1. Introduction to Linear Ordinary Differential Equations

[!Question] Question: What is a Linear Ordinary Differential Equation (ODE)? A linear ODE is an equation of the form

$$y^{(n)} = \sum_{i=0}^{n-1} a_i y^{(i)}$$

where $y^{(i)}$ is the i-th derivative of the function real/complex valued function $y(t)$ and a_i are either constants or functions of t. In this definition n gives the order of the ODE.

In an ODE the variable of interest is the unknown function y.

[!example] Example Linear First Order ODE ODE: $y' = 2y + 3$ Solution: $y = ce^{2t} - \frac{3}{2}$ >[!Calculation]- Calculation (Separation of Variables) >

$$\begin{aligned}
\frac{dy}{dt} &= 2y + 3 \\
> \frac{dy}{2dt} &= y + \frac{3}{2} \\
> \frac{1}{y+\frac{3}{2}}\, dy &= 2\, dt \\
> \int \frac{1}{y+\frac{3}{2}}\, dy &= \int 2\, dt \\
> \ln\left(y + \frac{3}{2}\right) &= 2t + c_0 \\
> y + \frac{3}{2} &= e^{2t}e^{c_0} \\
> y &= ce^{2t} - \frac{3}{2} >
\end{aligned}$$

Note, solving an ODE yields an integration constant c. Any choice of c will be a solution to the ODE.

![[ODE_example_1.png | center | 500]]

Figure: Direction Field of ODE with solutions at $y_0 = (-20, -15, -10, -5, 0, 5, 10, 15)$ with $y_0 = y(0)$

[!remark] The solution for a first order ODE characterizes infinitely many functions, dependent on the choice of the integration constant c.

[!theorem] Hypothesis: ODE LVQ Fundamental Principle 1. Let an ODE represent a prototype in an LVQ learning scheme 2. Let an (for now) arbitrary function represent a data point in the LVQ learning scheme

Main Idea: As previously remarked, an ODE **represents a set of functions prototyped by the ODE**. What if we can develop a learning scheme, such that * an ODE will be learned discriminating functions

[!Warning] Issues 1. How to initialize prototypes? 2. Given an ODE, how to determine which function out of the set of functions described by the ODE fits best to a datapoint (function)? 3. Time series data more often than not is a set of discrete points. How should we get an ODE from such a set of discrete data points? 4. How to compare an ODE (prototype) to another function (data point)? (Metric) 5. Given we solved the first issues, how to formulate an update rule?

2. Road to ODE LVQ

Issue 1 Prototype Inititialization

[!theorem] Initialization of Prototypes 1. Choose an arbitrary datapoint for a given class. 2. An ODE can be calculated given there is a function suitable for calculating its ODE

[!example] Example: Determining the ODE for a function $y(t) = \frac{c_1}{t} + c_2 t$ First, we take the derivative with respect to t

$$y' = -\frac{c_1}{t^2} + c_2$$

Solving for c_2 gives

$$c_2 = \frac{c_1}{t^2} + y'$$

Inserting this solution into the original equation gives

$$\begin{aligned} y &= \frac{c_1}{t} + \left(\frac{c_1}{t^2} + y'\right) t \\ &= \frac{c_1}{t} + \frac{c_1}{t} + y't \\ &= \frac{2c_1}{t} + y't \\ y - y't &= 2c_1 \\ yt - y't^2 &= 2c_1 \end{aligned}$$

Taking now again the derivative with respect to t yields

$$\begin{aligned} y + y't - 2y't - y''t^2 &= 0 \\ y - y't - y''t^2 &= 0 \end{aligned}$$

Issue 2 Function Choice in ODE

[!def] Definition Initial Value Problem ([[../../../Sources/nagy.pdf#page=18|Source]]) The initial value problem (IVP) is to find all solutions y to

$$y' = ay + b$$

that satisfy the initial condition

$$y(t_0) = y_0 \qquad \text{(Initial Condition)}$$

where a, b, t_0, y_0 are given constants. >[!note] >The differential equation has a unique solution with respect to the initial condition.

