Harkrishan Lal Vasudeva

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With contributions from Satish Shirali

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To
Siddhant, Ashira and Shrayus

Preface

Algebraic and topological structures compatibly placed on the same underlying set lead to the notions of topological semigroups, groups and vector spaces, among others. It is then natural to consider concepts such as continuous homomorphisms and continuous linear transformations between above-said objects. By an 'operator', we mean a continuous linear transformation of a normed linear space into itself.

Functional analysis was developed around the turn of the last century by the pioneering work of Banach, Hilbert, von Neumann, Riesz and others. Within a few years, after an amazing burst of activity, it was well developed as a major branch of mathematics. It is a unifying framework for many diverse areas such as Fourier series, differential and integral equations, analytic function theory and analytic number theory. The subject continues to grow and attracts the attention of some of the finest mathematicians of the era.

A generalisation of the methods of vector algebra and calculus manifests itself in the mathematical concept of a Hilbert space, named after the celebrated mathematician Hilbert. It extends these methods from two-dimensional and three-dimensional Euclidean spaces to spaces with any finite or infinite dimension. These are inner product spaces, which allow the measurement of angles and lengths; once completed, they possess enough limits in the space so that the techniques of analysis can be used. Their diverse applications attract the attention of physicists, chemists and engineers alike in good measure.

Chapter 1 establishes notations used in the text and collects results from vector spaces, metric spaces, Lebesgue integration and real analysis. No attempt has been made to prove the results included under the above topics. It is assumed that the reader is familiar with them. Appropriate references have, however, been provided.

Chapter 2 includes in some details the study of inner product spaces and their completions. The space $L^2(X, \mathrm{M}, \mu)$, where X, M and μ denote, respectively, a nonempty set, a σ-algebra of subsets of it and an extended nonnegative real-valued measure, has been studied. The theorem of central importance in the analysis due to Riesz and Fischer, namely that $L^2(X, \mathrm{M}, \mu)$ is a complete metric space, has been proved. So has been the result, namely, the space $A(\Omega)$ of holomorphic functions defined on a bounded domain Ω is complete. To make the book useful to probabilists, statisticians, physicists, chemists and engineers, we have included many applied topics: Legendre, Hermite, Laguerre polynomials, Rademacher functions, Fourier series and Plancherel's theorem. Such applications of the abstract theory are also of significance for the pure mathematician who wants to know the origin of the subject. This chapter also contains the study of linear functionals on Hilbert spaces; more specifically, Riesz Representation Theorem, the dual of a Hilbert space is itself a Hilbert space and the fact that these spaces constitute important examples of reflexive normed linear spaces. Applications of Hilbert space theory to different branches of mathematics, such as approximation theory (Müntz' Theorem), measure theory (Radon–Nikodým Theorem), Bergman kernel and conformal mapping (analytic function theory), are included in Chap. 2.

A major portion of this book is devoted to the study of operators in Hilbert spaces. It is carried out in Chaps. 3 and 4. The set of operators in a Hilbert space H, equipped with the uniform norm, is denoted by $\mathcal{B}(H)$. Some well-known classes of operators have been defined. Under compact operators, Fredholm theory has been discussed. The Mean Ergodic Theorem has been proved as an application at the end of Chap. 3. Spectrum of an operator is the key to the understanding of the operator. Properties of the spectrum of different classes of operators, such as normal operators, self-adjoint operators, unitaries, isometries and compact operators, have been discussed under appropriate headings. Here, the properties of the spectrum specific to the class of operators under consideration are studied. A large number of examples of operators together with their spectrum and its splitting into point spectrum, continuous spectrum, residual spectrum, approximate point spectrum and compression spectrum have been painstakingly worked out. It is expected that the treatment will aid the understanding of the reader. The treatment of polar decomposition of an operator is different from the ones available in books. Numerical range and numerical radius of an operator have been defined. The spectral radius and the numerical radius of an operator have been compared. Professor Ajit Iqbal Singh deserves special thanks for the help she rendered while this part was being written. Spectral theorems, which reveal almost everything about the operators, have been accorded special treatment in the text. After proving the spectral theorem for compact normal operators, spectral theorems for self-adjoint operators and normal operators have been proved. Here, we have been guided by the fundamental principle of pedagogy that repetition helps in imbibing rather subtle techniques needed for proving the spectral theorems. A bird's eye view of invariant subspaces with special attention to the Volterra operator is included. We close the chapter with a brief introduction to unbounded operators.

Chapter 5 contains important theorems followed by applications from Banach spaces.

The final chapter contains hints and solutions to the 166 problems listed under various sections. These are over and above the numerous detailed examples scattered all over the text.

Chandigarh, India Harkrishan Lal Vasudeva

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About the Author

Harkrishan Lal Vasudeva was a visiting professor of mathematics at Indian Institute of Science Education and Research, Mohali, India from 2010 to 2016. Earlier, he taught at Panjab University, Chandigarh, India, and held visiting positions at the University of Sheffield, the UK, and the University of Graz, Austria, for research projects. He has numerous research articles to his credit in various international journals and has co-authored several books, two of which have been published by Springer.

Chapter 1
Preliminaries

1.1 Vector Spaces

The important underlying structure in every Hilbert space is a vector space (linear space). The present section contains preparatory material on these spaces. The reader who is already familiar with their basic theory can pass directly to Sect. 1.2, for there is nothing in the present section which is particularly oriented to the study of Hilbert spaces.

Definition 1.1.1 Let X be a nonempty set of elements $x, y, z, \ldots$ and F be a **field** of scalars $\lambda, \mu, \nu, \ldots$. To each pair of elements x and y of X, there corresponds a third element $x + y$ in X, the sum of x and y, and to each $\lambda \in F$ and $x \in X$ corresponds the element $\lambda \cdot x$ or simply λx in X, called the scalar product of λ and x such that the operations of addition and multiplication satisfy the following rules:

(A1) $x + y = y + x$,
(A2) $x + (y + z) = (x + y) + z$,
(A3) there is a unique element 0 in X, called zero element, such that $x + 0 = x$ for all $x \in X$,
(A4) for each $x \in X$, there is a unique element $(-x)$ in X such that $x + (-x) = 0$,
(M1) $\lambda(x + y) = \lambda x + \lambda y$,
(M2) $(\lambda\mu)x = \lambda(\mu x)$ and
(M3) $1x = x$,

where $1 \in F$ is the identity in F, for all $\lambda, \mu \in F$ and $x, y, z \in X$.

Then, $(X, +, \cdot)$ satisfying properties (A1)–(A4) and (M1)–(M3) is called a **vector space over *F***. The elements of X are called **vectors** or **points**, and those of F are called **scalars**.

If F is the field of complex numbers $\mathbb{C}$ [resp. real numbers $\mathbb{R}$], then $(X, +, \cdot)$ is called a complex [resp. real] vector space or a **complex** [resp. **real**] linear space.

In what follows, $\mathbb{F}$ will denote the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers.

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Remarks 1.1.2

(i) It is more satisfying to apply the term vector space over F to the ordered triple $(X, +, \cdot)$, but if this sort of thing is done systematically in all mathematics, the terminology will become extremely cumbersome. In order to avoid this difficulty, we shall apply the term vector space over F to X, where it is understood that X is equipped with the operations '+' and '$\cdot$', the latter being scalar multiplication of elements of X by those of F.

(ii) We shall mainly restrict our attention to the 'complex' vector spaces. The strong motivational factor for this choice is that the complex numbers constitute an **algebraically closed** field; that is, a polynomial of degree n has precisely n roots (counting multiplicity) in the field of complex numbers, whereas the field of real numbers does not have this property. This property of complete factorisation of polynomials into linear factors is an appropriate setting for a satisfactory treatment of the theory of operators in a Hilbert space. It is also useful in dealing with the spaces of functions.

(iii) The **additive identity** element of the field will be denoted by 0 and so shall be the identity element of vector addition. It is unlikely that any confusion will result from this practice.

(iv) The following immediate consequences of the axioms of a vector space are easy to prove:

(a) The vector equation $x + y = z$, where y and z are given vectors in X, has one and only one solution;
(b) If $x + z = z$, then $x = 0$;
(c) $\lambda 0 = 0$ for every scalar λ;
(d) $0x = 0$ for every $x \in X$;
(e) If $\lambda x = 0$, then either $\lambda = 0$ or $x = 0$.
For given vectors x and y in X, the vector $x + (-y)$ is called the difference of x and y and is denoted by $x - y$.
(f) $(-\lambda)x = \lambda(-x) = -(\lambda x)$;
(g) $\lambda(x - y) = \lambda x - \lambda y$;
(h) $(\lambda - \mu)x = \lambda x - \mu x$.

(v) It is easy to check that $Y \subseteq X$ is a vector space over F if, and only if, $x, y \in Y$, $\lambda, \mu \in F$ imply $\lambda x + \mu y \in Y$.

Examples abound. We shall give at this point a few elementary ones: the real field or the complex field with usual operations is a real or complex vector space (scalar multiplication coinciding with the usual binary operation of multiplication). The complex field may also be considered as a real vector pace. The set of all n-tuples $x = (x_1, \ldots, x_n)$, $x_i \in F$, $i = 1, 2, \ldots, n,$ is a vector space $\mathbb{R}^n$ or $\mathbb{C}^n$, where $F = \mathbb{R}$ or $\mathbb{C}$. The set of all real or complex functions defined on some fixed set is a vector space, the operations being the usual ones. The vector space consisting of the zero vector only is called the **trivial vector space**.

The **Cartesian product** $X \times Y$ of vector spaces X and Y over the same field can be made into a vector space over that field in an obvious way.

Definition 1.1.3 A sequence of vectors $x_1, x_2, \ldots, x_n$ is said to be **linearly independent** if the relation

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0 \tag{1.1}$$

holds only in the trivial case when $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$; otherwise, the sequence $x_1, x_2, \ldots, x_n$ is said to be linearly dependent.

The left member of (1.1) is said to be a linear combination of the finite sequence $x_1, x_2, \ldots, x_n$. Thus, linear independence of the vectors $x_1, x_2, \ldots, x_n$ means that every nontrivial linear combination of these vectors is different from zero. If one of the vectors is equal to zero, then these vectors are evidently linearly dependent. In fact, if for some i, $x_i = 0$, then we obtain the nontrivial relation on taking $\lambda_i = 1$ and $\lambda_j = 0$, $1 \leq j \leq n, j \neq i$. A repetition of a vector in a sequence renders it linearly dependent. An arbitrary nonempty collection of vectors is said to be linearly independent if every finite sequence of distinct terms belonging to the collection is linearly independent.

Definition 1.1.4 A **basis** or a **Hamel basis** for a vector space X is a collection $\mathcal{B}$ of linearly independent vectors with the property that any vector $x \in X$ can be expressed as a linear combination of some subset of $\mathcal{B}$.

Remarks 1.1.5

(i) Observe that a linear combination of vectors in a collection is always a finite sum even though the collection may contain an infinite number of vectors. In fact, infinite sums do not make any sense until the notion of 'limit' of a sequence of vectors has been defined in X.

(ii) The space X is said to be **finite-dimensional**, more precisely, n-dimensional if $\mathcal{B}$ contains precisely n linearly independent vectors. In this case, any $(n + 1)$ elements of X are linearly dependent. If $\mathcal{B}$ contains arbitrarily many linearly independent vectors, then X is said to be **infinite-dimensional**. The trivial vector space has dimension zero.

(iii) Permuting the vectors in a sequence does not alter its linear independence.

(iv) If x and y are linearly dependent and both are nonzero, then each is a nonzero scalar multiple of the other.

Definition 1.1.6 A nonempty subset Y of a vector space X that is also a vector space with respect to the same operations of vector addition and scalar multiplication as in X is called a **vector subspace** (or a **linear subspace**). In other words, if $x, y \in Y$, $\lambda, \mu \in F$ imply $\lambda x + \mu y \in Y$, then Y is a vector subspace (or a linear subspace) of X.

One of the common methods of constructing a linear subspace Y is to consider the set of all finite linear combinations

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

of elements x_1, x_2, …, x_n of M, where M is a nonempty finite or infinite set of elements of X. This set Y is the smallest subspace of X that contains M. It is called the **linear span of** M or the **linear subspace** [or **manifold**] **spanned by** M, and we write $Y = [M]$.

Definition 1.1.7 Given two vector spaces X and Y (over the same field), we can form a new vector space V as follows: define vector operations on the Cartesian product of X and Y, the set of all ordered pairs $\langle x, y\rangle$, where $x \in X$ and $y \in Y$. We define

$$\lambda_1\langle x_1, y_1\rangle + \lambda_2\langle x_2, y_2\rangle = \langle \lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2\rangle.$$

The vector space V so formed is called the **external direct sum** of X and Y; we denote it by $X \oplus Y$. The vector $\langle x, 0\rangle$ in V, if identified with the vector $x \in X$, permits one to think of X as a subspace of V. Similarly, Y can be viewed as a subspace of V. The mapping $\langle x, y\rangle \to \langle x, 0\rangle$ [resp. $\langle 0, y\rangle$] is called the **projection** of $X \oplus Y$ onto X [resp. Y].

Let Y_1,Y_2, …, Y_n be subspaces of X. By $Y_1 + Y_2 + \cdots + Y_n$, we shall mean all sums $x_1 + x_2 + \cdots + x_n$, where $x_j \in Y_j$, $j = 1, 2, \ldots, n$. The spaces Y_1, Y_2, …, Y_n are said to be linearly independent if for any $i = 1, 2, \ldots, n$,

$$Y_i \cap (Y_1 + Y_2 + \cdots + Y_{i-1} + Y_{i+1} + \cdots + Y_n) = \{0\}.$$

If Y_1, Y_2, …, Y_n are linearly independent and $X = Y_1 + Y_2 + \cdots + Y_n$, the spaces $\{Y_i\}_{i=1}^n$ are said to form a **direct sum decomposition** of X, and we write

$$X = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n.$$

In case $\{Y_i : i = 1, 2, \ldots, n\}$ [or $\{Y_i\}_{i=1}^n$] constitute a direct sum decomposition of X, each element $x \in X$ can be uniquely written in the form $y_1 + y_2 + \cdots + y_n$, where $y_j \in Y_j$, $j = 1, 2, \ldots, n$.

Let Y be a subspace of a vector space X ($Y \subset X$). Let $x + Y = \{x + y : y \in Y\}$ for all $x \in X$, and let $X/Y = \{x + Y : x \in X\}$. The sets $x + Y$ are called the **cosets** of Y in X. We observe that $0 + Y = Y$. Obviously, $x_1 + Y = x_2 + Y$ if, and only if, $x_1 - x_2 \in Y$, and consequently, for each pair x_1, x_2, either $x_1 + Y = x_2 + Y$ or $(x_1 + Y) \cap (x_2 + Y) = \varnothing$. If $x_1 + Y = x_1' + Y$ and $x_2 + Y = x_2' + Y$, then $(x_1 + x_2) + Y = (x_1' + x_2') + Y$ and $\alpha x_1 + Y = \alpha x_1' + Y$. The vector space X/Y with addition and scalar multiplication defined as

$$(x_1 + Y) + (x_2 + Y) = (x_1 + x_2) + Y \quad \text{and} \quad \alpha(x + Y) = \alpha x + Y$$

for all x_1, $x_2 \in X$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$) is called the **quotient space of X modulo Y**.

Definition 1.1.8 Two vector spaces H and K are said to be **isomorphic** if there exists a bijective **linear map** between H and K, i.e. if there exists a bijective linear mapping $A : H \to K$ such that

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A(x_1) + \alpha_2 A(x_2),$$

for all $x_1, x_2 \in H$ and scalars α_1 and α_2.

1.2 Metric Spaces

A vector space is a purely algebraic object, and if the processes of analysis are to be meaningful in it, a measure of distance between any two of its vectors (or elements) must be defined. Many of the familiar analytical concepts such as convergence in $\mathbb{R}^3$ with the usual distance can be fruitfully generalised to inner product spaces (to be studied in Chap. 2).

Intuitively, one expects a distance to be a nonnegative real number, symmetric and to satisfy the triangle inequality. These considerations motivate the following definitions.

Definition 1.2.1 A nonempty set X, whose elements we call 'points' is said to be a **metric space** if with any two points x, y of X is associated with a real number $d(x, y)$, called the **distance** from x to y, such that

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if, and only if, $x = y$,
(ii) $d(x, y) = d(y, x)$ and
(iii) $d(x, z) \leq d(x, y) + d(y, z)$, for any $x, y, z \in X$ [**triangle inequality**].
The function d: $X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes nonnegative reals, with these properties is called a **distance function** or a **metric** on X.

It should be emphasised that a metric space is not the set of its points; it is, in fact, the pair (X, d) consisting of the set of its points together with the metric d.

$(\mathbb{R}, d)$ [resp. $(\mathbb{C}, d)$], where $d(x, y) = |x - y|, x, y \in \mathbb{R}$ [resp. $\mathbb{C}$] are examples of metric spaces.

Any nonempty subset of a metric space is itself a metric space if we restrict the metric to it and is called a **subspace**.

Certain standard notions from the topology of real numbers have natural generalisations to metric spaces.

Definition 1.2.2 By a **sequence** $\{x_n\}_{n\geq 1}$ in a metric space X is meant a mapping of $\mathbb{N}$, the set of natural numbers, into X. A sequence $\{x_n\}_{n\geq 1}$ in a metric space is said to **converge** to the point $x \in X$ if $\lim_n d(x_n, x) = 0$, and we write $\lim_n x_n = x$. This means: given any number $\varepsilon > 0$, there is an integer n_0 such that $d(x_n, x) < \varepsilon$ whenever $n \geq n_0$. It is easy to see that if $\lim_n x_n = x$ and $\lim_n x_n = y$, then $x = y$. In fact,

$$0 \le d(x, y) \le d(x, x_n) + d(x_n, y).$$

The element x is called the **limit** of the sequence $\{x_n\}_{n \ge 1}$.

A sequence $\{x_n\}_{n \ge 1}$ in X is said to be **Cauchy** if for every $\varepsilon > 0$, there is an integer n_0 such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \ge n_0$, and we write $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Note that every convergent sequence is Cauchy. In fact, if $\lim_n x_n = x$

$$d(x_n, x_m) \le d(x_n, x) + d(x, x_m) \to 0 \quad \text{as } n, m \to \infty.$$

The converse is however not true; that is, not every Cauchy sequence is convergent. In fact, the sequence $x_n = \frac{1}{n}$, $n = 1,2,\ldots$ in the open interval (0, 1) with the metric $d(x, y) = |x - y|$, $x, y \in (0, 1)$, is Cauchy but the only possible limit, namely, 0 does not belong to the interval.

In the metric space $(\mathbb{R}, d)$ [resp. $(\mathbb{C}, d)$], where $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$ [resp. $\mathbb{C}$], a sequence $\{x_n\}_{n \ge 1}$ is convergent if, and only if, it is Cauchy (this is the well-known Cauchy criterion of convergence).

An important class of metric spaces in which the analogue of the Cauchy criterion holds are known as 'complete' metric spaces. More precisely, we have the following definitions.

Definition 1.2.3 A metric space (X, d) is said to be **complete** in case every Cauchy sequence in the space converges. Otherwise, (X, d) is said to be **incomplete**.

Proposition 1.2.4 *Let (X, d) be a metric space. Then*

(a) $|d(x, y) - d(z, y)| \le d(x, z)$ *for all* $x, y, z \in X$;
(b) *If* $\lim_n d(x_n, x) = 0$ *and* $\lim_n d(y_n, y) = 0$ *then* $\lim_n d(x_n, y_n) = d(x, y)$;
(c) *If* $\{x_n\}_{n \ge 1}$ *and* $\{y_n\}_{n \ge 1}$ *are Cauchy sequences in* (X, d), *then* $d(x_n, y_n)$ *is a convergent sequence of real numbers.*

Proof

(a) By the triangle inequality,

$$d(x, y) \le d(x, z) + d(z, y).$$

Transposing $d(z, y)$, we get

$$d(x, y) - d(z, y) \le d(x, z). \tag{1.2}$$

Interchanging the roles of x and z, we get

$$d(z, y) - d(x, y) \le d(z, x),$$

that is,

$$-(d(x,y) - d(z,y)) \leq d(x,z). \tag{1.3}$$

Combining (1.2) and (1.3), the desired inequality follows.

(b) Using the triangle inequality for real numbers and (a), we have

$$\begin{aligned} |d(x,y) - d(x_n,y_n)| &\leq |d(x,y) - d(x_n,y)| + |d(x_n,y) - d(x_n,y_n)| \\ &\leq d(x,x_n) + d(y,y_n). \end{aligned} \tag{1.4}$$

Since $\lim_n d(x_n, x) = 0 = \lim_n d(y_n, y)$, it follows that $\lim_n d(x_n, y_n) = d(x, y)$.

(c) Again,

$$\begin{aligned} |d(x_n,y_n) - d(x_m,y_m)| &\leq |d(x_n,y_n) - d(x_m,y_n)| + |d(x_m,y_n) - d(x_m,y_m)| \\ &\leq d(x_n,x_m) + d(y_n,y_m) \end{aligned} \tag{1.5}$$

and the right-hand side of (1.5) tends to zero as $m, n \to \infty$ because $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are Cauchy sequences. Thus, the sequence $\{d(x_n, y_n)\}_{n\geq 1}$ is Cauchy, and since the real numbers are complete, it converges. □

Definition 1.2.5 Let x_0 be a fixed point of the metric space X and $r > 0$ be a fixed real number. Then, the set of all points x in X such that $d(x, x_0) < r$ is called the **open ball** with **centre** x_0 and **radius** r. We denote it by $S(x_0, r)$. Thus

$$S(x_0, r) = \{x \in X : d(x, x_0) < r\}. \tag{1.6}$$

We speak of a **closed ball** if the inequality in (1.6) is replaced by $d(x, x_0) \leq r$, and we denote the set by $\overline{S}(x_0, r)$. Thus

$$\overline{S}(x_0, r) = \{x \in X : d(x, x_0) \leq r\}. \tag{1.7}$$

A set O in a metric space is said to be **open** if it contains an open ball about each of its points. In other words, for every $x \in O$, there exists an $r > 0$ such that all y with $d(y, x) < r$ belong to O. A set F in a metric space X is **closed** if its complement $X \backslash F$ (or F^c) is open in X. An open ball is an open set in X, and a closed ball is a closed set in X.

An open ball is easily seen to be an open set. Indeed, if $y \in S(x_0, r)$, then the open ball about y with radius $r - d(y, x_0)$ is a subset of $S(x_0, r)$ because any z in the latter ball satisfies $d(z, y) < r - d(y, x_0)$ and therefore also satisfies $d(z, x_0) \leq d(z, y) + d(y, x_0) < (r - d(y, x_0)) + d(y, x_0) = r$.

It is immediate from the definition of an open set that it is a union of open balls with centres in the set. Conversely, any union of open balls is an open set, because any union of open sets is clearly an open set and open balls are open sets.

Let (X, d) be a metric space and $\varnothing \neq A \subseteq X$. Then, d can be restricted to A in the obvious sense, and it is trivial to check that the restriction d_A provides a metric on A. It is called the metric **induced on** A **by** d, or simply **induced metric** for short.

An open ball in A with reference to the induced metric is easily seen to be the intersection with A of an open ball in X [caution: the converse is false]. Together with the fact that intersection is distributive over union and the observation in the preceding paragraph, this implies that every open subset of A is the intersection with A of an open set in X. On the other hand, the intersection with A of an open ball in X having its centre in A is an open ball in A. This implies that the intersection with A of an open set in X is an open set in A. In summary, a subset of A is open with reference to the induced metric if, and only if, it is the intersection with A of an open set in X.

If (A, d_A) is complete, then A is a closed subset of X.

Definition 1.2.6 A **neighbourhood** of a point $x_0 \in X$ is any open ball in (X, d) with centre x_0.

We say that x_0 is an **interior point** of a set A if A contains a neighbourhood of x_0. The **interior** of a set A, denoted by A^o, consists of all interior points of A and can be easily seen to be the largest open set contained in A.

Definition 1.2.7 A point $x_0 \in X$ is called a **limit point** of set $A \subseteq X$ if every open ball with centre x_0 contains a point of A different from x_0.

It may be easily seen that x_0 is a limit point of A if, and only if, every open ball with centre x_0 contains an infinite sequence of distinct points of A which converges to x_0.

The **closure** of a subset $A \subseteq X$, denoted by $\overline{A}$, is the union of A and the set of all its limit points. $\overline{A}$ is the smallest closed set containing A, and A is closed if, and only if, $A = \overline{A}$.

The closure of the open unit ball in $\mathbb{C}$, $\{z : |z| < 1\}$, is the closed unit ball $\{z : |z| \leq 1\}$.

Definition 1.2.8 A mapping f from a metric space (X, d) to a metric space (X', d') is said to be **continuous at x_0** $\in X$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$. The function f is **continuous on X** if it is continuous at each point of X.

The mapping f is said to be **uniformly continuous on X** if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d'(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

The function f is continuous on X if $f^{-1}(V) = \{x \in X : f(x) \in V\}$, called 'inverse image of V', is open [resp. closed] when V is open [resp. closed] in X'.

Definition 1.2.9 Let f be a real-valued function defined on a metric space (X, d). The function is said to be **lower semi-continuous at x_0** $\in X$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f(x) > f(x_0) - \varepsilon$$

for all x satisfying the inequality $d(x, x_0) < \delta$.

Upper semi-continuity is defined by replacing the inequality displayed above by $f(x) < f(x_0) + \varepsilon$.

The function f is continuous at $x_0 \in X$ if, and only if, it is both upper semi-continuous and lower semi-continuous there.

Definition 1.2.10 A metric space X is said to be **separable** if in the space X there exists a sequence

$$\{x_1, x_2, \ldots, x_n, \ldots\} \tag{1.8}$$

such that for every $x \in X$ and every $\varepsilon > 0$ there is an element x_{n_0} of (1.8) with $d(x, x_{n_0}) < \varepsilon$.

A subset $A \subseteq X$, where X is a metric space, is said to be **dense** if $\overline{A} = X$. In view of this terminology, the definition of separability may be rephrased as follows: X is said to be *separable* if X contains a countable dense set.

Definition 1.2.11 A subset $K \subseteq X$, where X is a metric space, is said to be **bounded** if there exists an $M \geq 0$ such that $d(x, y) \leq M$ whenever x and y are points in K.

The following is an immediate consequence of the definition.

Proposition 1.2.12 *Let x_0 be a fixed point of the metric space X and $K \subseteq X$. Then K is bounded if, and only if, the numbers $d(x, x_0)$ are bounded as x varies over K.*

Proof Suppose $d(x, x_0) \leq M$ for all $x \in K$; if $x, y \in K,$ then

$$d(x, y) \leq d(x, x_0) + d(x_0, y) \leq 2M.$$

Thus K is bounded.

Conversely, suppose that K is bounded, say $d(x, y) \leq M$ for all $x, y \in K$. Fix any point $y_0 \in K$. Then

$$d(x, x_0) \leq d(x, y_0) + d(y_0, x_0) \leq M + d(y_0, x_0)$$

for all $x \in K$. □

We review briefly the basic facts about the completion of a metric space. For details, the reader may refer to 1–5 of [30].

Definition 1.2.13 Let (X, d) be an arbitrary metric space. A complete metric space (X^*, d^*) is said to be a **completion** of the metric space (X, d) if

(i) X is a subspace of X^* and
(ii) Every point of X^* is the limit of some sequence in X (i.e. X is dense in X^*).

For example, the space of real numbers is a completion of the space of rational numbers. It will follow upon using Theorem 1.2.15 that the real numbers form the only completion of the space of rational numbers.

Definition 1.2.14 Let (X, d) and (X', d') be two metric spaces. A mapping T from X to X' is an **isometry** if

$$d'(T(x), T(y)) = d(x, y)$$

for all $x, y \in X$. The mapping T is also called an **isometric imbedding** of X into X'. If, however, the mapping is onto, the spaces X and X' themselves, between which there exists an isometric mapping, are said to be **isometric**.

It may be noted that an isometry is always one-to-one.

Theorem 1.2.15 *Every metric space has a completion and any two completions are isometric to each other. Moreover, there is a unique isometry between them that reduces to the identity when restricted to the given metric space* [30].

Let (X, d) be a metric space and $Y \subseteq X$. A collection of open sets $\mathcal{G}$ in X is called an **open cover** of Y if for each $y \in Y$, there is a $G \in \mathcal{G}$ such that $y \in G$. A finite subcollection of $\mathcal{G}$ which is itself a cover is called a **finite subcover** of Y.

Definition 1.2.16 A metric space (X, d) is said to be **compact** *if every open cover contains a finite subcover.* A subset K of X is said to be a **compact subset** if the metric space formed by K with the restriction of d to it is compact. A subset of X is said to be **precompact** (or **relatively compact**) if its closure in X is compact.

A compact subset is always closed and therefore also precompact.

A closed subset of a compact metric space is compact. Also, a finite union of compact subsets is compact.

A subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ is compact if, and only if, it is closed as well as bounded.

The sequence criterion for compactness is: (X, d) is compact if, and only if, every sequence in X has a convergent subsequence.

Every compact metric space is bounded but not conversely.

A continuous image of a compact metric space is compact.

Definition 1.2.16A Given a positive ε, an **ε-net** for a subset K of a metric space is a subset Y of the metric space such that, for every $x \in K$, there exists $y \in Y$ such that $d(x, y) < \varepsilon$. A subset K is said to be **totally bounded** if for every positive ε, there exists a finite ε-net for K.

A subset of a metric space is totally bounded if, and only if, it is precompact. A subset of a complete metric space is compact if, and only if, it is closed and totally bounded. A closed subset of a complete metric space is compact if, and only if, it is totally bounded.

If A is a nonempty subset of a metric space X with metric d and $x \in X$, then the nonnegative number $d(x, A) = \inf\{d(x, a) : a \in A\}$ is called the distance from x to A. Clearly, $d(x, A) = 0$ if, and only if, $x \in \overline{A}$. The function ϕ defined on X by $\phi(x) = d(x, A)$ is continuous. In particular, for any $\alpha > 0$, the set $\{x \in X : d(x, A) \geq \alpha\}$ is closed. Moreover, ϕ vanishes at all points of $\overline{A}$ and nowhere else.

Theorem 1.2.17 *Given disjoint closed subsets A and B of a compact metric space X, there always exists a continuous function* $f\colon X \to [0, 1]$ *such that* $f(a) = 0$ *for every* $a \in A$ *and* $f(b) = 1$ *for every* $b \in B$ [29, Theorem 3.4.4 on p. 116].

Proposition 1.2.18 *Given a closed subset A of a metric space, its complement is the union of a sequence of closed subsets.*

Proof For each natural number n, take K_n to be $\{x \in X : d(x, A) \geq \frac{1}{n}\}$. Then, K_n is closed and therefore compact; it is also disjoint from A. But the union of all sets K_n is $\{x \in X : d(x, A) > 0\}$, which is precisely the complement of $\overline{A} = A$. □

Let (X_n, d_n), $n = 1, 2, \ldots$, be metric spaces with $d_n(X_n) = \sup\{d_n(x, y) : x, y \in X_n\} \leq 1$ for each n. For $x, y \in \prod_{n=1}^{\infty} X_n$, define

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n),$$

where $x = \{x_n\}_{n \geq 1}$ and $y = \{y_n\}_{n \geq 1}$. Observe that the series on the right converges because $2^{-n} d_n(x_n, y_n) \leq 2^{-n}$.

Then, d turns out to be a metric on $X = \prod_{n=1}^{\infty} X_n$ and (X, d) is called a **product metric space**. Also, convergence in this metric turns out to be the same as coordinatewise convergence.

Theorem 1.2.19 (Tychonoff) *(X, d) is compact if, and only if, each (X_n, d_n) is compact* [30].

Another important theorem in metric spaces is the theorem of Ascoli, also known as the Arzelà–Ascoli Theorem.

As usual, $C[0, 1]$ denotes the metric space of continuous functions defined on $[0, 1]$ with metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$

Definition 1.2.20 A nonempty subset K of $C[0, 1]$ is said to be **equicontinuous** if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every $f \in K$,

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon.$$

Theorem 1.2.21 (Ascoli) *Let K be a nonempty subset of $C[0, 1]$. Then the following are equivalent*:

(a) *The closure of K is compact*;
(b) *K is* **uniformly bounded** (*i.e. there exists $M > 0$ such that $|f(x)| \leq M$ for every $x \in [0, 1]$ and every $f \in K$*) *and equicontinuous* [30].

1.3 Lebesgue Integration

In this section, we shall review the theory of measure and integrable functions as developed by H. Lebesgue in 1902. His integral, though more complicated to develop and define than Riemann's, yet as a tool, is easier to use and has better properties. For example, problems which involve integration together with a limiting process are often awkward with the Riemann integral but are easily handled when Lebesgue integration is used.

Measure theory is based on the idea of generalising the length of an interval in $\mathbb{R}$, the area of a rectangle in $\mathbb{R}^2$, etc. to the measure of a subset. The more the 'measurable' subsets, the more the functions that can be integrated. A well-behaved measure, i.e. a measure with acceptable properties, is possible on a wide class of subsets. We begin with the following definitions.

Definition 1.3.1 Let X be a set and $\mathfrak{M}$ be a collection of subsets of X with the following properties:

(i) $X \in \mathfrak{M}$,
(ii) If $A \subseteq X$ and $A \in \mathfrak{M}$, then $X \backslash A \in \mathfrak{M}$ and
(iii) If $A_n \subseteq X$ and $A_n \in \mathfrak{M}$, $n = 1,2,\ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{M}$.

Such a system of sets is called a **σ-algebra**.

In case X is a metric space, there is a smallest σ-algebra containing all open subsets of it, and each member of this smallest σ-algebra is called a **Borel set** and the smallest σ-algebra containing all open subsets is called the **Borel field**.

Definition 1.3.2 Let μ be an extended real-valued function defined on $\mathfrak{M}$ such that

(i) $\mu(A) \geq 0$ for every $A \in \mathfrak{M}$ and
(ii) $A_n \in \mathfrak{M}$, $n = 1, 2, \ldots$ and $A_n \cap A_m = \emptyset$, $n \neq m$ implies

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Then μ is called a **positive measure** on $\mathfrak{M}$ or on X.

A **measure space** is a triple $(X, \mathfrak{M}, \mu)$, where X is a nonempty set, $\mathfrak{M}$ a σ-algebra of subsets of X and μ a positive measure on X. A subset A in the measure space $(X, \mathfrak{M}, \mu)$ is said to have **σ-finite measure** if A is a countable union of sets A_i, $i = 1, 2, \ldots$, with $\mu(A_i) < \infty$ and we say that μ is σ-finite on A. The measure μ is said to be σ-finite if it is σ-finite on X.

There exists a unique positive measure on the σ-algebra of Borel subsets of $\mathbb{R}^n$, which agrees with the volume when restricted to products of intervals. It is called the **Borel measure** in $\mathbb{R}^n$. There exists a σ-algebra of subsets of $\mathbb{R}^n$ larger than the Borel σ-algebra, the elements of which are called **Lebesgue measurable** subsets, and on which is defined a positive measure, which agrees with the Borel measure when restricted to the Borel σ-algebra. It is called the **Lebesgue measure** in $\mathbb{R}^n$ and has the following additional property of being **complete**: let E be a Lebesgue measurable set of measure 0 and F be any subset of E. Then, F is also Lebesgue measurable and hence has measure 0.

The Lebesgue measure on $\mathbb{R}^n$ is σ-finite. Let E be a measurable subset of $\mathbb{R}$ and μ be the Lebesgue measure on $\mathbb{R}$. There exists an open set O and a closed set F in $\mathbb{R}$

such that $F \subseteq E \subseteq O$, $\mu(O \backslash E) < \varepsilon$ and $\mu(E \backslash F) < \varepsilon$. This property is called the **regularity** of the Lebesgue measure μ.

Definition 1.3.3 Let $\mathfrak{M}$ be a σ-algebra of subsets of X. An extended real-valued function f defined on X is said to be **measurable** if $f^{-1}(O) = \{x \in X : f(x) \in O\}$, where O is an open subset of $\mathbb{R}$, is measurable and if the subsets $f^{-1}(\infty)$ and $f^{-1}(-\infty)$ are measurable. A complex-valued function $g + ih$ is measurable if, and only if, g and h are both measurable.

It can be shown that if $f = g + ih$ is measurable, then $f^{-1}(O) = \{x \in X : f(x) \in O\}$, where O is an open subset of $\mathbb{C}$, is measurable [26].

If f and g are extended real-valued measurable functions, then so are $f + g$ (provided it is defined), fg, αf ($\alpha \in \mathbb{R}$), $|f|$, $\max\{f, g\}$, $\min\{f, g\}$, where

$$(\mathbf{max}\{f,g\})(x) = \max\{f(x), g(x)\} \quad \text{for each} \quad x \in X.$$

If $\{f_n\}_{n \geq 1}$ is a sequence of extended real-valued measurable functions defined on X, then $\mathbf{sup}_n \boldsymbol{f_n}$, $\mathbf{inf}_n \boldsymbol{f_n}$, $\mathbf{limsup}_n \boldsymbol{f_n}$ ($=\lim_n \sup_{k \geq n} f_k$), $\mathbf{liminf}_n \boldsymbol{f_n}$, and the **pointwise limit** $\mathbf{lim}_n \boldsymbol{f_n}$, when it exists, are measurable.

For $A \subseteq X$, let χ_A denote the **characteristic function** of A, that is,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

It is measurable if, and only if, A is a measurable subset of X, i.e. $A \in \mathfrak{M}$. A **simple function** is a real-valued function on X whose range is finite. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the distinct values of such a function f, then

$$f = \sum_{j=1}^{m} \alpha_j \chi_{Aj}, \text{where } A_j = \{x \in X : f(x) = \alpha_j\}, j = 1, 2, \ldots, m.$$

Also, f is measurable if, and only if, $A_1, \ldots, A_m$ are measurable subsets of X.

Let $f : X \to [0, \infty]$ and for $n = 1, 2, \ldots$ consider the simple functions

$$s_n(x) = \begin{cases} (j-1)/2^n & \text{if } (j-1)/2^n \leq f(x) < j/2^n, \quad j = 1, 2, \ldots, n2^n \\ n & \text{if } f(x) \geq n. \end{cases}$$

Then, $0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq f(x)$ and $s_n(x) \to f(x)$ for each $x \in X$. If f is bounded, the sequence $\{s_n\}$ converges to f uniformly on X. If $f : X \to [-\infty, \infty]$, then by considering $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$, we see that there exists a sequence of simple functions converging to f at every point of X. Note that if f is measurable, then each of these simple functions is measurable.

Definition 1.3.4 Let $(X, \mathfrak{M}, \mu)$ be a measure space and suppose f is measurable.

(i) For f a simple function, say $f = \sum_{j=1}^{m} \alpha_j \chi A_j$, the **integral** of f over X is defined as

$$\int_X f \mathrm{d}\mu = \sum_{j=1}^{m} \alpha_j \mu(A_j),$$

the convention $0 \cdot \infty = 0$ being used.

(ii) For f extended real-valued and nonnegative, the **integral** of f over X is defined as

$$\int_X f \mathrm{d}\mu = \sup\left\{ \int_X g \mathrm{d}\mu : g \text{ is simple and } 0 \le g(x) \le f(x) \text{ for } x \in X \right\}.$$

(iii) Suppose that f is extended real-valued and $f = f^{+} - f^{-}$, where $f^{+} = \max\{f, 0\}$ and $f^{-} = -\min\{f, 0\}$. The **integral** of f over X is defined as

$$\int_X f \mathrm{d}\mu = \int_X f^{+} \mathrm{d}\mu - \int_X f^{-} \mathrm{d}\mu$$

provided at least one of the integrals on the right is finite.

(iv) The function f is said to be **integrable** (or **μ-integrable**) if $\int_X f^{+} \mathrm{d}\mu$ and $\int_X f^{-} \mathrm{d}\mu$ are both finite.

(v) Suppose f is complex-valued and the integrals of $\Re f$ and $\Im f$ are defined as in (iii) and are finite. Then set

$$\int_X f \mathrm{d}\mu = \int_X \Re f \mathrm{d}\mu + i \int_X \Im f \mathrm{d}\mu.$$

In this case, f is said to be **integrable** (or **μ-integrable**).

(vi) For a measurable set A, let χ_A be the characteristic function of A. If the integral of $f\chi_A$ can be defined as above, set

$$\int_A f \mathrm{d}\mu = \int_X f \chi_A \mathrm{d}\mu.$$

(vii) If the integral can be defined in this manner, we say that the **integral exists**.

When μ is Lebesgue measure on a bounded closed interval, the μ-integral defined above is the Lebesgue integral and coincides with the Riemann integral for all Riemann integrable functions.

It is sometimes convenient to denote $\int_A f \mathrm{d}\mu$ by $\int_A f(x) \mathrm{d}\mu(x)$.

Definition 1.3.5 Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space which is nontrivial in the sense that $0 < \mu(A) < \infty$ for some $A \in \mathfrak{M}$. For $p > 0$, we shall denote by $\boldsymbol{\tilde{L}^p(X, \mathfrak{M}, \mu)}$ the set of measurable complex-valued functions defined on X such that $\int_X |f|^p < \infty$.

With pointwise addition and scalar multiplication, $\tilde{L}^1(X, \mathfrak{M}, \mu)$ is a vector space of functions.

In order to introduce the set $\tilde{L}^\infty(X, \mathfrak{M}, \mu)$, we shall need the following: for a nonnegative measurable function g, let A be the set of all real numbers α such that

$$\mu(\{x \in X : g(x) > \alpha\}) = 0.$$

If $A = \varnothing$, put $\beta = \infty$. If $A \neq \varnothing$, put $\beta = \inf A$. Since

$$\{x \in X : g(x) > \beta\} = \bigcup_{n=1}^{\infty} \left\{x \in X : g(x) > \beta + \frac{1}{n}\right\}$$

and since the union of a countable collection of sets of measure zero is a set of measure zero, it follows that $\beta \in A$. We call β the **essential supremum** of g and write β = **ess sup** $\boldsymbol{g}$. The function g is said to be essentially bounded, if ess sup g is finite. The collection of measurable functions f for which ess sup $|f| < \infty$ will be denoted by $\tilde{L}^\infty(X, \mathfrak{M}, \mu)$.

The next three theorems involve interchange of integration with the limiting process for a sequence of functions.

Theorem 1.3.6 (Monotone Convergence Theorem) *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and assume that* $\{f_n\}_{n \geq 1}$ *is a monotone increasing sequence of nonnegative extended real-valued measurable functions. Then*

$$\lim_n \int_X f_n \mathrm{d}\mu = \int_X \lim_n f \mathrm{d}\mu.$$

The reader will note that each of the above integrals is defined (though not necessarily finite).

The following immediate consequence of the Monotone Convergence Theorem will be needed in Sect. 2.8.

Corollary 1.3.7 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and* $\{f_n\}_{n \geq 1}$ *be a sequence of nonnegative integrable functions, each defined on* X, *such that*

$$\sum_{k=1}^{\infty} \int_X f_k \mathrm{d}\mu < \infty.$$

Then $\sum_{k=1}^{\infty} f_k$ *is integrable and*

$$\int_X \left(\sum_{k=1}^{\infty} f_k \right) \mathrm{d}\mu = \sum_{k=1}^{\infty} \int_X f_k \mathrm{d}\mu < \infty.$$

Proof Since $\left\{ \sum_{k=1}^{n} f_k \right\}_{n \geq 1}$ is an increasing sequence of functions that converge to $\sum_{k=1}^{\infty} f_k$, by the Monotone Convergence Theorem, we conclude that

$$\int_X \left(\sum_{k=1}^{\infty} f_k \right) \mathrm{d}\mu = \sum_{k=1}^{\infty} \int_X f_k \mathrm{d}\mu < \infty$$

and hence $\sum_{k=1}^{\infty} f_k$ is integrable. □

Theorem 1.3.8 (Fatou's Lemma) *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and assume that* $\{f_n\}_{n \geq 1}$ *is a sequence of nonnegative extended real-valued measurable functions. Then*

$$\int_X \liminf{}_n f_n \mathrm{d}\mu \leq \liminf{}_n \int_X f_n \mathrm{d}\mu.$$

A complex-valued measurable function is integrable if, and only if, its real and imaginary parts are. Obviously, there can be no Monotone Convergence Theorem or Fatou's Lemma for complex-valued functions. Nonetheless, the following result, which is initially proved for real-valued functions by using Fatou's Lemma, can be extended to complex-valued functions without any difficulty [26].

Theorem 1.3.9 (Lebesgue Dominated Convergence Theorem) *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and assume that* $\{f_n\}_{n \geq 1}$ *is a sequence of complex measurable functions. Suppose that* $\lim_n f_n = f$. *If there is an integrable function* g *such that* $|f_n| \leq g$ $(n \geq 1)$, *then* f *is integrable and*

$$\lim{}_n \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

We finally recall the role played by the sets of measure zero. Let P be a property which a function is eligible to have at a point (e.g. continuity, positivity and the like). If f has the property P at all points outside some set of measure zero, then f is said to have the property P **almost everywhere** (abbreviated as a.e.).

For example, if f is a nonnegative measurable function defined on X and $\int_X f \mathrm{d}\mu = 0$, then $f = 0$ a.e.

For the definition of the spaces L^p, see Sect. 2.4.

Corollary 1.3.10 *Let* $\{f_n\}$ *be a sequence of complex-valued measurable functions on* X *such that* $\sum_{n=1}^{\infty} |f_n| \in L^1$ *(or equivalently,* $\sum_{n=1}^{\infty} \int_X |f_n| \mathrm{d}\mu < \infty$*). Then* $\sum_{n=1}^{\infty} f_n \in L^1$ *and* $\int_X \sum_{n=1}^{\infty} f_n \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X f_n \mathrm{d}\mu$.

Proposition 1.3.11 *If the function* $f \in L^1[0, 1]$ *and* $\int_0^1 f(x)x^n \mathrm{d}x = 0$ *for* $n = 0, 1, 2, \ldots$, *then* $f(x) = 0$ *a.e. on* $[0, 1]$.

Hint: Obviously, $\int_0^1 f(x)g(x)\mathrm{d}x = 0$ for every polynomial g and hence for every continuous function g by Weierstrass' Approximation Theorem. Since the characteristic function of $[0, t]$, where $t \in [0, 1]$, can be approximated pointwise by a sequence of continuous functions, one gets

$$\int_0^t f(x)\mathrm{d}x = 0 \ \text{ for every } \ t \in [0, 1],$$

using the Dominated Convergence Theorem. It follows from Corollary 1.3.10 that the integral of f over any open subset U of $[0, 1]$ vanishes. The result now follows on using the regularity of Lebesgue measure.

Proposition 1.3.12 *If the real- or complex-valued function* $f(t)$ *is integrable and* $\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-itx}\mathrm{d}t = 0$ *for all real* x, *then* $f(t) = 0$ *a.e.* [see Corollary (21.47) of 12].

Remark 1.3.13 If f is extended real-valued and if $\int_X f \mathrm{d}\mu$ is finite, then f is finite almost everywhere.

Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be σ-finite measure spaces. A *measurable rectangle* is any set of the form $A \times B$, where $A \in \mathcal{S}$ and $B \in \mathcal{T}$. $\mathcal{S} \times \mathcal{T}$ denotes the σ-algebra generated by the collection of measurable rectangles.

With each function f on $X \times Y$ and with each $x \in X$, we associate a function f_x defined on Y as $f_x(y) = f(x, y)$. Similarly, if $y \in Y$, f^y is the function on X such that $f^y(x) = f(x, y)$. Let f be an $(\mathcal{S} \times \mathcal{T})$-measurable function. Then, for each $x \in X$ [resp. $y \in Y$], the function f_x [resp. f^y] is $\mathcal{T}$-measurable [resp. $\mathcal{S}$-measurable].

With each subset Q of $X \times Y$ and with each $x \in X$, we associate a subset Q_x of Y defined as $Q_x = \{y \in Y : (x, y) \in Q\}$. Similarly, if $y \in Y$, Q^y is the subset of X such that $Q^y = \{x \in X : (x, y) \in Q\}$. Let Q be $(\mathcal{S} \times \mathcal{T})$-measurable. Then, for each $x \in X$ [resp. $y \in Y$], the set Q_x [resp. Q^y] is $\mathcal{T}$-measurable [resp. $\mathcal{S}$-measurable].

Let $Q \in \mathcal{S} \times \mathcal{T}$. If $\varphi(x) = \nu(Q_x)$ [resp. $\psi(y) = \mu(Q^y)$], then for every $x \in X$ [resp. $y \in Y$], the function φ is $\mathcal{S}$-measurable [resp. ψ is $\mathcal{T}$-measurable] and

$$\int_X \varphi \mathrm{d}\mu = \int_Y \psi \mathrm{d}\nu, \quad \text{i.e.,} \quad \int_X \nu(Q_x)\mathrm{d}\mu(x) = \int_Y \mu(Q^y)\mathrm{d}\nu(y).$$

The *product measure* $\mu \times \nu$ is given by

$$(\mu \times \nu)(Q) = \int_X \nu(Q_x)\mathrm{d}\mu(x) = \int_Y \mu(Q^y)\mathrm{d}\nu(y) \quad \text{for} \quad Q \in \mathcal{S} \times \mathcal{T}.$$

Theorem 1.3.14 (Fubini) *If* $f \in L^1(\mu \times \nu)$, *then* f_x *for almost all* $x \in X$ [*resp.* f^y *for almost all* $y \in Y$] *is in* $L^1(\nu)$ [*resp.* $L^1(\mu)$] *and*

$$\int_X \left(\int_Y f_x \mathrm{d}\nu \right) \mathrm{d}\mu = \int_{X \times Y} f \mathrm{d}(\mu \times \nu) = \int_Y \left(\int_X f^y \mathrm{d}\mu \right) \mathrm{d}\nu.$$

The above equality holds for nonnegative measurable functions as well.

1.4 Zorn's Lemma

A partially ordered set is a set S with a relation $x \leq y$ between ordered pairs of elements of S satisfying (i) $x \leq x$, (ii) $x \leq y$, $y \leq x$ implies $x = y$, and (iii) $x \leq y$, $y \leq z$ implies $x \leq z$. If every pair of elements of a subset $S' \subseteq S$ are comparable, that is $x \in S'$ and $y \in S'$ implies either $x \leq y$ or $y \leq x$, then S' is called a *totally ordered subset* of S (or a *chain*). An *upper bound* of a set $A \subseteq S$ is any $y \in S$ such that $x \leq y$ for all $x \in A$. A *maximal element* of S is a $y \in S$ such that $y \leq x$ implies $y = x$.

Zorn's Lemma *If S is a partially ordered set in which every totally ordered subset has an upper bound, then S has a maximal element.*

Remark This lemma is logically equivalent to the **axiom of choice**, that is one can be derived from the other and vice versa. This axiom says if $\{X_\alpha\}_{\alpha \in \Lambda}$, Λ any indexing set, is any family of sets, then there exists a set that contains exactly one element from each X_α.

For a discussion of the equivalence alluded to above and related material, the reader may consult J.L. Kelley [15].

1.5 Absolute Continuity

A real-valued function defined on $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ if, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \varepsilon$$

for every finite pairwise disjoint family $\{(c_i, d_i)\}$ of intervals with $\sum_{i=1}^{n} (d_i - c_i) < \delta$.

The following results are well known [25]:

(i) An absolutely continuous function is continuous.
(ii) The indefinite integral $\int_{[a,x]} f \mathrm{d}\mu$, $f \in L^1[a,b]$, is absolutely continuous.
(iii) If f is absolutely continuous, then f has a derivative almost everywhere.
(iv) Let f be an absolutely continuous function on $[a, b]$, and suppose that $f'(x) = 0$ a.e. Then, f is a constant.
(v) A function f on $[a, b]$ has the form

$$f(x) = f(a) + \int_a^x \varphi(\tau)\mathrm{d}\tau$$

for some $\varphi \in L^1[a, b]$ if, and only if, f is absolutely continuous on $[a, b]$. In this case, $\varphi'(x) = f(x)$ a.e. on $[a, b]$.

Chapter 2
Inner Product Spaces

2.1 Definition and Examples

In the study of vector algebra in $\mathbb{R}^n$, the notion of angle between two nonzero vectors is introduced by considering the inner (or dot) product. In fact, if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are any two vectors in the n-dimensional Euclidean space $\mathbb{R}^n$, then their inner product is defined by

$$(x, y) = \sum_{i=1}^{n} x_i y_i,$$

and this inner product is related to the norm by

$$(x, x) = \|x\|^2.$$

The familiar equation

$$(x, y) = \|x\|\|y\| \cos \theta$$

determines the angle θ between x and y. The vectors x and y are orthogonal if $(x, y) = 0$. This concept of orthogonality proves useful and lends itself to the generalisation to spaces of higher dimensions.

We introduce below the abstract notion of an inner product and show how a vector space equipped with an inner product reflects properties analogous to those enjoyed by the n-dimensional Euclidean space $\mathbb{R}^n$.

Recall that we denote by $\mathbb{F}$ either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers.

Definition 2.1.1 Let H be a vector space over $\mathbb{F}$. An **inner product** on H is a function $(\cdot,\cdot)$ from $H \times H$ into $\mathbb{F}$ such that for all $x, y, z \in H$ and $\lambda \in \mathbb{F}$,

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(i) $(x, y) = \overline{(y, x)}$,
(ii) $(x + z, y) = (x, y) + (z, y), \quad (\lambda x, y) = \lambda(x, y)$,
(iii) $(x, x) \geq 0$ and $(x, x) = 0$ if, and only if, $x = 0$.

An **inner product space** is a vector space with an inner product on it. Axiom (ii) for an inner product space can be expressed as follows: the inner product is linear in the first variable. In axiom (i), $\overline{(y, x)}$ denotes the complex conjugate of (y, x). Inner product spaces are also called **pre-Hilbert spaces**.

It is left to the reader to verify that when $\mathbb{F} = \mathbb{R}$ and $H = \mathbb{R}^n$, the usual inner product $(x, y) = \sum_{i=1}^{n} x_i y_i$ (described above) satisfies the foregoing definition.

The following proposition contains some immediate consequences of Definition 2.1.1.

Proposition 2.1.2 *For any x, y, z in an inner product space H and any $\lambda \in \mathbb{F}$, the following hold*:

(a) $(x, y + z) = (x, y) + (x, z)$;
(b) $(x, \lambda y) = \overline{\lambda}(x, y)$;
(c) $(0, y) = (x, 0) = 0$;
(d) $(x - y, z) = (x, z) - (y, z)$;
(e) $(x, y - z) = (x, y) - (x, z)$;
(f) *if* $(x, y) = (x, z)$ *for all* x, *then* $y = z$.

Proof (a) Using Definition 2.1.1(i) and (ii), we have

$$(x, y + z) = \overline{(y + z, x)} = \overline{(y, x) + (z, x)} = \overline{(y, x)} + \overline{(z, x)} = (x, y) + (x, z).$$

(c) $(0, y) = (0 + 0, y) = (0, y) + (0, y)$, on using Definition 2.1.1(ii), and hence $(0, y) = 0$.

(f) Suppose $(x, y) = (x, z)$ for all x. Then

$$(x, y - z) = (x, y) - (x, z) = 0$$

for all x; in particular, $(y - z, y - z) = 0$ and hence $y - z = 0$ by Definition 2.1.1(iii).

The proofs of (b), (d) and (e) are no different and are left to the reader. □

Examples 2.1.3

(i) Let $H = \mathbb{C}^n = \{x = (x_1, x_2, \ldots, x_n): x_i \in \mathbb{C}, 1 \leq i \leq n\}$ be the complex vector space of n-tuples. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, define

$$(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}. \tag{2.1}$$

It is routine to check that the formula (2.1) does define an inner product on $\mathbb{C}^n$ in the sense of Definition 2.1.1. This space is called ***n*-dimensional unitary space** and is denoted by $\mathbb{C}^n$. Indeed, the vectors (1, 0, …, 0), (0, 1, 0, …, 0), …, (0, 0, …, 0, 1) constitute a basis for $\mathbb{C}^n$.

(ii) Let ℓ_0 be the vector space of all sequences $x = \{x_n\}_{n\geq 1}$ of complex numbers, all of whose terms, from some index onwards, are zero (the index, of course, may vary with the sequence). If $x = \{x_n\}_{n\geq 1}$ and $y = \{y_n\}_{n\geq 1}$, define

$$(x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n}. \tag{2.2}$$

Since the sum on the right side of (2.2) is essentially finite, convergence is not an issue here. The axioms of Definition 2.1.1 are easily verified.

(iii) Let ℓ^2 denote the set of all complex sequences $x = \{x_n\}_{n\geq 1}$ which are square summable, that is,

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

The addition of vectors $x = \{x_n\}_{n\geq 1}$ and $y = \{y_n\}_{n\geq 1}$ and the scalar multiplication of $x = \{x_n\}_{n\geq 1}$ by a scalar $\lambda \in \mathbb{C}$ are defined by

$$x + y = \{x_n + y_n\}_{n\geq 1} \quad \text{and} \quad \lambda x = \{\lambda x_n\}_{n\geq 1}.$$

Since

$$|\alpha + \beta|^2 \leq 2\left|\alpha^2\right| + 2|\beta|^2$$

for α, $\beta \in \mathbb{C}$, it follows that

$$\sum_{n=1}^{m} |x_n + y_n|^2 \leq 2\sum_{n=1}^{m} |x_n|^2 + 2\sum_{n=1}^{m} |y_n|^2 \leq 2\sum_{n=1}^{\infty} |x_n|^2 + 2\sum_{n=1}^{\infty} |y_n|^2,$$

and hence

$$\sum_{n=1}^{\infty} |x_n + y_n|^2 \leq 2\sum_{n=1}^{\infty} |x_n|^2 + 2\sum_{n=1}^{\infty} |y_n|^2. \tag{2.3}$$

Thus, if $x = \{x_n\}_{n\geq 1}$ and $y = \{y_n\}_{n\geq 1}$ are in ℓ^2, it follows from (2.3) that $x + y \in \ell^2$. Also, if $x \in \ell^2$ and $\lambda \in \mathbb{C}$, then $\sum_{n=1}^{\infty} |\lambda x_n|^2 = |\lambda|^2 \sum_{n=1}^{\infty} |x_n|^2$ shows that $\lambda x \in \ell^2$. Consequently, ℓ^2 is a vector space over $\mathbb{C}$.

For $x = \{x_n\}_{n\geq 1}$ and $y = \{y_n\}_{n\geq 1}$ in ℓ^2, the series $\sum_{n=1}^{\infty} x_n \overline{y_n}$ converges absolutely. In fact,

$$|x_n\bar{y}_n| \le \frac{1}{2}\left(|x_n|^2 + |y_n|^2\right)$$

implies

$$\sum_{n=1}^{m} |x_n\,\bar{y}_n| \le \frac{1}{2}\sum_{n=1}^{m} |x_n|^2 + \frac{1}{2}\sum_{n=1}^{m} |y_n|^2 \le \frac{1}{2}\sum_{n=1}^{\infty} |x_n|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |y_n|^2,$$

and hence

$$\sum_{n=1}^{\infty} |x_n\,\bar{y}_n| \le \frac{1}{2}\sum_{n=1}^{\infty} |x_n|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |y_n|^2.$$

Now define

$$(x, y) = \sum_{n=1}^{\infty} x_n\,\bar{y}_n, \quad x, y \in \ell^2. \tag{2.4}$$

It is now easy to check that the axioms for an inner product are satisfied. Thus, ℓ^2 with the inner product defined in (2.4) is an inner product space.

(iv) Let ***C*[*a*, *b*]**, $-\infty < a < b < \infty$, be the vector space of all continuous complex-valued functions defined on $[a, b]$. Define

$$(f, g) = \int_a^b f(t)\,\overline{g(t)}\,\mathrm{d}t, \quad f, g \in C[a, b]. \tag{2.5}$$

Observe that $(f,f) = \int_a^b |f(t)|^2 \mathrm{d}t = 0$ implies $f(t) = 0$ for each $t \in [a, b]$, in view of the continuity of f. The other axioms in Definition 2.1.1 are consequences of the properties of integrals.

(v) Let $C^n[a, b]$ be the vector space of all n times continuously differentiable complex-valued functions defined on $[a, b]$. For $f, g \in C^n[a, b]$, define

$$(f, g) = \sum_{i=0}^{n} \int_a^b f^{(i)}(t)\overline{g^{(i)}(t)}\,\mathrm{d}t, \tag{2.6}$$

where $f^{(i)}(t)$ denotes the ith derivative of f, $1 \le i \le n$, and $f^{(0)}(t) = f(t)$, $t \in [a, b]$. Observe that $0 = (f,f) = \sum_{i=0}^{n} \int_a^b \left|f^{(i)}(t)\right|^2 \mathrm{d}t = 0$ implies $\sum_{i=0}^{n} \left|f^{(i)}(t)\right|^2 = 0$, $t \in [a, b]$, in view of the continuity of $\sum_{i=0}^{n} \left|f^{(i)}(t)\right|^2$, $t \in [a, b]$. This implies $f(t) = 0$ for each $t \in [a, b]$. The other axioms in Definition 2.1.1 are consequences of the properties of integrals.

(vi) Let RL^2 denote the space of rational functions (i.e. a ratio of two polynomials with complex coefficients) which are analytic on the unit circle

$$\partial D = \{z \in \mathbb{C} : |z| = 1\}$$

with the usual pointwise addition and scalar multiplication. The inner product is defined by

$$(f, g) = \frac{1}{2\pi i} \int_{\partial D} f(z)\overline{g(z)} \frac{dz}{z}, \tag{2.7}$$

where the integral is being taken in the anticlockwise direction around ∂D.

RH^2 is the subspace of RL^2 consisting of those rational functions which are analytic on the closed unit disc $\overline{D}$, where

$$D = \{z \in \mathbb{C} : |z| < 1\},$$

with inner product given by (2.7).

Thus, a rational function belongs to RL^2 if it has no pole of absolute value 1, and it belongs to RH^2 if it has no pole of absolute value less than or equal to 1.

Clearly, (2.7) satisfies the axioms in (i), (ii) and part of (iii). We need to check that $(f, f) > 0$ when $f \neq 0$. Indeed,

$$(f, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta, \tag{2.8}$$

using the parametrisation $z = e^{i\theta}$, $-\pi < \theta \leq \pi$. Since $f(e^{i\theta})$ is continuous on $[-\pi, \pi]$, the right-hand side of (2.8) is positive unless $f = 0$.

(vii) A **trigonometric polynomial** is a finite sum of the form

$$f(x) = a_0 + \sum_{n=1}^{k} a_n e^{i\lambda_n x}, \quad x \in [-\pi, \pi],$$

where $k \in \mathbb{N}$, $a_0, a_1, \ldots, a_k \in \mathbb{C}$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{N}$. It is clear that every trigonometric polynomial is of period 2π. The space TP of trigonometric polynomials is a vector space over $\mathbb{C}$ with respect to pointwise addition and scalar multiplication. If we define the inner product by

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)} dt, \tag{2.9}$$

then TP becomes an inner product space.

Problem Set 2.1

2.1.P1. For which values of $\alpha \in \mathbb{C}$ does the sequence $\{n^{-\alpha}\}_{n\geq 1}$ belong to ℓ^2?

2.1.P2. [Notations as in Example 2.1.3(vi)] Calculate the inner product of functions

$$f(z) = \frac{1}{z-\alpha} \quad \text{and} \quad g(z) = \frac{1}{z-\beta}, \quad \text{where} \quad |\alpha| < 1, |\beta| < 1.$$

2.1.P3. [Notations as in Example 2.1.3(vi)] Let $k_\alpha \in RL^2$ be defined by $k_\alpha(z) = (1 - \bar{\alpha}z)^{-1}$, where $|\alpha| \neq 1$. Show that for $f \in RH^2$,

$$(f, k_\alpha) = \begin{cases} f(\alpha) & \text{if } |\alpha| < 1 \\ 0 & \text{if } |\alpha| > 1 \end{cases}$$

2.1.P4. [Notations as in Example 2.1.3(vi)] Let $f \in RH^2$ and $\alpha \in D$. Show that

$$|f(\alpha)| \leq \frac{1}{\sqrt{1-|\alpha|^2}} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(e^{i\theta}\right) \right|^2 d\theta \right\}^{\frac{1}{2}}.$$

2.2 Norm of a Vector

Let X be a vector (linear) space over $\mathbb{F}$.

Definition 2.2.1 A **norm** $\|\cdot\|$ is a function from X into the nonnegative reals $\mathbb{R}^+$ satisfying

(i) $\|x\| = 0$ if, and only if, $x = 0$,
(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for each $\lambda \in \mathbb{F}$ and $x \in X$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. [**triangle inequality**]

We emphasise that, by definition, $\|x\| \geq 0$ for all $x \in X$.

If X is a linear space and $\|\cdot\|$ is a norm defined on X, then $d(x, y) = \|x - y\|$ indeed gives rise to a metric as a consequence of the foregoing Definition 2.2.1. The details are as follows.

That the distance $d(x, y)$ from a vector x to a vector y in H is strictly positive (that is, $d(x, y) \geq 0$ and equality holds if, and only if, $x = y$) follows from (i). The fact that $d(x, y) = d(y, x)$ follows from

$$\|x-y\| = \|-(y-x)\| = |-1|\|y-x\| = \|y-x\|,$$

in view of (ii). Also

$$d(x,z) = \|x-z\| = \|x-y+y-z\| \le \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$$

for all x, y and z. The reader will observe that (iii) has been used in proving the preceding inequality.

A linear space X equipped with a norm $\|\cdot\|$ is called a **normed linear space**. If the metric space (X, d), where $d(x, y) = \|x - y\|$, $x, y \in X$, is complete, then the normed linear space is said to be **complete** and is called a **Banach space**. These spaces are named after the great Polish mathematician Stefan Banach. $\mathbb{R}^n$, the real space of n-tuples $x = (x_1, x_2, \ldots, x_n)$ with each of the norms $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$, $\|x\|_\infty = \sup_i |x_i|$ is a Banach space. That $\|x\|_1$, $\|x\|_2$ and $\|x\|_\infty$ are norms can be verified, see [30]. So is the space $\mathbb{C}^n$ of complex n-tuples. The space $(\mathbb{C}^n, \|\cdot\|_2)$ is complete [see Example 2.3.4(i)]. That $\mathbb{C}^n$ with $\|\cdot\|_1$ and $\mathbb{C}^n$ with $\|\cdot\|_\infty$ are complete follows from the inequalities $\|\cdot\|_\infty \le \|\cdot\|_2 \le \|\cdot\|_1 \le n\|\cdot\|_\infty$, see [30].

Hilbert spaces are Banach spaces whose norms are derived from an inner product as detailed below.

Definition 2.2.2 In an inner product space H, the **norm** (or **length**) of a vector $x \in H$, denoted by $\|x\|$, is the nonnegative real number as defined by

$$\|x\| = \sqrt{(x,x)},$$

and is called the norm **induced by** the inner product on H.

We shall see below that this satisfies the conditions for being a norm as laid out in Definition 2.2.1.

The norm of an element $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in the unitary space $\mathbb{C}^n$ is

$$\|x\| = \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{\frac{1}{2}},$$

and that of an $x = \{\alpha_i\}_{i\ge 1}$ in ℓ^2 is

$$\|x\| = \left(\sum_{i=1}^{\infty} |\alpha_i|^2\right)^{\frac{1}{2}}.$$

The norm of an element $f \in C[a, b]$ [respectively, $f \in C^n[a, b]$] is

$$\|f\| = \left(\int_a^b |f(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \left[\text{resp.} \left(\sum_{i=0}^{n} \int_a^b \left|f^{(i)}(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right].$$

The norm of an element f in RL^2 or in RH^2 is

$$\|f\| = \left(\frac{1}{2\pi i} \int_{\partial D} |f(z)|^2 \frac{\mathrm{d}z}{z}\right)^{\frac{1}{2}} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f(\mathrm{e}^{i\theta})^2\right| \mathrm{d}\theta\right)^{\frac{1}{2}}.$$

Proposition 2.2.3 *In an inner product space H, $\|\cdot\|$ has the following properties: for $x, y \in H$ and $\lambda \in \mathbb{F}$,*

(a) $\|x\| \geq 0$ *and* $\|x\| = 0$ *if, and only if,* $x = 0$;
(b) $\|\lambda x\| = |\lambda|\|x\|$;
(c) **(Parallelogram Law)**

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2;$$

(d) **(Polarisation Identity in case $\mathbb{F} = \mathbb{C}$)**

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

Proof (a) is immediate from Definition 2.1.1(c) while (b) follows from

$$\|\lambda x\|^2 = (\lambda x, \lambda x) = |\lambda|^2 (x,x) = |\lambda|^2 \|x\|^2.$$

For $x, y \in H$, we have

$$\|x+y\|^2 = (x+y, x+y) = \|x\|^2 + \|y\|^2 + (x,y) + (y,x). \tag{2.10}$$

In the identity (2.10), replace y by $-y$ to obtain

$$\|x-y\|^2 = (x-y, x-y) = \|x\|^2 + \|y\|^2 - (x,y) - (y,x). \tag{2.11}$$

Adding (2.10) and (2.11), we get

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

This proves the Parallelogram Law.

In the identity (2.10), replace y by $-y$, iy and $-iy$:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - (x, y) - (y, x). \tag{2.12}$$

$$\|x + iy\|^2 = \|x\|^2 + \|y\|^2 - i(x, y) + i(y, x). \tag{2.13}$$

$$\|x - iy\|^2 = \|x\|^2 + \|y\|^2 + i(x, y) - i(y, x). \tag{2.14}$$

Multiply both sides of (2.12) by -1, (2.13) by i and (2.14) by $-i$ and add to (2.10) to obtain the following:

$$\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 = 4(x, y).$$

This completes the proof of the Polarisation Identity. □

Remark In Proposition 2.2.3, the assertions (a)–(c) are valid in real as well as complex inner product spaces, but (d) holds only in a complex inner product space.

Theorem 2.2.4 (Cauchy–Schwarz Inequality) *Let H be an inner product space and let $\|x\|$ denote the norm of $x \in H$. Then*

$$|(x, y)| \le \|x\|\|y\| \tag{2.15}$$

for x, $y \in H$, and equality holds if, and only if, x and y are linearly dependent.

Proof Choose a real number θ such that $e^{i\theta}(x, y) = |(x, y)|$. Let $\lambda = \alpha e^{i\theta}$, where $\alpha \in \mathbb{R}$. Then

$$(x - \bar{\lambda}y, x - \bar{\lambda}y) = \|x\|^2 + \left\|\bar{\lambda}y\right\|^2 - \lambda(x, y) - \bar{\lambda}(y, x). \tag{2.16}$$

The expression on the left side of (2.16) is real and nonnegative. Hence,

$$\|x\|^2 + \alpha^2\|y\|^2 - 2\alpha|(x, y)| \ge 0 \tag{2.17}$$

for every real α. If $\|y\| = 0$, then we must have $|(x, y)| = 0$, for otherwise (2.17) will be false for large positive values of α. If $\|y\| > 0$, take $\alpha = |(x, y)|/\|y\|^2$ in (2.17) and obtain

$$|(x, y)|^2 \le \|x\|^2\|y\|^2.$$

If x and y are linearly dependent, then we may write $y = kx$ or $x = ky$ for some $k \in \mathbb{F}$. Then

$$\begin{aligned} |(x, y)| &= |(x, kx)| = |k|\|x\|^2 \\ &= \|kx\|\|x\| = \|y\|\|x\|, \end{aligned}$$

that is, equality holds in (2.15).

On the other hand, suppose that $|(x,y)| = \|x\|\|y\|$. If $\|y\| = 0$, then $y = 0$ and x and y are linearly dependent. If $\|y\| \neq 0$, then

$$\begin{aligned}\left(x - \frac{(x,y)}{\|y\|^2}\,y, x - \frac{(x,y)}{\|y\|^2}\,y\right) &= \|x\|^2 + \frac{|(x,y)|^2}{\|y\|^2} - 2\Re\left(x, \frac{(x,y)}{\|y\|^2}\,y\right) \\ &= \|x\|^2 + \frac{|(x,y)|^2}{\|y\|^2} - 2\frac{|(x,y)|^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2}.\end{aligned}$$

Hence, $x - \frac{(x,y)}{\|y\|^2}y = 0$; that is, x and y are linearly dependent. □

Remark The above proof of the Cauchy–Schwarz Inequality is valid in the real case as well.

Applying Theorem 2.2.4 in specific spaces such as $\mathbb{C}^n$, ℓ^2 and $C[a, b]$, the following corollary results.

Corollary 2.2.5

(a) *If* $x_1, x_2, \ldots, x_n$ *and* $y_1, y_2, \ldots, y_n$ *are complex numbers*,

$$\left|\sum_{i=1}^{n} \alpha_i \overline{\beta_i}\right| \leq \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\beta_i|^2\right)^{\frac{1}{2}}.$$

(b) *If* $\{\alpha_i\}_{i\geq 1}$ *and* $\{\beta_i\}_{i\geq 1}$ *are square summable sequences of complex numbers*,

$$\left|\sum_{i=1}^{\infty} \alpha_i \overline{\beta_i}\right| \leq \left(\sum_{i=1}^{\infty} |\alpha_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\beta_i|^2\right)^{\frac{1}{2}}.$$

(c) *If* $f, g \in C[a, b]$, *then*

$$\left|\int_a^b f(t)\,\overline{g(t)}\,\mathrm{d}t\right| \leq \left(\int_a^b |f(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}.$$

In each case, equality holds if, and only if, the vectors involved are linearly dependent.

Theorem 2.2.6 (Triangle inequality) *In an inner product space H,*

$$\|x+y\| \le \|x\| + \|y\| \tag{2.18}$$

for all x, $y \in H$.

Proof For x, $y \in H$,

$$\begin{aligned}
\|x+y\|^2 &= (x+y, x+y) \\
&= \|x\|^2 + \|y\|^2 + (x,y) + (y,x) \\
&= \|x\|^2 + \|y\|^2 + 2\Re(x,y) \\
&\le \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|,
\end{aligned}$$

using the Cauchy–Schwarz Inequality (2.15). Thus,

$$\|x+y\|^2 \le (\|x\| + \|y\|)^2,$$

which implies

$$\|x+y\| \le \|x\| + \|y\|.$$

□

Applying Theorem 2.2.6 to specific inner product spaces, such as ℓ^2, RL^2 and $C[a, b]$, the following inequalities are obtained:

Corollary 2.2.7

(a) *If $\{x_i\}_{i\ge 1}$ and $\{y_i\}_{i\ge 1}$ are in ℓ^2, then*

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty} |y_i|^2\right)^{\frac{1}{2}}.$$

(b) *If f, g are in RL^2, then*

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta}) + g(e^{i\theta})|^2 d\theta\right)^{\frac{1}{2}} \le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta\right)^{\frac{1}{2}} + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(e^{i\theta})|^2 d\theta\right)^{\frac{1}{2}}.$$

(c) *If f, g are in C*[*a*, *b*], *then*

$$\left(\int_a^b |f(t)+g(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \leq \left(\int_a^b |f(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} + \left(\int_a^b |g(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}.$$

Corollary 2.2.8 *In an inner product space H*,

$$|\|x\| - \|y\|| \leq \|x-y\| \tag{2.19}$$

for all $x, y \in H$.

Proof For $x, y \in H$,

$$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$$

by Theorem 2.2.6. This implies

$$\|x\| - \|y\| \leq \|x-y\|. \tag{2.20}$$

On interchanging the roles of x and y, we get

$$\|y\| - \|x\| \leq \|y-x\| = \|x-y\|. \tag{2.21}$$

The inequality (2.19) follows upon combining (2.20) and (2.21). □

Problem Set 2.2

2.2.P1. Show that for x, y and $z \in X$, an inner product space, the following Apollonius Identity holds:

$$\|x-z\|^2 + \|y-z\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2.$$

2.2.P2. Show that the formula $(A, B) = \mathrm{trace}(B^*A)$ defines an inner product on the space $\mathbb{C}^{n\times n}$ of $n \times n$ complex matrices, where $n \in \mathbb{N}$ and B^* denotes the conjugate transpose of B. Determine $\|I_n\|$, where I_n is the identity matrix.

2.2.P3. Show that (i) $x = \left\{\frac{1}{n}\right\}_{n\geq 1} \in \ell^2$ and determine $\|x\|$; (ii) $x = \left\{2^{-n/2}\right\}_{n\geq 1} \in \ell^2$ and determine $\|x\|$.

2.2.P4. Prove that, for any function $f \in C[0, \pi]$,

$$\left|\int_0^{\pi} f(t)\sin t\,\mathrm{d}t\right| \le \sqrt{\frac{\pi}{2}}\left[\int_0^{\pi} |f(t)|^2 \mathrm{d}t\right]^{\frac{1}{2}},$$

and describe the nonzero functions for which equality holds.

2.2.P5. Suppose $\mu(X) = 1$ and f, g are positive measurable functions on X such that $fg \ge 1$. Prove that $\int_X f\,\mathrm{d}\mu \int_X g\,\mathrm{d}\mu \ge 1$.

2.2.P6. Let $X_1 = (C[0, 1], \|\cdot\|_\infty)$, where $\|x\|_\infty = \sup_{0 \le t \le 1}|x(t)|$, and let $X_2 = (C[0, 1], \|\cdot\|_2)$, where

$$\|x\|_2 = (x, x)^{\frac{1}{2}},\ (x, y) = \int_0^1 x(t)\overline{y(t)}\mathrm{d}t.$$

Show that the identity mapping id: $X_1 \to X_2$ is continuous, but its inverse id: $X_2 \to X_1$ is not.

2.2.P7. Let X be an inner product space. If $\|x\| = \|y\| = \frac{1}{2}\|x + y\|$, then show that $x = y$. The result fails to hold if $X = \mathbb{R}^n$ or $\mathbb{C}^n$ with norm $\|\cdot\|_1$ when $n > 1$.

2.2.P8. Let X be a vector space over $\mathbb{C}$. Let $(\cdot,\cdot)$ be a complex-valued function of two variables (x, y): $X \times X \to \mathbb{C}$ which has the following properties:

(a) $(\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y)$
(b) $(x, y) = \overline{(y, x)}$
(c) $(x, x) \ge 0$ and (x, x) may be zero for nonzero x.

Prove that the Cauchy–Schwarz Inequality still holds but without the rider about when equality holds.

2.2.P9. [Notations as in Example 2.1.3(vi)] Let $g(z) = \frac{1}{(z-\alpha)(z-\beta)}$, where α and β are distinct points in D. Using the Residue Theorem, show that

$$\|g\| = \frac{(1 - |\alpha\beta|^2)^{\frac{1}{2}}}{(1 - |\alpha|^2)^{\frac{1}{2}}(1 - |\beta|^2)^{\frac{1}{2}}|1 - \overline{\alpha}\beta|}.$$

2.2.P10. [Notations as in Example 2.1.3(vi)] Prove that for $\alpha \in D$,

$$F = \{f \in RH^2 : f(\alpha) = 0\}$$

is a closed linear subspace of RH^2.

2.3 Inner Product Spaces as Metric Spaces

We have seen in Proposition 2.2.3 and Theorem 2.2.6 that the norm induced by an inner product in H satisfies the following conditions of Definition 2.2.1:

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if, and only if, $x = 0$,
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in H$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in H$.

Remark 2.3.1 The inner product in H together with the metric $d(x, y) = \|x - y\|$ from the norm induced by the inner product is a metric space. As in any metric space, $d(x, y)$ is called the distance from x to y or between x and y.

We shall henceforth feel free to use, for inner product spaces, all metric concepts such as open and closed sets, convergence, continuity, uniform continuity, Cauchy sequence, completeness, dense sets and separability. Below we translate the general metric space concepts defined in Sect. 1.2 into inner product space terms.

It follows from (2.19) that the map $x \to \|x\|$ defined in H is continuous. In fact, it is uniformly continuous in view of (2.19).

Remarks 2.3.2

(i) A sequence $\{x_n\}_{n\geq 1}$ in a normed space or in an inner product space is Cauchy if for every $\varepsilon > 0$, there exists an integer n_0 such that

$$\|x_n - x_m\| < \varepsilon \text{ whenever } n, m \geq n_0.$$

(ii) Every convergent sequence is Cauchy; the converse is, however, not true; in fact, let $\{x_n\}_{n\geq 1}$, where $x_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots)$, $n = 1, 2, \ldots$ be a sequence in the inner product space ℓ_0 (see Example 2.1.3(ii)). Then the sequence $\{x_n\}_{n\geq 1}$ is Cauchy because

$$\|x_{m+p} - x_m\| = \left(\sum_{k=m+1}^{m+p} \frac{1}{k^2}\right)^{\frac{1}{2}}$$

can be made arbitrarily small by choosing m sufficiently large. However, the sequence does not converge to an element of the space. Assume the contrary, that is suppose $x_n \to x$, where $x = (\lambda_1, \lambda_2, \ldots, \lambda_N, 0, 0, \ldots)$. If $n \geq N$, then

$$\begin{aligned}\|x_n - x\|^2 &= \sum_{k=1}^{n} \left|\frac{1}{k} - \lambda_k\right|^2 + \sum_{k=n+1}^{\infty} |\lambda_k|^2 \\ &= \sum_{k=1}^{n} \left|\frac{1}{k} - \lambda_k\right|^2.\end{aligned}$$

On letting $n \to \infty$, we obtain $\sum_{k=1}^{\infty} \left|\frac{1}{k} - \lambda_k\right|^2 = 0$, which implies $\lambda_k = \frac{1}{k}$ for each k, contradicting the fact that x has finitely many nonzero terms.

(iii) The open [respectively, closed] ball with centre x_0 and radius ε is the set $\{x \in H\colon \|x - x_0\| < \varepsilon\}$ [respectively, $\{x \in H\colon \|x - x_0\| \le \varepsilon\}$]. In view of Proposition 1.2.10, a subset $K \subseteq H$ is bounded if, and only if, there exists an $M > 0$ such that $K \subseteq \{x\colon \|x\| \le M\}$.

Definition 2.3.3 An inner product space H is said to be **complete** if every Cauchy sequence in H converges. That is, if $\{x_n\}_{n\ge 1}$ is a sequence in H satisfying $\|x_n - x_m\| \to 0$ as $n, m \to \infty$, there exists an $x \in H$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. An inner product space which is complete is called a **Hilbert space**.

Every Hilbert space is a Banach space. The norm in a Hilbert space is derived from the inner product.

Examples 2.3.4

(i) The inner product space $H = \mathbb{C}^n$ with the metric given by

$$d(x,y) = \|x-y\| = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}, \tag{2.22}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{C}^n$, is a Hilbert space with metric as above or with inner product as in (i) of Examples 2.1.3.

We need to check that $\mathbb{C}^n$, with the metric defined in (2.22), is complete [see (i) of Examples 2.1.3]. Let $\left\{x^{(m)}\right\}_{m\ge 1} = \left(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\right)$ denote a Cauchy sequence in $\mathbb{C}^n$, i.e. $d(x^{(m)}, x^{(m')}) \to 0$ as $m, m' \to \infty$. Then for a given $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that

$$\left(\sum_{k=1}^{n} \left|x_k^{(m)} - x_k^{(m')}\right|^2\right)^{\frac{1}{2}} < \varepsilon \quad \text{for all} \quad m, m' \ge n_0(\varepsilon). \tag{2.23}$$

Hence $\left|x_k^{(m)} - x_k^{(m')}\right| < \varepsilon$ for all $m, m' \ge n_0(\varepsilon)$ and all $k = 1, 2, \ldots, n$. Upon fixing k and using the Cauchy Principle of Convergence, it follows that $\left\{x_k^{(m)}\right\}_{m\ge 1}$ converges to a limit x_k. Let $x = (x_1, x_2, \ldots, x_n)$ and $m \ge n_0(\varepsilon)$. It follows from (2.23) that

$$\sum_{k=1}^{n} \left|x_k^{(m)} - x_k^{(m')}\right|^2 < \varepsilon^2 \tag{2.24}$$

for all $m' \geq n_0(\varepsilon)$. Letting $m' \to \infty$ in (2.24), we have

$$\sum_{k=1}^{n} \left| x_k^{(m)} - x_k \right|^2 \leq \varepsilon^2$$

for all $m \geq n_0(\varepsilon)$. That is, $d(x^{(m)}, x) \to 0$ in $\mathbb{C}^n$.

(ii) The inner product space $H = \ell^2$ (see Example 2.1.3(iii)) is a Hilbert space.

We shall show that ℓ^2 with the metric

$$d(x, y) = \|x - y\| = \left(\sum_{k=1}^{\infty} |x_k - y_k|^2 \right)^{\frac{1}{2}} \tag{2.25}$$

is complete. Let $\{x^{(m)}\}_{m \geq 1} = \left(x_1^{(m)}, x_2^{(m)}, \ldots \right)$ denote a Cauchy sequence in ℓ^2. Then for a given $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that

$$\left(\sum_{k=1}^{\infty} \left| x_k^{(m)} - x_k^{(m')} \right|^2 \right)^{\frac{1}{2}} < \varepsilon \quad \text{for all} \quad m,\, m' \geq n_0(\varepsilon). \tag{2.26}$$

This implies $\left| x_k^{(m)} - x_k^{(m')} \right| < \varepsilon$ for all $m, m' \geq n_0(\varepsilon)$, i.e. for each k, the sequence $\left\{ x_k^{(m)} \right\}_{m \geq 1}$ is a Cauchy sequence of complex numbers. So by the Cauchy Principle of Convergence, $\lim_m x_k^{(m)} = x_k$, say. Let x be the sequence $(x_1, x_2, \ldots)$. It will be shown that $x \in \ell^2$ and $\lim_m x^{(m)} = x$. From (2.26), we have

$$\sum_{k=1}^{N} \left| x_k^{(m)} - x_k^{(m')} \right|^2 < \varepsilon^2 \tag{2.27}$$

for any positive integer N, provided $m, m' \geq n_0(\varepsilon)$. Letting $m' \to \infty$ in (2.27), we obtain

$$\sum_{k=1}^{N} \left| x_k^{(m)} - x_k \right|^2 \leq \varepsilon^2$$

for any positive integer N and all $m \geq n_0(\varepsilon)$. The sequence $\left\{ \sum_{k=1}^{N} \left| x_k^{(m)} - x_k \right|^2 \right\}_{N \geq 1}$ is a monotonically increasing sequence of nonnegative real numbers and is bounded above and, therefore, has a finite limit $\sum_{k=1}^{\infty} \left| x_k^{(m)} - x_k \right|^2$ which is less than or equal to ε^2. Hence

$$\left(\sum_{k=1}^{\infty}\left|x_k^{(m)}-x_k\right|^2\right)^{\frac{1}{2}}\le\varepsilon\quad\text{for all}\quad m,\, m'\ge n_0(\varepsilon). \tag{2.28}$$

Observe that

$$\left(\sum_{k=1}^{\infty}|x_k|^2\right)^{\frac{1}{2}}\le\left(\sum_{k=1}^{\infty}\left|x_k^{(m)}-x_k\right|^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left|x_k^{(m)}\right|^2\right)^{\frac{1}{2}},$$

using Corollary 2.2.7(a) and consequently $x\in\ell^2$. Moreover, $\lim_m x^{(m)}=x$ in ℓ^2 by (2.28).

It follows from Remark 2.3.2(ii) that ℓ_0 is an inner product space that is not complete.

Remarks 2.3.5

(i) The inner product space ℓ_0 of sequences all of whose terms, from some index onwards, are zero is dense in ℓ^2. In fact, let $x=(\alpha_1,\alpha_2,\ldots)$ be an element in ℓ^2 (not in ℓ_0) and $\varepsilon>0$ be given. Choose N such that

$$\sum_{k=N+1}^{\infty}|\alpha_k|^2<\varepsilon.$$

Then the sequence $y=(\alpha_1,\alpha_2,\ldots,\alpha_N,0,\ldots)$ is in the desired inner product space and is such that

$$\|x-y\|=\sum_{k=N+1}^{\infty}|\alpha|^2<\varepsilon.$$

This shows that each $x\in\ell^2$ (not in ℓ_0) is a limit point of the space ℓ_0 of sequences all of whose terms, from some index onwards, are zero.

(ii) It may be discerned from (i) above that ℓ_0 is not complete.

(iii) For $j=1,2,\ldots$, let $e_j=(0,\ldots,0,1,0,0,\ldots)$, where 1 occurs only in the jth place and

$$E=\{k_1e_1+\cdots+k_ne_n : n=1,2,\ldots,\Re k_j,\Im k_j\ \text{are rational}\}.$$

Since the rational numbers constitute a countable set, E is countable. We show that E is dense in ℓ^2. Let $(x_1,x_2,\ldots)\in\ell^2$ and $\varepsilon>0$. As $\sum_{j=1}^{\infty}\left|x_j\right|^2$ is finite, there is some N such that

$$\sum_{j=N+1}^{\infty} |x_j|^2 < \varepsilon^2/2.$$

Since the rational numbers are dense in $\mathbb{R}$, there are $k_1, \ldots, k_N$ in $\mathbb{C}$ with $\Re k_j$, $\Im k_j$ rational and

$$|x_j - k_j|^2 < \varepsilon^2/2N, \quad j = 1, 2, \ldots, N.$$

Consider $y = k_1 e_1 + \cdots + k_N e_N$ in E. Then

$$\|x - y\|^2 = \sum_{j=1}^{N} |x_j - k_j|^2 + \sum_{j=N+1}^{\infty} |x_j|^2 < \varepsilon^2/2 + \varepsilon^2/2 = \varepsilon^2.$$

Hence, $y \in S(x, \varepsilon)$. Thus, E is dense in ℓ^2. Consequently, ℓ^2 is a separable metric space.

Definition 2.3.6 Two Hilbert spaces H and K are said to be **isometrically isomorphic** if there exists a **linear isometry** between H and K, i.e. if there exists a bijective linear mapping $A{:}H \to K$ such that

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A(x_1) + \alpha_2 A(x_2),$$

$$(Ax_1, Ax_2) = (x_1, x_2)$$

for all $x_1, x_2 \in H$ and scalars α_1 and α_2.

Theorem 2.3.7 *For every inner product space X, there is a Hilbert space H such that X is a dense linear subspace of H, and for $x, y \in X$, the inner product (x, y) in X and in H is the same. The space H is unique up to a linear isometry; that is, if X is a dense linear subspace of a Hilbert space K, then there is a unique linear isometry $A{:}H \to K$ such that the restriction of A to X is the identity map.*

Proof Consider X as a metric space with the metric induced by the inner product on X, i.e. with $d(x, y) = \|x - y\| = (x - y, x - y)^{\frac{1}{2}}$. Let H be its completion. Let $x, y \in H$ and let $\{x_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ be sequences in X such that $x_n \to x$ and $y_n \to y$. Then for scalars λ and μ, the sequence $\{\lambda x_n + \mu y_n\}_{n \geq 1}$ is a Cauchy sequence in X. Now, if H is to be given a Hilbert space structure such that the inner product on H induces the metric on H, then

$$\lim_n(\lambda x_n + \mu y_n) = \lambda \lim_n x_n + \mu \lim_n y_n = \lambda x + \mu y.$$

Thus, $\lambda x + \mu y$ must be defined to be the limit of the Cauchy sequence $\{\lambda x_n + \mu y_n\}_{n \geq 1}$. It may be checked that the addition, scalar multiplication and the limits of the Cauchy sequences are well defined. It is now easy to check that with this definition of addition and scalar multiplication, H becomes a vector space. Now define $(x, y) = \lim_n (x_n, y_n)$; note that it is well defined. In fact, it is an inner product on H whose restriction to X agrees with the given inner product in X. With this inner product, H is a Hilbert space. The uniqueness can be easily verified. □

Problem Set 2.3

2.3.P1. Show that the space $(C[0, 1], \|\cdot\|_\infty)$, where $\|x\|_\infty = \sup_{0 \le t \le 1}|x(t)|$, is not an inner product space, hence not a Hilbert space.

Definition A **strictly convex norm** on a normed linear space is a norm such that, for all $x, y \in X$, $\|x\| = \|y\| = 1$, $y \neq x \Rightarrow \|x + y\| < 2$.

2.3.P2. (a) Show that the norm on a Hilbert space is strictly convex.
(b) Show that the norm $\|\cdot\|_\infty$ on $C[0, 1]$ is not strictly convex.
(c) Show that the norm $\|\cdot\|_1$ on $C[0, 1]$ is not strictly convex.

2.3.P3. Let H be the collection of all absolutely continuous functions $x:[0, 1] \to \mathbb{F}$ such that $x(0) = 0$ and $x' \in L^2[0, 1]$. If $(x, y) = \int_0^1 x'(t)\overline{y'(t)}\mathrm{d}t$ for $x, y \in H$, show that H is a Hilbert space.

2.3.P4. Let H be a Hilbert space over $\mathbb{R}$. Show that there is a Hilbert space K over $\mathbb{C}$ and a map $U:H \to K$ such that (i) U is linear (ii) $(Ux_1, Ux_2) = (x_1, x_2)$ for all $x_1, x_2 \in H$, (iii) for any $z \in K$, there are unique $x_1, x_2 \in H$ such that $z = Ux_1 + iUx_2$.

2.3.P5. (a) Suppose x and y are vectors in a normed space such that $\|x\| = \|y\|$. If there exists $t \in (0,1)$ such that $\|tx + (1 - t)y\| < \|x\|$, then show that this strict inequality holds for all $t \in (0, 1)$.
(b) Let x and y belong to a real or complex strictly convex normed space. If $\|x + y\| = \|x\| + \|y\|$ and $x \neq 0 \neq y$, show that there exists $\alpha > 0$ such that $y = \alpha x$.

2.3.P6. The set of all vectors $x = \{\eta_n\}_{n \ge 1}$ with $|\eta_n| \le \frac{1}{n}$, $n = 1, 2, \ldots$ in real ℓ^2 is called the *Hilbert cube*. Show that this set is compact in ℓ^2.

2.3.P7. Let $a = \{a_n\}_{n \ge 1}$ be a sequence of positive real numbers. Define $\ell_a^2 = \{x = (x_1, x_2, \ldots): x_i \in \mathbb{C}$ and $\sum_{i=1}^\infty a_i|x_i|^2 < \infty\}$. Define an inner product on ℓ_a^2 by $(x, y) = \sum_{i=1}^\infty a_i x_i \overline{y_i}$. Show that ℓ_a^2 is a Hilbert space.

2.3.P8. For a real number s, we define on $\mathbb{Z}$ a measure μ_s by setting

$$\mu_s(\{n\}) = \left(1 + n^2\right)^{s/2}, \quad n \in \mathbb{Z}.$$

Put $H^s = L^2(\mu_s)$. Prove that for $r < s$, we have $H^s \subseteq H^r$.

2.3.P9. (a) Find a sequence a of positive real numbers such that $(1, 1/2^2, 1/3^3, \ldots) \notin \ell_a^2$ [see Problem 2.3.P7].
(b) Find a sequence a of positive real numbers such that all $x = \{x_n\}_{n \ge 1}$ with $|x_n| = n^n$ are in ℓ_a^2.

2.3.P10. Let M be a closed subspace of a Hilbert space H, and let $y \in H$, $y \notin M$. If M' is the subspace spanned by M and y, then M' is closed. In particular, a finite-dimensional subspace must be closed.

2.4 The Space L^2 $(X, \mathfrak{M}, \mu)$

A class of spaces associated with $\widetilde{L}^p(X, \mathfrak{M}, \mu)$ for all p, $0 < p \le \infty$ [see Definition 1.3.5] is important in analysis. Here we are concerned only with cases $p = 1$ and $p = 2$. We shall see that there is a Hilbert space associated with $\widetilde{L}^2(X, \mathfrak{M}, \mu)$. Henceforth, the symbols $\mathfrak{M}$ and μ will be omitted.

Proposition 2.4.1 $\widetilde{L}^2(X)$ *is a vector space.*

Proof Suppose that $f \in \widetilde{L}^2(X, \mathfrak{M}, \mu)$ and $\alpha \in \mathbb{C}$. Then f is measurable and so αf is measurable; $|f|^2$ is integrable and so $|\alpha f|^2 = |\alpha|^2|f|^2$ is integrable. Thus, $\alpha f \in \widetilde{L}^2(X, \mathfrak{M}, \mu)$.

Suppose that $f, g \in \widetilde{L}^2(X)$. Then f and g are measurable and so $f + g$ is measurable. For all complex numbers α and β,

$$|\alpha + \beta|^2 \le 2|\alpha|^2 + 2|\beta|^2.$$

The above inequality may be seen to result on applying the Cauchy–Schwarz Inequality (Theorem 2.2.4) to the inner product $((\alpha, \beta), (1,1))$ in the inner product space $\mathbb{C}^2$. So, for all $x \in X$,

$$0 \le |f(x) + g(x)|^2 \le 2\left(|f(x)|^2 + |g(x)|^2\right).$$

The function on the right is integrable and therefore so is the measurable function $|f + g|^2$. This proves that $f + g \in \widetilde{L}^2(X)$. □

Definition 2.4.2 If f and g are two complex-valued measurable functions defined on X, let us write '$f \sim g$' if $\{x \in X: f(x) \ne g(x)\}$ is a null set (a measurable set of measure zero). One says that the functions f and g are **equivalent** or that $f - g$ is a **null function**.

Let $\mathcal{N}$ denote the set of all null functions, that is,

$$\mathcal{N} = \{f : f \sim 0\}.$$

The relation '$\sim$' is an equivalence relation on the set of all complex-valued measurable functions defined on X. As such, it partitions the set into disjoint equivalence classes, where a typical class, denoted by $[f]$, is given by

$$[f] = \{g : g \text{ measurable on } X \text{ and } g \sim f\}$$

and $g \in [f]$ would be called a **representative** of this class.

Note that $\mathcal{N}$ is a vector space of functions and $\mathcal{N} \subseteq \widetilde{L}^p(X)$ for all $p > 0$.

If f is a function in $\widetilde{L}^p(X)$, then $[f] = f + \mathcal{N}$ is the coset containing f, as defined towards the end of Sect. 1.1.

Definition 2.4.3 The space $\boldsymbol{L^2(X, \mathfrak{M}, \mu)}$ is the set of all equivalence classes of functions in $\widetilde{L}^2(X)$.

Thus, if $f \in \widetilde{L}^2(X)$, then $[f]$ is the corresponding member of $L^2(X, \mathfrak{M}, \mu)$. One says that f is a representative of the equivalence class $[f]$. The set just defined is what we intend to make into the promised Hilbert space associated with $\widetilde{L}^2(X, \mathfrak{M}, \mu)$.

Proposition 2.4.4 $L^2(X)$ *is a vector space*.

Proof $\mathcal{N}$ is a subspace of $\widetilde{L}^2(X)$, and $L^2(X)$ is actually the quotient space $\widetilde{L}^2(X, \mathfrak{M}, \mu)/\mathcal{N}$. □

We next define an inner product on the space $\widetilde{L}^2(X)$.

Proposition 2.4.5 *If f, $g \in \widetilde{L}^2(X)$* then $fg \in \widetilde{L}^1(X)$.

Proof Suppose f, $g \in \widetilde{L}^2(X)$. Then, f, g are measurable and so the product fg is measurable. The functions $|f|^2$ and $|g|^2$ are integrable and it follows from the inequality

$$|f(x)g(x)| \leq \frac{1}{2}\left(|f(x)|^2 + |g(x)|^2\right), \quad x \in X$$

that fg is also integrable. □

Now let us define (f, g) for f, $g \in \tilde{L}^2(X)$ by

$$(f, g) = \int_X f(x)\overline{g(x)}\mathrm{d}\mu(x) \quad \text{or} \quad \int_X (f\overline{g})\mathrm{d}\mu.$$

Note that if $g \in \widetilde{L}^2(X)$ then $\overline{g}$ (defined by $\overline{g}(x) = \overline{g(x)}$) is also a member of $\widetilde{L}^2(X, \mathfrak{M}, \mu)$, and so by Proposition 2.4.5, (f, g) is well defined. The reader can check that $(\cdot,\cdot)$ has all the properties of an inner product except one. If $(f, f) = 0$, then

$$0 = (f, f) = \int_X |f(x)|^2\mathrm{d}\mu(x) = \int_X |f|^2\mathrm{d}\mu$$

and therefore, $f \sim 0$; that is, f is a null function, but one cannot conclude that $f = 0$.

However, if $f \sim f'$ and $g \sim g'$, then $(f, g) = (f', g')$. In fact,

$$\begin{aligned}\left|\int_X (f\overline{g}) - \int_X f'\overline{g}'\right| &= \left|\int_X (f-f')\overline{g} + \int_X f'\left(\overline{g-g'}\right)\right| \\ &\le \left|\int_X (f-f')\overline{g}\right| + \left|\int_X f'\left(\overline{g-g'}\right)\right| \\ &\le (f-f',f-f')^{\frac{1}{2}}(g,g)^{\frac{1}{2}} + (f',f')^{\frac{1}{2}}(g-g',g-g')^{\frac{1}{2}} = 0,\end{aligned}$$

using the Cauchy–Schwarz Inequality 2.2.4. In view of the inequality proved above, the integral

$$\int_X (f\overline{g})$$

depends only on the equivalence classes $[f]$ and $[g]$ of the functions.

We can now define $(\cdot,\cdot)$ on $L^2(X, \mathfrak{M}, \mu)$.

Definition 2.4.6 For $[f]$, $[g] \in L^2(X)$, define

$$([f], [g]) = \int_X (f\overline{g})\mathrm{d}\mu,$$

where $f \in [f]$ and $g \in [g]$.

In view of the remarks preceding Definition 2.4.6, $(\cdot,\cdot)$ is unambiguously defined on $L^2(X, \mathfrak{M}, \mu)$.

Proposition 2.4.7 *The space $L^2(X)$ with inner product as in Definition 2.4.6 is an inner product space.*

Proof For $[f] \in L^2(X)$, if $([f], [f]) = 0$, then $\int_X |f|^2\mathrm{d}\mu = 0$ and hence $[f] = 0$, the zero of $L^2(X)$. The verification of the other axioms of an inner product is straightforward. □

Remark 2.4.8 We shall adopt the usual practice and abandon the notation $[f]$. The symbol f will be used to denote both a function in $\widetilde{L}^2(X)$ and the corresponding equivalence class of functions in $L^2(X)$. Working mathematicians tend to ignore the distinction between a function and its equivalence class. The correspondence between statements about $\widetilde{L}^2(X)$ and $L^2(X)$ is straightforward and gives rise to no confusion. In the subsequent discussions, it will always be clear whether calculations are in terms of functions or equivalence classes of functions.

Note that for $f \in L^2(X)$,

$$\|f\| = (f,f)^{\frac{1}{2}} = \left(\int_X |f(x)|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}} = \left(\int_X |f|^2 \mathrm{d}\mu\right)^{\frac{1}{2}}.$$

Now we can prove a result which is of central importance in analysis. It is often called the Riesz–Fischer Theorem.

Theorem 2.4.9 (Riesz–Fischer Theorem) $(L^2(X), (\cdot,\cdot))$ *is a Hilbert space*.

Proof Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $L^2(X)$; that is, for $\varepsilon > 0$ there exists an integer n_0 such that

$$\|f_n - f_m\| < \varepsilon \quad \text{whenever} \quad n, m \geq n_0.$$

Then there exists a subsequence $\{f_{n_i}\}_{i\geq 1}$, $n_1 < n_2 < \cdots$, such that

$$\|f_{n_{i+1}} - f_{n_i}\| = \left(\int_X |f_{n_{i+1}} - f_{n_i}|^2 \mathrm{d}\mu\right)^{\frac{1}{2}} < \frac{1}{2^i}, \quad i = 1, 2, \ldots$$

Indeed, if n_k has been selected, choose $n_{k+1} > n_k$ such that n, $m > n_{k+1}$ implies $\|f_n - f_m\| < \frac{1}{2^{k+1}}$.

Let

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|,$$

and

$$g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|.$$

Then by the triangle inequality (Theorem 2.2.6),

$$\|g_k\| = \left\|\sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|\right\| \leq \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\| < \sum_{i=1}^{k} \frac{1}{2^i} < 1 \text{ for} \quad k = 1, 2, \ldots$$

Hence, an application of Fatou's Lemma (Theorem 1.3.8) to $\{g_k^2\}_{k\geq 1}$ gives

$$\int_X g^2 = \int_X (\liminf_k g_k^2) \mathrm{d}\mu$$
$$\leq \liminf_k \int_X g_k^2 \mathrm{d}\mu$$
$$\leq 1,$$

that is, $\|g\| \leq 1$. In particular, $g(x) < \infty$ a.e. Indeed, if $E = \{x\colon |g(x)| = \infty\}$ has positive measure, then $\int_X g(x)^2 \mathrm{d}\mu(x) = \infty$. Therefore, the series

$$g = \sum_{i=1}^{\infty} \left(f_{n_{i+1}} - f_{n_i}\right). \tag{2.29}$$

converges absolutely for almost all x. Denote the sum of (2.29) by $f(x)$ for those x at which (2.29) converges. Put $f(x) = 0$ on the remaining set of measure zero. Since

$$f_{n_1} + \sum_{i=1}^{k-1} (f_{n_{i+1}} - f_{n_i}) = f_{n_k},$$

we see that

$$f(x) = \lim_i f_{n_i}(x) \text{ a.e.}$$

Having determined a function f which is the pointwise limit almost everywhere of $\{f_{n_i}\}_{i \geq 1}$, we have to show that f is the L^2-limit of $\{f_n\}_{n \geq 1}$, i.e. $\lim_n \|f_n - f\| = 0$, where $\|\cdot\|$ denotes the L^2-norm. Recall that

$$\|f_n - f_m\| < \varepsilon \quad \text{whenever} \quad n, m \geq n_0.$$

For $m > n_0$, another application of Fatou's Lemma shows that

$$\int_X |f - f_m|^2 \mathrm{d}\mu \leq \lim_i \int_X |f_{n_i} - f_m|^2 \mathrm{d}\mu \leq \varepsilon^2. \tag{2.30}$$

We conclude from (2.30) that $f - f_m \in L^2(X)$ and hence $f \in L^2(X)$ since $f = (f - f_m) + f_m$. Finally,

$$\|f - f_m\| \to 0 \text{ as } m \to \infty.$$

This completes the proof that $L^2(X, \mathfrak{M}, \mu)$ is a Hilbert space. □

Remark 2.4.10 In the course of the proof, we have shown that if $\{f_n\}_{n \geq 1}$ is a Cauchy sequence in $L^2(X)$ with limit f, then $\{f_n\}_{n \geq 1}$ has a subsequence which convergence pointwise almost everywhere to f.

The simple functions play an important role in $L^2(X)$.

Theorem 2.4.11 *Let S be the collection of all measurable simple functions on X vanishing outside subsets of finite measure. Then S is dense in* $L^2(X)$.

Proof Clearly, $S \subseteq L^2(X)$. Let $f \in L^2(X)$ and assume that $f \geq 0$. There exists a sequence $\{s_n\}_{n \geq 1}$ of measurable simple functions such that

$$0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq f(x) \quad \text{and} \quad s_n(x) \to f(x)$$

(see 1.3.3). Since $0 \leq s_n(x) \leq f(x)$, we have $s_n \in L^2(X)$ and hence $s_n \in S$. Since $|f - s_n|^2 \leq f^2$, the Dominated Convergence Theorem 1.3.9 shows that $\|f - s_n\| \to 0$ as $n \to \infty$. Then, f is in the L^2-closure of S. The general case when f is complex follows from the above considerations. □

Theorem 2.4.12 *Let* $X = \mathbb{R}$, $\mathfrak{M}$ *be the* σ*-algebra of measurable subsets of* $\mathbb{R}$ *and* μ *be the usual Lebesgue measure on* $\mathbb{R}$. *Then the set of all continuous functions vanishing outside subsets of finite measure is dense in* $L^2(X)$.

Proof Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. Then every bounded measurable (in particular continuous) function is square integrable on E. Consider now a nonempty closed subset F of E and its characteristic function χ_F. For $n = 1, 2, \ldots$, let

$$f_n(x) = \frac{1}{1 + n\,\mathrm{dist}(x, F)}, \quad x \in E,$$

where $\mathrm{dist}(x, F) = \inf\{|x - y|: y \in F\}$. Note that each f_n is continuous on E. Also, $f_n(x) = 1$ for all $x \in F$ and $f_n(x) \to 0$ as $n \to \infty$ for all $x \notin F$. Hence, $(\chi_F - f_n)(x) \to 0$ for all $x \in E$. Since $\mu(E) < \infty$ and $|f_n(x)| \leq 1$ for all $x \in E$, the Dominated Convergence Theorem shows that

$$\int_E |\chi_F - f_n|^2 d\mu \to 0 \text{ as } n \to \infty.$$

In case F is empty, χ_F is 0 everywhere and we may carry out the above argument with each f_n chosen to be 0 everywhere.

Now let E_0 be any measurable subset of $\mathbb{R}$ with $\mu(E_0) < \infty$. Let $\varepsilon > 0$ be given. Then, there exists a closed set $F \subseteq E_0$ such that $\mu(E_0 \backslash F) < \varepsilon$. This follows on using regularity of Lebesgue measure. Since

$$\begin{aligned} d(\chi_{E_0}, f_n) &= \|\chi_{E_0} - f_n\| \\ &\leq d(\chi_{E_0}, \chi_F) + d(\chi_F, f_n), \end{aligned}$$

and since

$$d(\chi_{E_0},\chi_F) = \left(\int_X |\chi_{E_0} - \chi_F|^2 \mathrm{d}\mu\right)^{\frac{1}{2}} = \mu(E_0 \backslash F)^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}},$$

it follows that

$$d(\chi_{E_0}, f_n) \to 0 \text{ as } n \to \infty.$$

We have thus proved that the characteristic functions of measurable subsets of finite measure can be approximated in L^2-norm by continuous functions which vanish outside sets of finite measure. The proof is now completed by using Theorem 2.4.11 and the triangle inequality. ☐

Problem Set 2.4

2.4.P1. For which real α does the function $f_\alpha(t) = t^\alpha \exp(-t)$, $t > 0$, belong to $L^2(0,\infty)$? What is $\|f_\alpha\|$ when defined?

2.4.P2. (a) Show that the subspace $M = \{x = \{x_n\}_{n\geq 1} \in \ell^2 : \sum_{n=1}^\infty \frac{1}{n} x_n = 0\}$ is closed in ℓ^2.

(b) Show that the subspace $M = \{x(t) \in L^2[1,\infty)\colon \int_1^\infty \frac{1}{t} x(t)\mathrm{d}t = 0\}$ is closed in $L^2[1,\infty)$.

2.4.P3. $L^p[0, 1]$, $1 \leq p < \infty$, $p \neq 2$, is not a Hilbert space.

2.5 A Subspace of $L^2(X, \mathfrak{M}, \mu)$

The following subspace of $L^2(X, \mathfrak{M}, \mu)$ will play an important role in the discussion of applications of Hilbert space tools to problems in analysis. Here $X = [a, b]$, $\mathfrak{M}$ is the σ-algebra of Lebesgue measurable subsets of $[a, b]$ and μ is the Lebesgue measure.

Example 2.5.1 Let $[a, b]$ be a closed subinterval of $\mathbb{R}$. Let $C[a, b]$ denote the space of complex-valued continuous functions defined on $[a, b]$ with inner product given by

$$(f,g) = \int_a^b f(t)\overline{g(t)}\mathrm{d}t, \quad f,g \in C[a,b]. \tag{2.31}$$

Then, $C[a, b]$ is an inner product space (see Example 2.1.3(iv)) which is dense in $L^2[a, b]$. Extend $f \in L^2[a, b]$ to $\mathbb{R}$ by setting $f = 0$ outside $[a, b]$. The extended function is defined on $\mathbb{R}$ and is in $L^2(\mathbb{R})$. There exists a continuous function g vanishing outside a set of finite measure such that $\|f - g\| < \varepsilon$ [see Theorem 2.4.12]. Consider the restriction of g to $[a, b]$, to be denoted by h. Then the given f is such that $\|f - h\| < \varepsilon$. Moreover, $C[a, b] \neq L^2[a, b]$ as the following argument shows.

If two functions differ at a point and are both continuous there, then they differ on a neighbourhood of that point. Consequently, they cannot be equivalent. It follows that the function

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in (\frac{1}{2}, 1] \\ 0 & \text{if} x = \frac{1}{2} \end{cases}$$

is not equivalent to any continuous function. This proves the assertion that $C[a, b] \neq L^2[a, b]$.

Thus, $C[a, b]$ is an inner product space which is not a Hilbert space.

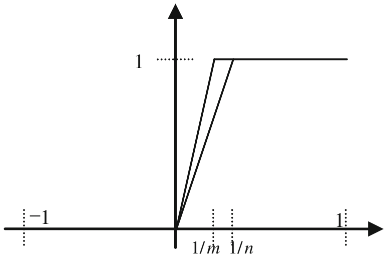
Remarks 2.5.2

(i) The reader will note that if g_1 and g_2 are continuous functions on $[a, b]$ and $g_1 \sim g_2$ then $g_1 = g_2$. So, if $f \in L^2[a, b]$ has a representative which is continuous, then that representative is unique. Thus, $C[a, b] \to L^2[a, b]$ is an injection. Moreover, uniform convergence of continuous functions implies convergence in L^2-norm. The aforementioned implication stems from the finiteness of the Lebesgue measure of $[a, b]$ and it fails if the bounded interval is replaced by an unbounded interval.

(ii) That the inner product space $C[a, b]$ with the inner product defined by (2.31) above is not complete can also be seen by exhibiting a Cauchy sequence in the space which converges to an element not lying in the space.

Let a and b be -1 and 1, respectively, and consider the sequence

$$f_n(t) = \begin{cases} 0 & -1 \leq t \leq 0 \\ nt & 0 < t \leq \frac{1}{n} \\ 1 & \frac{1}{n} < t \leq 1 \end{cases}.$$



Observe that $(m > n)$

$$\begin{aligned}\int_{-1}^{1} |f_n(t) - f_m(t)|^2 \mathrm{d}t &= \int_{0}^{1/m} (mt - nt)^2 \mathrm{d}t + \int_{1/m}^{1/n} (1 - nt)^2 \mathrm{d}t \\ &= (m-n)^2 \frac{1}{3m^3} + \left(\frac{1}{n} - \frac{1}{m}\right) \\ &\quad - n\left(\frac{1}{n^2} - \frac{1}{m^2}\right) + \frac{n^2}{3}\left(\frac{1}{n^3} - \frac{1}{m^3}\right) \\ &= \frac{(m-n)^2}{3m^2 n}.\end{aligned}$$

The right-hand side of the above equality tends to zero as $n, m \to \infty$. Thus, the sequence $\{f_n\}_{n \geq 1}$ is Cauchy. We next show that $f_n \to f$ in the L^2-norm, where $f(t) = 0$ for $-1 \leq t \leq 0$ and $f(t) = 1$ for $0 < t \leq 1$. In fact,

$$\int_{-1}^{1} |f_n(t) - f(t)|^2 \mathrm{d}t = \int_{0}^{1/n} (1 - nt)^2 \mathrm{d}t = \frac{1}{n} - \frac{1}{n} + \frac{1}{3n} = \frac{1}{3n} \to 0 \,\mathrm{as}\,\mathrm{n} \to \infty.$$

The limit function is not equivalent to any continuous function for reasons similar to those in Example 2.5.1.

Problem Set 2.5

2.5.P1. Prove that the system $\{1, t^3, t^6, \ldots\}$ has a dense linear span in the space $L^2[0, 1]$ as well as in $L^2[-1, 1]$.

2.6 The Hilbert Space $A(\Omega)$

Suppose $\Omega \subseteq \mathbb{C}$ is an arbitrary bounded domain whose boundary consists of smooth simple closed curves.

Consider the class of all holomorphic functions f in Ω for which the integral

$$\iint_{\Omega} |f|^2 \mathrm{d}m < \infty, \tag{2.32}$$

where dm is the two-dimensional Lebesgue measure, exists. In order to interpret the integral as a limit of the Riemann integral, we shall need the following:

Proposition 2.6.1 *Let Ω be a nonempty open subset of $\mathbb{C}$. Then there exists a sequence $\{K_n\}_{n\geq 1}$ of closed and bounded subsets of Ω such that Ω is their union. Moreover, the sets K_n can be chosen to satisfy the following conditions*:

(a) $K_n \subseteq K_{n+1}{}^{\mathrm{o}}$, $n = 1, 2, \ldots$;
(b) *every compact* (*closed and bounded*) *subset K of Ω is contained in K_n for some n.*

Proof For each positive integer n, let

$$K_n = \left\{ z : |z| \leq n \quad \text{and} \quad \operatorname{dist}(z, \mathbb{C}\backslash\Omega) \geq \frac{1}{n} \right\}.$$

Observe that K_n is both bounded and closed and hence compact or empty. Moreover, $K_n \subseteq \Omega$. Obviously,

$$K_n \subseteq \left\{ z : |z| < n+1 \quad \text{and} \quad \operatorname{dist}(z, \mathbb{C}\backslash\Omega) > \frac{1}{n+1} \right\} \subseteq K_{n+1}{}^{\mathrm{o}}.$$

This proves (a).

We next show that $\Omega = \bigcup_{n=1}^{\infty} K_n$. For, if $z \in \Omega$, then there exist m_1 and m_2 such that $|z| \leq m_1$ and $\operatorname{dist}(z, \mathbb{C}\backslash\Omega) \geq \frac{1}{m_2}$. Thus, $z \in K_n$ for some n. On the other hand, $K_n \subseteq \Omega$. This proves that $\Omega = \bigcup_{n=1}^{\infty} K_n$.

In view of (a), it follows that $\Omega = \bigcup_{n=1}^{\infty} K_n{}^{\mathrm{o}}$. If K is a compact subset of Ω, then the $K_n{}^{\mathrm{o}}$ form an open cover of K. By Definition 1.2.16, the compact set K is covered by finitely many $K_n{}^{\mathrm{o}}$. Consequently, K is contained in K_n for some large n. This completes the proof. □

On letting

$$\varphi_n(z) = \begin{cases} |f(z)|^2 & z \in K_n \\ 0 & z \in \Omega\backslash K_n \end{cases},$$

we see that φ_n is monotonically increasing and $\lim_n \varphi_n(z) = |f(z)|^2$, $z \in \Omega$. The Dominated Convergence Theorem now implies

$$\iint_{\Omega} \varphi_n \mathrm{d}m \to \iint_{\Omega} |f|^2 \mathrm{d}m \text{ as } n \to \infty,$$

that is,

$$\iint_{K_n} |f|^2 \mathrm{d}m \to \iint_{\Omega} |f|^2 \mathrm{d}m.$$

We now compute in terms of **Taylor coefficients** the integral (2.32) for $\Omega = \{z\colon |z| < R\}$ and a holomorphic function f defined in Ω.

The function f can be expanded in Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = f^{(n)}(0)/n!$, $n = 0, 1, 2, \ldots$

$$\begin{aligned}
\iint_{\Omega} |f|^2 \mathrm{d}m &= \iint_{\Omega} \left| \sum_{n=0}^{\infty} a_n z^n \right|^2 \mathrm{d}m \\
&= \iint \left(\sum_{n=0}^{\infty} a^n r^n \mathrm{e}^{in\theta} \right) \left(\sum_{k=0}^{\infty} \overline{a_k} r^k \mathrm{e}^{-ik\theta} \right) r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^R \int_0^{2\pi} \left(\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n \overline{a_k} r^{n+k+1} \mathrm{e}^{i(n-k)\theta} \right) \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^R \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n \overline{a_k} \int_0^{2\pi} \mathrm{e}^{i(n-k)\theta} \mathrm{d}\theta \, r^{n+k+1} \, \mathrm{d}r,
\end{aligned}$$

where term-by-term integration is valid because the power series representation converges uniformly on $|z| \leq r$ for $r < R$. Then

$$\iint_{\Omega} |f|^2 \mathrm{d}m = \int_0^R 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n+1} \mathrm{d}r = \pi \sum_{n=0}^{\infty} |a_n|^2 \frac{R^{2n+2}}{n+1}. \tag{2.33}$$

Definition 2.6.2 Let Ω be a bounded domain in the complex plane. The class of all holomorphic functions in Ω for which the integral $\iint_{\Omega} |f|^2 \mathrm{d}m$ is finite is denoted by $\boldsymbol{A(\Omega)}$ and is known as the **Bergman space**. Briefly,

$$\boldsymbol{A}(\Omega) = \{ f \text{ is holomorphic in } \Omega \text{ and } \iint_{\Omega} |f|^2 \mathrm{d}m < \infty \}$$

The integral is to be understood in the sense of $\lim_n \iint_{K_n} |f(z)|^2 \mathrm{d}m(z)$, where $\{K_n\}$ is a nondecreasing sequence of compact subsets of Ω whose union is Ω.

The following inequality will be useful in proving that $A(\Omega)$ is a Hilbert space.

Proposition 2.6.3 *Suppose $f \in A(\Omega)$ and $d_z = \operatorname{dist}(z, \partial\Omega)$. Then*

$$|f(z)|^2 \leq \left(\iint_{\Omega} |f|^2 \mathrm{d}m \right) / \pi d_z^2. \tag{2.34}$$

Proof Let D be the disc with centre z and radius d_z. Clearly,

$$\iint_{\Omega} |f|^2 \mathrm{d}m \geq \iint_{D} |f|^2 \mathrm{d}m.$$

It follows from the relation (2.33) above that

$$\iint_{\Omega} |f|^2 \mathrm{d}m \geq \pi |a_0|^2 d_z^2 = \pi |f(z)|^2 d_z^2.$$

So,

$$|f(z)|^2 \leq \left(\iint_{\Omega} |f|^2 \mathrm{d}m \right) / \pi d_z^2.$$

This completes the proof. □

The inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$ implies that

$$|af(z) + bg(z)|^2 \leq 2\left(|a|^2 |f(z)|^2 + |b|^2 |g(z)|^2 \right) \tag{2.35}$$

for any two functions $f, g \in A(\Omega)$; further, the identity

$$f\overline{g} = \frac{1}{2}|f + g|^2 + \frac{i}{2}|f + ig|^2 - \frac{1+i}{2}|f|^2 - \frac{1+i}{2}|g|^2 \tag{2.36}$$

holds.

Definition 2.6.4 *For $f, g \in A(\Omega)$, we write*

$$(f, g) = \iint_{\Omega} f\overline{g}\, \mathrm{d}m. \tag{2.37}$$

That the right side of (2.37) is finite follows from (2.35) and (2.36) above. It is easily verified that (2.37) defines an inner product on $A(\Omega)$.

Theorem 2.6.5 *With (f, g) defined as in (2.37), $A(\Omega)$ is a Hilbert space.*

Proof We show that $A(\Omega)$ is a complete inner product space. In view of (2.35), $A(\Omega)$ is closed under addition and scalar multiplication.

As usual, $A(\Omega)$ with norm defined by

$$\|f\| = (f,f)^{\frac{1}{2}} = \left(\iint_{\Omega} |f|^2 \,\mathrm{d}m\right)^{\frac{1}{2}},$$

becomes a normed space [see Definition 2.2.2 and Theorem 2.2.6]. It remains to show that $A(\Omega)$ is complete in this norm. Suppose $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in this norm, that is,

$$\|f_n - f_p\|^2 = \left(\iint_{\Omega} |f_n - f_p|^2 \,\mathrm{d}m\right) < \varepsilon$$

for $n, p \geq N$. For each compact set $K \subset \Omega$, it follows, on using Proposition 2.6.3, that

$$|f_n(z) - f_p(z)|^2 < \varepsilon/\pi d^2 \quad (z \in K),$$

where $d = \inf\{d(z,\omega): z \in K \text{ and } \omega \in \partial\Omega\}$. This means by Weierstrass' Theorem that on each compact subset, $K \subset \Omega$, the sequence $\{f_n\}_{n\geq 1}$ converges uniformly to a holomorphic function f:

$$f_n(z) \to f(z) \text{ as } n \to \infty \quad z \in K \subset \Omega.$$

Since

$$\iint_{K} |f_n - f_p|^2 \mathrm{d}m \leq \iint_{\Omega} |f_n - f_p|^2 \,\mathrm{d}m, \quad n, p \geq N,$$

it follows, on letting $p \to \infty$, that

$$\iint_{K} |f_n - f|^2 \mathrm{d}m \leq \varepsilon, \quad n \geq N,$$

for each compact set K; hence,

$$\iint_{K} |f_n - f|^2 \mathrm{d}m \leq \varepsilon, \quad n \geq N,$$

The last inequality implies $f \in A(\Omega)$ and $\|f_n - f\| \to 0$ as $n \to \infty$. This completes the proof. $\square$

Problem Set 2.6

2.6.P1. Let Ω be an arbitrary domain in $\mathbb{C}$ whose boundary consists of a finite number of smooth simple closed curves and let $A(\Omega)$ be the collection of all holomorphic functions $f{:}\Omega \to \mathbb{C}$ for which

$$\iint_{\Omega} |f(z)|^2 \mathrm{d}x\mathrm{d}y \left[\text{same as} \iint_{\Omega} |f|^2 \mathrm{d}m \text{ of Sect. 2.6}\right] < \infty$$

holds.

(a) Show that every $f \in A(\Omega)$, where $\Omega = \{z \in \mathbb{C}\colon 0 < |z| < 1\}$ has a removable singularity at $z = 0$;
(b) Show that if $\alpha \in \Omega$, then $\{f \in A(\Omega)\colon f(\alpha) = 0\}$ is closed in $A(\Omega)$.

2.7 Direct Sum of Hilbert Spaces

The definition of the external direct sum of vector spaces [Definition 1.1.7] is extended to any arbitrary family of vector spaces (each vector space is over the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ complex numbers) beginning with any finite such family. This procedure lays bare the intricacies involved and aids understanding.

The direct sum

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

of the vector spaces $H_1, H_2, \ldots, H_n$ is the set $H = H_1 \times H_2 \times \cdots \times H_n$, in which the addition and scalar multiplication are defined by the formula

$$\begin{aligned}(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),\\ \lambda(x_1, x_2, \ldots, x_n) &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n),\end{aligned}$$

where $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ are in $H_1 \times H_2 \times \ldots \times H_n$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

It is quite clear that H contains a subspace Y_i of H, where

$$Y_i = \{(x_1, x_2, \ldots, x_n) : x_j = 0 \quad \text{for} \quad j \neq i\},$$

which is isomorphic to H_i. It is sometimes convenient to refer to the space H_i itself as a subspace of H, and when such reference is made, it is the isomorphic space Y_i that is to be understood. The map of H into H_i given by

$$(x_1, x_2, \ldots, x_n) \to (0, 0, \ldots, x_i, 0, \ldots)$$

is a projection and is sometimes called projection of H onto H_i.

If H_1, H_2, …, H_n are Hilbert spaces, then H is the uniquely determined Hilbert space with inner product

$$((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum_{i=1}^{n} (x_i, y_i)_i, \tag{2.38}$$

where $(\cdot,\cdot)_i$ is the inner product in H_i. Then the norm in a direct sum of Hilbert spaces is given by

$$\|(x_1, x_2, \ldots, x_n)\| = |((x_1, x_2, \ldots, x_n), (x_1, x_2, \ldots, x_n))|^{\frac{1}{2}}. \tag{2.39}$$

To summarise, we state the following definition:

Definition 2.7.1 For each i = 1, 2, …, n, let H_i be a Hilbert space with inner product $(\cdot,\cdot)_i$. The **direct sum** of Hilbert spaces H_1, H_2, …, H_n is the vector space $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$ in which the inner product and the norm are defined by (2.38) and (2.39).

Henceforth, the subscripts i in the notation for the inner products and the norms will be omitted because the context will make it clear which one is intended.

Proposition 2.7.2 *With notations as above*, $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$ *is a Hilbert space*.

Proof It must be shown that (2.38) defines an inner product on H and H is complete with respect to the norm defined by (2.39). We shall check the second assertion only.

Let $\{(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})\}_{m \geq 1}$ be a Cauchy sequence in H, that is, $\sum_{i=1}^{n} \|x_i^{(m)} - x_i^{(\ell)}\|^2 \to 0$ as $m, \ell \to \infty$. For each k, $\|x_k^{(m)} - x_k^{(\ell)}\|^2 \leq \sum_{i=1}^{n} \|x_i^{(m)} - x_i^{(\ell)}\|^2$ shows that $\{x_k^{(m)}\}_{m \geq 1}$ is Cauchy in H_k. Since H_k is a Hilbert space, there exists x_k in H_k such that $x_k^{(m)} \to x_k$ as $m \to \infty$. Clearly, the vector $(x_1, x_2, \ldots, x_n)$ is in H. It will be shown that $(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \to (x_1, x_2, \ldots, x_n)$ in H. Let $\varepsilon > 0$ be given. For each k, there exists an integer m_k such that

$$\left\|x_k^{(m)} - x_k\right\| < \varepsilon/\sqrt{n} \quad \text{for} \quad m \geq m_k.$$

Consequently,

$$\sum_{k=1}^{n} \left\|x_k^{(m)} - x_k\right\|^2 < \varepsilon^2 \quad \text{for} \quad m \geq \max\{m_1, m_2, \ldots, m_n\}.$$

This completes the proof of the assertion made. □

We next define $H_1 \oplus H_2 \oplus \cdots$, also written as $\overset{\infty}{\underset{i=1}{\oplus}} H_i$, for a sequence of Hilbert spaces H_1, H_2, …. Let

$$H = \left\{ \{x_n\}_{n \geq 1} \colon x_n \in H_n, \quad n = 1, 2, \ldots \quad \text{and} \quad \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \right\}.$$

For $x = \{x_n\}_{n \geq 1}$ and $y = \{y_n\}_{n \geq 1}$ in H, define

$$(x, y) = \sum_{n=1}^{\infty} (x_n, y_n). \tag{2.40}$$

The sum on the right is seen to be finite by using the Cauchy–Schwarz Inequality for each H_i and then for ℓ^2. It can then be verified that $(\cdot,\cdot)$ is an inner product on H and the norm relative to the inner product is $\|x\| = (\sum_{n=1}^{\infty} \|x_n\|^2)^{\frac{1}{2}}$.

With this inner product, H can be shown to be a Hilbert space.

Proposition 2.7.3 *With notations as above*, $H = H_1 \oplus H_2 \oplus \cdots = \overset{\infty}{\underset{i=1}{\oplus}} H_i$ *is a Hilbert space*.

Proof It must be shown that (2.40) defines an inner product on H and H is complete with respect to the norm defined by

$$\|x\| = \left(\sum_{n=1}^{\infty} \|x_n\|^2 \right)^{\frac{1}{2}}.$$

For $x = \{x_n\}_{n \geq 1}$ and $y = \{y_n\}_{n \geq 1}$ in H,

$$\sum_{n=1}^{\infty} |(x_n, y_n)| \leq \sum_{n=1}^{\infty} \|x_n\| \, \|y_n\| \leq \left(\sum_{n=1}^{\infty} \|x_n\|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \|y_n\|^2 \right)^{\frac{1}{2}},$$

using the Cauchy–Schwarz Inequality twice. Hence, the series on the right of (2.40) converges absolutely. Consequently, $(\cdot,\cdot)$ is well defined. It is a routine exercise to show that $(\cdot,\cdot)$ is an inner product on H. It remains to show that H is a complete space. Suppose $\{x^{(m)}\}_{m \geq 1} = \{(x_1^{(m)}, x_2^{(m)}, \ldots)\}_{m \geq 1}$ is a Cauchy sequence in H, that is, $\|x^{(m)} - x^{(n)}\| \to 0$ as $m, n \to \infty$. For each k, $\|x_k^{(m)} - x_k^{(n)}\|^2 \leq \sum_{j=1}^{\infty} \|x_j^{(m)} - x_j^{(n)}\|^2 = \|x^{(m)} - x^{(n)}\|^2$, which shows that the sequence $\{x_k^{(1)}, x_k^{(2)}, \ldots\}$ of kth components is Cauchy. Since H_k is a Hilbert space, $x_k^{(n)} \to x_k$ as $n \to \infty$ for suitable x_k in H_k. It will be shown that $\sum_{n=1}^{\infty} \|x_k\|^2 < \infty$ and $x^{(n)} \to x$, where $x = \{x_k\}_{k \geq 1}$.

Given $\varepsilon > 0$. Let p be an integer such that $\|x^{(m)} - x^{(n)}\| < \varepsilon$ whenever $m, n \geq p$. For any positive integer r, one has

$$\sum_{k=1}^{r} \left\| x_k^{(m)} - x_k^{(n)} \right\|^2 \leq \left\| x^{(m)} - x^{(n)} \right\| \leq \varepsilon^2,$$

provided $m, n \geq p$. Letting $m \to \infty$,

$$\sum_{k=1}^{r} \left\| x_k - x_k^{(n)} \right\|^2 \leq \varepsilon^2$$

provided $n \geq p$. Since r is arbitrary,

$$\sum_{k=1}^{\infty} \left\| x_k - x_k^{(n)} \right\|^2 \leq \varepsilon^2 \tag{2.41}$$

provided $n \geq p$. In particular,

$$\sum_{k=1}^{\infty} \left\| x_k - x_k^{(p)} \right\|^2 \leq \varepsilon^2;$$

hence, the sequence $\{x_k - x_k^{(p)}\}_{k \geq 1}$ belongs to H. Consequently, the sequence

$$\{x_k\}_{k \geq 1} = \left\{ x_k - x_k^{(p)} + x_k^{(p)} \right\}_{k \geq 1}$$

belongs to H. It follows from (2.41) that $\|x - x^{(n)}\| \leq \varepsilon$ whenever $n \geq p$. Thus $x^{(n)} \to x$. □

Definition 2.7.4 If $H_1, H_2, \ldots$ are Hilbert spaces, the space H in Proposition 2.7.3 is called the **direct sum** of $H_1, H_2, \ldots$

For our next definition, a summation over an arbitrary (possibly uncountable) indexing set is to be understood in the following sense:

Suppose $S = \{x_\alpha \colon \alpha \in \Lambda\}$, where Λ is an indexing set, is a collection of elements from a normed linear space X. $\{x_\alpha \colon \alpha \in \Lambda\}$ is said to be **summable to $\boldsymbol{x} \in \boldsymbol{X}$**, written

$$\sum_{\alpha \in \Lambda} x_\alpha = x \quad \text{or} \quad \sum_{\alpha} x_\alpha = x,$$

if for all $\varepsilon > 0$, there exists some finite set of indices $J_0 \subseteq \Lambda$, such that for any finite set of indices $J \supseteq J_0$,

$$\left\| \sum_{\alpha \in J} x_\alpha - x \right\| < \varepsilon.$$

Definition 2.7.5 For each α in the index set Λ, let H_α be a Hilbert space. The **direct sum** $\oplus_\alpha \boldsymbol{H_\alpha}$ of Hilbert spaces H_α is defined to be the family of all functions $\{x_\alpha\}$ on Λ such that for each α, $x_\alpha \in H_\alpha$ and $\sum_{\alpha\in\Lambda} \|x_\alpha\|^2 < \infty$.

If $x, y \in H$, $(x, y) = \sum_\alpha (x_\alpha, y_\alpha)$ is an inner product on H, then H is a Hilbert space with respect to the norm $\|x_\alpha\| = \left(\sum_{\alpha\in\Lambda} \|x_\alpha\|^2\right)^{\frac{1}{2}}$. The proof is not included.

Permuting the index set Λ results in an isomorphic Hilbert space.

Let $X = C^n[a, b]$, the linear space of all scalar valued n times continuously differentiable functions on $[a, b]$. For x, y in X, define

$$(x, y) = \sum_{j=0}^{n} \int_a^b x^{(j)}(t)\overline{y^{(j)}(t)}\mathrm{d}t.$$

Clearly, $(\cdot,\cdot)$ is an inner product on X and

$$\|x\|^2 = (x, x) = \sum_{j=0}^{n} \int_a^b \left|x^{(j)}(t)\right|^2 \mathrm{d}t, \quad x \in X.$$

Let
$H = \{x \in C[a, b]$: $x^{(n-1)}$ is defined and absolutely continuous, $x^{(n)} \in L^2[a, b]\}$.
For x, $y \in H$, let

$$(x, y) = \sum_{j=0}^{n} \int_a^b x^{(j)}(t)\,\overline{y^{(j)}(t)}\,\mathrm{d}t.$$

It can be seen that $(\cdot,\cdot)$ is an inner product on H and

$$\|x\|^2 = (x, x) = \sum_{j=0}^{n} \int_a^b \left|x^{(j)}(t)\right|^2 \mathrm{d}t, \quad x \in H.$$

Theorem 2.7.6 *With notations as in the paragraph above*, *H is a Hilbert space and X is dense in H.*

Proof Consider the direct sum of $(n + 1)$ copies of $L^2[a, b]$, i.e.

$$\oplus_{(n+1)\text{copies}} L^2[a, b].$$

Then $\oplus_{(n+1)\text{copies}} L^2[a, b]$ is seen to be a Hilbert space by Proposition 2.7.2. Let $T{:}H \to \oplus_{n+1} L^2[a, b]$ be defined by

$$Tx = \left(x, x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right).$$

Observe that T is both linear and injective; moreover, it preserves inner products. We shall next show that $T(H)$ is a closed subspace of $\oplus_{n+1}L^2[a, b]$ and is consequently a Hilbert space. This will imply that H is a Hilbert space.

Let $\{x_k\}_{k\geq 1}$ be a Cauchy sequence in H and let

$$y = (y_0, y_1, \ldots, y_n) = \lim_{k\to\infty} Tx_k = \lim_{k\to\infty}\left(x_k, x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)}\right),$$

that is, $\{x_k^{(j)}\}_{k\geq 1}$ converges in $L^2[a, b]$ to y_j, $j = 0, 1, 2, \ldots, n$, where $x_k^{(0)}$ means x_k. Now, for $j = 1, 2, \ldots, n$ and each $t \in [a, b]$,

$$x_k^{(j-1)}(t) = x_k^{(j-1)}(a) + \int_a^t x_k^{(j)}(\tau)\mathrm{d}\tau. \tag{2.42}$$

Observe that

$$\left|\int_a^t y_j(\tau)\mathrm{d}\tau - \int_a^t x_k^{(j)}(\tau)\mathrm{d}\tau\right| \leq (b-a)^{\frac{1}{2}}\left\|y_j - x_k^{(j)}\right\| \tag{2.43}$$

for $j = 1, 2, \ldots, n$ and all $t \in [a, b]$. Therefore, the sequence of continuous functions $\left\{\int_a^t x_k^{(j)}(\tau)\mathrm{d}\tau\right\}_{k\geq 1}$ is uniformly convergent to the continuous function $\int_a^t y_j(\tau)\mathrm{d}\tau$. It is also convergent as a sequence in $L^2[a, b]$. But $\{x_k^{(j-1)}\}_{k\geq 1}$ is convergent in $L^2[a, b]$ to y_{j-1} and so by (2.42), the sequence $\{x_k^{(j-1)}(a)\}_{k\geq 1}$ of constant functions is convergent in $L^2[a, b]$. Therefore, the sequence $\{x_k^{(j-1)}(a)\}_{k\geq 1}$ is convergent in $\mathbb{C}$, and the function on the right of (2.42) is uniformly convergent to a continuous function. Thus, it follows that

$$y_{j-1}(t) = y_{j-1}(a) + \int_a^t y_j(\tau)\mathrm{d}\tau.$$

and hence, y_{j-1} is an absolutely continuous function. We have shown that $y_0 \in H$.

Consequently, $y = Tx$, where $x = y_0$.

We next show that X is dense in H.

Let $x \in H$. Then $x^{(n)} \in L^2[a, b]$ so that we can find a sequence $\{z_m\}_{m\geq 1}$ in $C[a, b]$ such that $\|z_m - x^{(n)}\|_2 \to 0$ as $m \to \infty$. Define recursively $u_m^{(1)}, u_m^{(2)}, \ldots, u_m^{(n)}$ by the formula

$$u_m^{(j)}(t) = \int_a^t u_m^{(j-1)}(\tau)\mathrm{d}\tau + x^{(n-j)}(a), \quad j = 1, 2, \ldots, n,$$

where $u_m^{(0)} = z_m$. Observe that $u_m^{(n)} \in X$. We claim that

$$\left\| u_m^{(j)} - x^{(n-j)} \right\|_2 \to 0 \quad \text{as} \quad m \to \infty.$$

The result is true for $j = 0$. Assume that it is true for $j \geq 0$. Then

$\left| u_m^{(j+1)}(t) - x^{n-(j+1)}(t) \right| \leq \left\| u_m^{(j)} - x^{(n-j)} \right\|_2 (b-a)^{\frac{1}{2}}$, using (2.42) and (2.43) above.

Hence, $\{u_m^{(j+1)}(t)\}_{m \geq 1}$ converges uniformly to $x^{n-(j+1)}$ for $t \in [a, b]$, so that $\|u_m^{(j+1)} - x^{n-(j+1)}\|_2 \to 0$ as $m \to \infty$. This completes the argument for $j = 1, 2, \ldots, n$. Consequently, $u_m^{(n)} \to x$ in H. The proof is now complete. □

2.8 Orthogonal Complements

In the familiar Euclidean space, we assign a length to each vector and to each pair of vectors an angle between them. The first notion has been made abstract in the definition of a norm. An appropriate notion of angle and the associated notion of orthogonality are introduced below. The introduction of the concept of orthogonality depends on the definition of inner product in a pre-Hilbert space.

Recall from Definition 2.1.1 that a real vector space H equipped with an inner product is called a real pre-Hilbert space. The angle θ between two nonzero vectors in a real pre-Hilbert space may be defined in a manner consistent with the properties of an inner product by means of the relation $(x, y) = \|x\| \|y\| \cos\theta$. Observe that the Cauchy–Schwarz Inequality then says that $|\cos\theta| \leq 1$. This definition is not satisfactory in a complex pre-Hilbert space, for (x, y) is in general a complex number. Nevertheless, if the condition $(x, y) = 0$ is taken as the definition of orthogonality (perpendicularity), then the concept is just as useful here as in the real case.

Definition 2.8.1 Let H be a pre-Hilbert space. Two vectors x and y in H are said to be **orthogonal** if $(x, y) = 0$; we write $x \perp y$.

Since $(x, y) = 0$ implies $(y, x) = 0$, we have $x \perp y$ if, and only if, $y \perp x$. It is also clear that $x \perp 0$ for every x. Also, the relation $(x, x) = \|x\|^2$ shows that 0 is the only vector orthogonal to itself.

Definition 2.8.2 A set M of nonzero vectors in a pre-Hilbert space H is said to be an **orthogonal set** if $x \perp y$ whenever x and y are distinct vectors of M. A set M of vectors in a pre-Hilbert space H is said to be **orthonormal** if

$$\text{(i)}\, M \text{ is orthogonal} \quad \text{and} \quad \text{(ii)}\, \|x\| = 1 \quad \text{for every} \quad x \in M.$$

An orthonormal set M of vectors in a pre-Hilbert space is called **complete** (or **maximal**) **orthonormal system** provided it is not a proper subset of some other orthonormal set.

Remarks 2.8.3

(i) If x is orthogonal to $y_1, y_2, \ldots, y_n$, then x is orthogonal to every linear combination of the y_k. In fact, if $x \perp y_k$ for all k and $y = \sum_{k=1}^{n} \lambda_k y_k$, then

$$(x, y) = \left(x, \sum_{k=1}^{n} \lambda_k y_k\right) = \sum_{k=1}^{n} \overline{\lambda_k}(x, y_k) = 0.$$

(ii) If $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ since $\|x + y\|^2 = (x + y, x + y) = \|x\|^2 + (x, y) + (y, x) + \|y\|^2 = \|x\|^2 + \|y\|^2$, using $(x, y) = 0 = (y, x)$.

(iii) An orthogonal subset M of H not containing the zero vector is linearly independent. Indeed, if $\sum_{k=1}^{n} \lambda_k y_k = 0$, where $y_1, y_2, \ldots, y_n$ are orthogonal, then on taking the inner product of the sum on the left-hand side with y_m, we find that $\lambda_m = 0$.

Examples 2.8.4

(i) The sequence $\{x_j\}_{j \geq 1}$, where $x_j = (0, 0, \ldots, 0, \lambda_j, 0, \ldots)$ and the scalar λ_j occurs at the jth place, in the space ℓ_0 of finitely nonzero sequences, is an orthogonal sequence (a sequence whose range is an orthogonal set).
The sequence $\{e_j\}_{j \geq 1}$, where $e_j = (0, 0, \ldots, 0, 1, 0, \ldots)$ and 1 occurs at the jth place is an orthonormal sequence, in the space ℓ_0 of finitely nonzero sequences.

(ii) Let $H = C[-\pi, \pi]$ and let $f_n(x) = \sin nx$, $n = 1, 2, \ldots$ and $g_n(x) = \cos nx$, $n = 1, 2, \ldots$. Since

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = 0 = \int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x, \quad n \neq m,$$

it follows that $\{f_n\}_{n \geq 1}$ and $\{g_n\}_{n \geq 1}$ are orthogonal sequences in $C[-\pi, \pi]$. In the space, the vectors

$$u_n(x) = \frac{1}{\sqrt{\pi}} \sin nx, \quad n = 1, 2, \ldots$$

form an orthonormal sequence and so do the vectors

$$v_0(x) = \frac{1}{\sqrt{2\pi}}, \quad v_n(x) = \frac{1}{\sqrt{\pi}}\cos nx, \quad n = 1, 2, \ldots.$$

Also, note that the vectors

$$v_0, v_1, u_1, v_2, u_2, \ldots$$

form an orthonormal sequence in $C[-\pi, \pi]$.

Recall that (f, u_k), $k = 1, 2, \ldots$ and (f, v_k), $k = 1, 2, \ldots$ are called Fourier coefficients of the function $f \in C[-\pi, \pi]$.

(iii) The sequence $\varphi_n(z) = \sqrt{\frac{n}{\pi}} z^{n-1}, n = 1, 2, \ldots$ is orthonormal in $A(D)$, where $D = \{z \in \mathbb{C}: |z| < 1\}$. In fact,

$$\begin{aligned}
(\varphi_n, \varphi_m) &= \iint_D \varphi_n \overline{\varphi_m}\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{\sqrt{nm}}{\pi} \int_0^1 \int_0^{2\pi} r^{n+m-1} \mathrm{e}^{i(n-m)\theta} \mathrm{d}r\, \mathrm{d}\theta \\
&= \frac{\sqrt{nm}}{\pi(n+m)} \int_0^{2\pi} \mathrm{e}^{i(n-m)\theta} d\theta \\
&= \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases}
\end{aligned}$$

The following definition generalises the notion of Fourier coefficients to any arbitrary infinite-dimensional pre-Hilbert space.

Definition 2.8.5 If $\{x_n\}_{n \geq 1}$ is an orthonormal sequence in a pre-Hilbert space H, then for any $x \in H$, (x, x_n) is called the **Fourier coefficient** of x with respect to $\{x_n\}_{n \geq 1}$. The **Fourier series** of x with respect to $\{x_n\}_{n \geq 1}$ is the series $\sum_{n=1}^{\infty} (x, x_n) x_n$.

In the Hilbert space ℓ^2, let

$$e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, \ldots), e_3 = (0, 0, 1, \ldots), \ldots.$$

Then $\{e_n\}_{n \geq 1}$ is an orthonormal sequence in ℓ^2. If $x = \{\lambda_j\}_{j \geq 1} \in \ell^2$, then $(x, e_j) = \lambda_j$ and $x = \sum_{j=1}^{\infty} (x, e_j) e_j$ is its Fourier series with respect to the orthonormal sequence $\{e_n\}_{n \geq 1}$. Observe that $\sum_{j=1}^{\infty} \left|(x, e_j)\right|^2 = \sum_{j=1}^{\infty} \left|\lambda_j\right|^2 < \infty$. That this result holds for any orthonormal sequence is a consequence of the following:

Theorem 2.8.6 (Bessel's Inequality) *Let x_1, x_2, …, x_n be orthonormal vectors in a pre-Hilbert space H. For every $x \in H$,*

$$\left\| x - \sum_{k=1}^{n} (x, x_k) x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x, x_k)|^2,$$

hence

$$\sum_{k=1}^{n} |(x, x_k)|^2 \leq \|x\|^2.$$

Proof For λ_1, λ_2, …, $\lambda_n \in \mathbb{C}$,

$$\left\| \sum_{k=1}^{n} \lambda_k x_k \right\|^2 = \left(\sum_{k=1}^{n} \lambda_k x_k, \sum_{k=1}^{n} \lambda_k x_k \right) = \sum_{k=1}^{n} |\lambda_k|^2.$$

So,

$$\begin{aligned} \left\| x - \sum_{k=1}^{n} \lambda_k x_k \right\|^2 &= \left(x - \sum_{k=1}^{n} \lambda_k x_k, x - \sum_{k=1}^{n} \lambda_k x_k \right) \\ &= \|x\|^2 - \sum_{k=1}^{n} \lambda_k (x_k, x) - \sum_{k=1}^{n} \overline{\lambda_k} (x, x_k) + \sum_{k=1}^{n} |\lambda_k|^2 \\ &= \|x\|^2 - \sum_{k=1}^{n} |(x_k, x)|^2 + \sum_{k=1}^{n} |(x, x_k) - \lambda_k|^2. \end{aligned}$$

In particular, if $\lambda_k = (x, x_k)$, then

$$\left\| x - \sum_{k=1}^{n} (x, x_k) x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x_k, x)|^2.$$

Since the left-hand side of the above equality is nonnegative, we get

$$\sum_{k=1}^{n} |(x, x_k)|^2 \leq \|x\|^2.$$

□

Since Bessel's Inequality holds for each n orthonormal vectors, it yields the following corollary:

Corollary 2.8.7 *If x_1, x_2, … is any orthonormal sequence of vectors, then for any x in the pre-Hilbert space H,*

$$\sum_{n=1}^{\infty} |(x, x_n)|^2 \leq \|x\|^2.$$

In particular, $(x, x_n) \to 0$ *as* $n \to \infty$.

Remarks 2.8.8

(i) As a special case of Corollary 2.8.7, we obtain the following inequality in the pre-Hilbert space $C[-\pi, \pi]$: For $f \in C[-\pi, \pi]$,

$$\sum_{n=1}^{\infty} |(f, u_n)|^2 + \sum_{n=0}^{\infty} |(f, v_n)|^2 \leq \int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x,$$

where $u_n(x) = \frac{1}{\sqrt{\pi}} \sin nx$, $v_0(x) = \frac{1}{\sqrt{2\pi}}$, $v_n(x) = \frac{1}{\sqrt{\pi}} \cos nx$, $n = 1, 2, \ldots$[see Example 2.8.4(ii)].

(ii) Let M denote the linear manifold spanned by orthonormal vectors $x_1, x_2, \ldots, x_n$. Then the proof of Theorem 2.8.6 shows that the distance $\left\|x - \sum_{k=1}^{n} \lambda_k x_k\right\|$ is minimised if we set $\lambda_k = (x, x_k)$, $k = 1, 2, \ldots, n$; i.e. $\left\|x - \sum_{k=1}^{n} (x, x_k) x_k\right\| \leq \left\|x - \sum_{k=1}^{n} \lambda_k x_k\right\|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary scalars.
Thus, $y = \sum_{k=1}^{n} (x, x_k) x_k$ is the vector in M which provides the 'best approximation' to the vector x in the pre-Hilbert space H. Also note that if $n > m$, then in the best approximation by the linear span of $x_1, x_2, \ldots, x_n$, the first m coefficients are precisely the same as required for the best approximation in the linear span of $x_1, x_2, \ldots, x_m$.

(iii) We set $z = x - y$, where $y = \sum_{k=1}^{n} (x, x_k)\, x_k$ provides the best approximation amongst the vectors in M, then $(z, x_k) = (x, x_k) - (y, x_k) = 0$ for $k = 1, 2, \ldots, n$. Hence $(z, y) = 0$. Thus, $x = y + z$, where y is a linear combination of $x_1, x_2, \ldots, x_n$ providing the best approximation to x and $z \perp x_k$, $k = 1, 2, \ldots, n$, is a decomposition of x. The decomposition is unique. Indeed, the vector in M providing the best approximation to $x \in H$ is unique. If $x = y_1 + z_1$ is the another decomposition of x, where y_1 provides the best approximation amongst the vectors in M and $z_1 \perp x_k$, $k = 1, 2, \ldots, n$, then $y + z = y_1 + z_1$ implies $y - y_1 = z_1 - z$, which in turn says $y = y_1$ and $z = z_1$, since y, y_1 are in M and z, z_1 are orthogonal to M.

It follows from Remark 2.8.3(iii) that every orthonormal sequence in H is linearly independent. Conversely, given any countable linearly independent sequence in H, we can construct an orthonormal sequence, keeping the span of the elements at each step of construction [see Theorem 2.8.9 below] in tact.

Theorem 2.8.9 (Gram–Schmidt orthonormalisation) *Let* $x_1, x_2, \ldots$ *be a linearly independent sequence in an inner product space* H. *Define* $y_1 = x_1$, $u_1 = \frac{x_1}{\|x_1\|}$ *and for* $n = 2, 3, \ldots$,

$$y_n = x_n - (x_n, u_1)u_1 - (x_n, u_2)u_2 - \cdots - (x_n, u_{n-1})u_{n-1}$$

and

$$u_n = \frac{y_n}{\|y_n\|}.$$

Then $\{u_1, u_2, \ldots\}$ *is an orthonormal sequence in H and for* $n = 1, 2, \ldots,$

$$\operatorname{span}\{u_1, u_2, \ldots, u_n\} = \operatorname{span}\{x_1, x_2, \ldots, x_n\}.$$

Proof As $\{x_1\}$ is a linearly independent set, $y_1 = x_1 \neq 0$ and $u_1 = \frac{x_1}{\|x_1\|}$ is such that $\|u_1\| = 1$ and $\operatorname{span}\{u_1\} = \operatorname{span}\{x_1\}$.

For $n \geq 1$, assume that we have defined $y_1, y_2, \ldots, y_n$ and $u_1, u_2, \ldots, u_n$ as stated above and proved that $\{u_1, u_2, \ldots, u_n\}$ is an orthonormal sequence satisfying $\operatorname{span}\{u_1, u_2, \ldots, u_n\} = \operatorname{span}\{x_1, x_2, \ldots, x_n\}$. Define

$$y_{n+1} = x_{n+1} - (x_{n+1}, u_1)u_1 - (x_{n+1}, u_2)u_2 - \cdots - (x_{n+1}, u_n)u_n.$$

Since the set $\{x_1, x_2, \ldots, x_{n+1}\}$ is linearly independent, x_{n+1} does not belong to $\operatorname{span}\{x_1, x_2, \ldots, x_n\} = \operatorname{span}\{u_1, u_2, \ldots, u_n\}$. Hence, $y_{n+1} \neq 0$ and let $u_{n+1} = \frac{y_{n+1}}{\|y_{n+1}\|}$. Then $\|u_{n+1}\| = 1$ and for $j \leq n$,

$$\begin{aligned}
\left(y_{n+1}, u_j\right) &= (x_{n+1}, u_j) - \sum_{k=1}^{n} (x_{n+1}, u_k)\left(u_k, u_j\right) \\
&= (x_{n+1}, u_j) - (x_{n+1}, u_j) \\
&= 0,
\end{aligned}$$

since $(u_k, u_j) = 0$ for all $k \neq j$, $k = 1, 2, \ldots, n$. Thus

$$\left(u_{n+1}, u_j\right) = \frac{(y_{n+1}, u_j)}{\|y_{n+1}\|} = 0 \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Hence, $\{u_1, u_2, \ldots, u_{n+1}\}$ is an orthonormal sequence. Moreover,

$$\begin{aligned}
\operatorname{span}\{u_1, u_2, \ldots, u_{n+1}\} &= \operatorname{span}\{x_1, x_2, \ldots, x_n, u_{n+1}\} \\
&= \operatorname{span}\{x_1, x_2, \ldots, x_{n+1}\}.
\end{aligned}$$

The argument is now complete in view of mathematical induction. □

Remarks 2.8.10

(i) The Gram–Schmidt orthonormalisation process as described above yields an orthonormal sequence which is unique.

Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be n-term ($n > 1$) orthogonal sequences of nonzero vectors in an inner product space. Suppose they have the same linear span and so do $e_1, \ldots, e_{n-1}$ and $f_1, \ldots, f_{n-1}$. Then the vectors e_n and f_n are scalar multiples of each other, as the following argument shows.

It is sufficient to argue that f_n is a scalar multiple of e_n.

Since f_n lies in the linear span of $e_1, \ldots, e_n$, there exist scalars $\lambda_1, \ldots, \lambda_n$ such that $f_n = \sum_{1 \le k \le n} \lambda_k e_k$. However, the vectors $e_1, \ldots, e_{n-1}$ lie in the linear span of $f_1, \ldots, f_{n-1}$. Therefore, the sum of the first $n - 1$ terms in the preceding sum can be written as $\sum_{1 \le k \le n-1} \lambda_k e_k = \sum_{1 \le k \le n-1} c_k f_k$ for some scalars $c_1, \ldots, c_{n-1}$. Thus,

$$f_n = \sum_{1 \le k \le n-1} c_k f_k + \lambda_n e_n. \tag{2.44}$$

By the orthogonality of $f_1, \ldots, f_n$, for $1 \le j \le n - 1$, we have

$$0 = (f_n, f_j) = \left(\sum_{1 \le k \le n-1} c_k f_k + \lambda_n e_n, f_j \right) = c_j (f_j, f_j) + \lambda_n \big(e_n, f_j\big).$$

But f_j lies in the linear span of $e_1, \ldots, e_{n-1}$ and e_n is orthogonal to this linear span. Therefore, $(e_n, f_j) = 0$ and the above equality becomes $c_j(f_j, f_j) = 0$. As each f_j is nonzero, it now follows that each $c_j = 0$ ($1 \le j \le n - 1$). Using this in (2.44), we get $f_n = \lambda_n e_n$.

If the vectors e_n and f_n have the same norm, then it further follows that the scalar λ_n has absolute value 1.

(ii) If $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ are the orthogonal sequences of nonzero vectors in an inner product space and

$$\mathrm{span}\{e_1, \ldots, e_n\} = \mathrm{span}\{f_1, \ldots, f_n\} \quad \text{for} \quad n = 1, 2, \ldots,$$

then e_n and f_n are scalar multiples of each other. If the vectors e_n and f_n have the same norm, then it further follows that the scalar factor has absolute value 1.

(iii) Let $Q_0, Q_1, \ldots$ be the sequence of polynomials obtained from the sequence of polynomials $1, t, t^2, \ldots$ (on the domain $[-1, 1]$) by orthonormalisation, and let $P_0, P_1, \ldots$ be the sequence of Legendre polynomials defined in 2.8.13 below. The first k functions in either sequence span the space of polynomials of degree at most $k - 1$. It follows from what has been proved above that each Q_n is a scalar multiple of P_n and vice versa. The value of the scalar can be obtained by comparing (a) the leading coefficients or (b) the constant terms or (c) the integrals over $[-1, 1]$.

(iv) The Gram–Schmidt procedure when applied to a finite sequence $\{x_1, x_2, \ldots, x_n\}$ of independent vectors leads to orthonormal vectors $\{u_1, u_2, \ldots, u_n\}$ such that

$$\text{span}\{u_1, u_2, \ldots, u_k\} = \text{span}\{x_1, x_2, \ldots, x_k\} \quad \text{for} \quad k = 1, 2, \ldots, n.$$

As an immediate consequence, we record the following:

Corollary 2.8.11 *If H is a pre-Hilbert space of dimension n, then it has a basis of orthonormal vectors.*

Theorem 2.8.12 *Every finite-dimensional pre-Hilbert space is complete and is, therefore, a Hilbert space.*

Proof By Corollary 2.8.11, there is a basis u_1, u_2, …, u_n of orthonormal vectors. If $x = \sum_{k=1}^{n} \lambda_k u_k$, then $\|x\|^2 = \sum_{k=1}^{n} |\lambda_k|^2$, using Remark 2.8.3(ii). The completeness follows as in Example 2.3.4(i). □

The following examples illustrate the orthogonalisation procedure.

Examples 2.8.13

(i) Let $H = \ell^2$. For $n = 1, 2, \ldots$, let $x_n = (1, 1, \ldots, 1, 0, 0, \ldots)$, where 1 occurs only in the first n places. The Gram–Schmidt orthonormalisation process yields

$$y_n = (0, 0, \ldots, 0, 1, 0, \ldots), \quad n = 1, 2, \ldots,$$

where 1 occurs only in the nth place.

The vector $y_1 = x_1 = (1, 0, 0, \ldots)$. The vector $y_2 = \frac{x_2 - (x_2, x_1)x_1}{\|x_2 - (x_2, x_1)x_1\|} = (0, 1, 0, \ldots)$. By induction, it can be shown that

$$y_n = (0, 0, \ldots, 0, 1, 0, \ldots), \quad n = 1, 2, \ldots,$$

where 1 occurs only in the nth place. The sequence of vectors $\{y_n\}_{n \geq 1}$ is an orthonormal sequence in ℓ^2.

The set of finite linear combinations of the sequence $\{y_n\}_{n \geq 1}$ is dense in ℓ^2. Let $x = \{\lambda_i\}_{i \geq 1} \in \ell^2$. Given $\varepsilon > 0$, there exists n_0 such that $n > n_0$ implies $\sum_{n_0 + 1 \leq i < \infty} |\lambda_i|^2 < \varepsilon$. Then the vector

$$y = \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_{n_0} y_{n_0}$$

is such that

$$\|x - y\|_2^2 = \sum_{n_0 + 1 \leq i < \infty} |\lambda_i|^2 < \varepsilon.$$

(ii) **Legendre polynomials**. Consider the sequence $\{1, t, t^2, \ldots\}$ of vectors in $L^2[-1, 1]$. Since any nontrivial finite linear combination $\sum_{i=1}^{n} a_{k_i} t^{k_i}$ is a polynomial of degree $m = \max_i k_i$, it has at most m zeros. This shows that the vectors $\{t^k\}_{k\geq 0}$ are linearly independent. We next calculate the first three orthonormal vectors by the Gram–Schmidt procedure.

Let $y_0(t) = x_0(t) = 1$, so that $\|y_0\|^2 = \int_{-1}^{1} \mathrm{d}s = 2$ and $u_0 = \frac{y_0(t)}{\|y_0\|} = \frac{1}{\sqrt{2}}$. Next,

$$y_1(t) = x_1(t) - (x_1, u_0)u_0(t) = t - \left(\int_{-1}^{1} \frac{s}{\sqrt{2}}\mathrm{d}s\right)\frac{1}{\sqrt{2}} = t,$$

so that $\|y_1\|^2 = \int_{-1}^{1} s^2\mathrm{d}s = \frac{2}{3}$ and $u_1(t) = \frac{y_1(t)}{\|y_1\|} = \sqrt{\frac{3}{2}}t$.

Further,

$$\begin{aligned} y_2(t) &= x_2(t) - (x_2, u_0)u_0(t) - (x_2, u_1)u_1(t) \\ &= t^2 - \left(\int_{-1}^{1} \frac{s^2}{\sqrt{2}}\mathrm{d}s\right)\frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} \sqrt{\frac{3}{2}}s^3\mathrm{d}s\right)\left(\sqrt{\frac{3}{2}}t\right) = t^2 - \frac{1}{3}, \end{aligned}$$

so that $\|y_2\|^2 = \int_{-1}^{1} \left(s^2 - \frac{1}{3}\right)^2\mathrm{d}s = \frac{8}{45}$ and

$$u_2(t) = \frac{y_2(t)}{\|y_2\|} = \sqrt{10}\,(3t^2 - 1)/4.$$

We shall next prove that the general form of these orthonormal polynomials is $\sqrt{n + \frac{1}{2}}P_n(t)$, where

$$P_n(t) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left((t^2 - 1)^n\right), \quad n = 0, 1, 2, \ldots. \tag{2.45}$$

The reader can check by using (2.45) that $P_0(t) = 1$, $P_1(t) = t$ and $P_2(t) = P_2(t) = \frac{1}{2}(3t^2 - 1)$ and consequently the first three normalised polynomials are $\frac{1}{\sqrt{2}}$, $\sqrt{\frac{3}{2}}t$ and $\frac{\sqrt{10}}{4}(3t^2 - 1)$. That the general form of these polynomials is $\left\{\sqrt{n + \frac{1}{2}}P_n(t)\right\}$ will be verified below. We begin by showing that

$$\int_{-1}^{1} P_n(t)P_m(t)\mathrm{d}t = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m \end{cases}.$$

For $m \neq n$,

$$
\begin{aligned}
2^{n+m}n!m!\int_{-1}^{1} P_n(t)P_m(t)\mathrm{d}t &= \int_{-1}^{1} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left((t^2-1)^n\right)\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left((t^2-1)^m\right)\mathrm{d}t \\
&= \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left((t^2-1)^n\right)\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left((t^2-1)^m\right)\Big|_{-1}^{1} \\
&\quad - \int_{-1}^{1} \frac{\mathrm{d}^{m+1}}{\mathrm{d}t^{m+1}}\left((t^2-1)^m\right)\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left((t^2-1)^n\right)\mathrm{d}t \\
&= -\int_{-1}^{1} \frac{\mathrm{d}^{m+1}}{\mathrm{d}t^{m+1}}\left((t^2-1)^m\right)\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left((t^2-1)^n\right)\mathrm{d}t
\end{aligned}
$$

since $\frac{\mathrm{d}^{n-k}}{\mathrm{d}t^{n-k}}((t^2-1)^n)$, $k = 1, 2, \ldots, n$, is zero at $t = \pm 1$. Hence, if $n > m$ and we continue the process of integration by parts, we obtain

$$
\pm \int_{-1}^{1} \frac{d^{n-m-1}}{\mathrm{d}t^{n-m-1}}((t^2-1)^n)\frac{d^{2m+1}}{\mathrm{d}t^{2m+1}}((t^2-1)^m)\mathrm{d}t,
$$

which is equal to zero (the second factor in the integrand is identically zero).

