Second Edition

Linear Algebra

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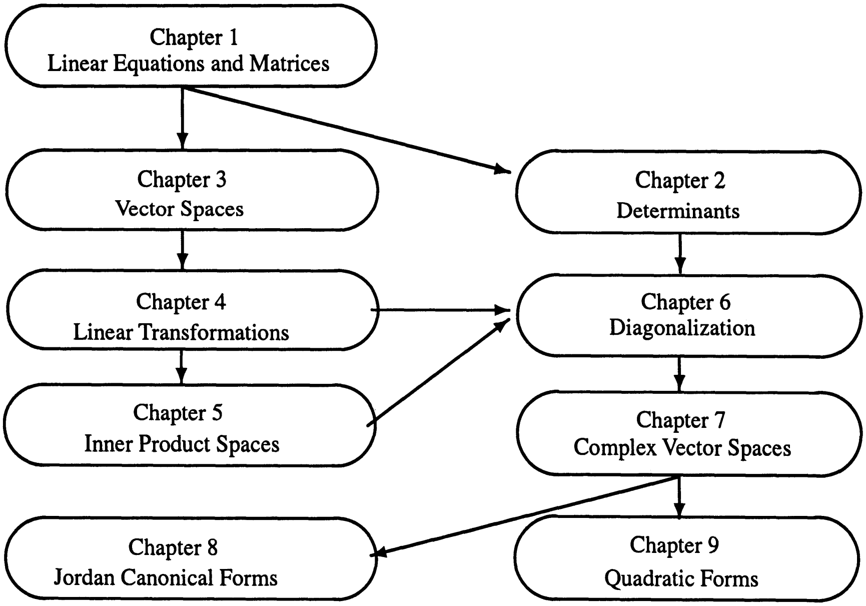
Preface to the Second Edition

This second edition is based on many valuable comments and suggestions from readers of the first edition. In this edition, the last two chapters are interchanged and also several new sections have been added. The following diagram illustrates the dependencies of the chapters.



The major changes from the first edition are the following.

(1) In Chapter 2, Section 2.5.1 "Miscellaneous examples for determinants" is added as an application.

(2) In Chapter 4, "A homogeneous coordinate system" is introduced for an application in computer graphics.

(3) In Chapter 5, Section 5.7 "Relations of fundamental subspaces" and Section 5.8 "Orthogonal matrices and isometries" are interchanged. "Least squares solutions," "Polynomial approximations" and "Orthogonal projection matrices" are collected together in Section 5.9–Applications.

(4) Chapter 6 is entitled "Diagonalization" instead of "Eigenvectors and Eigenvalues." In Chapters 6 and 8, "Recurrence relations," "Linear difference equations" and "Linear differential equations" are described in more detail as applications of diagonalizations and the Jordan canonical forms of matrices.

(5) In Chapter 8, Section 8.5 "The minimal polynomial of a matrix" has been added to introduce more easily accessible computational methods for A^n and e^A, with complete solutions of linear difference equations and linear differential equations.

(6) Chapter 8 "Jordan Canonical Forms" and Chapter 9 "Quadratic Forms" are interchanged for a smooth continuation of the diagonalization problem of matrices. Chapter 9 "Quadratic Forms" is extended to a complex case and includes many new figures.

(7) The errors and typos found to date in the first edition have been corrected.

(8) Problems are refined to supplement the worked-out illustrative examples and to enable the reader to check his or her understanding of new definitions or theorems. Additional problems are added in the last exercise section of each chapter. More answers, sometimes with brief hints, are added, including some corrections.

(9) In most examples, we begin with a brief explanatory phrase to enhance the reader's understanding.

This textbook can be used for a one- or two-semester course in linear algebra. A theory oriented one-semester course may cover Chapter 1, Sections 1.1–1.4, 1.6–1.7; Chapter 2 Sections 2.1–2.3; Chapter 3 Sections 3.1–3.6; Chapter 4 Sections 4.1–4.6; Chapter 5 Sections 5.1–5.4; Chapter 6 Sections 6.1–6.2; Chapter 7 Sections 7.1–7.4 with possible addition from Sections 1.8, 2.4 or 9.1–9.4. Selected applications are included in each chapter as appropriate. For a beginning applied algebra course, an instructor might include some of them in the syllabus at his or her discretion depending on which area is to be emphasized or considered more interesting to the students.