Similarly for second order ODEs etc. >[!theorem] Theorem Solutions to IVP of Second Order ODEs ([[../../../Sources/nagy.pdf#page=88 | Source]]) >If >- the functions a_0, a_1, b are continuous on a closed interval $I \subset \mathbb{R}$, >- the constant $t_0 \in I$, >- and $y_0, y_1 \in \mathbb{R}$ are arbitrary constants > >then there is a unique solution y, defined on I, of the initial value problem >

$$y'' + a_1(t)y' + a_0(t)y = b(t)$$

>with

$$y(t_0) = y_0 \quad y'(t_0) = y_1$$

[!remark] Remarks 1. Even if we would construct ODEs of higher orders, we could still find a matching function in the higher order ODE by utilizing the initial value problem. 2. Alex suggested that there might be other ways to pick a matching ODE, so this can still be changed if needed (needs more research) 3. This method for now allows a straight forward approach to pick a function from the ODE

Issue 3 Discrete Data

![[Figures/rs_1_fig_1.png|center|1000]]

Figure: How to find an ODE representing the sequence of discrete points

[!remark] Discussion: Many possibilities 1. Interpolation (Spline) 2. Possibly using Laplace transform and treat it as piecewise continuous functions (not confirmed yet) 3. Dirac Delta Expansion with Fourier Transform 4. My Approach: Discrete Fourier Transform (DFT)

[!properties] Properties of Sine and Cosine Functions The derivatives/anti-derivatives of sin and *cos* behave nicely, i.e.

$$\frac{d}{dt}\sin(ct) = c\cos(ct)$$
$$\frac{d}{dt}\cos(ct) = -c\sin(ct)$$

>[!note] >The notion *nice behavior* means sin and cos are very predictable in their derivatives/antiderivatives. The arguments inside the function stay constant and, the derivatives switch between sin and cos.

[!example] Example Step Function Let's approximate the step function the step function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < \pi \\ -1 & \text{if } \pi \leq x < 2\pi \end{cases}$$

by a partial Fourier sum $FS_N[f]_t$ with accuracy $N = 2$, we get

$$y = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \frac{1}{9}\sin(9x)\right)$$

We will omit the factor $\frac{4}{\pi}$ for simplification purposes

$$y = \sin(x) + \frac{1}{3}\sin(3x)$$

![[Figures/rs_1_fig_2.png]]

Figure: partial sum in green, step function in orange

Lets determine an ODE from the approximation and parameterize the first coefficient. Hence,

$$y = \underbrace{c_1 \sin(x)}_{\substack{\text{first}\\ \text{coefficient}}} + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \frac{1}{9}\sin(9x) \quad (1)$$

We determine the first derivative

$$y' = c_1 \cos(x) + \cos(3x) + \cos(5x) + \cos(7x) + \cos(9x)$$

We solve for c_1

$$c_1 = \frac{y' - \cos(3x) - \cos(5x) - \cos(7x) - \cos(9x)}{\cos(x)}$$

Inserting c_1 into equation (1) gives

$$\begin{aligned} y &= \frac{(y' - \cos(3x) - \cos(5x) - \cos(7x) - \cos(9x))\sin(x)}{\cos(x)} + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \frac{1}{9}\sin(9x) \\ &= \frac{y'\sin(x)}{\cos(x)} - \frac{\sin(x)(\cos(3x) + \cos(5x) + \cos(7x) + \cos(9x))}{\cos(x)} + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \frac{1}{9}\sin \end{aligned}$$

We now need to solve for y' to yield the ODE, the solution is

$$y' = \cos(3x) + \cos(5x) + \cos(7x) + \cos(9x) - \frac{\cos(x)}{\sin(x)}\left(\frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \frac{1}{9}\sin(9x) - y\right)$$

![[Figures/rs_1_fig_3.png]]

ODE with dependence on $\sin(x)$

If we solve for the 4-th coefficient ($\frac{1}{7}\sin(7x)$), we get

$$y' = \cos(x)+\cos(3x)+\cos(5x)+\cos(9x)-\frac{7\cos(7x)}{\sin(x)}\left(\sin(x)+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)+\frac{1}{9}\sin(9x)-y\right)$$

![[Figures/rs_1_fig_4.png]]

ODE with dependence on $\dfrac{1}{7}\sin(7x)$

[!Observation] Dependent on which frequency term we use, the ODE

[!question] Questions 1. Which parameter should be chosen to define the ODE, and could it be beneficial to use higher order ODEs? 2. How many should be chosen? 3. How would many parameters affect the construction?

Issue 4 + 5 Comparing ODE to Function (Metric) and updating prototype

[!observation] Idea 1. Find function with *initial value problem* (IVP) 2. Compare IVP determined function with data point function * using Dynamic Timewarping * using distance on $L^2[a,b]$ norm * using distance based on maximums norm 3. ??? * update IVP function u^*-> determine ODE -> update prototype with ODE from u^*