For $m = n$, we have

$$
\begin{aligned}
\int_{-1}^{1} P_n(t)^2\mathrm{d}t &= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1} (t^2-1)^n\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}((t^2-1)^n)\mathrm{d}t \\
&= \frac{(-1)^n}{2^{2n}(n!)^2}(2n)!\int_{-1}^{1} (t^2-1)^n\mathrm{d}t
\end{aligned}
\tag{2.46}
$$

since

$$
\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}((t^2-1)^n) = (2n)!.
$$

Setting $t = \cos\theta$ in (2.46) and using Wallis' formula,

$$
\int_{0}^{\pi/2} \sin^{2n+1}\theta\,\mathrm{d}\theta = \frac{2^n n!}{1\cdot 3\cdot\cdots\cdot(2n+1)},
$$

[which can be derived by integrating by parts repeatedly], it follows that

$$\int_{-1}^{1} P_n(t)^2 \mathrm{d}t = \frac{2}{2n+1}.$$

Thus $\left\{\sqrt{\frac{2n+1}{2}}P_n(t)\right\}_{n\geq 0}$ is an orthonormal sequence in $L^2[-1, 1]$.

It now follows readily that the functions $\left\{\sqrt{\frac{2n+1}{2}}P_n(t)\right\}_{n\geq 0}$ are obtained from $\{1, t, t^2, \ldots\}$ by the Gram–Schmidt orthonormalisation procedure since $P_n(t)$ is a polynomial of degree n. The essential uniqueness pointed out in Remark 2.8.10(ii), and the fact that the leading coefficients in $P_n(t)$ and in the nth polynomial obtained via orthonormalisation are both positive lead to the result.

(iii) **Hermite functions**. Consider the sequence of functions $\{f_n\}_{n\geq 0}$ on $\mathbb{R}$, where $f_n(t) = t^n\exp(-\frac{t^2}{2})$. Since

$$\int_0^1 t^{2n}\exp(-t^2)\mathrm{d}t < \int_0^1 t^{2n}\mathrm{d}t \quad \left(\exp\left(-t^2\right) < 1 \text{ for } 0 < t < 1\right) \tag{2.47}$$

and

$$\int_1^{\infty} t^{2n}\exp(-t^2)\,\mathrm{d}t < (n+1)!\int_1^{\infty}\frac{1}{t^2}\mathrm{d}t \quad \left(\mathrm{e}^{t^2} > \frac{t^{2n+2}}{(n+1)!}, t > 1\right), \tag{2.48}$$

it follows on using (2.47) and (2.48) that

$$\int_{-\infty}^{\infty} |t^n\exp(-\frac{t^2}{2})|^2\mathrm{d}t = \int_{-\infty}^{\infty} t^{2n}\exp(-t^2)\mathrm{d}t = 2\int_0^{\infty} t^{2n}\exp(-t^2)\mathrm{d}t$$

is finite. Thus, each $f_n \in L^2(\mathbb{R})$. Moreover, f_n's are linearly independent, because any nontrivial finite linear combination of functions f_n is a polynomial multiplied by $\exp(-\frac{t^2}{2})$, which is zero for no $t \in \mathbb{R}$ and any nonzero polynomial has at most finitely many zeros.

We next orthonormalise the functions $\{f_n\}_{n\geq 0}$, where $f_n(t) = t^n\exp(-\frac{t^2}{2})$, and obtain the first three orthonormal vectors.

To begin with, $(f_0, f_0) = \int_{-\infty}^{\infty}\exp(-t^2)\mathrm{d}t = \sqrt{\pi}$, using a well-known formula from advanced calculus. Thus

$$u_0(t) = \frac{\exp(-\frac{t^2}{2})}{\pi^{1/4}}.$$

The next orthonormal vector is

$$u_1(t) = \frac{t\,\exp(-\frac{t^2}{2}) - \left(t\,\exp(-\frac{t^2}{2}), \frac{\exp(-\frac{t^2}{2})}{\pi^{1/4}}\right)\frac{\exp(-\frac{t^2}{2})}{\pi^{1/4}}}{\left\|t\,\exp(-\frac{t^2}{2}) - \left(t\,\exp(-\frac{t^2}{2}), \frac{\exp(-\frac{t^2}{2})}{\pi^{1/4}}\right)\frac{\exp(-\frac{t^2}{2})}{\pi^{1/4}}\right\|}$$
$$= \frac{\sqrt{2}t\,\exp(-\frac{t^2}{2})}{\pi^{1/4}} = \frac{2t\,\exp(-\frac{t^2}{2})}{(2\pi^{1/2})^{1/2}}.$$

$$u_2(t) = \frac{t^2\exp(-\frac{t^2}{2}) - \left(t^2\exp(-\frac{t^2}{2}), u_0\right)u_0 - \left(t^2\exp(-\frac{t^2}{2}), u_1\right)u_1}{\left\|t^2\exp(-\frac{t^2}{2}) - \left(t^2\exp(-\frac{t^2}{2}), u_0\right)u_0 - \left(t^2\exp(-\frac{t^2}{2}), u_1\right)u_1\right\|}$$
$$= \frac{(4t^2-2)\exp(-\frac{t^2}{2})}{(\pi^{1/2}2^2 2!)^{1/2}}.$$

We shall next prove that the general form of these orthonormal functions is

$$v_n(t) = \frac{H_n(t)\exp(-\frac{t^2}{2})}{(2^n\cdot n!\pi^{1/2})^{1/2}},$$

where

$$H_n(t) = (-1)^n\exp\left(t^2\right)\exp^{(n)}\left(-t^2\right), \tag{2.49}$$

and the superscript '(n)' indicates the nth derivative of the function $t \to \exp(-t^2)$. The functions $\{H_n\}_{n\geq 0}$ are easily seen to be polynomials and are called Hermite polynomials. The degree of H_n is n, as shown in (2.50) below. The functions v_n are called Hermite functions.

For $n = 0$, 1 and 2, it can be verified that

$$H_0(t) = 1, \quad H_1(t) = 2t \quad \text{and} \quad H_2(t) = 4t^2 - 2.$$

We shall establish below that

$$H_n'(t) = 2nH_{n-1}(t), \quad n = 1, 2, \ldots. \tag{2.50}$$

In order to do so, we first prove by induction that

$$\exp^{(n+1)}\left(-t^2\right) = -2t\,\exp^{(n)}(-t^2) - 2n\,\exp^{(n-1)}\left(-t^2\right). \tag{2.51}$$

This is true for $n = 0$. Assume that it is true for $n = k - 1$. Then

$$\begin{aligned}
\exp^{(k+1)}\left(-t^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\exp^{(k)}\left(-t^2\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}(-2t\,\exp^{(k-1)}(-t^2) - 2(k-1)\exp^{(k-2)}\left(-t^2\right))\\
&= -2t\,\exp^{(k)}(-t^2) - 2\exp^{(k-1)}(-t^2) - 2(k-1)\exp^{(k-1)}\left(-t^2\right)\\
&= -2t\exp^{(k)}(-t^2) - 2k\exp^{(k-1)}\left(-t^2\right).
\end{aligned}$$

This proves (2.51) for all $n = 1, 2, \ldots$. Now, differentiating (2.49) and using (2.51), we obtain

$$\begin{aligned}
H_n'(t) &= (-1)^n\left\{2t\,\exp\left(t^2\right)\exp^{(n)}(-t^2) + \exp\left(t^2\right)\left(-2t\,\exp^{(n)}(-t^2) - 2n\,\exp^{(n-1)}\left(-t^2\right)\right)\right\}\\
&= (-1)^{n-1}2n\,\exp\left(t^2\right)\exp^{(n-1)}\left(-t^2\right)\\
&= 2n\,H_{n-1}(t).
\end{aligned}$$

This establishes (2.50).

The orthogonality of the Hermite functions may be obtained from

$$\int_{-\infty}^{\infty} H_m(t)H_n(t)\mathrm{e}^{-t^2}\mathrm{d}t = (-1)^n \int_{-\infty}^{\infty} H_m(t)\exp^{(n)}(-t^2)\,\mathrm{d}t.$$

For $n > m$, repeated integration by parts, using (2.50) and the fact that $\exp(-t^2)$ and all its derivatives vanish for $t = \pm\infty$, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} H_m(t)H_n(t)\mathrm{e}^{-t^2}\mathrm{d}t &= (-1)^{n-1}2m \int_{-\infty}^{\infty} H_{m-1}(t)\exp^{(n-1)}(-t^2)\mathrm{d}t\\
&= (-1)^{n-m}2^m m! \int_{-\infty}^{\infty} H_0(t)\exp^{(n-m)}(-t^2)\mathrm{d}t = 0.
\end{aligned}$$

For $n = m$,

$$\int_{-\infty}^{\infty} H_n(t)^2 \exp(-t^2)\mathrm{d}t = 2^n n! \int_{-\infty}^{\infty} H_0(t)\exp(-t^2)\mathrm{d}t$$
$$= 2^n n! \sqrt{\pi}.$$

Thus, the functions

$$v_n(t) = \frac{H_n(t)\exp(-\frac{t^2}{2})}{(2^n \cdot n! \pi^{1/2})^{1/2}}, \quad n = 0, 1, 2, \ldots \tag{2.52}$$

form an orthonormal sequence.

The reader can check using (2.52) that

$$v_j(t) = u_j(t), \quad j = 0, 1, 2.$$

The vector $H_n(t)\exp(-\frac{t^2}{2})$ is a linear combination of $f_0, \ldots, f_n$. Since the sets $\{H_k(t)\exp(-\frac{t^2}{2})\colon k = 0, \ldots, n\}$ and $\{f_k\colon k = 0, \ldots, n\}$ are linearly independent, it follows by Remark 2.8.10(ii) that $v_j = \pm u_j$ for all j. The ambiguity of sign can be removed by using the following observation. The leading coefficient of each H_n is positive in view of (2.50) and so is that of $f_n \exp(\frac{t^2}{2})$.

(iv) **Laguerre functions**. Consider the sequence of functions $\{f_n\}_{n \geq 0}$ on $(0,\infty)$, where $f_n(t) = t^n \exp(-\frac{t}{2})$. Since $\int_0^\infty t^{2n}\exp(-t)\mathrm{d}t = \Gamma(2n + 1)$, where $\Gamma(\cdot)$ is the gamma function and $\int_0^\infty \exp(-t)\mathrm{d}t = 1$, it follows that each $f_n \in L^2(0,\infty)$. Moreover, f_n's are linearly independent because any nontrivial finite linear combination of functions f_n is a polynomial multiplied by $\exp(-\frac{t}{2})$, which is zero for no $t \in (0,\infty)$ and any nonzero polynomial has at most finitely many zeros.

We next orthonormalise the functions $\{f_n\}_{n \geq 0}$, where $f_n(t) = t^n\exp(-\frac{t}{2})$, and obtain the first three orthonormal vectors.

$$(f_0, f_0) = \int_0^\infty \exp(-t)\mathrm{d}t = 1.$$

Thus, $u_0(t) = \exp\left(-\frac{t}{2}\right)$.

The next two orthonormal vectors are given as

$$u_1(t) = \frac{t\,\exp\left(-\frac{t}{2}\right) - \left(t\,\exp\left(-\frac{t}{2}\right), u_0\right)u_0}{\left\|t\,\exp\left(-\frac{t}{2}\right) - \left(t\,\exp\left(-\frac{t}{2}\right), u_0\right)u_0\right\|} = (t-1)\exp\left(-\frac{t}{2}\right), \text{ since}$$

$$(t, u_0) = 1 \text{and} \left\|t\,\exp\left(-\frac{t}{2}\right) - \left(t\,\exp\left(-\frac{t}{2}\right), u_0\right)u_0\right\| = 1.$$

$$u_2(t) = \frac{t^2\exp\left(-\frac{t}{2}\right) - \left(t^2\exp\left(-\frac{t}{2}\right), u_1\right)u_1 - \left(t^2\exp\left(-\frac{t}{2}\right), u_0\right)u_0}{\left\|t^2\exp\left(-\frac{t}{2}\right) - \left(t^2\exp\left(-\frac{t}{2}\right), u_1\right)u_1 - \left(t^2\exp\left(-\frac{t}{2}\right), u_0\right)u_0\right\|}$$

$$= \left(\frac{1}{2}t^2 - 2t + 1\right)\exp\left(-\frac{t}{2}\right), \text{ since}$$

$$\left(t^2\exp\left(-\frac{t}{2}\right), u_1(t)\right) = 4,\ \left(t^2\exp\left(-\frac{t}{2}\right), \exp\left(-\frac{t}{2}\right)\right) = 1 \text{ and}$$

$$\left\|t^2\exp\left(-\frac{t}{2}\right) - \left(t^2\exp\left(-\frac{t}{2}\right), u_1\right)u_1 - \left(t^2\exp\left(-\frac{t}{2}\right), u_0\right)u_0\right\| = 2.$$

We shall next prove that the general form of these orthonormal functions is

$$v_n(t) = \frac{1}{n!}\exp\left(-\frac{t}{2}\right)L_n(t), \tag{2.53}$$

where

$$L_n(t) = (-1)^n\exp(t)\frac{d^n}{\mathrm{d}t^n}(t^n\exp(-t)), \quad n = 0, 1, 2, \ldots.$$

Using Leibniz's formula for higher derivatives of a product, we have

$$L_n(t) = (-1)^n\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}n(n-1)\cdots(n-k+1)t^{n-k}. \tag{2.54}$$

The reader can check using (2.53) that $v_0(t) = \exp(-\frac{t}{2}) = u_0(t)$, $v_1(t) = (t-1)\exp(-\frac{t}{2}) = u_1(t)$ and $v_2(t) = (\frac{1}{2}t^2 - 2t + 1)\exp(-\frac{t}{2}) = u_2(t)$, where u_0, u_1 and u_2 are the orthonormalised vectors computed using the Gram–Schmidt orthonormalisation process.

We begin by showing that

$$\int_0^\infty \exp(-t)L_n(t)L_m(t)\mathrm{d}t = 0 \quad \text{for} \quad n > m.$$

For $m < n$,

$$\int_0^\infty \exp(-t)t^m L_n(t)\mathrm{d}t = (-1)^n \int_0^\infty t^m \frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^n \exp(-t))\mathrm{d}t$$
$$= (-1)^{n+m} m! \int_0^\infty \frac{\mathrm{d}^{n-m}}{\mathrm{d}t^{n-m}}(t^n \exp(-t))\mathrm{d}t = 0,$$

by repeated integration by parts. Also,

$$\int_0^\infty \exp(-t)L_n^2(t)\mathrm{d}t = (-1)^n \int_0^\infty \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^n \exp(-t))\right)L_n(t)\mathrm{d}t$$
$$= (-1)^n \int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^n \exp(-t)) \sum_{k=0}^n (-1)^{n-k}\binom{n}{k} n(n-1)\ldots(n-k+1)t^{n-k}\mathrm{d}t$$
$$= \int_0^\infty (-1)^{2n} t^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^n \exp(-t))\mathrm{d}t$$
$$= n! \int_0^\infty t^n \exp(-t)\mathrm{d}t = (n!)^2,$$

which shows that $\{v_n\}_{n \geq 0}$ is orthonormal.

The vector $L_n(t)\exp(-\frac{t}{2})$ is a linear combination of $f_0, \ldots, f_n$. Since the sets $\{L_k(t)\exp(-\frac{t}{2})\colon k = 0, \ldots, n\}$ and $\{f_k\colon k = 0, \ldots, n\}$ are linearly independent, it follows by Remark 2.8.10(ii) that $v_j = \pm u_j$ for all j. The ambiguity of sign can be removed by using the following observation. The leading coefficient of each L_n is positive in view of (2.54) and so is that of $f_n\exp(-\frac{t}{2})$.

(v) **Rademacher functions**. Consider the sequence $\{r_n\}$ of functions defined on the interval [0, 1] by

$$r_0(t) = 1, \quad r_k(t) = \mathrm{sgn}(\sin 2^k \pi t), \quad k = 1, 2, \ldots, \quad t \in [0, 1].$$

This sequence was introduced by Rademacher and the r_k are known as Rademacher functions. If the interval [0, 1] is divided into 2^k $(k \geq 1)$ equal parts, then $r_k(t)$ assumes on the interiors of those segments the values +1 and −1 alternately while at the endpoints, $r_k(t) = 0$.

The reader will note that $\|r_n\| = (\int_0^1 |r_n(t)|^2 \mathrm{d}t)^{\frac{1}{2}} = 1$, i.e. $r_n \in L^2[0, 1]$ and $\|r_n\| = 1$, $n = 0, 1, 2, \ldots$. To prove orthogonality, let $n > m \geq 0$. Let I be the open segment which lies between some two consecutive points of subdivision of the interval [0, 1] corresponding to the function r_m. Then, r_m has constant value +1 or −1 on I. Furthermore, I is composed of an even number, precisely 2^{n-m}, of intervals of equal length. On half of these intervals, $r_n(t)$ has value +1, whereas on the other half, $r_n(t)$ has value −1. Consequently,

$$\int_I r_m(t)r_n(t)\mathrm{d}t = \pm \int_I r_n(t)\mathrm{d}t = 0.$$

Summing up over all such segments I, we have

$$\int_0^1 r_m(t)r_n(t)\mathrm{d}t = 0.$$

Since the vectors of an orthonormal system in a pre-Hilbert space cannot be linearly dependent, it follows that the Rademacher sequence $\{r_n\}_{n\geq 0}$ of orthonormal functions in $L^2[0, 1]$ is a linearly independent sequence. Moreover, the function $f(t) = \cos 2\pi t$ is such that

$$\int_0^1 r_n(t)f(t)\mathrm{d}t = \sum_{k=1}^{2^n} (-1)^{k-1} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} f(t)\mathrm{d}t = 0,$$

since the kth term and the $(2^n - (k - 1))$th term are equal in magnitude and opposite in sign because $\cos 2\pi t = \cos 2\pi(1 - t)$, and consequently add up to 0.

Remark The sequence $\{r_n\}_{n\geq 0}$ converges for $t = 0, 1, \frac{k}{2^n}, 1 \leq k < 2^n, n = 1, 2, \ldots$, the points of subdivision of the interval [0, 1], and converges for no t other than these points of subdivision, for if $t \neq 0, 1$ or any of the points of subdivision, $\{r_n(t)\}_{n\geq 0}$ assumes the values +1 and −1 infinitely often. (For any t that is not of the form $\frac{k}{2^n}$, there exists an integer j such that $\frac{j}{2^n} < t < \frac{j+1}{2^n}$, that is, $j\pi < 2^n\pi t < (j + 1)\pi$: so, $r_n(t)$ is 1 if j is even and −1 if j is odd. As n increases, the parity of j keeps changing between even and odd [see Problem 2.8.P10.]). Thus, the sequence converges only on the set $\{0, 1, \frac{k}{2^n}: 1 \leq k < 2^n, n = 1, 2, \ldots\}$ of measure zero. However, the arithmetic averages of $\{r_n\}_{n\geq 0}$ converge to the zero function almost everywhere.

Lemma 2.8.14 *For distinct nonnegative integers $k_1, k_2, \ldots, k_n$,*

$$\int_0^1 r_{k_1} r_{k_2} \ldots r_{k_n} = 0.$$

Proof This is left as Problem 2.8.P11. □

Theorem 2.8.15 *Let* $\{r_n\}_{n\geq 1}$ *be the Rademacher functions. Then the sequence* $\{(\sum_{k=1}^n r_k)/n\}_{n\geq 1}$ *of arithmetic means converges to zero almost everywhere with respect to the Lebesgue measure on* $[0, 1]$.

Proof Set $f_n = [(\sum_{k=1}^n r_k)/n]^4$, $n = 1, 2, \ldots$. Observe that each f_n belongs to $L^1[0, 1]$. Indeed, $|f_n| \leq [(\sum_{k=1}^n |r_k|)/n]^4 = 1$ and $\int_0^1 f_n(t)\mathrm{d}t \leq 1$. Next, on using $r_k^2 = 1$, $k = 1, 2, \ldots$ (except at the finitely many points of subdivision), we have

$$
\begin{aligned}
n^4 f_n &= \left[\left(\sum_{k=1}^n r_k\right)^2\right]^2 \\
&= \left(\sum_{k=1}^n r_k^2 + 2\sum_{\substack{k,m=1\\k<m}}^n r_k r_m\right)^2 \\
&= \left(n + 2\sum_{\substack{k,m=1\\k<m}}^n r_k r_m\right)^2 \\
&= n^2 + 4n\sum_{\substack{k,m=1\\k<m}}^n r_k r_m + 4\left(\sum_{\substack{i,j=1\\i<j}}^n r_i r_j\right)\left(\sum_{\substack{k,m=1\\k<m}}^n r_k r_m\right) \\
&= n^2 + 4n\sum_{\substack{k,m=1\\k<m}}^n r_k r_m + 4\sum_{\substack{k,m=1\\k<m}}^n r_k^2 r_m^2 + 4\left(\sum_{j=1}^n 2r_j^2 \sum_{\substack{k,m=1\\k<m\\k,m\neq j}}^n r_k r_m + 2\sum_{\substack{k,m=1\\i,j=1\\k<m\\i<j\\i,j,k,m\,\mathrm{dist}}}^n r_i r_j r_k r_m\right) \\
&= n^2 + 4n\sum_{\substack{k,m=1\\k<m}}^n r_k r_m + 4((n-1) + (n-2) + \cdots + 2 + 1) + 8\sum_{j=1}^n \sum_{\substack{k,m=1\\k<m\\k,m\neq j}}^n r_k r_m \\
&\quad + 8\sum_{\substack{k,m=1\\i,j=1\\k<m\\i<j\\i,j,k,m\,\mathrm{dist}}}^n r_i r_j r_k r_m \\
&= n^2 + 2n(n-1) + 4n\sum_{\substack{k,m=1\\k<m}}^n r_k r_m + 8\sum_{j=1}^n \sum_{\substack{k,m=1\\k<m\\k,m\neq j}}^n r_k r_m + 8\sum_{\substack{k,m=1\\i,j=1\\k<m\\i<j\\i,j,k,m\,\mathrm{dist}}}^n r_i r_j r_k r_m.
\end{aligned}
\tag{2.55}
$$

Dividing both sides of (2.55) by n^4, integrating and using Lemma 2.8.14, we obtain

$$\int_0^1 f_n \mathrm{d}t = \frac{1}{n^2} + \frac{2n(n-1)}{n^4} < \frac{3}{n^2}.$$

Consequently,

$$\sum_{n=1}^{\infty} \int_0^1 f_n \mathrm{d}t < \infty.$$

By Corollary 1.3.7 and Remark 1.3.13, it follows that the sequence $\{f_n\}_{n \geq 1}$ converges to zero almost everywhere; that is, the sequence $\{(\sum_{k=1}^n r_k)/n\}_{n \geq 1}$ of arithmetic averages converges to zero almost everywhere with respect to Lebesgue measure. This completes the proof. □

Problem Set 2.8

2.8.P1. Using Bessel's Inequality, obtain the Cauchy–Schwarz Inequality.

2.8.P2. Give an example to show that strict inequality can hold in the Corollary 2.8.7 to Bessel's Inequality.

2.8.P3. Let $\{e_k\}_{k \geq 1}$ be any orthonormal sequence in an inner product space X. Show that for any x, $y \in X$,

$$\sum_{n=1}^{\infty} |(x, e_k)(y, e_k)| \leq \|x\| \|y\|.$$

2.8.P4. Let $\{e_k\}_{k \geq 1}$ be any orthonormal sequence in a Hilbert space H and let $M = \mathrm{span}\{e_k\}$. Show that for any $x \in H$, we have $x \in \overline{M}$ if, and only if, x can be represented by

$$x = \sum_{k=1}^{\infty} (x, e_k) e_k.$$

2.8.P5. Let $f(x)$ be a differentiable 2π-periodic function on $[-\pi, \pi]$ with derivative $f'(x) \in L^2[-\pi, \pi]$. Let f_n, $n \in \mathbb{Z}$, be the Fourier coefficients of $f(x)$ in the system $\{e^{inx}/\sqrt{2\pi}\}_{n \in \mathbb{Z}}$. Prove that $\sum_{n=-\infty}^{\infty} |f_n| < \infty$.

2.8.P6. Show that the system $\{1, t^2, t^4, \ldots\}$ is complete in the space $L^2[0, 1]$. It is not complete in $L^2[-1, 1]$.

2.8.P7. Find a nonzero vector in $\mathbb{C}^3$ orthogonal to $(1, 1, 1)$ and $(1, \omega, \omega^2)$, where $\omega = \exp(2\pi i/3)$.

2.8.P8. Let $\alpha \in \mathbb{C}$ be such that $|\alpha| \neq 1$. Find the Fourier coefficients of $f \in \mathrm{RL}^2$, where $f(z) = (z - \alpha)^{-1}$, with respect to the orthonormal sequence $\{e_j\}_{j=-\infty}^{\infty}$, $e_j(z) = z^j$.

2.8.P9. If the series $\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_k|^2 + |b_k|^2)$ converges, show that there exists a function $f \in L^2[0, 2\pi]$ having the a_k, b_k as its Fourier coefficients, i.e. the equations

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\mathrm{d}t, \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,\mathrm{d}t, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,\mathrm{d}t,$$
$$k = 1, 2, \ldots$$

are valid. This function is uniquely defined up to a set of measure zero; i.e. if there are two such functions, they differ only on a set of measure zero.

2.8.P10. Show that for any $t \in [0, 1]$ that is not of the form $\frac{k}{2^n}$ (i.e. t is not a 'dyadic rational') for any integers k and n, the parity of the (obviously unique) integer j such that $\frac{j-1}{2^n} < t < \frac{j}{2^n}$ keeps changing between even and odd as n increases.

2.8.P11. Prove Lemma 2.8.14: for distinct nonnegative integers $k_1, k_2, \ldots, k_n$, $\int_0^1 r_{k_1} r_{k_2} \ldots r_{k_n} = 0$.

2.8.P12. Show that completeness of the orthonormal set of Hermite functions in $L^2(-\infty,\infty)$ is equivalent to that of the orthonormal set of Laguerre functions in $L^2(0,\infty)$.

2.8.P13. Let X be a complex inner product space of dimension n. Show that X is isometrically isomorphic to $\mathbb{C}^n$ and is hence complete.

2.9 Complete Orthonormal Sets

Recall that a set M of vectors in a pre-Hilbert space is said to be orthogonal if $x \perp y$ whenever x and y are distinct vectors of M [Definition 2.8.1]. The orthogonal set M is said to be orthonormal if, in addition, $\|x\| = 1$ for every vector x in M.

An orthonormal set is said to be complete if it is a maximal orthonormal set [Definition 2.8.2]. We shall show that there are complete orthonormal sets in any nontrivial inner product space and discuss a few of the many important examples. One also speaks of complete orthogonal sets, which are defined analogously. The classical result of Riesz–Fischer and Parseval will be proved. These will lead to the identification of all infinite-dimensional Hilbert spaces.

We begin by showing that a nontrivial inner product space H ($H \neq \{0\}$) contains a complete orthonormal set.

Theorem 2.9.1 *Let H be an inner product space over $\mathbb{F}$ and let $H \neq \{0\}$. Then H contains a complete orthonormal set.*

Proof Let S denote the collection of all orthonormal sets in X. Since, for any nonzero vector x, the set $\{\frac{x}{\|x\|}\}$ is an orthonormal set; it follows that $S \neq \emptyset$. The collection S is partially ordered by inclusion. We wish to show that every totally ordered subset of S has an upper bound in S. It will then follow by Zorn's Lemma that S has a maximal element, namely a complete orthonormal set.

Let $T = \{A_\alpha\}_{\alpha \in \Lambda}$, where Λ is an indexing set, be any totally ordered subset of S. Then the set $\bigcup_\alpha A_\alpha$ is an upper bound for T; indeed, $A_\beta \subseteq \bigcup_\alpha A_\alpha$ for each β. We next show that $\bigcup_\alpha A_\alpha$ is orthonormal. Let x and y be any two distinct elements of $\bigcup_\alpha A_\alpha$ so that $x \in A_\beta$ and $y \in A_\gamma$ for some β and γ in the indexing set Λ. Since T is totally ordered, either $A_\beta \subseteq A_\gamma$ or $A_\gamma \subseteq A_\beta$. Supposing $A_\beta \subseteq A_\gamma$, it follows that $x, y \in A_\gamma$. So $x \perp y$ and $\|x\| = \|y\| = 1$. Thus, $\bigcup_\alpha A_\alpha$ is seen to be orthonormal.

By Zorn's Lemma [Sect. 1.3], S has a maximal element. This completes the proof. □

A slight modification of the proof of Theorem 2.9.1 yields the following corollary.

Corollary 2.9.2 *Let H be an inner product space over $\mathbb{F}$. If $E \subseteq H$ is an orthonormal set, then there exists a complete orthonormal set S such that $E \subseteq S$.*

The next result contains an alternate description of complete orthonormal sets.

Theorem 2.9.3 *Let H be an inner product space over $\mathbb{F}$. Suppose that $S \subseteq H$ is an orthonormal set. Then the following are equivalent*:

(a) *S is a complete orthonormal set*;
(b) *If $x \in H$ is such that $x \perp S$, then $x = 0$.*

Proof Suppose S is a complete orthonormal set. If $x \in H$ is such that $x \perp S$, and $x \neq 0$, then $S \cup \{\frac{x}{\|x\|}\}$ is an orthonormal set that properly contains S, contradicting the fact that S is a complete orthonormal set.

On the other hand, suppose $x \perp S$ implies $x = 0$. If S were not a complete orthonormal set, there would exist some orthonormal set $T \subseteq H$ such that T properly contains S. Hence, if $x \in T \backslash S$ then $\|x\| = 1$ and $x \perp S$. This contradicts the assumption that $x \perp S$ implies $x = 0$.

Therefore, the orthonormal set S is complete. □

So far we have considered examples of countable orthonormal sets in pre-Hilbert spaces. If a Hilbert space contains a countable complete orthonormal set, then it is said to be **separable**. This definition of separability is equivalent to Definition 1.2.10 as the next theorem shows.

Let S be a countable dense set in a Hilbert space $H \neq \{0\}$. By progressively reducing S, if necessary, it can be turned into a linearly independent set. The Gram–Schmidt orthonormalisation process applied to the linearly independent set renders it into an orthonormal set. This orthonormal set is in fact complete. More precisely, we have the following theorem.

Theorem 2.9.4 *Let $H \neq \{0\}$ be a Hilbert space that contains a countable dense subset S. Then H contains a countable complete orthonormal set that is obtained from S by the Gram–Schmidt orthonormalisation process. Thus H is separable.*

Let $H \neq \{0\}$ contain a countable, complete orthonormal set T, then H contains a countable dense set, namely the finite rational linear combinations of vectors in T.

Proof We assume, as we may, that $0 \notin S$. Enumerate the vectors in S as a sequence $\{x_n\}_{n \geq 1}$ and let $y_1 = x_{n_1}$, where $n_1 = 1$. If all the x_n for $n > n_1$ are scalar multiples of x_{n_1}, then the set $\{x_{n_1}\}$ is the linearly independent set obtained from S. Otherwise, let $y_2 = x_{n_2}$ be the first x_n which is not a scalar multiple of x_{n_1}. Then for $n < n_2$, x_n is a scalar multiple of x_{n_1}. If all the x_n for $n > n_2$ are expressible as linear combinations of x_{n_1} and x_{n_2}, then the set $\{x_{n_1}, x_{n_2}\}$ is the linearly independent set obtained from S. Otherwise, let $y_3 = x_{n_3}$ be the first x_n which is independent of x_{n_1}andx_{n_2}. Then for $n < n_3$, x_n is a linear combination of x_{n_1}andx_{n_2}. The proof continues inductively, and we thus obtain a finite or countably infinite linearly independent set $\{y_1, y_2, \ldots\} \subseteq S$. Let X be the smallest linear subspace of H containing $\{y_1, y_2, \ldots\}$. It is clear that $S \subseteq X$ since if $x_j \in S$ then x_j is a linear combination of $y_1, y_2, \ldots, y_k$, where k is chosen so that $n_k \leq j < n_{k+1}$. This says that X is dense in H. Orthonormalise $\{y_1, y_2, \ldots\}$ by the Gram–Schmidt procedure to obtain the orthonormal set $\{u_1, u_2, \ldots\}$. It remains to show that the orthonormal set $\{u_1, u_2, \ldots\}$ is complete.

Let $x \in H$ be such that $(x, u_k) = 0$ for $k = 1, 2, \ldots$. Then $(x, \sum_{k=1}^{n} \alpha_k u_k) = 0$ for all finite linear combinations of the u_n and so $(x, y) = 0$ for all $y \in X$. Let $\{z_n\}_{n \geq 1}$ be a sequence in X such that $\|x - z_n\| \to 0$ as $n \to \infty$. Then $\|x\|^2 = (x, x) - (x, z_n) = (x, x - z_n) \leq \|x\|\|x - z_n\| \to 0$ as $n \to \infty$.

Clearly, the closure of the rational linear combinations of the vectors of $T = \{x_k\}$ contains all possible linear combinations of T, i.e. contains $[T]$ and is hence the same as $[\overline{T}]$. Let $x \in H$. Now, $\sum_{k=1}^{n} (x, x_k)x_k \in [T]$. Using Bessel's Inequality [Theorem 2.8.6], it follows that $\sum_{n=1}^{\infty} (x, x_k)x_k$ converges to some $y \in H$. In fact, $y \in [\overline{T}]$. Suppose $y \neq x$. Then

$$(x - y, x_k) = (x, x_k) - (y, x_k) = (x, x_k) - (x, x_k) = 0.$$

Using the completeness of T, it follows that $x - y = 0$. Thus $x \in [\overline{T}]$. This completes the proof. □

However, there are Hilbert spaces which contain non-denumerable orthonormal sets and are, therefore, **nonseparable**. We give below examples of such Hilbert spaces.

Examples 2.9.5

(i) Consider the collection X of functions on $\mathbb{R}$ representable in the form

$$x(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$$

for arbitrary n, real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and complex coefficients $a_1, a_2, \ldots, a_n$. X is a vector space, and an inner product in X is defined by

$$(x, y) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\overline{y(t)}\mathrm{d}t.$$

If $y(t) = \sum_{k=1}^{n} b_k e^{i\mu_k t}$, then

$$\begin{aligned}(x, y) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{j=1}^{n}\sum_{k=1}^{m} a_j\overline{b_k}e^{i(\lambda_j-\mu_k)t}\mathrm{d}t \\ &= \sum_{j=1}^{n}\sum_{k=1}^{m} a_j\overline{b_k}\end{aligned} \tag{2.56}$$

since

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{i\lambda t}\mathrm{d}t = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda \neq 0. \end{cases}$$

The reader will note that the summation in (2.56) is taken over all j and k for which $\lambda_j = \mu_k$. X together with the inner product defined in (2.56) is an inner product space. This is known as the space of trigonometric polynomials on $\mathbb{R}$. Its completion H is a Hilbert space. The set $\{u_r(t) = e^{irt}: r \in \mathbb{R}\}$ is an uncountable orthonormal set in the Hilbert space H, where $H = \overline{X}$, the closure of X.

(ii) Let X be a nonempty set. Consider the Hilbert space $L^2(X, \mathfrak{S}, \mu)$, where $\mathfrak{S}$ denotes the collection of all subsets of X and μ is the counting measure on X; that is, if $E \in \mathfrak{S}$, $\mu(E)$ is equal to the number of points in E when E is finite and is infinite if E is infinite. The space $L^2(X, \mathfrak{S}, \mu)$ is denoted by $\ell^2(X)$.

Consider the subset of $\ell^2(X)$ consisting of all characteristic functions of one point sets in X, i.e. $\{\chi_{\{x\}}: x \in X\}$. Observe that $(\chi_{\{x\}}, \chi_{\{y\}}) = 0$ for $x \neq y$ and $\|\chi_{\{x\}}\| = 1$. Suppose now $x \neq y$ and consider the distance between $\chi_{\{x\}}$ and $\chi_{\{y\}}$:

$$\left\|\chi_{\{x\}} - \chi_{\{y\}}\right\|_2^2 = \sum_{z\in X} \left|\chi_{\{x\}} - \chi_{\{y\}}\right|^2 = 2.$$

Thus

$$\left\|\chi_{\{x\}} - \chi_{\{y\}}\right\|_2 = \sqrt{2}.$$

The open balls $S(\chi_{\{x\}}, 1/\sqrt{2})$ with centres $\chi_{\{x\}}$ and radii $1/\sqrt{2}$ are nonoverlapping, since no ball $S(\chi_{\{x\}}, 1/\sqrt{2})$ contains a point of the set $\{\chi_{\{x\}}: x \in X\}$ other than its centre. Now suppose that X is an uncountably infinite set. We claim that the space $\ell^2(X)$ is nonseparable. Suppose not and let $\{z_k\}$ be a countable dense set in $\ell^2(X)$. Each of the balls $S(\chi_{\{x\}}, 1/\sqrt{2})$ will contain a point z_k of the countable dense set. Since the balls are nonoverlapping, the points contained in different balls cannot be identical. We thus have an injective map from X into the countable dense set, which is not possible as X is uncountable.

In view of the examples above, we consider orthonormal sets which are not necessarily countable. We begin with the following definition that formalises the remarks above Definition 2.7.5.

Definition 2.9.6 Suppose $\{x_\alpha: \alpha \in \Lambda\}$, where Λ is an indexing set, is a collection of elements from a normed linear space X. $\{x_\alpha: \alpha \in \Lambda\}$ is said to be **summable to $\boldsymbol{x \in X}$**, written

$$\sum_{\alpha \in \Lambda} x_\alpha = x \quad \text{or} \quad \sum_{\alpha} x_\alpha = x,$$

if for all $\varepsilon > 0$, there exists some finite set of indices $J_0 \subseteq \Lambda$, such that for any finite set of indices $J \supseteq J_0$,

$$\left\|\sum_{\alpha \in J} x_\alpha - x\right\| < \varepsilon.$$

This notion of summability can be easily reconciled with the usual notion of summability of a series when Λ consists of the natural numbers.

Remarks 2.9.7

(i) Suppose $S = \{x_\alpha: \alpha \in \Lambda\}$, where Λ is an indexing set, is a collection of elements from the normed linear space $\mathbb{R}$. If $0 \leq x_\alpha < \infty$ for each $\alpha \in \Lambda$, then $\sum_\alpha x_\alpha$ is the supremum of the set of all finite sums $x_{\alpha_1} + x_{\alpha_2} + \cdots + x_{\alpha_n}$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct members of Λ. In this situation, the sum can be infinity, which is outside the space $\mathbb{R}$. However, if it is within the space, then the two notions of summation are identical. If $x_\alpha = \infty$ for some $\alpha \in \Lambda$, then the sum $\sum_\alpha x_\alpha$ is equal to infinity.

(ii) It is easy to check that if $\sum_\alpha x_\alpha = x$ and $\sum_\alpha y_\alpha = y$, then

$$\sum_\alpha (x_\alpha + y_\alpha) = x + y \quad \text{and} \quad \sum_\alpha \lambda x_\alpha = \lambda x, \quad \lambda \in \mathbb{F}.$$

The following proposition says though we are summing over an arbitrary indexing set, it is the sum over only a countable set of indices that matters.

Proposition 2.9.8 *Let X be a Banach space over $\mathbb{F}$ and suppose $\{x_j : j \in \Lambda\} \subseteq X$. The family $\{x_j : j \in \Lambda\}$ is summable if, and only if, for every $\varepsilon > 0$, there exists a finite set J_0 of indices such that $\left\| \sum_{j \in J} x_j \right\| < \varepsilon$ whenever J is a finite set of indices disjoint from J_0.*

If $\{x_j\}$ is summable, then the set of those indices for which $x_j \neq 0$ is countable.

Proof If $\{x_j\}$ is a summable family with sum x, then for every $\varepsilon > 0$, there exists a finite set J_0 such that $\left\| x - \sum_{j \in J_1} x_j \right\| < \varepsilon/2$ whenever $J_1 \supseteq J_0$ and is finite. It follows that if $J \cap J_0 = \emptyset$, then

$$\left\| \sum_{j \in J} x_j \right\| = \left\| \sum_{j \in J \cup J_0} x_j - \sum_{j \in J_0} x_j \right\| \leq \left\| \sum_{j \in J \cup J_0} x_j - x \right\| + \left\| x - \sum_{j \in J_0} x_j \right\| < \varepsilon.$$

The reader will note that we have not used the completeness of X in the above argument.

If, conversely, the condition is satisfied, then for every positive integer n, there exists a finite set J_n such that $\left\| \sum_{j \in J} x_j \right\| < 1/n$ whenever J is a finite set of indices and $J \cap J_n = \emptyset$. By replacing J_n by $J_1 \cup J_2 \cup \cdots \cup J_n$, $n = 1, 2, \ldots$, we see that there is a sequence $\{J_n\}$ of finite sets of indices which is increasing. If $n < m$, then

$$\left\| \sum_{j \in J_m} x_j - \sum_{j \in J_n} x_j \right\| = \left\| \sum_{j \in J_m \setminus J_n} x_j \right\| < 1/n$$

since $(J_m \setminus J_n) \cap J_n = \emptyset$. By the completeness of X, it follows that there exists x such that $\left\| \sum_{j \in J_n} x_j - x \right\| \to 0$. For $\varepsilon > 0$, there exists $n_0 > 2/\varepsilon$ such that $\left\| \sum_{j \in J_{n_0}} x_j - x \right\| < \varepsilon/2$. If J is any finite set of indices containing J_{n_0}, then

$$\left\| \sum_{j \in J} x_j - x \right\| \leq \left\| \sum_{j \in J_{n_0}} x_j - x \right\| + \left\| \sum_{j \in J \setminus J_{n_0}} x_j \right\| < \varepsilon/2 + 1/n_0 < \varepsilon.$$

Consequently, the family $\{x_j\}$ is summable with sum x.

Finally we show that $x_j = 0$ for all but countably many j. If j is an index which does not belong to $J_1 \cup J_2 \cup \ldots$, then $\|x_j\| < 1/n$ for every n. The reader will note that we have not used the completeness of X in this argument. This completes the proof. □

If the sequence $\{x_n\}_{n\geq 1}$ in a Hilbert space is orthogonal, then $\sum_{n=1}^{\infty} x_n$ converges if, and only if, $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. More generally, the following theorem holds.

Theorem 2.9.9 *Let H be a Hilbert space and let $\{x_j\colon j \in \Lambda\}$ be an orthogonal family in H, i.e. $x_j \perp x_k$ for $j \neq k$. Then $\sum_{j\in\Lambda} x_j$ converges if, and only if, $\sum_{j\in\Lambda} \|x_j\|^2 < \infty$. Moreover, if $\sum_{j\in\Lambda} x_j = x$, then $\|x\|^2 = \sum_{j\in\Lambda} \|x_j\|^2$.*

Proof If $\{x_j\}$ is summable, then for every positive number ε there exists a finite set J_0 such that $\left\|\sum_{j\in J} x_j\right\| < \varepsilon$ whenever $J \cap J_0 = \varnothing$, and consequently

$$\sum_{j\in J} \|x_j\|^2 = \left\|\sum_{j\in J} x_j\right\|^2 < \varepsilon^2$$

whenever $J \cap J_0 = \varnothing$.

If, conversely, $\sum_{j\in\Lambda} \|x_j\|^2 < \infty$, then for every positive ε there exists a finite set J_0 such that $\sum_{j\in J} \|x_j\|^2 < \varepsilon^2$ (consequently $\|\sum_{j\in J} x_j\|^2 < \varepsilon^2$) whenever $J \cap J_0 = \varnothing$. Summability now follows from the previous Theorem 2.9.8.

Observe that

$$\|x\|^2 = (x,x) = \left(\sum_{j\in\Lambda} x_j, x\right) = \sum_{j\in\Lambda}\left(x_j, \sum_{k\in\Lambda} x_k\right) = \sum_{j\in\Lambda}\sum_{k\in\Lambda}(x_j,x_k) = \sum_{j\in\Lambda}(x_j,x_j)$$
$$= \sum_{j\in\Lambda}\|x_j\|^2.$$

□

The following general form of Bessel's Inequality holds.

Theorem 2.9.10 (Bessel's Inequality) *Let $S = \{x_\alpha\colon \alpha \in \Lambda\}$ be an orthonormal set in an inner product space H and let $x \in H$. Then we have*

$$\sum_{\alpha\in\Lambda} |(x, x_\alpha)|^2 \leq \|x\|^2.$$

Proof The inequality in Theorem 2.8.6 implies that for each finite set $J \subseteq \Lambda$ of indices, we have

$$\sum_{\alpha\in J} |(x, x_\alpha)|^2 \leq \|x\|^2.$$

It now follows using Remark 2.9.7 that

$$\sum_{\alpha\in\Lambda}|(x,x_\alpha)|^2 = \sup\left\{\sum_{\alpha\in J}|(x,x_\alpha)|^2 : J\subseteq\Lambda,\ J \text{ finite}\right\}$$
$$\leq \|x\|^2.$$

□

Remark 2.9.11 The set $A = \{\alpha \in \Lambda: (x, x_\alpha) \neq 0\}$ is countable. By Bessel's Inequality, $\sum_{\alpha\in\Lambda}|(x,x_\alpha)|^2 \leq \|x\|^2$. The countability now follows from Theorem 2.9.8.

Theorem 2.9.12 *Let* $\{x_\alpha: \alpha \in \Lambda\}$ *be an orthonormal set in a Hilbert space* H. *For every* $x \in H$, *the vector* $y = \sum_{\alpha\in\Lambda}(x,x_\alpha)x_\alpha$ *exists in* H *and* $x - y \perp x_\alpha$ *for every* $\alpha \in \Lambda$.

Proof By Bessel's Inequality 2.9.10, there is a countable set of x_α for which $(x, x_\alpha) \neq 0$. Arrange them as a sequence $x_1, x_2, \ldots$ Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}\left\|\sum_{i=n}^{n+k}(x,x_i)x_i\right\|^2 &= \left(\sum_{i=n}^{n+k}(x,x_i)x_i, \sum_{j=n}^{n+k}(x,x_j)x_j\right)\\ &= \sum_{i=n}^{n+k}\sum_{j=n}^{n+k}(x,x_i)\overline{(x,x_j)}(x_i,x_j)\\ &= \sum_{i=n}^{n+k}|(x,x_i)|^2 < \varepsilon\end{aligned}$$

for large n and any positive integer k, again using Bessel's Inequality. It follows that the sequence of partial sums $\left\{\sum_{i=1}^{n}(x,x_i)x_i\right\}_{n\geq 1}$ is Cauchy in H, and H being a Hilbert space, $y = \sum_{i=1}^{\infty}(x,x_i)x_i$ exists in H and equals $\sum_{\alpha\in\Lambda}(x,x_\alpha)x_\alpha$. Note that the foregoing argument is valid whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, as in the latter case $\overline{(x,x_j)} = (x,x_j)$.

It remains to show that $x - y \perp x_\alpha$ for every $\alpha \in \Lambda$. For each n, let $y_n = \sum_{k=1}^{n}(x,x_k)x_k$. We first prove that $(x - y_n, x_\alpha) = 0$ for those α for which $(x, x_\alpha) = 0$ and any n. Note that x_α cannot appear in the representation of y_n for any n. Therefore, $(x_k, x_\alpha) = 0$ for all k and hence

$$(x - y_n, x_\alpha) = (x, x_\alpha) - \sum_{k=1}^{n}(x,x_k)(x_k,x_\alpha) = 0.$$

Next we prove $(x - y_n, x_\alpha) = 0$ for those α for which $(x, x_\alpha) \neq 0$ and sufficiently large n. Note that x_α must appear in the representation of y_n for sufficiently large n. Therefore,

$$(x - y_n, x_\alpha) = (x, x_\alpha) - \sum_{k=1}^{n} (x, x_k)(x_k, x_\alpha) = (x, x_\alpha) - (x, x_\alpha) = 0$$

for sufficiently large n.

Now, for sufficiently large n,

$$\begin{aligned} |(x - y, x_\alpha)| &\le |(x - y_n, x_\alpha)| + |(y_n - y, x_\alpha)| \\ &\le 0 + \|y_n - y\| \|x_\alpha\| \\ &= \|y_n - y\| \text{ (using orthonormality of } \{x_\alpha : \alpha \in \Lambda\}) \end{aligned}$$

Since $\|y_n - y\| \to 0$ as $n \to \infty$, it follows that $x - y \perp x_\alpha$ for every $\alpha \in \Lambda$. □

We next investigate the problem of writing an arbitrary element x in a Hilbert space H as a limit of linear combinations of elements of an orthonormal set. We begin with a definition.

Definition 2.9.13 Let H be a Hilbert space and $S = \{x_\alpha: \alpha \in \Lambda\}$ be an orthonormal set in H. We say that S is a **basis** (**orthonormal basis**) in H if for every $x \in H$, the following holds:

$$x = \sum_{\alpha \in \Lambda} (x, x_\alpha) x_\alpha.$$

The following theorem provides a characterisation of a basis in a Hilbert space.

Theorem 2.9.14 *If H is a Hilbert space, then $S = \{x_\alpha: \alpha \in \Lambda\}$ consisting of orthonormal vectors in H is a basis if, and only if, S is a complete orthonormal system of vectors.*

Proof Suppose S is a basis in H. Then if $x \in H$ satisfies $(x, x_\alpha) = 0$, $\alpha \in \Lambda$, then the definition of the basis gives

$$x = \sum_{\alpha \in \Lambda} (x, x_\alpha) x_\alpha = 0.$$

Thus, S is complete by Theorem 2.9.3.

On the other hand, suppose that S is complete in H. Let $\beta \in \Lambda$. Then for any $x \in H$, the sum $\sum_{\alpha \in \Lambda} (x, x_\alpha) x_\alpha$ exists by Theorem 2.9.12 and

$$\left(x - \sum_{\alpha \in \Lambda} (x, x_\alpha) x_\alpha, x_\beta \right) = (x, x_\beta) - \sum_{\alpha \in \Lambda} (x, x_\alpha)(x_\alpha, x_\beta) = (x, x_\beta) - (x, x_\beta) = 0,$$

using the fact that $S = \{x_\alpha: \alpha \in \Lambda\}$ consists of orthonormal vectors. Thus, the vector $x - \sum_{\alpha \in \Lambda} (x, x_\alpha) x_\alpha$, is orthogonal to x_β for every $\beta \in \Lambda$. The hypothesis, together with Theorem 2.9.3, now implies

$$x = \sum_{\alpha \in \Lambda} (x, x_\alpha) x_\alpha,$$

i.e. S is a basis in H. □

Examples 2.9.15

(i) The set $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, … is a complete orthonormal set (basis) in ℓ^2. Indeed, if $x = (x_1, x_2, \ldots) \in \ell^2$ and $(x, e_j) = 0, j = 1, 2, \ldots$, then $x_j = 0$, $j = 1, 2, \ldots$ and so $x = 0$. Moreover, if $x = (x_1, x_2, \ldots) \in \ell^2$ then $x = \sum_{i=1}^{\infty} (x, e_i) e_i$, where the partial sums converge in the ℓ^2-norm: $\left\| \sum_{i=1}^{n} (x, e_i) e_i - x \right\|^2 = \sum_{i=n+1}^{\infty} |x_i|^2$ is small for large n.

(ii) [Cf. Examples 2.9.5(ii)] Let X be a non-denumerable set. The set $\ell^2(X) = L^2(X, \mathfrak{S}, \mu)$, where $\mathfrak{S}$ denotes the collection of all subsets of X and μ is the counting measure on X, is a nonseparable Hilbert space. The set $\{\chi_{\{x\}}$: $x \in X\}$ of characteristic functions is an uncountable orthonormal set in $\ell^2(X)$. In fact, it is a complete orthonormal set. If $f \in \ell^2(X)$ and for $x \in X$, $(f, \chi_{\{x\}}) = 0$, then $\sum_{y \in X} f(y) \chi_{\{x\}}(y) = 0$, which implies $f(x) = 0$. So f is the identically zero function [see Theorem 2.9.3].

(iii) The Rademacher system is not complete. The function $f(x) = \cos 2\pi x$ is orthogonal to all the Rademacher functions [see Example 2.8.13(v)].

The following theorem provides various characterisations of complete orthonormal sets and helps decide which orthonormal sets are complete. Some of the characterisations have already been described.

Theorem 2.9.16 *Let $S = \{x_\alpha: \alpha \in \Lambda\}$ be an orthonormal set in Hilbert space H. Each of the following conditions implies the other five*:

(a) *S is a complete orthonormal set in H*;
(b) *$x \perp S$ implies $x = 0$*;
(c) *$x \in H$ implies $x = \sum_{\alpha \in \Lambda} (x, x_\alpha) x_\alpha$; that is, S is a basis in H*;
(d) $\|x\|^2 = \sum_{\alpha \in \Lambda} |(x, x_\alpha)|^2$ *for each $x \in H$*; (**Parseval's Identity**)
(e) *for $x, y \in H$, $(x, y) = \sum_{\alpha \in \Lambda} (x, x_\alpha) \overline{(y, x_\alpha)}$*;
(f) *$\overline{[S]} = H$; that is, the smallest subspace of H containing S is dense in H.*

The equality in (c) means that the right-hand side has only a countable number of nonzero terms, and every rearrangement of this series converges to x [Definition 2.9.6]. The equations in (d) and (e) are to be interpreted analogously.

Of course, (d) is a special case of (e).

Proof The equivalence of (a) and (b) has been proved [Theorem 2.9.3]. So also the equivalence of (a) and (c) [Theorem 2.9.14]. We shall prove that (b) ⇒ (f) ⇒ (d) ⇒ (e) ⇒ (b).

(b) implies (f). Let $M = \overline{[S]}$. Since $[S]$ is a subspace, so is M. (For $x, y \in M$, there exist sequences $\{x_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ such that $x_n \to x$ and $y_n \to y$; then

$x_n + y_n \to x + y$ and $\lambda x_n \to \lambda x$, $\lambda \in \mathbb{F}$.) Suppose $[S]$ is not dense in H. Then, $M \neq H$ so that there exists a nonzero vector x in H which is not in M. The vector $y = \sum_{\alpha\in\Lambda}(x, x_\alpha)x_\alpha$ exists in H and $x - y \perp x_\alpha$ for every $\alpha \in \Lambda$ [Theorem 2.9.12]. Moreover, $x \neq y$ since $y \in M$ and $x \notin M$ and hence $x - y \neq 0$. This contradicts (b).

(f) implies (d). Suppose (f) holds. For $x \in H$ and $\varepsilon > 0$, there exists a finite set $\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\}$ such that some linear combinations of these vectors have distance less than ε from x. By Remark 2.8.8(ii), the vector $z = \sum_{i=1}^{n}(x, x_{\alpha_i})x_{\alpha_i}$ provides the best approximation to the vector x in the linear span of $\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\}$; so $\|x - z\| < \varepsilon$, and hence $\|x\| < \|z\| + \varepsilon$ which implies $(\|x\| - \varepsilon)^2 < \|z\|^2 = \sum_{i=1}^{n}|(x, x_{\alpha_i})|^2 \leq \sum_{i=1}^{\infty}|(x, x_{\alpha_i})|^2$. Since $\varepsilon > 0$ is arbitrary, we obtain $\|x\|^2 \leq \sum_{i=1}^{\infty}|(x, x_{\alpha_i})|^2$. The result now follows using Bessel's Inequality 2.9.10.

(d) implies (e). Note that (d) can be written as

$$(x, x) = \left(\sum_{\alpha\in\Lambda}(x, x_\alpha)x_\alpha, \sum_{\alpha\in\Lambda}(x, x_\alpha)x_\alpha\right).$$

Fix x, $y \in H$. If (d) holds, then

$$(x + \lambda y, x + \lambda y) = \left(\sum_{\alpha\in\Lambda}(x + \lambda y, x_\alpha)x_\alpha, \sum_{\alpha\in\Lambda}(x + \lambda y, x_\alpha)x_\alpha\right)$$

for any scalar λ. Hence,

$$\bar{\lambda}(x, y) + \lambda(y, x) = \bar{\lambda}\left(\sum_{\alpha\in\Lambda}(x, x_\alpha)x_\alpha, \sum_{\alpha\in\Lambda}(y, x_\alpha)x_\alpha\right) + \lambda\left(\sum_{\alpha\in\Lambda}(y, x_\alpha)x_\alpha, \sum_{\alpha\in\Lambda}(x, x_\alpha)x_\alpha\right) \tag{2.57}$$

Taking $\lambda = 1$ and $\lambda = i$, (2.57) shows that the real and imaginary parts of (x, y) and $\left(\sum_{\alpha\in\Lambda}(x, x_\alpha)x_\alpha, \sum_{\alpha\in\Lambda}(y, x_\alpha)x_\alpha\right)$ are equal. Hence

$$\begin{aligned}(x, y) &= \left(\sum_{\alpha\in\Lambda}(x, x_\alpha)x_\alpha, \sum_{\alpha\in\Lambda}(y, x_\alpha)x_\alpha\right)\\ &= \sum_{\alpha\in\Lambda}(x, x_\alpha)\overline{(x_\alpha, y)}.\end{aligned}$$

(e) implies (b). Finally, if (b) fails to be true, there exists a vector $z \neq 0$ so that $(z, x_\alpha) = 0$ for all $\alpha \in \Lambda$. If $x = y = z$, the $\|z\|^2 = (x, y) \neq 0$ but $(z, x_\alpha)\overline{(x_\alpha, z)} = 0$. Hence (e) fails to hold. Thus, (e) implies (b) and the proof is complete. □

To deal with completeness of orthonormal sets in the next few examples, we will use their equivalent descriptions provided in Theorem 2.9.16.

Examples 2.9.17

(i) In the completion H of the inner product space of trigonometric polynomials [see Example 2.9.5(i)], the uncountable orthonormal set $\{u_r(t) = \exp(irt)$: $r \in \mathbb{R}\}$ is complete since $\overline{[\{u_r\}]} = H$ [equivalence of (a) and (d) in Theorem 2.9.16].

(ii) Let $H = L^2[-1, 1]$ and for $n = 0, 1, 2, \ldots$, let P_n denote the Legendre polynomial of degree n. Note that P_n is obtained by applying the Gram–Schmidt orthonormalisation process to the linearly independent vectors $\{1, t, t^2, \ldots, t^n\}$. Moreover,

$$\operatorname{span}\{1, t, t^2, \ldots, t^n\} = \operatorname{span}\{P_0, P_1, \ldots, P_n\}. \tag{2.58}$$

This is true for each n. Let $x \in H$ and $\varepsilon > 0$. By Example 2.5.1, there exists $y \in C[-1, 1]$ such that $\|x - y\| < \varepsilon$. By Weierstrass' Theorem, there is a polynomial $Q(t)$ such that $|y(t) - Q(t)| < \varepsilon$ for all $t \in [-1, 1]$. Then

$$\|y - Q\|_2^2 = \int_{-1}^{1} |y(t) - Q(t)|^2 \mathrm{d}t < 2\varepsilon^2.$$

Thus

$$\|x - Q\|_2 \le \|x - y\|_2 + \|y - Q\|_2 < (1 + \sqrt{2})\varepsilon.$$

This shows that the set of all polynomials on $[-1, 1]$ is dense in H.

In view of (2.58), the set $\{P_0, P_1, \ldots\}$ constitutes a complete orthonormal basis [Theorem 2.9.16(f)].

(iii) Let $H = L^2([-\pi, \pi], \frac{\mathrm{d}t}{2\pi})$ and for $n = 0, \pm 1, \pm 2, \ldots$,

$$u_n(t) = \mathrm{e}^{int}, \quad t \in [-\pi, \pi].$$

Then $\{u_n: n = 0, \pm 1, \pm 2, \ldots\}$ is an orthonormal set in H. Indeed,

$$(u_n, u_m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{i(n-m)t} \mathrm{d}t = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \ne m \end{cases}$$

The orthonormal set $\{u_n: n = 0, \pm 1, \pm 2, \ldots\}$ is usually called *trigonometric system*. For $x \in H$ and $n = 0, \pm 1, \pm 2, \ldots,$

$$(x, u_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-int} dt = \hat{x}(n),$$

where $\hat{x}(n)$ is the nth Fourier coefficient of x.

We shall show that if $\hat{x}(n) = 0$, $n = 0, \pm 1, \pm 2, \ldots$, then $x = 0$ a.e. This will prove that the trigonometric system is complete in $L^2([-\pi, \pi], \frac{dt}{2\pi})$.

Set

$$y(t) = \frac{1}{2\pi} \int_{-\pi}^{t} x(s) ds.$$

Since $L^2([-\pi, \pi], \frac{dt}{2\pi}) \subseteq L^1([-\pi, \pi], \frac{dt}{2\pi})$, it is evident that y is a well-defined absolutely continuous function on $[-\pi, \pi]$ [see 1–5]. In particular, $y \in L^2([-\pi, \pi], \frac{dt}{2\pi})$. Moreover, $y(-\pi) = 0$ and $y(\pi) = 0$, using the fact that $\hat{x}(0) = 0$ by hypothesis. Let a be any constant. On integrating by parts, we obtain

$$\int_{-\pi}^{\pi} [y(t) - a] e^{int} dt = 0, \quad n = \pm 1, \pm 2, \ldots. \tag{2.59}$$

Choose a so that (2.59) holds for $n = 0$ as well. Since $y(t) - a$ is a continuous periodic function, for $\varepsilon > 0$, there is a trigonometric polynomial

$$T(t) = \sum_{k=-n}^{n} c_k e^{ikt}$$

such that

$$\sup\{|y(t) - a - T(t)| : t \in [-\pi, \pi]\} < \varepsilon,$$

using Weierstrass' Theorem.

Now using (2.59) and the choice of a, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} |y(t)-a|^2 \mathrm{d}t &= \int_{-\pi}^{\pi} (y(t)-a)\left(\overline{y(t)}-\bar{a}-T(t)\right)\mathrm{d}t \\ &\leq \varepsilon \int_{-\pi}^{\pi} |y(t)-a| \mathrm{d}t \\ &\leq \varepsilon \left[\int_{-\pi}^{\pi} |y(t)-a|^2 \mathrm{d}t\right]^{\frac{1}{2}} \left[\int_{-\pi}^{\pi} \mathrm{d}t\right]^{\frac{1}{2}},\end{aligned}$$

which implies

$$\int_{-\pi}^{\pi} |y(t)-a|^2 \mathrm{d}t \leq 2\pi\varepsilon^2.$$

Thus, $y(t)$ is constant and $x(t) = 0$ almost everywhere. This completes the proof.

Remarks (a) In the above proof, we have used the fact that $x \in L^1\left([-\pi,\pi],\frac{\mathrm{d}t}{2\pi}\right)$ and have proved that if $\hat{x}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\mathrm{e}^{-int}\mathrm{d}t = 0$ for $n = 0, \pm 1, \pm 2, \ldots$, then $x = 0$ a.e.

(b) We put some of the results proved for abstract Hilbert spaces in the present setting of $L^2\left([-\pi,\pi],\frac{\mathrm{d}t}{2\pi}\right)$. For $x \in L^2\left([-\pi,\pi],\frac{\mathrm{d}t}{2\pi}\right)$, associate a function $\hat{x}$ defined on $\mathbb{Z}$, the set of integers. The Fourier series of x is

$$\sum_{n=-\infty}^{\infty} \hat{x}(n)\mathrm{e}^{int} \tag{2.60}$$

and its partial sums are

$$S_N = \sum_{n=-N}^{N} \hat{x}(n)\mathrm{e}^{int}, \quad N = 0, 1, 2, \ldots,$$

The Parseval identity asserts

$$\sum_{n=-\infty}^{\infty} \hat{x}(n)\overline{\hat{y}(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\overline{y(t)}\mathrm{d}t, \quad x, y \in L^2\left([-\pi,\pi],\frac{\mathrm{d}t}{2\pi}\right). \tag{2.61}$$

The Fourier series (2.60) converges to x in the L^2-norm:

$$\lim_N \|x - S_N\|_2 = 0. \tag{2.62}$$

(c) (**The Riemann–Lebesgue Lemma**) If $x \in L^2\left([-\pi,\pi],\frac{dt}{2\pi}\right)$, then

$$\int_{-\pi}^{\pi} x(t)e^{-int}dt \to 0 \quad \text{as} \quad n \to \pm\infty.$$

Indeed, the Parseval's identity (2.61) gives $\sum_{n=-\infty}^{\infty} |\hat{x}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x(t)|^2 dt < \infty$. (d) The relation (2.62) leads to the question whether the Fourier series of $x \in L^2\left([-\pi,\pi],\frac{dt}{2\pi}\right)$ tends to x pointwise. This is not true even for a continuous function, as was demonstrated by du Bois-Reymond in 1876. However, Fejér proved in 1900 that the Fourier series of a continuous function is Cesàro summable and the sum is the function itself. For a function $x \in L^2\left([-\pi,\pi],\frac{dt}{2\pi}\right)$, Lusin's conjecture that $\{S_N\}_{N\geq 0}$, where $S_N = \sum_{n=-N}^{N} \hat{x}(n)e^{int}$ converges to x pointwise a.e. was proved by Carleson in 1966.

(iv) A complete orthonormal system for the space $H = L^2(0,\infty)$ is given by the Laguerre functions

$$v_n(t) = \frac{1}{n!}\exp\left(-\frac{t}{2}\right)L_n(t), \tag{2.63}$$

where

$$L_n(t) = (-1)^n \exp(t)\frac{d^n}{dt^n}(t^n \exp(-t)), \quad n = 0,1,2,\ldots.$$

In fact, $\{v_n(t)\}_{n\geq 0}$ constitute an orthonormal set in H [Example 2.8.13(iv)] . In order to show that the system (2.63) is complete in H, it will be enough to show that if $f \in H$ and $\int_0^\infty f(t)\exp\left(-\frac{t}{2}\right)L_n(t)dt = 0$, $n = 0, 1, 2, \ldots$, then $f = 0$ a.e. Let

$$g(t) = f(t)\exp\left(-\frac{t}{2}\right), \quad 0 < t < \infty.$$

Since $f \in L^2(0,\infty)$ and $\exp(-\frac{t}{2}) \in L^2(0,\infty)$, it follows that $g \in L^1(0,\infty)$. Indeed, by the Cauchy–Schwarz Inequality,

$$\begin{aligned}\int_0^\infty |g(t)|\mathrm{d}t &= \int_0^\infty \left|f(t)\exp\left(-\frac{t}{2}\right)\right|\mathrm{d}t \\ &\le \left[\int_0^\infty |f(t)|^2\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_0^\infty \exp(-t)\mathrm{d}t\right]^{\frac{1}{2}} \\ &\le \|f\|_2.\end{aligned}$$

Each L_n is a polynomial of degree n [see (2.64) of Examples 2.8.13(iv)]. Therefore, each t^n is a linear combination of $L_0, \ldots, L_n$. Thus, we need only to show that

$$\int_0^\infty g(t)t^n\mathrm{d}t = 0, \quad n = 0, 1, 2, \ldots \text{implies}\, g(t) = 0 \quad \text{a.e.} \tag{2.64}$$

Now consider

$$\begin{aligned}\Phi(z) &= \int_0^\infty \exp(-tz)g(t)\mathrm{d}t \\ &= \int_0^\infty \exp(-tx)\exp(-ity)g(t)\mathrm{d}t, \quad \Re z > 0.\end{aligned} \tag{2.65}$$

We shall show that $\Phi(z)$ is holomorphic in $\Re z > 0$.

Since $g \in L^1(0,\infty)$ and $|\exp(-ity)| = 1$, the integral in (2.65) exists as a Lebesgue integral. Moreover, $\Phi(z)$ is continuous in $\Re z > 0$. Indeed, if $z_n \to z$ in $\Re z > 0$, then $g(t)\exp(-tz_n) \to g(t)\exp(-tz)$. Both the sequence of functions and the limit function are integrable and are dominated by the integrable function $g(t)$, an application of the Lebesgue Dominated Convergence Theorem 1.3.9 proves the assertion. If Δ denotes the boundary of any closed triangle in $\Re z > 0$, then

$$\begin{aligned}\oint_\Delta \Phi(z)\mathrm{d}z &= \oint_\Delta \left(\int_0^\infty g(t)\exp(-tz)\mathrm{d}t\right)\mathrm{d}z \\ &= \int_0^\infty g(t)\left(\oint_\Delta \exp(-tz)\mathrm{d}z\right)\mathrm{d}t \quad [\text{Fubini's Theorem}] \\ &= \int_0^\infty g(t)\cdot 0\mathrm{d}t \quad [\text{Cauchy's Theorem}] = 0.\end{aligned}$$

It now follows by using Morera's Theorem that $\Phi(z)$ is holomorphic in $\Re z > 0$.
On using integration by parts and induction on n, we see that

$$\int_0^\infty \exp(-t)t^{2n}\mathrm{d}t = (2n)! \leq (2^n n!)^2. \tag{2.66}$$

We next consider the series

$$\sum_{n=0}^\infty s^n \frac{1}{n!}\int_0^\infty g(t)t^n\mathrm{d}t, \quad s > 0, \tag{2.67}$$

and show that the series converges for $0 \leq s < \frac{1}{2}$ to the function $\Phi(s)$.

$$\begin{aligned}\left|\sum_{n=0}^\infty s^n \frac{1}{n!}\int_0^\infty g(t)t^n\mathrm{d}t\right| &\leq \sum_{n=0}^\infty s^n \frac{1}{n!}\int_0^\infty |f(t)|\exp\left(-\frac{t}{2}\right)t^n\mathrm{d}t \\ &\leq \sum_{n=0}^\infty s^n \frac{1}{n!}\|f\|_2 \cdot [(2n)!]^{1/2},\end{aligned}$$

using the Cauchy–Schwarz Inequality. From (2.66), we have

$$\leq \|f\|_2 \cdot \sum_{n=0}^\infty (2s)^n.$$

And the series on the right converges for $0 \leq s < \frac{1}{2}$.
Note that $\exp(-st)g(t) = \sum_{n=0}^\infty (-1)^n s^n \frac{1}{n!} g(t)t^n$, and

$$\begin{aligned}\sum_{n=0}^\infty \int_0^\infty \left|(-1)^n s^n \frac{1}{n!} g(t)t^n\right|\mathrm{d}t &\leq \sum_{n=0}^\infty \int_0^\infty s^n |g(t)|t^n\mathrm{d}t \\ &= \sum_{n=0}^\infty s^n \frac{1}{n!}\int_0^\infty |f(t)|\exp\left(-\frac{t}{2}\right)t^n\mathrm{d}t \\ &\leq \sum_{n=0}^\infty s^n \frac{1}{n!}\left[\int_0^\infty |f(t)|^2\mathrm{d}t \int_0^\infty t^{2n}\exp(-t)\mathrm{d}t\right]^{\frac{1}{2}} \quad \text{(Cauchy–Schwarz)} \\ &\leq \sum_{n=0}^\infty s^n \frac{1}{n!}\|f\|_2 \cdot 2^n \cdot n! \quad \text{(using (2.66))} \\ &= \sum_{n=0}^\infty \|f\|_2 (2s^n) < \infty \quad \text{for} \quad 0 \leq s < \frac{1}{2}.\end{aligned}$$

Then, on using Corollary 1.3.10,

$$\sum_{n=0}^{\infty}(-1)^n s^n \frac{1}{n!} g(t)t^n \in L^1(0,\infty)$$

and

$$\Phi(s) = \int_0^{\infty} \exp(-st)g(t)\mathrm{d}t = \sum_{n=0}^{\infty}\int_0^{\infty} (-1)^n s^n g(t)t^n \mathrm{d}t.$$

On using the hypothesis, we obtain $\Phi(s) = 0$ for all $s \geq 0$. Now,

$$\Phi(s) = \int_0^1 t^{s-1} g(-\ln t)\mathrm{d}t, \quad s \geq 0.$$

It may be observed that $t^{-1}g(-\ln t)$ is in $L^1[0, 1]$, using the substitution $u = -\ln t$. Using Proposition 1.3.11, it now follows that $g(-\ln t) = 0$ a.e. on (0,1), which implies [see Problem 2.9.P8] that $g(t) = 0$ a.e. on $(0,\infty)$. This completes the proof.

(v) A complete orthonormal system for the space $H = L^2(-\infty,\infty)$ is given by the Hermite functions

$$v_n(t) = \frac{H_n(t)\exp\left(-\frac{t^2}{2}\right)}{\left(2^n \cdot n!\pi^{1/2}\right)^{1/2}}, \quad n = 0, 1, 2, \ldots, \tag{2.68}$$

where

$$H_n(t) = (-1)^n \exp(t^2)\exp^{(n)}(-t^2).$$

In fact, $\{v_n(t)\}_{n\geq 0}$ constitute an orthonormal set in $L^2(-\infty,\infty)$ [Example 2.8.13 (iii)]. In order to show that the system (2.68) is complete in H, it will be enough to show that if $f \in L^2(-\infty,\infty)$ and $\int_{-\infty}^{\infty} f(t)\exp\left(-\frac{t^2}{2}\right)H_n(t)\mathrm{d}t = 0$, or equivalently, $\int_{-\infty}^{\infty} f(t)\exp\left(-\frac{t^2}{2}\right)t^n\mathrm{d}t = 0$, for $n = 0, 1, 2, \ldots$ implies $f = 0$ a.e. on $(-\infty,\infty)$. The equivalence follows exactly as in (iv) above, since H_n is a polynomial of degree n.

We consider the function

$$F(x) = \int_{-\infty}^{\infty} f(t)e^{-itx} \exp\left(-\frac{t^2}{2}\right) dt, \quad -\infty < x < \infty.$$

This integral exists, since $f \in L^2(-\infty,\infty)$, $\exp(-\frac{t^2}{2}) \in L^2(-\infty,\infty)$ and $|e^{-itx}| = 1$. In fact,

$$\begin{aligned}\int_{-\infty}^{\infty} \left| f(t)e^{-itx} \exp\left(-\frac{t^2}{2}\right)\right| dt &= \int_{-\infty}^{\infty} \left| f(t) \exp\left(-\frac{t^2}{2}\right)\right| dt \\ &\le \left[\int_{-\infty}^{\infty} |f(t)|^2 dt\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} \exp(-t^2) dt\right]^{\frac{1}{2}} < \infty,\end{aligned}$$

using the Cauchy–Schwarz Inequality.

We write

$$f(t)e^{-itx} = \sum_{n=0}^{\infty} (-i)^n \frac{x^n}{n!} f(t) t^n$$

and observe that

$$\begin{aligned}&\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{|x|^n}{n!} |f(t)||t^n| \exp\left(-\frac{t^2}{2}\right) dt \\ &= \int_{-\infty}^{\infty} |f(t)| \exp(|xt|) \exp\left(-\frac{t^2}{2}\right) dt \\ &= \int_{-\infty}^{\infty} |f(t)| \exp\left(-\frac{t^2}{4}\right) \exp(|xt|) \exp\left(-\frac{t^2}{4}\right) dt \\ &\le \left[\int_{-\infty}^{\infty} |f(t)|^2 \exp\left(-\frac{t^2}{2}\right) dt\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} \exp(2|xt|) \exp\left(-\frac{t^2}{2}\right) dt\right]^{\frac{1}{2}} < \infty,\end{aligned}$$

since

$$\int_{-\infty}^{\infty} |f(t)|^2 \exp\left(-\frac{t^2}{2}\right) dt \le \int_{-\infty}^{\infty} |f(t)|^2 dt = \|f\|_2^2,$$

and

$$\int_{-\infty}^{\infty} \exp(2|xt|)\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t = 2\int_{0}^{\infty} \exp(2|xt|)\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t$$
$$= 2\int_{0}^{\infty} \exp\left(-\frac{t^2}{2} + 2|x|t - 2x^2\right)\exp(2x^2)\mathrm{d}t$$
$$= 2\exp(2x^2)\int_{0}^{\infty} \exp\left(-\frac{1}{2}(t-2|x|)^2\right)\mathrm{d}t < \infty.$$

Using Corollary 1.3.10, it follows that

$$F(x) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-itx}\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t$$
$$= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} (-i)^n \frac{x^n}{n!} f(t)\exp\left(-\frac{t^2}{2}\right)t^n\mathrm{d}t$$
$$= \sum_{n=0}^{\infty} (-i)^n \frac{x^n}{n!}\int_{-\infty}^{\infty} f(t)\exp\left(-\frac{t^2}{2}\right)t^n\mathrm{d}t = 0,$$

i.e.

$$\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-itx}\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t = 0$$

for all real x. It follows on using Proposition 1.3.12 that $f = 0$ a.e., which was to be proved.

(vi) The set $\varphi_n(z) = \sqrt{\frac{n}{\pi}}z^{n-1}$, $n = 1, 2, \ldots$ is an orthonormal set in $H = A(D)$, where $D = \{z \in \mathbb{C}: |z| < 1\}$ [see 2.8.4(iii)]. We shall show that the Parseval formula

$$\sum_{n=1}^{\infty} |(f, \varphi_n)|^2 = \iint_{|z|<1} |f(z)|^2\mathrm{d}z, \quad f \in L^2(D)$$

holds, thereby establishing the completeness of $\{\varphi_n(z)\}_{n\geq 1}$ in $A(D)$ [Theorem 2.9.16].

For $f \in A(D)$, the Fourier coefficients are given by

$$\begin{aligned}\gamma_n = (f, \varphi_n) &= \sqrt{\frac{n}{\pi}} \iint_D f(z)\bar{z}^{n-1} \mathrm{d}x\mathrm{d}y \\ &= \lim_{r\to 1} \sqrt{\frac{n}{\pi}} \iint_{|z|<r} f(z)\bar{z}^{n-1} \mathrm{d}x\mathrm{d}y.\end{aligned}$$

On applying the complex Green's formula [29, p. 124], we obtain

$$\gamma_j = \lim_{r\to 1} \frac{1}{2i} \sqrt{\frac{n}{\pi}} \int_{|z|=r} f(z) \frac{\bar{z}^n}{n} \mathrm{d}z.$$

Since $|z|^2 = r^2$, we have $\bar{z} = r^2 z^{-1}$ and so

$$\gamma_n = \lim_{r\to 1} \frac{1}{\sqrt{n\pi}} \frac{r^{2n}}{2i} \int_{|z|=r} f(z) \frac{\mathrm{d}z}{z^n}. \tag{2.69}$$

Now if

$$f(z) = a_0 + a_1 z + \cdots, \quad |z| < 1$$

is the power series expansion of f, then

$$a_{n-1} = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^n} \mathrm{d}z. \tag{2.70}$$

From (2.69) to (2.70), we obtain

$$\begin{aligned}\gamma_n &= \lim_{r\to 1} \frac{1}{\sqrt{n\pi}} r^{2n} \pi a_{n-1} \\ &= \sqrt{\frac{\pi}{n}} a_{n-1}, \quad n = 1, 2, \ldots.\end{aligned} \tag{2.71}$$

Also,

$$\iint_{|z|<1} |f(z)|^2 \mathrm{d}x\mathrm{d}y = \pi \sum_{n=1}^{\infty} \frac{|a_{n-1}|^2}{n} \tag{2.72}$$

[see (2.70) above Definition 2.6.2].

From (2.71) to (2.72), it follows that Parseval's formula holds. The argument is therefore complete.

Theorem 2.9.18 *Any two complete orthonormal sets in a given Hilbert space $H \neq \{0\}$ have the same cardinal number.*

Proof Let H be a Hilbert space of dimension n and A be any complete orthonormal set in H. It consists of linearly independent vectors and therefore can have at most n vectors in it. We shall argue that it contains precisely n vectors: by Theorem 2.9.14, A is a basis. Since it is finite, it is a Hamel basis and must therefore contain precisely n vectors.

Let $A = \{x_\alpha: \alpha \in \Lambda\}$ and $B = \{y_\beta: \beta \in \Gamma\}$ be complete orthonormal sets in H. For any $x_\alpha \in A$, the set

$$B_{x_\alpha} = \{y_\beta \in B\colon (x_\alpha, y_\beta) \neq 0\}$$

must be countable [see Remark 2.9.11]. Clearly, $\bigcup_\alpha B_{x_\alpha} \subseteq B$. We next show that $B \subseteq \bigcup_\alpha B_{x_\alpha}$. Let $y_\beta \in B$. Suppose $y_\beta \in B_{x_\alpha}$ for no α. Then $(x_\alpha, y_\beta) = 0$ for all $\alpha \in \Lambda$. In other words, $y_\beta \perp A$. Since A is complete, it follows that $y_\beta = 0$, which is impossible since $\|y_\beta\| = 1$. Hence, $y_\beta \in B_{x_\alpha}$ for some x_α. Thus

$$B = \bigcup_\alpha B_{x_\alpha}.$$

It follows that $|B|$, the cardinality of B, satisfies $|B| \leq \aleph_0|A| = |A|$. Interchanging the roles of A and B, we also have $|A| \leq |B|$. This completes the proof. □

Definition 2.9.19 Let H be a Hilbert space. If $H \neq \{0\}$, we define the **orthogonal dimension** of H to be the unique cardinal number of a complete orthonormal set in H. If $H = \{0\}$, we say that H has orthogonal dimension 0.

If H is finite dimensional, then the orthogonal dimension of H is the cardinal of a Hamel basis.

Theorem 2.9.20 (Riesz–Fischer) *Let $\{x_\alpha\}_{\alpha\in A}$ be a complete orthonormal system in a Hilbert space H and $\ell^2(A) = L^2(A, \mathfrak{S}, \mu)$, where $\mathfrak{S}$ denotes the collection of all subsets of A and μ is counting measure on A. Then H is isometrically isomorphic to $\ell^2(A)$.*

Proof For $x \in H$, let $T(x)$ be that function on A such that

$$[T(x)](\alpha) = (x, x_\alpha), \quad \alpha \in A.$$

Then T maps H into $\ell^2(A)$, for $\sum_{\alpha\in A} |(x, x_\alpha)|^2 < \infty$ by Bessel's Inequality. Also, for $x, y \in H$, we have

$$\begin{aligned}[T(x+y)](\alpha) &= (x+y, x_\alpha) = (x, x_\alpha) + (y, x_\alpha)\\ &= [T(x)](\alpha) + [T(y)](\alpha), \quad \alpha \in A,\end{aligned}$$

i.e. $T(x + y) = T(x) + T(y)$. It is equally easy to show that $T(ax) = aT(x)$ for scalar a. Thus, T is linear. Using Theorem 2.9.16(e), we have

$$\begin{aligned}(T(x), T(y)) &= \sum_{\alpha\in A} [T(x)](\alpha)\overline{[T(y)](\alpha)} \\ &= \sum_{\alpha\in A} (x, x_\alpha)\overline{(y, x_\alpha)} \\ &= (x, y)\end{aligned}$$

and so T preserves inner products. It remains to show that the mapping $T{:}H \to \ell^2(A)$ is onto.

Let $f \in \ell^2(A)$. Then $\sum_{\alpha\in A} |f(\alpha)|^2 < \infty$. Let $\alpha_1, \alpha_2, \ldots$ be those α's for which $f(\alpha) \neq 0$. The condition $\sum_{\alpha\in A} |f(\alpha)|^2 < \infty$ becomes $\sum_{i=1}^{\infty} |f(\alpha_i)|^2 < \infty$. It follows from Theorem 2.9.9 that $x = \sum_{i=1}^{\infty} f(\alpha_i)\alpha_i$ is in H. Since $(x, x_{\alpha_j}) = f(\alpha_j)$, so $(x, x_{\alpha_j}) \neq 0$. For a fixed p and any $m \geq p$, we have

$$\begin{aligned}\left|(x, x_{\alpha_p}) - f(\alpha_p)\right| &= \left|(x, x_{\alpha_p}) - \sum_{i=1}^{m} f(\alpha_i)(x_{\alpha_i}, x_{\alpha_p})\right| \\ &\leq \left\|x - \sum_{i=1}^{m} f(\alpha_i)x_{\alpha_i}\right\| \left\|x_{\alpha_p}\right\| \to 0\end{aligned}$$

as $m \to \infty$. Therefore, $f(\alpha_j) = (x, x_{\alpha_j}) = (T(x))(\alpha_j)$ for all $j = 1, 2, \ldots$. The equality also holds for those α's for which $f(\alpha) = 0$. This completes the proof. □

Remarks 2.9.21

(i) The following form of the above Theorem was originally proved by Riesz and Fischer in 1907:

Let $\{\alpha_n\}_{n\in\mathbb{Z}}$ be in $\ell^2(\mathbb{Z})$, that is, $\sum_{n=-\infty}^{\infty} |\alpha_n|^2 < \infty$. Then, there exists a function f in $L^2([-\pi, \pi], \frac{dt}{2\pi})$, such that $\hat{f}(n) = \alpha_n, n \in \mathbb{Z}$, where $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$ is the nth Fourier coefficient of f with respect to the orthonormal basis $\{e^{-int}: n \in \mathbb{Z}\}$.

(ii) A Hilbert space is completely determined up to an isometric isomorphism by its orthogonal dimension, i.e. by the cardinality of its complete orthonormal basis. The space $L^2([-\pi, \pi], \frac{dt}{2\pi})$ is isometrically isomorphic to $\ell^2(\mathbb{Z})$ and hence also to $\ell^2(\mathbb{N})$.

Problem Set 2.9

2.9.P1. Let $\{e_n\}_{n\geq 1}$ and $\{\tilde{e}_n\}_{n\geq 1}$ be orthonormal sequences in a Hilbert space H and let $M_1 = \text{span}(e_n)$ and $M_2 = \text{span}(\tilde{e}_n)$. Show that $\overline{M_1} = \overline{M_2}$ if, and only if,

$$e_n = \sum_{m=1}^{\infty} \alpha_{nm} \tilde{e_m}, \quad \tilde{e_n} = \sum_{m=1}^{\infty} \overline{\alpha}_{nm} e_m, \quad \alpha_{nm} = (e_n, \tilde{e_m}).$$

2.9.P2. Let H be a Hilbert space. Then show that the following hold:

(a) If H is separable, every orthonormal set in H is countable.
(b) If H contains an orthonormal sequence which is complete in H, then H is separable.

2.9.P3. Let $A \subseteq [-\pi, \pi]$ and A be measurable. Prove that

$$\lim_{n\to\infty} \int_A \cos nt \, dt = \lim_{n\to\infty} \int_A \sin nt \, dt = 0.$$

2.9.P4. Let $n_1 < n_2 < n_3 < \cdots$ be positive integers and

$$E = \{x \in [-\pi, \pi] : \lim_k \sin n_k x \, \text{exists}\}.$$

Prove that $m(E) = 0$, where $m(E)$ denotes the Lebesgue measure of E.

2.9.P5. Let $e_j(z) = z^j$, $j \in \mathbb{Z}$. Show that $\{e_j\}_{j=-\infty}^{\infty}$ is an orthonormal sequence in RL^2 (notation as in Example 2.1.3(vi)).

2.9.P6. Let $\alpha_1, \alpha_2 \in D(0,1) = \{z: |z| < 1\}$ and $\alpha_1 \neq \alpha_2$. Show that the vectors

$$e_1(z) = \frac{\left(1 - |\alpha_1|^2\right)^{\frac{1}{2}}}{1 - \overline{\alpha_1} z} \quad \text{and} \quad e_2(z) = \frac{z - \alpha_1}{1 - \overline{\alpha_1} z} \frac{\left(1 - |\alpha_2|^2\right)^{\frac{1}{2}}}{1 - \overline{\alpha_2} z}$$

constitute an orthonormal system in RH^2 (notation as in Example 2.1.3 (vi)).

2.9.P7. Let $\{e_n\}_{n \geq 1}$ be an orthonormal basis in H. Show that for any orthonormal set $\{f_n\}_{n \geq 1}$, if

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty,$$

then $\{f_n\}_{n \geq 1}$ is an orthonormal basis.

2.9.P8. A real-valued function on an interval having a continuous nonvanishing derivative on the interior of its domain maps a set of (Lebesgue) measure zero into a set of measure zero. In case the domain is an open interval, in which case the range is also an open interval and an inverse exists, the inverse also has the same properties. [Examples of such a function on the domain $(0,\infty)$ would be $\exp(-x)$ and x^2.]

2.10 Orthogonal Decomposition and Riesz Representation

A result of particular interest about Hilbert space is the projection theorem, namely, if M is any closed subspace of a Hilbert space H, then H can be decomposed into the direct sum of M and its orthogonal complement (to be defined below). This important geometric property is one of the main reasons that Hilbert spaces are easier to handle than Banach spaces.

A characterisation of a bounded linear functional [see Definition 2.10.18 below] on a Hilbert space, known as the Riesz Representation Theorem, will be studied.

P.L. Chebyshev sought the approximation of arbitrary functions by linear combinations of given ones. He considered approximations in the spaces of continuous functions, L^p spaces, etc. These have a bearing on constrained optimisation.

We deal with the approximation problem in a pre-Hilbert space X; given a set of n linearly independent vectors $\{v_1, v_2, \ldots, v_n\}$ and an $x \in X$, to find a method of computing the minimum value of

$$\left\| x - \sum_{j=1}^{n} c_j v_j \right\|,$$

where $c_1, c_2, \ldots, c_n$ range over all scalars and to find the corresponding values of $c_1, c_2, \ldots, c_n$. The reader will learn that this is precisely the problem of finding the distance of x from the linear span of $\{v_1, v_2, \ldots, v_n\}$.

Recall that a set of M nonzero vectors in a pre-Hilbert space is said to be orthogonal if $x \perp y$ whenever x and y are distinct vectors of M.

Definition 2.10.1 Let X be a pre-Hilbert space and $x \in X$. We define

$$\boldsymbol{x}^{\perp} = \{y \in X : (x, y) = 0\}.$$

and if S is a subset of X,

$$\boldsymbol{S}^{\perp} = \{y \in X : (x, y) = 0 \quad \text{for all} \quad x \in S\}.$$

The symbol $x \perp$ [respectively, $S^{\perp}$] is read as x perp [respectively, S perp]. One writes $S^{\perp\perp}$ for the perp *of* $S^{\perp}$; thus $S^{\perp\perp} = (S^{\perp})^{\perp}$. The set $S^{\perp}$ is called the **orthogonal complement of** S.

Remarks 2.10.2

(i) Observe that $x^{\perp}$ is a subspace of X, since $(x, y) = 0$ and $(x, z) = 0$ imply $(x, \alpha y + \beta z) = 0$, where α, β are scalars. Also, $x^{\perp}$ is precisely the set of vectors where the continuous function $y \to (x, y)$ is zero. Hence, $x^{\perp}$ is a closed subspace of X. Since

$$S^{\perp} = \bigcap_{x \in S} x^{\perp},$$

it follows that $S^{\perp}$, being the intersection of closed subspaces of X, is itself a closed subspace of X.

(ii) $$S^{\perp} = (\overline{S})^{\perp}.$$

Let $y \in S^{\perp}$. Then $(x, y) = 0$ for all $x \in S$. Let $z \in \overline{S}$. Then there exists a sequence $\{z_n\}_{n \geq 1}$ in S such that $z_n \to z$. The continuity of the mapping $x \to (x, y)$ and the fact that $(z_n, y) = 0$ for $n = 1, 2, \ldots$ imply $(z, y) = 0$. Since $z \in \overline{S}$ is arbitrary, we conclude that $y \in (\overline{S})^{\perp}$.

On the other hand, if $y \in (\overline{S})^{\perp}$, then $(y, x) = 0$ for all $x \in \overline{S}$. Since $S \subseteq \overline{S}$, it follows that $(y, x) = 0$ for all $x \in S$, that is $y \in S^{\perp}$.

Proposition 2.10.3 *Let S and S_1 be subsets of an inner product space X. Then the following hold.*

(a) *$S^{\perp}$ is a closed subspace of X and $S \cap S^{\perp} \subseteq \{0\}$;*
(b) *$S \subseteq S^{\perp\perp}$;*
(c) *$S \subseteq S_1$ implies $S^{\perp} \supseteq S_1^{\perp}$;*
(d) *$S^{\perp} = S^{\perp\perp\perp}$.*

Proof

(a) In Remark 2.10.2(i), we have noted that $S^{\perp}$ is a closed subspace of X. If $x \in S \cap S^{\perp}$ then $x \perp x$, that is, $(x, x) = 0$, which implies $x = 0$.
(b) Let $x \in S$. For any $y \in S^{\perp}$, one has $(y, x) = 0$, so that $x \perp S^{\perp}$ and therefore $x \in S^{\perp\perp}$.
(c) If $x \in S_1^{\perp}$, then $(x, y) = 0$ for all $y \in S_1$. In particular, $(x, y) = 0$ for all $y \in S$, which implies $x \in S^{\perp}$.
(d) Applying (iii) to the relation $S \subseteq S^{\perp\perp}$, we have $(S^{\perp\perp})^{\perp} \subseteq S^{\perp}$. Also, $S^{\perp} \subseteq (S^{\perp})^{\perp\perp}$ by (b) above. Since $(S^{\perp\perp})^{\perp} = (S^{\perp})^{\perp\perp}$, as in each case one starts with S and perps three times, it follows that $(S^{\perp\perp})^{\perp} = S^{\perp}$. ☐

Example Let $S = \{f \in L^2[0, 1]: f(t) = 0 \text{ a.e. for } 0 \leq t \leq \frac{1}{2}\}$.
Then

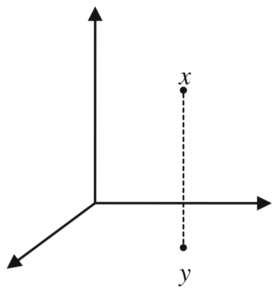
$$S^{\perp} = \left\{ g \in L^2[0,1] : g(t) = 0 \text{ a.e. on} \left[\frac{1}{2}, 1\right] \right\}$$

and

$$S^{\perp\perp} = \left\{ g \in L^2[0,1] : g(t) = 0 \text{ a.e. on} \left[0, \frac{1}{2}\right] \right\} = S.$$

Hint: To compute $S^{\perp}$, first show that $\int_x^1 g(t)\mathrm{d}t = 0$ for every $x \in [\frac{1}{2},1]$ and then use regularity of Lebesgue measure.

If x is a point lying outside a plane in $\mathbb{R}^3$, then there is a unique y in the plane which is closer to x than any other point of the plane. This assertion, when translated in the language of Hilbert spaces, yields rich dividends via the Riesz Representation Theorem below. The accompanying figure illustrates the situation when the plane is a coordinate plane. However, this need not always be the case.



Definition 2.10.4 A subset K of a vector space is **convex** if, for all $x, y \in K$, and all λ such that $0 < \lambda < 1$, the vector $\lambda x + (1 - \lambda)y$ belongs to K. The set of vectors $\{\lambda x + (1 - \lambda)y: 0 < \lambda < 1\}$ is the **line segment** joining x and y. The **convex hull** of a subset S of any vector space is the intersection of all convex subsets containing S and is denoted by **co(*S*)** or by **co*S***.

It is sometimes neater to work with an equivalent formulation of convexity as follows: for all $x, y \in K$, and all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, the vector $\alpha x + \beta y$ belongs to K.

It is easy to see that the intersection of any family of convex sets is again convex; in particular, any convex hull is a convex set. By using the alternative formulation of convexity, the convex hull of any finite set of vectors $\{x_1, x_2, \ldots, x_n\}$ is easily seen to consist of precisely those vectors which can be written as $\sum_{k=1}^n \lambda_k x_k$, where $0 \leq \lambda_k \leq 1$ for each k and $\sum_{k=1}^n \lambda_k = 1$. (Induction: we start with the vectors $x_1, x_2, \ldots, x_n$ and nonnegative $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying $\sum_{k=1}^n \lambda_k = 1$. If $\sum_{k=1}^{n-1} \lambda_k = 0$, then λ_k is 0 for $k = 1, \ldots, n-1$ and $\lambda_n = 1$, which together imply $\sum_{k=1}^n \lambda_k x_k$ is in the convex hull. Assume $\sum_{k=1}^{n-1} \lambda_k = \beta > 0$. Then $\sum_{k=1}^{n-1} (\lambda_k/\beta)x_k$ is in the convex hull by the induction hypothesis. Consequently, $\sum_{k=1}^n \lambda_k x_k = \beta \sum_{k=1}^{n-1} (\lambda_k/\beta)x_k + \lambda_n x_n$ is in the convex hull. Conversely, the vectors that can be written in this form obviously constitute a convex set that contains $\{x_1, x_2, \ldots, x_n\}$ and therefore contains the convex hull under reference.)

This description of the convex hull will now be used for arguing that it is compact when the vector space is normed.

When $n = 1$, there is nothing to prove. Assume as induction hypothesis that the convex hull of any n vectors is compact. Consider any set $\{x_1, x_2, \ldots, x_n, x\}$ of $n + 1$

vectors. Let $\{y_p\}_{p \geq 1}$ be a sequence in co$\{x_1, x_2, \ldots, x_n, x\}$. Each y_p can be written as $\sum_{k=1}^{n} \lambda_{k,p} x_k + \lambda_p x$, where $0 \leq \lambda_{k,p}, \lambda_p \leq 1$ for each k and $\sum_{k=1}^{n} \lambda_{k,p} + \lambda_p = 1$. If there are only finitely many such p, then we can assume that $1 - \lambda_p > 0$ for every p. It then follows that $\sum_{k=1}^{n} \lambda_{k,p}/(1 - \lambda_p) = 1$, each term in the sum being non-negative. This means $z_p = \sum_{k=1}^{n} \lambda_{k,p} x_k/(1 - \lambda_p)$ is in the convex hull of $\{x_1, x_2, \ldots, x_n\}$. By the induction hypothesis, $\{z_p\}$ has a subsequence $\{z_{p(q)}\}$ converging to a limit $z \in$ co$\{x_1, x_2, \ldots, x_n\}$. Now the bounded sequence $\{\lambda_{p(q)}\}$ in $\mathbb{R}$ has a convergent subsequence $\{\lambda_{p(q(r))}\}$, whose limit we shall denote by λ. Then $\{z_{p(q(r))}\}$ converges to z and therefore $\sum_{k=1}^{n} \lambda_{k,p(q(r))} x_k = (1 - \lambda_{p(q(r))}) z_{p(q(r))}$ forms a sequence converging to $(1 - \lambda)z$. As $y_{p(q(r))} = \sum_{k=1}^{n} \lambda_{k,p(q(r))} x_k + \lambda_{p(q(r))} x$, the subsequence $\{y_{p(q(r))}\}$ converges to $(1 - \lambda)z + \lambda x$, which belongs to co(co$\{x_1, x_2, \ldots, x_n\} \cup \{x\}$). The latter can easily be seen to be the same as co$\{x_1, x_2, \ldots, x_n, x\}$. This completes the induction proof that the convex hull of any finite set of vectors is compact.

If K is convex and x is any vector, then the convex hull co$(K \cup \{x\})$ is precisely $K_1 = \{\alpha x + \beta k\colon \alpha, \beta \geq 0, \alpha + \beta = 1\}$. The convexity of K_1 follows from the three computations

(a) $\alpha(\alpha_1 x + \beta_1 k_1) + \beta(\alpha_2 x + \beta_2 k_2) = (\alpha\alpha_1 + \beta\alpha_2)x + (\alpha\beta_1 k_1 + \beta\beta_2 k_2)$
$= (\alpha\alpha_1 + \beta\alpha_2)x + \gamma((\alpha\beta_1/\gamma)k_1 + (\beta\beta_2/\gamma)k_2)$, where $\gamma = 1 - (\alpha\alpha_1 + \beta\alpha_2)$ if nonzero;

(b) $(\alpha\beta_1/\gamma) + (\beta\beta_2/\gamma) = 1$ because

$$\alpha\beta_1 + \beta\beta_2 + (\alpha\alpha_1 + \beta\alpha_2) = \alpha(\alpha_1 + \beta_1) + \beta(\alpha_2 + \beta_2) = 1,$$

so that $\alpha\beta_1 + \beta\beta_2 = 1 - (\alpha\alpha_1 + \beta\alpha_2) = \gamma$,

(c) $\alpha\alpha_1 + \beta\alpha_2 \leq \alpha + \beta = 1$ when $0 \leq \alpha_1 \leq 1$ and $0 \leq \alpha_2 \leq 1$.

Regarding (a), we note that $\gamma = 0$ implies $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$, in which case $\alpha(\alpha_1 x + \beta_1 k_1) + \beta(\alpha_2 x + \beta_2 k_2) = x$. Once the convexity of K_1 is established, it is a trivial matter to see that it is the convex hull of K.

Theorem 2.10.5 (Closest point property) *Let K be a nonempty closed convex set in a Hilbert space H. For every $x \in H$, there is a unique point $y \in K$ which is closer to x than any other point of K, i.e.*

$$\|x - y\| = \inf_{z \in K} \|x - z\|.$$

Proof Let $\delta = \inf_{z \in K} \|x - z\|$. Since $K \neq \emptyset$, $\delta < \infty$, therefore for each $n \in \mathbb{N}$, there exists $y_n \in K$ such that

$$\|x - y_n\|^2 < \delta^2 + \frac{1}{n}. \tag{2.73}$$

We shall prove that $\{y_n\}_{n \geq 1}$ is a Cauchy sequence in K. Consider the vectors $x - y_n$ and $x - y_m$. By the Parallelogram Law [Proposition 2.2.3(c)],

$$\begin{aligned}\|(x - y_n) - (x - y_m)\|^2 + \|(x - y_n) + (x - y_m)\|^2 &= 2\left(\|x - y_n\|^2 + \|x - y_m\|^2\right)\\ &\leq 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right).\end{aligned}$$

Rearranging the left side, we obtain

$$\|y_n - y_m\|^2 + 4\left\|x - \frac{y_n + y_m}{2}\right\|^2 \leq 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right)$$

and hence

$$\|y_n - y_m\|^2 \leq 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right) - 4\left\|x - \frac{y_n + y_m}{2}\right\|^2.$$

Since K is convex, $y_n, y_m \in K$, we have $\frac{y_n + y_m}{2} \in K$, and hence

$$\left\|x - \frac{y_n + y_m}{2}\right\|^2 \geq \delta^2.$$

Consequently,

$$\begin{aligned}\|y_n - y_m\|^2 &\leq 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right) - 4\delta^2\\ &= 2\left(\frac{1}{n} + \frac{1}{m}\right).\end{aligned}$$

Thus, $\{y_n\}_{n \geq 1}$ is a Cauchy sequence and so converges to a limit $y \in H$. Since K is closed, $y \in K$ and therefore

$$\|x - y\| \geq \delta.$$

On letting $n \to \infty$ in (2.73), we obtain

$$\|x - y\| \leq \delta$$

and so

$$\|x - y\| = \delta.$$

We have proved that there is a closest point to x in K. It remains to show that it is unique. Suppose that $z \in K$ ($z \neq y$) is such that $\|x - z\| = \delta$. Then $\frac{y+z}{2} \in K$ so that

$$\left\|x - \frac{y+z}{2}\right\| \geq \delta.$$

On applying the Parallelogram Law [Proposition 2.2.3(c)] to $y - x$ and $z - x$, we get

$$\begin{aligned}\|y - z\|^2 &= 2\|y - x\|^2 + 2\|z - x\|^2 - \|y + z - 2x\|^2\\ &= 4\delta^2 - 4\left\|\frac{y+z}{2} - x\right\|^2 \leq 0.\end{aligned}$$

Hence $y = z$. □

Remarks 2.10.6

(i) If $x \in H$ is such that $x \in K$, then the vector nearest to x is x itself.

(ii) If K is not closed, the conclusion of Theorem 2.10.5 may not hold. In fact, in this situation, whether K is convex or not, there always exists a point in H having no closest approximation in K. Any point in the closure that does not belong to K will serve the purpose.

Let $H = \ell^2$ and $K = \{x = \{\lambda_k\}_{k\geq 1}$: $\lambda_k \neq 0$ for only finitely many k's and $\sum_{k=1}^{\infty} \lambda_k = 1\}$. K is convex. However, K is not closed. In fact, the sequence $y_1 = (1, 0, 0, \ldots)$, $y_2 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots\right)$, …, $y_n = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$, … is in K. However, the limit of the sequence $\{y_n\}_{n\geq 1}$ in ℓ^2 is the point $y = (0, 0, \ldots)$ of ℓ^2, which does not belong to K. According to the preceding paragraph, the point $y = (0, 0, \ldots)$ does not possess a closest point in K.

(iii) The conclusion of Theorem 2.10.5 fails to hold if H is not a Hilbert space. Let $X = \mathbb{R}^2$, the real Banach space with $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$. Consider the closed convex set $K = \{(x_1, x_2)$: $x_1 \geq 1\}$. The minimal distance of the origin from K is attained at each of the points of the line segment $\{(x_1, x_2)$: $x_1 = 1$ and $|x_2| \leq 1\}$. Even when the norm comes from an inner product and H is not complete, the existence part of the conclusion of Theorem 2.10.5 may fail to hold; an example of this will be given later in (ii) of Remarks 2.10.12. The uniqueness part however holds because its proof does not use completeness.

Corollary 2.10.7 *Every nonempty closed convex set K in a Hilbert space H contains a unique element of smallest norm.*

Proof Take $x = 0$ in Theorem 2.10.5. □

Example 2.10.8 Let $K = \{y = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n\colon \sum_{k=1}^{n} \lambda_k = 1\}$. K is a closed convex subset of $\mathbb{C}^n$. The unique vector $y_0 \in K$ of smallest norm is $y_0 = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$; indeed, if y_0 has two unequal components, then interchanging them leads to another vector in K with smallest norm; consequently, all components of y_0 are equal. The reader who is familiar with constrained optimisation can verify the claim made above independently of the corollary.

Corollary 2.10.9 *Let M be a closed subspace of a Hilbert space H. If x is a vector in H, and if $\delta = \inf\{\|x - z\|\colon z \in M\}$ then there exists a unique $y \in M$ such that $\delta = \|x - y\|$.*

Proof Every subspace of a vector space is convex. □

Theorem 2.10.10 *Let M be a closed subspace of a Hilbert space H and $x \in H$. If y denotes the unique element in M for which $\|x - y\| = \inf\{\|x - z\|\colon z \in M\}$, then $x - y$ is orthogonal to M. Conversely, if $y \in M$ is such that $x - y$ is orthogonal to M, then $\|x - y\| = \inf\{\|x - z\|\colon z \in M\}$.*

Proof Consider $z \in M$ with $\|z\| = 1$. Then $w = y + (x - y, z)z$ lies in M and we have

$$\begin{aligned}\|x - y\|^2 \le \|x - w\|^2 &= (x - w, x - w)\\ &= (x - y - (x - y, z)z, x - y - (x - y, z)z)\\ &= \|x - y\|^2 - |(x - y, z)|^2.\end{aligned}$$

This shows that $(x - y, z) = 0$, i.e. $x - y \perp z$. Since every vector in M is a scalar multiple of a vector in M of norm 1, it follows that $x - y \perp M$.

If $z \in M$, then $x - y$ is orthogonal to $y - z$, so that $\|x - z\|^2 = \|x - y + y - z\|^2 = \|x - y\|^2 + \|y - z\|^2 \ge \|x - y\|^2$. Thus, $\|x - y\| = \inf\{\|x - z\|\colon z \in M\}$. □

It may be noted that $x - y$ is closest to x in $M^\perp$.

Theorem 2.10.11 (Orthogonal Decomposition Theorem) *If M is a closed subspace of a Hilbert space H, then $H = M \oplus M^\perp$ and $M = M^{\perp\perp}$.*

Proof Let $x \in H$. Since M is a closed subspace of H, there exists a unique vector $y \in M$ such that $\|x - y\| = \inf\{\|x - z\|\colon z \in M\}$. So $x - y \perp M$ by Theorem 2.10.10. Hence, $x = y + (x - y)$, where $y \in M$ and $x - y \in M^\perp$. Since $M \cap M^\perp = \{0\}$, it follows that $H = M \oplus M^\perp$.

We already know that $M \subseteq M^{\perp\perp}$ [Proposition 2.10.3(b)]. On the other hand, let $x \in M^{\perp\perp}$. If $x = y + z$, where $y \in M$ and $z \in M^\perp$, then x and y are in $M^{\perp\perp}$. Hence, $z = x - y \in M^{\perp\perp}$ ($M^{\perp\perp}$ is a subspace of H). Since $z \in M^\perp$, it follows that $(z, z) = 0$, that is, $z = 0$ or $x = y$. This shows that $M^{\perp\perp} \subseteq M$. The proof is complete. □

Remarks 2.10.12

(i) If M is a closed subspace of a Hilbert space H and $x \in H$, then x can be uniquely expressed as

$$x = y + z,$$

where $y \in M$ and $z \in M^\perp$.

(ii) The condition that H is a Hilbert space for a closed subspace to satisfy $M = M^{\perp\perp}$ cannot be omitted. Let $\ell_0 \subseteq \ell^2$ be the inner product space consisting of sequences, each of which has only finitely many nonzero terms. Let $M = \{x = \{\lambda_k\}_{k\geq 1}: \lambda_k \neq 0$ for only finitely many k's and $\sum_{k=1}^{\infty} \frac{1}{k}\lambda_k = 0\}$. Clearly, M is a subspace of ℓ_0. Moreover, M is closed, as is proved below.

Let $\{x^{(n)}\}_{n\geq 1}$ be a sequence in M such that $x^{(n)} \to x$ in ℓ_0. By the Cauchy–Schwarz Inequality, it follows that

$$\left|\sum_{k=1}^{\infty} \frac{1}{k}\lambda_k\right|^2 = \left|\sum_{k=1}^{\infty} \frac{1}{k}\left(\lambda_k - \lambda_k^{(n)}\right)\right|^2 \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)\left(\sum_{k=1}^{\infty} \left|\lambda_k - \lambda_k^{(n)}\right|^2\right),$$

where $\lambda_k^{(n)}$ is the kth component of x_n. Consequently, $\sum_{k=1}^{\infty} \frac{1}{k}\lambda_k = 0$. So, $x \in M$.

We next show that $M^\perp = \{0\}$. Assume $0 \neq z \in \ell_0$ and $z \perp M$. Then there exists k such that $z = (x_1, \ldots, x_k, 0, \ldots)$ and $\sum_{i=1}^{k} |x_i|^2 \neq 0$. Let $\mu = -(k+1)\sum_{j=1}^{k} \frac{1}{j}x_j$. Then $w = (x_1, \ldots, x_k, \mu, 0, \ldots) \in M$ in view of the definition of μ. Hence, $z \perp w$, i.e. $(z, w) = 0$. But $(z, w) = \|z\|^2$. It follows that $z = 0$, contradicting the assumption on z.

Consequently, $M^{\perp\perp} = \ell_0 \neq M$.

We shall use the fact that $M^\perp = \{0\}$ to show that the closed convex subset M of the (incomplete) inner product space ℓ_0 has the property that *every* $x \in \ell_0$ that does not lie in M fails to have a nearest element in M. In particular, it will follow that the conclusion of Theorem 2.10.5 may fail to hold in the absence of completeness. Suppose $x \in \ell_0$ does not lie in M but has a closest element $y \in M$. It follows that $x - y \perp M$ exactly as in the first paragraph of the proof of Theorem 2.10.10, considering that completeness is not needed in that paragraph. Since we have shown that $M^\perp = \{0\}$, we infer that $x - y = 0$, which is a contradiction because $y \in M,$ whereas $x \notin M$.

(iii) If M is any linear subspace of H, then $\overline{M} = M^{\perp\perp}$. Observe that $M \subseteq M^{\perp\perp}$ [see Proposition 2.10.3(b)]. It follows that $\overline{M} \subseteq M^{\perp\perp}$, since $M^{\perp\perp}$ is a closed subspace of H. As $\mathrm{M} \subseteq \overline{M}$, it follows that $(\overline{M})^\perp \subseteq \mathrm{M}^\perp$ using Proposition 2.10.3(c). Another application of Proposition 2.10.3(c) yields $\mathrm{M}^{\perp\perp} \subseteq (\overline{M})^{\perp\perp} = \overline{M}$ since $\overline{M}$ is a closed subspace of H [Theorem 2.10.11].

(iv) If $H = M \oplus N$, $M \subseteq N^\perp$, then $M = N^\perp$ and is therefore closed, as we now show. Suppose $x \in N^\perp$ and $x \notin M$. The vector x has the representation $x = y + z$, where $y \in M$ and $z \in N$. Now

$$0 = (x, z) = (y, z) + (z, z)$$

which implies $z = 0$. Consequently, $x = y$, which is a contradiction because $x \notin M$ and $y \in M$.

(v) $S^{\perp\perp} = (S^{\perp})^{\perp}$ is the smallest closed subspace of the Hilbert space H which contains S.

(vi) Let $\{M_k\}_{k \geq 1}$ be a sequence of closed linear subspaces of a Hilbert space H. There exists a smallest closed linear subspace M such that $M_k \subseteq M$ for all k and it has the property that $x \perp M$ if, and only if, $x \perp M_k$ for all k. To see why, let $S = \{x \in H: x \in M_k \text{ for some } k\}$. Clearly $M_k \subseteq S$ for all k. Moreover, S is the smallest subset of H with this property. Set $M = S^{\perp\perp}$. If $\mathfrak{N}$ is a closed linear subspace such that $M_k \subseteq \mathfrak{N}$ for all k, then $S \subseteq \mathfrak{N}$. Hence, $M \subseteq \mathfrak{N}$ in view of (v). The assertion that $x \perp M$ if, and only if, $x \perp M_k$ for all k is proved by using the following observation:

$$M^{\perp} = S^{\perp\perp\perp} = S^{\perp}.$$

Example 2.10.13 Consider $\mathcal{F}_o = \{f \in L^2[-1, 1]: f(t) = -f(-t)\}$ and $\mathcal{F}_e = \{f \in L^2[-1, 1]: f(t) = f(-t)\}$.

The set $\mathcal{F}_o$ is an infinite-dimensional linear subspace of $L^2[-1, 1]$. [$f(t) = t^{2n-1}$, $n = 1, 2, \ldots$ are in $\mathcal{F}_o$; they are countably many and linearly independent.] Also, $\mathcal{F}_e$ is an infinite-dimensional subspace of $L^2[-1, 1]$. [$\mathcal{F}_e$ contains the functions $f(t) = t^{2n}$, $n = 0, 1, 2, \ldots$.]

For $f \in \mathcal{F}_o$ and $g \in \mathcal{F}_e$, the inner product

$$(f, g) = \int_{-1}^{1} f(t)\overline{g(t)}\mathrm{d}t = 0$$

since the function $f(t)\overline{g(t)}$ is odd. Hence $\mathcal{F}_o \perp \mathcal{F}_e$, i.e. $\mathcal{F}_o \subseteq \mathcal{F}_e^{\perp}$.

For any function $f \in L^2[-1, 1]$,

$$f_e(t) = \frac{f(t) + f(-t)}{2}, \quad f_o(t) = \frac{f(t) - f(-t)}{2} \quad \text{and} \quad f = f_e + f_o.$$

Moreover, this representation is unique, so that $L^2[-1, 1] = \mathcal{F}_o \oplus \mathcal{F}_e$. Since $\mathcal{F}_o \subseteq \mathcal{F}_e^{\perp}$, it follows that $\mathcal{F}_o = \mathcal{F}_e^{\perp}$. [See Remark 2.10.12(iv)].

The following proposition provides an alternate way of computing

$$\delta = \inf\{\|x - z\| : z \in M\},$$

where $x \in H$ and M is a closed subspace of H.

Proposition 2.10.14 *If M is a closed subspace of a Hilbert space H and $x \in H$, then*

$$\delta = \inf\{\|x-u\| : u \in M\} = \max\left\{|(x,z)| : z \in M^{\perp} \quad \text{and} \quad \|z\| = 1\right\}.$$

Proof Suppose $x \in H$. Then, x has a unique representation of the form

$$x = y + z, \quad y \in M \quad \text{and} \quad z \in M^{\perp}.$$

Let $w \in M^{\perp}$ and $\|w\| = 1$. Then

$$|(x,w)| = |(y+z,w)| = |(z,w)| \leq \|z\| = \|x-y\| = \delta \,[\text{Theorem } 2.10.10].$$

Hence, $\sup\{|(x, w)|: w \in M^{\perp}, \|w\| = 1\} \leq \delta$. The vector $w = \frac{x-y}{\delta}$ is in $M^{\perp}$ [Theorem 2.10.10] and $\|w\| = \frac{\|x-y\|}{\delta} = 1$. For this w,

$$(x,w) = \left(x, \frac{x-y}{\delta}\right) = \frac{1}{\delta}(x, x-y) = \frac{1}{\delta}(x-y, x-y) = \frac{1}{\delta}\|x-y\|^2 = \delta,$$

using the fact that $y \in M$ and $x - y \in M^{\perp}$.

This completes the proof. □

Let H be a Hilbert space and M *is* a closed subspace of H. The Orthogonal Decomposition Theorem 2.10.11 says that $H = M \oplus M^{\perp}$. Thus for each $x \in H$, there are unique $y \in M$ and $z \in M^{\perp}$ such that $x = y + z$. Note that $z = x - y$ is the unique vector in $M^{\perp}$ closest to x. The vector y is the **projection** of x on M and it follows that z is the projection of x on $M^{\perp}$. This sets up mappings from H onto M and from H onto $M^{\perp}$, respectively.

Theorem 2.10.15 *The mapping P_M:$H \to M$ defined by $P_M(x) = y$, $x \in H$, where y denotes the projection of x on M. The mapping P_M has the following properties*:

(a) *P_M is linear, i.e. $P_M(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 P_M(x_1) + \alpha_2 P_M(x_2)$, where α_1 and α_2 are scalars*;

(b) *If $x \in M$, then $P_M(x) = x$. Thus, P_M is idempotent, i.e. $P_M^2 = P_M$*;

(c) *If $x \in M^{\perp}$, then $P_M(x) = 0$*;

(d) *$(P_M(x), x) = \|P_M(x)\|^2 \leq \|x\|^2$ for all $x \in H$.*

Proof

(a) Let $x_i = y_i + z_i$, $i = 1, 2$ be the decomposition of x_i relative to M. Then

$$P_M(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 P_M(x_1) + \alpha_2 P_M(x_2).$$

(b) Since $x = x + 0$ is the unique decomposition of $x \in M$, it follows that $P_M(x) = x$. If $x \in H$, then $P_M(x) \in M$ and, by what has just been proved, $P_M(x) = P_M(P_M(x)) = P_M^2(x)$. Thus $P_M^2 = P_M$.

(c) If $x \in M^{\perp}$, then $x = 0 + x$ is the unique decomposition of x. Thus $P_M(x) = 0$.

(d) $(P_M(x), x) = (y, y + z)$, where $x = y + z$ is the unique decomposition of x. Now $y \perp z$ and therefore $(y, y + z) = (y, y) = (P_M(x), P_M(x)) = \|P_M(x)\|^2$. Also, $\|P_M(x)\|^2 = (y, y) \leq \|y\|^2 + \|z\|^2 = \|x\|^2$. □

Definition 2.10.16 The map P_M is called the **orthogonal projection on** $\boldsymbol{M}$.

Remarks 2.10.17

(i) The map P_M is often denoted by P when it is clear from the context on which subspace M the projection P_M is intended.
(ii) In Theorem 2.10.15(d), we have checked that

$$\|P_M(x)\| \leq \|x\| \quad \text{for} \quad \text{all}\, x \in H.$$

Since $P_M(x) = x$ for all $x \in M$, it follows that $\|P_M(x)\| = \|x\|$ and hence

$$\sup_{\|x\|=1}\|P_M(x)\| = 1\, \text{provided}\, M \neq \{0\}.$$

We now turn to the study of 'linear functionals' on Hilbert spaces.

Definition 2.10.18 A **linear functional** on a vector space X over a field $\mathbb{F}$ is a mapping $f{:}X \to \mathbb{F}$ which satisfies $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ for all $x, y \in X$ and all scalars λ, μ in $\mathbb{F}$.

Definition 2.10.19 Let X be a normed linear space over $\mathbb{F}$. A linear mapping $f{:}X \to \mathbb{F}$ is called a **bounded** linear functional on X if it maps bounded subsets of X into bounded subsets of $\mathbb{F}$, or equivalently, if there exists a constant K such that

$$|f(x)| \leq K\|x\|, \quad x \in X.$$

The equivalence is a consequence of the linearity of the functional.

The linear functional f is said to be a **continuous** linear functional if for a sequence $\{x_n\}_{n \geq 1}$ in X, $x_n \to x$ implies $f(x_n) \to f(x)$.

Let X^* denote the set of all bounded linear functionals on X. Define addition and scalar multiplication in X^* as follows:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)\,\text{and}\,(\alpha f_1)(x) = \alpha f_1(x) \quad \text{for} \quad f_1, f_2 \in X^* \quad \text{and} \quad \alpha \in \mathbb{F}.$$

It can be checked that $f_1 + f_2$ and αf_1 are in X^*.

Define a **norm on** $\boldsymbol{X^*}$ by setting

$$\|f\| = \sup_{x \neq 0}|f(x)|/\|x\| = \sup_{\|x\|=1}|f(x)|.$$

It can be checked that $\|\cdot\|$ is a norm on X^*. It is immediate from the definition of the norm in X^* that

$$|f(x)| \leq \|f\| \cdot \|x\|.$$

It will be proved later that X^* with the norm described above is complete.

Proposition 2.10.20 *A linear functional f:$X \to \mathbb{C}$ is bounded if, and only if, it is a continuous functional.*

Proof Indeed, if f is bounded, then

$$\begin{aligned} |f(x_n) - f(x)| &= |f(x_n - x)| \\ &\leq \|f\| \|x_n - x\| \to 0 \end{aligned}$$

as $x_n \to x$ and this implies the result in one direction.

Suppose $x_n \to 0$. Then $f(x_n) \to 0$. If f is not bounded, then for every $n \in \mathbb{N}$, there exists x_n with $\|x_n\| = 1$ and $|f(x_n)| \geq n$. But in this case, $|f(x_n/n)| \geq 1$, whereas $\|x_n/n\| \to 0$. □

Remark The study of continuous linear functionals will be taken up in more detail later in the book.

Proposition 2.10.21 *A linear functional f defined on X is continuous at $x = 0$, then it is continuous everywhere.*

Proof Suppose f is continuous at $x = 0$. Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that $\|x\| < \delta$ implies $|f(x)| < \varepsilon$. Therefore, for every $x \in X$, $\|x - y\| < \delta$ implies $|f(x) - f(y)| = |f(x-y)| < \varepsilon$. □

Remarks 2.10.22

(i) The point $x = 0$ could be replaced by any other point of X.
(ii) A slight modification of the argument in the proof above shows that f is uniformly continuous. Indeed, for every pair of points x, y in X, $\|x - y\| < \delta$ implies

$$|f(x) - f(y)| = |f(x-y)| < \varepsilon.$$

Thus, a linear functional is either uniformly continuous or everywhere discontinuous.

Theorem 2.10.23 *If X is a normed linear space, then X^* is a Banach space.*

Proof Let $\{f_n\}_{n \geq 1}$ be a Cauchy sequence of elements of X^*. This means that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $m,n \geq n_0$ implies

$$\|f_n - f_m\| < \varepsilon,$$

that is, for any $x \in X$,

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\| \|x\| < \varepsilon \|x\|. \tag{2.74}$$

In particular, for any $x \in X$, $\{f_n(x)\}_{n \ge 1}$ is a Cauchy sequence of scalars. So, the limit

$$\lim_n f_n(x) = f(x), \text{say},$$

must exist. The function f defined in this way is clearly linear. We next show that f is bounded. For any $j \in \mathbb{N}$ and an appropriate $n_1 \in \mathbb{N}$, we have

$$\left\|f_{n_1+j} - f_{n_1}\right\| < 1,$$

which implies

$$\left\|f_{n_1+j}\right\| < (1 + \|f_{n_1}\|),$$

or

$$\left|f_{n_1+j}(x)\right| < (1 + \|f_{n_1}\|)\|x\|.$$

On letting $j \to \infty$, we obtain

$$|f(x)| \le (1 + \|f_{n_1}\|)\|x\|.$$

Thus, f is a bounded linear functional in X. We next prove that $\{f_n\}_{n \ge 1}$ converges to f in the norm of X^*. Using (2.74) again, for $x \in X$ and $n \ge n_0$, we have

$$\lim_m |f_n(x) - f_m(x)| = |f_n(x) - \lim_m f_m(x)| \le \varepsilon \|x\|.$$

Hence

$$|f_n(x) - f(x)| \le \varepsilon \|x\|, \quad x \in X,$$

which implies $\|f_n - f\| \le \varepsilon$ for $n \ge n_0$. □

It is easy to write down the general linear functional on a finite-dimensional linear space. The description of continuous linear functionals on Banach spaces entails some efforts. However, not much effort is required to describe the continuous linear functionals on Hilbert spaces. We begin with some examples of continuous linear functionals.

Examples 2.10.24

(i) Let H be a Hilbert space of finite dimension and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis in H. If $x = \sum \alpha_k e_k, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, be any vector in H, then $f(x) = \sum \alpha_k f(e_k)$ is clearly a linear functional in H. Moreover,

$$\begin{aligned}
|f(x)| &= \left|\sum_{k=1}^{n} \alpha_k f(e_k)\right| \le \sum_{k=1}^{n} |\alpha_k||f(e_k)| \\
&\le \left(\sum_{k=1}^{n} |\alpha_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |f(e_k)|^2\right)^{\frac{1}{2}} \quad \text{[Cauchy–Schwarz Inequality]} \\
&= M\|x\|,
\end{aligned}$$

where $M = \left(\sum_{k=1}^{n} |f(e_k)|^2\right)^{\frac{1}{2}}$ and $\|x\| = \left(\sum_{k=1}^{n} |\alpha_k|^2\right)^{\frac{1}{2}}$; i.e. f is a bounded [continuous] linear functional.

(ii) Consider the Hilbert space $H = \ell^2$ of square summable sequences of scalars. For $y = \{y_n\}_{n \ge 1}$ in ℓ^2, define

$$f_y(x) = (x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

Observe that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} x_n \overline{y_n}\right| &\le \sum_{n=1}^{\infty} |x_n||y_n| \\
&\le \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{\frac{1}{2}} \\
&= \|x\|_2 \|y\|_2,
\end{aligned}$$

using the Cauchy–Schwarz Inequality. Thus, f_y is a bounded linear functional on ℓ^2 of norm at most $\|y\|_2$. For $y \in \ell^2$,

$$f_y(y) = (y, y) = \sum_{n=1}^{\infty} |y_n|^2 = \|y\|_2^2,$$

showing that $\|f_y\| = \|y\|_2$.

(iii) Consider the Hilbert space $L^2(X, \mathfrak{M}, \mu)$ of complex-valued measurable functions f defined on X for which $\int_X |f|^2 d\mu$ is finite. For $g \in L^2(X, \mathfrak{M}, \mu)$, define

$$f_g(h) = \int_X h\bar{g}\, \mathrm{d}\mu, \quad h \in L^2(X, \mathfrak{M}, \mu).$$

Observe that

$$\left|\int_X h\bar{g}\,\mathrm{d}\mu\right| \le \left(\int_X |h|^2\mathrm{d}\mu\right)^{\frac{1}{2}}\left(\int_X |g|^2\mathrm{d}\mu\right)^{\frac{1}{2}} = \|h\|_2\|g\|_2,$$

using Cauchy–Schwarz Inequality. Thus, f_g is a linear functional on $L^2(X, \mathfrak{M}, \mu)$ of norm at most $\|g\|_2$. For $g \in L^2(X, \mathfrak{M}, \mu)$,

$$f_g(g) = \int_X g\bar{g}\,\mathrm{d}\mu = \int_X |g|^2\mathrm{d}\mu = \|g\|_2^2,$$

showing that $\|f_g\| = \|g\|_2$.

(iv) Given a Hilbert space H and a vector $y \in H$, the function $f_y(x) = (x, y)$, $x \in H$, is a bounded linear functional on H of norm $\|y\|$. Indeed, f is clearly linear and $|f_y(x)| = |(x, y)| \le \|x\|\|y\|$ and so, $\|f_y\| \le \|y\|$. Furthermore, $|f_y(y)| = \|y\|^2$ and so $\|f_y\| = \|y\|$.

Note that (i) and (ii) are special cases of (iii) with μ as counting measure, and (iii) is a special case of (iv).

The existence of orthogonal decompositions implies that all bounded linear functionals on H can be obtained in this way.

Theorem 2.10.25 (Riesz Representation Theorem) *Let H be a Hilbert space over $\mathbb{C}$ and let $f \in H^*$, the space of all continuous linear functionals on H. Then there exists a unique vector $y \in H$ such that $f(x) = (x, y)$ for all $x \in H$.*

Moreover, the mapping $T{:}H \to H^*$ defined by $T(y) = f_y$, where $f_y(x) = (x, y)$, is onto, conjugate linear and isometric.

If H is a Hilbert space over $\mathbb{R}$, the mapping T is linear rather than conjugate linear.

Proof Let $f \in H^*$. If $f = 0$, choose $y = 0$. Then $f(x) = (x, y)$, $x \in H$. Furthermore, $y = 0$ is the only such element of H since $0 = f(y) = (y, y) = \|y\|^2$.

Suppose that $f \ne 0$ and let $W = \{x \in H{:}\ f(x) = 0\}$, known as the kernel of f and denoted by ker(f). Clearly, W is a linear subspace of H. Moreover, W is closed, so that W is a closed subspace of H. In fact, W is the inverse image of the closed set $\{0\}$ under the continuous linear functional f. Since $f \ne 0$, we have $W \ne H$. So, by the Orthogonal Decomposition Theorem 2.10.11, $H = W \oplus W^\perp$. Since $W^\perp \ne \{0\}$, there exists $y_0 \in W^\perp$, $y_0 \ne 0$. Clearly, $f(y_0) \ne 0$ as $y_0 \in W^\perp$. Let $y = \left[y_0\overline{f(y_0)}/\|y_0\|^2\right]$. For an arbitrary $x \in H$, we can form the element $x - [f(x)/f(y_0)]y_0 \in H$. Observe that

$$f(x - [f(x)/f(y_0)]y_0) = f(x) - [f(x)/f(y_0)]f(y_0) = 0.$$

So, $x - [f(x)/f(y_0)]y_0 \in W$. Consequently,

$$(x - [f(x)/f(y_0)]y_0, y_0) = 0, \text{i.e.}, (x, y_0) = [f(x)/f(y_0)]\|y_0\|^2,$$

which implies $f(x) = (x, y)$.

We next show that y is unique. Assuming the contrary, we have the equation

$$(x, y') = (x, y'')$$

for $x \in H$ where $y' \neq y''$. But this is impossible, since the substitution $x = y' - y''$ yields the contradiction $\|y' - y''\| = 0$. The fact that $\|f\| = \|y\|$ was proved in Example 2.10.24(iv).

The mapping $T{:}H \to H^*$ defined by $T(y) = f_y$, where $f_y(x) = (x, y)$, is conjugate linear: $T(\alpha y + \beta z) = f_{\alpha y+\beta z}$, where $f_{\alpha y + \beta z}(x) = (x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z) = \bar{\alpha} f_y(x) + \bar{\beta} f_z(x)$. Thus, $T(\alpha y + \beta z) = \bar{\alpha} f_y + \bar{\beta} f_z$.

The real case is left to the reader. This completes the proof. □

Remarks 2.10.26

(i) The functionals defined on ℓ^2 and $L^2(X, \mathbb{F}, \mu)$ in Examples 2.10.24(ii) and (iii) are the only continuous functionals on these spaces. To prove the statement without the use of the theorem will need quite an effort. The linear functionals defined in Example 2.10.24(i) are the only ones possible on that space.

(ii) The Riesz Representation Theorem has been proved for a Hilbert space. The hypothesis that the space is complete is essential for the theorem to hold. Consider the pre-Hilbert space ℓ_0 of finitely nonzero sequences. Define

$$f(x) = \sum_{n=1}^{\infty} \frac{x(n)}{n}, \quad x \in \ell_0 \quad \text{and} \quad x = \{x(n)\}_{n \geq 1}.$$

Clearly, f is linear and

$$|f(x)| = \left| \sum_{n=1}^{\infty} \frac{x(n)}{n} \right| \leq \left(\sum_{n=1}^{\infty} |x(n)|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} = M\|x\|,$$

where $M = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}}$, using the Cauchy–Schwarz Inequality. Thus, f is a bounded linear functional on ℓ_0. However, there exists no $y \in \ell_0$ for which $f(x) = (x, y)$. Indeed, for $x = e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$, where 1 occurs in the nth place, $f(x) = \frac{1}{n}$, while $(x, y) = \overline{y(n)}$, so that $y(n) = \frac{1}{n}$. Consequently, $y \notin \ell_0$.

Remark 2.10.27 In fact, every incomplete inner product space has a continuous linear functional that cannot be represented by an element of the space. Indeed, the linear functional defined by a vector in the completion but not in the incomplete space is the desired linear functional.

Let Y be a subspace of a normed linear space X and f be a bounded linear functional defined on Y. Then f can be extended to the whole of X, so that both the functional and its extension have the same norm [Theorem 5.3.2]. Apart from the fact that the procedure of extension is involved, the extension is not unique. However, the existence of an extension of a continuous linear functional defined on a subspace of a Hilbert space H to H is a direct consequence of the Riesz Representation Theorem 2.10.25. Moreover, the extension is unique.

Theorem 2.10.28 *Let H be a Hilbert space, Y a subspace of H and f a continuous linear functional defined on Y. Then, there exists a unique $F \in H^*$ such that $F|_Y = f$ and $\|f\|_Y = \|F\|_H$,*

$$\|f\|_Y = \sup_{\substack{\|x\|=1 \\ x\in Y}} |f(x)| \quad \text{and} \quad \|F\|_H = \sup_{\substack{\|x\|=1 \\ x\in H}} |F(x)|.$$

Proof Since f is linear and continuous, it follows that it is uniformly continuous on Y [Remark 2.10.22(ii)]. Hence, f can be extended to $\overline{Y}$, the closure of Y with the preservation of norm. This is shown as follows:

Let $x \in \overline{Y}$. There exists a sequence $\{x_n\}$ in Y such that $x_n \to x$ and, in view of the linearity and continuity of f, $\lim_n f(x_n)$ exists; moreover, it is independent of the sequence chosen. We define $f(x)$ to be $\lim_n f(x_n)$. Then $|f(x_n)| \leq \|f\|_Y \|x_n\|$ and this implies $|f(x)| \leq \|f\|_Y \|x\|$. Consequently, the norm of the extended f is at most the norm of the given f. The reverse inequality is trivial.

We may therefore assume without loss of generality that Y is a closed subspace of H. The Riesz Representation Theorem 2.10.25 asserts the existence of a unique element $y \in Y$ such that

$$f(x) = (x, y) \quad \text{for all} \quad x \in Y,$$

and

$$\|f\|_Y = \|y\|.$$

We now extend f to the whole of H by defining

$$F(x) = (x, y) \quad \text{for all} \quad x \in H,$$

i.e. $F(x) = 0$ if $x \in Y^\perp$ and $F|_Y = f$. It is clear from the definition of F that

$$\|F\|_H = \|y\| = \|f\|_Y.$$

We shall next show that any other extension of the linear functional f to the whole space increases the norm. Indeed, if F' is any other extension of f to the whole space, then

$$F'(x) = (x, z)$$

and

$$\|F'\| = \|z\|.$$

For $x \in Y$,

$$(x, y) = (x, z),$$

so that $y - z \perp Y$. Because $y \in Y$,

$$\|z\|^2 = \|y\|^2 + \|y - z\|^2,$$

which implies that

$$\|F'\| \geq \|f\|_Y,$$

where there is strict inequality if $y \neq z$. □

We note in passing that if $\overline{Y}$ is not the whole space H, then there exist extensions of arbitrarily large norm.

Let X be a normed linear space over $\mathbb{C}$. The space X^* of all bounded linear functionals on X is a Banach space. One can then consider continuous linear functionals on X^*, that is, the space $(X^*)^* = X^{**}$. By the preceding remark, X^{**} is again a Banach space. The element $x \in X$ defines a continuous linear functional on X^*; that is, x determines an element $\tau(x)$ of X^{**} defined by

$$\tau(x)(x^*) = x^*(x), \quad x^* \in X^*. \tag{2.75}$$

It is apparent that $\tau(x)$ is linear. Also, the inequality

$$|\tau(x)(x^*)| = |x^*(x)| \leq \|x^*\|\|x\|$$

shows that $\tau(x) \in X^{**}$ and $\|\tau(x)\| \leq \|x\|$. One learns in a course on Banach spaces that the mapping $\tau : X \to X^{**}$ defined by (2.75) above is an isometric isomorphism. The space is said to be **reflexive** if the mapping τ defined by (2.75) above is surjective. Not all Banach spaces are reflexive. However, all finite-dimensional normed linear spaces are. One of the distinguishing features of a Hilbert space is that it is reflexive. We begin by showing that H^*, the dual of a Hilbert space H, is itself a Hilbert space.

Theorem 2.10.29 *If H is a Hilbert space, then H^* is a Hilbert space. Moreover, there exists a conjugate linear map $T{:}H \to H^*$ which is one-to-one, onto, norm preserving and satisfies*

$$(f_1, f_2)_{H^*} = \left(T^{-1}f_2, T^{-1}f_1\right).$$

Proof By Theorem 2.10.23, H^* is a complete normed linear space. Consider the mapping $T{:}H \to H^*$ defined by

$$T(x)(y) = (y, x), \quad x, y \in H. \tag{2.76}$$

Note that T defined on H and given by (2.76) is conjugate linear, one-to-one, norm preserving and onto [Theorem 2.10.25]. Therefore, T^{-1} exists.

Define an inner product on H^* as follows: Given $f_1, f_2 \in H^*$, let

$$(f_1, f_2)_{H^*} = (T^{-1}f_2, T^{-1}f_1).$$

It is easy to check that this defines an inner product on H^*. It is related to the norm by $(f_1, f_1)_{H^*} = \|f_1\|^2_{H^*}$, because

$$(f_1, f_1)_{H^*} = (T^{-1}f_1, T^{-1}f_1) = \left\|T^{-1}f_1\right\|^2 = \|f_1\|^2_{H^*}.$$

Thus H^* is a Hilbert space. □

Theorem 2.10.30 *If H is a Hilbert space and $H^{**} = (H^*)^*$, then the mapping $\tau{:}\, H \to H^{**}(x \to \tau(x))$, where the defining equation for $\tau(x)$ is*

$$\tau(x)(f) = f(x) \quad f \in H^*,$$

is an isometric isomorphism between H and H^{**}. Thus H is reflexive.

Proof Let $T{:}H \to H^*$ and $S{:}\, H^* \to H^{**}$ be the conjugate linear maps assured by Theorem 2.10.29 (used twice). Since both are conjugate linear, one-to-one, norm preserving and onto, we know that the composition $ST{:}H \to H^{**}$ is linear, one-to-one, norm preserving and onto, which is to say that it is an isometric isomorphism between H^* and H^{**}. Thus, we need only to prove that

$$ST(x)(f) = f(x) \quad f \in H^*,$$

and set $\tau = ST$.

By Theorem 2.10.29, we also have

$$T(x)(y) = (y, x)_H, \quad x, y \in H, \tag{2.77}$$

$$S(g)(f) = (f, g)_{H^*}, \quad f, g \in H^*, \tag{2.78}$$

$$(f, g)_{H^*} = (T^{-1}g, T^{-1}f)_H, \quad f, g \in H^*. \tag{2.79}$$

We compute ST from here as follows:

$$\begin{aligned} ST(x)(f) &= S(Tx)(f) \\ &= (f, Tx)_{H^*} \quad \text{by (2.78)} \\ &= (T^{-1}(Tx), T^{-1}f)_H \quad \text{by (2.79)} \\ &= (x, T^{-1}f)_H = \big(T\big(T^{-1}f\big)\big)(x) \quad \text{by (2.77)} \\ &= f(x). \end{aligned}$$

□

The following result is an analogue of a familiar result from metric spaces.

Theorem 2.10.31 *In order that the linear span of a system M of vectors is dense in H, it is necessary and sufficient that a continuous linear functional $f \in H^*$ which vanishes for all $x \in M$ must be identically zero.*

Proof Necessity: suppose the linear span of M is dense in H, i.e. $[\overline{M}] = H$ and $f \in H^*$ vanishes for all $x \in M$. By linearity, f vanishes on $[M]$ and hence by continuity it vanishes on $[\overline{M}]$, which is the same as H.

Sufficiency: suppose the linear span of M is not dense in H, i.e. $[\overline{M}] \neq H$. Then there exists $y \neq 0$ such that $y \perp [\overline{M}]$. Then the linear functional f defined by $f(x) = (x, y)$ for all $x \in H$ vanishes for all $x \in M$ but is not identically zero because $f(y) = (y, y) \neq 0$. □

Problem Set 2.10

2.10.P1. Let $f \in RH^2$ have the series expansion $f = \sum_{j=0}^{\infty} a_j z^j$. Define $C_n(f) = a_n$. Show that C_n is a continuous linear functional on RH^2.

2.10.P2. Let $e_0(t) = 1$ and $e_1(t) = \sqrt{3}(2t - 1)$, $t \in [0, 1]$, be vectors in the Hilbert space $L^2[0, 1]$. Show that $e_0 \perp e_1$, $\|e_0\| = \|e_1\| = 1$. Compute the vector y in the linear span of $\{e_0, e_1\}$ closest to t^2 and also compute $\min\limits_{a,b} \int_0^1 \left|t^2 - a - bt\right|^2 dt$.

2.10.P3. Let $X = \mathbb{R}^2$. Find $M^\perp$

(a) $M = \{x\}$, where $x = (\xi_1, \xi_2) \neq 0$;
(b) M is a linearly independent set $\{x_1, x_2\} \subseteq X$.

2.10.P4. For any subset $M \neq \emptyset$ of a Hilbert space H, span(M) is dense in H if, and only if, $M^\perp = \{0\}$.

2.10.P5. (a) Prove that for any two subspaces M_1 and M_2 of a Hilbert space H, we have $(M_1 + M_2)^\perp = M_1^\perp \cap M_2^\perp$.

(b) Prove that for any two closed subspaces M_1 and M_2 of a Hilbert space H, we have

$$(M_1 \cap M_2)^\perp = \overline{M_1^\perp + M_2^\perp}.$$

2.10.P6. (a) Let K_1 and K_2 be the nonempty, closed and convex subsets of a Hilbert space H such that $K_1 \subseteq K_2$. Prove that, for all $x \in H$,

$$\|y_1 - y_2\|^2 \leq 2\left(d(x, K_1)^2 - d(x, K_2)^2\right),$$

where y_1 and y_2 are closest points of x in K_1 and K_2, respectively.

(b) Let $\{K_n\}_{n \geq 1}$ be an increasing sequence of nonempty closed convex subsets in H and let $K = \overline{\bigcup_n K_n}$. Prove that K is closed and convex. Also show that $\lim_n y_n = y$ for all $x \in H$, where y_n is the projection of x onto K_n, $n = 1, 2, \ldots$, and y is the projection of x onto K.

2.10.P7. Let M be a closed subspace of a Hilbert space H and $x_0 \in H$. Prove that $\min\{\|x_0 - x\|: x \in M\} = \max\{|(x_0, y)|: y \in M^\perp \text{ and } \|y\| = 1\}$.

2.10.P8. (a) Let a be a nonzero element of a Hilbert space H. Prove that, for all $x \in H$,

$$d(x, \{a\}^\perp) = \frac{|(x, a)|}{\|a\|}.$$

(b) Let $H = L^2[0, 1]$ and let

$$F = \left\{ f \in L^2[0,1] : \int_0^1 f(x)\mathrm{d}x = 0 \right\}.$$

Determine $F^\perp$. For $f(x) = \exp(x)$, determine $d(f, F)$.

2.10.P9. In the linear space $C[0, 1]$, consider the functional $F(x) = \int_0^1 x(t)f(t)\mathrm{d}t$, where f is a continuous function defined on $[0, 1]$. Show that $\|F\| = \int_0^1 |f(t)|\mathrm{d}t$.

2.10.P10. Let H be a Hilbert space and f be a nonzero continuous linear functional on H, i.e. $f \in H^*\backslash\{0\}$. Show that $\dim((\ker(f))^\perp) = 1$.

2.10.P11. Prove that if f is a linear functional on a Hilbert space H and $\ker(f)$ is closed, then f is bounded.

2.10.P12. Show that the subspace $M = \{x = \{x_n\}_{n \geq 1} \in \ell^2\colon \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x_n = 0\}$ is not a closed subspace of ℓ^2.

2.10.P13. Prove that the system $\sin nx$, $n = 1, 2, \ldots$, is complete in $L^2[0, \pi]$.

2.10.P14. Let K be a nonempty closed convex set in a Hilbert space H. Show that K contains a unique vector k of smallest norm and that $\Re(k, k - x) \leq 0$ for all $x \in K$. Moreover, if $k \in K$ satisfies $\Re(k, k - x) \leq 0$ for all $x \in K$, then k is the vector of smallest norm in K.

2.10.P15. Let y be a nonzero vector in a Hilbert space H and let

$$M = \{x \in H : (x, y) = 0\}.$$

What is $M^{\perp}$?

2.11 Approximation in Hilbert Spaces

Let H be a Hilbert space and $v_1, v_2, \ldots, v_k$ be linearly independent vectors in H. Suppose that $x \in H$. In linear approximation, it is required to find a method of computing the minimum value of the quantity

$$\left\| x - \sum_{j=1}^{n} \lambda_j v_j \right\|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ range over all scalars and also determine those values of $\lambda_1, \lambda_2, \ldots, \lambda_n$ for which the minimum is attained.

Let M be the closed linear space generated by linearly independent vectors $v_1, v_2, \ldots, v_n$ and $x \in H$. By Theorem 2.10.10, there exists a unique minimising vector $y \in M$ and $x - y \perp M$. Since $y \in M$, we have $(x - y, y) = 0$.

Denote $(v_j, v_i) = a_{ij}$ and $b_i = (x, v_i)$. If $y = \sum_{j=1}^{n} c_j v_j$ is the minimising vector, then

$$(x - y, v_i) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n,$$

which, written in full, reads as

$$b_i = \sum_{j=1}^{n} a_{ij} c_j, \quad i = 1, 2, \ldots, n.$$

Since the vectors $\{v_i\}$ are linearly independent, the matrix $[a_{ij}]$ is nonsingular. Consequently, the n linear equations in n unknowns c_1, c_2, …, c_n have a unique solution.

Let $\delta = \inf\left\{\left\|x - \sum_{j=1}^{n} \lambda_j v_j\right\| : \lambda_1, \lambda_2, \ldots, \lambda_n \text{ are scalars}\right\}$. Then

$$\delta^2 = \|x-y\|^2 = (x-y, x-y) = \left(x, x - \sum_{j=1}^{n} c_j v_j\right) = \|x\|^2 - \sum_{j=1}^{n} \overline{c_j} b_j \qquad (2.80)$$

If we replace v_1, v_2, …, v_n by an orthonormal set u_1, u_2, …, u_n, then $a_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. Hence, $c_j = b_j$, $j = 1, 2, \ldots, n$ and it follows from (2.80) that

$$\delta^2 = \|x\|^2 - \sum_{j=1}^{n} |b_j|^2 = \|x\|^2 - \sum_{j=1}^{n} \left|(x, u_j)\right|^2.$$

We have thus proved the following theorem.

Theorem 2.11.1 *Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal set in H and let $x \in H$. Then*

$$\left\|x - \sum_{k=1}^{n} (x, u_k) u_k\right\| \leq \left\|x - \sum_{k=1}^{n} \lambda_k u_k\right\|$$

for all scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$. Equality holds if, and only if, $\lambda_k = (x, u_k)$, $k = 1, 2, \ldots, n$. Moreover, $\sum_{k=1}^{n} \lambda_k u_k$ is the orthogonal projection of x onto the subspace M generated by $\{u_1, u_2, \ldots, u_n\}$, and if δ is the distance of x from M, then $\delta^2 = \|x\|^2 - \sum_{k=1}^{n} |(x, u_k)|^2$.

Remark 2.11.2 If the subspace M generated by $(n + 1)$ orthonormal vectors and it is desired to obtain the distance of $x \in H$ from M, then

$$\delta^2 = \|x-y\|^2 = \|x\|^2 - \sum_{k=1}^{n+1} |(x, u_k)|^2, \qquad (2.81)$$

where y is the orthogonal projection of x on M and is given by

$$y = \sum_{k=1}^{n+1} (x, u_k) u_k. \qquad (2.82)$$

The reader will notice that the first n components in the sums on the right of (2.81) and (2.82) remain unaltered when the dimension of the space is increased from n to $n + 1$. This exhibits the importance of orthonormalising the linearly independent vectors.

Example 2.11.3 Consider the real inner product space $C[-1, 1]$, the inner product (x, y), $x, y \in C[-1, 1]$ being defined by

$$(x, y) = \int_{-1}^{1} x(t)y(t)\mathrm{d}t.$$

Consider the three linearly independent vectors $1, t, t^2$ (the Wronskian of the vectors $1, t, t^2$ is $2 \neq 0$) in $C[-1, 1]$. The Gram–Schmidt orthonormalisation process yields

$$u_0(t) = \frac{1}{\sqrt{2}}, \quad u_1(t) = \sqrt{\frac{3}{2}}t, \quad u_2(t) = \sqrt{\frac{5}{2}}\frac{1}{2}\left(3t^2 - 1\right).$$

Let M_2 [respectively, M_3] be the linear space generated by $\{u_0, u_1\}$ [respectively $\{u_0, u_1, u_2\}$]. Consider $x(t) = e^t$ in $C[-1, 1]$. We shall compute the distance of x from M_2 and M_3.

$$(x, u_0) = \frac{1}{\sqrt{2}}\int_{-1}^{1} e^t\mathrm{d}t = \frac{e - e^{-1}}{\sqrt{2}},$$

$$(x, u_1) = \sqrt{\frac{3}{2}}\int_{-1}^{1} te^t\mathrm{d}t = \sqrt{6}\,\mathrm{e}^{-1}$$

$$(x, u_2) = \sqrt{\frac{5}{2}}\frac{1}{2}\int_{-1}^{1} \left(3t^2 - 1\right)e^t\mathrm{d}t = \sqrt{\frac{5}{2}}\left(e - 7e^{-1}\right).$$

Let y_2 and y_3 be the projections of $x(t) = e^t$ on the subspaces M_2 and M_3, respectively. Then

$$\begin{aligned} y_2 &= (x, u_0)u_0 + (x, u_1)u_1 \\ &= \frac{1}{\sqrt{2}}\frac{e - e^{-1}}{\sqrt{2}} + \sqrt{6}e^{-1}\sqrt{\frac{3}{2}}t \\ &= \frac{1}{2}(e - e^{-1}) + 3e^{-1}t \end{aligned}$$

and

$$\begin{aligned} y_3 &= (x, u_0)u_0 + (x, u_1)u_1 + (x, u_2)u_2 \\ &= \frac{1}{2}(e - e^{-1}) + 3e^{-1}t + \sqrt{\frac{5}{2}}(e - 7e^{-1})\sqrt{\frac{5}{2}}\frac{1}{2}\left(3t^2 - 1\right) \\ &= \frac{1}{2}(e - e^{-1}) + 3e^{-1}t + \frac{5}{4}\left(e - 7e^{-1}\right)\left(3t^2 - 1\right). \end{aligned}$$

If δ_2 [respectively, δ_3] denotes the distance of x from M_2 [respectively, M_3], then by Theorem 2.11.1,

$$\begin{aligned} \delta_2^2 &= \|x\|^2 - |(x, u_0)|^2 - |(x, u_1)|^2 \\ &= \frac{1}{2}\left(e^2 - e^{-2}\right) - \left(\frac{e - e^{-1}}{\sqrt{2}}\right)^2 - 6e^{-2} \\ &= 1 - 7e^{-2} \end{aligned}$$

and

$$\begin{aligned} \delta_3^2 &= \|x\|^2 = |(x, u_0)|^2 - |(x, u_1)|^2 - |(x, u_2)|^2 \\ &= \delta_2^2 - \frac{5}{2}\left(e - 7e^{-1}\right)^2 \\ &= 1 - 7e^{-2} - \frac{5}{2}\left(e^2 + 49e^{-2} - 14\right) \\ &= 36 - \frac{5}{2}e^2 - \frac{259}{2}e^{-2}. \end{aligned}$$

Problem Set 2.11

2.11.P1. Find $\min\limits_{a,b,c} \int_{-1}^{1} \left|t^3 - a - bt - ct^2\right|^2 \mathrm{d}t$ and $\max \int_{-1}^{1} t^3 g(t)\mathrm{d}t$, where g is subject to the restrictions

$$\int_{-1}^{1} g(t)\mathrm{d}t = \int_{-1}^{1} tg(t)\mathrm{d}t = \int_{-1}^{1} t^2 g(t)\mathrm{d}t = 0, \int_{-1}^{1} |g(t)|^2 \mathrm{d}t = 1.$$

2.11.P2. Find the point nearest to $(1, -1, 1)$ in the linear span of $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ in $\mathbb{C}^3$, where $\omega = \exp(2\pi i/3)$.

2.12 Weak Convergence

Let $\{x_n\}_{n \geq 1}$ be a sequence in a Hilbert space H. Recall that $\{x_n\}_{n \geq 1}$ converges to x in H if $\|x_n - x\| = (x_n - x, x_n - x)^{\frac{1}{2}} \to 0$ as $n \to \infty$, and we write $x_n \to x$. From now on it will be called *strong convergence* to distinguish it from *weak convergence*, to be introduced shortly. The relationship between the two types of convergence will be discussed. The concepts of strong convergence and weak convergence are identical in finite-dimensional spaces. A characterisation of weak convergence in special spaces will also find a mention below.

Definition 2.12.1 A sequence of vectors $\{x_n\}_{n \geq 1}$ **converges weakly to** a vector $\boldsymbol{x}$ and we write $x_n \rightharpoonup x$ ($x_n \xrightarrow{w} x$) if

$$\lim_{n \to \infty} (x_n, y) = (x, y)$$

for all

$$y \in H.$$

The concepts of a weakly Cauchy sequence and weak completeness are defined analogously.

Remarks 2.12.2

(i) A sequence cannot converge weakly to two different limits: assume that $x_n \xrightarrow{w} x_0$ and $x_n \xrightarrow{w} y_0$. Then

$$(x_n, y) \to (x_0, y) \text{ and } (x_n, y) \to (y_0, y)$$

for all $y \in H$. Consequently, $(x_0, y) = (y_0, y)$, or $(x_0 - y_0, y) = 0$, for all $y \in H$. If we choose $y = x_0 - y_0$, we obtain $(x_0 - y_0, x_0 - y_0) = 0$, which implies $x_0 = y_0$.

(ii) If $x_n \xrightarrow{w} x_0$, then every arbitrary subsequence $\{x_{n_k}\}_{k \geq 1}$ converges weakly to x_0.

(iii) Strong convergence of $\{x_n\}_{n \geq 1}$ to x_0 implies $x_n \xrightarrow{w} x_0$. Indeed, for $y \in H$, we have

$$|(x_n - x_0, y)| \leq \|x_n - x_0\| \|y\|,$$

by the Cauchy–Schwarz Inequality.

(iv) The converse of (iii) is, however, not true. Indeed, let $\{e_n\}_{n \geq 1}$ be an infinite orthonormal sequence of vectors in H. Since for any $y \in H$,

$$\sum_{n=1}^{\infty} |(y, e_n)|^2 \le \|y\|^2 \quad \text{(by Bessel's Inequality)},$$

therefore, $\lim_{n\to\infty}(e_n, y) = 0$. Thus, the sequence $\{e_n\}_{n\geq 1}$ converges weakly to the vector zero, but this sequence cannot converge strongly, since

$$\left\|e_i - e_j\right\|^2 = 2 \quad (i \neq j),$$

so that $\|e_i - e_j\| \nrightarrow 0$ as $i, j \to \infty$.

However, the following theorem holds:

Theorem 2.12.3 *If H is a finite-dimensional Hilbert space, strong convergence is equivalent to weak convergence.*

Proof Since we have already shown that, in any Hilbert space, strong convergence implies weak convergence [Remark 2.12.2(iii)], it is enough to show in this situation that weak convergence implies strong convergence. To this end, let $e_1, \ldots, e_k$ be an orthonormal basis for H and let

$$x_n \xrightarrow{w} x,$$

where

$$x_n = \alpha_1^{(n)} e_1 + \cdots + \alpha_k^{(n)} e_k$$

for $n = 1, \ldots, k$, and

$$x = \alpha_1 e_1 + \cdots + \alpha_k e_k.$$

Since $x_n \xrightarrow{w} x$, it follows that

$$\left(x_n, e_j\right) \to \left(x, e_j\right), \quad \text{i.e., } \alpha_j^{(n)} \to \alpha_j$$

for $j = 1, \ldots, k$. For any prescribed $\varepsilon > 0$, there must be an integer n_0 such that for all $n > n_0$ and for every $j = 1, \ldots, k$,

$$\left|\alpha_j^{(n)} - \alpha_j\right| < \varepsilon/k;$$

hence

$$\|x_n - x\|^2 = \left\|\sum_{j=1}^{k}\left(\alpha_j^{(n)} - \alpha_j\right)e_j\right\|^2 = \sum_{j=1}^{k}\left|\alpha_j^{(n)} - \alpha_j\right|^2 < \varepsilon.$$

Thus, $x_n \to x$ strongly. This completes the proof. □

The next result pinpoints the relationship between weak and strong convergences.

Theorem 2.12.4 *Let* $\{x_n\}_{n\geq 1}$ *be a sequence in a Hilbert space* H. *Then* $x_n \to x$ *if, and only if,* $x_n \xrightarrow{w} x$ *and* $\limsup_{n\to\infty}\|x_n\| \leq \|x\|$.

Proof Let $x_n \to x$. Then $x_n \xrightarrow{w} x$ [Remark 2.12.2(iii)]. Also, $\limsup_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|x_n\| = \|x\|$, since $0 \geq \|x_n - x\| \geq |\|x_n\| - \|x\||$.

Conversely, let $x_n \xrightarrow{w} x$ and $\limsup_{n\to\infty}\|x_n\| \leq \|x\|$. For each n, $0 \leq \|x_n - x\|^2 = (x_n - x, x_n - x) = \|x_n\|^2 + \|x\|^2 - 2\Re(x_n, x)$. Since

$$\limsup_{n\to\infty}\|x_n\| \leq \|x\| \text{and } \Re(x_n, x) \to \Re(x, x) = \|x\|^2,$$

we have

$$0 \leq \limsup_{n\to\infty}\|x_n - x\| \leq \|x\|^2 + \|x\|^2 - 2\|x\|^2 = 0,$$

so that $\lim_{n\to\infty}\|x_n - x\|$ exists and equals zero. □

The Riesz Representation Theorem enables us to prove the following analogue of the classical Bolzano–Weierstrass Theorem.

Theorem 2.12.5 *Any bounded sequence in* H *has a weakly convergent subsequence and the limit has the same bound.*

Proof Let $\{x_n\}_{n\geq 1}$ be a sequence in H and an $M > 0$ be such that $\|x_n\| \leq M$ for all n. We need to find a weakly convergent subsequence of $\{x_n\}_{n\geq 1}$.

By the Cauchy–Schwarz Inequality,

$$|(x_n, x_1)| \leq \|x_n\|\|x_1\| \leq M^2$$

for all n. The classical Bolzano–Weierstrass Theorem shows that the bounded sequence $\{(x_n, x_1)\}_{n\geq 1}$ has a convergent subsequence $\{(x_{n(1)}, x_1)\}_{n(1)\geq 1}$, say. Applying the preceding argument to the sequence $\{(x_{n(1)}, x_2)\}_{n(1)\geq 1}$, we extract a convergent subsequence $\{(x_{n(2)}, x_2)\}_{n(2)\geq 1}$.

Continuing inductively, we obtain for each k a convergent subsequence $\{(x_{n(k)}, x_k)\}_{n(k)\geq 1}$ of $\{(x_{n(k-1)}, x_k)\}_{n(k-1)\geq 1}$.

Consider now the diagonal sequence $\{z_p\}_{p\geq 1}$, where z_p denotes the pth term of the sequence $\{x_{n(p)}\}_{n(p)\geq 1}$. We show that for each $x \in H$, the sequence $\{(z_p, x)\}_{p\geq 1}$ of scalars converges.

If $x = x_m$ for some m, then for each $p > m$, $\{(z_p, x)\}_{p\geq 1}$ is a subsequence of the convergent sequence $\{(z_p, x_m)\}_{p\geq 1}$ and is, therefore, convergent. Hence, if $x \in$ span $\{x_1, x_2, \ldots\}$, then $\{(z_p, x)\}_{p\geq 1}$ converges in the field of scalars.

Let $x \in \overline{\text{span}\{x_1, x_2, \ldots\}}$. Consider a sequence $\{y_r\}_{r\geq 1}$ in span$\{x_1, x_2, \ldots\}$ such that $y_r \to x$ as $r \to \infty$. Then for all n, m and r, we have

$$\begin{aligned}|(z_n, x) - (z_m, x)| &= |(z_n - z_m, x)| \\ &\leq |(z_n - z_m, x - y_r)| + |(z_n - z_m, y_r)| \\ &\leq \|z_n - z_m\|\|x - y_r\| + |(z_n - z_m, y_r)| \\ &\leq 2M\|x - y_r\| + |(z_n - z_m, y_r)|.\end{aligned}$$

Since $\|x - y_r\| \to 0$ as $r \to \infty$ and $|(z_n - z_m, y_r)| \to 0$ as $n, m \to \infty$ for each r, we see that $\{(z_n, x)\}_{n\geq 1}$ is a Cauchy sequence of scalars and is, therefore, convergent. Next, let $x \perp \overline{\text{span}\{x_1, x_2, \ldots\}}$. Then $(z_n, x) = 0$ for all n, since z_n is in span$\{x_1, x_2, \ldots\}$. Thus $(z_n, x) \to 0$ as $n \to \infty$.

By the Orthogonal Decomposition Theorem 2.10.11,

$$H = \overline{\text{span}\{x_1, x_2, \ldots\}} \oplus \overline{\text{span}\{x_1, x_2, \ldots\}}^{\perp}.$$

Hence, $\{(z_n, x)\}_{n\geq 1}$ converges for each $x \in H$. Define

$$f(x) = \lim_{n\to\infty} (x, z_n), \quad x \in H. \tag{2.83}$$

Clearly, f is linear and

$$|f(x)| = \lim_{n\to\infty} |(x, z_n)| \leq M\|x\|$$

for all $x \in H$. Thus, f is a continuous linear functional on H satisfying $\|f\| \leq M$. By the Riesz Representation Theorem 2.10.25, there exists a unique $y \in H$ such that

$$f(x) = (x, y), \quad x \in H. \tag{2.84}$$

and $\|y\| = \|f\| \leq M$. On comparing (2.83) and (2.84), we obtain

$$\lim_{n\to\infty} z_n = y \text{ (weak)}.$$

This completes the proof. □

Every convergent sequence in a normed linear space X is bounded. This is easily seen as follows: let $\{x_n\}_{n\geq 1}$ be a sequence in X and suppose that $\lim_{n\to\infty} x_n = x$. For a given $\varepsilon > 0$, there exists an integer n_0 such that $n \geq n_0$ implies $\|x_n - x\| < \varepsilon$. But since $\|x_n\| - \|x\| \leq \|x_n - x\|$, this implies $\|x_n\| < \varepsilon + \|x\|$, $n \geq n_0$. It now follows that

$$\|x_n\| < \varepsilon + \|x\| + M,$$

where $M = \max\{\|x_k\|: 1 \leq k \leq n_0\}$.

Thus, the terms of a convergent sequence in a normed linear space, *a fortiori*, in a Hilbert space are bounded. The foregoing statement is true for a weakly convergent sequence.

Theorem 2.12.6 *If H is a Hilbert space and $x_n \xrightarrow{w} x$, then there exists a positive constant M such that*

$$\|x_n\| \leq M.$$

We discuss preliminary results needed for the proof of Theorem 2.12.6.

Definition 2.12.7 A real functional $p(x)$ in H is said to be **convex** if for all $x, y \in H$ and $\alpha \in \mathbb{C}$, the following hold:

$$p(x+y) \leq p(x) + p(y) \quad \text{and} \quad p(\alpha x) = |\alpha| p(x).$$

Observe that (i) $p(0) = 0$ (ii) $p(x - y) \geq |p(x) - p(y)|$ and $p(x) \geq 0$, where $x, y \in H$. Indeed, $p(0) = p(0 \cdot x) = 0 \cdot p(x) = 0$. Also, $p(x - y) + p(y) \geq p(x)$ and hence $p(x - y) \geq p(x) - p(y)$. Since $p(x - y) = |-1|\, p(y - x) \geq p(y) - p(x)$, it follows that $p(x - y) \geq |p(x) - p(y)|$. On setting $y = -x$, we obtain $p(2x) = 2p(x) \geq |p(x) - p(-x)| = 0$.

That a lower semi-continuous convex functional in a Hilbert space is bounded is the content of the Lemma below. In conjunction with the observation above, it will further follow that it is uniformly continuous.