In definitions, we use bold face for the word being defined, and sometimes an italic or shadowbox to emphasize a sentence or undefined or post-defined terminology.

Acknowledgement: The authors would like to express our sincere appreciation for the many opinions and suggestions from the readers of the first edition including many of our colleagues at POSTECH. The authors are also indebted to Ki Hang Kim and Fred Roush at Alabama State University and Christoph Dalitz at Höchschule Niederrhein for improving the manuscript and selecting the newly added subjects in this edition. Our thanks again go to Mrs. Kathleen Roush for grammatical corrections in the final manuscript, and also to the editing staff of Birkhäuser for gladly accepting the second edition for publication.

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Preface to the First Edition

Linear algebra is one of the most important subjects in the study of science and engineering because of its widespread applications in social or natural science, computer science, physics, or economics. As one of the most useful courses in undergraduate mathematics, it has provided essential tools for industrial scientists. The basic concepts of linear algebra are vector spaces, linear transformations, matrices and determinants, and they serve as an abstract language for stating ideas and solving problems.

This book is based on lectures delivered over several years in a sophomore-level linear algebra course designed for science and engineering students. The primary purpose of this book is to give a careful presentation of the basic concepts of linear algebra as a coherent part of mathematics, and to illustrate its power and utility through applications to other disciplines. We have tried to emphasize computational skills along with mathematical abstractions, which have an integrity and beauty of their own. The book includes a variety of interesting applications with many examples not only to help students understand new concepts but also to practice wide applications of the subject to such areas as differential equations, statistics, geometry, and physics. Some of those applications may not be central to the mathematical development and may be omitted or selected in a syllabus at the discretion of the instructor. Most basic concepts and introductory motivations begin with examples in Euclidean space or solving a system of linear equations, and are gradually examined from different points of view to derive general principles.

For students who have finished a year of calculus, linear algebra may be the first course in which the subject is developed in an abstract way, and we often find that many students struggle with the abstractions and miss the applications. Our experience is that, to understand the material, students should practice with many problems, which are sometimes omitted. To encourage repeated practice, we placed in the middle of the text not only many examples but also some carefully selected problems, with answers or helpful hints. We have tried to make this book as easily accessible and clear as possible, but certainly there may be some awkward expressions in several ways. Any criticism or comment from the readers will be appreciated.

We are very grateful to many colleagues in Korea, especially to the faculty members in the mathematics department at Pohang University of Science and Technology (POSTECH), who helped us over the years with various aspects of this book. For their valuable suggestions and comments, we would like to thank the students at POSTECH, who have used photocopied versions of the text over the past several years. We would also like to acknowledge the invaluable assistance we have received from the teaching assistants who have checked and added some answers or hints for the problems and exercises in this book. Our thanks also go to Mrs. Kathleen Roush who made this book much more readable with grammatical corrections in the final manuscript. Our thanks finally go to the editing staff of Birkhäuser for gladly accepting our book for publication.

Jin Ho Kwak
Sungpyo Hong

April 1997, Pohang, South Korea

Contents

Linear Algebra

1

Linear Equations and Matrices

1.1 Systems of linear equations

One of the central motivations for linear algebra is solving a system of linear equations. We begin with the problem of finding the solutions of a system of m linear equations in n unknowns of the following form:

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}\right.$$

where $x_1,\ x_2,\ \ldots,\ x_n$ are the unknowns and a_{ij}'s and b_i's denote constant (real or complex) numbers.

A sequence of numbers $(s_1, s_2, \ldots, s_n)$ is called a **solution** of the system if $x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n$ satisfy each equation in the system simultaneously. When $b_1 = b_2 = \cdots = b_m = 0$, we say that the system is **homogeneous**.

The central topic of this chapter is to examine whether or not a given system has a solution, and to find the solution if it has one. For instance, every homogeneous system always has at least one solution $x_1 = x_2 = \cdots = x_n = 0$, called the **trivial solution**. Naturally, one may ask whether such a homogeneous system has a nontrivial solution or not. If so, we would like to have a systematic method of finding all the solutions. A system of linear equations is said to be **consistent** if it has at least one solution, and **inconsistent** if it has no solution.

