converge weakly on V, i.e., for $\sum c_k u_k[\phi]$ to converge for any test function ϕ supported in V, and this is usually true under very mild conditions on the coefficients c_k. (Cf. the discussion of weak convergence of Fourier series in §9.3.) Once $\sum c_k u_k$ converges weakly, the equation $L\left(\sum c_k u_k\right) = \sum c_k L(u_k)$ holds automatically, since differentiation and multiplication by $C^{(\infty)}$ functions are continuous with respect to weak convergence. In this way one can justify the validity of the series solutions to boundary value problems that are derived in a formal, computational way.

More specifically, suppose we are interested in finding solutions u for something like the 1-dimensional heat equation or wave equation on an interval (a, b) with boundary conditions at a and b, say $B_1(u) = B_2(u) = 0$. The method of separation of variables leads to series solutions of the form

$$u(x,t) = \sum c_k f_k(t)\phi_k(x), \tag{9.42}$$

where the ϕ_k's are the normalized eigenfunctions for an appropriate Sturm-Liouville problem,

$$L(\phi_k) + \lambda\phi_k = 0, \qquad B_1(\phi_k) = B_2(\phi_k) = 0,$$

the f_k's are certain continuous functions of t, and the c_k's are chosen so as to satisfy initial conditions. Each term $c_k f_k(t)\phi_k(x)$ satisfies the original differential equation by construction. Hence, in order to show that $u(x, t)$ is a weak solution on the region $a < x < b$, $t > 0$, it suffices to show that the series (9.42) converges weakly there, i.e., that if $\psi(x, t)$ is any test function supported in this region then the numerical series

$$\sum c_k \int_0^\infty \int_a^b f_k(t)\phi_k(x)\psi(x,t)\,dx\,dt \tag{9.43}$$

converges.

The key to this convergence is the following calculation. For the moment we think if $\psi(x, t)$ as a function of x alone, with t as an extra parameter, and accordingly write $\psi_t(x) = \psi(x, t)$: then (9.43) can be written as

$$\sum c_k \int_0^\infty f_k(t)\langle\phi_k, \overline{\psi}_t\rangle\,dt. \tag{9.44}$$

We shall assume that the coefficients of the Sturm-Liouville operator L are $C^{(\infty)}$, so that L can be applied any number of times to ψ_t to yield a smooth function. Since $\psi_t(x)$ vanishes near the endpoints a and b, by Lagrange's identity we have

$$\langle \phi_k, L^m(\overline{\psi}_t)\rangle = \langle L^m(\phi_k), \overline{\psi}_t\rangle = \lambda_k^m \langle \phi_k, \overline{\psi}_t\rangle,$$

and hence

$$\left|\langle \phi_k, \overline{\psi}_t\rangle\right| \le |\lambda_k|^{-m} \|\phi_k\| \, \|L^m(\overline{\psi}_t)\| = C_m |\lambda_k|^{-m}. \tag{9.45}$$

Now, the eigenvalue λ_k is typically of the order of magnitude of k^2. If this is the case, the estimate (9.45), which is valid for any positive integer m, shows that $\langle \phi_k, \overline{\psi}_t\rangle$ tends to zero as $k \to \infty$ more rapidly than any negative power of k; moreover, it is nonzero only for t in a bounded interval. That being so, one needs only some very mild control over the coefficients c_k and the functions $f_k(t)$ to ensure the convergence of (9.44).

For example, for the vibrating string problem

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(l,t) = 0, \quad u(x,0) = g(x), \quad u_t(x,0) = 0$$

the series solution is

$$u(x,t) = \sum c_k \cos \frac{k\pi c t}{l} \sin \frac{k\pi x}{l}, \qquad c_k = \frac{2}{l} \int_0^l g(x) \sin \frac{k\pi x}{l}\, dx.$$

Here we have $f_k(t) = \cos(k\pi ct/l)$, $\phi_k(x) = \sin(k\pi x/l)$, and $\lambda_k = (k\pi/l)^2$. If g is (say) bounded on $[0,l]$, the coefficients c_k are also bounded, and clearly $|f_k(t)| \le 1$ for all k and t. Hence the preceding calculation shows that the integrated series (9.44) converges rapidly for any test function ψ; this is what we wished to show. The same sort of argument works equally well for many other boundary value problems.

Weak versus classical solutions

For some kinds of differential equations it turns out that all weak solutions are actually smooth functions and hence are classical solutions; a theorem to this effect is called a *regularity theorem.* If this is the case, one often obtains classical solutions by the two-step process of first finding weak solutions and then proving that they are smooth functions. Here the theory of distributions is an important technical tool even though it does not show up in the final result. On the other hand, when regularity theorems do not hold, the study of weak solutions is likely to reveal interesting phenomena connected with the differential equation that are not readily accessible by classical methods.

More precisely, a differential operator L with $C^{(\infty)}$ coefficients is called **hypoelliptic** on an open set V if, whenever f is a $C^{(\infty)}$ function on an open subset $W \subset V$, every weak solution of $L(u) = f$ on W is a $C^{(\infty)}$ function on W. Here are some examples.

(i) Every ordinary differential operator $L(u) = a_k u^{(k)} + \cdots + a_1 u' + a_0 u$ is hypoelliptic on any open set where the leading coefficient a_k does not vanish.
(ii) The Laplace operator ∇^2 is hypoelliptic on $\mathbf{R}^n$.
(iii) The heat operator $(\partial/\partial t) - k\nabla^2$ is hypoelliptic on $\mathbf{R}^{(n+1)}$.
(iv) A differential operator $L = \sum_{|\alpha|\le k} a_\alpha \partial^\alpha$ of order k is called **elliptic** on an open set V if the kth-order part of L is everywhere nondegenerate on V, i.e., if $\sum_{|\alpha|=k} a_\alpha(\mathbf{x})\boldsymbol{\xi}^\alpha \neq 0$ for all $\mathbf{x} \in V$ and all nonzero $\boldsymbol{\xi} \in \mathbf{R}^n$. It is a fundamental theorem that if L is elliptic on V, then L is hypoelliptic on V. (This is the origin of the word *hypoelliptic.*) Note that examples (i) and (ii) are special cases of this one, but (iii) is not.

The proofs of the foregoing regularity theorems are beyond the scope of this book, as is the question of what one can say about the smoothness of a solution u of $L(u) = f$ when L is hypoelliptic and f is not $C^{(\infty)}$ but satisfies some weaker smoothness conditions. See Folland [24]. Some partial results for the Laplace and heat operators are given in Exercises 3–4.

The wave operator $(\partial^2/\partial t^2) - c^2\nabla^2$ is *not* hypoelliptic. We have observed earlier that if g is any twice differentiable function on $\mathbf{R}$, then $u(x,t) = g(x-ct)$ is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ in one space dimension; the right hand side $f(x,t) \equiv 0$ is as smooth as possible, but u need not be $C^{(\infty)}$. (The same example works in higher dimensions; Take $u(\mathbf{x},t) = g(x_1 - ct)$.)

In fact, now that we have the notion of weak solution, the assumption that g is twice differentiable can be discarded. Suppose g is any locally integrable function on $\mathbf{R}$; then $u(x,t) = g(x-ct)$ is locally integrable on $\mathbf{R}^2$, and for any test function ϕ on $\mathbf{R}^2$,

$$(u_{tt} - c^2 u_{xx})[\phi] = u[\phi_{tt} - c^2\phi_{xx}] = \iint g(x-ct)\Big[\phi_{tt}(x,t) - c^2\phi_{xx}(x,t)\Big]\,dx\,dt.$$

If we make the change of variable $y = x - ct$, $z = x + ct$, $\psi(y,z) = \phi(x,t)$, then $\phi_{tt} - c^2\phi_{xx} = -4c^2\psi_{yz}$ (cf. Exercise 6, §1.1) and $dy\,dz = 2c\,dx\,dt$, so

$$(u_{tt} - c^2 u_{xx})[\phi] = -2c\iint g(y)\psi_{yz}(y,z)\,dy\,dz.$$

Performing the z-integration first, we have

$$\int \psi_{yz}(y,z)\,dz = \psi_y(y,z)\Big|_{z=-\infty}^{\infty} = 0,$$

and hence $(u_{tt} - c^2 u_{xx})[\phi] = 0$. Thus u is a weak solution of the wave equation that is nonclassical if g is not differentiable. (Actually, g could be taken to be any distribution on $\mathbf{R}$; see Exercise 5.)

The wave equation in n dimensions

In §7.5 we made an attempt to solve the wave equation in n space dimensions by means of the Fourier transform. We got as far as showing that the Fourier transform (with respect to the space variables) of the solution of the initial value problem

$$u_{tt} = c^2\nabla^2 u, \qquad u(\mathbf{x},0) = f(\mathbf{x}), \quad u_t(\mathbf{x},0) = g(\mathbf{x})$$

is

$$\widehat{u}(\boldsymbol{\xi},t) = \widehat{f}(\boldsymbol{\xi})\cos ct|\boldsymbol{\xi}| + \widehat{g}(\boldsymbol{\xi})\frac{\sin ct|\boldsymbol{\xi}|}{c|\boldsymbol{\xi}|}. \tag{9.46}$$

In order to complete the solution, we need to find the inverse Fourier transforms of $|c\boldsymbol{\xi}|^{-1}\sin ct|\boldsymbol{\xi}|$ and $\cos ct|\boldsymbol{\xi}|$. Actually, if we can do one of these we can do the other, since

$$\cos ct|\boldsymbol{\xi}| = \frac{d}{dt}\frac{\sin ct|\boldsymbol{\xi}|}{c|\boldsymbol{\xi}|}. \tag{9.47}$$

Hence, our task is to find the tempered distribution $K(\mathbf{x},t)$ such that

$$\widehat{K}(\boldsymbol{\xi},t) = \frac{\sin ct|\boldsymbol{\xi}|}{c|\boldsymbol{\xi}|}. \tag{9.48}$$

In the case $n = 3$, the one of greatest physical importance, we essentially solved this problem in Example 2 of §9.4, where we showed that $4\pi|\boldsymbol{\xi}|^{-1}\sin|\boldsymbol{\xi}|$ is the Fourier transform of surface measure on the unit sphere in $\mathbf{R}^3$. From this it follows easily that for any $t > 0$, $|c\boldsymbol{\xi}|^{-1}\sin ct|\boldsymbol{\xi}|$ is the Fourier transform of $(4\pi c^2 t)^{-1}$ times surface measure on the sphere of radius ct about the origin. Indeed, if we denote the element of surface measure by dS, we have

$$\frac{1}{4\pi c^2 t}\int_{|\mathbf{x}|=ct} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\,dS(\mathbf{x}) = \frac{t}{4\pi}\int_{|\mathbf{y}|=1} e^{-ict\boldsymbol{\xi}\cdot\mathbf{y}}\,dS(\mathbf{y}) = t\frac{\sin|ct\boldsymbol{\xi}|}{|ct\boldsymbol{\xi}|} = \frac{\sin ct|\boldsymbol{\xi}|}{c|\boldsymbol{\xi}|}.$$

(The first equation is true by the change of variable $\mathbf{x} = ct\mathbf{y}$, since dilation of the coordinates by a factor of ct produces a dilation of surface measure by a factor of $(ct)^2$.) It follows that the inverse Fourier transform of the second term in (9.46) is

$$\frac{1}{4\pi c^2 t}\int_{|\mathbf{y}|=ct} g(\mathbf{x}-\mathbf{y})\,dS(\mathbf{y}) = \frac{1}{4\pi c^2 t}\int_{|\mathbf{x}-\mathbf{y}|=ct} g(\mathbf{y})\,dS(\mathbf{y}).$$

Moreover, in view of (9.47), to compute the inverse Fourier transform of the first term we simply replace g by f and differentiate with respect to t. The result is

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\frac{1}{4\pi c^2 t}\int_{|\mathbf{x}-\mathbf{y}|=ct} f(\mathbf{y})\,dS(\mathbf{y}) + \frac{1}{4\pi c^2 t}\int_{|\mathbf{x}-\mathbf{y}|=ct} g(\mathbf{y})\,dS(\mathbf{y}).$$

From this formula one can see clearly how a disturbance starting at position $\mathbf{y}_0$ at time $t = 0$ propagates out to the sphere $|\mathbf{x}-\mathbf{y}_0| = ct$ at time t, traveling with speed c and decreasing in amplitude as it goes.

For $n = 1$ the solution u was calculated in Exercise 2, §7.3, so we henceforth assume $n > 1$. For $n = 2$ or $n \geq 4$ we do not have a ready-made formula for the inverse Fourier transform of $|c\boldsymbol{\xi}|^{-1}\sin ct|\boldsymbol{\xi}|$, but we can proceed as follows. For $\epsilon > 0$, let

$$G_\epsilon(\boldsymbol{\xi},t) = \frac{\sin ct|\boldsymbol{\xi}|}{c|\boldsymbol{\xi}|}e^{-\epsilon|\boldsymbol{\xi}|}.$$

$G_\epsilon(\boldsymbol{\xi}, t)$ is an integrable function of $\boldsymbol{\xi}$ for each $t > 0$ and $G_\epsilon(\boldsymbol{\xi}) \to |c\boldsymbol{\xi}|^{-1} \sin ct|\boldsymbol{\xi}|$ temperately as $\epsilon \to 0$, so to calculate K we can apply the inverse Fourier transform to G_ϵ and then take the limit as $\epsilon \to 0$. For this purpose, we observe that

$$G_\epsilon(\boldsymbol{\xi}, t) = \frac{e^{-(\epsilon - ict)|\boldsymbol{\xi}|} - e^{-(\epsilon + ict)|\boldsymbol{\xi}|}}{2ic|\boldsymbol{\xi}|} = \frac{1}{2ic} \int_{\epsilon - ict}^{\epsilon + ict} e^{-s|\boldsymbol{\xi}|}\, ds. \tag{9.49}$$

In Exercise 1 of §7.5 we sketched a proof that for $s > 0$,

$$\frac{1}{(2\pi)^n} \int e^{-s|\boldsymbol{\xi}|} e^{i\mathbf{x}\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \frac{\Gamma\Big(n+1)/2\Big)}{\pi^{(n+1)/2}} \frac{s}{\Big(|\mathbf{x}|^2 + s^2\Big)^{(n+1)/2}}.$$

By essentially the same argument, or simply by analytic continuation, this formula remains valid for all complex s with $\operatorname{Re} s > 0$ (for which $e^{-s|\boldsymbol{\xi}|}$ is still an integrable function of $\boldsymbol{\xi}$), provided that one interprets the square root on the right as the principal branch. Combining this result with (9.49) and reversing the order of integration (which is legitimate since the integrals converge absolutely), we have

$$\begin{aligned}
\frac{1}{(2\pi)^n} \int G_\epsilon(\boldsymbol{\xi}, t) e^{i\mathbf{x}\boldsymbol{\xi}}\, d\boldsymbol{\xi} &= \frac{1}{2ic(2\pi)^n} \int_{\epsilon - ict}^{\epsilon + ict} \int_{\mathbf{R}^n} e^{-s|\boldsymbol{\xi}|} e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}\, ds \\
&= \frac{\Gamma\Big((n+1)/2\Big)}{2ic\pi^{(n+1)/2}} \int_{\epsilon - ict}^{\epsilon + ict} \frac{s}{\Big(|\mathbf{x}|^2 + s^2\Big)^{(n+1)/2}}\, ds \\
&= \frac{\Gamma\Big((n+1)/2\Big)}{2ic(1-n)\pi^{(n+1)/2}} \left[\frac{1}{\Big(|\mathbf{x}|^2 + (\epsilon + ict)^2\Big)^{(n-1)/2}} - \frac{1}{\Big(|\mathbf{x}|^2 + (\epsilon - ict)^2\Big)^{(n-1)/2}} \right].
\end{aligned} \tag{9.50}$$

Now we wish to let $\epsilon \to 0$, and at this point a crucial distinction appears between the cases n odd and n even. If n is odd, the exponent $(n-1)/2$ is an integer, and as $\epsilon \to 0$ the two terms in brackets in (9.50) both become $\Big(|\mathbf{x}|^2 - (ct)^2\Big)^{(1-n)/2}$; so they cancel out *except* on the sphere $|\mathbf{x}| = ct$, where they both blow up. (More precisely, their difference tends uniformly to zero on the sets $|\mathbf{x}| \le ct - \delta$ and $|\mathbf{x}| \ge ct + \delta$ for any $\delta > 0$.) The result is that $K(\mathbf{x}, t) = \lim_{\epsilon \to 0} \mathscr{F}^{-1} G_\epsilon(\mathbf{x}, t)$ is a distribution supported on the sphere $|\mathbf{x}| = ct$. As we have already seen, when $n = 3$ it is $(4\pi t)^{-1}$ times surface measure on this sphere; for larger n it is more singular.

On the other hand, if n is even, the exponent $(n-1)/2$ is a half-integer. If $|\mathbf{x}| > ct$, the two terms in brackets in (9.50) both approach the positive number $\Big(|\mathbf{x}|^2 - (ct)^2\Big)^{(n-1)/2}$ as $\epsilon \to 0$, so once again they cancel out. But if $|\mathbf{x}| < ct$, $|\mathbf{x}|^2 + (\epsilon + ict)^2$ and $|\mathbf{x}|^2 + (\epsilon - ict)^2$ approach the negative number $|\mathbf{x}|^2 - (ct)^2$ from the upper and lower half-planes, respectively — that is, from opposite sides of the branch cut for the square root. Therefore, the two terms in brackets add up instead of canceling out. The result is a distribution whose support is not the sphere $|\mathbf{x}| = ct$ but the whole closed ball $|\mathbf{x}| \le ct$, and which agrees with the function

$$\frac{\Gamma\big((n+1)/2\big)}{c(1-n)\pi^{(n+1)/2}} \frac{(-1)^{(n/2)}}{\big((ct)^2 - |\mathbf{x}|^2\big)^{(n-1)/2}} \tag{9.51}$$

on the open ball $|\mathbf{x}| < ct$.

The physical significance of this is that in odd dimensions (except $n = 1$), a wave starting at a point $\mathbf{x}_0$ at time $t = 0$ reaches the points on the sphere $|\mathbf{x} - \mathbf{x}_0| = ct$ at time t and then departs, leaving no trace behind; but in even dimensions, such a wave reaches the sphere $|\mathbf{x} - \mathbf{x}_0| = ct$ at time t and then leaves a gradually decaying tail as it proceeds onward. It is lucky that we live in 3 dimensions; in an even-dimensional world, light and sound signals would not be so readily intelligible as they are!

The function (9.51) blows up along the sphere $|\mathbf{x}| = ct$, and when $n > 2$ it does so fast enough to be nonintegrable. Thus, when n is even and $n > 2$, $K = \lim_{\epsilon\to 0} \mathscr{F}^{-1} G_\epsilon$ is not a function, and care must be taken in interpreting the limit of (9.50) as a distribution. We shall not attempt to say any more about this case here, nor about the case when n is odd and $n > 3$. (The complete results can be found in Folland [24] or John [33].) However, when $n = 2$, (9.51) *is* an integrable function, and $\mathscr{F}^{-1} G_\epsilon(\mathbf{x}, t)$ converges to it temperately; hence it is the desired distribution K. Since $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, we therefore have

$$K(\mathbf{x}, t) = \begin{cases} (2\pi)^{-1}\big((ct)^2 - |\mathbf{x}|^2\big)^{-1/2} & \text{for } |\mathbf{x}| < ct, \\ 0 & \text{for } |\mathbf{x}| > ct, \end{cases}$$

and the solution u of the wave equation whose Fourier transform is given by (9.46) is

$$\frac{1}{2\pi c}\frac{\partial}{\partial t}\int_{|\mathbf{y}|<ct} \frac{f(\mathbf{x}-\mathbf{y})}{\big((ct)^2 - |\mathbf{y}|^2\big)^{1/2}}\, d\mathbf{y} + \frac{1}{2\pi c}\int_{|\mathbf{y}|<ct} \frac{g(\mathbf{x}-\mathbf{y})}{\big((ct)^2 - |\mathbf{y}|^2\big)^{1/2}}\, d\mathbf{y}.$$

The distribution $K(\mathbf{x}, t)$ given by (9.48) is itself a solution of the wave equation, with initial data $K(\mathbf{x}, 0) = 0$ and $K_t(\mathbf{x}, 0) = \delta(\mathbf{x})$. This gives another important example of a weak solution of the wave equation that is not a classical solution.

We conclude by remarking that there is a useful and elegant device for solving the wave equation in terms of superpositions of plane waves, known as the *Radon transform.* An introduction to the Radon transform can be found in Folland [24] or Walker [53].

EXERCISES

1. Show that the series solution of the problem

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u_x(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

given by formula (4.25) is a weak solution of the wave equation on the region $0 < x < l$, $t > 0$, whenever f and g are in $L^2(0, l)$.

2. Show that the Fourier series solution of the Dirichlet problem in a rectangle

$$u_{xx} + u_{yy} = 0, \qquad u(0, y) = u(l, y) = 0, \quad u(x, 0) = f_1(x), \quad u(x, L) = f_2(x)$$

(see §4.4) is a weak solution of Laplace's equation whenever f_1 and f_2 are in $L^2(0, l)$.

3. Suppose u is a tempered distribution on $\mathbf{R}^n$ and $f = \nabla^2 u$ is a Schwartz function. Prove that u is $C^{(\infty)}$. (Hint: let χ be as in Exercise 3, §9.2, and let $\check{\chi}$ be its inverse Fourier transform. Show that $u = u * \check{\chi} + v$ where $\widehat{v}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-2}\big[\chi(\boldsymbol{\xi}) - 1\big]\widehat{f}(\boldsymbol{\xi})$ and that $u * \check{\chi}$ and v are both $C^{(\infty)}$ functions.)
4. Suppose u is a tempered distribution on $\mathbf{R}^n \times \mathbf{R}$ and $f = (\partial u/\partial t) - \nabla^2 u$ is a Schwartz function. Prove that u is $C^{(\infty)}$. (See the hint for Exercise 3.)
5. Suppose g is a distribution on $\mathbf{R}$. Show how to make sense of $u(x, t) = g(x - ct)$ as a distribution on $\mathbf{R}^2$, and show that u is a weak solution of the wave equation. (Exercise 9 of §9.1 may be helpful.)
6. Show that the only weak solutions of the differential equation $F' = 0$ on $\mathbf{R}$ are the classical solutions, i.e., the constant functions. (If $F' = 0$, let ϕ_ϵ be as in Theorem 9.5, §9.2. Show that $(F * \phi_\epsilon)' = 0$, and then let $\epsilon \to 0$.)

CHAPTER 10
GREEN'S FUNCTIONS

Suppose L is a linear (ordinary or partial) differential operator with smooth coefficients, and suppose we are interested in solving the inhomogeneous equation $L(u) = f$ for a reasonably broad class of functions f. This can be done if we can solve the equation $L(u) = f$ in the special case when f is a delta function. Indeed, suppose that for each $\mathbf{y}$ we can find a distribution $G(\mathbf{x}, \mathbf{y})$ such that

$$L_{\mathbf{x}}\Big[G(\mathbf{x}, \mathbf{y})\Big] = \delta(\mathbf{x} - \mathbf{y}). \tag{10.1}$$

(Here the subscript $\mathbf{x}$ on L means that L is to be applied to G as a function of $\mathbf{x}$, whereas $\mathbf{y}$ is merely a parameter.) Formally, then, we can solve $L(u) = f$ by setting

$$u(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}, \tag{10.2}$$

because

$$L(u)(\mathbf{x}) = \int L_{\mathbf{x}}\Big[G(\mathbf{x}, \mathbf{y})\Big] f(\mathbf{y})\, d\mathbf{y} = \int \delta(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} = f(\mathbf{x}). \tag{10.3}$$

If we are interested in solving the equation $L(u) = f$ on a region D, the variables $\mathbf{x}$ and $\mathbf{y}$ should be restricted to D, and the integration in (10.2) and (10.3) is over D. We may also wish u to satisfy some linear, homogeneous boundary conditions on the boundary of D, say $B(u) = 0$. This may be accomplished by requiring $G(\mathbf{x}, \mathbf{y})$ to satisfy these conditions as a function of $\mathbf{x}$ for each $\mathbf{y}$, for then

$$B(u) = \int B_{\mathbf{x}}\Big[G(\mathbf{x}, \mathbf{y})\Big] f(\mathbf{y})\, d\mathbf{y} = 0.$$

Of course, this all needs to be made more precise. One needs to add conditions on f and on the $\mathbf{y}$-dependence of G so that the integral $\int G(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$ is well-defined and so that the interchange of integration and differentiation in (10.3) is valid. One also needs assurance that the distributions G are sufficiently regular for the boundary conditions $B(G) = 0$ to make sense. However, the basic idea is sound. A distribution $G(\mathbf{x}, \mathbf{y})$ satisfying (10.1) is called a **fundamental solution** or **Green's function** for the differential operator L. If boundary conditions

$B_{\mathbf{x}}\left[G(\mathbf{x}, \mathbf{y})\right] = 0$ are also imposed, G is called a **Green's function** for the boundary value problem $L(u) = f$, $B(u) = 0$.

The basic theory of Green's functions for ordinary differential operators is fairly simple, and we present it in §10.1. For partial differential operators a comprehensive treatment would require a book by itself (for example, Stakgold [48]); we shall restrict ourselves to the construction of a few important examples in §10.2. In the last two sections we use Green's functions for Sturm-Liouville problems to derive eigenfunction expansions, thereby completing the discussion in §3.5.

10.1 Green's functions for ordinary differential operators

Let L be a kth-order linear ordinary differential operator on an interval (a, b), where $-\infty \le a < b \le \infty$:

$$L(u) = p_k(x)u^{(k)} + p_{k-1}(x)u^{(k-1)} + \cdots + p_1(x)u' + p_0(x)u. \tag{10.4}$$

We assume that the coefficients $p_j(x)$ are all continuous on (a, b) and that the leading coefficient $p_k(x)$ is nonvanishing on (a, b). The fundamental existence theorem for ordinary differential equations then guarantees that the solutions of the homogeneous equation $L(u) = 0$ are of class $C^{(k)}$ and form a k-dimensional vector space. We fix a point $y \in (a, b)$ and investigate the solutions of the equation

$$L(u)(x) = \delta(x - y), \qquad x \in (a, b).$$

We first observe that, since $\delta(x - y) = 0$ except at $x = y$, $u(x)$ must agree with a solution $u_-(x)$ of the homogeneous equation on the interval $a < x < y$ and with another such solution $u_+(x)$ on the interval $y < x < b$. The question is how these solutions should fit together at $x = y$ in order to produce a delta function when L is applied.

If L is first-order, the answer is immediate from the discussion in §9.1. There we showed that if u is a function that is smooth except at $x = y$ and has a jump discontinuity at $x = y$, then the distribution derivative u' is the pointwise derivative of u plus $\left[u(y+) - u(y-)\right]\delta(x - y)$. Thus, in order to obtain $L(u) = p_1u' + p_0u = \delta(x - y)$, we have merely to choose u_- and u_+ so that

$$p_1(y)\left[u_+(y) - u_-(y)\right] = 1.$$

Since $p_1(y) \neq 0$ by assumption, and since the value $u(y)$ of a solution of the homogeneous equation at the given point y can be chosen arbitrarily, this is always possible.

A similar idea works in the kth-order case. Here we wish to produce a delta function not in u' but in $u^{(k)}$, and the way to do this is to require a jump discontinuity in $u^{(k-1)}$ at $x = y$. The lower-order derivatives $u^{(j)}$, $j < k - 1$, should be everywhere continuous; otherwise $u^{(k)}$ will involve not only the delta function but some of its derivatives. In other words, we require that

$$\begin{aligned} &u_+^{(j)}(y) = u_-^{(j)}(y) \quad \text{for } 0 \le j \le k-2, \\ &u_+^{(k-1)}(y) - u_-^{(k-1)}(y) = p_k(y)^{-1}. \end{aligned} \tag{10.5}$$

Again, these equations can always be satisfied, because $p_k(y) \neq 0$ and the values of a solution of the homogeneous equation and its first $k-1$ derivatives at the point y can be specified arbitrarily. We sum up the results in a theorem:

Theorem 10.1. *Let L be given by (10.4), where the coefficients p_j are continuous on (a,b) and $p_k(x) \neq 0$ for $x \in (a,b)$. Given $y \in (a,b)$, let $u_+(x)$ and $u_-(x)$ be solutions of $L(u) = 0$ that satisfy the relations (10.5). Then the function*

$$u(x) = \begin{cases} u_-(x) & \text{for } a < x < y < b, \\ u_+(x) & \text{for } a < y < x < b \end{cases} \tag{10.6}$$

satisfies $L(u) = \delta(x-y)$.

The general solution of $L(u) = 0$ contains k arbitrary constants, so we have a total of $2k$ degrees of freedom in choosing two solutions u_- and u_+. The equations (10.5) impose k constraints, so there remain k arbitrary constants in (10.6) that can be used to satisfy boundary conditions or initial conditions. (Note: These constants are constant only with respect to x; they may (and usually do) depend on y.) This is the general procedure for constructing Green's functions; we now investigate how it works in some important particular cases.

Green's functions for initial value problems

Let us assume (for simplicity) that the interval (a,b) is $\mathbf{R}$ and find the Green's function $G(x,y)$ for the initial value problem

$$L(u) = f, \qquad u(0) = u'(0) = \cdots = u^{(k-1)}(0) = 0. \tag{10.7}$$

$G(x,y)$ will be constructed from solutions u_+ and u_- of $L(u) = 0$ satisfying (10.5), and we also want G to satisfy the initial conditions (10.7). The only solution of $L(u) = 0$ satisfying these conditions is $u \equiv 0$. Hence, if $y > 0$ we must take $u_- \equiv 0$, whereas if $y < 0$ we must take $u_+ \equiv 0$; and then u_+ [resp. u_-] is completely determined by (10.5). Namely, if $y > 0$ the first $k-2$ derivatives of u_+ at y must vanish while $u^{(k-1)}(y)$ must equal $p_k(y)^{-1}$, and these initial values for u_+ at y uniquely determine u_+; likewise for u_- if $y < 0$. In other words, if for $y \in \mathbf{R}$ we denote by $v_y(x)$ the solution of the initial value problem

$$L(v_y) = 0, \quad v_y^{(j)}(y) = 0 \text{ for } j < k-1, \quad v_y^{(k-1)}(y) = p_k(y)^{-1}, \tag{10.8}$$

then $u_+ = v_y$ when $y > 0$ and $u_- = -v_y$ when $y < 0$. In short,

$$G(x,y) = \left\{ \begin{array}{rl} v_y(x) & \text{if } 0 < y < x \\ -v_y(x) & \text{if } x < y < 0 \\ 0 & \text{otherwise} \end{array} \right\} = v_y(x)\Big[H(x-y) - H(-y)\Big].$$

See Figure 10.1. (Note: If one is interested only in solving the differential equation on the half-line $[0,\infty)$, so that x and y are positive, one can write simply $G(x,y) = H(x-y)v_y(x)$.) The solution of the initial value problem (10.7) is then

$$u(x) = \int G(x,y)f(y)\,dy = \begin{cases} \int_0^x v_y(x)f(y)\,dy & \text{if } x > 0 \\ -\int_x^0 v_y(x)f(y)\,dy & \text{if } x < 0 \end{cases} = \int_0^x v_y(x)f(y)\,dy. \tag{10.9}$$

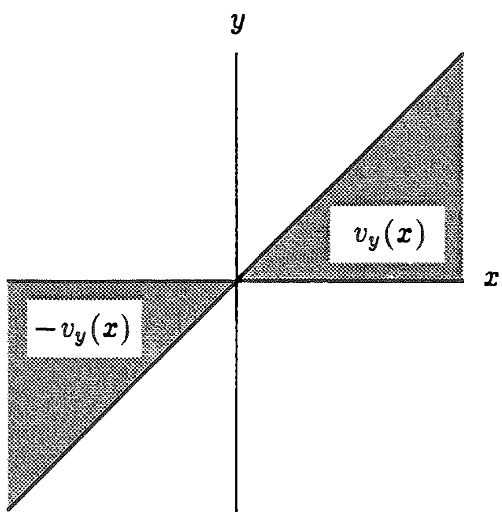


FIGURE 10.1. Schematic representation of the Green's function $G(x,y)$ for the initial value problem. In the unshaded regions, $G(x,y) = 0$.