Lemma 2.12.8 *Suppose $p(x)$ is a convex functional in a Hilbert space H and assume that $p(x)$ is lower semi-continuous. Then there exists $M > 0$ such that*

$$p(x) \leq M\|x\| \quad \text{for all} \quad x \in H.$$

Proof We first show that the functional $p(x)$ is bounded in the ball $S(0,1)$. We assume the contrary. Then $p(x)$ is unbounded in every ball, because every ball is obtained by dilation and/or translation of the ball $S(0,1)$. We choose a point $x_1 \in S(0,1)$ such that $p(x_1) > 1$. The lower semi-continuity of the functional $p(x)$ implies that there exists a ball $S(x_1,\rho_1) \subseteq S(0,1)$ with radius $\rho_1 < \frac{1}{2}$ in which $p(x) > 1$. By reducing the radius ρ_1, we may assume that $\overline{S}(x_1, \rho_1) \subseteq S(0,1)$. Since $p(x)$ is unbounded in every ball, in a similar manner, we obtain a point $x_2 \in S(x_1, \rho_1)$ and also a closed ball $\overline{S}(x_2, \rho_2) \subseteq S(x_1, \rho_1)$ with radius $\rho_2 < \frac{1}{2}\rho_1$, in which $p(x) > 2$. Continuing the process, we obtain an infinite sequence of balls

$$S(0,1) \supseteq \overline{S}(x_1,\rho_1) \supseteq \overline{S}(x_2,\rho_2) \supseteq \ldots,$$

for which $\rho_k < \frac{1}{2}\rho_{k-1}$ (k = 1, 2, … and $\rho_0 = 1$) and also $p(x) > n$ if $x \in \overline{S}(x_n, \rho_n)$. Observe that the sequence $\{x_n\}_{n\geq 1}$ of the centres of the balls $\overline{S}(x_n,\rho_n)$, n = 1, 2, … is Cauchy and since H is complete, $\lim_{n\to\infty}x_n$ exists and equals x, say. Then x lies in the intersection of the closed balls, and hence $p(x) > n$ for each n, which is a contradiction.

Let $x \in H$ be arbitrary. Then $x/2\|x\|$ is an element of H of norm $\frac{1}{2}$ and is, therefore, in $S(0,1)$. Now,

$$p(x/2\|x\|) \leq M_1,$$

where M_1 is an upper bound of p on $S(0,1)$, i.e. $p(x) \leq 2M_1\|x\|$. Take $M = 2M_1$. □

Corollary 2.12.9 *Let $p_k(x)$, k = 1, 2, … be a sequence of convex continuous functionals in H. If this sequence is bounded at each point $x \in H$, then the functional*

$$p(x) = \sup_k p_k(x)$$

is also convex and bounded, and hence continuous.

Proof Evidently, $p(x)$ is a convex functional. On the other hand, for each $x_0 \in H$ and each $\varepsilon > 0$, there exists N such that

$$p_N(x_0) > p(x_0) - \frac{1}{2}\varepsilon,$$

i.e.

$$p(x_0) - p_N(x_0) < \frac{1}{2}\varepsilon.$$

By continuity of the functional $p_N(x)$, there exists $\delta > 0$ such that

$$|p_N(x) - p_N(x_0)| < \frac{1}{2}\varepsilon$$

for $\|x - x_0\| < \delta$. But if $\|x - x_0\| < \delta$, then

$$\begin{aligned} p(x) - p(x_0) &> \sup_k p_k(x) - p_N(x_0) - \frac{1}{2}\varepsilon \\ &\geq p_N(x) - p_N(x_0) - \frac{1}{2}\varepsilon > -\varepsilon. \end{aligned}$$

This implies that the functional $p(x)$ is lower semi-continuous. By Lemma 2.12.8, it follows that $p(x)$ is bounded. Continuity now follows from the observation preceding the lemma. □

Every weakly convergent sequence of vectors in a Hilbert space is bounded. This is an immediate consequence of the following theorem.

Theorem 2.12.10 *Let $\{\Phi_k\}_{k\geq 1}$ be a sequence of continuous linear functionals defined on the Hilbert space H. Suppose that the numerical sequence $\{\Phi_k(x)\}_{k\geq 1}$ is bounded for each $x \in H$. Then the sequence $\{\|\Phi_k\|\}_{k\geq 1}$ of norms of the functionals is bounded.*

Proof For $x \in H$, define

$$P_k(x) = |\Phi_k(x)|, \quad k = 1, 2, \ldots.$$

Then $\{p_k\}_{k\geq 1}$ is convex and continuous. By Corollary 2.12.9, the convex functional

$$p(x) = \sup_k p_k(x)$$

is convex and bounded; i.e. there exists $M > 0$ such that

$$\sup_{\|x\|\leq 1} p(x) \leq M.$$

Consequently,

$$\begin{aligned}\|\Phi_k\| &= \sup_{\|x\|\leq 1} |\Phi_k(x)| \\ &= \sup_{\|x\|\leq 1} p_k(x) \\ &\leq \sup_{\|x\|\leq 1} \sup_k p_k(x) \\ &= \sup_{\|x\|\leq 1} p(x) \leq M.\end{aligned}$$

This completes the proof. □

Proof of Theorem 2.12.6 Let $\{x_n\}_{n\geq 1}$ be a weakly convergent sequence. Each vector x_n determines a functional $\Phi_n(x) = (x, x_n)$. Since the sequence $\{x_n\}_{n\geq 1}$ is weakly convergent, the numerical sequence $\{\Phi_n(x)\}_{n\geq 1}$ converges for each $x \in H$ and hence is bounded. Using Theorem 2.12.10, it follows that

$$\|\Phi_n\| \leq M, \quad n = 1, 2, \ldots.$$

As $\|\Phi_n\| = \|x_n\|$, $n = 1, 2, \ldots$ [Example 2.10.24(iv)], the result follows.

Definition 2.12.11 A sequence of vectors $\{x_n\}_{n\geq 1}$ in an inner product space is said to be **weakly Cauchy** if, for each $y \in H$,

$$\lim_{m,n}(x_m - x_n, y) = 0.$$

An inner product space is said to be **weakly complete** if every weakly Cauchy sequence converges to a weak limit in H.

Corollary 2.12.12 *Let H be a Hilbert space. Then H is weakly complete.*

Proof Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in the sense of weak convergence, that is, for each $y \in H$,

$$\lim_{m,n}(x_m - x_n, y) = 0.$$

It follows that the sequence $\{(x_n, y)\}_{n\geq 1}$ of scalars converges for each y in H. By Theorem 2.12.10, the sequence $\{x_n\}_{n\geq 1}$ is bounded:

$$\|x_n\| \leq M, \quad n = 1, 2, \ldots.$$

Therefore, the limit

$$\lim_{n\to\infty}(x, x_n)$$

defines a linear functional $\Phi(x)$ with norm less than or equal to M. By the Riesz Representation Theorem 2.10.25, $\Phi(x) = (x, z)$, where z is a unique element of the Hilbert space H. This element is the weak limit of the sequence $\{x_n\}_{n\geq 1}$. □

We give below two applications of Corollary 2.12.9.

Theorem 2.12.13 (F. Riesz) *If a functional Φ is defined everywhere on $L^2[a, b]$ by the formula*

$$\Phi(x) = \int_a^b x(t)y(t)\mathrm{d}t, \quad x \in L^2[a, b],$$

where y is a fixed measurable function defined on $[a, b]$, then Φ is a bounded linear functional on $L^2[a, b]$, so that $y \in L^2[a, b]$.

Proof Clearly, Φ is a linear functional on $L^2[a, b]$. Set

$$E_n = \{t : t \in [a, b] \cap [-n, n] \text{ and } |y(t)| \leq n\}$$

and

$$p_n(x) = \int_{E_n} |x(t)y(t)|\mathrm{d}t, \quad x \in L^2[a,b].$$

Then $\{p_n\}_{n \geq 1}$ is a sequence of convex functionals: indeed, for x, $z \in L^2[a, b]$ and $\alpha \in \mathbb{C}$,

$$\begin{aligned} p_n(x+z) &= \int_{E_n} |[x(t)+z(t)]y(t)|\mathrm{d}t \\ &\leq \int_{E_n} |x(t)y(t)|\mathrm{d}t + \int_{E_n} |z(t)y(t)|\mathrm{d}t \\ &= p_n(x) + p_n(z) \end{aligned}$$

and

$$p_n(\alpha x) = \int_{E_n} |\alpha x(t)y(t)|\mathrm{d}t = |\alpha| \int_{E_n} |x(t)y(t)|\mathrm{d}t = |\alpha| p_n(x).$$

Moreover,

$$\begin{aligned} p_n(x) &\leq n \int_{E_n} |x(t)|\mathrm{d}t \leq n \left(\int_{E_n} |x(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{E_n} \mathrm{d}t \right)^{\frac{1}{2}} \\ &= n\mu(E_n)^{\frac{1}{2}} \|x\|_2, \end{aligned}$$

using the Cauchy–Schwarz Inequality, where μ denotes the usual Lebesgue measure.

Thus, for $n = 1, 2, \ldots, p_n$ is a continuous convex functional on $L^2[a, b]$. The equality

$$p(x) = \lim_n p_n(x) = \lim_n \int_{E_n} |x(t)y(t)|\mathrm{d}t = \int_a^b |x(t)y(t)|\mathrm{d}t,$$

using the Monotone Convergence Theorem 1.3.6 shows that $p(x)$ is finite for any x in $L^2[a, b]$. By Corollary 2.12.9, the functional $p(x)$ is bounded; i.e. there exists $M > 0$ such that

$$p(x) \leq M\|x\|, \quad x \in L^2[a,b].$$

Thus

$$|\Phi(x)| \leq p(x) \leq M\|x\|, \quad x \in L^2[a,b],$$

i.e. Φ is a bounded linear functional on $L^2[a, b]$; so $y \in L^2[a, b]$ and $\|y\|_2 = \|\Phi\|$, using the definition of Φ and the Riesz Representation Theorem. □

Theorem 2.12.14 (Landau) *If Φ is a functional defined everywhere in ℓ^2 by means of the formula*

$$\Phi(x) = \sum_{k=1}^{\infty} a_k x_k, \quad x = \{x_k\}_{k\geq 1} \in \ell^2,$$

where $\{a_k\}_{k\geq 1}$ is some fixed sequence, then $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

Proof Define $p_n(x) = \sum_{k=1}^{n} |a_k x_k|, x = \{x_k\}_{k\geq 1} \in \ell^2$. Check that p_n, $n = 1, 2, \ldots$, is a continuous convex functional. Then the equality

$$p(x) = \lim_n p_n(x) = \lim_n \sum_{k=1}^{n} |a_k x_k| = \sum_{k=1}^{\infty} |a_k x_k|$$

implies that $p(x)$ is finite for any $x \in \ell^2$. So, by Corollary 2.12.9, the functional $p(x)$ is continuous; i.e. there exists $M > 0$ such that $p(x) \leq M\|x\||$, $x \in \ell^2$. Consequently,

$$|\Phi(x)| \leq \sum_{k=1}^{\infty} |a_k x_k| = p(x) \leq M\|x\|.$$

So, Φ is a bounded linear functional on ℓ^2. The form of Φ and the Riesz Representation Theorem imply that $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. □

Remark 2.12.15 Landau's Theorem may also be stated as follows: if $\sum_{k=1}^{\infty} a_k x_k$ converges for every $\{x_k\}_{k\geq 1}$ in ℓ^2, then $\sum_{k=1}^{\infty} |a_k|^2 < \infty$

Problem Set 2.12

2.12.P1. Show that for a sequence $\{x_n\}_{n\geq 1}$ in an inner product space X and $x \in X$, the conditions

$$\text{(i)} \;\; ||x_n|| \to ||x|| \quad \text{and} \quad \text{(ii)} \;\; (x_n, x) \to (x, x),$$

imply $x_n \to x$ in X.

2.12.P2. (**Banach–Saks**) Let $\{x_n\}_{n\geq 1}$ be a sequence in a Hilbert space converging weakly to $x \in H$. Prove that there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ such that the sequence $\{y_k\}_{k\geq 1}$ defined by

$$y_k = \frac{1}{k}(x_{n_1} + x_{n_2} + \cdots + x_{n_k})$$

converges strongly to x.

2.12.P3. (a) (**Mazur's Theorem**) Let $\{x_n\}_{n\geq 1}$ be a weakly convergent sequence in a Hilbert space H and let x be its weak limit. Prove that x lies in the closed convex hull of the range $\{x_n\colon n \geq 1\}$ of the sequence.

(b) Let C be a convex subset of a Hilbert space H. Prove that C is closed if, and only if, it contains the weak limit of every sequence of points in it.

2.12.P4. (a) Let H be a separable Hilbert space and let $\{e_n\}_{n\geq 1}$ be an orthonormal basis for H. Let $B = \{x \in H\colon \|x\| \leq 1\}$. For $x, y \in H$, let

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|(x-y, e_n)| \tag{2.85}$$

Show that d is a metric on B.

(b) Show that the topology generated by d is the same as the one given by the weak topology, i.e. $d(x_k, x) \to 0$ if, and only if, $x_k \xrightarrow{w} x$.

(c) Show that the metric space (B, d) is compact.

2.13 Applications

Müntz's Theorem

Weierstrass's Theorem for $C[0, 1]$ says, in effect, that all linear combinations of the functions

$$1, x, x^2, \ldots, x^n, \ldots \tag{2.86}$$

are dense in $C[0, 1]$. Instead of working with all positive powers of x, let us permit gaps to occur, and consider the infinite set of functions

$$1, x^{n_1}, x^{n_2}, \ldots, x^{n_k}, \ldots, \tag{2.87}$$

where n_k are positive integers satisfying $n_1 < n_2 < \cdots < n_k \cdots$. The result we shall prove is called Müntz's Theorem and asserts that the linear combinations of the functions (2.87) are dense in $C[0, 1]$ and hence in $L^2[0, 1]$ if, and only if, the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ diverges. The following will be needed in Sect. 2.13.

Definition Let $x_1, x_2, \ldots, x_n$ be any vectors in an inner product space X. Then the $n \times n$ matrix $G(x_1, x_2, \ldots, x_n)$ whose (i, j)th entry is (x_i, x_j), where $(\cdot,\cdot)$ is the inner product in X, is called the **Gram matrix** of the given finite sequence of vectors.

Its determinant is called their **Gram determinant**.

Proposition *The Gram matrix $G(x_1, x_2, \ldots, x_n)$ is nonsingular if, and only if, the vectors $x_1, x_2, \ldots, x_n$ are linearly independent.*

Proof Observe that for the given G, and any n-tuple of scalars $x = (\xi_1, \xi_2, \ldots, \xi_n)$, we have

$$xG = [\xi_1, \xi_2, \ldots, \xi_n] \begin{bmatrix} (x_1, x_1) & (x_1, x_2) & \cdots & (x_1, x_n) \\ \vdots & \vdots & \vdots & \vdots \\ (x_n, x_1) & (x_n, x_2) & \cdots & (x_n, x_n) \end{bmatrix}$$
$$= \left[\sum_{i=1}^{n} \xi_i (x_i, x_1) \quad \sum_{i=1}^{n} \xi_i (x_i, x_2) \quad \ldots \quad \sum_{i=1}^{n} \xi_i (x_i, x_n) \right].$$

So,

$$xGx^* = \left[\sum_{i=1}^{n} \xi_i (x_i, x_1) \quad \sum_{i=1}^{n} \xi_i (x_i, x_2) \quad \ldots \quad \sum_{i=1}^{n} \xi_i (x_i, x_n) \right] \begin{bmatrix} \bar{\xi}_1 \\ \vdots \\ \bar{\xi}_n \end{bmatrix}$$
$$= \sum_{i,j=1}^{n} (x_i, x_j) \xi_i \bar{\xi}_j = \left(\sum_{i=1}^{n} \xi_i x_i, \sum_{i=1}^{n} \xi_i x_i \right) = \left\| \sum_{i=1}^{n} \xi_i x_i \right\|^2$$

From the above equality, the result follows. □

Corollary *A necessary and sufficient condition for the vectors $x_1, x_2, \ldots, x_n$ to be linearly dependent is that*

$$\det G(x_1, x_2, \ldots, x_n) = 0.$$

Let M be the closed subspace generated by $x_1, x_2, \ldots, x_n$. Then H can be written as $M \oplus M^{\perp}$. If $y \in H$, then $y = z + w$, where $z \in M$ and $w \in M^{\perp}$, so that $y - z \in M^{\perp}$ [see Remark 2.10.12(i)]. The minimum distance δ from to the subspace M is $\delta = \|y - z\|$, where

$$z = \sum_{i=1}^{n} a_i x_i$$

[Theorem 2.10.10]. We wish to calculate the coefficients a_i, $i = 1, 2, \ldots, n$ and the minimal distance δ.

Since $y - z \perp x_j, j = 1, 2, \ldots, n$, we obtain a system of equations

$$\left(y - \sum_{i=1}^{n} a_i x_i, x_j \right) = 0, \quad j = 1, 2, \ldots, n,$$

which, when written in full, have the form

$$\left.\begin{array}{l} a_1(x_1, x_1) + a_2(x_2, x_1) + \cdots a_n(x_n, x_1) = (y, x_1) \\ a_1(x_1, x_2) + a_2(x_2, x_2) + \cdots a_n(x_n, x_2) = (y, x_2) \\ \qquad\qquad\qquad\qquad \cdots \\ a_1(x_1, x_n) + a_2(x_2, x_n) + \cdots a_n(x_n, x_n) = (y, x_n) \end{array}\right\} \tag{2.88}$$

and represent a system of equations in the unknowns a_i, $i = 1, 2, \ldots, n$. The matrix of its coefficients is precisely the transpose of the Gram matrix $G(x_1, x_2, \ldots, x_n)$. Since the vectors $x_1, x_2, \ldots, x_n$ are linearly independent, the matrix is nonsingular by Proposition 4.2.2 and the system has one and only one solution. Moreover, by Cramer' Rule, the unique solution is given by

$$a_i = \det G^{(i)} / \det G, \quad j = 1, 2, \ldots, n,$$

where $G^{(i)}$ is obtained from G by replacing its ith column by the column of constants (y, x_i).

Now,

$$\begin{aligned} \delta^2 &= \|y - z\|^2 = (y - z, y - z) \\ &= (y, y - z) \\ &= \|y\|^2 - \left(y, \sum_{i=1}^{n} a_i x_i \right), \end{aligned}$$

so that

$$\left(\sum_{i=1}^{n} a_i x_i, y \right) = \|y\|^2 - \delta^2. \tag{2.89}$$

We combine Eq. (2.89) with the system of Eq. (2.88) and write them in the form

$$\left.\begin{array}{c} a_1(x_1,x_1)+a_2(x_2,x_1)+\cdots a_n(x_n,x_1)-(y,x_1)=0 \\ a_1(x_1,x_2)+a_2(x_2,x_2)+\cdots a_n(x_n,x_2)-(y,x_2)=0 \\ \cdots \\ a_1(x_1,x_n)+a_2(x_2,x_n)+\cdots a_n(x_n,x_n)-(y,x_n)=0 \\ a_1(x_1,y)+a_2(x_2,y)+\cdots a_n(x_n,y)+\delta^2-(y,y)=0 \end{array}\right\}. \tag{2.90}$$

If we introduce a dummy value $a_{n+1} = 1$ as a coefficient of the elements of the last column, the (2.90) becomes a system of $n + 1$ homogeneous linear equations in the $n + 1$ variables $a_1, a_2, \ldots, a_n, a_{n+1}$ (= 1). The system (2.90) will possess a nontrivial solution if the determinant of the system vanishes, i.e.

$$\det\begin{bmatrix} (x_1,x_1) & (x_2,x_1) & \cdots & (x_n,x_1) & -(y,x_1) \\ (x_1,x_1) & (x_2,x_1) & \cdots & (x_n,x_2) & -(y,x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (x_1,y) & (x_2,y) & \cdots & (x_n,y) & \delta^2-(y,y) \end{bmatrix}=0.$$

This gives

$$\delta^2=\frac{\det G(x_1,x_2,\ldots,x_n,y)}{\det G(x_1,x_2,\ldots,x_n)}.$$

Apart from the above observations, the following lemmas will be needed in the proof of Müntz's Theorem.

Lemma *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive real numbers and A be the matrix whose (i, j) th entry is $a_{ij}=\frac{1}{\lambda_i+\lambda_j}$. Then*

$$\det A=2^{-n}\prod_{j=1}^{n}\frac{1}{\lambda_j}\prod_{1\le j<k\le n}\left(\frac{\lambda_j-\lambda_k}{\lambda_j+\lambda_k}\right)^2.$$

Proof If A is a 1×1 matrix, then $\det A=\frac{1}{2\lambda_1}=2^{-1}\prod_{j=1}^{n}\frac{1}{\lambda_j}$. Thus, the assertion is true for $n = 1$. Assume that the assertion is true for m; i.e. if A is an $m \times m$ matrix, then

$$\det A=2^{-m}\prod_{j=1}^{m}\frac{1}{\lambda_j}\prod_{1\le j<k\le m}\left(\frac{\lambda_j-\lambda_k}{\lambda_j+\lambda_k}\right)^2.$$

Consider the $(m + 1) \times (m + 1)$ matrix whose (i, j)th entry is $a_{ij}=\frac{1}{\lambda_i+\lambda_j}$. Its determinant, when written in full, takes the form

$$\begin{vmatrix} \frac{1}{\lambda_1+\lambda_1} & \frac{1}{\lambda_1+\lambda_2} & \cdots & \frac{1}{\lambda_1+\lambda_m} & \frac{1}{\lambda_1+\lambda_{m+1}} \\ \frac{1}{\lambda_2+\lambda_1} & \frac{1}{\lambda_2+\lambda_2} & \cdots & \frac{1}{\lambda_2+\lambda_m} & \frac{1}{\lambda_2+\lambda_{m+1}} \\ \vdots & & \ddots & & \vdots \\ \frac{1}{\lambda_{m+1}+\lambda_1} & \frac{1}{\lambda_{m+1}+\lambda_2} & \cdots & \frac{1}{\lambda_{m+1}+\lambda_m} & \frac{1}{\lambda_{m+1}+\lambda_{m+1}} \end{vmatrix}.$$

By subtracting the last row from each of the others, removing common factors, subtracting the last column from each of the others and again removing the common factors, we obtain

$$2^{-1}\frac{1}{\lambda_{m+1}}\frac{\prod_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^2}{\prod_{i=1}^{m}(\lambda_{m+1}+\lambda_i)^2}\begin{vmatrix} \frac{1}{\lambda_1+\lambda_1} & \frac{1}{\lambda_1+\lambda_2} & \cdots & \frac{1}{\lambda_1+\lambda_m} & 1 \\ \frac{1}{\lambda_2+\lambda_1} & \frac{1}{\lambda_2+\lambda_2} & \cdots & \frac{1}{\lambda_2+\lambda_m} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\lambda_{m+1}+\lambda_1} & \frac{1}{\lambda_{m+1}+\lambda_2} & \cdots & \frac{1}{\lambda_{m+1}+\lambda_m} & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

On expanding the determinant by the last row, we have

$$\det A = 2^{-m-1}\prod_{j=1}^{m+1}\frac{1}{\lambda_j}\prod_{1\le j<k\le m+1}\left(\frac{\lambda_j-\lambda_k}{\lambda_j+\lambda_k}\right)^2.$$

By induction, the result follows. □

Lemma *Let* $1, t^{n_1}, t^{n_2}, \ldots, 1 \le n_1 < n_2 < \cdots$ *be a set of functions defined on* $[0, 1]$. *The sequence* $\{t^{n_k}\}_{k\ge 1}$ *is total (finite linear combinations are dense) in* $C[0, 1]$, *if, and only if, it is complete in* $L^2[0, 1]$.

Proof Let $x \in C[0, 1]$. The inequality

$$\left[\int_0^1\left|x(t)-\sum_{i=1}^{k}a_it^{n_i}\right|^2\mathrm{d}t\right]^{\frac{1}{2}} \le \max_{0\le t\le 1}\left|x(t)-\sum_{i=1}^{k}a_it^{n_i}\right| \tag{2.91}$$

shows that the sequence is complete in $L^2[0, 1]$ if it is total in $C[0, 1]$.

Conversely, suppose that the sequence $\{t^{n_k}\}_{k\ge 1}$ is complete in $L^2[0, 1]$. In order to show that the finite linear combinations constitute a dense subset of $C[0, 1]$, it is enough to show that the inequality (2.91) in the reverse direction holds for the functions t^m, $m = 1, 2, \ldots$. Now

$$\begin{aligned}\left|t^m - \sum_{i=1}^{k} a_i t^{n_i}\right| &= m\left|\int_0^t \left(t^{m-1} - \sum_{i=1}^{k} b_i t^{n_i-1}\right)\mathrm{d}t\right| \\ &\leq m\int_0^1 \left|t^{m-1} - \sum_{i=1}^{k} b_i t^{n_i-1}\right|\mathrm{d}t \\ &\leq m\left(\int_0^1 \left|t^{m-1} - \sum_{i=1}^{k} b_i t^{n_i-1}\right|\mathrm{d}t\right)^{\frac{1}{2}},\end{aligned} \tag{2.92}$$

using the Cauchy–Schwarz Inequality. The above inequality proves the assertion. □

Remarks

(i) The function 1 must be added in the case of $C[0, 1]$ but is redundant in $L^2[0, 1]$. Indeed, if the function 1 is missing from $\{t^{n_k}\}_{k\geq 1}$, then the polynomial $\sum_{i=1}^{k} a_i t^{n_i}$ is itself zero at $t = 0$ and cannot, therefore, approximate the continuous function $x(t)$ for which $x(0) \neq 0$.

(ii) Since $(x^p, x^q) = \int_0^1 t^{p+q}\mathrm{d}t = \frac{1}{p+q+1}$, it follows that

$$\begin{aligned}\det G(t^{n_1}, t^{n_2}, \ldots, t^{n_k}) &= \begin{vmatrix} \frac{1}{n_1+n_1+1} & \frac{1}{n_1+n_2+1} & \cdots & \frac{1}{n_1+n_k+1} \\ \frac{1}{n_2+n_1+1} & \frac{1}{n_2+n_2+1} & \cdots & \frac{1}{n_2+n_k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n_k+n_2+1} & \frac{1}{n_k+n_2+1} & \cdots & \frac{1}{n_k+n_k+1} \end{vmatrix} \\ &= \frac{\prod_{i>j}(n_i - n_j)^2}{\prod_{i,j}(n_i + n_j + 1)}\end{aligned}$$

and analogously,

$$\det G(t^m, t^{n_1}, t^{n_2}, \ldots, t^{n_k}) = \frac{\prod_{i>j}(n_i - n_j)^2}{\prod_{i,j}(n_i + n_j + 1)} \cdot \frac{\prod_{i=1}^{k}(m - n_i)^2}{\prod_{i=1}^{k}(m + n_i + 1)^2} \cdot \frac{1}{2m+1}.$$

From this, it follows that

$$\frac{\det G(t^m, t^{n_1}, t^{n_2}, \ldots, t^{n_k})}{\det G(t^{n_1}, t^{n_2}, \ldots, t^{n_k})} = \frac{1}{2m+1}\prod_{i=1}^{k}\left(\frac{m - n_i}{m + n_i + 1}\right)^2.$$

(iii) The series $\sum_{i=1}^{\infty} \ln(1+a_i)$ and the series $\sum_{i=1}^{\infty} a_i$ converge or diverge simultaneously. This is because

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$$

and so, for any $\varepsilon > 0$,

$$(1-\varepsilon)a_i < \ln(1+a_i) < (1+\varepsilon)a_i.$$

(**Müntz's Theorem**) *A necessary and sufficient condition for the set*

$$t^{n_1}, t^{n_2}, \ldots, \quad 1 \le n_1 < n_2 < \cdots$$

to be complete in $L^2[0, 1]$ *is that*

$$\sum_{i=1}^{\infty} \frac{1}{n_i} = \infty.$$

Proof If one of the exponents n_i coincides with m, then for $k \ge i$,

$$\det G(t^m, t^{n_1}, t^{n_2}, \ldots, t^{n_k}) = 0$$

and hence, the minimal distance is zero. Thus, completeness holds if, and only if, for each $m \ge 1$ with $m \ne n_i$, $i = 1, 2, \ldots$, the minimal distance

$$\delta_k^2 = \frac{\det G(t^m, t^{n_1}, t^{n_2}, \ldots, t^{n_k})}{\det G(t^{n_1}, t^{n_2}, \ldots, t^{n_k})} \to 0 \quad \text{as} \quad k \to \infty. \tag{2.93}$$

Now,

$$\frac{\det G(t^m, t^{n_1}, t^{n_2}, \ldots, t^{n_k})}{\det G(t^{n_1}, t^{n_2}, \ldots, t^{n_k})} = \frac{1}{2m+1} \prod_{i=1}^{k} \left(\frac{m-n_i}{m+n_i+1} \right)^2. \tag{2.94}$$

In view of (2.94), the condition (2.93) becomes

$$\lim_{k\to\infty} \left[\sum_{i=1}^{k} \left(\ln\left(1 - \frac{m}{n_i}\right) - \ln\left(1 + \frac{m+1}{n_i}\right)\right)\right] = -\infty. \tag{2.95}$$

If the series $\sum_{i=1}^{\infty} \frac{1}{n_i}$ diverges, then by (iii) of Remarks 4.2.6

$$\sum_{i=1}^{\infty} \ln\left(1 - \frac{m}{n_i}\right) = -\infty, \quad \sum_{i=1}^{\infty} \ln\left(1 + \frac{m+1}{n_i}\right) = +\infty \tag{2.96}$$

and therefore, (2.95) is satisfied and hence so is (2.93). If, however, the series $\sum_{i=1}^{\infty} \frac{1}{n_i}$ converges, then the series (2.96) also converges, so that (2.95) is not satisfied and hence (2.93) does not hold. □

Radon–Nikodým Theorem

Definition Let (X,Σ), where X is a nonempty set and Σ is a σ-algebra of subsets of X, be a measurable space, and let ν, μ be finite nonnegative measures on (X, Σ). The measure ν is said to be **absolutely continuous** with respect to μ, in symbols, $\nu \ll \mu$, *if* $\nu(E) = 0$ for every $E \in \Sigma$ for which $\mu(E) = 0$.

For $h \in L^1(X, \Sigma, \mu)$, the integral

$$\nu(E) = \int_E h \, \mathrm{d}\mu, \quad E \in \Sigma$$

defines a measure on Σ which is clearly absolutely continuous with respect to μ. The point of the Radon–Nikodým Theorem is the converse: every $\nu \ll \mu$ is obtained in this way.

von Neumann showed how to derive this from the Riesz Representation Theorem for linear functionals on a Hilbert space.

(**Radon–Nikodým Theorem**) *Let* ν *and* μ *be finite nonnegative measures on* (X, Σ)*.If* $\nu \ll \mu$, *then there exists a unique nonnegative measurable function h such that*

$$\nu(E) = \int_E h \, \mathrm{d}\mu, \quad E \in \Sigma.$$

In particular, $h \in L^1(X, \Sigma, \mu)$.

Proof For any $E \in \Sigma$, put $\varphi(E) = \nu(E) + \mu(E)$. Since ν and μ are finite nonnegative measures, so is φ. Moreover,

$$\int_X x \, \mathrm{d}\varphi = \int_X x \, \mathrm{d}\nu + \int_X x \, \mathrm{d}\mu \tag{2.97}$$

holds for $x = \chi_E$, $E \in \Sigma$. Hence, (2.97) holds for simple functions and consequently for any nonnegative measurable function x.

Let H be the real Hilbert space $L^2(X, \Sigma, \varphi)$ with the norm $||x||^2 = \int_X |x|^2 \mathrm{d}\varphi$. For $x \in H$, the Cauchy–Schwarz Inequality gives

$$\left|\int_X x\,\mathrm{d}\nu\right| \le \int_X |x|\mathrm{d}\nu \le \int_X |x|\mathrm{d}\varphi \le \left(\int_X |x|^2\mathrm{d}\varphi\right)^{\frac{1}{2}} (\varphi(X))^{\frac{1}{2}} < \infty$$

since $\varphi(X) < \infty$. Thus, the mapping

$$L : x \to \int_X x\,\mathrm{d}\nu$$

is seen to be defined and finite on H. It is clear that $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$ for all α, β scalars and $x, y \in L^2(X, \Sigma, \varphi) = H$. Thus, L is a bounded linear functional on H and so, by Theorem 2.10.25, there is a function $y \in H$ such that

$$\int_X x\mathrm{d}\nu = \int_X xy\,\mathrm{d}\varphi = \int_X xy\,\mathrm{d}\nu + \int_X xy\,\mathrm{d}\mu,$$

where we have used (2.97) in the last equality. It is easy to discern that y is non-negative a.e. with respect to φ and hence with respect to μ and ν as well. This may be written as

$$\int_X x(1-y)d\nu = \int_X xy\,d\mu. \tag{2.98}$$

Let $E = \{s \in X\colon y(s) \ge 1\}$. Since $\chi_E \in L^2(X, \Sigma, \varphi)$, we apply (2.98) to $x = \chi_E$ to obtain

$$0 \le \mu(E) = \int_X \chi_E \mathrm{d}\mu \le \int_X \chi_E y\,\mathrm{d}\mu = \int_X \chi_E(1-y)\mathrm{d}\nu \le 0.$$

Thus, we have $\mu(E) = 0$ and since $\nu \ll \mu, \nu(E) = 0$.

Let $z = y\chi_{E^c}$. Then $z(s) \in [0,1)$ and $z = y$ a.e. with respect to both ν and μ. The equality (2.98) then becomes

$$\int_X x(1-z)\mathrm{d}\nu = \int_X xz\,\mathrm{d}\mu. \tag{2.99}$$

Consider any bounded, nonnegative, measurable function x. Let z be as above. Since both x and z are bounded and φ is a finite measure, the function $(1 + z + z^2 + \cdots + z^{n-1})x$ is in $L^2(X, \Sigma, \varphi)$ for every positive integer n and hence by (2.99)

$$\int_X \left(1+z+z^2+\cdots+z^{n-1}\right)x(1-z)\mathrm{d}\nu = \int_X \left(1+z+z^2+\cdots+z^{n-1}\right)xz\,\mathrm{d}\mu$$

holds. In view of the fact that $z(s) \neq 1$ for any s, the above equality can be written as

$$\int_X (1-z^n)x\,d\nu = \int_X \frac{z(1-z^n)}{1-z}x\,d\mu.$$

Since $0 \leq z(s) < 1$ for all $s \in X$, the sequences $(1-z^n)x$ and $\frac{z(1-z^n)}{1-z}x$ increase to x and $\frac{z}{1-z}x$, respectively, as $n \to \infty$. By the Monotone Convergence Theorem 1.3.6, we obtain

$$\int_X x\,\mathrm{d}\nu = \int_X \frac{z}{(1-z)}x\,\mathrm{d}\mu.$$

Now define $h = \frac{z}{1-z}$; then we have

$$\int_X x\,\mathrm{d}\nu = \int_X hx\,\mathrm{d}\mu.$$

In particular, for $E \in \mathfrak{M}$ and $x = \chi_E$, we obtain

$$\nu(E) = \int_E h\,\mathrm{d}\mu.$$

The uniqueness is obvious. □

Remarks

(i) The construction of h shows that $h \geq 0$.
(ii) The Radon–Nikodým Theorem is valid if ν and μ are σ-finite measures. For details, the reader may consult [26].

Bergman Kernel and Conformal Mappings

Let Ω be a bounded domain in the $z = x + iy$ plane, whose boundary consists of a finite number of smooth simple closed curves. The class of all holomorphic functions in Ω for which the integral $\iint_\Omega |f(z)|^2\mathrm{d}x\mathrm{d}y < \infty$ is denoted by $A(\Omega)$. The integral is understood as the limit of Riemann integrals

$$\lim_n \iint_{K_n} |f(z)|^2\mathrm{d}x\mathrm{d}y,$$

where $\{K_n\}_{n\geq 1}$ is a nondecreasing sequence of compact subsets of Ω whose union is Ω. It has been proved [see 2.6.2, 2.6.3, 2.6.4, 2.6.5] that $A(\Omega)$ is a Hilbert space.

Consider the linear functional $L(f) = f(\zeta)$, where $\zeta \in \Omega$ is fixed and $f \in A(\Omega)$. Observe that $|L(f)| = |f(\zeta)| \leq \frac{\|f\|}{\sqrt{\pi} d_\zeta}$, where $d_\zeta = \mathrm{dist}(\zeta, \partial\Omega)$ and $\partial\Omega$ denotes the boundary of Ω [see Proposition 2.6.3]. It follows on using Theorem 2.10.25 that there exists a uniquely determined $u_\zeta \in A(\Omega)$ such that

$$f(\zeta) = (f, u_\zeta), \quad f \in A(\Omega). \tag{2.100}$$

The traditional notation is $u_\zeta(z) = K(z, \zeta)$ and K is called the **Bergman kernel** of Ω. For each $\zeta \in \Omega$, the function has the reproducing property

$$f(\zeta) = (f, K(\cdot, \zeta)) = \iint\limits_{\Omega} f(z)\overline{K(z,\zeta)}\mathrm{d}x\mathrm{d}y, \quad f \in A(\Omega). \tag{2.101}$$

The following two assertions are immediate from (2.101).

(a) If one substitutes $f = K(\cdot, \zeta)$ in (2.101), one finds that

$$\begin{aligned}\|K(\cdot,\zeta)\|^2 &= \iint\limits_{\Omega} K(z,\zeta)\overline{K(z,\zeta)}\,\mathrm{d}x\mathrm{d}y \\ &= K(\zeta,\zeta), \quad \zeta \in \Omega.\end{aligned}$$

(b) For $z_1, z_2 \in \Omega$, the relation $K(z_1, z_2) = \overline{K(z_2, z_1)}$ holds. To see this, we let $f = K(\cdot, z_2)$ and $\zeta = z_1$ in (2.101) and we obtain

$$\begin{aligned}K(z_1,z_2) &= \iint\limits_{\Omega} K(z,z_2)\overline{K(z,z_1)}\mathrm{d}x\mathrm{d}y \\ &= \iint\limits_{\Omega} \overline{\overline{K(z,z_2)}K(z,z_1)}\mathrm{d}x\mathrm{d}y \\ &= \overline{K(z_2,z_1)}\end{aligned}$$

The relation between the kernel function and a certain minimum problem in $A(\Omega)$ is also important. Suppose $\zeta \in \Omega$ is fixed, and write

$$M = \{f \in A(\Omega) : f(\zeta) = 1\}.$$

There is exactly one solution $f_0 \in M$ such that $\min_{f\in M}\|f\| = \|f_0\|$. *Moreover, the function f_0 is connected with the Bergman kernel function as follows*:

$$f_0(z) = \frac{K(z,\zeta)}{K(\zeta,\zeta)} \quad \text{and} \quad K(z,\zeta) = \frac{f_0(z)}{\|f_0\|^2}.$$

Proof Since $A(\Omega)$ is a Hilbert space and $M \subseteq A(\Omega)$ is its closed subspace, the first assertion follows on using Corollary 2.10.7.

For each $f \in A(\Omega)$, we have $f(\zeta) = (f, K(\cdot,\zeta))$. For $f \in M$, on using the Cauchy–Schwarz inequality, we have

$$1 = (f, K(\cdot,\zeta)) \leq \|f\|\|K(\cdot,\zeta)\| = \|f\| \cdot \sqrt{K(\zeta,\zeta)}. \tag{2.102}$$

Equality in the above inequality occurs provided

$$f = f_0 = CK(\cdot,\zeta), \quad \text{where } C \text{ is a constant.} \tag{2.103}$$

Since $1 = f_0(\zeta) = CK(\zeta,\zeta)$ (therefore $C = \frac{1}{K(\zeta,\zeta)}$), it follows that

$$f_0(z) = \frac{K(z,\zeta)}{K(\zeta,\zeta)}.$$

This implies

$$K(z,\zeta) = f_0(z)K(\zeta,\zeta).$$

Also,

$$K(z,\zeta) = \frac{f_0(z)}{\|f_0\|^2},$$

since $\|f\|^2 = \|f_0\|^2 = \frac{1}{K(\zeta,\zeta)}$, using (2.102) and (2.103). □

Recall that the **Riemann Mapping Theorem** asserts: if Ω is a simply connected domain having more than one boundary point, then there exists a holomorphic function in Ω which maps Ω bijectively onto $D = \{z : |z| < 1\}$. If ζ is fixed, then the mapping function $f(z) = f(z, \zeta)$ for which $f(\zeta) = 0$ and $f'(\zeta) > 0$ is unique.

The mapping function f and the Bergman kernel K of Ω *are related as follows*:

$$f'(z) = \sqrt{\frac{\pi}{K(\zeta,\zeta)}}k(z,\zeta) \quad \text{and} \quad k(z,\zeta) = \frac{1}{\pi}f'(z)f'(\zeta), z \in \Omega$$

Proof Let Ω_r denote the subdomain of Ω which is mapped by f onto the disc $\{\omega : |\omega| < r\}$, where $r < 1$ and $\omega = f(z)$. Denote the boundary of Ω_r by γ_r. If $g \in A(\Omega)$, then $\frac{g(z)}{f(z)}$ has a simple pole at $z = \zeta$ and the residue at this pole is

$$\lim_{z \to \zeta} \frac{(z-\zeta)g(z)}{f(z)} = \frac{g(\zeta)}{f'(\zeta)}.$$

By the Residue Theorem,

$$\frac{g(\zeta)}{f'(z)} = \frac{1}{2\pi i}\int_{\gamma_r} \frac{g(z)}{f(z)}\mathrm{d}z = \frac{1}{2\pi i r^2}\int_{\gamma_r} \overline{f(z)}g(z)\mathrm{d}z.$$

since $|f(z)|^2 = r^2$ for $z \in \gamma_r$. Using Green's formula, we obtain

$$\frac{g(\zeta)}{f'(\zeta)} = \frac{1}{\pi r^2}\iint_{\Omega_r} \overline{f'(z)}g(z)\mathrm{d}x\mathrm{d}y.$$

Letting $r \to 1$, we get

$$g(\zeta) = \iint_{\Omega} \frac{\overline{f'(z)}f'(\zeta)}{\pi} g(z)\mathrm{d}x\mathrm{d}y.$$

In other words, the function

$$K(z,\zeta) = \frac{f'(z)f'(\zeta)}{\pi} \tag{2.104}$$

has the reproducing property for $A(\Omega)$ and is therefore the Bergman kernel. For $z = \zeta$, it follows that

$$K(\zeta,\zeta) = \frac{f'(\zeta)^2}{\pi},$$

which implies on using (2.104)

$$f'(z) = \sqrt{\frac{\pi}{K(\zeta,\zeta)}}k(z,\zeta).$$

This completes the proof. □

Remarks

(i) In only a few cases, it is possible to obtain a representation for the kernel function in closed form. It is easy to find a series representation with respect to some complete orthonormal system $\{\phi_j\}$, because by (2.101), the Fourier coefficients are

$$u_j = \left(K(\cdot,\zeta), \phi_j\right) = \overline{\phi_j(\zeta)}, \quad j = 1, 2, \ldots$$

and the Bergman kernel has the series representation [see Theorem 2.9.15(iii)]

$$K(z,\zeta) = \sum_{j=1}^{\infty} \overline{\phi_j(\zeta)}\,\phi_j(z), \quad z,\zeta \in \Omega.$$

(ii) Consider the special case where $\Omega = D\{z : |z| < 1\}$. According to (vi) of Examples 2.9.16, the set $\phi_n(z) = \sqrt{\frac{n}{\pi}} z^{n-1}$ $n = 1, 2, \ldots$ is an orthonormal system in $A(D)$. Thus,

$$K(z,\zeta) = \sum_{n=1}^{\infty} \frac{n}{\pi} \overline{\zeta^{n-1}} z^{n-1} = \frac{1}{\pi} \frac{1}{(1 - z\bar{\zeta})^2}, \quad z,\zeta \in \Omega.$$

is the kernel function of D. The series converges uniformly in $|\zeta| \le r, r < 1$.

The reproducing property becomes

$$f(\zeta) = \frac{1}{\pi} \iint_D \frac{f(z)}{(1 - \bar{z}\zeta)^2} \mathrm{d}x\mathrm{d}y.$$

Special Case of Browder Fixed Point Theorem

Let C be a nonempty convex, closed and bounded subset of a Hilbert space H and let T be a map from C into C such that

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all} \quad x, y \in C.$$

Then T has at least one fixed point.

Solution: for each $n \in \mathbb{N}$, let

$$T_n(x) = \frac{1}{n} a + \frac{n-1}{n} T(x),$$

where $a \in C$ is fixed. Then T_n is a contraction and therefore has a fixed point $x_n \in C$. Indeed, for $x, y \in C$, $\|T_n(x) - T_n(y)\| = \frac{n-1}{n} \|Tx - Ty\| \le \frac{n-1}{n} \|x - y\|$.

Since C is a bounded subset of H and $\{x_n\}_{n\ge1}$ is in C, it follows that there exists a subsequence $\{x_{n_j}\}_{j\ge1}$ such that $x_{n_j} \xrightarrow{w} x$, say [Theorem 3.1.5]. By the Banach–Saks Theorem [Problem 2.12.P2], $\{x_{n_j}\}_{j\ge1}$ has subsequence such that a sequence of certain convex combinations of its terms converges strongly to x. Consequently, $x \in C$ as C is convex and (strongly) closed. We shall prove that x is a fixed point of T.

For any point y in H, we note that

$$\left\|x_{n_j} - y\right\|^2 = \left\|x_{n_j} - x\right\|^2 + \|x - y\|^2 + 2\Re\left(x_{n_j} - x, x - y\right), \tag{2.105}$$

where $2\Re\left(x_{n_j} - x, x - y\right) \to 0$ as $j \to \infty$, since $x_{n_j} - x \to 0$ weakly in H.

Observe that

$$\begin{aligned} T(x_{n_j}) - x_{n_j} &= T(x_{n_j}) - T_{n_j}(x_{n_j}) = T(x_{n_j}) - \frac{1}{n_j}a - \frac{n_j - 1}{n_j}T(x_{n_j}) \\ &= \frac{1}{n_j}\left(T(x_{n_j}) - a\right) \to 0 \,\text{as}\, j \to \infty. \end{aligned} \tag{2.106}$$

Setting $y = T(x)$ in (2.105), we have

$$\lim_{j\to\infty}\left\{\left\|x_{n_j} - T(x)\right\|^2 - \left\|x_{n_j} - x\right\|^2\right\} = \|x - T(x)\|^2. \tag{2.107}$$

On the other hand, using the hypothesis,

$$\left\|T(x_{n_j}) - T(x)\right\| \le \left\|x_{n_j} - x\right\|.$$

Hence

$$\begin{aligned} \left\|x_{n_j} - T(x)\right\| &\le \left\|x_{n_j} - T(x_{n_j})\right\| + \left\|T(x_{n_j}) - T(x)\right\| \\ &\le \left\|x_{n_j} - T(x_{n_j})\right\| + \left\|x_{n_j} - x\right\|. \end{aligned}$$

Thus on using (2.106), we obtain

$$\limsup_{j\to\infty}\left[\left\|x_{n_j} - T(x)\right\| - \left\|x_{n_j} - x\right\|\right] \le 0$$

and therefore

$$\limsup_{j\to\infty}\left\{\left\|x_{n_j} - T(x)\right\|^2 - \left\|x_{n_j} - x\right\|^2\right\} \le 0,$$

which implies, on using (2.107),

$$\|x - T(x)\| = 0.$$

Chapter 3
Linear Operators

3.1 Basic Definitions

Let X and Y be finite-dimensional vector spaces over the same field $\mathbb{F}$. Recall that a mapping $T{:}X \to Y$ is called linear if $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{F}$. T is also called a linear operator or linear transformation. If $\dim(X) = n$ and $\dim(Y) = m$, we choose a basis $\{e_1, e_2, \ldots, e_n\}$ for X and a basis $\{f_1, f_2, \ldots, f_m\}$ for Y. An $m \times n$ matrix A of elements of $\mathbb{F}$ corresponds to a linear transformation $T{:}X \to Y$ in the following way: for each integer k, $1 \leq k \leq n$, there are unique elements $\tau_{1,k}, \tau_{2,k}, \ldots, \tau_{m,k}$ of $\mathbb{F}$ such that

$$Te_k = \sum_{j=1}^{m} \tau_{j,k} f_j. \tag{3.1}$$

Each point $x \in X$ has a unique representation in the form $x = \sum_{k=1}^{n} \xi_k e_k$, where $\xi_1, \xi_2, \ldots, \xi_n$ are in $\mathbb{F}$. Hence,

$$\begin{aligned} Tx &= \sum_{k=1}^{n} \xi_k Te_k \\ &= \sum_{k=1}^{n} \xi_k \left(\sum_{j=1}^{m} \tau_{j,k} f_j \right) \\ &= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \tau_{j,k} \xi_k \right) f_j. \end{aligned} \tag{3.2}$$

If $\eta_1, \eta_2, \ldots, \eta_m$ are the components of the vector Tx with respect to the basis $\{f_1, f_2, \ldots, f_m\}$, then $\eta_j = \sum_{k=1}^{n} \tau_{j,k} \xi_k$. In this sense, the matrix $A = [\tau_{j,k}]$ corresponds to the linear transformation T. It is also said that the matrix A **represents** the linear transformation T with respect to the aforementioned bases of X and Y.

H.L. Vasudeva, *Elements of Hilbert Spaces and Operator Theory*,
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Conversely, let $A = [\tau_{j,k}]$ be an $m \times n$ matrix of elements of $\mathbb{F}$. We can define a mapping $T{:}X \to Y$ in the following manner. Consider an $x \in X$. It has a unique representation in the form $x = \sum_{k=1}^{n} \xi_k e_k$, where $\xi_1, \xi_2, \ldots, \xi_n$ are in $\mathbb{F}$. Set

$$\eta_j = \sum_{k=1}^{n} \tau_{j,k}\xi_k, \quad j = 1, 2, \ldots, m. \tag{3.3}$$

and

$$Tx = \sum_{j=1}^{m} \eta_j f_j = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \tau_{j,k}\xi_k \right) f_j. \tag{3.4}$$

T is obviously linear. Our considerations show that a linear operator T determines a unique $m \times n$ matrix representing T with respect to a given basis for X and a given basis for Y, where the vectors of each basis are arranged in a fixed order, and conversely.

Questions about the system (3.3) can be formulated as questions about T. For example, for which $\eta_1, \eta_2, \ldots, \eta_m$ does the system (3.3) have a solution $\xi_1, \xi_2, \ldots, \xi_n$? This amounts to asking for a description of the range of T.

The most complete and satisfying results about (3.3) are obtained when $m = n$. Indeed, if $m = n$, the system (3.3) has a unique solution, if and only if, the matrix $[\tau_{j,k}]$ is nonsingular, equivalently, the linear operator T determined by the matrix $[\tau_{j,k}]$ is one-to-one (or onto). In particular, if $X = Y$, $e_j = f_j$, $j = 1, \ldots, n$, the operator T maps X to itself. If p is a polynomial, then $p(T)$ makes sense. The study of $p(T)$ can provide insight about T. For example, λ is an eigenvalue of T if, and only if, it is a root of the characteristic polynomial $\det(\lambda I - T)$.

Recall that if H is a Hilbert space and M is a closed subspace of H, then the mappings P_M from H onto M and $P_M{}^{\perp}$ onto $M^{\perp}$ are linear [see Theorem 2.10.15].

We give below a formal definition of a linear operator.

Definition 3.1.1 Let X and Y be linear spaces (vector spaces) over the same scalar field $\mathbb{F}$, say. A mapping T defined over a linear subspace D of X, written $D(T)$, and taking values in Y is said to be a **linear operator** if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \text{ for scalars } \alpha_1, \alpha_2 \text{ and } x_1, x_2 \text{ in } D.$$

The definition implies, in particular, that

$$T(0) = 0, \quad T(-x) = -T(x).$$

We denote

$$\mathrm{ran}(T) = \{y \in Y{:}y = Tx \text{ for some } x \text{ in } D(T)\}$$

and

$$\ker(T) = \{x \in D(T) : Tx = 0\}.$$

We call $D(T)$ the *domain*, $\operatorname{ran}(T)$ the *range* and $\ker(T)$ the *kernel,* respectively, of the operator T. A linear operator is also called a linear transformation with domain $D(T) \subseteq H$ into Y. If the range $\operatorname{ran}(T)$ is contained in the scalar field $\mathbb{F}$, then T is called a *linear functional* [see Definition 2.10.18] on $D(T)$. If a linear operator gives a one-to-one map ($x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$ or equivalently, $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$) of $D(T)$ onto $\operatorname{ran}(T)$, then the inverse map T^{-1} gives a linear operator on $R(T)$ onto $D(T)$:

$$T^{-1}Tx = x \quad \text{for} \quad x \in D(T) \ \text{ and}$$
$$TT^{-1}y = y \quad \text{for} \quad y \in \operatorname{ran}(T).$$

T^{-1} is called the **inverse operator** or, in short, the **inverse** of T.

The following proposition is an easy consequence of the linearity of T.

Proposition 3.1.2 *A linear operator T admits an inverse T^{-1} if, and only if,* $Tx = 0$ *implies* $x = 0$.

Proof Suppose $Tx = 0$ implies $x = 0$. Let $Tx_1 = Tx_2$. Since T is linear,

$$T(x_1 - x_2) = Tx_1 - Tx_2 = 0,$$

so that $x_1 = x_2$ by hypothesis.

Conversely, if T^{-1} exists, then $Tx_1 = Tx_2$ implies $x_1 = x_2$. Let $Tx = 0$. Since T is linear, $T0 = 0 = Tx$, so that $x = 0$ by hypothesis. □

Example 3.1.3 Let X be the vector space of all real-valued functions which are defined over $\mathbb{R}$ and have derivatives of all orders everywhere on $\mathbb{R}$. Define T: $X \to X$ by $y(t) = Tx(t) = x'(t)$. Then, $R(T) = X$. Indeed, for $y \in X$, we have $y = Tx$, where $x(t) = \int_0^t y(\tau)d\tau$. Since $Tx = 0$ for every constant function, T^{-1} does not exist.

Definition 3.1.4 Let T_1 and T_2 be linear operators with domains $D(T_1)$ and $D(T_2)$ both contained in a linear space X and ranges $R(T_1)$ and $R(T_2)$ both contained in a linear space Y. Then, $\boldsymbol{T_1 = T_2}$ if, and only if, $D(T_1) = D(T_2)$ and $T_1x = T_2x$ *for all* $x \in D(T_1) = D(T_2)$. If $D(T_1) \subseteq D(T_2)$ and $T_1x = T_2x$ *for all* $x \in D(T_1)$, T_2 is called an **extension** of T_1 and T_1 a **restriction** *of* T_2. We shall write $T_1 \subseteq T_2$.

We shall abbreviate "$D(T)$" to simply "D" when there is only one operator under consideration.

The following is a special case of bijective mappings between sets.

Proposition 3.1.5 *Let T:$X \to Y$ and S:$Y \to Z$ be bijective linear operators, where X, Y, Z are linear spaces over the same scalar field $\mathbb{F}$. Then, the inverse $(ST)^{-1}$:$Z \to X$ of the product (composition) of S and T exists and satisfies*

$$(ST)^{-1} = T^{-1}S^{-1}.$$

Remark 3.1.6 The identity map, a composition of linear maps and the inverse of a linear map (when it exists) are all linear.

3.2 Bounded and Continuous Linear Operators

Every linear functional is a linear transformation between the linear space and the one-dimensional scalar field underlying the linear space. The study of continuous linear functionals on inner product spaces and more specifically on Hilbert spaces has yielded many valuable results [Sect. 2.10]. It seems natural to attempt generalising the considerations to linear transformations (operators) from Hilbert space into itself. The interplay between algebraic notions and metric structure proves interesting and useful in applications.

Definition 3.2.1 Let X and Y be normed linear spaces and $T{:}D \to Y$ a linear operator, where $D \subseteq X$. T is said to be **continuous at** $x_0 \in D$ if $\lim_{x\to x_0} T(x) = Tx_0$. T is **continuous in *D*** if it is continuous at each point of D.

A linear operator is **bounded** *if*

$$\sup_{\substack{x\in D\\ ||x||\leq 1}} ||Tx|| < \infty$$

The left member of the above inequality is called the **norm** *of the operator T in D, provided it is finite, and is denoted by the symbol* $\|T\|$ *or sometimes by* $\|T\|_D$. *If* $M \geq \|T\|_D$, *then M is called a* **bound** *of T.*

The infimum of all bounds M is the norm $\|T\|_D$.

Remarks 3.2.2

(i) If $x \in D$ and $x \neq 0$, then by the definition of the norm of T,

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq \|T\|_D.$$

Hence, for any $x \in D$, $x \neq 0$, we have $\|Tx\| \leq \|T\|_D\|x\|$. However, it is easily seen that this inequality holds also when $x = 0$ (the two sides are both zero in this event), and therefore,

$$||Tx|| \leq ||T||_D||x|| \quad \text{for all } x \in D. \tag{3.5}$$

(ii) It follows from the relation (3.5) and linearity of T that T is uniformly continuous. Indeed, by (3.5),

$$\|Tx - Ty\| = \|T(x - y)\| \leq \|T\|\|x - y\| \text{for}\, x, y \in D.$$

(iii) From (3.5), it also follows that, if $x \in D$ and $\|x\| \leq 1$, then

$$||Tx|| \leq ||T|| \tag{3.6}$$

and the above inequality is strict if $\|x\| < 1$ and $\|T\| \neq 0$.

(iv) Now assume that $D \neq \{0\}$. Then, it follows from (3.5) and (3.6) and the equality $\|T(\alpha x)\| = |\alpha|\|Tx\|$ that $\|T\|$ can be defined as

$$||T|| = \sup_{\substack{x \in D \\ ||x||=1}} ||Tx|| \tag{3.7}$$

or equivalently by

$$||T|| = \sup_{\substack{x \in D \\ ||x|| \neq 0}} \frac{||Tx||}{||x||}. \tag{3.8}$$

Thus, if T is a bounded linear operator on $D \subseteq X$ and $D \neq \{0\}$, then

$$\|T\| = \sup_{\substack{x \in D \\ \|x\|=1}} \|Tx\| = \sup_{\substack{x \in D \\ \|x\| \leq 1}} ||Tx|| = \sup_{\substack{x \in D \\ \|x\| \neq 0}} \frac{\|Tx\|}{\|x\|}. \tag{3.9}$$

The following proposition gives equivalent conditions for the continuity of a linear operator from $D \subseteq X$ into Y.

Proposition 3.2.3 *Let X and Y be normed linear spaces over the same field of scalars and $D \subseteq X$ be the domain of the linear operator T from D into Y. Then, the following conditions are equivalent*:

(a) *T is continuous at a given $x_0 \in D$*;
(b) *T is bounded*; and
(c) *T is continuous everywhere and the continuity is uniform.*

Proof If $D = \{0\}$, there is nothing to prove.

(a) implies (b). Suppose T is continuous at $x_0 \in D$. Then for given $\varepsilon > 0$, there is a $\delta > 0$ such that $\|Tx - Tx_0\| < \varepsilon$ for all $x \in D$ satisfying $\|x - x_0\| < \delta$. We now take $y \neq 0$ in D and set

$$x = x_0 + \frac{\delta}{2\|y\|}y.$$

Then,

$$x - x_0 = \frac{\delta}{2\|y\|} y.$$

Hence, $\|x - x_0\| = \delta/2 < \delta$, so that we have $\|Tx - Tx_0\| < \varepsilon$. Since T is linear, we obtain

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \left\| T \frac{\delta}{2\|y\|} y \right\| = \frac{\delta}{2\|y\|} \|Ty\|$$

and this implies

$$\frac{\delta}{2\|y\|} \|Ty\| < \varepsilon.$$

Therefore, $\|Ty\| < \frac{2\varepsilon}{\delta} \|y\| = M\|y\|$, where $M = \frac{2\varepsilon}{\delta}$. Thus, T is bounded.

(b) implies (c). Suppose T is bounded and $M > 0$ a bound. Then for $x, y \in D$, we have $\|Tx - Ty\| = \|T(x - y)\| \leq M\|x - y\|$. Let $\varepsilon > 0$ and $\delta = \varepsilon/M$. Then, $\|x - y\| < \delta$ implies $\|T(x - y)\| < M\delta = \varepsilon$. Since $x, y \in D$ are arbitrary, T is uniformly continuous on D and hence continuous everywhere on D.

(c) implies (a). Trivial. □

Remark The terms *continuous linear operator* and *bounded linear operator* will be used interchangeably.

Many properties of linear functionals generalise easily to linear operators. The analogue of the dual space is the space of all continuous linear operators from a normed linear space X into a normed linear space Y (which may or may not be the same as X) and is denoted by $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$. Note that in this context $D = X$. We abbreviate $\mathcal{B}(X, X)$ as $\mathcal{B}(X)$.

First of all, $\mathcal{B}(X, Y)$ becomes a vector space if we define the sum $T_1 + T_2$ of two operators T_1, T_2 in $\mathcal{B}(X, Y)$ in a natural way,

$$(T_1 + T_2)x = T_1x + T_2x$$

and the product αT of $T \in \mathcal{B}(X, Y)$ and a scalar α by

$$(\alpha T)x = \alpha(Tx).$$

Since

$$\begin{aligned}
\|(T_1 + T_2)x\| &\leq \|T_1x\| + \|T_2x\| \\
&\leq \sup\{\|T_1x\| : x \in X \text{ and } \|x\| = 1\} + \{\sup\|T_2x\| : x \in X \text{ and } \|x\| = 1\} \\
&= \|T_1\| + \|T_2\|,
\end{aligned}$$

it follows that

$$\|T_1 + T_2\| \le \|T_1\| + \|T_2\| \quad \text{for} \quad T_1, T_2 \in \mathcal{B}(X, Y).$$

Similarly, it can be proved that

$$\|\alpha T\| = |\alpha| \|T\| \quad \text{for} \quad \alpha \in \mathbb{F} \quad \text{and} \quad T \in \mathcal{B}(X, Y).$$

It is immediate from Definition 3.2.1 that

$$\|T\| = 0 \quad \text{impllies} \quad T = O.$$

These imply that $\mathcal{B}(X, Y)$ is a normed vector space (linear space) over the scalar field $\mathbb{F}$.

Theorem 3.2.4 *If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.*

Proof Let $\{T_n\}_{n \ge 1}$ be a Cauchy sequence in $\mathcal{B}(X, Y)$. Then for any $x \in X$,

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \le \|T_n - T_m\| \|x\|,$$

so that $\{T_n x\}_{n \ge 1}$ is a Cauchy sequence in Y. Since Y is complete, the sequence converges, say $T_n x \to y$. Clearly, the limit y depends on x. This defines a map T: $X \to Y$, where $y = Tx = \lim_n T_n x$. The map T is a linear operator since

$$\begin{aligned} \lim_n T_n(\alpha_1 x_1 + \alpha_2 x_2) &= \lim_n(\alpha_1 T_n(x_1) + \alpha_2 T_n(x_2)) \\ &= \alpha_1 \lim_n T_n(x_1) + \alpha_2 \lim_n T_n(x_2) \end{aligned}$$

for scalars α_1, α_2 and x_1, x_2 in X.

We prove that T is bounded and $\|T_n - T\| \to 0$ as $n \to \infty$. The sequence $\{T_n\}_{n \ge 1}$, being Cauchy, is bounded, i.e., there exists an $M > 0$ such that $\|T_n\| \le M$, $n = 1, 2, \ldots$. For any $x \in X$, $\|T_n x\| \le \|T_n\| \|x\| \le M\|x\|$. Consequently,

$$\|Tx\| = \|\lim_n T_n x\| = \lim_n \|T_n x\| \le M\|x\|.$$

This proves that T is bounded. It remains to show that $\|T_n - T\| \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. There exists n_0 such that $m, n \ge n_0$ implies $\|T_n - T_m\| < \varepsilon$. Then,

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \le \|T_n - T_m\| \|x\| \le \varepsilon \|x\|$$

for $m, n \ge n_0$ and $x \in X$. Letting $m \to \infty$, we get

$$\|T_n x - Tx\| \le \varepsilon \|x\| \quad \text{for} \quad n \ge n_0 \quad \text{and} \quad x \in X.$$

This implies that $\|T_n - T\| \le \varepsilon$ for $n \ge n_0$, so that $\|T_n - T\| \to 0$ as $n \to \infty$. □

Remark If Y is one-dimensional and X is a normed linear space, we obtain Theorem 2.10.23.

Example 3.2.5

(i) (**Identity operator**) Let H be a Hilbert space. The identity operator I: $H \to H$ defined by $Ix = x$, $x \in H$, is linear and bounded with $\|I\| = 1$ when $H \neq \{0\}$.

(ii) (**Zero operator**) The zero operator on H defined by $Ox = 0$, $x \in H$, is linear and $\|O\| = 0$.

(iii) If H is a Hilbert space of finite dimension and T is a linear mapping on H into H, then T is continuous. For, let $e_1, e_2, \ldots, e_n$ be an orthonormal basis for H. If $x = \sum_{k=1}^{n} \xi_k e_k$ is any vector in H, then

$$\|Tx\| = \left\| \sum_{k=1}^{n} \xi_k (Te_k) \right\| \leq \sum_{k=1}^{n} |\xi_k| \|(Te_k)\| \leq \left(\sum_{k=1}^{n} |\xi_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \|(Te_k)\|^2 \right)^{\frac{1}{2}},$$

using Cauchy–Schwarz inequality. Thus, $\|Tx\| \leq M\|x\|$, where

$$M = \left(\sum_{k=1}^{n} \|Te_k\|^2 \right)^{\frac{1}{2}}$$

is independent of x.

(iv) Let T be a linear operator defined on a Hilbert space $H \neq \{0\}$ by the formula

$$Tx = \alpha x, \quad x \in H,$$

where $\alpha \in \mathbb{F}$ is fixed. Then,

$$\|Tx\| = \|\alpha x\| = |\alpha| \|x\|.$$

Consequently,

$$\|T\| = \sup_{\substack{x \in H \\ \|x\| = 1}} \|Tx\| = \sup_{\substack{x \in H \\ \|x\| = 1}} |\alpha| \|x\| = |\alpha|.$$

Thus, T is a bounded linear operator on H of norm $|\alpha|$.

(v) Let M be a closed subspace of a Hilbert space H and $x \in H$. Then, $x = y + z$, where $y \in M$ and $z \in M^{\perp}$ and this representation is unique [see Remark 2.10.12]. Define $T: H \to H$ by the formula

$$Tx = y, x \in H.$$

We know that T is linear and $\|Tx\|^2 = \|y\|^2 \leq \|y\|^2 + \|z\|^2 = \|y + z\|^2 = \|x\|^2$ [see Theorem 2.10.15]. Thus, T is a bounded linear operator on H and $\|T\| \leq 1$. Indeed, $\|T\| = 1$ when $M \neq \{0\}$; for $x \in M$, $Tx = x$, and hence, $\|Tx\| = \|x\|$. Recall that this operator is called the *projection* on M and is denoted by P_M [see Definition 2.10.16].

(vi) **(Multiplication operator)** Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and $H = L^2(X, \mathfrak{M}, \mu)$ be the Hilbert space of square integrable functions defined on X. For $y \in H$, an essentially bounded measurable function, define $Tx(t) = y(t)x(t)$, $x \in H$ and $t \in X$. Clearly, T is a bounded linear operator in H. Indeed,

$$\|Tx\|_2^2 = \int_X |y(t)|^2 |x(t)|^2 \mathrm{d}\mu(t) \leq \operatorname*{ess\,sup}_{t \in X} |y(t)|^2 \int_X |x(t)|^2 \mathrm{d}\mu(t) = \|y\|_\infty^2 \|x\|_2^2, \quad x \in H.$$

Thus, $\|T\| \leq \|y\|_\infty$. Indeed, $\|T\| = \|y\|_\infty$, as the following argument shows: if $\mu(X) = 0$, then $H = \{0\}$, $\|T\| = 0 = \|y\|_\infty$. Suppose $\mu(X) > 0$. If $\varepsilon > 0$, the σ-finiteness of the measure space implies that there is a measurable set $F \subseteq X$, $0 < \mu(F) < \infty$, such that $|y(t)| \geq \|y\|_\infty - \varepsilon$ on F. If $f = (\mu(F))^{-\frac{1}{2}} \chi_F$, then $f \in L^2(X, \mathfrak{M}, \mu)$ and $\|f\|_2 = 1$. So,

$$\begin{aligned}
\|Tf\|^2 &= \int_X |y(t)|^2 (\mu(F))^{-1} \chi_F(t) \mathrm{d}\mu(t) \\
&= (\mu(F))^{-1} \int_F |y(t)|^2 \mathrm{d}\mu(t) \\
&\geq (\mu(F))^{-1} \left(\|y\|_\infty - \varepsilon\right)^2 \mu(F) \\
&= \left(\|y\|_\infty - \varepsilon\right)^2,
\end{aligned}$$

which implies $\|T\| \geq \|y\|_\infty - \varepsilon$, as $\|f\|_2 = 1$. Since $\varepsilon > 0$ is arbitrary, we get $\|T\| \geq \|y\|_\infty$.

The operator T is called a **multiplication operator**.

(vii) Let H be a separable Hilbert space and $\{e_i\}_{i \geq 1}$ be an orthonormal basis in H. Define $T{:}H \to H$ as follows:

$$Te_i = e_{i+1}, \quad i = 1, 2, \ldots.$$

If $x \in H$, then $x = \sum_{k=1}^\infty \lambda_k e_k, \lambda_k \in \mathbb{F}$, $k = 1, 2, \ldots$, where $\sum_{k=1}^\infty |\lambda_k|^2 < \infty$. In particular, $\sum_{k=1}^\infty \lambda_k e_{k+1}$ is an element of H. Define $Tx = \sum_{k=1}^\infty \lambda_k Te_k = \sum_{k=1}^\infty \lambda_k e_{k+1}$. Clearly T is linear. Moreover,

$$\|Tx\|^2 = \left\| \sum_{k=1}^{\infty} \lambda_k e_{k+1} \right\|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 \|e_{k+1}\|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2.$$

On the other hand,

$$\|x\|^2 = \left\| \sum_{k=1}^{\infty} \lambda_k e_k \right\|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 \|e_k\|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2.$$

Thus, $\|Tx\| = \|x\|$, $x \in H$, i.e., T is a bounded linear operator on H of norm 1. The operator described above is called the **simple unilateral shift**.

(viii) Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and $k: X \times X \to \mathbb{C}$ be an $\mathfrak{M} \times \mathfrak{M}$ measurable function for which there are constants c_1 and c_2 such that

$$\int_X |k(s,t)| \mathrm{d}\mu(t) \le c_1 \quad \text{a.e. } [\mu],$$

$$\int_X |k(s,t)| \mathrm{d}\mu(s) \le c_2 \quad \text{a.e. } [\mu].$$

For $x \in L^2(\mu)$, set

$$(Kx)(s) = \int_X k(s,t) x(t) \mathrm{d}\mu(t).$$

We shall show that K is a bounded linear operator in $L^2(\mu)$ and $\|K\| \le (c_1 c_2)^{\frac{1}{2}}$.

$$\begin{aligned}
|(Kx)(s)| &\le \int_X |k(s,t)||x(t)| \mathrm{d}\mu(t) \\
&= \int_X |k(s,t)|^{\frac{1}{2}} |k(s,t)|^{\frac{1}{2}} |x(t)| \mathrm{d}\mu(t) \\
&\le \left[\int_X |k(s,t)| \mathrm{d}\mu(t) \right]^{\frac{1}{2}} \left[\int_X |k(s,t)||x(t)|^2 \mathrm{d}\mu(t) \right]^{\frac{1}{2}} \\
&\le c_1^{\frac{1}{2}} \left[\int_X |k(s,t)||x(t)|^2 \mathrm{d}\mu(t) \right]^{\frac{1}{2}} \quad \text{a.e. } [\mu].
\end{aligned}$$

Hence, by Fubini's Theorem (the function under the integral sign is nonnegative),

$$\begin{aligned}\int_X |(Kx)(s)|^2 \mathrm{d}\mu(s) &\le c_1 \int_X \int_X |k(s,t)||x(t)|^2 \mathrm{d}\mu(t)\mathrm{d}\mu(s) \\ &= c_1 \int_X |x(t)|^2 \int_X |k(s,t)|\mathrm{d}\mu(s)\mathrm{d}\mu(t). \\ &\le c_1 c_2 ||x||^2.\end{aligned}$$

The above argument shows that the formula used to define Kx is such that Kx is finite a.e. [μ] and $Kx \in L^2(\mu)$ and $\|Kx\|^2 \le c_1c_2\|x\|^2$.
The operator K described above is called an **integral operator** and the function k is called its **kernel**.

(ix) A particular instance of the integral operator described above is known as the **Volterra operator**. Let k:[0,1] × [0,1]→$\mathbb{F}$ be the characteristic function of the set $\{(s,t) \in [0,1] \times [0,1] : t < s\}$. The corresponding operator V: $L^2[0,1] \to L^2[0,1]$ is defined by

$$Vx(s) = \int_0^s x(t)\mathrm{d}t, \quad x \in L^2[0,1].$$

Then,

$$\begin{aligned}|Vx(s)|^2 &\le \left(\int_0^s |x(t)|\mathrm{d}t\right)^2 \\ &\le \left(\int_0^s \mathrm{d}t\right)\left(\int_0^s |x(t)|^2\mathrm{d}t\right) \\ &= s\left(\int_0^s |x(t)|^2\mathrm{d}t\right).\end{aligned}$$

Consequently,

$$
\begin{aligned}
\int_0^1 |Vx(s)|^2 \mathrm{d}s &= \int_0^1 s \int_0^s |x(t)|^2 \mathrm{d}t\mathrm{d}s \\
&\le \int_0^1 s \int_0^1 |x(t)|^2 \mathrm{d}t\mathrm{d}s \\
&= \int_0^1 s\mathrm{d}s \int_0^1 |x(t)|^2 \mathrm{d}t \\
&= \frac{1}{2}||x||_2^2.
\end{aligned}
$$

So,

$$||Vx||_2^2 \le \frac{1}{2}||x||_2^2.$$

Thus, V is a bounded linear operator of norm not exceeding $\frac{1}{\sqrt{2}}$.

(x) Let H be the Hilbert space $L^2[0,1]$ of square integrable functions defined on $[0,1]$ and $D = C^1[0,1]$ be the linear subspace of continuously differentiable functions. Define

$$T{:}D \to L^2[0,1], \quad D \subseteq L^2[0,1],$$

by the rule

$$Tx(t) = x'(t), \quad t \in [0,1].$$

Clearly, T is linear. However, T is not bounded. In fact, for the sequence $x_n(t) = \sin n\pi t$, we have $Tx_n(t) = n\pi\cos n\pi t$ and

$$\int_0^1 |Tx_n(t)|^2 \mathrm{d}t = (n\pi)^2 \int_0^1 \cos^2 n\pi t \mathrm{d}t = (n\pi)^2 \int_0^1 \frac{\cos 2n\pi t + 1}{2} \mathrm{d}t = \frac{1}{2}(n\pi)^2$$

Consequently, $\|Tx_n\|_2 = \frac{n\pi}{\sqrt{2}}$.

Also,

$$||x_n||^2 = \int_0^1 \sin^2 n\pi t \, \mathrm{d}t = \int_0^1 \frac{1 - \cos 2n\pi t}{2} \mathrm{d}t = \frac{1}{2}.$$

Thus,

$$\|T\| = \sup_{\substack{x \in D \\ \|x\| \neq 0}} \frac{||Tx||}{||x||} \geq \sup_n \frac{||Tx_n||}{||x_n||} = \sup_n (n\pi) = \infty$$

and hence, T is not bounded.

Problem Set 3.2

3.2.P1. Let $[\tau_{i,j}]_{i,j \geq 1}$ be an infinite matrix [that is, a double sequence $\{\tau_{i,j}\}_{i,j \geq 1}$ normally presented as an array] and $K^2 = \sum_{i,j=1}^{\infty} \left|\tau_{i,j}\right|^2 < \infty$. The operator T is defined on ℓ^2 by

$$T\left(\{x_i\}_{i \geq 1}\right) = \{y_i\}_{i \geq 1},$$

where

$$y_i = \sum_{j=1}^{\infty} \tau_{i,j} x_j, \quad i = 1, 2, \ldots .$$

Show that T is a bounded linear operator on ℓ^2.

3.2.P2. Let H be a separable Hilbert space and $\{e_i\}_{i \geq 1}$ be an orthonormal basis. Let $T{:}H \to H$ be a bounded linear operator. Show that T is defined by the matrix $[(Te_j,e_i)]_{\,i,j \geq 1}$.

3.2.P3. Let $[\tau_{i,j}]_{i,j \geq 1}$ be an infinite matrix such that

$$\alpha_1 = \sup_j \sum_{i=1}^{\infty} \left|\tau_{i,j}\right| < \infty \quad \text{and} \quad \alpha_\infty = \sup_i \sum_{j=1}^{\infty} \left|\tau_{i,j}\right| < \infty.$$

Show that there is an operator T on H such that

$$\left(Te_j, e_i\right) = \tau_{i,j} \quad \text{and} \quad ||T||^2 \leq \alpha_1 \alpha_\infty.$$

3.2.P4. If $\tau_{i,j} \geq 0$ (i, j = 1, 2,…), if $p_i > 0$ (i = 1, 2,…) and if α_1 and α_∞ are positive numbers such that

$$\sum_{i=1}^{\infty} \tau_{i,j} p_i \leq \alpha_1 p_j, \quad j = 1, 2, \ldots,$$

$$\sum_{j=1}^{\infty} \tau_{i,j} p_j \leq \alpha_\infty p_i, \quad i = 1, 2, \ldots,$$

then there exists an operator T on ℓ^2 with $(Te_j, e_i) = \tau_{i,j}$ and $\|T\|^2 \leq \alpha_1 \alpha_\infty$.

3.2.P5. Show that the matrix $\left[\frac{1}{i+j-1}\right]_{i,j\geq 1}$ defines a bounded linear operator on ℓ^2 with $\|T\| \leq \pi$ (The matrix is known as the Hilbert matrix.).

3.2.P6. Let $\{e_n\}_{n\geq 1}$ be the usual basis for ℓ^2 and $\{\alpha_n\}_{n\geq 1}$ be a sequence of scalars. Show that there is a bounded linear operator T on ℓ^2 such that $Te_n = \alpha_n e_n$ for all n if, and only if, $\{\alpha_n\}_{n\geq 1}$ is bounded. This type of operator is called a *diagonal operator*.

3.2.P7. **(Laplace transform)** Let $x(t)$ be a complex-valued function on $\mathbb{R}^+ = \{t \in \mathbb{R}: t \geq 0\}$. Its Laplace transform Lx is the function on $\mathbb{R}^+$ defined by

$$y(s) = (Lx)(s) = \int_0^\infty x(t)\mathrm{e}^{-st}\mathrm{d}t.$$

Show that the Laplace transform is a bounded linear map of $L^2(\mathbb{R}^+)$ into itself and $\|L\| = \sqrt{\pi}$.

3.2.P8. Find an operator T on $\mathbb{R}^2$ for which $(Tx, x) = 0$ for all x and $\|T\| = 1$.

3.2.P9. If $\mathfrak{M}$ is a total subset of a Hilbert space H and $S, T \in \mathcal{B}(H)$ are such that $Sx = Tx$ for all $x \in \mathfrak{M}$, then $S = T$.

3.2.P10. Let $H = L^2[0,1]$ and

$$k(s,t) = \begin{cases} 0 & \text{if} \quad 0 \leq s < t \leq 1 \\ \frac{1}{\sqrt{s-t}} & \text{if} \quad 0 \leq t < s \leq 1. \end{cases}$$

(That $k(s,t)$ is undefined when $s = t$ is of no consequence.) Define

$$(Kx)(s) = \int_0^1 k(s,t)x(t)\mathrm{d}t.$$

Show that K is a bounded linear operator of norm at most 2.

3.2.P11. Let $\{a_i\}_{i\geq 1}$ be a sequence of complex numbers. Define an operator D_a on ℓ^2 by

$$D_a x = \{a_i x_i\}_{i\geq 1} \quad \text{for} \quad x = \{x_i\}_{i\geq 1} \in \ell^2.$$

Prove that D_a is bounded if, and only if, $\{a_i\}_{i\geq 1}$ is bounded and in this case $\|D_a\| = \sup|a_i|$.

3.2.P12. Let H_1 and H_2 be Hilbert spaces. Define $H_1 \oplus H_2$ [see Sect. 2.7] to be the Hilbert space consisting of all pairs $\langle u_1, u_2\rangle$, $u_i \in H_i$, $i = 1, 2$,

$$\langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle,$$
$$\lambda\langle u_1, u_2\rangle = \langle \lambda u_1, \lambda u_2\rangle,$$

the inner product being defined by

$$(\langle u_1, u_2\rangle, \langle v_1, v_2\rangle) = (u_1, v_1)_{H_1} + (u_2, v_2)_{H_2}.$$

Given $A_1 \in \mathcal{B}(H_1)$ and $A_2 \in \mathcal{B}(H_2)$, define A on H by the matrix

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

i.e., $A\langle u_1, u_2\rangle = \langle A_1u_1, A_2u_2\rangle$. Prove that $A \in \mathcal{B}(H)$ and that

$$||A|| = \max\{||A_1||, ||A_2||\}.$$

3.2.P13. Let $\ell^2(\mathbb{Z})$ be the Hilbert space of all sequences $\{\xi_j\}_{j\in\mathbb{Z}}$ with $\sum_{j=-\infty}^{\infty} \left|\xi_j\right|^2 < \infty$ and the usual inner product. Define an operator $S{:}\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by the formula

$$S\left(\left\{\xi_j\right\}_{j\in\mathbb{Z}}\right) = \left\{\xi_{j-1}\right\}_{j\in\mathbb{Z}}.$$

Show that $\|Sx\| = \|x\|$ for any $x \in \ell^2(\mathbb{Z})$. Give a formula and a matrix representation for the operator S^n for $n \in \mathbb{Z}$.

3.3 The Algebra of Operators

For a normed linear space X and a Banch space Y, the space $\mathcal{B}(X, Y)$ of bounded linear operators from X to Y is a Banach space [Theorem 3.2.4] in the norm defined by

$$||T|| = \sup_{\substack{x\in X \\ ||x||=1}} ||Tx|| = \sup_{\substack{x\in X \\ ||x|| \leq 1}} ||Tx|| = \sup_{\substack{x\in X \\ ||x||\neq 0}} \frac{||Tx||}{||x||}.$$

In what follows, we shall assume that $X = Y = H$, a Hilbert space. The Banach space $\mathcal{B}(X, Y)$ is then denoted by $\mathcal{B}(H)$. It turns out that $\mathcal{B}(H)$ is a "Banach algebra".

Definition 3.3.1 An **algebra** A **over a field** $\mathbb{F}$ is a vector space over $\mathbb{F}$ such that to each ordered pair of elements $x, y \in A$ a unique product $xy \in A$ is defined, with the properties

$$\begin{aligned}
(xy)z &= x(yz) \\
x(y+z) &= xy + xz \\
(x+y)z &= xz + yz \\
\alpha(xy) &= (\alpha x)y = x(\alpha y)
\end{aligned}$$

for all $x, y, z \in A$ and $\alpha \in \mathbb{F}$.

Depending on whether $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, A is called a *real* or *complex algebra*. A is said to be **commutative** if the multiplication is commutative, that is, for all $x, y \in A$,

$$xy = yx.$$

A is called an *algebra with identity* if it contains an element $e \neq 0$ such that for all $x \in A$, we have

$$xe = ex = x.$$

The element e is called an *identity*. If A has an identity, it is unique.

It may be noted that $\mathbb{F}$ and $\mathcal{B}(H)$ are algebras with identity.

Definition 3.3.2 A **normed algebra** is a normed space which is an algebra such that for all $x, y \in A$,

$$\|xy\| \leq \|x\| \|y\|$$

and if A has an identity e,

$$\|e\| = 1.$$

A **Banach algebra** is a normed algebra which is complete considered as a normed space.

The space $C[a, b]$ of continuous functions defined on $[a, b]$ is a commutative Banach algebra in which the product is defined by

$$xy(t) = x(t)y(t)$$

and

$$\|x\| = \sup_{t \in [0,1]} |x(t)|.$$

The commutative algebra has an identity, namely the function 1.

Theorem 3.3.3 $(\mathcal{B}(H), \|\bullet\|)$, *where* $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$, $T \in \mathcal{B}(H)$, *is a Banach algebra with identity, provided that* $H \neq \{0\}$.

Proof Since

$$\|(ST)(x)\| = \|S(Tx)\| \leq \|S\| \|Tx\| \leq \|S\| \|T\| \|x\|, \quad S, T \in \mathcal{B}(H),$$

it follows that

$$\|ST\| \leq \|S\| \|T\|.$$

That $\mathcal{B}(H)$ is a Banach space has been checked in Theorem 3.2.4. The operator I is the identity and satisfies $\|I\| = 1$ when $H \neq \{0\}$. □

Remarks 3.3.4

(i) If the dimension of H is 2 or greater, the algebra $\mathcal{B}(H)$ is not commutative. For example,

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and}$$
$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

(ii) As in every algebra, T^n will denote the product of n factors all equal to T, $n = 1,2,\ldots$; T^0 is defined to be I, the identity operator. More generally, if $p(\lambda) = \sum_{j=0}^{n} \alpha_j \lambda_j$ is any polynomial, we shall use the symbol $p(T)$, $T \in \mathcal{B}(H)$, for the operator $\sum_{j=0}^{n} \alpha_j T^j$.

(iii) Let H be a Hilbert space different from $\{0\}$. We have seen that $\mathcal{B}(H)$ is a Banach algebra with identity I and norm $\|T\| = \sup\limits_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\|$.

From now on, the Hilbert space will always be assumed to contain nonzero vectors.

Definition 3.3.5 A sequence $\{T_n\}_{n \geq 1}$ in $\mathcal{B}(H)$ **converges** to $T \in \mathcal{B}(H)$ **in the uniform operator norm** if $\lim_n \|T_n - T\| = 0$.

There are two other modes of convergence: *strong operator convergence* and *weak operator convergence*.

Definition 3.3.6 The sequence $\{T_n\}_{n \geq 1}$ in $\mathcal{B}(H)$ **converges strongly** to $T \in \mathcal{B}(H)$ if, for each $x \in H$, $\lim_n \|T_n x - Tx\| = 0$. The sequence $\{T_n\}_{n \geq 1}$ in $\mathcal{B}(H)$ **converges weakly** to $T \in \mathcal{B}(H)$ if, for all $x, y \in H$, $\lim_n |(T_n x, y) - (Tx, y)| = 0$.

Clearly, uniform operator convergence implies strong operator convergence and strong operator convergence implies weak operator convergence. The reverse implications, namely that strong operator convergence implies uniform operator convergence and that weak operator convergence implies strong operator convergence, are not true in general [see Problem 3.8.P1].

These are some of the important modes of convergence in $\mathcal{B}(H)$. They will suffice for any developments we contemplate.

The inverses of certain operators will be of concern in later Sections. If $T \in \mathcal{B}(H)$, where H is of course a Hilbert space, and I is the identity operator, we shall be concerned with the operator $(T - \lambda I)^{-1}$, $\lambda \in \mathbb{C}$. When $H = \mathbb{C}^n$ and T is a linear operator on H, the set of λ's for which $(T - \lambda I)^{-1}$ does not exist are precisely the eigenvalues of T. When H is infinite-dimensional, the set of λ's for which $(T - \lambda I)^{-1}$ does not exist will turn out to be a nonempty compact subset of the complex plane. Assuming that $(T - \lambda I)^{-1}$ exists, in which case it is obviously linear, it will be of interest to know whether it is bounded.

The treatment of the above question leads us into what is known as 'spectral theory' or 'spectral analysis'.

Definition 3.3.7 Let $T \in \mathcal{B}(H)$. T is said to be **invertible in** $\mathcal{B}(\boldsymbol{H})$ if it has a set theoretic inverse T^{-1} and $T^{-1} \in \mathcal{B}(\boldsymbol{H})$.

It is known that when the set theoretic inverse T^{-1} of an operator $T \in \mathcal{B}(H)$ exists, it is in $\mathcal{B}(H)$ [Theorem 5.5.2].

The following fundamental proposition will be used to show that the collection of invertible elements in $\mathcal{B}(H)$ is an open set and inversion is continuous in the uniform operator norm.

Proposition 3.3.8 *If $T \in \mathcal{B}(H)$ and $\|I - T\| < 1$, then T is invertible and*

$$T^{-1} = \sum_{k=0}^{\infty} (I - T)^k,$$

where convergence takes place in the uniform operator norm. Moreover,

$$\|T^{-1}\| \leq \frac{1}{1 - \|I - T\|}.$$

Proof Set $\eta = \|I - T\| < 1$. Then for $n > m$, we have

$$\begin{aligned}\left\|\sum_{k=0}^{n} (I - T)^k - \sum_{k=0}^{m} (I - T)^k\right\| &= \left\|\sum_{k=m+1}^{n} (I - T)^k\right\| \leq \sum_{k=m+1}^{n} \|(I - T)\|^k \\ &= \sum_{k=m+1}^{n} \eta^k < \frac{\eta^{m+1}}{1 - \eta}.\end{aligned}$$

The sequence of partial sums $\left\{\sum_{k=0}^{n} (I - T)^k\right\}_{n \geq 0}$ is Cauchy. If $S = \sum_{k=0}^{\infty} (I - T)^k$, then

$$\begin{aligned}TS &= [I - (I - T)]\left(\sum_{k=0}^{\infty} (I - T)^k\right) \\ &= \lim_n [I - (I - T)] \sum_{k=0}^{n} (I - T)^k \\ &= \lim_n \left[I - (I - T)^{n+1}\right] = I.\end{aligned}$$

since $\lim_n \|(I - T)^{n+1}\| = 0$. Similarly, $ST = I$, so that T is invertible with $T^{-1} = S$. Moreover,

$$\|S\| = \lim_n \left\|\sum_{k=0}^{n} (I - T)^k\right\| \leq \lim_n \sum_{k=0}^{n} \|I - T\|^k = \frac{1}{1 - \|I - T\|}. \qquad \square$$

Let $\mathcal{G}$ denote the set of invertible elements in $\mathcal{B}(H)$.

Proposition 3.3.9 *If $T \in \mathcal{G}$ and $S \in \mathcal{B}(H)$ satisfies $\|S - T\| < \frac{1}{\|T^{-1}\|}$, then S is invertible. In particular, the set $\mathcal{G}$ is open in $\mathcal{B}(H)$. Moreover, the map $T \to T^{-1}$ defined on $\mathcal{G}$ is continuous.*

Proof Let $T \in \mathcal{G}$. Consider $\{S \in \mathcal{B}(H) : \|S - T\| < \frac{1}{\|T^{-1}\|}\}$. Then, $1 > \|T^{-1}\|\|S - T\| \geq \|T^{-1}S - I\|$. The preceding Proposition 3.3.8 implies that $T^{-1}S \in \mathcal{G}$, and hence, $S = T(T^{-1}S)$ is in $\mathcal{G}$ (the product of invertible elements is invertible). Thus, the ball of radius $\frac{1}{\|T^{-1}\|}$ about each of its elements T, namely $\{S \in \mathcal{B}(H) : \|S - T\| < \frac{1}{\|T^{-1}\|}\}$, is contained in $\mathcal{G}$. Consequently, $\mathcal{G}$ is an open subset of $\mathcal{B}(H)$.

It remains to show that the map $T \to T^{-1}$ is continuous on $\mathcal{G}$.

If $T \in \mathcal{G}$, then the inequality $\|T - S\| < \frac{1}{2\|T^{-1}\|}$ implies that $\|I - T^{-1}S\| < \frac{1}{2}$ and hence

$$\begin{aligned}
\|S^{-1}\| &= \|S^{-1}TT^{-1}\| \\
&\leq \|S^{-1}T\|\|T^{-1}\| \\
&= \|(T^{-1}S)^{-1}\|\|T^{-1}\| \\
&\leq \frac{1}{1 - \|I - T^{-1}S\|}\|T^{-1}\| \leq 2\|T^{-1}\|,
\end{aligned}$$

by Proposition 3.3.8. Thus, the inequality

$$\|T^{-1} - S^{-1}\| = \|T^{-1}(T - S)S^{-1}\| \leq 2\|T^{-1}\|^2\|T - S\|$$

shows that the map $T \to T^{-1}$ is continuous on $\mathcal{G}$. □

Remark 3.3.10 The reader is undoubtedly familiar with the equivalence of the following assertions when H is finite-dimensional:

(i) T is invertible;
(ii) T is injective;
(iii) T is surjective;
(iv) there exists $S \in \mathcal{B}(H)$ such that $TS = I$; and
(v) there exists $S \in \mathcal{B}(H)$ such that $ST = I$.

The above assertions are not equivalent in infinite-dimensional spaces. Let $H = \ell^2$ and T denote the "right shift":

$$T(\{x_i\}_{i \geq 1}) = (0, x_1, x_2, \ldots).$$

Then, T is injective but not surjective, and thus not invertible.

The operator S defined by

$$S\big(\{x_i\}_{i\geq 1}\big) = (x_2, x_3, x_4, \ldots)$$

is surjective but not injective and thus also not invertible. Moreover, $ST(\{x_i\}_{i\geq 1}) = S(0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$, which means $ST = I$. The reader may note that $TS \neq I$.

Furthermore, no operator in a ball of radius 1 around T is invertible. Indeed, if $\|T - A\| < 1$, then

$$\|I - SA\| = \|S(T - A)\| \leq \|S\|\|T - A\| < 1$$

since $\|S\| \leq 1$. This implies SA is invertible by Proposition 3.3.8. If A were invertible, so would be S; but this is not the case.

We next derive useful criteria for the invertibility of an operator.

Definition 3.3.11 An operator $T \in \mathcal{B}(H)$ is said to be **bounded below** if there exists an $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all x $\in H$.

An operator which is bounded below is clearly injective.

Theorem 3.3.12 An operator $T \in \mathcal{B}(H)$ *is invertible if, and only if, it is bounded below and has dense range.*

Proof If T is invertible, then the range of T is H and is therefore dense. Moreover,

$$\|Tx\| \geq \frac{1}{\|T^{-1}\|}\left\|T^{-1}Tx\right\| = \frac{1}{\|T^{-1}\|}\|x\|, \quad x \in H$$

and therefore, T is bounded below.

Conversely, if T is bounded below, there exists an $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in H$. Hence, if $\{Tx_n\}_{n\geq 1}$ is a Cauchy sequence in H, then the inequality

$$\|x_n - x_m\| \leq \frac{1}{\alpha}\|Tx_n - Tx_m\|$$

implies $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in H. Let $x = \lim_n x_n$. Then, $x \in H$ and $Tx = \lim_n Tx_n$; and hence, ran(T) is closed. Since ran(T) is dense in H, it follows that ran$(T) = H$. As T is bounded below, this implies T^{-1} is well defined. Moreover, if $y = Tx$, then

$$\left\|T^{-1}y\right\| = \|x\| \leq \frac{1}{\alpha}\|Tx\| = \frac{1}{\alpha}\|y\|. \quad \square$$

We proceed to study vector-valued functions, which will be needed in Sect. 4.3 below.

Definition 3.3.13 Let f be a function defined in a domain Ω of the complex plane whose values are in a complex Banach space X.

(a) $f(\zeta)$ is **strongly holomorphic** in Ω if the limit

$$\lim_{h\to 0}\frac{f(\zeta+h)-f(\zeta)}{h}$$

exists in the norm (of X) at every point ζ of Ω.

(b) $f(\zeta)$ is **weakly holomorphic** in Ω if for every bounded linear functional F on X, $F(f(\zeta))$ is holomorphic in Ω in the classical sense.

The words *holomorphic* and *analytic* will be used interchangeably, as is the usual practice.

Every strongly holomorphic function is weakly holomorphic. N. Dunford has proved the following surprising result.

Theorem 3.3.14 *Let $f{:}\Omega\to X$ be a weakly holomorphic function from Ω to X. Then, f is strongly holomorphic.*

Proof For a bounded linear functional F on X, $F(f(\zeta))$ is holomorphic in Ω; so, we can represent it by the Cauchy integral formula

$$F(f(\zeta))=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{F(f(z))}{z-\zeta}\mathrm{d}z,$$

where γ is a simple closed rectifiable curve around ζ in Ω. Hence, for small $|h|$,

$$\begin{aligned}&\frac{F(f(\zeta+h))-F(f(\zeta))}{h}-\frac{F(f(\zeta+k))-F(f(\zeta))}{k}\\&=\frac{1}{2\pi ih}\int\limits_{\gamma}F(f(z))\left[\frac{1}{z-\zeta-h}-\frac{1}{z-\zeta}\right]\mathrm{d}z-\frac{1}{2\pi ik}\int\limits_{\gamma}\left[\frac{1}{z-\zeta-k}-\frac{1}{z-\zeta}\right]\mathrm{d}z\\&=\frac{h-k}{2\pi i}\int\limits_{\gamma}\frac{F(f(z))}{(z-\zeta-h)(z-\zeta-k)(z-\zeta)}\mathrm{d}z.\end{aligned}$$

So,

$$\begin{aligned}&\frac{1}{h-k}\left(\frac{F(f(\zeta+h))-F(f(\zeta))}{h}-\frac{F(f(\zeta+k))-F(f(\zeta))}{k}\right)\\&\quad=\frac{1}{2\pi i}\int_{\gamma}\frac{F(f(z))}{(z-\zeta-h)(z-\zeta-k)(z-\zeta)}\mathrm{d}z.\end{aligned}\tag{3.10}$$

Since γ is compact and the function $F(f(\cdot))$ is a continuous function, $|F(f(z))|$ is bounded. For small enough $|h|$ and $|k|$, it now follows that the right-hand side of (3.10) is bounded. Hence, by the uniform boundedness principle [Theorem 5.4.6], there exists a constant $C>0$ such that

$$\left\| \frac{f(\zeta+h)-f(\zeta)}{h} - \frac{f(\zeta+k)-f(\zeta)}{k} \right\| \leq C|h-k|.$$

(Any element of X may be thought of as a linear functional on X^*). Since X is complete, it follows that the difference quotient of f tends to a limit as h tends to 0. Thus, $f(\zeta)$ is strongly analytic. □

A holomorphic function $f:\Omega \to X$ has a Taylor series representation at every $z \in \Omega$, i.e., for every $z \in \Omega$, there is an $r = r(z)$ such that $D(z,r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\} \subset \Omega$ and

$$f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - z)^n \tag{3.11}$$

for some a_0, a_1,… in X and all $\zeta \in D(z,r)$ with series (3.11) being absolutely convergent ($\sum_{k=0}^{\infty} ||a_n|||\zeta - z|^n < \infty$).

The other standard results concerning holomorphic functions remain valid in this more general setting. These results can be proved by the same method that is used for complex functions.

Also, the radius of convergence of (3.11) is lim inf$||a_n||^{\frac{1}{n}}$ just as in the classical case. Correspondingly, the Laurent series

$$g(\zeta) = \sum_{n=0}^{\infty} b_n\zeta^{-n} \tag{3.12}$$

has radius of convergence s = lim sup$||b_n||^{\frac{1}{n}}$ Indeed, if $|\zeta| > s$, then choosing $\varepsilon > 0$ such that $(1 + \varepsilon)s/|\zeta| < 1$, we have $||b_n||^{\frac{1}{n}} < (1 + \varepsilon)s$ for every sufficiently large n. Hence, $\|b_n\zeta^{-n}\| < ((1 + \varepsilon)s/|\zeta|)^n$ if n is sufficiently large, implying that (3.12) is absolutely convergent. Conversely, if $|\zeta| < s$, then there is an infinite sequence $n_1 < n_2 < \ldots$ such that $||b_{n_k}|| > |\zeta|^{n_k}$. But then $\|b_{n_k}\zeta^{n_k}\| > 1$ and so, (3.12) does not converge.

Problem Set 3.3

3.3.P1. Let H be a Hilbert space and let $T_1, T_2, T_3 \in \mathcal{B}(H)$. On $H^{(3)} = H \oplus H \oplus H$, define T by the matrix

$$T = \begin{bmatrix} 0 & T_3 & T_1 \\ 0 & 0 & T_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Prove that $T \in \mathcal{B}(H^{(3)})$. For $\alpha \in \mathbb{C}$, show that $(I - \alpha T)$ is invertible and find its inverse.

3.3.P2. Let $\mu = \{\mu_k\}_{k \geq 1}$ be a sequence of complex numbers with $\sup_k|\mu_k| < 1$. Prove that the following systems of equations have unique solutions in ℓ^2 for any $\{\eta_k\}_{k \geq 1} \in \ell^2$. Find the solutions for $\eta_k = \delta_{1k}, \mu_k = \frac{1}{2^{k-1}}$.

(a) $\xi_k - \mu_k\xi_{k+1} = \eta_k$, $k = 1, 2, \ldots$
(b) $\xi_k - \mu_k\xi_{k-1} = \eta_k$, $k = 2, 3, \ldots$ and $\xi_1 = 1$.

3.3.P3. Show that $T \in \mathcal{B}(H)$ is surjective if, and only if, T^* is bounded below.
3.3.P4. Show that if $T \geq O$ then $(I + T)^{-1}$ exists.

3.4 Sesquilinear Forms

In this section, a new kind of functional—a sesquilinear functional, or a sesquilinear form, will be introduced. On the pattern of linear functionals, the notion of bounded sesquilinear functionals is studied. A characterisation of such functionals is provided.

Definition 3.4.1 Let X be a vector space over $\mathbb{C}$. *A* **sesquilinear form** on X is a mapping B from $X \times X$ into the complex plane $\mathbb{C}$ with the following properties:

(i) $$B(x_1 + x_2, y) = B(x_1, y) + B(x_2, y),$$

(ii) $$B(x, y_1 + y_2) = B(x, y_1) + B(x, y_2),$$

(iii) $$B(\alpha x, y) = \alpha B(x, y) \text{and}$$

(iv) $$B(x, \beta y) = \bar{\beta} B(x, y).$$

for all x, x_1, x_2, y, y_1, y_2 in X and all scalars α, β in $\mathbb{C}$.

Thus, B is linear in the first argument and conjugate linear in the second argument. If X is a real vector space, then (iv) is simply

$$B(x, \beta y) = \beta B(x, y).$$

and B is called **bilinear**, since it is linear in each of the two arguments.

Definition 3.4.2 A **Hermitian form** B on a complex vector space X is a mapping from $X \times X$ into the complex plane $\mathbb{C}$ satisfying properties (i), (ii), (iii) and the additional property

(v) $$B(x, y) = \overline{B(y, x)}.$$

It is then obvious that B must also have the property (iv) above and thus be sesquilinear. However, a sesquilinear form need not be Hermitian, for example, $B(x, y) = i(x, y)$, where (x, y) on the right denotes an inner product in X. In this connection, see (ii) of Remark 3.4.4 below.

A sesquilinear form B on X is said to be *nondegenerate* if it has the following property:

(vi) If $x \in X$ is such that for all $y \in X$, $B(x, y) = 0$, then $x = 0$; if $y \in X$ is such that for all $x \in X$, $B(x, y) = 0$, then $y = 0$.

Example 3.4.3

(i) The inner product in any pre-Hilbert space is a nondegenerate Hermitian form. In particular, the usual inner product $(x, y) = \sum_{i=1}^{n} x_i\overline{y_i}$ is a nondegenerate Hermitian form on $\mathbb{C}^n$. But if we delete one or more terms in the preceding sum, it will define a degenerate Hermitian form on $\mathbb{C}^n$.
(ii) The form

$$B(x, y) = x_1\overline{y_1} - x_2\overline{y_2}$$

is a nondegenerate form on $\mathbb{C}^2$ (it would be degenerate when viewed as a form on $\mathbb{C}^n$, $n > 2$).

Remarks 3.4.4

(i) The property (iv) above is responsible for the name "sesquilinear"; the Latin word "sesquilinear" means one time and a half.
(ii) A sesquilinear form is Hermitian if, and only if, $B(x, x)$ is a real number for all x.

It follows in view of the property (v) with $y = x$ that $B(x, x) = \overline{B(x, x)}$, that is, $B(x, x)$ is real. On the other hand, we have

$$B(x + y, x + y) - B(x, x) - B(y, y) = B(x, y) + B(y, x). \tag{3.13}$$

Since the left-hand side of the above equality (3.13) is real for all x and y in X, it implies

$$\Im B(x, y) = -\Im B(y, x). \tag{3.14}$$

Apply (3.13) with iy in place of y. The left side must again be real and so must be the right-hand side, which is now, in view of the sesquilinearity,

$$i[-B(x, y) + B(y, x)].$$

Consequently, $\Re[-B(x, y) + B(y, x)] = 0$, which implies $\Re B(y, x) = \Re B(x, y)$. Hence, in view of (3.14), $B(x, y) = \overline{B(y, x)}$.We shall essentially be interested in *positive definite forms*. These are sesquilinear forms which satisfy the following condition:

$$\text{for all} \quad x \in X, x \neq 0, B(x, x) > 0.$$

In particular, positive definite sesquilinear forms are Hermitian. They are obviously nondegenerate.

Sesquilinear forms which satisfy the weaker condition, namely

$$\text{for all} \quad x \in X, x \neq 0, B(x, x) \geq 0$$

are called *nonnegative sesquilinear* forms.

We now present a result for sesquilinear forms generalising the Cauchy–Schwarz inequality for inner products.

Theorem 3.4.5 *Let B be a nonnegative sesquilinear form on the complex vector space X. Then,*

$$|B(x, y)|^2 \leq B(x, x)B(y, y) \quad \textit{for all} \quad x, y \in X.$$

Proof If $B(x, y) = 0$, the inequality is, of course, true. Suppose $B(x, y) \neq 0$. Then for arbitrary complex numbers α, β, we have

$$\begin{aligned} 0 &\leq B(\alpha x + \beta y, \alpha x + \beta y) \\ &= \alpha\bar{\alpha}B(x, x) + \alpha\bar{\beta}B(x, y) + \bar{\alpha}\beta B(y, x) + \beta\bar{\beta}B(y, y) \\ &= \alpha\bar{\alpha}B(x, x) + \alpha\bar{\beta}B(x, y) + \bar{\alpha}\beta\overline{B(x, y)} + \beta\bar{\beta}B(y, y) \end{aligned}$$

since B is nonnegative. Now let $\alpha = t$ be real and set $\beta = B(x, y)/|B(x, y)|$. Then,

$$\beta B(y, x) = |B(x, y)| \quad \text{and} \quad \beta\bar{\beta} = 1.$$

Hence,

$$0 \leq t^2 B(x, x) + 2t|B(x, y)| + B(y, y)$$

for an arbitrary real number t. Thus, the discriminant

$$4|B(x,y)|^2 - 4B(x,x)B(y,y) \le 0,$$

which completes the proof. □

Definition 3.4.6 Let H be a Hilbert space. The sesquilinear form B is said to be **bounded** if there exists some positive constant M such that

$$|B(x,y)| \le M\|x\|\|y\| \quad \text{for all} \quad x, y \in H.$$

The **norm** of B is defined by

$$\|B\| = \sup_{\|x\|=\|y\|=1} |B(x,y)| = \sup_{\substack{x\in H, y\in H \\ x\ne 0\ne y}} \frac{|B(x,y)|}{\|x\|\|y\|}.$$

Example 3.4.7

(i) If H is a Hilbert space, the sesquilinear form $B{:}H \times H \to \mathbb{C}$ defined by $B(x, y) = (x, y)$ is bounded by the Cauchy–Schwarz inequality. Moreover, $\|B\| = 1$. Indeed, $|B(x, y)| = |(x, y)| \le \|x\|\|y\|$, and so, $\|B\| \le 1$. For $y = x$, $|B(x, y)| = |(x, x)| = \|x\|^2 = 1$ if $\|x\| = 1$.

(ii) If H is a Hilbert space and $T{:}H \to H$ is a bounded linear operator, then $B(x, y) = (Tx, y)$ is a bounded sesquilinear form with $\|B\| = \|T\|$. Indeed, for $x, y \in H$, $\|x\| = \|y\| = 1$,

$$|B(x,y)| = |(Tx,y)| \le \|Tx\|\|y\| \le \|T\|;$$

hence,

$$\|B\| \le \|T\|.$$

On the other hand, for $y = Tx$,

$$\|B\| \ge \frac{|B(x,Tx)|}{\|x\|\|Tx\|} = \frac{\|Tx\|^2}{\|x\|\|Tx\|} = \frac{\|Tx\|}{\|x\|},$$

which implies

$$\|B\| \ge \|T\|.$$

(iii) A bounded sesquilinear form $B{:}H \times H \to \mathbb{C}$ is **jointly continuous** in both variables:

$$\begin{aligned}|B(x,y) - B(x_0,y_0)| &= |B(x-x_0, y-y_0) + B(x-x_0, y_0) + B(x_0, y-y_0)| \\ &\le \|B\|(\|x-x_0\|\|y-y_0\| + \|x-x_0\|\|y_0\| + \|x_0\|\|y-y_0\|).\end{aligned}$$

It is interesting that the Riesz Representation Theorem 2.10.25 yields a general representation of sesquilinear forms on Hilbert space.

Theorem 3.4.8 *Let H be a Hilbert space and $B(\cdot,\cdot):H \times H \to \mathbb{C}$ be a bounded sesquilinear form. Then, B has a representation*

$$B(x, y) = (Sx, y),$$

where $S:H \to H$ is a bounded linear operator. S is uniquely determined by B and has norm

$$\|S\| = \|B\|.$$

Proof For fixed x, the expression $\overline{B(x, y)}$ defines a linear functional in y whose domain is H. Then, the Theorem 2.10.25 of F. Riesz yields an element $z \in H$ such that

$$\overline{B(x, y)} = (y, z).$$

Hence,

$$B(x, y) = (z, y).$$

Here, z is unique but, of course, depends on $x \in H$. Define the mapping $S:H \to H$ by $Sx = z$, $x \in H$. Then,

$$B(x, y) = (Sx, y).$$

Since

$$B(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 B(x_1, y) + \alpha_2 B(x_2, y),$$

we have

$$(S(\alpha_1 x_1 + \alpha_2 x_2) - \alpha_1 S x_1 - \alpha_2 S x_2, y) = 0, \quad y \in H.$$

Since y is arbitrary,

$$S(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 S x_1 + \alpha_2 S x_2,$$

so that S is a linear operator. The domain of the operator S is the whole of H. Furthermore, since $|(Sx, y)| \leq \|Sx\|\|y\|$, we have

$$||B|| = \sup_{\substack{x \in H, y \in H \\ x \neq 0 \neq y}} \frac{|B(x, y)|}{||x||\,||y||} = \sup_{\substack{x \in H, y \in H \\ x \neq 0 \neq y}} \frac{|(Sx, y)|}{||x||\,||y||} \leq \sup_{x \neq 0} \frac{||Sx||}{||x||} = ||S||.$$

On the other hand,

$$||B|| = \sup_{\substack{x\in H, y\in H \\ x\neq 0\neq y}} \frac{|(Sx,y)|}{||x|| \, ||y||} \geq \sup_{\substack{x\in H \\ x\neq 0\neq Sx}} \frac{|(Sx,Sx)|}{||x|| \, ||Sx||} = \sup_{x\neq 0} \frac{||Sx||}{||x||} = ||S||.$$

It remains to check that S is unique. Suppose there is a linear operator $T{:}H\to H$ such that for x, $y \in H$, we have

$$B(x,y) = (Sx,y) = (Tx,y).$$

It then follows that

$$((S-T)x,y) = 0, x,y \in H.$$

Setting $y = (S - T)x$, we obtain $\|(S - T)x\| = 0$, that is, $Sx = Tx$ for each $x \in H$. Consequently, $S = T$. □

The following simple Theorem is often useful:

Theorem 3.4.9 *If a complex scalar function* $B{:}H \times H\to\mathbb{C}$, *where* H *denotes a Hilbert space*, *satisfies the following conditions*:

(i) $$B(x_1+x_2,y) = B(x_1,y)+B(x_2,y),$$

(ii) $$B(x,y_1+y_2) = B(x,y_1)+B(x,y_2),$$

(iii) $$B(\alpha x,y) = \alpha B(x,y),$$

(iv) $$B(x,\beta y) = \overline{\beta} B(x,y),$$

(v) $$|B(x,x)| \leq M\|x\|^2 \text{and}$$

(vi) $$|B(x,y)| = |B(y,x)|,$$

where M *is a constant*; x, x_1, x_2, y, y_1, y_2 *are arbitrary elements of* H; α, β *are scalars*, *then* B *is a bounded sesquilinear functional with* $\|B\| \leq M$.

Proof From (i)–(iv), it follows that

$$B(x,y)+B(y,x) = \frac{1}{2}[B(x+y,x+y) - B(x-y,x-y)].$$

This implies

$$|B(x,y)+B(y,x)| \leq \frac{1}{2}M\left[||x+y||^2+\|x-y\|^2\right] = M\left[||x||^2+\|y\|^2\right]. \tag{3.15}$$

Let $\|x\| \leq 1$, $\|y\| \leq 1$ and $y = \lambda z$, where λ is a complex number of absolute value 1, to be specified later. Then, (3.15) yields

$$\left|\overline{\lambda}B(x,z)+\lambda B(z,x)\right| \leq 2M. \tag{3.16}$$

Assume that $B(x, z) \neq 0$ and let

$$B(x,z) = |B(x,z)|\mathrm{e}^{i\gamma}, B(z,x) = |B(z,x)|\mathrm{e}^{i\delta}.$$

Then by (3.16) and (vi),

$$|B(x,z)|\left|\overline{\lambda}\mathrm{e}^{i\gamma}+\lambda\mathrm{e}^{i\delta}\right| \leq 2M.$$

Letting $\lambda = \mathrm{e}^{i(\gamma-\delta)/2}$, we find that

$$\overline{\lambda}\mathrm{e}^{i\gamma}+\lambda\mathrm{e}^{i\delta} = \mathrm{e}^{i(\gamma+\delta)/2}+\mathrm{e}^{i(\gamma+\delta)/2} = 2\mathrm{e}^{i(\gamma+\delta)/2},$$

which yields

$$|B(x,z)| \leq M, ||x|| \leq 1, ||z|| \leq 1.$$

As the relation obviously holds for $B(x,z) = 0$, the result follows. □

Corollary 3.4.10 *If the bounded sesquilinear functional B satisfies the condition*

$$|B(x,y)| = |B(y,x)|, x,y \in H,$$

then

$$||B|| = \sup_{\substack{x\in H \\ ||x||\neq 0}} \frac{|B(x,x)|}{||x||^2}.$$

Proof The supremum in question is obviously a possible value of M that satisfies (v) of Theorem 3.4.9. It follows that

$$||B|| \leq \sup_{\substack{x\in H \\ ||x||\neq 0}} \frac{|B(x,x)|}{||x||^2};$$

but on the other hand,

$$\sup_{\substack{x\in H\\ \|x\|\neq 0}} \frac{|B(x,x)|}{\|x\|^2} \leq \sup_{\substack{x\in H, y\in H\\ x\neq 0\neq y}} \frac{|B(x,y)|}{\|x\|\|y\|} = \|B\|$$ □

The following result, which is a special case of Corollary 3.4.10, plays an important role in the exposition of spectral theory given in subsequent pages.

Corollary 3.4.11 *If H is a Hilbert space, the norm of a Hermitian bounded sesquilinear form* $B{:}H \times H \to \mathbb{C}$ *is given by the formula*

$$\|B\| = \sup_{\substack{x\in H\\ \|x\|\neq 0}} \frac{|B(x,x)|}{\|x\|^2}.$$

Proof Indeed, a Hermitian bounded sesquilinear form B satisfies the condition $|B(x, y)| = |B(y, x)|$. □

Problem Set 3.4

3.4.P1. Let $B(\cdot,\cdot)$ be a bounded sesquilinear form on a Hilbert space H. Show that

(a) **(Parallelogram law)** For all $x, y \in H$,

$$B(x+y, x+y) + B(x-y, x-y) = 2B(x,x) + 2B(y,y);$$

(b) **(Polarisation identity)** For all $x, y \in H$,

$$4B(x,y) = B(x+y, x+y) - B(x-y, x-y) + iB(x+iy, x+iy) \\ -iB(x-iy, x-iy).$$

(c) $B = 0$ if, and only if, $B(x,x) = 0$ for all $x \in H$.

3.4.P2. A function f defined on a Hilbert space H is called a **quadratic form** if there exists a sesquilinear form B on $H \times H$ such that $f(x) = B(x, x)$. Show that a pointwise limit of quadratic forms is a quadratic form.

3.5 The Adjoint Operator

The study of bilinear forms on a Hilbert space H yields rich dividends. The algebra $\mathcal{B}(H)$ of bounded linear operators on H admits a canonical bijection $T \to T^*$ possessing pleasant algebraic properties. Moreover, many properties of T can be studied through the operator T^*. It also helps us to study three important classes of operators, namely self-adjoint, unitary and normal operators. These classes have been studied extensively, because they play an important role in various applications.

Definition 3.5.1 Let T be a bounded linear operator on a Hilbert space H. Then, the **Hilbert space adjoint** T^* of T is the operator

$$T^*:H \to H$$

such that for all x, $y \in H$,

$$(Tx, y) = (x, T^*y).$$

We first show that this definition makes sense and also prove that the adjoint operator has the same norm.

Theorem 3.5.2 *The Hilbert space adjoint T^* of T in* Definition 3.5.1 *exists*, *is unique and is a bounded linear operator with norm*

$$||T^*|| = ||T||.$$

Proof The formula

$$B(y, x) = (y, Tx), x, y \in H, \tag{3.17}$$

defines a bounded sesquilinear form on $H \times H$, because the inner product is a sesquilinear form and T is a bounded linear operator. Indeed, for y_1, y_2, x_1, x_2 in H and α, β scalars,

$$\begin{aligned} B(\alpha y_1 + \beta y_2, x) &= (\alpha y_1 + \beta y_2, Tx) \\ &= (\alpha y_1, Tx) + (\beta y_2, Tx) \\ &= \alpha(y_1, Tx) + \beta(y_2, Tx) \\ &= \alpha B(y_1, Tx) + \beta B(y_2, x) \end{aligned}$$

and

$$\begin{aligned} B(y, \alpha x_1 + \beta x_2) &= (y, T(\alpha x_1 + \beta x_2)) \\ &= (y, \alpha T x_1 + \beta T x_2) \\ &= \overline{\alpha}(y, Tx_1) + \overline{\beta}(y, Tx_2) \\ &= \overline{\alpha} B(y, x_1) + \overline{\beta} B(y, x_2). \end{aligned}$$

Moreover, B is bounded:

$$|B(x, y)| = |(x, Ty)| \le \|x\|\|Ty\| \le \|T\|\|x\|\|y\|. \tag{3.18}$$

This implies $\|B\| \le \|T\|$. Also,

$$\|B\| = \sup_{\substack{x\neq 0 \\ y\neq 0}} \frac{(y, Tx)}{\|y\|\|x\|} \ge \sup_{\substack{x\neq 0 \\ Tx\neq 0}} \frac{|(Tx, Tx)|}{\|Tx\|\|x\|} = \|T\|. \tag{3.19}$$

From (3.18) and (3.19), we conclude that

$$||B|| = ||T||. \tag{3.20}$$

From the representation Theorem 3.4.8 for bounded sesquilinear forms, we have

$$B(y, x) = (T^* y, x), \tag{3.21}$$

where we have replaced S of Theorem 3.4.8 by T^*.

The operator $T^*{:}H \to H$ is a uniquely defined bounded linear operator with norm

$$||T^*|| = ||B|| = ||T||.$$

The last equality is the assertion of (3.20). Note that

$$(y, Tx) = (T^* y, x) \tag{3.22}$$

follows on comparing (3.17) and (3.21). On taking conjugates in (3.22), we obtain

$$(Tx, y) = (x, T^* y).$$

This completes the proof. □

Remarks 3.5.3

(i) $T = O$ if and only if, $(Tx, y) = 0$ for all $x, y \in H$. $T = O$ means $Tx = 0$ for all $x \in H$ and this implies $(Tx, y) = (0, y) = 0$. On the other hand, $(Tx, y) = 0$ for all $x, y \in H$ implies $Tx = 0$ for all $x \in H$, which, by definition, says $T = O$.

(ii) $(Tx,x) = 0$ for all $x \in H$ if and only if, $T = O$. For $x = \alpha y + z \in H$,

$$\begin{aligned} 0 &= (T(\alpha y + z),\ \alpha y + z) \\ &= |\alpha|^2 (Ty, y) + \alpha(Ty, z) + \overline{\alpha}(Tz, y) + (Tz, z) \\ &= \alpha(Ty, z) + \overline{\alpha}(Tz, y) \end{aligned} \tag{3.23}$$

since $(Ty, y) = 0$ and $(Tz, z) = 0$. Setting $\alpha = 1$ and $\alpha = i$ in (3.23) gives

$$(Ty, z) + (Tz, y) = 0 \tag{3.24}$$

and

$$(Ty, z) - (Tz, y) = 0. \tag{3.25}$$

From (3.24) and (3.25), we get $(Ty, z) = 0$, which implies $T = O$ on using (i) above.

The following general properties of Hilbert space adjoint operators are frequently used in studying these operators.

Theorem 3.5.4 *If S, $T \in \mathcal{B}(H)$ and α is a scalar, then*

(a) $(\alpha S + T)^* = \overline{\alpha} S^* + T^*$
(b) $(ST)^* = T^* S^*$
(c) $(S^*)^* = S$
(d) *If S is invertible in $\mathcal{B}(H)$ and S^{-1} is its inverse, then S^* is invertible and* $((S^*)^{-}1) = ((S^{-}1)^*$.
(e) $||S^*S|| = ||SS^*|| = ||S||^2$
(f) $S^*S = 0$ *if, and only if,* $S = O$.

Proof

(a) By definition of the adjoint, for all $x, y \in H$,

$$\begin{aligned}(x, (\alpha S + T)^* y) &= ((\alpha S + T)x, y) \\ &= (\alpha Sx, y) + (Tx, y) \\ &= (x, \overline{\alpha} S^* y) + (x, T^* y) \\ &= (x, (\overline{\alpha} S^* + T^*)y).\end{aligned}$$

Hence, $(\alpha S + T)^* y = (\overline{\alpha} S^* + T^*)y$ for all $y \in H$, which implies $(\alpha S + T)^* = \overline{\alpha} S^* + T^*$.

(b) For $x, y \in H$,

$$\begin{aligned}(x, (ST)^* y) &= (ST(x), y) \\ &= (Tx, S^* y) \\ &= (x, T^* S^*(y)).\end{aligned}$$

Hence, $(ST)^* y = T^* S^*(y)$ for all $y \in H$, which implies (b).

(c) For $x, y \in H$,

$$((x, (S^*)^* y) = (S^* x, y) = (x, Sy).$$

Hence, $(S^*)^* y = Sy$ for all $y \in H$, which implies (c).

(d) If I denotes the identity operator in $\mathcal{B}(H)$, then $I^* = I$. Indeed, for $x, y \in H$,

$$(x, I^*y) = (Ix, y) = (x, y) = (x, Iy).$$

Hence, $I^*y = Iy$ for all $y \in H$, which implies $I^* = I$.
Suppose S is an invertible element in $\mathcal{B}(H)$. Then, $S^{-1}S = SS^{-1} = I$. Using (ii) above, we have $(S^{-1}S)^* = S^*(S^{-1})^* = I^*$. Since $I^* = I$, we get $S^*(S^{-1})^* = I$. Similarly, $(S^{-1})^*S^* = I$. Hence, $(S^*)^{-1} = (S^{-1})^*$.

(e) By the Cauchy–Schwarz inequality,

$$||Sx||^2 = (Sx, Sx) = (S^*Sx, x) \leq \|S^*Sx\|||x|| \leq |S^*S|||x||^2.$$

Taking the supremum over all x of norm 1, we obtain

$$||S||^2 \leq ||S^*S||.$$

Applying Theorem 3.5.2, we obtain

$$||S||^2 \leq ||S^*S|| \leq ||S^*||||S|| = ||S||^2.$$

Hence,

$$||S^*S|| = ||S||^2. \tag{3.26}$$

Replacing S by S^*, and using Theorem 3.5.2, we obtain

$$||SS^*|| = ||S||^2. \tag{3.27}$$

Thus, on using (3.26) and (3.27), the result follows.

(f) This is an immediate consequence of (e) above. □

Remarks 3.5.5

(i) The map $T \to T^*$ has properties very similar to the complex conjugation $z \to \bar{z}$ on $\mathbb{C}$. A new feature is the relation (b) of Theorem 3.5.4 which results from the noncommutativity of operator multiplication.

(ii) Since $\|T^*\| = \|T\|$, we have $\|T^* - S^*\| = \|(T - S)^*\| = \|T - S\|$, and it follows that the map $T \to T^*$ from $\mathcal{B}(H)$ to $\mathcal{B}(H)$ is continuous in the norm.

(iii) If H is a Hilbert space, we know that $\mathcal{B}(H)$ is a Banach algebra [Theorem 3.3.3]. Moreover, in view of Theorem 3.5.4, the mapping $T \to T^*$ of $\mathcal{B}(H)$ into itself is such that

(a) $$T^{**} = T$$

(b) $$(S + T)^* = S^* + T^*$$

(c) $$(\alpha S)^* = \overline{\alpha} T^*$$

(d) $$(ST)^* = T^* S^*$$

(e) $$\|T^* T\| = \|T\|^2.$$

It is immediate from (a) that the mapping $T \to T^*$ is both one-to-one and onto. It is useful to have the following general definition.

Definition 3.5.6 Let A be an algebra over $\mathbb{C}$. A mapping $a \to a^*$ of A into itself is called an **involution** if, for all $a, b \in A$ and all $\alpha \in \mathbb{C}$,

(i) $a^{**} = a$
(ii) $(a + b)^* = a^* + b^*$
(iii) $(\alpha a)^* = \overline{\alpha} a^*$
(iv) $(ab)^* = b^* a^*$.

*An algebra with an involution is called a *algebra. A normed algebra with an involution is called a* **normed *algebra**. *A Banach algebra A with an involution satisfying* $||a^* a|| = ||a||^2$ *is called a* ***C*-algebra***.

Observe that in a C^*-algebra,

$$||a||^2 = ||a^* a|| \leq \|a^*\| \|a\|,$$

which implies $||a|| \leq ||a^*||$ provided $a \neq 0$. Replacing a by a^* and using (a) of the above definition, we obtain $||a^*|| \leq ||a||$. Thus, $||a|| = ||a^*||$ for $a \in A$, since the equality is trivially true when $a = 0$.

In view of the observations [(iii) of Remark 3.5.5], it follows that $\mathcal{B}(H)$ is a C^*-algebra. Obviously, every *subalgebra of $\mathcal{B}(H)$, that is, a subalgebra containing adjoints, which is closed in the norm is also a C^*-algebra. Every C^*-algebra has the same "mathematical structure" as a subalgebra of $\mathcal{B}(H)$ for a suitable Hilbert space H; this is known as the **Gelfand–Naimark Theorem**. The study of such algebras constitutes an important area of research in functional analysis and is beyond the scope of the present text.

There is an interesting relationship between the range of an operator $T \in \mathcal{B}(H)$ and the kernel of its adjoint T^*. This relationship proves useful in deciding the invertibility of operators.

Theorem 3.5.7 *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space H. Then,* $T(\mathfrak{M}) \subseteq \mathfrak{N}$ if and only if, $T^*(\mathfrak{N}^\perp) \subseteq \mathfrak{M}^\perp$.

Proof Suppose $T(\mathfrak{M}) \subseteq \mathfrak{N}$ and let $y \in T^*(\mathfrak{N}^\perp)$. There exists $x \in \mathfrak{N}^\perp$ such that $y = T^*x$. For $z \in \mathfrak{M}$, $(y, z) = (T^*x, z) = (x, Tz) = 0$, since $x \in \mathfrak{N}^\perp$ and $Tz \in \mathfrak{N}$; thus, $y \perp \mathfrak{M}$.

If $T^*(\mathfrak{N}^\perp) \subseteq \mathfrak{M}^\perp$, then by the argument in the above paragraph, $T^{**}(\mathfrak{M}^{\perp\perp}) \subseteq \mathfrak{N}^{\perp\perp}$. Since $T^{**} = T$ and $\mathfrak{M}$ and $\mathfrak{N}$ are closed subspaces of H, it follows that $T(\mathfrak{M}) \subseteq \mathfrak{N}$. □

Theorem 3.5.8 *If* $T \in \mathcal{B}(H)$, *then* $\ker(T) = \ker(T^*T) = [\mathrm{ran}(T^*)]^\perp$ *and* $[\ker(T)]^\perp = \overline{[\mathrm{ran}(T^*)]}$.

Proof Clearly, $\ker(T) \subseteq \ker(T^*T)$. The reverse inclusion follows from the computation $\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x)$.

Now, $x \in \ker(T) \Leftrightarrow Tx = 0 \Leftrightarrow (Tx, y) = 0$ for all $y \in H \Leftrightarrow (x, T^*y) = 0$ for all $y \in H \Leftrightarrow x \in [\mathrm{ran}(T^*)]^\perp$. Thus, $\ker(T) = [\mathrm{ran}(T^*)]^\perp$.

It follows by (iii) of Remark 2.10.12 that $[\ker(T)]^\perp = [\mathrm{ran}(T^*)]^{\perp\perp} = \overline{[\mathrm{ran}(T^*)]}$. □

The following Theorem provides a criterion for the invertibility of $T \in \mathcal{B}(H)$.

Theorem 3.5.9 *If* $T \in \mathcal{B}(H)$ *is such that* T *and* T^* *are both bounded below, then* T *is invertible.*

Proof If T^* is bounded below, then $\ker(T^*) = \{0\}$. In view of Theorem 3.5.8, $[\mathrm{ran}(T)]^\perp = \{0\}$, which implies $\overline{[\mathrm{ran}(T)]} = [\mathrm{ran}(T)]^{\perp\perp} = \{0\}^\perp = H$. Thus, $\mathrm{ran}(T)$ is dense in H and the result now follows on using Theorem 3.3.12. □

In the following examples, we compute the adjoints of some well-known operators.

Example 3.5.10

(i) Let $H = \mathbb{C}^n$, the Hilbert space of finite dimension n and $\{e_1, e_2, \ldots, e_n\}$ be the standard orthonormal basis for H. Define $T:\mathbb{C}^n \to \mathbb{C}^n$ by setting

$$(Tx)_i = \sum_{j=1}^n \alpha_{i,j} x_j.$$

Clearly, T is linear and hence bounded [(iii) of Example 3.2.5].
Since the inner product in $\mathbb{C}^n$ is $(x, y) = \sum_{i=1}^n x_i \overline{y_i}$,

$$\begin{aligned}(Tx, y) &= \sum_{i=1}^n (Tx)_i \bar{y}_i \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{i,j} x_j \right) \bar{y}_i \\ &= \sum_{j=1}^n x_j \overline{\sum_{i=1}^n \bar{\alpha}_{i,j} y_i} \\ &= (x, T^*y),\end{aligned}$$

where $(T^*y)_j = \sum_{i=1}^n \overline{\alpha}_{i,j}y_i$. The adjoint of T is, therefore, represented by the usual conjugate transpose of the matrix representing T.

(ii) Let H be a separable Hilbert space and $\{e_n\}_{n\geq 1}$ constitute an orthonormal basis for H. By Problem 3.2.P2, each $T \in \mathcal{B}(H)$ is defined by a matrix $[\alpha_{i,j}]_{i,j\geq 1}$, where $\alpha_{i,j} = (Te_j, e_i)$, $i,j = 1,2,\ldots$. Since $T^* \in \mathcal{B}(H)$ and

$$\left(T^*e_j, e_i\right) = \left(e_j, Te_i\right) = \overline{\left(Te_i, e_j\right)} \quad \text{for} \quad i,j = 1,2,\ldots,$$

it follows that the matrix representing T^* is the conjugate of the transpose of the matrix $[\alpha_{i,j}]_{i\text{-}j\geq 1}$ representing T.

(iii) The adjoint of the operator $T \in \mathcal{B}(H)$ defined by $Tx = \alpha x$, $x \in H$ and $\alpha \in \mathbb{C}$, is the operator T^* defined by $T^*x = \overline{\alpha}x$, $x \in H$. Indeed, for $x, y \in H$, $(x, T^*y) = (Tx, y) = (\alpha x, y) = (x, \overline{\alpha}y)$. Thus, $(x, (T^* - \overline{\alpha}I)y) = 0$. Consequently, $T^* = \overline{\alpha}I$.

(iv) Let M be a closed subspace of a Hilbert space H and P_M the orthogonal projection on M. Moreover, $\|P_M\| = 1$ [(ii) of Remark 2.10.17].
The adjoint $P_M{}^*$ of P_M is P_M itself. Indeed, for $x_1, x_2 \in H$ with $x_i = y_i + z_i$, where $y_i \in M$ and $z_i \in M^{\perp}$, $i = 1, 2$, we have

$$\begin{aligned}(x_1, P_M{}^*x_2) &= (P_Mx_1, x_2) = (y_1, y_2 + z_2) = (y_1, y_2) = (y_1 + z_1, y_2) \\ &= (x_1, P_Mx_2),\end{aligned}$$

i.e., $(x_1,(P_M{}^* - P_M)x_2) = 0$, which implies $P_M{}^* = P_M$.

(v) Let $H = L^2(X, \mathfrak{M}, \mu)$, where $(X, \mathfrak{M}, \mu)$ is a σ-finite measure space and $y \in L^\infty(X, \mathfrak{M}, \mu)$ be an essentially bounded measurable function. A multiplication operator $T \in \mathcal{B}(H)$ [see Example (vi) of 3.2.5] has adjoint T^* which is also a multiplication operator. The defining relation for T^* is $(x, T^*z) = (Tx, z)$, $x, z \in H$. Consequently,

$$\int_X x(t)\overline{T^*z(t)}\mathrm{d}t = \int_X y(t)x(t)\overline{z(t)}\mathrm{d}t, \quad x, z \in H,$$

which implies

$$\int_X x(t)\overline{\left[T^*z(t) - \overline{y(t)}z(t)\right]}\mathrm{d}t = 0.$$

Since the above relation holds for all $x, z \in H$, it follows that $T^*z(t) = \overline{y(t)}z(t)$ in H. Thus, the adjoint T^* of the multiplication operator T is multiplication by the complex conjugate of y. In particular, if y is real-valued, then $T^* = T$.

(vi) Let H be a separable Hilbert space, $\{e_i\}_{i\geq 1}$ be an orthonormal basis in H and $T \in \mathcal{B}(H)$ be the simple unilateral shift [see (vii) of Example 3.2.5]. The defining relation for T^* is $(x,T^*y) = (Tx,\ y)$, $x,\ y \in H$. Now,

$$\begin{aligned}
(x, T^*y) &= \left(T\left(\sum_{k=1}^{\infty}\lambda_k e_k\right), \sum_{k=1}^{\infty}\mu_k e_k\right), \quad x = \sum_{k=1}^{\infty}\lambda_k e_k, y = \sum_{k=1}^{\infty}\mu_k e_k,\\
&= \left(\sum_{k=1}^{\infty}\lambda_k e_{k+1}, \sum_{k=1}^{\infty}(\mu_k e_k)\right)\\
&= \sum_{k=1}^{\infty}\lambda_k\bar{\mu}_{k+1}\\
&= \left(\sum_{k=1}^{\infty}\lambda_k e_k, \sum_{k=1}^{\infty}\mu_{k+1}e_k\right)\\
&= \left(\sum_{k=1}^{\infty}\lambda_k e_k, \sum_{k=1}^{\infty}\mu_k e_{k-1}\right).
\end{aligned}$$

As the above equality holds for all $x,\ y \in H$, it follows that

$$T^*y = \sum_{k=2}^{\infty}\mu_k e_{k-1}, \quad \text{where} \quad y = \sum_{k=1}^{\infty}\mu_k e_k.$$

In particular, $T^*e_k = e_{k-1}$, $k = 1, 2, \ldots$, where $e_0 = 0$. Thus, the adjoint of the simple unilateral shift is

$$T^*\left(\sum_{k=1}^{\infty}\mu_k e_k\right) = \sum_{k=2}^{\infty}\mu_k e_{k-1}.$$

This can also be described as

$$T^*e_1 = 0 \quad \text{and} \quad T^*e_i = e_{i-1}, \quad i = 2, 3, \ldots.$$

(vii) If K is the integral operator with kernel k as in (viii) of Example 3.2.5, then K^* is the integral operator with kernel $k^*(s,t) = \overline{k(t,s)}$. The defining relation for K^* is $(x,\ K^*y) = (Kx,\ y)$ for $x,\ y \in L^2(\mu)$. Now,

$$\begin{aligned}
(x, K^*y) = (Kx, y) &= \int_X\left(\int_X k(s,t)x(t)\mathrm{d}\mu(t)\right)\overline{y(s)}\mathrm{d}\mu(s)\\
&= \int_X\int_X x(t)k(s,t)\overline{y(s)}\mathrm{d}\mu(s)\mathrm{d}\mu(t).
\end{aligned}$$

The reversal of the order of integration is justified by Fubini's Theorem [Theorem 1.3.14]. As this holds for all x and y in $L^2(\mu)$, we must have

$$\overline{K^*y(t)} = \int_X k(s,t)\overline{y(s)}\mathrm{d}\mu(s)$$

for almost all t, or, interchanging the roles of s and t,

$$\overline{K^*y(s)} = \int_X k(t,s)\overline{y(t)}\mathrm{d}\mu(t),$$

$$\overline{K^*y(s)} = \int_X \overline{\overline{k(t,s)}y(t)\mathrm{d}\mu(t)}$$

for almost all s. Thus, K^* is the integral operator with kernel k^*, where $k^*(s,t) = \overline{k(t,s)}$.

Remarks 3.5.11 The Laplace transform $L{:}L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)$ with kernel $k(s, t) = \mathrm{e}^{-st}$ [Problem 3.2.P7] defined by

$$Lx(s) = \int_0^\infty x(t)\mathrm{e}^{-st}\mathrm{d}t$$

is such that $L^* = L$. Indeed, $k^*(s,t) = \overline{k(t,s)}\mathrm{e}^{-st}$.

Since $\|S^*S\| = \|S\|^2$, $S \in \mathcal{B}(H)$, it follows that $\|L\circ L\| = \|L\|^2$. The mapping $L\circ L$ is easily computed. For $x \in L^2(\mathbb{R}^+)$,

$$\begin{aligned}(L\circ L)x(r) &= \int_0^\infty Lx(s)\mathrm{e}^{-rs}\mathrm{d}s = \int_0^\infty\left(\int_0^\infty x(t)\mathrm{e}^{-st}\mathrm{d}t\right)\mathrm{e}^{-rs}\mathrm{d}s \\ &= \int_0^\infty x(t)\int_0^\infty \mathrm{e}^{-(r+t)s}\mathrm{d}s\mathrm{d}t, \text{ using Fubini's Theorem} \\ &= \int_0^\infty \frac{x(t)}{r+t}\mathrm{d}t.\end{aligned}$$

We have thus proved the following result.

The integral operator, called Hilbert–Hankel operator,

$$Hx(r) = \int_0^\infty \frac{x(t)}{r+t}\mathrm{d}t$$

is bounded as a map from $L^2(\mathbb{R}^+)$ to itself and its norm equals $\sqrt{\pi}$.

Problem Set 3.5

3.5.P1. Let $\{\mu_n\}_{n\geq 1}$ be a bounded sequence of complex numbers, $M = \sup\{|\mu_k| : k \geq 1\}$. Show that there exists one and only one operator T on a Hilbert space H such that

(a) $Te_k = \mu_k e_k$ for all k, where $\{e_k\}_{k\geq 1}$ is an orthonormal basis in H;

(b) $$T\left(\sum_{k=1}^\infty \lambda_k e_k\right) = \sum_{k=1}^\infty \lambda_k \mu_k e_k;$$

(c) $$\|T\| = M;$$

(d) $$T^* e_k = \overline{\mu}_k e_k \quad \text{for all } k;$$

(e) $$T^*\left(\sum_{k=1}^\infty \lambda_k e_k\right) = \sum_{k=1}^\infty \lambda_k \overline{\mu}_k e_k; \text{ and}$$

(f) $$T^*T = TT^*.$$

3.6 Some Special Classes of Operators

The adjoint operation in $\mathcal{B}(H)$ in a way extends the conjugation operation in the complex numbers. Unlike conjugation in complex numbers, the adjoint operation in $\mathcal{B}(H)$ does not preserve the product.

Those operators T for which $T^*T = TT^*$ have "decent" properties. Such operators and their suitable subsets will be studied in this section.

Definition 3.6.1 If $T \in \mathcal{B}(H)$, then

(a) T is **Hermitian** or **self-adjoint** if $T^* = T$;
(b) T is **unitary** if T is bijective and $T^* = T^{-1}$; and
(c) T is **normal** if $T^*T = TT^*$.

Remarks 3.6.2

(i) In the analogy between the adjoint and the conjugate, Hermitian operators become analogues of real numbers, unitaries are the analogues of complex numbers of absolute value 1. Normal operators are the true analogues of complex numbers: Note that

$$T = \frac{T + T^*}{2} + i\frac{T - T^*}{2i},$$

where $\frac{T+T^*}{2}$ and $\frac{T-T^*}{2i}$ are self-adjoint and $T^* = \frac{T+T^*}{2} - i\frac{T-T^*}{2i}$. The operators $\frac{T+T^*}{2}$ and $\frac{T-T^*}{2i}$ are called real and imaginary parts of T.

(ii) If T is self-adjoint or unitary, then T is normal. However, a normal operator need not be self-adjoint or unitary. First note that I, the identity operator in $\mathcal{B}(H)$, is self-adjoint. The operator $T = 2iI$ is such that $T^* = -2iI$; so, $TT^* = 4I = T^*T$, but $T^* \neq T$ and $T^{-1} = -\frac{1}{2}\,iI \neq T^*$.
From Examples 3.2.5 and 3.5.10, we can readily produce some infinite-dimensional operators satisfying conditions (a), (b) and (c) of Definition 3.6.1.

(iii) If $T \in \mathcal{B}(H)$, where H is a separable Hilbert space and T is defined by the matrix $M = [\alpha_{i,j}]_{i,j=1}^{\infty}$ with respect to an orthonormal basis $\{e_n\}_{n\geq 1}$ ($\alpha_{i,j} = (Te_j, e_i)$, $i,j = 1,2,\ldots$), then T^* is defined by $\overline{M}^t = [\overline{\alpha}_{j,i}]_{i,j=1}^{\infty}$ with respect to the same basis. Thus, T is self-adjoint if, and only if, $\alpha_{i,j} = \overline{\alpha}_{j,i}, i,j = 1,2\ldots$, that is, $\overline{M}^t = M$.

Since $\overline{M}^t M = [\Sigma_n \alpha_{n,j}\overline{\alpha}_{n,i}]_{i,j=1}^{\infty}$ and $M\overline{M}^t = [\Sigma_n \alpha_{i,n}\overline{\alpha}_{j,n}]_{i,j=1}^{\infty}$ with respect to the basis $\{e_n\}_{n\geq 1}$, it follows that T is unitary if, and only if, $\overline{M}^t M = I = M\overline{M}^t$, that is,

$$\sum_n \alpha_{n,j}\overline{\alpha}_{n,i} = \delta_{i,j} = \sum_n \alpha_{i,n}\overline{\alpha}_{j,n}$$

for all $i, j = 1, 2, \ldots$ where $\delta_{i,j}$ is 1 if $i = j$ and zero otherwise. This says that the columns of M form an orthonormal set in ℓ^2 and so do its rows. Next, T is normal if, and only if, $\overline{M}^t M = M\overline{M}^t$. This is certainly the case if M is a diagonal matrix.

(iv) If T denotes the operator of multiplication by $y \in L^{\infty}(\mu)$ (notations as in (vi) of Example 3.2.5 and (v) of Example 3.5.10), then T is normal; T is Hermitian if, and only if, y is real-valued; T is unitary if, and only if, $|y| = 1$ a.e.

(v) By (viii) of Example 3.2.5 and (vii) of Example 3.5.10, the integral operator K with kernel k is self-adjoint if, and only if, $k(s,t) = \overline{k(t,s)}$ a.e. $[\mu \times \mu]$.

(vi) [(vii) of Example 3.2.5 and (vi) of Example 3.5.10] If $T \in \mathcal{B}(\ell^2)$ is the simple shift, then $T^*Te_1 = T^*e_2 = e_1$ and $TT^*e_1 = T0 = 0$; so, $T^*T \neq TT^*$, that is, T is not a normal operator.

The following is an important and rather simple criterion for self-adjointness in the complex case.

Theorem 3.6.3 *Let $T \in \mathcal{B}(H)$. Then,*

(a) *If T is self-adjoint, (Tx, x) is real for all $x \in H$.*
(b) *If H is a complex Hilbert space and (Tx, x) is real for all $x \in H$, the operator T is self-adjoint.*

Proof For $x, y \in H$ and $T \in \mathcal{B}(H)$, $B(x, y) = (Tx, y)$ is a sesquilinear form. The conclusion now follows from (ii) of Remark 3.4.4. □

Remarks 3.6.4

(i) Part (b) of the preceding proposition is false if it is only assumed that H is a real Hilbert space. For example, if $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on $\mathbb{R}^2$, then $(Tx,x) = 0$ for all $x \in \mathbb{R}^2$. However, $T^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = T$.

(ii) If S and T are bounded self-adjoint operators on a Hilbert space H, then so is $\alpha S + \beta T$, where α and β are real numbers. Thus, the collection of all self-adjoint operators is a real vector space, which we shall denote by $S(H)$.

(iii) If $T \in \mathcal{B}(H)$, then T^*T and $T + T^*$ are self-adjoint.

(iv) If $S,T \in \mathcal{B}(H)$ are self-adjoint, then ST is self-adjoint if, and only if, $ST = TS$. Indeed, $(ST)^* = T^*S^* = TS$; so, $ST = (ST)^*$ if, and only if, $ST = TS$.

Sequences of self-adjoint operators occur in various problems. For them, the following holds:

Theorem 3.6.5 *Let $\{T_n\}_{n \geq 1}$ be a sequence of bounded self-adjoint linear operators on a Hilbert space H. Suppose $\{T_n\}_{n \geq 1}$ converges, say $\lim_n T_n = T$ (uniform norm), i.e. $\lim_n \|T_n - T\| = 0$. Then, the limit operator T is a bounded self-adjoint operator on H.*

Proof Clearly, T is a bounded linear operator. It is enough to show that $T^* = T$. It follows from Theorems 3.5.2 and 3.5.4 that

$$\|T_n{}^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|.$$

Therefore, $T^* = \lim_n T_n{}^* = \lim_n T_n = T$. □

The following result is important for the discussion of “spectral theory”.

Theorem 3.6.6 *If $T \in \mathcal{B}(H)$ is self-adjoint*,

$$||T|| = \sup\{|(Tx, x)| : \|x\| \le 1\} = \sup\{|(Tx, x)| : \|x\| = 1\}.$$

(The latter will be needed in 3.7.4.)

Proof Define $B(x, y) = (Tx, y)$, $x, y \in H$; B is a bounded sesquilinear form with $\|B\| = \|T\|$ [(ii) of Example 3.4.7]. Since $B(y, x) = (Ty, x) = (y, Tx) = \overline{(Tx, y)} = \overline{B(x, y)}$, B is Hermitian. Hence, by Corollary 3.4.11,

$$\begin{aligned} ||B|| &= \sup\left\{|B(x, x)|/\|x\|^2 : x \in H, x \neq 0\right\} \\ &= \sup\{|B(x, x)| : x \in H, ||x|| \le 1\}. \end{aligned}$$

□

Corollary 3.6.7 *If $T \in \mathcal{B}(H)$ is such that $T = T^*$ and $(Tx, x) = 0$ for all $x \in H$, then $T = O$.*

Remarks 3.6.8 The above Corollary is not true unless $T = T^*$. See (i) of Remark 3.6.4. However, if the Hilbert space under consideration is complex, then the hypothesis, namely $T = T^*$, can be deleted. In fact, the following holds.

Proposition 3.6.9 *If H is a complex Hilbert space and $T \in \mathcal{B}(H)$ is such that $(Tx, x) = 0$ for all $x \in H$, then $T = O$.*

Proof For $x, y \in H$, the following equality is easily verified:

$$\begin{aligned} (Tx, y) = \frac{1}{4}\{&(T(x+y), x+y) - (T(x-y), x-y) \\ &+ i(T(x+iy), x+iy) - i(T(x-iy), x-iy)\}. \end{aligned}$$

Since $(Tx, x) = 0$ for all $x \in H$, it follows that $(Tx, y) = 0$ for $x, y \in H$. Setting $y = Tx$, we obtain

$$||Tx|| = 0 \quad \text{for all} \quad x \in H, \text{ that is,}$$

$Tx = 0$ for all $x \in H$. Consequently, $T = O$. □

The notion of positive definite matrix is familiar from linear algebra; it has a natural generalisation to infinite dimensions.

Definition 3.6.10 Let $T \in \mathcal{B}(H)$ be such that $T^* = T$. If for each $x \in H$, $(Tx, x) \ge 0$, we say that T is **positive semidefinite**. If $(Tx, x) > 0$ for all nonzero $x \in H$, we say that T is **positive definite**. Alternatively, these are known as **positive** and **strictly positive** operators.

Remarks 3.6.11

(i) If T is any operator on a complex Hilbert space, then the condition $(Tx,x) \geq 0$ for all $x \in H$ implies T is self-adjoint. However, in a real Hilbert space, this is not true. Indeed, the operator T in $\mathbb{R}^2$ defined by the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is not self-adjoint but $(Tx, x) = x_1^2 + x_2^2 \geq 0$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ [See also (i) of Remark 3.6.4.].

(ii) We write $T \geq O$ to mean T is positive. The collection of all positive operators is a positive cone: if $S \geq O$, $T \geq O$, then for all nonnegative real numbers α and β, we have $\alpha S + \beta T \geq O$. This defines a partial order on the collection $S(H)$ of self-adjoint operators: $S \geq T$ if, and only if, $S - T \geq O$. Also, if $S_1 \geq T_1$ and $S_2 \geq T_2$, then $S_1 + S_2 \geq T_1 + T_2$.

(iii) If $T \in \mathcal{B}(H)$ is any operator, then T^*T and TT^* are positive. Indeed, $(T^*Tx, x) = (Tx, Tx) = \|Tx\|^2 \geq 0$ for all $x \in H$. The argument that TT^* is positive is similar.

(iv) If $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then it can be checked that $A \geq B$. Indeed, $A - B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$. However, the relation $A^2 \geq B^2$ is false. In fact, $A^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and $B^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and $A^2 - B^2 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ and this does not represent a positive operator, as can be easily verified by considering the vector $(1, -2)$.

(v) The multiplication operator $T{:}L^2[0,1] \to L^2[0,1]$ defined by

$$Tx(t) = tx(t), \quad 0 < t < 1$$

is a positive operator, since

$$(Tx, x) = \int_0^1 t|x(t)|^2 \mathrm{d}t \geq 0$$

for any $x \in L^2[0,1]$.

It was pointed out in (ii) of Remark 3.6.11 that the sum of positive operators is positive. Let us turn to products. From (iv) of Remark 3.6.4, we know that a product of bounded self-adjoint operators is self-adjoint if, and only if, the operators commute. We shall see below that the product of two positive operators is positive if, and only if, the operators commute.

Theorem 3.6.12 *If $S, T \in \mathcal{B}(H)$, where H is a complex Hilbert space, are such that $S \geq O$, $T \geq O$, then their product ST is positive if, and only if, $ST = TS$.*

Proof The "only if" part is trivial in view of (iv) of Remark 3.6.4.

To prove the "if" part, we suppose $ST = TS$ and show that $(STx,x) \geq 0$ for all $x \in H$. If $S = O$, the inequality holds. Let $S \neq O$. Set $S_1 = S/\|S\|$, $S_2 = S_1 - S_1^2,\ldots,S_{n+1} = S_n - S_n^2,\ldots$. Note that each S_i is self-adjoint. We shall show that, for each $i = 1,2,\ldots$, $O \leq S_i \leq I$.

For $i = 1$ and $x \in H$, $(S_1x, x) = ((S/\|S\|)x, x) = (Sx, x)/\|S\| \leq \|Sx\|\|x\|/\|S\| \leq \|x\|^2 = (x, x)$; so, $((I - S_1)x, x) \geq 0$. Thus, the result is true for $i = 1$.

Assume that $O \leq S_k \leq I$. Then, $(S_k^2(I - S_k)x,x) = ((I - S_k)S_kx,S_kx) \geq O$, that is, $S_k^2(I - S_k) \geq O$. Similarly, it can be shown that $S_k(I - S_k)^2 \geq O$. Consequently, $S_{k+1} = S_k^2(I - S_k) + S_k(I - S_k)^2 \geq O$ and $I - S_{k+1} = (I - S_k) + S_k^2 \geq O$ by the induction hypothesis and the fact that $S_k^2 \geq O$ whenever $S_k \geq O$. This completes the argument when $O \leq S_k \leq I$.

We now consider the general case. Observe that $S_1 = S_1^2 + S_2 = S_1^2 + S_2^2 + S_3 = \cdots = S_1^2 + S_2^2 + \cdots + S_n^2 + S_{n+1}$.

Since $S_{n+1} \geq O$, this implies

$$S_1^2 + S_2^2 + \cdots + S_n^2 = S_1 - S_{n+1} \leq S_1. \tag{3.28}$$

By the definition of $\leq$ and the fact that $S_i = S_i^*$, this means that

$$\sum_{i=1}^{n} ||S_ix||^2 = \sum_{i=1}^{n} (S_ix, S_ix) = \sum_{i=1}^{n} \left(S_i^2x, x\right) \leq (S_1x, x).$$

Since n is arbitrary, the infinite series $\sum_{i=1}^{\infty} ||S_ix||^2$ converges, which implies $\|S_ix\| \to 0$ and hence $S_ix \to 0$. By (3.28),

$$\left(\sum_{i=1}^{n} S_i^2x, x\right) = (S_1 - S_{n+1})x \to S_1x \quad \text{as} \quad n \to \infty. \tag{3.29}$$

Observe that all the S_i commute with T since they are the sums and products of $S_1 = \|S\|^{-1}S$ and S and T commute. Finally,

$$\begin{aligned}(STx, x) &= ||S||(S_1Tx, x) \\ &= ||S||(TS_1x, x) \\ &= ||S||\left(T\lim_n \sum_{i=1}^{n} S_i^2x, x\right) \\ &= ||S||\lim_n \sum_{i=1}^{n} (TS_i^2x, x) \\ &= ||S||\lim_n \sum_{i=1}^{n} (TS_ix, S_ix) \\ &\geq 0,\end{aligned}$$

using $S = \|S\|S_1$, (3.29) and, $T \geq O$. Thus,

$$(STx, x) \geq 0 \quad \text{for} \quad \text{all} \;\; x \in H \qquad \square$$

In (ii) of Remark 3.6.11, it was pointed out that the collection of positive operators on a Hilbert space H is a positive cone in $S(H)$. The positive cone induces a partial order in $S(H)$. This leads to the following definition.

Definition 3.6.13 Let $\{T_n\}_{n \geq 1}$ be a sequence of bounded linear self-adjoint operators defined in a Hilbert space H, i.e. $T_n \in \mathcal{B}(H)$, $n = 1,2,\ldots$. The sequence $\{T_n\}_{n \geq 1}$ is said to be **increasing** [resp. **decreasing**] if $T_1 \leq T_2 \leq \cdots$ [resp. $T_1 \geq T_2 \geq \cdots$].

An increasing [resp. decreasing] sequence $\{T_n\}_{n \geq 1}$ in $\mathcal{B}(H)$ has the following remarkable property. It follows from Theorem 3.4.8 proved above.

Theorem 3.6.14 *Let $\{T_n\}_{n \geq 1}$ be an increasing sequence of bounded linear self-adjoint operators on a Hilbert space H that is bounded from above, that is,*

$$T_1 \leq T_2 \leq \cdots \leq T_n \leq \cdots \leq \alpha I,$$

where α is a real number. Then, $\{T_n\}_{n \geq 1}$ is strongly convergent.

Proof For each $x \in H$, the sequence $\{(T_n x, x)\}_{n \geq 1}$ of real numbers is bounded from above by $\alpha\|x\|^2$. So, $\lim_n (T_n x, x)$ exists and equals $f(x)$, say. Being a limit of quadratic forms [see Problem 3.4.P2], this is again a quadratic form, that is, there exists a sesquilinear form $B(x, y)$ on H such that $f(x) = B(x, x)$. Clearly, B is bounded. By Theorem 3.4.8, there exists a self-adjoint operator T such that $f(x) = (Tx, x)$. It remains to show that $\lim_n \|(T_n - T)x\| = 0$ for each $x \in H$.

Without loss of generality, we may assume that $T_1 \geq O$ by replacing each T_i by $T_i - T_1$ and α by 2α. Then for $n > m$, we have $O \leq T_n - T_m \leq \alpha I$. This shows that

$$||T_n - T_m|| = \sup_{\|x\|=1}((T_n - T_m)x, x) \leq \alpha.$$

Using the generalised Cauchy–Schwarz inequality [Theorem 3.4.5 with $B(x, y) = (Ax, y)$, where A is a positive operator], we get for each x and $y = (T_n - T_m)x$,

$$\begin{aligned}
||T_n x - T_m x||^4 &= [((T_n - T_m)x, (T_n - T_m)x)]^2 \\
&= [((T_n - T_m)x, y)]^2 \\
&\leq ((T_n - T_m)x, x)((T_n - T_m)y, y) \\
&= ((T_n - T_m)x, x)\left((T_n - T_m)^2 x, (T_n - T_m)x\right) \\
&\leq ((T_n - T_m)x, x)\|T_n - T_m\|\|(T_n - T_m)x\|^2 \\
&\leq ((T_n - T_m)x, x)\alpha^3 ||x||^2.
\end{aligned}$$

Since $\lim_n (T_n x, x) = (Tx, x)$, it follows that $\lim_{n,m}((T_n - T_m)x, x) = 0$. So the left-hand side of the above inequality tends to zero as $n,m \to \infty$, i.e.

$$\lim_{n,m}||T_nx - T_mx|| = 0.$$

Hence, $\{T_nx\}_{n\geq 1}$ is a Cauchy sequence and $\lim_n T_nx = Bx$, say, exists. Obviously, Bx depends linearly on x. Moreover, $0 \leq (T_nx, x) \leq \alpha\,(x, x)$, and so, it follows that $0 \leq B(x, x) \leq \alpha\|x\|^2$, which implies that B is a bounded linear operator. □

Recall that if $T \in \mathcal{B}(H)$, $T^*T \geq O$ since $(T^*Tx, x) = \|Tx\|^2 \geq 0$ [(iii) of Remark 3.6.11]. Just as $|z| = \sqrt{\bar{z}z}$, we would like to define $|T| = \sqrt{T^*T}$. This requires the notion of square roots of positive operators. We begin with a Lemma.

Lemma 3.6.15 *The power series for the function* $\sqrt{1-z}$ *about* $z = 0$ *converges absolutely for all complex numbers in the unit disc* $\{z \in \mathbb{C}:|z| \leq 1\}$.

Proof Since the function $f(z) = \sqrt{1-z}$ is holomorphic in the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, it can be expanded in a Taylor series about $z = 0$:

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n, \quad \text{where} \quad \alpha_n = \frac{f^{(n)}(0)}{n!}.$$

Note that the series converges absolutely in the open unit disc and the derivatives at the origin are all negative:

$$f'(z) = -\frac{1}{2}(1-z)^{-\frac{1}{2}}, f''(z) = -\frac{1}{2^2}(1-z)^{-2+\frac{1}{2}}, \ldots, f^{(n)}(z) = -\frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{2^n}(1-z)^{-n+\frac{1}{2}}, \ldots.$$

So, the α_n are all negative for $n \geq 1$. Thus,

$$\begin{aligned}\sum_{k=0}^{n} |\alpha_k| &= 2 - \sum_{k=0}^{n} \alpha_k \\ &= 2 - \lim_{x\to 1-} \sum_{k=0}^{n} \alpha_k x^k \\ &\leq 2 - \lim_{x\to 1-} \sqrt{1-x} \\ &= 2,\end{aligned}$$

where $\lim_{x\to 1-}$ means that the limit is being taken as $x \to 1$ from the left. The sequence of partial sums { $\{\sum_{k=0}^{n} |\alpha_k|\}_{n\geq 1}$ on the left is increasing and is bounded above by 2. It follows that $\sum_{k=0}^{\infty} |\alpha_k| \leq 2$, which implies that the series $\sum_{k=0}^{\infty} \alpha_k z^k$ converges absolutely for $|z| = 1$. This proves the Lemma. □

Now consider the Cauchy product of the above power series with itself, which is

$$\sum_{k=0}^{\infty} \beta_k z^k, \quad \text{where} \ \ \beta_k = \sum_{j=0}^{k} \alpha_j \alpha_{k-j} \quad \text{for each} \ \ k.$$

It converges absolutely and its sum is the product $(\sqrt{1-z})^2 = 1 - z$. See Theorem 15 on p.51 of [28]. This means that, if

$$P_n(z) = \sum_{k=0}^{n} \beta_k z^k, \quad \text{and} \quad Q_n(z) = \sum_{k=0}^{n} \alpha_n z^n,$$

then $|Q_n(z)^2 - P_n(z)| \to 0$ as $n \to \infty$ for $|z| \leq 1$. By a computation (best avoided on paper), one can see that the polynomial $Q_n(z)^2 - P_n(z)$ has coefficients that are sums of products of only those α_j with $j \geq 1$. As noted in the course of the proof of the above Lemma, these α_j are all negative, and hence, the coefficients of $Q_n(z)^2 - P_n(z)$ are all positive. It follows for any bounded linear operator T that $\|Q_n(T)^2 - P_n(T)\| \leq |Q_n(\|T\|)^2 - P_n(\|T\|)|$. In particular, whenever $\|T\| \leq 1$, we have $\|Q_n(T)^2 - P_n(T)\| \to 0$ as $n \to \infty$. That is to say, the Cauchy product $\sum_{k=0}^{\infty} \beta_k T^k$ of $\sum_{k=0}^{\infty} \alpha_k T^k$ with itself converges in norm to $(\sum_{k=0}^{\infty} \alpha_k T^k)^2$, provided that $\|T\| \leq 1$.

On the other hand, since the Cauchy product $\sum_{k=0}^{\infty} \beta_k z^k$ of $\sum_{k=0}^{\infty} \alpha_k z^k$ with itself converges to $1 - z$ (as noted at the beginning of the preceding paragraph), the uniqueness of the power series of any holomorphic function implies that $\beta_0 = 1 = -\beta_1$ and $\beta_k = 0$ for $k > 1$. Therefore, $(\sum_{k=0}^{\infty} \alpha_k T^k)^2 = \sum_{k=0}^{\infty} \beta_k T^k = I - T$.

Theorem 3.6.16 *Let* $T \in \mathcal{B}(H)$ and $T \geq O$. *Then, there is a unique* $S \in \mathcal{B}(H)$ *with* $S \geq O$ and $S^2 = T$. *Furthermore,* S *commutes with every bounded operator which commutes with* T.

Proof If $T = O$, then take $S = O$. We may next assume, without loss of generality, that $\|T\| \leq 1$. Indeed, for any positive T and $x \in H$,

$$(Tx, x) \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|(x, x),$$

which implies

$$(T/\|T\|x, x) \leq (x, x), \quad x \in H$$

and therefore, $T/\|T\| \leq I$. Assuming we have already proved the Theorem for this case, we could then assert the existence of a positive operator S such that $S^2 = T/\|T\|$. From this, it follows that $\|T\|^{\frac{1}{2}}S$ is a positive square root of T.

Since $I - T$ is self-adjoint, it follows from (ii) of Example 3.4.7 and Corollary 3.4.11 that

$$\|I - T\| = \sup_{\|x\| \neq 0} \frac{|(I - T)x, x)|}{\|x\|^2} = \sup_{\|x\|=1} |((I - T)x, x)| \leq 1.$$

The above Lemma now implies that the series

$$I + \alpha_1(I - T) + \alpha_2(I - T)^2 + \cdots \tag{3.30}$$

converges in norm to an operator S. From what has been noted just before the statement of this Theorem, it also follows that $S^2 = I - (I - T) = T$. Furthermore, since $O \leq (I - T) \leq I$, we have

$$0 \leq ((I - T)^n x, x) \leq 1$$

for all $x \in H$ with $\|x\| = 1$. Thus,

$$\begin{aligned}
(Sx, x) &= 1 + \sum_{n=1}^{\infty} \alpha_n ((I - T)^n x, x) \\
&\geq 1 + \sum_{n=1}^{\infty} \alpha_n, \quad \text{using} \;\; \alpha_n < 0 \quad \text{for all} \;\; n \geq 1 \\
&= 0,
\end{aligned}$$

since the value of the sum of the series $1 + \sum_{n=1}^{\infty} \alpha_n z^n$ at $z = 1$, which is $1 + \sum_{n=1}^{\infty} \alpha_n$, is zero. Thus, $S \geq O$.

From here onwards, we do not need the restriction that $\|T\| \leq 1$. We next check that S commutes with every operator that commutes with T. Let $V \in \mathcal{B}(H)$ be such that $VT = TV$. Then, $V(I - T)^n = (I - T)^n V$ and consequently, $VS = SV$. It remains to show that S is unique.