For example, suppose that the system has only one linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

If $a_i = 0$ for $i = 1, \ldots, n$, then the equation becomes $0 = b$. Thus it has no solution if $b \neq 0$ (nonhomogeneous), or has infinitely many solutions (any n numbers x_i's can be a solution) if $b = 0$ (homogeneous).

In any case, if all the coefficients of an equation in a system are zero, the equation is vacuously trivial. In this book, when we speak of a system of linear equation, we always assume that not all the coefficients in each equation of the system are zero unless otherwise specified.

Example 1.1 The system of one equation in two unknowns x and y is

$$ax + by = c,$$

in which at least one of a and b is nonzero. Geometrically this equation represents a straight line in the xy-plane. Therefore, a point $P = (x, y)$ (actually, the coordinates x and y) is a solution if and only if the point P lies on the line. Thus there are infinitely many solutions which are all the points on the line.

Example 1.2 The system of two equations in two unknowns x and y is

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2. \end{cases}$$

Solution: (I) *Geometric method.* Since the equations represent two straight lines in the xy-plane, only three types are possible as shown in Figure 1.1.

Case (1) $\begin{cases} x - y = -1 \\ 2x - 2y = -2 \end{cases}$ Case (2) $\begin{cases} x - y = -1 \\ x - y = 0 \end{cases}$ Case (3) $\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$

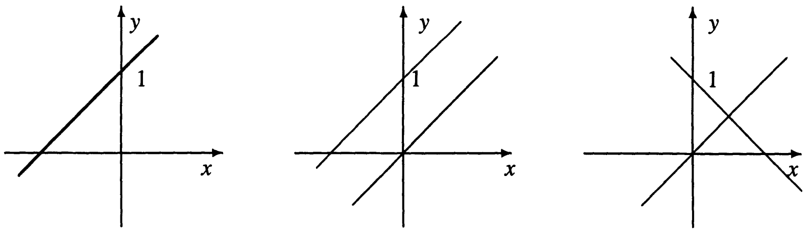

Figure 1.1. Three types of solution sets

Since a solution is a point lying on both lines simultaneously, by looking at the graphs in Figure 1.1, one can see that only the following three types of solution sets are possible:

(1) the straight line itself if they coincide,
(2) the empty set if the lines are parallel and distinct,
(3) only one point if they cross at a point.

(II) *Algebraic method.* Case (1) (two lines coincide): Let the two equations represent the same straight line, that is, one equation is a nonzero constant multiple of the other. This condition is equivalent to

$$a_2 = \lambda a_1, \ b_2 = \lambda b_1, \ c_2 = \lambda c_1 \quad \text{for some } \lambda \neq 0.$$

In this case, if a point (s, t) satisfies one equation, then it automatically satisfies the other too. Thus, there are infinitely many solutions which are all the points on the line.

Case (2) (two lines are parallel but distinct): In this case, $a_2 = \lambda a_1,\ b_2 = \lambda b_1$, but $c_2 \neq \lambda c_1$ for $\lambda \neq 0$. (Note that the first two equalities are equivalent to $a_1 b_2 - a_2 b_1 = 0$). Then no point (s, t) can satisfy both equations simultaneously, so that there are no solutions.

Case (3) (two lines cross at a point): Let the two lines have distinct slopes, which means $a_1 b_2 - a_2 b_1 \neq 0$. In this case, they cross at a point (the only solution), which can be found by the elementary method of *elimination* and *substitution*. The following computation shows how to do this:

Without loss of generality, one may assume $a_1 \neq 0$ by interchanging the two equations if necessary. (If both a_1 and a_2 are zero, the system reduces to a system of one variable.)

(1) Elimination: The variable x can be eliminated from the second equation by adding $-\dfrac{a_2}{a_1}$ times the first equation to the second, to get

$$\left\{\begin{array}{rcl} a_1 x \ + \qquad\qquad b_1 y & = & c_1 \\ (b_2 - \dfrac{a_2}{a_1} b_1) y & = & c_2 - \dfrac{a_2}{a_1} c_1. \end{array}\right.$$

(2) Since $a_1 b_2 - a_2 b_1 \neq 0$, y can be found by multiplying the second equation by a nonzero number $\dfrac{a_1}{a_1 b_2 - a_2 b_1}$ to get

$$\left\{\begin{array}{rcl} a_1 x \ + \ b_1 y & = & c_1 \\ y & = & \dfrac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}. \end{array}\right.$$