It is straightforward to verify directly that this formula gives the solution of (10.7). In the first place, v_y depends continuously on y, so the integral $\int_0^x v_y(x)f(y)\,dy$ is well-defined. Moreover, we have

$$u'(x) = \int_0^x v_y'(x)f(y)\,dy + v_x(x)f(x) = \int_0^x v_y'(x)f(y)\,dy$$

since $v_y(y) = 0$ for any y, in particular for $y = x$. The same calculation, applied successively to the derivatives of u, shows that

$$u^{(j)}(x) = \int_0^x v_y^{(j)}(x)f(y)\,dy \qquad (j \le k-1)$$

and

$$u^{(k)}(x) = \int_0^x v_y^{(k)}(x)f(y)\,dy + v_x^{(k-1)}(x)f(x) = \int_0^x v_y^{(k)}(x)f(y)\,dy + p_k(x)^{-1}f(x).$$

From this it is clear that $u^{(j)}(0) = 0$ for $j \leq k-1$ and that

$$L(u)(x) = \sum_0^k p_j(x)u^{(j)}(x) = \sum_0^k p_j(x)\int_0^x v_y^{(j)}(x)f(y)\,dy + f(x)$$
$$= \int_0^x L(v_y)(x)f(y)\,dy + f(x) = f(x).$$

Let us see how the formula (10.9) works in the simple special case where L is first-order and $p_1 \equiv 1$. That is, we consider the initial value problem

$$L(u) \equiv u' + pu = f, \qquad u(0) = 0.$$

The solutions of the homogeneous equation $u' + pu = 0$ are the constant multiples of

$$v_0(x) = \exp\left[-\int_0^x p(t)\,dt\right],$$

and in particular, the solution v_y satisfying $v_y(y) = 1$ is

$$v_y(x) = \frac{v_0(x)}{v_0(y)}.$$

Hence, formula (10.9) becomes

$$u(x) = v_0(x)\int_0^x \left[v_0(y)\right]^{-1} f(y)\,dy,$$

which is the solution obtained by the standard elementary technique of using the "integrating factor"

$$\exp\left[\int_0^x p(t)\,dt\right] = \left[v_0(x)\right]^{-1}.$$

Let us also examine the case where L has constant coefficients:

$$L(u) = p_k u^{(k)} + \cdots + p_1 u' + p_0 u \qquad (p_k, \ldots, p_0 \in \mathbf{C}).$$

L commutes with translations, so if $v(x)$ is a solution of $L(v) = 0$, then so is $v(x-y)$ for any $y \in \mathbf{R}$. In particular, if v_0 is the solution of

$$L(v_0) = 0, \quad v_0(0) = v_0'(0) = \cdots = v_0^{(k-2)}(0) = 0, \quad v_0^{(k-1)}(0) = p_k^{-1}, \qquad (10.10)$$

then the function $v_y(x)$ in (10.8) is simply given by

$$v_y(x) = v_0(x-y),$$

and formula (10.9) becomes

$$u(x) = \int_0^x v_0(x-y)f(y)\,dy. \qquad (10.11)$$

This is the solution of the initial value problem (10.7) that we obtained in §8.3 by means of the Laplace transform. Indeed, applying the Laplace transform to (10.11) and (10.10), we have

$$\mathscr{L}u = (\mathscr{L}v_0)(\mathscr{L}f) \quad \text{and} \quad (\mathscr{L}v_0)(z) = \frac{1}{p_k z^k + \cdots + p_1 z + p_0}.$$

The Laplace transform is perhaps the easiest technique for calculating v_0 and hence the Green's function. (Note: when one uses the Laplace transform, one is always implicitly working on the half-line $[0,\infty)$. However, the inverse Laplace transform of $(p_k z^k + \cdots + p_0)^{-1}$ is an exponential polynomial on $[0,\infty)$, and the solution v_0 of (10.10) is given by this exponential polynomial on the whole line. As we have observed in (10.9), the formula (10.11) then gives the desired solution $u(x)$ whether x is positive or negative.)

Example 1. Let $L(u) = u'' + \mu^2 u$, where μ is a complex constant. The Laplace transform of v_0 is

$$(\mathscr{L}v_0)(z) = \frac{1}{z^2 + \mu^2},$$

which has simple poles at $z = \pm i\mu$. Hence,

$$\begin{aligned} v_0(x) &= \left[\mathrm{Res}_{i\mu} + \mathrm{Res}_{-i\mu}\right] \frac{e^{xz}}{z^2 + \mu^2} \\ &= \frac{e^{i\mu x} - e^{-i\mu x}}{2i\mu} = \frac{1}{\mu}\sin \mu x. \end{aligned}$$

Thus the Green's function for the initial value problem is

$$G(x,y) = \frac{1}{\mu}\left[H(x-y) - H(-y)\right] \sin \mu(x-y)$$

and the solution of $L(u) = f$, $u(0) = u'(0) = 0$, is

$$u(x) = \frac{1}{\mu}\int_0^x f(y) \sin \mu(x-y)\, dy. \tag{10.12}$$

Green's functions for boundary value problems

We now consider differential operators L on a closed interval $[a,b]$ together with boundary conditions involving the two endpoints. We shall restrict attention to second-order operators and separated boundary conditions; however, the results can be extended to higher-order operators and more general boundary conditions. (For instance, see Exercise 7.) A comprehensive treatment can be found in Dunford-Schwartz [18] or Naimark [40].

Specifically, let

$$L(u) = p_2(x)u'' + p_1(x)u' + p_0(x)u,$$

where p_0, p_1, and p_2 are continuous on $[a, b]$ and $p_2 \neq 0$ on $[a, b]$. We shall study the boundary value problem

$$\begin{aligned} &L(u) = f, \\ &\alpha u(a) + \alpha' u'(a) = 0, \quad \beta u(b) + \beta' u'(b) = 0 \qquad (\alpha\alpha' \neq 0,\ \beta\beta' \neq 0). \end{aligned} \tag{10.13}$$

There is a potential difficulty here that does not arise in the case of initial value problems, which may be illustrated by the following example. Consider the boundary value problem

$$u'' = f, \qquad u'(0) = u'(1) = 0. \tag{10.14}$$

If we set

$$F_1(x) = \int_0^x f(t)\,dt, \qquad F_2(x) = \int_0^x F_1(t)\,dt,$$

the general solution if $u'' = f$ is

$$u(x) = F_2(x) + c_1 x + c_0.$$

We then have $u'(x) = F_1(x) + c_1$. Since $F_1(0) = 0$, the condition $u'(0) = 0$ forces $c_1 = 0$. But then we cannot have $u'(1) = 0$ unless it happens that $F_1(1) = \int_0^1 f(t)\,dt = 0$. That is, problem (10.14) has no solution unless $\int_0^1 f(t)\,dt = 0$, and in that case the solution is not unique because the constant c_0 is still arbitrary. In particular, since $\int_0^1 \delta(t - y)\,dt = 1$ for $0 < y < 1$, there is no solution with $f(x) = \delta(x - y)$, i.e., no Green's function.

This difficulty arises whenever the homogeneous equation $L(u) = 0$ has nonzero solutions that satisfy the boundary conditions in (10.13). (In the preceding example, the constant functions are such solutions.) This situation is just like the problem of solving systems of linear equations: If the $n \times n$ matrix A is nonsingular, the equation $A\mathbf{x} = \mathbf{y}$ has a unique solution $\mathbf{x}$ for any $\mathbf{y}$; but if the homogeneous equation $A\mathbf{x} = 0$ has nontrivial solutions, the equation $A\mathbf{x} = \mathbf{y}$ has either no solutions or infinitely many solutions, depending on what $\mathbf{y}$ is. *We therefore assume henceforth that there are no nonzero solutions of*

$$L(u) = 0, \qquad \alpha u(a) + \alpha' u'(a) = \beta u(b) + \beta' u'(b) = 0.$$

Under this hypothesis we shall construct the Green's function.

We can always find nontrivial solutions of $L(u) = 0$ satisfying *one* of the boundary conditions, because of the fundamental existence theorem for ordinary differential equations. Thus, let $v_a(x)$ and $v_b(x)$ be nonzero solutions of the problems

$$\begin{aligned} L(v_a) &= 0, \qquad \alpha v_a(a) + \alpha' v_a'(a) = 0, \\ L(v_b) &= 0, \qquad \beta v_b(b) + \beta' v_b'(b) = 0. \end{aligned}$$

and let $W(x)$ be their Wronskian:

$$W = v_a v_b' - v_b v_a'.$$

Then any constant multiple of v_a satisfies the boundary condition at a in (10.13), whereas any constant multiple of v_b satisfies the boundary condition at b. Conversely, the constant multiples of v_a [resp. v_b] are the only solutions of $L(u) = 0$ satisfying the boundary condition at a [resp. b]. This is because a function u satisfies the boundary condition at a if and only if the vector $\big(u(a), u'(a)\big)$ is proportional to $(\alpha', -\alpha)$, and a solution of $L(u) = 0$ is uniquely determined by the initial values $u(a)$ and $u'(a)$; likewise at b.

Moreover, since there are no nonzero solutions of $L(u) = 0$ satisfying both boundary conditions, v_a and v_b are linearly independent, and their Wronskian W is therefore nonvanishing.

Now, if $y \in (a, b)$, the Green's function $G(x, y)$ for problem (10.13) must satisfy the equation $L(u) = 0$ for $x < y$ and $x > y$, and it must satisfy the boundary conditions in (10.13). Hence we must have

$$G(x,y) = \begin{cases} C_a v_a(x) & \text{for } x < y \\ C_b v_b(x) & \text{for } x > y \end{cases}$$

for some constants C_a and C_b. Moreover, these functions must match up at $x = y$ according to the relations (10.5):

$$C_a v_a(y) = C_b v_b(y), \qquad C_b v_b'(y) - C_a v_a'(y) = p_2(y)^{-1}.$$

We can solve these two equations for the constants C_a and C_b:

$$C_a = \frac{v_b(y)}{p_2(y)W(y)}, \qquad C_b = \frac{v_a(y)}{p_2(y)W(y)}.$$

This procedure works because, as we observed above, $W(y) \neq 0$.

In short, we have

$$G(x,y) = \frac{v_a(x)v_b(y)}{p_2(y)W(y)} \quad \text{if } x < y, \qquad G(x,y) = \frac{v_b(x)v_a(y)}{p_2(y)W(y)} \quad \text{if } x > y,$$

or

$$G(x,y) = \frac{v_a(x_-)v_b(x_+)}{p_2(y)W(y)} \qquad \big(x_- = \min(x,y),\ x_+ = \max(x,y)\big). \tag{10.15}$$

The solution to (10.13) is therefore

$$u(x) = v_a(x)\int_x^b \frac{v_b(y)f(y)}{p_2(y)W(y)}\,dy + v_b(x)\int_a^x \frac{v_a(y)f(y)}{p_2(y)W(y)}\,dy. \tag{10.16}$$

Formula (10.15) can be simplified when the boundary value problem (10.13) is self-adjoint, as defined in §3.5. That is, suppose that the coefficients p_0, p_1, p_2 of L and the constants α, α', β, β' in (10.13) are real and that $p_1(x) = p_2'(x)$. The Wronskian W satisfies

$$\begin{aligned} p_2 W' &= p_2(v_a v_b')' - p_2(v_b v_a')' = p_2 v_a v_b'' - p_2 v_b v_a'' \\ &= v_a(-p_1 v_b' - p_0 v_b) - v_b(-p_1 v_a' - p_0 v_a) = -p_1(v_a v_b' - v_b v_a') \\ &= -p_1 W. \end{aligned}$$

Thus

$$(p_2 W)' = p_2 W' + p_2' W = p_2 W' + p_1 W = 0,$$

so $p_2 W$ is actually a constant C, and we have (with the notation of (10.15))

$$G(x,y) = \frac{v_a(x_-)v_b(x_+)}{C}.$$

The value of the constant C can be determined in several ways — for example, by evaluating $W(y)$ at some convenient point.

An interesting feature here is that G is symmetric in x and y: $G(x,y) = G(y,x)$. This can also be deduced on abstract grounds from the self-adjointness of the boundary value problem.

Example 2. Consider the boundary value problem

$$u'' + \mu^2 u = f, \qquad u'(0) = u'(l) = 0 \qquad (\mu \in \mathbf{C}).$$

Here $p_2(x) = 1$, $a = 0$, and $b = l$, and we can take

$$v_0(x) = \cos \mu x, \qquad v_l(x) = \cos \mu(x - l).$$

Then

$$\begin{aligned} W(x) = -\mu \cos \mu x \sin \mu(x-l) + \mu \sin \mu x \cos \mu(x-l) &= \mu \sin[\mu x - \mu(x-l)] \\ &= \mu \sin \mu l, \end{aligned}$$

so by (10.15),

$$G(x,y) = \frac{\cos \mu x_- \cos \mu(x_+ - l)}{\mu \sin \mu l} \qquad (x_- = \min(x,y),\ x_+ = \max(x,y)).$$

Observe that this makes sense only when μl is not an integer multiple of π. If $\mu = n\pi/l$, then the homogeneous problem $u'' + \mu^2 u = 0$, $u'(0) = u'(l) = 0$ has the nontrivial solution $u = \cos(n\pi x/l)$, and the Green's function does not exist.

EXERCISES

Find the Green's function $G(x, y)$ for the problems in Exercises 1–10. Exercises 1–3 are initial value problems; 5–7 are boundary value problems of the sort discussed in the text; 8 is a boundary value problem with periodic boundary conditions; and 9–11 deal with problems on an unbounded interval with a boundary condition at infinity (which work much like the problems in the text). Note that the differential equations in Exercises 4 and 5 are Euler equations.

1. $u'' + 4u' + 4u = f, \quad u(0) = u'(0) = 0$
2. $u'' + 9u' + 20u = f, \quad u(0) = u'(0) = 0$
3. $u^{(4)} + u = f, \quad u(0) = u'(0) = u''(0) = u'''(0) = 0$
4. $x^2u'' + 4xu' + 2u = f, \quad u(1) = u'(1) = 0 \quad (x > 0)$
5. $x^2u'' - 2xu' + 2u = f$ on $(1, 2)$, $\quad u(1) = u(2) = 0$
6. $u'' + \mu^2 u = f$ on $(0, \frac{1}{2}\pi)$, $\quad u(0) = u'(\frac{1}{2}\pi) = 0 \quad (\mu \neq 1, 3, 5, \ldots)$
7. $u'' + \mu^2 u = f$ on $(0, 1)$, $\quad u(0) = 0,\ u'(1) = -u(1) \quad (\tan\mu + \mu \neq 0)$
8. $u'' + \mu^2 u = f$ on $(0, 1)$, $\quad u(0) = u(1),\ u'(0) = u'(1) \quad$ ($\mu/2\pi$ not an integer) (Hint: Work directly from the formulas (10.5) and the boundary conditions. $e^{i\mu x}$ and $e^{-i\mu x}$ are probably easier to work with than $\cos\mu x$ and $\sin\mu x$.)
9. $u'' + \mu^2 u = f$ on $(0, \infty)$, $\quad u(0) = u(\infty) = 0 \quad (\operatorname{Im}\mu > 0)$
10. $u'' + \mu^2 u = f$ on $(0, \infty)$, $\quad u'(0) = u(\infty) = 0 \quad (\operatorname{Im}\mu > 0)$
11. $u'' + u' - 2u = f$ on $(-\infty, \infty)$, $\quad u(\pm\infty) = 0$
12. Verify directly that the function u defined by (10.16) satisfies (10.13).

10.2 Green's functions for partial differential operators

Our first object in this section is to find Green's functions for the heat, wave, and Laplace operators on all of $\mathbf{R}^n$ by means of the Fourier transform. In the absence of boundary conditions, the Green's function is far from unique, as one can add on any solution of the corresponding homogeneous equation. However, in each case physical considerations will single out one particular Green's function, which will also emerge naturally from the Fourier analysis.

At the outset, let us make the following general observation. If L is any differential operator with constant coefficients (such as the heat, wave, or Laplace operator), then L commutes with translations: If $g(\mathbf{x}) = f(\mathbf{x}-\mathbf{x}_0)$ then $(Lg)(\mathbf{x}) = (Lf)(\mathbf{x}-\mathbf{x}_0)$. In particular, if g satisfies the equation $L(g) = \delta(\mathbf{x})$, then $G(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}-\mathbf{y})$ satisfies $L_{\mathbf{x}}(G) = \delta(\mathbf{x}-\mathbf{y})$, i.e., G is a Green's function for the operator L on all of $\mathbf{R}^n$. The Green's functions we shall construct are all of this translation-covariant sort.

Let us begin with the heat equation. We wish to solve

$$g_t - k\nabla^2 g = \delta(\mathbf{x}, t) = \delta(\mathbf{x})\delta(t) \qquad (\mathbf{x} \in \mathbf{R}^n,\ t \in \mathbf{R}). \tag{10.17}$$

We shall think of g as representing the heat distribution in $\mathbf{R}^n$ in response to the injection of a unit quantity of heat at the origin at time $t = 0$. In particular, since nothing happens before time $t = 0$, we shall require that $g(\mathbf{x}, t) = 0$ for $t < 0$.

Taking the Fourier transform of (10.17) with respect to $\mathbf{x}$, we obtain

$$\widehat{g}_t + k|\boldsymbol{\xi}|^2\widehat{g} = \delta(t), \qquad \widehat{g}(\boldsymbol{\xi}, t) = 0 \text{ for } t < 0.$$

The methods of §10.1 quickly give the solution: For each $\boldsymbol{\xi}$, $\widehat{g}(\boldsymbol{\xi}, t)$ must vanish for $t < 0$, satisfy the homogeneous equation $\widehat{g}_t + k|\boldsymbol{\xi}|^2 g = 0$ for $t > 0$, and jump from 0 to 1 at $t = 0$. The unique function satisfying all these conditions is $\widehat{g}(\boldsymbol{\xi}, t) = H(t)e^{-kt|\boldsymbol{\xi}|^2}$, where H is the Heaviside function, and its inverse Fourier transform is

$$g(\mathbf{x}, t) = \frac{H(t)}{(4\pi kt)^{n/2}} e^{-|\mathbf{x}|^2/4kt}. \tag{10.18}$$

The corresponding solution of the general inhomogeneous equation $u_t - k\nabla^2 u = f$ is

$$u(\mathbf{x}, t) = (f * g)(\mathbf{x}, t) = \iint f(\mathbf{y}, s) g(\mathbf{x} - \mathbf{y},\, t - s)\, d\mathbf{y}\, ds.$$

Observe that this is just like the solution of the initial value problem $u_t = k\nabla^2 u$, $u(\mathbf{x}, 0) = f(\mathbf{x})$ obtained in §7.3 except that now the convolution is in both $\mathbf{x}$ and t.

Incidentally, one could also take the Fourier transform of (10.17) in both $\mathbf{x}$ and t. Denoting the full Fourier transform of g by $\mathscr{F}_{\mathbf{x},t} g$ to distinguish it from the Fourier transform in $\mathbf{x}$ denoted by $\widehat{g}$ above, we have

$$i\tau(\mathscr{F}_{\mathbf{x},t} g)(\boldsymbol{\xi}, \tau) + k|\boldsymbol{\xi}|^2(\mathscr{F}_{\mathbf{x},t} g)(\boldsymbol{\xi}, \tau) = 1, \quad \text{or} \quad (\mathscr{F}_{\mathbf{x},t} g)(\boldsymbol{\xi}, \tau) = \frac{1}{k|\boldsymbol{\xi}|^2 + i\tau}.$$

Inversion of the Fourier transform again yields the result (10.18); see Exercise 1.

Let us now turn to the Green's function for the wave equation:

$$G_{tt} - c^2\nabla^2 G = \delta(\mathbf{x}, t) = \delta(\mathbf{x})\delta(t). \tag{10.19}$$

Again, we think of G as representing the wave motion resulting from a sharp impulse at the origin at time $t = 0$, and accordingly we require that $G(\mathbf{x}, t) = 0$ for $t < 0$. Fourier transformation in $\mathbf{x}$ yields

$$\widehat{G}_{tt} + c^2|\boldsymbol{\xi}|^2\widehat{G} = \delta(t), \qquad \widehat{G}(\boldsymbol{\xi}, t) = 0 \text{ for } t < 0.$$

Thus the function $\gamma(t) = \widehat{G}(\boldsymbol{\xi}, t)$ must satisfy $\gamma'' + c^2|\boldsymbol{\xi}|^2\gamma = 0$ for $t > 0$, together with the initial conditions (10.5), namely, $\gamma(0) = 0$ and $\gamma'(0) = 1$. The solution is

$$\widehat{G}(\boldsymbol{\xi}, t) = H(t)\frac{\sin ct|\boldsymbol{\xi}|}{c|\boldsymbol{\xi}|}. \tag{10.20}$$

Apart from the factor $H(t)$, which does not enter into the Fourier inversion, G is the distribution that gives the solution of the initial value problem for the wave equation. We studied it in §9.5, and from the discussion there we can read off the formula for G in space dimensions 1, 2, and 3:

$$n = 1: \qquad G(x,t) = \begin{cases} 1 & \text{if } 0 \le |x| < ct, \\ 0 & \text{otherwise;} \end{cases}$$

$$n = 2: \qquad G(\mathbf{x},t) = \begin{cases} (2\pi)^{-1}\Big((ct)^2 - |\mathbf{x}|^2\Big)^{-1/2} & \text{if } 0 \le |\mathbf{x}| < ct, \\ 0 & \text{otherwise;} \end{cases}$$

$$n = 3: \qquad G(\mathbf{x},t) = \begin{cases} (4\pi c^2 t)^{-1}\sigma_{ct}(\mathbf{x}) & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\sigma_r(\mathbf{x})$ is surface measure on the sphere of radius r about the origin in $\mathbf{R}^3$; i.e., if $\phi \in C_0^{(\infty)}(\mathbf{R}^3)$, $\sigma_r[\phi]$ is the integral of ϕ over the sphere $|\mathbf{x}| = r$.

Here also it is instructive to consider the full Fourier transform $\mathscr{F}_{\mathbf{x},t}G$ of G in both $\mathbf{x}$ and t. Formally, applying $\mathscr{F}_{\mathbf{x},t}$ to (10.19) gives

$$(-\tau_2 + c^2|\boldsymbol{\xi}|^2)(\mathscr{F}_{\mathbf{x},t}G)(\boldsymbol{\xi},\tau) = 1, \quad \text{or} \quad (\mathscr{F}_{\mathbf{x},t}G)(\boldsymbol{\xi},\tau) = \frac{1}{c^2|\boldsymbol{\xi}|^2 - \tau^2}.$$

However, $1/\Big(c^2|\boldsymbol{\xi}|^2 - \tau^2\Big)$ is not a locally integrable function, so one must take care in interpreting it as a distribution. In fact, $\mathscr{F}_{\mathbf{x},t}G$ is the Fourier transform in t of the function $\widehat{G}$ given by (10.20), which we can calculate as follows. For $\epsilon > 0$, let

$$\widehat{G}_\epsilon(\boldsymbol{\xi},t) = e^{-\epsilon t}\widehat{G}(\boldsymbol{\xi},t) = H(t)\frac{e^{it(c|\boldsymbol{\xi}|+i\epsilon)} - e^{it(-c|\boldsymbol{\xi}|+i\epsilon)}}{2ic|\boldsymbol{\xi}|}.$$

$\widehat{G}_\epsilon(\boldsymbol{\xi},t)$ is an integrable function of t, and its Fourier transform $\Gamma_\epsilon(\boldsymbol{\xi},\tau)$ in t is easily calculated:

$$\begin{aligned}\Gamma_\epsilon(\boldsymbol{\xi},\tau) = \int_0^\infty e^{-i\tau t}\widehat{G}_\epsilon(\boldsymbol{\xi},t)\,dt &= \frac{1}{2c|\boldsymbol{\xi}|(c|\boldsymbol{\xi}| + i\epsilon - \tau)} - \frac{1}{2c|\boldsymbol{\xi}|(-c|\boldsymbol{\xi}| + i\epsilon - \tau)} \\ &= \frac{1}{c^2|\boldsymbol{\xi}|^2 - (\tau - i\epsilon)^2}.\end{aligned}$$

Clearly $\widehat{G}_\epsilon \to \widehat{G}$ temperately as $\epsilon \to 0$, so $\mathscr{F}_{\mathbf{x},t}G$ is the distribution limit as ϵ decreases to 0 of the function Γ_ϵ.

In the physics literature, the Green's function G defined by (10.20) is called the *retarded Green's function,* because the convolution integral

$$u(\mathbf{x},t) = (f * G)(\mathbf{x},t) = \int_{\mathbf{R}^n}\int_0^\infty f(\mathbf{x}-\mathbf{y},\, t-s)G(\mathbf{y},s)\,ds\,d\mathbf{y}$$

that solves $u_{tt} - c^2\nabla^2 u = f$ expresses u at time t in terms of the values of f at earlier times $t - s$. Although this is the most natural Green's function for most purposes in classical physics, other choices are possible. One is the *advanced Green's function*

$$G_{\text{adv}}(\mathbf{x},t) = G(\mathbf{x},-t),$$

which vanishes for $t > 0$ rather than $t < 0$ and gives a solution of $u_{tt} - \nabla^2 u = f$ whose values at time t are determined by the values of f at later times. Its full Fourier transform is

$$\mathscr{F}_{\mathbf{x},t} G_{\text{adv}} = \lim_{\epsilon \to 0,\ \epsilon > 0} h_\epsilon, \qquad h_\epsilon(\boldsymbol{\xi}, \tau) = \frac{1}{c^2|\boldsymbol{\xi}|^2 - (\tau + i\epsilon)^2}.$$

There is yet another Green's function, the *causal Green's function* or *Feynman propagator* G_F, that is of fundamental importance in quantum field theory. It is most easily defined in terms of the Fourier transform in both $\mathbf{x}$ and t:

$$\mathscr{F}_{\mathbf{x},t} G_F = \lim_{\epsilon \to 0,\ \epsilon > 0} \Phi_\epsilon, \qquad \Phi_\epsilon(\boldsymbol{\xi}, t) = \frac{1}{c^2|\boldsymbol{\xi}|^2 - t^2 - i\epsilon}.$$

An explanation of the significance of G_F is beyond the scope of this discussion (see Bogoliubov-Shirkov [9]). The point we wish to make, however, is that there are many Green's functions for the wave operator, whose Fourier transforms are distributions that agree with the function $(c^2|\boldsymbol{\xi}|^2 - \tau^2)^{-1}$ on the set where $|\tau| \neq c|\boldsymbol{\xi}|$. The difference between any two such Green's functions is a solution of the homogeneous wave equation, and its Fourier transform is a distribution supported on the cone $|\tau| = c|\boldsymbol{\xi}|$.

Finally, we turn to the Laplace equation in $\mathbf{R}^n$, whose Green's function satisfies

$$\nabla^2 N(\mathbf{x}) = \delta(\mathbf{x}). \tag{10.21}$$

We denote the Green's function by N in honor of Isaac Newton, because when $n = 3$ it represents the gravitational potential generated by a particle of mass $1/4\pi$ located at the origin. (It is also the electrostatic potential generated by a particle of charge $-1/4\pi$ located at the origin.)

In dimension $n = 1$ the equation (10.21) is an ordinary differential equation; it was analyzed in §10.1. The case $n = 2$ turns out to be somewhat special, so for the time being we shall assume $n \geq 3$. We shall need the following basic facts about integration in spherical coordinates in $\mathbf{R}^n$. (See Appendix 4. The reader is free to think only of the familiar case $n = 3$, but we do the general case in order to display the pattern; it is really no more difficult.) If $f\big(|\mathbf{x}|\big)$ is a radial function on $\mathbf{R}^n$, then

$$\int f\big(|\mathbf{x}|\big)\, d\mathbf{x} = \omega_n \int_0^\infty f(r) r^{n-1}\, dr, \tag{10.22}$$

where ω_n is the $(n-1)$-dimensional measure of the unit sphere $|\mathbf{x}| = 1$. (Of course $\omega_3 = 4\pi$; see Appendix 4 for the calculation of ω_n in general.) In particular,

$$\int_{a<|\mathbf{x}|<b} |\mathbf{x}|^{-c}\, d\mathbf{x} = \omega_n \int_a^b r^{n-c-1}\, dr,$$

which has a finite limit as $a \to 0$ when $c < n$ and a finite limit as $b \to \infty$ when $c > n$.

Now, taking the Fourier transform of (10.21), we obtain

$$-|\boldsymbol{\xi}|^2\widehat{N}(\boldsymbol{\xi}) = 1, \quad \text{or} \quad \widehat{N}(\boldsymbol{\xi}) = -|\boldsymbol{\xi}|^{-2}.$$

According to the preceding remarks, when $n \geq 3$ the singularity of $\widehat{N}$ at the origin is integrable, so $\widehat{N}$ is a tempered distribution; however, it is not an L^1 function because it is not integrable at infinity. Hence, calculating its inverse Fourier transform requires a bit of thought. We begin by noting that $\widehat{N}$ is radial and homogeneous of degree -2; hence N must be radial and homogeneous of degree $2-n$. (Cf. Exercise 1 of §9.1 and equation (9.30).) It is therefore reasonable to guess that $N(\mathbf{x})$ should be a constant multiple of $|\mathbf{x}|^{2-n}$. (Again, the singularity of this function at the origin is integrable, so it defines a tempered distribution.)

This guess turns out to be correct, and its validity can be established by either of two methods: (i) calculating the inverse Fourier transform of $-|\boldsymbol{\xi}|^{-2}$ explicitly, or (ii) showing directly that $\nabla^2|\mathbf{x}|^{2-n}$ is a constant times $\delta(\mathbf{x})$. We shall proceed by the second method here; the first one can be carried out by a clever use of Gaussian integrals that is explained in Exercise 4.