Suppose there is S', with $S' \geq O$ and $(S')^2 = T$. Then since

$$S'T = (S')^3 = TS',$$

S' commutes with T and thus with S. Therefore,

$$(S - S')S(S - S') + (S - S')S'(S - S') = \left(S^2 - S'^2\right)(S - S') = O. \tag{3.31}$$

Since both terms on the left of (3.31) are positive, they must both be zero; so their difference $(S - S')^3 = O$. Since $S - S'$ is self-adjoint, it follows that

$$\|S - S'\|^2 = \|(S - S')(S - S')\| = \left\|(S - S')^2\right\|$$

and $\|S - S'\|^4 = \|(S - S')^2\|^2 = \|(S - S')^4\|$, so $S - S' = O$. $\square$

Example 3.6.17

(i) In $L^2[0, 1]$, the multiplication operator

$$(Tx)(t) = tx(t), 0<t<1, \quad x \in L^2[0,1]$$

has the square root S, where

$$(Sx)(t) = \sqrt{t}x(t), 0<t<1, \quad x \in L^2[0,1].$$

(ii) For $a > 0$, the 2×2 matrix

$$T = \begin{bmatrix} a & 1 \\ 1 & a^{-1} \end{bmatrix}$$

is positive. Indeed,

$$\begin{aligned}(Tx, x) &= \left(\begin{bmatrix} ax_1 + x_2 \\ x_1 + a^{-1}x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ &= a|x_1|^2 + \overline{x_1}x_2 + x_1\overline{x_2} + a^{-1}|x_2|^2 \\ &= \left| \sqrt{a}x_1 + \sqrt{a^{-1}}x_2 \right|^2 \geq 0 \quad \text{for all vectors } (x_1, x_2) \in \mathbb{C}^2.\end{aligned}$$

In what follows, we shall determine the square root of the matrix T. The characteristic values are the roots of the equation $\det(\lambda I - T) = 0$. These roots are 0, $(a + a^{-1})$ and the corresponding eigenvectors are $\begin{bmatrix} a^{-1} \\ -1 \end{bmatrix}, \begin{bmatrix} a \\ 1 \end{bmatrix}$, respectively. If V is the matrix $\begin{bmatrix} a^{-1} & a \\ -1 & 1 \end{bmatrix}$, then $TV = \begin{bmatrix} 0 & a^2+1 \\ 0 & a+a^{-1} \end{bmatrix}$. Consequently, $V^{-1}TV = \begin{bmatrix} 0 & 0 \\ 0 & a+a^{-1} \end{bmatrix}$, where $V^{-1} = \frac{1}{a+a^{-1}} \begin{bmatrix} 1 & -a \\ 1 & a^{-1} \end{bmatrix}$. Hence, $T^{\frac{1}{2}} = V \begin{bmatrix} 0 & 0 \\ 0 & (a+a^{-1})^{1/2} \end{bmatrix} V^{-1} = \begin{bmatrix} (a+a^{-1})^{-1/2}a & (a+a^{-1})^{-1/2} \\ (a+a^{-1})^{-1/2} & (a+a^{-1})^{-1/2}a^{-1} \end{bmatrix}$.

(iii) Using (ii) above, we may guess that the square root of the matrix

$$\begin{bmatrix} T & I \\ I & T^{-1} \end{bmatrix} \in \mathcal{B}(H \oplus H),$$

where T is a positive invertible operator in $\mathcal{B}(H)$, is

$$\begin{bmatrix} (T+T^{-1})^{-1/2}T & (T+T^{-1})^{-1/2} \\ (T+T^{-1})^{-1/2} & (T+T^{-1})^{-1/2}T^{-1} \end{bmatrix}.$$

Note that $T + T^{-1}$ is invertible by Theorem 3.5.9, because it is self-adjoint and is also bounded below in view of the fact that

$$\|(T+T^{-1})x\|^2 = \|Tx\|^2 + \|T^{-1}x\|^2 + 2\|x\|^2 \geq 2\|x\|^2.$$

Now, it follows on using matrix multiplication that

$$\begin{aligned} \begin{bmatrix} (T+T^{-1})^{-1/2}T & (T+T^{-1})^{-1/2} \\ (T+T^{-1})^{-1/2} & (T+T^{-1})^{-1/2}T^{-1} \end{bmatrix}^2 &= (T+T^{-1})^{-1}\begin{bmatrix} T & I \\ I & T^{-1} \end{bmatrix}^2 \\ &= (T+T^{-1})^{-1}\begin{bmatrix} T^2+I & (T+T^{-1}) \\ (T+T^{-1}) & T^{-2}+I \end{bmatrix} \\ &= (T+T^{-1})^{-1}\begin{bmatrix} (T+T^{-1})T & (T+T^{-1}) \\ (T+T^{-1}) & (T+T^{-1})T^{-1} \end{bmatrix} \\ &= \begin{bmatrix} T & I \\ I & T^{-1} \end{bmatrix}. \end{aligned}$$

Theorem 3.6.18 *If $T \in \mathcal{B}(H)$ is self-adjoint and $n \in \mathbb{N}$, then $\|T^n\| = \|T\|^n$.*

Proof When $T = O$, there is nothing to prove. So we may take $\|T\|^m > 0$ for all $m \in \mathbb{N}$.
The case $n = 1$ is trivial. For $n = 2$, the desired equality follows from

$$\|T^2\| = \|T^*T\| = \|T\|^2.$$

This says that, when $k = 1$, the equality $\|T^{2^k}\| = \|T\|^{2^k}$ holds. Assume this for some $k \in \mathbb{N}$. Then,

$$\|T^{2^{k+1}}\| = \|(T^{2^k})^2\| = \|(T^{2^k})^*(T^{2^k})\| = \|T^{2^k}\|^2 = \left(\|T\|^{2^k}\right)^2 = \|T\|^{2^{k+1}}.$$

It follows by induction that

$$\|T^{2^k}\| = \|T\|^{2^k} \quad \text{for all } k \in \mathbb{N}.$$

Now consider an arbitrary $n \in \mathbb{N}$. Choose $k \in \mathbb{N}$ such that $n < 2^k$, and put $m = 2^k - n$. Then, $0 \leq \|T^m\| \leq \|T\|^m \neq 0$ and $0 \leq \|T^n\| \leq \|T\|^n$. If it were to be the case that $\|T^n\| < \|T\|^n$, then it would follow that

$$\left\|T^{2^k}\right\| = \|T^{n+m}\| \leq \|T^n\| \cdot \|T^m\| < \|T\|^n \|T\|^m = \|T\|^{n+m} = \|T\|^{2^k},$$

contradicting what was proved earlier by induction. Thus, $\|T^n\| = \|T\|^n$. Therefore, by induction, the desired equality must hold for all $n \in \mathbb{N}$. □

Theorem 3.6.19 *If $T \in \mathcal{B}(H)$ is positive, then the sesquilinear form defined by (Tx, y) is nonnegative and satisfies*

$$|(Tx, y)|^2 \leq (Tx, x)(Ty, y) \quad \text{for all} \;\; x, y \in H.$$

Proof It is trivial that (Tx, y) defines a nonnegative sesquilinear form. The inequality now follows from Theorem 3.4.5. □

As an application of the above Theorem, we show for a positive operator T and any positive integer k that

$$\left(T^2 x, x\right) \leq (Tx, x)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}} \left(T^{2^k+1} x, x\right)^{\frac{1}{2^k}}.$$

Taking $y = Tx$ in the inequality of Theorem 3.6.19, we get

$$\left(T^2 x, x\right)^2 \leq (Tx, x)\left(T^3 x, x\right)$$

and hence,

$$\left(T^2 x, x\right) \leq (Tx, x)^{\frac{1}{2}}\left(T^3 x, x\right)^{\frac{1}{2}}.$$

This means the inequality in question is true with $k = 1$. In order to prove it by induction, assume it true for some k. Taking $y = T^{2^k} x$ in the inequality of Theorem 3.6.19, we get

$$\left(T^{2^k+1} x, x\right)^2 \leq (Tx, x)\left(T^{2^k+1} x, T^{2^k} x\right) = (Tx, x)\left(T^{2^{k+1}+1} x, x\right).$$

Taking the root of order 2^{k+1} on both sides and combining with the induction hypothesis, we find that

$$\left(T^2 x, x\right) \leq (Tx, x)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}}} \left(T^{2^{k+1}+1} x, x\right)^{\frac{1}{2^{k+1}}}.$$

This completes the proof by induction. □

Problem Set 3.6

3.6.P1. If $H = \mathbb{C}^n$, then the set of invertible matrices is dense in the space of all matrices.

3.6.P2. Let $T \in \mathcal{B}(H)$, where $H = \mathbb{C}^n$ and $\{e_k{:}k = 1, 2, \ldots, n\}$ be an orthonormal basis for H. Then, T has the matrix representation $[\alpha_{ij}]$ and T^* has the representation $[\overline{\alpha}_{ji}]$ with respect to the given orthonormal basis. Show that if the basis is not orthonormal, then this relation between the matrix representations need not hold.

3.6.P3. Let $T{:}X{\to}X$ be a bounded linear operator on a complex inner product space X. If $(Tx,x) = 0$ for all $x \in X$, show that $T = O$. Show that this does not hold in the case of a real inner product space.

3.6.P4. Let the operator T: $\mathbb{C}^2{\to}\mathbb{C}^2$ be defined by $Tx = \langle \xi_1 + i\xi_2, \xi_1 - i\xi_2 \rangle$, where $x = \langle \xi_1, \xi_2 \rangle$. Find T^*. Show that we have $T^*T = TT^* = 2I$. Find $T_1 = \frac{1}{2}(T_1 + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$.

3.6.P5. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R, $0 < R \leq \infty$. If $A \in \mathcal{B}(H)$ and $\|A\| < R$, show that there is an operator $T \in \mathcal{B}(H)$ such that for any $x, y \in H$, $(Tx, y) = \sum_{n=0}^{\infty} a_n(A^n x, y)$. Moreover, T is unique. If $BA = AB$, then show that $BT = TB$. [When the sum of the series $\sum_{n=0}^{\infty} a_n z^n$ is denoted by $f(z)$, the operator T is denoted by $f(A)$.]

3.6.P6. Let H be a Hilbert space and $A \in \mathcal{B}(H)$. Define the operator B on $H \oplus H$ by

$$B = \begin{bmatrix} 0 & iA \\ -iA^* & 0 \end{bmatrix}.$$

Prove that B is self-adjoint and $\|B\| = \|A\|$.

3.6.P7. If $T \in \mathcal{B}(H)$, show that $T + T^* \geq O$ if, and only of, $(T + I)$ is invertible in $\mathcal{B}(H)$ and $\|(T - I)(T + I)^{-1}\| \leq 1$.

3.7 Normal, Unitary and Isometric Operators

The true analogues of complex numbers are the normal operators. The following Theorem gives a characterisation of these operators.

Theorem 3.7.1 *If* $T \in \mathcal{B}(H)$, *the following are equivalent*:

(a) T *is normal*;

(b) $\|Tx\| = \|T^*x\|$ *for all* $x \in H$.

If H *is a complex Hilbert space*, *then these statements are also equivalent to*:

(c) *The real and imaginary parts of* T *commute*, *i.e.*

$$T_1T_2 = T_2T_1, \quad \text{where} \quad T_1 = \frac{T+T^*}{2} \quad \text{and} \quad T_2 = \frac{T-T^*}{2i}.$$

Proof If $x \in H$, then

$$\begin{aligned}\|Tx\|^2 - \|T^*x\|^2 &= (Tx, Tx) - (T^*x, T^*x) \\ &= (T^*Tx, x) - (TT^*x, x) \\ &= ((T^*T - TT^*)x, x).\end{aligned}$$

Since $T^*T - TT^*$ is Hermitian, it follows on using Corollary 3.6.7 that (a) and (b) are equivalent.

We next show that (a) and (c) are equivalent:

$$\begin{aligned}T^*T &= (T_1 - iT_2)(T_1 + iT_2) = T_1^2 + i(T_1T_2 - T_2T_1) + T_2^2 \\ TT^* &= (T_1 + iT_2)(T_1 - iT_2) = T_1^2 + i(T_2T_1 - T_1T_2) + T_2^2.\end{aligned}$$

Hence, $T^*T = TT^*$ if, and only if, $T_1T_2 = T_2T_1$. □

For any operator T, we have $\|T^k\| \leq \|T\|^k$, where k is a positive integer. A strengthening of the preceding inequality holds for normal operators:

Theorem 3.7.2 *Let $T \in \mathcal{B}(H)$ satisfy $T^*T = TT^*$. Then,*

$$\left\|T^k\right\| = \|T\|^k \quad \text{for } k = 2^n,\ n = 1, 2, \ldots.$$

Proof For $n = 1$,

$$\begin{aligned}\left\|T^2\right\|^2 &= \left\|T^2\left(T^2\right)^*\right\| \quad [\text{Theorem 3.5.4(e)}] \\ &= \left\|T^2(T^*)^2\right\| \\ &= \left\|(TT^*)^2\right\| \quad [T^*T = TT^*] \\ &= \|TT^*\|^2 \quad [\text{Theorem 3.5.4(e)}] \\ &= \|T\|^4, \quad [\text{Theorem 3.5.4(e)}]\end{aligned}$$

which implies

$$\left\|T^2\right\| = \|T\|^2.$$

Suppose the result is true for $n = m$. Then,

$$\begin{aligned}\left\|T^{2^{m+1}}\right\|^2 &= \left\|T^{2^{m+1}}\left(T^{2^{m+1}}\right)*\right\| \quad \text{[Theorem 3.5.4(e)]}\\ &= \left\|T^{2^m}T^{2^m}\left(T^{2^m}\right)*\left(T^{2^m}\right)*\right\|\\ &= \left\|T^{2^m}\left(T^{2^m}\right)*T^{2^m}\left(T^{2^m}\right)*\right\| \quad [T*T = TT*]\\ &= \left\|\left(T^{2^m}\left(T^{2^m}\right)*\right)^2\right\| \quad \text{[Theorem 3.5.4(e)]}\\ &= \left\|\left(T^{2^m}\left(T^{2^m}\right)*\right)\right\|^2 \quad [\text{using the case } n=1]\\ &= \left\|T^{2^m}\right\|^{2^2} \quad \text{[induction hypothesis]}\\ &= \|T\|^{2^{m+2}}.\end{aligned}$$

Consequently,

$$\left\|T^{2^{m+1}}\right\| = \|T\|^{2^{m+1}}.$$

By induction, the proof is complete. □

If $T \in \mathcal{B}(H)$ is self-adjoint, it was proved in Theorem 3.6.6 that

$$\|T\| = \sup\{|(Tx,x)| : \|x\| = 1\}.$$

The norm of any bounded linear normal operator can be computed using the foregoing formula. We begin with the following:

Definition 3.7.3 For any $T \in \mathcal{B}(H)$,

$$\boldsymbol{\rho(T)} = \sup\{|(Tx,x)| : \|x\| = 1\}.$$

Proposition 3.7.4 *Let $T \in \mathcal{B}(H)$, where H is complex. Then,*

$$\|Tx\|^2 + \left|\left(T^2x,x\right)\right| \leq 2\rho(T)\|Tx\|\|x\| \tag{3.32}$$

for every $x \in H$.

Proof Let λ and θ be real numbers. Then for $x \in H$,

$$\begin{aligned}\|Tx\|^2 + e^{2i\theta}\left(T^2x,x\right) = &\frac{1}{2}(\lambda e^{2i\theta}T^2x + \lambda^{-1}e^{i\theta}Tx, \lambda e^{i\theta}Tx + \lambda^{-1}x)\\ &-\frac{1}{2}(\lambda e^{2i\theta}T^2x - \lambda^{-1}e^{i\theta}Tx, \lambda e^{i\theta}Tx - \lambda^{-1}x).\end{aligned}$$

Since $|(Tx,x)| \leq \rho(T)\|x\|^2$ for every $x \in H$, we have

$$
\begin{aligned}
\left|\|Tx\|^2 + e^{2i\theta}\left(T^2x, x\right)\right| &\le \frac{1}{2}\left|\left(\lambda e^{2i\theta}T^2x + \lambda^{-1}e^{i\theta}Tx, \lambda e^{i\theta}Tx + \lambda^{-1}x\right)\right| \\
&\quad + \frac{1}{2}\left|\left(\lambda e^{2i\theta}T^2x - \lambda^{-1}e^{i\theta}Tx, \lambda e^{i\theta}Tx - \lambda^{-1}x\right)\right| \\
&\le \frac{1}{2}\left|e^{i\theta}\left(T\left(\lambda e^{i\theta}Tx + \lambda^{-1}x\right), \left(\lambda e^{i\theta}Tx + \lambda^{-1}x\right)\right)\right| \\
&\quad + \frac{1}{2}\left|e^{i\theta}\left(T\left(\lambda e^{i\theta}Tx - \lambda^{-1}x\right), \left(\lambda e^{i\theta}Tx - \lambda^{-1}x\right)\right)\right| \\
&\le \frac{1}{2}\rho(T)\left(\left\|\lambda e^{i\theta}Tx + \lambda^{-1}x\right\|^2 + \left\|\lambda e^{i\theta}Tx + \lambda^{-1}x\right\|^2\right).
\end{aligned}
\tag{3.33}
$$

If $Tx \neq 0$, choosing $\lambda \neq 0$ such that $\lambda^2\|Tx\| = \|x\|$ and θ such that $e^{2i\theta}(T^2x, x) = |(T^2x, x)|$, we deduce from (3.33) that

$$
\begin{aligned}
\|Tx\|^2 + \left|\left(T^2x, x\right)\right| &\le \rho(T)\left(\lambda^2\|Tx\|^2 + \lambda^{-2}\|x\|^2\right) \\
&= \rho(T)(\|Tx\|\|x\| + \|Tx\|\|x\|) \\
&= 2\rho(T)\|Tx\|\|x\|.
\end{aligned}
$$

The inequality (3.32) is obviously true in case $Tx = 0$. □

The following proposition will also be needed.

Proposition 3.7.5 *If* $T \in \mathcal{B}(H)$, *then* $\|T\| \le 2\rho(T)$ *and* $\rho(T^2) \le \rho(T)^2$.

Proof From Proposition 3.7.4, we have

$$\|Tx\|^2 \le 2\rho(T)\|Tx\|\|x\| \quad \text{for all} \quad x \in H.$$

This implies

$$\|Tx\| \le 2\rho(T)\|x\|,$$

and so,

$$\|T\| \le 2\rho(T).$$

Let $x \in H$ be such that $\|x\| = 1$. Then, Proposition 3.7.4 gives

$$\|Tx\|^2 + \left\|\left(T^2x, x\right)\right\| \le 2\rho(T)\|Tx\|,$$

that is,

$$\|Tx\|^2 - 2\rho(T)\|Tx\| + \left|\left(T^2x, x\right)\right| \leq 0.$$

Therefore,

$$(\|Tx\| - \rho(T))^2 + \left|\left(T^2x, x\right)\right| \leq \rho(T)^2,$$

which implies

$$\left|\left(T^2x, x\right)\right| \leq \rho(T)^2.$$

Hence,

$$\rho\left(T^2\right) = \sup\left\{\left|\left(T^2x, x\right)\right| : \|x\| = 1\right\} \leq \rho(T)^2. \qquad \square$$

Corollary 3.7.6 $\rho(T^p) \leq \rho(T)^p$, *for* $p = 2^n$, $n = 1,2,\ldots$.

Proof By induction. $\square$

Theorem 3.7.7 *If* $T \in \mathcal{B}(H)$ *is a normal operator, then*

$$\|T\| = \sup\{|(Tx, x)| : \|x\| = 1\}.$$

Proof From the definition of ρ and the definition of norm, it follows that

$$\begin{aligned} \rho(T) &= \sup\{|(Tx, x)| : \|x\| = 1\} \\ &\leq \sup\{\|Tx\|\|x\| : \|x\| = 1\} \\ &= \sup\{\|Tx\| : \|x\| = 1\} = \|T\|. \end{aligned} \tag{3.34}$$

Since T is normal, we have

$$\|T^p\| = \|T\|^p \quad \text{for } p = 2^n, n = 1, 2, \ldots.$$

So,

$$\begin{aligned} \|T\| &= \|T^p\|^{\frac{1}{p}} \\ &\leq \left(2\rho(T^p)\right)^{\frac{1}{p}} \qquad \text{[Corollary 3.7.6]} \\ &= 2^{\frac{1}{p}}\rho(T). \end{aligned}$$

On letting $p \to \infty$, we obtain

$$\|T\| \leq \rho(T). \tag{3.35}$$

Combining (3.34) and (3.35), we get the desired expression for the norm of the operator T. □

Corollary 3.7.8 *Let $T \in \mathcal{B}(H)$ be self-adjoint. Then,*

$$\|T\| = \sup\{|(Tx, x)| : \|x\| = 1\}.$$

Proof Every self-adjoint operator $T \in \mathcal{B}(H)$ is normal. □

In three-dimensional Euclidean space $\mathbb{C}^3$, the simplest operator after that of projection is rotation of the space, which changes neither the length of the vectors nor orthogonality between pairs of them. We consider below the analogue of this operation in Hilbert space.

Definition 3.7.9 Let H be a Hilbert space and U be a bounded linear operator with domain H and range H. U is called **unitary** *if*

$$(Ux, Uy) = (x, y)$$

for all $x, y \in H$.

If $y = x$, then the defining relation for a linear unitary operator U takes the form $\|Ux\| = \|x\|$ for all $x \in H$, in particular, U is bounded and $\|U\| = 1$.

Theorem 3.7.10 *Let U be a unitary operator on a Hilbert space H. Then, U^{-1} exists and is unitary. Moreover, $U^{-1} = U^*$.*

Proof In order to show that U^{-1} exists, it is enough to show that U is injective, which follows from the fact that $\|Ux\| = \|x\|$ for all $x \in H$.

We next show that U^{-1} is unitary. Choose arbitrary $x, y \in H$ and let $x = U^{-1}x'$, $y = U^{-1}y'$. Then, $Ux = x'$ and $Uy = y'$. So, $(x', y') = (Ux, Uy) = (x, y) = (U^{-1}x', U^{-1}y')$, that is, U^{-1} is unitary.

It remains to show that $U^{-1} = U^*$. For $x, y \in H$, let $U^{-1}y = z$, so that $y = Uz$. Then, $(Ux, y) = (Ux, Uz) = (x, z) = (x, U^{-1}y)$. Also, $(Ux, y) = (x, U^*y)$. Consequently, $(x, U^*y) = (x, U^{-1}y)$ and this implies $U^*y = U^{-1}y$ for all $y \in H$. This proves the assertion. □

Corollary 3.7.11 *Let U be a bounded linear operator defined on H. Then, U is unitary if, and only if, $UU^* = U^*U = I$.*

Proof Indeed, for $x, y \in H$ and U a unitary operator,

$$(x, y) = \left(x, U^{-1}Uy\right) = (x, U^*Uy),$$

which implies $U^*U = I$. Similarly, $UU^* = I$.

On the other hand, if $UU^* = U^*U = I$, then U is invertible (hence has range H) and

$$(x, y) = (U^*Ux, y) = (Ux, Uy).$$ □

The following simple characterisation of unitary operators is often useful.

Theorem 3.7.12 *Let H be a Hilbert space and let $U \in \mathcal{B}(H)$. Then, U is unitary if, and only if,*

(a) $\|Ux\| = \|x\|$ *for all* $x \in H$

and

(b) *the range of U is dense in H.*

Proof Suppose U is unitary. It has been observed that $\|Ux\| = \|x\|$ for all $x \in H$, that is, (a) holds. Condition (b) is satisfied by virtue of the definition of a unitary operator.

Suppose that (a) and (b) hold. Then for $x, y \in H$ and $\alpha \in \mathbb{C}$,

$$(x + \alpha y, x + \alpha y) = (U(x + \alpha y), U(x + \alpha y)).$$

Since U is linear, the above equality leads to

$$\begin{aligned}
&(x, x) + |\alpha|^2(y, y) + \alpha(y, x) + \overline{\alpha}(x, y) \\
&= (Ux, Ux) + |\alpha|^2(Uy, Uy) + \alpha(Uy, Ux) + \overline{\alpha}(Ux, Uy)
\end{aligned}$$

and this implies

$$\alpha(y, x) + \overline{\alpha}(x, y) = \alpha(Uy, Ux) + \overline{\alpha}(Ux, Uy), \tag{3.36}$$

using (a). On taking $\alpha = 1$ and $\alpha = i$ in (3.36), we obtain

$$(y, x) + (x, y) = (Uy, Ux) + (Ux, Uy) \tag{3.37}$$

and

$$(y, x) - (x, y) = (Uy, Ux) - (Ux, Uy). \tag{3.38}$$

On subtracting (3.38) from (3.37), we get

$$(Ux, Uy) = (x, y) \tag{3.39}$$

for all $x, y \in H$.

By (a), U is bounded below. Therefore, by (b) and Theorem 3.3.12, U is invertible. Together with (3.39) and Definition 3.7.9, this entails that U is unitary. □

Example 3.7.13

(i) Let $\ell^2(\mathbb{Z})$ denote the Hilbert space consisting of the complex functions x on $\mathbb{Z}$ such that $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$. Define U on $\ell^2(\mathbb{Z})$ by $U(x)(n) = x(n-1)$ for $x \in \ell^2(\mathbb{Z})$. The operator U is called the bilateral shift. It is clearly linear and the following calculation

$$||Ux||^2 = \sum_{n=-\infty}^{\infty} |(Ux)(n)|^2 = \sum_{n=-\infty}^{\infty} |x(n-1)|^2 = \|x\|^2$$

for $x \in \ell^2(\mathbb{Z})$ shows that it is bounded with norm 1.
The defining relation for U^* is $(x, U^*y) = (Ux, y)$, $x, y \in H$.

$$\begin{aligned}(x, U^*y) = (Ux, y) &= \sum_{n=-\infty}^{\infty} (Ux)(n)\overline{y(n)} = \sum_{n=-\infty}^{\infty} x(n-1)\overline{y(n)} \\ &= \sum_{n=-\infty}^{\infty} x(n)\overline{y(n+1)}.\end{aligned}$$

Therefore, $U^*y(n) = y(n+1)$. An easy computation shows that $UU^* = U^*U = I$. Thus, U is a unitary operator.

(ii) Let $H = L^2[0, 2\pi]$. Define $U{:}H \to H$ by the formula $(Ux)(t) = e^{it}x(t)$ for $x \in L^2[0, 2\pi]$. Observe that U is onto. Indeed, if $y \in L^2[0, 2\pi]$, then $e^{-it}y(t) = z(t) \in L^2[0, 2\pi]$ and

$$(Uz)(t) = e^{it}z(t) = e^{it}(e^{-it}y(t)) = y(t).$$

Moreover,

$$||Ux||^2 = \int_0^{2\pi} |e^{it}x(t)|^2 dt = \int_0^{2\pi} |x(t)|^2 dt = ||x||^2.$$

Thus,

$$||Ux|| = ||x|| \quad \text{for } x \in L^2[0, 2\pi].$$

Consequently, U is a unitary operator on $L^2[0,2\pi]$.
More general than a unitary operator defined on H is an isometric operator.

Definition 3.7.14 Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$. The operator T is said to be **isometric** if $\|Tx\| = \|x\|$ for all x in H.

Remarks 3.7.15 (i) An isometry is a distance preserving transformation:

$$\|Tx - Ty\| = \|T(x - y)\| = \|x - y\| \quad \text{for all} \;\; x, y \in H.$$

In particular, T is injective.

(ii) Observe that a unitary operator H is isometric. However, not every isometric operator is unitary. The simple unilateral shift T discussed in (vii) of Example 3.2.5 is an isometry but is not unitary because it is obviously not a bijection. In fact, T is not even normal, because its adjoint is given [see (vi) of Example 3.5.10] by $T^*(\{x_i\}_{i\geq 1}) = (x_2, x_3, \ldots)$, and hence,

$$T^*T\left(\{x_i\}_{i\geq 1}\right) = T^*(0, x_1, x_2, \ldots) = (x_1, x_2, \ldots) = \{x_i\}_{i\geq 1},$$

so that $T^*T = I$, while

$$TT^*\left(\{x_i\}_{i\geq 1}\right) = T(x_2, x_3, \ldots) = (0, x_2, x_3, \ldots).$$

The following provides a characterisation of an isometry.

Proposition 3.7.16 *Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$. Then, the following are equivalent.*

(a) *T is an isometry*,
(b) $T^*T = I$ and
(c) $(Tx, Ty) = (x, y)$.

Proof (a) implies (b). Since $\|Tx\| = \|x\|$ for all $x \in H$, we have

$$(T^*Tx, x) = (Tx, Tx) = ||Tx||^2 = ||x||^2 = (x, x).$$

This implies $T^*T = I$, in view of Problem 3.6.P4.

(b) implies (c). $(Tx,Ty) = (T^*Tx, y) = (x, y)$.

(c) implies (a). This follows on taking $y = x$ in $(Tx,Ty) = (x, y)$. □

Theorem 3.7.17 *The range* ran(T) *of an isometric operator T defined on a complex Hilbert space is a closed linear subspace of T.*

Proof Clearly, ran(T) = $T(H)$ is a linear subspace of H. Suppose $y \in \overline{[\mathrm{ran}(T)]}$. We need to show that $y \in$ ran(T). Chose a sequence $\{y_n\}_{n\geq 1}$ in ran(T) such that $y_n \to y$ as $n \to \infty$. Note that $y_n = Tx_n$ for some x_n in H, $n = 1,2,\ldots$. Since $\|x_m - x_n\| = \|T(x_m - x_n)\| = \|y_m - y_n\| \to 0$ as $m,n \to \infty$, it follows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. Since H is complete, there exists x in H such that $x_n \to x$. By continuity of T, we have $Tx_n \to Tx$, i.e. $Tx = \lim_n Tx_n = \lim_n y_n = y$. Hence, $y = Tx$, so $y \in$ ran(T). □

The following Theorem is an alternative characterisation of unitary operators [see Corollary 3.7.11].

Theorem 3.7.18 *Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$. Then, the following are equivalent*:

(a) $T^*T = TT^* = I$,
(b) *T is a surjective isometry* and
(c) *T is a normal isometry*.

Proof (a) implies (b). From (a), it follows that $TT^* = I$. This ensures that T is surjective. It also follows that $T^*T = I$, and hence, by Proposition 3.7.16, T is an isometry.

(b) implies (c). Since T is an isometry, $(Tx,Ty) = (x, y)$ by Proposition 3.7.16. Being surjective, T must be unitary by Definition 3.7.9. Hence, $T^*T = TT^* = I$ by Corollary 3.7.11, so that T is normal.

(c) implies (a). Since T is an isometry, $T^*T = I$ by Proposition 3.7.16. Since T is normal, $T^*T = TT^* = I$. This completes the proof. □

Definition 3.7.19 Let S and T be bounded linear operators on a Hilbert space H. The operator S is said to be **unitarily equivalent to** T if there exists a unitary operator U on H such that

$$S = UTU^{-1} = UTU^*.$$

Remark 3.7.20 If T is self-adjoint or normal, then so is any operator S that is unitarily equivalent to T. The reason is as follows: $S^* = (UTU^*)^* = (U^*)^*T^*U^* = UTU^* = S$, using the hypothesis that $T = T^*$. A similar argument shows that if T is normal, then so is S.

Problem Set 3.7

3.7.P1. Show that the range of a bounded linear operator need not be closed.

3.7.P2. [See Problem 3.7.P1] Let $T{:}H \to H$ be a bounded linear operator on a Hilbert space H. Suppose there exists $M > 0$ such that $\|Tx\| \geq M\|x\|$ for any $x \in H$. Prove that the range of T is a closed subspace of H.

3.7.P3. Let $H = \mathbb{C}^2$ and T be the operator defined on H by the matrix $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$. Find $\|T\|$ and $r(T)$. Show that T is not a normal operator.

3.7.P4. Let $S = I + T^*T{:}H \to H$, where $T \in \mathcal{B}(H)$. Show that

(a) $S^{-1}{:}\mathrm{ran}(S) \to H$ exists,
(b) $\mathrm{ran}(S)$ is closed,
(c) $\mathfrak{N}(S)$ = kernel of $S = \{0\}$ and
(d) $\|S^{-1}\| \leq 1$.

3.7.P5. If $f(z) = \sum_{k=0}^{\infty} z^n/n!$ and $A \in \mathcal{B}(H)$ is such that $A = A^*$, show that $f(iA)$ is unitary.

3.7.P6. Recall from (v) of Example 2.1.3 that RH^2 denotes the space of rational functions which are analytic on the closed unit disc $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, with the usual addition and scalar multiplication and with inner product

$$(f, g) = \frac{1}{2\pi i} \int_{\partial D} f(z)\overline{g(z)} \frac{dz}{z}.$$

Define an operator U on the inner product space RH^2 by

$$Uf(z) = \frac{(1 - |\alpha|^2)^{1/2}}{1 - \bar{\alpha}z} f\left(\frac{z - \alpha}{1 - \bar{\alpha}z}\right) \quad \text{for all} \quad z \in \overline{D},$$

where $\alpha \in D$ is fixed. Show that U is an isometry: $\|Uf\| = \|f\|$ for all $f \in RH^2$.

3.7.P7. Let $T \in \mathcal{B}(H)$ be a normal operator. Assume that $T^m = O$ for some positive integer m. Show that $T = O$.

3.7.P8. Let $T \in \mathcal{B}(H)$ be normal. Show that T is injective if, and only if, T has dense range.

3.7.P9. (a) Give an example of an operator $S \in \mathcal{B}(H)$ such that $\ker(S) = \{0\}$ but ran (S) is not dense in H.

(b) Give an example of an operator $T \in \mathcal{B}(H)$ such that T is surjective but $\ker(T) \neq \{0\}$.

3.7.P10. Let H be a Hilbert space. Show that the set of all normal operators in $\mathcal{B}(H)$ is closed in $\mathcal{B}(H)$ in the operator norm.

3.7.P11. If T is a normal operator on the complex Hilbert space H and $S \in \mathcal{B}(H)$ is such that $TS = ST$, then $T^*S = ST^*$.

Let $T \in \mathcal{B}(H)$ be a self-adjoint operator on a complex Hilbert space $H \neq \{0\}$. Then, $\sigma(T) \in \mathbb{R}$. So, $\pm\, i \in \rho(T)$, the resolvent set of T. The operators $T \pm iI$ are invertible elements in $\mathcal{B}(H)$. Consider the operator

$$U = (T - iI)(T + iI)^{-1} = (T + iI)^{-1}(T - iI)$$

and the inverse operator

$$U^{-1} = (T + iI)(T - iI)^{-1} = (T - iI)^{-1}(T + iI).$$

The transformation U is called the **Cayley transform of** T.

3.7.P12. (a) Show that U is unitary and $U = I - 2i(T + iI)^{-1}$.

(b) Also show that $1 \in \rho(U)$ and

(c) $T = i(I + U)(I - U)^{-1} = i(I - U)^{-1}(I + U)$.

3.8 Orthogonal Projections

Let H be a Hilbert space and M a closed subspace of H. The orthogonal decomposition theorem [Theorem 2.10.11] says that $H = M \oplus M^{\perp}$, where $M^{\perp}$ denotes the orthogonal complement of M. Thus, for each $x \in H$, there exists a unique $y \in M$ and $z \in M^{\perp}$ such that $x = y + z$.

The concept of orthogonal projection operator P_M, or briefly, projection, was defined in Definition 2.10.16. It was proved in Theorem 2.10.15 that the mapping $P_M{:}H \to H$ has range M; its kernel is $M^{\perp}$ and P_M restricted to M is the identity operator on M. Also proved therein are the following:

(i) P_M is linear, bounded with norm 1;
(ii) P_M is self-adjoint; and
(iii) P_M is idempotent: $P_M^2 = P_M$.

Definition 3.8.1 Let $P \in \mathcal{B}(H)$. P is called an **orthogonal projection** if $P^* = P$ and $P^2 = P$.

Associated with any closed subspace M of H, the orthogonal projection operator P_M, or briefly, P, has the properties (i), (ii) and (iii), and also satisfies $\text{ran}(P_M) = M$, $\ker(P_M) = M^{\perp}$ [Theorem 2.10.15].

We now reverse the above trend that if $P \in \mathcal{B}(H)$ is such that $P^* = P$ and $P^2 = P$, then there exists a unique subspace M of H such that P is the associated orthogonal projection operator P_M.

Set

$$M = \{x \in H : Px = x\}.$$

Clearly, $M = \ker(I - P)$ and therefore a closed subspace.

We next show that $\text{ran}(P) = M$ and $\ker(P) = M^{\perp}$. Indeed, if $x \in H$, then $Px = P^2x = P(Px)$. Thus, $Px \in M$ for each $x \in H$, i.e. $PH \subseteq M$. On the other hand, if $x \in M$, then $x = Px \in PH$. Hence, $PH = M$. Also, if $Px = 0$, then for $z \in H$, $(x, P^*z) = (Px, z) = (0, z) = 0$, that is, $x \in (P^*H)^{\perp} = (PH)^{\perp} = M^{\perp}$. On the other hand, if $x \in M^{\perp}$, then $(Px, z) = (x, P^*z) = (x, Pz) = 0$ for each $z \in H$. Therefore, $Px = 0$ for $x \in M^{\perp}$.

Finally, for $x \in H$, we have $x = y + z$, where $y \in M$ and $z \in M^{\perp}$, and hence, $Px = Py + Pz = y$. Thus, P is the operator of orthogonal projection on M.

Combining the discussions in the paragraph above, we have the following Theorem.

Theorem 3.8.2 *Let $P \in \mathcal{B}(H)$. Then, P is a projection if, and only if,*

$$\{x \in H{:}Px = x\} = \ker(I - P) = \text{ran}(P) = \ker(P)^{\perp}.$$

Remarks 3.8.3

(i) The argument used to establish the above theorem shows that to each closed linear subspace M in H there corresponds a unique orthogonal projection

P such that $\text{ran}(P) = M$; to each orthogonal projection P there corresponds a closed linear subspace $M = \{x \in H{:}Px = x\} = \text{ran}(P)$. This enables us to replace geometric properties of subspaces in terms of algebraic properties of projections corresponding to them [see Theorems 3.8.4 and 3.8.5 below].

(ii) Every orthogonal projection is a positive operator: Indeed,

$$(Px, x) = (P^2x, x) = (Px, Px) = ||Px||^2 \geq 0.$$

(iii) Consider the operator P on $\mathbb{C}^2$ corresponding to the matrix $P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Observe that $P^2 = P$. Its range is $\{(x, 0){:}x \in \mathbb{C}\}$ and its kernel is $\{(x, -x){:}x \in \mathbb{C}\}$. However, P^* has matrix $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. So it is not an orthogonal projection.

(iv) Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space. For $y \in L^\infty(\mu)$, consider the operator T on $L^2(\mu)$ of multiplication by y:

$$Tx(t) = y(t)x(t), \quad x \in L^2(\mu), \quad t \in X.$$

[See (vi) of Example 3.2.5.] The operator T is bounded with $\|T\| = \|y\|_\infty$ and it is self-adjoint if, and only if, y is real-valued a.e. Observe that $T^2 = T$ if, and only if, $y^2 = y$ a.e., or y is equal a.e. to a characteristic function. Thus, if the operator of multiplication by a real-valued y is a projection, then it is an orthogonal projection.

We propose to show below in detail how the closed subspaces of a Hilbert space and the corresponding orthogonal projections are related to each other.

Theorem 3.8.4 *Let M and N be closed subspaces of a Hilbert space H, and P and Q denote the projections on M and N, respectively. Then,*

(a) *$I - P$ is the projection on $M^\perp$;*
(b) *$M \perp N$ if, and only if, $PQ = O$.*

Proof (a) Note that $(I - P)^* = I^* - P^* = I - P$ and $(I - P)^2 = I - 2P + P^2 = I - P$. Thus, $I - P$ is a projection operator. We next show that $\{x \in H : (I - P)x = x\} = M^\perp$. If $(I - P)x = x$, then $Px = 0$, which implies $x \in M^\perp$. On the other hand, if $x \in M^\perp$, then $Px = 0$, and hence, $(I - P)x = x$.

(b) Suppose that $PQ = O$. Then for $x \in M$ and $y \in N$,

$$(x, y) = (Px, Qy) = (x, PQy) = 0.$$

Therefore, $M \perp N$. Conversely, if $M \perp N$, then for any $x \in H$, $Qx \in N \subseteq M^\perp$; so, $PQx = 0$ for $x \in H$. Hence, $PQ = O$. □

Under the condition (b) of the above theorem, we speak of projections P and Q themselves as being orthogonal.

Theorem 3.8.5 *Let M and N be closed subspaces of a Hilbert space H. If P and Q denote projections on M and N, respectively, then the following are equivalent*:

(a) $M \subseteq N$;
(b) $P \leq Q$;
(c) $PQ = P$; and
(d) $QP = P$.

Proof (a) implies (c). If $M \subseteq N$, then $Px \in N$ for each $x \in H$. Therefore, $Q(Px) = Px$, $x \in H$; so $QP = P$. Also, $(QP)^* = P^*$, that is, $P^*Q^* = P^*$, which implies $PQ = P$.

(c) implies (b). Suppose $PQ = P$. Then for $x \in H$,

$$(Px, x) = (P^2x, x) = (Px, Px) = ||Px||^2 = ||PQx||^2 \leq ||Qx||^2 = (Qx, Qx) = (Qx, x).$$

Hence, $P \leq Q$.

(b) implies (a). Suppose that $P \leq Q$ and let $x \in M$. Then,

$$\begin{aligned}||x||^2 = ||Px||^2 &= (Px, Px) = (P^2x, x) = (Px, x) \leq (Qx, x)\\ &= (Q^2x, x) = (Qx, Qx) = ||Qx||^2 \leq ||x||^2.\end{aligned}$$

Hence, $\|Qx\| = \|x\|$. Now,

$$x = Qx + (I - Q)x,$$

and so,

$$||x||^2 = ||Qx||^2 + \|(I - Q)x\|^2$$

and this implies

$$\|(I - Q)x\| = 0,$$

since

$$||x||^2 = ||Qx||^2.$$

Consequently,

$$x = Qx,$$

i.e. $x \in N$.

(c) implies (d). Let $PQ = P$. Then, $P = P^* = (PQ)^* = Q^*P^* = QP$. Now let $QP = P$. Then, $P = P^* = (QP)^* = P^*Q^* = PQ$. □

The next few results give necessary and sufficient conditions for addition, subtraction and multiplication of projection operators to result in a projection operator.

Theorem 3.8.6 *Let $\{P_i\}_{i\geq 1}$ be a denumerable or finite family of projections and $\Sigma_i P_i = P$ in the sense of strong convergence. Then, a necessary and sufficient condition that P be a projection is that $P_j P_k = O$ whenever $j \neq k$. If this condition is satisfied and if, for each j, the range of P_j is M_j, then the range of P is $M = \Sigma_i M_i = \{x \in H{:}x = \Sigma_i x_i,\ x_i \in M_i,\ i = 1,2,\ldots\} = \overline{[\cup_k M_k]}$.*

Proof If the family $\{P_i\}_{i\geq 1}$ satisfies the condition, then

$$P^2 = (\Sigma_i P_i)(\Sigma_j P_j) = \Sigma_{i,j} P_i P_j = \Sigma_i P_i = P$$

and

$$(Px, y) = (\Sigma_i P_i x, y) = \Sigma_i (P_i x, y) = \Sigma_i (x, P_i y) = (x, \Sigma_i P_i y) = (x, Py)$$

for every pair x, y in H. In other words, the orthogonality of the family $\{P_i\}$ implies that P is idempotent and Hermitian, and hence, P is a projection.

If, conversely, P is a projection and if $x \in \operatorname{ran}(P_k)$ for some value of k, then

$$||x||^2 \geq ||Px||^2 = (Px, x) = \Sigma_i (P_i x, x) = \Sigma_i ||P_i x||^2 \geq ||P_k x||^2 = ||x||^2.$$

It follows that every term in the chain of inequalities is equal to every other term. From the equality

$$\sum_i ||P_i x||^2 = ||P_k x||^2,$$

we conclude that $P_i x = 0$ whenever $i \neq k$ and hence, $P_i(\operatorname{ran}(P_k)) = \{0\}$ whenever $i \neq k$. Thus, the family $\{P_i\}_{i\geq 1}$ satisfies the condition $P_j P_k = O$ whenever $j \neq k$.

We next show that $\operatorname{ran}(P) = \Sigma_i M_i$, where $M_i = \operatorname{ran}(P_i)$. For any $Px \in \operatorname{ran}(P)$, we have $Px = \Sigma_i P_i x \in \Sigma_i M_i$, because $P_i x \in M_i$. Thus, $\operatorname{ran}(P) \subseteq \Sigma_i M_i$. On the other hand, every $z \in \Sigma_i M_i$ is of the form $\Sigma_i x_i$, $x_i \in M_i$, so that $Pz = \Sigma_i P_i x_i = \Sigma_i x_i = z$, which implies $z \in \operatorname{ran}(P)$. Thus, $\Sigma_i M_i \subseteq \operatorname{ran}(P)$.

Finally, we show that $\operatorname{ran}(P) = \overline{[\cup_k M_k]}$. From the equality of $\|x\|$ and $\|Px\|$, we conclude that $x \in \operatorname{ran}(P)$ and hence, $M_k \subseteq \operatorname{ran}(P)$ for all k and it therefore follows that $\overline{[\cup_k M_k]} \subseteq \operatorname{ran}(P)$. On the other hand, $P_k x \in M_k$ for every vector x and every value of k; it follows that $Px = \Sigma_k P_k x \in \Sigma_k M_k \subseteq \overline{[\cup_k M_k]}$ for all x, i.e. $\operatorname{ran}(P) \subseteq \overline{[\cup_k M_k]}$. □

The useful fact about the product of projections is contained in the following.

Theorem 3.8.7 *The product of two projection operators P and Q is a projection operator if, and only if, $PQ = QP$. In this case, PQ is the projection on $M \cap N$, where M* [resp.N] *is the subspace of H on which P* [resp. Q] *is the projection.*

Proof Suppose that PQ is a projection. Then,

$$PQ = (PQ)^* = Q^*P^* = QP.$$

On the other hand, suppose that $PQ = QP = R$, say. Then,

$$R^2 = (PQ)(PQ) = PPQQ = P^2Q^2 = PQ = R$$

and for all pairs x, y in H,

$$(Rx, y) = (PQx, y) = (Qx, Py) = (x, QPy) = (x, Ry).$$

Thus, R is both self-adjoint and idempotent.

Finally, we show that the range of PQ is $M \cap N$.

For $x \in H$, let

$$y = PQx = QPx.$$

By the first representation, $y \in M$ and by the second representation, $y \in N$. Hence, $x \in M \cap N$, i.e. the range of PQ, ran$(PQ) \subseteq M \cap N$. If $x \in M \cap N$, then $PQx = x$. Thus, ran$(PQ) = M \cap N$. □

We finally treat the difference of projections.

Theorem 3.8.8 *The difference of two projections*, $P_1 - P_2$, *is a projection if, and only if*, $M_2 \subseteq M_1$, *where* M_1 [resp.M_2] *is the subspace of* H *on which* P_1 [resp.P_2] *is the projection. In this case*, ran$(P_1 - P_2) = M_1 \cap M_2^{\perp}$.

Proof Suppose $P_1 - P_2$ is an orthogonal projection. Then for $x \in H$,

$((P_1 - P_2)x,x) = ((P_1 - P_2)^2x,x) = ((P_1 - P_2)x,(P_1 - P_2)x)) = \|(P_1 - P_2)x\|^2 \geq 0$,

which proves that $M_2 \subseteq M_1$ [see Theorem 3.8.5]. On the other hand, suppose that $M_2 \subseteq M_1$. Then,

$$P_1P_2 = P_2 = P_2P_1 \quad [\text{Theorem 3.8.5}]. \tag{3.40}$$

Now,

$$(P_1 - P_2)^2 = P_1^2 - P_1P_2 - P_2P_1 + P_1^2 = P_1 - P_2$$

and

$$(P_1 - P_2)^* = P_1{}^* - P_2{}^* = P_1 - P_2.$$

Finally, we show that ran$(P_1 - P_2) = M_1 \cap M_2^{\perp}$. Since $P_1P_2 = P_2P_1$ by (3.40) above, it follows that

$$P_1(I - P_2) = P_1 - P_1P_2 = P_1 - P_2P_1 = (I - P_2)P_1.$$

Hence, by Theorem 3.8.7, $P_1(I - P_2)$ is an orthogonal projection with range given by

$$\text{ran}(P_1) \cap \text{ran}(I - P_2) = \text{ran}P_1 \cap \text{ran}(P_2)^{\perp}.$$

The proof is completed by observing that $P_1(I - P_2) = (I - P_2)P_1 = P_1 - P_2$. □

Let H be a finite-dimensional Hilbert space and $T \in \mathcal{B}(H)$ be such that $T^*T = TT^*$. The subspace M formed by the eigenvectors belonging to a certain eigenvalue is invariant under T, i.e. $T(M) \subseteq M$. In fact, $T(M^{\perp}) \subseteq M^{\perp}$ as well. Since T and T^* commute, it follows that $(T - \lambda I)$ and $(T^* - \overline{\lambda} I)$ commute. Therefore, they have the same kernel. This implies that $Ty = \lambda y$ if, and only if, $T^*y = \overline{\lambda} y$. Let $x \in M^{\perp}$ and $y \in M$. Then, $(Tx, y) = (x, T^*y) = (x, \overline{\lambda} y) = \lambda(x, y) = 0$. Consequently, $T(M^{\perp}) \subseteq M^{\perp}$. M is called a reducing subspace of T.

Although no analogous structure theory exists for operators on infinite-dimensional spaces, the notions of "invariant subspaces" and "reducing subspaces" do make sense.

Definition 3.8.9 A subspace M of a Hilbert space H is said to be **invariant** under a bounded linear operator $T \in \mathcal{B}(H)$ if $T(M) \subseteq M$. The subspace $M \subseteq H$ is said to **reduce** T if $T(M) \subseteq M$ and $T(M^{\perp}) \subseteq M^{\perp}$, i.e. if both M and $M^{\perp}$ are invariant under T. Then, M and $M^{\perp}$ are called **reducing subspaces** of T.

It can be easily checked that M reduces T if, and only if, M is invariant under both T and T^*.

The investigation of T is facilitated by considering $T|_M$ and $T|_{M^{\perp}}$ separately.

Note that the subspace $\{0\}$ and H are invariant under any $T \in \mathcal{B}(H)$. Also, $\ker(T)$ is always invariant under T; for $Tx = 0$ implies $T(Tx) = 0$.

Theorem 3.8.10 *Let P be the orthogonal projection onto the subspace M of H. Then, M is invariant under an operator $T \in \mathcal{B}(H)$ if, and only if, $TP = PTP$; M reduces T if, and only if, $TP = PT$.*

Proof For each $x \in H$, $Px \in M$. Suppose M is invariant under T. Then, $T(Px) \in M$, and hence, $PTPx = TPx$; so $PTP = TP$. Conversely, if $PTP = TP$, then for every $x \in M$, we have $Tx = TPx = PTPx$, and this is a vector in M. This proves that M is invariant under T.

It remains to show that M reduces T if, and only if, $TP = PT$.

M reduces T if, and only if, $TP = PTP$ and $T(I - P) = (I - P)T(I - P)$ if, and only if, $TP = PTP = PT$. This completes the proof. □

Problem Set 3.8

3.8.P1. Let $X = Y = \ell^2$.

(a) Define $T_n x = \frac{1}{n} x$ for all $x \in \ell^2$. Show that $\lim_n \|T_n\| = 0$.

(b) Let e_1, e_2, … be an orthonormal basis. Let P_n be the orthogonal projection on the linear span of $\{e_1, e_2, \ldots, e_n\}$, so that $I - P_n$ is the orthogonal projection on the complement of this space. Show that $P_n \to I$ in strong operator convergence, but not in the operator norm convergence.

(c) Let T: $\ell^2 \to \ell^2$ be defined as follows: $T((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots)$. Show that $T^n \to O$ weakly but not strongly. For $x, y \in \ell^2$,

$$(T^n x, y) = ((0, \ldots, 0, x_1, x_2, \ldots), (y_1, y_2, \ldots, y_n, y_{n+1}, \ldots))$$
$$= \sum_{k=1}^{\infty} x_k \bar{y}_{n+k}.$$

Definition. *A linear operator P in any linear space X is said to be a* **projection** *if* $P^2 = P$. (Note that we do not require a projection to be a bounded linear operator or to be self-adjoint.)

3.8.P2. Let P be a projection in X. Then,

(a) $I - P$ is a projection in X;
(b) $\mathrm{ran}(P) = \{x \in X : Px = x\}$;
(c) $\mathrm{ran}(P) = \ker(I - P)$;
(d) $X = \mathrm{ran}(P) \oplus \mathrm{ran}(I - P)$; and
(e) if P is bounded, then $\mathrm{ran}(P)$ and $\mathrm{ran}(I - P)$ are closed.

3.8.P3. Show that a projection P in a Hilbert space is an orthogonal projection iff $\mathrm{ran}(P) \perp \ker(P)$.

3.8.P4. Consider the Volterra operator V on $L^2[0,1]$ given by

$$Vx(s) = \int_0^s x(t)\mathrm{d}t, \quad x \in L^2[0,1].$$

Find V^* and show that $V + V^*$ is a projection on the space spanned by the vector 1.

3.9 Polar Decomposition

This section deals with an application of positivity defined in Definition 3.6.10 to obtain the "polar decomposition" of an operator, analogous to the representation of a complex number z as $|z|e^{i\theta}$ for some real θ. Does an analogue exist for operators? In order to answer this question, we need to define the analogues of $|z|$ and $e^{i\theta}$ amongst operators suitably.

Definition 3.9.1 For $T \in \mathcal{B}(H)$, we define

$$|T| = \sqrt{T^*T}.$$

Remarks 3.9.2

(i) The reader should note that $T^*T \geq O$ and therefore $\sqrt{T^*T}$ is uniquely defined and is positive.
(ii) It is true that $|\lambda T| = |\lambda||T|$, whenever $\lambda \in \mathbb{C}$ and $T \in \mathcal{B}(H)$.
(iii) If the square of an operator S is invertible, i.e. $S^2U = US^2 = I$ for some U, then we have $S(SU) = I = (US)S$. Also,

$$SU = \left(US^2\right)(SU) = (US)\left(S^2U\right) = US,$$

which is therefore an inverse of S. Now, if T is any invertible operator, then so is T^*T and consequently, $|T|$ is invertible.

The analogue in $\mathcal{B}(H)$ of the complex numbers of absolute value 1 is rather complicated. At first one might expect that unitary operator would suffice. A little reflection shows that this is not the case.

Example 3.9.3 Let T be the simple unilateral shift on ℓ^2. Then, as seen in Remark 3.7.15(ii), $T^*T = I$, so that $|T| = \sqrt{T^*T} = I$, but T is not unitary. So, if write $T = U|T|$ or $|T|U$, we must have $U = T$, which is not unitary.

Definition 3.9.4 An operator $T \in \mathcal{B}(H)$ is called a **partial isometry** if T is an isometry when restricted to the closed subspace $[\ker(T)]^\perp$, i.e. $\|Tx\| = \|x\|$ for every $x \in [\ker(T)]^\perp$.

Observe that $\|T\| \leq 1$. Every isometry is a partial isometry. Every orthogonal projection is a partial isometry.

The subspace $[\ker(T)]^\perp$ is called the **initial space** of T and ran(T) is called its **final space**. It is obvious that the initial space is always closed; we shall now show that the final space too is always closed, i.e. $\overline{[\mathrm{ran}(T)]} = \mathrm{ran}(T)$ when T is a partial isometry: let $x \in \overline{[\mathrm{ran}(T)]}$. Then, there exists a sequence $\{x_n\}_{n\geq 1}$ in H such that $Tx_n \to x$. For each n, there exist $y_n \in \ker(T)$ and $z_n \in [\ker(T)]^\perp$ such that $x_n = y_n + z_n$. Then, we have $Tz_{n\text{-}} = Tx_n$ and also

$$\begin{aligned} \|Tx_n - Tx_m\| &= \|T(y_n - y_m) + T(z_n - z_m)\| \\ &= \|T(z_n - z_m)\| \quad \text{because}\, y_n - y_m \in \ker(T) \\ &= \|z_n - z_m\| \quad \text{because}\, z_n - z_m \in [\ker(T)]^\perp. \end{aligned}$$

But $\{Tx_n\}_{n\geq 1}$ is a Cauchy sequence (since it converges to x), and by the above equality, $\{z_n\}_{n\geq 1}$ is also Cauchy sequence. Let $z_n \to z$. By continuity of T, we have $Tz = \lim_n Tz_n = \lim_n Tx_n = x$, which shows that $x \in \mathrm{ran}(T)$.

The following proposition is in order.

Proposition 3.9.5 *Let $U \in \mathcal{B}(H)$. Then, the following statements are equivalent*:

(a) *U is a partial isometry*;
(b) *U^* is a partial isometry*;
(c) *U^*U is a projection*; and
(d) *UU^* is a projection.*

Moreover, U^*U is a projection on $[\ker(U)]^{\perp}$ *and UU^* is a projection on* $\overline{[\operatorname{ran}(U)]} = \operatorname{ran}(U)$.

Proof (a) implies (c): to begin with, observe that for any $T \in \mathcal{B}(H)$, we have $\ker(T) = \ker(T^*T)$ by Theorem 3.5.8.

For $x \in H$,

$$((I - U^*U)x, x) = (x, x) - (U^*Ux, x) = ||x||^2 - ||Ux||^2 \geq 0,$$

since $\|U\| \leq 1$. Thus, $I - U^*U$ is a positive operator. Now if $x \perp \ker(U)$, then $\|Ux\| = \|x\|$, which implies that $((I - U^*U)x,x) = 0$. Since $\left\|(I - U^*U)^{\frac{1}{2}}x\right\|^2 = |((I - U^*U)x, x) = 0$, we have $(I - U^*U)x = 0$ or $U^*Ux = x$. On the other hand, U^*U obviously maps $\ker(U)$ into $\{0\}$. Consequently, $(U^*U)^2 = U^*U$. Since U^*U is self-adjoint, it follows by Theorem 3.8.2 that it is a projection onto the orthogonal complement of its own kernel. However, its kernel is the same as that of U. (Note that the orthogonal complement is by definition the initial space of U.)

(c) implies (a): if U^*U is a projection and $x \perp \ker(U^*U)$, then $U^*Ux = x$. Therefore,

$$||Ux||^2 = (Ux, Ux) = (U^*Ux, x) = (x, x) = ||x||^2$$

and hence, U preserves the norm on $[\ker(U^*U)]^{\perp}$. But as noted at the beginning, $\ker(U^*U) = \ker(U)$. Therefore, U is a partial isometry.

(b) implies (d) and (d) implies (b) follow by reversing the roles of U and U^*.

(c) implies (d): first observe that UU^* is self-adjoint. We shall show that

$$(UU^*)^2 = (UU^*U)U^* = UU^*.$$

It is enough to show that $UU^*U = U$. To this end, we note that this holds on $\ker(U)$. Since it has already been proved that (c) implies (a), we know that U is a partial isometry. Therefore, for x in $\ker(U)^{\perp}$, we have $\|Ux\| = \|x\|$, which implies $U^*Ux = x$ (see the proof of (a) implies (c)); thus, we have $UU^*U = U$ also on $\ker(U)^{\perp}$ and hence on all of H. □

Observe that Proposition 3.9.5 has the following consequence: if U is a partial isometry, then $\|Ux\| = \|x\|$ if, and only if, $x \in \operatorname{ran}(U^*U)$. Indeed, $\|Ux\|^2 = (Ux,Ux) = (U^*Ux,x) = (U^*UU^*Ux,x) = \|U^*Ux\|^2$, and it is true of any orthogonal projection P that $\|Px\| = \|x\|$ is equivalent to $x \in \operatorname{ran}(P)$.

We next prove the analogue of the decomposition $z = |z|e^{i\theta}$ for some θ. Theorem 3.5.8 will be used frequently without explicit mention.

Theorem 3.9.6 (Polar Decomposition) *Let $T \in \mathcal{B}(H)$. Then, there is a partial isometry U such that $T = U|T|$ and* $\ker(U) = \ker(T)$. *Moreover*, $\operatorname{ran}(U) = \overline{[\operatorname{ran}(T)]}$. *Amongst all bounded linear operators V such that $T = V|T|$, U is uniquely determined by the condition* $\ker(V) \supseteq \ker(T)$.

Proof Define U:ran$(|T|)\rightarrow$ran(T) by $U(|T|x) = Tx$. Since

$$||Tx||^2 = (Tx, Tx) = (x, T^*Tx) = (x, |T|^2x) = |||T|x|||^2, \tag{3.41}$$

it follows that U is well defined. Indeed, if we apply (3.41) to $x - y$, we deduce that if $|T|x = |T|y$ then $Tx = Ty$. The equality (3.41) also shows that U preserves norms and hence extends to a norm preserving linear mapping of $\overline{[\operatorname{ran}(|T|)]}$ onto $\overline{[\operatorname{ran}(T)]}$ such that $\ker(U) = \{0\}$. Extend U to all of H by defining it to be zero on $\overline{[\operatorname{ran}(|T|)]}^{\perp}$ $= \ker(|T|)$, so that it now has kernel equal to $\ker(|T|)$ but the same range as before, which is $\overline{[\operatorname{ran}(T)]}$. Observe that $T = U|T|$ on H. Furthermore, in view of (3.41), $|T|x = 0$ if, and only if, $Tx = 0$, so that $\ker(|T|) = \ker(T)$. Thus, $\ker(U) = \ker(T)$ and, as already noted, $\operatorname{ran}(U) = \overline{[\operatorname{ran}(T)]}$.

We next consider uniqueness.

If V is any linear operator with $V|T| = T$ and $\ker(V) \supseteq \ker(T)$, we note that $Vy = Uy$ for every $y \in \operatorname{ran}(|T|)$, so that $U = V$ on $\overline{[\operatorname{ran}(|T|)]}$. Since both operators are zero on $\overline{[\operatorname{ran}(|T|)]}^{\perp} = \ker(|T|) = \ker(T) \subseteq \ker(V)$, it follows that $V = U$. □

The preceding decomposition theorem is due to von Neumann.

The factorisation $T = U|T|$, where U is the unique partial isometry such that $T = U|T|$ and $\ker(U) = \ker(T)$ is called *the polar decomposition of* T and U is called *the partial isometry in the polar decomposition of* T.

The uniqueness argument in the last paragraph of the above proof begins by assuming that V satisfies $\ker(V) \supseteq \ker(T)$ as well as $V|T| = T$, but not that it is a partial isometry. Nevertheless, even a partial isometry V satisfying only $T = V|T|$ need not be unique. This is illustrated by (ii) of the Remarks below.

Remarks 3.9.7 (i) If $T \in \mathcal{B}(H)$ is invertible, the partial isometry in its polar decomposition is unitary, as we now show.

Since T is invertible, $\ker(T) = \{0\}$ and $\operatorname{ran}(T) = H$. Consequently, if $T = U|T|$ is the polar decomposition of T, then $\ker(U) = \ker(T) = \{0\}$ and $\operatorname{ran}(U) = \overline{[\operatorname{ran}(T)]} = H$. Hence, U is unitary.

(ii) If $y(t)$ is a complex measurable function on [0,1], there are complex measurable functions α on [0,1] such that $|\alpha(t)| = 1$ when $y(t) \neq 0$ and $y(t) = \alpha(t)|y(t)|$ everywhere. Then, the operator T of multiplication on $L^2[0,1]$ defined by

$$Tx(t) = y(t)x(t), \quad x \in L^2[0,1],$$

satisfies $T = V|T|$, where V is the operator of multiplication by α. Loosely speaking,

$$T = \alpha|y(t)|.$$

If y vanishes on a set Y of positive measure, then several such α are possible, amongst which several have the property that $|\alpha|$ is the characteristic function of some set. In case α is chosen (nonuniquely) so that $|\alpha|$ is the characteristic function of some set E, then V can be shown to be a partial isometry by arguing as follows. The kernel of V is $\{x \in L^2[0, 1]: x(t) = 0 \text{ a.e. on } E\}$ and its orthogonal complement is $\{x \in L^2[0, 1]: x(t) = 0 \text{ a.e. on } E^c\}$; we have to show for any x in this orthogonal complement that $\|Vx\| = \|x\|$, i.e. $\int_0^1 |\alpha(t)x(t)|^2 \mathrm{d}t = \int_0^1 |x(t)|^2 \mathrm{d}t$. Since $|\alpha|$ is the characteristic function of E, the former integral equals $\int_E |x(t)|^2 \mathrm{d}t$; since x vanishes a.e. on E^c, the latter integral also equals $\int_E |x(t)|^2 \mathrm{d}t$. Thus, the two integrals are equal and V is therefore a partial isometry.

What it takes for V to have the same kernel as T is that

$$\begin{aligned}\{x \in L^2[0,1] : x(t) = 0 \,\text{a.e. on}\, E\} &= \{x \in L^2[0,1] : y(t)x(t) = 0 \,\text{a.e. on}\, [0,1]\} \\ &= \{x \in L^2[0,1] : x(t) = 0 \,\text{a.e. on}\, Y^c\},\end{aligned}$$

or equivalently, the symmetric difference $(E \backslash Y^c) \cup (Y^c \backslash E)$ has measure zero. This amounts to saying that the characteristic function $|\alpha|$ of E must be equal a.e. to that of Y^c. In other words, α must be equal a.e. to 0 on Y and $y(t)/|y(t)|$ on Y^c. With this choice of α, the polar decomposition of T is $V|T|$.

It has been shown by Ichinose and Iwashita in [14] that a partial isometry such that $T = V|T|$ is unique if, and only if, either $\ker(T)$ or $\ker(T^*)$ is $\{0\}$. They have proved this for operators from one Hilbert space to another, but we shall confine ourselves to operators from a Hilbert space into itself. Our considerations carry over verbatim to the broader case. We begin with a preliminary remark.

Remark 3.9.8 The zero operator is a partial isometry. It is easy to see that, given a partial isometry $V \in \mathcal{B}(H)$ and any $x \in H$, the equality $\|Vx\| = \|x\|$ is equivalent to $x \in (\ker(V))^\perp$. Also, given any partial isometry V and any complex number λ of absolute value 1, the operator λV is a partial isometry with the same kernel as V. Distinct λ gives rise to distinct partial isometries λV unless $V = O$.

Proposition 3.9.9 *If $T \in \mathcal{B}(H)$ and V is a partial isometry such that $T = V|T|$, then*

(a) $V^*T = |T| = T^*V$;
(b) $V^*V|T| = |T|$ *and* $V|T|V^* = |T^*|$.

Proof (a) Since $T = V|T|$, we have $V^*T = V^*V|T|$. Therefore, in order to show that $V^*T = |T|$ it is sufficient to arrive at $\mathrm{ran}(|T|) \subseteq \mathrm{ran}(V^*V)$. We arrive at this by showing that $y = |T|x$ implies $\|Vy\| = \|y\|$ and using the observation just after Proposition 3.9.5:

$$\|Vy\|^2 = (V|T|x, V|T|x) = (Tx, Tx) = (T^*Tx, x) = \left(|T|^2x, x\right) = (|T|x, |T|x)$$
$$= \|y\|^2.$$

As $|T|$ is self-adjoint, it follows that $|T| = T^*V$ as well.

(b) It follows from (a) that $V^*V|T| = V^*T = |T|$. As for $V|T|V^*$, we note that it is positive and that its square is $V|T|V^*V|T|V^* = V|T|(V^*V|T|)V^* = V|T|^2V^* = (V|T|)(|T|V^*) = TT^*$. It is immediate from here that $V|T|V^* = |T^*|$. □

Theorem 3.9.10 *If* $T \in \mathcal{B}(H)$ *and either* $\ker(T)$ *or* $\ker(T^*)$ is $\{0\}$, *then there is a unique partial isometry* V *such that* $T = V|T|$.

Proof Existence has been established in Theorem 3.9.6. Uniqueness when $\ker(T) = \{0\}$ is a trivial consequence of the last part of that Theorem.

To prove uniqueness when $\ker(T^*) = \{0\}$, consider any partial isometries U and V such that $T = U|T|$ and $T = V|T|$. By Proposition 3.9.9(a), we have $T^*U = |T| = T^*V$. When $\ker(T^*) = \{0\}$, this equality leads to $U = V$ immediately. □

Theorem 3.9.11 *If* $T \in \mathcal{B}(H)$ *and there is a unique partial isometry* V *such that* $T = V|T|$, *then either* $\ker(T)$ *or* $\ker(T^*)$ *is* $\{0\}$.

Proof We prove the contrapositive that if $\ker(T) \neq \{0\} \neq \ker(T^*)$, then there exist several partial isometries V satisfying $T = V|T|$. Theorem 3.9.6 ensures that at least one such partial isometry U always exists and we show how to get others from it when $\ker(T) \neq \{0\} \neq \ker(T^*)$. Recall that Theorem 3.9.6 provides not only that

$$T = U|T|$$

but also that

$$\overline{[\mathrm{ran}(T)]} = \mathrm{ran}(U) \quad \text{and} \quad \ker(T) = \ker(U).$$

Since $\ker(T)$ and $\ker(T^*)$ must each have a one-dimensional subspace, there exists an isometry from the former one-dimensional subspace to the latter. Extend it to be an element of $\mathcal{B}(H)$ by defining it to be 0 on the orthogonal complement of the one-dimensional subspace and call it V. Then, V is a partial isometry, distinct from O; moreover,

$$(\ker(T))^{\perp} \subseteq \ker(V)$$

and

$$\mathrm{ran}(V) \subseteq \ker(T^*).$$

There are infinitely many possibilities for V because λV has the same properties when $|\lambda| = 1$. Since it is a partial isometry, V^*V is the projection on $(\ker(V))^{\perp}$. Since

$\ker(T^*) = \overline{[\operatorname{ran}(T)]}^{\perp} = (\operatorname{ran}(U))^{\perp} = \ker(U^*)$, the second of the above inclusions is equivalent to $\operatorname{ran}(V) \subseteq \ker(U^*)$, which is to say,

$$U^*V = O.$$

Besides, in the light of the fact that $\operatorname{ran}(|T|) \subseteq \ker(|T|)^{\perp} = (\ker(T))^{\perp}$ the first of the above inclusions leads to $\operatorname{ran}(|T|) \subseteq \ker(V)$, which can be rephrased as

$$V|T| = O.$$

Set $W = U + V$. It is enough to show that W is a partial isometry and that $W|T| = T$. The latter is an easy consequence of the equality $V|T| = O$:

$$W|T| = (U+V)|T| = U|T| + V|T| = U|T| + O = U|T| = T.$$

We can show that W is a partial isometry by merely arguing that W^*W is a projection [Proposition 3.9.5]. Keeping in mind that $U^*V = O$, so that $V^*U = O$ as well, we have

$$W^*W = (U^* + V^*)(U+V) = U^*U + U^*V + V^*U + V^*V = U^*U + V^*V.$$

But U^*U is the projection on $(\ker(U))^{\perp} = (\ker(T))^{\perp} \subseteq \ker(V)$. This means U^*U and V^*V are projections on mutually orthogonal subspaces. Therefore, their sum W^*W is a projection. This establishes that W is a partial isometry. ☐

We note in passing that every partial isometry W such that $W|T| = T$ must necessarily be of the form $U + V$, where V is a partial isometry which, as in the foregoing proof, satisfies $(\ker(T))^{\perp} \subseteq \ker(V)$ and $\operatorname{ran}(V) \subseteq \ker(T^*)$. For details, the reader is referred to [14].

Theorem 3.9.12 *If $T \in \mathcal{B}(H)$ and $n \in \mathbb{N}$, then* $\||T|^n\| = \|T\|^n$.

Proof Equality (1) in the proof of Theorem 3.9.6 justifies the case $n = 1$. For other values of n, the desired equality follows upon applying Theorem 3.6.18 to the self-adjoint operator $|T|$ and using the case when $n = 1$. ☐

Proposition 3.9.13 *If $T \in \mathcal{B}(H)$, then*

$$\ker(T^*T) = \ker(T) = \ker(|T|) \text{ and } \overline{[\operatorname{ran}(T^*T)]} = \overline{[\operatorname{ran}(|T|)]}.$$

Proof The first equality is a restatement of the first equality of Theorem 3.5.8. Applying it to $|T|$ in place of T, we get $\ker(|T|) = \ker(|T|^*|T|) = \ker(|T|^2) = \ker(T^*T)$. The last equality follows upon taking orthogonal complements and invoking the third equality in Theorem 3.5.8. ☐

Problem Set 3.9

3.9.P1. Let $T:\ell^2 \to \ell^2$ be defined by $((\xi_1, \xi_2, \ldots) \to (0, 0, \xi_3, \xi_4, \ldots)$. Without using general properties of projections, show that T is bounded and positive. Find the square root of T.

3.9.P2. Let $T \in \mathcal{B}(H)$ be self-adjoint and positive, where H denotes a complex Hilbert space. Show that

(a) $\left\|T^{\frac{1}{2}}\right\| = \|T\|^{\frac{1}{2}}$,

(b) $|(Tx, y)| \leq (Tx, x)^{\frac{1}{2}}(Ty, y)^{\frac{1}{2}}$and

(c) $\|Tx\| \leq \|T\|^{\frac{1}{2}}(Tx, x)^{\frac{1}{2}}$, so that $(Tx, x) = 0$ if, and only if, $Tx = 0$.

3.9.P3. (a) If $T \in \mathcal{B}(H)$ is a partial isometry and $x \in \text{ran}(T)$, show that T^*x is the unique element y of $[\ker(T)]^\perp$ such that $x = Ty$. Moreover, $\|T^*x\| = \|y\| = \|x\|$.
(b) Show that if $T \in \mathcal{B}(H)$ is a partial isometry, then so is T^*.

3.10 An Application

Mean Ergodic Theorem

Ergodic theory has its roots in the study of chaotic motion of small particles, such as pollen, suspended in a liquid. The chaotic motion was originally observed by the botanist R. Brown in 1862 and subsequently came to be called Brownian motion. The first result in connection with Brownian motion that led to major developments in mathematics was proved by Poincaré in 1890.

Let (X, Σ, μ) be a measure space and T be a measurable transformation of X into itself ($F \in \Sigma$ implies $T^{-1}(F) \in \Sigma$). The transformation is said to be *measure preserving* if $\mu(T^{-1}(E)) = \mu(E)$ for every $E \in \Sigma$. A point $x \in E$ is called *recurrent* with respect to E and T if $T^n x \in E$ for at least one positive integer n. Poincaré proved that almost every point of E is recurrent provided that $\mu(X) < \infty$. In fact, if $E \in \Sigma$ and $\mu(X) < \infty$, then for almost every $x \in E$, there are infinitely many n such that $T^n x \in E$, that is, almost every point of any measurable subset E returns to E infinitely many times. The question arises if such a point has a mean time of sojourn in E; more precisely if

$$\lim_n n^{-1} \sum_{k=0}^{n-1} \chi_E(T^k x)$$

exists where T^0 denotes the identity transformation. More generally, we may ask for which class of measurable functions $f(x)$

$$\lim_n n^{-1}\sum_{k=0}^{n-1} f(T^k x)$$

exists in some sense.

If we begin with a function f in $L^1(X, \Sigma, \mu)$, the associated function Uf given by $(Uf)(x) = f(Tx)$ belongs to $L^1(X, \Sigma, \mu)$ and has the norm as f. This is easy to see for characteristic functions, hence for simple functions and consequently for other functions, using the Monotone Convergence Theorem. Applying this to $|f|^2$, we conclude that U is also an isometry on $L^2(X, \Sigma, \mu)$. Note that the general term $f(T^k x)$ in the summation in the preceding paragraph can now be written as $(U^k f)(x)$.

The question raised above will now be answered in the general context of a Hilbert space for an operator U satisfying $\|U\| \leq 1$, not necessarily preserving the norm Riesz and Nagy [cf. 23, p. 454].

(Mean Ergodic Theorem) *Let H be a Hilbert space and U be a bounded linear operator in H with $\|U\| \leq 1$. If P is the orthogonal projection on the closed linear subspace $M = \{x \in H : Ux = x\}$, then*

$$\lim_n n^{-1}\sum_{k=0}^{n-1} U^k x = Px$$

for all $x \in H$.

Proof First, we shall prove that $Ux = x$ if, and only if, $U^*x = x$, where U^* denotes the adjoint of U. Observe that $\|U^*\| = \|U\| \leq 1$. Now $Ux = x$ implies

$$\begin{aligned} 0 \leq \|U^*x - x\|^2 &= \|U^*x\|^2 - (U^*x, x) - (x, U^*x) + \|x\|^2 \\ &= \|U^*x\|^2 - (x, Ux) - (Ux, x) + \|x\|^2 \\ &= \|U^*x\|^2 - (x, x) - (x, x) + \|x\|^2 \\ &= \|U^*x\|^2 - \|x\|^2 \leq 0, \end{aligned}$$

so, $U^*x = x$. Similarly, $U^*x = x$ implies $Ux = x$.

For any $x \in M$, the sums $n^{-1}\sum_{k=0}^{n-1} U^k x$ are all equal to x and so, converge to $x = Px$. Next, consider an element $x = y - Uy$, $y \in H$. For such an x, $\sum_{k=0}^{n-1} U^k x = y - U^n y$ and so, $\|n^{-1}\sum_{k=0}^{n-1} U^k x\| \leq 2n^{-1}\|y\| \to 0$ as $n \to \infty$. The collection

$$\{x \in H : x = y - Uy, y \in H\} \tag{3.42}$$

is clearly linear but not necessarily closed. Let z be any element in the closure K of the collection (3.42). Then, there is a sequence $x_p = y_p - Uy_p$ such that $x_p \to z$ as $p \to \infty$. Let $A_n = n^{-1}\sum_{k=0}^{n-1} U^k$. Then, $\|A_n\| \leq 1$ for all n and

$$\|A_n z\| \le \|A_n(z - x_p)\| + \|A_n x_p\| \le \|z - x_p\| + \|A_n x_p\|.$$

So, given $\varepsilon > 0$, there exists an integer p_0 such that $\|z - x_{p_0}\| < \frac{\varepsilon}{2}$. Also,

$$\begin{aligned}\|A_n x_{p_0}\| &= \|A_n y_{p_0} - A_n U y_{p_0}\| \\ &= n^{-1}\left\|\sum_{k=0}^{n-1} U^k y_{p_0} - \sum_{k=0}^{n-1} U^{k+1} y_{p_0}\right\| \\ &= n^{-1}\|y_{p_0} - U^n y_{p_0}\| \\ &\le 2n^{-1}\|y_{p_0}\| < \frac{\varepsilon}{2},\end{aligned}$$

provided n is sufficiently large. Therefore, $\lim_n A_n z = 0$ for $z \in K$.

We next show that $K^{\perp} = M$:

$v \in K^{\perp} \Leftrightarrow (v, y - Uy) = 0$ for all $y \Leftrightarrow (v, y) - (U^*v, y) = 0$ for all $y \Leftrightarrow (v - U^*v, y) = 0$ for all $y \Leftrightarrow v = U^*v \Leftrightarrow v \in M$.

Finally, $x \in H$ can be written as $x_1 + x_2$ with $x_1 \in M$ and $x_2 \in M^{\perp}$ ($= K$), so that $n^{-1}\sum_{k=0}^{n-1} U^k x$ converges to $x_1 + 0 = x_1 = Px$. This completes the proof. □

Chapter 4
Spectral Theory and Special Classes of Operators

4.1 Spectral Notions

As noted earlier, if H is a complex Hilbert space, $\mathcal{B}(H)$ is a C^*-algebra with identity [see Definition 3.5.6]. The invertibility of an operator $T \in \mathcal{B}(H)$ and its ramifications were discussed in 3.3.7–3.3.12. In what follows, we shall study the invertibility of the operators $\lambda I - T$, where $T \in \mathcal{B}(H)$, I is the identity operator and $\lambda \in \mathbb{C}$. The study of the distribution of the values of λ for which $\lambda I - T$ does not have an inverse is called 'spectral theory' for the operator.

The study of the complement of the set $\{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible in } \mathcal{B}(H)\}$, called the 'spectrum' of the operator T, is an important part of operator theory. In finite dimensions, it is the set of eigenvalues of T. In infinite dimensions, the operator $\lambda I - T$ may fail to be invertible in different ways. So, finding the spectrum is not an easy problem. It is definitely more complicated than in the finite-dimensional case.

Definition 4.1.1 If $T \in \mathcal{B}(H)$, we define the **spectrum** of T to be the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible in } \mathcal{B}(H)\}$$

and the **resolvent set** of T to be the set

$$\rho(T) = \mathbb{C} \backslash \sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible in } \mathcal{B}(H)\}.$$

$R(\lambda_0, T)$ denotes $(\lambda_0 I - T)^{-1}$ and is called the **resolvent** at λ_0 of T. Further, the **spectral radius** of T is defined by

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Examples 4.1.2

(i) For the identity operator $I \in \mathcal{B}(H)$, $\sigma(I) = \{1\}$, $\rho(T) = \mathbb{C}\backslash\{1\}$ and $r(I) = 1$.

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(ii) For an $n \times n$ matrix T, $\lambda I - T$ is not invertible if and only if $\det(\lambda I - T) = 0$. Thus, in the finite-dimensional case, $\sigma(I)$ is just the set of eigenvalues of T (since $\det(\lambda I - T)$ is an nth-degree polynomial whose roots are the eigenvalues of T).

(iii) Let $f : [a, b] \to \mathbb{C}$ be continuous, where $a < b$ are in $\mathbb{R}$. The multiplication operator

$$\left(T_f x\right)(T) = f(t)x(T), \quad a \le t \le b$$

is a bounded operator on $L^2[a,\ b]$. We argue in the next paragraph that $\sigma(T_f) = \operatorname{ran}(f) = \{\lambda \in \mathbb{C} : \text{there exists } t \in [a, b] \text{ for which } f(t) = \lambda\} = \{f(t) : t \in [a, b]\}$.

If $\lambda \notin \operatorname{ran}(f)$, then $(\lambda I - T_f)$ has a bounded inverse $T_{(\lambda - f)^{-1}}$ and so, $\lambda \notin \sigma(T_f)$. On the other hand, if $\lambda = f(t_0)$ for some $t_0 \in [a, b]$, then $\lambda \in \sigma(T_f)$. Otherwise, $(\lambda I - T_f)$ has a bounded inverse S. Pick an interval J_n about t_0 in $[a,\ b]$, of length $\delta_n > 0$, such that $|f(t) - \lambda| < \frac{1}{n}$ for $t \in J_n$, and define

$$g_n(T) = \begin{cases} \delta_n^{-1/2} & t \in J_n \\ 0 & \text{otherwise.} \end{cases}$$

Then, $(\lambda I - T_f)g_n \to 0$ as $n \to \infty$ because $\int |(\lambda I - T_f)g_n|^2\,dt \le \frac{1}{n^2}\delta_n^{-1}\delta_n = \frac{1}{n^2}$ but $S(\lambda I - T_f)g_n = g_n$ which has norm 1 for all n, contradicting the continuity of S.

Depending on the complications to the invertibility of the operator $\lambda I - T$, we classify $\sigma(T)$, the spectrum of T.

Recall that $\lambda I - T$ fails to be invertible if either $\operatorname{ran}(\lambda I - T) \ne H$ or $\ker(\lambda I - T) \ne \{0\}$ [Problem 3.3.P3].

Definition 4.1.3

(a) The **point spectrum** (**eigenspectrum**, **eigenvalues**) of $T \in \mathcal{B}(H)$ is defined to be the set

$$\boldsymbol{\sigma_p}(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \ne \{0\}\};$$

in other words, there is a nonzero vector x in H such that $(\lambda I - T)x = 0$, i.e. $\lambda I - T$ is not injective.

(b) The **continuous spectrum** $\sigma_c(T)$ is the set
$\boldsymbol{\sigma_c(T)} = \{\lambda \in \mathbb{C} : \lambda I - T$ is injective and $\operatorname{ran}(\lambda I - T)$ is dense in H but $(\lambda I - T)^{-1}$ is not bounded$\}$.

(c) The **residual spectrum** $\sigma_r(T)$ is the set
$\boldsymbol{\sigma_r(T)} = \{\lambda \in \mathbb{C} : \lambda I - T$ is injective and $\operatorname{ran}(\lambda I - T)$ is not dense in H and $(\lambda I - T)^{-1}$ exists as a bounded or unbounded operator$\}$.

Remarks 4.1.4

(i) The conditions in (a), (b) and (c) are mutually exclusive and exhaustive by Theorem 3.3.12. Thus, we have the following disjoint splitting of $\mathbb{C}$:

$$\mathbb{C} = \rho(T) \cup \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

and the union

$$\sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

comprises the spectrum of T.

(ii) If H is finite-dimensional and $T \in \mathcal{B}(H)$, then the two conditions $\ker(\lambda I - T) = \{0\}$ and $\operatorname{ran}(\lambda I - T) = H$ are equivalent. Hence, $\sigma(T) = \sigma_p(T)$ for every operator T on a finite-dimensional Hilbert space H. Consequently, in this case, $\sigma_c(T) = \varnothing = \sigma_r(T)$.

(iii) The multiplication operator $T_t : L^2[a,b] \to L^2[a,b]$ defined by $T_t(x(t)) = tx(t)$, $a \leq x \leq b$, is such that $\sigma_p(T_t) = \varnothing$. Indeed, the condition $(\lambda I - T_t)x = 0$ implies $(\lambda - t)x(t) = 0$ a.e. and so, $x(t) = 0$ a.e. It has been proved in Example (iii) of 4.1.2 that $\sigma(T_t) = [a,b]$. The domain of $(\lambda I - T_t)^{-1}$ is the set of all y's in $L^2[a, b]$ for which there exists an x in $L^2[a, b]$ satisfying $(\lambda I - T_t)x = y$, i.e. $\frac{y(t)}{\lambda - t}$ is in $L^2[a, b]$. We shall argue that the set $\{y \in L^2[a,b] : \frac{y(t)}{\lambda - t} \in L^2[a,b]\}$ is dense in $L^2[a, b]$. For an arbitrary $\delta > 0$, there exists an $\varepsilon > 0$ such that the function f_ε, where f_ε is 0 on $I = (\lambda - \varepsilon, \lambda + \varepsilon) \cap [a,b]$ and is f on its complement, satisfies the inequality

$$\int_a^b |f - f_\varepsilon|^2 = \int_I |f(t)|^2 \mathrm{d}t < \delta.$$

Moreover, the function $\frac{f_\varepsilon(t)}{\lambda - t}$ is in $L^2[a, b]$ since its L^2-norm is less than or equal to $\frac{1}{\varepsilon}$ times the L^2-norm of f.

But the set $\{y \in L^2[a,b] : \frac{y(t)}{\lambda - t} \in L^2[a,b]\}$ does not coincide with $L^2[a, b]$ as it does not contain the constant function 1. Thus, each $\lambda \in \sigma(T_t)$ is in $\sigma_c(T_t)$. It follows from (i) above that $\sigma_r(T_t) = \varnothing$.

Theorem 3.3.12[1] leads to yet another useful division of the spectrum into two parts, not necessarily disjoint. It is an immediate consequence of that Theorem that

[1]Note that the same theorem had made it possible earlier to divide the complement of the point spectrum into two *disjoint* parts.

$\lambda \in \sigma(T)$ if and only if either $\mathrm{ran}(\lambda I - T)$ is not dense in H or $(\lambda I - T)$ is not bounded below: there is no $\varepsilon > 0$ such that $\|(\lambda I - T)x\| \geq \varepsilon\|x\|$ for every $x \in H$. In the former case, λ is said to belong to the **compression spectrum $\boldsymbol{\sigma}_{\mathbf{com}}(\boldsymbol{T})$** of T, and in the latter case, λ is said to belong to the **approximate point spectrum $\boldsymbol{\sigma}_{ap}(\boldsymbol{T})$** of T. In other words,

$$\sigma_{\mathrm{com}}(T) = \{\lambda \in \mathbb{C} : \mathrm{ran}(\lambda I - T)\ \text{is not dense in}\ H\},$$

$$\begin{aligned}\sigma_{ap}(T) = \{\lambda \in \mathbb{C} &: \text{there is a sequence}\ \{x_n\}_{n\geq 1}\ \text{such that}\ \|x_n\| \\ &= 1\ \text{for every}\ n\ \text{and}\ \|(\lambda I - T)x_n\| \to 0\ \text{as}\ n \to \infty\}.\end{aligned}$$

Sometimes, $\{x_n\}_{n\geq 1}$ is called an approximate eigenvector corresponding to the approximate eigenvalue. Clearly,

$$\sigma_p(T) \subseteq \sigma_{ap}(T) \quad \text{and} \quad \sigma(T) = \sigma_{ap}(T) \cup \sigma_{\mathrm{com}}(T).$$

The reader will note that

$$\sigma_r(T) = \sigma_{\mathrm{com}}(T)\backslash\sigma_p(T),$$

which is to say the residual spectrum is the set of those points in the compression spectrum that are not eigenvalues. Also,

$$\sigma_{\mathrm{com}}(T) \cup \sigma_p(T) = \sigma_p(T) \cup \sigma_r(T)$$

and

$$\begin{aligned}\sigma_c(T) &= \sigma(T)\backslash(\sigma_{\mathrm{com}}(T) \cup \sigma_p(T)) \\ &= \sigma_{ap}(T)\backslash(\sigma_{\mathrm{com}}(T) \cup \sigma_p(T)) \\ &= \sigma_{ap}(T)\backslash(\sigma_p(T) \cup \sigma_r(T)).\end{aligned}$$

Problem Set 4.1

4.1.P1. For $T \in \mathcal{B}(H)$, show that (i) $\sigma_{\mathrm{com}}(T) \subset \overline{\sigma_p(T^*)}$ and (ii) $\sigma_p(T) \subseteq \overline{\sigma_{\mathrm{com}}(T^*)}$.

4.1.P2. Let $H = \ell^2$ and $\{e_k\}_{k\geq 1}$ be the standard orthonormal basis in ℓ^2. Any $x \in \ell^2$ has the representation $x = \sum_{n=1}^{\infty}(x, e_n)e_n = \sum_{n=1}^{\infty}\alpha_n e_n$, where $\alpha_n = (x, e_n)$, $n = 1, 2, \ldots$. Define $T : \ell^2 \to \ell^2$ by taking $Tx = \sum_{n=1}^{\infty}\frac{\alpha_n}{n+1}e_{n+1}$; in other words, $Te_1 = \frac{1}{2}e_2, Te_2 = \frac{1}{3}e_3, \ldots$. Show that T is a bounded linear operator, $0 \in \sigma_r(T)$ and any $\lambda \neq 0$ belongs to $\rho(T)$.

4.1.P3. Let $H = \ell^2$ and $\{e_k\}_{k\geq 1}$ be the standard orthonormal basis in ℓ^2. Any $x \in \ell^2$ has the representation $x = \sum_{n=1}^{\infty} (x, e_n)e_n = \sum_{n=1}^{\infty} \alpha_n e_n$, where $\alpha_n = (x, e_n), n = 1, 2, \ldots$. Consider a sequence of scalars $\{\lambda_n\}_{n\geq 1}$ such that $\lambda_n \to 1$ and no λ_n equals 1. Define $T : \ell^2 \to \ell^2$ by $Tx = \sum_{n=1}^{\infty} \alpha_n \lambda_n e_n$. Show that

(a) T is a bounded linear operator;
(b) $\{\lambda_n : n = 1, 2, \ldots\} \subseteq \sigma_p(T)$;
(c) $1 \in \sigma_c(T)$;
(d) $\lambda \neq \lambda_n$ for any n and $\lambda \neq 1$ implies $\lambda \in \rho(T)$;
(e) $\sigma_r(T) = \varnothing$.

4.1.P4. Show that if $A, B \in \mathcal{B}(H), \lambda \in \rho(AB)$ and $\lambda \neq 0$, then $\lambda \in \rho(BA)$ and

$$(\lambda I - BA)^{-1} = \lambda^{-1} I + \lambda^{-1} B(\lambda I - BA)^{-1} A.$$

Deduce that $\sigma(AB)$ and $\sigma(BA)$ have the same elements with one possible exception: the point zero. Show that the point zero is exceptional.

4.1.P5. Let $\mu = \{\mu_k\}_{k\geq 1}$ be a bounded sequence of complex numbers, $M = \sup_{k\geq 1} |\mu_k|$. Define $T : \ell^2 \to \ell^2$ by

$$T(x_1, x_2, \ldots) = (\mu_1 x_1, \mu_2 x_2, \ldots).$$

Show that $||T|| = \sup_{k\geq 1}|\mu_k| = M$. Show also that the eigenvalues of T are $\mu_1, \mu_2, \ldots$ and $\sigma(T) = \overline{\{\mu_k : k \geq 1\}}$. What is T^*?

4.1.P6. Let $T \in \mathcal{B}(H)$ be self-adjoint and x be a fixed unit vector in H. Suppose $||Tx|| = ||T||$. Show that x is an eigenvector of T^2 corresponding to the eigenvalue $\|T\|^2 (=\|T^2\|)$. Also, prove that $Tx = ||T||x$ or $Ty = -||T||y$, where $y = ||T||x - Tx \neq 0$.

4.1.P7. Let $T \in \mathcal{B}(H)$, where H is a complex Hilbert space. Show that the following statements are equivalent:

(a) There exists $\lambda \in \sigma_{ap}(T)$ such that $|\lambda| = \|T\|$;
(b) $||T|| = \sup_{||x||=1} |(Tx, x)|$.

4.1.P8. Let S and T denote a pair of self-adjoint operators in $\mathcal{B}(H)$. Then,

$$\max_{\nu\in\sigma(T)} \min_{\mu\in\sigma(S)} |\nu - \mu| \leq ||S - T||.$$

(The reader will note that by interchanging S and T, we also obtain

$$\max_{\mu\in\sigma(S)} \min_{\nu\in\sigma(T)} |\nu - \mu| \leq ||S - T||.)$$

4.2 Resolvent Equation and Spectral Radius

Let H be a finite-dimensional Hilbert space and $T \in \mathcal{B}(H)$. The set of λ's for which $\det(\lambda I - T) = 0$ comprise the spectrum of T. The fundamental theorem of algebra guarantees that $\sigma(T) \neq \emptyset$. For every bounded linear operator defined on a Hilbert space (finite- or infinite-dimensional), the spectrum $\sigma(T)$ is a nonempty, closed and bounded subset of the complex plane.

Theorem 4.2.1 (The resolvent equation) *For* $\lambda, \mu \in \rho(T)$,

$$R(\lambda, T) - R(\mu, T) = -(\lambda - \mu)R(\lambda, T)R(\mu, T).$$

Proof We have

$$\begin{aligned} R(\lambda, T) - R(\mu, T) &= (\lambda I - T)^{-1} - (\mu I - T)^{-1} \\ &= (\lambda I - T)^{-1}[(\mu I - T) - (\lambda I - T)](\mu I - T)^{-1} \\ &= -(\lambda - \mu)R(\lambda, T)R(\mu, T). \end{aligned}$$

□

The above relation has the consequence that

$$\begin{aligned} R(\lambda, T)R(\mu, T) &= -\frac{R(\lambda, T) - R(\mu, T)}{\lambda - \mu} \\ &= -\frac{R(\mu, T) - R(\lambda, T)}{\mu - \lambda} \\ &= R(\mu, T)R(\lambda, T). \end{aligned}$$

Thus, the family $\{R(\lambda, T) : \lambda \in \rho(T)\}$ is a commuting family, i.e. any two members of the family commute with each other.

Theorem 4.2.2 *Let* $T \in \mathcal{B}(H)$. *The resolvent set* $\rho(T)$ *of* T *is open, and the map* $\lambda \to R(\lambda, T) = (\lambda I - T)^{-1}$ *from* $\rho(T) \subseteq \mathbb{C}$ *to* $\mathcal{B}(H)$ *is strongly holomorphic in the sense of* Definition 3.3.13 (*understood with* $X = \mathcal{B}(H)$), *vanishing at* ∞. *For each* $x, y \in H$, *the map* $\lambda \to (R(\lambda, T)x, y) = ((\lambda I - T)^{-1}x, y) \in \mathbb{C}$ *is holomorphic on* $\rho(T)$, *vanishing at* ∞.

Proof Let $\lambda \in \rho(T)$. By definition, $\lambda I - T$ is invertible and thus belongs to the set $\mathcal{G}$ of all invertible elements of $\mathcal{B}(H)$. By the first part of Proposition 3.3.9, $\mathcal{G}$ is open. Therefore, some $\delta > 0$ has the property that any $S \in \mathcal{B}(H)$ which satisfies the inequality $\|S - (\lambda I - T)\| < \delta$ belongs to $\mathcal{G}$. If $\|\lambda - \mu\| < \delta$, then $S = \mu I - T$ clearly satisfies the inequality and therefore belongs to $\mathcal{G}$, so that $\mu \in \rho(T)$. This shows that $\rho(T)$ is open.

Since the map $\lambda \to (\lambda I - T)$ from $\rho(T)$ to $\mathcal{G}$ is continuous, it follows by the second part of Proposition 3.3.9 that the map $\lambda \to (\lambda I - T)^{-1}$ from $\rho(T) \subseteq \mathbb{C}$ to $\mathcal{B}(H)$ is also continuous. The resolvent identity of Theorem 4.2.1 now shows that the map is strongly holomorphic with derivative $-R(\lambda, T)^2$.

If $\|\lambda\| \to \infty$, then $I - \lambda^{-1}T \to I$ in the uniform operator norm, which implies $(I - \lambda^{-1}T)^{-1} \to I$ [by the second part of Theorem 3.3.9]. Consequently,

$$R(\lambda, T) = (\lambda I - T)^{-1} = \lambda^{-1}(I - \lambda^{-1}T)^{-1} \to O.$$

Being strongly holomorphic, the map is also weakly holomorphic. Now, for $x, y \in H$, the map from $\mathcal{B}(H)$ to $\mathbb{C}$ given by $S \to (Sx, y)$ is a linear functional on $\mathcal{B}(H)$. Hence, the map $\lambda \to (R(\lambda, T)x, y) = ((\lambda I - T)^{-1}x, y) \in \mathbb{C}$ is holomorphic, vanishing at ∞. □

Corollary 4.2.3 *For* $T \in \mathcal{B}(H), \sigma(T) = \mathbb{C}\backslash\rho(T)$ is a closed subset of $\mathbb{C}$.

Recall that the spectral radius of an operator $T \in \mathcal{B}(H)$ is defined to be

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Theorem 4.2.4 *Let* $T \in \mathcal{B}(H)$, *where* $H(\neq \{0\})$. *If* $|\lambda| > \|T\|$, *then* $\lambda \in \rho(T)$ *and*

$$R(\lambda, T) = (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1}T^n,$$

where convergence takes place in the uniform operator norm. Also, the spectrum $\sigma(T)$ *of* T *is a nonempty compact subset which lies in* $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$. *In particular, there exists* $\lambda \in \sigma(T)$ *such that* $|\lambda| = r(T)$.

Proof By Corollary 4.2.3, $\sigma(T)$ is a closed subset of $\mathbb{C}$. If $|\lambda| > \|T\|$, then $\|I - (I - \lambda^{-1}T)\| = \|\lambda^{-1}T\| < 1$, and by Proposition 3.3.8, $I - \lambda^{-1}T$ is invertible with $(I - \lambda^{-1}T)^{-1} = \sum_{n=0}^{\infty} (\lambda^{-1}T)^n$, convergence being in the uniform operator norm. This implies that $\lambda I - T = \lambda(I - \lambda^{-1}T)$ is invertible and $(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1}T^n$, convergence being in the uniform operator norm.

In particular, $|\lambda| > \|T\|$ implies $\lambda \notin \sigma(T)$. In other words, $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq ||T||\}$, showing that $\sigma(T)$ is bounded. Being closed, it is also compact.

We show that the assumption $\sigma(T) = \varnothing$ leads to a contradiction. $\sigma(T) = \varnothing$ implies $\rho(T) = \mathbb{C}$. Now, for every x, y in H, $(R(\lambda, T)x, y)$ is an entire function, which vanishes at ∞, and is therefore bounded. By Liouville's Theorem, $(R(\lambda, T)x, y)$ is constant and the value of this constant is zero. Since $(R(\lambda, T)x, y) = 0$ for every x, y in H implies $R(\lambda, T) = O$, it follows that

$$O = R(\lambda, T)(\lambda I - T) = I.$$

This is a contradiction. □

Theorem 4.2.5 (Gelfand's formula) *For any* $T \in \mathcal{B}(H)$, *the following limit exists*

$$\lim_{n\to\infty}||T^n||^{\frac{1}{n}}$$

and equals $r(T)$.

The following lemma will be needed in the proof of Gelfand's formula.

Lemma 4.2.6 *For* $T \in \mathcal{B}(H)$, $\lim_{n\to\infty}||T^n||^{\frac{1}{n}}$ exists and equals $\inf_n ||T^n||^{\frac{1}{n}}$. *Moreover*, $0 \leq \inf_n ||T^n||^{\frac{1}{n}} \leq \|T\|$.

Proof Set $\alpha = \inf_n ||T^n||^{\frac{1}{n}}$. Then, for $\varepsilon > 0$, there exists m such that $||T^m||^{\frac{1}{m}} < \alpha + \varepsilon$. Now, any $n \in \mathbb{N}$ can be written as $n = pm + q$, $0 \leq q < m$. So,

$$\|T^n\|^{\frac{1}{n}} = \|T^{pm+q}\|^{\frac{1}{n}} \leq \|T^m\|^{\frac{p}{n}}\|T\|^{\frac{q}{n}} < (\alpha+\varepsilon)^{m\frac{p}{n}}\|T\|^{\frac{q}{n}}.$$

Since $m\frac{p}{n} \to 1$ and $\frac{q}{n} \to 0$ as $n \to \infty$, it follows that

$$\limsup_n ||T^n||^{\frac{1}{n}} \leq \alpha + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we have

$$\limsup_n ||T^n||^{\frac{1}{n}} \leq \alpha.$$

Also, $\alpha \leq ||T^n||^{\frac{1}{n}}$ for every n and this implies $\alpha \leq \liminf_n ||T^n||^{\frac{1}{n}}$. Consequently, $\lim_{n\to\infty}||T^n||^{\frac{1}{n}}$ exists and equals $\inf_n ||T^n||^{\frac{1}{n}}$. Finally, $||T^n||^{\frac{1}{n}} \leq (\|T\|^n)^{\frac{1}{n}} = \|T\|$ implies $\alpha = \inf_n ||T^n||^{\frac{1}{n}} \leq \|T\|$. □

Proof of Gelfand's Formula Let $\lambda \in \mathbb{C}$ be such that $|\lambda| > \alpha = \inf_n \|T^n\|^{\frac{1}{n}}$. Then, there exists a positive integer m such that $|\lambda| > \|T^m\|^{\frac{1}{m}}$, i.e. $\|T^m\| < |\lambda^m|$, so that $\lambda^m \in \rho(T^m)$. Since

$$\begin{aligned} T^m - \lambda^m I &= (T - \lambda I)(T^{m-1} + \lambda T^{m-2} + \cdots + \lambda^{m-1} I) \\ &= (T^{m-1} + \lambda T^{m-2} + \cdots + \lambda^{m-1} I)(T - \lambda I), \end{aligned}$$

it follows that

$$(T - \lambda I)^{-1} = (T^m - \lambda^m I)^{-1}(T^{m-1} + \lambda T^{m-2} + \cdots + \lambda^{m-1} I)^{-1},$$

and so $\lambda \in \rho(T)$. Consequently, $r(T) \leq \alpha$. It remains to show that $r(T) \geq \alpha$. To this end, we proceed as follows:

Let $|\lambda| > r(T)$. Then, $\lambda \in \rho(T)$. The resolvent $R(\lambda, T)$ exists and is strongly holomorphic on $\rho(T)$ by Proposition 4.2.1. It therefore has a Laurent series around $\lambda = 0$, converging in the operator norm.

If $|\lambda| > \|T\|$, then by Theorem 4.2.4,

$$R(\lambda, T) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n,$$

which converges in the operator norm.

Since $|\lambda| > \|T\| \geq r(T)$, by uniqueness of Laurent series, it follows that

$$R(\lambda, T) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n \quad \text{if } |\lambda| > r(T).$$

Hence,

$$\lim_n ||\lambda^{-n-1} T^n|| = 0 \quad \text{if } |\lambda| > r(T),$$

and so, for any $\varepsilon > 0$, we must have

$$\begin{aligned} ||T^n|| &\leq \varepsilon |\lambda|^{n+1} \quad \text{for large } n \text{ and } |\lambda| > r(T). \\ &\leq (\varepsilon + |\lambda|)^{n+1}, \end{aligned}$$

which implies

$$||T^n||^{\frac{1}{n}} \leq (\varepsilon + |\lambda|)^{1+\frac{1}{n}} \quad \text{for large } n \text{ and } |\lambda| > r(T),$$

and hence,

$$\lim_{n\to\infty} ||T^n||^{\frac{1}{n}} \leq |\lambda| \quad \text{for } |\lambda| > r(T).$$

Consequently,

$$\lim_{n\to\infty} ||T^n||^{\frac{1}{n}} \leq r(T).$$

Using the Lemma proved above, we obtain Gelfand's formula:

$$r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}.$$

□

Remarks 4.2.7

(i) If $T \in \mathcal{B}(H)$ is such that $T^*T = TT^*$, then

$$r(T) = ||T||.$$

For a normal operator T, $\|T^p\| = \|T\|^p$ for $p = 2^n$, $n = 1, 2, \ldots$ [Theorem 3.7.2]. It follows that $||T^p||^{\frac{1}{p}} = ||T||$ for $p = 2^n$, $n = 1, 2, \ldots$, which implies that the limit of the subsequence $\{||T^p||^{\frac{1}{p}}\}_{p=2^n}$ of the convergent sequence $\{||T^n||^{\frac{1}{n}}\}_{n\geq 1}$ equals $\|T\|$; so $\lim_{n\to\infty}||T^n||^{\frac{1}{n}} = ||T||$. Hence, if T is normal, $r(T) = \|T\|$. Therefore, by Theorem 4.2.4, there exists $\lambda \in \sigma(T)$ such that $|\lambda| = \|T\|$. In particular, if the spectrum contains only real numbers [e.g. self-adjoint operators; see Theorem 4.4.2], then $|\lambda| = \pm\lambda$ and therefore either $\|T\| \in \sigma(T)$ or $-\|T\| \in \sigma(T)$.

(ii) For $T \in \mathcal{B}(H), \sigma(T) = \{0\}$ if and only if $\lim_{n\to\infty}||T^n||^{\frac{1}{n}} = 0$. Indeed, if $\lim_{n\to\infty}||T^n||^{\frac{1}{n}} = 0$, then $r(T) = 0$, which implies $\sigma(T) = \{0\}$. On the other hand, if $\sigma(T) = \{0\}$, then $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = 0$, i.e., $\lim_{n\to\infty}||T^n||^{\frac{1}{n}} = 0$.

(iii) An operator $T \in \mathcal{B}(H)$ is called **nilpotent** if there exists an $n \in \mathbb{N}$ such that $T^n = O$ and is called **quasinilpotent** if $\sigma(T) = \{0\}$.

Any normal quasinilpotent operator is the zero operator. Indeed, if T is normal, then $\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}} = \|T\|$. Since T is quasinilpotent, $\sigma(T) = \{0\}$. It then follows from (i) and (ii) above that $\|T\| = 0$, which implies $T = O$.

Problem Set 4.2

4.2.P1. (a) The analogue of Theorem 4.2.4 $[\sigma(T)) \neq \emptyset]$ fails for real spaces.

(b) Give an example to show that it is possible to have $r(T) = 0$ but $T \neq O$.

4.2.P2. Let $A, B \in \mathcal{B}(H)$ be bounded linear operators on a complex Hilbert space H such that $AB = BA$. Show that

$$r(AB) \leq r(A)r(B).$$

Give an example to show that commutativity cannot be dropped.

4.2.P3. Let $A, B \in \mathcal{B}(H)$ be bounded linear operators on a complex Hilbert space H such that $AB = BA$. Show that $r(A + B) \leq r(A) + r(B)$. Give an example to show that commutativity cannot be dropped.

4.3 Spectral Mapping Theorem for Polynomials

Let $T \in \mathcal{B}(H)$. To every polynomial $p(z) = \sum_{j=0}^{n} c_j z^j$, we can associate the operator $p(T) \in \mathcal{B}(H)$ defined by $\sum_{j=0}^{n} c_j T^j$. With $f(z) = \bar{z}$ and $f(z) = z^{-1}$, we can associate the operators $f(T) = T^*$ and $f(T) = T^{-1}$. The purpose of this section is to investigate the relationship between $\sigma(T)$ and the spectrum of the operators defined above. In fact, we have the following theorem.

Spectral Mapping Theorem 4.3.1 *Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Then,*

(a) $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$;

(b) *if T is invertible, then* $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$;

(c) *if $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial with complex coefficients and if $p(T)$ is defined by $\sum_{j=0}^{n} c_j T^j$, then $\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\} = p(\sigma(T))$.*

Proof

(a) Suppose $\lambda \notin \sigma(T)$. Then, $(\lambda I - T)^{-1}$ exists, so that $(\bar{\lambda} I - T^*)^{-1} = [(\lambda I - T)^*]^{-1} = [(\lambda I - T)^{-1}]^*$ exists [see Theorem 3.5.4(d)]. Thus, $\bar{\lambda} \notin \sigma(T^*)$. We have thus proved $\sigma(T^*) \subseteq \{\bar{\lambda} : \lambda \in \sigma(T)\}$. Applying this argument to T^*, we get $\sigma(T) \subseteq \{\bar{\lambda} : \lambda \in \sigma(T^*)\}$, that is, $\{\lambda : \lambda \in \sigma(T)\} \subseteq \{\bar{\lambda} : \lambda \in \sigma(T^*)\}$. Taking conjugates, we get $\{\bar{\lambda}\lambda \in \sigma(T)\} \subseteq \{\lambda : \lambda \in \sigma(T^*)\} = \sigma(T^*)$, so that $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.

(b) If T is invertible, then $0 \notin \sigma(T)$, so that $\{\lambda^{-1} : \lambda \in \sigma(T)\}$ is well defined. If $\lambda \notin \sigma(T)$ and $\lambda \neq 0$, then the equation

$$(\lambda^{-1} I - T^{-1}) = \lambda^{-1} T^{-1}(T - \lambda I) = -\lambda^{-1} T^{-1}(\lambda I - T)$$

shows $\lambda^{-1} \notin \sigma(T^{-1})$, for if $\lambda^{-1} \in \sigma(T^{-1})$, then either $\lambda \in \sigma(T)$ or $\lambda = 0$. In other words, $\sigma(T^{-1}) \subseteq \{\lambda^{-1} : \lambda \in \sigma(T)\}$. To prove the reverse inclusion, we apply the result to T^{-1}. Thus,

$$\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}.$$

(c) When p is the zero polynomial or has degree 1, this is obvious. Let $\lambda \in \sigma(T)$ and p be a polynomial of degree $n > 1$. Then, $p(z) - p(\lambda)$ is a polynomial of degree n with λ as a root and we can factor $p(z) - p(\lambda)$ as $(z - \lambda)q(z)$, where q is a polynomial of degree $n - 1$. Then,

$$p(T) - p(\lambda)I = (T - \lambda I)q(T) = B, \text{ say.}$$

If B were invertible, then the equation $BB^{-1} = B^{-1}B = I$ can be written as

$$(T - \lambda I)q(T)B^{-1} = B^{-1}q(T)(T - \lambda I) = I.$$

This would mean $T - \lambda I$ is invertible, which is not possible if $\lambda \in \sigma(T)$. Thus, B is not invertible, i.e. $p(\lambda) \in \sigma(p(T))$. So, $p(\sigma(T)) \subseteq \sigma(p(T))$.

Let $\lambda \in \sigma(p(T))$. Factorise the polynomial $p(z) - \lambda$ into linear factors and write

$$p(T) - \lambda I = c(T - \lambda_1 I) \cdots (T - \lambda_n I).$$

Since $p(T) - \lambda I$ is not invertible, one of the factors $T - \lambda_j I$ is not invertible. Thus, $\lambda_j \in \sigma(T)$, and also, $p(\lambda_j) - \lambda = 0$. This shows that $\lambda = p(\lambda_j)$ for some $\lambda_j \in \sigma(T)$. Hence, $\sigma(p(T)) \subseteq p(\sigma(T))$. This completes the proof. □

Example 4.3.2 [(ix) of Examples 3.2.5]. The Volterra integral operator

$$V : L^2[0,1] \to L^2[0,1]$$

defined by

$$Vx(s) = \int_0^s x(t)\mathrm{d}t, \quad x \in L^2[0,1]$$

is a bounded linear operator of norm not exceeding $\frac{1}{\sqrt{2}}$. We shall show that $r(V) = 0$ and 0 is not an eigenvalue of V. Now,

$$\begin{aligned} V^2x(s) = V(V(x)(s)) &= \int_0^s (Vx)(t)\mathrm{d}t = \int_0^s \int_0^t x(u)\mathrm{d}u\,\mathrm{d}t \\ &= \int_0^s x(u)\left(\int_u^s \mathrm{d}t\right)\mathrm{d}u = \int_0^s (s-u)x(u)\mathrm{d}u. \end{aligned}$$

Proceeding as above, one can show that

$$V^n x(s) = \frac{1}{(n-1)!}\int_0^s (s-u)^{n-1}x(u)\mathrm{d}u;$$

so

$$\begin{aligned} \|V^n x\|_2^2 &= \int_0^1 |V^n x(s)|^2 \mathrm{d}s = \left(\frac{1}{(n-1)!}\right)^2 \int_0^1 \left|\int_0^s (s-u)^{n-1}x(u)\mathrm{d}u\right|^2 \mathrm{d}s \\ &\leq \left(\frac{1}{(n-1)!}\right)^2 \int_0^1 \left(\int_0^s (s-u)^{n-1}|x(u)|\mathrm{d}u\right)^2 \mathrm{d}s \\ &\leq \left(\frac{1}{(n-1)!}\right)^2 \int_0^1 \left(\int_0^s |x(u)|^2\mathrm{d}u\right)\left(\int_0^s (s-u)^{2n-2}\mathrm{d}u\right)\mathrm{d}s, \end{aligned}$$

using the Cauchy–Schwarz inequality,

$$\leq \left(\frac{1}{(n-1)!}\right)^2 ||x||_2^2.$$

Thus,

$$||V^n x|| \leq \left(\frac{1}{(n-1)!}\right) ||x||.$$

Consequently, $||V^n|| \leq \frac{1}{(n-1)!}$, which implies

$$r(V) = \lim_{n\to\infty} ||V^n||^{\frac{1}{n}} \leq \lim_{n\to\infty} \left[\frac{1}{(n-1)!}\right]^{\frac{1}{n}} = 0.$$

The spectrum of V is thus a single point 0. Moreover, 0 is not an eigenvalue of V; for if $Vx = 0$, then $\int_0^s x(u)\mathrm{d}u = 0$ for every $s \in [0, 1]$ and this implies $x = 0$ a.e. Since $0 \in \sigma(V)$, V is not invertible. Since the range $\{\int_0^s x(u)\mathrm{d}u : x \in L^2[0, 1]\}$ of V is dense in $L^2[0, 1]$ (see below), it follows that $0 \in \sigma_c(V)$.

The range of V consists of continuous functions on [0, 1] vanishing at 0 and differentiable a.e. We shall show that they are dense in $L^2[0, 1]$. We need consider only real functions. By the Stone–Weierstrass Theorem [13, Theorem 7.34 of Chap. II], they are uniformly dense in the algebra of all real continuous functions vanishing at 0. It is sufficient therefore to argue that this algebra is L^2-dense in the algebra of *all* continuous real functions.

Let f be any real continuous function on [0, 1], $f(0) \neq 0$, and let $\varepsilon > 0$ be given. There exists a positive $\delta_1 < 1$ such that on the interval $[0, \delta_1]$, we have $|f(x)| \leq 2|f(0)|$. Choose a positive $\delta < \delta_1$ such that it also satisfies the inequality

$$\delta < \frac{\varepsilon^2}{16|f(0)|^2}.$$

Since $0 < \delta < \delta_1$, the inequality $|f(x)| \leq 2|f(0)|$ holds on $[0, \delta]$ as well. Now, consider the continuous function g defined to agree with f on $[\delta, 1]$ and have a straight-line graph from the origin to the point $(\delta, f(\delta))$ on the graph of f. Then, $g(0) = 0$ and g satisfies $|g(x)| \leq 2|f(0)|$ on $[0, \delta]$, and hence,

$$|f - g| \leq 4\,|f(0)| \quad \text{on } [0, \delta].$$

Moreover,

$$|f - g| \quad \text{is zero on } [\delta, 1].$$

It follows that $\int_{[0,1]} |f - g|^2 \leq 16|f(0)|^2\delta$, which is less than ε^2 by choice of δ. Thus, $\|f - g\| < \varepsilon$ in $L^2[0, 1]$.

Proposition 4.3.3 *Let* $T \in \mathcal{B}(H)$. *Then,* (a) $\sigma_p(T^*) = \overline{\sigma_{\text{com}}(T)}$, (b) $\sigma(T^*) = \sigma_{ap}(T^*) \cup \overline{\sigma_{ap}(T)}$, (c) $\sigma_{\text{com}}(T^*) \subseteq \overline{\sigma_p(T)} \subseteq \overline{\sigma_{ap}(T)}$ *and* (d) $\sigma_r(T) = \overline{\sigma_p(T^*)} \backslash \sigma_p(T)$, *where the bar signifies complex conjugation* [not closure].

Proof If $\lambda \in \sigma_p(T^*)$, then $\lambda I - T^*$ has a nonzero kernel, and therefore, $\text{ran}(\overline{\lambda} I - T)$ has a nonzero orthogonal complement, i.e. $\lambda \in \overline{\sigma_{\text{com}}(T)}$; both these implications are reversible. This proves (a).

The operator $\lambda I - T^*$ is not invertible if and only if one of $\lambda I - T^*$ and $\overline{\lambda} I - T$ is not bounded from below [Theorem 3.5.9]. In other words, $\lambda \in \sigma(T^*)$ if and only if either $\lambda \in \sigma_{ap}(T^*)$ or $\overline{\lambda} \in \sigma_{ap}(T)$. This means

$$\sigma(T^*) = \sigma_{ap}(T^*) \cup \overline{\sigma_{ap}(T)}.$$

This proves (b).

If $\lambda \in \sigma_{\text{com}}(T^*)$, then by definition, $\lambda I - T^*$ does not have dense range, and therefore, $\overline{\lambda} I - T$ has a nontrivial kernel [Theorem 3.5.8], i.e. $\overline{\lambda} \in \sigma_p(T)$. But $\sigma_p(T) \subseteq \sigma_{ap}(T)$. This proves (c).

By Remark 4.1.4, $\sigma_r(T) = \sigma_{\text{com}}(T) \backslash \sigma_p(T) = \overline{\sigma_p(T^*)} \backslash \sigma_p(T)$ by part (a). This proves (d). □

Proposition 4.3.4 *Let* $T \in \mathcal{B}(H)$. *Then,* $\sigma_{ap}(T)$ *is a closed subset of* $\mathbb{C}$.

Proof Let $\lambda \notin \sigma_{ap}(T)$. Then, $\lambda I - T$ is bounded below. So there exists some $\varepsilon > 0$ such that $\|(\lambda I - T)x\| \geq \varepsilon\|x\|$. Also, for all μ, $\|(\lambda I - T)x\| \leq \|(\mu I - T)x\| + \|(\lambda - \mu)x\|$ for all $x \in H$. It follows that $(\varepsilon - |\lambda - \mu|)\|x\| \leq \|(\mu I - T)x\|$ for all μ and all $x \in H$. For $|\lambda - \mu|$ sufficiently small, the preceding inequality implies $\mu I - T$ is bounded below. Hence, the complement of $\sigma_{ap}(T)$ is open. □

Our next result shows that $\sigma_{ap}(T)$ is not empty.

Theorem 4.3.5 *If* $T \in \mathcal{B}(H)$, *then* $\partial\sigma(T) \subseteq \sigma_{ap}(T)$.

Proof Let $\lambda \in \partial\sigma(T))$, and let $\{\lambda_n\}_{n \geq 1}$ be a sequence in the resolvent set $\rho(T)$ such that $\lambda_n \to \lambda$. We claim that $\|(\lambda_n I - T)^{-1}\| \to \infty$ as $n \to \infty$. Suppose this is false. By passing to a subsequence if necessary, there is a constant M such that $\|(\lambda_n I - T)^{-1}\| \leq M$ for all n. Choose n sufficiently large so that $|\lambda_n - \lambda| < M^{-1} \leq \|(\lambda_n I - T)^{-1}\|^{-1}$. It follows on using Proposition 3.3.9 that $\lambda I - T$ is invertible, a contradiction.

Let $\|x_n\| = 1$ satisfy $\alpha_n = \|(\lambda_n I - T)^{-1} x_n\| > \|(\lambda_n I - T)^{-1}\| - \frac{1}{n}$. Then, $\alpha_n \to \infty$ as $n \to \infty$. Put $y_n = \alpha_n^{-1}(\lambda_n I - T)^{-1} x_n$; then, $\|y_n\| = 1$. Now,

$$\begin{aligned}(\lambda I - T)y_n &= (\lambda_n I - T)y_n + (\lambda - \lambda_n)y_n \\ &= \alpha_n^{-1}x_n + (\lambda - \lambda_n)y_n.\end{aligned}$$

Thus,

$$||(\lambda I - T)y_n|| \leq \alpha_n^{-1} + |\lambda - \lambda_n|,$$

so that $\|(\lambda I - T)y_n\| \to 0$ as $n \to \infty$; so $\lambda \in \sigma_{ap}(T)$. □

We now work out in detail an example which illustrates the various kinds of spectra.

Example 4.3.6 Let T be the simple unilateral shift on ℓ^2 defined by

$$T(\lambda_1, \lambda_2, \ldots) = (0, \lambda_1, \lambda_2, \ldots), \quad \{\lambda_i\}_{i \geq 1} \in \ell^2.$$

As seen in (vi) of Examples 3.5.10, the adjoint T^* of T, called the left shift operator, acts on ℓ^2 by

$$T^*(\lambda_1, \lambda_2, \ldots) = (\lambda_2, \lambda_3, \ldots), \quad \{\lambda_i\}_{i \geq 1} \in \ell^2.$$

It has been observed [Example (vii) of 3.2.5] that $\|Tx\| = \|x\|$ for $x \in \ell^2$, and hence, $\|T\| = 1$. Since $\|T^*\| = \|T\|$ [Theorem 3.5.2], it follows that $\|T^*\| = 1$. Consequently, $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma(T^*) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

In what follows, $\sigma(T), \sigma_p(T), \sigma_c(T), \sigma_r(T), \sigma_{ap}(T), \sigma_{\text{com}}(T)$ and their analogues for T^* will be characterised.

(i) Suppose $|\lambda| < 1$. The vector $x_\lambda = (1, \lambda, \lambda^2, \ldots)$ is in ℓ^2 and satisfies $(\lambda I - T^*)x_\lambda = 0$. Thus, all such λ are in the point spectrum of T^*. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq \sigma_p(T^*).$$

Since the spectrum of an operator is a bounded closed subset of $\mathbb{C}$, it follows that $\sigma(T^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. In view of Theorem 4.3.1(a), we have $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. This characterises $\sigma(T)$ and $\sigma(T^*)$.

(ii) From Theorem 4.3.5, $\partial\sigma(T^*) \subseteq \sigma_{ap}(T^*)$. Since $\sigma_p(T^*) \subseteq \sigma_{ap}(T^*)$ by definition, we have $\sigma(T^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \partial\sigma(T^*) \subseteq \sigma_p(T^*) \cup \sigma_{ap}(T^*) = \sigma_{ap}(T^*) \subseteq \sigma(T^*)$, where we have used (i) for the first inclusion. Thus, we have shown that $\sigma(T^*) = \sigma_{ap}(T^*)$.

(iii) It may be remarked that no λ satisfying $|\lambda| = 1$ is in $\sigma_p(T^*)$. Indeed, if $x = \{x_i\}_{i \geq 1}$, $x \neq 0$, is such that $T^* x = \lambda x$, then $(x_2, x_3, \ldots) = (\lambda x_1, \lambda x_2 \ldots)$, which implies $x_{n+1} = \lambda x_n$ for $n \geq 1$. So, $x_{n+1} = \lambda^n x_1$, $n \geq 1$. Hence, $x_1(1, \lambda, \lambda^2, \ldots) \in \ell^2$. Since $|\lambda| = 1$, the vector $x_1(1, \lambda, \lambda^2, \ldots) \in \ell^2$ if and only if $x_1 = 0$, which implies $x = 0$, a contradiction.

We next consider the spectrum of T.

(i) $\sigma_p(T) = \emptyset$. Indeed, if $\{\xi_n\}_{n\geq 1} \in \ell^2$ and $(\lambda I - T)(\{\xi_n\}) = 0$, $\lambda \neq 0$, then $0 = \lambda\xi_1$, $\xi_1 = \lambda\xi_2$, $\xi_2 = \lambda\xi_3$, …, implying that $\xi_1 = 0$, $\xi_2 = 0$, ….

(ii) $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If $|\lambda| < 1$, and $x \in \ell^2$, then $\|(T - \lambda I)x\| \geq |\|Tx\| - |\lambda|\|x\|| \geq |(1 - |\lambda|)\|x\||$, which implies $\lambda \notin \sigma_{ap}(T)$. Consequently, $\sigma_{ap}(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. It follows in view of Theorem 4.3.5 that $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

(iii) By Proposition 4.3.3, $\sigma_p(T^*) = \overline{\sigma_{\text{com}}(T)}$. It follows that $\sigma_{\text{com}}(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

(iv) $\sigma_c(T) = \sigma(T)\backslash(\sigma_{\text{com}}(T) \cup \sigma_p(T)) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}\backslash\{\lambda \in \mathbb{C} : |\lambda| < 1\} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ since $\sigma_p(T)) = \emptyset$.

(v) $\sigma_r(T) = \sigma(T)\backslash(\sigma_c(T)) \cup \sigma_p(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

We have thus proved the following:

$\sigma(T^*) = \sigma_{ap}(T^*)$, since $\sigma_{\text{com}}(T^*) = \overline{\sigma_p(T)} = \emptyset$ by Proposition 4.3.3 and (i) of paragraph above.

Also,

$$\sigma(T^*) = \sigma_p(T^*) \cup \sigma_c(T^*) \cup \sigma_r(T^*), \quad \text{where } \sigma_p(T^*) = \{\lambda \in \mathbb{C} : |\lambda| < 1\},$$

$$\sigma_c(T^*) = \sigma(T^*)\backslash(\sigma_{\text{com}}(T^*)) \cup \sigma_p(T^*) = \sigma(T^*)\backslash\sigma_p(T^*) = \{\lambda \in \ : |\lambda| = 1\}$$

and

$$\sigma_r(T^*) = \emptyset.$$

We summarise below the decomposition of the spectrum of T:

$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{\text{com}}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{\lambda \in \mathbb{C} : |\lambda| < 1\}$

and

$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) = \emptyset \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

4.4 Spectrum of Various Classes of Operators

Let H be a complex Hilbert space and $\mathcal{B}(H)$ denote the algebra of bounded linear operators on H. Normal operators and their suitable subsets such as self-adjoint operators and unitary operators have been studied in Sect. 3.7 and so have been the isometric operators. The spectral properties of a member of the class are somewhat simpler to describe than those of a general member of $\mathcal{B}(H)$. We begin with normal operators.

Theorem 4.4.1 *Every point in the spectrum of a normal operator is an approximate eigenvalue.*

Proof If $T \in \mathcal{B}(H)$ is a normal operator and $\lambda \in \mathbb{C}$, then so is $\lambda I - T$. So, for each $x \in H$,

$$\begin{aligned} ||(\lambda I - T)x|| &= ||(\lambda I - T)^*x|| \quad \text{[Theorem 3.7.1]} \\ &= ||(\overline{\lambda} I - T^*)x||. \end{aligned}$$

Thus, λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*. This means $\overline{\sigma_p(T^*)} = \sigma_p(T)$. Now, by Proposition 4.3.3, $\sigma_p(T^*) = \overline{\sigma_{\text{com}}(T)}$, and hence,

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{\text{com}}(T) = \sigma_{ap}(T) \cup \overline{\sigma_p(T^*)}.$$

Since we have shown that $\overline{\sigma_p(T^*)} = \sigma_p(T)$, the above equality leads to

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T),$$

from which we get $\sigma(T) = \sigma_{ap}(T)$ since $\sigma_p(T) \subseteq \sigma_{ap}(T)$ by definition. □

The following theorem, which is a consequence of the one above, is important in its own right.

Theorem 4.4.2 [cf. Problem 3.8.P2] *The spectrum of every self-adjoint operator $T \in \mathcal{B}(H)$ is a subset of $\mathbb{R}$. In particular, the eigenvalues of T, if any, are real. Furthermore, if T is a positive operator, then the spectrum of T is nonnegative, and eigenvalues, if any, are also nonnegative.*

Proof Let $\lambda = \mu + i\nu$, where μ and ν are real and $\nu \neq 0$ be a complex number. If T is a self-adjoint operator and $x \in H$, then

$$\begin{aligned} ||(\lambda I - T)x||^2 &= ((\lambda I - T)x, (\lambda I - T)x) \\ &= ((\overline{\lambda} I - T)(\lambda I - T)x, x) \\ &= |\lambda|^2(x, x) - 2\mu(Tx, x) + ||Tx||^2 \\ &= ||(\mu I - T)x||^2 + v^2||x||^2 \\ &\geq v^2||x||^2; \end{aligned}$$

So, $\lambda I - T$ is bounded below. This means that λ is not an approximate eigenvalue and hence cannot be in the spectrum $\sigma(T)$ of T by Theorem 4.4.1. Consequently, $\sigma(T) \subseteq \mathbb{R}$.

Assume that T is positive and $\lambda < 0$. Then,

$$\begin{aligned}||(\lambda I - T)x||^2 &= ((\lambda I - T)x, (\lambda I - T)x) \\ &= ((\lambda I - T)^2 x, x) \\ &= \lambda^2(x,x) - 2\lambda(Tx,x) + ||Tx||^2 \\ &\geq \lambda^2||x||^2 \quad \text{since}\, \lambda < 0.\end{aligned}$$

So, $\lambda I - T$ is bounded below. This means that λ is not an approximate eigenvalue and hence cannot be in the spectrum $\sigma(T)$ of T by Theorem 4.4.1. ☐

The second assertion of the above theorem is trivial to prove directly. For a self-adjoint operator, (Tx, x) is real. If $\lambda \in \sigma_p(T)$, then there exists a nonzero x such that $(Tx, x) = (\lambda x, x) = \lambda(x, x)$.

In light of Theorem 4.4.2 and Remark 4.2.7(i), a self-adjoint operator T must have the property that either $\|T\| \in \sigma(T)$ or $-\|T\| \in \sigma(T)$.

Theorem 4.4.3 *Let $\mathcal{B}(H)$ denote the algebra of bounded linear operators on a complex Hilbert space H. Suppose $T \in \mathcal{B}(H)$ satisfies the equality $TT^* = T^*T$, i.e. T is normal. Then,*

(a) $\sigma_p(T) = \overline{\sigma_p(T^*)}$;
(b) *eigenvectors corresponding to distinct eigenvalues, if any, are orthogonal*;
(c) $\sigma_r(T) = \varnothing$.

Proof

(a) Since T is normal, for $\lambda \in \mathbb{C}$, $\|(\lambda I - T)x\| = \|(\lambda I - T)^* x\|$ for each $x \in H$. It follows that $\ker(\lambda I - T) \neq \{0\}$ if and only if $\ker(\bar{\lambda} I - T^*) \neq \{0\}$, that is $\sigma_p(T) = \overline{\sigma_p(T^*)}$.
(b) Let λ, μ be distinct eigenvalues of T and $x, y \in H$ corresponding eigenvectors. Then, $Tx = \lambda x$ and $Ty = \mu y$. It follows from (a) that

$$\lambda(x,y) = (\lambda x, y) = (Tx, y) = (x, T^*y) = (x, \bar{\mu} y) = \mu(x,y).$$

Noting that $\lambda \neq \mu$, we deduce that $(x, y) = 0$, which says x is orthogonal to y.
(c) For any $T \in \mathcal{B}(H)$, $\sigma_r(T) = \overline{\sigma_p(T^*)} \backslash \sigma_p(T)$ by Proposition 4.3.3(d). It therefore follows upon using (a) above that $\sigma_r(T) = \varnothing$ when T is normal. ☐

The spectrum of a self-adjoint operator can be characterised in more detail. Recall that the spectrum $\sigma(T)$ of an operator $T \in \mathcal{B}(H)$ is a nonempty compact subset of $\mathbb{C}$. In the present case, we have the following.

Theorem 4.4.4 *The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator T on a complex Hilbert space H lies in the closed interval $[m, M]$ on the real axis, where* $m = \inf_{\|x\|=1}(Tx, x)$ and $M = \sup_{\|x\|=1}(Tx, x)$.

Proof The fact that $T = T^*$ implies (Tx, x) is real for each $x \in H$. Indeed, for $x \in H$, we have $\overline{(Tx, x)} = (x, Tx) = (Tx, x)$.

The spectrum $\sigma(T)$ lies on the real axis [Theorem 4.4.2]. We show that any real number $M + \varepsilon$ with $\varepsilon > 0$ belongs to the resolvent set $\rho(T)$. For every $x \in H$, $x \neq 0$, and $v = \|x\|^{-1}x$, we have $x = \|x\|v$ and

$$(Tx, x) = ||x||^2(Tv, v) \le ||x||^2\sup_{||v||=1}(Tv, v) = ||x||^2M.$$

Hence, $-(Tx, x) \geq -\|x\|^2M$. On applying the Cauchy–Schwarz inequality to $((\lambda I - T)x, x)$, where $\lambda = M + \varepsilon$, $\varepsilon > 0$, we obtain

$$\begin{aligned}\|(\lambda I - T)x\|||x|| &\ge ((\lambda I - T)x, x) = -(Tx, x) + \lambda(x, x)\\ &\ge (-M + \varepsilon)||x||^2\\ &\ge \varepsilon||x||^2.\end{aligned}$$

This implies

$$\|(\lambda I - T)x\| \ge \varepsilon||x||, \quad x \in H.$$

Consequently, $\lambda \notin \sigma_{ap}(T) = \sigma(T)$, and hence, $\lambda \in \rho(T)$.

The argument when $\lambda < m$ is similar and is therefore not included. □

Let $T \in \mathcal{B}(H)$, where H is a Hilbert space over the field $\mathbb{C}$ of complex numbers, and $T = T^*$. In the theorem above, we defined

$$m = \inf_{||x||=1}(Tx, x)$$

and

$$M = \sup_{||x||=1}(Tx, x).$$

The numbers m and M are related to the norm $\|T\|$ of T. The following theorem has already been proved using Example 3.4.7(ii) and Corollary 3.4.11 [see Theorem 3.6.6]. An independent proof is desirable.

Theorem 4.4.5 *For* $T \in \mathcal{B}(H)$, $T = T^*$, we have

$$||T|| = \max\{|m|, |M|\} = \sup\{|(Tx, x)| : \|x\| = 1\}.$$

Proof Denote the supremum by α. By the Cauchy–Schwarz inequality,

$$\sup_{||x||=1}|(Tx, x)| \le \sup_{||x||=1}||Tx||||x|| = ||T||,$$

so that $\alpha \leq \|T\|$. It remains to prove that $\|T\| \leq \alpha$. If $Tx = 0$ for all $x \in H$ with $\|x\| = 1$, then $\|T\| = \sup_{\|x\|=1}\|Tx\| = 0$. In this case, the proof is complete. Let $x \in H$ be such that

$\|x\| = 1$ and $Tx \neq 0$. Set $v = ||Tx||^{\frac{1}{2}}x$ and $w = ||Tx||^{-\frac{1}{2}}Tx$. Then, $\|v\|^2 = \|w\|^2 = \|Tx\|$. If $y_1 = v + w$ and $y_2 = v - w$, then

$$\begin{aligned}(Ty_1, y_1) - (Ty_2, y_2) &= 2\{(Tv, w) + (Tw, v)\} \\ &= 2\{(Tx, Tx) + \left(T^2x, x\right)\} \\ &= 4||Tx||^2.\end{aligned}$$

Now, for every $y \neq 0$, and $z = \|y\|^{-1}y$, we have $\|y\|z = y$ and

$$|(Ty, y)| = \|y\|^2|(Tz, z)| \leq \|y\|^2\sup_{||z||=1}|(Tz, z)| = \alpha\|y\|^2. \tag{4.1}$$

By the triangle inequality in $\mathbb{C}$,

$$\begin{aligned}|(Ty_1, y_1) - (Ty_2, y_2)| &\leq |(Ty_1, y_1)| + |(Ty_2, y_2)| \\ &\leq \alpha\left\{||y_1||^2 + ||y_2||^2\right\} \\ &= 2\alpha\left\{||v||^2 + ||w||^2\right\} \\ &= 4\alpha||Tx||.\end{aligned} \tag{4.2}$$

From (4.1) and (4.2), we obtain

$$4||Tx||^2 \leq 4\alpha||Tx||,$$

which implies

$$||Tx|| \leq \alpha.$$

This completes the proof. □

The bounds for $\sigma(T)$ in Theorem 4.4.4 cannot be tightened.

Theorem 4.4.6 *If $T \in \mathcal{B}(H)$ is self-adjoint, then m and M, where m and M are as in Theorem 4.4.4, are in the spectrum $\sigma(T)$ of T.*

Proof We show that $M \in \sigma_{ap}(T) = \sigma(T)$. The proof that $m \in \sigma(T)$ is similar and is, therefore, not included.

By the Spectral Mapping Theorem 4.3.1, $M \in \sigma(T)$ if and only if $M + k \in \sigma(T + kI)$, where k is a real constant. Without loss of generality, we may assume $0 \leq m \leq M$. By Theorem 4.4.5,

$$M = \sup_{||x||=1}(Tx, x) = ||T||.$$

By the definition of supremum, there is a sequence $\{x_n\}_{n\geq 1}$ of vectors in H such that

$$\|x_n\| = 1, (Tx_n, x_n) > M - \delta_n, \quad \delta_n \geq 0 \text{ and } \delta_n \to 0.$$

Then, $\|Tx_n\| \leq \|T\| \, \|x_n\| = \|T\| = M$, and since T is self-adjoint

$$\begin{aligned}\|Tx_n - Mx_n\|^2 &= (Tx_n - Mx_n, Tx_n - Mx_n) \\ &= \|Tx_n\|^2 - 2M(Tx_n, x_n) + M^2\|x_n\|^2 \\ &\leq M^2 - 2M(M - \delta_n) + M^2 \\ &= 2M\delta_n \to 0.\end{aligned}$$

It follows by definition that $M \in \sigma_{ap}(T) = \sigma(T)$. This completes the proof. □

Remark If $T \in \mathcal{B}(H)$ is a nonzero self-adjoint operator and $m + M \geq 0$, then $M > 0$ since $m \leq M$ and the bounds m and M cannot both be 0 by Corollary 3.6.7. Therefore, $|m| \leq |M|$. Hence, by Theorem 4.4.5 and Theorem 4.4.6, $\|T\| = |M| = M \in \sigma(T)$. On the other hand, if $m + M < 0$, then $m < 0$, and hence, $\|T\| = |m| = -m$, so that $-\|T\| = m \in \sigma(T)$.

We now consider a subset of scalars which is closely related to the spectrum $\sigma(T)$ of a bounded linear operator T defined on a complex Hilbert space H.

Definition 4.4.7 The **numerical range** of a bounded linear operator T defined on a complex Hilbert space H is the set

$$\boldsymbol{W}(\boldsymbol{T}) = \{(Tx, x) : \|x\| = 1\}.$$

The reader will note that $\|x\| = 1$, not $\|x\| \leq 1$. The numerical range of T is the range of the restriction to the unit sphere $\{x \in H : \|x\| = 1\}$ of the quadratic form (Tx, x) associated with T.

The following properties of the numerical range are easy to discern:

(a) $W(aI + bT) = a + bW(T)$, where a and b are complex numbers;
(b) $W(T)$ is real if T is self-adjoint;
(c) $W(U^*TU) = W(T)$ if U is unitary.

Since $|(Tx, x)| \leq \|T\| \, \|x\|^2$ for every $x \in H$, we see that $|k| \leq \|T\|$ for all $k \in W(T)$. In particular, $W(T)$ is a bounded subset of $\mathbb{C}$. It, however, may not be closed. For example, if $H = \ell^2$ and $T \in \mathcal{B}(H)$ is defined by $Tx = \sum_{n=1}^{\infty} \alpha_n e_n/n$, where $x = \sum_{n=1}^{\infty} \alpha_n e_n$. Then, $(Te_n, e_n) = \frac{1}{n} \in W(T)$ for each n but $(Te_n, e_n) \to 0 \notin W(T)$. However, the numerical range $W(T)$ of $T \in \mathcal{B}(H)$ is a convex subset of $\mathbb{C}$ as we shall later prove.

Examples 4.4.8

(i) Let $T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{C}^2$, where $|u|^2 + |v|^2 = 1$. Now,
$$(Tx, x) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right) = \left(\begin{bmatrix} u \\ 0 \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right) = |u|^2.$$

So,

$$W(T) = [0, 1].$$

(ii) Let $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{C}^2$, where $|u|^2 + |v|^2 = 1$. Now,

$$(Tx, x) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right) = \left(\begin{bmatrix} 0 \\ u \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right) = u\bar{v}.$$

$|u\bar{v}| \leq \frac{1}{2}(|u|^2 + |v|^2) = \frac{1}{2}$ and equality holds if and only if $|u| = |v| = \frac{1}{\sqrt{2}}$. In other words, the numerical range of the operator under consideration lies within the closed disc centred at 0 and having radius $\frac{1}{2}$. We proceed to show that the numerical range is in fact the entire disc.
Consider any complex number $X + iY$ lying this disc; then, $X^2 + Y^2 \leq \frac{1}{4}$. Our claim is that there exist complex numbers u and v such that $u\bar{v} = X + iY$ and $|u|^2 + |v|^2 = 1$. Observe that as r ranges over [0, 1], the product $r^2(1 - r^2)$ ranges over $[0, \frac{1}{4}]$, taking the maximum value $\frac{1}{4}$ when $r = \frac{1}{\sqrt{2}}$. Using this observation about the product $r^2(1 - r^2)$, we obtain a number $r \in [0, 1]$ such that $r^2(1 - r^2) = X^2 + Y^2$. Taking s to be $(1 - r^2)^{\frac{1}{2}}$, we can write $r^2 + s^2 = 1$ and $X^2 + Y^2 = r^2s^2$. From the latter of these equalities, we have $X + iY = rse^{i\psi}$ for some ψ. Now, choose θ and ϕ in any manner so long as $\psi = \theta - \phi$ and set $u = re^{i\theta}$ and $v = se^{i\phi}$. Then, $|u|^2 + |v|^2 = r^2 + s^2 = 1$ and $u\bar{v} = rse^{i(\theta - \phi)} = rse^{i\psi} = X + iY$. This proves our claim. (Note that the numerical range has turned out to be convex.)

(iii) Let $T = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. We shall demonstrate that the numerical range of the operator T in $\mathbb{C}^2$ is the set of all complex numbers $X + iY$ such that

$$\frac{\left(X - \frac{1}{2}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{Y^2}{\left(\frac{1}{2}\right)^2} \leq 1.$$

The author is indebted to Professor Ajit Iqbal Singh for the elegant argument given below.

Lemma A *If A and B are any two real numbers, not both zero, then the quadratic equation*

$$\left(A^2 + B^2\right)t^2 - (2A + 1)t + 2 = 0$$

has a real root if and only if

$$\frac{\left(A-\frac{1}{2}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{B^2}{\left(\frac{1}{2}\right)^2}\leq 1.$$

Proof The discriminant of the quadratic equation is

$$(2A+1)^2-8\left(A^2+B^2\right),$$

which can be put into the form

$$-2\left(\frac{\left(A-\frac{1}{2}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{B^2}{\left(\frac{1}{2}\right)^2}-1\right).$$

Therefore, the quadratic equation, which has real coefficients, has a real root if and only if

$$\frac{\left(A-\frac{1}{2}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{B^2}{\left(\frac{1}{2}\right)^2}\leq 1.$$

□

Lemma B *A complex number* $X+iY$ *is of the form* $(d+1)/(|d|^2+1)$, *where* d *is a complex number if and only if its real and imaginary parts* X *and* Y *satisfy*

$$\frac{\left(X-\frac{1}{2}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{Y^2}{\left(\frac{1}{2}\right)^2}\leq 1.$$

Proof Only if part: Assume $X+iY=(d+1)/(|d|^2+1)$, where d is a complex number. Then, $d=(X+iY)(|d|^2+1)-1$ and $\overline{d}=(X-iY)(|d|^2+1)-1$, so that

$$|d|^2=\left(X^2+Y^2\right)\left(|d|^2+1\right)^2-2X(|d|^2+1)+1.$$

Put $t=|d|^2+1$. Then, t is real and the above equation is a quadratic in t with real coefficients, namely

$$\left(X^2+Y^2\right)t^2-(2X+1)t+2=0. \qquad (*)$$

The required inequality now follows by Lemma A.

If part: Assume that $X + iY$ is any complex number such that its real and imaginary parts X and Y satisfy the inequality in question. If $X^2 + Y^2 = 0$, then $X + iY = 0$, and choosing $d = -1$ leads to $X + iY = (d+1)/(|d|^2+1)$. So, suppose $X^2 + Y^2 \neq 0$. Using X and Y, set up the quadratic equation (*), which obviously has real coefficients. By Lemma A, it must have a real solution. In what follows, the symbol 't' will denote any one real solution. Obviously, $t \neq 0$. Consider the complex number d defined in terms of the nonzero number t and the given complex number $X + iY$ as

$$d = (X + iY)t - 1, \quad \text{or equivalently,} \quad d = (Xt - 1) + iYt.$$

This complex number d has the property that

$$X + iY = (d+1)/t \tag{$**$}$$

and

$$\begin{aligned} |d|^2 &= |(Xt-1) + iYt|^2 = (Xt-1)^2 + Y^2t^2 = X^2t^2 - 2Xt + 1 + Y^2t^2 \\ &= \left(X^2 + Y^2\right)t^2 - 2Xt + 1. \end{aligned}$$

Hence,

$$\begin{aligned} |d|^2 + 1 &= \left(X^2 + Y^2\right)t^2 - 2Xt + 2 \\ &= t \quad \text{in view of (4.3).} \end{aligned}$$

When this is combined with (**), the required equality $X + iY = (d + 1)/(|d|^2 + 1)$ springs forth. □

With the above Lemma B in hand, we can now prove that the numerical range of the operator T in $\mathbb{C}^2$ given by the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ is the set of all complex numbers $X + iY$ such that

$$\frac{\left(X - \frac{1}{2}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{Y^2}{\left(\frac{1}{2}\right)^2} \leq 1.$$

To see why this is so, let $x = \begin{bmatrix} p \\ q \end{bmatrix} \in \mathbb{C}^2$, where $|p|^2 + |q|^2 = 1$. Now,

$$(Tx, x) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \right) = \left(\begin{bmatrix} 0 \\ p+q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \right) = (p+q)\overline{q}.$$

If $q = 0$, then this is 0. Now, suppose $q \neq 0$. Then, $p = dq$, where $d \in \mathbb{C}$. Then, $(Tx, x) = (d+1)|q|^2$. Also, $(|d|^2+1)|q|^2 = 1$. So, $|q|^2 = 1/(|d|^2+1)$, and hence, $(Tx, x) = (d+1)/(|d|^2+1)$, which is independent of q. Now, $d = -1$ implies $(Tx, x) = 0$.

Therefore, the numerical range can be characterised as consisting of all values of $(d + 1)/(|d|^2 + 1)$ as d ranges over all complex numbers (keeping in mind that the values of (Tx, x) when d is not available—i.e. when $q = 0$—are generated by $d = -1$). This characterisation reduces the matter to Lemma B.

(iv) Let T be the left unilateral shift defined on ℓ^2 by $Tx = \sum_{n=1}^{\infty} x_{n+1}e_n$, where $x = \sum_{n=1}^{\infty} x_n e_n$. Then, $(Tx, x) = \sum_{n+1}^{\infty} x_{n+1}\overline{x}_n$. Taking m to be the smallest index for which $x_m \neq 0$ (such an m must exist when $||x|| = 1$), we get

$$|(Tx, x)| \leq \frac{1}{2}\left[|x_m|^2 + 2|x_{m+1}|^2 + 2|x_{m+2}|^2 + \cdots\right] = \frac{1}{2}\left[2 - |x_m|^2\right] < 1.$$

It follows that $W(T)$ is contained in the open unit disc with centre 0.

Conversely, let $z = re^{i\theta}$ with $0 \leq r < 1$. Consider the vector

$$x = \sum_{n=1}^{\infty} r^{n-1}\sqrt{1-r^2}\mathrm{e}^{-i(n-1)\theta}e_n.$$

Observe that $||x|| = 1$ and $(Tx, x) = re^{i\theta}$.

The numerical range $W(T)$ of a bounded linear operator T belonging to $\mathcal{B}(H)$, where H is a complex Hilbert space, has decent properties, some of which are easy to prove.

Theorem 4.4.9 *Let H be a Hilbert space over $\mathbb{C}$ and $T \in \mathcal{B}(H)$. Then, $W(T) = \{(Tx, x) : ||x|| = 1\}$ has the following properties*:

(a) $\lambda \in W(T)$ *if and only if* $\overline{\lambda} \in W(T^*)$;
(b) [Hausdorff–Toeplitz] $W(T)$ *is a convex subset of* $\mathbb{C}$;
(c) $\sigma(T) \subseteq \overline{W(T)}$;
(d) *if T is normal, then the convex hull of the spectrum $\sigma(T)$ of T*, $\mathrm{co}(\sigma(T)) = \overline{W(T)}$.

Proof

(a) Let $x \in H$ with $\|x\| = 1$. Then, $\overline{(Tx, x)} = (x, Tx) = (T^*x, x)$. Thus, $(Tx, x) \in W(T)$ if and only if $\overline{(Tx, x)} \in W(T^*)$.
(b) Let $\xi = (Tx, x)$ and $\eta = (Ty, y)$ for unit vectors x and y in H. We want to prove that every point of the segment joining ξ and η is in $W(T)$. If $\xi = \eta$, the problem is trivial. Suppose $\xi \neq \eta$. Choose complex numbers a and b such that $a\xi + b = 1$ and $a\eta + b = 0$. Indeed, $a = 1/(\xi - \eta)$ and $b = -\eta/(\xi - \eta)$ are the desired complex numbers.

For any complex numbers u and v, it is easy to verify that

$$W(uT+vI)=\{uw+v : w\in W(T)\}=uW(T)+v.$$

Consequently, the set $\{0, 1\}$ is contained in $W(aT + bI)$. It will suffice to show that the interval $(0, 1)$ is included in $W(aT + bI)$. If any $t \in (0, 1)$ can be shown to be of the form $a(Tz, z) + b$, where $||z|| = 1$, then

$$\begin{aligned} a(Tz,z)+b=t&=t(a\xi+b)+(1-t)(a\eta+b)\\ &=a(t\xi+(1-t)\eta)+b, \end{aligned}$$

which implies $t\xi + (1 - t)\eta = (Tz, z) \in W(T)$.
So, there is no loss of generality in assuming that $\xi = 1$ and $\eta = 0$, i.e. $(Tx, x) = 1$ and $(Ty, y) = 0$, and showing that $[0, 1] \subseteq W(T)$. It follows that x and y are linearly independent, because otherwise, x would be a scalar multiple of y, and hence, (Tx, x) would also be zero. Write

$$T=T_1+iT_2,$$

$T_1=\frac{T+T^*}{2}$ and $T_2=\frac{T-T^*}{2i}$ are Hermitian. Now,

$$\begin{aligned} 1=(Tx,x)=(T_1x,x)+i(T_2x,x)\Rightarrow(T_1x,x)=1,(T_2x,x)=0,\\ 0=(Ty,y)=(T_1y,y)+i(T_2y,y)\Rightarrow(T_1y,y)=0,(T_2y,y)=0. \end{aligned}$$

If x is replaced by λx, $\lambda \in \mathbb{C}$, where $|\lambda|=1$, the value of (Tx, x) remains unaltered and $(T_2\lambda.x,y)=\lambda(T_2x,y)$. Furthermore, we may assume that (T_2x, y) is purely imaginary. Indeed, $\lambda=i\overline{\mu}/|\mu|$, where $\mu = (T_2x, y)$, has the desired property.
Set $z(t) = tx + (1 - t)y$, $0 \le t \le 1$. Since x and y are linearly independent, $z(t) = 0$ for no t.
Since

$$(T_2z(T),z(T))=t^2(T_2x,x)+t(1-t)((T_2x,y)+\overline{(T_2x,y)})+(1-t)^2(T_2y,y),$$

for all t, it follows from the relations $(T_2x, x) = 0 = (T_2y, y)$ and $\Re(T_2x, y) = 0$, that $(T_2z(t), z(t)) = 0$. Hence, $(Tz(t), z(t))$ is real for all t. So, the function

$$t\to(Tz(T),z(T))/||z(T)||^2$$

is real-valued and continuous on $[0, 1]$ and its values at 0 and 1 are, respectively, 0 and 1. Hence, the range of the function contains every $t \in [0, 1]$.

(c) Let $\lambda \in \sigma_p(T)$. Then, $Tx = \lambda x$ for some $x \in H$ with $||x|| = 1$. Since $(Tx, x) = (\lambda x, x) = \lambda(x, x) = \lambda||x||^2 = \lambda$, we see that $\lambda \in W(T)$. Next, let $\lambda \in \overline{\sigma(T)}$. Note that $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{\text{com}}(T)$ [Remarks 4.1.4] $= \sigma_{ap}(T) \cup \overline{\sigma_p(T^*)}$ [Proposition 4.3.3(a)]. So, $\lambda \in \sigma_{ap}(T)$ or $\overline{\lambda} \in \sigma_p(T^*)$. If $\lambda \in \sigma_{ap}(T)$ then there is a sequence $\{x_n\}_{n\geq 1}$ in H such that $||x_n|| = 1$ and $Tx_n - \lambda x_n \to 0$ as $n \to \infty$. Since

$$\begin{aligned}|(Tx_n, x_n) - \lambda| &= |(T - \lambda I)x_n, x_n)| \\ &\leq \|(T - \lambda I)x_n\| \| |x_n| | \to 0 \text{ as } n \to \infty,\end{aligned}$$

we see that $\lambda = \lim_n(Tx_n, x_n)$, and hence, $\lambda \in \overline{W(T)}$.

Also, if $\overline{\lambda} \in \sigma_p(T^*)$, then we have seen above that $\lambda \in W(T^*)$, and hence, $\overline{\lambda} \in W(T)$ by (a) above. This completes the proof.

(d) For a proof of this, we refer the reader to [3]. □

Remark 4.4.10 If $T \in \mathcal{B}(H)$ is self-adjoint, then (Tx, x) is real. Indeed, $(Tx, x) = (\overline{x, Tx}) = (\overline{Tx, x})$. Consequently, $W(T) \subseteq \mathbb{R}$. If $m = \inf_{||x||=1}(Tx, x)$ and $M = \sup_{||x||=1}(Tx, x)$, then $\overline{W(T)} \subseteq [m, M]$. Since $W(T)$ is convex [(b) above], so is $\overline{W(T)}$. Therefore, $\overline{W(T)} \subseteq [m, M]$. Now, $[m, M] = \text{co}\sigma(T)$ in view of Theorem 4.4.4 and Theorem 4.4.6. Hence, $\text{co}\sigma(T) = \overline{W(T)}$, i.e. for a self-adjoint T, (d) holds.

The numerical range, like the spectrum, associates a set of complex numbers with each operator $T \in \mathcal{B}(H)$; it is a set-valued function. The smallest disc centred at the origin that contains the numerical range has radius given by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup\{|(Tx, x)| : \|x\| = 1\},$$

called the **numerical radius** of T. [Cf. Definition 3.7.3.]

In Theorem 3.7.7, it was proved that for a normal operator, the norm is the same as its numerical radius.

Observe that $w(T)$ is a vector space norm on $\mathcal{B}(H)$. That is, $0 \leq w(T)$ for every $T \in \mathcal{B}(H)$ and $0 < w(T)$ whenever T is not zero; $w(\alpha T) = |\alpha| w(T)$ and $w(T + S) \leq w(T) + w(S)$ for every $\alpha \in \mathbb{C}$ and every S and T in $\mathcal{B}(H)$. The numerical radius will now be shown to be equivalent to the operator norm of $\mathcal{B}(H)$ and to dominate the spectral radius.

Proposition 4.4.11 *For any* $T \in \mathcal{B}(H)$, *we have* $0 \leq r(T) \leq w(T) \leq \|T\| \leq 2w(T)$.

Proof Since $\sigma(T) \subseteq \overline{W(T)}$ by Theorem 4.4.9(c), we have

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} \leq \sup\{|\lambda| : \lambda \in \overline{W(T)}\} = \sup\{|\lambda| : \lambda \in W(T)\}.$$

So,

$$r(T) \le w(T).$$

Moreover,

$$w(T) = \sup\{|(Tx, x)| : \|x\| = 1\} \le \sup\{||Tx|| : ||x|| = 1\} = ||T||.$$

Note that

$$\begin{aligned} |(Tz, z)| &= |(Tz/||z||, z/||z||)| \cdot ||z||^2 \\ &\le \sup\{|(Tu, u)| : ||u|| = 1\} \cdot ||z||^2 \\ &= w(T)||z||^2 \quad \text{for every } z \in H. \end{aligned}$$

By the parallelogram law,

$$\begin{aligned} 4|(Tx, y)| &= |(T(x+y), x+y) - (T(x-y), x-y) \\ &\quad + i(T(x+iy), x+iy) - i(T(x-iy), x-iy)| \\ &\le w(T)\left(||x+y||^2 + ||x-y||^2 + ||x+iy||^2 + ||x-iy||^2\right) \\ &= 4w(T)\left(||x||^2 + ||y||^2\right) \\ &\le 8w(T) \quad \text{whenever } ||x|| = 1 = ||y||. \end{aligned}$$

Therefore,

$$||T|| = \sup\{|(Tx, y)| : \|x\| = 1 = \|y\|\} \le 2w(T).$$

□

Remark It is known that if T^2 is the zero operator, then $||T|| = 2w(T)$. See [28] and references therein.

If α is any positive number, then the vector space norm $w_\alpha(T) = \alpha w(T)$ is an algebra norm (i.e. satisfies $w_\alpha(ST) \le w_\alpha(S)w_\alpha(T)$) if and only if $\alpha \ge 4$. See [11].

The inequality $\|T^*T + TT^*\| \le 4w(T) \le 2\|T^*T + TT^*\|$ has been proved in Kittaneh [18].

If $T \in \mathcal{B}(H)$, then $|(Tx, y)|^2 \le (|T|x, x)(|T^*|y, y)$ and $2w(T) \le \|T\| + \|T^2\|^{\frac{1}{2}}$. The second of these is due to Kittaneh [17].

If S, $T \in \mathcal{B}(H)$ are positive, then $\|S^{\frac{1}{2}}T^{\frac{1}{2}}\| \le \|ST\|^{\frac{1}{2}}$ and

$$2||S + T|| \le \|S\| + \|T\| + \left((\|S\| - \|T\|)^2 + 4\|S^{\frac{1}{2}}T^{\frac{1}{2}}\|^2\right)^{\frac{1}{2}}.$$

The second of these is due to Kittaneh [16].

We now turn to the properties of the spectrum of unitary and isometric operators.

Recall that $U \in \mathcal{B}(H)$, the algebra of all bounded linear operators on a complex Hilbert space H, is unitary if and only if $UU^* = U^*U = I$. Moreover, $\|U\| = 1 = \|U^*\|$. Therefore, $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and so is $\sigma(U^*)$. Note that $0 \notin \sigma(U)$ since U, by definition, is invertible. If $|\lambda| < 1$, then $\lambda I - U = \lambda(U^* - \frac{1}{\lambda}I)U$. Since $\frac{1}{\lambda}$ is not in the closed unit disc, the operator $\lambda(U^* - \frac{1}{\lambda}I)$ is invertible, and hence, so is $\lambda I - U$. Thus, $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. The unitary operator U is normal, so $\sigma_r(U) \neq \emptyset$ [Theorem 4.4.3].

Examples 4.4.12

(i) (Bilateral shift; (i) of Examples 3.7.13). The operator $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is defined by the rule

$$U(x)(n) = x(n-1), \quad x \in \ell^2(\mathbb{Z}).$$

and its adjoint $U^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is defined by

$$U^*(x)(n) = x(n+1), \quad x \in \ell^2(\mathbb{Z}).$$

$UU^* = U^*U = I$. In other words, U^{-1} exists and $U^{-1} = U^*$. Also, $\|U\| = \|U^*\| = 1$ and so $\sigma(U)$ and $\sigma(U^*)$ are contained in the closed unit disc. From the paragraph above, it follows that $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. From the normality of U, $\sigma(U) = \sigma_{ap}(U)$[Theorem 4.4.1]. We next show that each λ with $|\lambda| = 1$ is an approximate eigenvalue of U.
For fixed θ in $[0, 2\pi]$ and $n \in \mathbb{N}$, let x_n be the vector in $\ell^2(\mathbb{Z})$ defined by

$$x_n(k) = \begin{cases} (2n+1)^{-\frac{1}{2}}\mathrm{e}^{-ik\theta}, & |k| \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Note that $||x_n||^2 = (2n+1)^{-1}\sum_{k=-n}^{n} 1 = 1$. Also, $(2n+1)^{\frac{1}{2}}x_n(k)$ and $(2n+1)^{\frac{1}{2}}Ux_n(k)$ are, respectively,

$$(\ldots,0,\mathrm{e}^{in\theta},\mathrm{e}^{i(n-1)\theta}, \ldots, \mathrm{e}^{i\theta}, 1, \mathrm{e}^{-i\theta},\mathrm{e}^{-2i\theta}, \ldots, \mathrm{e}^{-i(n-1)\theta}, \mathrm{e}^{-in\theta}, 0, \ldots),$$

$$(\ldots,0, 0, \mathrm{e}^{in\theta}, \mathrm{e}^{i(n-1)\theta},\ldots,\mathrm{e}^{i\theta}, 1, \mathrm{e}^{-i\theta},\mathrm{e}^{-2i\theta}, \ldots, \mathrm{e}^{-i(n-1)\theta},\mathrm{e}^{-in\theta}, 0, \ldots).$$

Consequently, $(2n+1)^{\frac{1}{2}}(U - \mathrm{e}^{i\theta}I)x_n$ is

$$(\ldots,0, -\mathrm{e}^{i(n+1)\theta},0,0,\ldots,0, \mathrm{e}^{-in\theta},0,\ldots),$$

where the only two nonzero entries are in positions $-n$ and $n + 1$. Therefore, $||(U - e^{i\theta}I)x_n||^2 = \frac{2}{2n+1}$, so that $\lim_n (U - e^{i\theta}I)x_n = 0$.
Thus, each $e^{i\theta}, \theta \in [0, 2\pi]$ is an approximate eigenvalue of U.
It may be argued that $\sigma_p(U) = \varnothing$ as is done below. Let λ be an eigenvalue, so that $|\lambda| = 1$, and let $x \in \ell^2(\mathbb{Z})$ be a corresponding nonzero eigenvector. Since $(Ux)(n) = x(n - 1)$ and $(\lambda x)(n) = \lambda(x(n))$, we have $x(n - 1) = \lambda(x(n))$, and hence, $x(-n) = \lambda^n x(0)$ for any $n \geq 0$. This implies $||x||^2 \geq \sum_{n=0}^{\infty} |x(-n)|^2 = |x(0)|^2 \sum_{n=0}^{\infty} |\lambda|^{2n}$, which leads to $x(0) = 0$. Therefore, $x(n) = 0$ for all nonpositive n; by similar considerations, we can show that $x(n) = 0$ for positive n as well. Hence, the contradiction that $x = 0$.

(ii) (Multiplication Operator) Let $H = L^2[0, 2\pi]$. The multiplication operator $U:H \to H$ is defined by the formula $(Ux)(t) = e^{it}x(t)$, $x \in H$. Note that U is unitary [(ii) of Examples 3.7.13]. So, $\sigma(U) \subseteq \{\lambda \subseteq \mathbb{C} : |\lambda| = 1\}$. It follows from (iii) of Examples 4.1.2 that $\sigma(U) = \{e^{it} : t \in [0, 2\pi]\}$. From the fact that U is normal, each point of the spectrum is an approximate eigenvalue [Theorem 4.4.1].

The functions $e^{-int}, n \in \mathbb{Z}$, form an orthonormal basis of $L^2[0, 2\pi]$. If we identify $L^2[0, 2\pi]$ with $\ell^2(\mathbb{Z})$ in terms of the basis, then the multiplication operator U gets identified with the bilateral shift. Thus, (ii) is really the 'same' example as (i). Therefore, the operators have the same spectrum of each kind. In particular, the point spectrum of the multiplication operator is empty, a fact which can of course be deduced directly from the definition of the operator as well.

Recall that an operator $V \in \mathcal{B}(H)$, the algebra of operators on a complex Hilbert space H, is an isometry if $||Vx|| = ||x||$ for each $x \in H$. The norm $||V||$ of V is 1. So, $\sigma(V) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. There exist isometries whose spectrum coincides with the unit disc. In fact, if V is the simple unilateral shift and V^* denotes the adjoint of V [Example 4.3.6], then $\sigma(V^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma(V) = \{\overline{\lambda} : \lambda \in \sigma(V^*)\} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

The next result shows that the eigenvalues of an isometry, if any, lie on the unit circle and the eigenspaces corresponding to distinct eigenvalues are orthogonal.

Theorem 4.4.13 *Let $V \in \mathcal{B}(H)$ be an isometry. Then,*

(a) *Every $\lambda \in \sigma_p(V)$ lies on the unit circle.*
(b) *If $\mathfrak{M}_\lambda$ and $\mathfrak{M}_\mu$ are eigenspaces of V corresponding to λ and μ, respectively, then $\lambda \neq \mu$ implies $\mathfrak{M}_\lambda \perp \mathfrak{M}_\mu$.*

Proof

(a) Let $\lambda \in \sigma_p(V)$. Then, there exists an $x \in H$, $x \neq 0$, such that $Vx = \lambda x$. Now, $||x||^2 = ||Vx||^2 = (Vx, Vx) = |\lambda|^2||x||^2$, so that $|\lambda|^2 = 1$, and hence, $|\lambda| = 1$.