(3) Substitution: Now, x is solved by substituting the value of y into the first equation, and we obtain the solution to the problem:

$$\left\{\begin{array}{rcl} x & = & \dfrac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1} \\ y & = & \dfrac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}. \end{array}\right.$$

Note that the condition $a_1 b_2 - a_2 b_1 \neq 0$ is necessary for the system to have only one solution. □

In Example 1.2, the original system of equations has been transformed into a simpler one through certain operations, called *elimination* and *substitution*, which is just the solution of the given system. That is, if (x, y) satisfies the original system of equations, then it also satisfies the simpler system in (3), and vice-versa. As in Example 1.2, we will see later that any system of linear equations may have either *no solution, exactly one solution, or infinitely many solutions.* (See Theorem 1.6.)

Note that an equation $ax + by + cz = d$, $(a, b, c) \neq (0, 0, 0)$, in three unknowns represents a plane in the 3-space $\mathbb{R}^3$. The solution set includes

$$
\begin{array}{ll}
\{(x, y, 0) \mid ax + by = d\} & \text{in the } xy\text{-plane,} \\
\{(x, 0, z) \mid ax + cz = d\} & \text{in the } xz\text{-plane,} \\
\{(0, y, z) \mid by + cz = d\} & \text{in the } yz\text{-plane.}
\end{array}
$$

One can also examine the various possible types of the solution set of a system of three equations in three unknown. Figure 1.2 illustrates three possible cases.

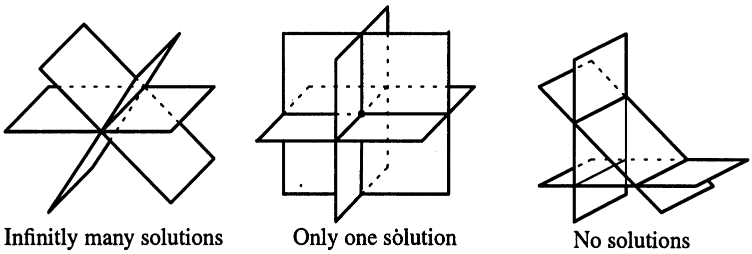

Figure 1.2. Three planes in $\mathbb{R}^3$

Problem 1.1 For a system of three linear equations in three unknowns

$$
\left\{
\begin{array}{rcrcrcl}
a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\
a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\
a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3,
\end{array}
\right.
$$

describe all the possible types of the solution set in the 3-space $\mathbb{R}^3$.

1.2 Gaussian elimination

A basic idea for solving a system of linear equations is to transform the given system into a simpler one, keeping the solution set unchanged, and Example 1.2 shows an idea of how to do it. In fact, the basic operations used in Example 1.2 are essentially only the following three operations, called **elementary operations**:

(1) multiply a nonzero constant throughout an equation,
(2) interchange two equations,
(3) add a constant multiple of an equation to another equation.

It is not hard to see that none of these operations alters the solutions. That is, if x_i's satisfy the original equations, then they also satisfy those equations altered by the three operations, and vice-versa.

Moreover, each of the three elementary operations has its *inverse* operation which is also an elementary operation:

(1′) multiply the equation with the reciprocal of the same nonzero constant,
(2′) interchange two equations again,
(3′) add the negative of the same constant multiple of the equation to the other.

Therefore, by applying a finite sequence of the elementary operations to the given original system, one obtains another new system, and by applying these inverse operations in reverse order to the new system, one can recover the original system. Since none of the three elementary operations alters the solutions, the two systems have the same set of solutions. In fact, a system may be solved by transforming it into a simpler system using the three elementary operations finitely many times.

These arguments can be formalized in mathematical language. Observe that in performing any of these three elementary operations, only the coefficients of the variables are involved in the operations, while the variables $x_1, x_2, \ldots, x_n$ and the equal sign "=" are simply repeated. Thus, keeping the places of the variables and "=" in mind, we just pick up the coefficients only from the given system of equations and make a rectangular array of numbers as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

This matrix is called the **augmented matrix** of the system. The term *matrix* means just any rectangular array of numbers, and the numbers in this array are called the *entries* of the matrix. In the following sections, we shall discuss matrices in general. For the moment, we restrict our attention to the augmented matrix of a system.