To find the distribution derivative $\nabla^2|\mathbf{x}|^{2-n}$, we shall approximate $|\mathbf{x}|^{2-n}$ by smooth functions and calculate their Laplacians. Namely, for $\epsilon > 0$ we set

$$F_\epsilon(\mathbf{x}) = \left(|\mathbf{x}|^2 + \epsilon^2\right)^{(2-n)/2}.$$

Since

$$\frac{\partial|\mathbf{x}|^2}{\partial x_j} = \frac{\partial}{\partial x_j}(x_1^2 + \cdots + x_n^2) = 2x_j,$$

we have

$$\frac{\partial F_\epsilon}{\partial x_j} = (2-n)x_j\left(|\mathbf{x}|^2 + \epsilon^2\right)^{-n/2} \tag{10.23}$$

and

$$\frac{\partial^2 F_\epsilon}{\partial x_j^2} = (2-n)\left(|\mathbf{x}|^2 + \epsilon^2\right)^{-n/2} - n(2-n)x_j^2\left(|\mathbf{x}|^2 + \epsilon^2\right)^{-(n+2)/2},$$

and hence

$$\begin{aligned}\nabla^2 F_\epsilon = \sum_1^n \frac{\partial^2 F_\epsilon}{\partial x_j^2} &= n(2-n)\left(|\mathbf{x}|^2 + \epsilon^2\right)^{-n/2} - n(2-n)|\mathbf{x}|^2\left(|\mathbf{x}|^2 + \epsilon^2\right)^{-(n+2)/2} \\ &= n(2-n)\epsilon^2\left(|\mathbf{x}|^2 + \epsilon^2\right)^{-(n+2)/2}.\end{aligned}$$

We observe that

$$\nabla^2 F_\epsilon(\mathbf{x}) = \epsilon^{-n}\Phi(\epsilon^{-1}\mathbf{x}) \quad \text{where } \Phi(\mathbf{x}) = n(2-n)\left(|\mathbf{x}|^2 + 1\right)^{-(n+2)/2},$$

which has the standard form of an approximate identity except that $\int \Phi(\mathbf{x})\,d\mathbf{x}$ is not equal to 1. However, this can be remedied by dividing Φ by the constant

$C = \int \Phi(\mathbf{x})\,d\mathbf{x}$. We then have (by Theorem 9.2 of §9.2) that $C^{-1}\nabla^2 F_\epsilon$ converges weakly to δ as $\epsilon \to 0$. Since $F_\epsilon(\mathbf{x})$ converges weakly to $|\mathbf{x}|^{2-n}$ by the dominated convergence theorem, it follows that $\nabla^2(C^{-1}|\mathbf{x}|^{2-n}) = \delta(\mathbf{x})$.

It remains to evaluate the constant C. By (10.22), we have

$$C = \int \Phi(\mathbf{x})\,d\mathbf{x} = n(2-n)\omega_n \int \frac{r^{n-1}}{(r^2+1)^{(n+2)/2}}\,dr.$$

This integral can be evaluated by the substitution

$$s = \frac{r^2}{r^2+1} = 1 - \frac{1}{r^2+1}, \qquad ds = \frac{2r\,dr}{(r^2+1)^2}$$

which gives

$$\frac{r^{n-1}}{(r^2+1)^{(n+2)/2}}\,dr = \frac{r^{n-2}}{2(r^2+1)^{(n-2)/2}}\frac{2r\,dr}{(r^2+1)^2} = \frac{s^{(n-2)/2}}{2}\,ds,$$

so that

$$C = (2-n)\omega_n \frac{n}{2}\int_0^1 s^{(n-2)/2}\,ds = (2-n)\omega_n s^{n/2}\Big|_0^1 = (2-n)\omega_n.$$

Thus, if we take

$$N(\mathbf{x}) = \frac{1}{(2-n)\omega_n |\mathbf{x}|^{n-2}},$$

we have $\nabla^2 N = \delta$. In particular, for $n = 3$, $N(\mathbf{x}) = -1/4\pi|\mathbf{x}|$ is (except for a factor of $\pm 4\pi$) the familiar Newtonian or Coulomb potential.

This procedure breaks down when $n = 2$: In this case $|\mathbf{x}|^{2-n}$ is the constant 1, and its Laplacian vanishes identically. However, starting from equation (10.23), the preceding calculations are valid even for $n = 2$ if we omit the factor $2 - n$. Hence, if we can find a function $F_\epsilon(\mathbf{x})$ such that

$$\frac{\partial F_\epsilon}{\partial x_j} = \frac{x_j}{|\mathbf{x}|^2 + \epsilon^2}, \tag{10.24}$$

we will have $\nabla^2 F_\epsilon \to C\delta$ where

$$C = 2\int_{\mathbf{R}^2} \frac{d\mathbf{x}}{(|\mathbf{x}|^2+1)^2} = 2\pi \int_0^\infty \frac{2r\,dr}{(r^2+1)^2} = -\frac{2\pi}{r^2+1}\Big|_0^\infty = 2\pi.$$

But (10.24) is easily solved for F_ϵ by integration:

$$F_\epsilon(\mathbf{x}) = \frac{1}{2}\log\left(|\mathbf{x}|^2 + \epsilon^2\right) = \log\left(|\mathbf{x}|^2 + \epsilon^2\right)^{1/2}.$$

As $\epsilon \to 0$, $F_\epsilon(\mathbf{x}) \to \log|\mathbf{x}|$ weakly, so the Green's function is $(2\pi)^{-1}\log|\mathbf{x}|$.

We sum up the results in a theorem.

Theorem 10.2. *Let $N(\mathbf{x})$ be defined on $\mathbf{R}^n$ by*

$$N(\mathbf{x}) = \frac{1}{(2-n)\omega_n |\mathbf{x}|^{n-2}} \quad (n \geq 3), \qquad N(\mathbf{x}) = \frac{1}{2\pi} \log|\mathbf{x}| \quad (n = 2),$$

where ω_n is the $(n-1)$-dimensional measure of the unit sphere $|\mathbf{x}| = 1$ in $\mathbf{R}^n$. Then $\nabla^2 N = \delta$.

Let us return for a moment to the Fourier analysis with which we began. For $n \geq 3$, it is indeed true that function N of Theorem 10.2 is the inverse Fourier transform of $-|\boldsymbol{\xi}|^{-2}$: see Exercise 4. When $n = 2$, $|\boldsymbol{\xi}|^{-2}$ is not locally integrable near the origin, and in order to interpret it as a distribution one must define a "finite part" of the divergent integral $\int \phi(\boldsymbol{\xi})|\boldsymbol{\xi}|^{-2}\,d\boldsymbol{\xi}$ as in §9.3. If this is done suitably, the distribution thus defined is the Fourier transform of $(2\pi)^{-1}\log|\mathbf{x}|$; see Exercise 5.

Green's functions with boundary conditions

We begin with a few general remarks about finding Green's functions for the heat, wave, or Laplace operators on a region D subject to homogeneous boundary conditions on the boundary of D. In the first place, such a Green's function will no longer have the form $G(\mathbf{x},\mathbf{y}) = g(\mathbf{x}-\mathbf{y})$, because the boundary conditions are not translation-invariant. However, if we denote by $g_0(\mathbf{x}-\mathbf{y})$ the free-space Green's function constructed above and by $G(\mathbf{x},\mathbf{y})$ the Green's function subject to the boundary conditions $B\Big[G(\mathbf{x},\mathbf{y})\Big] = 0$, the difference $h(\mathbf{x},\mathbf{y}) = g_0(\mathbf{x}-\mathbf{y}) - G(\mathbf{x},\mathbf{y})$ will satisfy the homogeneous differential equation and the inhomogeneous boundary conditions $B\Big[h(\mathbf{x},\mathbf{y})\Big] = B\Big[g_0(\mathbf{x}-\mathbf{y})\Big]$. In this way, solving the inhomogeneous equation with homogeneous boundary conditions can be reduced to solving the homogeneous equation with inhomogeneous boundary conditions. (This reduction may or may not be useful in practice.)

Situations where the Green's function with boundary conditions can be written down explicitly in closed form are extremely rare. One such is the Dirichlet problem for the half-space $\{\mathbf{x} : x_n > 0\}$:

$$\nabla^2_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \text{ for } x_n, y_n > 0, \qquad G(\mathbf{x},\mathbf{y}) = 0 \text{ for } x_n = 0,\ y_n > 0.$$

The solution is simply

$$G(\mathbf{x},\mathbf{y}) = N(\mathbf{x}-\mathbf{y}) - N(\mathbf{x}-\mathbf{y}'), \qquad \mathbf{y}' = (y_1, \ldots, y_{n-1}, -y_n),$$

where N is the free-space Green's function of Theorem 10.2. To see that this works, observe that

$$\nabla^2_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{x}-\mathbf{y}'),$$

and this agrees with $\delta(\mathbf{x}-\mathbf{y})$ except at the point $\mathbf{x} = \mathbf{y}'$ which (for $y_n > 0$) is outside the region $x_n > 0$. Moreover, $N(\mathbf{x}-\mathbf{y})$ depends only on $|\mathbf{x}-\mathbf{y}|$, and this is unchanged if y_n is replaced by $-y_n$ when $x_n = 0$; hence $G(\mathbf{x},\mathbf{y}) = 0$ when $x_n = 0$.

The physical interpretation is clear: think of $N(\mathbf{x}-\mathbf{y})$ as the electrostatic potential due to a charge at $\mathbf{y}$. To make the potential zero on the plane $x_n = 0$, put an equal and opposite charge at the point $\mathbf{y}'$ on the other side of that plane. This is an instance of Kelvin's "method of images." A similar construction works for the Dirichlet problem for a ball; see Exercises 6 and 7.

In Chapter 4 we discussed a method for solving inhomogeneous equations with homogeneous boundary conditions by means of Fourier series or other Sturm-Liouville expansions. This technique can also be used to write down the Green's function for such problems in terms of infinite series. For example, let us consider the inhomogeneous heat equation on an interval $[0, l]$ with initial temperature zero and both ends held at temperature zero:

$$u_t - ku_{xx} = F(x,t), \qquad u(x,0) = u(0,t) = u(l,t) = 0. \tag{10.25}$$

We solved this problem in §4.2 by expanding everything in a Fourier sine series, obtaining

$$u(x,t) = \sum_1^\infty \sin\frac{n\pi x}{l} \int_0^t \beta_n(s) e^{-n^2\pi^2 k(t-s)/l^2}\, ds$$

where

$$F(x,t) = \sum_1^\infty \beta_n(t) \sin n\pi x/l.$$

(See equations (4.16) and (4.17).) Since the Fourier coefficients $\beta_n(t)$ of F are given by

$$\beta_n(t) = \frac{2}{l}\int_0^l F(y,t)\sin\frac{n\pi y}{l}\,dy,$$

this can be rewritten as

$$\begin{aligned} u(x,t) &= \frac{2}{l}\int_0^t\int_0^l \sum_1^\infty e^{-n^2\pi^2 k(t-s)/l^2} \sin\frac{n\pi x}{l}\sin\frac{n\pi y}{l} F(y,s)\,dy\,ds \\ &= \int_0^\infty\int_0^l G(x,t;y,s)F(y,s)\,dy\,ds \end{aligned}$$

where

$$G(x,t;y,s) = \frac{2}{l}\sum_1^\infty H(t-s)e^{-n^2\pi^2k(t-s)/l^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi y}{l}. \tag{10.26}$$

This G is therefore the Green's function.

The formula (10.26) can also be derived by taking $F(x,t)$ in (10.25) to be the delta function $\delta(x-y)\delta(t-s)$ and solving according to the method of §4.2. Namely, the Fourier coefficients $\beta_n(t)$ of this delta function are

$$\beta_n(t) = \frac{2}{l}\int_0^l \delta(x-y)\delta(t-s)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\delta(t-s)\sin\frac{n\pi y}{l},$$

so if we write $u(x,t) = \sum_1^\infty b_n(t)\sin(n\pi x/l)$ in (10.25), the coefficients $b_n(t)$ must satisfy

$$b_n'(t) + \frac{n^2\pi^2 k}{l^2} b_n(t) = \frac{2}{l}\delta(t-s)\sin\frac{n\pi y}{l}, \qquad b_n(0) = 0.$$

The solution to this, for $t > 0$, is

$$b_n(t) = \frac{2}{l} H(t-s) e^{-n^2\pi^2 k(t-s)/l^2} \sin\frac{n\pi y}{l},$$

which gives (10.26).

The same ideas apply to other boundary value problems. In summary, Sturm-Liouville expansions can be used either to solve an inhomogeneous differential equation with homogeneous boundary conditions directly or to compute the Green's function, which in turn yields the solution of the equation. The preferability of one formulation over the other is a matter of choice. The advantage of the Green's function is that it expresses the solution of the general equation $L(u) = f$ in a simple way, namely, $u(\mathbf{x}) = \int G(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$, in terms of the solution of the specific equation $L(u) = \delta(\mathbf{x}-\mathbf{y})$.

For more information on Green's functions for boundary value problems, see Stakgold [48].

EXERCISES

1. Show that the Fourier transform in both $\mathbf{x}$ and t of the function $g(\mathbf{x},t)$ defined by (10.18) is $(k|\boldsymbol{\xi}|^2 + i\tau)^{-1}$. (Hint: Take the Fourier transform first in $\mathbf{x}$, then in t.)
2. Let L be a partial differential operator in the variables $x_1, \dots x_n$ with constant coefficients. Suppose $\{K_t\}_{t>0}$ is a family of tempered distributions depending smoothly on t such that

$$\partial_t^m K_t = L(K_t) \text{ for } t > 0, \quad \lim_{t\to 0} \partial_t^j K_t(\mathbf{x}) = \begin{cases} 0 & \text{for } j \le m-2, \\ \delta(\mathbf{x}) & \text{for } j = m-1, \end{cases}$$

where $\partial_t = (\partial/\partial t)$ and the limits are in the sense of temperate convergence.

a. Show that if f is a Schwartz function, $u(x,t) = f * K_t(x)$ solves the initial value problem

$$\partial_t^m u = L(u), \quad \partial_t^j u(\mathbf{x},0) \text{ for } j \le m-2, \quad \partial_t^{m-1} u(\mathbf{x},0) = f(\mathbf{x}).$$

b. If $G(\mathbf{x},t) = H(t)K_t(\mathbf{x})$, show that $(\partial_t^m - L)(G) = \delta(\mathbf{x})\delta(t)$.

(Note that the fundamental solutions for the heat and wave operators discussed in the text are examples of this construction.)

3. Show that the formula $N(\mathbf{x}) = |\mathbf{x}|^{2-n}/(2-n)\omega_n$ of Theorem 10.2 also gives a Green's function for the Laplacian in dimension $n = 1$ if we take $\omega_1 = 2$. (This is as it should be: The "unit sphere" in $\mathbf{R}$ consists of the two points ± 1, and the zero-dimensional measure of a set is the number of points in the set.)

4. Applying the characterization of the Fourier transform for tempered distributions, $F[\widehat{\phi}] = \widehat{f}[\phi]$, to $F(\mathbf{x}) = e^{-t|\mathbf{x}|^2/2}$, one sees that for any Schwartz function ϕ on $\mathbf{R}^n$,

$$\int e^{-t|\mathbf{x}|^2/2}\widehat{\phi}(\mathbf{x})\,d\mathbf{x} = \left(\frac{2\pi}{t}\right)^{n/2}\int e^{-|\boldsymbol{\xi}|^2/2t}\phi(\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

Suppose $0 < \alpha < n$. Multiply both sides of this equation by $t^{[(n-\alpha)/2]-1}$ and integrate in t from 0 to ∞. Reverse the order of integration on both sides, and substitute $t \to t^{-1}$ on the right side, to show that

$$\int |\mathbf{x}|^{\alpha-n}\widehat{\phi}(\mathbf{x})\,d\mathbf{x} = \frac{2^{\alpha}\pi^{n/2}\Gamma(\frac{1}{2}\alpha)}{\Gamma\left(\frac{1}{2}(n-\alpha)\right)}\int |\boldsymbol{\xi}|^{-\alpha}\phi(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

or in other words,

$$\mathscr{F}\left[|\mathbf{x}|^{\alpha-n}\right] = \frac{2^{\alpha}\pi^{n/2}\Gamma(\frac{1}{2}\alpha)}{\Gamma\left(\frac{1}{2}(n-\alpha)\right)}|\boldsymbol{\xi}|^{-\alpha} \qquad (0 < \alpha < n).$$

In particular, if $n \geq 3$ the inverse Fourier transform of $-|\boldsymbol{\xi}|^{-2}$ is

$$-\frac{\Gamma(\frac{1}{2}n-1)}{4\pi^{n/2}|\mathbf{x}|^{n-2}} = \frac{\Gamma(\frac{1}{2}n)}{(2-n)2\pi^{n/2}|\mathbf{x}|^{n-2}} = \frac{1}{(2-n)\omega_n|\mathbf{x}|^{n-2}}.$$

5. On $\mathbf{R}^2$, let

$$f_{\alpha}(\mathbf{x}) = -\frac{\Gamma(1-\frac{1}{2}\alpha)}{2^{\alpha}\pi\Gamma(\frac{1}{2}\alpha)}\left(|\mathbf{x}|^{\alpha-2}-1\right) \qquad (0 < \alpha < 2).$$

a. Deduce from the result of Exercise 4 that

$$\widehat{f}_{\alpha}(\boldsymbol{\xi}) = -|\boldsymbol{\xi}|^{-\alpha} + \frac{\pi\Gamma(1-\frac{1}{2}\alpha)}{2^{\alpha-2}\Gamma(\frac{1}{2}\alpha)}\delta(\boldsymbol{\xi}).$$

b. Show that $f_{\alpha}(\mathbf{x}) \to (2\pi)^{-1}\log|\mathbf{x}|$ temperately as $\alpha \to 2-$. (Hint: Use the fact that $\Gamma(1-\frac{1}{2}\alpha) = 2\Gamma(2-\frac{1}{2}\alpha)/(2-\alpha)$.)

c. Show (using the hint of part (b)) that

$$\lim_{\alpha\to 2-}\left[\frac{\pi\Gamma(1-\frac{1}{2}\alpha)}{2^{\alpha-2}\Gamma(\frac{1}{2}\alpha)} - \int_{|\boldsymbol{\xi}|<1}\frac{d\boldsymbol{\xi}}{|\boldsymbol{\xi}|^{\alpha}}\right] = 2\pi\left[\log 2 + \Gamma'(1)\right].$$

d. Deduce that the Fourier transform F of $(2\pi)^{-1}\log|\mathbf{x}|$ is given by

$$F[\phi] = -\int_{|\boldsymbol{\xi}|<1}\frac{\phi(\boldsymbol{\xi})-\phi(\mathbf{0})}{|\boldsymbol{\xi}|^2}\,d\boldsymbol{\xi} - \int_{|\boldsymbol{\xi}|>1}\frac{\phi(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}\,d\boldsymbol{\xi} + 2\pi\left[\log 2 + \Gamma'(1)\right]\phi(0).$$

6. Let $B = \{\mathbf{x} : |\mathbf{x}| < 1\}$ be the unit ball and $S = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the unit sphere in $\mathbf{R}^n$, $n \geq 3$.
   a. Show that if $\mathbf{x} \in S$ and $\mathbf{y} \neq \mathbf{0}$, then $|\mathbf{x} - \mathbf{y}| = \Big| |\mathbf{y}|\mathbf{x} - |\mathbf{y}|^{-1}\mathbf{y} \Big|$. (Hint: $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$.)
   b. Let N be as in Theorem 10.2. Show that

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}) &= N(\mathbf{x} - \mathbf{y}) - |\mathbf{y}|^{2-n} N\Big(\mathbf{x} - |\mathbf{y}|^{-2}|\mathbf{y}\Big) \\ &= \frac{1}{(2-n)\omega_n}\left[|\mathbf{x} - \mathbf{y}|^{2-n} - \Big| |\mathbf{y}|\mathbf{x} - |\mathbf{y}|^{-1}\mathbf{y} \Big|^{2-n}\right] \end{aligned}$$

   is the Green's function for the Dirichlet problem on B, that is,

$$\nabla_{\mathbf{x}}^2 G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in B), \qquad G(\mathbf{x}, \mathbf{y}) = 0 \quad (\mathbf{x} \in S,\ \mathbf{y} \in B).$$

7. In this problem we identify the plane $\mathbf{R}^2$ with $\mathbf{C}$. Let $D = \{z : |z| < 1\}$ be the unit disc and $C = \{z : |z| = 1\}$ be the unit circle.
   a. If $|w| < 1$, show that the fractional linear map $f(z) = (z - w)/(\overline{w}z - 1)$ maps D and C onto themselves and maps w to 0.
   b. By Theorem 10.2, $N(z) = (2\pi)^{-1} \log|z|$ satisfies $\nabla^2 N = \delta$, and $N(z) = 0$ for $|z| = 1$. Part (a) then suggests that the Green's function for the Dirichlet problem on D, namely, the solution of

$$\nabla_z^2 G(z, w) = \delta(z - w) \quad (z, w \in D), \qquad G(z, w) = 0 \quad (z \in C,\ w \in D),$$

   should be

$$G(z, w) = \frac{1}{2\pi} \log\left|\frac{z - w}{\overline{w}z - 1}\right| = \frac{1}{2\pi}\Big[\log|z - w| - \log|z - \overline{w}^{-1}| - \log|\overline{w}|\Big].$$

   Verify this.

8. Find the Green's function for the vibrating string problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x, t) \quad \text{for } 0 < x < l,\ t > 0, \\ u(x, 0) = u_t(x, 0) &= u(0, t) = u(l, t) = 0 \end{aligned}$$

   in the form of a Fourier series.

9. Find the Green's function for the Dirichlet problem on the unit square,

$$\nabla^2 u = f, \quad u(x_1, 0) = u(x_1, 1) = u(0, x_2) = u(1, x_2) = 0,$$

   in the form of a double Fourier series. (We denote the coordinates by (x_1, x_2) rather than (x, y) so that (y_1, y_2) can be the point where the Green's function has its singularity.)

10.3 Green's functions and regular Sturm-Liouville problems

In this section we study the Green's function for the inhomogeneous Sturm-Liouville problem

$$u'' + p(x)u + \lambda u = f \text{ for } a < x < b, \tag{10.27}$$

$$u'(a) = \alpha u(a), \qquad u'(b) = \beta u(b), \tag{10.28}$$

and use it to derive the completeness of the eigenfunctions for the associated homogeneous problem. Here $p(x)$ is a real-valued continuous function on $[a, b]$ and α and β are real constants.

Problem (10.27–28) is of a rather special type, and we restrict our attention to it in order to simplify the exposition. However, the ideas we are about to present work in greater generality:

(i) The general Sturm-Liouville equation

$$[r(x)u']' + p(x)u + \lambda w(x)u = f$$

can be reduced to (10.27) by a change of independent and dependent variables known as the *reduction to Liouville normal form.* The exact formulas are rather complicated and of little practical importance in general, so we shall omit them; the interested reader may find them in Birkhoff-Rota [6]. The main point is that the results we prove for (10.27) can be extended to the more general case.

(ii) The boundary condition $u'(a) = \alpha u(a)$ can be replaced by the condition $u(a) = 0$ (which is the limiting case of the former condition as $\alpha \to \infty$), and likewise the condition $u'(b) = \beta u(b)$ can be replaced by $u(b) = 0$. The arguments and calculations are almost the same, although some of the formulas turn out a little differently. The reader is invited to work out these cases in Exercises 4–8.

(iii) The same techniques can also be applied to other (nonseparated) self-adjoint boundary conditions. See, for example, Exercise 3.

(iv) The method of Green's functions also yields expansion theorems for singular Sturm-Liouville problems. We shall present some examples in §10.4.

The Green's function for (10.27-28) was calculated in §10.1. We review the results here, but with slightly different notation to take account of the dependence of the relevant functions on the complex parameter λ. Let $v_a(x;\lambda)$ and $v_b(x;\lambda)$ be the solutions of the initial value problems

$$\begin{aligned} &v_a'' + pv_a + \lambda v_a = 0, \qquad v_a(a;\lambda) = 1, \quad v_a'(a;\lambda) = \alpha, \\ &v_b'' + pv_b + \lambda v_b = 0, \qquad v_b(b;\lambda) = 1, \quad v_b'(b;\lambda) = \beta. \end{aligned} \tag{10.29}$$

(Here and below, primes denote differentiation with respect to x.) Thus v_a satisfies the boundary condition at a in (10.28), whereas v_b satisfies the boundary condition at b. The Wronskian

$$W = v_a v_b' - v_b v_a'$$

satisfies

$$W' = v_a v_b'' - v_b v_a'' = -v_a(p+\lambda)v_b + v_b(p+\lambda)v_a = 0$$

because of the differential equation (10.27), so W is independent of x and is a function only of λ. By equation (10.15), then, the Green's function is given by

$$G(x,y;\lambda) = \frac{v_a(x_-;\lambda)v_b(x_+;\lambda)}{W(\lambda)} \\ \Big(x_- = \min(x,y),\ x_+ = \max(x,y)\Big). \tag{10.30}$$

The Green's function fails to exist when $W(\lambda) = 0$. This happens precisely when v_a and v_b are constant multiples of one another, i.e., when the solution of $u'' + pu + \lambda u = 0$ that satisfies one of the boundary conditions also satisfies the other one, or in other words, when there is a nonzero solution of $u'' + pu + \lambda u = 0$ that satisfies both boundary conditions. But such solutions are precisely the eigenfunctions for the Sturm-Liouville problem, so the zeros of $W(\lambda)$ are the eigenvalues.

We need to invoke the following fact from the general theory of ordinary differential equations, proved in Appendix 5: *If the coefficients of a linear ordinary differential equation depend analytically on a complex parameter λ, then the solution satisfying a fixed set of initial conditions also depends analytically on λ.* In our case this means that v_a and v_b are entire analytic functions of λ, and hence so is the Wronskian $W(\lambda)$. Moreover, it then follows from (10.30) that $G(x,y;\lambda)$ is an entire analytic function of λ except for poles at the zeros of $W(\lambda)$, i.e., the eigenvalues.

The crucial step in deriving eigenfunction expansions is to calculate the residues of the Green's function at its poles. For this it will be important to know that the poles are all simple, that is, that the zeros of W are all simple.

Lemma 10.1. *The zeros of W are all simple; that is, if $W(\lambda_0) = 0$, then $W(\lambda)$ vanishes only to first order as $\lambda \to \lambda_0$.*

Proof: W can be evaluated by computing $v_a v_b' - v_b v_a'$ at any convenient point x in $[a,b]$, say $x = b$:

$$W(\lambda) = v_a(b;\lambda)v_b'(b;\lambda) - v_b(b;\lambda)v_a'(b;\lambda) = \beta v_a(b;\lambda) - v_a'(b;\lambda).$$

It follows that for any λ and λ_0,

$$W(\lambda_0)v_a(b;\lambda) - W(\lambda)v_a(b;\lambda_0) = v_a'(b;\lambda)v_a(b;\lambda_0) - v_a'(b;\lambda_0)v_a(b;\lambda).$$

On the other hand,

$$v_a'(a;\lambda)v_a(a;\lambda_0) - v_a'(a;\lambda_0)v_a(a;\lambda) = \alpha - \alpha = 0,$$

so

$$W(\lambda_0)v_a(b;\lambda) - W(\lambda)v_a(b;\lambda_0) = v_a'(x;\lambda)v_a(x;\lambda_0) - v_a'(x;\lambda_0)v_a(x;\lambda)\Big|_a^b$$
$$= \int_a^b \left[v_a''(x;\lambda)v_a(x;\lambda_0) - v_a''(x;\lambda_0)v_a(x;\lambda)\right] dx.$$

But

$$v_a''(x;\lambda) = -\lambda v_a(x;\lambda) - p(x)v_a(x;\lambda), \quad v_a''(x;\lambda_0) = -\lambda_0 v_a(x;\lambda_0) - p(x)v_a(x;\lambda_0),$$

and hence

$$W(\lambda_0)v_a(b,\lambda) - W(\lambda)v_a(b,\lambda_0) = (\lambda - \lambda_0)\int_a^b v_a(x;\lambda)v_a(x;\lambda_0)\,dx.$$

In particular, if $W(\lambda_0) = 0$, then

$$\frac{W(\lambda)}{\lambda - \lambda_0}v_a(b;\lambda_0) = -\int_a^b v_a(x;\lambda)v_a(x;\lambda_0)\,dx.$$

As $\lambda \to \lambda_0$ the right-hand side approaches the constant $-\int_a^b v_a(x;\lambda_0)^2 dx$, which is negative since λ_0 and $v_a(x;\lambda_0)$ are real. Thus $v_a(b;\lambda_0) \neq 0$ and $W(\lambda)/(\lambda - \lambda_0)$ approaches a nonzero constant, i.e., W vanishes to first order. ∎

We need one more ingredient before coming to the main results: some asymptotic formulas for v_a, v_b, W, and G as $|\lambda| \to \infty$. These are obtained by the following trick. By a slight modification of equation (10.12), the solution of the initial value problem

$$u'' + \mu^2 u = f, \qquad u(a) = u'(a) = 0$$

is

$$u(x) = \frac{1}{\mu}\int_a^x f(y)\sin\mu(x-y)\,dy,$$

from which it follows easily that the solution of the initial value problem

$$u'' + \mu^2 u = f, \qquad u(a) = 1, \quad u'(a) = \alpha$$

is

$$u(x) = \cos\mu(x-a) + \frac{\alpha}{\mu}\sin\mu(x-a) + \frac{1}{\mu}\int_a^x f(y)\sin\mu(x-y)\,dy.$$

But by (10.29), when $f(x) = -p(x)v_a(x;\mu^2)$ the solution to this initial value problem is $v_a(x;\mu^2)$. Therefore, $v_a(x;\mu^2)$ satisfies the integral equation

$$v_a(x;\mu^2) = \cos\mu(x-a) + \frac{\alpha}{\mu}\sin\mu(x-a) - \frac{1}{\mu}\int_a^x p(y)v_a(y;\mu^2)\sin\mu(x-y)\,dy. \tag{10.31}$$

In exactly the same way, one sees that $v_b(x;\mu^2)$ satisfies the integral equation

$$v_b(x;\mu^2) = \cos\mu(b-x) + \frac{\beta}{\mu}\sin\mu(b-x) - \frac{1}{\mu}\int_x^b p(y)v_a(y;\mu^2)\sin\mu(y-x)\,dy. \tag{10.32}$$

With this in hand, we now derive the asymptotic formulas for v_a, v_b, and their derivatives. In each case the function in question is represented as a principal term plus an error term whose order of magnitude is roughly $1/\mu$ times the principal term and which is therefore relatively small when μ is large.

Lemma 10.2. *For $a \le x \le b$ and $\mu \in \mathbf{C}$ we have*

$$\begin{aligned} v_a(x;\mu^2) &= \cos\mu(x-a) + \mu^{-1}E_1(x;\mu), \quad v_a'(x;\mu^2) = -\mu\sin\mu(x-a) + E_2(x;\mu), \\ v_b(x;\mu^2) &= \cos\mu(b-x) + \mu^{-1}E_3(x;\mu), \quad v_b'(x;\mu^2) = \mu\sin\mu(b-x) + E_4(x;\mu), \\ W(\mu^2) &= \mu\sin\mu(b-a) + E_5(\mu), \end{aligned}$$

where the error terms $E_j(x;\mu)$ satisfy

$$\begin{aligned} |E_j(x;\mu)| &\le Ce^{|\operatorname{Im}\mu|(x-a)} \qquad (j=1,2), \\ |E_j(x;\mu)| &\le Ce^{|\operatorname{Im}\mu|(b-x)} \qquad (j=3,4), \\ |E_5(\mu)| &\le Ce^{|\operatorname{Im}\mu|(b-a)}, \end{aligned}$$

with C independent of x and μ.