(b) Let $x \in \mathfrak{M}_\lambda$ and $y \in \mathfrak{M}_\mu$. Then, by Proposition 3.7.16,

$$(x, y) = (Vx, Vy) = (\lambda x, \mu y) = \lambda\bar{\mu}(x, y).$$

So, $(1 - \lambda\bar{\mu})(x, y) = 0$. Since $1 - \lambda\bar{\mu} \neq 0$ (if $\lambda\bar{\mu} = 1$, then $\mu = \lambda|\mu|^2 = \lambda$), it follows that $(x, y) = 0$. This completes the proof. □

Problem Set 4.4

4.4.P1. Let $A \in \mathcal{B}(H)$ be self-adjoint and $\lambda \in \mathbb{C}$ be a complex number such that $\Im\lambda \neq 0$. Show that

$$||(A - \lambda I)x|| \geq |\Im\,\lambda|||x|| \quad \text{for all}\, x \in H.$$

Hence or otherwise, show that the spectrum of A is real.

4.5 Compact Linear Operators

A typical example of an unbounded operator is the differential operator studied in Example (x) of 3.2.5. [An unbounded linear operator is defined on a dense linear subspace of the space under consideration.] The theory developed for bounded linear operators is not applicable to differential operators. In order to overcome this difficulty in part, the results of bounded linear operators are applied to the inverse operators of differential operators after restricting the latter to a subspace on which they are injective. The inverse of the linear differential operator cited above is the familiar Volterra operator in (ix) of 3.2.5: $Vx(s) = \int_0^s x(t)\mathrm{d}t$. These inverse operators are not only bounded, but in addition possess a special property called 'compactness'. Compact operators are also called completely continuous operators. Most of the statements about these operators are generalisations of the statements about linear operators in finite-dimensional spaces.

The use of linear operator methods to prove some of Fredholm's results on linear integral equations of the form

$$(T - \lambda I)x(s) = y(s), \quad \text{where } Tx(s) = \int_a^b k(s, t)x(t)\mathrm{d}t,$$

λ being a parameter, y and k given functions and x the unknown function was pioneered by F. Riesz in 1916. The concept of linear spaces had not been formulated by then and Riesz worked with integral equations. His techniques generalise directly and can be applied to compact (or completely continuous) operators.

Compact linear operators are defined as follows.

Definition 4.5.1 Let X and Y be normed linear spaces. A linear operator $T{:}X \to Y$ is called a **compact operator** (or **completely continuous operator**) if it maps the unit ball $B = \{x \in X : \|x\| \leq 1\}$ of X onto a precompact (i.e. having compact closure) subset of Y.

Since T is linear, this means that for every bounded subset M of X, the closure $\overline{T(M)}$ is a compact subset of Y.

The sequence criterion for compactness in a metric space tells us that T is compact if and only if for a bounded sequence $\{x_n\}_{n\geq 1}$ in X, the sequence $\{Tx_n\}_{n\geq 1}$ in Y has a convergent subsequence.

The following lemma shows that a compact linear operator is continuous, whereas the converse is generally not true [see Remark 4.5.3(i)].

Lemma 4.5.2 *Let X and Y be normed linear spaces. Then, every compact linear operator $T{:}X \to Y$ is bounded and hence continuous.*

Proof The unit sphere $S = \{x \in X : ||x|| = 1\}$ is bounded. Since T is a compact operator, $\overline{T(S)}$ is compact. It is therefore bounded, that is,

$$\sup_{||x||=1} ||Tx|| < \infty.$$

Thus, T is a bounded linear operator and is therefore continuous. □

Remarks 4.5.3

(i) We show that the identity operator on an infinite-dimensional normed linear space is not compact.

Let X be an infinite-dimensional normed linear space and $\{x_1, x_2, \ldots\}$ denote linearly independent vectors in X. We claim that there exist y_n, $n = 1, 2, \ldots$, satisfying the properties

$$||y_n|| = 1 \quad \text{for all}\, n,$$

$$y_n \in M_n = [\{x_1, x_2, \ldots, x_n\}], \tag{4.3}$$

the linear span of $\{x_1, x_2, \ldots, x_n\}$,

$$||y_{n+1} - x|| \geq \frac{1}{2} \quad \text{for all}\, x \in M_n. \tag{4.4}$$

To prove this claim, set

$$y_1 = \frac{x_1}{||x_1||} \in M_1.$$

Note that M_1 being a finite-dimensional subspace of X is closed. By Riesz Lemma 5.2.11, there exists a vector $y_2 \in M_2$ with $\|y_2\| = 1$ such that

$$||y_2 - x|| \geq \frac{1}{2} \quad \text{for all } x \in M_1.$$

Continuing in this manner, we obtain y_1, y_2,… satisfying the conditions specified in (4.3) and (4.4). Now, consider the sequence $\{y_n\}_{n\geq 1}$. It is clear that it is bounded $(||y_n|| = 1, n = 1, 2, \ldots)$. Its image under the identity operator is the sequence itself. In view of (4.4), the sequence under consideration satisfies

$$||y_n - y_m|| \geq \frac{1}{2} \quad \text{for all} \quad n \neq m$$

and therefore cannot have a convergent subsequence.

The above remark essentially says that in an infinite-dimensional normed space, the unit ball is never compact.

(ii) In case of some normed linear spaces, it is possible to prove the above result without appealing to the Riesz lemma, as the following example shows.

Let $X = \ell^p$, and e_k = (0, 0, …, 0, 1, 0, …), where 1 occurs at the kth place, k = 1, 2, …. Then, $\{e_n\}_{n\geq 1}$ is a bounded sequence in ℓ^p and $||e_n||$ = 1, n = 1, 2, …. However, $\{Ie_n\}_{n\geq 1}$ has no convergent subsequence. Indeed,

$$\left|\left|Ie_n - Ie_m\right|\right|_p = \left|\left|e_n - e_m\right|\right|_p = 2^{1/p} \quad \text{for } n \neq m.$$

A similar argument works in any infinite-dimensional Hilbert space. Let $\{e_k\}_{k\geq 1}$ be any orthonormal sequence; then, $||e_k|| = 1$ for every k and $||e_n - e_m||_p = \sqrt{2}$ whenever $n \neq m$.

(iii) If either X or Y is finite-dimensional, then every $T \in \mathcal{B}(X, Y)$ is compact. Suppose dim(Y) < ∞. Let $\{x_n\}_{n\geq 1}$ be a bounded sequence in X. Then, the inequality $||Tx_n|| \leq \|T\|\|x_n\|$ shows that $\{Tx_n\}_{n\geq 1}$ is bounded. Since dim (Y) < ∞, it follows that $\{Tx_n\}_{n\geq 1}$ has a convergent subsequence. Now, suppose dim(X) < ∞. Note that dim(TX) $\leq$ dim(X). The result therefore follows from what has just been proved.

Definition 4.5.4 Let $T \in \mathcal{B}(X, Y)$. The **rank** of T is defined to be the dimension of the range ran(T) of T. If the range is finite-dimensional, we say that T has **finite rank**.

The rank is a purely algebraic concept.

In (iii) of the remark above, we have noted that finite rank operators in $\mathcal{B}(X,Y)$ are compact. We write $\mathcal{B}_0(X,Y)$ for the collection of all compact operators from X to

Y and $\mathcal{B}_{00}(X,Y)$ for the collection of all finite rank operators from X to Y. We abbreviate $\mathcal{B}_0(X,X)$ as $\mathcal{B}_0(X)$.

The reader will note that $\mathcal{B}_{00}(X,Y)\subseteq\mathcal{B}_0(X,Y)\subseteq\mathcal{B}(X,Y)$.

Examples 4.5.5

(i) Let X be a normed linear space, z a vector in X and f a bounded linear functional on X. We define $T:X\to X$ by

$$Tx = f(x)z, \quad x \in X.$$

T is linear since f is a linear functional. Moreover, T is bounded. Indeed,

$$\|Tx\| = \|f(x)z\| \le \|f\|\|x\|\|z\|,$$

which implies

$$\|T\| \le \|f\|\|z\|.$$

Since T is of rank 1, it follows that T is a completely continuous operator.

(ii) Let $X = C[0, 1]$, the space of continuous functions on [0, 1] with $\|x\| = \sup\{|x(t)| : 0 \le t \le 1\}$. Let $k(x, y)$ be a continuous kernel on $[0, 1] \times [0, 1]$. Define the integral operator T by

$$(Tx)(s) = \int_0^1 k(s,t)x(t)\mathrm{d}t, \quad x \in C[0,1].$$

Then, T will be shown to be a compact operator.

Let $\{x_n\}_{n\ge1}$ be a sequence in X with $\|x_n\| \le 1$ for all n. We shall show that $\{Tx_n\}_{n\ge1}$ has a convergent subsequence. For this, we shall use Ascoli's Theorem [see Theorem 1.2.21]. Since $\|Tx_n\| \le \|T\|$, the sequence $\{Tx_n\}_{n\ge1}$ is bounded. We shall show that it is equicontinuous. Since k is uniformly continuous, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|s_1 - s_2| < \delta$ implies $|k(s_1,t) - k(s_2,t)| < \varepsilon$ for all $t \in [0, 1]$. Thus, for $|s_1 - s_2| < \delta$, we have

$$|Tx_n(s_1) - Tx_n(s_2)| \le \int_0^1 |k(s_1,t) - k(s_2,t)||x_n(t)|\mathrm{d}t \le \varepsilon \int_0^1 |x_n(t)|\mathrm{d}t \le \varepsilon.$$

Thus, the sequence $\{Tx_n\}_{n\ge1}$ is equicontinuous. So, by Ascoli's Theorem, it has a convergent subsequence.

If we take k to be the characteristic function of the set $\{(s,t) \in [0, 1] \times [0, 1] : t < s\}$, which is patently discontinuous, the above argument does not apply. But we still have an operator in $C[0, 1]$, called the Volterra operator, just like its counterpart in $L^2[0, 1]$. Since $|(Tx)(s)| = |\int_0^s k(s,t)x(t)\mathrm{d}t| \le s||x|| \le 1$, not only is T bounded with norm at most 1, but also satisfies $|Tx(s_1) - Tx(s_2)| \le |s_1 - s_2| \cdot ||x||$, which has the consequence that T maps a bounded set in $C[0, 1]$ into an equicontinuous set. By Ascoli's Theorem, T is compact.

(iii) Let k be a complex function belonging to $L^2([0, 1] \times [0, 1])$. We define the transformation T on $L^2[0, 1]$ by

$$(Tx)(s) = \int_0^1 k(s,t)x(t)\mathrm{d}t, \quad x \in L^2[0,1].$$

The computation

$$\begin{aligned}\int_0^1 |(Tx)(s)|^2 \mathrm{d}s &= \int_0^1 \left| \int_0^1 k(s,t)x(t)\mathrm{d}t \right|^2 \mathrm{d}s \\ &\le \int_0^1 \left\{ \int_0^1 |k(s,t)|^2 \mathrm{d}t \right\} \left\{ \int_0^1 |x(t)|^2 \mathrm{d}t \right\} \mathrm{d}s \\ &\le ||x||^2 \int_0^1 \int_0^1 |k(s,t)|^2 \mathrm{d}t\mathrm{d}s,\end{aligned}$$

using the Cauchy–Schwarz inequality, shows that T is a bounded linear operator in $x \in L^2[0, 1]$ with

$$||T|| \le \left\{ \int_0^1 |k(s,t)|^2 \mathrm{d}s\mathrm{d}t \right\}^{\frac{1}{2}} = ||k||_{L^2([0,1]\times[0,1])}.$$

We shall show that T is a compact operator.

Let $\Phi : L^2([0,1] \times [0,1]) \to \mathcal{B}(L^2([0,1])$ be defined by

$$\Phi(k) = T.$$

We have shown above that Φ is a bounded linear operator with norm at most $||k||_{L^2([0,1]\times[0,1])}$.

Let $\{f_i(s)\}$ be an orthonormal basis in $L^2[0,\ 1]$. Then, $\{f_i(s)f_j(t)\}$ is an orthonormal basis in $L^2([0,1]\times[0,1])$; so, $k(s,t)=\sum_{i,j=1}^{\infty}a_{i,j}f_i(s)f_j(t)$, where the series converges in the norm of $L^2([0,\ 1]\times[0,\ 1])$. Let $k_n(s,t)=\sum_{i,j=1}^{n}a_{i,j}f_i(s)f_j(t)$. Then, $||k-k_n||_{L^2([0,1]\times[0,1])}\to 0$ as $n\to\infty$. Considering the operator

$$(K_nx)(s)=\int_0^1 k_n(s,t)x(t)\mathrm{d}t,\quad x\in L^2[0,1],$$

it follows that $\|\Phi(k)-\Phi(k_n)\|_{op}\to 0$ as $n\to\infty$. This completes the argument.

In the case of the Volterra operator, $k(s,\ t)$ equals 1 if $0\ \le\ t\ \le\ s$ and equals 0 if $s<t\ \le\ 1$. Therefore, it is a compact operator in $L^2[0,\ 1]$. Since its range includes all polynomials with constant term 0, it is not of finite rank. In fact, its range is dense, as has been shown in Example 4.3.2. Thus, we have $\mathcal{B}_{00}(X,Y)\subset\mathcal{B}_0(X,Y)\subset\mathcal{B}(X,Y)$ when $X=Y=L^2([0,\ 1]$. Moreover, its spectrum consists of only 0, which is not an eigenvalue [see Example 4.3.2].

Recall that the uniform limit of a sequence of continuous functions is continuous. The following similar result for completely continuous operators holds.

Theorem 4.5.6 *Let $\{T_n\}_{n\ge1}$ be a sequence of completely continuous operators mapping a normed linear space X into a Banach space Y be such that*

$$\lim_n||T_n-T||=0.$$

Then, T is a completely continuous operator.

Proof Let $B_0=\{x\in X:||x||\le1\}$ be the unit ball in X. Since T_n is compact, the set $T_n(B_0)$ in Y is precompact (i.e. $\overline{T_n(B_0)}$ is compact). Given $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $||T_n-T||<\varepsilon/3$. By compactness, we can cover $T_n(B_0)$ with a finite number m of balls $B(T_nx_j,\varepsilon/3)$, where $x_1,x_2,\ldots,x_m$ are in B_0. Suppose $x\in B_0$ and let $j\ \le\ m$ be such that $||T_nx-T_nx_j||<\varepsilon/3$. By the triangle inequality,

$$\begin{aligned}\left|\left|Tx-Tx_j\right|\right| &\le\|Tx-T_nx\|+\left\|T_nx-T_nx_j\right\|+\left|\left|T_nx_j-Tx_j\right|\right|\\ &\le 2\|T_n-T\|+\left\|T_nx-T_nx_j\right\|<\varepsilon.\end{aligned}$$

Therefore,

$$T(B_0)\subseteq\bigcup_{j=1}^{m}B(Tx_j,\varepsilon),$$

and $T(B_0)$ is precompact. □

Corollary 4.5.7 *If $T \in \mathcal{B}(X, Y)$ and there exists a sequence $T_n \in \mathcal{B}_{00}(X, Y)$ such that $||T_n - T|| \to 0$ as $n \to \infty$, then $T \in \mathcal{B}_0(X,Y)$.*

Proof

$$\mathcal{B}_{00}(X, Y) \subseteq \mathcal{B}_0(X, Y).$$

□

Remark

(i) The above corollary provides a frequently used sufficient condition for an operator to be compact; namely, it is sufficient that it be the norm limit of a sequence of finite rank operators. The necessity of this condition has been shown to be false by *P*. Enflo [8].

(ii) If X and Y are Hilbert spaces, the following statement also holds [see Problem 4.5.P6]: if $T \in \mathcal{B}(X, Y)$ is compact and ran(T) = Y, then there exists a sequence $T_n \in \mathcal{B}_{00}(X, Y)$ such that $||T_n - T|| \to 0$ as $n \to \infty$.

Lemma 4.5.8 *The set $\mathcal{B}_0(X, Y)$ of all compact linear operators is a closed linear subspace of $\mathcal{B}(X, Y)$.*

Proof Let S and T be in $\mathcal{B}_0(X, Y)$ and $\alpha, \beta \in \mathbb{C}$. If $\{x_n\}_{n \geq 1}$ is a bounded sequence in X, then $\{Tx_n\}_{n \geq 1}$ has a convergent subsequence $\{Tx_{n_k}\}_{k \geq 1}$, say, and $\{Sx_{n_k}\}_{k \geq 1}$ has also a convergent sequence $\{Sx_{n_{k(j)}}\}_{j \geq 1}$, say. The sequence $\{Tx_{n_{k(j)}}\}_{j \geq 1}$ converges because it is a subsequence of a convergent sequence. It is now clear that the sequence $\{\alpha Tx_{n_{k(j)}} + \beta Sx_{n_{k(j)}}\}_{j \geq 1}$ converges.

Using Theorem 4.5.6, we conclude that $\mathcal{B}_0(X, Y)$ is a closed linear subspace of $\mathcal{B}(X, Y)$. □

Theorem 4.5.9 *If S and T are linear operators mapping a normed linear space X into itself, where S is completely continuous and T is bounded, then ST and TS are completely continuous operators.*

Proof Suppose B is a bounded set in X and consider

$$ST(B) = S(T(B)).$$

Since B is bounded and T is a bounded linear operator, it follows $T(B)$ is a bounded subset of X. S being completely continuous implies $S(T(B))$ is precompact. Therefore, ST is a completely continuous operator.

Consider now $TS(B) = T(S(B))$. By the complete continuity of S, $S(B)$ is precompact, i.e. $\overline{S(B)}$ is compact. Since the continuous image of a compact set is a compact set, and T is continuous, it follows that $T(\overline{S(B)})$ is compact. Note that $T(S(B)) \subseteq T(\overline{S(B)})$. This implies $T(S(B))$ is precompact. The complete continuity of TS has now been proved. □

Remark Combining this result with Lemma 4.5.8 and Theorem 4.5.9, we see that $\mathcal{B}_0(X)$, the class of all completely continuous operators on X is what is known as a 'closed two-sided ideal' in the algebra $\mathcal{B}(X)$ of all bounded linear operators on X. In particular, the square of a compact operator is compact. The converse, however, is not true, as the following example shows.

In ℓ^2, let $\{e_k\}$ be the standard basis, i.e. $e_k = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$, where the only nonzero entry is 1 and occurs in the kth position. Define the operator T in ℓ^2 by setting $Te_k = (1 - (-1)^k)e_{2k}$. Certainly, T maps into ℓ^2 because convergence of $\Sigma|x_k|^2$ implies that of $\Sigma(1-(-1)^k)^2|x_k|^2$. Clearly, $T^2 = O$. But T is not compact, because it maps the bounded sequence $e_1, e_3, e_5, \ldots$ into the sequence $2e_2, 2e_6, 2e_{10}, \ldots$, which can have no convergent subsequence in view of the fact that $||2e_m - 2e_n|| = 2\sqrt{2}$ when $n \neq m$.

The simple unilateral shift is not compact because it maps the bounded sequence $e_1, e_2, e_3, \ldots$ into the sequence $e_2, e_3, e_4, \ldots$.

It is easy to see that $\mathcal{B}_{00}(X)$, the class of all finite rank operators on X, is a two-sided ideal in $\mathcal{B}(X)$. Suppose $T \in \mathcal{B}_{00}(X)$ and $S \in \mathcal{B}(X)$. Since the range of the product TS is contained in that of T, it is surely finite-dimensional. To see why ST also has finite-dimensional range, we first note that $T(X)$ is finite-dimensional, and therefore, any linear image of it is also finite-dimensional. It follows that the linear image $S(T(X)) = ST(X)$ is finite-dimensional.

We shall use this to show that if X is a Hilbert space H, then the adjoint T^* of a finite rank operator $T \in \mathcal{B}_{00}(H)$ and its absolute value $|T|$ are also of finite rank. Let P be the orthogonal projection on the range of T. Then, $PT = T$ and $P^* = P$. Therefore, $T^* = T^*P^* = T^*P$, which must have finite rank, because P does. Also, T^*T must have finite rank, which means $\mathrm{ran}(T^*T)$ is finite-dimensional and hence closed. It follows by the last part of Theorem 3.9.13 that $|T|$ has finite rank.

It turns out that the closure of $\mathcal{B}_{00}(H)$ is precisely $\mathcal{B}_0(H)$; so, there is no question of $\mathcal{B}_{00}(H)$ being closed unless it equals $\mathcal{B}_0(H)$, which we know it does not when $H = L^2([0, 1]$.

We recall the following definition for the benefit of the reader.

Let H be a Hilbert space, $\{x_n\}_{n\geq 1}$ a sequence of elements of H and $x \in H$. If, for all elements $y \in H$, the sequence (x_n, y) of scalars converges to (x, y) as $n \to \infty$, then $\{x_n\}_{n\geq 1}$ is said to converge weakly to x and we write:

$$x_n \xrightarrow{w} x.$$

Also, x is called the weak limit of the sequence.

The following equivalent criterion of a compact operator in a Hilbert space holds.

Theorem 4.5.10 *A bounded linear operator in a Hilbert space is compact if and only if it maps every weakly convergent sequence into a strongly convergent sequence.*

Proof Suppose that $T \in \mathcal{B}(H)$ is a compact operator and let $\{x_n\}_{n \geq 1}$ be a sequence in H such that $x_n \xrightarrow{w} x$. If possible, suppose that $\{Tx_n\}_{n \geq 1}$ does not converge strongly to Tx. Then, there exists $\varepsilon > 0$ and an increasing sequence n_1, n_2, … such that $\|Tx_{n_k} - Tx\| \geq \varepsilon$, $k = 1, 2, \ldots$. As the sequence $\{x_n\}_{n \geq 1}$ converges weakly, it follows [Theorem 2.12.6] that $\|x_n\| \leq M$, $n = 1, 2, \ldots$, for some suitable $M > 0$, and hence, by compactness of T, the sequence $\{Tx_{n_k}\}_{k \geq 1}$ has a subsequence $\{Tx_{n_{k_j}}\}_{j \geq 1}$ such that $Tx_{n_{k_j}} \to y$ as $j \to \infty$ strongly. Since strong convergence implies weak convergence, $Tx_{n_{k_j}} \xrightarrow{w} y$ as $j \to \infty$. Also, $x_{n_{k_j}} \xrightarrow{w} x$ as $j \to \infty$; so $Tx_{n_{k_j}} \xrightarrow{w} Tx$ as $j \to \infty$. Thus, $y = Tx$ and $Tx_{n_{k_j}} \to Tx$ as $j \to \infty$ strongly. Moreover,

$$\left\| Tx_{n_{k_j}} - Tx \right\| \geq \varepsilon, \quad j = 1, 2, \ldots.$$

This contradiction completes the argument.

Conversely, suppose that $\{x_n\}_{n \geq 1}$ is a bounded sequence in H. Then, it contains a weakly convergent subsequence $\{x_{n_k}\}_{k \geq 1}$ [Theorem 2.12.5]. By hypothesis, $\{Tx_{n_k}\}_{k \geq 1}$ converges in H; consequently, T is compact. □

As an illustration of the use of the above theorem, we show that if $\{e_n\}_{n \geq 1}$ is an orthonormal sequence, not necessarily complete, in a Hilbert space H and T compact operator, then $||Te_n|| \to 0$. First consider an arbitrary subsequence, which we shall continue to call $\{e_n\}_{n \geq 1}$ for ease of notation. For any $x \in H$, the sum $\sum_{n=1}^{\infty} |(x, e_n)|^2$ must converge by Bessel's inequality [Theorem 2.8.6]. So, $(x, e_n) \to 0$ as $n \to \infty$, i.e. the sequence $\{e_n\}_{n \geq 1}$ converges to 0 weakly. Using the fact that T is continuous, we find that $Te_n \to 0$ weakly. As T is also compact, it follows by Theorem 4.5.10 that $\{Te_n\}_{n \geq 1}$ converges strongly to some $y \in H$. Since strong convergence implies weak convergence, we know that $\{Te_n\}_{n \geq 1}$ converges weakly to y, and hence, $y = 0$. Thus, $\{Te_n\}_{n \geq 1}$ converges strongly to 0.

The next result can be rephrased as saying that the class of compact operators in a Hilbert space is closed under taking adjoints.

Theorem 4.5.11 *The adjoint of a compact operator is compact.*

Proof Let $\{x_n\}_{n \geq 1}$ be a sequence in H such that $||x_n|| \leq M, n = 1, 2, \ldots$ and $M > 0$. If $y_n = T^*x_n, n = 1, 2, \ldots$, then $\{y_n\}_{n \geq 1}$ is also a bounded sequence in H. Since T is compact, the sequence $\{Ty_n\}_{n \geq 1}$ has a convergent subsequence $\{Ty_{n_j}\}_{j \geq 1}$, say. For all i, j,

$$\begin{aligned}\left\|y_{n_i}-y_{n_j}\right\|^2 &= \left\|T^*x_{n_i}-T^*x_{n_j}\right\|^2\\ &= \left(T^*x_{n_i}-T^*x_{n_j}, T^*x_{n_i}-T^*x_{n_j}\right)\\ &= \left(TT^*\left(x_{n_i}-x_{n_j}\right), \left(x_{n_i}-x_{n_j}\right)\right)\\ &\le \left\|TT^*\left(x_{n_i}-x_{n_j}\right)\right\|\left\|x_{n_i}-x_{n_j}\right\|\\ &\le 2M\left\|Ty_{n_i}-Ty_{n_j}\right\|.\end{aligned}$$

This implies that the sequence $\{y_{n_j}\}_{j\geq 1}$ is a Cauchy sequence in H, and since H is complete, it converges in H. Consequently, T^* is a compact operator. □

We shall show that when T is compact, the operator $I - T$ has the following feature of operators in a finite-dimensional space: it is onto if and only if it is one-to-one:

Theorem 4.5.12 *Let T be a compact operator. Then,*

$$\operatorname{ran}(I-T) = H \Leftrightarrow \ker(I-T) = \{0\}.$$

Proof Suppose ran$(I - T) = H$ but ker$(I - T) \neq \{0\}$. Then, there exists a nonzero vector $x_1 \in \ker(I - T)$. Since ran$(I - T) = H$, we can obtain a sequence $\{x_n\}_{n\geq 1}$ of nonzero vectors in H such that

$$(I-T)x_{n+1} = x_n \quad \text{for every } n.$$

Then, $(I-T)^n x_{n+1} = x_1 \neq 0$ but $(I-T)^{n+1}x_{n+1} = (I-T)x_1 = 0$. To put it another way,

$$x_{n+1} \notin \ker(I-T)^n \quad \text{but} \quad x_{n+1} \in \ker(I-T)^{n+1}.$$

Combined with the obvious inclusions $\ker(I-T)^n \subseteq \ker(I-T)^{n+1}$ for every n, this yields the strict inclusions

$$\ker(I-T)^n \subset \ker(I-T)^{n+1} \quad \text{for every } n.$$

Each of these kernels is closed, and therefore, each $\ker(I-T)^n$ is a proper closed subspace of the Hilbert space $\ker(I-T)^{n+1}$. Hence, there exists a sequence $\{y_n\}_{n\geq 1}$ of unit vectors such that

$$y_n \in \ker(I-T)^n \quad \text{and} \quad y_{n+1} \perp \ker(I-T)^n \;\; \text{for every } n.$$

Then, surely,

$$\|y_{n+1} - x\| \geq 1 \quad \text{for any } x \in \ker(I-T)^n.$$

For indices $p > q$, we have

$$y_q + (I - T)y_p - (I - T)y_q \in \ker(I - T)^{p-1},$$

because

$$\begin{aligned}(I - T)^{p-1}(y_q + (I - T)y_p - (I - T)y_q) &= (I - T)^{p-1}y_q + (I - T)^p y_p - (I - T)^p y_q \\ &= (I - T)^{p-1}y_q + 0 + 0 \\ &= (I - T)^{p-1-q}(I - T)^q y_q = 0.\end{aligned}$$

Therefore, $||y_p - (y_q + (I - T)y_p - (I - T)y_q)|| \geq 1$, i.e. $||Ty_p - Ty_q|| \geq 1$ when $p > q$. But this means that although the set $\{y_n : n \geq 1\}$ is bounded, the set $\{Ty_n : n \geq 1\}$ cannot contain a Cauchy sequence. This contradicts the compactness of T and thereby shows that ran$(I - T) = H \Rightarrow \ker(I - T) = \{0\}$.

For the converse, suppose that $\ker(I - T) = \{0\}$. By Theorem 3.5.8, the orthogonal complement of $\ker(I - T)$ is the closure of the range of $I - T^*$. Therefore, ran$(I - T^*)$ is dense. However, by Theorem 4.5.11, T^* is also compact, and hence, by Problem 4.5.P14, ran$(I - T^*)$ is closed. Thus, ran$(I - T^*) = H$. By the compactness of T^*, what has already been proved above implies that $\ker(I - T^*) = \{0\}$. Invoking Theorem 3.5.8 once again in exactly the same manner as above, we find that ran$(I - T) = H$. ☐

In the presence of the additional hypothesis that T is self-adjoint, the above result is a trivial consequence of Theorem 3.5.8 and Problem 4.5.P14. However, much more can be said in that situation [see (Problem 4.5.P15)].

An *alternative* between two specified statements is as assertion to the effect that precisely one among two statements holds, i.e. one of them holds but not both. A little reflection shows that this is the same as saying that one holds if and only if the other does not. For the sceptical reader, we show the corresponding simple computation in Boolean algebra, wherein $\wedge$ denotes conjunction, $\vee$ denotes disjunction and $'$ denotes negation. Recall that $P \Rightarrow Q$ is the same as $P' \vee Q$. The computation is as follows:

$$(P \vee Q) \wedge (P \wedge Q)' = (P \vee Q) \wedge (P' \vee Q') = (P' \Rightarrow Q) \wedge (Q \Rightarrow P') = P' \Leftrightarrow Q.$$

Thus, the equivalence of any two statements can be restated as an alternative between one of the statements and the negation of the other. Conventionally, the equivalence asserted by Theorem 4.5.12 is expressed as an alternative and named after the discoverer, who originally put it forth in 1903 in the context of integral equations:

Theorem 4.5.13 (Fredholm Alternative) *For a compact operator T in a Hilbert space H, precisely one of the following holds*:

(a) *For every $y \in H$, there exists $x \in H$ such that $x - Tx = y$*;
(b) *There exists a nonzero $x \in H$ such that $x - Tx = 0$.*

Proof Immediate from Theorem 4.5.12. □

Theorem 4.5.14 *For a compact operator T in a Hilbert space H, the dimensions of* $\ker(I - T)$ *and* $\ker(I - T^*)$ *are the same.*

Proof Both the dimensions in question are finite because of the compactness of T and hence of T^* [Prop. 4.8.3]. Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ be orthonormal bases of $\ker(I - T)$ and $\ker(I - T^*)$, respectively. It is sufficient to prove that assuming $m > n$ leads to a contradiction. With this in view, assume that $m > n$.

Set up the operator S defined as

$$Sx = Tx + \sum_{j=1}^{n} (x, x_j)y_j, \quad x \in H.$$

Obviously, S is compact. We contend that $\ker(I - S) = \{0\}$. Considering that $y_1, \ldots, y_n$ are orthonormal and lie in $\ker(I - T^*)$, we obtain for any $x \in H$

$$\begin{aligned}((I - S)x, y_k) &= ((I - T)x, y_k) - (x, x_k)\\ &= (x, (I - T^*)y_k) - (x, x_k)\\ &= -(x, x_k), \quad 1 \le k \le n.\end{aligned}$$

Now, let $x \in \ker(I - S)$. Then, the above n equalities lead to the n orthogonality relations

$$(x, x_k) = 0, \quad 1 \le k \le n.$$

These imply on the one hand that

$$x \in \ker(I - T)^{\perp},$$

because $\{x_1, \ldots, x_n\}$ is an orthonormal basis of $\ker(I - T)$. On the other hand, since it follows from the definition of S that $(I - T)x = \sum_{j=1}^{n} (x, x_j)y_j$, the same n orthogonality relations imply this time around that

$$x \in \ker(I - T).$$

But $\ker\,(I - T)^{\perp} \cap \ker(I - T) = \{0\}$, and it follows that $x = 0$. This validates our contention that $\ker(I - S) = \{0\}$.

As S is compact, Theorem 4.5.12 now tells us that $\mathrm{ran}(I - S) = H$. In particular, $y_{n+1} = (I - S)z$ for some $z \in H$. Recalling the definition of S, we obtain

$$(I - T)z = \sum_{n=1}^{n} (z, x_j)y_j + y_{n+1}.$$

Considering that $y_1,\ldots,y_{n+1}$ are orthonormal and $y_{n+1} \in \ker(I - T^*)$, we now arrive at the contradiction that

$$\begin{aligned}
0 &= (z, (I - T^*)y_{n+1}) \\
&= ((I - T)z, y_{n+1}) \\
&= \left(\sum_{j=1}^{n} (z, x_j) y_j + y_{n+1}, y_{n+1} \right) \\
&= (y_{n+1}, y_{n+1}) \\
&= 1.
\end{aligned}$$

□

Since integral operators are compact, the Fredholm alternative and Theorem 4.5.14 have direct implications regarding solutions of integral equations; in fact, they are generalisations of Fredholm's results on the latter. For an explicit formulation in terms of integral equations, the reader is referred to Limaye [21, p. 339] or Riesz and Nagy [24, p. 164].

Theorems 4.5.12, 4.5.13 and 4.5.14 can be further generalised even to Banach spaces, but the matter will not be taken up in this book.

Problem Set 4.5

4.5.P1. Show that the operator $K{:}L^2[a, b] \to L^2[a, b]$ defined by

$$(K\varphi)(t) = \int_a^t \varphi(s)\mathrm{d}s, \quad \varphi \in L^2[a, b]$$

does not have finite rank.

4.5.P2. Show that the operator $K : L^2[a, b] \to L^2[a, b]$ defined by

$$(Kf)(t) = \sum_{j=1}^{n} \varphi_j(t) \int_0^t \psi_j(s)\mathrm{d}s,$$

where $f = \sum_{j=1}^{n} \varphi_j \otimes \psi_j$ and φ_j, ψ_j are in $L^2[a, b]$, is of finite rank.

4.5.P3. (a) Let the operator $K : L^2[0, 1] \to L^2[0, 1]$ be given by

$$Kx(t) = \int_0^1 k(t, s)x(s)\mathrm{d}\mu(S),$$

where $k(t, s) = \max\{t, s\}$, $0 \leq t, s \leq 1$. Prove that K is self-adjoint, is compact and has denumerably many negative eigenvalues with 0 as the only accumulation point.

(b) Suppose in part (a), the function k is changed to be $k(t, s) = \min\{t, s\}$. Prove that K is self-adjoint, is compact and has positive eigenvalues. (The reader can check that they are denumerably many and have 0 as the only accumulation point.)

4.5.P4. Take V as in Problem 3.8.P4, and find the eigenvalues of the operator V^*V on $L^2[0, 1]$. Prove that $||V|| = 2\pi^{-1}$.

4.5.P5. Let V be the Volterra operator on $L^2[0, 1]$ [see Example (ix) of 3.2.5]. Prove by induction that

$$(V^n(x)(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} x(s)\mathrm{d}s.$$

Hence, solve the integral equation

$$y(t) = \sin t + \int_0^t y(s)\mathrm{d}s. \tag{4.5}$$

4.5.P6. (a) Let H and K be Hilbert spaces and $T \in \mathcal{B}_0(H, K)$. Show that ran($T$) is separable.

(b) Let $\{e_k\}_{k\geq 1}$ be an orthonormal basis for $\overline{[\mathrm{ran}(T)]}$. If P_n:$K \to K$ is the orthogonal projection onto the closed linear subspace generated by $\{e_k\}_{1\leq k\leq n}$, then show that $P_nT \to PT$ (unif), where P is the orthogonal projection on $\overline{[\mathrm{ran}(T)]}$.

4.5.P7. Let H be a separable Hilbert space with basis $\{e_n\}_{n\geq 1}$. Let $\{\alpha_n\}_{n\geq 1}$ be a sequence of complex numbers with $M = \sup|\alpha_n| < \infty$. Define an operator T on ℓ^2 by

$$Tx = (\alpha_1 x_1, \alpha_2 x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^2.$$

Prove that T is compact if and only if $\lim_n \alpha_n = 0$.

4.5.P8. Let $\{a_j\}_{j\geq 1}$ be a sequence of complex numbers with $\sum_{j=1}^{\infty} |a_j| < \infty$. Define an operator T on ℓ^2 by

$$Tx = \left(\sum_{i=1}^{\infty} a_i x_i, \sum_{i=1}^{\infty} a_{i+1} x_i, \ldots, \sum_{i=1}^{\infty} a_{i+n-1} x_i, \ldots\right),$$
$$x = (x_1, x_2, \ldots) \in \ell^2.$$

Show that T is compact.

4.5.P9. Let $[\tau_{ij}]_{i,j\geq 1}$ be an infinite matrix and $\sum_{i,j=1}^{\infty}|\tau_{ij}|^2 < \infty$ and operator T be defined on ℓ^2 by

$$T(\{x_i\}_{i\geq 1}) = \{y_i\}_{i\geq 1},$$

where

$$y_i = \sum_{j=1}^{\infty} \tau_{ij}x_j, \quad i = 1, 2, \ldots.$$

Show that T is a compact linear operator on ℓ^2.

4.5.P10. Define $T : \ell^2 \to \ell^2$ by $Tx = T(x_1, x_2, \ldots) = (\sum_{j=1}^{\infty} \tau_{1j}x_j, \sum_{j=1}^{\infty} \tau_{2j}x_j, \ldots)$, where $\tau_{ik} = 0$ for $|i - k| > 1$. Then, T is compact if and only if $\lim_{i,k}\tau_{i,k} = 0$. Observe that the matrix defining T has the form

$$\begin{bmatrix} \tau_{11} & \tau_{12} & 0 & 0 & \cdots \\ \tau_{12} & \tau_{22} & \tau_{23} & 0 & \cdots \\ 0 & \tau_{32} & \tau_{33} & \tau_{34} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}.$$

This matrix may be expressed as

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & \cdots \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}.$$

Such a matrix is called a Jacobi matrix. The condition $\lim_{i,k}\tau_{i,k} = 0$ is equivalent to $\lim_k\alpha_k = 0 = \lim_k\beta_k = \lim_k\gamma_k$.

4.5.P11. Prove that the mapping T defined on ℓ^2 by

$$Tx = (\xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \ldots), \quad x = (\xi_1, \xi_2, \xi_3, \ldots) \in \ell^2$$

has range contained in ℓ^2 and is compact.

4.5.P12. Let T be a compact operator on X, i.e. $T \in \mathcal{B}_0(X)$, and suppose $\lambda \neq 0$ is not an eigenvalue of T. Show that $\lambda \notin \sigma(T)$.

4.5.P13. Construct an example of a compact operator which has no proper value.

4.5.P14. Let $T \in \mathcal{B}(H)$, H a complex Hilbert space, be compact and $\lambda \neq 0$ a complex number. Then, ran$(T - \lambda I)$ is closed.

4.5.P15. **(Fredholm Alternative)** Let $T \in \mathcal{B}(H)$, H a complex Hilbert space, be compact and self-adjoint. If λ is an eigenvalue of T, we denote by $\mathfrak{N}_\lambda(T)$ the eigenspace of T associated with λ and by P_λ the orthogonal projection of H onto $\mathfrak{N}_\lambda(T)$. Then, one of the following holds:

(a) If λ is not an eigenvalue of T, then the equation

$$Tx - \lambda x = y \tag{4.6}$$

with $y \in H$ has a unique solution. The unique solution x is given by

$$x = (T - \lambda I)^{-1} y = \sum_{\mu \in \sigma_p(T)} (\mu - \lambda)^{-1} P_\mu y.$$

(b) If λ is an eigenvalue of T, then the Eq. (4.6) has infinitely many solutions for $y \in \mathfrak{N}_\lambda(T)^\perp$ and no solution otherwise. In the first case, the solutions are given by

$$x = z + \sum_{\substack{\mu \in \sigma_p(T) \\ \mu \neq \lambda}} (\mu - \lambda)^{-1} P_\mu y.$$

with $z \in \mathfrak{N}_T(\lambda)$.

4.5.P16. Let $H = L^2[0, 1]$. For $x \in H$, let

$$Tx(s) = \int_0^1 k(s,t)x(t)\mathrm{d}t,$$

and

$$k(s,t) = \begin{cases} (1-s)t & 0 \leq t \leq s \leq 1 \\ s(1-t) & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $x \in H$ and $0 \neq \lambda \in \mathbb{C}$ be such that $Tx = \lambda x$. Then, for all $s \in [0, 1]$,

$$\lambda x(s) = Tx(s) = \int_0^s (1-s)tx(t)\mathrm{d}t + \int_s^1 s(1-t)x(t)\mathrm{d}t. \tag{4.7}$$

Show that $Tx(s) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[\int_0^1 x(t) \sin n\pi t \, \mathrm{d}t\right] \sin n\pi s$. Use the Fredholm alternative to determine the solution of the operator equation $Tx - \lambda x = y, y \in H$.

4.5.P17. Let $\{a_j\}_{j\geq 1}$ be a sequence of complex numbers such that $\sum_{j=1}^{\infty} |a_j| < \infty$. Define an operator on ℓ^2 by the matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Prove that A is compact.

4.6 Hilbert–Schmidt Operators

Problem 3.2.P1 and Example (viii) of 3.2.5 provide sufficient conditions on infinite matrices and kernels to induce bounded linear operators on a Hilbert space. In fact, Example (viii) of 3.2.5 is a continuous analogue of Problem 3.2.P3. These are typical illustrations of a class of operators—the Hilbert–Schmidt operators. We shall show that if T is a Hilbert–Schmidt operator in a Hilbert space H, then so its adjoint T^*. These operators constitute a two-sided ideal in $\mathcal{B}(H)$, the algebra of bounded linear operators in H. Every Hilbert–Schmidt operator is a compact operator. The converse is, however, not true. The class of Hilbert–Schmidt operators is defined as follows.

Definition 4.6.1 Let T $\in \mathcal{B}(H)$ be an operator on a Hilbert space H, and let $\{x_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis for H. If $\Sigma_{\gamma\in\Gamma}||Tx_\gamma||^2 < \infty$, then T is called a **Hilbert–Schmidt operator**.

The set of all Hilbert–Schmidt operators on H will be denoted by HS.

In this definition of the class HS, a particular orthonormal basis was used. The following lemma shows that the class HS depends only upon the Hilbert space and not upon the basis.

Lemma 4.6.2 *Let $T \in \mathcal{B}(H)$ be an operator on a Hilbert space H. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ and $\{y_\gamma\}_{\gamma\in\Gamma}$ be orthonormal bases for H. Then,*

$$\Sigma_{\gamma\in\Gamma}||Tx_\gamma||^2 = \Sigma_{\beta\in\Gamma}||T^*y_\beta||^2 = \Sigma_{\alpha\in\Gamma}\Sigma_{\beta\in\Gamma}|(Tx_\alpha, y_\beta)|^2.$$

Whenever any one of them is summable, so are the others and their sum is the same, independent of $\{x_\gamma\}_{\gamma\in\Gamma}$ *and* $\{y_\gamma\}_{\gamma\in\Gamma}$.

Proof By using Parseval's equality [Theorem 2.9.16], $||Tx_\alpha||^2 = \Sigma_{\beta\in\Gamma}|(Tx_\alpha, y_\beta)|^2$. Thus,

$$\begin{aligned}\Sigma_{\alpha\in\Gamma}||Tx_\alpha||^2 &= \Sigma_{\alpha\in\Gamma}\Sigma_{\beta\in\Gamma}|(Tx_\alpha, y_\beta)|^2\\ &= \Sigma_{\beta\in\Gamma}\Sigma_{\alpha\in\Gamma}|(Tx_\alpha, y_\beta)|^2\\ &= \Sigma_{\beta\in\Gamma}\Sigma_{\alpha\in\Gamma}|(x_\alpha, T^*y_\beta)|^2\\ &= \Sigma_{\beta\in\Gamma}\left\|T^*y_\beta\right\|^2,\end{aligned}$$

using Parseval's equality again. Thus, if either of the sums is finite, so are the others. Moreover, the sum is independent of the basis used. □

Remarks 4.6.3

(i) In a sum of uncountably many nonnegative terms, only countably many terms are nonzero [Proposition 2.9.8]. It is therefore legitimate to interchange the order of summation in the above argument.

(ii) The quantity $(\Sigma_{\alpha\in\Gamma}||Tx_\alpha||^2)^{\frac{1}{2}}$ is called the **Hilbert–Schmidt norm** of $T \in \mathcal{B}(H)$ and is denoted by $\|T\|_{\mathrm{HS}}$:

$$||T||_{\mathrm{HS}} = \left(\Sigma_{\gamma\in\Gamma}||Tx_\gamma||^2\right)^{\frac{1}{2}}.$$

Since the equality $\|TX\|^2 = (Tx, Tx) = (T^*Tx, x) = (|T|^2x, x) = (|T|x, |T|x) = \||T|x\|^2$ holds for all $x \in H$, it follows that $T \in HS$ if and only if $|T| \in HS$ and that $||T||_{\mathrm{HS}} = \||T|\|_{\mathrm{HS}}$. It also follows that $||T|| = \||T|\|$.

(iii) $||T|| \le \|T\|_{\mathrm{HS}}$. The case $||T||_{\mathrm{HS}} = \infty$ is obvious. To prove our assertion when $||T||_{\mathrm{HS}} < \infty$, it is sufficient to show that $||Tx|| \le \|T\|_{\mathrm{HS}}$ for all x with $\|x\| = 1$. That is easy. Choose a basis $\{x_\gamma\}_{\gamma\in\Gamma}$ with x as one of its elements. Then,

$$||Tx|| \le \left(\Sigma_{\gamma\in\Gamma}||Tx_\gamma||^2\right)^{\frac{1}{2}} = ||T||_{\mathrm{HS}}.$$

(iv) $T \in HS$ if and only if $T^* \in HS$. Moreover, $||T||_{\text{HS}} = ||T^*||_{\text{HS}}$. If $\{x_\gamma\}_{\gamma\in\Gamma}$ is a complete orthonormal basis in H, then it follows from Lemma 4.6.2 that

$$\Sigma_{\gamma\in\Gamma}\left\|Tx_\gamma\right\|^2 = \Sigma_{\gamma\in\Gamma}\left\|T^*x_\gamma\right\|^2 \tag{4.8}$$

by taking the two orthonormal systems therein to be the same. If $\Sigma_{\gamma\in\Gamma}\|Tx_\gamma\|^2 < \infty$, then $\Sigma_{\gamma\in\Gamma}\|T^*x_\gamma\|^2 < \infty$ and is independent of the choice of complete orthonormal basis $\{x_\gamma\}_{\gamma\in\Gamma}$. So, $T \in HS$ implies $T^* \in HS$. On the other hand, if $T^* \in HS$, then on replacing T by T^* in (4.8), we have $T = T^{**} \in HS$ and $\Sigma_{\gamma\in\Gamma}\|Tx_\gamma\|^2 < \infty$ for every complete orthonormal basis $\{x_\gamma\}_{\gamma\in\Gamma}$ in H and the sum is independent of the choice of the complete orthonormal basis. That $||T||_{\text{HS}} = ||T^*||_{\text{HS}}$ is obvious from (4.8).

(v) If S and T are Hilbert–Schmidt operators, then so is their sum and

$$||S+T||_{\text{HS}} \le ||S||_{\text{HS}} + ||T||_{\text{HS}}.$$

Indeed,

$$\begin{aligned} ||S+T||_{\text{HS}} &= \left(\Sigma_{\gamma\in\Gamma}\left\|(S+T)x_\gamma\right\|^2\right)^{\frac{1}{2}} \\ &\le \left(\Sigma_{\gamma\in\Gamma}\left(\left\|Sx_\gamma\right\| + \left\|Tx_\gamma\right\|\right)^2\right)^{\frac{1}{2}} \\ &\le \left(\Sigma_{\gamma\in\Gamma}\left(\left\|Sx_\gamma\right\|\right)^2\right)^{\frac{1}{2}} + \left(\Sigma_{\gamma\in\Gamma}\left(\left\|Tx_\gamma\right\|\right)^2\right)^{\frac{1}{2}} \\ &= ||S||_{\text{HS}} + ||T||_{\text{HS}}. \end{aligned}$$

The other properties of a norm are obvious.

(vi) If $S \in HS$ and $T \in \mathcal{B}(H)$, then $ST \in HS$ and $TS \in HS$. Moreover, both $||ST||_{\text{HS}}$ and $||TS||_{\text{HS}}$ are less than or equal to $||T||\,||S||_{\text{HS}}$.
Since $||TSx_\gamma||^2 \le ||T||^2\|Sx_\gamma\|^2$, where $\{x_\gamma\}_{\gamma\in\Gamma}$ is a complete orthonormal basis in H, it follows that

$$\Sigma_{\gamma\in\Gamma}\left\|TSx_\gamma\right\|^2 \le \|T\|^2\Sigma_{\gamma\in\Gamma}\left\|Sx_\gamma\right\|^2$$

and this implies

$$||TS||_{\text{HS}} \le \|T\|\|S\|_{\text{HS}}.$$

Also,

$$\|ST\|_{\text{HS}} = \|(ST)^*\|_{\text{HS}} = \|T^*S^*\|_{\text{HS}} \le \|T^*\|\|S^*\|_{\text{HS}} = \|T\|\|S\|_{\text{HS}}.$$

(vii) If T is an operator of rank 1, then the range of T is generated by a single vector, z say. Then, $Tx = \lambda(x)z$, where λ is a linear functional on H and is therefore given by $\lambda(x) = (x, y)$ for some fixed vector y. Consequently, $Tx = (x, y)z$. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a complete orthonormal basis in H. Then,

$$\Sigma_{\gamma\in\Gamma}\left\|Tx_\gamma\right\|^2 = \Sigma_{\gamma\in\Gamma}\left\|\left(x_\gamma, y\right)z\right\|^2 = \Sigma_{\gamma\in\Gamma}\left|\left(x_\gamma, y\right)\right|^2\|z\|^2 = \|y\|^2\|z\|^2,$$

using Parseval's identity. Thus, any operator of rank 1 is Hilbert–Schmidt. It now follows from Remark 4.6.3(v) that any operator of finite rank is Hilbert–Schmidt.

We next show that Hilbert–Schmidt operators are compact operators. Indeed, they are limits of finite rank operators in the Hilbert–Schmidt norm $\|\cdot\|_{HS}$ and therefore also in the operator norm.

Theorem 4.6.4 *Every Hilbert–Schmidt operator is compact and is the limit of a sequence of operators with finite-dimensional range in the sense of the Hilbert–Schmidt norm as well as the operator norm.*

Proof Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a complete orthonormal basis in H, and let T be a Hilbert–Schmidt operator. Since

$$||T||_{\mathrm{HS}}^2 = \Sigma_{\gamma\in\Gamma}\left\|Tx_\gamma\right\|^2 < \infty,$$

only a countable number of the elements $\|Tx_\gamma\|^2$ are different from zero. Moreover, for every integer n, there is a finite subset $J_n \subseteq \Gamma$ such that

$$\Sigma_{\gamma\notin J_n}\left\|Tx_\gamma\right\|^2 < \frac{1}{n^2}.$$

For each n, let the linear operator T_n be defined by the formula $T_nx_\gamma = Tx_\gamma$ if $\gamma \in J_n$ and $T_nx_\gamma = 0$ if $\gamma \notin J_n$. Then, the range of T_n is finite-dimensional. Also,

$$||T - T_n||_{\mathrm{HS}}^2 = \Sigma_{\gamma\notin J_n}\left\|Tx_\gamma\right\|^2 < \frac{1}{n^2}$$

and so,

$$||T - T_n|| \le ||T - T_n||_{\mathrm{HS}} < \frac{1}{n}.$$

Hence, T is the limit of the sequence $\{T_n\}_{n\geq 1}$ in $\|\bullet\|_{\mathrm{HS}}$ and therefore also in the operator norm. It follows upon using Corollary 4.5.7 that T is compact. □

Problem 4.6.P3 proves that the space of Hilbert–Schmidt operators is complete with respect to the norm $\|\bullet\|_{\mathrm{HS}}$ and that it is actually a Hilbert space with respect to a suitable inner product.

The first of the following examples shows that not every compact operator is Hilbert–Schmidt.

Examples 4.6.5

(i) Let $\{x_n\}_{n\geq 1}$ be a complete orthonormal basis in a separable Hilbert space H, and let T be the operator defined by the equations

$$Tx_n = n^{-\frac{1}{2}}x_n, n = 1, 2, \ldots.$$

Since $\sum_{n=1}^{\infty}\|Tx_n\|^2 = \sum_{n=1}^{\infty}(n^{-\frac{1}{2}}x_n, n^{-\frac{1}{2}}x_n) = \sum_{n=1}^{\infty}n^{-1}$ and since $\lim_n n^{-1} = 0$, it follows on using Problem 4.5.P7 that T is a compact operator. It is, however, not Hilbert–Schmidt since $\sum_{n=1}^{\infty}n^{-1} = \infty$.

(ii) Let $\{x_n\}_{n\geq 1}$ be a complete orthonormal basis in a separable Hilbert space H, and let T be the operator defined by the equations

$$Tx_n = n^{-1}x_n, n = 1, 2, \ldots.$$

Since $\sum_{n=1}^{\infty}\|Tx_n\|^2 = \sum_{n=1}^{\infty}(n^{-1}x_n, n^{-1}x_n,) = \sum_{n=1}^{\infty}n^{-2} < \infty$, it is a Hilbert–Schmidt operator.

(iii) Let $H = \ell^2$ and $T \in \mathcal{B}(H)$ be defined by the matrix $[\tau_{i,j}]_{i,j\geq 1}$ with respect to the standard orthonormal basis $\{e_j\}_{j\geq 1}$. Then,

$$\sum_{j=1}^{\infty}\|Te_j\|^2 = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\|\tau_{i,j}e_i\|^2 = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}|\tau_{i,j}|^2.$$

Hence, T is a Hilbert–Schmidt operator if and only if $|\tau_{i,j}|^2 < \infty$. In that case, T is compact [see Problem 4.7.P1]. As we have seen in (ii) of Examples 3.5.10, the operator T^* is defined by $[\overline{\tau_{j,i}}]_{i,j\geq 1}$ and is therefore also a Hilbert–Schmidt operator. This is also a consequence of (iv) of Remarks 4.6.3.

(iv) For other examples of Hilbert–Schmidt operators, see Problems 4.7.P3 and 4.7.P4.

Problem Set 4.6

4.6.P1. Show that the Volterra integral operator

$$(Tf)(t) = \int_0^t f(s)\mathrm{d}s, \quad f \in L^2[0,1]$$

as a map on $L^2[0, 1]$ is a Hilbert–Schmidt operator and hence compact.

4.6.P2. Suppose $(X, \mathfrak{M}, \mu)$ is a measure space and $k \in L^2(\mu \times \mu)$. Let $K{:}L^2(\mu) \to L^2(\mu)$ be the integral operator with kernel k defined by

$$Kf(s) = \int k(s,t)f(t)d\mu(t), \quad f \in L^2(\mu).$$

Show that $K : L^2(\mu) \to L^2(\mu)$ is a Hilbert–Schmidt operator.

Let H be a Hilbert space and HS(H) denote the class of Hilbert–Schmidt operators. For S, $T \in$ HS(H) and $\{x_j\}_{j\in J}$ an orthonormal basis in H, define

$$(S,T) = \Sigma_j(Sx_j, Tx_j).$$

It is known that (S, T) is independent of the chosen basis. The following equalities are easy to verify:

(i) $\overline{(S,T)} = (T,S)$.

(ii) $(cS,T) = c(S,T), \quad c \in \mathbb{C}$.

(iii) $(S,cT) = \overline{c}(S,T), \quad c \in \mathbb{C}$.

(iv) $(S_1 + S_2, T) = (S_1, T) + (S_2, T), \qquad S_1, S_2 \in$ HS(H).

(v) $(S, T_1 + T_2) = (S, T_1) + (S, T_2), \qquad T_1, T_2 \in$ HS(H).

(vi) $(S, S) \geq 0$, $(S, S) = 0$ only for $S = O$.

(vii) $(S^*, T^*) = \overline{(S,T)}$.

(viii) $(XS,T) = (S,X^*T), \quad X \in \mathcal{B}(H)$.

(ix) $(SX,T) = (S,TX^*), \quad X \in \mathcal{B}(H)$.

The first six assertions say that (HS(H),(,)) is an inner product space. The following theorem shows that is a Hilbert space.

4.6.P3. The Hilbert–Schmidt class of operators is complete with respect to the metric $||S - T||_{\mathrm{HS}} = (\Sigma_j||(S-T)x_j||^2)^{\frac{1}{2}}$ induced by the inner product $(S,T) = \Sigma_j(Sx_j, Tx_j)$.

4.7 The Trace Class

Let H be a finite-dimensional complex Hilbert space and $\mathcal{B}(H)$ be the space of all linear transformations in H. If $\{x_1, x_2, \ldots, x_n\}$ is an orthonormal basis for H and $T \in \mathcal{B}(H)$, then T is completely determined by its values at $x_1, x_2, \ldots, x_n$. Thus, let $Tx_i = \sum_{j=1}^{n} a_{ij}x_j$. Then, T corresponds in a natural way to the matrix $[a_{ij}]$ [see 3.1], which we denote by $\tilde{T}$. The trace of T, tr(T), is defined by $\text{tr}(T) = \sum_{i=1}^{n} a_{ii}$, the sum of the diagonal elements of $\tilde{T} = [a_{ij}]$.

Specifically, tr(T) appears to have little to do with the eigenvalues of T. However, the characteristic polynomial $\det(\lambda I - \tilde{T})$ of T, where I denotes the identity matrix, is a polynomial of the form $\lambda^n - C_1\lambda^{n-1} + \cdots + (-1)^n C_n$, whose roots are the eigenvalues of T, counting multiplicities. Thus, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of T, counting multiplicities, we obtain $C_1 = \sum_{i=1}^{n} \lambda_i$. A remarkable result of linear algebra is the trace formula, which says that the sum of the eigenvalues of the square matrix $\widetilde{T} = [a_{ij}]$ equals the trace of T:

$$\text{tr}(T) = \sum_{i=1}^{n} \lambda_i. \tag{4.9}$$

These and other results are described in 4.7.P1 and 4.7.P2.

In 1959, Lidskii showed that the relation (4.9) is valid also for a large class of compact operators in a Hilbert space, namely ‘trace class’ operators. This is called Lidskii’s Theorem. The proof is somewhat intricate and is, therefore, not included. However, we shall prove the result for self-adjoint trace class operators [Problem 4.7.P8].

Let H be a complex Hilbert space (now infinite-dimensional) and $\mathcal{B}(H)$ denote the set of all bounded linear operators on H. Applying Lemma 4.6.2 to the square root of $|T|$ yields the following:

Proposition 4.7.1 *Let $T \in \mathcal{B}(H)$ be an operator on a Hilbert space H, and let $|T| = (T^*T)^{\frac{1}{2}}$. Let $\{x_j\}_{j\in J}$ be an orthonormal basis for H. The sum $\Sigma_{j\in J}(|T|x_j, x_j) = \Sigma_{j\in J}\||T|^{\frac{1}{2}}x_j\|^2$ is independent of the choice of the basis.*

Definition 4.7.2 An operator $T \in \mathcal{B}(H)$, where H is a Hilbert space, is **trace class** if there is a basis $\{x_j\}_{j\in J}$ such that $\Sigma_{j\in J}(|T|x_j, x_j)$ is finite. The set of trace class operators in H is denoted by **TC(*H*)** or simply by **TC** if the Hilbert space H is understood.

In view of Proposition 4.7.1, an operator $T{:}H \to H$ is trace class if and only if $\sum_{j\in J} (|T|x_j, x_j)$ is finite for every choice of orthonormal basis $\{x_j\}_{j\in J}$ and

$$||T||_{\text{tr}} = \sum_{j \in J} \left(|T|x_j, x_j\right)$$

is well defined, depending only on the operator T. The number $\|T\|_{\text{tr}}$ is called the **trace norm** of T. That it is actually a norm is proved in Problem 4.7.P4.

Since $T^*T = |T|^2 = |T|^*|T|$, we have $|(|T|)| = |T|$, and hence, T is trace class if and only if $|T|$ is; moreover, $\|T\|_{\text{tr}} = \||T|\|_{\text{tr}}$.

Remarks 4.7.3

(i) Observe that an operator T on H is a Hilbert–Schmidt operator if and only if $|T|^2$ is trace class, and moreover, $\|T\|_{\text{HS}}^2 = (\||T|^2\|_{\text{tr}})$. Indeed,

$$\begin{aligned} ||T||_{\text{HS}} &= \left(\sum_{j \in J} ||Tx_j||^2\right)^{\frac{1}{2}} = \left(\sum_{j \in J} \left(Tx_j, Tx_j\right)\right)^{\frac{1}{2}} \\ &= \left(\sum_{j \in J} \left(T^*Tx_j, x_j\right)\right)^{\frac{1}{2}} = \left(\sum_{j \in J} \left(|T|^2 x_j, x_j\right)\right)^{\frac{1}{2}} = \left(\||T|^2\|_{\text{tr}}\right)^{\frac{1}{2}}, \end{aligned}$$

where $\{x_j\}_{j \in J}$ is any orthonormal basis for H. We know from (ii) of Remarks 4.6.3 that $\|T\|_{\text{HS}} = \||T|\|_{\text{HS}}$. Therefore, $\||T|\|_{\text{HS}} = (\||T|^2\|_{\text{tr}})^{\frac{1}{2}}$.

(ii) T is trace class if and only if $|T|^{\frac{1}{2}}$ is Hilbert–Schmidt, and moreover, $|T|_{\text{tr}} = \||T|^{\frac{1}{2}}\|_{\text{HS}}^2$. Indeed,

$$||T||_{\text{tr}} = \sum_{j \in J} (|T|x_j, x_j) = \sum_{j \in J} (|T|^{\frac{1}{2}}x_j, |T|^{\frac{1}{2}}x_j) = \sum_{j \in J} \left\||T|^{\frac{1}{2}}x_j\right\|^2 = \left\||T|^{\frac{1}{2}}\right\|_{\text{HS}}^2.$$

It can also be seen as a consequence of (i) and the observation made right above it.

(iii) It follows from (ii) that $T \in \text{TC}$ implies $|T|^{\frac{1}{2}} \in HS$, which implies $|T| \in HS$ in view of (vi) of Remarks 4.6.3, which in turn implies $T \in HS$ by (ii) of Remarks 4.6.3. Thus, $\text{TC} \subseteq HS$. We shall see in Example 4.7.4 below that the inclusion is strict. On combining the inequality $||T||_{\text{HS}} \le ||T||_{\text{tr}}^{\frac{1}{2}}||T||^{\frac{1}{2}}$ proved in Problem 4.7.P5 with the inequality $\|T\| \le \|T\|_{\text{HS}}$ proved in (iii) of Remarks 4.6.3, we get $||T||_{\text{HS}}^{\frac{1}{2}} \le ||T||_{\text{tr}}^{\frac{1}{2}}$, and hence,

$$\|T\| \le ||T||_{\text{HS}} \le ||T||_{\text{tr}}.$$

Using this inequality with (ii), we obtain $||T||_{\text{tr}} \le \||T|^{\frac{1}{2}}\|_{\text{tr}}^2$. In particular, if $|T|^{\frac{1}{2}}$ is trace class, then T is also trace class. Moreover, we have

$$\|T\| \le \|T\|_{\mathrm{HS}} = \left(\left\||T|^2\right\|_{\mathrm{tr}}\right)^{\frac{1}{2}} \le \|T\|_{\mathrm{tr}} = \||T|\|_{\mathrm{tr}} \le \left\||T|^{\frac{1}{2}}\right\|_{\mathrm{tr}}^2.$$

The inequality $\|T\|_{\mathrm{HS}} \le \|T\|_{\mathrm{tr}}^{\frac{1}{2}}\|T\|^{\frac{1}{2}}$ in combination with (ii) yields $\||T|^2\|_{\mathrm{tr}} \le \|T\|_{\mathrm{tr}}\|T\|$, which bears a resemblance to the inequality $\|T^2\|_{\mathrm{HS}} \le \|T\|_{\mathrm{HS}}\|T\|$ based on (vi) of Remarks 4.6.3. If T is normal, then $|T^2| = |T|^2$ Hence,

$$\left\|T^2\right\|_{\mathrm{tr}} = \left\||T|^2\right\|_{\mathrm{tr}} \le \|T\|_{\mathrm{tr}}\|T\| \quad \text{for normal } T.$$

Let T be the projection on an n-dimensional subspace. For any orthonormal basis of the Hilbert space that extends an orthonormal basis $x_1, \ldots, x_n$ of the subspace, we have $(|T|^2x, x) = 1 = (T^*Tx, x)$ if x is one of the x_n and 0 otherwise. Therefore, $\|T\|_{\mathrm{HS}} = \sqrt{n}$ and $\|T\|_{\mathrm{tr}} = n$, while $\|T\| = 1$. Thus, the ratios $\|T\|_{\mathrm{tr}}/\|T\|_{\mathrm{HS}}$ and $\|T\|_{\mathrm{HS}}/\|T\|$ can be arbitrarily large in an infinite-dimensional Hilbert space.

Example 4.7.4 (cf. Problem 3.2.P6). Let $\{\alpha_n\}_{n\ge 1}$ be any bounded sequence of complex numbers. Define an operator T on ℓ^2 by

$$Tx = \sum_{n=1}^{\infty} \alpha_n (x, e_n) e_n, \quad \text{where} \quad x = \sum_{n=1}^{\infty} (x, e_n) e_n,$$

and $\{e_n\}_{n\ge 1}$ is the usual basis for ℓ^2. Then, as in Problem 4.6.P3,

$$\|T\|_{\mathrm{tr}} = \sum_{n=1}^{\infty} (|T|e_n, e_n) = \sum_{n=1}^{\infty} |\alpha_n| < \infty \text{ provided } \{\alpha_n\}_{n\ge 1} \in \ell^1,$$

and

$$\|T\|_{\mathrm{HS}} = \left[\Sigma_{n=1}^{\infty}\left(|T|^2 e_n, e_n\right)\right]^{\frac{1}{2}} = \left[\Sigma_{n=1}^{\infty}(Te_n, Te_n)\right]^{\frac{1}{2}} = \left(\Sigma_{n=1}^{\infty}|\alpha_n|^2\right)^{\frac{1}{2}} < \infty$$

provided $\{\alpha_n\}_{n\ge 1} \in \ell^2$.

Thus, a diagonal operator T on ℓ^2 defined by a sequence $\alpha = \{\alpha_n\}_{n\ge 1}$ is trace class [resp. Hilbert–Schmidt] provided $\alpha \in \ell^1$ [resp. $\alpha \in \ell^2$]. When $\alpha_n = \frac{1}{n}$, we have $T \in HS$ but $T \notin \mathrm{TC}$, which shows that $\mathrm{TC} \subset HS$. In particular, an inequality of the form $\|T\|_{\mathrm{tr}} \le k\|T\|_{\mathrm{HS}}$ with k a constant cannot hold at least if $T \notin \mathrm{TC}$. We shall demonstrate that it cannot hold even for operators in TC and in fact not even for operators of finite rank. Fix any positive integer m, and choose α so that $\alpha_n = 1$ for

$n \le m$ and $\alpha_n = 0$ for $n > m$. Then, T is of finite rank, $||T||_{\text{tr}} = m$ and $||T||_{\text{HS}} = \sqrt{m}$, and hence, $\|T\|_{\text{tr}}/\|T\|_{\text{HS}} = \sqrt{m}$.

The following proposition shows that every trace class operator T is the product of two Hilbert–Schmidt operators and so is $|T|$.

Proposition 4.7.5 *If* $T \in \mathcal{B}(H)$ *the following statements are equivalent*:

(a) $T \in \text{TC}$,
(b) $|T|^{\frac{1}{2}} \in HS$,
(c) *T is the product of two Hilbert–Schmidt operators,*
(d) *$|T|$ is the product of two Hilbert–Schmidt operators.*

Proof Let $T = W|T|$ and $|T| = W^*T$ be the polar decomposition of T, where W denotes a partial isometry [Proposition 3.9.9].

The equivalence of (a) and (b) has already been recorded in (ii) of Remarks 4.7.3.

(b) implies (c). Here, $T = (W|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}})$, and from (b) and (v) of Remarks 4.6.3, both of these are in HS.
(c) implies (d). If $T = BC$, where B and C are Hilbert–Schmidt operators $|T| = (W^*B)C$. By (v) of Remarks 4.6.3, W^*B is Hilbert–Schmidt.
(d) implies (a). Suppose $|T| = BC$, where B and C are Hilbert–Schmidt operators. For any orthonormal basis $\{x_j\}_{j\in J}$, $(|T|x_j, x_j) = (Cx_j, B^*x_j) \le \|Cx_j\|\|B^*x_j\|$. Hence,

$$\Sigma_j\left(|T|x_j, x_j\right) \le \Sigma_j\|Cx_j\|\|B^*x_j\| \le \left[\Sigma_j\|Cx_j\|^2\right]^{\frac{1}{2}}\left[\Sigma_j\|B^*x_j\|^2\right]^{\frac{1}{2}} = \|C\|_{\text{HS}}\|B\|_{\text{HS}}.$$

□

The next proposition leads to the definition of the trace of an operator $T \in \text{TC}$.

Proposition 4.7.6 *Let T be trace class and $\{x_j : j \in J\}$ be a given orthonormal basis for H. Then, the family $\{|(Tx_j, x_j)|\}$ of nonnegative numbers is summable. Consequently*, $\{(Tx_j, x_j)\}$ *is also summable*; *moreover, its sum is independent of the choice of basis.*

Proof Since C is in HS if and only if C^* is in HS, we may assume that $T = C^*B$ with both B and C in HS. Then,

$$\left(Tx_j, x_j\right) = \left(Bx_j, Cx_j\right).$$

Clearly,

$$\left|\left(Bx_j, Cx_j\right)\right| \le \frac{1}{2}\left(||Bx_j||^2 + ||Cx_j||^2\right)$$

and therefore,

$$\Sigma_j\left|\left(Bx_j, Cx_j\right)\right| \le \frac{1}{2}\left(\Sigma_j\left|\left|Bx_j\right|\right|^2 + \Sigma_j\left|\left|Cx_j\right|\right|^2\right) = \frac{1}{2}\left(||B||_{\mathrm{HS}}^2 + ||C||_{\mathrm{HS}}^2\right).$$

Thus, the family $\{|(Tx_j, x_j)|\}$ is summable. Since

$$\Re\left(Bx_j, Cx_j\right) = \frac{1}{4}(||B + C)x_j||^2 - ||(B - C)x_j||^2),$$

we also have

$$\Re\Sigma_j\left(Bx_j, Cx_j\right) = \frac{1}{4}\left(\|B + C\|_{\mathrm{HS}}^2 - \|(B - C\|_{\mathrm{HS}}^2\right).$$

The right-hand side of the above equality is clearly independent of $\{x_j : j \in J\}$. Replacing B by iB, we see that

$$\Im\Sigma_j\left(Bx_j, Cx_j\right) = -\Re\Sigma_j i\left(Bx_j, Cx_j\right)$$

is also independent of $\{x_j : j \in J\}$. Moreover,

$$\Sigma_j\left(Tx_j, x_j\right) = \Re\Sigma_j(Bx_j, Cx_j) + i\Im\Sigma_j\left(Bx_j, Cx_j\right)$$

The independence of $\Sigma_j(Tx_{j,}x_j)$ from the choice of basis is now immediate. □

We note that $\Sigma_j|(Tx_{j,}x_j)|$ need not be independent of the choice of basis. Consider the operator in ℓ^2 given in terms of the standard basis $\{e_j\}$ by $Te_1 = e_2$, $Te_2 = e_1$ and $Te_j = 0$ for $j > 2$. Another orthonormal basis $\{f_j\}$ is obtained by taking $f_1 = (e_1 + e_2)/\sqrt{2}, f_2 = (e_1 - e_2)/\sqrt{2}$ and $f_j = e_j$ for $j > 2$. A simple computation shows that $\Sigma_j|(Te_j, e_j)| = 0$ but $\Sigma_j|(Tf_j, f_j)| = 2$.

Another example (in ℓ^2) that is worth noting is $Te_1 = e_2, Te_2 = -e_1, Te_j = 0$ for $j > 2$. This operator has the property that $(Tx, x) = 0$ for all $x \in \ell^2$, so that $\Sigma_j|(Tx_j, x_j)| = 0$ for any basis $\{x_j\}$, although $T \neq O$.

In view of Proposition 4.7.6, it is possible to define the trace of an operator T in trace class.

Definition 4.7.7 If $T \in \mathcal{B}(H)$ is in trace class and $\{x_j : j \in J\}$ is an orthonormal basis, define

$$\mathbf{tr}(\boldsymbol{T}) = \Sigma_j\left(Tx_{j,}x_j\right)$$

The number tr(T) is called the **trace** of T.

Remarks 4.7.8

(i) By Proposition 4.7.6, the definition of tr(T) does not depend on the choice of basis.

(ii) If $T \in \mathcal{B}(H)$ then $||T||_{\text{tr}} = |||T|||_{\text{tr}} = \text{tr}(|T|)$ In case $T \geq O$, we have $T = |T|$, and hence, $||T||_{\text{tr}} = \text{tr}(T)$

(iii) If $\dim(H) < \infty$ then tr(T) is precisely the sum of the diagonal elements of the matrix representation of T; indeed, if $Tx_i = \Sigma_j a_{ij} x_j$, then $\Sigma_i (Tx_{i,} x_i) = \sum a_{ii}$, where $\{x_i\}_{1 \leq i \leq n}$ is a basis for H.

The next proposition enumerates the basic properties of trace class operators.

Proposition 4.7.9 *Let T be a trace class operator and X any bounded linear operator, both defined on a Hilbert space H. Then,*

(a) $T \in$ TC *if and only if* $T^* \in$ TC;

(b) $T \in$ TC *implies* $\alpha T \in$ TC *for any complex number* α;

(c) *XT and TX are in trace class for any* $X \in \mathcal{B}(H)$;

(d) *If S and T are in trace class, then so is $S + T$.*

Proof

(a) holds since $T = BC$ if and only if $T^* = C^*B^*$ and B is Hilbert–Schmidt if and only if B^* is Hilbert–Schmidt.

(b) Since $B \in HS$ implies $\alpha B \in HS$ and

$$\|\alpha B\|_{\text{HS}} = |\alpha| \|B\|_{\text{HS}},$$

it follows that if $T = BC$ with both B and C in HS, then $\alpha T = (\alpha B)C$ is trace class.

(c) Let $T = BC$ with B and C in HS. Then, CX and XB are also in HS [Remarks 4.6.3(vi)]. Consequently, TX and XT are trace class.

(d) We use the polar decomposition for $S + T$. Then,

$$|S + T| = W^*(S + T) = W^*S + W^*T.$$

By (c) above, $W^*S \in$ TC and so does W^*T, and thus, $\Sigma_j (W^*Sx_j, x_j)$ and $\Sigma_j (W^*Tx_j, x_j)$ are well defined. Since

$$\left(|S + T| x_j, x_j\right) = (W^*Sx_j, x_j) + \left(W^*Tx_j, x_j\right),$$

$$\Sigma_j \left(|S + T| x_j, x_j\right) = \Sigma_j (W^*Sx_j, x_j) + \Sigma_j \text{S}_j \left(W^*Tx_j, x_j\right) < \infty,$$

we have using Proposition 4.7.6. Consequently, $S + T$ is trace class. $\square$

Remark The trace class operators constitute an ideal in $\mathcal{B}(H)$.

Proposition 4.7.10 *Let H be a Hilbert space and $\{x_j : j \in J\}$ be an orthonormal basis for H. Let T be a trace class operator. Then,*

(a) $\mathrm{tr} : \mathrm{TC} \to \mathbb{C}$ *is a positive linear functional, i.e. if* $T \in \mathrm{TC}$, $T \geq O$ *and* $T \neq O$, *then* $\mathrm{tr}(T) > 0$;
(b) $\mathrm{tr}(T^*) = \overline{\mathrm{tr}(T)}$;
(c) $|\mathrm{tr}(T)| \leq \mathrm{tr}(|T|)$;
(d) $\mathrm{tr}(TX) = \mathrm{tr}(XT)$ *for any* $X \in \mathcal{B}(H)$;
(e) $|\mathrm{tr}(XT)| = |\mathrm{tr}(Tx)| \leq \|X\| \cdot \mathrm{tr}(|T|) = \|X\|\|T\|_{\mathrm{tr}}$ *for any* $X \in \mathcal{B}(H)$;
(f) $||T^*||_{\mathrm{tr}} = \mathrm{tr}(|T^*|) = \mathrm{tr}(|T|) = \|T\|_{\mathrm{tr}}$;
(g) $||XT||_{\mathrm{tr}} \leq \|X\|||T||_{\mathrm{tr}}$ *and* $\|TX\|_{\mathrm{tr}} \leq \|X\|\|T\|_{\mathrm{tr}}$ *whenever* $X \in \mathcal{B}(H)$;
(h) *Every finite rank operator is trace class*;
(i) *Every trace class operator is compact.*

Proof Let $\{x_j\}_{j \in J}$ be an orthonormal basis for H.

(a) It is clear that the trace is a linear functional on trace class operators on H and is positive. If $T > O$, then there exists $j \in J$ such that $(Tx_j, x_j) > 0$, and consequently, $\mathrm{tr}(T) = \Sigma_j(Tx_j, x_j) \geq (Tx_j, x_j) > 0$. On the other hand, if $\Sigma_j(Tx_j, x_j) = 0$ for $T \geq O$, then $0 = \Sigma_j(Tx_j, x_j) = \Sigma_j\||T|^{\frac{1}{2}}x_j\|^2$, which implies $|T|^{\frac{1}{2}}x_j = 0$ for each $j \in J$; so, $Tx_j = 0$ for each j. Hence, $T = O$.
(b) $\mathrm{tr}(T^*) = \Sigma_j(T^*x_j, x_j) = \Sigma_j(x_j, Tx_j) = \Sigma_j\overline{(Tx_j, x_j)} = \overline{\Sigma_j(Tx_j, x_j)} = \overline{\mathrm{tr}(T)}$
(c) $|\mathrm{tr}(T)| = |\Sigma_j(Tx_j, x_j)| \leq \Sigma_j|(Tx_j, x_j)| = \Sigma_j|(W|T|x_j, x_j)|$, where $T = W|T|$ is the polar decomposition of T; so,

$$\begin{aligned}
|\mathrm{tr}(T)| &= \Sigma_j\left|\left(|T|^{\frac{1}{2}}x_j, |T|^{\frac{1}{2}}W^*x_j\right)\right| \\
&\leq \left(\Sigma_j\left\||T|^{\frac{1}{2}}x_j\right\|^2\right)^{\frac{1}{2}}\left(\Sigma_j\left\||T|^{\frac{1}{2}}W^*x_j\right\|^2\right)^{\frac{1}{2}} \\
&\leq \left\||T|^{\frac{1}{2}}\right\|_{\mathrm{HS}}\left\||T|^{\frac{1}{2}}W^*\right\|_{\mathrm{HS}} \\
&\leq \left\||T|^{\frac{1}{2}}\right\|_{\mathrm{HS}}^2 = \||T|\|_{\mathrm{tr}}.
\end{aligned}$$

(d) If $T \in \mathrm{TC}$ and $X \in \mathcal{B}(H)$ then TX and XT are in trace class [Proposition 4.7.9 (c)]. We write $T = C^*B$ with B and C in HS. Then,

$$\begin{aligned}
\mathrm{tr}(Tx) = \mathrm{tr}(C^*BX) &= \Sigma_j\left(BXx_j, Cx_j\right) = \overline{\Sigma_j\left(X^*B^*x_j, C^*x_j\right)} = \overline{\Sigma_j\left(B^*x_j, XC^*x_j\right)} \\
&= \Sigma_j\left(Bx_j, CX^*x_j\right) = \mathrm{tr}(XT),
\end{aligned}$$

using (b) above and (vii) of the preamble to Problem 4.6.P3.

(e) Let $T = W|T|$ be the polar decomposition of T, where W is the partial isometry:

$$\operatorname{tr}(XT) = \operatorname{tr}(XW|T|) = \Sigma_j|(|T|^{\frac{1}{2}}x_j, |T|^{\frac{1}{2}}W^*X^*x_j).$$

Applying the Cauchy–Schwarz inequality to the right-hand side of the above equation, we get

$$\begin{aligned}
|\operatorname{tr}(XT)| &\leq \left(\Sigma_j\left\||T|^{\frac{1}{2}}x_j\right\|^2\right)^{\frac{1}{2}}\left(\Sigma_j\left\||T|^{\frac{1}{2}}W^*X^*x_j\right\|^2\right)^{\frac{1}{2}} \\
&= \left\||T|^{\frac{1}{2}}\right\|_{\text{HS}}\left\||T|^{\frac{1}{2}}W^*X^*\right\|_{\text{HS}} \\
&\leq \left\||T|^{\frac{1}{2}}\right\|_{\text{HS}}^2||W^*X^*|| \\
&\leq \left\||T|^{\frac{1}{2}}\right\|_{\text{HS}}^2||X|| \\
&\leq \|T\|_{\text{tr}}\|X\|.
\end{aligned}$$

(f) Let $T = W|T|$ be the polar decomposition of T, where W denotes the partial isometry. Then, $TT^* = W|T|^2W^*$, and so by the uniqueness of the square root [Theorem 3.6.16], $|T^*| = W|T|W^*$. Hence, $\|T^*\|_{\text{tr}} = \operatorname{tr}(|T^*|) = \operatorname{tr}(W|T|W^*)$. By part (e), this is just $\operatorname{tr}(W^*W|T|)$ and $W^*W|T| = |T|$ since W^*W is the projection of H onto the closure of $\operatorname{ran}(|T|)$.

(g) Let $T = W|T|$ be the polar decomposition of T and $XT = V|XT|$ be the polar decomposition of XT. By Proposition 3.9.9(a), $|XT| = V^*(XT) = V^*XW|T|$. It therefore follows from (i) of Remarks 4.7.8 and part (e) above that

$$||XT||_{\text{tr}} = \operatorname{tr}(|XT|) = \operatorname{tr}(V^*XW|T|) \leq \|V^*XW\|\||T|\|_{\text{tr}} \leq \|V^*\|\|X\|\|W\|\||T|\|_{\text{tr}}.$$

But $||V^*|| \leq 1$ and $||W|| \leq 1$, because both are partial isometries. Hence,

$$\|XT\|_{\text{tr}} \leq \|X\|\|T\|_{\text{tr}}.$$

By applying this to the trace class operator T^* [see Theorem 4.7.9(a)] and using part (f) twice, we get

$$\|TX\|_{\text{tr}} = \|X^*T^*\|_{\text{tr}} \leq \|X^*\|\|T^*\|_{\text{tr}} = \|X\|\|T\|_{\text{tr}}.$$

(h) As noted in Section 14, if T is of finite rank, then so is $|T|$. Choose a finite orthonormal basis $x_1, \ldots, x_n$ for the finite-dimensional range of $|T|$, and extend it to an orthonormal basis $\{x_j\}_{j\in J}$ of H. Then, for any e in the basis, $(|T|e, e) = 0$ unless e is among the vectors $x_1, \ldots, x_n$. Thus, only finitely many terms in the sum $\Sigma_{j\in J}(|T|x_j, x_j)$ are nonzero.

(i) If $T \in \mathrm{TC}(H)$ and $\{x_j\}_{j \in J}$ is an orthonormal basis, choose $j_1, j_2, \ldots, j_n$ in J such that $\Sigma_{j \neq j_1, j_2, \ldots, j_n}(|T|x_j, x_j) < \varepsilon$. Define the finite rank operator T_n by letting $T_n = T$ on $\{x_k : k = j_1, j_2, \ldots, j_n\}$ and $T = O$ on $\{x_k : k \neq j_1, j_2, \ldots, j_n\}$. Then,

$$\|T - T_n\|_{\mathrm{tr}} = \mathrm{tr}(|T - T_n|) = \Sigma_j\big(|T - T_n|x_j, x_j\big) = \Sigma_{j \neq j_1, j_2, \ldots, j_n}\big(|T|x_j, x_j\big) < \varepsilon.$$

Also,

$$||T - T_n|| \leq ||T - T_n||_{\mathrm{tr}}.$$

So T can be approximated in the operator norm by finite rank operators. Consequently, every trace class operator is compact. [Alternatively, we can argue using 4.7.3(iii) that a trace class operator is in HS and therefore compact by 4.6.4.] □

The following lemma of Lidskii [20, pp. 43–46] enables us to show that the Volterra operator is not trace class.

Lemma 4.7.11 *Let T be a trace class operator that has no eigenvalues except possibly for* 0. *Then,* tr(T) = 0.

The Volterra operator $V : L^2[0,1] \to L^2[0,1]$ has no eigenvalues, as shown earlier [Example 4.3.2]. If it were trace class, its trace would be 0, and hence, $\sum_j (Ve_j, e_j) = 0$ for any orthonormal basis $\{e_j\}$. However, for the particular orthonormal basis obtained from $1, \cos(2\pi nt), \sin(2\pi nt)$, an elementary computation shows that the sum $\sum_j (Ve_j, e_j)$ works out to be 1/2.

Problem Set 4.7

4.7.P1. Let H be a finite-dimensional complex Hilbert space and $T \in \mathcal{B}(H)$. Show that tr(T) is independent of the basis used to define it. If $S, T \in \mathcal{B}(H)$ are any two operators, then show that tr(ST) = tr(TS).

4.7.P2. (a) Show that the trace of an operator $T \in \mathcal{B}(H)$, where H is a finite-dimensional Hilbert space, is equal to the negative of the coefficient of λ^{n-1} in the characteristic polynomial of T, i.e. tr(T) = sum of the eigenvalues of the square matrix T.

(b) The trace of a nilpotent operator is zero.

4.7.P3. Let φ and ψ be two vectors in a Hilbert space H. The symbol $\varphi \otimes \overline{\psi}$ represents a transformation in H, whose defining equation is

$$(\varphi \otimes \overline{\psi})f = (f, \psi)\varphi.$$

Show that

(i) $\|\varphi \otimes \overline{\psi}\| = \|\varphi\|\|\psi\|$;
(ii) $\|\varphi \otimes \overline{\psi}\|_{HS} = \|\varphi\|\|\psi\|$;
(iii) $\|\varphi \otimes \overline{\psi}\|_{tr} = \|\varphi\|\|\psi\|$.

4.7.P4. TC(H), where H is a Hilbert space, is an ideal in $\mathcal{B}(H)$ and $\|\cdot\|_{tr}$ is a norm on TC(H).

4.7.P5. Show that $\mathrm{TC}(H) \subseteq \mathrm{HS}(H)$, i.e. every trace class operator is Hilbert–Schmidt.

4.7.P6. If T is a compact operator and $\alpha_1, \alpha_2, \ldots$ are the eigenvalues of $|T|$, each repeated as often as its multiplicity, then $T \in \mathrm{TC}(H)$ if and only if $\{\alpha_n\} \in \ell^1$ and in this case, $||T||_{tr} = \Sigma\alpha_n$.

4.7.P7. Let H be a Hilbert space and TC denote the class of trace class operators. For $T \in$ TC, $||T|| = \Sigma_{j \in J}(|T|x_j, x_j)$ denotes the trace norm of T, where $\{x_j\}_{j \in J}$ is an orthonormal basis for H. Then, $(\mathrm{TC}, ||T||_{tr})$ is a Banach space.

4.7.P8. Let T be a self-adjoint trace class operator. Show that tr(T) is the sum of the eigenvalues of T, each counted as often as its multiplicity.

4.8 Spectral Decomposition for Compact Normal Operators

In 4.5, compact linear operators were studied in the setting of normed linear spaces. Since working in the structurally powerful setting of Hilbert spaces usually makes life a little easier, it is hardly surprising that we obtain a spectral decomposition theorem for a bounded compact normal operator in such a situation.

Many of the results including the spectral decomposition theorem that holds for a linear transformation on finite-dimensional spaces go through unscathed in the infinite-dimensional case, provided the additional hypothesis of compactness of the operator is imposed. Some results in this spirit will be proved in this section.

It will be helpful to recall the spectral decomposition theorem for a linear transformation on a finite-dimensional Hilbert space so that the reader can find her way through the details.

Let T be a normal operator on a finite-dimensional complex Hilbert space. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T, M_j be the eigenspaces associated with λ_j, i.e. $M_j = \{x \in H : (T - \lambda_j I)x = 0\}$, and P_j the orthogonal projections of H on M_j, $j = 1, 2, \ldots, k$ (k an integer not greater than dim$H$). Then,

(i) $M_i \perp M_j$, for $i \neq j$;
(ii) $P_i \neq O$ for $i = 1, 2, \ldots, k$ and $P_i P_j = O$ for $i \neq j$;
(iii) $\sum_{i=1}^{k} P_i = I$;
(v) $\sum_{i=1}^{k} \lambda_i P_i = T$ [2, Chap. 1].

Throughout this section, T will denote a compact linear operator on an infinite-dimensional complex Hilbert space. The general properties of the spectrum of an operator in a Hilbert space H have been studied in 4.1–4.4. Specifically, it was noted in Theorem 4.2.4 that the spectrum of any bounded linear operator is nonempty and is contained in $\{\lambda \in \mathbb{C} : |\lambda| \leq ||T||\}$. For a compact operator, we can always identify one point of the spectrum.

Proposition 4.8.1 *For every compact operator T, $0 \in \sigma(T)$.*

Proof If not, then T^{-1} exists and is bounded. By Theorem 4.5.9, it follows that $I = T^{-1}T$ is a compact operator. This contradicts (i) of Remark 4.5.3. □

Remark 4.8.2 There are compact operators whose spectrum consists of the single point 0 and it may not be an eigenvalue [see Example 4.3.2 and 4.5.5(iii)].

Proposition 4.8.3 *If T is a compact operator and λ is a nonzero scalar, then $\mathfrak{N}_T(\lambda) = \{x \in H : Tx = \lambda x\}$ the null space of $(T - \lambda I)$ is finite-dimensional.*

Proof We shall prove the contrapositive of this result. Suppose, for $\lambda \neq 0$, $\mathfrak{N}_T(\lambda)$ is infinite-dimensional. Then, we can select an infinite collection of linearly independent vectors x_1, x_2, … from $\mathfrak{N}_T(\lambda)$; moreover, in view of the Gram–Schmidt orthonormalisation process, this set of vectors can be assumed to be an orthonormal set. For $n \neq m$,

$$||Tx_n - Tx_m||^2 = ||\lambda x_n - \lambda x_m||^2 = |\lambda|^2 ||x_n - x_m|| = 2|\lambda|^2,$$

which means that no subsequence of $\{Tx_n\}_{n\geq 1}$ can possibly be a Cauchy sequence. This precludes the possibility of T being compact and proves the proposition. □

Definition 4.8.4 Suppose λ is an eigenvalue for the operator T. If $\mathfrak{N}_T(\lambda)$ has finite dimension n, λ is said to be an eigenvalue of **multiplicity *n***. If $\mathfrak{N}_T(\lambda)$ is infinite-dimensional, λ is said to have **infinite multiplicity**. If λ is not an eigenvalue of T, i.e. if $\mathfrak{N}_T(\lambda) = \varnothing$, it is convenient to say that λ is an eigenvalue of **multiplicity zero**.

Remark 4.8.5 In this terminology, Proposition 4.8.3 asserts that every nonzero proper value of a compact operator has finite multiplicity. It was pointed out in Remark 4.8.2 that there are compact operators having no eigenvalue at all. However, every compact normal operator has at least one eigenvalue. The following theorem will facilitate the proof of the assertion.

Theorem 4.8.6 *Let T be a compact operator on a Hilbert space H and* λ *a nonzero approximate eigenvalue for T, i.e.* $\lambda \in \sigma_{ap}(T)$. *Then,* λ *is an eigenvalue for T.*

Proof By assumption, there exists a sequence $\{x_n\}_{n\geq 1}$ of vectors such that $||x_n|| = 1$ for every n and $||Tx_n - \lambda x_n|| \to 0$ as $n \to \infty$. Since T is compact, we may suppose, after passing to a subsequence, that Tx_n is convergent, say $Tx_n \to y$ as $n \to \infty$. Since

$$\|y - \lambda x_n\| \leq \|y - Tx_n\| + \|Tx_n - \lambda x_n\| \to 0 \text{ as } n \to \infty.$$

It follows that $\lambda x_n \to y$. Then $T(\lambda x_n) = \lambda Tx_n \to Ty$. Thus,

$$Ty = \lim_n T(\lambda x_n) = \lambda \lim_n Tx_n = \lambda y.$$

Also,

$$\|y\| = \|\lim_n \lambda x_n\| = \lim_n |\lambda| \|x_n\| = |\lambda| > 0.$$

Hence, λ is an eigenvalue for T, and y is the corresponding eigenvector. $\square$

Remark 4.8.7 Since $\sigma_p(T) \subseteq \sigma_{ap}(T)$ in general, the above Theorem 4.8.6 implies that for a compact operator T, $\sigma_{ap}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$. That is, if $\lambda \neq 0$, then $\lambda \in \sigma_{ap}(T)$ if and only if $\lambda \in \sigma_p(T)$.

Recall that if T is normal, then $\sigma(T) = \sigma_{ap}(T)$ [see Theorem 4.4.1]. In what follows, we show that if T is both compact and normal, then $\sigma(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$, i.e. the nonzero elements in the spectrum of a compact normal operator have to be eigenvalues.

Theorem 4.8.8 [Cf. (i) of Remarks 4.2.7] *If T is a compact operator which satisfies the normality condition, i.e.* $T^*T = TT^*$, *then*

(a) $\sigma(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$;
(b) $\sigma_p(T) \neq \emptyset$ *and there is some* $\lambda \in \sigma_p(T)$ *such that* $|\lambda| = ||T|| = r(T)$. *Obviously,* $\lambda \in \partial\sigma(T)$.

Proof

(a) Since T is normal, we know that [Theorem 4.4.1]

$$\sigma_{ap}(T) = \sigma(T),$$

so that

$$\sigma(T)\backslash\{0\} = \sigma_{ap}(T)\backslash\{0\}.$$

By Theorem 4.8.6 however,

$$\sigma_{ap}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}.$$

The two equalities above imply that

$$\sigma_{ap}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}.$$

(b) If $T = O$, then $0 \in \sigma_p(T)$ and H is the corresponding eigenspace. Moreover, $0 = ||T||$. On the other hand, if T is nonzero, then $||T|| > 0$, and since T is normal, $r(T) = ||T||$ [by (i) of Remarks 4.2.7], which implies $\sigma(T)\backslash\{0\} \neq \emptyset$. Also, there must be some $\lambda \in \sigma(T)$ such that $|\lambda| = \|T\|$ [Theorem 4.2.4]. Since $\lambda \neq 0$, $\lambda \in \sigma_p(T)$. This completes the proof. □

Alternative argument when $T \neq O$, using Theorem 4.3.5 instead of Theorem 4.4.1: By Theorem 4.2.4, there exists $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$. Obviously, $\lambda \in \partial\sigma(T)$, which is a subset of $\sigma_{ap}(T)$ by Theorem 4.3.5. By Remark 4.2.7(i), $r(T) = ||T||$. So, there exists $\lambda \in \sigma_{ap}(T)$ such that $|\lambda| = \|T\|$ Since $\sigma_{ap}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$ by Theorem 4.8.6 and $\lambda \neq 0$, we must have $\lambda \in \sigma_{ap}(T)$.

Remark As shown in Example 4.5.5, the Volterra operator is compact and has spectrum $\{0\}$. It therefore follows from the above theorem that it is not normal.

Corollary 4.8.9 *If $T \in \mathcal{B}(H)$ is compact and satisfies the additional hypothesis $T = T^*$, i.e. T is self-adjoint, then $\sigma_p(T) \neq \emptyset$ and $||T||$ or $-||T||$ is an eigenvalue of T.*

Proof If T is compact and self-adjoint, Theorem 4.8.8 applies. So there exists some $\lambda \in \sigma_p(T)$ such that $|\lambda| = ||T||$. Using the fact that T is self-adjoint, it follows that $\lambda = \overline{\lambda}$ [Theorem 4.4.2], i.e. λ is real. Consequently, either $||T||$ or $-||T||$ is in the eigenspectrum of T. □

Let M be a subspace of a Hilbert space H and $T \in \mathcal{B}(H)$. Recall [Definition 3.8.9] that M is said to be an invariant subspace of T if $T(M) \subseteq M$ and is called a reducing subspace if $T(M) \subseteq M$ and also $T(M^\perp) \subseteq M^\perp$.

A convenient equivalent description of closed invariant subspaces is provided by the following result.

Proposition 4.8.10 *The closed subspace M is invariant under T if and only if $M^\perp$ is invariant under T T^*.*

Proof Suppose $M^\perp$ is invariant under T^*. Then, for $x \in M$, *and* $y \in M^\perp$,

$$(x, T^*y) = 0 = (Tx, y).$$

This implies that

$$Tx \in M^{\perp\perp} = M.$$

Hence, M is invariant under T. The converse follows similarly. □

Our next result gives an equivalent description of when M reduces T.

Proposition 4.8.11 *The closed subspace M reduces T if and only if M is invariant under T and T^*.*

Proof Suppose M reduces T, i.e. M is invariant under T and $M^{\perp}$ is invariant under T. The last statement means, by Proposition 4.8.10, that $M^{\perp\perp} = M$ is invariant under T^*. On the other hand, suppose that M is invariant under T and T^*. Since M is invariant under T^*, again by Proposition 4.8.10, $M^{\perp}$ is invariant under $T^{**} = T$. Thus, M reduces T. □

The following result concerning a normal operator holds even if the operator is not compact.

Proposition 4.8.12 *If $T \in \mathcal{B}(H)$ satisfies the condition $TT^* = T^*T$ and $\lambda \in \mathbb{C}$, then* $\ker(T - \lambda I) = \ker(T - \overline{\lambda}I)^*$, i.e. $\mathfrak{N}_T(\lambda) = \mathfrak{N}_{T^*}(\overline{\lambda})$. *Moreover,* $\mathfrak{N}_T(\lambda)\ (= \ker(T - \lambda I))$ *is a reducing subspace of T.*

Proof Since T is a normal operator, so is $T - \lambda I$. Hence, for $x \in H$, $\|(T - \lambda I)x\| = \|(T - \overline{\lambda}I)^*x\|$ [Theorem 3.7.1]. Thus, $\mathfrak{N}_T(\lambda) = \mathfrak{N}_{T^*}(\overline{\lambda})$. In view of Proposition 4.8.11, it remains to show $\mathfrak{N}_T(\lambda)$ is invariant under T and T^*. If $x \in \mathfrak{N}_T(\lambda)$, then on the one hand, $Tx = \lambda x$, which implies $Tx \in \mathfrak{N}_T(\lambda)$; on the other hand, $x \in \mathfrak{N}_{T^*}(\overline{\lambda})$, so that $T^*x - \overline{\lambda}x = 0$, which implies $T^*x \in \mathfrak{N}_T(\lambda)$. The proof is now complete. □

Theorem 4.8.13 *Let $T \in \mathcal{B}(H)$ be a compact normal operator on a complex Hilbert space H and $\mathfrak{N}_T(\lambda) = \ker(T - \lambda I)$ denote eigensubspaces of T. Then, $\mathfrak{N}_T(\lambda)$ are a total family; that is, if $x \perp \mathfrak{N}_T(\lambda)$ for every scalar λ, then $x = 0$. That is, the only vector that is orthogonal to every eigenvector of T is the zero vector.*

Proof Let M be the smallest closed subspace of H such that $\mathfrak{N}_T(\lambda) \subseteq M$ for all λ. Let $\mathfrak{N} = M^{\perp}$. We need to show that $\mathfrak{N} = \{0\}$.

Since each $\mathfrak{N}_T(\lambda)$ reduces T [Proposition 4.8.12], $T(M) \subseteq M$ and $T^*(M) \subseteq M$. It follows that $T(\mathfrak{N}) \subseteq \mathfrak{N}$ and $T^*(\mathfrak{N}) \subseteq \mathfrak{N}$ [Proposition 4.8.11]. Thus, $\mathfrak{N}$ reduces T.

Assume to the contrary that $\mathfrak{N} \neq \{0\}$ and let $R = T|_{\mathfrak{N}}$. Then, R is a normal operator in $\mathfrak{N}$. We assert that R is a compact operator in the Hilbert space $\mathfrak{N}$. For a sequence $\{x_n\}_{n\geq 1}$ in $\mathfrak{N}$, $||x_n|| \leq 1$, $Rx_n = Tx_n$ and $\{Tx_n\}_{n\geq 1}$ has a convergent subsequence $\{Tx_{n(k)}\}_{k\geq 1}$; it follows that R is a compact operator.

Thus, R is a compact normal operator in the Hilbert space $\mathfrak{N} \neq \{0\}$. By Theorem 4.8.8, there exists a vector $x \neq 0$ in $\mathfrak{N}$ and a scalar λ such that $Tx = \lambda x$. Then, $x \in \mathfrak{N}_T(\lambda) \subseteq M$. But $x \in \mathfrak{N} = M^{\perp}$. Hence, $x \perp x$, so that $x = 0$. This contradiction completes the argument. □

The next result will show that the proper values of a compact normal operator can be enumerated in a finite or infinite sequence.

Theorem 4.8.14 *Let $T \in \mathcal{B}(H)$ be a compact normal operator on a complex Hilbert space H. Then, $\sigma_p(T)$ is at most countable (it could be empty) and* 0 *is its only possible limit point.*

Proof We contend that for any $\varepsilon > 0$, there are at most finitely many points in $\{\lambda \in \sigma_p(T) : ||T|| \geq |\lambda| \geq \varepsilon\}$.

Since with the exception of 0, any $\lambda \in \sigma_p(T)$ can be found in $\bigcup_{n=1}^{\infty} \{\lambda \in \sigma_p(T) : |\lambda| \geq \frac{1}{n}\}$. The theorem will follow once the contention is established. The contention is proved by a contradiction. Thus, we now suppose that for $\varepsilon > 0$, there exists an infinite sequence $\{\lambda_k\}_{k\geq 1}$ of distinct proper values of T such that $\varepsilon \leq |\lambda_k| \leq ||T||$. Passing to a subsequence, we may assume that $\lambda_k \to \lambda$, where $\varepsilon \leq |\lambda| \leq ||T||$. Let $Tx_k = \lambda_k x_k$, $||x_k|| = 1$. Since T is compact, $\{Tx_k\}$ has a convergent subsequence. We denote this subsequence by $\{Tx_k\}_{k\geq 1}$ and assume that $Tx_k \to y$ for suitable $y \in H$. Thus, $\lambda_k x_k \to y$, which implies $x_k \to \lambda^{-1} y$.

Hence, $\|x_n - x_m\| = \|\lambda_n^{-1} y - \lambda_m^{-1} y\| \to 0$ as $m, n \to \infty$. But $\{x_n\}_{n\geq 1}$ is an orthonormal [Theorem 4.4.3] sequence; hence, $\|x_n - x_m\|^2 = 2$ whenever $m \neq n$. This contradiction completes the argument. □

Corollary 4.8.15 *If $T \in \mathcal{B}(H)$ is compact and normal, then $\sigma(T)$ is at most countable.*

Proof This follows immediately from Theorems 4.4.1, 4.8.6 and 4.8.14. □

We proceed to prove a special case of the spectral decomposition theorem, namely when H is an infinite-dimensional Hilbert space and T is bounded and normal with finite-dimensional range. Since T is normal, its residual spectrum is empty [Theorem 4.4.3(c)]; since it is bounded and ran(T) is finite-dimensional, it is compact [Remarks (3.14.3(iii)]. We next note that zero cannot belong to its continuous spectrum. For zero to belong to $\sigma_c(T)$, the range ran(T) must be dense in H. Since ran(T) is finite-dimensional, and therefore closed, and H is infinite-dimensional, we see that this cannot happen. Therefore, $\sigma_c(T) = \emptyset$. This means that $\sigma(T) = \sigma_p(T)$. The next theorem shows that $\sigma_p(T)$ can have only finitely many points and establishes the spectral decomposition theorem.

Theorem 4.8.16 *Let $T \in \mathcal{B}(H)$ be a normal compact operator on a complex Hilbert space H. The following conditions on T are equivalent*:

(a) ran(T) *is finite dimensional*;
(b) *T has only finitely many distinct proper values $\lambda_1, \lambda_2, \ldots, \lambda_n$.*

Moreover, in this case,

$$T = \sum_{k=1}^{n} \lambda_k P_k,$$

where P_k denotes the projection of H onto $\mathfrak{N}_T(\lambda_k)$.

Proof (a) implies (b). Assume to the contrary that there is an infinite sequence $\lambda_1, \lambda_2, \ldots, \lambda_n \ldots$ of distinct proper values with associated proper vectors $x_1, x_2, \ldots, x_n, \ldots$. Since proper vectors associated with distinct proper values of a normal operator are orthogonal, they are linearly independent. We can write $T(\lambda_n^{-1} x_n) = x_n$ for all but at most one n; it follows that ran(T) is infinite-dimensional. This contradicts (a).

(b) implies (a). Now, suppose $\sigma_p(T) = \lambda_1, \lambda_2, \ldots, \lambda_n$. We show that ran($T$) is finite-dimensional. Since T is normal, proper vectors associated with distinct proper values are orthogonal, which implies that the subspaces $\mathfrak{N}_T(\lambda_k), k = 1, 2, \ldots, n$, are mutually orthogonal. If M denotes the space spanned by these subspaces, then

$$M = \mathfrak{N}_T(\lambda_1) \oplus \mathfrak{N}_T(\lambda_2) \oplus \cdots \oplus \mathfrak{N}_T(\lambda_n).$$

We assert that $H = M$. Being a direct sum of finitely many closed subspaces, M is closed, and it will therefore suffice to show that $M^\perp = \{0\}$. If $x \perp M$, then $x \perp \mathfrak{N}_T(\lambda_k)$ for every k, which implies $x = 0$ by Theorem 4.8.13. This completes the argument that $H = M$. It now follows that ran(T) = $T(M)$. The reader will note that $T(M)$ is the T-image of the direct sum of only those $\mathfrak{N}_T(\lambda_k)$ for which $\lambda_k \neq 0$. However, these are finite in number, and each is finite-dimensional. Therefore, $T(M)$ is finite-dimensional.

Finally, we show that $T = \sum_{k=1}^n \lambda_k P_k$, where P_k denotes the projection of H onto $\mathfrak{N}_T(\lambda_k)$.

As $H = M$, given any vector $x \in H$, we can write $x = \sum_{k=1}^n y_k$ with $y_k \in \mathfrak{N}_T(\lambda_k)$. If $j \neq k$, then $y_k \in \mathfrak{N}_T(\lambda_j)^\perp$; hence, $P_j y_k = 0$, where P_j denotes the projection on the eigenspace corresponding to the eigenvalue λ_j. It follows that $P_j x = \sum_{k=1}^n P_j y_k = y_j$. This shows that $\sum_{k=1}^n P_j = I$. Also,

$$Tx = \sum_{k=1}^n T y_k = \sum_{k=1}^n \lambda_k y_k = \sum_{k=1}^n \lambda_k P_k x;$$

hence,

$$T = \sum_{k=1}^n \lambda_k P_k.$$

This completes the proof. □

From now on, it will be assumed that $T \in \mathcal{B}(H)$ is both compact and normal. We further assume that ran(T) is infinite-dimensional. The following are easy consequences of Theorem 4.8.14:

(i) The distinct proper values can be enumerated in an infinite sequence of scalars $\lambda_1, \lambda_2, \ldots$.
(ii) Suppose $\{\lambda_n\}_{n \geq 1}$ is any sequence of distinct proper values of T. Given $\varepsilon > 0$, there are at most finitely many λ_n satisfying the inequality $|\lambda_n| \geq \varepsilon$;

hence, there exists an index N such that $|\lambda_n| < \varepsilon$ for all $n \geq N$. This says that $\lambda_n \to 0$ as $n \to \infty$.

(iii) The eigenvalues (nonzero) can be arranged to satisfy $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$. Indeed, each of the sets $\Lambda_n = \{\frac{1}{n} \leq |\lambda| \leq ||T||\}$, where λ is a proper value of T, contains at most finitely many scalars, and every nonzero proper value belongs to some Λ_n. Moreover, $\Lambda_1 \subseteq \Lambda_2 \subseteq \ldots$ First write down these proper values which are in Λ_1, then those which are in Λ_2 but not in Λ_1, and so on. Thus, the eigenvalues can be arranged according to the diminishing absolute value.

Let $T \in \mathcal{B}(H)$ be compact and normal, and $\lambda_1, \lambda_2, \ldots$. denote the sequence of distinct nonzero proper values enumerated as in the preceding paragraph. Let P_k denote the projection whose range is the eigensubspace $\mathfrak{N}_T(\lambda_k)$ and P_0 be the projection whose range is the null space of T, i.e. $\{x \in H : Tx = 0\}$. Theorem 4.8.18 shows that for every $x \in H$, $Tx = \sum_{k=1}^{\infty} \lambda_k P_k x$, i.e. $||Tx - \sum_{k=1}^{n} \lambda_k P_k x|| \to 0$ as $n \to \infty$.

Theorem 4.8.17 Suppose $\{\mathfrak{N}_j\}_{j \geq 1}$ is a sequence of closed linear subspaces of H such that $\mathfrak{N}_j \perp \mathfrak{N}_k$, $j \neq k$. If $x \perp \cup_j \mathfrak{N}_j \Rightarrow x = 0$, then $H = \overline{[\cup_j \mathfrak{N}_j]} = \sum_j \oplus \mathfrak{N}_j$ and conversely.

Proof Suppose the condition, namely $x \perp \cup_j \mathfrak{N}_j \Rightarrow x = 0$, holds. Clearly, we can write

$$H = \overline{[\cup_j \mathfrak{N}_j]} \oplus \overline{[\cup_j \mathfrak{N}_j]}^{\perp}.$$

If $x \in \overline{[\cup_j \mathfrak{N}_j]}^{\perp}$, then certainly, $x \perp \cup_j \mathfrak{N}_j$, which by the given condition implies $x = 0$. Hence, $\overline{[\cup_j \mathfrak{N}_j]}^{\perp} = \{0\}$, which implies $H = \overline{[\cup_j \mathfrak{N}_j]}$. It remains to show that $\overline{[\cup_j \mathfrak{N}_j]} = \sum_j \oplus \mathfrak{N}_j$.

Clearly, $\sum_j \oplus \mathfrak{N}_j \subseteq \overline{[\cup_j \mathfrak{N}_j]}$. To prove the reverse inequality, consider any $x \in \overline{[\cup_j \mathfrak{N}_j]}$. Since each $\mathfrak{N}_j$ is closed,

$$H = \mathfrak{N}_j \oplus \mathfrak{N}_j^{\perp}.$$

So, $x = x_j + y_j$, where $x_j \in \mathfrak{N}_j$ and $y_j \in \mathfrak{N}_j^{\perp}$. Consider any j for which $x_j \neq 0$. For such j,

$$\left(x, \frac{x_j}{||x_j||}\right) = \left(x_j + y_j, \frac{x_j}{||x_j||}\right) = \frac{(x_j, x_j)}{||x_j||} = ||x_j||.$$

Therefore, by Bessel's inequality

$$\sum_j \|x_j\|^2 = \sum_j \left| \left(x, \frac{x_j}{\|x_j\|} \right) \right|^2 \le \|x\|^2.$$

It follows that $\sum_j \|x_j\|^2 < \infty$ even when those j for which $x_j = 0$ are included in the summation.

Given $\varepsilon > 0$, there exists n_0 such that

$$\sum_{j=n_0+1}^{\infty} \|x_j\|^2 < \varepsilon \quad \text{and hence} \quad \left\| \sum_{j=n_0+1}^{\infty} x_j \right\|^2 < \varepsilon.$$

Therefore, $\sum_{j=1}^{\infty} x_j$ is convergent.

Let $w = \Sigma x_j$. Then, $w \in \Sigma_j \oplus \mathfrak{N}_j \subseteq \overline{[\cup_j \mathfrak{N}_j]}$. Consider any $y \in \mathfrak{N}_k$. We have

$$\begin{aligned}
(x - w, y) &= \left(x_k + y_k - \Sigma_j x_j, y\right) \\
&= (x_k, y) - \left(\Sigma_j x_j, y\right) \\
&= (x_k, y) - \Sigma_j \left(x_j, y\right) \\
&= (x_k, y) - (x_k, y) \ \text{because}\, x_j \in \mathfrak{N}_j \,\text{and}\, y \in \mathfrak{N}_k. \\
&= 0.
\end{aligned}$$

Thus, $x - w \perp \mathfrak{N}_k$ for all k. In view of the hypothesis, it follows that $x = w$. Since $w \in \sum_j \oplus \mathfrak{N}_j$, we obtain $x \in \sum_j \oplus \mathfrak{N}_j$.

Conversely, if $H = \Sigma_j \oplus \mathfrak{N}_j$, any $x \in H$ can be written as $x = \Sigma_j x_j$, where $x_j \in \mathfrak{N}_j$. If $x \perp \mathfrak{N}_j$ for every j, then $(x, x) = (\Sigma_j x_j, x) = \Sigma_j (x_j, x) = 0$, and therefore, $x = 0$. This completes the proof. □

Theorem 4.8.18 (Spectral Theorem for Completely Continuous Normal Operators) *Let $T \in \mathcal{B}(H)$, H a complex Hilbert space, be compact and normal. Then,*

(1) $H = \mathfrak{N}_T(0) \oplus \sum_{\substack{\lambda \in \sigma_p(T) \\ \lambda \neq 0}} \oplus \mathfrak{N}_T(\lambda)$.

(2) $\overline{[\mathrm{ran}(T)]} = \overline{[\mathrm{ran}(T^*)]} = \sum_{\substack{\lambda \in \sigma_p(T) \\ \lambda \neq 0}} \oplus \mathfrak{N}_T(\lambda) = \sum_{\substack{\lambda \in \sigma_p(T) \\ \lambda \neq 0}} \oplus \mathfrak{N}_{T^*}(\overline{\lambda})$.

(3) $T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda$ and $I = \sum_{\lambda \in \sigma(T)} P_\lambda$, where P_λ denotes the projection of H onto $\mathfrak{N}_T(\lambda)$.

Proof

(1) Since T is normal, $\{\mathfrak{N}_T(\lambda) : \lambda \in \sigma(T)\}$ is an orthogonal family of closed subspaces. By Theorem 4.8.13, since T is compact and normal,

$$x \perp \mathfrak{N}_T(\lambda) \quad \text{for all } \lambda \Rightarrow x = 0.$$

This implies on using Theorem 4.8.17 that

$$H = \overline{[\bigcup_{\lambda\in\sigma(T)} \mathfrak{N}_T(\lambda)]} = \sum_{\lambda\in\sigma(T)} \oplus \mathfrak{N}_T(\lambda)$$

and this proves (1).

(2) Using the fact that for normal transformations, ker(T) = ker(T*). we see that

$$\overline{[\mathrm{ran}(T)]} = \ker(T^*)^{\perp} = (\ker(T))^{\perp} = \overline{[\mathrm{ran}(T^*)]}.$$

This proves the first equality in (2). It also shows that we have the orthogonal sum decomposition

$$H = \ker(T) + \overline{[\mathrm{ran}(T)]},$$

whereas by (1) above,

$$H = \mathfrak{N}_T(0) \oplus \sum_{\substack{\lambda\in\sigma_p(T)\\ \lambda\neq 0}} \oplus\mathfrak{N}_T(\lambda).$$

Since the orthogonal complement of a subspace is unique [observe that $H = H_1 \oplus H_2$ implies $H_2 = H_1^{\perp}$ and that $\ker(T) = \mathfrak{N}_T(0)$], it follows that

$$\overline{[\mathrm{ran}(T)]} = \sum_{\substack{\lambda\in\sigma_p(T)\\ \lambda\neq 0}} \oplus\mathfrak{N}_T(\lambda),$$

which implies, using the fact that T* is also completely continuous and normal, that

$$\overline{[\mathrm{ran}(T^*)]} = \sum_{\substack{\lambda\in\sigma_p(T)\\ \lambda\neq 0}} \oplus\mathfrak{N}_{T^*}(\overline{\lambda}).$$

Since T is normal,

$$\mathfrak{N}_T(\lambda) = \mathfrak{N}_{T^*}(\overline{\lambda}).$$

Therefore, the last equality in (2) follows.

(3) Since the family $\{\mathfrak{N}_T(\lambda) : \lambda \in \sigma(T)\}$ is an orthogonal family, the countable collection $\{P_\lambda\}_{\lambda \in \sigma(T)}$ must be an orthogonal family of orthogonal projections. Moreover, ΣP_λ is an orthogonal projection on $\overline{[\bigcup_{\lambda\in\sigma(T)} \mathfrak{N}_T(\lambda)]} = \sum_{\lambda\in\sigma(T)} \oplus \mathfrak{N}_T(\lambda) = H$. As the orthogonal projection on the whole space is the identity operator, we have

$$\sum_{\lambda\in\sigma(T)} P_\lambda = I.$$

Since every $x \in H$ can be written as

$$x = \sum_{\lambda\in\sigma(T)} x_\lambda,$$

where $x_\lambda \in \mathfrak{N}_T(\lambda)$, and T is continuous, it follows that

$$Tx = \sum_{\lambda\in\sigma(T)} Tx_\lambda = \sum_{\lambda\in\sigma(T)} \lambda x_\lambda = \sum_{\lambda\in\sigma(T)} \lambda P_\lambda x,$$

which implies

$$T = \sum_{\lambda\in\sigma(T)} \lambda P_\lambda.$$

□

Problem Set 4.8

4.8.P1. **(Spectral Theorem for compact self-adjoint operators)** Let $T \in \mathcal{B}(H)$ be a compact self-adjoint operator. For each eigenvalue λ, let $\mathfrak{N}_\lambda(T) = \{x \in H : (T - \lambda I)x = 0\}$ be the eigenspace of T corresponding to λ. Then, prove the following:

(a) Its nonzero eigenvalues (necessarily real) are of finite multiplicity and form either a finite or a countably infinite sequence $\lambda_1, \lambda_2, \ldots$; in the latter case, $\lambda_n \to 0$;

(b) If $x_1, x_2, \ldots$ is a corresponding orthonormal sequence of proper vectors (i.e. $Tx_i = \lambda_i x_i$ for $i = 1, 2, \ldots$), then $T = \Sigma_i \lambda_i P_i$, where P_i denotes the projection of H onto $\mathfrak{N}_{\lambda_i}(T)$;

(c) T admits a complete orthonormal family of eigenvectors.

4.9 Spectral Measure and Integral

Let T be a compact normal operator on an infinite-dimensional Hilbert space H. It was proved in Theorem 4.8.18 that

$$T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda, \quad I = \sum_{\lambda \in \sigma(T)} P_\lambda,$$

where P_λ denotes the orthogonal projection on $\mathfrak{N}_T(\lambda) = \{x \in H : Tx = \lambda x\}$. The fact that T is compact ensures that $\sigma(T)$ is countable. This fact is not crucial once we know how to deal with uncountable sums. In particular, we know how to deal with uncountable weighted sums of projections: $Tx = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda x, x \in H$. Recall that even in this case, the above sum has only a countable number of nonzero vectors. What really brings compactness into play is that a compact normal operator has nonempty point spectrum, and more than that, it has enough eigensubspaces to span H:

$$H = \mathfrak{N}_T(0) \oplus \sum_{\substack{\lambda \in \sigma_p(T) \\ \lambda \neq 0}} \oplus \mathfrak{N}_T(\lambda).$$

However, a normal (noncompact) operator may have an empty point spectrum [Example 4.4.12(i)], or it may have eigenspaces but not enough to span the whole space. The spectral theorem for normal (noncompact) operators can be proved, provided that point spectrum is replaced with the full spectrum, which is never empty. For a proper statement of the spectral theorem for a plain normal operator, a sound knowledge of measure theory is required, which we have assumed all along.

Definition 4.9.1 Let X be a nonempty set and Σ be any σ-algebra of subsets of X. Let $\mathcal{P}(H)$ denote the collection of all orthogonal projection operators in a complex Hilbert space H. A **spectral measure** in H is a mapping $P:\Sigma \to \mathcal{P}(H)$ with the following properties:

(i) If $\{E_n\}_{n \geq 1}$ is a countable family of mutually disjoint elements of Σ, then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n), \tag{4.10}$$

where the series on the right converges strongly in the sense of Definition 3.3.6;

(ii) $P(X) = I$, *the identity operator in* H.

Remarks 4.9.2

(i) If $\{E_k\}_{k\geq 1}$ is a countably infinite collection of pairwise disjoint sets in Σ, then the above identity (4.10) means that for each $x \in H$,

$$\left\|\sum_{k=1}^{n} P(E_k)x - P\left(\bigcup_{k=1}^{\infty} E_k\right)x\right\| \to 0 \quad \text{as } n \to \infty.$$

(ii) $P(\varnothing) = O$. Let $\{E_n\}_{n\geq 1}$ be a sequence of empty sets, i.e. $E_n = \varnothing$ for every n. Then, for $x \in H$,

$$P\left(\bigcup_{n=1}^{\infty} E_n\right)x = \sum_{n=1}^{\infty} P(E_n)x,$$

which implies $P(\varnothing)x = \sum_{n=1}^{\infty} P(\varnothing)x$; since the right-hand side converges and each term of the series is $P(\varnothing)x$, it follows that $P(\varnothing)x = O$ for each $x \in H$, and hence, $P(\varnothing) = O$.

(iii) For $\{E_k\}_{1\leq k\leq n}$ disjoint subsets of Σ,

$$P\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} P(E_k).$$

Let $E_{n+1} = E_{n+2} = \cdots = \varnothing$. Then, for $x \in H$,

$$P\left(\bigcup_{k=1}^{n} E_k\right)x = P\left(\bigcup_{k=1}^{\infty} E_k\right)x = \sum_{k=1}^{\infty} P(E_k)x = \sum_{k=1}^{n} P(E_k)x,$$

using (ii) above; that is, P is finitely additive.

(iv) If E_1 and E_2 are in Σ and $E_1 \subseteq E_2$, then $P(E_1) \leq P(E_2)$. Observe that $E_2 = E_1 \cup (E_2 \backslash E_1)$. It follows from (iii) above that $P(E_2) = P(E_1) + P(E_2 \backslash E_1)$. So, for all $x \in H$, we have

$$\begin{aligned}([P(E_2) - P(E_1)]x, x) &= (P(E_2 \backslash E_1)x, x) = (P(E_2 \backslash E_1)x, P(E_2 \backslash E_1)x) \\ &= ||P(E_2 \backslash E_1)x||^2.\end{aligned}$$

This implies $P(E_1) \leq P(E_2)$.

(v) Since $E \in \Sigma$ is such that $E \subseteq X$, it follows on using (iv) and $P(X) = I$ that $||P(E)|| \leq 1$, that is, P is a bounded map from Σ to $\mathcal{P}(H)$. The reader will note that this is a property of any orthogonal projection.

(vi) If E_1 and E_2 are in Σ, then $P(E_1 \cap E_2) = P(E_2)P(E_1) = P(E_1)P(E_2)$. Observe that $P(E_1 \cap E_2) \leq P(E_1) \leq P(E_1 \cup E_2)$ by (iv) above. Therefore,

$$P(E_1 \cap E_2)P(E_1) = P(E_1 \cap E_2) \quad \text{and} \quad P(E_1 \cup E_2)P(E_1) = P(E_1).$$

Multiplying by $P(E_1)$ in the equality (which can be obtained as in the case of real measures)

$$P(E_1 \cup E_2) + P(E_1 \cap E_2) = P(E_1) + P(E_2),$$

we obtain

$$P(E_1) + P(E_1 \cap E_2) = P(E_1) + P(E_2)P(E_1),$$

which implies

$$P(E_1 \cap E_2) = P(E_2)P(E_1) = P(E_1)P(E_2).$$

(vii) If $E_1 \cap E_2 = \varnothing$, it follows on using (ii) and (vi) above that

$$P(E_2)P(E_1) = P(E_1)P(E_2) = O.$$

(viii) For x,y in H, $(P(E)x,y)$ is a complex measure on the σ-algebra Σ. This is an immediate consequence of the fact that P is a spectral measure.

Examples 4.9.3

(i) Let (X, Σ, μ) be an arbitrary but fixed measure space, where μ is a complex measure. For $S \in \Sigma$, the function χ_S is essentially bounded with $||\chi_S||_\infty \leq 1$. Let M_{χ_S} be the operator of multiplication by χ_S in $L^2(X)$. Then, the mapping $P(S) = M_{\chi_S}$ from Σ into $\mathcal{B}(L^2(X))$ defines a spectral measure in $H = L^2(X)$ as we now argue:
In (iv) of Remarks 3.8.3, it was observed that the multiplication operator determined by a characteristic function is an orthogonal projection. Since $P(X)x = \chi_X(x) = x$ for every $x \in H$, it follows that $P(X) = I$. If $\{E_k\}_{k\geq 1}$ is a countably infinite collection of pairwise disjoint sets in Σ and $E = \bigcup_k E_k$, then $\chi_E = \Sigma_k \chi_{E_k}$, and hence,

$$P\left(\bigcup_k E_k\right)x = \chi_E x = \Sigma_k \chi_{E_k} x = \Sigma_k P(E_k)x,$$

i.e. P is countably additive.

(ii) Let $\{\mu_n\}$ be a sequence of measures on (X, Σ), and let $H = \oplus L^2(\mu_n)$. If P_{μ_n} is the projection-valued measure in $L^2(\mu_n)$ as in (i) above, then $P = \oplus P_{\mu_n}$ is a spectral measure in H.

(iii) Let X be any set, Σ the set of all subsets of X and H any separable Hilbert space. Fix an orthonormal basis $\{e_1, e_2, \dots\}$ for H and a sequence $\{x_n\}_{n\geq 1}$ in X indexed by the orthonormal basis. Define $P(E)$ to be the orthogonal projection onto the closed subspace generated by $\{e_n : x_n \in E\}$. Then, P can be shown to be a spectral measure in H.

Since the closed subspace generated by the sequence $\{e_n\}_{n\geq 1}$ is H, it follows that $P(X) = I$. The countable additivity of P can be proved as in (i) above.

Proposition 4.9.4 *If P is a spectral measure in H, and $x,y \in H$, then*

$$\mu_{x,y}(E) = (P(E)x, y)$$

defines a countably additive complex measure on Σ with total variation measure $|\mu_{x,y}|$, where $|\mu_{x,y}|(E) = \sup\{\sum_{j=1}^{\infty} |\mu_{x,y}(E_j)| : \{E_j\}$ is a partition of $E\}$, *with* $|\mu_{x,y}|(E) \leq \|x\|\|y\|$ *for all E.* (The concept of total variation is formally defined in the first paragraph of 4.10.)

Proof Let $\{E_n\}_{n\geq 1}$ be a family of pairwise disjoint sets in Σ. Then, $\{P(E_n)\}_{n\geq 1}$ is a family of mutually orthogonal projections such that

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n).$$

$$\mu_{x,y}\left(\bigcup_{n=1}^{\infty} E_n\right) = \left(P\left(\bigcup_{n=1}^{\infty} E_n\right)x, y\right) = \sum_{n=1}^{\infty} (P(E_n)x, y) = \sum_{n=1}^{\infty} \mu_{x,y}(E_n).$$

This shows that $\mu_{x,y}$ is countably additive.

We next estimate the total variation $|\mu_{x,y}|$. Let E be any set in Σ and $\{E_n\}_{n\geq 1}$ be any partition of E, i.e. a family of pairwise disjoint sets in Σ with union E. Let also $\alpha_1,\alpha_2,\ldots,\alpha_n$ be complex numbers such that $|\alpha_j| = 1$ and $|(P(E_j)x, y)| = \alpha_j(P(E_j)x, y)$. Then,

$$\sum_{j=1}^{n} \left|\mu_{x,y}(E_j)\right| = \sum_{j=1}^{n} \alpha_j(P(E_j)x, y) = \left(\sum_{j=1}^{n} \alpha_j P(E_j)x, y\right) \leq \left\|\sum_{j=1}^{n} \alpha_j P(E_j)x\right\| \|y\|.$$

But the vectors $\{\alpha_j P(E_j)x : 1 \leq j \leq n\}$ are pairwise orthogonal. Therefore,

$$\left\|\sum_{j=1}^{n} \alpha_j (P(E_j)x\right\|^2 = \sum_{j=1}^{n} \left\|(P(E_j)x\right\|^2 \leq \sum_{j=1}^{\infty} \left\|(P(E_j)x\right\|^2 = \left\|P\left(\bigcup_{j=1}^{\infty} E_j\right)x\right\| \leq \|x\|^2.$$

Hence,

$$\sum_{j=1}^{n} \left|\mu_{x,y}(E_j)\right| \leq \|x\|\|y\|.$$

It follows from here that

$$\sum_{j=1}^{\infty}\left|\mu_{x,y}(E_j)\right| \leq \|x\|\|y\|.$$

The desired conclusion follows on taking the supremum over all partitions of E. □

Let (X,Σ) be an arbitrary but fixed measurable space. Following the familiar ideas of Lebesgue integration, we define an integral $\int f \mathrm{d}P$, where f is a bounded measurable function defined on X and P denotes a spectral measure in a Hilbert space H.

Let $B(X)$ denote the space of all scalar-valued bounded measurable functions defined on X. For $f \in B(X)$, the norm $\|f\|$ of f is defined to be $\sup\{|f(x)| : x \in X\}$. By a *simple* function s, we always understand a function of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

where χ_{E_i} is the characteristic function of $E_i \in \Sigma$, the E_i are disjoint, nonempty and $\cup_i E_\mathrm{i} = X$. The above representation of X in which the E_i are disjoint and α_i (i = 1, 2,…,n) are distinct is called the canonical representation of s. It is evidently unique, because $E_i = \{x : s(x) = \alpha_i\}$; moreover, $\{\alpha_i : i = 1, 2, \ldots, n\}$ is the range of s.

Let H be a Hilbert space. Recall that a spectral measure for (X, Σ, H) is a function $P : \Sigma \to \mathcal{P}(H)$, the collection of all orthogonal projections defined on H, satisfying (i) and (ii) of Definition 4.9.1.

Definition 4.9.5 If $s = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, where E_i are disjoint nonempty measurable sets that cover X and $\alpha_i \in \mathbb{C}$ are distinct (i.e. $\sum_{i=1}^{n} \alpha_i \chi_{E_i}$ is the canonical representation of s), the **spectral integral** of s is defined by

$$\int s \mathrm{d}P = \sum_{j=1}^{n} \alpha_i P(E_i),$$

and for $E \in \Sigma$,

$$\int_E s(x)\mathrm{d}P(x) = \sum_{i=1}^{n} \alpha_i P(E_i \cap E).$$

In view of the fact that $E_i = \{x : s(x) = \alpha_i\}$ and $\{\alpha_i : i = 1, 2, \ldots, n\}$ is the range of s, the summation defining the integral can also be written as $\Sigma a P(A_a)$, where the summation is taken over all a in the range of s and $A_a = \{x : s(x) = a\}$. Using this, it is readily verified that if $\sum_{i=1}^{n} \alpha_i \chi_{E_i} = \sum_{j=1}^{m} \beta_j \chi_{F_j}$, then $\int s \mathrm{d}P = \sum_{i=1}^{n} \alpha_i P(E_i) = \sum_{j=1}^{m} \beta_j P(F_j)$, where the F_j are disjoint. Indeed, the set $A_a = \{x : s(x) = a\} = \cup F_j$, where the union is taken over all those j such that $\beta_j = a$. Hence, $aP(A_a) = \sum_{\beta_j = a} \beta_j P(F_j)$, and so,

$$\int s\mathrm{d}P = \sum_{i=1}^{n} \alpha_i P(E_i) = \Sigma a P(A_a) = \Sigma \sum_{\beta_j = a} \beta_j P\big(F_j\big) = \sum_{j=1}^{m} \beta_j P\big(F_j\big).$$

Remarks 4.9.6

(i) The integral is written also as $\int_X s(x)\mathrm{d}P(x)$. It may be noted that the integral is an operator on H.

(ii) If s and t are simple functions and α, β are scalars, then

$$\int_X (\alpha s + \beta t)(x)\mathrm{d}P(x) = \alpha \int_X s(x)\mathrm{d}P(x) + \beta \int_X t(x)\mathrm{d}P(x).$$

Let $\{A_i\}$ and $\{B_j\}$ be sets occurring in the canonical representations of s and t. Then,

$$\alpha\, s + \beta\, t = \alpha_i \sum_{i=1}^{n} \alpha_i \chi_{A_i} + \beta \sum_{j=1}^{m} \beta_j \chi_{B_j} = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha\alpha_i + \beta\beta_j)\chi_{A_i \cap B_j}$$

$$\left(\chi_{A_i} = \chi_{A_i \cap X} = \sum_{j=1}^{m} \chi_{A_i \cap B_j}\right).$$

Since the collection of sets obtained by taking the intersections $A_i \cap B_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, form a finite disjoint collection of measurable sets, we get from Definition 4.9.5 and the paragraph following it,

$$\begin{aligned}
\int_X (\alpha s + \beta t)(x)\mathrm{d}P(x) &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha\alpha_i + \beta\beta_j) P(A_i \cap B_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha\alpha_i + \beta\beta_j) P(A_i) P(B_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha\alpha_j P(A_i) P(B_j) + \sum_{i=1}^{n} \sum_{j=1}^{m} \beta\beta_j P(A_i) P(B_j) \\
&= \sum_{i=1}^{n} \alpha\alpha_i P(A_i) P(\cup_j B_j) + \sum_{j=1}^{m} \beta\beta_j P(\cup_i A_i) P(B_j) \\
&= \sum_{i=1}^{n} \alpha\alpha_i P(A_i) + \sum_{j=1}^{m} \beta\beta_j P\big(B_j\big) \\
&= \alpha \int_X s(x)\mathrm{d}P(x) + \beta \int_X t(x)\mathrm{d}P(x),
\end{aligned}$$

where we have used the fact that $\cup_i A_i = \cup_j B_j = X$ and $P(X) = I$.

(iii) For a simple function s, $\|\int_X s\mathrm{d}P\| \le \sup\{|s(x)| : x \in X\}$.
Observe that for any $y \in H$,

$$\begin{aligned}\left\|\sum_{i=1}^{n} \alpha_i P(E_i)y\right\|^2 &= \left(\sum_{i=1}^{n} \alpha_i P(E_i)y, \sum_{i=1}^{n} \alpha_i P(E_i)y\right)\\ &= \sum_{i=1}^{n} (|\alpha_i|^2 P(E_i)y, y), \text{ using orthogonality of the } P(E_i)\\ &\le \sup_i|\alpha_i|^2 \sum_{i=1}^{n} (P(E_i)y, y) \le \sup|s(x)|^2\|y\|^2.\end{aligned}$$

Consequently,

$$\left\|\int_X s\mathrm{d}P\right\| \le \sup\{|s(x)| : x \in X\}.$$

(iv) For simple functions s and t,

$$\int_X st\mathrm{d}P = \int_X s\mathrm{d}P \int_X t\mathrm{d}P.$$

With notations as in (iii) above,

$$st = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j\chi_{A_i\cap B_j}.$$

Therefore,

$$\begin{aligned}\int_X st\mathrm{d}P &= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j P(A_i\cap B_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j P(A_i)P(B_j)\\ &= \sum_{i=1}^{n} \alpha_i P(A_i) \sum_{j=1}^{m} \beta_j P(B_j)\\ &= \int_X s\mathrm{d}P \int_X t\mathrm{d}P.\end{aligned}$$

(v) For a simple function s,

$$\left(\int_X s\mathrm{d}P\right)^* = \int_X \overline{s}\mathrm{d}P.$$

With notations as in (iii) above,

$$\left(\int_X s\mathrm{d}P\right)^* = \left(\sum_{i=1}^n \alpha_i P(A_i)\right)^* = \sum_{i=1}^n \overline{\alpha_i} P(A_i) = \int_X \overline{s}\mathrm{d}P.$$

(vi) If $x \in H$ is such that $\|x\| = 1$, s is a simple function, and μ_x is the measure associated with P by the formula

$$\mu_x(E) = (P(E)x, x) = ||P(E)x||^2,$$

then

$$\left\|\left(\int_X s\mathrm{d}P\right)x\right\|^2 = \int_X |s|^2 d\mu_x.$$

Indeed, using the notations as in (iii) above, we have

$$\begin{aligned}\left\|\left(\int_X s\mathrm{d}P\right)x\right\|^2 &= \left\|\sum_{i=1}^n \alpha_i P(E_i)x\right\|^2 = \left(\sum_{i=1}^n \alpha_i P(E_i)x, \sum_{i=1}^n \alpha_i P(E_i)x\right) = \sum_{i=1}^n \left(|\alpha_i|^2 P(E_i)x, x\right) \\ &= \sum_{i=1}^n |\alpha_i|^2 \mu_x(E_i) = \int_X |s|^2 d\mu_x.\end{aligned}$$

Now, let f be any bounded measurable function on X. Then, there exists a sequence $\{s_n\}_{n\geq 1}$ of simple functions such that s_n converges uniformly to f. It follows from Remark 4.9.6(iii) that

$$\left\|\int_X s_n\mathrm{d}P - \int_X s_m\mathrm{d}P\right\| \leq \sup\{|s_n(x) - s_m(x)| : x \in X\}.$$

The right-hand side tends to zero as n and m tend to ∞. Thus, $\{\int_X s_n \mathrm{d}P\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{B}(H)$. Hence, it has a limit

$$T = \lim_{n\to\infty} \int\limits_X s_n \mathrm{d}P,$$

where the limit is taken in the norm of $\mathcal{B}(H)$. We next show that the limit is independent of the sequence $\{s_n\}_{n\geq 1}$ chosen to approximate f.

Let $\{t_n\}_{n\geq 1}$ be another sequence of simple functions such that $\lim_n t_n = f$ (uniformly). Then,

$$\left\| \int\limits_X s_n \mathrm{d}P - \int\limits_X t_n \mathrm{d}P \right\| = \left\| \int\limits_X (s_n - t_n) \mathrm{d}P \right\| \leq \sup\{|s_n(x) - t_n(x)| : x \in X\},$$

using (iii) of Remarks 4.9.6. The right-hand side of the above inequality tends to zero as $n \to \infty$.

Thus, we may define the **spectral integral** $\int_X f \mathrm{d}P$ by putting

$$\int\limits_X f \mathrm{d}P = \lim_{n\to\infty} \int\limits_X s_n \mathrm{d}P,$$

where $\{s_n\}_{n\geq 1}$ is any sequence of simple functions converging uniformly to f.

The following proposition is an analogue of the observations made in Remark 4.9.6.

Proposition 4.9.7 *Let f and g be bounded measurable functions defined on X, i.e. $f, g \in B(X)$, and α, β be scalars. Then, the following holds:*

(a) $\int_X (\alpha f + \beta g)\mathrm{d}P = \alpha \int_X f \mathrm{d}P + \beta \int_X g \mathrm{d}P$;
(b) $\int_X fg \mathrm{d}P = \int_X f \mathrm{d}P \int_X g \mathrm{d}P$;
(c) $(\int_X f \mathrm{d}P)^* = \int_X \bar{f} \mathrm{d}P$;
(d) *If y is a unit vector in H and μ_y is the measure associated with P according to the formula $\mu_y(E) = (P(E)y, y) = \|P(E)y\|^2$, then*

$$\left\| \left(\int\limits_X f \mathrm{d}P \right) y \right\|^2 = \int\limits_X |f|^2 d\mu_y;$$

(e) $\|\int_X f \mathrm{d}P\| \leq \sup\{|f(x)| : x \in X\}$.

Proof

(a) Let $\{f_n\}_{n\geq 1}$ [resp. $\{g_n\}_{n\geq 1}$] be a sequence of simple functions such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$[resp. $\|g_n - g\|_\infty \to 0$ as $n \to \infty$]. Then,

$$\|\alpha f_n + \beta g_n - \alpha f - \beta g\|_\infty \leq |\alpha|\|f_n - f\|_\infty + \beta||g_n - g||_\infty \to 0 \text{ as } n \to \infty.$$

It follows from (ii) of Remarks 4.9.6 that

$$\int_X (\alpha f_n + \beta g_n)\mathrm{d}P = \alpha \int_X f_n \mathrm{d}P + \beta \int_X g_n \mathrm{d}P.$$

On taking limits as $n \to \infty$, we obtain the desired equality.

(b) Let $\{f_n\}_{n\geq 1}$[resp.$\{g_n\}_{n\geq 1}$] be as in (a). Then, the following inequality, namely

$$\|f_n g_n - fg\|_\infty = ||(f_n - f)g_n + f(g_n - g)||_\infty \leq ||g_n||_\infty\|f_n - f\|_\infty + \|f\|_\infty||g_n - g||_\infty$$

implies $\lim_n\|f_n g_n - fg\|_\infty = 0$. So,

$$O = \lim_n \int_X (f_n g_n - fg)\mathrm{d}P = \lim_n \int_X (f_n g_n)\mathrm{d}P - \int_X (fg)\mathrm{d}P. \tag{4.11}$$

Also, on using (iv) of Remarks 4.9.6, we obtain

$$\begin{aligned}\lim_n \int_X (f_n g_n)\mathrm{d}P &= \lim_n\left(\int_X f_n \mathrm{d}P \int_X g_n \mathrm{d}P\right) \\ &= \left(\lim_n \int_X f_n \mathrm{d}P\right)\left(\lim_n \int_X g_n \mathrm{d}P\right) \\ &= \int_X f \mathrm{d}P \int_X g \mathrm{d}P.\end{aligned} \tag{4.12}$$

Consequently, using (4.11) and (4.12), we obtain

$$\int_X (fg)\mathrm{d}P = \int_X f \mathrm{d}P \int_X g \mathrm{d}P.$$

(c) Let $\{f_n\}_{n\geq 1}$ be a sequence of simple functions such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Then, $\|\overline{f}_n - \overline{f}\|_\infty \to 0$ as $n \to \infty$. For a simple function h, it has been proved that

$$\left(\int_X h\mathrm{d}P\right)^* = \int_X \overline{h}\mathrm{d}P.$$

From the definition of the spectral integral ($f \in B(X)$),

$$\begin{aligned}\left(\int_X f\mathrm{d}P\right)^* &= \left(\lim_n \int_X f_n\mathrm{d}P\right)^* \\ &= \lim_n \left(\int_X f_n\mathrm{d}P\right)^* \\ &= \lim_n \int_X \overline{f}_n\mathrm{d}P \\ &= \int_X \overline{f}\mathrm{d}P.\end{aligned}$$

(d) Observe that the formula holds for simple functions. The desired equality follows by taking limits as above.

(e) Let $\{f_n\}_{n\geq 1}$ be a sequence of simple functions such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given. There exists an $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \text{implies } \|f_n - f\|_\infty < \varepsilon \quad \text{and} \quad \left\|\int_X f\mathrm{d}P - \int_X f_n\mathrm{d}P\right\| < \varepsilon$$

For $n \geq n_0$, using (iii) of Remarks 4.9.6, we have

$$\begin{aligned}\left\|\int_X f\mathrm{d}P\right\| &\leq \left\|\int_X (f - f_n)\mathrm{d}P\right\| + \left\|\int_X f_n\mathrm{d}P\right\| \\ &\leq \varepsilon + \sup\{|f_n(x)| : x \in X\} \leq 2\varepsilon + \sup\{|f(x)| : x \in X\}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

Remarks 4.9.8

(i) We have defined the spectral integral $\int_X f\mathrm{d}P$ over the whole space X. If E is a measurable set, we put

$$\int_E f\mathrm{d}P = \int_X (\chi_E f)\mathrm{d}P.$$

Other conventions of ordinary integrals suitably modified are adopted here too.

(ii) The integral defined above is a normal operator on H. If f is real, the operator is self-adjoint and it is unitary if $|f(x)| = 1$ for all $x \in X$.

(iii) The mapping $\rho : B(X) \to \mathcal{B}(H)$ defined by $\rho(f) = \int_X f\mathrm{d}P$ is a continuous *homomorphism of the Banach *algebra $B(X)$ into the Banach *algebra $\mathcal{B}(H)$. The only property that remains to be checked is the continuity of ρ. The remaining properties have already been verified in Theorem 4.9.7. Let $\{f_n\}_{n\geq 1}$ be a sequence in $B(X)$ such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Then,

$$\left\|\int_X f_n\mathrm{d}P - \int_X f\mathrm{d}P\right\| = \left\|\int_X (f_n - f)\mathrm{d}P\right\| \leq \|f_n - f\|_\infty.$$

Problem Set 4.9

4.9.P1. Let P_1 be a spectral measure on (X, Σ) with values in a Hilbert space H_1. Let $U : H_1 \to H_2$ be a unitary transformation from H_1 onto H_2. For all measurable sets $E \in \Sigma$, set $P_2(E) = UP_1(E)U^*$. Show that P_2 is a spectral measure in H_2.

4.9.P2. [Cf. Proposition 2.7.3] Let $\{H_n\}_{n\geq 1}$ be a sequence of Hilbert spaces and $H = \oplus_n H_n$ be their direct sum. Let P_n be a spectral measure on (X, Σ) with values in the Hilbert space H_n. If $x = (x_1, x_2, \ldots)$ is an element of H, define

$$P(E)x = (P_1(E)x_1, P_2(E)x_2, \ldots), E \in \Sigma.$$

Then, P is a spectral measure on H. (We say that P is the direct sum of P_1, P_2,… and write $P = \Sigma \oplus_n P_n$.)

4.9.P3. Let (X, Σ) be a measurable space, and let $E \to P(E)$ be a mapping of Σ into $\mathcal{P}(H)$, where $\mathcal{P}(H)$ denotes the collection of all projections in the complex Hilbert space H. For each unit vector x in H, let

$$\mu_x(E) = (P(E)x, x) = ||P(E)x||^2.$$

Then, P is a spectral measure if and only if for every $x \in H$, μ_x is a probability measure on (X, Σ).

4.10 Spectral Theorem for Self-adjoint Operators

Let $T \in \mathcal{B}(H)$, where H is a complex Hilbert space, satisfy $T = T^*$. In this case,

$$||T|| = \sup\{|(Tx, x)| : \|x\| = 1\} = \max\{|m|, |M|\},$$

where

$$m = \inf\{(Tx, x) : ||x|| = 1\}, M = \sup\{(Tx, x) : ||x|| = 1\}.$$

We know that the purely real spectrum of T is contained in $[-\|T\|, \|T\|]$. A strengthening of this is $\sigma(T) \subseteq [m, M]$. Moreover, m,M are in $\sigma(T)$.

The reader will benefit from reviewing relevant portions of 4.4 on the spectra of various classes of operators.

Let X be a set and $\mathfrak{M}$ be a σ-algebra of subsets of X. Let μ be a complex-valued function defined on $\mathfrak{M}$ such that

$$A_n \in \mathfrak{M}, n = 1, 2, \ldots \quad \text{and} \quad A_n \cap A_m = \varnothing, n \neq m \text{ implies}$$
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Then, μ is called a **complex measure** on $\mathfrak{M}$ or on X. The **total variation measure** $|\mu|$ is defined as

$$|\mu|(E) = \sup \sum_{n=1}^{\infty} |\mu(E_n)|, \quad E \in \mathfrak{M},$$

where the sup is taken over all countable disjoint collections $\{E_n\}$ of measurable sets whose union is E. It can be shown that $|\mu|$ is indeed a positive measure and is finite.

$C(\sigma(T))$ will denote the space of complex-valued continuous bounded functions f on the compact space $\sigma(T)$. We shall be using the following Riesz Representation Theorem; namely, every bounded linear functional Φ on $C(\sigma(T))$ can be described as the integral of $f \in C(\sigma(T))$ with respect to a uniquely determined complex measure $m(S)$ defined on the Borel subsets S of $\sigma(T)$, and $\|\Phi\| = |\mu|(\sigma(T))$. [The foregoing theorem and related matters can be found in [25, Chap. 6].]

The following will be needed in the proof of the spectral theorem.

Let p be a polynomial with complex coefficients,

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n.$$

Then, for a self-adjoint operator T in $\mathcal{B}(H)$, by $p(T)$ we understand

$$p(T) = a_0 I + a_1 T + \cdots + a_n T^n.$$

Note that

$$p(T)^* = (a_0 I + a_1 T + \cdots + a_n T^n)^* = \overline{a}_0 I + \overline{a}_1 T + \cdots + \overline{a}_n T^n = \overline{p}(T).$$

Thus, the mapping $\varphi : \mathbb{C}[p(\lambda)] \to \mathcal{B}(H)$ given by $\varphi(p) = p(T)$, which is obviously an algebra homomorphism, satisfies $\varphi(\overline{p}) = p(T)^*$.

According to the Spectral Mapping Theorem [4.3.1(c)],

$$\sigma(p(T)) = p(\sigma(T)).$$

Observe that $p(T)$ is a normal operator. From this equality and Remark 4.2.7(i) that the spectral radius of a normal operator equals its norm, it follows that

$$\begin{aligned} \|p(T)\| = r(p(T)) &= \sup\{|\lambda| : \lambda \in \sigma(P(T)\} \\ &= \sup\{|p(\lambda)| : \lambda \in \sigma(T)\}. \end{aligned} \tag{4.13}$$

Let $f(\lambda)$ be any continuous complex-valued function on $\sigma(T)$, the spectrum of T. By the Stone–Weierstrass Approximation Theorem, f can be approximated uniformly on $\sigma(T)$ by restrictions of polynomials with complex coefficients. So, there exists a sequence $\{p_n\}_{n\geq 1}$ of polynomials such that

$$\lim_{n\to\infty} \sup\{|f(\lambda) - p_n(\lambda)| : \lambda \in \sigma(T)\} = 0.$$

It follows that $\{p_n\}_{n\geq 1}$ is a Cauchy sequence:

$$\lim_{m,n\to\infty} \sup\{|p_n(\lambda) - p_m(\lambda)| : \lambda \in \sigma(T)\} = 0.$$

From (4.13), we obtain

$$\lim_{m,n\to\infty} \|p_n(T) - p_m(T)\| = \lim_{m,n\to\infty} \sup\{|p_n(\lambda) - p_m(\lambda)| : \lambda \in \sigma(T)\} = 0.$$

Since the algebra $\mathcal{B}(H)$ of bounded linear operators is complete, $\lim_{n\to\infty} p_n(T)$ exists. We denote this limit by $\boldsymbol{f(T)}$. It is quite obviously unique, because if $\{q_n\}_{n\geq 1}$ were to be another such sequence of polynomials, then so would the sequence $\{r_n\}_{n\geq 1}$ obtained by alternating p_n with q_n, so that $\lim_{n\to\infty} p_n(T)$ and $\lim_{n\to\infty} q_n(T)$ would both have to agree with $\lim_{n\to\infty} r_n(T)$. The properties of the map $f \to f(T)$ are summarised below. Part (d) is the **Spectral Mapping Theorem for continuous functions and self-adjoint operators**.

Theorem 4.10.1 *The mapping* $\varphi : f \to f(T)$ *from* $C(\sigma(T))$ *to* $\mathcal{B}(H)$ *is an isometric isomorphism of* $C(\sigma(T))$ *into the algebra* $\mathcal{B}(H)$. *Here, f and g are in* $C(\sigma(T))$.

(a) $(f+g)(T) = f(T) + g(T), (fg)(T) = f(T)(g(T), (\alpha f)(T) = \alpha(f(T))$ for complex α;
(b) $f(T)^* = \overline{f}(T)$;
(c) $\|f(T)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}$; if $f \geq 0$, then $f(T) \geq 0$; and $|f|(T) = |f(T)|$;
(d) $\sigma(f(T)) = f(\sigma(T))$ **(Spectral Mapping Theorem)**.

Proof Observe that *f*(*T*) is normal since $p_n(T)$'s are always normal [see Problem 3.7.P10].

(a) These hold when *f* and *g* are polynomials, and therefore, they hold for uniform limits of polynomials.
(b) It follows on taking limits in the equality $p(T)^* = \overline{p}(T)$ that $f(T)^* = \overline{f}(T)$. In particular, if *f* is real-valued, then *f*(*T*) is self-adjoint.
(c) Since *f*(*T*) is the uniform limit of $p_n(T)$,

$$||f(T)|| = \|\lim_n p_n(T)\| = \lim_n \|p_n(T)\| \tag{4.14}$$

and since $f(\lambda)$ is the uniform limit of $p_n(\lambda)$,

$$\sup\{|f(\lambda)| : \lambda \in \sigma(T)\} = \lim_n \sup\{|p_n(\lambda)| : \lambda \in \sigma(T)\}. \tag{4.15}$$

(4.14) and (4.15) together with (4.13) above yield

$$\|f(T)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}.$$

If $f \geq 0$, then $f = g^2$ for some $g \in C_{\mathbb{R}}(\sigma(T))$, and hence, $f(T) = g(T)^2 \geq 0$. Thus, for any $f \in C_{\mathbb{R}}(\sigma(T))$, we have $|f|(T) \geq 0$. Since $(|f|(T))^2 = |f|^2(T) = f^2(T) = f(T)^2$, it follows that $|f|(T) = |f(T)|$.
This completes the proof of (c).

(d) Let $\lambda' \notin f(\sigma(T))$. Then, there is no $\lambda \in \sigma(T)$ for which $f(\lambda) = \lambda'$, which means that the function $f - \lambda' \in C(\sigma(T))$ does not vanish anywhere on $\sigma(T)$, so that it has a continuous reciprocal $g \in C(\sigma(T))$. But then, the operator *g*(*T*) is an inverse for $(f - \lambda')(T) = f(T) - \lambda' I$, and hence, $\lambda' \notin \sigma(f(T))$. This shows that $\sigma(f(T)) \subseteq f(\sigma(T))$. We proceed to prove the reverse inclusion. Consider $\lambda' \in f(\sigma(T))$. This means $\lambda' = f(\lambda)$ for some $\lambda \in \sigma(T)$. We must show that $f(T) - \lambda' I$ is not invertible. Let $\{p_n\}_{n \geq 1}$ be a sequence of polynomials converging uniformly to f on $\sigma(T)$. Then, $\{p_n - p_n(\lambda)\}_{n \geq 1}$ is a sequence of polynomials converging uniformly to $f - f(\lambda)$ on $\sigma(T)$. It follows by parts (a) and (c) that the operator sequence $\{p_n(T) - p_n(\lambda)I\}_{n \geq 1}$ converges in norm to $f(T) - f(\lambda)I = f(T) - \lambda' I$. By Theorem 4.3.1, and the set on the right-hand side always has 0 in it. Consequently, none of the operators $p_n(T) - p_n(\lambda)I$ is invertible. Hence, by Theorem 3.3.9, their norm limit is also not invertible. However, their norm limit has already been shown to be $f(T) - \lambda' I$. Thus, $f(T) - \lambda' I$ is not invertible.

$$\sigma(p_n(T) - p_n(\lambda)I) = ((p_n - p_n(\lambda))(\sigma(T)) \qquad \square$$

We next construct a linear functional on $C(\sigma(T))$. For $x, y \in H$, define

$$F_{x,y}(f) = (f(T)x, y), \quad f \in C(\sigma(T)).$$

Using (c) of Theorem 4.10.1,

$$\begin{aligned} \left|F_{x,y}(f)\right| &= |(f(T)x, y)| \leq \|f(T)\| \|x\| \|y\| \\ &\leq \sup\{|f(\lambda)| : \lambda \in \sigma(T)\} \|x\| \|y\|. \end{aligned}$$

By the Riesz Representation Theorem [25, Chap. 6], there exists a uniquely determined complex measure $\mu_{x,y}$ such that

$$(f(T)x, y) = F_{x,y}(f) = \int f(\lambda) d\mu_{x,y}(\lambda), \tag{4.16}$$

where the integral is over $\sigma(T)$. Moreover, $|\mu_{x,y}|(\sigma(T)) = \|F_{x,y}\|$. The manner of dependence of $\mu_{x,y}$ on x and y is described in the theorem below.

Theorem 4.10.2 *Let $\mu_{x,y}$ be the measure on $\sigma(T)$ defined by (4.16). Then,*

(a) $\mu_{x,y}$ *is linear in x and conjugate linear in y*;
(b) $\mu_{x,y}$ *is conjugate symmetric, i.e.* $\mu_{x,y} = \overline{\mu_{y,x}}$;
(c) $\|F_{x,y}\| \leq \|x\| \|y\|$ *and the total variation* $|\mu_{x,y}| \leq \|x\| \|y\|$;
(d) *the measures* $\mu_{x,x}$ *are real and nonnegative-valued.*

Proof

(a) Observe that

$$\begin{aligned} F_{x+z,y}(f) &= (f(T)(x+z), y) = (f(T)x, y) + (f(T)z, y) \\ &= \int f(\lambda) d\mu_{x,y}(\lambda) + \int f(\lambda) d\mu_{z,y}(\lambda) = F_{x,y}(f) + F_{z,y}(f) \\ &= (F_{x,y} + F_{z,y})(f) \quad \text{for all} f \in C(\sigma(T)). \end{aligned}$$

So,

$$\int f(\lambda) d\mu_{x+z,y}(\lambda) = \int f(\lambda) d\mu_{x,y}(\lambda) + \int f(\lambda) d\mu_{z,y}(\lambda),$$

which implies

$$\int f(\lambda)d(\mu_{x+z,y} - \mu_{x,y} - \mu_{z,y})(\lambda) = 0$$

for all $f \in C(\sigma(T))$. Consequently,

$$\mu_{x+z,y} = \mu_{x,y} + \mu_{z,y},$$

i.e. $\mu_{x,y}$ is additive in x. The other assertions relating to linearity are proved similarly.

(b) Observe that $(f(T))^* = \bar{f}(T), f \in C(\sigma(T))$. So, for $x, y \in H$,

$$(f(T)x, y) = \left(x, \bar{f}(T)y\right) = \overline{(\bar{f}(T)y, x)}$$

which implies

$$\int f(\lambda)d\mu_{x,y}(\lambda) = \int \overline{\bar{f}(\lambda)d\mu_{y,x}(\lambda)} = \int f(\lambda)d\overline{\mu_{y,x}}(\lambda), \quad f \in C(\sigma(T)).$$

Thus,

$$\mu_{x,y} = \overline{\mu_{y,x}}.$$

(c) Since

$$\left|F_{x,y}(f)\right| = |(f(T)x, y)| \le \|f(T)\|\|x\|\|y\| = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}\|x\|\|y\|$$

for $x, y \in H$ and $f \in C(\sigma(T))$, it follows that

$$\left\|F_{x,y}\right\| \le \|x\|\|y\|.$$

As the total variation of the measure is the norm of the functional it represents, (c) follows.

(d) For a nonnegative function f, the operator $f(T)$ is positive by Theorem 4.10.1 (c). This shows that the linear functional $F_{x,x}(f) = (f(T)x,x)$ is positive. Consequently, the measure $\mu_{x,x}$ representing $F_{x,x}$ is nonnegative-valued. □

According to Theorem 4.10.2 above, $\mu_{x,y}(S)$ where S is a Borel subset of $\sigma(T)$ is a bounded, conjugate symmetric sesquilinear form on H. We conclude from Theorem 3.4.8 that for each S, there is a bounded self-adjoint operator $P(S)$ such that

$$\mu_{x,y}(S) = (P(S)x, y). \tag{4.17}$$

Theorem 4.10.3 *The family of operators* $\{P(S) : S$ *is a Borel subset of* $\sigma(T)\}$ *has the following properties*:

(a) $P(S)^* = P(S)$;
(b) $\|P(S)\| \leq 1$;
(c) $P(\varnothing) = O, P(\sigma(T)) = I$;
(d) *If* $S_1 \cap S_2 = \varnothing$, *then* $P(S_1 \cup S_2) = P(S_1) + P(S_2)$;
(e) *Each* $P(S)$ *commutes with every operator that commutes with* T; *in particular, it commutes with* T;
(f) *Each* $P(S)$ *is an orthogonal projection. If* S_1 *and* S_2 *are such that* $S_1 \cap S_2 = \varnothing$, *then* $\operatorname{ran}(P(S_1)) \perp \operatorname{ran}(P(S_2))$;
(g) *All orthogonal projections* $P(S_1)$, $P(S_2)$ *commute*;
(h) $P(S)$ *is strongly countably additive in the sense of* Definition 3.3.6.

In particular, P *is a spectral measure*.

Proof

(a) We have $\mu_{x,y}(S) = (P(S)x, y)$ and $\overline{\mu_{y,x}}(S) = \overline{(P(S)y, x)} = (x, P(S)y) = (P(S)^*x, y)$.
(b) In (c) of Theorem 4.10.2, it was proved that the total variation $|\mu_{x,y}| \leq \|x\|\|y\|$. So,

$$|(P(S)x, y)| \leq \|x\|||y||,$$

which implies

$$||P(S)|| = \sup\{|(P(S)x, y)| : \|x\| = 1 = \|y\|\} \leq 1.$$

(c) Since $\mu_{x,y}(\varnothing) = 0$, it follows from (4.17) that $(P(\varnothing)x, y) = 0$ for all $x, y \in H$, i.e. $P(\varnothing) = O$.
Setting $f(\lambda) = 1$ everywhere, we have $f(T) = I$ and (4.16) gives

$$(x, y) = \int d\mu_{x,y} = (P(\sigma(T))x, y), \quad x, y \in H,$$

which implies

$$P(\sigma(T)) = I.$$

(d) It follows from the additivity of the measure $\mu_{x,y}$.
(e) First, let R be any operator that commutes with T. It follows that R commutes with every polynomial in T and therefore with $f(T)$. So,

$$(f(T)Rx, y) = (Rf(T)x, y) = (f(T)x, R^*y).$$

The functional on the left side of the above equation is represented by the measure $\mu_{Rx,y}$, whereas the functional on the right is represented by the measure μ_{x,R^*y}. Since the functionals are the same, so are the measures which represent them:

$$\mu_{Rx,y} = \mu_{x,R^*y} = \overline{\mu_{R^*y,x}}. \tag{4.18}$$

On using (4.18), we obtain

$$(P(S)Rx, y) = \overline{(P(S)R^*y, x)} = \overline{(R^*y, P(S)x)} = (P(S)x, R^*y) = (RP(S)x, y).$$

Since this holds for all x and y, we have

$$P(S)R = RP(S)$$

for all Borel subsets S of $\sigma(T)$.

(f) Let S_1 and S_2 be Borel subsets of $\sigma(T)$. Then,

$$\mu_{x,y}(S_1 \cap S_2) = (P(S_1 \cap S_2)x, y). \tag{4.19}$$

Also,

$$\begin{aligned} \mu_{x,y}(S_1 \cap S_2) &= \int \chi_{S_1 \cap S_2} d\mu_{x,y} = \int \chi_{S_1}\chi_{S_2} d\mu_{x,y} \\ &= \int \chi_{S_1} d\mu_{P(S_2)x,y} = \mu_{P(S_2)x,y}(S_1) = (P(S_1)P(S_2)x, y). \end{aligned} \tag{4.20}$$

From (4.19) and (4.20), it follows that

$$P(S_1 \cap S_2) = P(S_1)P(S_2). \tag{4.21}$$

In particular,

$$P(S)^2 = P(S) \quad \text{for all Borel subsets } S \text{ of } \sigma(T).$$

In view of (a), each $P(S)$ is an orthogonal projection. Consequently,

$$\operatorname{ran}(P(S_1)) \perp \operatorname{ran}(P(S_2))$$

provided $S_1 \cap S_2 = \varnothing$.

(g) Interchanging the roles of S_1 and S_2 in (4.20) above, we have

$$P(S_1 \cap S_2) = P(S_2)P(S_1)$$

for all Borel subsets S_1 and S_2 of $\sigma(T)$.

(h) Let $\{S_i\}_{i\geq 1}$ be a sequence of pairwise disjoint Borel subsets of $\sigma(T)$. Observe that $P(\bigcup_{i=1}^n S_i)$ is an increasing sequence of self-adjoint operators bounded above by I, and therefore, by Theorem 3.6.14, it converges strongly. On the other hand, in view of part (d), we have $P(\bigcup_{i=1}^n S_i) = \sum_{i=1}^n P(S_i)$. Thus, the strong limit

$$\lim_n \sum_{i=1}^n P(S_i) = \sum_{i=1}^\infty P(S_i)$$

exists. It remains to prove that the strong limit equals $P(\bigcup_{i=1}^\infty S_i)$.

For x,y in H,

$$\mu_{x,y}\left(\bigcup_{i=1}^\infty S_i\right) = \sum_{i=1}^\infty \mu_{x,y}(S_i) = \sum_{i=1}^\infty (P(S_i)x, y) \tag{4.22}$$

Also,

$$\mu_{x,y}\left(\bigcup_{i=1}^\infty S_i\right) = \left(P\left(\bigcup_{i=1}^\infty S_i\right)x, y\right). \tag{4.23}$$

On comparing (4.22) and (4.23), and using the fact that $\sum_{i=1}^\infty P(S_i)$ is a strong limit and therefore also a weak limit, we obtain

$$\left(P\left(\bigcup_{i=1}^\infty S_i\right)x, y\right) = \sum_{i=1}^\infty (P(S_i)x, y) = \left(\sum_{i=1}^\infty P(S_i)x, y\right)$$

which implies

$$\left(\left[P\left(\bigcup_{i=1}^\infty S_i\right) - \sum_{i=1}^\infty P(S_i)\right]x, y\right) = 0.$$

Since this holds for all y in H, we obtain

$$P\left(\bigcup_{i=1}^\infty S_i\right)x = \sum_{i=1}^\infty P(S_i)x, \quad x \in H.$$

Theorem 4.10.4 (Spectral Theorem for Self-adjoint Operators) *Let H be a complex Hilbert space and $T \in \mathfrak{B}(H)$ satisfy $T^* = T$. Then, the spectral measure P on $\sigma(T)$ defined in* (4.17) *satisfies*

$$f(T) = \int\limits_{\sigma(T)} f(\lambda) \mathrm{d}P(\lambda) \tag{4.24}$$

for all continuous functions f on $\sigma(T)$. In particular,

$$I = \int\limits_{\sigma(T)} \mathrm{d}P(\lambda) \quad \textit{and} \quad T = \int\limits_{\sigma(T)} \lambda \, \mathrm{d}P(\lambda)\cdot$$

Proof Note that (4.16) leads to

$$\int \chi_S(\lambda) d\mu_{x,y}(\lambda) = \left(\int \chi_S(\lambda) \mathrm{d}P(\lambda)x, y\right).$$

It follows that the same holds for simple functions and hence for their uniform limits. Thus,

$$\int f(\lambda) d\mu_{x,y}(\lambda) = \left(\int f(\lambda) \mathrm{d}P(\lambda)x, y\right),$$

where f is any continuous function. In view of (4.16), the left-hand side equals $(f(T)x,y)$. Consequently,

$$(f(T)x, y) - \left(\int f(\lambda) \mathrm{d}P(\lambda)x, y\right) = 0.$$

Since this holds for all vectors x and y, the desired result, namely

$$f(T) = \int\limits_{\sigma(T)} f(\lambda) \mathrm{d}P(\lambda)$$

holds. □

Theorem 4.10.5 (Version of the Spectral Theorem) *Let T be a self-adjoint operator in a separable Hilbert space H. Then, there exists a measure μ on the Borel sets of $X = \sigma(T)$ and a real-valued bounded measurable function $\phi \in L^\infty(X, \mu)$ such that T is unitarily equivalent to the operator M_ϕ on $L^2(X, \mu)$, that is, there exists an isometric isomorphism $U : H \to L^2(X, \mu)$ such that $UTU^{-1} = M_\phi$.*

Proof We first consider the case when there exists a cyclic vector x [see Halmos [12]]. (A vector $x \in H$ is called a **cyclic vector** for a self-adjoint operator $T \in \mathcal{B}(H)$ if the closed linear span of $\{x, Tx, T^2x, \ldots\}$ is dense in H.) Without loss of generality, we may assume that $||x|| = 1$.

For each real polynomial p, we write

$$L(p) = (p(T)x, x).$$

Clearly, L is a linear functional; since

$$\begin{aligned} |L(p)| = |(p(T)x, x)| \leq ||p(T)|| ||x||^2 &= \sup\{|\lambda| : \lambda \in \sigma(p(T))\} \\ &= \sup\{|p(\lambda)| : \lambda \in \sigma(T)\}, \end{aligned}$$

(where we have used the Spectral Mapping Theorem 4.3.1 in the final step), the functional L is bounded for polynomials. It follows by the Weierstrass Theorem that L has a bounded extension to all the real-valued continuous functions on $\sigma(T)$. To prove L is positive, observe that if p is a real polynomial, then

$$L(p^2) = (p(T)^2x, x) = \|p(T)x\|^2 \geq 0. \tag{$*$}$$

If f is an arbitrary nonnegative continuous function on $\sigma(T)$, then f can be uniformly approximated by squares of real polynomials. Therefore,

$$\left|L(f) - L(p^2)\right| \leq ||L|| \sup\left\{|f - p^2|\right\}.$$

Hence, $L(f) \geq 0$, using ($*$) above.

(The above paragraph could be replaced by the following: for each continuous function $f \in C(\sigma(T))$, we write

$$L(f) = (f(T)x, x),$$

where $f(T)$ is to be understood as in Theorem 4.10.1. Clearly, L is a linear functional on $C(\sigma(T))$; since

$$\begin{aligned} |L(f)| &= |(f(T)x, x)| \leq \|f(T)\| ||x||^2 \\ &= \sup\{|\lambda| : \lambda \in \sigma(f(T))\} \text{ by Theorem 4.10.1(c)} \\ &= \sup\{|f(\lambda)| : \lambda \in \sigma(T)\} \text{ by Theorem 4.10.1(d)}, \end{aligned}$$

the functional L is bounded. By the second assertion of Theorem 4.10.1(c), L is positive.)

The Riesz Representation Theorem yields the existence of a finite positive measure μ defined on the σ-algebra Σ of Borel subsets of $\sigma(T)$ such that

$$(p(T)x, x) = \int_{\sigma(T)} p d\mu$$

for every real polynomial p. In particular, $\mu(\sigma(T)) = (x, x) = 1$.

For each polynomial q, possibly complex, write

$$Uq = q(T)x.$$

Since T is self-adjoint, $q(T)^* = \overline{q}(T)$ is a polynomial in T and so is $q(T)^*q(T) = |q|^2(T)$. It follows that

$$\int_{\sigma(T)} |q|^2 d\mu = (\overline{q}(T)q(T)x, x) = (q(T)^*q(T)x, x)$$
$$= (q(T)x, q(T)x) = \|q(T)x\|^2 = \|Uq\|^2.$$

This means that the linear transformation from the subset of $L^2(X, \mu)$ consisting of polynomials into H is an isometry, and hence, it has a unique isometric extension that maps $L^2(X, \mu)$ into H. The assumption that x is a cyclic vector implies that the range of U is in fact dense in, and hence equal to, the entire space H (range of an isometry is closed).

It remains only to prove that the $U^{-1}TU = M_\phi$ for some $\phi \in L^\infty(X, \mu)$. Write $\phi(\lambda) = \lambda, \lambda \in \sigma(T)$. It may be noted that $||\phi||_\infty \leq ||T||$. Given a complex polynomial q, put

$$\widetilde{q}(\lambda) = \lambda\, q(\lambda) = \phi(\lambda)q(\lambda).$$

Then,

$$U^{-1}TUq = U^{-1}Tq(T)x = U^{-1}\widetilde{q}(T)x = U^{-1}U\widetilde{q} = \widetilde{q}.$$

In other words, $U^{-1}TU$ agrees on polynomials with the multiplication induced by ϕ. That is enough to conclude that $U^{-1}TU$ is equal to that multiplication. This proves the result when there is a cyclic vector.

The general case can be reduced to this special case as follows.

Let x be any unit vector in H, and let S be the closed linear span of $\{x, Tx, T^2x, \ldots\}$. If $S = H$, the theorem reduces to the case considered above. If $S \neq H$, then S is a reducing subspace for T. Let $\mathfrak{S}$ be the collection of all families of mutually orthogonal closed subspaces of the form described above, partially ordered by inclusion. Each totally ordered subcollection $\mathfrak{S}_1$ of $\mathfrak{S}$ has an upper bound, namely the union of all families in the subcollection, as we now show. The union is surely a collection of subspaces of the form in question; consider any two subspaces in the union $\bigcup \mathfrak{S}_1$; they must belong to some two families belonging to the subcollection $\mathfrak{S}_1$; since $\mathfrak{S}_1$ is totally ordered, one of the two families must contain the other, so that both subspaces belong to the larger of the two families, which means they are mutually orthogonal. Thus, $\mathfrak{S}_1$ has an upper bound. By Zorn's lemma, $\mathfrak{S}$ must contain a maximal family $\mathfrak{S}_2$, i.e. a family of mutually orthogonal subspaces of the form described above that is not contained in any other such family.

We claim that the direct sum K of the subspaces in the maximal family $\mathfrak{S}_2$ is H.

If $K \neq H$, then H contains a nonzero vector y orthogonal to K. Then, the closed linear span of $\{y, Ty, T^2y, \ldots\}$ is orthogonal to K, which contradicts the maximality of $\mathfrak{S}_2$.

Since H is separable, $\mathfrak{S}_2$ is a countable collection $\{S_n\}_{n=1}^{\infty}$ of pairwise orthogonal subspaces of H such that

$$H = \bigoplus_{n=1}^{\infty} S_n.$$

Each S_n reduces T, and $T|_{S_n}$ has a cyclic vector for each n.

We have proved that for each n, there is an $L^2(X_n, \mu_n)$, where $X_n = \sigma(T|_{S_n})$, and a ϕ_n in $L^\infty(X_n, \mu_n)$ such that $T|_{S_n}$ is unitarily equivalent to M_{ϕ_n} on $L^2(X_n, \mu_n)$. We denote the relevant unitary map from $L^2(X_n, \mu_n)$ to S_n by U_n. Choose $x_n \in S_n$ such that $\|x_n\|^2 = 1/2^n$ and the linear span of $\{x_n, Tx_n, T^2x_n, \ldots\}$ is dense in S_n. For such a choice of x_n, $\mu_n(X_n) = 1/2^n$. Let $\phi = \phi_n$ on X_n. We claim that T is unitarily equivalent to $\bigoplus_{n=1}^{\infty} M_{\phi_n}$ on the space $\bigoplus_{n=1}^{\infty} L^2(X_n, \mu_n)$.

Let us regard X_n as subsets of distinct replicas of the complex plane, so that $X_n \cap X_m = \varnothing$ for $n \neq m$. Set $X = \bigcup_{n=1}^{\infty} X_n$, and suppose Σ consists of all sets of the form $M = \bigcup_{n=1}^{\infty} M_n$, where $M_n \subseteq X_n$ is a Borel subset of X_n. We define $\mu(M) = \sum_{n=1}^{\infty} \mu_n(M_n)$. Note that the series is dominated by the convergent series $\sum_{n=1}^{\infty} (1/2^n)$ and is therefore convergent. Then, (X, μ) is a finite measure space, and $L^2(X_n, \mu_n)$ is isomorphic to $\bigoplus_{n=1}^{\infty} L^2(X_n, \mu_n)$. Let ϕ be defined on X by setting $\phi(x) = \phi_n(x)$ for $x \in X_n$. Clearly, $\phi \in L^\infty(X, \mu)$ because ϕ_n are uniformly bounded by $\|T\|$. Moreover, M_ϕ defined on $L^2(X, \mu)$ is unitarily equivalent to $\bigoplus_{n=1}^{\infty} M_{\phi_n}$(via the isomorphism alluded to above) and hence to T (via the composition with $\bigoplus_{n=1}^{\infty} U_n$). □

The notations in Theorem 4.10.6 are as in Theorem 4.10.5.

Let M be a Borel subset of $\sigma(T)$ and $E(M)$ be the multiplication induced by $\chi_M \circ \phi$. Then, E is indeed a spectral measure:

(i) $E(\varnothing) = O$,

(ii) $E(\bigcup_{n=1}^{\infty} S_i) = \sum_{i=1}^{\infty} E(S_i)$, S_i pairwise disjoint Borel subsets of $\sigma(T)$, is easy to verify. E is called the **spectral measure of T**. The integral version of the spectral theorem, which is an easy consequence of Theorem 4.10.5, is the following.

Theorem 4.10.6 (Integral Version of the Spectral Theorem) *If T is a multiplication operator on $L^2(X, \mu)$ induced by the real-valued essentially bounded function ϕ, then $T = \int \lambda\, dE(\lambda)$, where E is the spectral measure of T.*

Proof Fix $x, y \in L^2(X, \mu)$ and write

$$\nu(M) = (E(M)x, y)$$

for each Borel set M; it is to be proved that

$$(Tx, y) = \left(\int \lambda\, dE(\lambda)x, y\right) = \int \lambda\, d\nu(\lambda)\cdot$$

By Theorem 4.10.5, $((E(M)x, y) = \int (\chi_M \circ \phi)x\bar{y}d\mu$ and $\nu(M) = \int \chi_M d\nu$. It follows that

$$\int (\chi_M \circ \phi)x\bar{y}d\mu = \int \chi_M d\nu$$

for all Borel sets M. This implies that

$$\int (f \circ \phi)x\bar{y}d\mu = \int f d\nu,$$

when f is a simple function and hence when $f \in L^\infty(X, \mu)$. In particular, this result holds for $f(\lambda) = \lambda$. □

We next define the support of a spectral measure.

Definition 4.10.7 Let P be a spectral measure on a Hausdorff topological space X with its Borel σ-algebra Σ, i.e. generated by the open subsets of X. Let E be the union of all open sets G for which $P(G) = O$. The set $X \backslash E$ is called the **support** of P and is written as **supp P**.

Proposition 4.10.8 *Let P be the projection-valued measure associated with the self-adjoint operator T as in* (4.16). *Then,* supp $P = \sigma(T)$.

Proof By definition, supp $(P) \subseteq \sigma(T)$. Suppose $U = \sigma(T) \backslash \mathrm{supp}(P)$. Our claim is that U is empty. If every open $G \subseteq \sigma(T)$ such that $P(G) = O$ is shown to be empty, then by the definition of support, our claim that U is empty will be established. In order to show this by contradiction, suppose that there exists a nonempty open $G \subseteq \sigma(T)$ such that $P(G) = O$. Choose any

$$t \in G \subseteq \sigma(T).$$

Then, there is an open interval $(t-\varepsilon, t+\varepsilon)$ such that $(t-\varepsilon, t+\varepsilon) \cap \sigma(T) \subseteq G$. Considering that $(t-\varepsilon, t+\varepsilon) \cap \sigma(T)$ is the same set as $\{\lambda \in \sigma(T) : |t-\lambda| < \varepsilon\}$, this implies

$$P(\{\lambda \in \sigma(T) : |t-\lambda| < \varepsilon\}) \leq P(G) = O.$$

Now, let ξ be an arbitrary unit vector, and consider the measure $\mu_{\xi,\xi}$ as in (4.17), namely

$$\mu_{\xi,\xi}(S) = (P(S)\xi, \xi).$$

By Theorem 4.10.2(d), it is a nonnegative measure and

$$\begin{aligned} \mu_{\xi,\xi}(\sigma(T)) &= (P(\sigma(T))\xi, \xi) \\ &= (\xi, \xi) \quad \text{by Theorem 4.10.3(c)} \\ &= 1 \quad \text{since } \xi \text{ is a unit vector.} \end{aligned}$$

Also,

$$\mu_{\xi,\xi}(\{\lambda \in \sigma(T) : |t-\lambda| < \varepsilon\}) = (P(\{\lambda \in \sigma(T) : |t-\lambda| < \varepsilon\})\xi, \xi) = 0.$$

Therefore, $\int_{\sigma(T)} (\lambda - t)^2 d\mu_{\xi,\xi}(\lambda) \geq \varepsilon^2$. On the other hand,

$$\begin{aligned} \|(T-tI)\xi\|^2 = ((T-tI)\xi, (T-tI)\xi) &= \left((T-tI)^2\xi, \xi\right) \\ &= \int\limits_{\sigma(T)} (\lambda - t)^2 d\mu_{\xi,\xi}(\lambda) \quad \text{by (4.19).} \end{aligned}$$

Consequently,

$$\|(T-tI)\xi\|^2 \geq \varepsilon^2.$$

This holds for an arbitrary unit vector ξ, and it follows that $t \notin \sigma(T)$, in contradiction to the choice of t. This contradiction shows that there cannot exist a nonempty open $G \subseteq \sigma(T)$ such that $P(G) = O$. As noted in the very first paragraph of the proof, this establishes our claim that U is empty. $\square$

4.11 Spectral Mapping Theorem For Bounded Normal Operators

We give below an elementary proof of the Spectral Mapping Theorem for polynomials of normal operators:

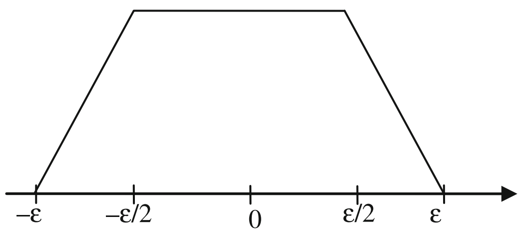
$$\sigma(p(N, N^*)) = \{p(\lambda, \overline{\lambda}) : \lambda \in \sigma(N)\}$$

for any normal operator N and any complex polynomial p in two variables. The following lemma of [32] will be needed in the proof of the above-mentioned theorem.

Lemma 4.11.1 (Whitley) *Let N be a normal operator whose spectrum contains* 0. *Given* $\varepsilon > 0$, *there exists a closed nonzero subspace M with the property that every operator which commutes with N^*N is reduced by M and the restriction $N|_M$ of the operator N to M satisfies* $\|T|_M\| \leq \varepsilon$.

Proof Let $A = N^*N$. Since $0 \in \sigma(N)$ and N is normal, by Theorem 4.4.1, there exists a sequence $\{x_n\}$ of unit vectors such that $Nx_n \to 0$. Thus, $Ax_n \to 0$ and the self-adjoint operator A is such that $0 \in \sigma(A)$. Given $\varepsilon > 0$, we consider the continuous function defined for all real t by

$$f(t) = \begin{cases} 1 & |t| \leq \varepsilon/2 \\ 2(1 - |t/\varepsilon|) & \varepsilon/2 \leq |t| \leq \varepsilon \\ 0 & |t| \geq \varepsilon \end{cases}$$



The symbol $f(A)$ has the same meaning as in Sect. 4.10.

Let $M = \{x \in H : f(A)x = x\}$, a closed subspace. As we have noted, if B is any operator which commutes with A, then B commutes with $f(A)$. Consequently, for any $x \in M$, $Bx = Bf(A)x = f(A)Bx$, which shows that M is invariant under B. Since

$$B^*A = B^*N^*N = (N^*NB)^* = (AB)^* = (BA)^* = AB^*,$$

M is invariant under B^* and so reduces B. For any $x \in M$ satisfying $\|x\| = 1$,

$$\|Ax\| = \|Af(A)x\| \leq \|Af(A)\| = \sup\{|\lambda f(\lambda)| : \lambda \in \sigma(A)\} \leq \varepsilon.$$

Indeed, $\sup\{|\lambda f(\lambda)| : \lambda \in \mathbb{R}\} \leq \varepsilon$ in view of the specific choice of *f*. Thus, $||Nx||^2 = (Ax, x) \leq \varepsilon$, and so, $||N|_M|| \leq \varepsilon^{\frac{1}{2}}$. It remains to show that $M \neq \{0\}$. We compute

$$\|(I - f(A))f(2A)\| = \sup\{|(1 - f(\lambda))f(2\lambda)| : \lambda \in \sigma(A)\} = 0$$

since $f(\lambda) = 1$ whenever $f(2\lambda) \neq 0$. Hence, every element in the range of the operator $f(2A)$ lies in M, and this range is not $\{0\}$ because $\|f(2A)\| = \sup\{|f(2\lambda)| : \lambda \in \sigma(A)\} \geq |f(0)| = 1$. □

Theorem 4.11.2 (Spectral Mapping Theorem for Normal Operators (and polynomials)) *If N is a normal operator and* $p(\cdot,\cdot)$ *a polynomial in two variables with complex coefficients, then*

$$\sigma(p(N, N^*)) = \left\{p(\lambda, \overline{\lambda}) : \lambda \in \sigma(N)\right\}.$$

Proof We write $p(\lambda, \overline{\lambda}) = \sum_{n,m} a_{n,m}\lambda^n\overline{\lambda}^m$. Let $\lambda \in \sigma(N)$. Then, there are vectors x_j of norm 1 with $(\lambda I - N)x_j \to 0$. By the normality of N, $(\overline{\lambda}I - N^*)x_j \to 0$. Then,

$$\begin{aligned}
\left(p(N, N^*) - p(\lambda, \overline{\lambda})I\right)x_j &= \Sigma_{n,m}a_{n,m}\left(N^nN - \lambda^n\overline{\lambda}^mI\right)x_j \\
&= \Sigma_{n,m}a_{n,m}\left(N^n\left(N - \overline{\lambda}^mI\right)x_j + \overline{\lambda}^m(N^n - \lambda^nI)x_j\right) \\
&= \Sigma_{n,m}a_{n,m}\Bigg(N^n\left(N{-}1 + \cdots + \overline{\lambda}^{m-1}I\right)\left(N^* - \overline{\lambda}I\right)x_j \\
&\qquad + \overline{\lambda}^m\left(N^{n-1} + \cdots + \lambda^{n-1}I\right)(N - \lambda I)x_j\Bigg),
\end{aligned}$$

which converges to 0 as $j \to \infty$. We conclude that $p(\lambda, \overline{\lambda})$ is in $\sigma(p(N, N^*)$.

Now, choose $\mu \in \sigma(p(N, N^*))$. The operator $B = p(N, N^*) - \mu I$ is normal and has 0 in its spectrum. By Lemma 4.11.1, for each n, there is a closed nonzero subspace M_n with $\|B|_{M_n}\| \leq \frac{1}{n}$ which reduces any operator that commutes with B^*B; in particular, it reduces N. Note that $(N|_{M_n})^* = N^*|_{M_n}$ because M_n reduces N. It therefore follows that $(N^*N)|_{M_n} = N^*|_{M_n}N|_{M_n} = (N|_{M_n})^*(N|_{M_n}) = (N|_{M_n})(N|_{M_n})^* = (NN^*)|_{M_n}$. Consequently, $N|_{M_n}$ is normal.

We know $\sigma(N|_{M_n})$ is nonvoid, and so we choose $\lambda_n \in \sigma(N|_{M_n})$. For this λ_n, there is a vector y_n of norm 1 in M_n with $\|(\lambda_nI - N)y_n\| \leq \frac{1}{n}$. The sequence $\{\lambda_n\}$ is bounded by $\|N\|$ and so contains a subsequence (denoted by $\{\lambda_n\}$ again) converging to some λ. Thus, $|\lambda_n - \lambda|$ and $\|(\lambda_nI - N)y_n\|$ both converge to 0. From the elementary inequality $\|(\lambda_nI - N)y_n\| \leq |\lambda_n - \lambda| + \|(\lambda_nI - N)y_n\|$, we now obtain $(\lambda I - N)y_n \to 0$, so that the point λ must in $\sigma(N)$. As we saw in the beginning of the proof, $(\lambda I - N)y_n \to 0$ implies that $\{(p(N, N^*) - p(\lambda, \overline{\lambda})I)y_n\}$ must also converge to 0. Recall that the vector y_n is in the subspace M_n, $||y_n|| = 1$ and

$\|B|_{M_n}\| \le \frac{1}{n}$; so $\{By_n\}$ converges to 0. By the definition of B, this means $\{(p(N,N^*)-\mu I)y_n\}$ converges to 0. The two convergence facts that have just been deduced, when combined with the elementary observation that

$$\big(p(\lambda,\overline{\lambda}) - \mu\big)y_n = (p(N,N^*) - \mu I)y_n - \big(p(N,N^*) - p(\lambda,\overline{\lambda})I\big)y_n,$$

lead to the conclusion that $(p(\lambda,\overline{\lambda}) - \mu)y_n$ converges to 0. But $\|y_n\| = 1$, and therefore, we get $\mu = p(\lambda,\overline{\lambda})$. This completes the proof. □

The following will be needed in the proof of the spectral theorem.

Let $p(\cdot,\cdot)$ be a polynomial $p(\lambda,\overline{\lambda}) = \Sigma_{n,m}a_{n,m}\lambda^n\overline{\lambda}^m$ with complex coefficients $a_{n,m}$. Then, for a normal operator N in $\mathcal{B}(H)$, by $p(N,N^*)$ we understand

$$p(N,N^*) = \Sigma_{n,m}a_{n,m}N^nN^{*m}.$$

We shall be regarding $p(\cdot,\cdot)$ as a function on $\mathbb{C}$ or on a subset of it. Therefore, in order that $p(N,N^*)$ be well defined, it is essential to know that the function $p(\cdot,\cdot)$ cannot be identically 0 unless each coefficient is individually 0. This is far from obvious because polynomials such as $\lambda - \overline{\lambda}$ and $1 - \overline{\lambda}\lambda$ vanish on infinite subsets of $\mathbb{C}$. However, it is true that if $p(\lambda,\overline{\lambda})$ vanishes on the whole of $\mathbb{C}$, then each coefficient is 0, as we shall prove presently. Only after this is proved, will $p(N,N^*)$ be well defined.

Suppose $p(\lambda,\overline{\lambda}) = \Sigma_{n,m}a_{n,m}\lambda^n\overline{\lambda}^m$ vanishes for every $\lambda \in \mathbb{C}$. Take an arbitrary $\alpha > 0$. Considering that $p(\lambda,\overline{\lambda})$ vanishes on the circle of radius $\alpha^{\frac{1}{2}}$, on which $\lambda\overline{\lambda} = \alpha$, we have $\Sigma_{n,m}a_{n,m}\lambda^n(\alpha^m/\lambda^m) = 0$ for every $\lambda \in \mathbb{C}$ such that $\lambda\overline{\lambda} = \alpha$. Now,

$$\begin{aligned}\Sigma_{n,m}a_{n,m}\lambda^n(\alpha^m/\lambda^m) &= \Sigma_{n,m}a_{n,m}\alpha^m\lambda^{n-m} = \Sigma_{k,m}a_{m+k,m}\alpha^m\lambda^k \\ &= \lambda^{-k_0}\Sigma_{k,m}a_{m+k,m}\alpha^m\lambda^{k+k_0},\end{aligned}$$

where k_0 can be so chosen as to guarantee that every $k + k_0$ in the (finite) sum is nonnegative. Then, the ordinary polynomial in λ given by

$$\Sigma_{k,m}a_{m+k,m}\alpha^m\lambda^{k+k_0},$$

wherein the coefficient of λ^{k+k_0} is $\Sigma_m a_{m+k,m}\alpha^m$, has the property that it vanishes on the circle $\lambda\overline{\lambda} = \alpha$. However, this circle has infinitely many points on it, and an ordinary polynomial has at most finitely many roots unless each coefficient is 0. Therefore, the coefficient $\Sigma_m a_{m+k,m}\alpha^m$ must be 0 for each k. This in turn has the consequence that the ordinary polynomial $p_k(\zeta) = \Sigma_m a_{m+k,m}\zeta^m$ vanishes for $\zeta = \alpha$ and each k. Since $\alpha > 0$ is arbitrary, it follows that each $p_k(\zeta)$ vanishes on the right half of the real axis, which has infinitely many points on it. It follows that each coefficient of each $p_k(\zeta)$ is 0, i.e. $a_{m+k,m} = 0$ for every m and every k. This concludes the proof that every $a_{n,m}$ is 0.

We note in passing that the above argument can be easily modified to prove that if $p(\lambda, \overline{\lambda})$ vanishes on an open disc, then every coefficient is 0, because there have to be infinitely many positive α such that the circle $\lambda\overline{\lambda} = \alpha$ has infinitely many points lying in the disc.

Denote by $\mathcal{P}$ the algebra over $\mathbb{C}$ of all functions p with domain $\mathbb{C}$ that are given by a polynomial $p(\lambda, \overline{\lambda})$. It is closed under complex conjugation, and moreover,

$$p(N, N^*)^* = \left(\Sigma_{n,m} a_{n,m} N^n N\right)^* = \Sigma_{n,m} \overline{a}_{n,m} N N^m = \overline{p}(N, N^*).$$

Thus, the mapping $\varphi : \mathcal{P} \to \mathcal{B}(H)$ given by $\varphi(p) = p(N, N^*)$, which is obviously an algebra homomorphism, satisfies $\varphi(\overline{p}) = p(N, N^*)^*$.

According to the Spectral Mapping Theorem (4.11.2),

$$\sigma(p(N, N^*)) = \left\{p(\lambda, \overline{\lambda}) : \lambda \in \sigma(N)\right\}.$$

Observe that $p(N, N^*)$ is a normal operator. From this equality and Remark 4.2.7(i) that the spectral radius of a normal operator equals its norm, it follows that

$$\begin{aligned} \|p(N, N^*)\| &= r(p(N, N^*)) = \sup\{|\lambda| : \lambda \in \sigma(p(N, N^*))\} \\ &= \sup\left\{\left|p(\lambda, \overline{\lambda})\right| : \lambda \in \sigma(N)\right\}. \end{aligned} \tag{4.25}$$

The class of restrictions to $\sigma(N)$ of the functions of $\mathcal{P}$ is an algebra over $\mathbb{C}$, which separates points, contains constants and is closed under complex conjugation. Therefore, the Stone–Weierstrass approximation theorem is applicable, and consequently, any continuous complex-valued function $f(\lambda)$ on $\sigma(N)$ can be approximated uniformly on $\sigma(N)$ by such functions. So, there exists a sequence $\{p_n(\lambda, \overline{\lambda})\}_{n \geq 1}$ of polynomials such that

$$\lim_{n \to \infty} \sup\left\{\left|f(\lambda) - p_n(\lambda, \overline{\lambda})\right| : \lambda \in \sigma(N)\right\} = 0.$$

It follows that $\{p_n\}_{n \geq 1}$ is a Cauchy sequence:

$$\lim_{m,n \to \infty} \sup\left\{\left|p_n(\lambda, \overline{\lambda}) - p_m(\lambda, \overline{\lambda})\right| : \lambda \in \sigma(N)\right\} = 0.$$

From (4.25), we obtain

$$\begin{aligned} \lim_{m,n \to \infty} \|p_n(N, N^*) - p_m(N, N^*)\| &= \lim_{m,n \to \infty} \sup\left\{\left|p_n(\lambda, \overline{\lambda}) - p_m(\lambda, \overline{\lambda})\right| : \lambda \in \sigma(N)\right\} \\ &= 0. \end{aligned}$$

Since the algebra $\mathcal{B}(H)$ of bounded linear operators is complete, $\lim_{n \to \infty} p_n(N, N^*)$ exists. We denote this limit by $\boldsymbol{f(N)}$. It is quite obviously unique, because if $\{q_n\}_{n \geq 1}$ were to be another such sequence of polynomials, then so would the sequence $\{r_n\}_{n\geq 1}$ obtained by alternating p_n with q_n, so that $\lim_{n\to\infty}p_n(N,N^*)$ and $\lim_{n\to\infty}q_n(N,N^*)$ would both have to agree with $\lim_{n\to\infty}r_n(N,N^*)$. The properties of the map $f\to f(N)$ are summarised below. Part (d) is the **Spectral Mapping Theorem for continuous functions and normal operators**.

Theorem 4.11.3 *Let N be a normal operator in $\mathcal{B}(H)$. The mapping $\varphi : f\to f(N)$ from $C(\sigma(N))$ to $\mathcal{B}(H)$ is an isometric isomorphism of $C(\sigma(N))$ into the algebra $\mathcal{B}(H)$. Here, f and g are in $C(\sigma(N))$.*

(a) $(f+g)(N)=f(N)+g(N), (fg)(N)=f(N)(g(N), (\alpha f)(N)=\alpha(f(N))$ for complex α;
(b) $f(N)^*=\overline{f}(N)$;
(c) $\|f(N)\|=\sup\{(|f(\lambda)|):\lambda\in\sigma(N)\}$; if $f\geq 0$, then $f(N)\geq O$;
(d) $\sigma(f(N))=f(\sigma(N))$ (**Spectral Mapping Theorem**)

Proof Observe that $f(N)$ is normal since $p_n(N,N^*)$'s are always normal [see Problem 3.7.P10].

(a) These hold when f and g are polynomials, and therefore, they hold for uniform limits of polynomials.
(b) It follows on taking limits in the equality $p(N,N^*)^*=\overline{p}(N,N^*)$ that $f(N)^*=\overline{f}(N)$. In particular, if f is real-valued, then $f(N)$ is self-adjoint.
(c) Since $f(N)$ is the uniform limit of $p_n(N,N^*)$,

$$\|f(N)\|=\|\lim_n p_n(N,N^*)\|=\lim_n\|p_n(N,N^*)\| \tag{4.26}$$

and since $f(\lambda)$ is the uniform limit of $p_n(\lambda,\overline{\lambda})$,

$$\sup\{|f(\lambda)|:\lambda\in\sigma(N)\}=\lim_n\sup\left\{\left|p_n(\lambda,\overline{\lambda})\right|:\lambda\in\sigma(N)\right\}. \tag{4.27}$$

(4.26) and (4.27) together with (4.25) above yield

$$\|f(N)\|=\sup\{|f(\lambda)|:\lambda\in\sigma(N)\}.$$

If $f\geq 0$, then $f=g^2$ for some $g\in C_{\mathbb{R}}(\sigma(N))$, and hence, $f(N)=g(N)^2\geq O$. Thus, for any nonnegative $f\in C_{\mathbb{R}}(\sigma(N))$, we have $f(N)\geq O$.
This completes the proof of (c).
(d) Let $\lambda'\notin f(\sigma(N))$. Then, there is no $\lambda\in\sigma(N)$ for which $f(\lambda)=\lambda'$, which means that the function $f-\lambda'\in C(\sigma(N))$ does not vanish anywhere on $\sigma(N)$, so that it has a continuous reciprocal $g\in C(\sigma(N))$. But then the operator $g(N)$ is an inverse for $(f-\lambda')(N)=f(N)-\lambda' I$, and hence, $\lambda'\notin\sigma(f(N))$. This shows that $\sigma(f(N))\subseteq f(\sigma(N))$. We proceed to prove the reverse inclusion.

Consider $\lambda' \in f(\sigma(N))$. This means $\lambda' = f(\lambda)$ for some $\lambda \in \sigma(N)$. We must show that $f(N) - \lambda' I$ is not invertible. Let $\{p_n\}_{n\geq 1}$ be a sequence of polynomials converging uniformly to f on $\sigma(N)$. Then, $\{p_n - p_n(\lambda, \overline{\lambda})\}_{n\geq 1}$ is a sequence of polynomials converging uniformly to $f - f(\lambda)$ on $\sigma(N)$. It follows by parts (a) and (c) that the operator sequence $\{p_n(N, N^*) - p_n(\lambda, \overline{\lambda})I\}_{n\geq 1}$ converges in norm to $f(N) - f(\lambda)I = f(N) - \lambda' I$. By Theorem 4.11.2,

$$\sigma\big(p_n(N, N^*) - p_n(\lambda, \overline{\lambda})I\big) = \big(p_n - p_n\big(\lambda, \overline{\lambda}\big)\big)(\sigma(N))$$

and the set on the right-hand side always has 0 in it. Consequently, none of the operators $p_n(N, N^*) - p_n(\lambda, \overline{\lambda})I$ is invertible. Hence, by Theorem 3.3.9, their norm limit is also not invertible. However, their norm limit has already been shown to be $f(N) - \lambda' I$. Thus, $f(N) - \lambda' I$ is not invertible. □

Example 4.11.4 Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and $H = L^2(X, \mathfrak{M}, \mu)$ be the Hilbert space of square integrable functions defined on X. For $\phi \in L^\infty(X, \mathfrak{M}, \mu)$, define $(M_\phi x)(T) = \phi(T)x(T), x \in H, t \in X$. Then, M_ϕ is a bounded linear operator and $\|M_\phi\| = \|\phi\|_\infty$ [see (vi) of Examples 3.2.5], and its spectrum, as we show below, is what is called the **essential range** of ϕ (denoted by **ess ran**(ϕ)); that is to say

$$\sigma(M_\phi) = \text{ess ran}(\phi) = \{\lambda : \mu\{x : |\phi(x) - \lambda| < \varepsilon\} > 0 \,\text{for all}\, \varepsilon > 0\}.$$

In particular, the essential range is compact and hence a Borel set in $\mathbb{C}$.

The proof of the above equality follows.

(A) If $\lambda \notin \sigma(M_\phi)$, then $\lambda \notin$ ess ran(ϕ).
Consider any $\lambda \notin \sigma(M_\phi)$. Then, the operator $M_\phi - \lambda I$, which maps L^2 into itself, has a bounded inverse. In particular, it is bounded below. That is, there exists $\varepsilon > 0$ such that

$$\big\|\big(M_\phi - \lambda I\big)f\big\| \geq \varepsilon \|f\| \quad \text{for all}\, f \in L^2(X, \mathfrak{M}, \mu). \tag{4.28}$$

Let $A \subseteq X$ be the set $\{x \in X : |\phi(x) - \lambda| < \frac{\varepsilon}{2}\}$. We shall show that $\mu(A) = 0$, which means precisely that $\lambda \notin$ ess ran(ϕ) as per the definition of ess ran(ϕ) . Suppose to the contrary that $\mu(A) > 0$. Then, select f to be the characteristic function χ_A. This ensures that

$$\|f\|^2 = \mu(A) > 0.$$

On the other hand, $|\phi(x) - \lambda| \cdot \chi_A(x) < \frac{\varepsilon}{2}$ for all x, and therefore,

$$\left\|\left(M_\phi - \lambda I\right)f\right\|^2 = \int [|\phi - \lambda|\chi_A]^2 d\mu \le \frac{\varepsilon^2}{4}\,\mu(A) = \frac{\varepsilon^2}{4}\,\|f\|^2 > 0.$$

This contradicts (4.28), thereby showing that $\mu(A) = 0$ as desired.

(B) If $\lambda \in$ ess ran(ϕ), then $\lambda \notin \sigma(M_\phi)$.

Suppose $\lambda \notin$ ess ran(ϕ). This means there exists $\varepsilon > 0$ such that $\mu\{x \in X : |\phi(x) - \lambda| < \varepsilon\} = 0$. Therefore, not only is $\mu\{x \in X : |\phi(x) - \lambda| = 0\} = 0$, so that the function $1/(\phi - \lambda)$ is defined almost everywhere, but also is $\mu\{x \in X : 1/|\phi(x) - \lambda| > 1/\varepsilon\} = 0$. Consequently $\|1/|\phi - \lambda|\|_\infty \le \frac{1}{\varepsilon}$. It follows that $1/(\phi - \lambda) \in L^\infty(X, \mathfrak{M}, \mu)$ and hence gives rise to a bounded linear multiplication operator $M_{1/(\phi-\lambda)}$. It is obvious that this multiplication operator is the inverse of $M_{\phi-\lambda}$, which is the same operator as $M_\phi - \lambda I$. Consequently, $\lambda \notin \sigma(M_\phi)$.

This completes the proof.

Also, $(M_\phi)^* = M_{\overline{\phi}}$, where $\overline{\phi}(t) = \overline{\phi(t)}$ for all $t \in X$ [see (v) of Examples 3.5.10]. Thus, $M_\phi(M_\phi)^* = (M_\phi)^* M_\phi = M_{|\phi|^2}$ and M_ϕ is normal [see (iv) of Remarks 3.6.2].

4.12 Spectral Theorem for Bounded Normal Operators

The notion of spectral measure and that of spectral integral were introduced in Sect. 4.9 with a view to establishing a version of the spectral theorem for self-adjoint [normal] operators analogous to the spectral theorem for such operators in finite-dimensional Hilbert spaces, which states that they have diagonal matrices with respect to an appropriately chosen orthonormal basis [2, Chap 1].

The spectral theorem for self-adjoint operators was obtained in Sect. 4.10 [see Theorem 4.10.4]. It is proposed to obtain a similar theorem for normal operators.

We use now the map $f \to f(N)$ from $C(\sigma(N))$ to $\mathcal{B}(H)$ described in Theorem 4.11.3 to construct the functional

$$L_{x,y}(f) = (f(N)x, y), \quad x, y \in H \tag{4.29}$$

on $C(\sigma(N))$. By the Riesz Representation Theorem, there exists a uniquely determined complex measure $\mu_{x,y}$ on $\sigma(N)$ such that

$$(f(N)x, y) = \int_{\sigma(N)} f(\lambda)\, d\mu_{x,y}(\lambda). \tag{4.30}$$

Since $L_{x,y}$ depends on x and y, so does the measure $\mu_{x,y}$; the manner of its dependence on x and y reflects the manner of dependence of $L_{x,y}$ on x and y. We list the details in the theorem below.

Theorem 4.12.1 *Let $\mu_{x,y}$ be the measure defined on $\sigma(N)$ by (4.30) above.*

(a) *$\mu_{x,y}$ depends sesquilinearly on x and y, that is, linearly on x and conjugate linearly on y;*
(b) *$\mu_{x,y}$ is conjugate symmetric in x and y, that is, $\mu_{x,y} = \overline{\mu_{y,x}}$;*
(c) $|\mu_{x,y}| \leq \|x\|\|y\|$;
(d) *the measure $\mu_{x,x}$ is real and nonnegative.*

Proof Same as 4.10.2. □

According to Theorem 4.10.2, $[x, y] = \mu_{x,y}(S)$, where S is a Borel subset of $\sigma(N)$, is a bounded conjugate symmetric sesquilinear functional of x and y. It follows from Theorem 3.4.8 that for each S, there is a bounded self-adjoint operator $P(S)$ such that

$$\mu_{x,y}(S) = (P(S)x, y).$$

The family of operators has the following properties.

Theorem 4.12.2 (Cf. Theorem 4.10.3) *The family of operators $\{P(S) : S$ is a Borel subset of $\sigma(N)\}$ has the following properties:*

(a) $P(S)^* = P(S)$;
(b) $\|P(S)\| \leq 1$;
(c) $P(\varnothing) = O, P(\sigma(N)) = I$;
(d) *If $S_1 \cap S_2 = \varnothing$, then $P(S_1 \cup S_2) = P(S_1) + P(S_2)$;*
(e) *Each $P(S)$ commutes with every operator that commutes with N; in particular, it commutes with N;*
(f) *Each $P(S)$ is an orthogonal projection. If S_1 and S_2 are such that $S_1 \cap S_2 = \varnothing$, then $\mathrm{ran}(P(S_1)) \perp \mathrm{ran}(P(S_2))$;*
(g) *All orthogonal projections $P(S_1), P(S_2)$ commute;*
(h) *$P(S)$ is strongly countably additive in the sense of* Definition 3.3.6.

In particular, P is a spectral measure.

Proof Same as Theorem 4.10.3. □

Theorem 4.12.3 (Spectral Theorem for Normal Operators) *Let $N \in \mathcal{B}(H)$ be such that $N^*N = NN^*$. Then, there is a uniquely defined spectral measure P on $\sigma(N)$ such that*

$$f(N) = \int_{\sigma(N)} f(\lambda) \mathrm{d}P(\lambda)$$

for all continuous functions f on $\sigma(N)$. *In particular,*

$$I = \int_{\sigma(N)} \mathrm{d}P(\lambda) \quad \textit{and} \quad N = \int_{\sigma(N)} \lambda \, \mathrm{d}P(\lambda).$$

Proof Same as Theorem 4.10.4. □

Recall that a vector $x \in H$ is called a *cyclic vector* for an operator $N \in \mathcal{B}(H)$ if the set of all vectors of the form $p(N,N^*)x$, where p is a polynomial in two variables, is dense in H.

Theorem 4.12.4 *If N is a normal operator on a separable Hilbert space H, then there exists a finite measure space* $(X, \mathfrak{M}, \mu)$ *and* $\phi \in L^\infty(x)$ *such that N is unitarily equivalent to the operator* M_ϕ *on* $L^2(X)$, *i.e. there exists a unitary operator* $U : L^2(X) \to H$ *such that* $N = UM_\phi U^{-1}$. (Here, U being unitary means it is bijective and $(Ux,Uy) = (x,y)$ for all $x, y \in L^2(X)$.)

Proof We first consider the case when N has a cyclic vector $\xi \in H$. Assume without loss of generality that $\|\xi\| = 1$.

For each continuous function $f \in C(\sigma(N))$, we write

$$F(f) = (f(N)\xi, \xi),$$

where $f(N)$ is to be understood as in Theorem 4.11.3. Clearly, F is a linear functional on $C(\sigma(N))$; since

$$\begin{aligned} |F(f)| &= |(f(N)\xi, \xi)| \le \|f(N)\| \|\xi\|^2 \\ &= \sup\{|\lambda| : \lambda \in \sigma(f(N))\} \text{ by Theorem 4.11.3} \\ &= \sup\{|f(\lambda)| : \lambda \in \sigma(N)\} \text{ by Theorem 4.11.3,} \end{aligned}$$

the functional F is bounded. By the second assertion of Theorem 4.11.3, F is positive.

Now, the Riesz Representation Theorem [25, Chap. 6] implies that there exists a regular positive finite Borel measure μ on the σ-algebra $\mathfrak{M}$ of Borel subsets of $\sigma(N)$ such that

$$F(g) = \int_{\sigma(N)} g d\mu \quad \text{for all } g \in C(\sigma(N)).$$

We shall show that N is unitarily equivalent to M_ϕ on $L^2(\sigma(N))$, where ϕ is given by $\phi(\lambda) = \lambda$.

Define U on the set of all polynomials $p(\lambda, \overline{\lambda}) \in L^2(\sigma(N))$ into H by

$$U\big(p(\lambda, \overline{\lambda})\big) = p(N, N^*)\xi.$$

Then,

$$\begin{aligned}\int_{\sigma(N)} \left|p(\lambda, \overline{\lambda})\right|^2 d\mu &= \int_{\sigma(N)} p(\lambda, \overline{\lambda})\overline{p(\lambda, \overline{\lambda})} d\mu \\ &= F\Big(p(\lambda, \overline{\lambda})\overline{p(\lambda, \overline{\lambda})}\Big) \\ &= (p(N, N^*)[p(N, N^*)]^*\xi, \xi) \\ &= \|p(N, N^*)\xi\|^2.\end{aligned}$$

Since the linear manifold $\{p(N, N^*)\xi : p \text{ is a polynomial}\}$ is dense in H, U has a unique extension to a unitary operator from $L^2(\sigma(N))$ to H. Also, for any polynomial p,

$$U^{-1}NUp = U^{-1}Np(N, N^*)\xi = U^{-1}\big(U\big(M_\lambda p(\lambda, \overline{\lambda})\big)\big) = M_\lambda p(\lambda, \overline{\lambda}).$$

This completes the proof in the case that N has a cyclic vector. We shall reduce the general case to this special case.

Let x be any unit vector in H, and let S be the closed linear span of $\{p(N, N^*)x : p \text{ is a polynomial}\}$. If $S = H$, the theorem reduces to the case considered above. If $S \neq H$, then S is a reducing subspace for N. Let $\mathfrak{S}$ be the collection of all families of mutually orthogonal closed subspaces of the form described above, partially ordered by inclusion. Each totally ordered subcollection $\mathfrak{S}_1$ of $\mathfrak{S}$ has an upper bound, namely the union of all families in the subcollection, as we now show. The union is surely a collection of subspaces of the form in question; consider any two subspaces in the union $\cup\ \mathfrak{S}_1$; they must belong to some two families belonging to the subcollection $\mathfrak{S}_1$; since $\mathfrak{S}_1$ is totally ordered, one of the two families must contain the other, so that both subspaces belong to the larger of the two families, which means they are mutually orthogonal. Thus, $\mathfrak{S}_1$ has an upper bound. By Zorn's lemma, $\mathfrak{S}$ must contain a maximal family $\mathfrak{S}_2$, i.e. a family of mutually orthogonal subspaces of the form described above that is not contained in any other such family.

We claim that the direct sum K of the subspaces in the maximal family $\mathfrak{S}_2$ is H.

If $K \neq H$, then H contains a nonzero vector y orthogonal to K. Then, the closed linear span of $\{p(N, N^*)y : p \text{ is a polynomial}\}$ is orthogonal to K, which contradicts the maximality of $\mathfrak{S}_2$.

Since H is separable, $\mathfrak{S}_2$ is a countable collection $\{S_n\}_{n=1}^{\infty}$ of pairwise orthogonal subspaces of H such that

$$H = \bigoplus_{n=1}^{\infty} S_n.$$

Each S_n reduces N, and $N|_{S_n}$ has a cyclic vector for each n.

We have proved that for each n, there is an $L^2(X_n, \mu_n)$, where $X_n = \sigma(N|S_n)$, and a ϕ_n in $L^\infty(X_n, \mu_n)$ such that $N|_{S_n}$ is unitarily equivalent to M_{ϕ_n}on $L^2(X_n, \mu_n)$. We denote the relevant unitary map from $L^2(X_n, \mu_n)$ to S_n by U_n. Choose $x_n \in S_n$ such that $\|x_n\|^2 = 1/2^n$ and the linear span of $\{p(N, N^*)x_n : p \text{ is a polynomial}\}$ is dense in S_n. For such a choice of X_n, $\mu_n(X_n) = 1/2^n$. Let $\phi = \phi_n$ on X_n. We claim that N is unitarily equivalent to $\bigoplus_{n=1}^{\infty} M_{\phi_n}$ on the space $\bigoplus_{n=1}^{\infty} L^2(X_n, \mu_n)$.

Let us regard X_n as subsets of distinct replicas of the complex plane, so that $X_n \cap X_m = \varnothing$ for $n \neq m$. Set $X = \cup_{n=1}^{\infty} X_n$, and suppose Σ consists of all sets of the form $M = \cup_{n=1}^{\infty} M_n$, where $M_n \subseteq X_n$ is a Borel subset of X_n. We define $\mu(M) = \sum_{n=1}^{\infty} \mu_n(M_n)$. Note that the series is dominated by the convergent series $\sum_{n=1}^{\infty} (1/2^n)$ and is therefore convergent. Then, (X, μ) is a finite measure space, and $L^2(X, \mu)$ is isomorphic to $\bigoplus_{n=1}^{\infty} L^2(X_n, \mu_n)$. Let ϕ be defined on X by setting $\phi(x) = \phi_n(x)$ for $x \in X_n$. Clearly, $\phi \in L^\infty(X, \mu)$ because ϕ_n are uniformly bounded by $\|N\|$. Moreover, M_ϕ defined on $L^2(X, \mu)$ is unitarily equivalent to $\bigoplus_{n=1}^{\infty} M_{\phi_n}$ (via the isomorphism alluded to above) and hence to N (via the composition with $\bigoplus_{n=1}^{\infty} U_n$).□

The notations below are as in Example 4.11.4.

Let M be a Borel subset of ess ran(ϕ) and $P(M)$ be the multiplication induced by $\chi_M \circ \phi$. Then, P is indeed a spectral measure:

(i) $P(\varnothing) = O$,

(ii) $P\left(\bigcup_{n=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} P(S_i)$, $\quad S_i$ pairwise disjoint Borel subsets of ess ran(ϕ),

as is easy to verify. P is called the spectral measure of the multiplication operator M_ϕ. The integral version of the spectral theorem, which is an easy consequence of Theorem 4.12.4, is the following.

Theorem 4.12.5 (Spectral Theorem) *If N is a normal operator on a Hilbert space H, then there is a unique spectral measure P such that $N = \int \lambda \, dP(\lambda)$.*

Proof By Theorem 4.12.4, we may assume that $N = M_\phi$ on $L^2(X, \mathfrak{M}, \mu)$, where $\phi \in L^\infty(X, \mathfrak{M}, \mu)$, X being the essential range of ϕ, i.e. the spectrum of M_ϕ and $\mathfrak{M}$ the σ-algebra of Borel sets. For each Borel set S, define $P(S)$ as multiplication by $\chi_S \circ \phi$. It has been observed in Examples 4.9.3 that P is a spectral measure and in

4.11.4 that the support is the essential range of ϕ, i.e. the spectrum of M_ϕ. We next show that

$$M_\phi = \int \lambda \, \mathrm{d}P(\lambda).$$

For a fixed Borel set $S \in \mathfrak{M}$ and $f \in L^2(X, \mathfrak{M}, \mu)$,

$$\int \chi_s \mathrm{d}(P(\lambda)f, f) = (P(S)f, f) = \int (\chi_s \circ \phi) f\bar{f} d\mu.$$

Thus, for a measurable simple function $\psi = \sum \alpha_i \chi_{S_i}$,

$$\int_{\mathbb{C}} \psi \, d(P(\lambda)f, f) = \int_X (\psi \circ \phi)|f^2| d\mu$$

and therefore, the relation holds for the identity function $h(\lambda) = \lambda$. Thus,

$$\int_{\mathbb{C}} \lambda \, d(P(\lambda)f, f) = \int_X h|f^2| d\mu = (M_\phi f, f).$$

Using the polarisation identity [Proposition 2.2.3(d)] yields

$$\int_{\mathbb{C}} \lambda \, d(P(\lambda)f, g) = (M_\phi f, g)$$

for all f and g in $L^2(X, \mathfrak{M}, \mu)$.

The uniqueness is easily proved as follows.

Suppose that $N = \int \lambda \, \mathrm{d}P(\lambda) = \int \lambda \, dQ(\lambda)$. Then, $N^* = \int \bar{\lambda} \mathrm{d}P(\lambda) = \int \bar{\lambda} dQ(\lambda)$, using Proposition 4.9.7(c). It follows on using Proposition 4.9.7(d) that

$$\int p(\lambda, \bar{\lambda}) d(P(\lambda)x, x) = \int p(\lambda, \bar{\lambda}) d(Q(\lambda)x, x),$$

where x is a fixed vector in H and p is any polynomial. The Stone–Weierstrass Theorem implies that $\int f(\lambda) d(P(\lambda)x, x) = \int f(\lambda) d(Q(\lambda)x, x)$ for all continuous functions f on the union of the supports of the measures P and Q. Since the measures $(P(\lambda)x, x)$ and $(Q(\lambda)x, x)$ are finite Borel measures and thus are regular [26], it follows that

$$(P(\lambda)(S)x, x) = (Q(\lambda)(S)x, x) \quad \text{for all Borel sets } S.$$

$\square$

Corollary 4.12.6 *If $N = \int \lambda \, \mathrm{d}P(\lambda)$ and S is a Borel subset, then $P(S)H$ and $(P(S)H)^{\perp}$ are invariant under N and the spectrum of the restriction of N to $P(S)H$ is contained in $\overline{S}$.*

Proof Since $P(S) = \int \chi_S \mathrm{d}P_\lambda$, it follows from Theorem 4.12.5 that $P(S)H$ and $(P(S)H)^{\perp}$ are invariant under N. Moreover, $\sigma(N|_{P(S)H})$ is the essential range of the product of the functions $\chi_S \circ \phi$ and ϕ restricted to $\phi^{-1}(S)$ and is therefore contained in $\overline{S}$. □

4.13 Invariant Subspaces

If M is a closed subspace of a Banach space X which is neither $\{0\}$ nor the whole space, it is called a *nontrivial* subspace.

Let T be a bounded linear operator on X. Recall that a subspace M is said to be *invariant under T*, or *T-invariant* if $Tx \in M$ whenever $x \in M$. We are interested in nontrivial invariant subspaces of $T \in \mathcal{B}(X)$.

One of the most important and most difficult unsolved problems of operator theory in Hilbert spaces is the problem of invariant subspaces. The question is simple to state: Does every operator on an infinite-dimensional complex separable Hilbert space H have an invariant subspace? All the adjectives used to describe the Hilbert space are crucial here, as the following examples show.

Examples 4.13.1

(i) The operator $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ acting on $\mathbb{R}^2$ has no nontrivial invariant subspace. Suppose T has an invariant subspace M. Then, M must be of dimension 1. Let $M = \{\alpha x : \alpha \in \mathbb{R}\}$. Since M is T-invariant, $Tx = \lambda x$ for some $\lambda \in \mathbb{R}$. Hence, $T^2 x = T(Tx) = \lambda(Tx) = \lambda^2 x$. But

$$T^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I,$$

where I denotes the identity operator. So, $-x = \lambda^2 x$, which implies $\lambda^2 = -1$ if $x \neq 0$. However, this cannot happen if λ is real. Thus, the problem has a negative answer if the Hilbert space under consideration is a real Hilbert space.

(ii) The problem has an affirmative answer in finite-dimensional spaces (of dimension greater than 1) over the complex field.
Let $H = \mathbb{C}^n$ be n-dimensional space over $\mathbb{C}$, $n > 1$, and $T \in \mathcal{B}(H)$ be represented by a matrix. The characteristic polynomial $\det(\lambda I - T)$ is a polynomial of degree n and hence has a root α, say. Then, $\det(\alpha I - T) = 0$, that is the matrix $\alpha I - T$ is not invertible and so, is not injective. Thus,

ker($\alpha I - T$) $\neq \{0\}$. Since $n > 1$, ker($\alpha I - T$) is a subspace M such that $\{0\} \neq M \neq H$. Now, $x \in M$ implies $Tx = \alpha x \in M$. Consequently, M is an invariant subspace of T.

(iii) The problem has an affirmative answer in a nonseparable Hilbert space too. Let $T \in \mathcal{B}(H)$ where H is a nonseparable Hilbert space and x be any nonzero vector in H. Consider the orbit of x under T, namely $\{x, Tx, T^2x, \ldots\}$. The closed linear subspace M generated by $\{x, Tx, T^2x, \ldots\} \neq \{0\}$ is invariant under T. Moreover, it is separable. Indeed, all finite complex (with rational real and imaginary parts) linear combinations of $\{x, Tx, T^2x, \ldots\}$ constitute a countable dense subset of M. Being separable, M cannot be the whole of H. Thus, M is an invariant subspace of T.

(iv) Using the spectral theorem for normal operators, it can be concluded that normal operators have invariant subspaces [Corollary 4.12.6].

However, for a bounded nonnormal operator on an infinite-dimensional complex separable Hilbert space, the invariant subspace problem remains a recalcitrant open question.

It is helpful to weaken the problem and seek the answer to the question: Does every bounded linear operator on an infinite-dimensional complex separable Hilbert space have an invariant linear manifold (not necessarily closed)? Surprisingly, the answer is positive and easy. See [29]. If on the other hand, in an attempt to get a counterexample when 'Hilbert space' is replaced by 'Banach space', there has been some progress both in the positive direction and in the negative direction. The earliest nontrivial result is due to Aronszajn and Smith [1]:

Theorem 1. Every compact operator on an infinite-dimensional complex separable Banach space has an invariant subspace.

The result has been generalised in [4], but the generalisation is still clearly tied to compactness:

Theorem 2. Let X be an infinite-dimensional complex separable Banach space and $T \in \mathcal{B}(X)$ be such that $p(T)$ is compact for some nonzero complex polynomial $p(z)$. Then, T has an invariant subspace.

In 1973, V. I. Lomonosov proved a very powerful result [22]. When it appeared, it caused great excitement, both for the strength of its conclusion and for the simplicity of its proof.

We shall need the following definition.

Definition 4.13.2 Let $\mathcal{A}$ denote the set of all $S \in \mathcal{B}(X)$ that commute with $T \in \mathcal{B}(X)$. $\mathcal{A}$ is called the **commutant** of T and is a subalgebra of $\mathcal{B}(X)$. An closed subspace of X that is S-invariant for every $S \in \mathcal{A}$ is called a **hyperinvariant subspace** for T.

Since T commutes with itself, every hyperinvariant subspace is an invariant subspace.

In the proof of Lomonosov's Theorem, we need a basic result concerning the closed convex hull of compact sets.

Theorem 4.13.3 (Mazur) *Let X be a Banach space and A be a compact subset of X. The closed convex hull* $K = \overline{\text{co}}A$ *is compact.*

Proof We need to show that K is totally bounded, i.e. that for every $\varepsilon > 0$, it contains a finite ε-net. Since A is compact, it contains a finite $\frac{1}{4}\varepsilon$-net, $\{x_1, x_2, \ldots, x_n\}$, say. That is, $\{x_1, x_2, \ldots, x_n\} \subseteq A$ and for every $x \in A$, there is an x_i such that $\|x - x_i\| < \frac{1}{4}\varepsilon$; so,

$$A \subseteq \bigcup_{j=1}^{n} B\left(x_j, \frac{1}{4}\varepsilon\right).$$

The set $C = \text{co}\{x_1, x_2, \ldots, x_n\}$ is easily seen to be compact; so, it contains a finite $\frac{1}{4}\varepsilon$-net, $\{y_1, y_2, \ldots, y_m\}$, say; so,

$$C \subseteq \bigcup_{i=1}^{m} B\left(y_i, \frac{1}{4}\varepsilon\right).$$

If $w \in \overline{\text{co}}A$, there is a z in $\text{co}A$ with $\|w - z\| < \frac{1}{4}\varepsilon$, where $z = \Sigma_{p=1}^{\ell} \alpha_p a_p$, $a_p \in A$, $\alpha_p \geq 0$ and $\sum_p \alpha_p = 1$. Now, for each a_p, there is an $x_{j(p)}$ with $\|a_p - x_{j(p)}\| < \frac{1}{4}\varepsilon$. Therefore,

$$\left\| z - \sum_{p=1}^{\ell} \alpha_p x_{j(p)} \right\| = \left\| \sum_{p=1}^{\ell} \alpha_p \left(a_p - x_{j(p)}\right) \right\| \leq \sum_{p=1}^{\ell} \alpha_p \|a_p - x_{j(p)}\| < \frac{1}{4}\varepsilon.$$

But $\Sigma_p \alpha_p x_{j(p)} \in C$, so there is a y_i with $\|\Sigma_p \alpha_p x_{j(p)} - y_i\| < \frac{1}{4}\varepsilon$. By the triangle inequality,

$$\|w - y_i\| \leq \|w - z\| + \left\| z - \sum_{p=1}^{\ell} \alpha_p x_{j(p)} \right\| + \left\| \sum_{p=1}^{\ell} \alpha_p x_{j(p)} - y_i \right\| < \frac{3}{4}\varepsilon < \varepsilon.$$

Consequently,

$$\overline{\text{co}}A \subseteq \bigcup_{i=1}^{m} B(y_i, \varepsilon),$$

showing that $\overline{\text{co}}A$ is totally bounded. This completes the proof. □

Theorem 4.13.4 (Lomonosov) *Let T be a nontrivial compact operator on an infinite-dimensional complex separable Banach space X. Then, T has a hyperinvariant subspace.*

Proof Let $\mathcal{A} = \{S \in \mathcal{B}(x) : ST = TS\}$ be the commutant of T. Note that $\mathcal{A}$ is an algebra. If there exists a nonzero $\lambda \in \sigma(T)$, then the eigenspace $\ker(\lambda I - T)$ is invariant under all $S \in \mathcal{A}$. If $x \in \ker(\lambda I - T)$, then $(\lambda I - T)Sx = S(\lambda I - T)x = S(0) = 0$, so that $Sx \in \ker(\lambda I - T)$. The argument in this case is complete. It is required to prove the theorem only in the case $\sigma(T) = \{0\}$. We may assume without loss of generality that $\|T\| = 1$. Indeed, one only needs to replace T by $T/\|T\|$. Since $T \neq O$, we may choose x_0 in X such that $\|Tx_0\| > 1$. Then, $\|x_0\| > 1$. Let $B = \{x \in X : \|x - x_0\| < 1\}$ be the open ball of radius 1 centred at x_0. Note that $0 \notin \overline{B}$. Since $\|T\| = 1$ and $\|Tx_0\| > 1$, the closure $\overline{T(B)}$ does not contain the vector 0. In fact, the positive number $\|Tx_0\| - 1$ has the property that $\|Tx\| > \|Tx_0\| - 1$ whenever $x \in B$. This is because $x \in B$ implies $\|x - x_0\| < 1$, which implies $\|Tx - Tx_0\| < 1$, which in turn implies $\|Tx\| \geq \|Tx_0\| - \|Tx - Tx_0\| > \|Tx_0\| - 1$.

For each nonzero vector $y \in X$, consider the set $\mathcal{A}y = \{Sy : S \in \mathrm{A}\}$. This is a nonzero, not necessarily closed, linear subspace invariant under $\mathcal{A}$ ($\mathcal{A}$ is a subalgebra of ($\mathcal{B}(X)$). If we show that for some nonzero y, the space $\mathcal{A}y$ is not dense in X, then its closure is a nontrivial hyperinvariant subspace for T. Suppose to the contrary that for every nonzero y, the space $\mathcal{A}y$ is dense in X. Then, in particular, for every nonzero y, there exists S in $\mathcal{A}$ such that $\|Sy - x_0\| < 1$. In other words, $y \in S^{-1}(B)$ for some $S \in \mathcal{A}$. Note that $S^{-1}(B)$ is open since B is open. So, the family $\{S^{-1}(B) : S \in \mathrm{A}\}$ is an open cover for $X \backslash \{0\}$ and hence for the set $\overline{T(B)}$. Since the set is compact (because T is compact), there is a finite set $\{S_1, S_2, \ldots, S_n\}$ in $\mathcal{A}$ such that

$$\overline{T(B)} \subseteq \bigcup_{i=1}^{n} S_i^{-1}(B).$$

In particular, $Tx_0 \in S_i^{-1}(B)$ for some i_1, $1 \leq i_1 \leq n$. This means that $S_{i_1} Tx_0 \in B$ and $TS_{i_1}Tx_0 \in T(B)$. So, $TS_{i_1}Tx_0 \in S_{i_2}^{1}(B)$ for some i_2, $1 \leq i_2 \leq n$. This means that $S_{i_2}TS_{i_1}Tx_0 \in B$. Continuing this process m times, we see that

$$S_{i_m}TS_{i_{m-1}} \cdots S_{i_2}TS_{i_1}Tx_0 \in B$$

and since T commutes with the S_{i_j},

$$S_{i_m} \cdots S_{i_1}TS_{i_1}T^m x_0 \in B. \tag{$*$}$$

All the S_{i_j} are from the finite set $\{S_1, S_2, \ldots, S_n\}$. Let $c = \max\{\|S_i\| : 1 \leq i \leq n\}$. Then,

$$\|S_{i_m} \cdots S_{i_1}T^m\| \leq c^m \|T^m\| = \|(cT)^m\|.$$

The operator cT has spectral radius 0. So, by the spectral radius formula, $\|(cT)^m\|^{\frac{1}{m}}$ converges to zero, and hence, $\|(cT)^m\|$ converges to zero. Thus,

$$\|S_{i_m}\cdots S_{i_1}T^m x_0\| \to 0 \quad \text{as}\, m \to \infty.$$

Hence, from (*), the point $0 \in \overline{B}$. This is a contradiction. □

An immediate consequence of Lomonosov's Theorem is the earliest nontrivial result in this direction.

Corollary 4.13.5 (Aronszajn–Smith) *Every compact operator on an infinite-dimensional complex separable Banach space has an invariant subspace.*

Corollary 4.13.6 (Bernstein–Robinson; Halmos) *Let X be an infinite-dimensional complex separable Banach space and $T \in \mathcal{B}(X)$ be such that p(T) is compact for some polynomial p(λ). Then, T has an invariant subspace.*

Proof If $p(T) \neq O$, then Lomonosov's Theorem applies. If $p(T) = O$, let $p(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n, a_n \neq 0$. For $x \neq 0$ in X, consider the closed linear span M of the vectors $\{x, Tx, T^2x, \ldots, T^{n-1}x\}$. Since $T^n = -a_n^{-1}(a_0 I + a_1 T + \cdots + a_{n-1}T^{n-1})$, it follows that M is invariant under T. As $x \in M$, $M \neq \{0\}$; moreover, $\dim(M) < \infty$. Consequently, $M \neq X$. □

Remark In 1966, Bernstein and Robinson published a proof of this result (which had not previously been known) using nonstandard analysis. This was promoted as a great triumph for nonstandard analysis. However, Halmos immediately waded through the nonstandard analysis and presented a clean standard proof, showing that Bernstein and Robinson were clever mathematicians but that nonstandard analysis did not really make this proof easier.

The invariant subspace problem for Banach spaces remained an open question for long period up to mid-1985, when it was solved in the negative. P. Enflo [9] showed the existence of a Banach space and an operator T defined on it having no invariant subspace. His work (not published but circulated) eventually appeared in 1987. C. J. Read in 1984 did the same thing. Later, B. Beauzamy (1985) sorted out Enflo's ideas and gave an exposition and simplification of his construction of the space and the operator defined on it. C. J. Read in 1986 gave a self-contained exposition showing that there is a bounded operator on ℓ^1 having no invariant subspace.

The deep study carried out by P. Enflo, C. J. Read and B. Beauzamy did not completely settle the matter. Which Banach spaces, or at least reflexive Banach spaces, X have the property that there is a bounded linear operator defined in X with no invariant subspace?

It has been pointed out in the foregoing that the question 'Does every operator $T \in \mathcal{B}(H)$ have an invariant subspace?' is unanswered. However, for certain specific operators, the results which have been obtained in this direction are complete. We shall illustrate this fact with an example.

Let $V : L^2[0,1] \to L^2[0,1]$ be the Volterra operator,

$$Vf(x) = \int_0^x f(t)\mathrm{d}t, \quad f \in L^2[0,1]. \tag{4.31}$$

The Volterra integral operator is a bounded linear operator of norm not exceeding $1/\sqrt{2}$ [see (ix) of Examples 3.2.5]. Its spectral radius $r(V)$ is 0, and 0 is not an eigenvalue of V [Example 4.3.2]; also, the operator under consideration is compact [Problem 4.7.P3].

If $H_\alpha = \{f \in L^2[0,1] : f(t) = 0 \text{ a.e. for } 0 \le t \le \alpha\}$, it is clear that the H_α form closed subspaces of $L^2[0, 1]$ which are invariant under V. It will be shown that such subspaces are the only closed invariant subspaces of the operator V. Brodskii [6] and Donoghue [7] have independently proved this result.

The following discussion will facilitate the proof of the theorem. If $f, g \in L^1(\mathbb{R})$, then the convolution of f and g is defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y. \tag{4.32}$$

It is well known that the convolution of two L^1-functions is an L^1-function and convolution is both commutative and associative, i.e.

$$f * g = g * f; (f * g) * h = f * (g * h) \text{ for all } f, g, h \in L^1(\mathbb{R});$$

moreover,

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1.$$

For any measurable complex-valued function ϕ on $\mathbb{R}$, let

$$\ell_\phi = \sup\{\ell : \phi(x) = 0 \text{ a.e.} \quad \text{for } x < \ell\}.$$

Obviously, $\phi = \psi$ a.e. implies $\ell_\phi = \ell_\psi$, and therefore, we can legitimately speak of ℓ_ϕ for any $\phi \in L^1(\mathbb{R})$. Also, $\ell_\phi > -\infty$ if and only if $\phi(x)$ is zero a.e. for large negative x.

It follows that the integrand $f(y)g(x-y) = 0$ a.e. if either $y < \ell_f$ or $x - y < \ell_g$, in particular, if $x < \ell_f + \ell_g$. Accordingly,

$$\ell_{f*g} \ge \ell_f + \ell_g. \tag{4.33}$$

The Titchmarsh Convolution Theorem says that the sign of equality holds in (4.33):

$$\ell_{f*g} = \ell_f + \ell_g.$$

The Volterra operator will be expressed as a convolution. Extend f to all of $\mathbb{R}$ by setting $f = 0$ outside the interval [0, 1], and let

$$h(s) = \begin{cases} 0 & \text{for } s < 0 \\ 1 & \text{for } s > 0. \end{cases}$$

be the Heaviside function. Then, for all x in [0, 1],

$$Vf(x) = h * f(x).$$

Indeed,

$$\begin{aligned} h * f(x) &= \int_{-\infty}^{\infty} h(y)f(x-y)\mathrm{d}y = \int_{0}^{\infty} f(x-y)\mathrm{d}y = -\int_{x}^{-\infty} f(u)\mathrm{d}u \\ &= \int_{-\infty}^{x} f(u)\mathrm{d}u = \int_{0}^{x} f(u)\mathrm{d}u, \end{aligned}$$

since f is zero outside the interval [0, 1].

It can be checked that the powers of V applied to f are given by

$$V^n f(x) = h_n * f(x), \quad 0 \le x \le 1,$$

where

$$h_{n+1}(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^n}{n!} & \text{for } x > 0. \end{cases}$$

The following lemma will be needed in the proof of the theorem:

Lemma 4.13.7 *Let f be any function in $L^2[0, 1]$. Then, the set of functions f, Vf, V^2f, … spans $L^2[\ell, 1]$, where $\ell = \ell_f$.*

Proof Suppose $g \in L^2[0, 1]$ is such that $(V^n f, g) = 0$ for $n = 0, 1, 2, \ldots$, where $(\cdot, \cdot)$ denotes the inner product in $L^2[0, 1]$. This condition may be written as $(h_n * f, g) = 0$ for $n = 0, 1, 2, \ldots$.

Define $\tilde{g}$ in $L^2[-1, 0]$ by $\tilde{g}(x) = \overline{g(-x)}$. For any $k \in L^2[0, 1]$,

$$\begin{aligned}(k,g) &= \int_0^1 k(x)\overline{g(x)}\mathrm{d}x \\ &= \int_0^1 k(x)\tilde{g}(-x)\mathrm{d}x \\ &= k*\tilde{g}(0),\end{aligned}$$

where the functions involved are assumed to have been extended outside their domains of definition to be 0. Using associativity and commutativity of convolution, we have

$$0 = (h_n*f,g) = ((h_n*f)*\widetilde{g})(0) = (h_n*(f*\widetilde{g}))(0) = ((f*\widetilde{g})*h_n)(0) = \left(\widetilde{h_n}, f*\widetilde{g}\right) \tag{4.34}$$

Note that $f*\widetilde{g}$ is supported by $[-1, 1]$ and $\widetilde{h}_n(x) = \frac{x^{n-1}}{(n-1)!}$ on the interval $[-1, 0]$ and zero for $x > 0$. Since by the Weierstrass Approximation Theorem, polynomials are dense in $L^2[-1, 0]$, it follows from (4.34) that $f*\widetilde{g} = 0$ on $[-1, 0]$. We can express this as $\ell_{f*\widetilde{g}} \geq 0$. According to Titchmarsh's Theorem, it follows that

$$\ell_f + \ell_{\widetilde{g}} \geq 0,$$

which implies that $\widetilde{g}(s) = 0$ for $s < -\ell_f$. In view of the definition of $\tilde{g}$, this is the same as $g(s) = 0$ for $s > \ell_f$. So $(V^n f, g) = 0$ for $n = 0, 1, 2, \ldots$. This implies

$$(\mathrm{span}\{V^n f\})^\perp \subseteq L^2[0,\ell],$$

i.e. $L^2[0,\ell]^\perp \subseteq (\mathrm{span}\{V^n f\})$. On the other hand, since each $V^n f = 0$ on $[0,\ell]$, it follows that $\{V^n f : n = 0,1,2,\ldots\}$ spans $L^2[\ell,1]$. □

Theorem 4.13.8 *Let* $H_\alpha = \{f \in L^2[0,1] : f(t) = 0$ a.e. for $0 \leq t \leq \alpha\}$. *These are the only invariant subspaces of the Volterra operator* (4.31).

Proof Let Y be a closed invariant subspace of V in $L^2[0, 1]$ and $f \in Y$. Then, Vf, $V^2 f$, … are all in Y, and so, by Lemma 4.13.7, $Y \supseteq L^2[\ell_f, 1]$. It follows that $Y \supseteq L^2[\alpha, 1]$, where

$$\alpha = \inf\{\ell_f : f \in Y\}.$$

On the other hand, by the definition of α, $f(x) = 0$ for $x < \alpha$ for any $f \in Y$. This shows that $Y = L^2[\alpha, 1] = H_\alpha$. □

4.14 Unbounded Operators

Throughout the text except Example 3.2.5(x), we have considered bounded linear operators defined for all elements of a Hilbert space H. The linear operators which are not defined for all elements of H and are not bounded arise naturally both in analysis and in mathematical physics. The example cited above is the simplest of this phenomenon. We recall the example below for the benefit of the reader. Let $H = L^2[0,1]$ and $C^1[0,1]$ be the subspace of H consisting of continuously differentiable functions. For $x \in C^1[0,1]$ let $Tx = x'$, the derivative of x. Then, $T : C^1[0,1] \to L^2[0,1]$ is unbounded [see Example 3.2.5(x)], and the domain of T, denoted by $D(T)$, is a proper subspace of $L^2[0,1]$.

In this section, we propose to extend some of the results for bounded operators on a Hilbert space H in Chap. 3 to possibly unbounded operators defined on a dense subspace of H. We begin with the following:

Definition 4.14.1 Let H be a Hilbert space. By an operator on H, we shall mean a linear mapping T whose domain $D(T)$ is a subspace of H and whose range ran (T) lies in H.

It is not assumed that T is bounded or continuous. However, if an operator T defined on a subspace $D(T)$ of a Hilbert space is bounded ($||T||_{D(T)} = \sup_{\substack{x \in D(T) \\ ||x|| \le 1}} ||Tx|| < \infty$), then T has a unique linear extension $\widetilde{T}$ to $\overline{D(T)}$ by continuity. If $\overline{D(T)}$ is not H, we extend $\widetilde{T}$ to the whole of H. Indeed, if $x \in H$ with $x = y + z$, $y \in \overline{D(T)}, z \in \overline{D(T)}^{\perp}$, we let $\widetilde{T}x = \widetilde{T}y$. Then, $\widetilde{T}$ is a bounded linear extension of T to H and T is the restriction of $\widetilde{T}$ to $D(T)$ [cf. 2.10.28]. So, in the case of such transformations, we can assume, without loss of generality, that they are defined on the whole of H.

Let H be a Hilbert space over $\mathbb{C}$. Recall that $H \times H$ denotes the set of all ordered pairs $(x, y), x, y \in H$. With the usual addition and scalar multiplication, $H \times H$ is a vector space over $\mathbb{C}$. This vector space equipped with the norm $\|(x, y)\| = \|x\| + \|y\|$ is a Hilbert space.

Definition 4.14.2 The **graph** $G(T)$ of an operator T in H is the subspace of $H \times H$ which consists of ordered pairs (x, Tx), $x \in D(T)$. An operator S is called an **extension** of T if $D(T) \subseteq D(S)$ and $Sx = Tx$ for all $x \in D(T)$. This happens if and only if $G(T) \subseteq G(S)$ and the inclusion is expressed in the form

$$T \subseteq S.$$

The following example shows that $\overline{G(T)} \supseteq G(S)$ when S is an extension of T.

Example 4.14.3 Let $H = L^2(\mathbb{R}), C_0^1(\mathbb{R})$ [resp. $C_0^\infty(\mathbb{R})$] denote once [resp. infinitely] continuously differentiable functions with compact support.

Let $Tx = ix'$ with $x \in D(T) = C_0^\infty(\mathbb{R})$
and

$$Sx = ix' \text{with}\, x \in D_S = C_0^1(\mathbb{R}).$$

Note that S is an extension of T. We show that $\overline{G(T)} \supseteq G(S)$.

First, we introduce the approximate identity $\{y_\varepsilon(T)\}$. Let $y(t)$ be any positive infinitely differentiable function with support in $(-1, 1)$, so that $\int_{-\infty}^{\infty} y(t)\mathrm{d}t = 1$. Define $y_\varepsilon(T) = \varepsilon^{-1}y(t/\varepsilon)$. If $x \in D(S)$, set

$$x_\varepsilon(s) = \int_{-\infty}^{\infty} y_\varepsilon(s-t)x(t)\mathrm{d}t.$$

Then,

$$\begin{aligned}|x_\varepsilon(s) - x(s)| &\le \int_{-\infty}^{\infty} y_\varepsilon(s-t)|x(t)-x(s)|\mathrm{d}t \\ &\le \left(\sup_{|t-s|\le\varepsilon} |x(t)-x(s)|\right)\int_{-\infty}^{\infty} y_\varepsilon(s-t)\mathrm{d}t \\ &= \sup_{|t-s|\le\varepsilon} |x(t)-x(s)|.\end{aligned}$$

Considering that x has compact support, it is uniformly continuous. It follows that $x_\varepsilon \to x$ uniformly and hence in $L^2(\mathbb{R})$, since the x_ε have support in a fixed compact set. Similarly,

$$\begin{aligned}i\frac{\mathrm{d}}{\mathrm{d}s}x_\varepsilon(s) &= \int_{-\infty}^{\infty} i\frac{\mathrm{d}}{\mathrm{d}s}y_\varepsilon(s-t)x(t)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} -i\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)y_\varepsilon(s-t))x(t)\mathrm{d}t, \\ \left(\frac{\mathrm{d}}{\mathrm{d}s}y_\varepsilon(s-t) = \varepsilon^{-1}\frac{\mathrm{d}}{\mathrm{d}s}y_\varepsilon((s-t)/\varepsilon) = -\frac{\mathrm{d}}{\mathrm{d}t}y_\varepsilon(s-t)\right) \\ &= \int_{-\infty}^{\infty} y_\varepsilon(s-t)i\frac{\mathrm{d}}{\mathrm{d}t}x(t)\mathrm{d}t \xrightarrow{L^2(\mathbb{R})} i\frac{\mathrm{d}}{\mathrm{d}s}x(s).\end{aligned}$$

Since $x_\varepsilon(t)$ has compact support and is infinitely differentiable, $x_\varepsilon \in C_0^\infty(\mathbb{R})$. Thus, $x_\varepsilon \in D(T)$ for each $\varepsilon > 0$. We have thus proved:

$x_\varepsilon \to x$ in $L^2(\mathbb{R})$ and $Tx_\varepsilon \to Sx$ for any $x \in D(S)$. Thus, the closure of the graph of T contains the graph of S.

A **closed operator** in H is one whose graph is a closed subset of $H \times H$. Equivalently, an unbounded operator T is closed if the following holds: whenever $x_n \in D(T), x_n \to x$ and $Tx_n \to y$, we have $x \in D(T)$ and $Tx = y$.

We give below an example of an unbounded operator whose graph is not closed.

Example 4.14.4 (Cf. Example 3.2.5(x)) The operator $Tx = x'$ with $D(T) = C^1[0,1] \subseteq L^2[0,1]$ is not closed. Its closure $\tilde{T}$ is the operator with domain

$$D(\tilde{T}) = \{x \in C[0,1] : x \text{ is absolutely continuous and } x' \in L^2[0,1]\}.$$

Consider

$$y(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} \\ 1 & \frac{1}{2} < t \le 1 \end{cases}$$

and

$$y_n(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ n(t - \frac{1}{2} + \frac{1}{n}) & \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1 & \frac{1}{2} < t \le 1 \end{cases} \quad \text{for } n = 2, 3, \ldots.$$

It is easy to verify that $y_n \to y$ in $L^2[0,1]$. Define

$$x(t) = \int_0^t y(s)\mathrm{d}s \quad \text{and} \quad x_n(t) = \int_0^t y_n(s)\mathrm{d}s,$$

then

$$x'(t) = y(t) \text{ for all } t \text{ except when } t = \frac{1}{2}$$
$$x_n'(t) = y_n(t) \text{ for all } t \in [0,1].$$

Also,

$$|x_n(t) - x(t)| \le \int_0^t |y_n(s) - y(s)|\mathrm{d}s \le \|y_n - y\|_1 \le \|y_n - y\|_2.$$

Hence, $x_n \to x$ uniformly on [0, 1] and consequently in $L^2[0,1]$. Thus, $x_n \in D(T)$, $x_n \to x$ and $x_n' = y_n \to y$ in $L^2[0,1]$. Clearly, x is not differentiable at $t = \frac{1}{2}$. Indeed, $x'_{\frac{1}{2}+} = 1$ and $x'_{\frac{1}{2}-} = 0$. Hence, $x \notin D(T)$. This shows that $G(T)$ is not closed, i.e. T is not a closed operator.

On the other hand, if $x_n \in D(\tilde{T}), x_n \to x \,\text{in}\, L^2[0,1]$ and $\tilde{T}x_n \to y$ in $L^2[0,1]$, then $x(t) = k + \int_0^t y(s)\mathrm{d}s,\ t \in [0,1]$ is in $D(\tilde{T})$ and $\tilde{T}(x) = x' = y \,\text{in}\, L_2[0,1]$.

This shows that $\tilde{T}$ is a closed operator.

Remark By the Closed Graph Theorem [5.5.7], $T \in \mathcal{B}(H)$ if and only if $D(T)$ = H and T is closed.

Adjoint of an operator Recall that if T is a bounded linear operator defined everywhere in H, each element $y \in H$ uniquely determines an element $y^* \in H$ such that

$$(Tx, y) = (x, y^*)$$

for all $x \in H$. This defines an operator on H called the adjoint of T, denoted by T^*, such that $T^*y = y^*$ [see Definition 3.5.1].

We wish to replicate the above procedure, i.e. it is desired to associate with an unbounded operator T in H with domain $D(T)$, a Hilbert space adjoint T^*. We consider once again the scalar product

$$(Tx, y), \quad x \in D(T). \tag{4.35}$$

(The subset of H over which y varies will be specified later.)

It is no longer possible to assert that for each element y, the expression (4.35) as a function of $x \in D(T)$ can be written in the form

$$(x, y^*).$$

In general, there exist some pairs y and y^* for which

$$(Tx, y) = (x, y^*), \quad x \in D(T). \tag{4.36}$$

One such pair for which (4.36) is satisfied is $y = y^* = 0$. The existence of pairs y and y^* for which (4.36) holds for each $(Tx, y), x \in D(T)$ is not sufficient to define an operator T^*, which is an adjoint of T. It is also necessary that the element y^* be uniquely determined by the element y. The last requirement is fulfilled if and only if $\overline{D(T)} = H$. Indeed, if $D(T)$ is not dense in H, then there is a nonzero vector $z \perp \overline{D(T)}$. Then, (4.36) implies that

$$(Tx, y) = (x, y^* + z), \quad x \in D(T).$$

On the other hand, if $\overline{D(T)} = H$ and if for each $x \in D(T)$,

$$(Tx, y) = (x, y_1^*) = (x, y_2^*), \quad x \in D(T),$$

then

$$(x, y_1{}^* - y_2{}^*) = 0, \quad x \in D(T)$$

and this implies $y_1{}^* = y_2{}^*$.

Definition 4.14.5 Let T be an unbounded operator in H which is densely defined, i.e. $\overline{D(T)} = H$. The operator T^* with domain $D(T^*)$

$$D(T^*) = \{y \in H : \text{there exists}\, y^* \in H\, \text{for which}\, (Tx, y) = (x, y^*), x \in D(T)\}$$

is then defined by

$$(Tx, y) = (x, y^*), \quad x \in D(T), y \in D(T^*).$$

The operator T^* with domain $D(T^*)$ is called the **adjoint** of T with domain $D(T)$.
It is clear that the operator T^* is also linear; in fact, if $y_1, y_2 \in D(T^*)$, then for $\alpha_1, \alpha_2 \in \mathbb{C}$, we have

$$\begin{aligned}(Tx, \alpha_1 y_1 + \alpha_2 y_2) &= (Tx, \alpha_1 y_1) + (Tx, \alpha_2 y_2)\\ &= \overline{\alpha_1}(Tx, y_1) + \overline{\alpha_2}(Tx, y_2)\\ &= \overline{\alpha_1}(x, T^* y_1) + \overline{\alpha_2}(x, T^* y_2)\\ &= (x, \alpha_1 T^* y_1) + (x, \alpha_2 T^* y_2)\\ &= (x, \alpha_1 T^* y_1 + \alpha_2 T^* y_2).\end{aligned}$$

This shows that T^* is defined for $\alpha_1 y_1 + \alpha_2 y_2$ and

$$T^*(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 T^* y_1 + \alpha_2 T^* y_2.$$

Remarks 4.14.6

(i) If the transformation T is bounded, then as seen before [see Theorem 3.5.2], T^* is everywhere defined and bounded. Moreover, $\|T^*\| = \|T\|$.

(ii) In the general case, the domain of T^* need not be the whole space. In fact, $D(T^*)$ may reduce to zero.
Let $H = \ell^2$. Consider the double indexing $e_{k,j}, k, j = 1, 2, \ldots$ of the standard orthonormal basis $\{e_1, e_2, \ldots\}$ of ℓ^2. Let $T(e_{k,j}) = e_k$ for $k, j = 1, 2, \ldots$ and extend T linearly to the span of $\{e_{k,j}, k, j = 1, 2, \ldots\} = D(T)$. If $y \in D(T^*)$, then for each $k = 1, 2, \ldots$,

$$y(k) = (y, e_k) = (y, Te_{k,j}) = (T^* y, e_{k,j}),$$

which tends to zero as $j \to \infty$, so that $y = 0$.

(iii) The operator T^* is closed: if $\{y_n\}_{n\geq 1}$ is a sequence of elements in $D(T^*)$ such that $y_n \to y$ and $T^* y_n \to z$, then

$$(Tx, y) = \lim_n (Tx, y_n) = \lim_n (x, T^* y_n) = (x, z), \quad x \in D(T)$$

using the continuity of the inner product. From the above chain of equalities, it follows that $y \in D(T^*)$ and $T^* y = z$.

(iv) If $S \subseteq T$, then $S^* \supseteq T^*$. Indeed, $y \in D(T^*)$ if and only if

$$(Tx, y) = (x, T^* y) \quad \text{for all}\, x \in D(T).$$

Since $D(S) \subseteq D(T)$, we have $Tx = Sx$ for $x \in D(S)$ and $(Sx, y) = (x, T^* y)$ for all $x \in D(S)$. This implies $y \in D(S^*)$ and $S^* y = T^* y$.

(v) If the operator T^{**} exists, then $T \subset T^{**}$.
Let $x \in D(T)$. Then, for $y \in D(T^*)$, we have
$(T^* y, x) = (y, Tx)$. This shows that $x \in D(T^{**})$ and $T^{**} x = Tx$.

(vi) Let $T : D(T) \subseteq H \to \operatorname{ran}(T) \subseteq H$ be one-to-one. Then, T has an inverse, which maps the elements of ran(T) into $D(T)$. The inverse is denoted by T^{-1}, and $T^{-1} y = x$ if and only if $Tx = y$, where $y \in \operatorname{ran}(T)$. Moreover,

$$D\left(T^{-1}\right) = \operatorname{ran}(T) \text{ and } \operatorname{ran}\left(T^{-1}\right) = D(T).$$

Assume that $D(T)$ and $D(T^{-1})$ are dense in H. Then, the operators T^* and $(T^{-1})^*$ exist. We shall show that

$$(T^*)^{-1} = \left(T^{-1}\right)^*.$$

Let $x \in D(T)$ and $y \in D((T^{-1})^*)$. Then,

$$(x, y) = \left(T^{-1} Tx, y\right) = \left(Tx, \left(T^{-1}\right)^* y\right),$$

which implies

$$\left(T^{-1}\right)^* y \in D(T^*)$$

and

$$T^* \left(T^{-1}\right)^* y = y. \tag{4.37}$$

On the other hand, if $x \in D(T^{-1})$ and $z \in D(T^*)$, then

$$(x, z) = \left(TT^{-1} x, z\right) = \left(T^{-1} x, T^* z\right).$$

It follows that

$$T^*z \in D\big((T^{-1})^*\big)$$

and

$$\big(T^{-1}\big)^*T^*z = z. \tag{4.38}$$

Relations (4.37) and (4.38) imply

$$(T^*)^{-1} = \big(T^{-1}\big)^*.$$

Before proceeding further, we need define algebraic operations on unbounded operators. These are defined just as with bounded operators. However, the domains are carefully watched. Here are the natural definitions: if S and T are unbounded operators with domains $D(S)$ and $D(T)$, respectively, then

$$(S+T)x = Sx + Tx, \quad D(S+T) = D(S) \cap D(T)$$
$$ST(x) = S(Tx), \quad D(ST) = \{x \in D_T : Tx \in D(S)\}.$$

It may be noted that $D(S+T) = \{0\}$ if $D(S) \cap D(T) = \{0\}$ and $D(ST) = \{0\}$ if $\operatorname{ran}(T) \cap D(S) = \{0\}$.

The proposition below describes the relationship between the algebraic operations above and their adjoints.

Proposition 4.14.7 *If S, T, $S + T$ and ST are densely defined, then*

(a) $(S+T)^* \supseteq S^* + T^*$ *and* $(ST)^* \supseteq T^*S^*$;
(b) *If, in addition,* $S \in \mathcal{B}(H)$, *then*

$$(S+T)^* = S^* + T^* \text{ and } (ST)^* = T^*S^*.$$

Proof

(a) We show that $(ST)^* \supseteq T^*S^*$. Suppose $x \in D(ST)$ and $y \in D(T^*S^*)$. Then,

$$(Tx, S^*y) = (x, T^*S^*y)$$

since $x \in D(ST)$ and $S^*y \in D(T^*)$. Also,

$$(STx, y) = (Tx, S^*y)$$

because $Tx \in D(S)$ and $y \in D(S^*)$. Hence,

$$(STx, y) = (x, T^*S^*y),$$

which says that $y \in D((ST)^*)$. This proves the assertion.

(b) Under the additional hypothesis $S \in \mathcal{B}(H)$, we show that $(ST)^* \subseteq T^*S^*$.

Observe that $S^* \in \mathcal{B}(H)$ implies $D(S^*) = H$

$$(Tx, S^*y) = (STx, y) = (x, (ST)^*y)$$

and for every $x \in D(ST)$. Hence, $S^*y \in D(T^*)$, i.e. $y \in D(T^*S^*)$, and therefore, the proof of the assertion is complete. The proofs of $(S + T)^* \supseteq S^* + T^*$ and that of $(S + T)^* \subseteq S^* + T^*$ in case $S \in \mathcal{B}(H)$ are similar and are, therefore, not included. □

Definition 4.14.8 An operator T in H is said to be **symmetric** if

$$(Tx, y) = (x, Ty)$$

for all $x \in D(S)$ and $y \in D(T)$.

Note that densely defined symmetric operators are thus exactly those that satisfy

$$T \subset T^*.$$

If $T = T^*$, then T is said to be **self-adjoint**.

One of the very early theorems of functional analysis is due to Hellinger and Toeplitz.

Theorem 4.14.9 (Hellinger and Toeplitz) *If T is a linear operator in a Hilbert space H with domain $D(T) = H$ and satisfies the relation $(x, Ty) = (Tx, y)$ for all x and y in H, then T is bounded.*

Proof We will prove that the graph $G(T)$ of T is closed. Suppose that $(x_n, Tx_n) \to (x, y)$. We need only prove that $(x, y) \in G(T)$, i.e. $y = Tx$. For any z in H,

$$(z, y) = \lim_n (z, Tx_n) = \lim_n (Tz, x_n) = (Tz, x) = (z, Tx).$$

Thus, $y = Tx$ and $G(T)$ is closed. A use of the Closed Graph Theorem [5.5.7] completes the proof. □

Example 4.14.10 Let $H = L^2[0, 1]$ and $AC[0, 1]$ the set of absolutely continuous functions on [0, 1] whose derivatives are in $L^2[0, 1]$. Let $T_k, k = 1, 2, 3, \ldots$ be the operators $T_k = i\frac{\mathrm{d}}{\mathrm{d}_t}$ with domains

$$\begin{aligned}
D(T_1) &= \{x : x \in AC[0, 1]\} \\
D(T_2) &= \{x : x \in AC[0, 1] \text{ and } x(0) = x(1)\} \\
D(T_3) &= \{x : x \in AC[0, 1] \text{ and } x(0) = 0 = x(1)\}.
\end{aligned}$$

Since the polynomials are dense on $C[0, 1]$ and $C[0, 1]$ is dense in $L^2[0, 1]$. it follows that each $D(T_k)(k = 1, 2, 3, \ldots)$ is dense in $L^2[0, 1]$.

We claim $T_1{}^* = T_3, T_2{}^* = T_2$ and $T_3{}^* = T_1$. For $x \in D(T_1), y \in D(T_3)$, we have using integration by parts and the fact that $y(1) = 0 = y(0)$

$$(T_1x, y) = (x, T_3y);$$

this shows that $T_1{}^*$ extends T_3, i.e. $T_3 \subseteq T_1{}^*$.

Similarly for $x, y \in D(T_2)$, we have $T_2 \subseteq T_2{}^*$, and for $x \in D(T_3), y \in D(T_1)$, we have $T_1 \subseteq T_3{}^*$.

On the other hand, for $y \in D(T_j{}^*)$, $z(t) = \int_0^t T_j{}^* y(s)\mathrm{d}s$ and for $x \in D(T_j), j = 1, 2, 3$, we have

$$\begin{aligned} \int_0^1 ix'\overline{y} = \left(T_jx, y\right) = \left(x, T_j{}^*y\right) = \int_0^1 x\overline{z}' \\ = x(1)\overline{z(1)} - \int_0^1 x'\overline{z}. \end{aligned} \tag{4.39}$$

For $j = 1$ or 2, $D(T_j)$ contains no nonzero constants, so that (4.39) implies $z(1) = 0$. For $j = 3$, $x(1) = 0$. It follows in all cases that

$$iy - z \in \operatorname{ran}\left(T_j\right)^{\perp}. \tag{4.40}$$

Since $\operatorname{ran}(T_1) = L^2$, $iy = z$ if $j = 1$ and since $z(1) = 0$, in that case, $y \in D(T_3)$. Thus, $T_1{}^* \subseteq T_3$.

We next deal with the case $j = 2$ or 3. In this case, $\operatorname{ran}(T_j)$ consists of all $u \in L^2$ such that $\int_0^1 u = 0$. (Indeed, $T_jx = ix'$ and $\int_0^1 x' = x(1) - x(0) = 0$.)

Thus, $\operatorname{ran}(T_2)^{\perp} = \operatorname{ran}(T_3)^{\perp}$ and each equals the space generated by constants. Hence, it follows that $iy - z$ is constant. Thus, y is absolutely continuous and $y \in L^2$, i.e. $y \in D(T_1)$. Thus, $T_3{}^* \subseteq T_1$.

If $j = 2$, then $z(1) = 0$; hence, $y(0) = y(1)$ and $y \in D(T_3)$. Thus, $T_2{}^* \subseteq T_3$.

Remarks 4.14.11

(i) The above example shows the sensitivity of $D(T^*)$ vis-à-vis $D(T)$.

(ii) Since the operators T_1, T_2, and T_3 are adjoints of some operators, and since the adjoint of an operator is always closed [Remarks 4.14.7 (iii)], it follows that each of the operators T_1, T_2, and T_3 is closed.

(iii) A direct proof of the fact that each of the operators T_1, T_2, and T_3 is closed is possible (see [25, pp. 308–311]).

(iv) The operators T_2 and T_3 are symmetric. Indeed,

$$\left(T_j x, y\right) - \left(x, T_j y\right) = i\int_0^1 [x'\overline{y} + x\overline{y}'] = ix\overline{y}\big|_0^1 = 0,$$

where $x, y \in D(T_j)$, $j = 2$ or 3.

The following theorem provides conditions which ensure that a symmetric operator is self-adjoint.

Theorem 4.14.12 *Suppose T is a densely defined operator in H and it is symmetric. Then, the following holds*:

(a) *If $D(T) = H$, then T is self-adjoint and $T \in \mathcal{B}(H)$.*
(b) *If T is self-adjoint and injective, then* ran(T) *is dense in H and T^{-1} is self-adjoint.*
(c) *If $\overline{\text{ran}(T)} = H$, then T is injective.*
(d) *If* ran(T) = H, *then T is self-adjoint, and $T^{-1} \in \mathcal{B}(H)$.*

Proof

(a) Since T is symmetric, it follows that $T \subseteq T^*$. By hypothesis, $D(T) = H$; therefore, $T = T^*$. Hence, T is closed. It therefore follows from [5.5.7] that T is continuous.
(b) Suppose $\overline{\text{ran}(T)} \neq H$. There exists $y \perp \overline{\text{ran}(T)}$. Then, $x \to (Tx, y) = 0$ is continuous for all $x \in D(T)$; hence, $y \in D(T^*) = D(T)$ and $(x, Ty) = (Tx, y) = 0$ for all $x \in D(T)$. Thus, $Ty = 0$, which implies $y = 0$ since T is assumed to be one-to-one. This shows that $\overline{\text{ran}(T)} = H$. T^{-1} is therefore densely defined with $D(T^{-1}) = \text{ran}(T)$ and $(T^{-1})^*$ exists. From [Remarks 4.14.7 (vi)], it follows that $(T^{-1})^* = (T^*)^{-1} = (T)^{-1}$ since T is self-adjoint.
(c) Suppose $Tx = 0$ for some $x \in D(T)$. Then, $(x, Ty) = (Tx, y) = 0$ for every $y \in D(T)$. Thus, $x \perp \text{ran}(T)$. This implies $x = 0$ since ran(T) is dense in H.
(d) Since ran(T) = H, (c) implies T is one-to-one and $D(T^{-1}) = H$. If $x \in H$ and $y \in H$, then $x = Tz$ and $y = Tw$ for some z and w in $D(T)$, so that

$$\left(T^{-1}x, y\right) = (z, Tw) = (Tz, w) = \left(x, T^{-1}y\right).$$

Hence, T^{-1} is symmetric. The result now follows on using Theorem 4.14.9 of Hellinger and Toeplitz. □

Recall that $H \times H$ denotes the Hilbert space of all ordered pairs $\{(x, y) : x, y \in H\}$ with usual addition, scalar multiplication and inner product defined in Definition 2.7.1. In particular, the norm in $H \times H$ is given by

$$\|(x, y)\| = \|x\| + \|y\|$$

If T is a closed operator in H, then the set

$$F = \{(x, Tx) : x \in D(T)\} \tag{4.41}$$

is a closed subset of $H \times H$. By the projection theorem (2.10.11), we have $H \times H = F \oplus F^{\perp}$ with $F \cap F^{\perp} = \{0\}$. Observe that $(u, y) \in F^{\perp}$ if and only if for all $x \in D(T)$,

$$0 = ((x, Tx), (u, y)) = (x, u) + (Tx, y),$$

i.e. $(Tx, y) = (x, -u)$, which says that $y \in D(T^*)$ and $u = -T^*y$. Thus,

$$F^{\perp} = \{(-T^*y, y) : y \in D(T^*)\}, \tag{4.42}$$

so that

$$H \times H = \{(x, Tx) : x \in D(T)\} \oplus \{(-T^*y, y) : y \in D(T^*)\}.$$

Theorem 4.14.13 *If T is a densely defined closed operator in H, then $D(T^*)$ is dense in H and $T^{**} = T$.*

Proof $D(T^*)$ is dense in H. Let $u \perp D(T^*)$. Then, for all $y \in D(T^*)$, using (4.41) and (4.42), we have

$$((0, u), (-T^*y, y)) = (0, -T^*y) + (u, y) = 0.$$

Hence, $(0, u) \in (F^{\perp})^{\perp} = F$ [see Theorem 2.10.11], i.e. $u = T(0) = 0$. It remains to show that $T^{**} = T$. It follows in view of Remark 4.14.7 (v) that $T \subset T^{**}$ and $T^{**}(x) = Tx$ for all $x \in D(T)$. On the other hand, let $z \in D(T^{**})$. Then, $((z, T^{**}z), (-T^*y, y)) = (z, -T^*y) + (T^{**}z, y) = (T^{**}z, -y) + (T^{**}z, y) = 0$, i.e. $(z, T^{**}z) \perp (-T^*y, y)$ for every $y \in D(T^*)$, which implies $(z, T^{**}z) \in (F^{\perp})^{\perp} = F$. Hence, $z \in D(T)$, using (4.41), and $Tz = T^{**}z$. Thus, $T^{**} = T$. This completes the proof. □

The next result gives the basic criterion for self-adjointness.

Theorem 4.14.14 *Let T be a densely defined symmetric operator in a Hilbert space H. Then, the following three statements are equivalent*:

(a) *T is self-adjoint*;
(b) *T is closed and* $\ker(T^* \pm iI) = \{0\}$;
(c) $\operatorname{ran}(T \pm iI) = H$.

Proof

(a) ⇒ (b) Suppose that T is a self-adjoint operator and there is a $y \in D(T^*) = D(T)$ such that $T^*y = -iy$. Then, $Ty = iy$ and

$$i(y,y) = (iy,y) = (Ty,y) = (y,T^*y) = (y,iy) = -i(y,y),$$

so $y = 0$. A similar proof shows that $T^*y = -iy$ can have no nonzero solution.

(b) ⇒ (c) Since $T^*y = -iy$ has no solutions, ran$(T - iI)$ must be dense. Otherwise, if $y \in \text{ran}(T - iI)^{\perp}$, we would have $((T - iI)x, y) = 0$ for all $x \in D(T)$ so $y \in D(T^*)$ and $(T - iI)^*y = (T^* + iI)y = 0$, which is impossible since $(T^* + iI)y = 0$ has no solution. Reversing this last argument, it can be checked that if ran$(T - iI)$ is dense, then $\ker(T^* + iI) = \{0\}$. Since ran$(T - iI)$ is dense, we need only show it is closed to conclude that ran$(T - iI) = H$. But for all $x \in D(T)$,

$$\|(T - iI)x\|^2 = \|Tx\|^2 + \|x\|^2.$$

Thus, if $x_n \in D(T)$ and $(T - iI)x_n \to y_0$, we conclude that x_n converges to some x_0, and Tx_n converges too. Since T is closed, $x_0 \in D(T)$ and $(T - iI)x_0 = y_0$. Thus, ran$(T - iI)$ is closed, so ran$(T - iI) = H$. Similarly, ran$(T + iI) = H$.

(c) ⇒ (a) Let $y \in D(T^*)$. Since ran$(T - iI) = H$, there is an $x \in D(T)$ such that $(T - iI)x = (T^* - iI)y$. $D(T) \subseteq D(T^*)$, so $x - y \in D(T^*)$ and

$$(T^* - iI)(x - y) = 0.$$

Since ran$(T + iI) = H$, $\ker(T - iI) = \{0\}$, so $x = y$. This proves that $D(T^*) = D(T)$, so that T is self-adjoint.

In what follows, we discuss the spectrum of an unbounded operator.

Let $T{:}D(T) \subseteq H \to H$ be a linear operator with $\overline{D(T)} = H$. A point $\lambda \in \mathbb{C}$ is a **regular point** if the operator $(T - \lambda I)^{-1}$ defined on H exists and is bounded. The set of regular points of T is denoted by $\rho(T)$. In symbols,

$$\rho(T) = \left\{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \in \mathcal{B}(H)\right\}.$$

The **spectrum** of $T, \sigma(T)$ is

$$\sigma(T) = \mathbb{C} \backslash \rho(T),$$

i.e. $\sigma(T)$ consists of all nonregular points of T. The following characterisation of regular points is almost immediate.

Let T be a closed operator. A point $\lambda \in \mathbb{C}$ is regular if and only if $\ker(T - \lambda I) = \{0\}$, $\operatorname{ran}(T - \lambda I) = H$, since under these circumstances, $(T - \lambda I)^{-1}$ is a closed operator formally defined on all of H, and therefore, by the Closed Graph Theorem [5.5.7], it is bounded.

The definitions of **point spectrum**, $\sigma_p(T)$, **continuous spectrum**, $\sigma_c(T)$ and **residual spectrum**, $\sigma_r(T)$, are the same for unbounded operators as they are for bounded operators. If T is closed, then

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

and the union on the right side of the above equality is disjoint.

Remarks 4.14.15 An operator T in H is said to be invertible if $T : D(T) \subseteq H \to H, (\overline{D(T)} = H)$ is bijective and its inverse (denoted by T^{-1}) is a bounded linear operator in H.

(i) $\rho(T)$ is an open subset, and $\sigma(T)$ is a closed subset of $\mathbb{C}$.
Let T be an invertible operator in H. Then, $U = T^{-1} \in \mathcal{B}(H)$ and $I - \lambda U$ is invertible provided $|\lambda| < 1/\|U\|$ [see Proposition 3.3.8]. Observe that $TU = I$ and $UTx = x$ for every $x \in D(T)$. Therefore,

$$(T - \lambda I)U(I - \lambda U)^{-1} = I \text{ and } U(I - \lambda U)^{-1}(T - \lambda I)x = x$$
$$\text{for every } x \in D(T).$$

Thus, for all λ satisfying $|\lambda| < 1/\|U\|$, $(T - \lambda I)$ is a bijective linear map from $D(T - \lambda I) = D(T)$ to H and its inverse $U(I - \lambda U)^{-1} \in \mathcal{B}(H)$, i.e. $\lambda \in \rho(T)$. Consequently, $\rho(T)$ is an open subset of $\mathbb{C}$ and $\sigma(T)$ is a closed subset of $\mathbb{C}$.

(ii) $\operatorname{ran}(T - \lambda I)$ is not dense in H if and only if $\overline{\lambda} \in \sigma_p(T^*)$. Indeed,

$$((T - \lambda I)x, y) = \left(x, \left(T^* - \overline{\lambda} I\right)y\right) \quad \text{and} \quad y \notin \overline{\operatorname{ran}(T - \lambda I)}$$

means $(T^* - \overline{\lambda} I)y = 0$. On the other hand, if $(T^* - \overline{\lambda} I)y = 0$, then $y \perp \overline{\operatorname{ran}(T - \lambda I)}$. Moreover, we have

$$(\operatorname{ran}(T - \lambda I))^{\perp} = \ker\left(T^* - \overline{\lambda} I\right).$$

(iii) Let T be symmetric. Then, $\lambda \in \sigma_p(T)$ implies λ is real. Indeed, $\lambda(x, x) = (\lambda x, x) = (Tx, x) = (x, Tx) = (x, \lambda x) = \overline{\lambda}(x, x), x \in D(T)$.

(iv) If $\lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \sigma_p(T)$ and x_1, x_2 are such that $Tx_i = \lambda_i x_i$, i = 1, 2, then they are orthogonal, i.e. $(x_1, x_2) = 0$

$$\lambda_1(x_1, x_2) = (\lambda_1 x_1, x_2) = (Tx_1, x_2) = (x_1, Tx_2) = (x_1, \lambda_2 x_2) = \lambda_2(x_1, x_2),$$

which implies $(\lambda_1 - \lambda_2)(x_1, x_2) = 0$. Consequently, $(x_1,\ x_2) = 0$ since $\lambda_1 \neq \lambda_2$.

(v) Let $z = u + iv$ with $u, v \in \mathbb{R}$, $v \neq 0$, and let T be a closed symmetric operator. Then, ran$(T - zI)$ is a closed subspace of H.
Observe that if T is symmetric, then so is $T - uI$ for $u \in \mathbb{R}$. For $x \in D(T)$, we have

$$\begin{aligned} \|Tx - zx\|^2 &= \|(T - uI)x - ivx\|^2 \\ &= \|(T - uI)x\|^2 - ((T - uI)x, ivx) - (ivx, (T - uI)x) + v^2(x, x) \\ &= \|(T - uI)x\|^2 + v^2\|x\|^2, \end{aligned}$$

which implies

$$\|Tx - zx\| \geq |v|\|x\|.$$

Assume that

$$y_n = Tx_n - zx_n \to y.$$

From the last inequality, it follows that

$$\|y_n - y_m\| \geq |v|\|x_n - x_m\|$$

and hence, there exists x such that $x_n \to x$. Since T is closed, the operator $(T - zI)$ is closed, and therefore, $x \in D(T)$ and $y = (T - zI)x$, i.e. $y \in D(T)$.

(vi) For a self-adjoint operator T in H, $\sigma(T) \subseteq \mathbb{R}$.
Let $\lambda \in \mathbb{C}$ and $\Im\lambda \neq 0$. We prove that $\lambda \in \rho(T)$. For $x \in D(T)$ [see (v) above],

$$\|(T - \lambda I)x\|^2 = \|(T - (\Re\lambda)I)x\|^2 + (\Im\lambda)^2\|x\|^2,$$

which implies

$$\|(T - \lambda I)x\|^2 \geq (\Im\lambda)^2\|x\|^2.$$

We next show that ran$(T - \lambda I)$ is dense in H. Consider $y \in H$ such that $((T - \lambda I)x, y) = 0$ for all $x \in D(T)$. This implies

$$(Tx, y) = \lambda(x, y) = (x, \overline{\lambda}y) \quad \text{for all } x \in D(T).$$

It follows that $y \in D(T^*)$ and $T^*y = \overline{\lambda}y$. As T is self-adjoint, $Ty = T^*y = \overline{\lambda}y$ Consequently,

$$\overline{\lambda}(y, y) = (\overline{\lambda}y, y) = (Ty, y) \in \mathbb{R}, \quad \text{since } T \text{ is symmetric,}$$

which is impossible unless $y = 0$ (since $\Im\lambda \neq 0$). Thus, $(T - \lambda I)$ is bounded below and has dense range. The use of the Lemma 4.14.16 completes the proof.

Lemma 4.14.16 *Let $T : D(T) \subseteq H \to H$ be a linear operator. Then, the following holds*:

(a) *Suppose there is an $m > 0$ such that $||Tx|| \geq m||x||$ for all $x \in D(T)$. Then, T is closed if and only if* ran(T) *is closed.*
(b) *Assume that T is closed. Then, $T^{-1} \in \mathcal{B}(H)$ if and only if* ran(T) *is dense in H and there is an $m > 0$ such that $||Tx|| \geq m||x||$ for all $x \in D(T)$.*

Proof

(a) The condition $\|Tx\| \geq m\|x\|$ shows that T is injective and so has an inverse T^{-1} with $D(T^{-1}) = \text{ran}(T)$ and $\|T^{-1}y\| \leq m^{-1}\|y\|$ for $y \in D(T^{-1})$. Therefore, T^{-1} is bounded on ran(T), and hence, T^{-1} is closed if and only if $\text{ran}(T) = D(T^{-1})$ is closed.
(b) If $T^{-1} \in \mathcal{B}(H)$, then $\|T^{-1}x\| \leq \|T^{-1}\| \, \|x\|$, and with $y = T^{-1}x$, this proves that $\|Ty\| \geq \|T^{-1}\|^{-1}\|y\|$ for $y \in D(T)$. To prove the converse, note that ran(T) is closed, since it is dense in H, ran(T) = H. As T is injective, $T^{-1} \in \mathcal{B}(H)$. □

(vii) Let $T{:}D(T) \to H$ be a linear transformation where $\overline{D(T)} = H$. Assume T is self-adjoint. Then, $\sigma_r(T) = \emptyset$.

Suppose $\lambda \in \sigma_r(T) \subseteq \sigma(T) \subseteq \mathbb{R}$. Since $\lambda = \overline{\lambda}$, $\lambda \in \sigma_r(T)$ it follows that $\overline{\text{ran}(T - \lambda I)} \neq H$, $\ker(T^* - \overline{\lambda} I) = \ker(T - \lambda I) = (\text{ran}(\overline{T - \lambda I}))^{\perp} \neq H^{\perp} = \{0\}$. The fact that $\ker(T - \lambda I) \neq \{0\}$ is equivalent to $\lambda \in \sigma_p(T)$. Since $\sigma_p(T) \cap \sigma_r(T) = \emptyset$, it follows that $\sigma_r(T) = \emptyset$.

Remark 4.14.17 We give below an example of a closed densely defined symmetric operator in H such that its spectrum is not contained in $\mathbb{R}$.

Consider the operator T_3 of Example 4.14.10. Recall that T_3 is a closed densely defined symmetric operator. We show that $\sigma(T_3) = \mathbb{C}$ since ran($T_3 - \lambda I$) is not all of H.

Let $y \in \text{ran}(T_3 - \lambda I)$. Then, there exists an absolutely continuous function x on [0, 1] such that $x' \in L^2[0, 1]$ and

$$ix' - \lambda x = y, x(0) = 0 = x(1).$$

Solving the differential equation, we see that for all $t \in [0, 1]$,

$$x(t) = \alpha e^{-i\lambda t} - ie^{-i\lambda t}\int\limits_0^t e^{i\lambda s}y(s)\mathrm{d}s$$

for some $\alpha \in \mathbb{C}$. Using the conditions $x(0) = 0 = x(1)$, we obtain $\alpha = 0$ and $\int_0^1 e^{i\lambda s}y(s)\mathrm{d}s = 0$, respectively. In particular, for $y(s) = e^{-i\lambda s}$ for some $s \in [0, 1]$, we get a contradiction.

Example 4.14.18 Let $H = L^2(\mathbb{R})$ and $D(T) = \{x \in L^2(\mathbb{R}) : tx(t) \in L^2(\mathbb{R})\}$. $D(T)$ is dense in $L^2(\mathbb{R})$ since it contains the set of all continuous functions which vanish outside a finite interval (the interval depending on the function). For $x \in D(T)$, define $Tx(t) = tx(t)$. It is evident that T is an unbounded symmetric operator (indeed, with x having support near ∞ or $-\infty$, we can make $\|Tx\|$ as large as we like while keeping $\|x\| = 1$).

Let $y \in D(T^*)$ and $z = T^*y$. Then, for each $x \in D(T)$,

$$(Tx, y) = (x, z).$$

Equivalently,

$$\int\limits_{-\infty}^{\infty} tx(t)\overline{y(t)}\mathrm{d}t = \int\limits_{-\infty}^{\infty} x(t)\overline{ty(t)}\mathrm{d}t = \int\limits_{-\infty}^{\infty} x(t)\overline{z(t)}\mathrm{d}t$$

or

$$\int\limits_{-\infty}^{\infty} x(t)\left\{\overline{ty(t)} - \overline{z(t)}\right\}\mathrm{d}t = 0.$$

The last equation holds, in particular, for any function $x \in L^2(\mathbb{R})$ which vanishes outside a finite interval. Therefore, for any finite α and β,

$$\int\limits_{\alpha}^{\beta} x(t)\left\{\overline{ty(t)} - \overline{z(t)}\right\}\mathrm{d}t = 0.$$

It follows that $z(t) = ty(t) \in L^2(\mathbb{R})$ for almost all t in $(-\infty, \infty)$. Hence,

$$y \in D(T) \quad \text{and} \quad z = Ty.$$

This shows that $T^* \subseteq T$. Since $T \subseteq T^*$, by the symmetry of T, we have $T = T^*$.

From the equation $T = T^*$, it follows, in particular, that T is closed.

We next show that the point spectrum of T is empty and the continuous spectrum of T is $\mathbb{R}$.

(i) The operator of multiplication by the independent variable does not have an eigenvector.
Since

$$Tx = \lambda x$$

implies that

$$\int_{-\infty}^{\infty} |t-\lambda|^2 |x(t)|^2 \mathrm{d}t = 0.$$

We have $x(t) = 0$ a.e., i.e. $x = 0$. Consequently, $\sigma_p(T)$ is empty.

(ii) All points of $\mathbb{R}$ are in the continuous spectrum of T since

$$(T-\lambda I)D(T), \quad \lambda \in \mathbb{R}$$

consists of all functions $y(t)$ in $L^2(\mathbb{R})$ which remain in $L^2(\mathbb{R})$ after division by $t-\lambda$, i.e. $y \in \operatorname{ran}(T-\lambda I)$ if and only if $y \in L^2(\mathbb{R})$ and $\frac{y(t)}{t-\lambda} \in L^2(\mathbb{R})$. It is evident that $\operatorname{ran}(T-\lambda I)$ is dense in $L^2(\mathbb{R})$. But it does not coincide with $L^2(\mathbb{R})$ because, for instance, it does not contain a function equal to 1 in a neighbourhood of $t=\lambda$. Thus, the continuous spectrum of T is $\mathbb{R}$.

Example 4.14.19 Let $H = L^2(\mathbb{R})$ and $C_0^1(\mathbb{R})[\text{resp.}\, C_0^\infty(\mathbb{R})]$ denote the once [resp. infinitely] differentiable functions with compact support. Let

$$Tx = ix' \quad \text{with}\, x \in D(T) = C_0^\infty(\mathbb{R})$$

and

$$T_1 x = ix' \quad \text{with}\, x \in D(T_1) = C_0^1(\mathbb{R}).$$

Note that T_1 is an extension of T. We shall show that

$$\overline{G(T)} = G(T_1).$$

First, we introduce the approximate identity. Let $y(t)$ be any positive, infinitely differentiable, even function with support in $(-1,1)$, such that $\int_{-\infty}^{\infty} y(t)\mathrm{d}t = 1$. Define $y_\varepsilon(t) = \varepsilon^{-1} y(t/\varepsilon)$. If $x \in D(T_1)$, set

$$x_\varepsilon(t) = \int_{-\infty}^{\infty} y_\varepsilon(t-u)x(u)\mathrm{d}u.$$

Then,

$$
\begin{aligned}
|x_\varepsilon(t) - x(t)| &\le \int_{-\infty}^{\infty} y_\varepsilon(t-u)|x(u) - x(t)|\mathrm{d}u \\
&\le \left(\sup_{|t-u|\le\varepsilon} |x(u) - x(t)|\right) \int_{-\infty}^{\infty} y_\varepsilon(t-u)\mathrm{d}u \\
&= \sup_{|t-u|\le\varepsilon} |x(u) - x(t)|.
\end{aligned}
$$

Since x has compact support, it is uniformly continuous, which implies that $x_\varepsilon \to x$ uniformly. Since the x_ε have support in a fixed compact set, $x_\epsilon \to x \,\text{in}\, L^2(\mathbb{R})$. Similarly,

$$
\begin{aligned}
i\frac{\mathrm{d}}{\mathrm{d}t} x_\varepsilon(t) &= \int_{-\infty}^{\infty} i\frac{\mathrm{d}}{\mathrm{d}t} y_\varepsilon(t-u)x(u)\mathrm{d}u \\
&= \int_{-\infty}^{\infty} -i\left(\frac{\mathrm{d}}{\mathrm{d}u} y_\varepsilon(t-u)\right)x(u)\mathrm{d}u \\
&= \int_{-\infty}^{\infty} y_\varepsilon(t-u)i\frac{\mathrm{d}}{\mathrm{d}u}x(u)\mathrm{d}u \to i\frac{\mathrm{d}}{\mathrm{d}u}x(u).
\end{aligned}
$$

Since $y_\varepsilon(t)$ has compact support and is infinitely differentiable, $y_\varepsilon \in C_0^\infty(\mathbb{R})$. Thus, $y_\varepsilon \in D_T$ for each $\varepsilon > 0$. We have thus shown:

$$
x_\varepsilon \to x \,\text{in}\, L^2(\mathbb{R}) \quad \text{and} \quad Tx_\varepsilon \to T_1 x \quad \text{for any}\, x \in D(T_1).
$$

Thus, $\overline{G(T)} \supseteq G(T_1)$. Since $G(T_1) \supseteq G(T)$, it follows that $\overline{G(T)} = G(T_1)$.

Our next example shows that $D(T^*)$ may not be dense.

Example 4.14.20 Let x be a bounded measurable function such that $x \notin L^2(\mathbb{R})$. Let $D(T) = \{y \in L^2(\mathbb{R}) \int_{-\infty}^{\infty} |x(t)y(t)|\mathrm{d}t < \infty\}$. $D(T)$ certainly contains all the L^2-functions with compact support so that $D(T)$ is dense in $L^2(\mathbb{R})$. Let $z \in L^2(\mathbb{R})$ be fixed. Define

$$
Ty = (y, x)z, \quad y \in D(T).
$$

Suppose $w \in D(T^*)$. Then,

$$(y, T^*w) = (Ty, w) = ((y,x)z, w) = (y,x)(z,w) = \left(y, \overline{(z,w)}x\right).$$

Therefore, $T^*w = \overline{(z,w)}x = (w,z)x$. Since $x \notin L^2(\mathbb{R})$, $(w, z) = 0$. Thus, any $w \in D(T^*)$ is orthogonal to z, so $D(T^*)$ is not dense. In fact, it is just the vectors orthogonal to z and on that domain, T^* is the zero operator.

Remarks 4.14.21

(i) The spectrum of an operator in H is not necessarily bounded. Indeed, $\sigma(T_3)$, as shown in Remark 4.14.15, is $\mathbb{C}$. This holds even for a self-adjoint operator.
(ii) The operator T_2 of Example 4.14.9 is self-adjoint, and its spectrum is not bounded.

For $n = 0, \pm 1, \pm 2, \ldots, T_2 - 2n\pi I$ is not injective. Indeed, $x_n = \exp(-2n\pi it), n = 0, \pm 1, \pm 2, \ldots$ are such that $T_2 x_n = ix_n' = i(-2n\pi i)x_n = 2n\pi x_n$.

$$\sigma(T_2) \supseteq \sigma_p(T_2) \supseteq \{2n\pi : n = 0, \pm 1, \pm 2, \ldots\}.$$

Example 4.14.22 The following is an example of a closed, densely defined operator with an empty spectrum. Let $H = L^2[0, 1]$ and $AC[0, 1]$ be the set of absolutely continuous functions on $[0, 1]$, whose derivatives are in H. Define $T_4 : D(T_4) \subseteq H \to H$ by $T_4 x = ix'$, where $D(T_4) = \{x \in AC[0,1] : x(0) = 0\}$. Consider $\lambda \in \mathbb{C}$. Then, the operator

$$S_\lambda y(t) = i \int_0^t \mathrm{e}^{-\mathrm{i}\lambda(\mathrm{t-s})} \mathrm{y(s)ds}$$

satisfies

$$(T_4 - \lambda I)S_\lambda = I \quad \text{and} \quad S_\lambda(T_4 - \lambda I) \text{ is identity on } D(T_4).$$

Thus, $(T_4 - \lambda I)$ is bijective. So, S_λ is bounded.

In what follows, we shall show how the spectral theorem for bounded self-adjoint operators, which we developed in Theorem 4.10.5, can be extended to unbounded self-adjoint operators. The theorem of the spectral decomposition of a general self-adjoint operator has been proved in several ways. Riesz and Nagy [24] give a proof by taking the limit of a sequence of bounded operators, but the technicalities are rather unpleasant. The proof can also be deduced reasonably quickly from the unitary case, via the Cayley transform.

Theorem 4.14.23 (Spectral Theorem—Multiplication operator form)

Let T be a densely defined self-adjoint operator on a separable Hilbert space H, i.e. $\overline{D(T)} = H$. Then, there is a measure space $(X, \mathfrak{M}, \mu)$ with μ a finite measure, a unitary operator $U{:}H \to L^2$ and a real-valued function f defined on X which is finite a.e. such that

(a) $\psi \in D(T)$ *if, and only if*, $f(\cdot)(U\psi)(\cdot) \in L^2$;
(b) *if* $\phi \in U[D(T)]$ *then* $(UTU^{-1}\phi)(t) = f(t)\phi(t)$.

Proof From Theorem 4.14.14, we have $(T + iI)$ and $(T - iI)$ are one-to-one and ran $(T \pm iI) = H$. Since $(T \pm iI)$ are closed, so are $(T \pm iI)^{-1}$. It follows from [see Theorem 5.5.7] that they are bounded. Observe that

$$2i(T+iI)^{-1}(T-iI)^{-1} = (T-iI)^{-1} - (T+iI)^{-1} = 2i(T-iI)^{-1}(T+iI)^{-1} \tag{4.43}$$

and

$$\begin{aligned}\Big((T-iI)\psi, (T+iI)^{-1}(T+iI)\phi\Big) &= ((T-iI)\psi, \phi) = (\psi, (T+iI)\phi)\\ &= \Big((T-iI)^{-1}(T-iI)\psi, (T+iI)\phi\Big).\end{aligned} \tag{4.44}$$

Since ran$(T \pm iI) = H$, it follows on using (4.43) and (4.44) that

$$\Big((T+iI)^{-1}\Big)^* = (T-iI)^{-1}.$$

So,

$$\begin{aligned}\Big((T+iI)^{-1}\Big)^*(T+iI)^{-1} &= (T-iI)^{-1}(T+iI)^{-1} = (T+iI)^{-1}(T-iI)^{-1}\\ &= (T+iI)^{-1}\Big((T+iI)^{-1}\Big)^*,\end{aligned}$$

i.e. $(T+iI)^{-1}$ is a normal operator. Applying the spectral theorem for bounded normal operators [see Theorem 4.12.4], we find that there exists a finite measure space $(X, \mathfrak{M}, \mu)$ with μ a finite measure, a unitary operator $U{:}H \to L^2$ and a measurable bounded complex-valued function $g(t)$ such that

$$U(T+iI)^{-1}U^{-1}\phi(t) = g(t)\phi(t) \quad \text{for all } \phi \in L^2.$$

Since $\ker((T + iI)^{-1})$ is $\{0\}$, $g(t) \neq 0$ a.e., and so, the function $f(t) = g(t)^{-1} - i$ is finite a.e. Now, suppose $\psi \in D(T)$. Then, $\psi = (T+iI)^{-1}\phi$ for some $\phi \in H$ and $U\psi = U(T+iI)^{-1}U^{-1}U\phi = gU\phi$. Since fg is bounded, we conclude that $f(U\psi) = fgU\phi \in L^2$. Conversely, if $f(U\psi) \in L^2$, then there is a $\phi \in H$ such that $U\phi = (f+i)U\psi$. Thus, $gU\phi = g(f+i)U\psi = U\psi$, so that $\psi = (T+iI)^{-1}\phi$, which shows that $\psi \in D(T)$. This proves (a).

For the proof of (b), observe that if $\psi \in D(T)$ then $\psi = (T+iI)^{-1}\phi$ for some $\phi \in H$. Therefore, $T\psi = \phi - i\psi$.

$$\begin{aligned}(UT\psi)(t) &= (U\phi)(t) - i(U\psi)(t) \\ &= \left(g(t)^{-1} - i\right)(U\psi)(t) \\ &= f(t)(U\psi)(t).\end{aligned}$$

Observe that $U\phi(t) = g(t)^{-1}(U\psi)(t)$. Indeed, $U\psi = U(T+iI)^{-1}\phi = g(t)(U\phi)(t)$ as in the paragraph above.

It remains to show that f is real-valued. If $\Im f > 0$ on a set of nonzero measure, there is a bounded set B in $\Im z > 0$ such that $S = \{x : f(x) \in B\}$ has a nonzero measure. Let χ denote the characteristic function of S. Then, $f\chi \in L^2$ and $\Im(\chi, f\chi) > 0$. This is a contradiction.

Since T is self-adjoint and f is uniquely determined by T, the multiplication operator determined by f must be self-adjoint, i.e. $(\chi, f\chi)$ is real. $\square$

Chapter 5
Banach Spaces

5.1 Definition and Examples

The abstract theory of normed linear spaces was systematically developed by the Polish mathematician Stefan Banach and others in 1920–22. His epoch-making monograph 'Théorie des Operations Linéaires' was published in 1932. It gives an excellent account of the work of many celebrated mathematicians in the creation of functional analysis including Hilbert, Fréchet and Riesz. These mathematicians were concerned with function spaces earlier in the twentieth century.

A Banach space is a complete normed vector space; i.e., it is a vector space that allows the computation of vector length, distance between vectors, and is complete in the sense that every Cauchy sequence has a well-defined limit in the space.

We begin by recalling the definition of a Banach space and derive some of its easy consequences. We then describe some of the representative examples of the space.

Let X be a vector space over the field $\mathbb{F}$, where $\mathbb{F}$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A norm $\|\cdot\|$ on X is a function which assigns to each element x in X a nonnegative real number and has the following properties:

(i) $\|x\| = 0$ if, and only if, $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $\alpha \in \mathbb{F}$ and $x \in X$;
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all x and y in X.

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Property (iii) is called the **triangle inequality**.

A vector space with a norm is called a normed vector space (or a normed linear space).

From the norm arises a metric on X given by

$$d(x,y) = ||x-y||, \quad x,y \in X.$$

If the metric space (X, d) is complete, we say that X is a Banach space [see Definition 2.2.1 and the discussion following it].

Remarks 5.1.1

(i) It follows from the triangle inequality that the mapping $x \to \|x\|$ is uniformly continuous. Indeed, for x, y in X,

$$||x|| = ||x-y+y|| \le \|x-y\| + ||y||,$$

which implies

$$||x|| - ||y|| \le ||x-y||. \tag{5.1}$$

Interchanging the roles of x and y, we obtain

$$|||x|| - \|y\|| \le ||x-y||.$$

(ii) The function $a{:}X \times X \to X$ defined by $a(x,y) = x + y$ is continuous and so is the function $s{:}\mathbb{F} \times X \to X$ defined by $s(\alpha, x) = \alpha x$. Indeed,

$$\|a(x,y) - a(x_0,y_0)\| = \|x+y-x_0-y_0\| \le \|x-x_0\| + ||y-y_0||$$

and

$$\begin{aligned}\|s(x,y) - s(x_0,y_0)\| &= \|\alpha x - \alpha x_0\| = \|(\alpha-\alpha_0)x + \alpha_0(x-x_0)\| \\ &= |\alpha-\alpha_0|||x|| + |\alpha_0|||x-x_0||.\end{aligned}$$

(iii) The distance $d(x, y) = \|x - y\|$, x, $y \in X$, defined in the paragraph above satisfies the following:

(a) $d(x+z, y+z) = d(x,y), \quad x,y,z \in X,$
and

(b) $d(\alpha x, \alpha y) = |\alpha| d(x,y), \quad \alpha \in \mathbb{F} \quad \text{and} \quad x,y \in X.$

Example 5.1.2

(i) The absolute value $|\cdot|$ is a norm on $\mathbb{F}$, and with this norm, $\mathbb{F}$ is a Banach space.

(ii) The Euclidean space $\mathbb{F}^n$ of n-tuples $x = (x_1, x_2, \ldots, x_n)$ with the norm $||x||_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$ and metric $d(x,y) = ||x-y||_2 = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}$, where $y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}^n$, is a complete metric space and hence a Banach space [see (i) of Example 2.3.4].

(iii) With notations as in (ii) above, set

$$||x||_1 = \sum_{i=1}^n |x_i| \quad \text{and} \quad ||x||_\infty = \sup_i |x_i|.$$

It is easy to verify that $\|x\|_1$ and $\|x\|_\infty$ are norms on $\mathbb{F}^n$. With metrics defined by

$$d_1(x,y) = \sum_{i=1}^n |x_i - y_i| \quad \text{and} \quad d_\infty(x,y) = \max_i\{|x_i - y_i| : 1 \le i \le n\},$$

$\mathbb{F}^n$ becomes a metric space $(\mathbb{F}^n, d_1)$ and $(\mathbb{F}^n, d_\infty)$, respectively. Since

$$\left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}} \le \sum_{i=1}^n |x_i| \le n \cdot \max_i\{|x_i| : 1 \le i \le n\} \le n \cdot \left(\sum_{i=1}^n ||x_i||^2\right)^{\frac{1}{2}},$$

it follows that convergence of a sequence in one metric implies convergence in the other metrics and vice versa. Consequently, $(\mathbb{F}^n, \|\cdot\|_1)$ and $(\mathbb{F}^n, \|\cdot\|_\infty)$ are Banach spaces as well. In order to prove that $\mathbb{F}^n$ together with the norm $||x||_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$, $p \ge 1$ (p different from 1, 2 and ∞), is a Banach space, we shall need the following generalisation of the inequality between the arithmetic and geometric means.

Lemma 5.1.3 *Let* α *and* β *be nonnegative real numbers, and suppose* $0 < \lambda < 1$. *Then* $\alpha^\lambda \beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta$ *with equality only if* $\alpha = \beta$.

Proof Consider $\varphi:\mathbb{R}^+ \to \mathbb{R}$ defined by

$$\varphi(t) = (1-\lambda) + \lambda t - t^\lambda.$$

Then

$$\varphi'(t) = \lambda(1 - t^{\lambda-1}).$$

Since $\lambda - 1 < 0$, we have $\varphi'(t) < 0$ for $t < 1$ and $\varphi'(t) > 0$ for $t > 1$. Thus, for $t \neq 1$, we have

$$\varphi(t) > \varphi(1) = 0.$$

Hence

$$(1 - \lambda) + \lambda t \geq t^{\lambda}$$

with equality only for $t = 1$.

If $\beta = 0$, the inequality is trivial, and if $\beta \neq 0$, the Lemma follows on substituting $\frac{\alpha}{\beta}$ for t. □

The following inequalities which follow immediately from the Lemma above are known as Hölder's and Minkowski's inequalities, respectively.

Definition 5.1.4 If p and q are nonnegative real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, we call p and q a pair of **conjugate exponents**. When $p = 1$, $q = \infty$. Consequently, 1 and ∞ are also regarded as conjugate exponents.

Theorem 5.1.5 (Hölder's Inequality) *Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \ldots, n$ and suppose that $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}. \tag{5.2}$$

In the special case when $p = q = 2$, the above inequality reduces to

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}.$$

[Corollary 2.2.5(i)].

Proof Assume that $\sum_{i=1}^{n} x_i^p \neq 0 \neq \sum_{i=1}^{n} y_i^q$ and $\sum_{i=1}^{n} x_i^p = 1 = \sum_{i=1}^{n} y_i^q$. In this case, the inequality (5.2) reduces to

$$\sum_{i=1}^{n} x_i y_i \leq 1. \tag{5.3}$$

To obtain (5.3), we put successively $\alpha = x_i^p, \beta = y_i^q$, $(i = 1, 2, \ldots, n)$, $\lambda = \frac{1}{p}$, $1 - \lambda = \frac{1}{q}$ in the Lemma above and add to obtain

$$\sum_{i=1}^{n} x_i y_i \leq \frac{1}{p} \sum_{i=1}^{n} x_i^p + \left(1 - \frac{1}{p}\right) \sum_{i=1}^{n} y_i^q = 1.$$

The general case can be reduced to the foregoing special case if we take x_i' and y_i' the numbers

$$x_i' = \frac{x_i}{\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}}, \quad y_i' = \frac{y_i}{\left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}}},$$

for which the condition $\sum_{i=1}^n x_i'^p = 1 = \sum_{i=1}^n y_i'^q$ holds. It follows by what we have proved in the paragraph above that

$$\sum_{i=1}^{n} x_i' y_i' \leq 1.$$

This is equivalent to (5.3). This completes the proof. □

Remark 5.1.6 For $p = 1$ and $q = \infty$, the inequality (5.3) becomes

$$\sum_{i=1}^{n} x_i y_i \leq (\sup_i y_i) \sum_{i=1}^{n} x_i.$$

Theorem 5.1.7 (Minkowski's Inequality) *Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \ldots, n$ and suppose that $p \geq 1$. Then*

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}}. \tag{5.4}$$

Proof For $p = 1$, the inequality (5.4) is evident. So, assume that $p > 1$. We write $\sum_{i=1}^n (x_i + y_i)^p$ in the form

$$\sum_{i=1}^{n} (x_i + y_i)^p = \sum_{i=1}^{n} x_i (x_i + y_i)^{p-1} + \sum_{i=1}^{n} y_i (x_i + y_i)^{p-1}. \tag{5.5}$$

Let q be the conjugate index of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Apply Hölder's inequality (5.3) to the two sums on the right-hand side of (5.5) and obtain

$$\sum_{i=1}^{n}(x_i+y_i)^p \le \left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}(x_i+y_i)^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{n}y_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}(x_i+y_i)^{(p-1)q}\right)^{\frac{1}{q}}$$
$$= \left[\left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}y_i^p\right)^{\frac{1}{p}}\right]\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{\frac{1}{q}}$$

because $(p - 1)q = p$.

Dividing both sides by $(\sum_{i=1}^{n}(x_i+y_i)^p)^{\frac{1}{q}}$, we obtain (5.4) in the case $\sum_{i=1}^{n}(x_i+y_i)^p \neq 0$. In the case $\sum_{i=1}^{n}(x_i+y_i)^p = 0$, the inequality is self-evident. ☐

Remarks 5.1.8

(i) Set $d_p(x,y) = (\sum_{i=1}^{n}(x_i-y_i)^p)^{\frac{1}{p}}$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{F}^n$. Then, d_p is a metric on $\mathbb{F}^n$. It is enough to check that $d_p(x, y) \le d_p(x, z) + d_p(z, y)$, $x, y, z \in \mathbb{F}^n$. Indeed,

$$d_p(x,z) = \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}}, \quad d_p(z,y) = \left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{p}},$$

where $a_i = x_i - z_i$ and $b_i = z_i - y_i$, $i = 1, 2, \ldots, n$, and

$$d_p(x,y) = \left(\sum_{i=1}^{n}|a_i+b_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{p}},$$

using Minkowski's inequality,

$$= d_p(x,z) + d_p(z,y).$$

(ii) $(\mathbb{F}^n, d_p)$ is a complete metric space and hence a Banach space. The proof of the statement is as in Example 2.3.4(i).

The next examples of Banach spaces are called L^p-spaces. The fact that the vital property of completeness can be achieved is a direct consequence of the civilised behaviour of the integral when limits are taken.

We shall deal with a measure space $(X, \mathfrak{M}, \mu)$, where μ is a σ-finite measure defined on a σ-algebra $\mathfrak{M}$ of subsets of a set X, with proviso that $0 < \mu(A) < \infty$ for some $A \in \mathfrak{M}$.

Definition 5.1.9 Let f be measurable. Suppose that $p \geq 1$. Set

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

and

$$\|f\|_\infty = \text{ess sup}|f| = \inf\{M : \mu\{t : |f(t)| > M\} = 0\}.$$

The space $\tilde{L}^p$ (or $\tilde{L}^p(\boldsymbol{X})$ or $\tilde{L}^p(\boldsymbol{\mu})$ if X or μ needs emphasis) *is defined to be the set of all measurable functions f such that* $\|f\|_p < \infty$. *We call* $\|f\|_p < \infty$ *the* L^p-**norm** *of* f.

If μ is Lebesgue measure in $\mathbb{R}^n$, we write $\tilde{L}^p(\mathbb{R}^n)$ instead of $\tilde{L}^p(\mu)$. If μ is a counting measure on A, it is customary to write $\ell^p(A)$ or simply ℓ^p. If A is countable, the elements of ℓ^p are sequences $x = \{x_i\}_{i \geq 1}$ and

$$||x||_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

If $p = 2$, this agrees with Example 2.10.24(ii).

In order to prove inequalities about the L^p-norm, we need to prove Hölder Inequality and Minkowski Inequality.

Theorem 5.1.10 *Assume that* $p \geq 1$ *and let* q *be the conjugate index. With notations of* Definition 5.1.9 *above, the following hold for any measurable f and g*:

(a) $\|fg\| \leq \|f\|_p \|g\|_q$ (Hölder's Inequality)

(b) $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ (Minkowski's Inequality).

Proof (a) The case $p = 1$, $q = \infty$ is straightforward and is left to the reader.

Assume that $\|f\|_p = \|g\|_q = 1$. Apply Lemma 5.1.3 with $\alpha = |f(t)|^p, \beta = |g(t)|^q, \lambda = \frac{1}{p}$ and $1 - \lambda = \frac{1}{q}$. Then

$$|f(t)g(t)| \leq \lambda |f(t)|^p + (1 - \lambda)|g(t)|^q \tag{5.6}$$

and integrating both sides, we have

$$\int_X |fg| \mathrm{d}\mu \leq \lambda \int_X |f|^p \mathrm{d}\mu + (1 - \lambda) \int_X |g|^q \mathrm{d}\mu = 1 \tag{5.7}$$

The inequality (a) is trivial if $\|f\|_p = 0$ or $\|g\|_q = 0$. Let f and g be elements of L^p and L^q with $\|f\|_p \neq 0$ or $\|g\|_q \neq 0$. Then, $f/\|f\|_p$ and $g/\|g\|_q$ both have norm 1. Substituting in (5.7) gives

$$\frac{1}{\|f\|_p\|g\|_q}\int_X |fg|\mathrm{d}\mu = \int_X \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}\mathrm{d}\mu \leq 1,$$

i.e.

$$\|fg\|_1 \leq \int_X |fg|\mathrm{d}\mu \leq \int_X \|f\|_p\|g\|_q.$$

This proves (a).

(b) To prove (b) ($p > 1$), write

$$|f+g|^p \leq (|f|+|g|)^p \leq [2\max\{|f|,|g|\}]^p = 2^p\max\{|f|^p,|g|^p\} \leq 2^p(|f|^p+|g|^p).$$

This estimate shows that $|f + g|^p \in L^1$, i.e. $f + g \in L^p$.

Using (a) implies

$$\begin{aligned}\|f+g\|_p^p &= \int_X |f+g|^p\mathrm{d}\mu \leq \int_X |f+g|^{p-1}|f|\mathrm{d}\mu + \int_X |f+g|^{p-1}|g|\mathrm{d}\mu \\ &\leq \left(\int_X |f|^p\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_X |f+g|^{(p-1)q}\mathrm{d}\mu\right)^{\frac{1}{q}} + \left(\int_X |g|^p\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_X |f+g|^{(p-1)q}\mathrm{d}\mu\right)^{\frac{1}{q}} \\ &= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p/q}.\end{aligned}$$

The inequality

$$\|f+g\|_p^{p-p/q} \leq \|f\|_p + \|g\|_p$$

thus holds. Observing that $p - \frac{p}{q} = 1$, we obtain Minkowski's Inequality. The inequality (b) is trivial for $p = 1$. Indeed,

$$\int_X |f+g|\mathrm{d}\mu \leq \int_X |f|d\mu + \int_X |g|\mathrm{d}\mu. \qquad \square$$

Remarks 5.1.11

(i) $\tilde{L}^p$ may be thought of as the continuous analogue of the sequence space ℓ^p with integration replacing summation. In particular, if $X = \{1, 2, \ldots, n\}$ and $\mu(j) = 1$ for $j = 1, 2, \ldots, n$, then Hölder and Minkowski inequalities are the ones proved in Theorems 5.1.5 and 5.1.7.

(ii) When $p = 2$ and $q = 2$, the inequality (a) is called the **Cauchy–Schwarz inequality**.

(iii) To get equality in (a), it is necessary and sufficient that we have

$$\frac{|f(t)|}{\|f\|_p}\frac{|g(t)|}{\|g\|_q}=\frac{1}{p}\frac{|f(t)|^p}{\|f\|_p^p}+\frac{1}{q}\frac{|g(t)|^q}{\|g\|_q^q}$$

for almost all $t \in X$. This happens if, and only if,

$$\frac{|f|^p}{\|f\|_p^p}=\frac{|g|^q}{\|g\|_q^q} \quad \text{a.e., using Lemma 5.1.3.}$$

Thus, equality in (a) holds if, and only if, there exist constants A and B such that

$$A|f|^p = B|g|^q \text{ a.e.}$$

where A and B are not both zero.

(iv) Equality holds in (b) if, and only if, there are nonnegative real numbers A and B such that

$$Af = Bg \text{ a.e.},$$

where A and B are not both zero.

(v) $1 \le p < \infty$, $\ell^p = \{\{x_i\}_{i\ge 1} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$. For $\{x_i\}_{i\ge 1} \in \ell^p$ and $\{y_i\}_{i\ge 1} \in \ell^q$ with $\frac{1}{p}+\frac{1}{q}=1$, in view of (i), Hölder's Inequality takes the form

$$\sum_{i=1}^{\infty}|x_iy_i| \le \left\|\{x_i\}_{i\ge 1}\right\|_p\left\|\{y_i\}_{i\ge 1}\right\|_q$$

and Minkowski's Inequality takes the form

$$\left\|\{x_i+y_i\}_{i\ge 1}\right\|_p \le \left\|\{x_i\}_{i\ge 1}\right\|_p + \left\|\{y_i\}_{i\ge 1}\right\|_p.$$

(vi) From Minkowski's Inequality, it follows that if f and g are in $\tilde{L}^p$, then so is $f + g$. Consequently, $\tilde{L}^p$ is a linear space with $|f|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$. Clearly, $\|\alpha f\|_p = |\alpha|\|f\|_p$, $\alpha \in \mathbb{F}$ and $f \in \tilde{L}^p$ and, as observed above, $\|f + g\|_p \le \|f\|_p + \|g\|_p$, $f \in \tilde{L}^p$ and $g \in \tilde{L}^p$. Unfortunately, $\|f\|_p = 0$ does not satisfy the requirement that $f \equiv 0$. Indeed, $\|f\|_p = 0$ implies $f = 0$ a.e. The difficulty is overcome by considering equivalence classes of functions [Definitions 2.4.2 and 2.4.3]. With this artifice, $\tilde{L}^p(X,\mathfrak{M},\mu)$ becomes a normed linear space, which is denoted by $L^p(X,\mathfrak{M},\mu)$. That it is a Banach space is the content of the following theorem.

Theorem 5.1.12 (Riesz–Fischer) *For* $1 \leq p < \infty$, L^p *is a normed linear space over* $\mathbb{F}$, *where we identify the functions* f *and* g *if* $f(t) = g(t)$ *for* μ*-almost all* $t \in X$. *Moreover*, L^p *is a Banach space*, *i.e. with the metric* $d(f, g) = \|f - g\|_p$, L^p *is a complete metric space*.

Proof For $p = 2$, it is Theorem 2.4.9. The proof for $p \neq 2$ is the same as in Theorem 2.4.9. □

Another representative example of a Banach space is the following.

Let X be a Hausdorff topological space, and let $C_b(X)$ denote the set of bounded continuous complex-valued functions defined on X. For f and $g \in C_b(X)$ and $\alpha \in \mathbb{C}$, we define

$$(f+g)(x) = f(x) + g(x);$$
$$(\alpha f)(x) = \alpha f(x).$$

With these operations, $C_b(X)$ is a vector space over the complex field $\mathbb{C}$. Define

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

The number $\|f\|_\infty$ is called the norm of f. The symbol $\|f\|_\infty$ has also been used while defining the norm of essentially bounded functions; the context will make it clear which meaning is intended. The following properties of norm are easily verified:

(a) $\|f\|_\infty = 0$ if, and only if, $f = 0$;
(b) $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$;
(c) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

We define a metric d on $C_b(X)$ by $d(f, g) = \|f - g\|_\infty$. The properties of metric such as

$$d(f,g) = 0 \text{ if, and only if, } f = g$$
$$d(f,g) = d(g,f) \text{ and}$$
$$d(f,h) \leq d(f,g) + d(g,h)$$

follow immediately from the properties (a)–(c) of the norm. Convergence with respect to the metric d is just uniform convergence. An important property of this metric is that $C_b(X)$ is complete with respect to it.

Theorem 5.1.13 *If* X *is a Hausdorff topological space*, *then* $C_b(X)$ *is a complete metric space*.

Proof Let $\{f_n\}_{n \geq 1}$ be a Cauchy sequence in $C_b(X)$. Then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| = d(f_n, f_m)$$

for each $x \in X$. Hence, $\{f_n(x)\}_{n \geq 1}$ is a Cauchy sequence of complex numbers for each $x \in X$, so we may define $f(x) = \lim_n f_n(x)$. We need to show that f is in $C_b(X)$ and that $\lim_n \|f - f_n\| = 0$. To this end, given $\varepsilon > 0$, choose N such that n, $m \geq N$ implies $\|f_n - f_m\| < \varepsilon$. For x_0 in X, there exists a neighbourhood U of x_0 such that $|f_n(x_0) - f_n(x)| < \varepsilon$ for all x in U. Therefore

$$\begin{aligned}|f(x_0) - f(x)| &\leq \lim_n |f_n(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + \lim_n |f_N(x) - f(x)| \\ &\leq 3\varepsilon,\end{aligned}$$

which implies f is continuous. Further, for $n \geq N$ and x in X, we have

$$|f_n(x) - f(x)| = |f_n(x) - \lim_m f_m(x)| = \lim_m |f_n(x) - f_m(x)| \leq \lim_m \|f_n - f_m\| \leq \varepsilon.$$

Thus, $\lim_n \|f_n - f\| = 0$. Moreover,

$$\|f\| = \|f - f_N + f_N\| \leq \|f - f_N\| + \|f_N\| < \infty,$$

which implies $f \in C_b(X)$. □

Corollary 5.1.14 $C_0(X)$, *the space of continuous functions on X that vanish at* ∞ *is a closed subspace of* $C_b(X)$ *and is therefore a Banach space.*

$f \in C_0(X)$ if for every $\varepsilon > 0$, there exists a compact set K such that $K = \{x \in X : |f(x)| \geq \varepsilon\}$.

Proof Clearly, $C_0(X)$ is a linear subspace of $C_b(X)$. Indeed, if f, $g \in C_0(X)$ and $\varepsilon > 0$, then

$$A_f = \{x : |f(x)| \geq \varepsilon/2\} \quad \text{and} \quad A_g = \{x : |g(x)| \geq \varepsilon/2\}$$

are compact subsets of X and

$$\{x : |f(x) + g(x)| \geq \varepsilon\} \subseteq A_f \cup A_g, \tag{*}$$

($\varepsilon \leq |f(x) + g(x)| \leq |f(x)| + |g(x)|$ implies either $|f(x)| \geq \varepsilon/2$ or $|g(x)| \geq \varepsilon/2$).

Since the set $\{x : |f(x) + g(x)| \geq \varepsilon\}$ is a closed subset of X, it follows that it is compact, using (*) above. This implies $C_0(X)$ is a linear subspace of $C_b(X)$.

We next show that $C_0(X)$ is a closed subspace of $C_b(X)$. Consider a sequence $\{f_n\}_{n \geq 1}$ in $C_0(X)$, and suppose that $f_n \to f$ in $C_b(X)$. Let $\varepsilon > 0$ be given. Then, there exists an integer N such that $\|f_n - f\| < \varepsilon/2$ for all $n \geq N$. This implies $|f_n(x) - f(x)| < \varepsilon/2$ for all $n \geq N$ and all $x \in X$. If $|f(x)| \geq \varepsilon$, then

$$\varepsilon < |f(x) - f_n(x) + f_n(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \varepsilon/2 + |f_n(x)|$$

for all $n \geq N$ and all $x \in X$, so $|f_n(x)| \geq \varepsilon/2$ for all $n \geq N$ and all $x \in X$. Thus

$$\{x \in X : |f(x)| \geq \varepsilon\} \subseteq \{x \in X : |f_n(x)| \geq \varepsilon/2\}$$

and this implies $f \in C_0(X)$. □

Remarks 5.1.15

(i) $X = \mathbb{R}$, $C_0(\mathbb{R})$ is the set of all continuous functions $f:\mathbb{R} \to \mathbb{C}$ such that $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$.
(ii) If X is compact, then $C_0(X) = C_b(X) = C(X)$.
(iii) $X = \mathbb{N}$, $C_b(\mathbb{N}) = \ell^\infty = \{x = \{x_n\}_{n\geq 1} : \sup_n |x_n| < \infty\}$ and $C_0(\mathbb{N}) = c_0 = \{x = \{x_n\}_{n\geq 1} : |x_n| \geq \varepsilon$ for only finitely many $n\}$. In other words, c_0 consists of all sequences that converge to zero.

5.2 Finite-Dimensional Spaces and Riesz Lemma

Recall that a linearly independent subset B of a vector space X is a basis in X if every element of X can be expressed as a finite linear combination of elements of B, i.e. B generates X. The set B is called a **Hamel basis** in X. It is well known that every vector space has a Hamel basis.

Although a given vector space has many Hamel bases, each of them has the same number (cardinal) of elements. This cardinal number is called the **dimension** of X [10, §2.4].

The notion of Hamel basis is of little use in studying Banach spaces since it is not related to any topological property such as continuity and convergence. In what follows, it will be seen that if X is a Banach space, then either $\dim X < \infty$ or $\dim X \geq c$, where c denotes the cardinality of the continuum. Thus, there is no Banach space whose Hamel basis is countable. We begin with the following definition:

Let $\{x_n\}_{n\geq 1}$ be a sequence of elements in a Banach space X. We say that the series $\sum_{n=1}^{\infty} x_n$ converges if the sequence $\{s_k\}_{k\geq 1}$, $s_k = \sum_{n=1}^{k} x_n$, of its partial sums has a limit in X.

Definition 5.2.1 *By a* **Schauder basis** *in X we mean a sequence $\{x_n\}_{n\geq 1}$ of unit vectors in X such that every $x \in X$ has a unique representation $x = \sum_{n=1}^{\infty} a_n x_n$,* i.e. $\left\| x - \sum_{n=1}^{k} a_n x_n \right\| \to 0 \, as \, k \to \infty$.

Remarks 5.2.2

(i) Note that the order in which the elements are enumerated is important in the definition.

(ii) If $\{x_n\}_{n\geq 1}$ is a Schauder basis for a Banach space X, then the collection of finite sums $\sum_{k=1}^{n} a_k x_k$ for which a_k are scalars with rational real and imaginary parts are dense in X. So X is separable. Thus, a nonseparable Banach space does not have a Schauder basis.

Example 5.2.3

(i) For each $k \in \mathbb{N}$, we denote by $e_k = (0, 0, \ldots, 0, 1, 0, \ldots)$, i.e. e_k has all its entries zero except at the kth place, where the entry is 1. Consider the space ℓ^p, and let $x = \{x_k\}_{k\geq 1} \in \ell^p$. Write $y_n = x - (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$.

$$\therefore \|y_n\| = \left(\sum_{k=n+1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \to 0 \quad \text{because} \quad \sum_{k=1}^{\infty} |x_k|^p \text{ converges.}$$

This implies $x = \sum_{k=1}^{\infty} x_k e_k$. Moreover, this representation is unique. Indeed, if $x = \sum_{k=1}^{\infty} \lambda_k e_k$, then

$$\begin{aligned}\left\|\sum_{k=1}^{n} (\lambda_k - x_k)e_k\right\|_p &= \left\|x - \sum_{k=1}^{n} x_k e_k - x + \sum_{k=1}^{n} \lambda_k e_k\right\|_p \\ &\leq \left\|x - \sum_{k=1}^{n} x_k e_k\right\|_p + \left\|x - \sum_{k=1}^{n} \lambda_k e_k\right\|_p \to 0 \text{ as } n \to \infty,\end{aligned}$$

whence
$|\lambda_1 - x_1|^p + \cdots + |\lambda_n - x_n|^p \to 0$ as $n \to \infty$, which implies $\lambda_k = x_k$ for all $k \in \mathbb{N}$.

(ii) Let ℓ^∞ denote the collection of all bounded sequences of complex numbers, i.e. all infinite sequences

$$x = (x_1, x_2, \ldots) = \{x_i\}_{i\geq 1}$$

such that

$$\sup_i |x_i| < \infty.$$

Define addition and scalar multiplication pointwise, and set

$$||x||_\infty = \sup\{|x_i| : i \in \mathbb{N}\}.$$

It is not difficult to verify that ℓ^∞ is a Banach space with respect to this norm. Moreover,

$$c_0 = \{x \in \ell^\infty : \lim_{n\to\infty} x_n = 0\}$$

is a closed subspace of ℓ^∞. Indeed, if $\{x^{(n)}\}_{n\geq 1}$, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots)$, converges to x, say, then

$$|x_k| \leq \left|x_k - x_k^{(n)}\right| + \left|x_k^{(n)}\right| \leq \left|\left|x^{(n)} - x\right|\right|_\infty + \left|x_k^{(n)}\right|.$$

The right-hand side is small since $x^{(n)} \to x$ and $x^{(n)} \in c_0$. Consequently, c_0, being a closed subspace of the Banach space ℓ^∞, is itself a Banach space.

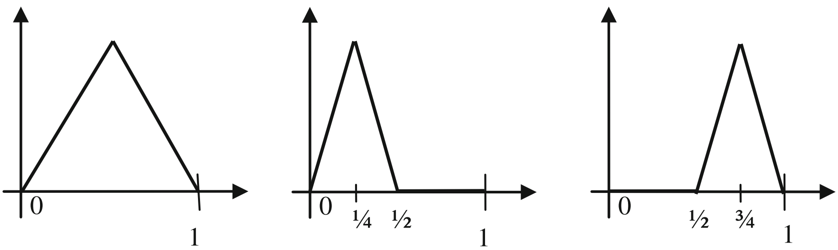
We next show that $e_k = (0, 0, \ldots, 0, 1, 0, \ldots)$, where 1 occurs at the kth place, is a Schauder basis for c_0. Let $x = \{x_k\}_{k\geq 1} \in c_0$. Then, $\lim_{k\to\infty} x_k = 0$. Write

$$y_n = x - (x_1, x_2, \ldots, x_n, 0, 0, \ldots) = x - \sum_{k=1}^{n} x_k e_k = (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots).$$
$$\therefore \quad ||y_k|| = \sup_{n\geq k+1}|x_n| \to 0 \text{ as } k \to \infty, \text{because } x \in c_0.$$

This implies $x = \sum_{k=1}^{\infty} x_k e_k$. Moreover, this representation is unique [see (i) above].

(iii) Consider the space $C[0, 1]$ of continuous real-valued functions defined on [0, 1]. The space $C[0, 1]$ is a Banach space with pointwise addition, scalar multiplication and norm defined by $\|f\|_\infty = \sup\{|f(t)|: 0 \leq t \leq 1\}$ [Remark 5.1.15(ii)]. Schauder constructed the following basis for $C[0, 1]$.

Let $\{r_i : i \geq 1\}$ be the enumeration of dyadic rationals in [0, 1]: $0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \ldots$. Let $f_1(t) \equiv 1, f_2(t) = t$, and for $n > 2$, define f_n as follows: $f_n(r_j) = 0$ if $j < n$, $f_n(r_n) = 1$, and f_n is linear between any two dyadic rationals amongst the first n dyadic rationals. Thus, f_3, f_4 and f_5 look like



f_3 f_4 f_5

Observe that for each n, $\|f_n\|_\infty = \sup\{|f_n(t)| : 0 \leq t \leq 1\} = |f_n(r_n)| = 1$. The sequence $\{f_n\}_{n\geq 1}$ consists of linearly independent vectors in $C[0,1]$. Indeed, if $\{f_{i_k}\}_{k=1}^{m}$ is any finite sequence of vectors in $\{f_n\}_{n\geq 1}$ and if

$$\sum_{k=1}^{m} a_{i_k} f_{i_k} = 0, \quad i_1 < i_2 < \cdots < i_m,$$

then the value of the sum at r_{i_m} is $a_{i_m} f_{i_m}(r_{i_m}) = 0$ which implies $a_{i_m} = 0$. Proceeding as above, we obtain $a_{i_1} = 0 = a_{i_2} = \cdots = a_{i_{m-1}}$.

We next show that $g \in C[0,1]$ has the form $g = \sum_{i=1}^{\infty} a_i f_i$. Indeed, if this is the representation, then $g(0) = a_1$, $g(1) = \sum_{i=1}^{\infty} a_i f_i(1) = a_1 f_1(1) + a_2 f_2(1) = a_1 + a_2$, which implies $a_2 = g(1) - a_1$ and, for $n > 2$,

$$g(r_n) = \sum_{i=1}^{\infty} a_i f_i(r_n) = \sum_{i=1}^{n-1} a_i f_i(r_n) + a_n,$$

which implies

$$a_n = g(r_n) - \sum_{i=1}^{n-1} a_i f_i(r_n).$$

Let $S_n = \sum_{i=1}^{n} a_i f_i$, where a_i are as determined above; S_n, being a finite linear combination of continuous functions, is itself a continuous function. Moreover,

$$\begin{aligned} S_n(r_j) &= \sum_{i=1}^{j-1} a_i f_i(r_j) + a_j f_j(r_j) + \sum_{i=j+1}^{n} a_i f_i(r_j) \\ &= \sum_{i=1}^{j-1} a_i f_i(r_j) + g(r_j) - \sum_{i=1}^{j-1} a_i f_i(r_j) \\ &= g(r_j), \quad (j = 1, 2, \ldots, n). \end{aligned}$$

Consequently, S_n is the piecewise linear function; i.e., if r_{j_1}, r_{j_2} are any two consecutive points of $\{r_j\}_{1 \le j \le n}$, then

$$S_n(\lambda r_{j_1} + (1-\lambda) r_{j_2}) = \lambda g(r_{j_1}) + (1-\lambda) g(r_{j_2}), \quad 0 \le \lambda \le 1.$$

Let $\varepsilon > 0$ be given. Since g is uniformly continuous on $[0, 1]$, there exists $\delta > 0$ such that $|g(t') - g(t'')| < \varepsilon$ whenever $t', t'' \in [0, 1]$ and $|t' - t''| < \delta$. Since the sequence $\{r_j\}_{j \ge 1}$ is dense in $[0, 1]$, there exists a positive integer $n_0 = n_0(\delta)$ such that for $n > n_0$, we have $\max |r_{j_1} - r_{j_2}| < \delta$, where the max is taken over all couples of consecutive points $\{r_j\}_{1 \le j \le n}$. Now, let $t = \lambda r_{j_1} + (1-\lambda) r_{j_2}$, where r_{j_1}, r_{j_2} are two consecutive points of $\{r_j\}_{1 \le j \le n}$, $n > n_0$ satisfying $t \in [r_{j_1}, r_{j_2}]$. Then by

$$\begin{aligned}|g(t)-S_n(t)| &= \left|g(t)-\lambda g(r_{j_1})-(1-\lambda)g(r_{j_2})\right|\\ &\le \left|\lambda\big(g(t)-g(r_{j_1})\big)+(1-\lambda)\big(\big(g(t)-g(r_{j_2})\big)\right|\\ &\le \max_{t',t''\in[r_{j_1},r_{j_2}]}|g(t')-g(t'')|, \quad n>n_0(\delta)\\ &< \varepsilon,\end{aligned}$$

which implies

$$||g-S_n||<\varepsilon \quad \text{for}\, n>n_0(\delta),$$

since $n_0(\delta)$ is independent of t.

Remark 5.2.4

(i) The space ℓ^∞ is not separable.

Let A be the set of elements of ℓ^∞ of the form $x = (x_1, x_2, \ldots)$ for which each x_i is either zero or 1. It is well known that A is uncountable. Let $x = \{x_i\}_{i\ge 1}$ and $\{y_i\}_{i\ge 1}$ be two distinct elements of A. Then

$$d(x,y)=\sup\{|x_i-y_i| : i=1,2,\ldots\}.$$

Suppose, if possible, that E_0 is a countable everywhere dense subset of ℓ^∞. Consider the balls of radius $\frac{1}{3}$ with centres at the points of E_0. Their union is the entire space ℓ^∞ because E_0 is everywhere dense and, in particular, contains A. Since the balls are countable in number, while A is not, in at least one ball, there must be two distinct elements x and y of A. Let x_0 denote the centre of such a ball. Then

$$1=d(x,y)\le d(x,x_0)+d(x_0,y)<\frac{1}{3}+\frac{1}{3}<1,$$

which is, however, impossible. Consequently, ℓ^∞ is not separable.

(ii) ℓ^∞ cannot have a Schauder basis [(ii) of Remark 5.2.3].

Does every separable Banach space have a Schauder basis? This long-standing problem was negatively settled by P. Enflo [8].

The space $\mathbb{F}^n$ is a Banach space [see (i) of Example 2.3.4]. It will be seen presently that $\mathbb{F}^n$ is the prototype of all n-dimensional normed linear spaces. We begin the definition of equivalent norms.

Definition 5.2.5 Suppose X is a vector space over $\mathbb{F}$ and suppose that $\|\cdot\|$ and $\|\cdot\|'$ are each norm on X. $\|\cdot\|$ is said to be equivalent to $\|\cdot\|'$, $\|\cdot\| \sim \|\cdot\|'$, if there exist positive constants α and β such that

$$\alpha||x||\le||x||'\le\beta||x|| \quad \text{for all} \;\; x\in X.$$

It is not difficult to see that this is an equivalence relation on the set of all norms over a given space.

If two norms are equivalent, then certainly if $\{x_n\}_{n\geq 1}$ is Cauchy with respect to either one of them, it must also be Cauchy with respect to the other.

Example 5.2.6 On $\mathbb{F}^n$, $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent. Indeed,

$$\left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{n}|x_i| \leq n\cdot \max_i\{|x_i| : 1\leq i\leq n\} \leq n\cdot\left(\sum_{i=1}^{n}\left||x_i|^2\right|\right)^{\frac{1}{2}},$$

[see (iii) of Example 5.1.2].