Proof: Since v_a, v_b, and their derivatives are jointly continuous in x and μ, they are bounded by some constant for $a \le x \le b$ and $|\mu| \le 1$ or $|\mu| \le 2\int_a^b |p(y)|\,dy$, and the same is true of the functions $\cos\mu(x-a)$, $\sin\mu(x-a)$, etc. Hence, when μ satisfies one of these inequalities, the conclusions of Lemma 10.2 are true provided one takes the constant C large enough. We shall therefore assume that μ is greater than both 1 and $2\int_a^b |p(y)|\,dy$. Let us set

$$F_\mu(x) = e^{-|\operatorname{Im}\mu|(x-a)}v_a(x;\mu^2), \qquad M_\mu = \max_{a\le x\le b}|F_\mu(x)|.$$

Then (10.31) can be restated as

$$\begin{aligned} F_\mu(x) =& e^{-|\operatorname{Im}\mu|(x-a)}\Big[\cos\mu(x-a) + \frac{\alpha}{\mu}\sin\mu(x-a)\Big] \\ &- \frac{1}{\mu}\int_a^x p(y)F_\mu(y)e^{-|\operatorname{Im}\mu|(x-y)}\sin\mu(x-y)\,dy. \end{aligned}$$

Since $|\sin z| \le e^{|\operatorname{Im} z|}$ and $|\cos z| \le e^{|\operatorname{Im} z|}$ for all $z \in \mathbf{C}$, we see by taking absolute values on both sides that

$$M_\mu \le 1 + \left|\frac{\alpha}{\mu}\right| + \frac{M_\mu}{|\mu|}\int_a^b |p(y)|\,dy \le 1 + |\alpha| + \frac{M_\mu}{2},$$

by the assumption on μ. Therefore $M_\mu \le 2(1+|\alpha|)$, or in other words,

$$|v_a(x;\mu^2)| \le 2(1+|\alpha|)e^{|\operatorname{Im}\mu|(x-a)}. \tag{10.33}$$

Now, according to (10.31), $v_a(x;\mu^2) = \cos\mu(x-a) + \mu^{-1}E_1(x;\mu)$ where

$$E_1(x;\mu) = \alpha\sin\mu(x-a) - \int_a^x p(y)v_a(y;\mu^2)\sin\mu(x-y)\,dy,$$

so by (10.33), we have

$$|E_1(x;\mu)| \le |\alpha| e^{|\operatorname{Im}\mu|(x-a)} + 2(1+|\alpha|)\int_a^x |p(y)| e^{|\operatorname{Im}\mu|(y-a)} e^{|\operatorname{Im}\mu|(x-y)}\,dy$$
$$\le \left(|\alpha| + 2(1+|\alpha|)\int_a^b |p(y)|\,dy\right) e^{|\operatorname{Im}\mu|(x-a)}.$$

This gives the desired estimate for E_1. Moreover, if we differentiate (10.31) in x we obtain $v_a'(x;\mu^2) = -\mu\sin\mu(x-a) + E_2(x;\mu)$ where

$$E_2(x;\mu) = \alpha\cos\mu(x-a) - \int_a^x p(y)v_a(y;\mu^2)\cos\mu(x-y)\,dy,$$

and another application of (10.33) then gives the desired estimate for E_2. The estimates for E_3 and E_4 are derived in an entirely similar way, using (10.32) instead of (10.31).

Finally, we have

$$\begin{aligned} W(\mu^2) &= v_a(x;\mu^2)v_b'(x;\mu^2) - v_b(x;\mu^2)v_a'(x;\mu^2) \\ &= \mu\cos\mu(x-a)\sin\mu(b-x) + \mu\sin\mu(x-a)\cos\mu(b-x) + E_5 \\ &= \mu\sin\mu(b-a) + E_5, \end{aligned}$$

where

$$\begin{aligned} E_5 =& E_4\cos\mu(x-a) + E_1\sin\mu(b-x) + E_1E_4 \\ & - E_2\cos\mu(b-x) + E_3\sin\mu(x-a) - E_2E_3. \end{aligned}$$

All the products on the right are dominated by $e^{|\operatorname{Im}\mu|(b-a)}$, so we are done. ∎

Lemma 10.3. *There exist positive constants C_1 and C_2 such that*

$$|G(x,y;\mu^2)| \le C_1|\mu|^{-1}$$

provided that (i) $|\mu| \ge C_2$, and (ii) either $\operatorname{Re}\mu = (k+\frac{1}{2})\pi(b-a)^{-1}$ for some integer k or $|\operatorname{Im}\mu| \ge (b-a)^{-1}$.

Proof: The numerator of $G(x,y;\mu^2)$ in formula (10.30) is

$$v_a(x_-;\mu^2)v_b(x_+;\mu^2),$$

and by Lemma 10.2 we have

$$|v_a(x_-;\mu^2)v_b(x_+;\mu^2)| \le Ce^{|\operatorname{Im}\mu|(x_- - a + b - x_+)} \le Ce^{|\operatorname{Im}\mu|(b-a)} \qquad (10.34)$$

since $x_- - x_+ < 0$. The denominator is $W(\mu^2)$, and again by Lemma 10.2 we have

$$|W(\mu^2)| \ge |\mu\sin\mu(b-a)| - Ce^{|\operatorname{Im}\mu|(b-a)}.$$

Now, $|\sin(s+it)|^2 = \cosh^2 t - \cos^2 s$ for any real s and t. Since $\cosh t \geq \frac{1}{2}e^{|t|}$ it follows that $|\sin(s+it)| \geq \frac{1}{4}e^{|t|}$ provided that either $\cos s = 0$ or $|t| > 1$. Hence, if we take $C_2 = 8C$, under the stated conditions on μ we have

$$|W(\mu^2)| \geq \left(\frac{1}{4}|\mu| - C\right) e^{|\operatorname{Im}\mu|(b-a)} \geq \frac{1}{8}|\mu| e^{|\operatorname{Im}\mu|(b-a)}.$$

Combining this with (10.34), we obtain the desired result:

$$|G(x,y;\mu^2)| = \left|\frac{v_a(x_-;\mu^2)v_b(x_+;\mu^2)}{W(\mu^2)}\right| \leq \frac{8C}{|\mu|}.$$ ∎

Finally we are ready to state and prove the main results, which we present as two theorems.

Theorem 10.3. *The set of eigenvalues of the Sturm-Liouville problem*

$$u'' + p(x)u + \lambda u = 0, \qquad u'(a) = \alpha u(a), \quad u'(b) = \beta u(b)$$

is infinite. The eigenvalues are all real, and they can be arranged in a sequence $\{\lambda_n\}_1^\infty$ *with* $\lambda_1 < \lambda_2 < \cdots$ *and* $\lim_{n\to\infty}\lambda_n = +\infty$.

Proof: The eigenvalues are the zeros of W, and we proved that they are all real in §3.5. Hence we need to examine the behavior of $W(\mu^2)$ when μ^2 is real, i.e., when μ is real or pure imaginary. By Lemma 10.2 we have

$$W(\mu^2) = \mu\sin\mu(b-a) + E_5(\mu), \qquad |E_5(\mu)| \leq Ce^{|\operatorname{Im}\mu|(b-a)}$$

for some $C > 0$. If μ is pure imaginary, say $\mu = it$, we have

$$|W(-t^2)| \geq |t\sinh t(b-a)| - Ce^{|t|(b-a)}.$$

Since $\sinh s \approx \frac{1}{2}e^s$ for s large and positive, we see that

$$|W(-t^2)| \geq (\tfrac{1}{4}|t| - C)e^{|t|(b-a)} > 0$$

for $|t| > 4C$. Hence there are no eigenvalues λ with $\lambda < -(4C)^2$.

On the other hand, if μ is real, then $W(\mu^2)$ is real (since it is the Wronskian of the real functions v_a and v_b), and we have

$$W(\mu^2) = \mu\Big[\sin\mu(b-a) + E_6(\mu)\Big], \qquad |E_6(\mu)| = |\mu^{-1}E_5(\mu)| \leq C|\mu|^{-1}.$$

Thus if $|\mu| > 2C$ then $W(\mu^2)/\mu$ equals $\sin\mu(b-a)$ plus an error of magnitude less than $\frac{1}{2}$. But $\sin(k+\frac{1}{2})\pi = (-1)^k$, so for all sufficiently large integers k, $W(\mu^2)$ will have opposite signs at $\mu = (k-\frac{1}{2})\pi/(b-a)$ and $\mu = (k+\frac{1}{2})\pi/(b-a)$, and hence it will have a zero in between. This proves that W has infinitely many zeros. We have seen that they are all real and greater than $-(4C)^2$; and since W is an entire analytic function, there can be only finitely many of them in any bounded interval. It follows that they can be arranged in an increasing sequence $\{\lambda_n\}$ with $\lim\lambda_n = +\infty$. ∎

Remark. This argument actually gives more precise asymptotic information about the λ_n's — namely, for large n, $\lambda_n = \mu_n^2$ where μ_n is approximately a zero of $\sin\mu(b-a)$, that is, an integer multiple of $\pi/(b-a)$.

Theorem 10.4. *Let $\{\lambda_n\}_1^\infty$ be the eigenvalues of the regular Sturm-Liouville problem*

$$u'' + p(x)u + \lambda u = 0, \qquad u'(a) = \alpha u(a), \quad u'(b) = \beta u(b),$$

and let ϕ_n be a real eigenfunction with eigenvalue λ_n.

(a) For any $\lambda \in \mathbf{C}$ that is not one of the eigenvalues,

$$G(x, y; \lambda) = \sum_1^\infty \operatorname{Res}_{\zeta=\lambda_n} \frac{G(x, y; \zeta)}{\lambda - \zeta} = \sum_1^\infty c_n \frac{\phi_n(x)\phi_n(y)}{\lambda - \lambda_n},$$

where the c_n's are certain constants and the series converges uniformly for $x, y \in [a, b]$.

(b) If u is of class $C^{(2)}$ on $[a, b]$ and satisfies the boundary conditions $u'(a) = \alpha u(a)$, $u'(b) = \beta u(b)$, and c_n is as in part (a), then

$$u(x) = \sum_1^\infty a_n \phi_n(x) \quad \textit{where} \quad a_n = c_n \langle u, \phi_n \rangle = c_n \int_a^b u(y)\phi_n(y)\, dy.$$

(c) $\{\phi_n\}_1^\infty$ is an orthogonal basis for $L^2(a, b)$, and $\|\phi_n\|^2 = c_n^{-1}$.

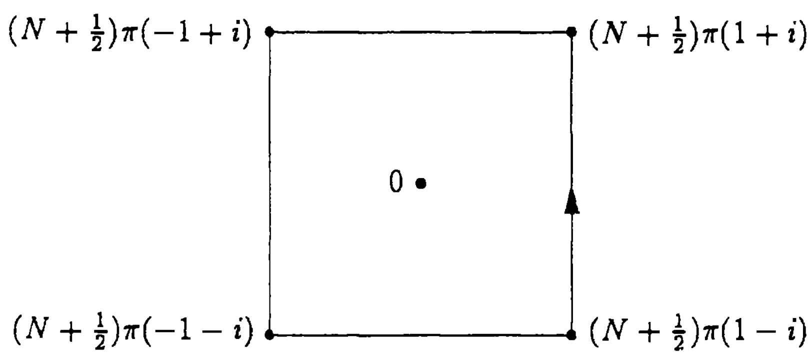


FIGURE 10.2. The contour in the proof of Theorem 10.4.

Proof: We begin with (a). Let γ_N be the square contour of side length $(2N + 1)\pi$ centered at the origin, as in Figure 10.2, and suppose λ is not an eigenvalue. By Lemma 10.3, if N is sufficiently large (in particular, $N > 2|\lambda|$) we have

$$\left| \frac{G(x, y; \zeta)}{\zeta - \lambda} \right| \le \frac{C|\zeta|^{-1/2}}{|\zeta - \lambda|} \le \frac{C'}{N^{3/2}} \quad \text{for } \zeta \text{ on } \gamma_N,$$

so

$$\left| \int_{\gamma_N} \frac{G(x, y; \zeta)}{\zeta - \lambda} d\zeta \right| \le \frac{C'}{N^{3/2}} (\text{length of } \gamma_N) = \frac{(8N + 4)\pi}{N^{3/2}},$$

which tends to zero as $N \to \infty$, uniformly in x and y. Moreover, $G(x, y; \zeta)/(\zeta - \lambda)$ is an analytic function of ζ in the whole plane except for poles at λ and at the eigenvalues λ_n, so by the residue theorem,

$$0 = \lim_{N \to \infty} \frac{1}{2\pi i} \int_{\gamma_N} \frac{G(x, y; \zeta)}{\zeta - \lambda} d\zeta = \operatorname{Res}_\lambda \frac{G(x, y; \zeta)}{\zeta - \lambda} + \sum_1^\infty \operatorname{Res}_{\lambda_n} \frac{G(x, y; \zeta)}{\zeta - \lambda},$$

the convergence being uniform in x and y. The residue at λ is simply $G(x,y;\lambda)$, whereas in view of Lemma 10.1 and formula (10.30), the residue at λ_n is

$$\frac{v_a(x_-;\lambda_n)v_b(x_+;\lambda_n)}{(\lambda_n-\lambda)W'(\lambda_n)} \qquad \left(x_-=\min(x,y),\ x_+=\max(x,y),\ W'=\frac{dW}{d\lambda}\right).$$

But $v_a(x;\lambda_n)$ and $v_b(x;\lambda_n)$ are both constant multiples of $\phi_n(x)$, so no matter whether $x<y$ or $x>y$ we have

$$v_a(x_-;\lambda_n)v_b(x_+;\lambda_n)=C_n\phi_n(x)\phi_n(y)$$

for some constant C_n. Therefore, the residue is

$$c_n\frac{\phi_n(x)\phi_n(y)}{(\lambda_n-\lambda)}, \quad \text{where} \quad c_n=\frac{C_n}{W'(\lambda_n)}.$$

This proves (a).

Next, suppose u is of class $C^{(2)}$ on $[a,b]$, $u'(a)=\alpha u(a)$, and $u'(b)=\alpha u(b)$. Pick a number λ that is not an eigenvalue, and set $f=u''+pu+\lambda u$. Then u is the solution of the boundary value problem (10.27-28), and hence $u(x)=\int_a^b G(x,y;\lambda)f(y)\,dy$. Therefore, by part (a),

$$u(x)=\int_a^b\sum_1^\infty c_n\frac{\phi_n(x)\phi_n(y)}{\lambda-\lambda_n}f(y)\,dy=\sum_1^\infty\frac{c_n\phi_n(x)}{\lambda-\lambda_n}\int_a^b\phi_n(y)f(y)\,dy.$$

Moreover,

$$\begin{aligned}\int_a^b\phi_n(y)f(y)\,dy&=\int_a^b\phi_n(y)\Big[u''(y)+p(y)u(y)+\lambda u(y)\Big]\,dy\\&=\int_a^b\Big[\phi_n''(y)+p(y)\phi_n(y)+\lambda\phi_n(y)\Big]u(y)\,dy\\&=(\lambda-\lambda_n)\int_a^b\phi_n(y)u(y)\,dy.\end{aligned}$$

(The second equality comes from integration by parts, where the boundary terms vanish because u and ϕ_n both satisfy (10.28); the third one is true because $\phi_n''+p\phi_n+\lambda_n\phi_n=0$.) This proves (b).

As for (c), we proved that $\{\phi_n\}_1^\infty$ is an orthogonal set in §3.5. Moreover, we have just proved that any $u\in L^2(a,b)$ satisfying the conditions of part (b) can be expanded in terms of the ϕ_n's. It is easy to see that every $u\in L^2(a,b)$ can be approximated in the L^2 norm by such functions. (For example, first convolve u with a smooth function to make it $C^{(2)}$, then multiply it by a smooth function that equals 1 on $[a+\epsilon,\ b-\epsilon]$ and vanishes to second order at a and b.) From this it follows easily, as in the proof of Theorem 3.6 of §3.3, that every $u\in L^2(a,b)$ can be expanded in terms of the ϕ_n's, that is, that the ϕ_n's form a basis. Finally, we know that the coefficients a_n in the expansion $u=\sum a_n\phi_n$ must be given by $a_n=\langle u,\phi_n\rangle/\|\phi_n\|^2$. On comparing this with part (b) we see that $\|\phi_n\|^2=c_n^{-1}$. ∎

Theorem 10.4 proves not only the completeness of the eigenfunctions for regular Sturm-Liouville problems but provides an efficient way of computing the normalization constants, once one has a formula for the Green's function. Namely, the proof of part (a) shows that

$$\operatorname{Res}_{\lambda=\lambda_n} G(x,y;\lambda) = c_n\phi_n(x)\phi_n(y)$$

where ϕ_n is a real eigenfunction with eigenvalue λ_n, and the norm of ϕ_n is simply $c_n^{-1/2}$.

Example. Consider the Sturm-Liouville problem associated to Fourier cosine series:

$$u'' + \lambda u = 0, \qquad u'(0) = u'(l) = 0.$$

At the end of §10.1 we calculated the Green's function for this problem, namely,

$$G(x,y;\lambda) = \frac{\cos\lambda^{1/2}x_- \cos\lambda^{1/2}(x_+ - l)}{\lambda^{1/2}\sin l\lambda^{1/2}}.$$

(The ambiguity in the sign of the square root is immaterial, as both numerator and denominator are even functions of $\lambda^{1/2}$.) The poles of G occur at $\lambda_n = (n\pi/l)^2$, $n = 0, 1, 2, \ldots$. Since $\lambda^{1/2}\sin l\lambda^{1/2} \approx l\lambda$ for λ near 0 and $\cos 0 = 1$, the residue at 0 is simply l^{-1}; the eigenfunction is $\phi_0(x) = 1$ and $\|\phi_0\|^2 = l$. Also, since the derivative of $\sin l\lambda^{1/2}$ is $\frac{1}{2}l\lambda^{-1/2}\cos l\lambda^{1/2}$, for $n > 0$ the residue at $(n\pi/l)^2$ is

$$\frac{\cos(n\pi x_-/l)\cos\left[(n\pi x_+/l) - n\pi\right]}{\frac{1}{2}l\cos n\pi} = \frac{2}{l}\cos\frac{n\pi x}{l}\cos\frac{n\pi y}{l}.$$

The eigenfunction is $\phi_n(x) = \cos(n\pi x/l)$, and $\|\phi_n\|^2 = l/2$. Of course in this case $\|\phi_n\|$ can be computed easily by elementary calculus, but this method works also in more complicated situations.

EXERCISES

1. Derive the eigenvalues and normalized eigenfunctions for the problem $u'' + \lambda u = 0$, $u(0) = u'(\frac{1}{2}\pi) = 0$, from the Green's function in Exercise 6, §10.1.
2. Derive the eigenvalues and normalized eigenfunctions for the problem $u'' + \lambda u = 0$, $u(0) = 0$, $u'(1) = -u(1)$, from the Green's function in Exercise 7, §10.1.
3. Derive the eigenvalues and a set of normalized eigenfunctions for the problem $u'' + \lambda u = 0$, $u(0) = u(1)$, $u'(0) = u'(1)$ from the Green's function in Exercise 8, §10.1. (The boundary conditions here are of a different type from those considered in the text. In this problem, for each eigenvalue except 0 the eigenspace is 2-dimensional. If ϕ_1 and ϕ_2 are an orthogonal basis for the eigenspace, the corresponding residue of the Green's function has the form $c_1\phi_1(x)\overline{\phi_1(y)} + c_2\phi_2(x)\overline{\phi_2(y)}$. The resulting eigenfunction expansion

is the Fourier series on $[0,1]$; one obtains the exponential or trigonometric form by taking $\phi_1(x)$ and $\phi_2(x)$ to be $e^{i\mu x}$ and $e^{-i\mu x}$ or $\cos\mu x$ and $\sin\mu x$, $\lambda = \mu^2$.)

Exercises 4–8 form a sequence whose purpose is to derive the results of this section for the Sturm-Liouville problem $u'' + pu + \lambda u = 0$ in which the boundary conditions (10.28) are replaced by one of the following sets of conditions:

$$\text{(I)}\quad u(a) = 0,\ u'(b) = \beta u(b) \qquad \text{(II)}\quad u'(a) = \alpha u(a),\ u(b) = 0$$
$$\text{(III)}\quad u(a) = u(b) = 0$$

In cases I and III the function v_a in (10.29) is redefined to be the solution of

$$v_a'' + pv_a + \lambda v_a = 0, \qquad v_a(a;\lambda) = 0, \quad v_a'(a;\lambda) = 1,$$

and in cases II and III the function v_b is redefined to be the solution of

$$v_b'' + pv_b + \lambda v_b = 0, \qquad v_b(b;\lambda) = 0, \quad v_b'(b;\lambda) = 1.$$

4. Prove Lemma 10.1 in cases I–III. (The proof is essentially unchanged in case I. In cases II and III, show that $W(\lambda) = v_a(b;\lambda)$ and hence that
$$W(\lambda_0)v_a'(b;\lambda) - W(\lambda)v_a'(b;\lambda_0) = (\lambda_0 - \lambda)\int_a^b v_a(x;\lambda)v_a(x;\lambda_0)\,dx.\Big)$$
5. Show that in cases I and III, (10.31) is replaced by
$$v_a(x;\mu^2) = \mu^{-1}\sin\mu(x-a) - \mu^{-1}\int_a^x p(y)v_a(y;\mu^2)\sin\mu(x-y)\,dy$$
and deduce as in Lemma 10.2 that
$$v_a(x;\mu^2) = \mu^{-1}\sin\mu(x-a) + \mu^{-2}E_1(x;\mu^2),$$
$$v_a'(x;\mu^2) = \cos\mu(x-a) + \mu^{-1}E_2(x;\mu^2)$$
where $|E_j(x;\mu^2)| \le Ce^{|\operatorname{Im}\mu|(x-a)}$. (Hint: The estimate $M_\mu \le 2(1+|\alpha|)$ in the proof turns into $M_\mu \le 2/|\mu|$ here.) Obtain also the corresponding results for v_b in cases II and III.
6. Show that $W(\mu^2) = -\cos\mu(b-a) + \mu^{-1}E_5(\mu)$ in cases I and II, and $W(\mu^2) = -\mu^{-1}\sin\mu(b-a) + \mu^{-2}E_5(\mu)$ in case III, where (in all cases) $|E_5(\mu)| \le Ce^{|\operatorname{Im}\mu|(b-a)}$.
7. Show that Lemma 10.3 is valid in case III as it stands, and is valid in cases I and II if the condition $\operatorname{Re}\mu = (k+\frac{1}{2})\pi/(b-a)$ is replaced by $\operatorname{Re}\mu = k\pi/(b-a)$.
8. Show that Theorems 10.3 and 10.4 remain valid in cases I–III.

10.4 Green's functions and singular Sturm-Liouville problems

The technique of the previous section for deriving the eigenfunction expansion associated to a regular Sturm-Liouville problem can be briefly summarized as follows. First calculate the Green's function $G(x, y; \lambda)$, and then find the eigenfunction expansion of G by integrating $G(x, y; \zeta)/(\lambda - \zeta)$ around a suitable large contour. Once this is accomplished, the expansion of a general $u \in L^2(a, b)$ follows without difficulty as in the proof of parts (b) and (c) of Theorem 10.4.

The same technique can also be applied to singular Sturm-Liouville problems, once one has calculated the Green's function $G(x, y; \lambda)$. G is always an analytic function of λ except along the real axis. It may turn out to be analytic in the whole plane except for poles at the eigenvalues, and in this case one obtains a complete set of eigenfunctions just as in the regular case. On the other hand, G may also have other singularities, such as branch cuts, along portions of the real axis. In this case one can still obtain an eigenfunction expansion by choosing a contour of integration that wraps around the singularities, but it will end up as an integral rather than, or in addition to, a series.

A complete account of these matters is beyond the scope of this book. Here we shall merely work out two important examples that illustrate these phenomena. The method of contour integrals for deriving eigenfunction expansions is developed in greater generality in Titchmarsh [52] and Stakgold [48]; see also Dunford-Schwartz [18] and Naimark [40] for other techniques.

Example 1. Our first example leads to the Fourier-Bessel series discussed in §5.4. The Bessel equation

$$x^2 u''(x) + x u(x) + (\lambda x^2 - \nu^2) u(x) = 0 \qquad (\nu \geq 0) \tag{10.35}$$

is of Sturm-Liouville type when written in the form

$$\frac{d}{dx}\Big[x u'(x)\Big] - \frac{\nu^2}{x} u(x) + \lambda x u(x) = 0 \qquad (\nu \geq 0). \tag{10.36}$$

We consider this equation on the interval $[0, 1]$, with the boundary conditions

$$u(1) = 0, \qquad \lim_{x \to 0} u(x) \text{ exists and is finite.}$$

We make two remarks before proceeding further. First, our methods apply equally well when the condition $u(1) = 0$ is replaced by $u'(1) + cu(1) = 0$. Second, equation (10.36) can be put into the form $u'' + pu + \lambda u = 0$ considered in §10.3 by the substitution $u(x) = x^{-1/2} v(x)$, which transforms (10.36) into

$$v''(x) + \frac{\frac{1}{4} - \nu^2}{x^2} v(x) + \lambda v(x) = 0. \tag{10.37}$$

(This calculation was essentially performed at the beginning of §5.3.) However, we shall work directly with (10.36). The calculations would be essentially the same if we used (10.37) but with some extra factors of $x^{1/2}$ that drop out at the end when the substitution is reversed.

Henceforth we set

$$\lambda = \mu^2.$$

The solutions of (10.36) are then linear combinations of $J_\nu(\mu x)$ and $Y_\nu(\mu x)$. The Wronskian $J_\nu Y_\nu' - Y_\nu J_\nu'$ was shown in Exercise 1 of §5.1 to equal $2/\pi x$, and it follows that the Wronskian of $J_\nu(\mu x)$ and $Y_\nu(\mu x)$ is

$$\begin{aligned} J_\nu(\mu x)\frac{d}{dx}Y_\nu(\mu x) - Y_\nu(\mu x)\frac{d}{dx}J_\nu(\mu x) &= \mu[J_\nu Y_\nu' - Y_\nu J_\nu'](\mu x) \\ &= \mu\frac{2}{\pi\mu x} = \frac{2}{\pi x}. \end{aligned} \tag{10.38}$$

In order to construct the Green's function, we define solutions $v_0(x;\lambda)$ and $v_1(x;\lambda)$ of (10.36) that satisfy the boundary conditions at 0 and 1, respectively. v_0 must be a multiple of $J_\nu(\mu x)$, and we shall take

$$v_0(x;\mu^2) = \mu^{-\nu}J_\nu(\mu x) = \sum_0^\infty \frac{(-1)^j\mu^{2j}}{j!\Gamma(\nu+j+1)}\left(\frac{x}{2}\right)^{\nu+2j}$$

The reason for dividing by μ^ν is to make $v_0(x;\lambda)$ an entire analytic function of $\lambda = \mu^2$ with no branch point at 0. For v_1 we take

$$v_1(x;\mu^2) = J_\nu(\mu)Y_\nu(\mu x) - Y_\nu(\mu)J_\nu(\mu x).$$

It is not obvious from inspection that this depends only on μ^2 rather than μ or that it is an entire analytic function of μ^2. However, we clearly have $v_1(1;\mu^2) = 0$, and by (10.38),

$$v_1'(1;\mu^2) = \mu[J_\nu(\mu)Y_\nu'(\mu) - Y_\nu(\mu)J_\nu'(\mu)] = \frac{2}{\pi}.$$

Hence v_1 is an analytic function of $\lambda = \mu^2$ because the differential equation (10.36) depends analytically on λ and the initial conditions at $x = 1$ are fixed. Moreover, by (10.38), the Wronskian of v_0 and v_1 is

$$W(x;\lambda) = W[v_0, v_1] = \mu^{-\nu}J_\nu(\mu)W[J_\nu(\mu x),\, Y_\nu(\mu x)] = \mu^{-\nu}J_\nu(\mu)\frac{2}{\pi x}.$$

Now the Green's function for (10.36) is given by formula (10.15):

$$\begin{aligned} G(x,y;\lambda) &= \frac{v_0(x_-;\lambda)v_1(x_+;\lambda)}{yW(y;\lambda)} \\ &= \frac{\pi}{2}\frac{J_\nu(\mu x_-)\Big[J_\nu(\mu)Y_\nu(\mu x_+) - Y_\nu(\mu)J_\nu(\mu x_+)\Big]}{J_\nu(\mu)}. \end{aligned} \tag{10.39}$$

(Note: Here it is important that G is the Green's function for (10.36) rather than (10.35). The Green's function for (10.35) is $G(x,y;\lambda)/y$.)

The singularity of $G(x,y;\lambda)$ at $\lambda = 0$ is easily seen to be removable. (The quotient $J_\nu(\mu x)/J_\nu(\mu)$ tends to x^ν as $\lambda \to 0$, and $v_1(x;\lambda)$ is analytic at $\lambda = 0$.) Hence, G is an analytic function of λ in the whole complex plane except for simple poles at the points $\lambda = \mu^2 \neq 0$ where $J_\nu(\mu) = 0$. These points are the eigenvalues, which we know in advance to be real. Since $i^{-\nu} J_\nu(it) > 0$ for all real $t \neq 0$, as is obvious from its Taylor expansion, J_ν has no purely imaginary zeros. Hence the eigenvalues are all positive, and we can write them in an increasing sequence $\{\lambda_n = \mu_n^2\}_1^\infty$ where μ_n is the nth positive zero of J_ν.

Moreover, from the asymptotic properties of $J_\nu(z)$ and $Y_\nu(z)$ discussed in §5.3 and §8.6, or rather their extension to the case where z is a complex variable, one sees that $G(x,y;\lambda)$ is dominated by $|\lambda|^{-1/2}$, except near the poles. Without going into all the details, the idea is as follows. We have

$$J_\nu(z) \sim \sqrt{2/\pi z}\,\cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi), \qquad Y_\nu(z) \sim \sqrt{2/\pi z}\,\sin(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi).$$

If we replace J_ν and Y_ν by the expressions on the right in (10.39), we obtain

$$\frac{\pi}{2}\,\frac{\sqrt{2/\pi^3\mu^3 x_- x_+}\,\cos(\mu x_- - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\sin\mu(1 - x_+)}{\sqrt{2/\pi\mu}\,\cos(\mu - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)},$$

which is bounded by a constant times $1/\mu\sqrt{x_- x_+}$ except near the points $\mu = \frac{1}{2}\nu\pi + \frac{1}{4}\pi + (k + \frac{1}{2})\pi$. (The factor $1/\sqrt{x_- x_+}$ necessitates a little extra care in establishing convergence near the origin but is essentially harmless; it is related to the fact that the eigenfunctions are orthogonal with respect to the measure $x\,dx$ rather than linear measure dx.)

With this estimate in hand, one can integrate $G(x,y;\zeta)/(\lambda - \zeta)$ around a sequence of large squares about the origin, as in the proof of Theorem 10.4(a), to obtain

$$\begin{aligned} G(x,y;\lambda) &= \sum_1^\infty \operatorname{Res}_{\zeta=\mu_n^2} \frac{G(x,y;\zeta)}{\lambda - \zeta} \\ &= -\frac{\pi}{2} \sum_1^\infty \operatorname{Res}_{\zeta=\mu_n^2} \frac{Y_\nu(\zeta^{1/2}) J_\nu(\zeta^{1/2} x_-) J_\nu(\zeta^{1/2} x_+)}{(\lambda - \zeta) J_\nu(\zeta^{1/2})}. \end{aligned}$$

Since

$$\frac{d}{d\zeta} J_\nu(\zeta^{1/2}) = \frac{J_\nu'(\zeta^{1/2})}{2\zeta^{1/2}},$$

this gives

$$G(x,y;\lambda) = -\pi \sum_1^\infty \frac{\mu_n Y_\nu(\mu_n) J_\nu(\mu_n x) J_\nu(\mu_n y)}{(\lambda - \mu_n^2) J_\nu'(\mu_n)}.$$

But from the recurrence formula $zJ'_\nu(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z)$ it follows that $J'_\nu(\mu_n) = -J_{\nu+1}(\mu_n)$, and by (10.38) we have

$$\frac{2}{\pi\mu_n} = -Y_\nu(\mu_n)J'_\nu(\mu_n) = J_{\nu+1}(\mu_n)Y_\nu(\mu_n), \quad \text{or} \quad -\pi\mu_n Y_n(\mu_n) = \frac{2}{J_{\nu+1}(\mu_n)}.$$

Therefore,

$$G(x,y;\lambda) = \sum_1^\infty \frac{2J_\nu(\mu_n x)J_\nu(\mu_n y)}{(\lambda - \mu_n^2)J_{\nu+1}(\mu_n)^2}. \tag{10.40}$$

The Fourier-Bessel expansion of an arbitrary $u \in L^2_w(0,1)$, where $w(x) = x$, now follows from (10.40) as in the proof of Theorem 10.4(b, c). Suppose to begin with that u is of class $C^{(2)}$ on $[0,1]$ and that $u(0) = u(1) = 0$, pick a number λ that is not an eigenvalue, and set $f = (xu')' - (\nu^2/x)u + \lambda xu$. (The extra boundary condition $u(0) = 0$ ensures the continuity of f at 0.) Then

$$u(x) = \int_0^1 G(x,y;\lambda)f(y)\,dy = \sum_1^\infty \frac{2J_\nu(\mu_n x)}{(\lambda - \mu_n)^2 J_{\nu+1}(\mu_n)^2}\int_0^1 J_\nu(\mu_n y)f(y)\,dy.$$

Since $J_\nu(\mu_n y)$ satisfies Bessel's equation

$$[yJ'_\nu(\mu_n y)]' - \frac{\nu^2}{y}J_\nu(\mu_n y) + \lambda y J_\nu(\mu_n y) = (\lambda - \mu_n^2)yJ_\nu(\mu_n y),$$

an integration by parts gives

$$\int_0^1 J_\nu(\mu_n y)f(y)\,dy = (\lambda - \mu_n^2)\int_0^1 yJ_\nu(\mu_n y)u(y)\,dy.$$

Therefore,

$$u(x) = \sum_1^\infty c_n J_\nu(\mu_n x) \quad \text{where } c_n = \frac{2}{J_{\nu+1}(\mu_n)^2}\int_0^1 u(y)J_\nu(\mu_n y)y\,dy.$$

A routine limiting argument then shows that this expansion is valid for an arbitrary $u \in L^2_w(0,1)$. Except for some technical details, therefore, we have proved the first part of Theorem 5.3 of §5.4; the second part can be established in exactly the same way. (See Exercise 3.)

Example 2. To illustrate what happens in situations where the eigenfunction expansion involves an integral rather than a sum, we shall derive the Fourier expansion of a function $u \in L^2(\mathbf{R})$ from the Sturm-Liouville problem

$$u''(x) + \lambda u(x) = 0, \qquad -\infty < x < \infty, \tag{10.41}$$

subject to the "boundary conditions" that solutions should be in $L^2(\mathbf{R})$. But (10.41) has *no* nonzero L^2 solutions for any value of λ. Indeed, the solutions of

(10.41) are of the form $u(x) = c_1 e^{i\mu x} + c_2 e^{-i\mu x}$ where $\lambda = \mu^2$. Since $e^{i\mu x}$ fails to decay at $-\infty$ or $+\infty$ according as $\operatorname{Im}\mu \geq 0$ or $\operatorname{Im}\mu \leq 0$, the integral $\int |u(x)|^2\,dx$ will diverge at either $-\infty$ or $+\infty$ (or both) no matter what c_1, c_2, and μ are, unless $c_1 = c_2 = 0$.

We can, however, try to find an L^2 Green's function, that is, to solve

$$G_{xx}(x,y;\lambda) + \lambda G(x,y;\lambda) = \delta(x-y) \tag{10.42}$$

subject to the condition that G should be in $L^2(\mathbf{R})$ as a function of x for each y. This is not possible if $\lambda \in [0,\infty)$, for the same reason as before: G must satisfy (10.41) for $x < y$ and $x > y$, and for $\lambda \geq 0$ the solutions of (10.41) do not decay at either $-\infty$ or $+\infty$. However, for any other λ there is one solution of (10.41) that decays at $-\infty$ and another one that decays at $+\infty$, and we can fit them together to solve (10.42). Specifically, if $\lambda \in \mathbf{C}\setminus[0,\infty)$, we denote by μ the square root of λ with positive imaginary part, and we set

$$v_-(x;\lambda) = e^{-i\mu x}, \qquad v_+(x;\lambda) = e^{i\mu x} \qquad (\lambda = \mu^2, \quad \operatorname{Im}\mu > 0).$$

Then $v_- \in L^2(-\infty,y)$ and $v_+ \in L^2(y,\infty)$ for any y, and their Wronskian is

$$v_-(x;\lambda)v_+'(x;\lambda) - v_+(x;\lambda)v_-'(x;\lambda) = 2i\mu.$$

The desired Green's function is then given by the usual prescription (10.15):

$$G(x,y;\lambda) = \frac{v_-(x_-;\lambda)v_+(x_+;\lambda)}{2i\mu} = \frac{e^{i\mu|x-y|}}{2i\mu}.$$

Since $\operatorname{Im}\mu > 0$, this is indeed exponentially decaying as $|x| \to \infty$.

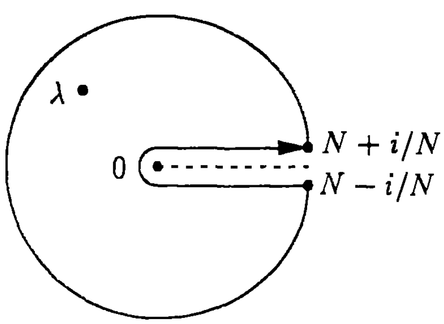


FIGURE 10.3. The contour γ_N in Example 2.

G is an analytic function of λ except along the positive real axis, where it has a branch cut: As λ approaches a point $a \in (0,\infty)$ from above or below, μ will approach the positive or negative square root of a. Rather than integrating $G(x,y;\zeta)/(\zeta-\lambda)$ over a large square as we did in the previous cases, therefore, we shall integrate it over the contour γ_N in Figure 10.3. Since $|G(x,y;\zeta)| \leq 1/2|\zeta|^{1/2}$, the integral over the circular part of γ_N will vanish as $N \to \infty$, whereas the integral over the horizontal parts becomes an integral over the upper and lower edges of the branch cut. Thus, if we denote the positive square root of $\zeta \in [0,\infty)$ by ξ, by the Cauchy integral formula we have

$$\begin{aligned} G(x,y;\lambda) &= \lim_{N\to\infty} \frac{1}{2\pi i} \int_{\gamma_N} \frac{G(x,y;\zeta)}{\zeta - \lambda}\, d\zeta \\ &= \frac{1}{2\pi i} \int_0^\infty \frac{e^{i\xi|x-y|}}{(2i\xi)(\xi^2 - \lambda)}\, d(\xi^2) - \frac{1}{2\pi i} \int_0^\infty \frac{e^{-i\xi|x-y|}}{(-2i\xi)(\xi^2 - \lambda)}\, d(\xi^2) \\ &= \frac{1}{2\pi} \int_0^\infty \frac{e^{i\xi|x-y|}}{\lambda - \xi^2}\, d\xi + \frac{1}{2\pi} \int_0^\infty \frac{e^{-i\xi|x-y|}}{\lambda - \xi^2}\, d\xi = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{i\xi|x-y|}}{\lambda - \xi^2}\, d\xi. \end{aligned}$$

Moreover, the substitution $\xi \to -\xi$ shows that

$$\int_{-\infty}^\infty \frac{e^{i\xi|x-y|}}{\lambda - \xi^2}\, d\xi = \int_{-\infty}^\infty \frac{e^{-i\xi|x-y|}}{\lambda - \xi^2}\, d\xi,$$

so the sign of $x - y$ is immaterial and we have

$$G(x,y;\lambda) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{i\xi(x-y)}}{\lambda - \xi^2}\, d\xi.$$

The Fourier inversion formula follows easily from this. Suppose that u is of class $C^{(2)}$ on $\mathbf{R}$ and (to make life easy) that u vanishes outside some bounded interval. Pick $\lambda \notin [0,\infty)$ and set $f = u'' + \lambda u$. Then u is the unique solution of $u'' + \lambda u = f$ that vanishes at $\pm\infty$, and hence

$$u(x) = \int G(x,y;\lambda) f(y)\, dy = \frac{1}{2\pi} \iint \frac{e^{i\xi(x-y)}}{\lambda - \xi^2} f(y)\, d\xi\, dy.$$

We reverse the order of integration and integrate by parts,

$$\int e^{-i\xi y} f(y)\, dy = \int e^{-i\xi y}[u''(y) + \lambda u(y)]\, dy = (\lambda - \xi^2) \int e^{-i\xi y} u(y)\, dy,$$

to obtain the Fourier integral representation of u:

$$u(x) = \frac{1}{2\pi} \int e^{i\xi x} \left(\int e^{-i\xi y} u(y)\, dy \right) d\xi.$$

One final comment: in the examples we have presented in this chapter, the eigenfunction expansion associated to a Sturm-Liouville problem involves either a discrete sum or an integral. However, in many singular Sturm-Liouville problems, the expansion involves both these ingredients. Typically such problems arise from the Schrödinger equations of quantum-mechanical systems that possess both bound and unbound states. We have discussed one such example, the hydrogen atom, in §6.5.

EXERCISES

1. Derive the Fourier sine expansion (§7.4) of a function on $[0,\infty)$ from the Green's function in Exercise 9, §10.1.
2. Derive the Fourier cosine expansion (§7.4) of a function on $[0,\infty)$ from the Green's function in Exercise 10, §10.1.
3. Work out Example 1 with the boundary condition $u(1) = 0$ replaced by $u'(1) = 0$. Proceed as follows: Let $v_0(x;\mu^2) = \mu^{-\nu} J_\nu(\mu x)$ as in the text, and let
$$v_1(x;\mu^2) = \mu\Big[J_\nu'(\mu)Y_\nu(\mu x) - Y_\nu'(\mu)J_\nu(\mu x)\Big].$$
   a. Show that $v_1(1;\mu^2) = -2/\pi$ and $v_1'(1;\mu^2) = 0$, and hence that v_1 is an analytic function of $\lambda = \mu^2$.
   b. Show that the Wronskian of v_0 and v_1 is $2\mu^{-\nu+1}J_\nu'(\mu)/\pi x$ and that the Green's function is
$$G(x,y;\mu^2) = \frac{\pi}{2}\frac{J_\nu(\mu x_-)\Big[J_\nu'(\mu)Y_\nu(\mu x_+) - Y_\nu'(\mu)J_\nu(\mu x_+)\Big]}{J_\nu'(\mu)}.$$
   c. Show that the singularity at $\lambda = 0$ of $G(x,y;\lambda)$ is removable if $\nu > 0$ and that the residue at $\lambda = 0$ is 2 if $\nu = 0$. (Hint: $v_1(x;\lambda)$ is analytic at $\lambda = 0$. For the second assertion, use Exercise 5 of §V.1.)
   d. Let $\{\mu_n\}_1^\infty$ be the positive zeros of J_ν'. Show that the residue of $G(x,y;\lambda)$ at $\lambda = \mu_n^2$ is
$$\frac{2\mu_n^2}{\mu_n^2-\nu^2}\frac{J_\nu(\mu_n x)J_\nu(\mu_n y)}{J_\nu(\mu)^2}.$$
   (Hint: From Bessel's equation, $\mu_n^2 J_\nu''(\mu_n) + (\mu_n^2 - \nu^2)J_\nu(\mu_n) = 0$, and from (10.38), $Y_\nu'(\mu_n) = 2/\pi\mu_n J_\nu(\mu_n)$.)
   e. Deduce part (b) of Theorem 5.3, §5.4, in the case $c = 0$.

APPENDIX 1
SOME PHYSICAL DERIVATIONS

The wave, heat, and Laplace equations, and various modifications of them, arise in many contexts in mathematical physics. It is hardly within the scope of this book to provide a complete catalogue of these contexts, let alone a discussion of the physical principles involved in them. Here we shall limit ourselves to a brief indication of how the differential equations are derived from physical laws in a few important situations.

The equations of electromagnetism

The fundamental equations of the classical (nonquantum) theory of electricity and magnetism are *Maxwell's equations,* which relate the electric field $\mathbf{E}$, the magnetic field $\mathbf{B}$, the charge density ρ, the current density $\mathbf{J}$, and the speed of light c. (c is an absolute constant; the other four quantities are functions of position $\mathbf{x}$ and time t.) In Gaussian cgs units, the equations are

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad & \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \\ \nabla \cdot \mathbf{E} &= 4\pi\rho, & \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{A1.1}$$

An explanation of the physics underlying these equations may be found in many textbooks; Purcell [43] is a good one.

Maxwell's equations imply that the electric and magnetic fields satisfy inhomogeneous wave equations. Indeed, for any vector field $\mathbf{F}$ we have

$$\nabla^2\mathbf{F} = -\nabla \times (\nabla \times \mathbf{F}) + \nabla(\nabla \cdot \mathbf{F}),$$

in which $\nabla^2\mathbf{F}$ denotes the vector field obtained by applying the Laplacian to $\mathbf{F}$ componentwise. If we combine this identity with Maxwell's equations and the fact that differentiation in $\mathbf{x}$ and differentiation in t commute with each other, we find that

$$\nabla^2\mathbf{E} = -\nabla \times \left(-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) + 4\pi\nabla\rho = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \mathbf{J}}{\partial t} + 4\pi\nabla\rho,$$

and

$$\nabla^2\mathbf{B} = -\nabla \times \left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}\right) = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} - \frac{4\pi}{c}\nabla \times \mathbf{J}.$$

That is, $\mathbf{E}$ and $\mathbf{B}$ satisfy the inhomogeneous wave equations

$$\begin{aligned} \mathbf{E}_{tt} - c^2\nabla^2\mathbf{E} &= -4\pi(c^2\nabla\rho + \mathbf{J}_t), \\ \mathbf{B}_{tt} - c^2\nabla^2\mathbf{B} &= 4\pi c\nabla \times \mathbf{J}. \end{aligned}$$

In regions where there are no charges or currents, $\mathbf{E}$ and $\mathbf{B}$ satisfy the homogeneous wave equation.

Gravitational and electrostatic potentials

Newton's law of gravitation may be expressed as follows: A mass m located at a point $\mathbf{y}$ induces a *gravitational field* $\mathbf{F}(\mathbf{x})$ at all other points $\mathbf{x}$ whose magnitude is proportional to $|\mathbf{y}-\mathbf{x}|^{-2}$ and whose direction points from $\mathbf{x}$ to $\mathbf{y}$. In other words,

$$\mathbf{F}(\mathbf{x}) = C\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3},$$

where by an appropriate choice of units we can take $C = 1$. (The gravitational force felt by a mass M at $\mathbf{x}$ is then $M\mathbf{F}(\mathbf{x})$.) More generally, the field induced by a distribution of mass throughout a region of space with mass density $\rho(\mathbf{y})$ is found by dividing the region up into infinitesimal pieces and adding up the fields induced by the masses in each piece, leading to

$$\mathbf{F}(\mathbf{x}) = \iiint \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}\rho(\mathbf{y})\,d\mathbf{y}.$$

With $\mathbf{F}$ given by this formula, it is easily verified by differentiation under the integral that

$$\mathbf{F}(\mathbf{x}) = -\nabla u(\mathbf{x})$$

where u is the *gravitational potential* defined by

$$u(\mathbf{x}) = -\iiint \frac{\rho(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y} = -\iiint \frac{\rho(\mathbf{x}+\mathbf{y})}{|\mathbf{y}|}\,d\mathbf{y}. \tag{A1.2}$$

(The second equation follows by the substitution $\mathbf{y} \to \mathbf{x}+\mathbf{y}$.)

From formula (A1.2) it follows that u satisfies the inhomogeneous Laplace equation

$$\nabla^2 u(\mathbf{x}) = 4\pi\rho(\mathbf{x}).$$

Briefly, the derivation is as follows. We shall assume (to make the argument easier) that the mass density $\rho(\mathbf{y})$ is twice continuously differentiable and vanishes outside some bounded set. Then

$$\nabla^2 u(\mathbf{x}) = -\iiint \frac{\nabla^2\rho(\mathbf{x}+\mathbf{y})}{|\mathbf{y}|}\,d\mathbf{y} = -\lim_{\epsilon\to 0}\iiint_{|\mathbf{y}|>\epsilon} \frac{\nabla^2\rho(\mathbf{x}+\mathbf{y})}{|\mathbf{y}|}\,d\mathbf{y}.$$

(Note that $\nabla^2\rho(\mathbf{x}+\mathbf{y})$ is the same whether we apply ∇^2 in $\mathbf{x}$ or $\mathbf{y}$.) Now let

$$\phi(\mathbf{y}) = |\mathbf{y}|^{-1}.$$

A short calculation shows that $\nabla^2\phi(\mathbf{y}) = 0$ for $\mathbf{y} \neq \mathbf{0}$, so

$$\begin{aligned}\nabla^2 u(\mathbf{x}) &= \lim_{\epsilon\to 0}\iiint_{|\mathbf{y}|>\epsilon}\left[\rho(\mathbf{x}+\mathbf{y})\nabla^2\phi(\mathbf{y}) - \phi(\mathbf{y})\nabla^2\rho(\mathbf{x}+\mathbf{y})\right]d\mathbf{y}\\ &= \lim_{\epsilon\to 0}\iiint_{|\mathbf{y}|>\epsilon}\nabla\cdot\left[\rho(\mathbf{x}+\mathbf{y})\nabla\phi(\mathbf{y}) - \phi(\mathbf{y})\nabla\rho(\mathbf{x}+\mathbf{y})\right]d\mathbf{y}.\end{aligned}$$

Thus, by the divergence theorem,

$$\nabla^2 u(\mathbf{x}) = -\lim_{\epsilon\to 0} \iint_{|\mathbf{y}|=\epsilon} \left[\rho(\mathbf{x}+\mathbf{y})\frac{\partial\phi}{\partial n}(\mathbf{y}) - \phi(\mathbf{y})\frac{\partial\rho}{\partial n}(\mathbf{x}+\mathbf{y})\right] d\mathbf{y}.$$

Here $\partial/\partial n$ denotes the outward normal derivative on the sphere $|\mathbf{y}| = \epsilon$, and $d\mathbf{y}$ denotes the element of area on this sphere. But $(\partial\phi/\partial n)(\mathbf{y}) = -|\mathbf{y}|^{-2}$, so when $|\mathbf{y}| = \epsilon$ we have $\phi(\mathbf{y}) = \epsilon^{-1}$ and $(\partial\phi/\partial n)(\mathbf{y}) = -\epsilon^{-2}$. Thus

$$\begin{aligned}\nabla^2 u(\mathbf{x}) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\iint_{|\mathbf{y}|=\epsilon}\left[\rho(\mathbf{x}+\mathbf{y}) - \epsilon\frac{\partial\rho}{\partial n}(\mathbf{x}+\mathbf{y})\right] d\mathbf{y}\\ &= 4\pi\lim_{\epsilon\to 0}\left[\text{mean value of } \rho(\mathbf{x}+\mathbf{y}) - \epsilon\frac{\partial\rho}{\partial n}(\mathbf{x}+\mathbf{y}) \text{ on } |\mathbf{y}| = \epsilon\right].\end{aligned}$$

As $\epsilon \to 0$, this tends to $4\pi\rho(\mathbf{x})$, as we claimed.

Exactly the same argument shows that the electrostatic potential u generated by a distribution of charges with charge density ρ satisfies $\nabla^2 u = -4\pi\rho$. (The minus sign is there because the electric field of a positive charge points away from the charge whereas the gravitational field of a mass points toward the mass.) The same result can be derived in more generality from Maxwell's equations (A1.1). Indeed, if the magnetic field is static ($\mathbf{B}_t = 0$) we have $\nabla \times \mathbf{E} = 0$. This implies that $\mathbf{E}$ is the gradient of a potential function $-u$, and then

$$\nabla^2 u = \nabla\cdot(\nabla u) = -\nabla\cdot\mathbf{E} = -4\pi\rho.$$

The vibrating string

Consider a perfectly elastic and flexible string stretched along the segment $[0, l]$ of the x-axis, with mass density $\rho(x)$. The string is allowed to vibrate in a direction perpendicular to the x-axis, say vertically. Let $u(x, t)$ denote the vertical displacement of the string at position x and time t. We aim to show that the wave equation (or a modified form when ρ is not constant) provides a good model for the motion of the string. As we shall see, the wave equation does not express a fundamental physical law in this situation; rather, it is an approximation that is valid when the displacement u, and more importantly its slope u_x, is small.

Consider the small segment of the string on the interval $[x, x + \Delta x]$. The vertical motion of this segment is determined by Newton's second law $F = ma$, where $m = \rho(x)\Delta x$ is the mass of the segment, $a = u_{tt}$ is the acceleration, and F is the total vertical force acting on the segment. The force F comes from the tension forces $\mathbf{T}_1$ and $\mathbf{T}_2$ pulling on the left and right ends of the segment (see Figure A1.1). Since the string is assumed flexible (so that it offers no resistance to bending), the tension forces are tangent to the string. Moreover, we shall make the assumption (valid for small displacements) that the magnitude of the tension forces is a constant T. Thus, the vertical components of the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ are $T\sin\alpha_1$ and $T\sin\alpha_2$ where α_j is the directed angle from the positive x-axis to $\mathbf{T}_j$. Thus, Newton's law $F = ma$ becomes

$$\rho(x)u_{tt}(x,t)\Delta x = T\sin\alpha_1 + T\sin\alpha_2. \qquad \text{(A1.3)}$$

On the other hand, it is clear from Figure A1.1 that $\tan\alpha_2$ is the slope of the graph of u at $x+\Delta x$, whereas $\tan\alpha_1$ is the negative of the slope at x; that is,

$$\tan\alpha_1 = -u_x(x,t), \qquad \tan\alpha_2 = u_x(x+\Delta x,\, t). \tag{A1.4}$$

We now use the approximation $\sin\theta \approx \tan\theta$, valid when θ is small, which yields (approximately)

$$\rho(x)u_{tt}(x,t)\Delta x = T\tan\alpha_1 + T\tan\alpha_2 = T\Big[u_x(x+\Delta x,\, t) - u_x(x,t)\Big].$$

Dividing through by Δx and letting $\Delta x \to 0$, we obtain

$$\rho(x)u_{tt}(x,t) = Tu_{xx}(x,t). \tag{A1.5}$$

This is the wave equation for a string with variable mass density. If the density ρ is constant, we obtain the standard wave equation

$$u_{tt} = c^2 u_{xx}, \qquad c^2 = \rho^{-1}T.$$

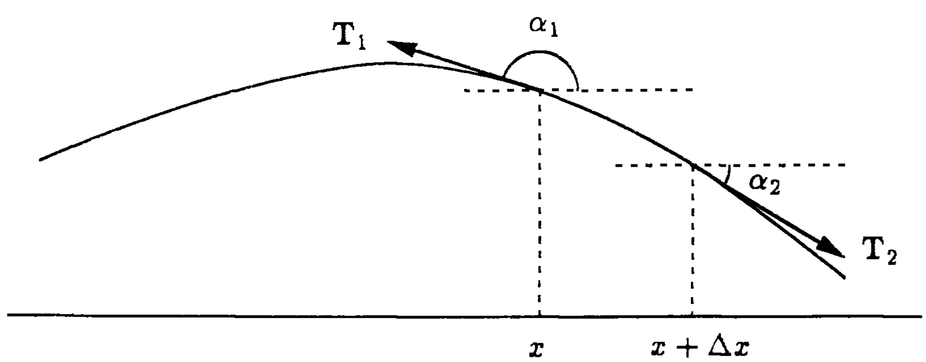


FIGURE A1.1. The curve represents a portion of a vibrating string; the vertical scale is greatly exaggerated for the sake of legibility.

The preceding derivation is the one found in many books, but it may leave the reader feeling a bit uneasy. We used the approximation $\sin\theta \approx \tan\theta$; why could we not equally well have made the approximation $\alpha_1 \approx \alpha_2$? α_1 and α_2 are clearly almost equal when Δx is small, but setting $\alpha_1 = \alpha_2$ would turn the wave equation into $u_{tt} = 0$, which is wrong!

Very well, let us dispense with the approximation $\sin\theta \approx \tan\theta$. The exact formula for sines in terms of tangents is

$$\sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$$

If we use this formula in (A1.3) together with the relations (A1.4), on letting $\Delta x \to 0$ as before we obtain the nonlinear equation

$$\rho(x)u_{tt}(x,t) = T\frac{\partial}{\partial x}\frac{u_x(x,t)}{\sqrt{1+u_x(x,t)^2}} = T\frac{u_{xx}(x,t)}{[1+u_x(x,t)^2]^{3/2}}.$$

We now arrive at the linear equation (A1.5) by making the assumption that the slope u_x is everywhere close enough to zero so that its square is negligible.

For a more thorough and rigorous development of wave equations for vibrating strings, see Antman [1].

Heat flow

We shall derive a model for the diffusion of heat in a region of space occupied by a substance whose composition may change from point to point, based on an intuitive notion of heat as an incompressible fluid that can flow throughout the region. The result will be a partial differential equation for the temperature $u(\mathbf{x},t)$ at position $\mathbf{x}$ and time t. The basic physical assumptions are as follows.

First, if the temperature at the point $\mathbf{x}$ is changed by an amount Δu, the amount of thermal energy in a small volume ΔV centered at $\mathbf{x}$ will change by an amount proportional to Δu and to ΔV:

$$\Delta E = \sigma(\mathbf{x})\Delta u\,\Delta V.$$

The coefficient $\sigma(\mathbf{x})$ is the *specific heat density* at $\mathbf{x}$. (In terms of more commonly used physical quantities, the specific heat density is the product of the specific heat [or heat capacity] and the mass density.) It follows that the rate of change of the total thermal energy in a region D is

$$\iiint_D \sigma(\mathbf{x})u_t(\mathbf{x},t)\,d\mathbf{x}.$$

The next assumption is that the amount of thermal energy in D can change only by the flow of heat across the surface S that bounds D or by the creation of heat within D. Thus, if we denote by $F(\mathbf{x},t)$ the rate at which heat is being produced at the point $\mathbf{x}$ at time t, we have

$$\iiint_D \sigma(\mathbf{x})u_t(\mathbf{x},t)d\mathbf{x} = (\text{flux of heat across } S \text{ into } D) + \iiint_D F(\mathbf{x},t)\,d\mathbf{x}. \quad \text{(A1.6)}$$

Finally, heat flows from hotter to colder regions, and it does so at a rate proportional to the difference in temperature. More precisely, the flux of heat per unit area in the direction of the unit vector $\mathbf{n}$ at the point $\mathbf{x}$ is proportional to the negative of the directional derivative of the temperature in the direction $\mathbf{n}$ at $\mathbf{x}$. The proportionality constant is called the *thermal conductivity* at $\mathbf{x}$ and is denoted by $K(\mathbf{x})$. Thus, the total flux of heat across the closed surface S into the interior region D is

$$-\iint_S K(\mathbf{x})\frac{\partial u}{\partial n}(\mathbf{x},t)\,d\mathbf{x}$$

where $\partial/\partial n$ denotes the inward-pointing normal derivative along S and $d\mathbf{x}$ denotes the element of area on S. But by the divergence theorem, this equals

$$\iiint_D \nabla \cdot \Big[K(\mathbf{x})\nabla u(\mathbf{x}, t)\Big]\, d\mathbf{x}.$$

Combining this with (A1.6), we see that

$$\iiint_D \sigma(\mathbf{x}) u_t(\mathbf{x}, t)\, d\mathbf{x} = \iiint_D \Big(\nabla \cdot \Big[K(\mathbf{x})\nabla u(\mathbf{x}, t)\Big] + F(\mathbf{x}, t)\Big)\, d\mathbf{x}.$$

But if this is to hold for arbitrary regions D, the integrands on the left and right must be equal:

$$\sigma(\mathbf{x}) u_t(\mathbf{x}, t) = \nabla \cdot \Big[K(\mathbf{x})\nabla u(\mathbf{x}, t)\Big] + F(\mathbf{x}, t).$$

This is the general form of the heat equation for nonuniform bodies. If the body is uniform, so that the specific heat density σ and the thermal conductivity K are constant, we have

$$u_t(\mathbf{x}, t) = k\nabla^2 u(\mathbf{x}, t) + F(\mathbf{x}, t), \qquad k = \frac{K}{\sigma},$$

which is the inhomogeneous form of the standard heat equation. (Of course the homogeneous equation $u_t = k\nabla^2 u$ is the special case $F = 0$.) In this situation the constant k is called the *diffusivity.*

Remark. We have assumed that σ and K are independent of the temperature u. This may or may not be a reasonable assumption in a particular physical situation. If it is not, the differential equation for heat flow becomes nonlinear.

APPENDIX 2
SUMMARY OF COMPLEX VARIABLE THEORY

This appendix is meant to provide a reasonably coherent but very abbreviated synopsis of the results from complex variable theory that are used in this book. It is not meant to be an outline for the logical development of the subject, which would require a somewhat different organization of the topics. For more details we refer the reader to books on complex analysis such as Boas [8] or Fisher [23].

Elementary functions of a complex variable

The exponential and trigonometric functions of a complex variable z are defined by

$$\exp z = e^z = \sum_0^\infty \frac{z^n}{n!}, \qquad \cos z = \sum_0^\infty \frac{(-1)^n z^{2n}}{(2n)!}, \qquad \sin z = \sum_0^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

(The other trigonometric functions are defined in the usual way in terms of cosine and sine: $\tan z = (\sin z)/(\cos z)$, etc.) It follows immediately from these equations that

$$e^{iz} = \cos z + i \sin z, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The familiar identities for the exponential and trigonometric functions of a real variable remain valid in the complex domain. For example, one has the addition formulas

$$e^{z+w} = e^z e^w, \qquad e^{-z} = 1/e^z,$$
$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \quad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w,$$

the Pythagorean identity,

$$\sin^2 z + \cos^2 z = 1,$$

and the differentiation formulas

$$\frac{d}{dz} e^z = e^z, \qquad \frac{d}{dz} \sin z = \cos z, \qquad \frac{d}{dz} \cos z = -\sin z.$$

The hyperbolic functions

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

are related to the trigonometric functions by

$$\cosh iz = \cos z, \qquad \cos iz = \cosh z, \qquad \sinh iz = i \sin z, \qquad \sin iz = i \sinh z.$$

If $z = x + iy$ with x and y real, we can use the addition formulas together with these equations to express e^z, $\sin z$, and $\cos z$ in terms of real functions of real variables:

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} = e^x(\cos y + i \sin y), \\ \cos(x \pm iy) &= \cos x \cos iy \mp \sin x \sin iy = \cos x \cosh y \mp i \sin x \sinh y, \\ \sin(x \pm iy) &= \sin x \cos iy \pm \cos x \sin iy = \sin x \cosh y \pm i \cos x \sinh y. \end{aligned}$$

If z is a nonzero complex number, a *logarithm* of z is any number w such that $z = e^w$. Writing $z = x + iy$ in polar coordinates as

$$z = r \cos\theta + ir \sin\theta = re^{i\theta},$$

we see that $\log r + i\theta$ is a logarithm of z. In order to make this into a well-defined function, we must restrict θ to an interval of length 2π, say $\theta \in (\theta_0, \theta_0 + 2\pi]$. The resulting function

$$f(re^{i\theta}) = \log r + i\theta, \qquad \theta_0 < \theta \le \theta_0 + 2\pi$$

is called a **branch** of the logarithm function, and the ray $\theta = \theta_0$ along which f is discontinuous is called the **branch cut.** The **principal branch** is the one defined by $\theta_0 = -\pi$, i.e., $-\pi < \theta \le \pi$. The notation $\log z$ is commonly used to denote any branch of the logarithm function; of course one must be careful to specify which branch one is using.