In what follows, we shall show that if X is finite-dimensional, then all norms on X are equivalent. We begin with the following observation.

Let $x_1, x_2, \ldots, x_n$ be orthonormal vectors in $\mathbb{F}^n$. Then for all scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$\|\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n\|_2^2=|\alpha_1|^2+|\alpha_2|^2+\cdots+|\alpha_n|^2.$$

A substitute for the above observation is the following Lemma.

Lemma 5.2.7 *Let* $\{x_1,x_2,\ldots,x_n\}$ *be linearly independent vectors in any normed linear space X. Then there exists a constant* $C>0$ *such that, for all scalars* $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$\|\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n\|\geq C(|\alpha_1|+|\alpha_2|+\cdots+|\alpha_n|), \tag{5.8}$$

i.e. the norm of any linear combination $\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n$ of linearly independent vectors $x_1, x_2,\ldots, x_n$ cannot be too small.

Proof Dividing both sides of inequality (5.8) by $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$ yields

$$\left\|\frac{\alpha_1}{\sum|\alpha_i|}x_1+\frac{\alpha_2}{\sum|\alpha_i|}x_2+\ldots+\frac{\alpha_n}{\sum|\alpha_i|}x_n\right\|\geq C \tag{5.9}$$

i.e. if $\Sigma|\alpha_i| = 1$, then (5.9) reduces to

$$\|\alpha_ix_i+\cdots+\alpha_nx_n\|\geq C. \tag{5.10}$$

If (5.10) were not true, for each positive integer m, there would exist $\alpha_1^{(m)}, \alpha_2^{(m)}, \ldots, \alpha_n^{(m)}$ such that

$$\left\|\alpha_1^{(m)}x_1+\alpha_2^{(m)}x_2+\cdots+\alpha_n^{(m)}x_n\right\|<\frac{1}{m} \tag{5.11}$$

The finite sequence $\left(\alpha_1^{(m)}, \alpha_2^{(m)}, \ldots, \alpha_n^{(m)}\right)$ indexed by m is a bounded sequence in $\mathbb{F}^n$. So, by the Heine–Borel Theorem, it has a convergent subsequence. The limit of the subsequence is $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\sum_i^n |a_i| = 1$, say. Since $\{x_j\}_{j=1}^n$ are linearly independent, this means

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \neq 0.$$

This contradicts (5.11) since (5.11) implies $\alpha^{(m)}{}_1 x_1 + \alpha^{(m)}{}_2 x_2 + \cdots + \alpha^{(m)}{}_n x_n \to 0$ as $m \to \infty$. □

Theorem 5.2.8 *Any two norms on a finite-dimensional vector space X are equivalent.*

Proof Let X be an n-dimensional vector space with basis $\{x_j\}_{j=1}^n$. If $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$, set

$$||x||_1 = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|.$$

It can be checked that $\|\cdot\|_1$ is a norm on X. Let $\|\cdot\|$ be an arbitrary norm on X. It suffices to show that $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. By Lemma 5.2.7, there exists a constant C such that

$$||x|| \geq C||x||_1.$$

On the other hand, if $C' = \max\|x_j\|$, then

$$||x|| \leq \sum_{j=1}^n |\alpha_j| \, ||x_j|| \leq C' \sum_{j=1}^n |\alpha_j| = C' \|x\|_1$$

Thus, $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. □

Corollary 5.2.9 *Every finite-dimensional normed linear space is a Banach space.*

Proof Since $(\mathbb{F}^n, ||\cdot||_\infty)$ is a Banach space and if a finite-dimensional space is complete in one norm, then it is complete in every other norm [Theorem 5.2.8], the Corollary follows. □

Corollary 5.2.10 *Every finite-dimensional subspace of a normed space is closed and complete.*

Proof Let X be a finite-dimensional subspace of a normed linear space Y. By Corollary 5.2.9, X is a Banach space. Let $y \in Y$ be a limit point of X. Then, there exists a sequence in X converging to y. The sequence must be Cauchy and hence has a limit in X since X is complete. It follows that $y \in X$. □

In $(\mathbb{C}^n, \|\cdot\|_2)$, where $\|\cdot\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$, the unit ball $\{x : \|x\|_2 \leq 1\}$ is a compact subset of $\mathbb{C}^n$ (Heine–Borel Theorem), and it follows that any closed ball (any centre and any radius) is compact.

Fredéric Riesz proved that the compactness of the unit ball characterises finite-dimensional normed spaces. We shall prove this by making use of the following Lemma, also due to Riesz.

Lemma 5.2.11 (Riesz) *Let Y be a closed proper subspace of a normed linear space X and* $\varepsilon > 0$ *be given. Then there exists an* $x_\varepsilon \in X$ *such that*

$$\mathrm{dist}(x_\varepsilon, Y) = \inf\{\|x_\varepsilon - x\| : x \in Y\} > 1 - \varepsilon.$$

Proof Choose any vector u not in Y. Put $d = \mathrm{dist}(u,Y) = \inf\{\|u - x\| : x \in Y\}$. Since Y is closed, $d > 0$. Hence by definition of d, there exists an $x_0 \in Y$ such that

$$d \le ||u - x_0|| < d(1 + \varepsilon)$$

and put

$$x_\varepsilon = \frac{(u - x_0)}{\|u - x_0\|}.$$

Then, $\|x_\varepsilon\| = 1$ and if $x \in Y$, then

$$||x_\varepsilon - x|| = \frac{1}{\|u - x_0\|}\|(\|u - x_0\|)x - u + x_0\| = \frac{1}{\|u - x_0\|}\|u - x_1\|,$$

where $x_1 = ||u - x_0||x + x_0$. Note that for each $x \in Y$, the vector x_1 is also in Y. So, by the definition of d,

$$\|x_\varepsilon - x\| \ge \frac{d}{\|u - x_0\|} > \frac{1}{1 + \varepsilon} > 1 - \varepsilon.$$

This completes the argument. □

If X is finite-dimensional, its unit sphere $\{x : \|x\| = 1\}$ is compact. Using this observation, we shall prove the following Corollary.

Corollary 5.2.12 *Let Y be a proper subspace of a finite-dimensional normed linear space X. Then there exists a point x in the unit sphere* $S(X) = \{x : \|x\| = 1\}$ *whose distance from Y is* 1, i.e. $\mathrm{dist}(x,Y) = 1$.

Proof By Riesz' Lemma 5.2.11, for all $n \in \mathbb{N}$, there exists x_n, $\|x_n\| = 1$, satisfying

$$\text{dist}(x_n, Y) > 1 - \frac{1}{n}.$$

The sequence $\{x_n\}_{n\geq 1}$ is in the compact set $S(X)$. So, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ such that $x_{n_k} \to x \in S(X)$. On taking limits in

$$\text{dist}(x_{n_k}, Y) > 1 - \frac{1}{n_k}$$

and using the fact that $x \to \text{dist}(x,Y)$ is a continuous function, it follows that

$$\text{dist}(x, Y) \geq 1. \tag{5.12}$$

On the other hand,

$$\text{dist}(x, Y) = \inf\{||x - y|| : y \in Y\} \leq ||x|| = 1. \tag{5.13}$$

On combining (5.12) and (5.13), we obtain the desired conclusion. □

The following characterisation of finite-dimensional normed linear spaces follows from Riesz' Lemma.

Corollary 5.2.13 (Riesz) *If X is a normed linear space with the property that closed bounded sets are compact or that the unit sphere is compact, then X is finite-dimensional.*

Proof If $x_1 \in X$ and x_1 does not generate X, there is x_2 with $\|x_2\| = 1$ so that $||x_2 - x_1|| > \frac{1}{2}$. If x_1 and x_2 do not generate X, there is x_3, $\|x_3\| = 1$, so that $||x_3 - x_1|| > \frac{1}{2}$ and $||x_3 - x_2|| > \frac{1}{2}$. This process must terminate since otherwise we would obtain a sequence of points on the unit sphere $S(X) = \{x : \|x\| = 1\}$ which has no convergent subsequence. This contradicts the assumption that closed bounded subsets of X are compact or that the unit sphere is compact. This completes the proof. □

Remark 5.2.14

(i) The closed bounded subsets in any finite-dimensional normed linear space are compact.
Combining the above remark with the Corollary proved above, we have the following assertion:

(ii) A normed linear space is finite-dimensional if, and only if, the closed bounded subsets are compact.

(i) Let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be finite-dimensional subspaces of a normed space, with all inclusions proper. Then, there are unit vectors $x_1, x_2, \ldots$ such that $x_n \in X_n$ and $d(x_n, X_{n-1}) = 1$ for $n \geq 2$.

In particular, an infinite-dimensional normed space contains an infinite sequence $\{x_n\}_{n\geq 1}$ of vectors satisfying $||x_n - x_m|| \geq 1$ for $n \neq m$.

To find x_n, apply the above Corollary 5.2.12 to the pair (X_n, X_{n-1}).
Riesz' Lemma has been used in Remark 4.5.3(i).

5.3 Linear Functionals and Hahn–Banach Theorem

A mapping T of a vector space X into a vector space Y is called a linear mapping or a linear operator if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

for all $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{F}$. If X and Y are normed linear spaces, we call a linear operator T bounded if there is a constant M such that, for all x, we have $\|Tx\| \leq M\|x\|$. We call the infimum of such M the norm of T and denote it by $\|T\|$. Thus,

$$||T|| = \sup_{x \in X, x \neq 0} \frac{||Tx||}{||x||}.$$

Since $T(\alpha x) = \alpha T(x)$, we have also

$$||T|| = \sup_{\|x\|=1} ||Tx|| = \sup_{\|x\| \leq 1} ||Tx||.$$

The Sects. 3.1 and Subsections 3.2.1–3.2.6 contain the essential information we need in the sequel. For the benefit of the reader, we reproduce important results therein (without proof):

(i) Let X and Y be normed linear spaces over the same field $\mathbb{F}$ of scalars and $D \subseteq X$ be the domain of the linear operator T from D into Y. Then, the following conditions are equivalent:

   (a) T is continuous at a given $x_0 \in D$;
   (b) T is bounded;
   (c) T is continuous everywhere and the continuity is uniform [this is Proposition 3.2.3].

$\mathcal{B}(X, Y)$ denotes the space of all bounded linear operators from a normed linear space X into a normed linear space Y.

(ii) If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space with addition, scalar multiplication and norm of the operator as defined below: for $T, T_1, T_2 \in \mathcal{B}(X, Y)$, $\alpha \in \mathbb{F}$,

$$(T_1 + T_2)x = T_1x + T_2x$$
$$(\alpha T)x = \alpha(Tx)$$
$$\|T\| = \sup_{\|x\|=1}\|Tx\|,$$

$x \in X$.

(iii) If Y is 1-dimensional and X is a normed linear space, we obtain Theorem 2-10.23. In this case, $\mathcal{B}(X, Y)$ is denoted by X^*, i.e. $X^* = \mathcal{B}(X, \mathbb{F})$. The space X^* is called the conjugate [adjoint, dual] space of X. The conjugate space X^{**} of the space X^* is called the second conjugate space of X. Elements of X^* are called bounded linear functionals.

(iv) Proposition 2.10.21 and the following Remark 2.10.22 contain the analogue of (i) for bounded linear functionals.

(v) Let X be a normed linear space. There is a so-called natural mapping of X into X^{**} defined as follows. For $x \in X$, define $\hat{x}$ on X^* by the rule $\hat{x}(f) = f(x)$, $f \in X^*$. Simple computations show that $\hat{x}$ is a linear functional on X^*. Indeed,

$$\hat{x}(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha\hat{x}(f) + \beta\hat{x}(g)$$

when f, $g \in X^*$ and $\alpha, \beta \in \mathbb{F}$. Moreover, $\hat{x}$ is bounded, since $|\hat{x}(f)| = |f(x)| \le \|f\|\|x\|$. Thus, the norm of $\hat{x}$ satisfies $\|\hat{x}\| \le \|x\|$. It will follow, using Corollary 5.3.7 of the Hahn–Banach Theorem, that there exists $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$. Hence

$$|\hat{x}(f)| = |f(x)| = \|x\|.$$

It follows that

$$\|\hat{x}\| = \|x\|.$$

The mapping $x \to \hat{x}$ of X into X^{**} preserves norms.
The space X is said to be reflexive if $X = X^{**}$, where the equality is understood in the sense of isomorphism (algebraic isomorphism and norm preserving) under the canonical (natural) mapping $x \to \hat{x}$. Thus, reflexivity occurs if, and only if, the above mapping carries X onto X^{**}.

(vi) Theorem 2.10.30 shows that every Hilbert space is reflexive.

So far, we do not know that for every nonzero normed linear space X there is at least one nonzero linear functional, i.e. $X^* \neq \{0\}$. We shall answer this question with the aid of the Hahn–Banach Theorem, which is next on our programme.

Definition 5.3.1 (Cf. Definition 2–12.7). Let X be a linear space. A real-valued function p defined on X is said to be a **convex functional** if

(i) $p(x + y) \leq p(x) + p(y)$ (subadditive)

and

(ii) $p(\alpha x) = \alpha p(x)$ (positive homogeneous)

for all $x,y \in X$ and nonnegative real numbers α.

Every convex functional p is a convex function; indeed, if $x,y \in X$ and $0 \leq \alpha \leq 1$, then $p(\alpha x + (1-\alpha)y) \leq p(\alpha x) + p((1-\alpha)y) \leq \alpha p(x) + (1-\alpha)p(y)$.

The first question with which we shall be concerned is that of extending a linear functional f from a subspace of X to the whole of X in such a manner that that the various properties of the functional are preserved, i.e. the extended linear functional F is defined on all of X and agrees with f on the subspace. The principal result in this direction is the following.

Theorem 5.3.2 (Hahn–Banach) *Let $M \neq \{0\}$ be a subspace of a real linear space X, p a convex functional on X and f a linear functional defined on M such that, for all $x \in M$,*

$$f(x) \leq p(x).$$

Then there exists a linear functional F defined on all of X extending f (i.e. $F(x) = f(x)$ for all $x \in M$) such that

$$F(x) \leq p(x), \quad x \in X.$$

Proof Let x_1 be a vector in $X \backslash M$. Let N denote the space spanned by M and x_1, i.e.

$$N = [M \cup \{x_1\}] = \{x + \lambda x_1 : x \in M, \lambda \in \mathbb{R}\}.$$

We first extend f to N and denote the extension of f to N by h. To do this, it suffices to define $h(x_1)$ appropriately. It is required that $h(x + \lambda x_1) \leq p(x + \lambda x_1)$ for all $\lambda \in \mathbb{R}$ and $x \in M$. Dividing by $|\lambda|$, the desired inequality can be written as

$$h\left(\frac{x}{|\lambda|} + \frac{\lambda}{|\lambda|}x_1\right) \leq p\left(\frac{x}{|\lambda|} + \frac{\lambda}{|\lambda|}x_1\right),$$

which is equivalent to

$$h\left(\frac{x}{-\lambda} - x_1\right) \leq p\left(\frac{x}{-\lambda} - x_1\right) \quad \text{if } \lambda < 0.$$

and

$$h\left(\frac{x}{\lambda} + x_1\right) \leq p\left(\frac{x}{\lambda} + x_1\right) \quad \text{if } \lambda > 0.$$

Setting $\frac{x}{-\lambda} = y$, the two inequalities can be expressed as

$$h(y - x_1) \le p(y - x_1) \quad \text{and} \quad h(-y + x_1) \le p(-y + x_1) \quad \text{for all} \quad y \in M.$$

Thus,

$$-p(-x_1 + y) + f(y) \le h(x_1) \le f(y) + p(-y + x_1).$$

So, a value can be selected for $h(x_1)$ such that the resultant h on M has the requisite properties if, and only if,

$$\sup_{y \in M}\{-p(-x_1 + y) + f(y)\} \le \inf_{z \in M}\{p(-z + x_1) + f(z)\}.$$

However, for $y,z \in M$, we have

$$f(y) - f(z) = f(y - z) \le p(y - z) \le p(x_1 - z) + p(y - x_1),$$

so that

$$-p(-x_1 + y) + f(y) \le p(-z + x_1) + f(z).$$

Therefore, f can be extended to h defined on N such that $h(x + \lambda x_1) \le p(x + \lambda x_1)$ for all $\lambda \in \mathbb{R}$ and $x \in M$.

Thus, we have obtained an extension of f to N. Observe that this extension is not unique since it depends on c, where

$$\sup_{y \in M}\{-p(-x_1 + y) + f(y)\} \le c \le \inf_{z \in M}\{p(-z + x_1) + f(z)\}.$$

We propose to obtain a 'maximal' extension of f. To this end, let $\mathcal{F}$ be the collection of all ordered pairs $(Y, \hat{h})$, where Y is a subspace of X that contains M and $\hat{h}$ is a linear functional defined on Y extending f defined on M and is dominated by p, i.e. $\hat{h}(y) \le p(y)$ for all $y \in Y$. Define a partial order $\le$ on $\mathcal{F}$ by requiring $(Y_1, \hat{h}_1) \le (Y_2, \hat{h}_2)$ if Y_2 is a linear space that contains Y_1 and $\hat{h}_2 = \hat{h}_1$ on Y_1. Let $\mathcal{G} = \left\{(Y_\alpha, \hat{h}_\alpha)\right\}_{\alpha \in \Lambda}$ be a totally ordered indexed subfamily of $\mathcal{F}$. Then, the pair (Y,g), where $Y = \cup_\alpha Y_\alpha$ and $g(x) = \hat{h}_\alpha(x)$ for $x \in Y_\alpha$, is an element of $\mathcal{F}$ and is an upper bound of $\mathbb{G}$. Therefore by Zorn's Lemma [see Sect. 1.4], $\mathcal{F}$ has a maximal element. Let $\left(Y_\infty, \hat{h}_\infty\right)$ be one such maximal element. If $Y_\infty \ne X$, we could again extend $\left(Y_\infty, \hat{h}_\infty\right)$ by adding one element as before. This would contradict the maximality of $\left(Y_\infty, \hat{h}_\infty\right)$. Thus, $Y_\infty = X$, and if we put $\hat{h}_\infty = F$, then F is a linear functional on X with the requisite properties. □

Remarks 5.3.3

(i) The crux of the Hahn–Banach Theorem is that the extended functional is still dominated by p. If this requirement were dispensed with, we could have obtained an extension as follows:

Take any Hamel basis for M; enlarge it to a Hamel basis for X. Define F arbitrarily on the new basis elements, and ensure that F is linear on X.

(ii) The Hanh–Banach Theorem has a wide range of applications, many of them involving an appropriate choice of the convex functional. The first of these is the following corollary:

Corollary 5.3.4 *Let X be a real normed linear space and M be a linear subspace of X. If $f \in M^*$, then there exists $F \in X^*$ such that F extends f and $\|F\| = \|f\|$.*

Proof Define p on X by $p(x) = \|f\|\|x\|$. Then, p is a convex functional on X and satisfies the inequality $f(x) \leq |f(x)| \leq p(x)$ for all $x \in M$. Apply the Hahn–Banach Theorem 5.3.2 to obtain a linear functional F defined on X extending f and satisfying $F(x) \leq p(x)$ for all $x \in X$. Clearly, $F \in X^*$ and $\|F\| \leq \|f\|$. But we also have

$$\begin{aligned} ||F|| &= \sup\{|F(x)| : x \in X \text{ and } ||x|| \leq 1\} \\ &\geq \sup\{|F(x)| : x \in M \text{ and } ||x|| \leq 1\} \\ &= \sup\{|f(x)| : x \in M \text{ and } ||x|| \leq 1\}. \end{aligned}$$

Thus,

$$||F|| = \|f\|.$$ □

Theorem 5.3.5 (Hahn–Banach Theorem for complex normed linear spaces) *Let X be a complex normed linear space and let M be a linear subspace of X. If $f \in M^*$, then there exists $F \in X^*$ (F an extension of f) such that $\|F\| = \|f\|$.*

Proof For each $x \in M$, write $f(x) = f_1(x) + if_2(x)$, where f_1 and f_2 are real-valued. It can easily be verified that f_1 and f_2 are real linear functional on M, i.e. $f_j(x + y) = f_j(x) + f_j(y)$ and $f_j(\alpha x) = \alpha f_j(x)$ for $\alpha \in \mathbb{R}$. Moreover, f_1 and f_2 are bounded and $\|f_j\| \leq \|f\|$. Now, regarding X and M as real linear spaces (multiplication is allowed by reals only) and applying the Hahn–Banach Theorem 5.3.2, we obtain a bounded linear functional F_1 on X such that F_1 extends f_1 and $\|F_1\| = \|f_1\|$. We next define F on X by

$$F(x) = F_1(x) - iF_1(ix).$$

Observe that

$$\begin{aligned} iF(x) &= iF_1(x) + F_1(ix) \\ &= F_1(ix) - iF_1(i(ix)) \\ &= F(ix), \end{aligned}$$

i.e. F is a complex linear functional. We next check that F extends f. Indeed, for $x \in M$, on equating real and imaginary parts in the following equality

$$\begin{aligned} F_1(ix) + if_2(ix) &= f_1(ix) + if_2(ix) = f(ix) = if(x) = -f_2(x) + if_1(x) \\ &= -f_2(x) + iF_1(x), \end{aligned}$$

we have $F_1(ix) = -f_2(x)$ and therefore

$$F(x) = F_1(x) - iF_1(ix) = f_1(x) + if_2(x) = f(x).$$

It remains to show that F is bounded and $\|F\| = \|f\|$. Let $x \in X$ be arbitrary and write $F(x) = r\exp(i\vartheta)$, where $r \geq 0$ and $\vartheta \in \mathbb{R}$. Then, we have

$$\begin{aligned} |F(x)| = r = \exp(-i\vartheta)F(x) &= F(\exp(-i\vartheta)x) = F_1(\exp(-i\vartheta)x) \\ &\leq \|F_1\|\|x\| \leq \|f_1\|\|x\| \leq \|f\|\|x\|. \end{aligned}$$

This proves that F is bounded and $\|F\| \leq \|f\|$. It is easy to discern that

$$\|F\| = \sup_{x\in X,\|x\|=1}|F(x)| \geq \sup_{x\in M,\|x\|=1}|F(x)| = \sup_{x\in M,\|x\|=1}|f(x)| = \|f\|.$$

Consequently, $\|F\| = \|f\|$. □

Corollary 5.3.6 *Let M be a closed subspace of a normed linear space X and let x_1 be a vector in X such that $d(x_1,M) = \delta > 0$. Then there exists a linear functional f on X with the following properties*:

$$\|f\| = 1, f(x_1) = \delta \quad \text{and} \quad f(x) = 0 \quad \text{for all} \;\; x \in M.$$

Proof Let $N = [M \cup \{x_1\}]$. Every vector in N can be uniquely written as $y = x + \alpha x_1, x \in M, \alpha \in \mathbb{F}$. Define $f_1(y) = f_1(x + \alpha x_1) = \alpha\delta$. Then, f_1 is linear on N, $f_1(x_1) = \delta$ and $f_1(x) = 0$ for all $x \in M$. We want to show that $\|f_1\| = 1$.

For $x \in M$ and $\alpha \neq 0$, we have

$$|f_1(x + \alpha x_1)| = |\alpha|\delta \leq |\alpha|\left\|x_1 + \frac{x}{\alpha}\right\| = \|x + \alpha x_1\|.$$

So, $\|f_1\| \leq 1$. Observe that for each $x \in M$, $|f_1(x - x_1)| = |\delta| = \delta$. Choose a sequence $\{x_n\}_{n\geq 1}$ in M such that $\|x_n - x\| \to \delta$. For this sequence,

$$|f_1(x_n - x)|/\|x_n - x\| \to 1.$$

Hence $\|f_1\| = 1$. □

Corollary 5.3.7 *For each nonzero vector x in a normed linear space X, there exists a linear functional f on X such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Proof Let $M = \{\lambda x : \lambda \in \mathbb{F}\}$. Define h on M by $h(\lambda x) = \lambda\|x\|$. Then, $\|h\| = 1$ and an extension to X given by the Hahn–Banach Theorem has the desired properties. □

Remarks 5.3.8 One of the consequences of the above Corollary is that X^* is not a trivial vector space, i.e. $X^* \neq \{0\}$ if $X \neq \{0\}$. In fact, X^* separates points on X. This means that if $x_1 \neq x_2$ in X, there exists $f \in X^*$ such that $f(x_1) \neq f(x_2)$. Indeed, if $x_0 = x_2 - x_1$, then $x_0 \neq 0$ and then by Corollary 5.3.7, there exists $f \in X^*$such that $\| f \| = 1$, $f(x_0) = \|x_2 - x_1\| \neq 0$, which implies $f(x_1) \neq f(x_2)$. □

A consequence of Corollary 5.3.7 is the following Proposition.

Proposition 5.3.9 *Let X be a Banach space. If the dual space X* is separable, then X is separable.*

Proof Let $\{f_n : n = 1,2,\ldots\}$ be a countable dense subset of the unit sphere in X^*, i.e.. $\|f_n\| = 1$, $n = 1,2,\ldots$. For each n, choose an $x_n \in X$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{5}{8}$. This is possible since $\|f_n\| = 1$.

Let M be the closed subspace of X generated by the vectors x_1,x_2,….

Let $M \neq X$. Then with $x_0 \in X\backslash M$, there is an f, $\| f \| = 1$, which vanishes on M but not at x_0. Thus for $n = 1, 2, \ldots$,

$$\frac{5}{8} \leq |f_n(x_n)| = |f_n(x_n) - f(x_n)| = |(f_n - f)(x_n)| \leq \|f_n - f\|,$$

which is impossible, since $\{f_n\}_{n \geq 1}$ is dense in the unit sphere in X. □

Remark 5.3.10 The set of linear combinations of the vectors $\{e_k\}_{k \geq 1}$, where $e_k = (0,0,\ldots,0,1,0,\ldots)$ (1 occurs at the kth place) is a dense subset of ℓ^1. So, ℓ^1 is separable [see Example 5.2.3(i)]. We have observed in Remark 5.2.4(i) that ℓ^∞ is not separable. Consequently, the converse of Proposition 5.3.9 is false. Of course, we need to prove that $(\ell^1)^* = \ell^\infty$. It is the content of the following Proposition.

Proposition 5.3.11 *The dual* $(\ell^1)^*$ *of* ℓ^1 *is* ℓ^∞, i.e. $(\ell^1)^* = \ell^\infty$.

Proof Let $x = (x_1, x_2, \ldots) \in \ell^1$, $y = (y_1, y_2, \ldots) \in \ell^\infty$. Define a functional f_y on ℓ^1 by setting $f_y(x) = \sum_{k=1}^\infty x_k y_k$. Then, $f_y \in \ell^\infty$ since $|f_y(x)| = |\sum_{k=1}^\infty x_k y_k| \leq \|y\|_\infty \|x\|_1$ and hence

$$\|f_y\| \leq \|y\|_\infty. \tag{5.14}$$

Let $f \in (\ell^1)^*$. From the continuity and linearity of f, it follows that

$$f(x) = f\left(\sum_{k=1}^\infty x_k e_k\right) = \sum_{k=1}^\infty x_k y_k,$$

where $e_k = (0, 0, \ldots,0, 1, 0, \ldots)$ (1 occurs at the kth place) and $y_k = f(e_k)$. Now, consider the vector $y = (y_1, y_2, \ldots)$. Set $z = (0, \ldots, \exp(-i\arg y_n), 0, \ldots)$, where the nonzero term appears at the nth place. We have that

$$||z|| = 1 \text{ and } f(z) = y_n \exp(-i \arg y_n) = |y_n| \leq \|f\| \text{ for every } n.$$

Hence, $\sup_n |y_n| \leq \|f\|$. Therefore, $y \in \ell^\infty$ and $\|y\|_\infty \leq \|f\|$. Starting with this $y \in \ell^\infty$, we can generate f_y as in the paragraph above, and obtain inequality (5.14); however, $f_y = f$. Consequently, $\|f\| \leq \|y\|_\infty$. This implies $\|y\|_\infty = \|f\|$. □

Remark 5.3.12 The dual $(\ell^\infty)^*$ is not ℓ^1. For, if $(\ell^\infty)^* = \ell^1$ and ℓ^1 being separable, it would imply by Proposition 5.3.9 that ℓ^∞ is separable, which is false [Remark 5.2.4(i)]. It follows that ℓ^1 is not a reflexive Banach space.

We end this Section with an application of the Hahn–Banach Theorem.

Let ℓ^∞ be the space of all bounded sequences of real numbers with norm $\|\{x_k\}\| = \sup_k |x_k|$ and c the closed subspace of convergent sequences. If $x \in c$, then $\ell(x) = \lim_k x_k$ is defined. It follows, using elementary properties of limits of sequences, that ℓ is a linear functional on c and $\ell(e) = 1$, where $e = (1,1,\ldots)$. Moreover, ℓ is continuous when c is equipped with the norm $\|\cdot\|_\infty$. Indeed,

$$|\ell(x)| = |\lim_n x_n| = \lim_n |x_n| \leq ||x||_\infty,$$

which implies $\|\ell\| \leq 1$ and since $\ell(e) = 1$, it follows that $\|\ell\| = 1$. Also,

$$\ell(x) = \ell(\tau(x)) \quad \text{for all} \quad x \in c, \quad \text{where} \quad \tau(x)(j) = x_{j+1} \quad \text{for} \quad j = 1, 2, \ldots.$$

We shall show that ℓ admits a Hahn–Banach extension to ℓ^∞ possessing all the properties of ℓ enunciated above.

Theorem 5.3.13 *Let ℓ^∞ be the space of all real bounded sequences. Then there exists $L \in (\ell^\infty)^*$ such that for any $x = \{x_n\}_{n \geq 1}$ in ℓ^∞,*

(a) $\liminf x_n \leq L(x) \leq \limsup x_n$ *and, in particular, if the limit* $\lim_n x_n$ *exists, then* $L(x) = \lim_n x_n$;

(b) $L(\{x_n\}) = L(\{x_{n+1}\})$.

The value $L(x)$ is called **a Banach limit** of x.

Proof (b) Consider the subspace $M = \{y = \{y_j\}_{j \geq 1} : y_j = x_{j+1} - x_j, \{x_j\}_{j \geq 1} \in \ell^\infty\}$ of the space ℓ^∞. Since $1 = \|e\| = d(e,0)$, where $e = (1,1,\ldots)$, it follows that $d(e, \overline{M}) \leq 1$. Assume that $d(e, \overline{M}) < 1$; i.e., there exists $\{x_j\}_{j \geq 1} \in \ell^\infty$ such that $|(x_{j+1} - x_j) - 1| \leq \varepsilon$ for every j. This implies $x_{j+1} \geq x_j + (1 - \varepsilon)$, contradicting the fact that $\{x_j\}_{j \geq 1}$ is a bounded sequence.

We have thus proved that $d(e, \overline{M}) = 1$. By Corollary 5.3.6, there exists $L \in (\ell^\infty)^*$ such that $L(e) = 1$, $\|L\| = 1$ and $L(\overline{M}) = 0$. The latter implies $L(\{x_n\}) = L(\{x_{n+1}\})$. This completes the proof of (b).

(a) We first show that for any $\{x_n\}_{n \geq 1} \in \ell^\infty$, $L(x) \leq \sup_n x_n$. Assume that $x_n \geq 0$ for every n. Since $\|L\| = 1$, we have $L(x) \leq \sup_n |x_n| = \sup_n x_n$. In the general case, choose λ large enough. Since $L(e) = 1$, it follows that $L(x) + \lambda = L(x + \lambda e) \leq$

$\sup_n(x_n + \lambda) = \lambda + \sup_n x_n$. Thus, $L(x) \leq \sup_n x_n$ for any sequence $x = \{x_n\}_{n \geq 1} \in \ell^\infty$.

Now consider $\lim\sup x_n = \inf_n(\sup_{k \geq n} x_k)$. For any $\varepsilon > 0$, there exists n_0 such that $\sup_{k \geq n_0} x_k < \lim\sup x_n + \varepsilon$. Set $x^{(n_0)} = (x_{n_0+1}, x_{n_0+2}, \cdots)$. We have

$$L(x) = L\left(x^{(n_0)}\right) \leq \sup_{k \geq n_0} x_k \leq \lim\sup x_n + \varepsilon,$$

which implies

$$L(x) \leq \lim\sup x_n.$$

On replacing x by $-x$ and cancelling (-1), we get

$$L(x) \geq \lim\inf x_n.$$

Consequently,

$$\lim\inf x_n \leq L(x) \leq \lim\sup x_n, \quad x \in \ell^\infty.$$

□

5.4 Baire Category Theorem and Uniform Boundedness Principle

There are several ways of proving the existence of an entity. All of these ways are difficult, no matter how one approaches the problem. The most difficult is the constructive type of proof. The example due to van der Waerden of a continuous nowhere differentiable function is the easiest to work with (Example 2.4.8 of [25]). The other is to use a contrapositive argument as is done in Euclid's proof of the existence of an infinite number of primes. Another tool that is often employed in an existence proof is the notion of 'category'. More specifically, we shall be concerned with a Theorem of Baire referred to in the literature as 'Baire's Category Theorem'. For an excellent account of this, the reader will benefit from consulting Sect. 2.4 of [25]. Another useful tool for existence proofs is Zorn's Lemma [see Sect. 1.4].

In what follows, we shall be concerned with Baire's Category Theorem, which belongs to the realm of metric spaces. The uniform boundedness principle (also called the Banach–Steinhaus Theorem) is still another consequence of the Theorem of Baire.

Definition 5.4.1 Let (X,d) be a metric space. A subset $Y \subseteq X$ is said to be **nowhere dense** if $(\bar{Y})^\circ$ is empty, i.e. $\bar{Y}$ contains no interior points. A subset $F \subseteq X$ is said to be of **category I** if it is a countable union of nowhere dense subsets. Subsets that are not of category I are said to be of **category II**.

Remarks 5.4.2

(i) A subset Y of X is nowhere dense if, and only if, the complement $(\bar{Y})^{\mathrm{c}}$ is dense in X. It is a consequence of the fact that a subset $Y \subseteq X$ is dense in X if, and only if, Y^{c} has an empty interior.

(ii) The notion of being nowhere dense is not the negation of being everywhere dense, i.e. not being nowhere dense does not imply that the set is everywhere dense. For an example, let $Y = \{x \in \mathbb{R} : 1 < x < 2\}$. Then, $(\bar{Y})^{\circ} = Y \neq \varnothing$, i.e. Y is not nowhere dense and $(\bar{Y})^{\mathrm{c}} = (-\infty, 1) \cup (2, \infty)$, which is not dense in $\mathbb{R}$.

A subset is nowhere dense if, and only if, the complement of its closure is everywhere dense. This will be used in the proof of Baire's Category Theorem.

(iii) Since a denumerable union of denumerable sets is again a denumerable set, it follows that if Y_1, Y_2, ... are each of category I, then so must be the union $\cup_i Y_i$.

(iv) If $X = Y_1 \cup Y_2$ and it is known that Y_1 is of category I while X is of category II, then in view of (ii) above, Y_2 must be of category II.

(v) A subset of a nowhere dense set is nowhere dense and therefore a subset of a set of category I is again of category I.

Example 5.4.3

(i) In a discrete metric space, the only nowhere dense set is the empty set.

(ii) An empty subset of a metric space is of category I. Also, the set $\mathbb{Q}$ of rationals in $\mathbb{R}$ is a set of category I. Indeed, if $x_1, x_2, \ldots$ is an enumeration of the rationals, each $\{x_i\}$ is closed and $\{x_i\}^{\mathrm{o}} = \varnothing$, and it follows that $\cup_i \{x_i\}$, the set of all rationals in $\mathbb{R}$, is of category I.

(iii) (Cantor set) Divide the closed interval $I = [0,1]$ into three equal parts by the points $\frac{1}{3}$ and $\frac{2}{3}$, and remove the open interval $I_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$ from I. Divide each of the remaining two closed intervals $J_{1,1} = \left[0, \frac{1}{3}\right]$ and $J_{1,2} = \left[\frac{2}{3}, 1\right]$ into three equal parts by the points $\frac{1}{9}, \frac{2}{9}$ and $\frac{7}{9}, \frac{8}{9}$, respectively, and remove the open intervals $I_{2,1} = \left(\frac{1}{9}, \frac{2}{9}\right)$ and $I_{2,2} = \left(\frac{7}{9}, \frac{8}{9}\right)$. Now, divide each of the remaining four closed intervals $J_{2,1} = \left[0, \frac{1}{9}\right]$, $J_{2,2} = \left[\frac{2}{9}, \frac{1}{3}\right]$, $J_{2,3} = \left[\frac{2}{3}, \frac{7}{9}\right]$ and $J_{2,4} = \left[\frac{8}{9}, 1\right]$ into three equal parts and remove the middle third open intervals. Continue this process indefinitely. The open set G removed in this way from $I = [0,1]$ is the union of the disjoint open intervals

$$G = \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \cdots.$$

The complement of G in [0,1], denoted by P, is called the **Cantor set**. Clearly, P is a closed subset of [0, 1]. We next show that P contains no open interval.

Observe that at the first stage, two closed disjoint intervals $J_{1,1}$ and $J_{1,2}$ each having length less than $\frac{1}{2}$ remain. Let $P_1 = J_{1,1} \cup J_{1,2}$. Thus, P_1 contains no interval of length $\frac{1}{2}$ or greater. At the second stage, four closed disjoint intervals $J_{2,1}$, $J_{2,2}$, $J_{2,3}$, $J_{2,4}$, each having length less than $\frac{1}{2^2}$ remain. Let $P_2 = J_{2,1} \cup J_{2,2} \cup J_{2,3} \cup J_{2,4}$. At the nth stage, $P_n = J_{n,1} \cup J_{n,2} \cup \cdots \cup J_{n,2^n}$, each having length less than $\frac{1}{2^n}$ and $P \subseteq P_n$ for each $n \in \mathbb{N}$. As this process is continued indefinitely, the Cantor set contains no interval of positive length. Consequently, P is a nowhere dense set.

The reader who has not seen this construction earlier is invited to use an induction argument to strengthen the proof outlined above.

Baire's Category Theorem says that a complete metric space cannot be the union of a countable number of nowhere dense sets.

Theorem 5.4.4 (Baire's Category Theorem) *Any complete metric space is of category II.*

Proof We assume the contrary, i.e. suppose that (X, d) is a complete metric space and

$$X = \cup_{n \geq 1} E_n,$$

where each E_n is nowhere dense, i.e. $(\overline{E_n})^c$ is dense for each $n \in \mathbb{N}$. So, we can assert that each of the sets $(\overline{E_n})^c$ is nonempty. In the case of $(\overline{E_1})^c$, let $x_1 \in (\overline{E_1})^c$. Since $(\overline{E_1})^c$ is open, there exists $r > 0$ such that $S(x_1, r) \subseteq (\overline{E_1})^c$. For $\varepsilon_1 < r$, we have

$$\overline{S}(x_1, \varepsilon_1) \subseteq S(x_1, r) \subseteq \left(\overline{E_1}\right)^c \subseteq (E_1)^c.$$

This in turn implies

$$\overline{S}(x_1, \varepsilon_1) \cap E_1 = \varnothing.$$

We make the following induction hypothesis. There exist balls $S(x_k, \varepsilon_k)$ for $k = 1, 2, \ldots, n-1$ such that

$$\overline{S}(x_k, \varepsilon_k) \cap E_k = \varnothing,\ x_k \in \left(\overline{E_k}\right)^c,$$

and

$$\varepsilon_k \leq \frac{1}{2}\varepsilon_{k-1} \quad \text{for} \ \ k = 1, 2, \ldots, n-1.$$

Using this information, we can construct the nth ball with the above properties. To this end, choose

$$x_n \in S(x_{n-1}, \varepsilon_{n-1}) \cap \left(\overline{E_n}\right)^c.$$

Such an element must exist, because otherwise

$$S(x_{n-1}, \varepsilon_{n-1}) \subseteq \overline{E_n}$$

and this implies $x_{n-1} \in (\overline{E_n})^{\mathrm{o}}$, contradicting the fact that $(\overline{E_n})^{\mathrm{o}} = \varnothing$. Since the intersection $S(x_{n-1}, \varepsilon_{n-1}) \cap (\overline{E_n})^{\mathrm{c}}$ is open, there exists $\varepsilon > 0$ such that

$$S(x_n, \varepsilon) \subseteq S(x_{n-1}, \varepsilon_{n-1}) \cap \left(\overline{E_n}\right)^{\mathrm{c}}.$$

Now, we choose positive $\varepsilon_n < \min\{\varepsilon, \frac{1}{2}\varepsilon_{n-1}\}$. Then

$$\overline{S}(x_n, \varepsilon_n) \subseteq S(x_n, \varepsilon) \subseteq S(x_{n-1}, \varepsilon_{n-1}) \cap \left(\overline{E_n}\right)^{\mathrm{c}},$$

which says

$$\overline{S}(x_n, \varepsilon_n) \cap E_n = \varnothing.$$

As we also have

$$\varepsilon_n < \frac{1}{2}\varepsilon_{n-1},$$

the nth ball with requisite properties has been constructed.

As $\overline{S}(x_n, \varepsilon_n) \subseteq \overline{S}(x_{n-1}, \varepsilon_{n-1})$, the balls $\{\overline{S}(x_n, \varepsilon_n)\}_{n \geq 1}$ form a nested sequence of nonempty closed balls in a complete metric space X with diameters tending to zero. By Theorem 1.2.22, there exists $x_0 \in \cap_{n \geq 1} \overline{S}(x_n, \varepsilon_n)$. Since $\overline{S}(x_n, \varepsilon_n) \cap E_n = \varnothing$ for each n, we have $x_0 \in E_n^c$ for each n. However, $\cap_{n \geq 1} E_n^c = \varnothing$. This contradiction shows that X is not of category I. This completes the proof. □

Remarks 5.4.5

(i) The irrationals in $\mathbb{R}$ constitute a set of category II. In fact, $\mathbb{R}$ is a complete metric space and $\mathbb{R} = \mathbb{Q} \cup \{\text{irrationals}\}$ and $\mathbb{Q}$ is a set of category I imply that irrationals in $\mathbb{R}$ constitute a set of category II [Remarks 5.4.2(iv)].

(ii) A interval of positive length is of category II. For, if a nonempty interval is of category I, then so is each of its translates. Since $\mathbb{R}$ is countable union of such translates, it follows that $\mathbb{R}$ is of category I, contradicting Baire's Theorem 5.4.4.

We now turn to one of the most important properties of Banach spaces, the so-called Principle of Uniform Boundedness. This principle was discovered by Lebesgue in 1908 in investigations on Fourier series.

Theorem 5.4.6 (Uniform Boundedness Principle) *Let X be a Banach space, Y a normed linear space and $\{\Lambda_\alpha\}$ be a subset of $\mathcal{B}(X, Y)$, where α ranges over some set A such that, for each $x \in X$,*

$$\sup_\alpha \|\Lambda_\alpha x\| < \infty.$$

Then

$$\sup_\alpha \|\Lambda_\alpha\| < \infty.$$

Proof Since each Λ_α is continuous and since the norm in Y is a continuous function on Y [see Proposition 3.2.3 and Remarks 5.1.1(i)], each function $x \to \|\Lambda_\alpha x\|$ is continuous on X so that the set $\{x \in X : \|\Lambda_\alpha x\| \leq n\}$ is closed in X. Hence for each $n = 1, 2, \ldots,$

$$F_n = \{x \in X : \|\Lambda_\alpha x\| \leq n \quad \text{for every} \quad \alpha \in A\},$$

being the intersection of closed sets, is closed in X.

Let $x \in X$. Then by the hypothesis, $\|\Lambda_\alpha x\| \leq n$ for all $\alpha \in A$ and some positive integer n, we have,

$$X = \cup_n F_n.$$

Since X is complete, by Baire's Category Theorem 5.4.4, there exists an $n_0 \in \mathbb{N}$ such that F_{n_0} is not nowhere dense in X. Consequently, there exists a nonempty open ball $B(x_0,r_0)$ such that

$$B(x_0, r_0) \subseteq F_{n_0},$$

which implies

$\|\Lambda_\alpha x\| \leq n_0$ for every $x \in B(x_0,r_0)$ and all $\alpha \in A$. In particular, $\|\Lambda_\alpha x_0\| \leq n_0$ for all $\alpha \in A$.

Note that

$$B(x_0, r) = x_0 + rB(0, 1).$$

Therefore, for $z \in B(0,1)$, $z = \frac{x - x_0}{r}$, $x \in B(x_0,r)$.

Consequently,

$$\|\Lambda_\alpha z\| = \left\|\Lambda_\alpha \left(\frac{x - x_0}{r}\right)\right\| \leq \frac{1}{r}(\|\Lambda_\alpha x\| + \|\Lambda_\alpha x_0\|) \leq \frac{2n_0}{r}$$

for all $z \in B(0,1)$ and all $\alpha \in A$. This implies $\|\Lambda_\alpha\| \leq \frac{2n_0}{r}$ for all $\alpha \in A$, i.e.

$$\sup_\alpha \|\Lambda_\alpha\| \leq \frac{2n_0}{r}.$$

This completes the argument. □

An interesting application of the uniform boundedness principle is to prove that there are continuous functions of period 2π whose Fourier series diverge at a given point.

Theorem 5.4.7 *There are continuous functions in* $C[-\pi, \pi]$ *whose Fourier series diverge at a given point in* $[-\pi, \pi]$.

Let $X = C[-\pi, \pi]$ be the Banach space of continuous functions ($f(-\pi) = f(\pi)$ for each $f \in X$) with pointwise addition, scalar multiplication and $\| f \|_\infty = \sup\{|f(t)|: -\pi \leq t \leq \pi\}$. The Fourier coefficients of a function f in X are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt. \tag{5.15}$$

The Fourier series of f is the series

$$\sum_{n=-\infty}^{\infty} a_n e^{int}. \tag{5.16}$$

One of the basic questions in the study of Fourier series is whether (5.16) converges at each $t \in [-\pi, \pi]$, and if so, is its sum equal to $f(t)$?

Associated with the Fourier series (5.16) is the sequence of its partial sums

$$\begin{aligned} S_n(f;t) &= \sum_{k=-n}^{n} a_k e^{ikt}, \quad t \in [-\pi, \pi] \\ &= \left(\sum_{k=-n}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx\right)e^{ikt} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sum_{k=-n}^{n} e^{ik(t-x)}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_n(t-x)dx, \end{aligned}$$

where

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}.$$

The problem is to determine whether $\lim_n S_n(f; t) = f(t)$ for every $f \in C[-\pi, \pi]$. We shall see that the uniform boundedness principle answers this question negatively.

Put

$$S^*(f;t) = \sup_n |S_n(f;t)|.$$

To begin with, set $t = 0$ and define

$$\Lambda_n f = S_n(f;0), \quad f \in C[-\pi,\pi], \quad n = 1,2,\ldots.$$

Now

$$|\Lambda_n f| = S_n(f;0) = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(-x) \mathrm{d}x \right| \leq \|f\|_\infty \cdot ||D_n||_1,$$

which implies

$$||\Lambda_n|| \leq ||D_n||_1.$$

We claim that $\|\Lambda_n\| \to \infty$ as $n \to \infty$. This will be shown by proving $\|\Lambda_n\| = \|D_n\|_1$ and $\|D_n\|_1 \to \infty$ as $n \to \infty$.

Observe that

$$\mathrm{e}^{ix/2} D_n(x) - \mathrm{e}^{-ix/2} D_n(x) = 2i \sin\left(n + \frac{1}{2}\right)x$$

and therefore

$$D_n(x) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}.$$

Since $|\sin x| \leq |x|$ for all real x, we have

$$\begin{aligned}
||D_n||_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x} \right| \mathrm{d}x \\
&> \frac{2}{\pi} \int_{0}^{\pi} \left| \sin(n + \frac{1}{2}x) \right| \frac{\mathrm{d}x}{x} \\
&= \frac{2}{\pi} \int_{0}^{(n+\frac{1}{2})\pi} |\sin u| \frac{\mathrm{d}u}{u} \\
&> \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin u| \mathrm{d}u \\
&= \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} \to \infty \text{ as } n \to \infty.
\end{aligned}$$

We have thus shown that $\|D_n\|_1 \rightarrow \infty$ as $n \rightarrow \infty$.

Next, fix n and put $g(x) = 1$ if $D_n(x) \geq 0$ and $g(x) = -1$ if $D_n(x) < 0$. There exists $f_j \in C[-\pi, \pi]$ such that $-1 \leq f_j(x) \leq 1$ and $f_j(x) \rightarrow g(x)$ for almost all x as $j \rightarrow \infty$. (Let g be a measurable function defined on $[-\pi, \pi]$ and assume $|g| \leq 1$. Then, there exists a sequence f_j of continuous functions defined on $[-\pi, \pi]$, $|f_j| \leq 1$ and $g(x) = \lim_j f_j(x)$ a.e.)

$$\lim_j \Lambda_n(f_j) = \lim_j \frac{1}{2\pi} \int_{-\pi}^{\pi} f_j(-x) D_n(x) \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(-x) D_n(x) \mathrm{d}x = ||D_n||_1.$$

We have thus shown that $\|\Lambda_n\| \rightarrow \infty$ as $n \rightarrow \infty$. By the uniform boundedness principle, $S^*(f;0) = \infty$. Thus, there is an $f \in C[-\pi,\pi]$ whose Fourier series diverges at $t = 0$. We choose $t = 0$ just for convenience. It is clear that the same result holds for every other t.

The pointwise limit of continuous functions is not continuous. The sequence $\{f_n\}_{n \geq 1}$, where $f_n(x) = x^n$, $x \in [0,1]$, consists of continuous functions defined on $[0,1]$. The function f, where $f(x) = 0$ if $0 \leq x < 1$ and $f(1) = 1$, is the pointwise limit of the sequence and is not continuous. However, if the functions involved are functionals, the limit functional is continuous.

Proposition 5.4.8 *Let $\{f_n\}_{n \geq 1}$ be a sequence of bounded linear functionals on a Banach space X, i.e. $\{f_n\}_{n \geq 1} \subseteq X^*$. Suppose that for each x, $f_n(x)$ converges to a limit $f(x)$. Then f is a bounded linear functional.*

Proof It is easy to verify that f is linear. For each x, the sequence $\{f_n(x)\}_{n \geq 1}$ is convergent, hence bounded, i.e. there exists a nonnegative constant M_x such that

$$\sup_n |f_n(x)| = M_x < \infty.$$

Hence by the uniform boundedness principle, there exists a constant M such that

$$\sup_n \sup_{\|x\| \leq 1} |f_n(x)| \leq M.$$

Hence

$$\sup_{\|x\| \leq 1} |f(x)| \leq M.$$ □

The following application of Baire's Category Theorem is also known as the uniform boundedness principle.

Theorem 5.4.9 *Let $\mathcal{F}$ be a family of real-valued continuous functions on a complete metric space X. Suppose that for each $x \in X$, there is a number M_x such that $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. Then there is a nonempty open ball B and a constant M such that*

$$|f(x)| \le M$$

for all $f \in \mathcal{F}$ *and all* $x \in B$.

Proof For each integer m, let

$$E(m,f) = \{x \in X : |f(x)| \le m\}.$$

This set is closed, since $E(m, f)^c = \{x \in X : |f(x)| > m\}$ and f is continuous. It follows that

$$E_m = \cap_{f \in \mathcal{F}} E(m,f),$$

being the intersection of closed sets, is closed. Now, $X = \cup_m E_m$. For, if $x \in X$, there is an M_x such that $|f(x)| \le M_x$ for all $f \in \mathcal{F}$, i.e. there is an integer m such that $|f(x)| \le m$ for all $f \in \mathcal{F}$. This implies $x \in E_m$.

Since X is a complete metric space, there is a set E_m which is not nowhere dense. Since this E_m is closed, it must contain some ball B. But for every $x \in B$, we have

$$|f(x)| \le m \text{ for all } f \in \mathcal{F}. \qquad \square$$

5.5 Open Mapping and Closed Graph Theorems

The next group of results, the Open Mapping Theorem and the Closed Graph Theorem, ranks in importance with the Hahn–Banach Theorem and the Principle of Uniform Boundedness. These ideas are due to Stefan Banach; their validity is far from being intuitively clear.

A continuous linear transformation T of a Banach space X onto a Banach space Y is called a **homomorphism** if it is an open mapping, i.e. if it takes open sets into open sets. Thus, a one-to-one homomorphism of X onto Y is an isomorphism. The mapping $T:X \to Y$ then has a continuous inverse. In this section, we shall show that all continuous linear transformations are homomorphisms. We begin by proving the following Lemma.

Lemma 5.5.1 *Let X and Y be Banach spaces and* $T:X \to Y$ *be a bounded linear mapping of X onto Y. Then the image by T of the unit ball in X contains a ball about the origin in Y.*

Proof Let $B_n = \{x \in X : \|x\| < \frac{1}{2^n}\}$, $n = 0, 1, 2, \ldots$. Since T is onto, $X = \cup_{k \ge 1} kB_1$. Indeed, if $x \in X$, we may write $x = k(x/k)$, where $k = [2\|x\|] + 1$—the square bracket denotes the integral part of $2\|x\|$. Then

$$\left\|\frac{x}{k}\right\| = \left\|\frac{x}{[2\|x\|]+1}\right\| = \frac{\|x\|}{[2\|x\|]+1} \leq \frac{\|x\|}{[2\|x\|]} \leq \frac{\|x\|}{2\|x\|} = \frac{1}{2},$$

i.e. $x/k \in B_1$ and so $x \in kB_1$. Therefore

$$X = \cup_{k \geq 1} kB_1.$$

Since T is onto, $TX = Y$, we get

$$Y = \cup_{k \geq 1} kT(B_1).$$

But Y is a complete metric space and so is not of the first category. Consequently, $T(B_1)$ cannot be nowhere dense and $\overline{T(B_1)}$ contains some ball, say

$$\{y \in Y : ||y - y_0|| < \delta\}.$$

Then $\overline{T(B_1)} - y_0$ contains the ball $\{y \in Y: \|y\| < \delta\}$. But

$$T(B_1) - y_0 \subseteq \overline{T(B_1)} - \overline{T(B_1)} \subseteq 2\overline{T(B_1)} \subseteq \overline{T(B_0)}.$$

Thus, $\overline{T(B_0)}$ contains a ball about the origin of radius δ and so by the linearity of $T, \overline{T(B_n)}$ contains a ball about the origin of radius $\delta/2^n$.

We next show that $T(B_0)$ contains a ball of radius $\delta/2$ about the origin. Let $y \in Y$ be arbitrary with $\|y\| < \delta/2$. Since $y \in \overline{T(B_1)}$, we choose $x_1 \in B_1$ such that

$$\|y - T(x_1)\| < \delta/4.$$

Since $y - T(x_1) \in \overline{T(B_2)}$, we can choose $x_2 \in B_2$ such that

$$\|y - T(x_1) - T(x_2)\| < \delta/8.$$

We continue this process and choose $x_n \in B_n$ with

$$\left\|y - \sum_{k=1}^{n} T(x_k)\right\| < \delta/2^{n+1}.$$

Since $||x_k|| < 1/2^k, \sum_{k=1}^{\infty} x_k$ is absolutely convergent and $x = \sum_{k=1}^{\infty} x_k \in B_0(||x|| = || \sum_{k=1}^{\infty} x_k|| \leq \sum_{k=1}^{\infty} ||x_k|| \leq \sum_{k=1}^{\infty} \left(1/2^k\right) = 1)$. Moreover,

$$T(x) = T\left(\sum_{k=1}^{\infty} x_k\right) = \sum_{k=1}^{\infty} T(x_k) = y.$$

Thus, $y \in T(B_0)$ and so

$$\{y : ||y|| < \delta/2\} \subseteq T(B_0).$$

This completes the proof. □

Theorem 5.5.2 (Open Mapping–Inverse Mapping Theorem) *Let X and Y be Banach spaces and T a continuous linear transformation of X onto Y. Then T is a continuous homomorphism. Thus, in particular, if T is one-to-one, it is an isomorphism; i.e., T has a continuous inverse.*

Proof Let U be any nonempty open subset of X and y be any point of $T(U)$. Then, there exists an $x \in U$ such that $y = Tx$. Since U is open, there is an $\varepsilon > 0$ such that $x + B(0,\varepsilon) \subseteq U$. By Lemma 5.5.1, we find that there is a $\delta > 0$ for which $B(0,\delta) \subseteq T(B(0,\varepsilon))$. Therefore, $y + B_\delta \subseteq T(x) + T(B(0,\varepsilon)) = T(x + B(0,\varepsilon)) \subseteq T(U)$, and $T(U)$ is open.

The last part is an immediate consequence of the Theorem 5.5.2. □

Corollary 5.5.3 *Let X be a vector space with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, each of which makes it a Banach space. Suppose there exists a constant α such that $\|x\|_1 \leq \alpha\|x\|_2$ for all $x \in X$. Then there exists a constant β such that $\|x\|_2 \leq \beta\|x\|_1$ for all $x \in X$.*

Proof The identity map of $(X, \|\cdot\|_2)$ onto $(X, \|\cdot\|_1)$ is one-to-one continuous linear transformation and so must be an isomorphism by Theorem 5.5.2. Therefore, the inverse mapping must be bounded, i.e. there exists a constant β such that $\|x\|_2 \leq \beta\|x\|_1$ for all $x \in X$. □

An application of the Open Mapping Theorem is the following result.

Theorem 5.5.4 *Let X be a Banach space with Schauder basis $\{x_n\}_{n\geq 1}$. Let $a_n(x)$ be the coefficients of x in this basis, i.e., let $x = \sum_{n=1}^{\infty} a_n(x)x_n$. Then each $a_n(x)$ is a bounded linear functional on X.*

Proof Let Y be the vector space of all sequences $\{a_n\}_{n\geq 1}$, for which $\sum_{n=1}^{\infty} a_n x_n$ converges in X. Introduce a norm in Y by

$$||y|| = \sup_n \left\| \sum_{i=1}^{n} a_i x_i \right\|.$$

With the above definition of norm, Y is a normed vector space. We show that it is a Banach space. Let $y_m = \{a_n^{(m)}\}$, $m = 1,2,\ldots$ be a Cauchy sequence in Y. Let $\varepsilon > 0$ be given. There is an integer n_0 such that $m, k \geq n_0$ implies

$$||y_m - y_k|| = \sup_n \left\| \left(\sum_{i=1}^{n} a_i^{(m)} - \sum_{i=1}^{n} a_i^{(k)} \right) x_i \right\| < \varepsilon$$

and so,

$$\left\| \left(\sum_{i=1}^{n} a_i^{(m)} - \sum_{i=1}^{n} a_i^{(k)} \right) x_i \right\| < \varepsilon \tag{5.17}$$

for m, $k \geq n_0$. Thus

$$\left\| \left(a_n^{(m)} - a_n^{(k)} \right) x_n \right\| \leq \left\| \sum_{i=1}^{n} \left(a_i^{(m)} - a_i^{(k)} \right) x_i \right\| + \left\| \sum_{i=1}^{n-1} \left(a_i^{(m)} - a_i^{(k)} \right) x_i \right\| < 2\varepsilon$$

and therefore for every n

$$\left| a_n^{(m)} - a_n^{(k)} \right| < \frac{2\varepsilon}{||x_n||} \quad \text{for } \; m, k \geq n_0.$$

Hence, the scalar sequence $\{a_n^{(m)}\}$, $m = 1, 2, \ldots$ converges to an a_n, and this is true for every n. In order to complete the argument that Y is a Banach space, we need to check that

$$y = (a_1, a_2, \ldots) \in Y \quad \text{and} \quad \lim_n y_n = y \, \text{in} \, Y.$$

In the inequality (5.17), we let $k \to \infty$ and obtain for $m \geq n_0$ and for every n,

$$\left\| \sum_{i=1}^{n} (a_i^{(m)} - a_i) x_i \right\| \leq \varepsilon. \tag{5.18}$$

Putting

$$S_n^{(m)} = \sum_{i=1}^{n} a_i^{(m)} x_i, S_n = \sum_{i=1}^{n} a_i x_i$$

and taking into account (5.18), we obtain

$$||S_{n+p} - S_n|| \leq \left\| S_{n+p}^{(m)} - S_n^{(m)} \right\| + 2\varepsilon$$

for $m \geq n_0$ and arbitrary n and p.

Let ω be given. We first choose ε so that $2\varepsilon < \omega/2$. Since the series $\sum_{i=1}^{\infty} a_i^{(n)} x_i$ converges, there exists an n_1 for a fixed $m \geq n_0$ such that, for every $n \geq n_1$ and every $p > 0$,

$$\left|\left|S_{n+p} - S_n\right|\right| < \omega$$

i.e. the series

$$\sum_{i=1}^{\infty} a_i x_i$$

converges and consequently, $y = (a_1, a_2, \ldots) \in Y$. Besides this, (5.18) implies that for $m \geq m_0$,

$$\sup_n \left|\left|\sum_{i=1}^{n} (a_i^{(m)} - a_i)x_i\right|\right| \leq \varepsilon,$$

i.e.

$$||y - y_m|| \leq \varepsilon.$$

This proves Y is a Banach space.

Consider the transformation $T{:}Y \to X$ for which $y = (a_1, a_2, \ldots) \in Y$ is mapped into $T(y) = \sum_{n=1}^{\infty} a_n x_n$. T is evidently linear, one-to-one and onto. It is bounded since $\sup_n || \sum_{i=1}^{n} a_i x_i || \geq \lim_n || \sum_{i=1}^{n} a_i x_i ||$. By the Open Mapping Theorem 5.5.2, the inverse T^{-1} is bounded. Now,

$$\begin{aligned} |a_n(x)| = |a_n| = \|a_n x_n\| &= \left\|\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n-1} a_i x_i\right\| \\ &\leq 2\sup_n \left\|\sum_{i=1}^{n} a_i x_i\right\| = 2\|y\| = 2\left\|T^{-1}x\right\| \\ &\leq 2\left\|T^{-1}\right\|\|x\|. \end{aligned}$$

This proves the boundedness of $a_n(x)$. □

There are transformations which do not have all the desirable properties of continuous linear transformations. However, they do arise quite frequently. Many operators defined in terms of differentiation (ordinary or partial) are discontinuous; thus, the powerful tools that we have developed are of little use in the situation. These operators have another property which leads to the solution of the problem at hand. This type of linear transformation—the closed linear transformation—is discussed below.

Let X and Y be vector spaces over the same scalar field. $X \oplus Y$ denotes the external direct sum of X and Y [Definition 1.1.7]. It is the collection of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. With coordinatewise addition and scalar multiplication, $X \oplus Y$ is a vector space.

If X and Y are normed linear spaces, we define

$$\|(x, y)\| = \|x\| + \|y\|.$$

It is indeed a norm. [This does not agree with (2.39) just above Definition 2.7.1. However, these norms are equivalent. See Example 5.2.6.] It is clear that $(x_n, y_n)\to(x,y)$ if, and only if, $x_n\to x$ and $y_n\to y$ ($\|(x_n, y_n) - (x, y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\| + \|y_n - y\|$). Hence, if X and Y are Banach spaces, then so is $X \oplus Y$. The reader will benefit by reviewing Sect. 2.7 entitled 'Direct sum of Hilbert spaces'.

Definition 5.5.5 Let X and Y be vector spaces over the same scalar field and $T: X\to Y$ be a linear transformation. The ***graph of T*** is the set $G(T) = \{(x,Tx) : x \in X\}$, which is a subset of $X \oplus Y$.

Let X and Y be normed linear spaces. A linear transformation $T{:}X\to Y$ is said to have a closed graph if whenever $x_n\to x$ in X and $T(x_n)\to y$ in Y, we have $Tx = y$, i.e. the graph of T as a set of ordered pairs $\{(x, Tx) : x \in X\}$ is a closed subset of $X \oplus Y$.

It is trivial that if T is continuous, then T has a **closed graph**. The converse is however not true.

Example 5.5.6

(i) [cf. Example 3.2.7(x)]. Consider the Banach space $(C[0, 1], \|\cdot\|_\infty)$ and the normed space $(C^1[0, 1], \|\cdot\|_\infty)$. Define the mapping $T{:}C^1[0, 1]\to C[0, 1]$ by

$$(Tx)(t) = x'(t), \quad x \in C^1[0,1], \quad t \in [0,1].$$

Then, T is linear. We show that T is not bounded. Let $\{x_n\}_{n\geq 1}$ be a sequence in $C^1[0,1]$, where $x_n(t) = t^n$, $n\in \mathbb{N}$, so that $(Tx_n)(t) = nt^{n-1}$. Then, $\|x_n\|_\infty = \sup_n\{t^n : t \in [0,1]\} = 1$,

$$||Tx_n||_\infty = \sup_n\{nt^{n-1} : t \in [0,1]\} = n \quad \text{for every } n.$$

This implies $\|Tx_n\|_\infty = n = n\|x_n\|_\infty$. Therefore

$$||T|| = \sup_{x\in C[0,1],\, x\neq 0} \frac{||Tx||_\infty}{||x||_\infty} \geq \sup_{1\leq n<\infty} \frac{||T_{x_n}||_\infty}{||x_n||_\infty} = \infty,$$

i.e.T is not bounded.

We next show that the graph of T is closed. Let $\{x_n\}_{n\geq 1}$ be a sequence in $C^1[0,1]$ with $\lim_n x_n = x$ and $\lim_n Tx_n = y$. Then for each $t \in [0,1]$, we have

$$\int_0^t y(\tau)d\tau = \lim_n \int_0^t (Tx_n)(\tau)d\tau = \lim_n \int_0^t x_n'(\tau)d\tau = \lim_n(x_n(t) - x(0))$$
$$= x(1) - x(0).$$

Since $y \in C[0,1]$, it follows from the Fundamental Theorem of Calculus that x is differentiable on $[0,1]$ and

$$y(t) = x'(t), \quad t \in [0,1],$$

which implies $x \in C^1[0,1]$ and $Tx = y$; i.e., T is closed.

(ii) Let $T{:}C[0,1] \subseteq L^2[0,1] \to L^2[0,1]$ be the operator defined by $(Tx)(t) = x(0)\cdot t$. This operator is not closed. Indeed, take

$$x_n(t) = \begin{cases} 1 - nt & \text{if} \quad 0 \le t \le \frac{1}{n} \\ 0 & \text{if} \quad \frac{1}{n} \le t \le 1 \end{cases}.$$

Observe that $x_n(t) \to 0$ as $n \to \infty$ in $L^2[0,1]$ but $(Tx_n)(t) = t \neq 0$, which shows that $Tx_n \not\to 0$ as $n \to \infty$.

Theorem 5.5.7 (Closed Graph) *Let X and Y be Banach spaces and T be a linear transformation from X into Y. Then T is bounded if, and only if, its graph is a closed subspace of $X \oplus Y$.*

Proof Let $G(T)$ be closed. As a closed subspace of the Banach space $X \oplus Y$, $G(T)$ is itself a Banach space.

The mappings P_1 and P_2 defined in $G(T)$ by

$$P_1(x, Tx) = x$$

and

$$P_2(x, Tx) = Tx$$

for all $x \in X$, are continuous. Indeed,

$$\|P_1(x, Tx)\| = \|x\| \le \|x\| + \|Tx\| = \|(x, Tx)\|$$

and

$$\|P_2(x, Tx)\| = \|Tx\| \le \|x\| + \|Tx\| = \|(x, Tx)\|.$$

Since T is single valued and the domain of T is X, P_1 is one-to-one and onto. It follows from Theorem 5.5.2 that P_1^{-1} is continuous. Clearly, $T = P_2 \circ P_1^{-1}$ and so is continuous. □

We finally give a simple application of the Closed Graph Theorem.

Example 5.5.8 Assume that $a = \{a_n\}_{n\geq 1}$ is such that $\sum_{k=1}^{\infty} |a_k x_k| < \infty$ for all $x = \{x_n\}_{n\geq 1}$ in ℓ^1, and write $Tx = \{a_k x_k\}, x \in \ell^1$ so that T is a map on ℓ^1 into ℓ^1. Clearly, T is linear. We show that T is continuous on ℓ^1 by showing that the graph $G(T)$ of T is closed. Let $(x, y) \in \overline{G(T)}$, so that there exists $x^{(n)} \to x$ as $n \to \infty$ in ℓ^1. Hence for each $k \in \mathbb{N}$, as $n \to \infty$,

$$x_k^{(n)} \to x_k \quad \text{and} \quad a_k x_k^{(n)} \to y_k.$$

Consequently, $\lim_n a_k x_k^{(n)} = a_k x_k = y_k$ for each $k \in \mathbb{N}$, which implies $Tx = y$, so that $(x,y) \in G(T)$, whence $G(T)$ is closed. It follows from the Closed Graph Theorem 5.5.7 that T is continuous.

Chapter 6
Hints and Solutions

6.1 Problem Set 2.1

2.1.P1. Observe that $|n^{-\alpha}|^2 = |n^{-2\alpha}| = |n^{-2\Re\alpha - 2i\Re\alpha}| = n^{-2\Re\alpha}$ since $|n^{-2i\Im\alpha}| = |\exp(-2i\Im\alpha \ln n)| = .$ So, $\sum_{n=1}^{\infty} |n^{-\alpha}|^2 = \sum_{n=1}^{\infty} \left|n^{-2\Re\alpha}\right|^2 < \infty$ if and only if $2\Re\alpha > 1$, that is, $\Re\alpha > \frac{1}{2}$.

2.1.P2.

$$(f, g) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{1}{z - \alpha} \frac{1}{\bar{z} - \bar{\beta}} \frac{dz}{z},$$

where the integral is taken in the anticlockwise direction. Since $z\bar{z} = 1$ on ∂D,

$$\begin{aligned}
(f, g) &= \frac{1}{2\pi i} \int\limits_{\partial D} \frac{1}{z - \alpha} \frac{1}{1 - \bar{\beta} z} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D} \frac{1}{z - \alpha} h(z) dz, \quad \text{where } h(z) = \frac{1}{1 - \bar{\beta} z}.
\end{aligned}$$

Observe that $h(z)$ is analytic in $\overline{D}$, the closure of the disc, since $|\beta| < 1$. By Cauchy's integral formula,

$$(f, g) = h(\alpha) = \frac{1}{1 - \bar{\beta}\alpha}.$$

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2.1.P3. For $\alpha \in D$,

$$(f, k_\alpha) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{1 - \alpha\bar{z}} \frac{dz}{z},$$

where the integral is taken in the anticlockwise direction;

$$= \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - \alpha} dz$$

$$= \begin{cases} f(\alpha) & \text{if } |\alpha| < 1 \\ 0 & \text{if } |\alpha| > 1 \end{cases},$$

using Cauchy's integral formula and Cauchy's integral theorem.
2.1.P4. In view of Problem 2.1.P3, for $\alpha \in D$,

$$|f(\alpha)| = |(f, k_\alpha)| = \left| \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{1 - \alpha\bar{z}} \frac{dz}{z} \right| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\theta}) e^{i\theta} d\theta}{e^{i\theta} - \alpha} \right|, \quad \text{since } z\bar{z} = 1$$

$$\leq \frac{1}{2\pi} \left\{ \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta \right\}^{\frac{1}{2}} \left\{ \int_{-\pi}^{\pi} \frac{1}{|e^{i\theta} - \alpha|^2} d\theta \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta \right\}^{\frac{1}{2}} \frac{1}{\sqrt{(1 - |\alpha|^2)}}$$

since $\frac{1}{|e^{i\theta} - \alpha|^2} = \frac{1}{(z-\alpha)(\bar{z}-\bar{\alpha})}$, $d\theta = \frac{dz}{iz}$, and

$$\int_{-\pi}^{\pi} \frac{1}{|e^{i\theta} - \alpha|^2} d\theta = \frac{1}{i} \int_{\partial D} \frac{1}{z - \alpha} \frac{1}{1 - \bar{\alpha} z} dz = 2\pi(1 - |\alpha|^2)^{-1}$$

6.2 Problem Set 2.2

2.2.P1.

$$\frac{1}{2}\|x - y\|^2 + 2\left\| z - \frac{1}{2}(x + y) \right\|^2 = \frac{1}{2}[(x, x) - (x, y) - (y, x) + (y, y)]$$

$$+ 2\left[(z, z) - \frac{1}{2}(z, x + y) - \frac{1}{2}(x + y, z) \right.$$

$$\left. + \frac{1}{4}\{(x, x) + (x, y) + (y, x) + (y, y)\} \right]$$

$$= (x, x) + (y, y) + 2(z, z) - (z, x) - (x, z) - (z, y) - (y, z)$$

$$= \|x - z\|^2 + \|y - z\|^2.$$

This identity generalises the theorem with this name in plane geometry: If ABC is a triangle and D is the midpoint of the side BC, then $(AB)^2 + (AC)^2 = 2(AD)^2 + (BD)^2$.

2.2.P2. We need to check that $(A, A) > 0$ when $A \neq O$. Indeed, $(A, A) = \text{trace}(A^* A) = \sum_{i,j=1}^{n} |a_{i,j}|^2 > 0$, where $A = [a_{ij}]_{i,j=1}^{n} \neq O$.

$$||I_n||^2 = (I_n, I_n) = \text{trace}(I_n{}^* I_n) = n. \text{ So, } ||I_n|| = \sqrt{n}.$$

2.2.P3. (i) Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, it follows that $x = \left\{\frac{1}{n}\right\}_{n \geq 1} \in \ell^2$. On writing the Fourier series for the function $f(x) = x^2/2$, $-\pi < x \leq \pi$, it can be checked that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$. So, $\|x\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$, and hence, $\|x\| = \pi/\sqrt{6}$. (ii) $\sum_{n=1}^{\infty} \left(2^{-n/2}\right)^2 = \sum_{n=1}^{\infty} 2^{-n} = 1$. So, $x = \{2^{-n/2}\}_{n \geq 1} \in \ell^2$ and $\|x\| = 1$.

2.2.P4.

$$\begin{aligned} \left| \int_0^{\pi} f(t) \sin t \, \mathrm{d}t \right| &\leq \int_0^{\pi} |f(t)||\sin t| \mathrm{d}t \\ &\leq \left[\int_0^{\pi} |f(t)|^2 \mathrm{d}t \right]^{\frac{1}{2}} \left[\int_0^{\pi} |\sin t|^2 \mathrm{d}t \right]^{\frac{1}{2}}. \end{aligned} \tag{6.1}$$

by Corollary 2.2.5(c). Since $\sin t \geq 0$ in $[0, \pi]$,

$$\int_0^{\pi} |\sin t|^2 \mathrm{d}t = \int_0^{\pi} \sin^2 t \mathrm{d}t = \int_0^{\pi} \frac{1 - \cos 2t}{2} \mathrm{d}t = \frac{\pi}{2}.$$

Hence,

$$\int_0^{\pi} f(t) \sin t \, \mathrm{d}t \leq \sqrt{\frac{\pi}{2}} \left[\int_0^{\pi} |f(t)|^2 \mathrm{d}t \right]^{\frac{1}{2}}.$$

For nonzero f, equality holds if and only if $f(t) = k \sin t$, where $k \neq 0$ [the last part of Corollary 2.2.5(c)].

2.2.P5. Since $1 \leq f^{\frac{1}{2}} g^{\frac{1}{2}}$, it follows on using Corollary 2.2.5(c) that

$$1 \leq \left(\int_X f^{\frac{1}{2}} g^{\frac{1}{2}} \mathrm{d}\mu \right)^2 \leq \int_X f \mathrm{d}\mu \int_X g \mathrm{d}\mu.$$

2.2.P6. For x, $y \in X$,

$$\|x-y\|_2^2 = \int_0^1 |x(t)-y(t)|^2 \mathrm{d}t \leq \|x-y\|_\infty^2.$$

The above inequality shows that the identity mapping id:$X_1 \to X_2$ is uniformly continuous. However, the identity map id:$X_2 \to X_1$ is not continuous, as the following example shows.

The sequence $\{x_n\}_{n \geq 1}$, where $x_n(t) = t^n$, converges to 0 in X_2, while $\|x_n\|_\infty = 1$ for all n.

2.2.P7. Indeed, by the parallelogram law, we have

$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = \frac{1}{2}\|x+y\|^2 + \frac{1}{2}\|x+y\|^2 - \|x+y\|^2 = 0,$$

and this implies $x = y$.

However, let $X = \mathbb{R}^n$ or $\mathbb{C}^n$. For $x = (1, 0, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$, $\|x\|_1 = \|y\|_1 = 1$ and $\frac{1}{2}\|x + y\|_1 = 1$, but $x \neq y$.

2.2.P8. If $(x, y) = 0$, then there is nothing to prove. Assume $(x, y) \neq 0$. Set $\theta = (x, y)/|(x, y)|$ and λ be a real number. We then have

$$0 \leq (\overline{\theta}x + \lambda y, \overline{\theta}x + \lambda y) = (x,x) + \lambda(y, \overline{\theta}x) + \lambda(\overline{\theta}x, y) + \lambda^2(y,y).$$

Since $(y, \overline{\theta}x) = \theta(y,x) = \theta\overline{(x,y)} = |(x,y)|$ and $(\overline{\theta}x, y) = \overline{\theta}(x,y) = |(x,y)|$, we obtain

$$(x,x) + 2\lambda|(x,y)| + \lambda^2(y,y) \geq 0 \quad \text{for } \lambda \in \mathbb{R}.$$

This implies

$$|(x,y)|^2 \leq (x,x)(y,y).$$

In order to see why the rider does not hold, consider $\mathbb{C}^2$, wherein we set

$$(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = x_2\overline{y}_2.$$

It is easy to check that (a), (b) and (c) hold. However,

$$|(\langle 1,1\rangle, \langle 0,1\rangle)|^2 = 1 = (\langle 1,1\rangle, \langle 1,1\rangle) \cdot (\langle 0,1\rangle, \langle 0,1\rangle),$$

though neither vector is a scalar multiple of the other.

2.2.P9. $\|g\|^2 = \frac{1}{2\pi i}\int_{\partial D} \frac{1}{(z-\alpha)(z-\beta)(\overline{z}-\overline{\alpha})(\overline{z}-\overline{\beta})}\frac{\mathrm{d}z}{z}.$

$$\text{Integrand} = \frac{z\bar{z}}{(z-\alpha)(z-\beta)(\bar{z}-\bar{\alpha})(\bar{z}-\bar{\beta})}\frac{1}{z}$$
$$= \frac{z\bar{z}}{(z-\alpha)(z-\beta)(1-\bar{\alpha}z)(1-\bar{\beta}z)}\frac{1}{\bar{z}}$$
$$= \frac{z}{(z-\alpha)(z-\beta)(1-\bar{\alpha}z)(1-\bar{\beta}z)}.$$

Since $|\alpha| < 1$ and $|\beta| < 1$, the only poles of the integrand in D are α and β. Moreover, they are simple poles.

$$\text{Res}_{z=\alpha}\text{Integrand} = \frac{\alpha}{(\alpha-\beta)(1-|\alpha|^2)(1-\alpha\bar{\beta})},$$

$$\text{Res}_{z=\beta}\text{Integrand} = \frac{\beta}{(\beta-\alpha)(1-\bar{\alpha}\beta)(1-|\beta|^2)}.$$

$$\text{Sum of the residues} = \frac{1-|\alpha\beta|^2}{\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right)|1-\bar{\alpha}\beta|^2}.$$

So,

$$||g|| = \frac{\left(1-|\alpha\beta|^2\right)^{\frac{1}{2}}}{\left(1-|\alpha|^2\right)^{\frac{1}{2}}\left(1-|\beta|^2\right)^{\frac{1}{2}}|1-\bar{\alpha}\beta|}$$

2.2.P10. Let $\{f_n\}_{n\geq 1}$ be a sequence in F, i.e. $f_n(\alpha) = 0$, $n = 1, 2, \ldots$. Assume that $f_n \to f$ in RH^2. Then,

$$(f, k_\alpha) = (f - f_n, k_\alpha) + (f_n, k_\alpha) = (f - f_n, k_\alpha).$$

On applying Cauchy–Schwarz inequality, we get

$$|(f, k_\alpha)| \leq \|f - f_n\|_2 \|k_\alpha\|_2 \to 0 \quad \text{as } n \to \infty,$$

since

$$||k_\alpha||_2^2 = \frac{1}{2\pi i}\int_{\partial D} \frac{1}{1-\bar{\alpha}z}\frac{1}{1-\alpha\bar{z}}\frac{\mathrm{d}z}{z}$$
$$= \frac{1}{2\pi i}\int_{\partial D} \frac{1}{1-\bar{\alpha}z}\frac{1}{z-\alpha}\,\mathrm{d}z = \frac{1}{1-|\alpha|^2},$$

using Cauchy's integral formula and $|\alpha| < 1$.

6.3 Problem Set 2.3

2.3.P1. We show that $\|x\|_\infty = \sup_{0\le t\le 1}|x(t)|$ does not satisfy the parallelogram law. Consider $x(t) = 1$, $0 \le t \le 1$ and $y(t) = t$, $0 \le t \le 1$. We then have $\|x\|_\infty = 1$ and $\|y\|_\infty = 1$, whereas $\|x + y\|_\infty = \sup_{0\le t\le 1}|1 + t| = 2$ and $\|x - y\|_\infty = 1$. Hence, $\|x+y\|_\infty{}^2 + \|x-y\|_\infty{}^2 = 5 \neq 4 = 2\|x\|_\infty{}^2 + 2\|y\|_\infty{}^2$.

2.3.P2. (a) For all x, y, $y \neq x$, of norm 1, the parallelogram law gives

$$\begin{aligned}||x+y||^2 &= 2||x||^2 + ||y||^2 - ||x-y||^2\\ &= 4 - ||x-y||^2 < 4.\end{aligned}$$

Hence, $\|x + y\| < 2$.

(b) For $x(t) = 1$ and $y(t) = t$, $0 \le t \le 1$, we have $\|x\|_\infty = 1 = \|y\|_\infty$ and $\|x + y\|_\infty = \sup_t|1 + t| = 2$. This shows that the norm $\|\cdot\|_\infty$ on $C[0, 1]$ is not strictly convex.

(c) For $x(t) = 2t$ and $y(t) = 2 - 2t$, $0 \le t \le 1$, we have $||x||_1 = t^2|_0^1 = 1$, $||y||_1 = (2t - t^2)|_0^1 = 1$ and $||x+y||_1 = (2t)|_0^1 = 2$. This shows that the norm $\|\cdot\|_1$ on $C[0, 1]$ is not strictly convex.

2.3.P3. Observe that

$$\begin{aligned}|(x,y)| &= \left|\int_0^1 x'(t)\,\overline{y'(t)}\mathrm{d}t\right|\\ &\le \int_0^1 |x'(t)||y'(t)|\mathrm{d}t\\ &\le \left(\int_0^1 |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^1 |y'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} < \infty,\end{aligned}$$

using the Cauchy–Schwarz inequality. It can be easily checked that $(x,y) = \int_0^1 x'(t)\overline{y'(t)}\mathrm{d}t$ is an inner product on H.

Also,

$$||x||^2 = \int_0^1 |x'(t)|^2\mathrm{d}t, \quad x \in H.$$

Let $\{x_n\}_{n\ge 1}$ be a Cauchy sequence in H. Then, $\{x_n{}'\}_{n\ge 1}$ is a Cauchy sequence in $L^2[0, 1]$. Since $L^2[0, 1]$ is complete, $x_n{}' \to z$ in $L^2[0, 1]$. Note that

$$x_n(t) = \int_0^t x_n{}'(s)\mathrm{d}s, \quad t \in [0, 1],$$

using the fundamental theorem for Lebesgue integration. Accordingly, define

$$x(t) = \int_0^t z(s)\mathrm{d}s.$$

Then, x is absolutely continuous on $[0, 1]$, $x(0) = 0$ and $x'(t) = z(t) \in L^2[0, 1]$, so that $x \in H$. Thus, H is complete.

2.3.P4. Let $K = \{\langle x, y\rangle : x, y \in H\}$, where $\langle x, y\rangle$ denotes an ordered pair. Define

$$\left.\begin{array}{r}\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle \\ (\alpha + i\beta)\langle x, y\rangle = \langle \alpha x - \beta y, \alpha y + \beta x\rangle\end{array}\right\}, \tag{6.2}$$

where $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ are in K and $\alpha + i\beta \in \mathbb{C}$. Note that this yields $i\langle x, 0\rangle = \langle 0, x\rangle$. It is easy to verify that K with the operations of addition and scalar multiplication defined by (6.2) is a vector space over $\mathbb{C}$. We next define an inner product on K. For $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle$ in K,

$$(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = (x_1, x_2) + (y_1, y_2) + i(y_1, x_2) - i(x_1, y_2). \tag{6.3}$$

Note that (6.3) defines an inner product on K and it extends the original inner product on H: If we set $y_1 = 0$ and $y_2 = 0$, then (6.3) becomes

$$(\langle x_1, 0\rangle, \langle x_2, 0\rangle) = (x_1, x_2).$$

Since

$$\|\langle x, y\rangle\|^2 = (\langle x, y\rangle, \langle x, y\rangle) = \|x\|^2 + \|y\|^2,$$

it follows that convergence in K is coordinate-wise. As H is complete, so is K. Thus, K is a Hilbert space.

Define $U{:}H \to K$ by $Ux = \langle x, 0\rangle$. Clearly, U is linear and $(Ux_1, Ux_2) = (\langle x_1, 0\rangle, \langle x_2, 0\rangle) = (x_1, x_2)$, $x_1, x_2 \in H$. Finally, if $z \in K$, then $z = \langle x_1, x_2\rangle$ for some x_1 and x_2 in H. Moreover, $z = \langle x_1, 0\rangle + \langle 0, x_2\rangle = \langle x_1, 0\rangle + i\langle x_2, 0\rangle = Ux_1 + iUx_2$.

Remark K is called the **complexification** of H.

2.3.P5. (a) Denote by s the number in $(0, 1)$ such that $\|sx + (1 - s)y\| < \|x\|$. We have to show that $\|tx + (1 - t)y\| < \|x\|$ for $t \in (0, 1)$ even if $t \neq s$. We consider the two cases $0 < t < s$ and $s < t < 1$ separately.

First suppose $0 < t < s$. Putting $u = (s - t)/s$, we have $u \in (0, 1)$ and

$$t = s(1-u), 1-t = 1-s(1-u) = 1-u-s(1-u)+u = (1-u)(1-s)+u.$$

Therefore,

$$tx+(1-t)y = s(1-u)x+((1-u)((1-s)+u)y = (1-u)(sx+(1-s)y)+uy,$$

so that

$$\|tx+(1-t)y\| \le (1-u)\|sx+(1-s)y\|+u\|y\| < (1-u)\|x\|+u\|x\| = \|x\|.$$

Now, suppose $s < t < 1$. Putting $u = (t-s)/(1-s)$, we have $u \in (0, 1)$ and

$$t = (1-s)u+s = s(1-u)+u, 1-t = 1-(1-s)u-s = (1-s)(1-u).$$

Therefore,

$$tx+(1-t)y = (s(1-u)+u)x+(1-s)(1-u)y = (1-u)(sx+(1-s)y)+ux,$$

so that

$$\|tx+(1-t)y\| \le (1-u)\|sx+(1-s)y\|+u\|x\| < (1-u)\|x\|+u\|x\| = \|x\|.$$

Remark The crux of the above argument is that in any real or complex linear space, if z is a 'proper' convex combination of two vectors, then any other proper convex combination of the same two vectors is a proper convex combination of one of them and z. The reader may wish to draw a figure to appreciate why we have considered two cases and the motivation for our choices of u.

The result is much easier to prove in an inner product space: The function of $t \in \mathbb{R}$ given by $\|tx+(1-t)y\|^2 - \|x\|^2$, which is the same as $\|t(x-y)+y\|^2 - \|x\|^2$, is easily seen to be a quadratic with leading coefficient $\|x-y\|^2 > 0$, so that it has to be negative when t lies between its roots, which are obviously 0 and 1.

(b) Put $u = x/\|x\|$ and $v = y/\|y\|$. Then, $\|u\| = \|v\| = 1$. Now, set $t = \frac{\|x\|}{\|x\|+\|y\|}$. It follows from the given equality that

$$1 = \frac{\|x+y\|}{\|x\|+\|y\|} = \left\| \frac{x}{\|x\|+\|y\|} + \frac{y}{\|x\|+\|y\|} \right\| = \|tu+(1-t)v\|.$$

By the given strict convexity and (a) above, we obtain $u = v$ and hence the desired conclusion. Remark. The converse is also true.

2.3.P6. Note that $\prod_{n=1}^{\infty}[-\frac{1}{n}, \frac{1}{n}]$ is compact by the theorem in the product metric [see Theorem 1.2.19]. Let $C = \{x = \{\{\eta_n\}_{n\ge 1} \in \ell^2 : |\eta_n| \le \frac{1}{n}, n = 1, 2, \ldots\}$, the Hilbert cube. Clearly, $C \subseteq \prod_{n=1}^{\infty}[-\frac{1}{n},\frac{1}{n}]$. We shall show that C is closed in $\prod_{n=1}^{\infty}[-\frac{1}{n},\frac{1}{n}]$. Let

$x = \{x_n\}_{n \geq 1}$ be a limit point of C in the product metric, and let $\{x^{(m)}\}_{m \geq 1}$ be a sequence in C such that $\lim_m x^{(m)} = x$, i.e. $\lim_m x_n^{(m)} = x_n$ for each n. Moreover,

$$|x_n| = \left|\lim_m x_n^{(m)}\right| = \lim_m \left|x_n^{(m)}\right| \leq \frac{1}{n}.$$

It remains to show that, on C, coordinate-wise convergence is equivalent to convergence in ℓ^2. Clearly, convergence in ℓ^2 implies coordinate-wise convergence. On the other hand,

$$\begin{aligned} \left\|x^{(m)} - x\right\|_2^2 = \sum_{n=1}^{\infty} \left|x_n^{(m)} - x_n\right|^2 &= \sum_{n=1}^{k} \left|x_n^{(m)} - x_n\right|^2 + \sum_{n=k+1}^{\infty} \left|x_n^{(m)} - x_n\right|^2 \\ &\leq \sum_{n=1}^{k} \left|x_n^{(m)} - x_n\right|^2 + \sum_{n=k+1}^{\infty} \left(\frac{2}{n}\right)^2. \end{aligned}$$

Given any $\varepsilon > 0$, by choosing k large enough, one can make the second term on the right less than $\varepsilon/2$; since convergence is coordinate-wise, by choosing m large enough the first term can also be made less than $\varepsilon/2$. This completes the argument.

2.3.P7. The series in the definition of the inner product on ℓ_a^2 converges: Indeed,

$$\begin{aligned} \left|a_j x_j \bar{y}_j\right| &= \left(\sqrt{a_j}|x_j|\right)\left(\sqrt{a_j}|y_j|\right) \leq \frac{1}{2}\left(a_j|x_j|^2 + a_j|y_j|^2\right), \\ \sum_{j=1}^{\infty} \left|a_j x_j \bar{y}_j\right| &\leq \frac{1}{2}\sum_{j=1}^{\infty}\left(a_j|x_j|^2 + a_j|y_j|^2\right). \end{aligned}$$

It may be easily checked that ℓ_a^2 is a vector space over $\mathbb{C}$ and (x, y) defines an inner product on it. It remains to prove completeness.

Let $\{x^{(m)}\}_{m \geq 1}$ be a Cauchy sequence in ℓ_a^2. Since

$$\begin{aligned} \left\|x^{(n)} - x^{(m)}\right\| &= \left(\sum_{k=1}^{\infty} a_k \left|x^{(n)}{}_k - x^{(m)}{}_k\right|^2\right)^{\frac{1}{2}} \\ &\geq a_k^{\frac{1}{2}} \left|x^{(n)}{}_k - y^{(n)}{}_k\right|, \end{aligned}$$

it follows that $\left\{x^{(n)}{}_k\right\}$ is a Cauchy sequence of scalars. Let $x_k = \lim_n x^{(n)}{}_k$. Now, we check that $x = \{x_k\} \in \ell_a^2$. For every $j \geq 1$, we have

$$\begin{aligned}\sum_{k=1}^{j} a_k|x_k|^2 &= \lim_n \sum_{k=1}^{j} a_k\left|x^{(n)}{}_k\right|^2 \\ &\leq \lim_n \sum_{k=1}^{\infty} a_k\left|x^{(n)}{}_k\right|^2 \\ &= \lim_n\left|\left|x^{(n)}\right|\right|^2 \leq M\end{aligned}$$

since Cauchy sequences are bounded. On letting $j \to \infty$, we get $\sum_{k=1}^{\infty} a_k|x_k|^2 \leq M$, that is, $x \in \ell_a^2$.

Let $\varepsilon > 0$ be given. There exists n_0 such that for n, $m \geq n_0$, we have

$$\sum_{k=1}^{j} a_k\left|x^{(n)}{}_k - x^{(m)}{}_k\right|^2 \leq \left|\left|x^{(n)} - x^{(m)}\right|\right|^2 \leq \varepsilon.$$

Let $n \geq n_0$ be fixed and let $m \to \infty$. Then,

$$\sum_{k=1}^{j} a_k\left|x^{(n)}{}_k - x_k\right|^2 \leq \varepsilon.$$

On letting $j \to \infty$, we obtain

$$\left|\left|x^{(n)} - x\right|\right|^2 = \sum_{k=1}^{\infty} a_k\left|x^{(n)}{}_k - x_k\right|^2 \leq \varepsilon \quad \text{for } n \geq n_0.$$

This completes the proof.

Remark The above result is also a consequence of Theorem 2.4.9 by appropriately choosing the measure on the set of natural numbers.

2.3.P8. For $f \in H^s$, $\sum_{n=-\infty}^{\infty} |f(n)|^2(1+n^2)^{s/2} < \infty$. Let $r < s$. To show $f \in H^r$, we need to check that $\sum_{n=-\infty}^{\infty} |f(n)|^2(1+n^2)^{r/2} < \infty$. Observe that $(1+n^2)^{s/2} = (1+n^2)^{(s-r)/2}(1+n^2)^{r/2}$ and $(1+n^2)^{(s-r)/2} \geq 1$, since $(s - r)/2 > 0$. Consequently, $(1+n^2)^{r/2} \leq (1+n^2)^{s/2}$, and hence,

$$\sum_{n=-\infty}^{\infty} |f(n)|^2\left(1+n^2\right)^{r/2} \leq \sum_{n=-\infty}^{\infty} |f(n)|^2\left(1+n^2\right)^{s/2} < \infty.$$

Remark We have actually proved that $\|f\|_{H^r} \leq \|f\|_{H^s}$. Thus, the identity map from H^s to H^r is continuous. However, its inverse is not.

2.3.P9. (a) Let a be the sequence $(1,\ 2^4,\ 3^6,\ \ldots)$. Then, $\sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{\infty} n^{2n}(1/n^{2n}) = \infty$.

(b) Let a be the sequence $\{1/n^{2n+2}\}_{n\geq 1}$. Then,

$$\sum_{n=1}^{\infty} a_n|x_n|^2 = \sum_{n=1}^{\infty} (1/n^{2n+2})n^{2n} = \sum_{n=1}^{\infty} (1/n^2) < \infty.$$

2.3.P10. Let z be a limit point of M'. Then, $z = \lim_n(x_n + \lambda_n y)$, where $x_n \in M$ and λ_n are scalars. Since convergent sequences in a metric space are bounded, there exists an $\eta > 0$ such that $\|x_n + \lambda_n y\| < \eta$ for $n = 1, 2, \ldots$. If it were true that $|\lambda_n|\to\infty$, we should have

$$\left\|\lambda_n^{-1}x_n + y\right\| < \eta/\lambda_n \to 0,$$

so that $-y \in M$, since M is closed. But $y \notin M$. Hence, $\{\lambda_n\}_{n\geq 1}$ has a Cauchy subsequence $\{\lambda_{n_i}\}_{i\geq 1}$ converging to some λ and so $\{x_{n_i}\}_{i\geq 1}$, being a difference of two Cauchy sequences $\{x_{n_i} + \lambda_{n_i}y\}$ and $\{\lambda_{n_i}y\}$, is itself a Cauchy sequence in H and converges to some $x \in M$. Thus, $z = x + \lambda y \in M'$. Consequently, M' contains all its limit points.

6.4 Problem Set 2.4

2.4.P1. Since $\exp(-2t) < 1$ for $0 < t < 1$, therefore, $t^{2\alpha}\exp(-2t) < t^{2\alpha}$ if $0 < t < 1$, and $\int_0^1 t^{2\alpha}\exp(-2t)\mathrm{d}t < \int_0^1 t^{2\alpha}\mathrm{d}t < \infty$ if and only if $2\alpha > -1$, i.e. $\alpha > -\frac{1}{2}$ (for $\alpha \leq -\frac{1}{2}$, the former integral diverges by the comparison test). Also, for $1 < t < \infty$, we have $\exp(2t) > \frac{(2t)^n}{n!} > \frac{(2t)^{2\alpha+2}}{n!}$ by choosing $n > 2\alpha + 2$. Therefore, $t^{2\alpha}\exp(-2t) < \frac{n!}{2^{2\alpha+2}t^2} < \frac{n!}{t^2}$ since $2\alpha + 2 > 1$. Consequently, $\int_1^{\infty} t^{2\alpha}\exp(-2t)\mathrm{d}t < \infty$. Thus, f_α belongs to $L^2(0, \infty)$ if and only if $\alpha > -\frac{1}{2}$. Moreover,

$$\begin{aligned}\|f_\alpha\|^2 &= \int_0^{\infty} t^{2\alpha}\exp(-2t)\mathrm{d}t = \frac{1}{2}\int_0^{\infty}\left(\frac{u}{2}\right)^{2\alpha}\exp(-u)\,\mathrm{d}u = \frac{1}{2^{2\alpha+1}}\int_0^{\infty} u^{2\alpha}\exp(-u)\mathrm{d}u \\ &= \frac{1}{2^{2\alpha+1}}\int_0^{\infty} u^{2\alpha+1-1}\exp(-u)\mathrm{d}u = \frac{1}{2^{2\alpha+1}}\Gamma(2\alpha+1).\end{aligned}$$

2.4.P2. (a) The vector $y = \{\frac{1}{n}\}_{n\geq 1}$ is in ℓ^2. Observe that $M = \{\, x \in \ell^2 : (x, y) = 0\}$. Since the inner product is continuous, it follows that M is a closed subspace of ℓ^2.

(b) The function $y(t) = \frac{1}{t}$ is such that $\int_0^{\infty} \frac{1}{t^2}\mathrm{d}t < \infty$, i.e. $y(t) \in L^2[1,\infty)$. Since the inner product is continuous, it follows that $M = \{x(t) \in L^2[1,\infty) : (x, y) = 0\}$ is a closed subspace of $L^2[1,\infty)$.

2.4.P3. Suppose $p = 1$. Consider the functions $f(t) = t$ and $g(t) = 1 - t$. We have $\|f\|_1 = \frac{1}{2}$, $\|g\|_1 = \frac{1}{2}$ and $\|f + g\|_1 = 1$, $\|f - g\|_1 = \frac{1}{2}$. It follows that the L^1 norm does not satisfy the parallelogram law. For $p \geq 1$, the parallelogram law for the same two functions will hold if and only if

$$\frac{\log(p+1)}{p} = \frac{\log 3}{2}.$$

The function f on the left is strictly decreasing for $p \geq 1$, as can be verified by computing the derivative of the product $p^2 f'(p)$. Consequently, the above equality holds if and only if $p = 2$.

6.5 Problem Set 2.5

2.5.P1. We first prove that every $f \in C[0, 1]$ can be uniformly approximated by a polynomial of the form $\sum_{k=1}^{n} a_k t^{3k}$. Consider the function $g(t) = f(t^{\frac{1}{3}})$. Let ε > 0 be arbitrary. By the Weierstrass theorem, there exists a polynomial such that

$$\max_{0 \leq t \leq 1} |g(t) - P(t)| < \varepsilon.$$

Setting $t = x^3$, we obtain

$$\begin{aligned}\max_{0 \leq t \leq 1} |g(t) - P(t)| &= \max_{0 \leq x \leq 1} \left|g(x^3) - P(x^3)\right| \\ &= \max_{0 \leq x \leq 1} \left|f(x) - P(x^3)\right| < \varepsilon.\end{aligned}$$

It follows that for the polynomial Q given by $Q(x) = P(x^3)$, we have

$$\|f - Q\|_2 = \left(\int_0^1 \left|f(t) - P(t^3)\right|^2 \mathrm{d}t \right)^{\frac{1}{2}} < \varepsilon. \tag{6.4}$$

The reader will note that the polynomial Q is a finite linear combination of the system under reference.

Let $h \in L^2[0, 1]$ and ε > 0 be given. There exists $f \in C[0, 1]$ such that

$$||h - f||_2 < \varepsilon \text{ [Example 2.5.1]}. \tag{6.5}$$

Equations (6.4) and (6.5) together complete the proof of the assertion. The same argument applies to $L^2[-1, 1]$.

6.6 Problem Set 2.6

2.6.P1. (a) Let $0 \leq r < R < 1$, and let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of f in the annulus $r < |z| < R$. Then,

$$\iint_{r<|z|<R} |f(z)|^2 \mathrm{d}x\mathrm{d}y = \int_r^R \int_0^{2\pi} \left(\sum a_n \rho^n e^{in\theta}\right)\left(\sum a_m \rho^m e^{-im\theta}\right) \rho \mathrm{d}\rho \mathrm{d}\theta$$

$$\begin{aligned} &= 2\pi \int_r^R \sum |a_n|^2 \rho^{2n+1} \mathrm{d}\rho \\ &= 2\pi \sum |a_n|^2 \left(\frac{R^{2n+2} - r^{2n+2}}{2n+2}\right) \\ &= \pi \sum |a_n|^2 \left(\frac{R^{2n+2} - r^{2n+2}}{n+1}\right). \end{aligned} \tag{6.6}$$

Since f is analytic in the annulus $0 < |z| < 1$ and the left-hand side of (6.6) is finite, it follows that $a_n = 0$ for $n < 0$. In this case, the point $z = 0$ is a removable singularity of f.(b) Let $\{f_n\}_{n\geq 1}$ be a sequence in $A(\Omega)$ such that $f_n(\alpha) = 0$ and $f \in A(\Omega)$ be such that $\int\int_\Omega |f_n - f|^2 \mathrm{d}x\mathrm{d}y \to 0$ as $n \to \infty$. Since by Proposition 2.6.3

$$|f_n(\alpha) - f(\alpha)|^2 \leq \iint_\Omega \left(\| f_n - f\|^2 \mathrm{d}x\mathrm{d}y\right)/\pi d_\alpha^2,$$

where $d_\alpha = \mathrm{dist}(\alpha, \partial\Omega)$, and since the right-hand side of the above inequality tends to zero as $n \to \infty$ and $f_n(\alpha) = 0$ for each n, it follows that $f(\alpha) = 0$.

6.7 Problem Set 2.8

2.8.P1. For $x, y \in H$, $y \neq 0$, let $e = y/\|y\|$. We have from Bessel's inequality,

$$|(x,e)|^2 \leq ||x||^2. \tag{6.7}$$

On substituting for e in (6.7) above, we obtain

$$|(x, y/\|y\|)|^2 \leq \|x\|^2, \quad \text{i.e., } |(x,y)| \leq \|x\|\|y\|.$$

2.8.P2: Let $\{e_k\}_{k\geq 1}$, where $e_k = (0, 0, \ldots, 0, 1, 0, \ldots)$ is the sequence in ℓ^2 which has 1 in the kth place and zeroes elsewhere. Define $f_n = e_{n+1}$, $n = 1, 2, \ldots$. Then, $\{f_n\}_{n\geq 1}$ is an orthonormal sequence in ℓ^2. Let $x = \{x_k\}_{k\geq 1}$ be a vector in ℓ^2 for which $x_1 \neq 0$. Then, $(x, f_n) = (x, e_{n+1}) = x_{n+1}$, $n = 1, 2, \ldots$. Now,

$$\sum_{n=1}^{\infty} |(x, f_n)|^2 = \sum_{n=1}^{\infty} |x_{n+1}|^2 < \sum_{n=1}^{\infty} |x_n|^2 = ||x||^2.$$

2.8.P3. Indeed, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_{n=1}^{\infty} |(x, e_k)(y, e_k)| &= \sum_{k=1}^{\infty} |(x, e_k)||(y, e_k)| \\ &\leq \left(\sum_{k=1}^{\infty} |(x, e_k)|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |(y, e_k)|^2\right)^{\frac{1}{2}} \\ &\leq \|x\|\|y\|.\end{aligned}$$

The last inequality results on using Bessel's inequality.

2.8.P4. Suppose $x = \sum_{k=1}^{\infty} (x, e_k)e_k$. Observe that $\sum_{k=1}^{n} (x, e_k)e_k$ is in M for every n. Now,

$$\begin{aligned}\left\| x - \sum_{k=1}^{\infty} (x, e_k)e_k \right\|^2 &= \left\| \sum_{k=n+1}^{\infty} (x, e_k)e_k \right\|^2 \\ &= \sum_{k=n+1}^{\infty} ||(x, e_k)e_k||^2 \\ &= \sum_{k=n+1}^{\infty} |(x, e_k)|^2 \to 0 \text{ as } n \to \infty,\end{aligned}$$

using Bessel's inequality. So, $x \in \overline{M}$.

Conversely, suppose $x \in \overline{M}$. Consider the sequence $\left\{\sum_{k=1}^{n} (x, e_k)e_k\right\}_{n \geq 1}$ in M. The sequence is Cauchy: Indeed, for $n > m$,

$$\begin{aligned}\left\| \sum_{k=1}^{n} (x, e_k)e_k - \sum_{k=1}^{m} (x, e_k)e_k \right\|^2 &= \sum_{k=m+1}^{n} \|(x, e_k)e_k\|^2 \\ &= \sum_{k=m+1}^{n} |(x, e_k)|^2 \to 0 \quad \text{as } m \to \infty.\end{aligned}$$

Thus, the sequence $\left\{\sum_{k=1}^{n} (x, e_k)e_k\right\}_{n \geq 1}$ converges and the limit $\sum_{k=1}^{\infty} (x, e_k)e_k = y$, say, is in $\overline{M}$. It can be checked that $x - y$ is orthogonal to every e_j. Consequently, $y = x$.

2.8.P5. For a differentiable periodic function f and $n \neq 0$, we have

$$\begin{aligned} f_n &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{f(x)\mathrm{e}^{-inx}}{-in}\bigg|_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} f'(x)\mathrm{e}^{-inx}\mathrm{d}x \right\} \\ &= \frac{1}{in\sqrt{2\pi}} \int_{-\pi}^{\pi} f'(x)\mathrm{e}^{-inx}\mathrm{d}x = \frac{1}{in}[f']_n, \end{aligned}$$

where $[f']_n$ is the nth Fourier coefficient of f' in the system $\{\mathrm{e}^{inx}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |f_n| &= |f_0| + \sum_{n=1}^{\infty} |f_n| + \sum_{n=1}^{\infty} |f_{-n}| = |f_0| + \sum_{n=1}^{\infty} \frac{1}{n}|f'|_n + \sum_{n=1}^{\infty} \frac{1}{n}[f']_{-n} \\ &\leq |f_0| + \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} [f']_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} [f']_{-n}^2\right)^{\frac{1}{2}} \\ &\leq |f_0| + 2\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \|f'\|_2 < \infty. \end{aligned}$$

2.8.P6. Observe that span$\{1, t^2, t^4, \ldots\} \subseteq$ the closed subspace of even functions in $L_e^2[-1,1] \neq L^2[-1,1]$.

2.8.P7. Let (x_1, x_2, x_3) be a vector orthogonal to the given vectors. Then,

$$x_1 + x_2 + x_3 = 0 \tag{6.8}$$

and

$$x_1 + \overline{\varpi}x_2 + \overline{\varpi}^2 x_3 = 0 \tag{6.9}$$

On subtracting these equations, we get $(1 - \overline{\varpi})(x_2 + x_3(1 + \overline{\varpi})) = 0$, which implies

$$x_2 + x_3(1 + \overline{\varpi}) = 0 \tag{6.10}$$

since $1 - \overline{\varpi} \neq 0$. On multiplying (6.8) by $\overline{\varpi}$, and subtracting (6.9) from it, we obtain

$$x_1(\overline{\varpi} - 1) + x_3(\overline{\varpi} - \overline{\varpi}^2) = 0,$$

which implies

$$x_1 = \overline{\omega} x_3. \tag{6.11}$$

Setting $x_1 = 1$, we obtain $x_3 = \omega$ from (6.11), and hence, $x_2 = \omega^2$ from (6.10), using the facts that $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$. Hence, $(1, \omega^2, \omega)$ is a nonzero vector orthogonal to the given vectors.

2.8.P8

Case 1 $|\alpha| < 1$. Let $D = \{z : |z| < 1\}$. The nth Fourier coefficient is given by

$$\begin{aligned}(f(z), z^n) &= \frac{1}{2\pi i} \int\limits_{\partial D} f(z)\overline{z}^n \frac{dz}{z} \\ &= \frac{1}{2\pi i} \int\limits_{\partial D} \frac{1}{z-\alpha} z^{-n-1} dz \quad \text{since } z\overline{z} = 1. \\ &= \begin{cases} 0 & n \geq 0 \text{ and } |\alpha| < 1 \\ \alpha^{-n-1} & n < 0 \text{ and } |\alpha| < 1 \end{cases}. \end{aligned} \tag{6.12}$$

For $n \geq 0$, the integrand in (6.12) has a pole at $z = 0$ of order $n + 1$ and a simple pole at $z = \alpha$. The sum of the residues at these poles is zero. For $n < 0, z^{-n-1}$ is holomorphic in D. The result follows upon using Cauchy's integral theorem.

Case 2 $|\alpha| > 1$.

$$(f(z), z^n) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{(\overline{z})^{n+1}}{z-\alpha} dz = \begin{cases} 0 & \text{if } n < 0 \\ -\alpha^{-n-1} & \text{if } n \geq 0 \end{cases}. \tag{6.13}$$

Indeed, for $n \geq 0$, $(z-\alpha)^{-1} = -\alpha^{-1}(1 - \frac{z}{\alpha})^{-1} = -\alpha^{-1} \sum_{j=0}^{\infty} (\frac{z}{\alpha})^j$. So, $(\overline{z})^{n+1}(z - \alpha)^{-1} = -\alpha^{-1}(\overline{z})^{n+1} \sum_{j=0}^{\infty} (\frac{z}{\alpha})^j$. Consequently, $\text{Res}_{z=0}(\overline{z})^{n+1}(z-\alpha)^{-1} = -\alpha^{-n-1}$. And for $n < 0$, the integrand in (6.13) is holomorphic in D. The result follows by Cauchy's integral theorem.

2.8.P9. In fact, the sequence $\{s_n\}_{n \geq 1}$, where

$$s_n = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt),$$

converges by Cauchy's test:

$$\frac{1}{\pi} \int_0^{2\pi} |s_m(t) - s_n(t)|^2 dt = \sum_{k=m+1}^{\infty} \left(|a_k|^2 + |b_k|^2\right).$$

Hence, by the completeness of $L^2[0, 2\pi]$, there exists a function $f \in L^2[0, 2\pi]$ for which the following holds:

$$\lim_{n\to\infty}\int_0^{2\pi}|f(t)-s_n(t)|^2 dt = 0.$$

Obviously, this function has the Fourier coefficients a_k, b_k.

2.8.P10. Suppose this is not so. Then, j is odd from a certain value of n onwards or is even. First, we consider the former possibility. So, there exists an integer $m > 0$ such that

$$p \geq 0, \frac{j-1}{2^{m+p}} < t < \frac{j}{2^{m+p}} \quad \text{implies}\, j \text{ is odd.} \tag{6.14}$$

In particular, $\frac{k-1}{2^m} < t < \frac{k}{2^m}$, where k is odd (and depends only on m and t). This can be written as

$$\frac{2^p(k-1)}{2^{m+p}} < t < \frac{2^p(k-1)+1}{2^{m+p}}, \quad \text{where } p = 0. \tag{6.15}$$

We shall argue by induction that the double inequality in (6.15) holds for every integer $p \geq 0$. Assume that it holds for some integer $p \geq 0$. By rewriting the double inequality as $\frac{2^{p+1}(k-1)}{2^{m+p+1}} < t < \frac{2^{p+1}(k-1)+2}{2^{m+p+1}}$ and using the fact t is not a dyadic rational, we infer that either $\frac{2^{p+1}(k-1)}{2^{m+p+1}} < t < \frac{2^{p+1}(k-1)+1}{2^{m+p+1}}$ or $\frac{2^{p+1}(k-1)+1}{2^{m+p+1}} < t < \frac{2^{p+1}(k-1)+2}{2^{m+p+1}}$. The latter of these is ruled out by (6.14). So, the former must hold. However, the former is precisely the double inequality in (6.15) with $p + 1$ in place of p. This completes the induction argument. The double inequality just established implies that $0 < t - \frac{2^p(k-1)}{2^{m+p}} < \frac{1}{2^{m+p}}$, which is the same as $0 < t - \frac{(k-1)}{2^m} < \frac{1}{2^{m+p}}$ for every integer $p \geq 0$. By taking the limit as $p \to \infty$, we arrive at a contradiction.

Now, consider the possibility that j is even from a certain value of n onwards. The reason here is analogous to the above, except that (6.15) has to be modified to (6.17) below. There exists an integer $m > 0$ such that

$$p \geq 0, \frac{j-1}{2^{m+p}} < t < \frac{j}{2^{m+p}} \quad \text{implies}\, j \text{ is even.} \tag{6.16}$$

In particular, $\frac{k-1}{2^m} < t < \frac{k}{2^m}$, where k is even (and depends only on m and t). This can be written as

$$\frac{2^p k - 1}{2^{m+p}} < t < \frac{2^p k}{2^{m+p}}, \quad \text{where } p = 0. \tag{6.17}$$

We shall argue by induction that the double inequality in (6.17) holds for every integer $p \geq 0$. Assume that it holds for some integer $p \geq 0$. By rewriting the double inequality as $\frac{2^{p+1}k-2}{2^{m+p+1}} < t < \frac{2^{p+1}k}{2^{m+p+1}}$ and using the fact t is not a dyadic rational, we infer that either $\frac{2^{p+1}k-2}{2^{m+p+1}} < t < \frac{2^{p+1}k-1}{2^{m+p+1}}$ or $\frac{2^{p+1}k-1}{2^{m+p+1}} < t < \frac{2^{p+1}k}{2^{m+p+1}}$. The former of these is ruled out by (6.16). So, the latter must hold. However, the latter is precisely the double inequality in (6.17) with $p + 1$ in place of p. This completes the induction argument. The double inequality just established implies that $0 < \frac{2^p k}{2^{m+p}} - t < \frac{1}{2^{m+p}}$, which is the same as $0 < \frac{k}{2^m} - t < \frac{1}{2^{m+p}}$ for every integer $p \geq 0$. By taking the limit as $p \to \infty$, we arrive at a contradiction.

2.8.P11. We may assume that the k_i are in an increasing order and that $n > 1$.

On each of the subintervals of the form $(\frac{j-1}{2^{k_{n-1}}}, \frac{j}{2^{k_{n-1}}})$, $r_{k_1} r_{k_2} \ldots r_{k_{n-1}}$ assumes that value 1 or −1. Any such interval is the union of the $2^{k_n - k_{n-1}}$ intervals $(\frac{k-1}{2^{k_k}}, \frac{k}{2^{k_k}})$, where $2^{k_n - k_{n-1}}(j-1) + 1 \leq k \leq 2^{k_n - k_{n-1}} j$, and the set of common endpoints of consecutive ones. Since r_{k_n} alternates in sign from one interval to the next, it follows that the product $r_{k_1} r_{k_2} \ldots r_{k_n}$ does the same on these consecutive intervals. The fact that the intervals are even in number now implies that the integral of the product over $(\frac{j-1}{2^{k_{n-1}}}, \frac{j}{2^{k_{n-1}}})$ vanishes. Since this is true for each j, the desired conclusion follows.

2.8.P12. As shown in the text, completeness of the orthonormal set of Hermite functions in $L^2(-\infty, \infty)$ is equivalent to the assertion that

$$\text{if } F \in L^2(-\infty, \infty) \text{ and } \int_{-\infty}^{\infty} F(u) \exp\left(-\frac{u^2}{2}\right) u^n \mathrm{d}u = 0, \quad n = 0, 1, 2, \ldots, \text{ then } F = 0 \text{ a.e.} \tag{6.18}$$

and completeness of the orthonormal set of Laguerre functions in $L^2(0, \infty)$ is equivalent to

$$\text{if } f \in L^2(0, \infty) \text{ and } \int_0^{\infty} f(t) \exp\left(-\frac{t}{2}\right) t^n \mathrm{d}t = 0, \quad n = 0, 1, 2, \ldots, \text{ then } f = 0 \text{ a.e.} \tag{6.19}$$

It is therefore sufficient to show that (6.18) and (6.19) are equivalent.

We shall first show that (6.18) implies (6.19).

With this in view, consider any $f \in L^2(0, \infty)$ satisfying the hypothesis that

$$\int_0^{\infty} f(t) \exp(-\tfrac{t}{2}) t^n \mathrm{d}t = 0, \quad n = 0, 1, 2, \ldots.$$

Define an even function F on $(-\infty, \infty)$ by

$$F(u) = f(2u^2)|u| \exp\left(-\frac{u^2}{2}\right).$$

Using the substitution $t = 2u^2$, we have

$$\int_{-\infty}^{\infty} |F(u)|^2 \mathrm{d}u = 2\int_0^{\infty} |F(u)|^2 \mathrm{d}u = 2\int_0^{\infty} \left|f\left(2u^2\right)\right|^2 u^2 \exp(-u^2)\mathrm{d}u$$
$$= 2\int_0^{\infty} |f(t)|^2 \exp\left(-\frac{t}{2}\right)\frac{\sqrt{t}}{\sqrt{2}}\frac{1}{4}\mathrm{d}t = \frac{1}{2}\int_0^{\infty} |f(t)|^2 \exp\left(-\frac{t}{2}\right)\frac{\sqrt{t}}{\sqrt{2}}\mathrm{d}t$$
$$< \infty \text{ because } \exp\left(-\frac{t}{2}\right)\frac{\sqrt{t}}{\sqrt{2}} \text{ is bounded on } (0, \infty).$$

So, $F \in L^2(-\infty, \infty)$. Since $\exp(-\frac{u^2}{2})u^n \in L^2(-\infty, \infty)$ for every nonnegative integer n, it follows that the product $F(u)\exp(-\frac{u^2}{2})u^n$ is integrable over $(-\infty, \infty)$. Moreover, for odd n, its integral over $(-\infty, \infty)$ is zero, i.e.

$$\int_{-\infty}^{\infty} F(u)\exp\left(-\frac{u^2}{2}\right)u^{2n+1}\mathrm{d}u = 0, \quad n = 0, 1, 2, \ldots. \tag{6.20}$$

For even n, which we may denote by $2n$, we use the same substitution as above (i.e. $t = 2u^2$) and obtain

$$\int_{-\infty}^{\infty} F(u)\exp\left(-\frac{u^2}{2}\right)u^{2n}\mathrm{d}u = 2\int_0^{\infty} F(u)\exp\left(-\frac{u^2}{2}\right)u^{2n}\mathrm{d}u = 2\int_0^{\infty} f(2u^2)\, u\exp(-u^2)u^{2n}\mathrm{d}u$$
$$= 2\int_0^{\infty} f(t)\exp\left(-\frac{t}{2}\right)\frac{t^n}{2^n}\frac{1}{4}\mathrm{d}t = 0 \text{ according to the hypothesis about } f.$$

Together with (6.20), this shows that F satisfies the hypothesis of (6.18) and hence also its conclusion that $F = 0$ a.e. on $(-\infty, \infty)$. It follows that $f(2u^2) = 0$ a.e. on $(0, \infty)$. Using Problem 2.9.P8, we obtain $f = 0$ a.e. on $(0, \infty)$. This completes the proof that (6.18) implies (6.19).

Next, we shall prove that (6.19) implies (6.18).

With this in view, consider any $F \in L^2(-\infty, \infty)$ satisfying the hypothesis that

$$\int_{-\infty}^{\infty} F(u)\exp\left(-\frac{u^2}{2}\right)u^n\mathrm{d}u = 0, \quad n = 0, 1, 2, \ldots.$$

Clearly, the function $F(-u)$ also belongs to $L^2(-\infty, \infty)$ and satisfies the same hypothesis. Therefore, so do the even and odd parts of F. If each of these parts can be shown to be zero a.e., it will instantly follow that $F = 0$ a.e. This means we need to cover only the two cases when F is even and F is odd.

Case when F is even: Define a function f on $(0, \infty)$ by

$$f(t) = F(\sqrt{2t})t^{\frac{1}{2}}\exp\left(-\frac{t}{2}\right).$$

Using the substitution $u^2 = 2t$, we have

$$\int_0^\infty |f(t)|^2 \mathrm{d}t = \int_0^\infty \left|F(\sqrt{2t})\right|^2 t\exp(-t)\mathrm{d}t = \int_0^\infty |F(u)|^2 \frac{u^2}{2}\exp\left(-\frac{u^2}{2}\right)u\,\mathrm{d}u < \infty$$

because $\frac{u^2}{2}\exp(-\frac{u^2}{2})u$ is bounded on $(0, \infty)$. So, $f \in L^2(0, \infty)$. Since $\exp(-\frac{t}{2})t^n \in L^2(0, \infty)$ for every nonnegative integer n, it follows that the product $f(t)\exp(-\frac{t}{2})t^n$ is integrable over $(0, \infty)$. Moreover, using the same substitution as above ($u^2 = 2t$), we have

$$\begin{aligned}\int_0^\infty f(t)\exp\left(-\frac{t}{2}\right)t^n\mathrm{d}t &= \int_0^\infty F\left(\sqrt{2t}\right)t^{\frac{1}{2}}\exp(-t)t^n\mathrm{d}t = \int_0^\infty F(u)\frac{u}{\sqrt{2}}\exp\left(-\frac{u^2}{2}\right)u^{2n}u\,\mathrm{d}u \\ &= \frac{1}{2^n\sqrt{2}}\int_0^\infty F(u)\exp\left(-\frac{u^2}{2}\right)u^{2n+2}\mathrm{d}u \\ &= \frac{1}{2^{n+1}\sqrt{2}}\int_{-\infty}^\infty F(u)\exp\left(-\frac{u^2}{2}\right)u^{2n+2}\mathrm{d}u \text{ because } F \text{ is even} \\ &= 0 \text{ by the hypothesis on } F.\end{aligned}$$

Thus, f satisfies the hypothesis of (6.19) and hence also its conclusion that $f = 0$ a.e. on $(0, \infty)$. It follows that $F(\sqrt{2t}) = 0$ a.e. on $(0, \infty)$. Using Problem 2.9.P8, we obtain $F = 0$ a.e. on $(0, \infty)$. As F is even, we further obtain $F = 0$ a.e. on $(-\infty, \infty)$.

Case when F is odd: Define a function f on $(0, \infty)$ by

$$f(t) = F(\sqrt{2t})\exp\left(-\frac{t}{2}\right).$$

Using the substitution $u^2 = 2t$, we have

$$\int_0^\infty |f(t)|^2 \mathrm{d}t = \int_0^\infty \left|F(\sqrt{2t})\right|^2 \exp(-t)\mathrm{d}t = \int_0^\infty |F(u)|^2\exp\left(-\frac{u^2}{2}\right)u\,\mathrm{d}u < \infty$$

because $\exp(-\frac{u^2}{2})u$ is bounded on $(0, \infty)$. So, $f \in L^2(0, \infty)$. Since $\exp(-\frac{t}{2})t^n \in L^2(0, \infty)$ for every nonnegative integer n, it follows that the product $f(t)\exp(-\frac{t}{2})t^n$ is integrable over $(0, \infty)$. Moreover, using the same substitution as above ($u^2 = 2t$), we have

$$\int_0^\infty f(t)\exp\left(-\frac{t}{2}\right)t^n \mathrm{d}t = \int_0^\infty F(\sqrt{2t})\exp(-t)t^n \mathrm{d}t = \int_0^\infty F(u)\exp\left(-\frac{u^2}{2}\right)\frac{1}{2^n}u^{2n}u\,\mathrm{d}u$$
$$= \frac{1}{2^n}\int_0^\infty F(u)\exp\left(-\frac{u^2}{2}\right)u^{2n+1}\mathrm{d}u$$
$$= \frac{1}{2^{n+1}}\int_{-\infty}^\infty F(u)\exp\left(-\frac{u^2}{2}\right)u^{2n+1}\mathrm{d}u \qquad \text{because } F \text{ is odd}$$
$$= 0 \qquad \text{by the hypothesis on } F.$$

Thus, f satisfies the hypothesis of (6.19) and hence also its conclusion that $f = 0$ a.e. on $(0, \infty)$. It follows that $F(\sqrt{2t}) = 0$ a.e. on $(0, \infty)$. Using Problem 2.9.P8, we obtain $F = 0$ a.e. on $(0, \infty)$. As F is odd, we further obtain $F = 0$ a.e. on $(-\infty, \infty)$.

2.8.P13. Let an orthonormal basis for X be $u_1, u_2, \ldots, u_n$. This is possible by (iv) of Remarks 2.8.10. Any vector $x \in X$ has the form $x = \sum_{i=1}^n \alpha_i u_i$. Define the linear transformation $A : X \to \mathbb{C}^n$ given by

$$Ax = (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

A is clearly linear and bijective. Moreover, for $y = \sum_{i=1}^n \beta_i u_i$, $Ay = (\beta_1, \beta_2, \ldots, \beta_n)$; observe that

$$(Ax, Ay) = ((\alpha_1, \alpha_2, \ldots, \alpha_n), (\beta_1, \beta_2, \ldots, \beta_n)) = \sum_{i=1}^\infty \alpha_i \overline{\beta_i}$$

and

$$\left(\sum_{i=1}^n \alpha_i u_i, \sum_{i=1}^n \beta_i u_i\right) = \sum_{i=1}^n \alpha_i \overline{\beta_i}.$$

Thus, A preserves inner products. So A is an isometry between X and $\mathbb{C}^n$. By Example 2.3.4(i), $\mathbb{C}^n$ is complete and hence so is its isometric image X.

6.8 Problem Set 2.9

2.9.P1. Suppose $\overline{M}_1 = \overline{M}_2$. Then, $e_n \in \overline{M}_2$, and hence, by Problem 2.8.P4,

$$e_n = \sum_{m=1}^{\infty} (e_n, \widetilde{e_m})\widetilde{e_m} = \sum_{m=1}^{\infty} \alpha_{nm}\widetilde{e_m}.$$

Similarly,

$$\widetilde{e_n} = \sum_{m=1}^{\infty} (\widetilde{e_n}, e_m)e_m = \sum_{m=1}^{\infty} \overline{\alpha}_{mn}e_m.$$

The converse is trivial.

2.9.P2. (a) Suppose H is separable, B any dense set in H and M any orthonormal set. Then, any two distinct elements x, y in M are at a distance of $\sqrt{2}$ from each other: indeed,

$$||x - y||^2 = (x - y, x - y) = ||x||^2 + ||y||^2 = 2.$$

Consequently, the balls $S(x, \sqrt{2}/3)$ and $S(y, \sqrt{2}/3)$ are disjoint. Since B is dense, there exist a and b in B such that $a \in S(x, \sqrt{2}/3)$ and $b \in S(y, \sqrt{2}/3)$. Moreover, $a \neq b$ since $S(x, \sqrt{2}/3)$ and $S(y, \sqrt{2}/3)$ are disjoint. If M were uncountable, we would have uncountably many such pairwise disjoint neighbourhoods, one for each x in M, so that B would be uncountable. Since B was any dense set, this would imply that H would not contain a dense set which is countable. This contradicts the separability of H.

(b) Let $\{e_k\}_{k\geq 1}$ be a complete orthonormal sequence in H. Let

$$S = \left\{ \gamma_1^{(n)}e_1 + \cdots + \gamma_n^{(n)}e_n : \gamma_k^{(n)} = a_k^{(n)} + ib_k^{(n)}, a_k^{(n)}, b_k^{(n)} \text{ are rational}, \ n = 1, 2, \ldots \right\}.$$

Then, S is countable. We shall show that S is dense in H. Let $x \in H$ and $\varepsilon > 0$ be given. The vector x can be expressed as

$$x = \sum_{k=1}^{\infty} (x, e_k)e_k.$$

Therefore, there exists an $n \in \mathbb{N}$ such that

$$\left\| x - \sum_{k=1}^{n} (x, e_k)e_k \right\| < \frac{\varepsilon}{2}.$$

For each (x, e_k), there exists γ_k with rational real and imaginary parts such that

$$\left\| \sum_{k=1}^{n} [(x, e_k) - \gamma_k]e_k \right\| < \frac{\varepsilon}{2}.$$

Set $s = \sum_{k=1}^{n} \gamma_k e_k$. Then,

$$\begin{aligned}\|x - s\| &= \left\| x - \sum_{k=1}^{n} \gamma_k e_k \right\| \\ &\leq \left\| x - \sum_{k=1}^{n} (x, e_k) e_k \right\| + \left\| \sum_{k=1}^{n} [(x, e_k) - \gamma_k] e_k \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This shows that S is dense in H.

2.9.P3. Let $x = \chi_A$. Then, $x \in L^2([-\pi, \pi], \frac{dt}{2\pi})$. The result now follows on using the Riemann–Lebesgue lemma [see Remark (c) under Example 2.9.17(iii)].

2.9.P4. Let

$$E_1 = \{x \in E \,:\, \lim_k 2\sin^2 n_k x - 1 \geq 0\} \text{ and } E_2 = \{x \in E : \lim_k 2\sin^2 n_k x - 1 < 0\}.$$

Observe that E_1 and E_2 are measurable and $E = E_1 \cup E_2$. Now,

$$0 = -\lim_k \int_{E_1} \cos 2n_k x \mathrm{d}x = \lim_k \int_{E_1} (2\sin^2 n_k x - 1)\mathrm{d}x = \int_{E_1} \lim_k (2\sin^2 n_k x - 1)\mathrm{d}x,$$

using the Lebesgue's dominated convergence theorem.

This implies $2\sin^2 n_k x - 1 = 0$ a.e. on E_1. Similarly, $2\sin^2 n_k x - 1 = 0$ a.e. on E_2. Hence,

$$\lim_k \sin n_k x = \frac{1}{\sqrt{2}} \text{ a.e. on } E_{11} \cup E_{21},$$

where $E_{11} = \{x \in E_1 : \lim_k \sin n_k x = \frac{1}{\sqrt{2}}\}$ and $E_{21} = \{x \in E_2 : \lim_k \sin n_k x = \frac{1}{\sqrt{2}}\}$.

Assume that $m(E_{11} \cup E_{21}) > 0$. Then, using the dominated convergence theorem, we get

$$\lim_k \int_{E_{11} \cup E_{21}} \sin n_k x \,\mathrm{d}x = \int_{E_{11} \cup E_{21}} \lim_k \sin n_k x \,\mathrm{d}x = \frac{1}{\sqrt{2}} m(E_{11} \cup E_{21}) \neq 0.$$

This contradicts the Riemann–Lebesgue lemma. The subsets of E_1 and E_2 where $\lim_k \sin n_k x = -\frac{1}{\sqrt{2}}$ can be treated similarly.

2.9.P5. Set $z = e^{i\theta}$. Then, $dz = ie^{i\theta}d\theta$, $\frac{dz}{z} = id\theta$ and

$$(e_j(z), e_k(z)) = \frac{1}{2\pi i} \int_{\partial D} z^j z^{-k} \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i} \int_{\partial D} z^{j-k} \frac{\mathrm{d}z}{z} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)\theta} \mathrm{d}\theta,$$

which is 1 if $j = k$ and 0 otherwise.

2.9.P6. Observe that

$$\begin{aligned}
||e_1(z)||^2 &= (e_1(z), e_1(z)) \\
&= \left(1 - |\alpha_1|^2\right) \frac{1}{2\pi i} \int_{\partial D} \frac{1}{1 - \overline{\alpha_1} z} \frac{1}{1 - \alpha_1 \overline{z}} \frac{dz}{z} \\
&= \left(1 - |\alpha_1|^2\right) \frac{1}{2\pi i} \int_{\partial D} \frac{(1 - \overline{\alpha_1} z)^{-1}}{z - \alpha_1} dz \\
&= \left(1 - |\alpha_1|^2\right) \left(1 - |\alpha_1|^2\right)^{-1} = 1
\end{aligned}$$

since the function $1 - \overline{\alpha}_1 z$ is holomorphic in $D(0, 1)$. Also,

$$\begin{aligned}
||e_2(z)||^2 &= (e_2(z), e_2(z)) \\
&= \frac{1}{2\pi i} \int_{\partial D} \left[\frac{z - \alpha_1}{1 - \overline{\alpha_1} z} \frac{\left(1 - |\alpha_2|^2\right)}{1 - \overline{\alpha_2} z} \frac{\overline{z} - \overline{\alpha_1}}{(1 - \alpha_1 \overline{z})} \frac{1}{(1 - \alpha_2 \overline{z})} \right] \frac{dz}{z} \\
&= \frac{(1 - |\alpha_2|^2)}{2\pi i} \int_{\partial D} \left[\frac{1}{(\overline{z} - \overline{\alpha_1})} \frac{1}{(1 - \overline{\alpha_2} z)} \frac{\overline{z} - \overline{\alpha_1}}{(z - \alpha_2)} \right] dz \\
&= \frac{(1 - |\alpha_2|^2)}{2\pi i} \int_{\partial D} \frac{(1 - \overline{\alpha_2} z)^{-1}}{z - \alpha_2} dz \\
&= (1 - |\alpha_2|^2)(1 - |\alpha_2|^2)^{-1} = 1.
\end{aligned}$$

It remains to show that $(e_1(z), e_2(z)) = 0$. Now,

$$\begin{aligned}
(e_1(z), e_2(z)) &= (1 - |\alpha_1|^2)^{\frac{1}{2}} (1 - |\alpha_2|^2)^{\frac{1}{2}} \frac{1}{2\pi i} \int_{\partial D} \frac{1}{(1 - \overline{\alpha_1} z)} \frac{\overline{z} - \overline{\alpha_1}}{(1 - \alpha_1 \overline{z})} \frac{1}{(1 - \alpha_2 \overline{z})} \frac{dz}{z} \\
&= A \frac{1}{2\pi i} \int_{\partial D} \frac{dz}{(z - \alpha_1)(z - \alpha_2)}, \quad \text{where } A = (1 - |\alpha_1|^2)^{\frac{1}{2}} (1 - |\alpha_2|^2)^{\frac{1}{2}} \\
&= A \frac{1}{2\pi i(\alpha_1 - \alpha_2)} \int_{\partial D} \frac{1}{z - \alpha_1} - \frac{1}{z - \alpha_2} dz = A \frac{1}{\alpha_1 - \alpha_2} (1 - 1) \\
&= 0
\end{aligned}$$

2.9.P7. We shall show that if $f_0 \perp f_n$, $n = 1, 2, \ldots$, then $f_0 = 0$. If $f_0 \neq 0$, then $\{f_0, f_1, f_2, \ldots\}$ is an orthogonal set of nonzero vectors and therefore linearly independent. Choose a positive integer n such that

$$\sum_{j=n+1}^{\infty} \left\|e_j - f_j\right\|^2 < 1;$$

it will turn out that for this n, the vectors $f_0, f_1, f_2, \ldots, f_n$ are linearly dependent. Write

$$g_k = \sum_{j=1}^{n} (f_k, e_j)e_j, \quad k = 0, 1, 2, \ldots, n.$$

Since each g_k belongs to span $(\{e_1, e_2, \ldots, e_n\})$ and since the elements $g_0, g_1, \ldots, g_n$ are $n + 1$ in number, it follows that the g_k are linearly dependent. So, there exist α_0, $\alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\sum_{k=0}^{n} \alpha_k g_k = 0.$$

This implies that

$$\begin{aligned} 0 &= \sum_{k=0}^{n} \alpha_k \sum_{j=1}^{n} (f_k, e_j)e_j \\ &= \sum_{j=1}^{n} \left(\sum_{k=0}^{n} \alpha_k f_k, e_j \right) e_j \end{aligned}$$

Since the e_j are linearly independent, it follows that the coefficients in the last sum must vanish, i.e. if

$$h = \sum_{k=0}^{n} \alpha_k f_k,$$

then $h \neq 0$ because the $\alpha_0, \alpha_1, \ldots, \alpha_n$ are not all zero and $f_0, f_1, f_2, \ldots, f_n$ are linearly independent, and also $h \perp e_1, e_2, \ldots, e_n$. Therefore, by Parseval's equality,

$$\begin{aligned} ||h||^2 &= \sum_{j=1}^{n} \left|(h, e_j)\right|^2 + \sum_{j=n+1}^{\infty} \left|(h, e_j)\right|^2 \\ &= \sum_{j=n+1}^{\infty} \left|(h, e_j)\right|^2 \\ &= \sum_{j=n+1}^{\infty} \left|(h, e_j) - (h, f_j)\right|^2, \end{aligned}$$

(h belongs to the linear span of $f_0, f_1, f_2, \ldots, f_n$ so that $h \perp f_j$, $j > n$)

$$\leq ||h||^2 \sum_{j=n+1}^{\infty} \left\|e_j - f_j\right\|^2 < \|h\|^2,$$

considering that $h \neq 0$.

2.9.P8. Since a single point forms a set of measure zero, we need to consider only the case when the domain is an open interval. Any open interval is a countable union of compact subintervals, and it is therefore sufficient to prove the result for a compact interval and a function having a continuous nonvanishing derivative on it (including endpoints). In this situation, the derivative is bounded and the required conclusion follows from the definition of Lebesgue measure and the Lagrange mean value theorem.

The second statement is obvious from elementary analysis.

6.9 Problem Set 2.10

2.10.P1. Let $f^{(k)}$ be a sequence in RH^2 such that $f^{(k)} \to f$ in RH^2. Then, $C_n(f^{(k)}) = a_n^{(k)}$ and

$$\begin{aligned}\left|a_n{}^{(k)} - a_n\right| &= \left|C_n\left(f^{(k)}\right) - C_n(f)\right| = \left|\left(f^{(k)}, z^n\right) - (f, z^n)\right| = \left|\left(f^{(k)} - f, z^n\right)\right| \\ &\leq \left\|f^{(k)} - f\right\|_2 \|z^n\|_2 = \left\|f^{(k)} - f\right\|_2 \to 0 \text{ as } k \to \infty.\end{aligned}$$

This completes the proof. [This also follows from (iv) of Examples 2.10.24 upon noting that $C_n(f) = (f, g)$, where $g(z) = z^n$.]

2.10.P2. Observe that $(e_0, e_0) = \int_0^1 1\mathrm{d}t = 1, (e_1, e_1) = 3\int_0^1 (2t-1)^2 \mathrm{d}t = 1$, and $(e_0, e_1) = \int_0^1 \sqrt{3}(2t-1)\mathrm{d}t = 0$. So, e_0 and e_1 constitute an orthonormal system in $L^2[0, 1]$.

We first determine the vector y closest to t^2 in the linear span of $\{e_0, e_1\}$. By Theorem 2.11.1,

$$\begin{aligned}y(t) &= \left(t^2, e_0\right)e_0 + \left(t^2, e_1\right)e_1 \\ &= \int_0^1 t^2 \mathrm{d}t + \left(\int_0^1 t^2 \cdot \sqrt{3}(2t-1)\mathrm{d}t\right)\sqrt{3}(2t-1) \\ &= \frac{1}{3} + 3(2t-1)\left(\frac{1}{2} - \frac{1}{3}\right) = t - \frac{1}{6}.\end{aligned}$$

Finally,

$$\min_{a,b}\int_0^1 \left|t^2-a-bt\right|^2 \mathrm{d}t=\int_0^1\left(t^2-t+\frac{1}{6}\right)^2\mathrm{d}t=\frac{1}{180}.$$

Remark If it is required only to find the min, then we can also proceed as follows by Theorem 2.11.1. The required min is

$$(t^2,t^2)-\left|(t^2,e_0)\right|^2-\left|(t^2,e_1)\right|^2.$$

Now, $(t^2,t^2)=\int_0^1 t^4\mathrm{d}t=\frac{1}{5}$, $(t^2,e_0)=\int_0^1 t^2\mathrm{d}t=\frac{1}{3}$ and $(t^2,e_1)=\int_0^1 t^2\cdot\sqrt{3}(2t-1)\mathrm{d}t=\frac{\sqrt{3}}{6}$. Therefore, $(t^2, t^2)-|(t^2, e_0)|^2-|(t^2, e_1)|^2=\frac{1}{5}-\frac{1}{9}-\frac{3}{36}=\frac{1}{180}$.

2.10.P3. (a) $M^\perp=\{(\eta_1,\eta_2)\in\mathbb{R}^2:((\eta_1,\eta_2),(\xi_1,\xi_2))=0\}$. So, $\eta_1\xi_1+\eta_2\xi_2=0$, that is, $\eta_1=(-\xi_2/\xi_1)\eta_2$, assuming $\xi_1\neq 0$. Consequently, $M^\perp=\{\alpha(-\xi_2,\xi_1):\alpha\in\mathrm{R}\}$. When $\xi_1=0$, the vectors orthogonal to $(0,\xi_2)$ are $\alpha(-\xi_2,\xi_1),\alpha\in\mathbb{R}$ with $\xi_1=0$, so that the same description of $M^\perp$ is valid.

(b) $M^\perp=\{y\in\mathbb{R}^2:(y,x_1)=0 \text{ and } (y,x_2)=0\}$. If $y=(y(1),y(2))$, $x_1=(x_1(1), x_1(2))$ and $x_2=(x_2(1),x_2(2))$, then

$$\begin{aligned}&y(1)x_1(1)+y(2)x_1(2)=0,\\&y(1)x_2(1)+y(2)x_2(2)=0,\\&y(1)[x_1(1)x_2(2)-x_2(1)x_1(2)]=0.\end{aligned}$$

This implies $y(1)=0$, since x_1 and x_2 are linearly independent. Similarly, $y(2)=0$. Thus, $M^\perp=\{0\}$.

2.10.P4. Let $x\in M^\perp$ and let $V=\mathrm{span}(M)$ be dense in H. Then, $x\in\overline{V}$. There exists sequence $\{x_n\}_{n\geq 1}$ in V such that $x_n\to x$. Since $x\in M^\perp$ and $M^\perp\perp V$, we have $(x_n, x)=0$. Now,

$$\begin{aligned}|(x,x)|&=|(x-x_n+x_n,x)|\leq|(x-x_n,x)|+|(x_n,x)|\to 0 \text{ as } n\to\infty, \text{using} (x_n,x)\\&=0.\end{aligned}$$

Therefore, $x=0$. Since $x\in M^\perp$ was arbitrary, it follows that $M^\perp=\{0\}$.

Conversely, suppose that $M^\perp=\{0\}$. If $x\perp V$, then $x\perp M$, so that $x\in M^\perp$, and hence, $x=0$. Consequently, $V^\perp=\{0\}$. Noting that V is a subspace of H and $H=\overline{V}\oplus\overline{V}^\perp$, it follows that $H=\overline{V}[V\subseteq\overline{V} \text{ and } \overline{V}^\perp\subseteq V^\perp=\{0\}]$.

2.10.P5. (a) If $x\in M_1^\perp\cap M_2^\perp$, then $(y,x)=0$ and $(z,x)=0$ for any $y\in M_1$ and $z\in M_2$. Hence, $(y+z,x)=0$, which means $x\in(M_1+M_2)^\perp$. On the other hand, $M_1\subseteq M_1+M_2$ and $M_2\subseteq M_1+M_2$; so $(M_1+M_2)^\perp\subseteq M_1^\perp\cap M_2^\perp$.

(b) From (a), it follows that $\overline{M_1+M_2}=(M_1^\perp\cap M_2^\perp)^\perp$. Replacing M_1 by $M_1^\perp$ and M_2 by $M_2^\perp$, we obtain the required equality.

2.10.P6. Apply the parallelogram law to the vectors $x - y_1$ and $x - y_2$:

$$\begin{aligned}\|y_1 - y_2\|^2 &= 2\|x - y_1\|^2 + 2\|x - y_2\|^2 - 4\left\|x - \frac{y_1 + y_2}{2}\right\|^2 \\ &= 2(\|x - y_1\|^2 - \|x - y_2\|^2) + 4\left\{\|x - y_2\|^2 - \left\|x - \frac{y_1 + y_2}{2}\right\|^2\right\}.\end{aligned}$$

Since the term in the curly brackets is nonpositive, the required inequality follows. (b) Clearly, K is closed. Since the sequence $\{K_n\}_{n \geq 1}$ is an increasing sequence of convex sets, $\bigcup_n K_n$ is convex. Indeed, for x, $y \in \bigcup_n K_n$, there exist n_1 and n_2 such that $x \in K_{n_1}$ and $y \in K_{n_2}$. Without loss of generality, we may assume $K_{n_1} \subseteq K_{n_2}$; so x and y are in K_{n_2}. Since K_{n_2} is convex, it follows that $\lambda x + (1 - \lambda)y \in K_{n_2}$, $0 \leq \lambda \leq 1$. K being the closure of the convex set $\bigcup_n K_n$ is itself convex.

We next show that $\lim_n d(x, K_n) = d(x, K)$. Since $d(x, K) = d(x, \bigcup_n K_n)$, it is therefore enough to show $\lim_n d(x, K_n) = d(x, \bigcup_n K_n)$. Observe that $\{d(x, K_n)\}_{n \geq 1}$ is a decreasing sequence of positive real numbers and is bounded below by $d(x, \bigcup_n K_n)$. If this were not the greatest lower bound, then for any $\varepsilon > 0$, there would exist K_m such that $d(x, K_m) < d(x, \bigcup_n K_n) + \varepsilon$. Hence, $\lim_n d(x, K_n) = d(x, \bigcup_n K_n) = d(x, K)$. Finally,

$$\begin{aligned}\|y_n - y\| &\leq 2\left(d(x, K_n)^2 - d(x, K)^2\right) \\ &= 2(d(x, K_n) + d(x, K))(d(x, K_n) - d(x, K)).\end{aligned}$$

Since the second factor on the right-hand side of the above inequality tends to zero as $n \to \infty$, the assertion that $\lim_n y_n = y$ is proved.

Remark Let $\{K_n\}_{n \geq 1}$ be a decreasing sequence of nonempty closed convex subsets in H such that $K = \bigcap_{n=1}^{\infty} K_n$ is nonempty. Then, it can be proved by a similar argument that $\lim_n y_n = y$.

2.10.P7. Let $x_0 = x + y$, where $x \in M$ and $y \in M^{\perp}$, be the decomposition of x_0. Then,

$$\|x_0 - x\| = \min\{\|x_0 - m\| : m \in M\}.$$

Thus,

$$\|y\| = \min\{\|x_0 - m\| : m \in M\}.$$

We shall show that $\|y\| = \max\{|(x_0, \omega)| : \omega \in M^{\perp} \text{ and } \|\omega\| = 1\}$.

$$|(x_0, \omega)| = |(x + y, \omega)| = |(y, \omega)| \leq \|y\|\|\omega\| = \|y\|.$$

So,

$$\sup_{\omega \in M^{\perp}, \|\omega\|=1} |(x_0, \omega)| \le ||y||. \tag{6.21}$$

Observe that $y/\|y\| \in M^{\perp}$, $\|y/\|y\|\| = 1$ and $\|(x_0, y/\|y\|) = \|y\|$. It therefore follows that the inequality in (6.21) is an equality and the sup is a max.

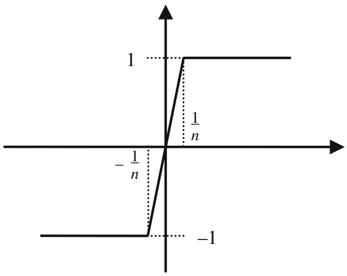
2.10.P8. (a) is a direct consequence of Problem 2.10.P7.

(b) By definition, $F = \{1\}^{\perp}$. It follows by (v) of Remarks 2.10.12 that $F^{\perp} = \{1\}^{\perp\perp}$ is the smallest closed subspace of H which contains $\{1\}$, i.e. span$\{1\}$.By (a) above,

$$d(f, F) = d\left(f, \{1\}^{\perp}\right) = |(f, 1)|,$$

where $(f, 1) = \int_0^1 \exp(x)\mathrm{d}x = e - 1$.

2.10.P9. Observe that F is a linear functional and



$$|F(x)| = \left|\int_0^1 x(t)f(t)\mathrm{d}t\right| \le \int_0^1 |x(t)||f(t)|\mathrm{d}t \le \sup_t |x(t)| \int_0^1 |f(t)|\mathrm{d}t.$$

This shows that F is a bounded linear functional on $C[0, 1]$ of norm at most $\int_0^1 |f(t)|\mathrm{d}t$. To obtain the reverse inequality, consider the function $y_n(t) = u_n(f(t))$, where

$$u_n(\tau) = \begin{cases} -1 & \tau \le -\frac{1}{n} \\ 1 & \tau \ge \frac{1}{n} \\ \text{linear} & \left[-\frac{1}{n}, \frac{1}{n}\right] \end{cases}.$$

It is obvious that $y_n(t)$ is continuous and $\max_t |y_n(t)| \le 1$. Moreover, the product $y_n(t)f(t)$ is a continuous nonnegative function and $y_n(t)f(t) = |f(t)|$ for $|f(t)| \ge \frac{1}{n}$. Thus, we have

$$
\begin{aligned}
F(y_n) &= \int_0^1 y_n(t)f(t)\mathrm{d}t \\
&= \int_{|f(t)|\geq\frac{1}{n}} y_n(t)f(t)\mathrm{d}t + \int_{|f(t)|<\frac{1}{n}} y_n(t)f(t)\mathrm{d}t \\
&\geq \int_{|f(t)|\geq\frac{1}{n}} y_n(t)f(t)\mathrm{d}t \\
&= \int_0^1 |f(t)|\mathrm{d}t - \int_{|f(t)|<\frac{1}{n}} |f(t)|\mathrm{d}t \\
&= \int_0^1 |f(t)|\mathrm{d}t - \frac{1}{n}.
\end{aligned}
$$

So, $\int_0^1 |f(t)|\mathrm{d}t - \frac{1}{n} \leq F(y_n) \leq ||F||$ and since n is arbitrary, $\int_0^1 f(t)\mathrm{d}t \leq ||F||$.

2.10.P10. By the Riesz representation theorem, f is represented by some nonzero $\phi \in H$. Now, $\ker(f) = \{\phi\}^{\perp}$, and by (v) of Remarks 2.10.12, $(\ker(f))^{\perp}$ is the closed linear span of $\{\phi\}$, which is just the span of ϕ and is hence one dimensional.

Remarks (i) If H is an inner product space and f is a linear functional on H, then $\dim((\ker(f))^{\perp}) = 0$ or 1. Suppose $x, y \in \ker(f)^{\perp}$ and are nonzero. Then, neither belongs to $\ker(f)$, so that $f(x) \neq 0 \neq f(y)$. But

$$f(y)x - f(x)y \in \ker(f) \cap \ker(f)^{\perp}$$

and therefore, $f(y)x - f(x)y = 0$. Since $f(x) \neq 0 \neq f(y)$, it follows that x and y are linearly dependent.

(ii) Let H be a Hilbert space and $f_1, \ldots, f_n$ be n linearly independent continuous linear functionals on H. Then, $\dim\left(\left(\bigcap_{j=1}^{n} \ker(f_j)\right)^{\perp}\right) = n$, considering that a finite-dimensional subspace must be closed [see Problem 2.3.P10].

2.10.P11. If $f = 0$, then there is nothing to prove. Suppose $f \neq 0$. Then, for every $x \in H$, $x = \alpha e + y$, where $y \in \ker(f)$, $f(e) = 1$ and $\alpha = f(x)$. Assume that $x_n = \alpha_n e + y_n \to \alpha_0 e + y_0 = x$, say. Observe that the sequence $\{\alpha_n\}$ is bounded; otherwise, there is a subsequence $\{\alpha_{n_i}\}_{i\geq 1}$ of $\{\alpha_n\}_{n\geq 1}$ such that $0 < |\alpha_{n_i}| \to \infty$. The sequence $\{x_n\}_{n\geq 1}$ is bounded, and therefore,

$$\frac{y_{n_i}}{\alpha_{n_i}} = e - \frac{x_{n_i}}{\alpha_{n_i}} \to e \in \ker(f),$$

because $\ker(f)$ is closed, which is a contradiction.

We shall now show that $\alpha_n \to \alpha_0$. If not, then, in view of the boundedness of $\{\alpha_n\}$, there exists a subsequence $\alpha_{n_k} \to a \neq \alpha_0$, and hence, $y_{n_k} = x_{n_k} - \alpha_{n_k} e \to x - ae = y'$, say. Since ker($f$) is closed, $y' \in \ker(f)$. Thus, we have $\alpha_0 e + y_0 = ae + y'$, which means $(\alpha_0 - a)e = y' - y_0$ with $\alpha_0 - a \neq 0$. This is a contradiction. Thus, $f(x_n) = \alpha_n \to \alpha_0 = f(x)$. This completes the argument.

2.10.P12. Consider the functional $f(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x_n$ defined on the subspace L consisting of vectors $x = \{x_n\}_{n \geq 1} \in \ell^2$ for which the series converges. Observe that L contains the space of sequences with all but finitely many elements equal to zero. L is clearly dense in ℓ^2 [Remarks 2.3.5(i)], $M = \ker(f)$, and hence, $\dim(L/M) = 1$.

Assume that M is closed. Using Problem 2.3.P10, it follows that L is closed, and since L is dense in ℓ^2, we conclude that $L = \ell^2$. Thus, the functional is defined on ℓ^2 and has a closed kernel. Consequently, f is continuous [Problem 2.10.P11]. By the Riesz representation theorem, we obtain $f(x) = \sum_{n=1}^{\infty} \alpha_n x_n$ with $\alpha = (\alpha_1, \alpha_2, \ldots) \in \ell^2$. Setting $x = e_k = (0, 0, \ldots, 0, 1, 0, \ldots)$, where 1 occurs at the kth place, in the representation of f, we obtain $\alpha_n = \frac{1}{\sqrt{n}}$, which implies $\left\{\frac{1}{\sqrt{n}}\right\}_{n \geq 1} \in \ell^2$, a contradiction.

2.10.P13. The system $\{\sin nx\}_{n \geq 1}$ is an orthogonal system in $L^2[0, \pi]$. We need to show that if $f \in L^2[0, \pi]$ and $\int_0^{\pi} f(x) \sin nx \, dx = 0$, $n = 1, 2, \ldots$, then $f = 0$ a.e.

Define a function $\tilde{f}$ as follows:

$$\tilde{f}(x) = \begin{cases} f(x) & 0 \leq x \leq \pi \\ -f(-x) & -\pi \leq x < 0 \end{cases}.$$

Since $\tilde{f}$ is an odd function, we obtain

$$\int_{-\pi}^{\pi} \tilde{f}(x) dx = 0, \int_{-\pi}^{\pi} \tilde{f}(x) \cos nx \, dx = 0, \quad n = 1, 2, \ldots.$$

Moreover,

$$\int_{-\pi}^{\pi} \tilde{f}(x) \sin nx \, dx = 2 \int_{0}^{\pi} \tilde{f}(x) \sin nx \, dx = 2 \int_0^{\pi} f(x) \sin nx \, dx = 0.$$

Since the system $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \sin nx, \frac{1}{\sqrt{2\pi}} \cos nx\}_{n \geq 1}$ is complete in $L^2[-\pi, \pi]$ as can be checked using arguments seen in Example 2.9.16(iii), we obtain $\tilde{f} = 0$ a.e. on $[-\pi, \pi]$. Consequently, $f = 0$ a.e. on $[0, \pi]$.

2.10.P14. Corollary 2.10.7 establishes the existence of the unique vector in K of the smallest norm.

Let $x \in K$ and $t \in [0, 1]$. Then, the point $(1 - t)k + tx \in K$ (which is convex), so

$$||(1-t)k+tx||^2 \geq ||k||^2,$$

i.e.

$$||k||^2 + t^2||x-k||^2 - 2t \cdot \Re(k, k-x) \geq ||k||^2,$$

which implies (on dividing by t)

$$t||x-k||^2 - 2\Re(k, k-x) \geq 0.$$

On letting $t \to 0$, we obtain

$$\Re(k, k-x) \leq 0.$$

To prove the final assertion, observe that, for $x \in K$,

$$||x||^2 = \|x-k+k\|^2 = \|k\|^2 + ||x-k||^2 + 2\Re(x-k, k) \geq ||k||^2.$$

So, k is the unique vector of the smallest norm.

2.10.P15. Define a linear functional f_y on H by

$$f_y(x) = (x, y).$$

Clearly, f_y is linear. Since the inner product is continuous, it follows that f_y is a continuous linear functional. Moreover, $\|f_y\| = \|y\|$ [Example 2.10.24(iv)], $\ker(f_y) = \{x \in H : f_y(x) = (x, y) = 0\} = M$. Using Problem 2.3.P10, $\dim((\ker(f))^{\perp}) = 1$. Consequently, $M^{\perp} = \mathbb{C}y$.

Remark The above result holds if H is an inner product space which is not necessarily complete, as the following argument shows.

It is trivial that $M^{\perp} \supseteq \mathbb{C}y$. We claim the reverse (converse). Consider a vector x in the pre-Hilbert space such that $(y, z) = 0 \Rightarrow (x, z) = 0$. Set $z = \frac{(x,y)}{||y||^2} y - x$. Then, (by an easy computation), $(y, z) = \overline{(x, y)} - (y, x) = 0$. Therefore, $(x, z) = 0$, which leads (by an easy computation again) to $\frac{|(x,y)|^2}{||y||^2} - ||x||^2 = 0$. By the 'rider' to the Cauchy–Schwarz inequality, it follows that $x \in \mathbb{C}y$.

Alternatively, use Remark (i) after Problem 2.10.P10.

6.10 Problem Set 2.11

2.11.P1. Let u_0, u_1, and u_2 be the orthonormal vectors obtained on orthonormalising 1, t, t^2 respectively. Then,

$$u_0(t) = y_0(t)/\|y_0\| = \frac{1}{\sqrt{2}}, \quad \text{where } y_0(t) = 1 \text{ and } \|y_0\| = \left(\int_{-1}^{1} 1\,\mathrm{d}t\right)^{\frac{1}{2}} = \sqrt{2},$$

$$y_1(t) = t - (t, u_0)u_0 = t - \left(\int_{-1}^{1} \frac{t}{\sqrt{2}}\mathrm{d}t\right)\frac{1}{\sqrt{2}} = t,$$

$$\|y_1\| = \left(\int_{-1}^{1} t^2\mathrm{d}t\right)^{\frac{1}{2}} = \left(\frac{2}{3}\right)^{\frac{1}{2}}.$$

So,

$$u_1(t) = y_1(t)/\|y_1\| = \left(\frac{3}{2}\right)^{\frac{1}{2}} t,$$

$$y_2(t) = t^2 - \left(\int_{-1}^{1} \frac{t^2}{\sqrt{2}}\mathrm{d}t\right)\frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} \left(\frac{3}{2}\right)^{\frac{1}{2}} t^3\mathrm{d}t\right)\left(\frac{3}{2}\right)^{\frac{1}{2}} t$$
$$= t^2 - \frac{1}{3}$$

and

$$\|y_2\| = \left(\int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 \mathrm{d}t\right)^{\frac{1}{2}} = \frac{2}{15}\sqrt{10}, \text{ so}$$

$$u_2(t) = y_2(t)/\|y_2\| = \frac{15}{2\sqrt{10}}\left(t^2 - \frac{1}{3}\right) = \frac{3}{4}\sqrt{10}\cdot\left(t^2 - \frac{1}{3}\right).$$

Let $x(t) = t^3$. Then,

$$(x, u_0) = \int_{-1}^{1} \frac{t^3}{\sqrt{2}}\mathrm{d}t = 0,$$

$$(x, u_1) = \int_{-1}^{1} t^3\left(\left(\frac{3}{2}\right)^{\frac{1}{2}} t\right)\mathrm{d}t = \left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{2}{5},$$

$$(x, u_2) = \int_{-1}^{1} t^3 \cdot \frac{3}{4}\sqrt{10}\cdot\left(t^2 - \frac{1}{3}\right)\mathrm{d}t = 0.$$

By Theorem 2.11.1,

$$\min_{a,b,c} \int_{-1}^{1} \left|t^3 - a - bt - ct^2\right|^2 \mathrm{d}t = ||x||^2 - |(x, u_0)|^2 - |(x, u_1)|^2 - |(x, u_2)|^2, \text{ where } x(t) = t^3,$$
$$= \frac{2}{7} - \frac{3}{2}\frac{4}{25} = \frac{2}{7} - \frac{6}{25} = \frac{8}{175}.$$

Using Problem 2.10.P7, it follows that max $\int_{-1}^{1} t^3 g(t)\mathrm{d}t$, where $g(t)$ satisfies the given restrictions, equals $\frac{8}{175}$.

2.11.P2. Let $x = (1, -1, 1)$. The orthonormal vectors corresponding to $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ are $u_1 = \frac{1}{\sqrt{3}}(1, \omega, \omega^2)$ and $u_2 = \frac{1}{\sqrt{3}}(1, \omega^2, \omega)$, respectively. Then, the required vector is

$$\begin{aligned}
(x, u_1)u_1 + (x, u_2)u_2 &= \frac{1}{3}(1 - \overline{\omega} + \overline{\omega}^2)(1, \omega, \omega^2) + \frac{1}{3}(1 - \overline{\omega}^2 + \overline{\omega})(1, \omega^2, \omega) \\
&= \frac{1}{3}\{\left(1 - \overline{\omega} + \overline{\omega}^2, \omega - \omega\overline{\omega} + \omega\overline{\omega}^2, \omega^2 - \omega^2\overline{\omega} + \omega^2\overline{\omega}^2\right) \\
&\quad + \left(1 - \overline{\omega}^2 + \overline{\omega}, \omega^2 - \omega^2\overline{\omega}^2 + \omega^2\overline{\omega}, \omega - \omega\overline{\omega}^2 + \omega\overline{\omega}\right)\} \\
&= \frac{1}{3}(2, \omega + \omega^2 - \omega\overline{\omega} - \omega^2\overline{\omega}^2 + \omega\overline{\omega}^2 + \omega^2\overline{\omega}, \omega^2 \\
&\quad + \omega - \omega^2\overline{\omega} - \omega\overline{\omega}^2 + \omega^2\overline{\omega}^2 + \omega\overline{\omega}) \\
&= \frac{1}{3}(2, -1 - 2 - 1, -1 + 1 + 1 + 1) \\
&= \frac{1}{3}(2, -4, 2) = \frac{2}{3}(1, -2, 1).
\end{aligned}$$

6.11 Problem Set 2.12

2.12.P1.

$$\begin{aligned}
||x_n - x||^2 &= (x_n - x, x_n - x) = -(x_n, x) + (x, x) - (x, x_n) + (x_n, x_n) \\
&\leq |(x_n, x) - (x, x)| + |(x, x_n) - (x, x)| + |(x_n, x_n) - (x, x)| \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

since $(x_n, x) \to (x, x)$ and $(x_n, x_n) \to (x, x)$.

Problems 2.12.P2, 2.12.P3 and 2.12.P4 relate to weak convergence, discussed in Chap. 2.

2.12.P2. Without loss of generality, we may assume that $\{x_n\}_{n \geq 1}$ converges weakly to 0. Indeed,

$$(x_n, y) \to (x, y) \text{ if, and only if, } (x_n - x, y) \to 0.$$

We next extract a subsequence $\{x_{n_k}\}_{k \geq 1}$ of $\{x_n\}_{n \geq 1}$ satisfying the following: For $k \geq 2$,

$$|(x_{n_1}, x_{n_k})| \leq \frac{1}{k}, |(x_{n_2}, x_{n_k})| \leq \frac{1}{k}, \ldots, |(x_{n_{k-1}}, x_{n_k})| \leq \frac{1}{k}.$$

Let $x_{n_1} = x_1$. Assume that $\{x_{n_2}, x_{n_3}, \ldots, x_{n_{k-1}}\}$, where $n_2 < n_3 < \cdots < n_{k-1}$, have been selected. Let $\varepsilon = \frac{1}{k}$. Since $\{x_n\}$ converges weakly to zero, there exist $n_k > n_{k-1}$ such that $|(y, x_{n_k})| \leq \frac{1}{k}$ for $y = x_{n_1}, x_{n_2}, \ldots, x_{n_{k-1}}$. In particular,

$$\left|(x_{n_j}, x_{n_k})\right| \leq \frac{1}{k} \quad \text{for } j = 1, 2, \ldots, k-1.$$

Now,

$$\begin{aligned}\|y_k\|^2 &= \frac{1}{k^2}\left(\sum_{j=1}^{k} x_{n_j}, \sum_{j=1}^{k} x_{n_j}\right) \leq \frac{1}{k^2}\sum_{j=1}^{k} \left\|x_{n_j}\right\|^2 + \frac{2}{k^2}\sum_{1 \leq l < m \leq k} |(x_{n_l}, x_{n_m})| \\ &\leq \frac{kM^2}{k^2} + \frac{2}{k^2}(k-1) < \frac{M^2+2}{k} \to 0 \text{ as } k \to \infty,\end{aligned}$$

where M is the bound of the sequence [see Theorem 4.1.6]. Thus, $y_k \to 0$ strongly.

2.12.P3. (a) Let C be the closed convex hull of $\{x_n : n \geq 1\}$. By Problem 2.12.P2, there exists a subsequence $\{x_{n_k}\}_{k \geq 1}$ of the sequence $\{x_n\}_{n \geq 1}$ such that $\{y_k\}_{k \geq 1}$, where $y_k = \frac{1}{k}(x_{n_1} + x_{n_2} + \cdots + x_{n_k})$, converges to x strongly. Since $\{y_k\}_{k \geq 1}$ is in C, which is closed, the limit x is in C.

(b) Suppose C is closed and let $\{x_n\}_{n \geq 1}$ be a weakly convergent sequence of elements in C. By part (a), the limit is in C.

Conversely, let y be a limit point of C. Then, there exists a sequence $\{y_n\}_{n \geq 1}$ in C such that $\lim_n y_n = y$. Since strong convergence implies weak convergence, it follows that $y_n \xrightarrow{w} y$. By hypothesis, $y \in C$. [The reader will observe that the convexity hypothesis has not been used.]

2.12.P4. (a) Observe that the series on the right of (1) converges. In fact,

$$2^{-n}|(x-y, e_n)| \leq 2^{-n}\|x-y\|$$

and the series $\sum_{n=1}^{\infty} 2^{-n}$ converges.

It is immediate that $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$. Also, $d(x, y) = d(y, x)$. For $x, y, z \in B$,

$$d(x,z) \le d(x,y) + d(y,z).$$

Indeed,

$$\begin{aligned}|(x-z,e_n)| &= |(x-y+y-z,e_n)| \\ &\le |(x-y,e_n)| + |(y-z,e_n)|\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{k} 2^{-n}|(x-z,e_n)| &\le \sum_{n=1}^{k} 2^{-n}|(x-y,e_n)| + \sum_{n=1}^{k} 2^{-n}|(y-z,e_n)| \\ &\le \sum_{n=1}^{\infty} 2^{-n}|(x-y,e_n)| + \sum_{n=1}^{\infty} 2^{-n}|(y-z,e_n)| \\ &= d(x,y) + d(y,z).\end{aligned}$$

So, $d(x, z) \le d(x, y) + d(y, z)$.

(b) Suppose $x_k \xrightarrow{w} x$. Observe that $2^{-n}|(x_k - x, e_n)| \le 2^{-n+1}$. Hence, by the Weierstrass M-test, the series defining $d(x_k, x)$ is uniformly convergent in k. So, for $\varepsilon > 0$, there exists an integer n_0 such that $\sum_{n=n_0+1}^{\infty} 2^{-n}|(x_k - x, e_n)| < \frac{\varepsilon}{2}$ for all k. Using this ε and n_0 and using the given weak convergence, we obtain an integer k_0 such that $k \ge k_0$ implies $\sum_{n=1}^{n_0} |(x_k - x, e_n)| < \frac{\varepsilon}{2}$. So,

$$\sum_{n=1}^{\infty} 2^{-n}|(x_k - x, e_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } k \ge k_0;$$

i.e. $d(x_k, x) \to 0$ as $k \to \infty$.

On the other hand, suppose that $d(x_k, x) \to 0$ as $k \to \infty$. Fix an arbitrary n_0 and consider any $\varepsilon > 0$. Then, there exists an integer k_0 such that

$$\sum_{n=1}^{\infty} 2^{-n}|(x_k - x, e_n)| < \varepsilon/2^{n_0} \quad \text{for all } k \ge k_0.$$

In particular,

$$|(x_k - x, e_{n_0})| < \varepsilon \quad \text{for all } k \ge k_0.$$

Since n_0 is arbitrary, the above statement holds for any finite linear combination of the e_m. Let y be an arbitrary element of H and $\varepsilon' > 0$. If z denotes a finite linear combination of the e_m satisfying $\|y - z\| < \varepsilon'/4$, then

$$|(x_k - x, y)| \le |(x_k - x, z)| + |(x_k - x, y - z)| \le |(x_k - x, z)| + \|x_k - x\|\|y - z\| < \varepsilon'$$

for sufficiently large k, using the fact that x_k and x are in B. This means $(x_k - x, y) \to 0$, i.e. $x_k \xrightarrow{w} x$.

Remark The 'only if' part of the above result is not valid if B is replaced by the entire Hilbert space. Counterexample: Let $x_k = 2^{k/2}e_k$ and $x = 0$. Then, $d(x_k, x) = \sum_{n=1}^{\infty} 2^{-n}|(x_k - x, e_n)| = \sum_{n=1}^{\infty} 2^{-n}\left|(2^{k/2}e_k, e_n)\right| = 2^{-k/2} \to 0$ as $k \to \infty$. But $\|x_k\| = 2^{k/2} \to \infty$ as $k \to \infty$, which implies by Theorem 4.1.6 that $\{x_k\}$ cannot converge weakly.

(c) [This is a direct consequence of (b) and Theorem 4.1.5, but here is a direct poof.] Let $\{x_n\}_{n\geq 1}$ be an infinite sequence in B. We shall extract a subsequence which converges to a point in B. Since $|(x_k, e_1)| \leq \|x_k\| \cdot \|e_1\|$ for each k, the sequence $\{(x_k, e_1)\}_{k\geq 1}$ is bounded. Hence, there exists a subsequence $x_{11}, x_{12}, x_{13}, \ldots$ of $\{x_n\}_{n\geq 1}$ such that $\{(x_{1k}, e_1)\}_{k\geq 1}$ converges. Again, $\{(x_{1k}, e_2)\}_{k\geq 1}$ is a bounded sequence, and hence, there is a subsequence $\{x_{2k}\}$ of the sequence$\{x_{1k}\}$ such that $\{(x_{2k}, e_1)\}$ and $\{(x_{2k}, e_2)\}$ converge. This procedure may now be repeated inductively so that at the kth step, we obtain $\{x_{jk}\}$, a subsequence of $\{x_{(j-1)k}\}$, such that $\{(x_{jk}, e_\alpha)\}$ converges when e_α is in the set $\{e_1, e_2, \ldots, e_j\}$. Now, arrange all these subsequences in an infinite array

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \tag{6.22}$$

having the property that the sequence in any row except the first is a subsequence of the sequence in the preceding row, and in addition, $\{(x_{nk}, e_\alpha)\}$ converges when e_α is in the set $\{e_1, e_2, \ldots, e_n\}$. Now, consider the sequence

$$x_{11}, x_{22}, x_{33}, \ldots.$$

on the diagonal of the array (6.22). For $n \geq n_0$, these x_{nn} form a subsequence of the sequence in the n_0th row. Hence, the diagonal sequence

$$x_{11}, x_{22}, x_{33}, \ldots \tag{6.23}$$

is such that $\{(x_{nn}, e_\alpha)\}$ converges when e_α is in the set $\{e_1, e_2, \ldots, e_n\}$. Since n_0 is arbitrary, we see that the diagonal sequence (6.23) is such that $\{(x_{nn}, e_\alpha)\}$ converges when e_α is any member of the orthonormal basis. Consequently, $x_{nn} \xrightarrow{w} x$, say. Since

$$d(x_{mm}, x) = \sum_{n=1}^{\infty} 2^{-n}|(x_{mm} - x, e_n)|,$$

it follows that $\{x_{mm}\}_{m\geq 1}$ converges to x in the metric, and since B is closed in the metric, it follows that $x \in B$. Thus, B is compact.

6.12 Problem Set 3.2

3.2.P1.

$$\begin{aligned}\left\|T(\{x_i\}_{i\geq 1})\right\|^2 &= \sum_{i=1}^{\infty}|y_i|^2 = \sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty}\tau_{i,j}\,x_j\right|^2 \\ &\leq \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|\tau_{i,j}||x_j|\right)^2 \\ &\leq \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|\tau_{i,j}|^2\sum_{j=1}^{\infty}|x_j|^2\right) \\ &= \sum_{j=1}^{\infty}|x_j|^2\sum_{i,j=1}^{\infty}|\tau_{i,j}|^2 \\ &= K^2||x||^2.\end{aligned}$$

So,

$$||T|| \leq K.$$

3-2.P2. For $x \in H$,

$$x = \sum_{j=1}^{\infty} x_j e_j, \tag{6.24}$$

where

$$x_j = (x, e_j), \quad j = 1, 2, \ldots$$

and

$$Tx = \sum_{i=1}^{\infty}(Tx, e_i)e_i. \tag{6.25}$$

Since T is bounded, it follows from the first relation that

$$Tx = \sum_{j=1}^{\infty} x_j Te_j = \sum_{j=1}^{\infty}(x, e_j)Te_j. \tag{6.26}$$

Substituting on the right-hand side of (6.26) the value of Te_j from (6.25), we obtain

$$Tx = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(x, e_j\right)\left(Te_j, e_i\right)e_i.$$

Set $\tau_{i,j} = (Te_j, e_i)$. Thus, if $x = \sum_{i=1}^{\infty} x_i e_i$ and $y = \sum_{i=1}^{\infty} y_i e_i$ and $y = Tx$, we get that $y_i = \sum_{j=1}^{\infty}\left(\tau_{i,j}, x_j\right)$. Hence, the sequence $\{x_i\}_{i\geq 1}$ is mapped by T to $\{y_i\}_{i\geq 1}$, where $y_i = \sum_{j=1}^{\infty}\left(Te_j, e_i\right)x_j$, $i = 1, 2, \ldots$. Consequently, T is defined by the matrix $[(Te_j, e_i)]_{i,j\geq 1}$.

Remark The matrix representing the bounded linear operator T may fail to satisfy the condition of Problem 3.2.P1. Indeed, if $I{:}H \to H$ is the identity operator, the matrix representing I is such that

$$\tau_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Consequently, $\sum_{i,j=1}^{\infty}\left|\tau_{i,j}\right|^2 = \infty$.

The next problem gives sufficient conditions under which a matrix $[\tau_{i,j}]$ defines a bounded operator on H with respect to an orthonormal basis $\{e_i\}_{i\geq 1}$ in H.

3.2.P3. Fix $x \in H$. For each $i, j = 1, 2, \ldots$

$$\begin{aligned}\sum_{j=1}^{\infty}\left|\tau_{i,j}(x, e_j)\right| &\leq \sup_j\left|(x, e_j)\right|\sum_{j=1}^{\infty}\left|\tau_{i,j}\right| \\ &\leq \sup_j\left|(x, e_j)\right|\sup_i\sum_{j=1}^{\infty}\left|\tau_{i,j}\right| \\ &= \alpha_\infty \sup_j\left|\left(x, e_j\right)\right| \\ &= \alpha_\infty\|x\| < \infty.\end{aligned}$$

Let $f_i(x) = \sum_{j=1}^{\infty}\tau_{i,j}(x, e_j)$. Then, $\sum_{i=1}^{\infty} f_i(x)e_i$ converges in H. Indeed,

$$\begin{aligned}\left\|\sum_{i=1}^{\infty} f_i(x)e_i\right\|^2 &= \sum_{i=1}^{\infty}|f_i(x)|^2 \\ &\leq \sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty}\tau_{i,j}\left(x, e_j\right)\right|^2 \\ &\leq \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|\tau_{i,j}|^{\frac{1}{2}}|\tau_{i,j}|^{\frac{1}{2}}\left|\left(x, e_j\right)\right|\right)^2 \\ &\leq \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|\tau_{i,j}|\right)\left(\sum_{j=1}^{\infty}|\tau_{i,j}|\left|\left(x, e_j\right)\right|^2\right) \\ &\leq \alpha_\infty\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\tau_{i,j}|\left|\left(x, e_j\right)\right|^2 \\ &\leq \alpha_1\alpha_\infty\sum_{j=1}^{\infty}\left|\left(x, e_j\right)\right|^2 = \alpha_1\alpha_\infty\|x\|^2.\end{aligned}$$

So, if we set

$$Tx = \sum_{i=1}^{\infty} f_i(x)e_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tau_{i,j}(x, e_j)e_i,$$

then T is a bounded linear operator of norm not exceeding $\sqrt{(}\alpha_1\alpha_\infty)$.

Remark The above conditions, namely $\alpha_1 < \infty$ and $\alpha_\infty < \infty$, are not necessary for a matrix to define a bounded linear operator on H. Define

$$\tau_{i,j} = \begin{cases} \frac{1}{j} & \text{if } i = 1 \text{ or } j = 1 \\ 0 & \text{if } i,j > 1 \end{cases}$$

The matrix $[\tau_{i,j}]_{i,j\geq 1}$, when written in full, is

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In this case, $\alpha_1 = \sup_j \sum_{i=1}^{\infty} |\tau_{i,j}| = \infty = \alpha_\infty = \sup_i \sum_{i=1}^{\infty} |\tau_{i,j}|$. Since $\sum_{i,j=1}^{\infty} |\tau_{i,j}|^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} + \sum_{i=2}^{\infty} \frac{1}{i^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} - 1 < \infty$, it follows from Problem 3.2.P1 that $[\tau_{i,j}]_{i,j\geq 1}$ defines a bounded linear operator.

3.2.P4. If $\{\xi_n\}_{n\geq 1}$ is a finitely nonzero sequence of complex numbers (i.e. $\xi_n = 0$ for n sufficiently large), then

$$\begin{aligned}
\sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} \tau_{i,j}\xi_j \right|^2 &= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} (\sqrt{\tau}_{i,j}\sqrt{p_j}) \frac{\sqrt{\tau_{i,j}}\xi_j}{\sqrt{p_j}} \right|^2 \\
&\leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \tau_{i,j}p_j \right) \left(\sum_{j=1}^{\infty} \frac{\tau_{i,j}|\xi_j|^2}{p_j} \right) \\
&\leq \sum_{i=1}^{\infty} \alpha_\infty p_i \left(\sum_{j=1}^{\infty} \frac{\tau_{i,j}|\xi_j|^2}{p_j} \right) \\
&\leq \alpha_\infty \sum_{j=1}^{\infty} \frac{|\xi_j|^2}{p_j} \sum_{i=1}^{\infty} \tau_{i,j}p_j \\
&\leq \alpha_\infty \sum_{j=1}^{\infty} \frac{|\xi_j|^2}{p_j} \alpha_1 p_j \\
&\leq \alpha_1\alpha_\infty \sum_{j=1}^{\infty} |\xi_j|^2.
\end{aligned}$$

These inequalities imply that the operator T on ℓ^2 defined by

$$T(\xi_1, \xi_2, \ldots) = \left(\sum_{j=1}^{\infty} \tau_{1,j}\xi_j, \sum_{j=1}^{\infty} \tau_{2,j}\xi_j, \ldots \right)$$

satisfies the condition on a dense subset of ℓ^2 and hence on all of ℓ^2.

3.2.P5. Apply Problem 3.2.P4 with $p_i = \frac{1}{\sqrt{i-\frac{1}{2}}}$. Since the Hilbert matrix is symmetric, it is enough to verify one of the two inequalities of Problem 3.2.P4.

$$\begin{aligned}
\sum_{i=1}^{\infty} \tau_{i,j}p_i &= \sum_{i=1}^{\infty} \frac{1}{(i-\frac{1}{2})+(j-\frac{1}{2})} \frac{1}{\sqrt{i-\frac{1}{2}}} \\
&< \int_0^{\infty} \frac{\mathrm{d}x}{(x+j-\frac{1}{2})\sqrt{x}} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}x}{(x^2+j-\frac{1}{2})} \\
&= \int_0^{\infty} \frac{2\mathrm{d}x}{x^2+(\sqrt{j-\frac{1}{2}})^2} \\
&= \frac{2}{\sqrt{j-\frac{1}{2}}} \tan^{-1} \frac{x}{\sqrt{j-\frac{1}{2}}}\Bigg|_0^{\infty} = \frac{\pi}{\sqrt{j-\frac{1}{2}}}.
\end{aligned}$$

So, $\sum_{i=1}^{\infty} \tau_{i,j}\, p_i \leq \pi p_j$. This completes the argument.

3.2.P6. Define

$$Tx = \sum_{n=1}^{\infty} \alpha_n (x, e_n) e_n, \text{ where } x = \sum_{n=1}^{\infty} (x, e_n) e_n \in \ell^2.$$

Clearly, T is linear. Moreover,

$$\begin{aligned}
||Tx||^2 &= \left(\sum_{n=1}^{\infty} \alpha_n (x, e_n) e_n, \sum_{n=1}^{\infty} \alpha_n (x, e_n) e_n \right) \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 |(x, e_n)|^2 \\
&\leq \sup_n |\alpha_n|^2 \sum_{n=1}^{\infty} |(x, e_n)|^2 = \sup_n |\alpha_n|^2 \|x\|^2.
\end{aligned}$$

Thus, T is a bounded linear operator on ℓ^2 and $\|T\| \le \sup_n|\alpha_n|$.

Since $Te_k = \alpha_k e_k$ and $\|e_k\| = 1$, $\|T\| \ge \|Te_k\| = \|\alpha_k e_k\| = |\alpha_k|$ for all k; therefore, $\|T\| \ge \sup_n|\alpha_n|$.

Suppose T is a bounded linear operator on ℓ^2 such that $Te_k = \alpha_k e_k$. Then, $|\alpha_k| = \|\alpha_k e_k\| = \|Te_k\| \le \|T\|$, which implies $\sup_k|\alpha_k| \le \|T\|$, i.e. the sequence $\{\alpha_n\}_{n\ge 1}$ is bounded.

3.2.P7. We estimate $|y(s)|$ by the Cauchy–Schwarz inequality:

$$\begin{aligned}|y(s)|^2 &\le \left(\int_0^\infty |x(t)|e^{-st}dt\right)^2 = \left(\int_0^\infty \left(|x(t)|e^{-\frac{1}{2}st}t^{\frac{1}{4}}\right)\left(e^{-\frac{1}{2}st}t^{-\frac{1}{4}}\right)dt\right)^2 \\ &\le \int_0^\infty |x(t)|^2 e^{-st}t^{\frac{1}{2}}dt \int_0^\infty e^{-st}t^{-\frac{1}{2}}dt.\end{aligned} \tag{6.27}$$

By the change of variables formula, we write the last integral as

$$\int_0^\infty e^{-st}t^{-\frac{1}{2}}dt = \int_0^\infty e^{-u}u^{-\frac{1}{2}}du \cdot s^{-\frac{1}{2}} = \alpha\, s^{-\frac{1}{2}}, \tag{6.28}$$

where

$$\alpha = \int_0^\infty e^{-u}u^{-\frac{1}{2}}du = \int_0^\infty e^{-x^2}x^{-1}2x dx = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}.$$

Substituting (6.28) into (6.27) gives

$$|y(s)|^2 \le \sqrt{\pi}\, s^{-\frac{1}{2}} \int_0^\infty |x(t)|^2 e^{-st}t^{\frac{1}{2}}dt.$$

So,

$$\|y\|^2 = \int_0^\infty |y(s)|^2 ds \le \sqrt{\pi}\int_0^\infty\int_0^\infty |x(t)|^2 e^{-st}t^{\frac{1}{2}}s^{-\frac{1}{2}}dtds. \tag{6.29}$$

Interchanging the order of integration and changing variables in the s-integral, we get

$$\int_0^\infty e^{-st}t^{-\frac{1}{2}}s^{-\frac{1}{2}}ds = \int_0^\infty e^{-u}u^{-\frac{1}{2}}du = \sqrt{\pi}.$$

Hence, we get from (6.29) that

$$\|y\|^2 \le \pi \|x\|^2;$$

this implies

$$\begin{aligned} ||L|| &= \sup_{||x||=1} ||Lx|| \\ &\le \sqrt{\pi}. \end{aligned} \tag{6.30}$$

Let $\varepsilon > 0$ be given. Using (6.28), we can choose a and b such that for $x(t) = 1/\sqrt{t}$ for $a < t < b$ and zero outside this interval, we get

$$y(s) = \int_0^\infty x(t)\mathrm{e}^{-st}\mathrm{d}t = \int_a^b \mathrm{e}^{-st}t^{-\frac{1}{2}}\mathrm{d}t > (\sqrt{\pi} - \varepsilon)s^{-\frac{1}{2}}.$$

For this choice of x,

$$||x||^2 = \ln(b/a)$$

and

$$\begin{aligned} ||Lx||^2 = \|y\|^2 &= \int_0^\infty |y(s)|^2 \mathrm{d}s \ge \int_a^b |y(s)|^2 \mathrm{d}s > (\sqrt{\pi} - \varepsilon)^2 \ln(b/a) \\ &= (\sqrt{\pi} - \varepsilon)^2 ||x||^2. \end{aligned}$$

Combining this with (6.30), we obtain the desired result.

3.2.P8. Let $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ be the operator on $\mathbb{R}^2$. Then, $T \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$. Hence, $\left(T\begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha \\ 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \begin{bmatrix} \alpha \\ 0 \end{bmatrix}\right) = 0,$

$$||T|| = \sup_{\substack{x \in \mathbb{R}^2 \\ ||x|| \ne 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in \mathbb{R}^2 \\ ||x|| \ne 0}} \frac{\left\| \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|} = 1.$$

3.2.P9. Let $\mathrm{N} = \{x \in H\colon (S - T)(x) = 0\}$. By assumption, $\mathfrak{M} \subseteq \mathfrak{N}$. As $\mathfrak{M}$ is total, so is $\mathfrak{N}$. Moreover, $\mathfrak{N}$ is a closed subspace of H. So, $\mathfrak{N} = H$.

3.2.P10. It is necessary to show that $Kx \in L^2[0, 1]$, but this will follow from the argument that demonstrates the boundedness of K. If $x \in L^2[0, 1]$,

$$
\begin{aligned}
|(Kx)(s)| &\le \int_0^1 |k(s,t)||x(t)|\mathrm{d}t \\
&= \int_0^1 |k(s,t)|^{\frac{1}{2}}|k(s,t)|^{\frac{1}{2}}|x(t)|\mathrm{d}t \\
&\le \left(\int_0^1 |k(s,t)|\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^1 |k(s,t)||x(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le 2^{\frac{1}{2}}\left(\int_0^1 |k(s,t)||x(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}
\end{aligned}
$$

since

$$
\int_0^1 |k(s,t)|\mathrm{d}t = \int_0^s |k(s,t)|\mathrm{d}t + \int_s^1 |k(s,t)|\mathrm{d}t = \int_0^s |k(s,t)|\mathrm{d}t
$$

$$
= \int_0^s \frac{1}{\sqrt{s-t}}\mathrm{d}t = \ 2\mathrm{s}^{\frac{1}{2}} < 2.
$$

Now,

$$
\int_0^1 |(Kx)(s)|^2\mathrm{d}s = 2\int_0^1\int_0^1 |k(s,t)||x(t)|^2\mathrm{d}t\mathrm{d}s = 2\int_0^1 |x(t)|^2\mathrm{d}t\int_0^1 |k(s,t)|\mathrm{d}s \le 4\|x\|_2^2,
$$

since

$$
\int_0^1 |k(s,t)|\mathrm{d}s = \int_0^t |k(s,t)|\mathrm{d}s + \int_t^1 |k(s,t)|\mathrm{d}s = \int_t^1 \frac{1}{\sqrt{s-t}}\mathrm{d}s = 2(1-t)^{\frac{1}{2}} < 2.
$$

3.2.P11. Suppose that $\{a_i\}_{i\geq 1}$ is bounded. Clearly, D_a is a linear operator on ℓ^2.

$$\begin{aligned}||D_a x|| &= \left(\sum_{i=1}^{\infty}|a_i x_i|^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty}|a_i|^2|x_i|^2\right)^{\frac{1}{2}}\\ &\le \sup|a_i|\left(\sum_{i=1}^{\infty}|x_i|^2\right)^{\frac{1}{2}}\\ &\le M||x||,\end{aligned}$$

where $M = \sup|a_i|$.

Consider the vector $e_i = (0, 0, \ldots, 0, 1.0, 0 \ldots)$, where the 1 appears at the ith place. Clearly, $\|e_i\| = 1$ and $\|D_a e_i\| = |a_i|$. It follows that $\|D_a\| = \sup_{\|x\|=1}\|D_a x\| \ge \sup_i\|D_a e_i\| = \sup|a_i|$.

Assume that D_a is bounded. Then,

$$|a_i| = ||D_a e_i|| \le ||D_a||.$$

So,

$$\sup|a_i| \le ||D_a||.$$

Consequently, $\{a_i\}_{i\ge 1}$ is bounded.

3.2.P12. It is clear that A is linear. It suffices to show that A is bounded and that it has the norm claimed for it.

$$||\langle u_1, u_2\rangle||^2 = (\langle u_1, u_2\rangle, \langle u_1, u_2\rangle) = (u_1, u_1)_{H_1} + (u_2, u_2)_{H_2} = ||u_1||^2 + ||u_2||^2.$$

Thus,

$$||A\langle u_1, u_2\rangle||^2 = ||A_1 u_1||^2 + ||A_2 u_2||^2 \le ||A_1||^2||u_1||^2 + ||A_2||^2||u_2||^2.$$

Hence,

$$||A|| \le \max\{||A_1||, ||A_2||\}.$$

Consequently, $A \in \mathcal{B}(H)$.

If $\|A_1\| > \|A_2\|$, then for all $\varepsilon > 0$, there exists $x \in H$, such that $\|A_1 x\| > (1 - \varepsilon)\|x\|\|A_1\|$. Now, considering the vector $\langle x, 0\rangle$, we get

$$||A\langle x, 0\rangle|| = ||A_1 x|| \ge (1-\varepsilon)||x||\,||A_1|| = (1-\varepsilon)||\langle x, 0\rangle||\,||A_1||.$$

Thus, $\|A\| \ge \|A_1\|$. Similarly, $\|A\| \ge \|A_2\|$, and hence, $\|A\| \ge \max\{\|A_1\|, \|A_2\|\}$.

3.2.P13. For $x, y \in \ell^2(\mathbb{Z})$, $x = \{\xi_j\}_{j\in\mathbb{Z}}$ and $y = \{\eta_j\}_{j\in\mathbb{Z}}$, $(x, y) = \sum_{j=-\infty}^{\infty}\xi_j\overline{\eta}_j$. Note that

$$|(x,y)| = \left|\sum_{j=-\infty}^{\infty} \xi_j \overline{\eta}_j\right| \le \sum_{j=-\infty}^{\infty} |\xi_j||\eta_j| \le \left(\sum_{j=-\infty}^{\infty} |\xi_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=-\infty}^{\infty} |\eta_j|^2\right)^{\frac{1}{2}} < \infty.$$

$$||Sx||^2 = \sum_{j=-\infty}^{\infty} \left|\xi_{j-1}\right|^2 = \sum_{j=-\infty}^{\infty} \left|\xi_j\right|^2 = \|x\|^2.$$

For any $x \in \ell^2(\mathbb{Z})$, we have

$$\begin{aligned}
&Sx = S\Big(\{\xi_j\}_{j\in\mathbb{Z}}\Big) = \{\xi_{j-1}\}_{j\in\mathbb{Z}},\\
&S^2x = S(\{\xi_{j-1}\}_{j\in\mathbb{Z}}) = \{\xi_{j-2}\}_{j\in\mathbb{Z}},\\
&\dots\\
&S^n = \{\xi_{j-n}\}_{j\in\mathbb{Z}},\\
&\dots.
\end{aligned}$$

If $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$,

$$\left(S^n e_j, e_i\right) = \left(e_{j-n}, e_i\right) = \delta_{j-n,i}, \quad \text{where } \delta_{i,j} = 1 \text{ if } i = j \text{ and } 0 \text{ otherwise.}$$

Hence, the matrix representation of S^n in this basis has the form

$$S^n = \begin{bmatrix}
\ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\cdots & 1 & 0 & \cdots & 0 & \cdots \\
\cdots & 0 & 1 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cdots & 0 & 0 & \cdots & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{bmatrix},$$

where each of the 1's has column number that exceeds the row number by n.

6.13 Problem Set 3.3

3.3.P1. For $x = \{x_i\}_{1\le i\le 3} \in H^{(3)}$, the norm of x is defined by

$$||x|| = \left(\sum_{i=1}^{3} ||x_i||^2\right)^{\frac{1}{2}}.$$

Now,

$$\begin{aligned} Tx &= \begin{bmatrix} 0 & T_3 & T_1 \\ 0 & 0 & T_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= (T_3x_2 + T_1x_3, T_2x_3, 0). \end{aligned}$$

So,

$$\begin{aligned} ||Tx|| &= \left(||T_3x_2 + T_1x_3||^2 + ||T_2x_3||^2 \right)^{\frac{1}{2}} \\ &\leq \|T_3x_2 + T_1x_3\| + \|T_2x_3\| \\ &\leq M||x||, \end{aligned}$$

where $M = 2\sup_{1 \leq i \leq 3}\|T_i\|$.

Thus, T is a bounded linear operator on $H^{(3)}$. Note that

$$T^2x = \begin{bmatrix} 0 & T_3 & T_1 \\ 0 & 0 & T_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_3x_2 + T_1x_3 \\ T_2x_3 \\ 0 \end{bmatrix} = (T_3T_2x_3, 0, 0)$$

and

$$T^3x = \begin{bmatrix} 0 & T_3 & T_1 \\ 0 & 0 & T_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_3T_2x_3 \\ 0 \\ 0 \end{bmatrix} = (0, 0, 0).$$

Thus, $T^3 = O$. Therefore,

$$(I - \alpha T)(I + \alpha T + \alpha^2 T^2) = I - \alpha^3 T^3 = I.$$

Also,

$$(I + \alpha T + \alpha^2 T^2)(I - \alpha T) = I.$$

Thus, for any $\alpha \in \mathbb{C}$, the operator $(I - \alpha T)^{-1}$ exists and equals $I + \alpha T + \alpha^2 T^2$.

3.3.P2. The systems of equations (a) and (b) are equivalent to the equation $Tx = y$, where $T = I - M$ with

$$M = \begin{bmatrix} 0 & \mu_1 & 0 & 0 & \cdots \\ 0 & 0 & \mu_2 & 0 & \cdots \\ 0 & 0 & 0 & \mu_3 & \cdots \end{bmatrix}$$

in (a) and

$$M = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \mu_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

in (b). In both cases, we have $\|M\| = \sup_k|\mu_k| < 1$, and hence, $I - M$ is invertible and each of the systems has a unique solution. If $\eta_k = \delta_{1k}$, $\mu_k = \frac{1}{2^{k-1}}$, then in case (a),

$$\xi_k - \mu_k\xi_{k+1} = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}.$$

$k = 1$, $\xi_1 - \xi_2 = 0$. Also, for $k = 1$, $\xi_k = \mu_k\xi_{k+1}$. So, $\xi_2 = \frac{1}{2}\xi_3 = \frac{1}{2}\frac{1}{2^2}\xi_4 = \frac{1}{2}\frac{1}{2^2}\frac{1}{2^3}\xi_5 = \cdots \to 0$ as $n \to \infty$. Consequently, $\xi_2 = 0$. Similarly, $\xi_3 = \xi_4 = \cdots = 0$. Hence, the solution in case (a) is $\xi_1 = 1$, $\xi_2 = \xi_3 = \cdots = 0$.

In case (b), $\xi_1 = 1$ and $\xi_k - \mu_k\xi_{k-1} = \eta_k$, $k = 2, 3, \ldots$.

Now, $\xi_2 - \frac{1}{2}\xi_1 = 0$. So, $\xi_2 = \frac{1}{2}$, $\xi_3 = \frac{1}{2^3}$, $\xi_4 = \frac{1}{2^3}\frac{1}{2^3} = \frac{1}{2^6}$, …, $\xi_k = \frac{1}{2^{k-1}}\xi_{k-1} = \frac{1}{2^{k-1}}\frac{1}{2^{k-2}}\xi_{k-2} = \frac{1}{2^{(k-1)+(k-2)}}\xi_{k-2} = \cdots = \frac{1}{2^{k(k-1)/2}}$.

3.3.P3. Suppose first that T^* is not bounded below. Then, there exists a sequence $\{x_n\}_{n\geq 1}$ in H such that $\|T^*x_n\| < \|x_n\|/n$ for $n = 1, 2, \ldots$. Let $y_n = nx_n/\|x_n\|$, so that $\|T^*y_n\| < \|y_n\|/n = n\|x_n\|/\|x_n\|(1/n) = 1$. We now show that the sequence $\{y_n\}_{n\geq 1}$ is weakly bounded in H. Consider $y \in H$. Since T is surjective, there exists an $x \in H$ such that $Tx = y$. Then,

$$|(y_n, y)| = |(y_n, Tx)| = |(T^*y_n, x)| \leq \|T^*y_n\|\|x\| < \|x\|.$$

Observe that $\|y_n\| = \sup_{\|y\|=1}|(y_n, y)| \leq \|x\|$. But $\|y_n\| = n \to \infty$. This contradiction proves that $T^*$is bounded below.

Conversely, assume that T^* is bounded below: $\alpha\|x\| \leq \|T^*x\|$ for all $x \in H$ and some $\alpha > 0$. We show that ran(T^*) is closed in H. If $T^*x_n \to y$, then the inequality

$$\alpha\|x_n - x_m\| \leq \|T^*x_n - T^*x_m\|$$

shows that $\{x_n\}$ is Cauchy in H. If $x_n \to x$, then $T^*x_n \to T^*x$. Hence, $y = T^*x \in$ ran (T^*). Thus, ran(T^*) is a Hilbert space. We next show that ran(T) = H. Let $y \in H$. Define F : ran$(T^*) \to \mathbb{C}$ by $F(T^*w) = (w, y)$, $w \in H$. The functional F is well defined since T^* is bounded below and hence injective. Clearly, F is linear. Also, for $z = T^*w$,

$$|F(z)| = |F(T^*w)| = |(w, y)| \leq \|w\|\|y\| \leq \frac{1}{\alpha}\|z\|\|y\|,$$

so that F is a bounded linear functional on ran(T^*). By the Riesz representation theorem 2.10.25, there exists a unique $x \in$ ran(T^*) such that $F(z) = (z, x)$ for all $z \in$ ran(T^*). Hence, for $w \in H$, we have

$$(w, y) = F(T^*w) = (T^*w, x) = (w, Tx).$$

Thus, $y = Tx$.

3.3.P4. Let $(I + T)x = 0$ for some $x \neq 0$ in H. Then, $-x = Tx$. Since $T \geq O$,

$$0 \leq (Tx, x) = -(x, x) = -||x||^2 \leq 0,$$

which implies $(x, x) = 0$, that is, $x = 0$. Thus, $\ker(I + T) = \{0\}$; consequently, the range of $I + T$ is dense. We next show that $I + T$ is bounded below. Moreover,

$$\|(I+T)x\|^2 = \|x\|^2 + \|Tx\|^2 + 2(Tx, x) \geq \|x\|^2 \text{ since } (Tx, x) \geq 0.$$

Therefore, $(I + T)^{-1}$ exists [Theorem 3.3.12].

6.14 Problem Set 3.4

3.4.P1. For all $x, y \in H$,

$$B(x+y, x+y) = B(x, x) + B(x, y) + B(y, x) + B(y, y) \tag{6.31}$$

Replacing y by $-y$, iy and $-iy$ yields

$$B(x-y, x-y) = B(x, x) - B(x, y) - B(y, x) + B(y, y), \tag{6.32}$$

$$B(x+iy, x+iy) = B(x, x) - iB(x, y) + iB(y, x) + B(y, y), \tag{6.33}$$

$$B(x-iy, x-iy) = B(x, x) + iB(x, y) - iB(y, x) + B(y, y). \tag{6.34}$$

Addition of (6.31) and (6.32) yields (a); an appropriate combination of (6.31–6.34) yields (b). Finally, (c) follows from (b), using (a).

Using Theorem 3.4.8, the above polarisation identity takes the form

$$4(Sx, y) = (S(x+y), x+y) - (S(x-y), x-y) + i(S(x+iy), x+iy) \\ - i(S(x-iy), x-iy)).$$

3.4.P2. Let $\{f_n\}_{n \geq 1}$ be a sequence of quadratic forms such that $f_n \to f$ pointwise, that is, $f_n(x) \to f(x)$ for each vector $x \in H$. If B_n denotes the sesquilinear form associated with f_n, $n = 1, 2, \ldots$, then for $x, y \in H$,

$$\begin{aligned} B_n(x, y) &= \frac{1}{4}(B_n(x+y, x+y) - B_n(x-y, x-y) + iB_n(x+iy, x+iy) \\ &\quad - iB_n(x-iy, x-iy)) \\ &= \frac{1}{4}(f_n(x+y) - f_n(x-y) + if_n(x+iy) - if_n(x-iy)) \end{aligned} \tag{6.35}$$

In view of the hypothesis, the right-hand side of (6.30) converges to

$$\frac{1}{4}(f(x+y)-f(x-y)+if(x+iy)-if(x-iy)). \tag{6.36}$$

As the limit of a sesquilinear form is a sesquilinear form (this can be verified easily), it follows that (6.36) defines a sesquilinear form. Hence, f defines a quadratic form.

6.15 Problem Set 3.5

3.5.P1. For $x \in H$, $x=\sum_{k=1}^{\infty}\lambda_k e_k$, say. Since $\sum_{k=1}^{\infty}|\lambda_k\mu_k e_k|^2 \le M^2\sum_{k=1}^{\infty}|\lambda_k|^2 = M^2\|x\|^2$, one can write $Tx=\sum_{k=1}^{\infty}\lambda_k\mu_k x_k$. The linearity of T results from the observation that if $x_n \to x$ and $y_n \to y$, then $\lambda x_n + \mu y_n \to \lambda x + \mu y$. Since $\|Tx\|^2 \le M^2\|x\|^2$, T is continuous and $\|T\| \le M$.

Clearly, $Te_k = \mu_k e_k$; since $\|e_k\| = 1$, $\|T\| \ge \|Te_k\| = \|\mu_k e_k\| = |\mu_k|$ for all k, and hence, $\|T\| \ge M$. This proves (a), (b) and (c).

Suppose $x=\sum_{k=1}^{\infty}\lambda_k e_k$ and $T^*x=\sum_{k=1}^{\infty}\nu_k e_k$. For all k, $\nu_k = (T^*x, e_k) = (x, Te_k) = (x,\mu_k e_k) = \overline{\mu}_k(x, e_k) = \overline{\mu}_k\lambda_k$. This proves (d) and (e).

For the proof of (f),

$T^*Te_k = |\mu_k|^2 e_k = TT^*e_k$. Since $\{e_k\}_{k\ge 1}$ is an orthonormal basis for H, the result follows in view of Problem 3.2.P9 above.

6.16 Problem Set 3.6

3.6.P1. $A = \frac{A+A^*}{2} + i\frac{A-A^*}{2i} = S+iT$, where S and T are Hermitian. Since $S = S^*$, there exists a unitary matrix U and a diagonal matrix D such that $S = UDU^*$. If D has zero at k places ($k < n$), replace these zeros by ε/k, $\varepsilon > 0$. Let D' be the diagonal matrix obtained from D by replacing the zero entries on the diagonal by ε/k. Observe that D' is nonsingular and $\|S - UD'U^*\| = \|UDU^* - UD'U^*\| = \|D - D'\| < \varepsilon$. A similar argument applies to T.

3.6.P2. [For matrix representation of T, see Linop-1.] Let $H = \mathbb{C}^2$ and A be defined by $A(x_1, x_2) = (x_2, x_1)$, where $(x_1,x_2) \in \mathbb{C}^2$. Then, $A^* = A$. Indeed,

$$\begin{aligned}(A^*(x_1,x_2),(y_1,y_2)) &= ((x_1,x_2),A(y_1,y_2))\\ &= ((x_1,x_2),(y_2,y_1))\\ &= x_1\overline{y}_2 + x_2\overline{y}_1\\ &= ((x_2,x_1),(y_1,y_2)).\end{aligned}$$

So, $A^*(x_1, x_2) = (x_2, x_1)$. Consider $e_1 = (1, 1)$ and $e_2 = (0, 1)$. Then, $\{e_1, e_2\}$ is a basis for $\mathbb{C}^2$ which is not orthonormal. Now,

$$\begin{aligned} Ae_1 &= A(1,1) = (1,1) = (1,1) + 0 \cdot (0,1), \\ Ae_2 &= A(0,1) = (1,0) = (1,1) - (0,1). \end{aligned}$$

Thus, A has the matrix representation

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Recall that $A^* = A$. Therefore, if the matrix representation of A^* were to be the conjugate transpose of the matrix representation of A, then the above matrix would have to be equal to its own conjugate transpose, which it is not.

3.6.P3. For $x, y \in X$, the following equality is easily verified:

$$\begin{aligned} (Tx, y) = \frac{1}{4}\{&(T(x+y), x+y) - (T(x-y), x-y) \\ &+ i(T(x+iy), x+iy) - i(T(x-iy), x-iy)\}. \end{aligned}$$

Since $(Tx, x) = 0$ for all $x \in X$, it follows that $(Tx, y) = 0$ for $x, y \in X$. Setting $y = Tx$, we obtain

$$\|Tx\| = 0 \quad \text{for all } x \in X, \text{ that is,}$$

$Tx = 0$ for all $x \in X$. Consequently, $T = O$.

Let $X = \mathbb{R}^2$ and T be the operator given by $T(a, b) = (-b, a)$, i.e. rotation through an angle $\frac{\pi}{2}$. Then, for an arbitrary $x = (a, b) \in \mathbb{R}^2$, we have

$$(Tx, x) = ((a,b), (-b,a)) = a(-b) + ba = 0.$$

But T is not the zero operator. (Note that if a and b were allowed to be complex, then (Tx, x) would be $a(-\overline{b}) + b\overline{a}$, which need not be 0.)

3.6.P4. $\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 + i\xi_2 \\ \xi_1 - i\xi_2 \end{bmatrix}$. So, $T = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$ and $T^* = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, $T^*x = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 + \xi_2 \\ -i\xi_1 + i\xi_2 \end{bmatrix}$,

$$T^*T = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = TT^*,$$

$$T_1 = \frac{1}{2}\left(\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & 1+i \\ 1-i & 0 \end{bmatrix},$$

$$T_2 = \frac{1}{2i}\left(\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\right) = \frac{1}{2i}\begin{bmatrix} 0 & i-1 \\ 1+i & -2i \end{bmatrix}.$$

3.6.P5. Suppose $\|A\| = r < R$. Consider the sequence $\{\sum_{k=0}^{\infty} a_k A^k\}_{n\geq 0}$. Since

$$\left\|\sum_{k=0}^{n} a_k A^k\right\| \leq \sum_{k=0}^{n} |a_k| \|A\|^k = \sum_{k=0}^{n} |a_k| r^k,$$

and the sequence of scalars $\left\{\sum_{k=0}^{n} |a_k| r^k\right\}_{n\geq 0}$ is convergent by hypothesis, it follows that the series $\sum_{k=0}^{\infty} a_k A^k$ converges absolutely to some element $T \in \mathcal{B}(H)$. Also, for x, $y \in H$,

$$\begin{aligned}(Tx, y) &= \left(\lim_{n\to\infty} \sum_{k=0}^{n} a_k A^k x, y\right) \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left(A^k x, y\right).\end{aligned}$$

Uniqueness is trivial.

Suppose $BA = AB$. Then, $(\sum_{k=0}^{n} a_k A^k)B = B(\sum_{k=0}^{n} a_k A^k)$. The result now follows by taking limits on both sides.

3.6.P6. For $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $H \oplus H$, where x_1, x_2, y_1, y_2 are in H, we have

$$\begin{aligned}(Bx, y) &= \left(\begin{bmatrix} 0 & iA \\ -iA^* & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right), \\ &= (iAx_2, y_1) + (-iA^*x_1, y_2) \\ &= (x_2, -iA^*y_1) + (x_1, iAy_2) \\ &= \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} 0 & iA \\ -iA^* & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right), \\ &= (x, By).\end{aligned}$$

Moreover,

$$\begin{aligned}\|Bx\|^2 &= \|iAx_2\|^2 + \|-iA^*x_1\|^2 \\ &\leq \max\{\|A\|, \|A^*\|\}^2 \|x\|^2 = \|A\|^2 \|x\|^2,\end{aligned}$$

and hence,

$$\|B\| \leq \|A\|. \tag{6.37}$$

For the reverse inequality, consider the vector

$$\tilde{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}.$$

For this vector,

$$\|B\tilde{x}\| = \|Ax_2\| \leq \|B\|\|\tilde{x}\| = \|B\|\|x_2\|$$

and hence,

$$\|A\| \leq \|B\|. \tag{6.38}$$

On combining (6.37) and (6.38), we obtain the desired inequality.

3.6.P7. Suppose $T + T^* \geq O$. Then, for $x \in H$,

$$\begin{aligned}\|(T+I)x\|^2 &= (Tx+x, Tx+x) \\ &= \|Tx\|^2 + (Tx,x) + (x,Tx) + \|x\|^2 \\ &= \|Tx\|^2 + ((T+T^*)x,x) + \|x\|^2 \\ &\geq \|x\|^2, \text{ using the hypothesis.}\end{aligned}$$

Since $T^* + (T^*)^* = T + T^* \geq O$, we may replace T by T^*, thus obtaining

$$\|(T+I)^*x\|^2 \geq \|x\|^2.$$

These two inequalities show that $T + I$ and $(T + I)^*$ are bounded below. By Theorem 3.5.9, it follows that $(T + I)$ is invertible.

Again, for $x \in H$,

$$\begin{aligned}\|(T-I)x\|^2 &= (Tx-x, Tx-x) \\ &= \|Tx\|^2 - (Tx,x) - (x,Tx) + \|x\|^2 \\ &= \|Tx\|^2 - ((T+T^*)x,x) + \|x\|^2 \\ &\leq \|Tx\|^2 + ((T+T^*)x,x) + \|x\|^2 \\ &\leq \|(T+I)x\|^2,\end{aligned}$$

which implies

$$\left\|(T-I)(T+I)^{-1}y\right\| \leq \left\|(T+I)(T+I)^{-1}y\right\| = \|y\|.$$

Hence,

$$\left\|(T-I)(T+I)^{-1}\right\| \le 1.$$

On the other hand, assume that $(T + I)^{-1}$ exists and $\|(T - I)(T + I)^{-1}\| < 1$. Observe that for $x \in H$,

$$\begin{aligned}\|(T-I)x\| &= \left\|(T-I)(T+I)^{-1}(T+I)x\right\| \\ &\le \left\|(T-I)(T+I)^{-1}\right\|\|(T+I)x\| \\ &\le \|(T+I)x\|.\end{aligned} \tag{6.39}$$

Now,

$$\begin{aligned}\|(T+I)x\|^2 - \|(T-I)x\|^2 &= \|Tx\|^2 + \|x\|^2 + ((T+T^*)x,x) \\ &\quad - \|Tx\|^2 - \|x\|^2 + ((T+T^*)x,x) \\ &= 2((T+T^*)x,x).\end{aligned} \tag{6.40}$$

From (6.39) and (6.40), we obtain $T + T^* \ge O$.

6.17 Problem Set 3.7

3.7.P1. Consider the multiplication operator $T{:}L^2[0, 1] \to L^2[0, 1]$ defined by

$$(Tx)(t) = tx(t), x \in L^2[0,1].$$

Consider the sequence $\{x_n(t)\}_{n \ge 1}$, where $x_n(t) = t^{1/n-1/2} \in L^2[0,1]$. The sequence $y_n(t) = t^{1/n+1/2}$, which is in the range of the operator T, converges in the L^2-norm to $t^{1/2}$. However, this function is not in the range of T, since any function that is a.e. equal to $t^{-1/2}$ does not belong to $L^2[0, 1]$.

3.7.P2. Let $Tx_n \to y$. Then, the sequence $\{Tx_n\}_{n \ge 1}$ is Cauchy in H, and from the inequality,

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \ge M\|x_n - x_m\|,$$

it follows that $\{x_n\}_{n \ge 1}$ is a Cauchy sequence in H. Hence, $x_n \to x$, and since the operator is continuous, we get $Tx_n \to Tx$ and y is in the range of T.

3.7.P3. $\left\|T \begin{bmatrix} \xi \\ \eta \end{bmatrix}\right\| = \left\|\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}\right\| = \left\|\begin{bmatrix} n\eta \\ 0 \end{bmatrix}\right\| = n|\eta| \le n\left(|\xi|^2 + |\eta|^2\right)^{\frac{1}{2}} = n\left\|\begin{bmatrix} \xi \\ \eta \end{bmatrix}\right\|$ for all ξ and η in $\mathbb{C}$. Therefore, $\|T\| \le n$. For the vector $(0, \eta)$ in $\mathbb{C}^2$,

$$\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \eta \end{bmatrix} = \begin{bmatrix} n\,\eta \\ 0 \end{bmatrix}.$$

So, $\|T(0, \eta)\| = n|\eta| = n\|(0, \eta)\|$. Consequently, $\|T\| = n$.

Observe that $T^2 = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $T^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $n \geq 2$. Consequently, $\|T^n\| = 0$, so that $r(T) = \lim_n \|T^n\|^{\frac{1}{n}} = 0$.

$$T^*T = \begin{bmatrix} 0 & 0 \\ 0 & n^2 \end{bmatrix}, TT^* = \begin{bmatrix} n^2 & 0 \\ 0 & 0 \end{bmatrix} \neq T^*T.$$

3.7.P4. (a) It is enough to show that S is one to one. For $x \in H$,

$$\begin{aligned} \|Sx\|^2 &= \|(I + T^*T)x\|^2 \\ &= \|x\|^2 + \|T^*Tx\|^2 + 2\|Tx\|^2 \\ &\geq \|x\|^2. \end{aligned}$$

In particular, $\|Sx\| \geq \frac{1}{2}\|x\|$, $x \in H$. Thus, S is one to one, and hence, S^{-1}:ran$(S) \to H$ exists.

(b) Let $y \in \overline{\text{ran}(S)}$ and choose $x_n \in H$ such that $\{Sx_n\}_{n \geq 1}$ converges to y. In particular, $\{Sx_n\}_{n \geq 1}$ is Cauchy and so for $\varepsilon > 0$, there exists n_0 such that $m,n \geq n_0$ implies

$$\|Sx_n - Sx_m\| = \|S(x_n - x_m)\| < \varepsilon.$$

By (a), $\|x_n - x_m\| \leq \|Sx_n - Sx_m\| < \varepsilon$ and so $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in H. There is an $x \in H$ such that $\lim_n x_n = x$. By continuity of S, $\lim_n Sx_n = Sx$. Thus, $y = Sx$ and so ran(S) is closed.

(c) If $x \in H$ is such that $x \neq 0$ and $x \in \ker(S)$, then

$$0 = \|Sx\| \geq \|x\| = 0,$$

which implies $x = 0$, a contradiction.

(d) Observe that ran$(S)^\perp$ = ker(S^*) = ker(S), since S is self-adjoint. Thus, ran$(S)^\perp = \{0\}$, which implies ran$(S) = H$. Now,

$$\begin{aligned} &\|x\| \leq \|Sx\| \\ &\left\|SS^{-1}x\right\| = \|x\| \leq \|Sx\| \\ &\left\|S^{-1}Sx\right\| \leq \|Sx\| \\ &1 \geq \frac{\left\|S^{-1}(Sx)\right\|}{\|Sx\|} \Rightarrow \left\|S^{-1}\right\| \leq 1. \end{aligned}$$

3.7.P5. Observe that $\left(\sum_{k=0}^{n}(iA)^k/k!\right)^* = \sum_{k=0}^{n}(-iA)^k/k!$, since $A = A^*$. The series representing $\exp z$ is uniformly and absolutely convergent. It follows that $\sum_{n=0}^{\infty}(iA)^n/n!$ is absolutely convergent. Hence,

$$\left(\sum_{n=1}^{\infty}(iA)^n/n!\right)\left(\sum_{m=0}^{\infty}(iA)^m/m!\right)^* = I = \left(\sum_{m=1}^{\infty}(iA)^m/m!\right)^*\left(\sum_{n=1}^{\infty}(iA)^n/n!\right).$$

3.7.P6. Set $\frac{z-\alpha}{1-\overline{\alpha}z} = \omega$. Then, $\frac{d\omega}{dz} = \frac{(1-|\alpha|^2)}{(1-\overline{\alpha}z)^2}$ and

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial D}|Uf(z)|^2 &= \frac{1}{2\pi i}\int\limits_{\partial D}\left|\frac{(1-|\alpha|^2)}{(1-\overline{\alpha}z)^2}f\left(\frac{z-\alpha}{1-\overline{\alpha}z}\right)^2\right|\frac{dz}{z}\\ &= \frac{1-|\alpha|^2}{2\pi i}\int\limits_{|\omega|=1}\frac{1}{|1-\overline{\alpha}z|^2}|f(\omega)|^2\frac{(1-\overline{\alpha}z)^2}{(1-|\alpha|^2)}\frac{d\omega}{z}\\ &= \frac{1}{2\pi i}\int\limits_{|\omega|=1}|f(\omega)|^2\frac{1-\overline{\alpha}z}{z(1-\alpha\overline{z})}d\omega\\ &= \frac{1}{2\pi i}\int\limits_{|\omega|=1}|f(\omega)|^2\frac{d\omega}{\omega}\end{aligned}$$

which implies

$$\|Uf\| = \|f\| \quad \text{for all} f \in RH^2.$$

3.7.P7. Let k be a positive integer such that $m \leq 2^k$. Note that if $T^m = O$, then $T^{2^k} = O$. Since T is normal,

$\|T\|^2 = \|T^2\|$ [Theorem 3.7.2]

An induction argument shows that

$$\|T\| = \left\|T^{2^k}\right\|^{1/2^k},$$

which, on using $T^{2^k} = O$, implies $\|T\| = 0$, that is, $T = O$.

3.7.P8. If T is normal, then by Theorem 3.7.1, $\|Tx\| = \|T^*x\|$ for all $x \in H$; so ker $(T) = \ker(T^*)$.

Suppose $\overline{\operatorname{ran}(T)} = H$. Now, by Theorem 3.5.8, we have $\overline{\operatorname{ran}(T)} = \ker(T^*)^{\perp}$. Combining the preceding two equalities with the one in the preceding paragraph, it follows that $H = \ker(T)^{\perp}$, which implies $\ker(T) = \{0\}$, that is, T is injective.

On the other hand, suppose that $\ker(T) = \{0\}$. Since $\ker(T) = \ker(T^*)$, we have $\ker(T^*) = \{0\}$. Therefore, by Theorem 3.5.8,

$$\overline{\operatorname{ran}(T)} = \ker(T^*)^{\perp} = H.$$

3.7.P9. (a) Let $H = \ell^2$; let S denote the right shift

$$S(\{x_i\}_{i \ge 1}) = (0, x_1, x_2, \ldots).$$

Obviously, $\ker(S) = \{0\}$. Since $e_1 = (1, 0, 0, \ldots)$ is orthogonal to the range of S, ran (S) is not dense in H.

(b) By Theorem 3.5.8, the adjoint of any operator satisfying the conditions in part (a) will have the desired properties. In the particular case of the right shift S, it turns out that $\ker(S^*) = \operatorname{span}\{e_1\}$, as we argue below.

The adjoint of the right shift is [see Example 3.5.10(vi)]

$$S^*x = \sum_{k=2}^{\infty} \mu_k e_{k-1} \quad \text{where } x = \sum_{k=1}^{\infty} \mu_k e_k.$$

Note that $S^*e_i = e_{i-1}$ for $i > 1$ and $S^*e_1 = 0$, i.e. $e_1 \in \ker(S^*)$. If $x \neq e_1$ is in ker (S^*), i.e. $S^*x = 0$, then

$$0 = \sum_{k=2}^{\infty} \mu_k e_{k-1}, \quad \text{where } x = \sum_{k=1}^{\infty} \mu_k e_k.$$

Since $\{e_k\}_{k \ge 1}$ are orthonormal, it follows that $\mu_k = 0$ for $k \ge 2$. Consequently, $x = \mu_1 e_1$, which shows that $\ker(S^*) = \operatorname{span}\{e_1\}$.

3.7.P10. Let $T, S \in \mathcal{B}(H)$ be normal operators. Then,

$$(T^* - S^*)(T - S) + (T^* - S^*)S + S^*(T - S) = T^*T - S^*S \tag{6.41}$$

Since $\|T^*\| = \|T\|$, $T \in \mathcal{B}(H)$, and for a normal operator T, $\|T^*T\| = \|T\|^2$, it follows from (6.41) that

$$\|T^*T - S^*S\| \le \|T - S\|^2 + 2\|S\|\|T - S\|. \tag{6.42}$$

Let $\{T_n\}_{n \ge 1}$ be a sequence of normal operators in $\mathcal{B}(H)$ that converges in the operator norm to $T \in \mathcal{B}(H)$, say. Since

$$\begin{aligned}\|T^*T - TT^*\| &= \|T^*T - T_n{}^*T_n + T_nT_n{}^* - TT^*\| \\ &\le \|T^*T - T_n{}^*T_n\| + \|T_nT_n{}^* - TT^*\| \\ &\le 2(\|T - T_n\|^2 + 2\|T\|\|T_n - T\|),\end{aligned}$$

using (6.42). Letting $n \to \infty$, we conclude that T is normal.

3.7.P11. It follows from the hypothesis $TS = ST$ that $T^kS = ST^k$ for each $k \ge 0$. So, if $p(z)$ is a polynomial, $p(T)S = Sp(T)$. Since for a fixed z, $\exp(i\bar{z}T)$ is a limit of polynomials, it follows that $\exp(i\bar{z}T)S = S\exp(i\bar{z}T)$ [see Problem 3.6.P5]. Therefore, we have

$$S = \exp(i\bar{z}T)\,S\exp(-i\bar{z}T). \tag{6.43}$$

Define

$$f(z) = \exp(izT^*)S\exp(-izT^*).$$

Because $\exp(A + B) = \exp(A)\exp(B)$ when A and B commute, the fact that T is normal implies

$$f(z) = \exp[i(\bar{z}T + zT^*)]S\exp[-i(\bar{z}T + zT^*)],$$

using (6.43).

For every $z \in \mathbb{C}$, $\bar{z}T + zT^*$ is self-adjoint. It follows that $\exp[i(\bar{z}T + zT^*)]$ and $\exp[-i(\bar{z}T + zT^*)]$ are unitary [see Problem 3-7.P5]. Therefore, $\|f(z)\| \le \|S\|$. Since f is an entire function, by Liouville's theorem, f is a constant. On differentiation with respect to z, we get

$$0 = f'(z) = iT^*\exp(izT^*)S\exp(-izT^*) + \exp(izT^*)S\exp(-izT^*)(-iT^*).$$

Putting $z = 0$, we obtain $T^*S = ST^*$.

3.7.P12. (a) Observe that for every $x \in H$,

$$\begin{aligned}\|(T \pm iI)x\|^2 &= ((T \pm iI)x, (T \pm iI)x) = \|Tx\|^2 \mp i(Tx, x) \pm i(Tx, x) + \|x\|^2 \\ &= \|Tx\|^2 + \|x\|^2,\end{aligned}$$

using the fact that $T = T^*$. If $(T \pm iI)x = 0$, then $\|Tx\|^2 + \|x\|^2 = 0$, which implies $\|x\|^2 = 0$. Thus, the mappings $T \pm iI$ are injective. We next show that $\text{ran}(T + iI) = H$. Suppose there exists a nonzero $z \in H$ such that $z \perp \text{ran}(T + iI)$. Then, for $x \in H$, $0 = ((T + iI)x,z) = (Tx,z) + i(x,z)$, which implies $(Tx,z) = (x,iz)$. Therefore, $T^*z = iz$. Since T is self-adjoint, this is the same as $Tz = iz$, i.e. $(T - iI)z = 0$. Since $T - iI$ is injective, it follows that $z = 0$, a contradiction. The bijectivity of $T \pm iI$ is also immediate from the fact that $\pm i \notin \sigma(T) \subseteq \mathbb{R}$.

Claim U is norm-preserving. Let y be any vector in H; denote by v and w the vectors

$$v = (T + iI)^{-1}y, \quad w = Uy. \tag{6.44}$$

Then, $(T + iI)v = y$ and $w = (T - iI)(T + iI)^{-1}y = (T - iI)v$. Therefore,

$$\|y\|^2 = \|(T+iI)v\|^2 = \|Tv\|^2 + \|v\|^2 \tag{6.45}$$

and similarly,

$$\|w\|^2 = ((T-iI)v, (T-iI)v) = \|Tv\|^2 + \|v\|^2 \tag{6.46}$$

From (6.44), (6.45) and (6.46), we get

$$\|Uy\|^2 = \|y\|^2.$$

Theorem 3.7.12 then implies that U is unitary.

(b) Note that $(T - iI) = -2iI + (T + iI)$. So,

$$U = (T-iI)(T+iI)^{-1} = (-2iI + (T+iI))(T+iI)^{-1} = I - 2i(T+iI)^{-1}.$$

(c) Let $x \in H$ and

$$y = (T+iI)x \quad \text{so that } Uy = (T-iI)x. \tag{6.47}$$

Adding and subtracting the two equations in (6.47), we get

$$(I+U)y = 2Tx \quad \text{and} \quad (I-U)y = 2iIx,$$

from which we see that $(I - U)y = 0$ implies $x = 0$, and consequently, by (6.47), $y = 0$. Hence, $I - U$ is injective. In fact, $(I - U)^{-1}$ exists by (b). We then have

$$(I+U)(I-U)^{-1}(2ix) = (I+U)y = 2Tx,$$

which implies

$$T = i(I+U)(I-U)^{-1}.$$

On taking adjoints and observing that $T = T^*$ and $U^{-1} = U^*$, we obtain the other expression for T as well.

6.18 Problem Set 3.8

3.8.P1. (a) $\|T_n x\| = \frac{1}{n}\|x\|$. So, $\sup_{\|x\|=1}\|T_n x\| = \frac{1}{n}$. Consequently, $\|T_n\| = \frac{1}{n}$, and hence, $\lim_n \|T_n\| = \lim_n \frac{1}{n} = 0$. So, $\{T_n\}_{n\geq 1}$ converges to zero in the norm.

(b) For $x \in \ell^2$, we have $x = \sum (x, e_i)e_i$ and $x - P_n x = \sum_{i=n+1}^{\infty} (x, e_i)e_i$. So, $\|x - P_n x\|^2 = \sum_{i=n+1}^{\infty} |(x, e_i)|^2 \to 0$ as $n \to \infty$ (using Theorem 2.9.9). This shows that $P_n \to I$ strongly. Now,

$$\|I - P_n\| = \sup_{\|x\|=1} \|x - P_n x\| = \sup_{\|x\|=1} |(x, e_i)| \le \sup_{\|x\|=1} \|x\| = 1.$$

For $x = e_{n+1}$, we have $\|x\| = 1$ and $\|(I - P_n)x\| = 1$. Therefore, $\|I - P_n\| = 1$, and hence, $P_n \not\to I$ in the norm.

(c) For $x, y \in \ell^2$,

$$(T^n x, y) = ((0, \ldots, 0, x_1, x_2, \ldots), (y_1, y_2, \ldots, y_n, y_{n+1}, \ldots)) = \sum_{k=1}^{\infty} x_k \overline{y}_{n+k}.$$

So,

$$|(T^n x, y) = \sum_{k=1}^{\infty} \left|x_k \overline{y}_{n+k}\right| \le \sum_{k=1}^{\infty} |x_k||y_{n+k}| \le \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |y_{n+k}|^2\right)^{\frac{1}{2}}.$$

As $n \to \infty$, the last factor $\left(\sum_{k=1}^{\infty} |y_{n+k}|^2\right)^{\frac{1}{2}} \to 0$. Hence, $T^n \to O$ weakly. However, $\|T^n x\| = \|x\|$ for all x and all n. Consequently, $T^n \not\to O$ strongly.

3.8.P2. (a) $(I - P)^2 = I - 2P + P^2 = I - P$.

(b) It is obvious that $\{x \in X : Px = x\} \subseteq \text{ran}(P)$. On the other hand, let $x \in \text{ran}(P)$. Then, $x = Py$ for some $y \in X$, and hence, $Px = P(Py) = Py = x$. This proves (b).

(c) This follows from (b) and from the observation that $(I - P)x = 0$ if and only if $x = Px$.

(d) For each $x \in X$, we have $x = Px + (I - P)x$. Thus,

$$X = \text{ran}(P) + \text{ran}(I - P).$$

If $x \in \text{ran}(P) \cap \text{ran}(I - P)$, then (b) applied to P and $I - P$ gives $x = Px = (I - P)x$, and hence,

$$x = Px = P((I - P)x) = (P - P^2)x = 0.$$

This shows that $\text{ran}(P) \cap \text{ran}(I - P) = \{0\}$, and hence, $X = \text{ran}(P) \oplus \text{ran}(I - P)$.

(e) By (c) applied to P and $I - P$, we have $\text{ran}(P) = \ker(I - P)$ and $\text{ran}(I - P) = \ker(P)$, and hence, $\text{ran}(P)$ and $\text{ran}(I - P)$ are closed subspaces because the kernel of any bounded linear operator is closed.

Recall that a projection P on a Hilbert space H is called an orthogonal projection if $P = P^*$, i.e.

$$(Px, y) = (x, Py) \text{ for all } x, y \in H.$$

3.8.P3. (a) Suppose P is an orthogonal projection. Then, for all x, $y \in H$,

$$(Px, (I-P)y) = (x, P(I-P)y) = (x, (P-P^2)x) = 0,$$

i.e.

$$\operatorname{ran}(P) \perp \operatorname{ran}(I-P), \quad \text{or } \operatorname{ran}(P) \perp \ker(P).$$

(b) Now, suppose $\operatorname{ran}(P) \perp \ker(P)$. Then, for all $x, y \in H$

$$(Px, y) = (Px, (P + (1-P))y) = (Px, Py) \text{ since } (Px,(1-P)y) = 0 \text{ by hypothesis.}$$

On the other hand,

$$\begin{aligned}(x, Py) &= ((P + (1-P))x, Py) \\ &= (Px, Py) \text{ since } ((1-P))x), Py) = 0 \text{ by hypothesis.}\end{aligned}$$

Hence,

$$(Px, y) = (x, Py) \quad \text{for all } x, y \in H,$$

that is, $P = P^*$.

3.8.P4. The defining relation for V^* is $(x, V^*y) = (Vx, y)$ for x, $y \in L^2[0, 1]$.

$$\begin{aligned}(x, V^*y) = (Vx, y) &= \int_0^1 (Vx)(s)\overline{y(s)}\mathrm{d}s = \int_0^1 \left(\int_0^s x(t)\mathrm{d}t\right)\overline{y(s)}\mathrm{d}s \\ &= \int_0^1 \int_t^1 x(t)\overline{y(s)}\mathrm{d}s\,\mathrm{d}t = \int_0^1 x(t)\overline{\int_t^1 y(s)\mathrm{d}s}\mathrm{d}t,\end{aligned}$$

which implies

$$(V^*y)(t) = \int_t^1 y(s)\mathrm{d}s, \text{ so}$$

$$(V + V^*)(x)(s) = \int_0^s x(t)\mathrm{d}t + \int_s^1 x(t)\mathrm{d}t = \int_0^1 x(t)\mathrm{d}t.$$

In particular, when x is a constant function, say A, we have $(V + V^*)(x)(s) = \int_0^1 A\mathrm{d}t = A$.

Clearly, $V + V^*$ is self-adjoint and $(V + V^*)^2 = (V + V^*)$. Indeed,

$$\begin{aligned}\left((V+V^*)^2x\right)(s) &= (V+V^*)((V+V^*)x)(s)\\ &= (V+V^*)\left(\int_0^1 x(t)\mathrm{d}t\right)\\ &= \int_0^1 x(t)\mathrm{d}t\\ &= ((V+V^*)x)(s).\end{aligned}$$

6.19 Problem Set 3.9

3.9.P1. Let $x = \{\xi_i\}_{i\geq 1}$ and $y = \{\eta_i\}_{i\geq 1}$ be the elements of ℓ^2. Then, $\|Tx\|^2 = \|(0, 0, \xi_3, \xi_4, \ldots)\|^2 = \sum_{k=3}^{\infty}|\xi_k|^2 \leq \sum_{k=1}^{\infty}\|\xi_k\|^2 = \|x\|^2$. This shows that T is a bounded linear operator of norm at most 1.

$(Tx, y) = \sum_{k=3}^{\infty}\xi_k\overline{\eta}_k = \overline{\sum_{k=3}^{\infty}\overline{\xi}_k\eta_k} = \overline{(Ty, x)} = (x, Ty)$, which shows that T is self-adjoint. Moreover, $(Tx, x) = \sum_{k=3}^{\infty}|\xi_k|^2 \geq 0$.

Define $S:\ell^2 \to \ell^2$ by $Sx = (0, 0, \xi_3, \xi_4, \ldots)$, $x = \{\xi_i\}_{i\geq 1} \in \ell^2$. Then, $S^2(x) = S(S(x)) = S(0, 0, \xi_3, \xi_4, \ldots) = (0, 0, \xi_3, \xi_4, \ldots) = Tx$. Thus, $S = \sqrt{T}$.

3.9.P2. (a) Let $S = T^{\frac{1}{2}}$. Note that S is self-adjoint.

$\|S^2\| = \|S\|^2$, [Theorem 3.5.4(e)], which implies $\left\|T^{\frac{1}{2}}\right\| = \|T\|^{\frac{1}{2}}$.

(b) For $x,y \in H$, using Theorem 3.4.5, we have

$$\begin{aligned}\|(Tx, y)\|^2 &= |B(x, y)|^2\\ &\leq B(x, x)B(y, y)\\ &= (Tx, x)(Ty, y),\end{aligned}$$

which implies the required inequality.

(c) $\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*\||(Tx, x)| = \|T\||(Tx, x)|$, which implies the required inequality. It is clear from the inequality that $(Tx, x) = 0$ if and only if $Tx = 0$.

3.9.P3. (a) Let $x = Tz$ and y be the orthogonal projection of z on $[\ker(T)]^{\perp}$. Then, y is the unique element of $[\ker(T)]^{\perp}$ such that $x = Ty$. Since T is a partial isometry, $\|y\| = \|Ty\| = \|x\|$. It remains only to show that $T^*x = y$. This is equivalent to $(T^*x, u) = (y, u)$ for all $u \in H$, or

$$(x, Tu) = (y, u) \quad \text{for all}\, u \in H.$$

This is true for $u \in \ker(T)$, because $y \in [\ker(T)]^\perp$. We need only show that it is true for $u \in [\ker(T)]^\perp$. So, consider any such u. Then, all four vectors $y \pm u$, $y \pm iu$ are in $[\ker(T)]^\perp$. Since T is a partial isometry, it follows that

$$\begin{aligned}(T(y \pm u), T(y \pm u)) = (y \pm u, y \pm u) \,\text{and}\, &(T(y \pm iu), T(y \pm iu)) \\ &= (y \pm iu, y \pm iu).\end{aligned}$$

By the polarisation identity, (Ty, Tu) is the same linear combination of the above four left-hand sides as (y, u) is of the above four right-hand sides. Consequently, $(Ty, Tu) = (y, u)$. Since $Ty = x$, this is the same as the equality that was to be proved.

(b) Let $x \in [\ker(T^*)]^\perp$. We have to prove that $\|T^*x\| = \|x\|$. In general, $[\ker(T^*)]^\perp = \overline{[\mathrm{ran}(T)]}$. Since T is a partial isometry, we know that $\mathrm{ran}(T)$ is closed [see comments after Definition 3.9.3]; therefore, $x \in \mathrm{ran}(T)$. From part (a), it follows that $\|T^*x\| = \|x\|$.

6.20 Problem Set 4.1

4.1.P1. (i) Let $M = \overline{[\mathrm{ran}(\lambda I - T)]}$ and $\lambda \in \sigma_{\mathrm{com}}(T)$. Then, M is a proper subspace of H. Hence, there exists a nonzero functional defined on H which vanishes on M [Theorem 2.10.31]. So, there exists $y \in H$ such that $((\lambda I - T)x, y) = 0$ for all $x \in H$. This implies $(x, (\bar{\lambda} I - T^*)y) = 0$ for all $x \in H$. Hence, y is an eigenvector and $\bar{\lambda}$ an eigenvalue of T^*.

(ii) If $\lambda \in \sigma_{\mathrm{p}}(T)$, then there exists a nonzero vector x in H such that $(\lambda I - T)x = 0$. Hence, for all y in H,

$$((\lambda I - T)x, y) = 0, \quad \text{that is, } (x, (\bar{\lambda} I - T^*)y) = 0$$

for all y in H [see Theorem 2.10.25]. This says that $x \perp \{(\bar{\lambda} I - T^*)y : y \in H\}$. If $\{(\bar{\lambda} I - T^*)y : y \in H\}$ were dense in H, then $x = 0$. This contradiction proves the assertion.

4.1.P2. $||Tx||^2 = \left(\sum_{n=1}^\infty \frac{\alpha_n}{(n+1)} e_{n+1}, \sum_{n=1}^\infty \frac{\alpha_n}{(n+1)} e_{n+1}\right) = \sum_{n=1}^\infty \frac{|\alpha_n|^2}{(n+1)^2} \le \sum_{n=1}^\infty |\alpha_n|^2 = ||x||^2$. This shows that T is a bounded linear operator. If $Tx = 0$, then $\sum_{n=1}^\infty \frac{\alpha_n}{(n+1)} e_{n+1} = 0$, which implies $\sum_{n=1}^\infty \left|\frac{\alpha_n}{(n+1)}\right|^2 = 0$, and hence, $\alpha_n = 0$, $n = 1, 2, \ldots$. Thus, $x = 0$. Consequently, T is one to one and T^{-1} exists on $\mathrm{ran}(T)$. Clearly, $e_1 \notin \mathrm{ran}(T)$ and

$$\overline{\operatorname{ran}(T)} = \overline{[\{e_2, e_3, \ldots\}]} \neq \ell^2,$$

i.e. the range is not dense in ℓ^2. Thus, it follows that $0 \in \sigma_r(T)$.

$$\begin{aligned}\|T^m x\| &= \left\| \sum_{n=1}^{\infty} \frac{\alpha_n}{(n+1)(n+2)\cdots(n+m)} \cdot e_{n+m} \right\| \\ &= \left(\sum_{n=1}^{\infty} \frac{|\alpha_n|^2}{\prod_{k=1}^{m} (n+k)^2} \right)^{\frac{1}{2}} \leq \sup_n \frac{1}{\prod_{k=1}^{m} (n+k)} \|x\|.\end{aligned}$$

Therefore,

$$\|T^m\| \leq \sup_n \frac{1}{\prod_{k=1}^{m} (n+k)}.$$

Also,

$$\frac{1}{\prod_{k=1}^{m} (n+k)} = \|T^m e_n\| \leq \|T^m\| \text{ for each } n,$$

and hence,

$$\sup_n \frac{1}{\prod_{k=1}^{m} (n+k)} \leq \|T^m\|.$$

Consequently,

$$||T^m|| = \sup_n \frac{1}{\prod_{k=1}^{m} (n+k)} = \frac{1}{(m+1)!},$$

which implies $\lim_{m\to\infty} ||T^m||^{1/m} = \lim_{m\to\infty} \left(\frac{1}{(m+1)!} \right)^{\frac{1}{m}} = 0$, i.e. $r(T) = 0$.

4.1.P3. (a) Since $\{\lambda_n\}_{n \geq 1}$ is convergent, it is bounded, by M, say. We can write

$$\begin{aligned}||Tx||^2 = (Tx, Tx) &= \left(\sum_{n=1}^{\infty} \alpha_n \lambda_n e_n, \sum_{n=1}^{\infty} \alpha_n \lambda_n e_n \right) = \sum_{n=1}^{\infty} |\alpha_n|^2 |\lambda_n|^2 \leq M^2 \sum_{n=1}^{\infty} |\alpha_n|^2 \\ &= M^2 \|x\|^2.\end{aligned}$$

Thus, T is a bounded linear operator on ℓ^2.

(b) Since $Te_n = \lambda_n e_n$ for $n = 1, 2, \ldots$, the required conclusion follows.

(c) Consider the operator $I - T$. Suppose $(I - T)x = 0$ for some $x \in \ell^2$. This implies $x = Tx$ or $\sum_{n=1}^{\infty} \alpha_n (1 - \lambda_n) e_n = 0$, which, by the orthonormality of

$\{e_k\}_{k\geq 1}$, says $\sum_{n=1}^{\infty}|\alpha_n|^2|1-\lambda_n|^2 = 0$. Since $1 - \lambda_n \neq 0$ for any n (hypothesis), we must have $\alpha_n = 0$ for $n = 1, 2, \ldots$. Hence, $x = 0$. Thus, $I - T$ is one to one. It follows that $(I - T)^{-1}$ exists on the range of $I - T$. Also,

$$(I-T)e_n = e_n - Te_n = (1-\lambda_n)e_n.$$

Consequently, $\|(I - T)e_n\| = |1 - \lambda_n| \to 0$ as $n \to \infty$. This shows that $I - T$ is not bounded below, and hence, $(I - T)^{-1}$ is not bounded.

We next check that $\mathrm{ran}(I - T)$ is dense in H. Indeed,

$$(I-T)\left(\frac{e_n}{1-\lambda_n}\right) = e_n,$$

and therefore, $e_n \in \mathrm{ran}(I - T)$ for $n = 1, 2, \ldots$, and hence, $\overline{[\{e_1, e_2, \ldots\}]} = H$. (c) is hence proved.

(d) λ is a scalar such that $\lambda \neq \lambda_n$ for any n and $\lambda \neq 1$. For any such λ, there must be a real number α such that

$$|\lambda - \lambda_n| > \alpha > 0 \quad \text{for all}\, n.$$

(Observe that 1 is the only limit point of $\{\lambda_n\}_{n\geq 1}$.)

Consider the operator $\lambda I - T$. We shall show that $\lambda I - T$ is bounded below and has dense range. For $x \in \ell^2$,

$$(\lambda I - T)x = \lambda x - Tx = \lambda\sum_{n=1}^{\infty}\alpha_n e_n - \sum_{n=1}^{\infty}\alpha_n\lambda_n e_n = \sum_{n=1}^{\infty}\alpha_n(\lambda - \lambda_n)\, e_n,$$

and

$$\|(\lambda I - T)x\|^2 = \sum_{n=1}^{\infty}|\alpha_n|^2|\lambda-\lambda_n|^2 \geq \alpha^2\sum_{n=1}^{\infty}|\alpha_n|^2 = \alpha^2\|x\|^2.$$

Therefore, $\lambda I - T$ is bounded below.

For $y = \sum_{n=1}^{\infty}\beta_n e_n \in \ell^2$, $(\lambda I - T)x = y$ implies $\sum_{n=1}^{\infty}\alpha_n(\lambda - \lambda_n)e_n = \sum_{n=1}^{\infty}\beta_n e_n$. Choose $\alpha_n = \frac{\beta_n}{\lambda-\lambda_n}$, and observing that $\sum_{n=1}^{\infty}|\alpha_n|^2 = \sum_{n=1}^{\infty}\frac{|\beta_n|^2}{|\lambda-\lambda_n|^2} \leq \frac{1}{\alpha^2}\sum_{n=1}^{\infty}|\beta_n|^2 < \infty$, it follows that $x \in \ell^2$ and $\lambda I - T$ is onto.

We have thus proved the following: For scalar λ such that $\lambda \neq \lambda_n$ for any n and $\lambda \neq 1$, we have $\lambda \in \rho(T)$. Hence, $\sigma(T) = \{\lambda_n : n = 1, 2, \ldots\} \cup \{1\}$, so $\sigma_p(T) = \{\lambda_n : n = 1, 2, \ldots\}$, $\sigma_c(T) = \{1\}$ and $\sigma_r(T) = \varnothing$.

4.1.P4. Let X denote $(\lambda I - AB)^{-1}$. Then,

$$\begin{aligned}(\lambda I - BA)(\lambda^{-1}I + \lambda^{-1}BXA) &= I - \lambda^{-1}BA + BXA - \lambda^{-1}BABXA \\ &= I - \lambda^{-1}BA + \lambda^{-1}B(\lambda I - AB)XA \\ &= I - \lambda^{-1}BA + \lambda^{-1}BA = I.\end{aligned}$$

Similarly, $(\lambda^{-1}I + \lambda^{-1}BXA)(\lambda I - BA) = I$. So, $\lambda \in \rho(BA)$.

We next show that $\sigma(AB)$ and $\sigma(BA)$ have the same elements with one possible exception: the point zero. Suppose $\lambda \notin \sigma(AB)$ and $\lambda \neq 0$. Then, $\lambda \in \rho(AB)$. The argument in the paragraph above shows that $\lambda \in \rho(BA)$, that is, $\lambda \notin \sigma(BA)$. So, $\lambda \neq 0$, $\lambda \in \sigma(BA) \Rightarrow \lambda \in \sigma(AB)$. The reverse implication follows on interchanging the roles of A and B. Consequently, $\sigma(AB)$ and $\sigma(BA)$ have the same elements with one possible exception: the point zero.

Let T denote the simple unilateral shift defined on the separable Hilbert space H in (vii) of Examples 3.2.5 and T^* denote the adjoint of T [(vi) of Examples 3.5.10]. An easy computation shows that $TT^* \neq I$, but $T^*T = I$:

$$\begin{aligned}T^*T(x_1, x_2, \ldots) &= T^*(0, x_1, x_2, \ldots) = (x_1, x_2, \ldots) \\ TT^*(x_1, x_2, \ldots) &= T(x_2, x_3, \ldots) = (0, x_2, x_3, \ldots).\end{aligned}$$

Thus, $0 \in \sigma(TT^*)$, but $0 \notin \sigma(T^*T)$.

4.1.P5. The operator T is a bounded linear operator with $\|T\| = M$. The adjoint T^* of T is given by $T^*(x_1, x_2, \ldots) = (\overline{\mu}_1 x_1, \overline{\mu}_2 x_2, \ldots)$. Moreover, $T^*T = TT^*$ [see Problem 3.5.P1].

Denote the standard orthonormal basis of ℓ^2 by $\{e_i\}_{i\,\geq 1}$, where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots)$ and 1 occurs at the ith place. If $x \in \ell^2$, then

$$(\lambda I - T)x = ((\lambda - \mu_1)x_1, (1 - \mu_2)x_2, \ldots).$$

Such an operator is invertible if and only if $\inf_j|\lambda - \mu_j| > 0$, that is, $\lambda \notin \overline{\{\mu_1, \mu_2, \ldots\}}$. Thus, $\sigma(T) = \overline{\{\mu_1, \mu_2, \ldots\}}$.

Each of the numbers μ_j is an eigenvalue of T, since $Te_j = \mu_j e_j$.

Assume that $\mu_0 \in \overline{\{\mu_1, \mu_2, \ldots\}}$ and $\mu_0 \neq \mu_k$ for $k = 1, 2, \ldots$. Then, μ_0 is not an eigenvalue of T. Indeed, if

$$(\mu_0 I - T)x = ((\mu_0 - \mu_1)x_1, (\mu_0 - \mu_2)x_2, \ldots) = 0,$$

then $x_1 = 0$, $x_2 = 0$, ….

As the operator T is normal, the residual spectrum $\sigma_r(T) = \varnothing$ [see Theorem 4.4.3]. Consequently, $\mu_0 \in \sigma_c(T)$. This may be seen by a direct argument as follows. For any e_j, $j = 1, 2, \ldots$,

$$e_j = \frac{1}{(\mu_0 - \mu_j)}(\mu_0 - \mu_j)e_j = \frac{1}{(\mu_0 - \mu_j)}(\mu_0 I - T)e_j \in \operatorname{ran}(\mu_0 I - T).$$

Hence, $\overline{\operatorname{ran}(\mu_0 I - T)} = \overline{\operatorname{span}\{e_1, e_2, \ldots\}} = \ell^2$.

Consequently, $\mu_0 \in \sigma_c(T)$.

4.1.P6. From the hypothesis that $\|Tx\| = \|T\|$ and that $T^* = T$, we have $(T^2x,x) = (Tx, Tx) = \|Tx\|^2 = \|T\|^2 = (\|T\|^2x,x)$.

The above chain of equalities implies $((\|T\|^2I - T^2)x,x) = 0$. Using the fact that $\|T\|^2I - T^2$ is a positive operator and hence has a square root, we obtain $(\|T\|^2 I - T^2)x = 0$, which implies that x is an eigenvector of T^2 corresponding to the eigenvalue $\|T\|^2$ $(= \|T^2\|)$. Now, $(\|T\|I + T)(\|T\|I - T)x = 0$. If $(\|T\|I - T)x = 0$, then $Tx = \|T\|x$. Otherwise, $(\|T\|I + T)y = 0$, where $y = \|Tx\| - Tx \neq 0$.

4.1.P7. (a) implies (b) Assume that $\lambda \in \sigma_{ap}(T)$ is such that $|\lambda| = \|T\|$. We shall show that

$$|\lambda| \in \overline{\{|(Tx,x)|: \|x\| = 1\}}.$$

It will then follow that

$$\|T\| = |\lambda| \leq \sup_{\|x\|=1} |(Tx,x)| \leq \sup_{\|x\|=\|y\|=1} |(Tx,y)| = \|T\|$$

and the result will be proved.

Since $\lambda \in \sigma_{ap}(T)$, there must exist $x_n \in H$, $\|x_n\| = 1$, such that

$$\|Tx_n - \lambda x_n\| \to 0 \text{ as } n \to \infty.$$

Since $\|x_n\| = 1$, we can write

$$\begin{aligned}|(Tx_n, x_n)| - \lambda &= |(Tx_n, x_n) - \lambda(x_n, x_n)| \\ &= |(Tx_n - \lambda x_n, x_n)| \\ &\leq \|Tx_n - \lambda x_n\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

which implies

$$(Tx_n, x_n) \to \lambda \text{ as } n \to \infty.$$

This completes the proof that (a) implies (b).

(b) implies (a). If (b) holds, there must exist vectors x_n, $\|x_n\| = 1$, such that

$$|(Tx_n, x_n)| \to \|T\| \tag{6.48}$$

The sequence $\{(Tx_n,x_n)\}_{n\geq 1}$ is therefore a bounded sequence of complex numbers and hence has a convergent subsequence $\{(Tx_{n_k}, x_{n_k})\}_{k\geq 1}$ such that

$$(Tx_{n_k}, x_{n_k}) \to \lambda.$$

It follows in view of (6.48) that

$$|\lambda| = \|T\|.$$

In order to complete the proof, it is enough to show that

$$\|Tx_{n_k} - \lambda x_{n_k}\| \to 0.$$

Now,

$$\begin{aligned}\|Tx_{n_k} - \lambda x_{n_k}\|^2 &= \|Tx_{n_k}\|^2 - (Tx_{n_k}, \lambda x_{n_k}) - (\lambda x_{n_k}, Tx_{n_k}) + |\lambda|^2 \\ &\leq \|T\|^2 \|x_{n_k}\|^2 - \overline{\lambda}(Tx_{n_k}, x_{n_k}) - \lambda(x_{n_k}, Tx_{nk}) + |\lambda|^2.\end{aligned}$$

Hence,

$$\lim_{k\to\infty} \|Tx_{n_k} - \lambda x_{n_k}\|^2 \leq |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0.$$

This completes the proof.

Remark For a normal operator T, $\|T\| = \sup_{\|x\|=1}|(Tx, x)|$ [Theorem 3.7.7]. Consequently, for a normal operator T, there exists $\lambda \in \sigma_{ap}(T)$ such that $|\lambda| = \|T\|$, using Problem 4.1.P7 above. This is also a consequence of Remark 4.2.7 and Theorem 4.4.1.

4.1.P8. Set $\|S - T\| = d$. Suppose $\max_{\nu\in\sigma(T)} \min_{\mu\in\sigma(S)}|\nu - \mu| > d$. Then, for some $\nu \in \sigma$ $\nu \in \sigma(T), \min_{\mu\in\sigma(S)}|\nu - \mu| > d$. Such a ν must belong to the resolvent set $\rho(S)$ and so $(S - \nu I)^{-1}$ exists. According to the spectral mapping theorem (4.3.1),

$$\sigma((S - \nu I)^{-1}) = (\sigma(S) - \nu)^{-1}.$$

It follows that

$$\left|\sigma((S - \nu I)^{-1})\right| < d^{-1}.$$

Since $S - \nu I$ is self-adjoint, its spectral radius, according to the (i) of Remarks 4.2.7, equals its norm. So, it follows that

$$\left\|(S - \nu I)^{-1}\right\| < d^{-1}.$$

Next,

$$(T - \nu I) = S - \nu I + T - S = (S - \nu I)(I + (S - \nu I)^{-1}(T - S)).$$

The second factor on the right is of the form $I + K$, where $K = (S - \nu I)^{-1}(T - S)$, so that $\|K\| \le d^{-1}d = 1$. It therefore follows that $I + K$ is invertible. The first factor, $(S - \nu I)$, too is invertible, and hence, so is their product $T - \nu I$. This implies that $\nu \notin \sigma(T)$. This contradiction completes the argument.

6.21 Problem Set 4.2

4.2.P1. (a) Let $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then, $\lambda I - T = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$, $\det(\lambda I - T) = \lambda^2 + 1$, which has no real roots. So, $\sigma(T)) = \varnothing$.

(b) or $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $T^2 = O$ and $\sigma(T) = \{0\}$. So, $r(T) = 0$, but $T \neq O$.

4.2.P2. On using commutativity, we have

$$(AB)^n = A^n B^n \text{ for } n = 1, 2, \ldots.$$

So,

$$\|(AB)^n\| \le \|A^n\| \|B^n\|,$$

which implies

$$\|(AB)^n\|^{1/n} \le \|A^n\|^{1/n} \|B^n\|^{1/n}.$$

Hence, on taking limits, we obtain $r(AB) = \lim_n \|(AB)^n\|^{1/n} \le \lim_n \|A^n\|^{1/n} \lim_n \|B^n\|^{1/n} = r(A)r(B)$.

Consider $N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $N^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\sigma(N) = \{0\} = \sigma(N^*)$. Also, $NN^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $N^*N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq NN^*$. So, $\sigma(NN^*) = \{0, 1\} = \sigma(N^*N)$. Hence, $r(NN^*) = r(N^*N) \neq 0$, whereas $r(N) = 0$ and $r(N^*) = 0$.

4.2.P3. It follows on using commutativity that

$$(A + B)^n = \sum_{k=0}^{n} {}^nC_k A^k B^{n-k}.$$

Let $p > r(A)$ and $q > r(B)$. Then, there exists m such that for $n > m$,

$$\|A^n\| < p^n \text{ and } \|B^n\| < q^n.$$

Consider $n > 2m$ and put $a = \|A\|$, $b = \|B\|$. Then,

$$(A+B)^n = \sum_{k=0}^{m} {}^nC_k A^k B^{n-k} + \sum_{k=m+1}^{n-m} {}^nC_k A^k B^{n-k} + \sum_{k=n-m+1}^{n} {}^nC_k A^k B^{n-k}$$

$$\|(A+B)^n\| \le \left(\sum_{k=0}^{m} {}^nC_k a^k q^{n-k} + \sum_{k=m+1}^{n-m} {}^nC_k p^k q^{n-k} + \sum_{k=n-m+1}^{n} {}^nC_k p^k b^{n-k} \right)$$

$$\le \sum_{k=0}^{m} {}^nC_k p^k q^{n-k} \left(\frac{a}{p}\right)^k + \sum_{k=m+1}^{n-m} {}^nC_k p^k q^{n-k} + \sum_{k=n-m+1}^{n} {}^nC_k p^k q^{n-k} \left(\frac{b}{q}\right)^{n-k}$$

If $M = \max\{\max_{0\le k\le m} \left(\frac{a}{p}\right)^k, 1, \max_{n-m+1\le k\le n} \left(\frac{b}{q}\right)^{n-k}\}$, then

$$\|(A+B)^n\| \le M(p+q)^n,$$

which implies

$$\|(A+B)^n\|^{1/n} \le M^{1/n}(p+q).$$

So,

$$r(A+B) \le p+q.$$

Hence,

$$r(A+B) \le \inf_{p > \|A^n\|^{\frac{1}{n}}} p + \inf_{q > \|B^n\|^{\frac{1}{n}}} q,$$

i.e.

$$r(A+B) \le r(A) + r(B).$$

Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, $\sigma(A) = \{0\}$ and $\sigma(B) = \{0\}$; $A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\sigma(A+B) = \{1, -1\}$. Consequently,

$$r(A+B) = \sup\{|\lambda| : \lambda \in \sigma(A+B)\} = 1,$$

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} = 0 \text{ and } r(B) = \sup\{|\lambda| : \lambda \in \sigma(B)\} = 0.$$

It may be noted that $AB \neq BA$.

6.22 Problem Set 4.4

4.4.P1. Let $\lambda = \alpha + i\beta$, where $\beta \neq 0$. Then,

$$\begin{aligned}\|(A-\lambda I)x\|^2 &= \|Ax-\alpha x-i\beta x\|^2 \\ &= (Ax-\alpha x-i\beta x, Ax-\alpha x-i\beta x) \\ &= (Ax-\alpha x, Ax-\alpha x)-(i\beta x, Ax-\alpha x)+(Ax-\alpha x, -i\beta x)+|\beta|^2\|x\|^2 \\ &= \|Ax-\alpha x\|^2+|\beta|^2\|x\|^2,\end{aligned}$$

since $(i\beta x, Ax - \alpha x) = (Ax - \alpha x, -i\beta x)$ because $A = A^*$. Thus,

$$\|(A-\lambda I)x\| \geq |\Im\lambda|\|x\|. \tag{6.49}$$

We must also show that $A - \lambda I$ is invertible. In other words, the spectrum of A is a subset of $\mathbb{R}$. From (6.49), it follows that $A - \lambda I$ is bounded below. Since $(A-\lambda I)^* = A-\overline{\lambda}I$, it follows from (6.49) again that

$$\|(A-\lambda I)^*x\| = \left\|(A-\overline{\lambda}I)x\right\| \geq \left|\Im\overline{\lambda}\right|\|x\| = |\Im\lambda|\|x\|,$$

that is, $(A - \lambda I)^*$ is bounded below. It follows from Theorem 3.5.9 that $(A - \lambda I)^{-1}$ exists. Consequently, λ is not a point of the spectrum of A.

6.23 Problem Set 4.5

4.5.P1. For any differentiable function $\varphi(t)$ with $\varphi(a) = 0$, we have $\varphi(t) = \int_a^t \varphi'(s)ds$, i.e. $\varphi(t) = (K\varphi')(t)$. Hence, $\mathrm{ran}(K) \supseteq \{\varphi \in L^2[a, b] : \varphi$ is differentiable and $\varphi(a) = 0\}$, and therefore, K does not have finite rank.

4.5.P2. Range$(K) \subseteq \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$, and span$\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ has dimension $\leq n$.

4.5.P3. (a) Since $k(t,s) = \overline{k(s,t)}$ and $k(t, s)$ is a continuous function, the operator is self-adjoint [Example (vii) of 3.5.10] and compact [Example 4.5.5(ii)]. We shall next show that there are denumerably many negative eigenvalues with 0 as the only accumulation point.

Assume that

$$\lambda y = Ky \quad \text{where } y \neq 0$$

Then,

$$\lambda y(t) = t\int_0^t y(s)\mathrm{d}s + \int_t^1 sy(s)\mathrm{d}s. \tag{6.50}$$

Differentiating (6.50) twice, we obtain

$$\begin{aligned}\lambda y'(t) &= ty(t) + \int_0^t y(s)\mathrm{d}s - ty(t) = \int_0^t y(s)\mathrm{d}s \\ \lambda y''(t) &= y(t).\end{aligned} \tag{6.51}$$

Clearly, $\lambda \neq 0$ because $y \neq 0$. Thus, we have the differential equation $y'' = y/\lambda$ with boundary conditions $y'(0) = 0$ and $y(1) = y'(1)$.

The first boundary condition results on substituting $t = 0$ in (6.51). For the second boundary condition, substitute $t = 1$ in (6.50) and (6.51) to obtain

$$\lambda y(1) = \int_0^1 y(s)\mathrm{d}s \quad \text{and} \quad \lambda y'(1) = \int_0^1 y(s)\mathrm{d}s,$$

which, on comparing, yields $y(1) = y'(1)$.

Set $1/\lambda = \mu$.

Case (i). $\mu > 0$. Then, $y(t) = c_1e^{\sqrt{\mu}t} + c_2e^{-\sqrt{\mu}t}$ and $0 = y'(0) = \sqrt{\mu}c_1 - \sqrt{\mu}c_2$, which implies $c_1 = c_2 = C$, say. Since $y \neq 0$, we know that $C \neq 0$. Also, $y(1) = ce^{\sqrt{\mu}} + ce^{-\sqrt{\mu}}$ and $y'(1) = C\sqrt{\mu}e^{\sqrt{\mu}} - C\sqrt{\mu}e^{-\sqrt{\mu}}$, which implies

$$C\cosh(\sqrt{\mu}) = C\sqrt{\mu}\sinh(\sqrt{\mu}),$$

i.e.

$$\coth(\sqrt{\mu}) = \sqrt{\mu}.$$

This equation has a unique solution, μ_0, say. Thus, there is a unique positive eigenvalue $\lambda_0 = 1/\mu_0$.

Case (ii). $\mu < 0$. Setting $\mu = -\eta^2$, we have

$$y(t) = c_1\cos\eta t + c_2\sin\eta t \quad \text{and} \quad y'(t) = \eta(-c_1\sin\eta t + c_2\cos\eta t).$$

Setting $t = 0$, we obtain $0 = y'(0) = \eta c_2$ and

$$y(t) = c_1\cos\eta t, \quad y'(t) = -\eta c_1\sin\eta t.$$

Substituting $t = 1$, we get

$$\cos\eta = -\eta\sin\eta,\ \text{ i.e., } -\eta = \cot\eta.$$

The last equation has denumerably many solutions η forming a sequence tending to ∞, and thus, we get denumerably many eigenvalues $-1/\eta^2$, with 0 as the only accumulation point.

(b) Since $k(t,s) = \overline{k(s,t)}$ and $k(t,s)$ is continuous, the operator is self-adjoint [Example (vii) of 3.5.10] and compact [Example 4.5.5(ii)]. Hence, the spectrum of K consists of 0 and real eigenvalues. Let λ be an eigenvalue and y be the corresponding eigenvector. Then,

$$\lambda y(t) = \int_0^t sy(s)\mathrm{d}s + t\int_t^1 y(s)\mathrm{d}s. \tag{6.52}$$

Differentiating the above equation twice, we get

$$\lambda y'(t) = ty(t) + \int_t^1 y(s)\mathrm{d}s - ty(t) = \int_t^1 y(s)\mathrm{d}s \tag{6.53}$$

$$\lambda y''(t) = -y(t). \tag{6.54}$$

Clearly, $\lambda \neq 0$, because otherwise $y = 0$ and so $\ker(K) = \{0\}$. We have the differential equation $\lambda y''(t) + y(t) = 0$. It follows from (6.52) and (6.53) that $y(0) = 0$ and $y'(1) = 0$.

We need to show that $\lambda > 0$. Multiplying (6.54) by $\bar{y}$ and integrating, we obtain

$$\lambda\int_0^1 y''(t)\bar{y}(t)\mathrm{d}t + \|y\|^2 = 0.$$

On integrating by parts, we get

$$\lambda\left\{ y'(t)\bar{y}(t)\,|_0^1 - \int_0^1 \bar{y}'(t)y'(t)\mathrm{d}t \right\} + \|y\|^2 = 0.$$

Using the boundary condition, it follows that

$$-\lambda\int_0^1 |y'(t)|^2\mathrm{d}t + \|y\|^2 = 0.$$

4.5.P4. $(V^*Vx)(s) = V^*((Vx)(s)) = \int_s^1 (Vx)(t)\mathrm{d}t = \int_s^1 \int_0^t x(u)\mathrm{d}u\,\mathrm{d}t.$

Since V is compact, so is V^*V. Moreover, the operator is self-adjoint. Thus, $\sigma(V^*V)$ is real. Let λ be an eigenvalue of V^*V, so that there exists a nonzero x such that $\lambda x = V^*Vx$. Then,

$$\lambda x(s) = \int_s^1 \int_0^t x(u)\mathrm{d}u\,\mathrm{d}t. \tag{6.55}$$

Since the right-hand side is continuous with respect to s whatever $x \in L^2[0, 1]$ may be, it is a consequence of (6.55) that the function x satisfying it must be continuous. Keeping this in view and taking the derivative twice, we obtain

$$\begin{aligned} \lambda x'(s) &= -\int_0^s x(u)\mathrm{d}u \\ \lambda x''(s) &= -x(s). \end{aligned} \tag{6.56}$$

Clearly, $\lambda \neq 0$, because otherwise $x = 0$, which is a contradiction. Thus, x satisfies the differential equation $\lambda x'' + x = 0$ with boundary conditions $x'(0) = x(1) = 0$, because (6.55) implies $x(1) = 0$ and (6.56) implies $x'(0) = 0$. Observe that $(\lambda x, x) = (V^*Vx, x) = \|Vx\|^2$. This implies $\lambda > 0$ since $\lambda \neq 0$. The solution of the differential equation is $x(s) = c_1\cos \frac{1}{\sqrt{\lambda}}\, s + c_2\sin \frac{1}{\sqrt{\lambda}}\, s$. From the boundary conditions, it follows that $c_2 = 0$, $\frac{1}{\sqrt{\lambda}} = \pi(k - \frac{1}{2})$, $k = 1, 2, \ldots$. Thus, the eigenvalues of the operator are $\lambda_k = \frac{4}{\pi^2(2k-1)^2}$, $k = 1, 2, \ldots$. Hence, in view of (i) of Remark 4.2.7, $\|V^*V\| = \frac{4}{\pi^2}$. It follows that $\|V\| = \|V^*V\|^{1/2} = 2\pi^{-1}$. Note that $\sigma_p(T)$ is countable and 0 is its only limit point [cf. Theorem 4.8.14].

Although the above solution has not used any alternative representation of V^*V, it is of some interest to note that it is possible to obtain one that involves only integrals over intervals in $\mathbb{R}$:

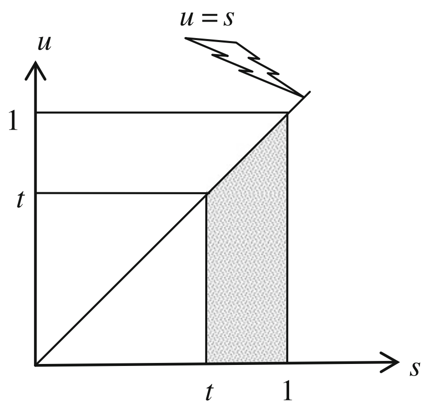
$$\begin{aligned} V^*Vx(t) &= \int_t^1 Vx(s)\mathrm{d}s = \int_t^1 \int_0^s x(u)\mathrm{d}u\mathrm{d}s, \quad 0 \le u \le s,\ t \le s \le 1 \\ &= \int_0^1 \int_0^s x(u)\mathrm{d}u\mathrm{d}s - \int_0^t \int_0^s x(u)\mathrm{d}u\mathrm{d}s, \\ &= \int_0^1 \left(\int_u^1 \mathrm{d}s \right) x(u)\mathrm{d}u - \int_0^t \left(\int_u^t \mathrm{d}s \right) x(u)\mathrm{d}u \\ &= \int_0^1 (1-u)x(u)\mathrm{d}u - \int_0^t (t-u)x(u)\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 x(u)\mathrm{d}u - \int_0^1 ux(u)\mathrm{d}u + \int_0^t ux(u)\mathrm{d}u - \int_0^t tx(u)\mathrm{d}u \\
&= \int_0^1 x(u)\mathrm{d}u - \int_t^1 ux(u)\mathrm{d}u - t\int_0^t x(u)\mathrm{d}u \\
&= \int_0^t x(u)\mathrm{d}u + \int_t^1 x(u)\mathrm{d}u - \int_t^1 ux(u)\mathrm{d}u - t\int_0^t x(u)\mathrm{d}u \\
&= (1-t)\int_0^t x(u)\mathrm{d}u + \int_0^1 (1-u)x(u)\mathrm{d}u.
\end{aligned}
$$

One can also arrive at this alternative representation by methods of two-variable integral calculus. The integral $\int_t^1 \int_0^s x(u)\mathrm{d}u\,\mathrm{d}s$ equals a double integral over the domain $0 \le u \le s, t \le s \le 1$. This domain is a trapezium and is the union of the domains given by $t \le s \le 1, 0 \le u \le t$ and by $u \le s \le 1, t \le u \le 1$, a fact that can either be guessed by drawing a figure (as one does in two-variable integral calculus) or proved by wrestling with inequalities. Its description as a union has the consequence that reversing the order of integration leads to

$$
\int_t^1 \int_0^s x(u)\mathrm{d}u\mathrm{d}s = \int_0^t \left(\int_t^1 x(u)\mathrm{d}s \right) \mathrm{d}u + \int_t^1 \left(\int_u^1 x(u)\mathrm{d}s \right) \mathrm{d}u,
$$

which is immediately seen to agree with the alternative representation we have obtained above.



It is also possible to get $V^*Vx(t) = (1-t)\int_0^t x(u)\mathrm{d}u \int_t^1 (1-u)x(u)\mathrm{d}u$ in another way. It is sufficient to derive this for any continuous x. Since $\int_0^s x(u)\mathrm{d}u$ is continuous in s, we first compute from $y(t) = V^*Vx(t) = \int_t^1 \int_0^s x(u)\mathrm{d}u\,\mathrm{d}s$ that

$$\begin{aligned} y'(t) &= -\int\limits_0^t x(u)\mathrm{d}u \\ &= -\int\limits_0^t x(u)\mathrm{d}u + (1-t)x(t) - (1-t)x(t). \end{aligned}$$

The assumed continuity of x now leads to

$$\begin{aligned} y'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(1-t)\cdot\int\limits_0^t x(u)\mathrm{d}u + (1-t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^t x(u)\mathrm{d}u) - (1-t)x(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t)\int\limits_0^t x(u)\mathrm{d}u\right] + \frac{\mathrm{d}}{\mathrm{d}t}\int_t^1 (1-u)x(u)\mathrm{d}u. \end{aligned}$$

Since $y(1) = 0$, it follows from here that

$$y(t) = (1-t)\int\limits_0^t x(u)\mathrm{d}u + \int\limits_t^1 (1-u)x(u)\mathrm{d}u.$$

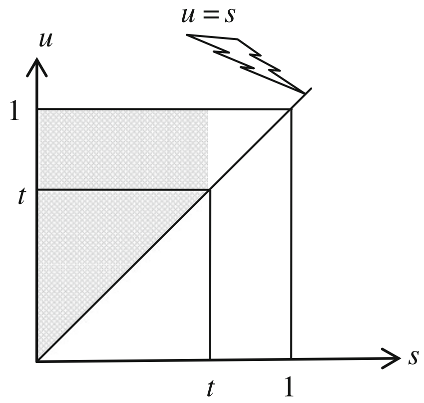
An alternative representation involving only integrals over intervals in $\mathbb{R}$ can also be obtained for $VV^*x(t)$ as follows:

$$\begin{aligned} VV^*x(t) &= \int\limits_0^t V^*x(s)\mathrm{d}s = \int\limits_0^t \int\limits_s^1 x(u)\mathrm{d}u\mathrm{d}s \\ &= \int\limits_0^t \int\limits_0^1 x(u)\mathrm{d}u\mathrm{d}s - \int\limits_0^t \int\limits_0^s x(u)\mathrm{d}u\mathrm{d}s \\ &= t\int\limits_0^1 x(u)\mathrm{d}u - \int\limits_0^t \left(\int\limits_u^t \mathrm{d}s\right) x(u)\mathrm{d}u \\ &= t\int\limits_0^1 x(u)\mathrm{d}u - \int\limits_0^t (t-u)x(u)\mathrm{d}u \\ &= t\int\limits_t^1 x(u)\mathrm{d}u + \int\limits_0^t ux(u)\mathrm{d}u. \end{aligned}$$

Here again, the integral $\int_0^t \int_s^1 x(u)\mathrm{d}u\,\mathrm{d}s$ equals a double integral over the domain $s \le u \le 1, 0 \le s \le t$. This domain is a trapezium and is the union of the domains given by $0 \le s \le u$, $0 \le u \le t$ and by $0 \le s \le t$, $t \le u \le 1$, a fact that can either be guessed by drawing a figure (as one does in two-variable integral calculus) or proved by wrestling with inequalities. Its description as a union has the consequence that reversing the order of integration leads to

$$\int_0^t \int_s^1 x(u)\mathrm{d}u\mathrm{d}s = \int_0^t \left(\int_t^1 x(u)\mathrm{d}s \right) \mathrm{d}u + \int_t^1 \left(\int_u^1 x(u)\mathrm{d}s \right) \mathrm{d}u,$$

which is immediately seen to agree with the alternative representation we have obtained above.



Also, for continuous x, the equality $y(t) = VV^*x(t) = \int_0^t \int_s^1 x(u)\mathrm{d}u\,\mathrm{d}s$ leads to

$$\begin{aligned} y'(t) &= -\int_t^1 x(u)\mathrm{d}u = \int_t^1 x(u)\mathrm{d}u - tx(t) + tx(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(t) \cdot \int_t^1 x(u)\mathrm{d}u + t \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_t^1 x(u)\mathrm{d}u + tx(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[t \cdot \int_t^1 x(u)\mathrm{d}u \right] + \frac{d}{\mathrm{d}t} \int_0^t ux(u)\mathrm{d}u. \end{aligned}$$

Since $y(0) = 0$, it follows from here that

$$y(t) = t\int_t^1 x(u)\mathrm{d}u + \int_0^t ux(u)\mathrm{d}u.$$

4-5.P5. It has been seen [see Example 4.3.2] that $r(V) = 0$ and

$$(V^n)(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}x(s)\mathrm{d}s.$$

Since $1 \notin \sigma(V)$, it follows that $(I - V)^{-1}$ exists and

$$(I-V)^{-1} = \sum_{n=0}^{\infty} V^n.$$

The series on the right converges absolutely in $\mathcal{B}(H)$ since $\sum_{n=0}^{\infty}\|V\|^n$ is a convergent geometric series because $\|V\| < 1$. Consequently,

$$(I-V)^{-1}y = \sum_{n=0}^{\infty}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)\mathrm{d}s = \int_0^t \mathrm{e}^{t-s}y(s)\mathrm{d}s.$$

$$\left(I + \sum_{n=1}^{\infty} V^n\right)\sin t = \sin t + \sum_{n=1}^{\infty} V^n \sin t$$

The integral equation to be solved can be written in the form

$$(I - V)y = \sin t.$$

Therefore,

$$\begin{aligned} y = (I-V)^{-1}\sin t &= \sin t + \sum_{n=1}^{\infty}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\cdot \sin s\,\mathrm{d}s \\ &= \sin t + \int_0^t \sum_{n=1}^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}\cdot \sin s\,\mathrm{d}s \\ &= \sin t + \int_0^t \mathrm{e}^{t-s}\sin s\,\mathrm{d}s. \end{aligned}$$

4.5.P6. (a) Let $B_n = \{x \in H: \|x\| < n\}$. Then, $H = \bigcup_{n=1}^{\infty} B_n$ because the norm of any vector in H is less than some n. Note that

$$T(H) = \bigcup_n T(B_n) \quad \text{and} \quad \overline{T(H)} \supseteq \bigcup_n \overline{T(B_n)}.$$

The complete continuity of T implies $\overline{T(B_n)}$ is compact and hence separable, for each n. It now follows that ran(T) is separable.

(b) Let P denote the orthogonal projection on $\overline{[\text{ran}(T)]}$:

$$Px = \sum_{n=1}^{\infty} (x, e_n)e_n.$$

Observe that

$$\begin{aligned} \|(P - P_n)x\|^2 &= \left\| \sum_{k=n+1}^{\infty} (x, e_k)e_k^2 \right\| = \sum_{k=n+1}^{\infty} \|(x, e_k)e_k\|^2 \\ &= \sum_{k=n+1}^{\infty} |(x, e_k)|^2 \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $P_n \to P$ (strongly). We show that $P_nT \to PT$ uniformly. Set

$$||P_nT - PT|| = \|(P_n - P)T\| = \alpha_n.$$

Note that α_n is a decreasing sequence of nonnegative numbers. So, $\alpha_n \to \alpha$, and $\alpha \geq 0$. By the definition of α_n, there exists a unit vector x_n in the orthogonal complement of the closed linear span of $\{e_k\}_{1 \leq k \leq n}$ such that $\|Tx_n\| \geq \frac{1}{2}\alpha_n$. Since $P_n \uparrow P$, the sequence $x_n \xrightarrow{w} 0$. Since T is compact, $Tx_n \to 0$. Hence, $\alpha_n \to 0$. Thus, PT is the uniform limit of P_nT.

4.5.P7. In Problem 3.2.P9, it has been shown that T is a bounded linear operator in ℓ^2 and $\|T\| = M$. It remains to show that T is compact if and only if $\alpha_n \to 0$ as $n \to \infty$.

Let P_n be the projection of H onto the closed linear subspace generated by (e_1, e_2, …, e_n). Then, $TP_n \in \mathcal{B}_{00}(H)$. Moreover, $T_n = T - TP_n$ is such that $T_ne_j = \alpha_je_j$ if $j > n$ and $T_ne_j = 0$ if $j \leq n$. So, $\|T - TP_n\| = \sup_{j>n}|\alpha_j|$ by Problem 3.2.P9. If $\alpha_n \to 0$, then $\|T - TP_n\| \to 0$ and T is compact since it is the limit of a sequence of finite rank operators.

Conversely, if T is compact, then we shall argue that $\alpha_n \to 0$. Suppose not. Then, there exists $\varepsilon > 0$ such that $\varepsilon \leq |\alpha_n| \leq \|T\|$ for infinitely many n. Passing to a subsequence, we may assume $\alpha_n \to \alpha$, where $\varepsilon \leq |\alpha| \leq \|T\|$. Now, $Te_n = \alpha_ne_n$. Again passing to a subsequence, we may assume that $Te_n \to y$ for a suitable vector y (using compactness of T). It now follows that $e_n = \alpha_n^{-1}\alpha_ne_n \to \alpha^{-1}y$. This contradicts the fact that $\{e_n\}$ is an orthonormal sequence and hence cannot have a convergent subsequence.

4.5.P8. Let us set $T_kx = (a_kx_k, a_kx_{k-1}, \ldots, a_kx_1, 0, 0, \ldots)$ and $B_n = \sum_{i=1}^{\infty} T_i$. Observe that

$$\|T_k x\| = \left(\left|\sum_{i=1}^{k} a_k x_i\right|^2\right)^{\frac{1}{2}} = |a_k| \left(\sum_{i=1}^{k} |x_i|^2\right)^{\frac{1}{2}} \leq |a_k| \|x\|,$$

and for $e_1 = (1, 0, 0, \ldots)$, $\|T_k e_1\| = |a_k| = |a_k| \|e_1\|$. So, $\|T_k\| = |a_k|$. For $m,n \in \mathbb{N}$,

$$\begin{aligned} \|B_n x - B_m x\| &= \left\|\sum_{i=m+1}^{n} T_i x\right\| \leq \|x\| \sum_{i=m+1}^{n} \|T_i\| \\ &\leq \|x\| \sum_{i=m+1}^{n} |a_i| \to 0 \text{ and } m, n \to \infty. \end{aligned}$$

Consequently, the series $\sum_{i=1}^{\infty} T_i$ converges in $\mathcal{B}(H)$. The operator T being its sum is bounded.

Next, each B_n has finite rank and further

$$\|(T - B_n)x\| = \left\|\sum_{i=n+1}^{\infty} T_i x\right\| \leq \|x\| \sum_{i=n+1}^{\infty} |a_i|.$$

Hence,

$$\|T - B_n\| \leq \sum_{i=n+1}^{\infty} |a_i| \to 0 \quad \text{as } n \to \infty,$$

which implies T is a compact operator.

4.5.P9. In Problem 3.2.P1, it has been proved that T is a bounded linear operator on ℓ^2. It remains to check that T is compact.

For each $\varepsilon > 0$, there exists an integer $p = p(\varepsilon)$ such that

$$\sum_{i=p+1}^{\infty} \sum_{j=1}^{\infty} \left|\tau_{ij}\right|^2 \leq \varepsilon^2.$$

Consider the operator T_ε defined by

$$T_\varepsilon x = \left(\sum_{j=1}^{\infty} \tau_{1j} x_j, \sum_{j=1}^{\infty} \tau_{2j} x_j, \ldots, \sum_{j=1}^{\infty} \tau_{pj} x_j, 0, \ldots\right),$$

where $x = \{x_i\}_{i \geq 1} \in \ell^2$. Note that T_ε is an operator of finite rank. Also, on using the Cauchy–Schwarz inequality, we have

$$\left\|(T - T_\varepsilon)x^2\right\| = \sum_{i=p+1}^{\infty}\left|\sum_{j=1}^{\infty}\tau_{ij}x_j\right|^2 \le \sum_{i=p+1}^{\infty}\sum_{j=1}^{\infty}\left|\tau_{ij}\right|^2\|x\|^2 \le \varepsilon^2\|x\|^2.$$

Being the limit of finite rank operators, T is compact.

Remarks (i) The convergence of $\sum_{i,j=1}^{\infty}\left|\tau_{ij}\right|^2$ is only a sufficient condition for the compactness of the matrix operator.

(ii) In the next example, we give a necessary and sufficient condition on the infinite matrix $[\tau_{ij}]_{i,j\ge 1}$ so that it represents a compact operator.

4.5.P10. Assume that the operator T determined by the matrix is compact. Let $\{e_i\}_{i\ge 1}$ be the standard orthonormal basis for ℓ^2. Then, $e_i \to 0$ (weakly). Since T is compact, it follows that $\{Te_i\}_{i\ge 1}$, where

$$Te_i = \beta_{i-1}e_{i-1} + \alpha_i e_i + \gamma_i e_{i+1}(\beta_0 = 0, \quad i = 1, 2, \ldots)$$

must converge strongly. Suppose that the matrix T does not satisfy the condition in question. We select a sequence $\{i_k\}_{k\ge 1}$ of integers such that

$$i_k \ge i_{k-1} + 3$$

and

$$\left|\beta_{i_k-1}\right|^2 + |\alpha_{i_k}|^2 + \left|\gamma_{i_k}\right|^2 \to \delta > 0,$$

where $\delta \le \infty$. Then,

$$\begin{aligned}\|Te_{i_n} - Te_{i_m}\|^2 &= \left\|\beta_{i_n-1}e_{i_n-1} + \alpha_{i_n}e_{i_n} + \gamma_{i_n}e_{i_n+1} - \beta_{i_m-1}e_{i_m-1} - \alpha_{i_m}e_{i_m} - \gamma_{i_m}e_{i_m+1}\right\|^2\\ &= \left|\beta_{i_n-1}\right|^2 + |\alpha_{i_n}|^2 + \left|\gamma_{i_n}\right|^2 + \left|\beta_{i_m-1}\right|^2 + |\alpha_{i_m}|^2 + \left|\gamma_{i_m}\right|^2 \to 2\delta \ne 0.\end{aligned}$$

This contradicts the strong convergence of the sequence $\{Te_i\}_{i\ge 1}$.

We next prove the sufficiency of the condition. Let $\lim_i\alpha_i = 0$, $\lim_i\beta_i = 0$, $\lim_i\gamma_i = 0$ and $\{x^{(n)}\}_{n\ge 1}$ converge to x weakly. Observe that

$$\begin{aligned}Tx^{(n)} &= \sum_{k=1}^{\infty}x^{(n)}{}_k Te_k = \sum_{k=1}^{\infty}x^{(n)}{}_k(\beta_{k-1}e_{k-1} + \alpha_k e_k + \gamma_k e_{k+1})\\ &= \sum_{k=1}^{\infty}(\beta_k x^{(n)}{}_{k+1} + \alpha_k x^{(n)}{}_k + \gamma_{k-1}x^{(n)}{}_{k-1})e_k, \quad \text{where } \gamma_0 = 0.\end{aligned}$$

So,

$$\begin{aligned}\left\|Tx^{(n)} - Tx^{(m)}\right\|^2 &= \sum_{k=1}^{\infty}\left|\beta_k\left(x^{(n)}{}_{k+1} - x^{(m)}{}_{k+1}\right) + \alpha_k\left(x^{(n)}{}_{k} - x^{(m)}{}_{k}\right) + \gamma_{k-1}\left(x^{(n)}{}_{k-1} - x^{(m)}{}_{k-1}\right)\right|^2 \\ &= \sum_{k=1}^{q}\left|\beta_k\left(x^{(n)}{}_{k+1} - x^{(m)}{}_{k+1}\right) + \alpha_k\left(x^{(n)}{}_{k} - x^{(m)}{}_{k}\right) + \gamma_{k-1}\left(x^{(n)}{}_{k-1} - x^{(m)}{}_{k-1}\right)\right|^2 \\ &\quad + \sum_{k=q+1}^{\infty}\left|\beta_k\left(x^{(n)}{}_{k+1} - x^{(m)}{}_{k+1}\right) + \alpha_k\left(x^{(n)}{}_{k} - x^{(m)}{}_{k}\right) + \gamma_{k-1}\left(x^{(n)}{}_{k-1} - x^{(m)}{}_{k-1}\right)\right|^2.\end{aligned}$$

The first term on the right tends to zero for fixed q as m and $n \to\infty$. The second term on the right can be made small for all m and n by taking q sufficiently large. If q is sufficiently large and $k > q$, then

$$|\beta_k| < \varepsilon, |\alpha_k| < \varepsilon \quad \text{and } |\gamma_k| < \varepsilon.$$

Therefore,

$$\sum_{k=q+1}^{\infty}\left|\beta_k\left(x^{(n)}{}_{k+1} - x^{(m)}{}_{k+1}\right) + \alpha_k\left(x^{(n)}{}_{k} - x^{(m)}{}_{k}\right) + \gamma_{k-1}\left(x^{(n)}{}_{k-1} - x^{(m)}{}_{k-1}\right)\right|^2 \le 9\varepsilon^2\left\|x^{(n)} - x^{(m)}\right\|^2.$$

Since every weakly convergent sequence is bounded, $\|x^{(m)}\| \le M$ for a suitable M and for n = 1, 2, …, it follows that the right-hand side of the above inequality is at most $36\varepsilon^2M^2$. Consequently, $\{Tx^{(n)}\}_{n\ge1}$ is a Cauchy sequence in ℓ^2 and hence converges.

4.5.P11. T is linear. If $x = \{\xi_j\}_{j\ge1} \in \ell^2$, then $y = Tx = \{\eta_j\}_{j\ge1}$, where for each j = 1, 2, …, $\eta_j = \frac{1}{j}\xi_j$, is in ℓ^2. Indeed,

$$\|Tx\|^2 = \sum_{j=1}^{\infty}|\eta_j|^2 = \sum_{j=1}^{\infty}\left|\frac{1}{j}\xi_j\right|^2 \le \sum_{j=1}^{\infty}|\xi_j|^2 = \|x\|^2.$$

Let

$$T_nx = \left(\xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \ldots, \frac{1}{n}\xi_n, 0, 0, \ldots\right), \quad x = \{\xi_j\}_{j\ge1} \in \ell^2.$$

T_n is linear and has finite-dimensional range. Furthermore,

$$\|(T - T_n)x\|^2 = \sum_{j=n+1}^{\infty}|\eta_j|^2 = \sum_{j=n+1}^{\infty}\frac{1}{j^2}|\xi_j|^2 \le \frac{1}{(n+1)^2}\sum_{j=n+1}^{\infty}|\xi_j|^2 \le \frac{\|x\|^2}{(n+1)^2},$$

which implies

$$\|(T - T_n)\| \leq \frac{1}{n+1}.$$

Hence, $T_n \to T$ and T is compact [Corollary 4.5.7].

The following results are needed in the next problem.

Let X be a Banach space. If $T \in \mathcal{B}(X)$, we define the spectrum of T to be the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible in } \mathcal{B}(X)\}$$

and the resolvent set of T to be

$$\rho(T) = \mathbb{C} \backslash \sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible in } \mathcal{B}(X)\}.$$

The other notions connected with the spectrum of $T \in \mathcal{B}(X)$, such as the point spectrum (eigenspectrum, eigenvalues), continuous spectrum, residual spectrum, approximate point spectrum and compression spectrum, are defined analogously [see Sect. 4.1]. For convenience, we recall the definition of eigenvalue:

$$\lambda \in \mathbb{C} \text{ is said to be an eigenvalue of } T \text{ if } \ker(\lambda I - T) \neq \{0\}.$$

(i) Let X denote a normed linear space and Y a proper closed subspace of X. Then, for every $\varepsilon > 0$, there is a point x_0 in the unit sphere of X such that

$$d(x_0, Y) = \inf\{\|x_0 - y\| : y \in Y\} \geq 1 - \varepsilon.$$

[5, Theorem 4.8](ii) Let $S,T \in \mathcal{B}(X)$ be such that $S + T = I$ and $SX \subseteq Y$, where Y is a closed proper subspace of X. Then, for every $\varepsilon > 0$, there is a point x_0 with $\|x_0\| \leq 1$ such that

$$d(Tx_0, TY) \geq 1 - \varepsilon.$$

Proof By (i) above, there exists an x_0 in the unit sphere of X ($\|x_0\| = 1$) such that $d(x_0,Y) \geq 1 - \varepsilon$. As

$$Tx_0 = x_0 - Sx_0, Sx_0 \in Y \text{ and } TY = (I - S)Y = Y - SY \subseteq Y,$$

we have

$$d(Tx_0, TY) \geq d(x_0 - Sx_0, Y) = d(x_0, Y) \geq 1 - \varepsilon.$$

(iii) Let $T \in \mathcal{B}_0(X)$, and let $S = I - T$. Then, SX is a closed subspace of X [5, Lemma 5, p. 191].

4.5.P12. By replacing T by T/λ, it suffices to prove the result for $\lambda = 1$. (Indeed, $\lambda \neq 0$ is an eigenvalue of T if and only if there exists an $x \in X$, $x \neq 0$, such that $(\lambda I - T)x = 0$ if and only if $(I - T/\lambda)x = 0$.)

Let then $T \in \mathcal{B}_0(X)$, $S = I - T$ and $\ker(S) = \{0\}$. We have to show that S is invertible in $\mathcal{B}(X)$. Let us prove that $SX = X$. Set $Y_n = S^n X$, $n = 0, 1, 2, \ldots$, so that $Y_0 = X \supseteq Y_1 \supseteq Y_2 \supseteq \ldots$. By (iii), the subspaces Y_n are closed. Let us show that $Y_n = Y_{n+1}$ for some n. Otherwise, $Y_0 \supset Y_1 \supset Y_2 \supset \ldots$, and all the inclusions are proper. Then, by (ii) above, we can find the elements y_n with $\|y_n\| \leq 1$ such that $(Ty_n, Ty_{n+1}) > \frac{1}{2}$. In particular, $\|Ty_n - Ty_m\| > \frac{1}{2}$ if $n \geq m$. So $\{Ty_n\}_{n \geq 1}$ has no convergent subsequence. This contradicts the compactness of T. We claim that $Y_0 = Y_1$. Suppose this is not the case. Then, there is an m such that $Y_{m-1} \neq Y_m = Y_{m+1}$. Let $u \in Y_{m-1} \backslash Y_m$. Since $Su \in Y_m = Y_{m+1} = SY_m$, it follows that there exists a $v \in Y_m$ such that $Su = Sv$, which implies $S(u - v) = 0$, so $0 \neq u - v \in \ker(S)$. This contradicts our assumption that $\ker(S) = \{0\}$. Consequently, $Y_1 = Y_0$, i.e. $SX = X$.

The bounded map $S{:}X \to X$ is a bijective map of the Banach space onto itself. It follows, using the inverse mapping theorem (5.5.2), that S is invertible.

4.5.P13. Choose a space with a complete orthonormal sequence of vectors $\{\varphi_i\}$ and a bounded sequence of nonzero numbers $\{\lambda_i\}$. The operator defined by

$$Tx = \sum_i \lambda_i (x, \varphi_i) \varphi_{i+1}$$

has no proper value, i.e. the equality $Tx = \mu x$ for complex μ admits $x = 0$ as the only solution. For $x = \Sigma_i a_i \varphi_i$, our equality amounts to

$$Tx = \sum_i a_i T\varphi_i = \sum_i a_i \lambda_i \varphi_{i+1} = \mu \sum_i a_i \varphi_i. \tag{6.57}$$

We distinguish two possibilities:

1. $\mu = 0$. Then, the left side of (6.57) is also zero. Hence, $a_i \lambda_i = 0$, and therefore, $a_i = 0$ (since $\lambda_i \neq 0$).
2. $\mu \neq 0$. Comparing corresponding coefficients on both sides of (6.57), we get

$$\mu a_1 = 0, \lambda_1 a_1 = \mu a_2, \lambda_2 a_2 = \mu a_3, \ldots.$$

Hence, $a_1 = 0$, and therefore, $a_2 = 0$, $a_3 = 0$. However, when $\lambda_i \to 0$, the operator is compact [Problem 4.5.P7] and therefore provides the required example.

4.5.P14. The space $\mathfrak{N}_\lambda(T) = \ker(T - \lambda I)$ is finite-dimensional. Hence, it is a direct summand, that is, there exists a closed subspace W such that

$$H = \mathfrak{N}_\lambda(T) \oplus W.$$

Note that

$$\operatorname{ran}(T-\lambda I)=(T-\lambda I)H=(T-\lambda I)W.$$

If $(T-\lambda I)$ were not bounded from below on W, then λ would be an approximate eigenvalue and hence an eigenvalue of T (since T is compact; see Theorem 4.8.6) with eigenvector in $\mathfrak{N}_\lambda(T)\cap W$. This is not possible, since $\mathfrak{N}_\lambda(T)\cap W=\{0\}$. So, $(T-\lambda I)$ is bounded from below on W, that is, there exists an $\alpha>0$ such that $\|(T-\lambda I)w\|\geq|\alpha|\|w\|$ for all $w\in W$. Let $\{w_n\}_{n\geq 1}$ be a sequence in W, and suppose that $\{(T-\lambda I)w_n\}_{n\geq 1}$ converges to y. For all n, m,

$$\|(T-\lambda I)(w_n-w_m)\|\geq|\alpha|\|w_n-w_m\|$$

and hence, $\{w_n\}_{n\geq 1}$ is a Cauchy sequence. Since W is closed, $\{w_n\}_{n\geq 1}$ converges to a limit $w\in W$, say. Hence, $y=(T-\lambda I)w$ is in $(T-\lambda I)W$. This shows that $\operatorname{ran}(T-\lambda I)$ is closed.

4.5.P15. Let f be a bounded function on the set $\sigma_p(T)$. We define an operator $f(T)$ on H by

$$f(T)x=\sum_{\lambda\in\sigma_p(T)}f(\lambda)P_\lambda x,\quad x\in H.$$

The right-hand side makes sense because the convergence condition of Theorem 2.9.8 is fulfilled, as we now argue. Since the eigenspaces $\mathfrak{N}_\lambda(T)$ are pairwise orthogonal, it follows from Parseval's identity [Theorem 2.9.15(iii)] that

$$\|f(T)x\|^2=\sum_{\lambda\in\sigma_p(T)}|f(\lambda)|^2\|P_\lambda x\|^2$$

and

$$\|x\|^2=\sum_{\lambda\in\sigma_p(T)}\|P_\lambda x\|^2.$$

It follows that

$$\|f(T)x\|=\sup_{\lambda\in\sigma_p(T)}|f(\lambda)|.$$

(Indeed, $\|f(T)x\|^2\leq\sup_{\lambda\in\sigma_p(T)}|f(\lambda)|^2\|x\|^2$. On the other hand, there exists $\lambda_0\in\sigma_p(T)$ such that $|f(\lambda_0)|>\sup_{\lambda\in\sigma_{p(T)}}|f(\lambda)|-\varepsilon$. Apply $f(T)$ to $P_{\lambda_0}(x)$, etc.)

In particular, if $\mu\neq 0$ is not an eigenvalue of T (so $\mu\in\mathbb{R}\backslash\{0\}$), we have $(T-\mu I)^{-1}x=\sum_{\lambda\in\sigma_p(T)}(\lambda-\mu)^{-1}P_\lambda x$ for all $x\in H$.

Suppose now that $\mu\neq 0$ is an eigenvalue of T. Then, $\operatorname{ran}(T-\mu I)$ is closed (see Problem 4-5.P2) and so equals $(\ker(T-\mu I))^\perp$ [see Theorem 3.5.8]. The operator T induces on $(\ker(T-\mu I))^\perp$ a compact self-adjoint operator whose set of eigenvalues is $\sigma_p(T)\backslash\{\mu\}$. We apply (a) to this induced operator and conclude that for $x \in (\ker(T - \mu I))^{\perp}$, the following is valid for all $\tilde{y} \in (\ker(T - \mu I))^{\perp}$:

$$T\tilde{y} - \mu\tilde{y} = x \quad \Leftrightarrow \quad \tilde{y} = \sum_{\lambda \in \sigma_p(T)} (\lambda - \mu)^{-1} P_\lambda x.$$

Next, if $y \in H$, we can write

$y = \tilde{y} + z$ with $\tilde{y} \in (\ker(T - \mu I))^{\perp}$ and $z \in \ker(T - \mu I)$.

It follows that $Ty - \mu y = x \in (\ker(T - \mu I))^{\perp}$ if and only if there exists $z \in (\ker(T - \mu I))$ such that

$$y = z + \sum_{\substack{\lambda \in \sigma_p(T) \\ \lambda \neq \mu}} (\lambda - \mu)^{-1} P_\lambda x.$$

4.5.P16. Putting $s = 0$ and $s = 1$ in the integral equation, we note that $x(0) = 0 = x(1)$. Since $tx(t)$ and $(1 - t)x(t)$ are integrable functions of $t \in [0, 1]$, it follows that the right-hand side of the integral equation is an absolutely continuous function of $s \in [0, 1]$. Hence, x is an absolutely continuous function of $s \in [0, 1]$. This implies $tx(t)$ and $(1 - t)x(t)$ are continuous functions of $t \in [0, 1]$. Thus,

$$\begin{aligned}\lambda x'(s) &= (1 - s)sx(s) - \int_0^s tx(t)\mathrm{d}t - s(1 - s)x(s) + \int_s^1 (1 - t)x(t)\mathrm{d}t \\ &= -\int_0^s tx(t)\mathrm{d}t + \int_s^1 (1 - t)x(t)\mathrm{d}t.\end{aligned}$$

This shows that $x'(s)$ is continuously differentiable and we have

$$\lambda x''(s) = -sx(s) - (1 - s)x(s) = -x(s).$$

The differential equation $\lambda x'' + x = 0$ has a nonzero solution satisfying $x(0) = 0 = x(1)$ if and only if $\lambda = 1/n^2\pi^2$, $n = 1, 2, \ldots$, and in that case, its most general solution is given by $x(s) = c \sin n\pi s$, $s \in [0, 1]$, where c is a constant. Let $\lambda_n = 1/n^2\pi^2$, $n = 1, 2, \ldots$ and $x_n(s) = \sin n\pi s$, $s \in [0, 1]$. Thus, each λ_n is an eigenvalue of T and the corresponding eigenspace $\mathfrak{N}_T(\lambda_n) = \{x : (T - \lambda_n I)x = 0\}$, the linear space generated by $x_n(s)$.

Now, 0 is not an eigenvalue of T. For if $Tx(s) = 0$ for some $x \in H$, then on differentiating $Tx(s)$ with respect to s twice, we conclude that $x(s) = 0$ for all $s \in [0, 1]$. On the other hand, T being compact [Example 4.5.5(ii)], $0 \in \sigma(T)$. Thus,

$$\sigma_p(T) = \{1/n^2\pi^2 : n = 1, 2, ...\} \text{ and } \sigma(\mathrm{T}) = \{0\} \cup \{1/n^2\pi^2 : n = 1, 2, ...\}.$$

Consequently, for $x \in H$ and $s \in [0, 1]$,

$$Tx(s) = \Sigma\lambda_n(x, x_n)x_n,$$

that is,

$$(1-s)\int_0^s tx(t)\mathrm{d}t + s\int_s^1 (1-t)x(t)\mathrm{d}t = \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[\int_0^1 x(t)\sin n\pi t\,\mathrm{d}t\right]\sin n\pi s,$$

where the series converges absolutely and uniformly [using Weierstrass M-test] for $s \in [0, 1]$.

For a nonzero constant λ and $y \in L^2[0, 1]$, consider the integral equation

$$\left[(1-s)\int_s^1 tx(t)\mathrm{d}t + s\int_s^1 (1-t)x(t)\mathrm{d}t\right] - \lambda x(s) = y(s), s \in [0, 1].$$

If $\lambda \neq 1/n^2\pi^2 : n = 1, 2, \ldots$, then the unique solution of the integral equation [by Fredholm alternative (a)] is

$$x(s) = (T - \lambda I)^{-1}y(s) = \sum_{n=1}^{\infty}(1 - n^2\pi^2\lambda)^{-1}n^2\pi^2(y, x_n)x_n(s), s\in[0, 1].$$

If $\lambda = 1/n_0^2\pi^2)$ for some $n_0 = 1, 2, \ldots$, then the integral equation has infinitely many solutions if $x \in \mathfrak{N}_T(1/n_0^2\pi^2)$ and no solution otherwise. In the first case, the solutions are given by

$$x(s) = z + \sum_{\substack{\lambda \in \sigma_p(T) \\ \lambda \neq n_0^2\pi^2}} ((n^2\pi^2)^{-1} - \lambda)^{-1}(y, x_n)x_n(s), s \in [0, 1], z \in \mathfrak{N}_T(n_0^2\pi^2).$$

4.5.P17. Let $A_1 = \begin{bmatrix} a_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & a_2 & 0 & \cdots \\ a_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$,

$$A_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_k & 0 & \cdots \\ 0 & 0 & \cdots & a_k & 0 & 0 & \cdots \\ \vdots & \vdots & ⋰ & 0 & 0 & 0 & \cdots \\ 0 & a_k & \cdots & 0 & 0 & 0 & \cdots \\ a_k & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and $B_n = \sum_{i=1}^{n} A_i$. We obtain $\|A_k\| = |a_k|$, and moreover, $\|(B_n x - B_m x)\| = \left\|\sum_{i=m+1}^{n} A_i x\right\| \leq \|x\| \sum_{i=m+1}^{n} \|A_i\| = \|x\| \sum_{i=m+1}^{n} |a_i| \to 0$. Hence, $\|B_n - B_m\| \to 0$ as $m, n \to \infty$, and thus, the series $\sum_{i=1}^{\infty} A_j$ converges in $\mathcal{B}(H)$. The operator A, being its sum, is bounded. Since each B_n is of finite rank and furthermore

$$\|(A - B_n)x\| = \left\|\sum_{i=n+1}^{\infty} A_i x\right\| \leq \|x\| \sum_{i=n+1}^{\infty} \|A_i\| = \|x\| \sum_{i=n+1}^{\infty} |a_i| \to 0 \text{ as } n \to \infty,$$

which implies A is compact.

6.24 Problem Set 4.6

4.6.P1. Clearly, T is linear.

$$\begin{aligned}
||Tf||_2^2 &= \int_0^1 |Tf(x)|^2 \mathrm{d}x = \int_0^1 \left|\int_0^x F(t)\mathrm{d}t\right|^2 \mathrm{d}x \\
&\leq \int_0^1 \left(\int_0^x |f(t)|\mathrm{d}t\right)^2 \mathrm{d}x \leq \int_0^1 \left(\int_0^x 1^2 \mathrm{d}t \int_0^x |f(t)|^2 \mathrm{d}t\right) \mathrm{d}x \\
&\leq \int_0^1 x\left(\int_0^1 |f(t)|^2 \mathrm{d}t\right) \mathrm{d}x \leq \frac{1}{2}||f||_2^2.
\end{aligned}$$

Thus, T is a bounded linear operator on $L^2[0, 1]$.

The vectors $e_k(t) = \exp(i2\pi kt)$, $k = 0, \pm 1, \pm 2, \ldots$, constitute a complete orthonormal sequence in $L^2[0, 1]$. We shall show that $\sum_{-\infty}^{\infty} ||Te_k||^2 < \infty$. Now,

$$(Te_k)(t) = \int_0^t \exp(i2\pi ks)\mathrm{d}s = [\exp(i2\pi kt) - 1]/2\pi ik$$

and

$$(Te_k, Te_k) = ([\exp(i2\pi kt) - 1]/2\pi ik, [\exp(i2\pi kt) - 1]/2\pi ik) = \frac{1}{2\pi^2 k^2}.$$

So,

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \|Te_k\|^2 = 2\sum_{k=1}^{\infty} \frac{1}{2\pi^2 k^2} = \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{6}.$$

Also,

$$\|Te_0\|^2 = (t, t) = \frac{1}{3}.$$

Consequently,

$$\sum_{k=-\infty}^{\infty} \|Te_k\|^2 = \frac{1}{2} < \infty.$$

4.6.P2. In Example (viii) of 3.2.5, it has been proved that K is a bounded liner operator with $\|K\|_2 \leq \|k\|_2$.

Let $\{e_n\}_{n \geq 1}$ be an orthonormal basis for $L^2(\mu)$, and for m, n = 1, 2, …, let

$$\omega_{m,n}(s,t) = e_m(s)\overline{e_n(t)}, \quad s, t \in X.$$

$\{\omega_{m,n}\}$ is an orthonormal system for $L^2(\mu \times \mu)$. For m, n = 1, 2, …,

$$\begin{aligned}(k, \omega_{m,n}) &= \int\int k(s,t)\overline{e_m(s)}e_n(t)d(\mu\times\mu)(s,t) \\ &= \int\left[\int k(s,t)e_n(t)d\mu(t)\overline{e_m(s)}d\mu(s)\right] \\ &= (Ke_n, e_m),\end{aligned}$$

using Fubini's theorem. By Parseval's formula,

$$\sum_{n=1}^{\infty}\|Ke_n\|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|(Ke_n, e_m)|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|(k, \omega_{m,n})\right|^2 = \|K\|_2^2 < \infty.$$

Hence, K is a Hilbert–Schmidt operator.

Remark In the solution of the above problem, we have used the fact that if $\{e_n\}_{n \geq 1}$ is an orthonormal basis for $L^2(\mu)$, then $\left\{e_m(s)\overline{e_n(t)}\right\}$ is an orthonormal basis for $L^2(\mu \times \mu)$. A proof of this statement is as follows.

$$(\omega_{m,n}, \omega_{p,q}) = (e_m, e_p)(e_q, e_n) = 0$$

since $(e_m, e_p) = \delta_{m,p}$ and $(e_q, e_n) = \delta_{q,n}$, where $\delta_{i,j}$ is the Kronecker delta. This proves that the set $\{\omega_{m,n}\}$ is an orthonormal set in $L^2(\mu \times \mu)$.

Consider any $x \in H$ such that $(x, \omega_{m,n}) = 0$ for all $m, n = 1, 2, \ldots$. For fixed $s \in X$, let $x_s(t) = x(s,t)$, $t \in X$. Now,

$$(x,x) = \iint |x(s,t)|^2 d(\mu \times \mu)(s,t)$$
$$= \int \left(\int |x_s(t)|^2 d\mu(t) \right) d\mu(s).$$

By Fubini's theorem, it follows that $x_s \in L^2(\mu)$ for almost every s, and since $\{e_n\}_{n \geq 1}$ is an orthonormal basis for $L^2(\mu)$, we have

$$\int |x_s(t)|^2 d\mu(t) = \|\overline{x}_s\|_2^2 = \sum_{n=1}^{\infty} |(\overline{x}_s, e_n)|^2,$$

Using Parseval's formula. Thus,

$$(x,x) = \int \left[\sum_{n=1}^{\infty} |(\overline{x}_s, e_n)|^2 \right] d\mu(s),$$

using the monotone convergence theorem.

Fix $n = 1, 2, \ldots$ and let $y_n(s) = (e_n, \overline{x})$, $s \in X$. Then, $y_n \in L^2(\mu)$, and for all $m = 1, 2, \ldots$,

$$0 = \left(x, \omega_{m,n}\right) = \iint x(s,t)\overline{e_m(s)}e_n(t) d(\mu \times \mu)(s,t)$$
$$= \int \left[\int x(s,t) e_n(t) d\mu(t) \right] \overline{e_m(s)} d\mu(s)$$
$$= \int y_n(s) \overline{e_m(s)} d\mu(s).$$

Since $\{e_m\}_{m \geq 1}$ is an orthonormal basis for $L^2(\mu)$, it follows that $y_n(s) = 0$. As this is true for each n, we see that

$$(x,x) = \sum_{n=1}^{\infty} \|y_n\|_2^2 = 0.$$

Thus, $x = 0$.

4.6.P3. Let $\{T_n\}_{n \geq 1}$ be a Cauchy sequence in HS. Then, $\|T_n - T_m\|_{\mathrm{HS}} \to 0$ as m, $n \to \infty$. Since $\|T_n - T_m\| \leq \|T_n - T_m\|_{\mathrm{HS}}$, the sequence $\{T_n\}_{n \geq 1}$ converges uniformly, and thus, it must converge uniformly to some operator T. We prove that $T \in \mathrm{HS}$ and $\|T - T_n\|_{\mathrm{HS}} \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, choose N so that $\|T_n - T_m\|_{\mathrm{HS}} < \varepsilon$ for $m,n \geq N$. This implies that $\sum_{j \in J} \left\|(T_n - T_m)x_j\right\|^2 \leq \varepsilon^2$ for $m, n \geq N$ and any finite set J of indices. Consequently, $T_n - T \in \mathrm{HS}$, and therefore, also $T \in \mathrm{HS}$. Moreover, $\|T - T_n\|_{\mathrm{HS}} \le \varepsilon$ for all $n \ge N$.

$$\sum_{j\in j} \left\|(T_n - T)x_j\right\|^2 \le \varepsilon^2 \quad \text{for } n \ge N \text{ and any finite set } J \text{ of indices.}$$

6.25 Problem Set 4.7

4.7.P1. We shall first prove the second assertion. Let $\{x_1, x_2, \ldots, x_n\}$ be an orthonormal basis for H, and let $Sx_i = \sum_{j=1}^n a_{ij}x_j$, $Tx_i = \sum_{j=1}^n b_{ij}x_j$. Then,

$$STx_i = S\left(\sum_{j=1}^n b_{ij}x_j\right) = \sum_{j=1}^n b_{ij}\left(\sum_{k=1}^n a_{jk}x_k\right)$$

and

$$TSx_i = T\left(\sum_{j=1}^n a_{ij}x_j\right) = \sum_{j=1}^n a_{ij}\left(\sum_{k=1}^n b_{jk}x_k\right).$$

Hence,

$$\mathrm{tr}(ST) = \sum_{i=1}^n \sum_{j=1}^n b_{ij}a_{ji} = \mathrm{tr}(TS).$$

To establish the first assertion, let $\{y_1, y_2, \ldots, y_n\}$ be any other orthonormal basis for H. Then, the linear operator U defined by $y_i = Ux_i$, $i = 1, 2, \ldots, n$ is a one-to-one map of H into itself. We shall calculate tr(T) relative to the basis $\{y_1, y_2, \ldots, y_n\}$. Observe that $SU^{-1}y_i = Sx_i = \sum_{j=1}^n a_{ij}x_j = U^{-1}(\sum_{j=1}^n a_{ij}y_j)$, and so,

$$USU^{-1}y_i = \sum_{j=1}^n a_{ij}y_j.$$

From this, it follows that tr(USU^{-1}) calculated using the basis $\{y_1, y_2, \ldots, y_n\}$ is $\sum_{j=1}^n a_{ii}$. Moreover, tr(USU^{-1}) = tr(S) = tr($U^{-1}US$), using the fact proved above that tr(ST) = tr(TS); both traces have been computed with respect to the basis $\{y_1, y_2, \ldots, y_n\}$.

4.7.P2. (a) If $\{x_1, x_2, \ldots, x_n\}$ is an orthonormal basis for H and $Tx_i = \sum_{j=1}^n a_{ij}x_j$,

$$\det(\lambda I - T) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}.$$

The characteristic polynomial of T is of the form

$$\lambda^n - C_1\lambda^{n-1} + \cdots + C_n,$$

whose roots are the eigenvalues of T. Thus, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of T (counting multiplicities), we obtain $C_1 = \sum_{i=1}^n \lambda_i$. On the other hand, from the definition of the determinant, $C_1 = \sum_{i=1}^n a_{ii}$, i.e. $\mathrm{tr}(T) = \sum_{i=1}^n \lambda_i$. Since $a_{ij} = (Tx_i, x_j)$, we also have $\mathrm{tr}(T) = \sum_{i=1}^n (Tx_i, x_i)$.

(b) Since T is nilpotent, there exists some positive integer n such that $T^n = O$, and it follows that $r(T) = 0$, which implies $\sigma(T) = \{0\}$. In view of the fact that H is finite-dimensional, 0 is the only eigenvalue. Consequently, by (a) above, the trace is 0.

4.7.P3. (i) $\|\varphi \otimes \overline{\psi}\| = \sup\{\|(f, \psi)\varphi\| : \|f\| = 1\} \leq \|\varphi\|\|\psi\|$. Also, $\left\|(\varphi \otimes \overline{\psi})\frac{\psi}{\|\psi\|}\right\| = \frac{1}{\|\psi\|}\|(\psi, \psi)\varphi\| = \|\varphi\|\|\psi\|$.

So,

$$\|\varphi \otimes \overline{\psi}\| = \|\varphi\|\|\psi\|.$$

(ii) $\left\|\sum_j (\varphi \otimes \overline{\psi})x_j\right\|^2 = \left\|\sum_j (x_j, \psi)\varphi\right\|^2 \leq \left\|\sum_j (x_j, \psi)\right\|^2 \|\varphi\|^2 = \|\psi\|^2\|\varphi\|^2$, where $\{x_j : j \in J\}$ denotes an orthonormal basis for H. So,

$$\|\varphi \otimes \overline{\psi}\|_{\mathrm{HS}} \leq \|\varphi\|\|\psi\|.$$

Now, consider a basis (orthonormal) which contains $\psi/\|\psi\|$ as a vector. Then,

$$\|(\varphi \otimes \overline{\psi})\psi/\|\psi\|\|^2 \leq \left\|\sum_j (\varphi \otimes \overline{\psi})x_j\right\|^2 = \|\varphi \otimes \overline{\psi}\|_{\mathrm{HS}}^2.$$

In view of (i), the LHS of the above inequality equals $\|\varphi\|^2\|\psi\|^2$ and this implies

$$\|\varphi \otimes \overline{\psi}\|_{\mathrm{HS}} \geq \|\varphi\|\|\overline{\psi}\|.$$

Thus, $\varphi \otimes \overline{\psi}$ is a Hilbert–Schmidt operator and $\|\varphi \otimes \psi\|_{\mathrm{HS}} = \|\varphi\|\|\psi\|$.

(iii) Set $x_1 = \psi/\|\psi\|$. Then, $\|x_1\| = 1$. Moreover,

$$\left|\varphi\otimes\overline{\psi}\right|^2 x_1 = \left(\varphi\otimes\overline{\psi}\right)^*\left(\varphi\otimes\overline{\psi}\right)x_1 = \left(\psi\otimes\overline{\varphi}\right)\left(\varphi\otimes\overline{\psi}\right)x_1$$
$$= \left(\psi\otimes\overline{\varphi}\right)(x_1,\psi)\varphi = (\varphi,\varphi)(\psi,\psi)x_1.$$

This shows that x_1 is an eigenvector of $\left|\varphi\otimes\overline{\psi}\right|^2$ with eigenvalue $\|\psi\|^2\|\varphi\|^2$. Also, every vector orthogonal to x_1 is an eigenvector of $\left|\varphi\otimes\overline{\psi}\right|^2$ corresponding to the eigenvalue 0. Choose a basis $\{x_j\}$ in H having x_1 as one of its elements. Then,

$$\left|\varphi\otimes\overline{\psi}\right|x_1 = \|\psi\|\|\varphi\|x_1 \quad \text{and } \left|\varphi\otimes\overline{\psi}\right|x_j = 0 \quad \text{for } j\neq 1.$$

Consequently,

$$\left\|\varphi\otimes\overline{\psi}\right\|_{\text{tr}} = \sum_j \left(\left(\varphi\otimes\overline{\psi}\right)x_j, x_j\right) = \left(\left(\varphi\otimes\overline{\psi}\right)x_1, x_1\right) = \|\varphi\|\|\psi\|.$$

4.7.P4. That TC(H) is an ideal in $\mathcal{B}(H)$ has been proved [Proposition 4.7.9(c)]. It remains to prove that $\|\cdot\|_{\text{tr}}$ is a norm on TC(H).

(i) $\|T\|_{\text{tr}} = \sum\limits_j \left(|T|x_j, x_j\right) \geq 0$, where $\{x_j : j\in J\}$ is an orthonormal basis for H. $\|T\|_{\text{tr}} = 0$ if and only if $T = O$. Suppose $\|T\|_{\text{tr}} = 0$. Then, $\sum\limits_j \left(|T|^{1/2}x_j, |T|^{1/2}x_j\right) = 0$, which implies $T = O$.

(ii) For $\alpha\in\mathbb{C}, T\in\text{TC}(H), \|\alpha T\|_{\text{tr}} = \sum_j \left(|\alpha T|x_j, x_j\right) = |\alpha|\sum_j \left(|T|x_j, x_j\right) = |\alpha||T|_{\text{tr}}$.

(iii) Let $S = W|S|$, $T = W_1|T|$ and $|S+T| = W_2{}^*(S+T)$, where W, W_1 and W_2 are partial isometries. Then,

$$|S+T| = W_2{}^*S + W_2{}^*T = W_2{}^*W|S| + W_2{}^*W_1|T|.$$

Hence,

$$\begin{aligned}
\|S+T\|_{\text{tr}} &= \sum_j \left(|S+T|x_j, x_j\right)\\
&= \sum_j \left(W_2{}^*W|S|x_j, x_j\right) + \sum_j \left(W_2{}^*W_1|T|x_j, x_j\right)\\
&= \sum_j \left(|S|x_j, W^*W_2x_j\right) + \sum_j \left(|T|x_j, W_1{}^*W_2x_j\right)\\
&= \sum_j \left(|S|^{\frac12}x_j, |S|^{\frac12}W^*W_2x_j\right) + \sum_j \left(|T|^{\frac12}x_j, |T|^{\frac12}W_1{}^*W_2x_j\right)\\
&\leq \left[\sum_j \left\||S|^{\frac12}x_j\right\|^2\right]^{\frac12}\left[\sum_j \left\||S|^{\frac12}W^*W_2x_j\right\|^2\right]^{\frac12} + \left[\sum_j \left\||T|^{\frac12}x_j\right\|^2\right]^{\frac12}\left[\sum_j \left\||T|^{\frac12}W_1{}^*W_2x_j\right\|^2\right]^{\frac12}\\
&\leq \left\||S|^{\frac12}\right\|_{\text{HS}}^2 + \left\||T|^{\frac12}\right\|_{\text{HS}}^2\\
&= \|S\|_{\text{tr}} + \|T\|_{\text{tr}}.
\end{aligned}$$

Remark The above calculation shows that $S+T\in \mathrm{TC}(H)$ if $S\in \mathrm{TC}(H)$ and $T\in \mathrm{TC}(H)$.

4.7.P5. Suppose $T \in \mathrm{TC}(H)$. It follows from Proposition 4.7.5 that since $|T|^{\frac{1}{2}} \in \mathrm{HS}(H)$. Let $T = W|T|$ be the polar decomposition of T. Then, $T = (W|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}) \in \mathrm{HS}(H)$ since the class of Hilbert–Schmidt operators is an ideal in $\mathcal{B}(H)$. Observe that for every $x \in H$,

$$\begin{aligned}\|Tx\|^2 &= (Tx, Tx) = (T^*Tx, x) = \left(|T|^2x, x\right)\\ &= \||T|x\|^2 = \left\||T|^{\frac{1}{2}}|T|^{\frac{1}{2}}x\right\|^2 \le \left\||T|^{\frac{1}{2}}\|^2\||T|^{\frac{1}{2}}x\right\|^2 \le \|T\|(|T|x, x)\end{aligned}$$

since

$$\begin{aligned}\left\||T|^{\frac{1}{2}}\right\|^2 &= \sup_{\|x\|=1}\left(|T|^{\frac{1}{2}}x, |T|^{\frac{1}{2}}x\right) = \sup_{\|x\|=1}(|T|x, x) \le \sup_{\|x\|=1}\||T|x\|\\ &= \||T|\| \le \|T\|.\end{aligned}$$

So,

$$\begin{aligned}\|T\|_{\mathrm{HS}} &= (\Sigma_j\|Tx_j\|^2)^{\frac{1}{2}}\\ &\le \|T\|^{\frac{1}{2}}(\Sigma_j(|T|x_j, x_j))^{\frac{1}{2}}\\ &\le \|T\|_{\mathrm{HS}}^{\frac{1}{2}}(\|T\|_{\mathrm{tr}})^{\frac{1}{2}},\end{aligned}$$

which implies

$$\|T\|_{\mathrm{HS}} \le \|T\|^{\frac{1}{2}}(\|T\|_{\mathrm{tr}})^{\frac{1}{2}}.$$

Since $\|T\| \le \|T\|_{\mathrm{HS}}$, this further implies

$$\|T\|_{\mathrm{HS}} \le \|T\|_{\mathrm{tr}}.$$

4.7.P6. Let $T = W|T|$ be the polar decomposition of T. Since $|T| = W^*T$, it is a compact positive operator, and the spectral theorem [Problem 4-8.P1] implies that $|T|$ can be diagonalised. Let $\{x_n\}_{n\ge 1}$ be an orthonormal basis of H, consisting of eigenvectors of $|T|$ such that $|T| = \Sigma\alpha_nP_n$, $\alpha_n \ge 0$ is the diagonalisation of $|T|$. If $T \in \mathrm{TC}(H)$, then $\infty > \mathrm{tr}(|T|) = \|T\|_{\mathrm{tr}} = \Sigma(|T|x_n,x_n) = \Sigma\alpha_n$. Conversely, if $\sum\alpha_n < \infty$, then it is easily seen that $|T|$ is trace class, and thus, $T = W|T| \in \mathrm{TC}(H)$ since $\mathrm{TC}(H)$ is an ideal.

4.7.P7. Follow the proof for Problem 4.6.P3.

4.7.P8. Recall that if T is a compact self-adjoint operator on H, then

$$T = \Sigma_n\lambda_nP_n,$$

where λ_n are the distinct nonzero eigenvalues of T (countable in number), P_n is the projection of H onto $\ker(T - \lambda_n I)$, $P_nP_m = P_mP_n = O$ if $n \neq m$, and each λ_n is real [Theorem 4.4.2]. If $\{x_n\}_{n\geq 1}$ denote the orthonormal basis of eigenvectors of T corresponding to λ_n, then

$$\mathrm{tr}(T) = \Sigma_n(Tx_n, x_n) = \Sigma_n\lambda_n.$$

This completes the argument.

6.26 Problem Set 4.8

4.8.P1. (a) By remark following Theorem 4.4.5, $||T||$ or $-||T||$ is a nonzero eigenvalue of T, that is, T has an eigenvalue λ_1 $|\lambda_1| = ||T|| > 0$ with. Let $x_1, ||x_1|| = 1$, be an eigenvector of T corresponding to λ_1, and let H_1 be the subspace orthogonal to the span $[\{x_1\}]$. The restriction T_1 of T to H_1 has its range in H_1 (for $x \in H_1$, we have $(Tx, x_1) = (x, Tx_1) = (x, \lambda_1 x_1) = \lambda_1(x, x_1) = 0$) and thus may be considered as a compact self-adjoint operator on H_1. If $T_1 \neq O$, by a repetition of the above argument, T_1 has an eigenvalue λ_2 such that $|\lambda_2| = ||T_1|| > 0$. Clearly,

$$\|\lambda_1\| = \|T\| \geq \|T_1\| = |\lambda_2|.$$

Let $x_2 \in H_1$ be an eigenvector of T_1 (and hence of T) of norm 1 corresponding to λ_2. It follows that $(x_2, x_1) = 0$ and $Tx_2 = \lambda_2 x_2$.

In the case T is not identically zero on the subspace H_2 orthogonal to both x_1 and x_2, H_2 is a Hilbert space and $T_2|_{H_2}$ is a compact self-adjoint operator on H_2. If $T_2 \neq O$, we construct again a vector x_3 subject to the conditions $||x_3|| = 1$, $(x_3, x_i) = 0$, $i = 1, 2$ and

$$Tx_3 = T_2x_3 = \lambda_3 x_3,$$

where $|\lambda_3| = ||T_2||$ and $|\lambda_2| \geq |\lambda_3|$. Continuing this process, we obtain a finite or infinite orthonormal sequence $x_1, x_2, \ldots$ of proper vectors of T corresponding to the nonzero proper values $\lambda_1, \lambda_2, \ldots$; we have $|\lambda_1| \geq |\lambda_2| \geq \ldots$. We show that $\lambda_n \to 0$ as $n \to \infty$. Assume for a moment that there is some $\delta > 0$ such that $|\lambda_{n_j}| \geq \delta$ for $j = 1, 2, \ldots$ with $n_1 < n_2 < \ldots$. If $j \neq k$, then $\left\|Tx_{n_j} - Tx_{n_k}\right\|^2 = \left\|\lambda_{n_j}x_{n_j} - \lambda_{n_k}x_{n_k}\right\|^2 = \left|\lambda_{n_j}\right|^2 + |\lambda_{n_k}|^2 \geq 2\delta^2$. But this is not possible, since $\left\{x_{n_j}\right\}$ is a bounded sequence and T is a compact operator.

In the case of a finite sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$, we have $T = O$ on the orthogonal complement of the Hilbert space spanned by $x_1, x_2, \ldots, x_n$. If $\lambda_1, \lambda_2, \ldots$ is an infinite sequence, then $Tx = \mu x$ with $||x|| = 1$ and $(x, x_i) = 0$, $i = 1, 2, \ldots$ implies $|\mu| = (Tx, x) \leq |\lambda_i|$ for every i, and hence, $\mu = 0$. It follows that the so-constructed finite or infinite sequence $\lambda_1, \lambda_2, \ldots$ contains all the nonzero proper values of T, each proper value appearing in the sequence as often as its multiplicity demands. Since $|\lambda_i| > 0$, the multiplicity of each proper value is finite.

(b) Now, let $\{\psi_i\}$ be any corresponding orthonormal sequence of proper vectors, that is, $T\psi_i = \lambda_i\psi_i$, $i = 1, 2, \ldots$. The subspace H' spanned by the ψ_i reduces T. Moreover, $T = O$ on $H'^{\perp}$. Thus,

$$T = \sum \lambda_i P_i.$$

We extend $\{\psi_i\}$ to a complete orthonormal family in our space H by adding $\{\omega_j\}$. Then, $T\omega_j = 0$ for all j, i.e. ω_j is a proper vector of T corresponding to the proper value 0.

(c) The family $\{\psi_i\} \cup \{\omega_j\}$ is a complete orthonormal family of eigenvectors. This completes the proof.

6.27 Problem Set 4.9

4.9.P1. Observe that $P_2(E)^2 = UP_1(E)U^*UP_1(E)U^* = UP_1(E)^2U^*$, using $U^*U = I$. Since $P_1(E)^2 = P_1(E)$, it follows that $P_2(E)^2 = P_2(E)$. Also, $P_2(E)^* = (UP_1(E)U^*)^* = UP_1(E)U^* = P_2(E)$, since $P_1(E)^* = P_1(E)$.

If $\{E_n\}_{n\geq 1}$ is a countable family of mutually disjoint measurable sets, then

$$\begin{aligned} P_2\left(\bigcup_{k=1}^{\infty} E_k\right) &= UP_1\left(\bigcup_{k=1}^{\infty} E_k\right)U^* \\ &= U\sum_{k=1}^{\infty} P_1(E_k)U^* \\ &= \sum_{k=1}^{\infty} UP_1(E_k)U^* \\ &= \sum_{k=1}^{\infty} P_2(E_k). \end{aligned}$$

4.9.P2. For $x, y \in H$, $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$,

$$\begin{aligned} (P(E)x, y) &= ((P_1(E)x_1, P_2(E)x_2, \ldots), (y_1, y_2, \ldots)) \\ &= \sum_{k=1}^{\infty} (P_k(E)x_k, y_k) = \sum_{k=1}^{\infty} (P_k(E)^*x_k, y_k) \\ &= ((P_1(E)^*x_1, P_2(E)^*x_2, \ldots), (y_1, y_2, \ldots)) \\ &= (P(E)^*x, y), \end{aligned}$$

which implies $P(E) = P(E)^*$ for all measurable subsets E. Similarly, it can be shown that $P(E)^2 = P(E)$ and P is countably additive (since P_1, P_2, … are countably additive.

4.9.P3. It is easy to see that μ_x is a probability measure if P is a spectral measure and $\mu_x(X) = ||P(X)x||^2 = ||x||^2 = 1$. Indeed, for a family $\{E_i\}_{i\geq 1}$ of pairwise disjoint sets in Σ with $\bigcup_i E_i = E$, we have

$$\mu_x(E) = \left(P\left(\bigcup_i E_i\right)x, x\right) = \Sigma_i(P(E_i)x, x) = \Sigma_i\mu_x(E_i).$$

Conversely, suppose that μ_x is a probability measure. If E_1 and E_2 are two disjoint measurable sets, then for all $x \in H$,

$$(P(E_1 \cup E_2)x, x) = \mu_x(E_1 \cup E_2) = \mu_x(E_1) + \mu_x(E_2) = (P(E_1)x, x) + (P(E_2)x, x).$$

Hence,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

Since $P(\cdot)$ is a projection operator, this means $P(E_1)\perp P(E_2)$ whenever $E_1 \cap E_2 = \varnothing$.

Now, let $\{E_n\}_{n\geq 1}$ be a family of mutually disjoint sets. Then, $\{P(E_n)\}_{n\geq 1}$ is a family of mutually orthogonal projections. Hence, the series $\Sigma_n P(E_n)$ converges strongly to a projection [Theorem 3.9.6]. So, we have

$$\left(\sum_{n=1}^{\infty} P(E_n)x, x\right) = \sum_{n=1}^{\infty}(P(E_n)x, x) = \sum_{n=1}^{\infty}\mu_x(E_n) = \mu_x\left(\bigcup_{n=1}^{\infty} E_n\right)$$
$$= \left(P\left(\bigcup_{n=1}^{\infty} E_n\right)x, x\right).$$

Thus, $P(\cdot)$ is countably additive on Σ. This shows that P is a projection-valued measure.

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