Once one has specified a branch of the logarithm, one defines complex powers of complex numbers by

$$z^a = \exp(a \log z).$$

If a is an integer, this definition of z^a coincides with the elementary algebraic one. Otherwise, the function z^a has more than one branch, and each branch has discontinuities along its branch cut.

Analytic functions

If U is an open set in the complex plane, a function $f : U \to \mathbf{C}$ is called **analytic** if its derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{A2.1}$$

exists at every point of U. f is said to be **analytic at a point** z_0 if it is analytic in some open set containing z_0, and **entire** if it is analytic in the whole complex plane. Since the complex number Δz is allowed to approach zero in any fashion whatever, analyticity is a much stronger condition than real differentiability. In particular, since the limit in (A2.1) must be the same when Δz approaches 0 along the real or the imaginary axis, one sees that an analytic function must satisfy the **Cauchy-Riemann equation**

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}. \tag{A2.2}$$

Conversely, if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous on an open set U and satisfy (A2.2) there, then f is analytic on U.

The functions z^n, e^z, $\cos z$, and $\sin z$ are analytic in the whole complex plane. Sums, products, and compositions of analytic functions are analytic, as are quotients as long as the denominator is nonzero, and the usal rules of calculus are valid.

Nonelementary analytic functions are generally constructed from the elementary ones by limiting processes such as infinite series or integrals. In particular, we have the following general procedures for manufacturing analytic functions.

(i) *Power series.* Given a sequence $\{a_n\}_0^\infty$ of complex numbers, we can form the series $\sum_0^\infty a_n(z-z_0)^n$. It is easily seen that the set of z for which this series converges is a disc $D = \{z : |z-z_0| < r\}$ centered at z_0 $(0 < r \leq \infty)$, perhaps together with some or all of the points on the circle $\{z : |z-z_0| = r\}$ when $r < \infty$. The sum of such a series is always an analytic function on the disc D, and its derivative can be calculated by termwise differentiation of the series.

(ii) *Integrals.* Suppose $f(z,t)$ is a function defined for z in an open set $U \subset \mathbf{C}$ and t in a real interval $[a,b]$, $-\infty < a < b < \infty$. Suppose that f is jointly continuous in z and t, and analytic in z for each fixed t. Then the integral $F(z) = \int_a^b f(z,t)\,dt$ is an analytic function of $z \in U$, and its derivative can be calculated by differentiating under the integral sign: $F'(z) = \int_a^b (\partial f/\partial z)(z,t)\,dt$. The same is true for integrals over an infinite interval ($a = -\infty$ and/or $b = \infty$) provided that the integral converges uniformly for z in any compact subset of U.

The facts in the preceding two paragraphs are special cases of the following general theorem.

Weierstrass's Uniform Convergence Theorem. *Suppose $\{f_n\}$ is a sequence of analytic functions on an open set $U \subset \mathbf{C}$ that converges uniformly on all compact subsets of U to a limit f. Then f is analytic on U, and $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on compact subsets of U for all $k \geq 0$.*

Power series

We have just pointed out that power series $\sum_0^\infty a_n(z-z_0)^n$ define analytic functions in their disc of convergence. One of the most fundamental results of complex function theory is that the converse is also true.

Theorem. *Every analytic function possesses derivatives of all orders. If f is analytic in a disc D centered at z_0, then f can be expanded in a power series in D. The coefficients of the series are uniquely determined by f and z_0, and they are given* *by Taylor's formula:*

$$f(z) = \sum_0^\infty a_n(z - z_0)^n \quad \textit{for } z \in D, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Among the important consequences of this theorem are the following:

(i) *If f is analytic in a connected open set U and $f \equiv 0$ on some disc $D \subset U$, then $f \equiv 0$ on U.* Indeed, if f vanishes on D, then $f^{(n)}(z_0) = 0$ for all n and all $z_0 \in D$. By the theorem, f vanishes on every disc in U centered at any point of D, hence on very disc in U centered at any point in one of *these* discs, and so forth. Since U is connected, it follows easily that $f \equiv 0$ on U.

(ii) *If f is analytic on a connected open set U and not identically zero, then the zeros of f are isolated.* Indeed, if $z_0 \in U$ and $f(z_0) = 0$, the coefficients a_n in the Taylor series for f about z_0 cannot all vanish (by (i)), so there is a smallest integer N such that $a_N \neq 0$. (We say that f has a **zero of order** N at z_0.) But then $f(z) = (z - z_0)^N g(z)$ where $g(z) = \sum_0^\infty a_{N+n}(z - z_0)^n$. g is continuous at z_0 and $g(z_0) = a_N \neq 0$, so $g(z) \neq 0$ for z near z_0. The same is then true of f except at $z = z_0$.

(iii) *If f and g are analytic on a connected open set U and $f = g$ on some set that has a limit point $z_0 \in U$, then $f \equiv g$ on U.* If not, $f - g$ would have a nonisolated zero at z_0.

If f is analytic in a punctured disc $\widetilde{D} = \{z : 0 < |z - z_0| < r\}$ about z_0 but not at z_0, f is said to have an **isolated singularity** at z_0. In this case, it is still possible to expand f in a power series about z_0, but one must allow negative powers of $z - z_0$:

$$f(z) = \sum_{-\infty}^\infty a_n(z - z_0)^n, \qquad 0 < |z - z_0| < r. \tag{A2.3}$$

Again this series is uniquely determined by f and z_0; it is called the **Laurent series** of f about z_0. There are three cases to consider:

(i) $a_n = 0$ for all $n < 0$. In this case f can be made analytic at z_0 by (re)defining $f(z_0)$ to be a_0, and the singularity at z_0 is called **removable**.

(ii) $a_n = 0$ for $n \le -N$ but $a_{-N} \neq 0$, where $N \ge 1$. In this case the singularity at z_0 is called a **pole of order** N. It is not hard to show that f has a pole of order N at z_0 if and only if $1/f$ has a zero of order N at z_0.

(iii) $a_n \neq 0$ for infinitely many negative n. In this case the singularity at z_0 is called **essential**.

The **residue** of f at an isolated singularity z_0 is the coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent expansion (A2.3); it is denoted by $\mathrm{Res}_{z_0} f$. If the singularity is a pole of order 1, the residue is given simply by

$$\mathrm{Res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$

If the singularity is a pole of order $N > 1$, the residue can be computed by applying Taylor's formula to the function $g(z) = (z - z_0)^N f(z)$:

$$\operatorname{Res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} \Big[(z - z_0)^N f(z) \Big].$$

If the singularity is essential, there is no general method for computing the residue except to find the Laurent series.

Contour integrals

For purposes of integration theory, curves in the complex plane can best be described in parametric form. Thus, let us define a **contour** in **C** to be a continuous map $\gamma : [a, b] \to \mathbf{C}$ from the real inverval $[a, b]$ into **C** whose derivative $\gamma'(t)$ exists and is nonzero at all but finitely many values of t and is piecewise continuous in the sense of §2.1. If γ is a contour, we say that a function f is **analytic on** γ if f is analytic on some open set containing the range $\{\gamma(t) : a \le t \le b\}$ of γ. In this case, we define the integral of f over γ to be

$$\int_\gamma f(z)\, dz = \int_a^b f\big(\gamma(t)\big) \gamma'(t)\, dt.$$

If γ is a contour and $\phi : [c, d] \to [a, b]$ is a $C^{(1)}$ map with $\phi(c) = a$, $\phi(d) = b$, and $\phi'(s) > 0$ for $s \in [c, d]$ (so that $\phi(s)$ is an increasing function of s), then $\gamma \circ \phi$ is another contour, but it is essentially the same as γ in the following senses. First, the points $\gamma(t)$ and $\gamma \circ \phi(s)$ trace out the same set in the complex plane as t runs over $[a, b]$ and s runs over $[c, d]$, and in the same order. Second, it follows easily from the chain rule that $\int_\gamma f(z)\, dz = \int_{\gamma \circ \phi} f(z)\, dz$ for any analytic function f. We therefore regard γ and $\gamma \circ \phi$ as different parametrizations of the same contour.

On the other hand, if $\phi : [c, d] \to [a, b]$ is a $C^{(1)}$ map with $\phi(c) = b$, $\phi(d) = a$, and $\phi'(s) < 0$ (so that $\phi(s)$ is a decreasing function of s), then the ranges of γ and $\gamma \circ \phi$ are the same but are traced out in the opposite order, and $\int_{\gamma \circ \phi} f(z)\, dz = -\int_\gamma f(z)\, dz$. In this situation we say that the contours γ and $\gamma \circ \phi$ are the same except for having **opposite orientations.**

A **simple closed contour** is a contour $\gamma : [a, b] \to \mathbf{C}$ such that $\gamma(a) = \gamma(b)$ but $\gamma(s) \ne \gamma(t)$ unless $\{s, t\} = \{a, b\}$. Any simple closed contour γ divides the complex plane into two regions, the interior and exterior regions of γ. γ is called **positively oriented** if the interior region lies on the left with respect to the direction of motion along γ as t increases, that is, if the vector $i\gamma'(t)$ obtained by rotating the tangent vector $\gamma'(t)$ counterclockwise through a right angle always points into the interior of γ at the point $\gamma(t)$. Simple closed contours are usually assumed to be positively oriented unless one specifies otherwise.

Suppose U is a connected open set in **C** that is bounded by finitely many simple closed contours, as in Figure A2.1. U will be in the interior of one of these contours, which we call γ_0, and in the exterior of the others, which we call

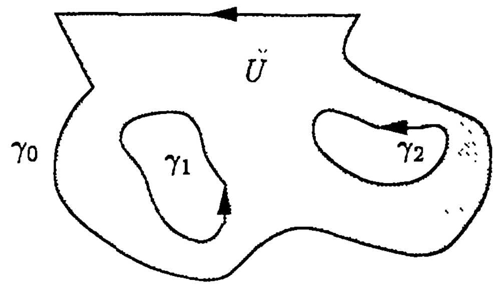


FIGURE A2.1. A region bounded by simple closed contours.

$\gamma_1, \dots, \gamma_k$. (In particular, U may be the whole interior of γ_0, in which case the other γ_j's are absent.) We assume that the contours γ_j, $0 \le j \le k$, are positively oriented; thus the "oriented boundary" of U consists of $\gamma_0, -\gamma_1, \dots - \gamma_k$ where $-\gamma_j$ denotes γ_j *with the opposite orientation.* With this understanding, the main theorems concerning integrals of analytic functions are as follows.

Cauchy's Theorem. *Let U and $\gamma_0, \dots, \gamma_k$ be as described. If f is analytic on U and on the γ_j's, then*

$$\int_{\gamma_0} f(z)\,dz - \sum_1^k \int_{\gamma_j} f(z)\,dz = 0.$$

This follows immediately from Green's theorem and the Cauchy-Riemann equation (A2.2):

$$\int_{\gamma_0} f(z)\,dz - \sum_1^k \int_{\gamma_j} f(z)\,dz = \iint_U \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy = \iint_U 0\,dx\,dy = 0.$$

The Residue Theorem. *Suppose γ is a simple closed contour with interior region U. If f is analytic on γ and on U except for singularities at $z_1, \dots, z_k \in U$, then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_1^k \operatorname{Res}_{z_j} f.$$

This result is obtained by applying Cauchy's theorem to f on the region $\widetilde{U}$ obtained by excising small, disjoint discs $D_1, \dots, D_k$ centered at the points $z_1, \dots, z_k$ from U. If γ_j is the circle that bounds D_j, Cauchy's theorem gives $\int_\gamma f(z)\,dz = \sum \int_{\gamma_j} f(z)\,dz$. On the other hand, $\int_{\gamma_j} f(z)\,dz$ can be evaluated by parametrizing γ_j as $\gamma_j(t) = z_j + re^{it}$, $0 \le t \le 2\pi$, and using the Laurent expansion (A2.3) of f about z_j:

$$\int_{\gamma_j} f(z)\,dz = \sum_{-\infty}^{\infty} \int_0^{2\pi} a_n r^n e^{int} ire^{it}\,dt.$$

All these integrals vanish except for the one with $n = -1$, which equals $2\pi i a_{-1}$. (Cf. the discussion of Fourier coefficients in §2.1.)

The Cauchy Integral Formula. *Suppose γ is a simple closed contour with interior region U and exterior region V. If g is analytic on γ and on U, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z-a}\,dz = \begin{cases} g(a) & \text{if } a \in U, \\ 0 & \text{if } a \in V. \end{cases}$$

This is the special case of the residue theorem in which $f(z) = g(z)/(z-a)$. By differentiating both sides with respect to a, we obtain the following.

The Cauchy Integral Formula for Derivatives. *Suppose γ is a simple closed contour with interior region U and exterior region V. If g is analytic on γ and on U, then for all $k \geq 0$,*

$$\frac{k!}{2\pi i}\int_\gamma \frac{g(z)}{(z-a)^{k+1}}\,dz = \begin{cases} g^{(k)}(a) & \text{if } a \in U, \\ 0 & \text{if } a \in V. \end{cases}$$

APPENDIX 3
THE GAMMA FUNCTION

The **gamma function** $\Gamma(z)$ is defined in the complex half-plane $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{A3.1}$$

(The reader who wishes to think of z as a real variable is free to do so; the complex nature of z is not essential for most of the following calculations.) The condition $\operatorname{Re} z > 0$ is needed for the integral to be convergent at $t = 0$; the factor e^{-t} ensures the convergence at infinity no matter what z is. It is easily verified that the integral converges uniformly in any vertical strip $\delta \le \operatorname{Re} z \le N$ and that one can perform differentiation under the integral sign; hence $\Gamma(z)$ is an analytic function in the right half-plane $\operatorname{Re} z > 0$. The most important basic property of the gamma function is the following.

The First Functional Equation.

$$\Gamma(z+1) = z\Gamma(z). \tag{A3.2}$$

Proof: Write out the integral for $\Gamma(z+1)$ and integrate by parts:

$$\begin{aligned}\Gamma(z+1) = \int_0^\infty t^z e^{-t}\, dt &= \lim_{\delta\to 0,\ N\to\infty} -t^z e^{-t}\Big|_\delta^N + \int_0^\infty z t^{z-1} e^{-t}\, dt \\ &= 0 + z\Gamma(z).\end{aligned}$$

∎

Equation (A3.2) can be iterated:

$$\Gamma(z+2) = (z+1)\Gamma(z+1), \quad \Gamma(z+3) = (z+2)\Gamma(z+2), \quad \Gamma(z+4) = (z+3)\Gamma(z+3),$$

and so forth. Consequently,

$$\Gamma(z+n) = z(z+1)\cdots(z+n-1)\Gamma(z). \tag{A3.3}$$

In particular, since

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = 1,$$

by taking $z = 1$ in (A3.3) we see that

$$\Gamma(n+1) = 1 \cdot 2 \cdots n\Gamma(1) = n!$$

Thus $\Gamma(z)$, or rather $\Gamma(z+1)$, provides a natural extension of the factorial function to numbers other than integers.

We can rewrite (A3.3) as

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}. \tag{A3.4}$$

The right side of (A3.4) is well defined for $\operatorname{Re} z > -n$ except for poles at $z = 0, -1, \ldots, -n+1$. Consequently, it can be taken as a *definition* of $\Gamma(z)$ for $\operatorname{Re} z > -n$. The functional equation ensures that these definitions (for different values of n) are all consistent with each other and with the original definition on their common domains; therefore, we obtain an extension of $\Gamma(z)$ to a meromorphic function in the whole plane, analytic except for simple poles at the nonpositive integers. See Figure A3.1.

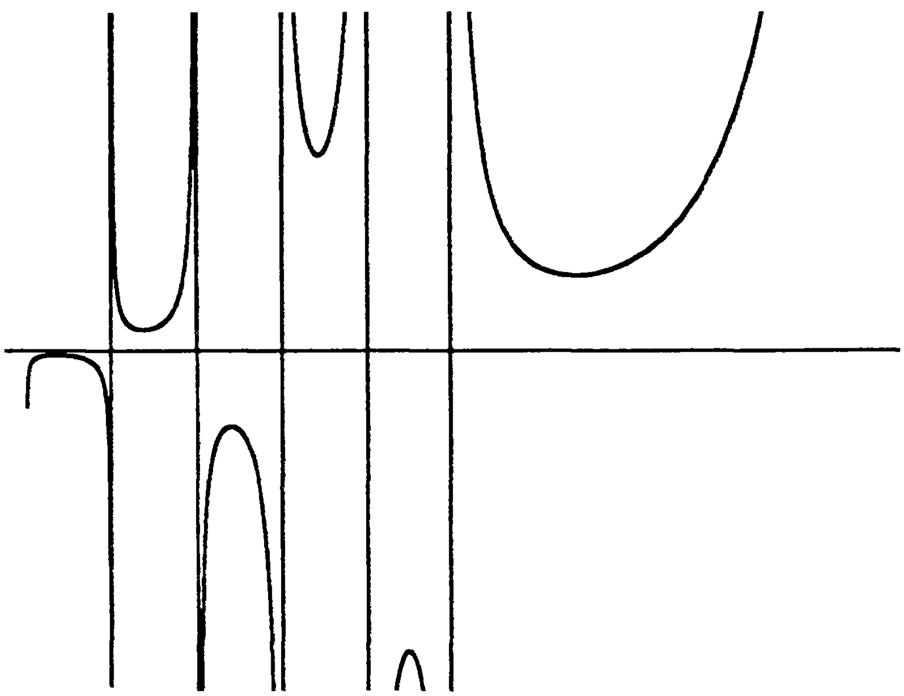


FIGURE A3.1. Graph of $\Gamma(x)$ on the interval $-5 < x < 5$, showing the vertical asymptotes at the nonpositive integers.

The importance of the gamma function arises partly from its connection with factorials, partly from the functional equation, and partly because the integral (A3.1) (and related integrals) tend to occur often in practice. Another integral that turns up frequently is the so-called **beta function**

$$\mathrm{B}(z,w) = \int_0^1 u^{z-1}(1-u)^{w-1}\,du.$$

This integral converges when $\operatorname{Re} z$ and $\operatorname{Re} w$ are both positive. It can be evaluated in terms of the gamma function, as follows.

Theorem. *If* $\operatorname{Re} z > 0$ *and* $\operatorname{Re} w > 0$,

$$\mathrm{B}(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{A3.5}$$

Proof: We write out $\Gamma(z)\Gamma(w)$ as an iterated integral and manipulate it:

$$\begin{aligned}
\Gamma(z)\Gamma(w) &= \int_0^\infty \int_0^\infty t^{z-1} s^{w-1} e^{-t-s} ds\, dt \\
&= \int_0^\infty \int_t^\infty t^{z-1} (r-t)^{w-1} e^{-r} dr\, dt \qquad (r = s+t) \\
&= \int_0^\infty \int_0^r t^{z-1} (r-t)^{w-1} e^{-r} dt\, dr \qquad \text{(reverse order of integration)} \\
&= \int_0^\infty \int_0^1 (ru)^{z-1} [r(1-u)]^{w-1} e^{-r} r\, du\, dr \qquad (u = t/r) \\
&= \int_0^\infty r^{z+w-1} e^{-r} dr \int_0^\infty u^{z-1} (1-u)^{w-1} du \\
&= \Gamma(z+w)\mathrm{B}(z, w).
\end{aligned}$$

∎

From this result, we obtain another important property of the gamma function.

The Second Functional Equation.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{A3.6}$$

Proof: If $0 < \operatorname{Re} z < 1$, we can take $w = 1 - z$ in (A3.5) and obtain

$$\Gamma(z)\Gamma(1-z) = \mathrm{B}(z, 1-z) = \int_0^1 u^{z-1}(1-u)^{-z} du = \int_0^1 \left(\frac{1}{u} - 1\right)^{-z} \frac{du}{u},$$

which by the substitution $u = 1/(1+v)$ becomes

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{dv}{v^z(1+v)}.$$

This integral can be evaluated by standard contour-integral techniques (one makes a branch cut for the function $1/v^z(1+v)$ along the positive real axis and integrates it over the contour in Figure A3.2), and its value is found to be $\pi/\sin \pi z$. (See, for instance, Boas [8], §11A.) Thus (A3.6) is proved when $0 < \operatorname{Re} z < 1$. But both sides of (A3.6) are analytic functions on the whole complex plane except for poles at the integers; since they agree for $0 < \operatorname{Re} z < 1$, they must therefore agree everywhere. ∎

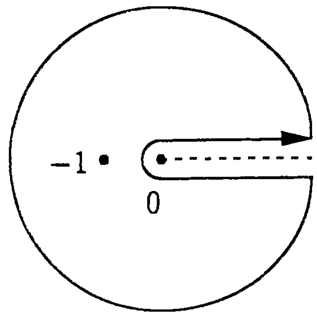


FIGURE A3.2. The contour in the proof of the second functional equation.

Corollary A3.1. *For* $n = 0, 1, 2, \ldots,$

$$\Gamma(n + \tfrac{1}{2}) = \tfrac{1}{2}(\tfrac{3}{2}) \cdots (n - \tfrac{1}{2})\sqrt{\pi}.$$

Proof: Taking $z = \frac{1}{2}$ in (A3.6) we see that $\Gamma(\frac{1}{2})^2 = \pi$, which proves the assertion for $n = 0$ since formula (A3.1) shows that $\Gamma(\frac{1}{2})$ is positive. The general case then follows by taking $z = \frac{1}{2}$ in (A3.3). ∎

Corollary A3.2. $\Gamma(z)$ *is never zero, and* $1/\Gamma(z)$ *is an entire analytic function.*

Proof: $\pi/\sin \pi z$ is never zero (otherwise $\sin \pi z$ would have poles), so any zeros of $\Gamma(z)$ must be canceled by poles of $\Gamma(1 - z)$. But the latter occur when z is a positive integer, and we already know that $\Gamma(n) = (n - 1)! \neq 0$. Since $\Gamma(z)$ is nonzero and analytic everywhere except for its poles, $1/\Gamma(z)$ is analytic everywhere with zeros at the poles of $\Gamma(z)$. ∎

The second functional equation can often be used to simplify formulas involving the gamma function. It also provides a means of extending $\Gamma(z)$ from its original domain to the whole complex plane in a single step rather than the piecemeal way in which we did it; namely,

$$\Gamma(z) = \frac{\pi}{\Gamma(1 - z)\sin \pi z}.$$

The right side can be defined by the integral (A3.1) if $\operatorname{Re}(1 - z) > 0$, i.e., if $\operatorname{Re} z < 1$. Consequently, this equation gives the extension of $\Gamma(z)$ to the half-plane $\operatorname{Re} z < 1$.

We conclude with one further useful formula.

The Duplication Formula. *For all* $z \in \mathbf{C}$ *we have*

$$\Gamma(2z)\sqrt{\pi} = 2^{2z-1}\Gamma(z)\Gamma(z + \tfrac{1}{2}).$$

Proof: Since both sides of the formula are meromorphic functions in the whole complex plane, it suffices to prove the formula for $\operatorname{Re} z > 0$ In this case, taking $w = z$ in (A3.5) and observing that $u(1-u)$ is symmetric about $u = \frac{1}{2}$, we see that

$$\frac{\Gamma(z)^2}{\Gamma(2z)} = \int_0^1 \Big[u(1-u)\Big]^{z-1} du = 2\int_0^{1/2} \Big[u(1-u)\Big]^{z-1} du.$$

By the substitution

$$u = \tfrac{1}{2}(1 - v^{1/2}), \quad du = -\tfrac{1}{4}v^{-1/2}dv, \quad u(1-u) = \tfrac{1}{4}(1-v)$$

and another application of (A3.5), this becomes

$$\frac{\Gamma(z)^2}{\Gamma(2z)} = 2^{1-2z}\int_0^1 v^{-1/2}(1-v)^{z-1}dv = 2^{1-2z}\frac{\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(z+\frac{1}{2})}.$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the desired result follows by solving this equation for $\Gamma(2z)$. ∎

Further information about the gamma function can be found in Erdélyi et al. [21], Hochstadt [30], and Lebedev [36].

APPENDIX 4
CALCULATIONS IN POLAR COORDINATES

In this appendix we derive the formula for the Laplace operator in the standard polar coordinate systems in the plane and in 3-space, and make a few remarks about integration in polar coordinates in n-space.

The Laplacian in polar coordinates

Polar coordinates (r, θ) in $\mathbf{R}^2$ are related to Cartesian coordinates (x, y) by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

If u is a function of class $C^{(2)}$ on a region in $\mathbf{R}^2$, by the chain rule we have

$$\begin{aligned} u_r &= \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x\cos\theta + u_y\sin\theta, \\ u_\theta &= \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -u_x r\sin\theta + u_y r\cos\theta. \end{aligned} \tag{A4.1}$$

For future reference we note that the equations (A4.1) can be solved simultaneously for u_y, yielding

$$u_y = u_r\sin\theta + r^{-1}u_\theta\cos\theta. \tag{A4.2}$$

Now, differentiating the equations (A4.1) by the chain rule once more, we obtain

$$\begin{aligned} u_{rr} &= (u_{xx}\cos\theta + u_{xy}\sin\theta)\cos\theta + (u_{xy}\cos\theta + u_{yy}\sin\theta)\sin\theta \\ &= u_{xx}\cos^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta \end{aligned}$$

and

$$\begin{aligned} u_{\theta\theta} &= -(-u_{xx}r\sin\theta + u_{xy}r\cos\theta)r\sin\theta - u_x r\cos\theta \\ &\quad + (-u_{xy}r\sin\theta + u_{yy}r\cos\theta)r\cos\theta - u_y r\sin\theta \\ &= r^2(u_{xx}\sin^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta) - r(u_x\cos\theta + u_y\sin\theta). \end{aligned}$$

The last equation can be rewritten as

$$r^{-2}u_{\theta\theta} = u_{xx}\sin^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta - r^{-1}u_r,$$

and upon adding this to u_{rr} we find that

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = u_{xx} + u_{yy}.$$

In other words, the Laplacian of u is given in polar coordinates by

$$\nabla^2 u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}. \tag{A4.3}$$

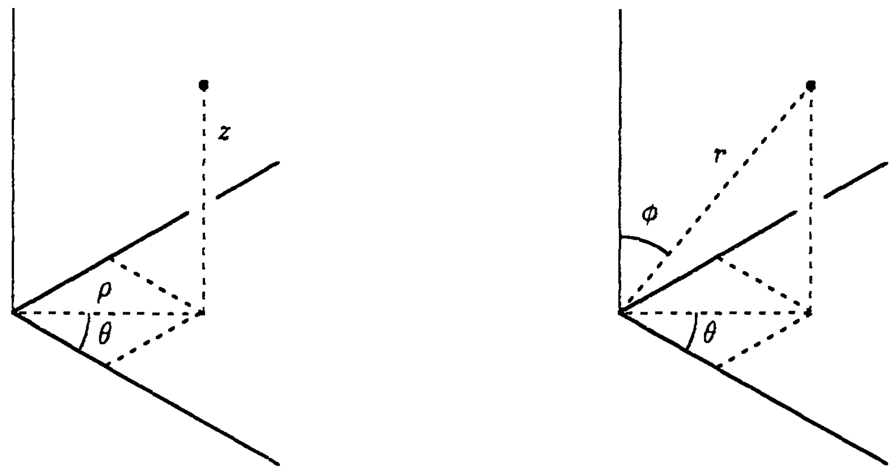


FIGURE A4.1. Cylindrical coordinates (left) and spherical coordinates (right).

The Laplacian in cylindrical and spherical coordinates

Cylindrical coordinates (ρ, θ, z) in $\mathbf{R}^3$ are obtained by using polar coordinates (ρ, θ) in the xy-plane and leaving z unchanged; that is, they are related to Cartesian coordinates (x, y, z) by

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta, \qquad z = z.$$

(See Figure A4.1. We use ρ instead of r here because we are going to use r for the radial variable in spherical coordinates.) Since holding x and y fixed is the same as holding ρ and θ fixed, partial derivatives with respect to z have the same meaning in Cartesian or cylindrical coordinates. It therefore follows immediately from (A4.3) that the Laplacian is given in cylindrical coordinates by

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = u_{\rho\rho} + \rho^{-1} u_\rho + \rho^{-2} u_{\theta\theta} + u_{zz}. \tag{A4.4}$$

Spherical coordinates (r, θ, ϕ) in $\mathbf{R}^3$ are related to Cartesian coordinates (x, y, z) by

$$x = r \cos\theta \sin\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\phi$$

and to cylindrical coordinates (ρ, θ, z) by

$$\rho = r \sin\phi, \qquad \theta = \theta, \qquad z = r \cos\phi.$$

Thus θ is the longitude on a sphere about the origin and ϕ is the co-latitude, i.e., the angle from the north pole. (See Figure A4.1.) θ can be any real number but should be restricted to an interval of length 2π such as $(-\pi, \pi]$ or $[0, 2\pi)$ to give unique cooordinates. On the other hand, ϕ is quite specifically restricted to the interval $[0, \pi]$.

The Laplacian in spherical coordinates can be calculated from the preceding results without any further tedious calculations by the following observations. In passing from cylindrical to spherical coordinates, the longitude θ is unchanged, and the cylindrical variables (z, ρ) are related to the spherical variables (r, ϕ) by the equations

$$z = r\cos\phi, \qquad \rho = r\sin\phi.$$

These equations are identical, except for the names of the variables, to the equations relating polar and Cartesian coordinates in the plane. By relabeling the variables in formulas (A4.2) and (A4.3), therefore, we see that

$$u_\rho = u_r\sin\phi + r^{-1}u_\phi\cos\phi, \qquad u_{zz} + u_{\rho\rho} = u_{rr} + r^{-1}u_r + r^{-2}u_{\phi\phi}.$$

Substitution of these formulas into (A4.4) gives

$$\begin{aligned}\nabla^2 u &= u_{\rho\rho} + u_{zz} + \rho^{-1}u_\rho + \rho^{-2}u_{\theta\theta}\\ &= u_{rr} + r^{-1}u_r + r^{-2}u_{\phi\phi} + \rho^{-1}u_r\sin\phi + (r\rho)^{-1}u_\phi\cos\phi + \rho^{-2}u_{\theta\theta},\end{aligned}$$

and one more application of the formula $\rho = r\sin\phi$ gives the final result:

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\tan\phi}u_\phi + \frac{1}{r^2}u_{\phi\phi} + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}. \tag{A4.5}$$

This rather ugly formula can be cleaned up a bit by observing that

$$u_{rr} + \frac{2}{r}u_r = \frac{1}{r}(ru)_{rr} = \frac{1}{r^2}(r^2u_r)_r, \qquad \frac{1}{\tan\phi}u_\phi + u_{\phi\phi} = \frac{1}{\sin\phi}(u_\phi\sin\phi)_\phi.$$

Hence

$$\begin{aligned}\nabla^2 u &= \frac{1}{r}(ru)_{rr} + \frac{1}{r^2\sin\phi}(u_\phi\sin\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}\\ &= \frac{1}{r^2}(r^2u_r)_r + \frac{1}{r^2\sin\phi}(u_\phi\sin\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}.\end{aligned} \tag{A4.6}$$

Integration in polar coordinates

We recall from calculus that the element of area in the plane is given in Cartesian and polar coordinates by

$$dA = dx\,dy = r\,dr\,d\theta.$$

Thus, integration over the plane is reduced to 1-dimensional integrals by the formulas

$$\iint_{\mathbf{R}^2}(\cdots)\,dA = \int_{\infty}^{\infty}\int_{-\infty}^{\infty}(\cdots)\,dx\,dy = \int_0^{2\pi}\int_0^{\infty}(\cdots)\,r\,dr\,d\theta.$$

In three dimensions, the element of volume is given in Cartesian, cylindrical, and spherical coodinates by

$$dV = dx\,dy\,dz = \rho\,d\rho\,d\theta\,dz = r^2 \sin\phi\,dr\,d\theta\,d\phi.$$

Thus,

$$\begin{aligned}\iiint_{\mathbf{R}^3}(\cdots)\,dV &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}(\cdots)\,\rho\,d\rho\,d\theta\,dz \\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}(\cdots)\,r^2\sin\phi\,dr\,d\theta\,d\phi.\end{aligned}$$

There are versions of polar or spherical coordinates in $\mathbf{R}^n$ for any n, consisting of a radial variable $r = |\mathbf{x}|$ and $n-1$ angular variables $\theta_1, \ldots, \theta_{n-1}$. The exact formulas for the angular variables are somewhat messy and rarely useful. For most purposes the important thing is that the n-dimensional element of volume $d\mathbf{x}$ is given by

$$d\mathbf{x} = r^{n-1}\,dr\,dS$$

where dS is the element of $(n-1)$-dimensional measure on the unit (hyper)sphere $|\mathbf{x}| = 1$ in $\mathbf{R}^n$. (Of course, $dS = d\theta$ when $n = 2$ and $dS = \sin\phi\,d\theta\,d\phi$ when $n = 3$. The factor of r^{n-1} is there because the surface measure of a sphere of radius r is proportional to r^{n-1}.) In particular, the integral of a function that depends only on the distance $|\mathbf{x}|$ from the origin is given by

$$\int_{\mathbf{R}^n} f\big(|\mathbf{x}|\big)\,d\mathbf{x} = \omega_n \int_0^{\infty} f(r) r^{n-1}\,dr, \tag{A4.7}$$

where ω_n is the $(n-1)$-dimensional surface area of the unit sphere.

ω_n can be calculated in terms of the gamma function (see Appendix 3) by the following device. We first observe that by the substitution $t = s^{1/2}$,

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = 2\int_0^{\infty} e^{-t^2}\,dt = \int_0^{\infty} s^{-1/2}e^{-s}\,ds = \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Now we calculate the integral of $e^{-|\mathbf{x}|^2}$ over $\mathbf{R}^n$ in two ways. In Cartesian coordinates we have $e^{-|\mathbf{x}|^2} = e^{-x_1^2}\cdots e^{-x_n^2}$, so the integral reduces to the product of n equal 1-dimensional integrals:

$$\int_{\mathbf{R}^n} e^{-|\mathbf{x}|^2}\,d\mathbf{x} = \left(\int_{\infty}^{\infty} e^{-t^2}dt\right)^n = \pi^{n/2}.$$

On the other hand, by (A4.7) and the substitution $r = s^{1/2}$,

$$\int_{\mathbf{R}^n} e^{-|\mathbf{x}|^2}\,d\mathbf{x} = \omega_n \int_0^{\infty} e^{-r^2} r^{n-1}\,dr = \frac{\omega_n}{2}\int_0^{\infty} s^{(n/2)-1}e^{-s}\,ds = \frac{\omega_n}{2}\Gamma(n/2).$$

On comparing these two equations, we find that

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{A4.8}$$

Finally, we mention one other consequence of (A4.7) that is useful for comparison tests in determining the convergence or divergence of multiple integrals: if $0 < a < \infty$ and $0 < c < \infty$,

$$\int_{|\mathbf{x}|<c} |\mathbf{x}|^{-a} d\mathbf{x} = \omega_n \int_0^c r^{n-1-a} dr < \infty \quad \Longleftrightarrow \quad a < n, \tag{A4.9}$$

$$\int_{|\mathbf{x}|>c} |\mathbf{x}|^{-a} d\mathbf{x} = \omega_n \int_c^\infty r^{n-1-a} dr < \infty \quad \Longleftrightarrow \quad a > n. \tag{A4.10}$$

APPENDIX 5
THE FUNDAMENTAL THEOREM OF ORDINARY DIFFERENTIAL EQUATIONS

The fundamental existence and uniqueness theorem of ordinary differential equations asserts that, under suitable hypotheses, an nth-order ordinary differential equation admits a unique solution y such that $y, y', \dots y^{(n-1)}$ have specified values at a particular point. A precise statement and proof can be found, for example, in Birkhoff-Rota [6]. In this appendix we present a proof of the fundamental theorem for the case of second-order linear equations. This is all we need for the purposes of this book, and the proof is considerably simpler than in the general case.

We shall consider the second-order linear equation

$$r(x)y'' + q(x)y' + p(x)y = f(x) \tag{A5.1}$$

on the closed interval $[a, b]$, with the assumption that p, q, and r are continuous functions on $[a, b]$ with either real or complex values and that $r(x) \neq 0$ for $x \in [a, b]$. (Similar results also hold for differential equations on an open interval (a, b); one simply applies the following arguments on an arbitrary closed subinterval of (a, b).) Since $r(x) \neq 0$, we can divide (A5.1) through by $r(x)$ and rewrite it in the form

$$y'' = \alpha(x)y' + \beta(x)y + \gamma(x),$$

where $\alpha = -q/r$, $\beta = -p/r$, and $\gamma = f/r$. For an equation in this form, the fundamental existence and uniqueness theorem is as follows. For use in §10.3, we have also incorporated the theorem on analytic dependence on parameters into it.

Theorem. *Suppose α, β, and γ are continuous functions (with real or complex values) on $[a, b]$. For any constants $c, c' \in \mathbf{C}$ and any $x_0 \in [a, b]$ there is a unique solution y of the initial value problem*

$$y'' = \alpha(x)y' + \beta(x)y + \gamma(x), \qquad y(x_0) = c, \ y'(x_0) = c', \tag{A5.2}$$

on the interval $[a, b]$. Moreover, if α, β, and γ depend analytically on a complex parameter λ, so does the solution y.

The remainder of this appendix is devoted to the proof of this theorem. The first step is the following lemma, which in effect gives the solution for the simple special case $\alpha = \beta = 0$.

Lemma A5.1. *If g is a continuous function on $[a,b]$, $x_0 \in [a,b]$, and $c, c' \in \mathbf{C}$, the function*

$$h(x) = \int_{x_0}^{x} (x-t)g(t)\,dt + c'(x-x_0) + c$$

is twice differentiable and satisfies

$$h'(x) = \int_{x_0}^{x} g(t)\,dt + c', \quad h''(x) = g(x), \quad h(x_0) = c, \quad h'(x_0) = c'.$$

Moreover, if g depends analytically on a complex parameter λ, so does h.

Proof: The formulas for h and its derivatives are obvious once one notes that

$$\frac{d}{dx}\int_{x_0}^{x}(x-t)g(t)\,dt = (x-x)g(x) + \int_{x_0}^{x}\frac{d(x-t)}{dx}g(t)\,dt = \int_{x_0}^{x} g(t)\,dt.$$

The assertion about analytic dependence is also obvious, as one can differentiate with respect to λ under the integral. ∎

Returning to the general problem (A5.2), we propose to obtain a solution y as the sum of an infinite series $\sum_1^\infty y_n$ whose terms are defined as follows. The first one is given by

$$y_1(x) = \int_{x_0}^{x}(x-t)\gamma(t)\,dt + c'(x-x_0) + c,$$

and for $n > 1$, y_n is defined inductively in terms of y_{n-1} by

$$y_n(x) = \int_{x_0}^{x}(x-t)\Big[\alpha(t)y'_{n-1}(t) + \beta(t)y_{n-1}(t)\Big]\,dt. \tag{A5.3}$$

Thus, by Lemma A5.1,

$$\begin{aligned} &y_1'' = \gamma(x), \qquad y_1(x_0) = c, \quad y_1'(x_0) = c', \\ &y_n'' = \alpha(x)y'_{n-1} + \beta(x)y_{n-1}, \qquad y_n(x_0) = y_n'(x_0) = 0 \qquad (n > 1), \end{aligned} \tag{A5.4}$$

and if α, β, and γ depend analytically on λ, so do the functions y_n.

To establish the convergence of the series $\sum_1^\infty y_n$, we need an estimate on the size of y_n. To begin with, since y_1, y_1', α, and β are continuous functions on $[a,b]$, there are constants A and B such that

$$|y_1(x)| \le A, \quad |y_1'(x)| \le A, \quad |\alpha(x)| + |\beta(x)| \le B \qquad (a \le x \le b). \tag{A5.5}$$

Moreover, if these functions depend analytically on λ, the constants A and B can be chosen so that the estimates (A5.5) hold for all λ in a given compact set K.

Lemma A5.2. *If (A5.5) holds, then*

$$|y_n(x)| \le \frac{A(LB)^{n-1}|x-x_0|^{n-1}}{(n-1)!}, \qquad |y_n'(x)| \le \frac{A(LB)^{n-1}|x-x_0|^{n-1}}{(n-1)!}$$

for all n, where $L = \max(b-a, 1)$.

Proof: The assertion is true for $n = 1$ by (A5.5). We therefore proceed by induction to deduce the estimate for y_n $(n > 1)$ from the estimate for y_{n-1}. By the definition (A5.3) of y_n and the inductive hypothesis, for $x_0 \le x \le b$ we have

$$|y_n(x)| \le \int_{x_0}^{x} (x-t)\Big[|\alpha(t)| + |\beta(t)|\Big] \frac{A(LB)^{n-2}(t-x_0)^{n-2}}{(n-2)!}\, dt.$$

But $|\alpha(t)| + |\beta(t)| \le B$ and $x - t \le x - x_0 \le b - a \le L$, so

$$|y_n(x)| \le \int_{x_0}^{x} LB \frac{A(LB)^{n-2}}{(n-2)!}(t-x_0)^{n-2} dt = \frac{A(LB)^{n-1}}{(n-1)!}.$$

A slight modification of this calculation shows that the desired estimate also holds for $a \le x \le x_0$. But, by Lemma A5.1, (A5.3) also implies that

$$y_n'(x) = \int_{x_0}^{x} \Big[\alpha(t)y_{n-1}'(t) + \beta(t)y_{n-1}(t)\Big]\, dt.$$

Hence the same estimate holds for y_n' but with one less factor of L (because of the absence of the factor $x - t$). But since $L \ge 1$ by definition, the estimate is still true with the extra L thrown in. ∎

Since $|x - x_0| \le b - a \le L$ for $x \in [a, b]$, it follows from Lemma A5.2 that

$$|y_n(x)| \le \frac{A(L^2B)^{n-1}}{(n-1)!}, \qquad |y_n'(x)| \le \frac{A(L^2B)^{n-1}}{(n-1)!}$$

The convergence of the series $\sum_1^\infty (L^2B)^{n-1}/(n-1)!$ (to e^{L^2B}, in fact) therefore implies that the series $\sum_1^\infty y_n$ and $\sum_1^\infty y_n'$ converge absolutely and uniformly for $x \in [a, b]$ (and λ in a compact set K). Moreover, by (A5.4) we have

$$\begin{aligned}\sum_1^\infty y_n'' &= y_1'' + \alpha(x)\sum_2^\infty y_{n-1}' + \beta(x)\sum_2^\infty y_{n-1} \\ &= \gamma(x) + \alpha(x)\sum_1^\infty y_n' + \beta(x)\sum_1^\infty y_n,\end{aligned} \tag{A5.6}$$

so the twice differentiated series also converges absolutely and uniformly. Therefore, the function

$$y = \sum_1^\infty y_n$$

exists on $[a, b]$ and is twice differentiable in x (and analytic in λ, since the uniform limit of analytic functions is analytic by Weierstrass's theorem on uniform convergence). Moreover, (A5.6) says that

$$y'' = \alpha(x)y' + \beta(x)y + \gamma(x),$$

whereas by (A5.4),

$$y(x_0) = y_1(x_0) + \sum_2^\infty y_n(x_0) = c, \qquad y'(x_0) = y_1'(x_0) + \sum_2^\infty y_n'(x_0) = c'.$$

In short, y solves the original problem (A5.2).

This completes the proof of existence. Uniqueness now follows easily. Indeed, suppose $y_{(1)}$ and $y_{(2)}$ are both solutions of (A5.2), and let $y = y_{(1)} - y_{(2)}$. Then y satisfies

$$y'' = \alpha(x)y' + \beta(x)y, \qquad y(x_0) = y'(x_0) = 0.$$

Pick an arbitrary point $x_1 \in [a, b]$ and let z be a solution of the problem

$$z'' = \alpha(x)z' + \beta(x)z, \qquad z(x_1) = y(x_1),\ z'(x_1) = y'(x_1) + 1.$$

(We have just seen that such a solution exists.) Then the Wronskian $W = yz' - zy'$ satisfies $W(x_0) = 0$ and $W(x_1) = y(x_1)$. However,

$$W' = (yz' - zy')' = yz'' - zy'' = y(\alpha z' + \beta z) - z(\alpha y' + \beta y) = \alpha(yz' - zy') = \alpha W,$$

so if we set $A(x) = \int_{x_0}^x \alpha(t)\,dt$, we have

$$(e^{-A}W)' = e^{-A}(W' - A'W) = e^{-A}(W' - \alpha W) = 0.$$

Thus $W = Ce^A$ for some constant C. The condition $W(x_0) = 0$ forces $C = 0$; but then $y(x_1) = W(x_1) = 0$. In short, y vanishes identically on $[a, b]$, so $y_{(1)} = y_{(2)}$. The proof is complete.

ANSWERS TO THE EXERCISES

Section 1.2

1. $c_1 + c_2 = 1$.
2. c. $c = 1$ or 0.
4. Linear if $G(x,t,u) = g(x,t)u + h(x,t)$; homogeneous if $G(x,t,u) = g(x,t)u$.
5. e. $\dfrac{\sin(2\pi x)\sinh(2\pi y)}{\sinh 2\pi} - \dfrac{\sin(3\pi x)\sinh(3\pi y)}{\sinh 3\pi} + \dfrac{2\sin(\pi x)\sinh\pi(1-y)}{\sinh\pi}$.

Section 1.3

1. a. $X'' = cX$, $Y' = -cyY$. b. $x^2X'' + xX' = cX$, $Y'' = -(1+c)Y$.
   c. Not possible. d. $X'' = c(X' + X)$, $Y' = -cY$.
2. a. $X'' = c_1xX$, $Y'' = c_2yY$, $Z'' = -(c_1 + c_2)Z$.
   b. $Y'' = c_1Y$, $Z'' = c_2Z$, $x^2X'' + xX' + (c_1 + c_2x^2)X = 0$.
3. $u(x,t) = 2\sin\pi x\cos 3\pi t - 3\sin 4\pi x\cos 12\pi t$.
4. $u(x,t) = 3 - 4e^{-2t/5}\cos 2x$; $t_0 = 10\log 10 + \frac{5}{2}\log 4 \approx 26.5$.
7. $u_n(x,t) = e^{-(2n+1)^2\pi^2kt/4l^2}\sin(2n+1)\pi x/2l$.

Section 2.2

1. a. Not piecewise continuous. b. Continuous. c. Continuous and piecewise smooth. d. Piecewise smooth. e. Continuous and piecewise smooth. f. Piecewise continuous.
2. 6. 0. 7. $\frac{1}{2}$. 12. $(4a)^{-1}$. 18. $\cosh b\pi$.

Section 2.3

3. $\displaystyle\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^4} = \frac{\pi|\theta|^3}{24} - \frac{\pi^2\theta^2}{16} + \frac{\pi^4}{96}$ on $(-\pi,\pi)$.
7. a. 12. b. ∞. c. 0.

Section 2.4

1. 1; $\quad\displaystyle\frac{4}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$.
2. $\displaystyle\frac{\pi}{2} + \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2}$; $\quad\displaystyle 2\sum_1^\infty \frac{\sin n\theta}{n}$.
3. $\displaystyle\frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1}$; $\quad\sin\theta$.
4. $\cos\theta$; $\quad\displaystyle\frac{8}{\pi}\sum_1^\infty \frac{n\sin 2n\theta}{4n^2-1}$.

5. $\dfrac{\pi^2}{3} + 4\sum_1^\infty \dfrac{(-1)^n}{n^2}\cos n\theta;\qquad 2\pi\sum_1^\infty \dfrac{(-1)^{n+1}}{n}\sin n\theta - \dfrac{8}{\pi}\sum_1^\infty \dfrac{\sin(2n-1)\theta}{(2n-1)^3}.$

6. $\dfrac{\pi}{4} - \dfrac{2}{\pi}\sum_1^\infty \dfrac{\cos(4n-2)\theta}{(2n-1)^2};\qquad \dfrac{4}{\pi}\sum_1^\infty (-1)^{n+1}\dfrac{\sin(2n-1)\theta}{(2n-1)^2}.$

7. $\dfrac{4}{\pi}\sum_1^\infty \dfrac{1}{2n-1}\sin\dfrac{(2n-1)x}{6}.$

8. $\dfrac{1}{2} + \dfrac{4}{\pi^2}\sum_1^\infty \dfrac{\cos(2n-1)\pi x}{(2n-1)^2}.$

9. $\dfrac{4}{\pi}\sum_1^\infty \dfrac{(-1)^{n+1}}{2n-1}\cos\dfrac{(2n-1)\pi x}{4}.$

10. $\dfrac{8l^2}{\pi^3}\sum_1^\infty \dfrac{1}{(2n-1)^3}\sin\dfrac{(2n-1)\pi x}{l}.$

11. $(e-1)\sum_{-\infty}^\infty \dfrac{e^{2\pi inx}}{1-2\pi in}.$

Section 2.5

1. a. $50 - \dfrac{400}{\pi^2}\sum_1^\infty \dfrac{1}{(2n-1)^2}\exp\dfrac{-(2n-1)^2\pi^2(1.1)t}{10^4}\cos\dfrac{(2n-1)\pi x}{100}.$
   b. $u(0,60) = 10,\quad u(10,60) = 12,\quad u(40,60) = 40.$
   c. $T = 3600$ (1 hour) is more than enough.
2. 10 terms are needed at $t = 60$; $T = 396{,}000$.
3. a. $\dfrac{400}{\pi}\sum_1^\infty \dfrac{1}{2n-1}\exp\dfrac{-(2n-1)^2\pi^2(1.1)t}{10^4}\sin\dfrac{(2n-1)\pi x}{100}.$
   b. $u(50,30) = 100$, $u(50,60) = 100$, $u(50,300) = 90$, $u(50,3600) = 3$.
4. a. $\dfrac{8m}{\pi^2}\sum_1^\infty \dfrac{(-1)^{n+1}}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{x}\cos\dfrac{(2n-1)\pi ct}{l}.$
5. a. $\dfrac{2ml^2}{\pi^2 a(l-a)}\sum_1^\infty \dfrac{1}{n^2}\sin\dfrac{n\pi a}{l}\sin\dfrac{n\pi x}{l}\cos\dfrac{n\pi ct}{l}.$
6. $\dfrac{4l}{c\pi^2}\sum_1^\infty \dfrac{(-1)^{n+1}}{(2n-1)^2}\sin\dfrac{(2n-1)\pi\delta}{l}\sin\dfrac{(2n-1)\pi x}{l}\sin\dfrac{(2n-1)\pi ct}{l}.$
7. a. $10 - 7e^{-.576x}\cos(2\pi t - .576x) - 5e^{-11.0x}\cos(730\pi t - 11.0x).$
   b. 21 cm; 4.6 m.

Section 3.1

3. a. $\mathbf{y}_3 = (10, 15i, -13)$ (or any nonzero multiple thereof).
   b. $\|\mathbf{y}_1\| = \sqrt{38}$, $\|\mathbf{y}_2\| = \sqrt{13}$, $\|\mathbf{y}_3\| = \sqrt{494}$.
   c. $(1,2,3i) = \frac{1}{38}(2-9i)\mathbf{y}_1 + \frac{1}{13}(4-3i)\mathbf{y}_2 + \frac{1}{494}(10-69i)\mathbf{y}_3$; $(0,1,0) = -\frac{3i}{38}\mathbf{y}_1 + \frac{2}{13}\mathbf{y}_2 - \frac{15i}{494}\mathbf{y}_3$.

4. b. $(1,0,0,0) = \frac{1}{3}\mathbf{u}_1 + \frac{1}{5}(2+4i)\mathbf{u}_2 + \frac{1}{15}(4-2i)\mathbf{u}_3$;
$(2, 10-i, 10-9i, -3) = 6\mathbf{u}_1 - 5\mathbf{u}_2 - 15i\mathbf{u}_3 + 3i\mathbf{u}_4$.

5. $c_j = \langle \mathbf{a}, \mathbf{u}_j \rangle$.

Section 3.2

3. $f_2(x) = 3x^2 - 1$ or any constant multiple thereof.

Section 3.3

8. $f(x) = \dfrac{4}{\pi}\displaystyle\sum_1^\infty \frac{1}{2n-1}\sin(n-\tfrac{1}{2})\frac{\pi x}{l} = \frac{4}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{2n-1}\cos(n-\tfrac{1}{2})\frac{\pi x}{l}$;

$$g(x) = \frac{8l}{\pi^2}\sum_1^\infty \frac{(-1)^{n+1}}{(2n-1)^2}\sin(n-\tfrac{1}{2})\frac{\pi x}{l}$$
$$= \frac{4l}{\pi^2}\sum_1^\infty \frac{(-1)^{n+1}(2n-1)\pi - 2}{(2n-1)^2}\cos(n-\tfrac{1}{2})\frac{\pi x}{l}.$$

10. a. $\dfrac{\pi^4}{90}$. b. $\dfrac{\pi^6}{960}$. c. $\dfrac{\pi}{4}(\coth\pi - \pi\,\mathrm{csch}^2\,\pi)$. d. $\dfrac{a^2(\pi-a)^2}{6}$.

Section 3.4

2. $f_1(x) = x - 1$ and $f_2(x) = \frac{1}{2}x^2 - 2x + 1$ will do.
3. $\|f_n\|^2 = \pi/(n+1)$.
7. a. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\cos x$. b. $2\sin x - \sin 2x$. c. $2\sin x - \dfrac{4}{\pi}\cos x$.

Section 3.5

1. $c\bar{c}' = r(a)/r(b)$.
3. $(2/l)^{1/2}\sin(n-\frac{1}{2})\pi x/l$, $n = 1,2,3,\ldots$.
4. $(2/l)^{1/2}\cos(n-\frac{1}{2})\pi x/l$, $n = 1,2,3,\ldots$.
5. If $\beta < 0$, the eigenvalues are the numbers $\lambda_n = \nu_n^2$ where the ν_n's are the positive solutions of $\tan\nu l = -\beta/\nu$, and the corresponding normalized eigenfunctions are
$$\phi_n(x) = \sqrt{\frac{2|\beta|}{l|\beta| + \sin^2\nu_n l}}\cos\nu_n x.$$
If $\beta = 0$, the eigenvalues are $\lambda_n = (n\pi/l)^2$, i.e., the squares of the solutions of $\tan\nu l = 0$, including $\lambda_0 = 0$. The normalized eigenfunctions are $\phi_n(x) = (2/l)^{1/2}\cos(n\pi x/l)$ $(n > 0)$ and $\phi_0(x) = l^{-1/2}$. If $\beta > 0$, the eigenvalues are the squares of the positive solutions ν_n of the equation $\tan\nu l = -\beta/\nu$, together with the square of the unique positive solution μ_0 of $\tanh\mu l = \beta/\mu$. The normalized eigenfunctions are
$$\phi_n(x) = \sqrt{\frac{2\beta}{l\beta - \sin^2\nu_n l}}\cos\nu_n x, \quad \phi_0(x) = \sqrt{\frac{2\beta}{l\beta + \sinh^2\mu_0 l}}\cosh\mu_0 x.$$

6. If $\alpha > 0$, the eigenvalues are $\lambda_n = \nu_n^2$ where the ν_n's are the positive solutions of $\tan \nu l = \alpha/\nu$, and the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2\alpha}{l\alpha + \sin^2 \nu_n l}} \cos \nu_n(l - x).$$

If $\alpha \leq 0$, there are modifications to be made as in Exercise 5.

7. The eigenvalues are $\lambda_n = \nu_n^2$ where the ν_n's are the positive solutions of $\tan \nu = -\nu$, and the normalized eigenfunctions are $\phi_n(x) = c_n \sin \nu_n x$, $c_n = [2/(1 + \cos^2 \nu_n)]^{1/2}$.
9. $w(x) = r(x)^{-1} \exp \int^x r^{-1}(t) q(t)\, dt$.
10. $\lambda_n = \left(\dfrac{n\pi}{\log b}\right)^2, \quad \phi_n(x) = \sqrt{\dfrac{2}{\log b}} \sin \dfrac{n\pi \log x}{\log b},$

$$g(x) = \sum_1^\infty \frac{4}{\pi(2n-1)} \sin \frac{(2n-1)\pi \log x}{\log b}.$$

11. $\lambda_n = \dfrac{1}{4} + \left(\dfrac{n\pi}{\log b}\right)^2, \qquad \phi_n(x) = \sqrt{\dfrac{2}{\log b}} x^{-1/2} \sin \dfrac{n\pi \log x}{\log b}.$

Section 4.2

In Exercises 1–4, let

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{(2n-1)\pi x}{2l}\, dx.$$

1. a. $u(x,t) = \displaystyle\sum_1^\infty b_n \exp \frac{-(n-\frac{1}{2})^2 \pi^2 kt}{l^2} \sin(n - \tfrac{1}{2}) \frac{\pi x}{l}$. b. $b_n = \dfrac{200}{\pi(2n-1)}$.
2. $u(x,t) = C + \displaystyle\sum_1^\infty \left(b_n - \frac{4C}{\pi(2n-1)}\right) \exp \frac{-(n-\frac{1}{2})^2 \pi^2 kt}{l^2} \sin(n - \tfrac{1}{2}) \frac{\pi x}{l}.$
3. $u(x,t) = Ax + \displaystyle\sum_1^\infty \left(b_n + \frac{(-1)^n 8Al}{(2n-1)^2 \pi^2}\right) \exp \frac{-(n-\frac{1}{2})^2 \pi^2 kt}{l^2} \sin(n - \tfrac{1}{2}) \frac{\pi x}{l}.$
4. $$u(x,t) = \frac{R(2lx - x^2)}{2k} + \sum_1^\infty \left(b_n - \frac{16Rl^2}{k\pi^3(2n-1)^3}\right) \exp \frac{-(n-\frac{1}{2})^2 \pi^2 kt}{l^2} \sin(n - \tfrac{1}{2}) \frac{\pi x}{l}.$$
5. $u(x,t) = (e^{-2t} - e^{-kt})(\sin x)/(k-2)$ if $k \neq 2$, $u(x,t) = te^{-2t} \sin x$ if $k = 2$.
6. $u(x,t) = \dfrac{4R}{\pi} \displaystyle\sum_{n \text{ odd}} \frac{e^{-ct} - e^{-n^2\pi^2 kt/l^2}}{n((n^2\pi^2 k/l^2) - c)} \sin \frac{n\pi x}{l}$ if $c \neq n^2\pi^2 k/l^2$ for any odd n.

   If $c = N^2\pi^2 k/l$ with N odd, the coefficient of $\sin(N\pi x/l)$ is $N^{-1} te^{-N^2\pi^2 kt/l^2}$.
7. b. $u(x,t) = Rt$. c. $u(x,t) = R(1 - e^{-ct})/c$.

8. $u(x,t)=\sum_1^\infty \frac{200\sin^2\nu_n l}{(lb+\sin^2\nu_n l)\cos\nu_n l}e^{-\nu_n^2 kt}\cos\nu_n x$ where $\nu_1,\nu_2\ldots$ are the positive solutions of $\tan\nu l=b/\nu$.
9. For each t, let $\sum a_n(t)\phi_n(x)$ be the expansion of $f(x,t)$ in terms of the eigenfunctions $\phi_n(x)$. Then

$$u(x,t)=\sum_1^\infty \phi_n(x)e^{-\lambda_n t}\int_0^t e^{\lambda_n s}a_n(s)\,ds.$$

10. b. $u(x,t)=\dfrac{le^{-ht}}{2}-\dfrac{4l}{\pi^2}\sum_1^\infty \dfrac{e^{-[h+(2n-1)^2\pi^2kt/l^2]t}}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{l}.$

c. $u(x,t)=100\dfrac{\sinh\alpha x}{\sinh\alpha l}+200\pi\sum_1^\infty \dfrac{(-1)^n ne^{-n^2\pi^2kt/l^2}}{\alpha^2l^2+n^2\pi^2}\sin\dfrac{n\pi x}{l},\quad \alpha=\sqrt{\dfrac{h}{k}}.$

Section 4.3

2. $u(x,t)=\dfrac{bl}{2}-\dfrac{4bl}{\pi^2}\sum_1^\infty \dfrac{1}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{l}\cos\dfrac{(2n-1)\pi ct}{l}.$
3. $u(x,t)=\dfrac{gx(x-l)}{2c^2}+\dfrac{4l^2g}{\pi^3c^2}\sum_1^\infty \dfrac{1}{(2n-1)^3}\sin\dfrac{(2n-1)\pi x}{l}\cos\dfrac{(2n-1)\pi ct}{l}.$
5. $u(x,t)=\sum_1^\infty (a_n\cos\lambda_n t+b_n\sin\lambda_n t)\sin\dfrac{n\pi x}{l},\quad \lambda_n^2=\dfrac{n^2\pi^2c^2}{l^2}+a^2.$
6. $u(x,t)=\sum_1^\infty e^{-kt}(a_n\cos\lambda_n t+b_n\sin\lambda_n t)\sin\dfrac{n\pi x}{l},\quad \lambda_n^2=\dfrac{n^2\pi^2c^2}{l^2}-k^2.$ If $k>\pi c/l$, some of the λ_n's will be imaginary, and $\cos\lambda_n t$ and $\sin\lambda_n t$ will in effect be hyperbolic functions. In this case, $e^{-kt}\cos\lambda_n t$ and $e^{-kt}\sin\lambda_n t$ are exponentially decreasing in t since $|\lambda_n|<k$.
7. $v(x,t)=\dfrac{2k}{\pi}\sum_1^\infty \dfrac{(-1)^{n+1}}{n(n^2c^2-k^2)}(k\sin kt-nc\sin nct)\sin nx.$ If $k=Nc$, the Nth term of the series is $(-1)^N(\sin Nct+Nct\cos Nct)(\sin Nx)/2N^2c$.

Section 4.4

1. $u(x,y)=\dfrac{8l^2}{\pi^3}\sum_1^\infty \dfrac{1}{(2n-1)^3\sinh(2n-1)\pi}\sin\dfrac{(2n-1)\pi x}{l}\sinh\dfrac{(2n-1)\pi y}{l}.$
2. $u(x,y)=\dfrac{y}{2}-\dfrac{4l}{\pi^2}\sum_1^\infty \dfrac{1}{(2n-1)^2\sinh(2n-1)\pi}\cos\dfrac{(2n-1)\pi x}{l}\sinh\dfrac{(2n-1)\pi y}{l}.$
3. $u(x,y)=C+\sum_1^\infty \dfrac{la_n}{n\pi\sinh n\pi}\cos\dfrac{n\pi x}{l}\cosh\dfrac{n\pi y}{l}$, where the a_n's are the coefficients of the Fourier cosine series for f (with $a_0=0$).
4. $u(x,y)=\sum_1^\infty c_n e^{-n\pi y/l}\sin\dfrac{n\pi x}{l},\quad c_n=\dfrac{2}{l}\int_0^l f(x)\sin\dfrac{n\pi x}{l}\,dx.$

5. a. $u(r,\theta) = \sum_{-\infty}^{\infty} c_n \frac{r^n + r_0^{2n} r^{-n}}{1 + r_0^{2n}} e^{in\theta}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta.$

   b. $u(r,\theta) = 1 + 2\frac{r^2 + r_0^2}{r(1 + r_0^2)} \sin\theta.$

7. Let $g(r) = \sum c_n \sin \frac{n\pi \log r}{\log r_0}$ and $h(r) = \sum d_n \sin \frac{n\pi \log r}{\log r_0}$. Then $u(r,\theta) = \sum_1^{\infty} (a_n e^{n\pi\theta/\log r_0} + b_n e^{-n\pi\theta/\log r_0}) \sin \frac{n\pi \log r}{\log r_0}$, where $a_n + b_n = c_n$ and $a_n e^{n\pi\beta/log r_0} + b_n e^{-n\pi\beta/\log r_0} = d_n$.

8. a. $u(r,\theta) = \sum_1^{\infty} b_n r^{n\pi/\beta} \sin \frac{n\pi\theta}{\beta}, \quad b_n = \frac{2}{\beta}\int_0^{\beta} f(\theta) \sin \frac{n\pi\theta}{\beta}\, d\theta.$

Section 4.5

2. Like Example 1, but replace
$$\left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \pi c t \sqrt{\frac{m^2}{l^2} + \frac{n^2}{L^2}} \quad \text{by} \quad e^{-kt} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} t \sqrt{\frac{(m\pi c)^2}{l^2} + \frac{(n\pi c)^2}{L^2} - k^2}.$$

3. $u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (a_{mn} \cos \lambda_{mn} t + b_{mn} \sin \lambda_{mn} t) \sin \frac{m\pi x}{l} \cos \frac{n\pi y}{L}$ where $\lambda_{mn}^2 = \frac{(m\pi c)^2}{l^2} + \frac{(n\pi c)^2}{L^2}$. There are some extra frequencies arising from the terms with $n = 0$.

4. $u(x,y,z,t) = \sum_{m,n=1}^{\infty} a_{mn} \exp\left[-\left(\frac{m^2}{l_1^2} + \frac{n^2}{l_2^2}\right) \pi^2 k t\right] \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2}$ where
$$a_{mn} = \frac{4}{l_1 l_2} \int_0^{l_1} \int_0^{l_2} f(x,y) \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2}\, dy\, dx.$$

5. $u(x,y,z) = \frac{128}{\pi} \sum \frac{(-1)^{n_1+1} \sin n_1 x \sin(2n_2 - 1)y \sin(2n_3 - 1)z}{n_1(2n_2 - 1)(2n_3 - 1)[n_1^2 + (2n_2 - 1)^2 + (2n_3 - 1)^2]}$, the sum being over all positive integers n_1, n_2, and n_3.

6. If $\rho(x,y,z) = \sum c_{n_1 n_2 n_3} \exp \pi i(n_1 x + n_2 y + n_3 z)/l$ with $c_{000} = 0$, then
$$u(x,y,z) = \frac{4l^2}{\pi} \sum_{(n_1,n_2,n_3) \neq (0,0,0)} \frac{c_{n_1 n_2 n_3}}{n_1^2 + n_2^2 + n_3^2} \exp \frac{\pi i(n_1 x + n_2 y + n_3 z)}{l}.$$

Section 5.4

1. $2b^2 \sum \frac{\lambda_k^2 - 4}{\lambda_k^3 J_1(\lambda_k)} J_0(\lambda_k x/b).$

2. $8b^2 \sum \frac{J_0(\lambda_k x/b)}{\lambda_k^3 J_1(\lambda_k)}.$

3. $2b \sum \left[\frac{1}{\lambda_k J_1(\lambda_k)} - \frac{1}{\lambda_k^3 J_1(\lambda_k)^2} \int_0^{\lambda_k} J_0(s)\, ds\right] J_0(\lambda_k x/b).$

4. $\sum \frac{J_1(\lambda_k/2)}{\lambda_k J_1(\lambda_k)^2} J_0(\lambda_k x/b).$

5. $2\sum \frac{\lambda_k J_1(\lambda_k)}{(\lambda_k^2+c^2)J_0(\lambda_k)^2} J_0(\lambda_k x/b) \quad (c \neq 0)$. When $c=0$ we have $J_1(\lambda_k) = -J_0'(\lambda_k) = 0$ for all k. The eigenvalue zero must be included, and the expansion of f is simply $f(x)=1$.

6. $2\sum \frac{J_1(\lambda_k x)}{\lambda_k J_2(\lambda_k)}$.

7. $2\sum \frac{\lambda_k J_{\nu+1}(\lambda_k)}{(\lambda_k^2-\nu^2)J_\nu(\lambda_k)^2} J_\nu(\lambda_k x)$.

8. $\sum \frac{4J_1(\lambda_k/2)-\lambda_k J_0(\lambda_k/2)}{(\lambda_k^2-1)J_1(\lambda_k)^2} J_1(\lambda_k x/2)$.

9. $\|\phi_k\|^2 = lJ_1(\lambda_k)^2$.

Section 5.5

1. $u(r,t) = 2A\sum \frac{\lambda_k J_1(\lambda_k)}{(\lambda_k^2+b^2c^2)J_0(\lambda_k)^2} J_0\left(\frac{\lambda_k r}{b}\right) e^{-\lambda_k^2 t/b^2}$,
$\lambda_k J_0'(\lambda_k) + bcJ_0(\lambda_k) = 0$.

2. $u(r,t) = B + \frac{2\rho(A-B)}{(\rho+\delta)} \sum \frac{J_1\big(\lambda_k\rho/(\rho+\delta)\big)}{\lambda_k J_1(\lambda_k)^2} J_0\left(\frac{\lambda_k r}{\rho+\delta}\right) e^{-\lambda_k^2 t/(\rho+\delta)^2}$,
$J_0(\lambda_k) = 0$.

3. a. $u(r,\theta,t) = 2a\cos\theta \sum \frac{\lambda_k J_2(\lambda_k)}{(\lambda_k^2-1)J_1(\lambda_k)^2} J_1(\lambda_k r)e^{-\lambda_k^2 t} + b, \quad J_1'(\lambda_k) = 0$.

   b. $u(r,\theta,t) = 2a\cos\theta \sum \frac{J_1(\lambda_{k,1} r)}{\lambda_{k,1}J_2(\lambda_{k,1})} e^{-\lambda_{k,1}^2 t} + 2b\sum \frac{J_0(\lambda_{k,0} r)}{\lambda_{k,0}J_1(\lambda_{k,0})} e^{-\lambda_{k,0}^2 t}$,
   $J_1(\lambda_{k,1}) = 0, \quad J_0(\lambda_{k,0}) = 0$.

4. a. $v(r) = a(1-r^2)/4$.

   b. $u(r,t) = \frac{a}{4}(1-r^2) - 2a\sum \frac{J_0(\lambda_k r)}{\lambda_k^3 J_1(\lambda_k)} e^{-\lambda_k^2 t}, \quad J_0(\lambda_k) = 0$.

5. $u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} (a_{kn}\cos n\theta + b_{kn}\sin n\theta) J_n\left(\frac{\lambda_{k,n} r}{b}\right) \sinh\left(\frac{\lambda_{k,n} z}{b}\right)$,
$b_{kn} = \frac{2}{b^2\pi\sinh\lambda_{k,n}} \int_{-\pi}^{\pi}\int_0^b g(r,\theta) \frac{J_n(\lambda_{k,n} r)}{J_{n+1}(\lambda_{k,n})^2} \sin n\theta\, r\,dr\,d\theta$, and similarly for a_{kn}, where $J_n(\lambda_{k,n}) = 0$.

6. $u(r,z) = a_0 z + \sum_1^{\infty} a_k J_0(\lambda_k r)\sinh\lambda_k z, \quad J_0'(\lambda_k) = 0, \quad a_0 = 2\int_0^1 rf(r)\,dr$,
$a_k = \frac{2}{J_0(\lambda_k)^2\sinh\lambda_k}\int_0^1 rf(r)J_0(\lambda_k r)\,dr \quad (k>0)$.

7. $u(r,\theta,z,t) = \sum (a_{mnk}\cos n\theta + b_{mnk}\sin n\theta) J_n(\lambda_{k,n} r)\sin m\pi z\cos\mu_{mnk}ct$,
$\mu_{mnk}^2 = m^2\pi^2 + \lambda_{k,n}^2, \quad J_n'(\lambda_{k,n}) = 0$.

Section 5.6

1. $u(r,\theta,z) = \sum (a_{mn}\cos m\theta + b_{mn}\sin n\theta) I_m\left(\frac{n\pi r}{l}\right) \sin\frac{n\pi z}{l}$, where

$b_{mn} = \dfrac{2}{\pi l I_m(n\pi b/l)} \displaystyle\int_0^l \int_{-\pi}^{\pi} h(\theta, z) \sin m\theta \sin \frac{n\pi z}{l}\, d\theta\, dz$, and similarly for a_{mn}.

3. $a_n = 0, \quad b_{2n} = 0, \quad b_{2n-1} = \dfrac{8l^3}{c\pi^4(2n-1)^4}$.
4. $u(z,t) = \sum \left(a_k \cos \dfrac{\lambda_k ct}{2\sqrt{l}} + b_k \sin \dfrac{\lambda_k ct}{2\sqrt{l}}\right) J_0(\lambda_k \sqrt{z/l}), \quad J_0(\lambda_k) = 0.$

Section 6.2

3. $y = a_0\left[1 + \sum_1^\infty c_k x^{2k}\right] + a_1\left[x + \sum_1^\infty d_k x^{2k+1}\right]$ where a_0, a_1 are arbitrary and

$$c_k = (-1)^k \frac{\nu(\nu-2)\cdots(\nu-2k+2)(\nu+1)(\nu+3)\cdots(\nu+2k-1)}{(2k)!},$$
$$d_k = (-1)^k \frac{(\nu-1)(\nu-3)\cdots(\nu-2k+1)(\nu+2)(\nu+4)\cdots(\nu+2k)}{(2k+1)!}.$$

6. $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x), \quad x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x), \quad x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{7}{35}P_0(x).$
7. $\displaystyle\sum_0^\infty (-1)^k \frac{(4k+3)(2k)!}{(2k+2)2^{2k}(k!)^2} P_{2k+1}.$
8. $\displaystyle\frac{1}{4}P_0 + \frac{1}{2}P_1 + \sum_1^\infty \left[\frac{P_{2k+2}(0)}{2(4k+3)} - \frac{(4k+1)P_{2k}(0)}{(4k+3)(4k-1)} + \frac{P_{2k-2}(0)}{2(4k-1)}\right] P_{2k}.$

Section 6.3

1. $u = \frac{1}{4} + \frac{1}{2}r\cos\phi + \sum_1^\infty c_k r^{2k} P_{2k}(\cos\phi)$ where

$$c_k = \frac{P_{2k+2}(0)}{2(4k+3)} - \frac{(4k+1)P_{2k}(0)}{(4k+3)(4k-1)} + \frac{P_{2k-2}(0)}{2(4k-1)}.$$

2. In spherical coordinates, $u = \frac{1}{3} - \frac{1}{3}r^2 P_2(\cos\phi) + \frac{1}{6}r^2(\cos 2\theta)P_2^2(\cos\theta)$, or $u = \frac{1}{3} - \frac{1}{6}r^2(3\cos^2\phi - 1) + \frac{1}{2}r^2(\cos 2\theta)\sin^2\phi$. In Cartesian coordinates, $u = \frac{1}{3}(1 + 2x^2 - y^2 - z^2)$.
3. $u = \sum_{m,n} c_{mn} r^{-n-1} e^{im\theta} P_n^{|m|}(\cos\phi)$ where c_{mn} is given by (6.26).
4. $u = \sum c_n r^{2n+1} P_{2n+1}(\cos\phi)$, $c_n = (4n+3)\int_0^{\pi/2} f(\phi) P_{2n+1}(\cos\phi)\sin\phi\, d\phi$. When $f(\phi) \equiv 1$, $c_n = (-1)^n(4n+3)(2n)!/(2n+2)2^{2n}(n!)^2$.
5. $u = \sum c_n r^{2n} P_{2n}(\cos\phi)$ where $c_n = (4n+1)\int_0^{\pi/2} f(\phi) P_{2n}(\cos\phi)\sin\phi\, d\phi$.
6. $u = \sum \left[c_{mn} \dfrac{r^n - a^{2n+1}r^{-n-1}}{b^n - a^{2n+1}b^{-n-1}} + d_{mn} \dfrac{r^n - b^{2n+1}r^{-n-1}}{a^n - b^{2n+1}a^{-n-1}}\right] e^{im\theta} P_n^{|m|}(\cos\phi)$ where c_{mn} and d_{mn} are the coefficients for the expansions of g and f in spherical harmonics given by (6.26).
7. $u = \sum r^{-1/2} J_{n+(1/2)}(\mu_l^n r) e^{im\theta} P_n^{|m|}(\cos\phi)[a_{lmn}\cos\mu_l^n ct + b_{lmn}\sin\mu_l^n ct]$ where $\mu_1^n, \mu_2^n, \ldots$ are the positive zeros of $J_{n+(1/2)}$.
8. $u = \sum c_n e^{-n(n+1)kt} P_n(x)$ where $c_n = \frac{1}{2}(2n+1)\int_{-1}^1 f(s) P_n(s)\, ds$.

Section 6.4

2. $y = a_0 y_0 + a_1 y_1$ where

$$y_0 = 1 + \sum_1^\infty \frac{(-\lambda)(4-\lambda)\cdots(4k-4-\lambda)}{(2k)!}x^{2k},$$
$$y_1 = x + \sum_1^\infty \frac{(2-\lambda)(6-\lambda)\cdots(4k-2-\lambda)}{(2k+1)!}x^{2k+1}.$$

4. $\dfrac{(2m)!}{2^{2m}}\displaystyle\sum_0^m \frac{H_{2k}}{(2k)!(m-k)!}$.

5. $e^{a^2/4}\displaystyle\sum_0^\infty \frac{a^n}{2^n n!}H_n$.

6. $\dfrac{1}{2}H_0 + \dfrac{1}{2\sqrt{\pi}}\displaystyle\sum_0^\infty \frac{(-1)^k}{2^{2k}(2k+1)k!}H_{2k+1}$.

9. $u(s,t,z) = \sum c_{mn}\dfrac{h_n(i\sqrt{2m\pi}\,t)}{h_n(i\sqrt{2m\pi})}h_n(\sqrt{2m\pi}\,s)\sin m\pi z$ where

$$c_{mn} = \frac{\sqrt{2m}}{2^{n-1}n!}\int_0^1\int_{-\infty}^\infty f(s,z)h_n(\sqrt{2m\pi}\,s)\sin m\pi z\,ds\,dz.$$

Section 6.5

1. a. $1 + \displaystyle\sum_1^\infty \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{n!(\alpha+1)(\alpha+2)\cdots(\alpha+n)}x^n$.

   b. $x^{-\alpha} + \displaystyle\sum_1^\infty \frac{(-\alpha-\lambda)(1-\alpha-\lambda)\cdots(n-1-\alpha-\lambda)}{n!(1-\alpha)(2-\alpha)\cdots(n-\alpha)}x^{n-\alpha}$.

5. $\Gamma(\nu+\alpha+1)\Gamma(\nu+1)\displaystyle\sum_0^\infty \frac{(-1)^n L_n^\alpha}{\Gamma(\alpha+n+1)\Gamma(\nu-n+1)}$

6. $(b+1)^{-\alpha-1}\displaystyle\sum_0^\infty \left(\frac{b}{b+1}\right)^n L_n^\alpha$

Section 7.1

1. a. L^1. b. L^2. c. Neither. d. Both.

3. a. $f*f(x) = x+2$ if $-2 \le x \le 0$, $= 2-x$ if $0 \le x \le 2$, $= 0$ otherwise. $f*f*f(x) = \frac{1}{2}(x+3)^2$ if $-3 \le x \le -1$, $= 3-x^2$ if $-1 \le x \le 1$, $= \frac{1}{2}(3-x)^2$ if $1 \le x \le 3$, $= 0$ otherwise.

   b. $f_\epsilon * g(x) = 2x^3 + (8\epsilon^2 - 2)x$.

4. $f*g(x) = \sqrt{\pi/3}\,e^{-2x^2/3}$.

Section 7.3

1. $u(x) = g * e^{-|x|} = e^{-x} \int_{-\infty}^{x} e^{y} g(y)\,dy + e^{x} \int_{x}^{\infty} e^{-y} g(y)\,dy.$

4. $\dfrac{1}{2b}\left[\displaystyle\int \frac{\sin(\pi y/b) f(t)\,dt}{\cosh[\pi(x-t)/b] - \cos(\pi y/b)} + \int \frac{\sin(\pi y/b) g(t)\,dt}{\cosh[\pi(x-t)/b] + \cos(\pi y/b)}\right].$

5. $u(r,z) = \dfrac{1}{\pi}\displaystyle\int \frac{e^{iz\xi}\sin l\xi}{\xi}\frac{I_0(r\xi)}{I_0(a\xi)}\,d\xi$ (I_0 = modified Bessel function of order zero).

7. $f(t) = \displaystyle\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) e^{i(b-a)(\Omega t - n\pi)/2\Omega} \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}.$

Section 7.4

1. a. $\dfrac{\xi}{\xi^2 + k^2}$. b. $\dfrac{k}{\xi^2 + k^2}$. c. $\dfrac{2}{(\xi^2+1)^2}$. d. $\dfrac{2\xi}{(\xi^2+1)^2}$.

4. (First part) $u(x,t) = \dfrac{1}{2\sqrt{\pi k t}} \displaystyle\int_0^{\infty} f(y)\left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}\right] dy.$

5. $u(x,y) = \displaystyle\int_0^{\infty} K(x,y,z) f(z)\,dz + \int_0^{\infty} K(y,x,z) g(z)\,dz$ where
$K(s,t,z) = \dfrac{1}{2\pi}\left[\dfrac{t}{(s-z)^2 + t^2} - \dfrac{t}{(s+z)^2+t^2}\right].$

6. $u(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \frac{\cos x\xi \cosh y\xi}{(1+\xi^2)\cosh \xi}\,d\xi.$

7. $u(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \frac{\xi \sin x\xi \sinh y\xi}{(1+\xi^2)\sinh \xi}\,d\xi.$

8. $u(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \frac{\sin c\xi \cos x\xi \cosh(y\sqrt{\xi^2+h})}{\xi \cosh\sqrt{\xi^2+h}}\,d\xi.$

9. a. $f(r) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \widetilde{f}(\nu)\sin(\nu \log r)\,d\nu$, where $\widetilde{f}(\nu) = \displaystyle\int_0^1 f(r)\sin(\nu\log r)\frac{dr}{r}.$

b. $u(r,\theta) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left[A(\nu)e^{\nu\theta} + B(\nu)e^{-\nu\theta}\right]\sin(\nu\log r)\,d\nu$ where
$A(\nu) = \dfrac{\widetilde{g}(\nu) - e^{-\nu b}\widetilde{f}(\nu)}{e^{\nu b} - e^{-\nu b}},\quad B(\nu) = \dfrac{e^{\nu b}\widetilde{f}(\nu) - \widetilde{g}(\nu)}{e^{\nu b} - e^{-\nu b}}$ ($\widetilde{f}$, $\widetilde{g}$ as in part (a)).

Section 7.5

1. a. $u(x,y) = \dfrac{-1}{4\pi^2}\displaystyle\iint \frac{\widehat{f}(\xi,\eta)}{\xi^2 + 2\eta^2 - 3i\xi + 4} e^{i(\xi x + \eta y)}\,d\xi\,d\eta.$

b. $u(x,y) = \dfrac{1}{4\pi^2}\displaystyle\iint \frac{\widehat{f}(\xi,\eta)}{\xi^4 + \eta^2 + 2} e^{i(\xi x + \eta y)}\,d\xi\,d\eta.$

5. $4\pi(1+|\xi|^2)^{-1}$; $u = f * K$ where $K(\mathbf{x}) = e^{-|\mathbf{x}|}/4\pi|\mathbf{x}|.$

Section 8.1

1. a. $\dfrac{a}{z^2 - a^2}$. b. $\dfrac{z}{z^2-a^2}$.

2. $\dfrac{z^2+2}{z(z^2+4)}$.
3. a. $(\sqrt{\pi}/2a)e^{z^2/4a^2}\operatorname{erfc}(z/2a)$. b. $z^{-1}e^{z^2/4a^2}\operatorname{erfc}(z/2a)$.
4. a. $e^{az}E_1(az)$. b. $a^{-1} - ze^{az}E_1(az)$.
5. $\dfrac{1}{z^2} - \dfrac{e^{-z}(2z+1)}{z^2(z+1)}$.
6. $\sqrt{\pi/z}\, e^{a/4z}\operatorname{erfc}(\sqrt{a/4z})$.
7. $\dfrac{\Gamma(n+\alpha+1)(z-1)^n}{n!z^{n+\alpha+1}}$.
8. a. $1+e^{-2t}$. b. $1-e^{-2t}-2te^{-2t}$. c. $1-\cos t$.
9. $(a-b)^{-1}(e^{at}-e^{bt})$ if $b \neq a$, te^{at} if $b = a$.
10. a. $\sin t$. b. $\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}t^{a+b-1}$. c. $\frac{2}{3}\sin t - \frac{1}{3}\sin 2t$.
12. a. $\dfrac{1}{z^2} - \dfrac{\pi}{z\sinh \pi z}$. b. $\dfrac{1}{z}\tanh\dfrac{\pi z}{2}$. c. $\dfrac{1}{z^2}\tanh\dfrac{\pi z}{2}$.
13. $\dfrac{1}{(z-1)\sqrt{z}}$ and $\dfrac{1}{z\sqrt{z+1}}$.
14. a. $\dfrac{\sqrt{\pi a}}{2z^{3/2}}e^{-a/4z}$. b. $\pi\operatorname{erf}(\sqrt{a/4z})$. c. $\sqrt{\pi/z}\, e^{-a/4z}$.
15. $z^{-1}e^{-a/4z}$.
16. $\dfrac{2^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}(z^2+1)^{-\nu-(1/2)}$.

Section 8.2

2. $e^{-2t} - 2te^{-2t} + 2e^{-4t}$.
3. $3 + e^t\cos 2t + 4e^t\sin 2t$.
4. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\sum_1^\infty \dfrac{\cos(2n-1)t}{(2n-1)^2}$.
5. $1 + \dfrac{4}{\pi}\sum_1^\infty \dfrac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2 t/4}$.
6. $\mathscr{L}^{-1}[e^{-a\sqrt{z}}] = \dfrac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$, $\mathscr{L}^{-1}[z^{-1}e^{-a\sqrt{z}}] = \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$.
7. $E_1(t)$.
8. (See 6.)

Section 8.3

1. $u(t) = \dfrac{1}{2(\omega^2-4)}(\omega\sin 2t - 2\sin\omega t)$ if $\omega \neq 2$, $u(t) = \frac{1}{8}(\sin 2t - 2t\cos 2t)$ if $\omega = 2$.
2. $u(t) = \int_0^t f(t-s)se^{-2s}\,ds + c_0e^{-2t} + (c_1+2c_0)te^{-2t}$.
3. $u(t) = e^{-t}\sin t + \frac{1}{2}H(t-\pi)\left[1 + e^{\pi-t}\cos t + e^{\pi-t}\sin t\right] - \frac{1}{2}H(t-2\pi)\left[1 - e^{2\pi-t}\cos t - e^{2\pi-t}\sin t\right]$.

4. $u(t) = \int_0^t f(t-s)[\cosh s - 1]\,ds + 1 - \sinh t$.
5. $u(t) = -t + \frac{1}{2}\sinh t + \frac{1}{2}\sin t$.
6. $u_1(t) = e^t[c_1\cos 2t + (c_1 - c_2)\sin 2t]$, $u_2(t) = e^t[c_2\cos 2t + (2c_1 - c_2)\sin 2t]$.
7. $u_1(t) = 5e^{-t} - 2e^{4t} + 2te^{4t}$, $u_2(t) = 5e^{-t} + 3e^{4t} - 3te^{4t}$.
8. $u_1(t) = \sin t + 1 + t + \frac{1}{2}t^2$, $u_2(t) = \cos t - t - \frac{1}{2}t^2$.
9. a. $u(t) = ct^2 - t$. b. $u(t) = c_1t^2 + c_2t^{-1} - t$.
10. $u(t) = c_1e^t + c_2(t+1)$.
11. $N_4(t) = A\left[1 - \dfrac{c_2c_3e^{-c_1t}}{(c_2-c_1)(c_3-c_1)} - \dfrac{c_1c_3e^{-c_2t}}{(c_1-c_2)(c_3-c_2)} - \dfrac{c_1c_2e^{-c_3t}}{(c_1-c_3)(c_2-c_3)}\right]$.

Section 8.4

1. $u(x,t) = \dfrac{x}{\sqrt{4\pi k}}\displaystyle\int_0^t f(t-s)e^{-as}s^{-3/2}e^{-x^2/4ks}\,ds$.
2. $u(x,t) = \operatorname{erfc}\dfrac{x}{\sqrt{4kt}}$ for $0 < t < 1$, $u(x,t) = \operatorname{erfc}\dfrac{x}{\sqrt{4kt}} - \operatorname{erfc}\dfrac{x}{\sqrt{4k(t-1)}}$ fc
$t \geq 1$.

Section 8.5

1. $u(t) = 2a^{-2}(\cosh at - 1)$.
2. $u(t) = f(t) + \frac{1}{2}\int_0^t f(t-s)[\sinh s - \sin s]\,ds$.
3. $u(t) = \dfrac{ae^{-t}}{\sqrt{a^2-1}}\sin t\sqrt{a^2-1}$ if $a > 1$, $u(t) = te^{-t}$ if $a = 1$, $u(t) = \dfrac{ae^{-t}}{\sqrt{1-a^2}}\sinh t\sqrt{1-a^2}$ if $a < 1$.
4. $u = \pm\sqrt{30}\,t^2e^{-3t}$.
6. $u(t) = \int_0^t[f'''(s) + f''(s)]\operatorname{erf}\sqrt{s}\,ds$.
8. a. $r(t) = cN_0$ b. $r(t) = 2cN_0 - (c-1)N_0e^{-t}$.
9. When $\alpha = \frac{1}{2}$, the curves are straight lines.

Section 8.6

1. $e^{z^2}\operatorname{erfc} z \sim \dfrac{1}{\sqrt{\pi}}\displaystyle\sum_0^\infty \frac{(-1)^n(2n)!}{2^{2n}n!}z^{-2n-1}$.
2. a. $1/2\sqrt{\pi}$.
3. $\Gamma(a,z) \sim z^ae^{-z}\left[1 + \sum_1^\infty(a-1)\cdots(a-n)z^{-n-1}\right]$.
4. $\displaystyle\int_0^\infty \frac{e^{-zt}}{t^2+1}\,dt \sim \sum_0^\infty(-1)^n(2n)!z^{-2n-1}$.
5. $H_\nu^{(1)}(w) \sim e^{i[z-(\nu\pi/2)-(\pi/4)]}\sqrt{\dfrac{2}{\pi z}}\displaystyle\sum_0^\infty \frac{\Gamma(k+\frac{1}{2}+\nu)\Gamma(k+\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)\,k!}(2iw)^{-k}$,
$H_\nu^{(2)}(w) \sim e^{-i[z-(\nu\pi/2)-(\pi/4)]}\sqrt{\dfrac{2}{\pi z}}\displaystyle\sum_0^\infty \frac{\Gamma(k+\frac{1}{2}+\nu)\Gamma(k+\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)\,k!}(-2iw)^{-k}$.

Section 9.1

4. $f'(x) = 2xH(2-x) + 2H(x-1) + 2\delta(x-1) - 4\delta(x-2)$.

Section 9.4

6. a. $-\xi^{-2}(\frac{2}{3}e^{2i\xi} - 2e^{i\xi/2} + \frac{7}{3}e^{-i\xi} - e^{-2i\xi})$.
   b. $-\xi^{-2}(-2e^{i\xi} + 3 + \frac{1}{2}e^{-i\xi} - 2e^{-2i\xi} + \frac{1}{2}e^{-3i\xi})$.
7. $i\xi^{-3}(2e^{2i\xi} - 2e^{i\xi} - 8i\xi + 6 - 6e^{-i\xi})$.

Section 10.1

1. $G(x,y) = (x-y)e^{-2(x-y)}\left[H(x-y) - H(-y)\right]$.
2. $G(x,y) = \left[e^{-4(x-y)} - e^{-5(x-y)}\right]\left[H(x-y) - H(-y)\right]$.
3. $G(x,y) = v(x-y)\left[H(x-y) - H(-y)\right]$ where $v(x) = a^3\left[e^{ax}(\sin ax - \cos ax) + e^{-ax}(\sin ax + \cos ax)\right]$, $a = 2^{-1/2}$.
4. $G(x,y) = (x^{-1} - yx^{-2})\left[H(x-y) - H(1-y)\right]$.
5. $G(x,y) = \dfrac{(x-x^2)(2-y)}{y^3}$ for $x < y$, $= \dfrac{(2x-x^2)(1-y)}{y^3}$ for $x > y$.

In the following formulas, $x_- = \min(x,y)$ and $x_+ = \max(x,y)$.

6. $G(x,y) = -\sin\mu x_- \cos\mu(x_+ - \frac{1}{2}\pi)/\mu\cos\frac{1}{2}\mu\pi$.
7. $G(x,y) = (\sin\mu x_-)[\sin\mu(x_+ - 1) - \mu\cos\mu(x_+ - 1)]/(\mu\sin\mu + \mu^2\cos\mu)$.
8. $G(x,y) = \dfrac{e^{i\mu(x_+ - x_-)}}{2i\mu(1-e^{i\mu})} - \dfrac{e^{i\mu(x_- - x_+)}}{2i\mu(1-e^{-i\mu})} = \dfrac{\cos\mu(x_+ - x_- - \frac{1}{2})}{2\mu\sin\frac{1}{2}\mu}$.
9. $G(x,y) = -\mu^{-1}(\sin\mu x_-)e^{i\mu x_+}$.
10. $G(x,y) = (i\mu)^{-1}(\cos\mu x_-)e^{i\mu x_+}$.
11. $G(x,y) = -\frac{1}{3}e^{x-y}$ for $x < y$, $= -\frac{1}{3}e^{-2(x-y)}$ for $x > y$.

Section 10.2

8. $G(x,t;y,s) = H(t-s)\displaystyle\sum_1^\infty \frac{2}{n\pi c}\sin\frac{n\pi c(t-s)}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi y}{l}$.
9. $G(\mathbf{x},\mathbf{y}) = -\dfrac{4}{\pi^2}\displaystyle\sum_{m,n=1}^\infty \frac{1}{m^2+n^2}\sin m\pi x_1 \sin m\pi y_1 \sin n\pi x_2 \sin n\pi y_2$.

Section 10.3

1. $\lambda_n = (2n-1)^2$ for $n = 1,2,3,\ldots$; $\phi_n(x) = \sin(2n-1)x$, $\|\phi_n\|^2 = \frac{1}{4}\pi$.
2. $\lambda_n = \mu_n^2$ where $\mu_1, \mu_2, \ldots$ are the positive solutions of $\tan\mu + \mu = 0$; $\phi_n(x) = \sin\mu_n x$; $\|\phi_n\|^2 = \frac{1}{2}(\cos^2\mu_n + 1)$.

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