Within an augmented matrix, the horizontal and vertical subarrays

$$[a_{i1}\ a_{i2}\ \cdots\ a_{in}\ b_i] \quad \text{and} \quad \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

are called the i-th *row* (matrix), which represents the i-th equation, and the j-th *column* (matrix), which are the coefficients of j-th variable x_j, of the augmented matrix, respectively. The matrix consisting of the first n columns of the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the **coefficient matrix** of the system.

One can easily see that there is a one-to-one correspondence between the columns of the coefficient matrix and variables of the system. Note also that the last column $[b_1\ b_2\ \cdots\ b_m]^T$ of the augmented matrix represents homogeneity of the system and so no variable corresponds to it.

Since each row of the augmented matrix contains all the information of the corresponding equation of the system, we may deal with this augmented matrix instead of handling the whole system of linear equations, and the elementary operations may be applied to an augmented matrix just like they are applied to a system of equations.

But in this case, the elementary operations are rephrased as the **elementary row operations** for the augmented matrix:

(1st kind)	multiply a nonzero constant throughout a row,
(2nd kind)	interchange two rows,
(3rd kind)	add a constant multiple of a row to another row.

The *inverse* row operations which are also elementary row operations are

(1st kind)	multiply the row by the reciprocal of the same constant,
(2nd kind)	interchange two rows again,
(3rd kind)	add the negative of the same constant multiple of the row to the other.

Definition 1.1 Two augmented matrices (or systems of linear equations) are said to be **row-equivalent** if one can be transformed to the other by a finite sequence of elementary row operations.

Note that, if a matrix B can be obtained from a matrix A by these elementary row operations, then one can obviously recover A from B by applying the inverse elementary row operations to B in reverse order. Therefore, the two systems have the same solutions:

Theorem 1.1 *If two systems of linear equations are row-equivalent, then they have the same set of solutions.*

The general procedure for finding the solutions will be illustrated in the following example:

Example 1.3 Solve the system of linear equations:

$$\left\{\begin{array}{rcrcrcr} & & 2y & + & 4z & = & 2 \\ x & + & 2y & + & 2z & = & 3 \\ 3x & + & 4y & + & 6z & = & -1. \end{array}\right.$$

Solution: One could work with the augmented matrix only. However, to compare the operations on the system of linear equations with those on the augmented matrix, we work on the system and the augmented matrix in parallel. Note that the associated augmented matrix for the system is

$$\left[\begin{array}{rrrr} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{array}\right].$$

(1) Since the coefficient of x in the first equation is zero while that in the second equation is not zero, we interchange these two equations:

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ & & 2y & + & 4z & = & 2 \\ 3x & + & 4y & + & 6z & = & -1 \end{array}\right. \qquad \left[\begin{array}{rrrr} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 & 4 & 6 & -1 \end{array}\right].$$

(2) Add -3 times the first equation to the third equation:

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ & & 2y & + & 4z & = & 2 \\ & - & 2y & & & = & -10 \end{array}\right. \qquad \left[\begin{array}{rrrr} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & 0 & -10 \end{array}\right].$$

Thus, the first variable x is eliminated from the second and the third equations. In this process, the coefficient 1 of the first unknown x in the first equation (row) is called the first **pivot**.

Consequently, the second and the third equations have only the two unknowns y and z. Leave the first equation (row) alone, and the same elimination procedure can be applied to the second and the third equations (rows): The pivot to eliminate y from the last equation is the coefficient 2 of y in the second equation (row).

(3) Add 1 times the second equation (row) to the third equation (row):

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ & & 2y & + & 4z & = & 2 \\ & & & & 4z & = & -8 \end{array}\right. \qquad \left[\begin{array}{rrrr} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -8 \end{array}\right].$$

The elimination process (i.e., (1): row interchange , (2): elimination of x from the last two equations (rows), and then (3): elimination of y from the last equation (row)) done so far to obtain this result is called **forward elimination**. After this forward elimination, the leftmost nonzero entries in the nonzero rows are called the pivots. Thus the pivots of the second and third rows are 2 and 4, respectively.

(4) Normalize nonzero rows by dividing them by their pivots. Then the pivots are replaced by 1:

$$\begin{cases} x + 2y + 2z = 3 \\ \quad\;\; y + 2z = 1 \\ \qquad\quad\;\; z = -2 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

The resulting matrix on the right-hand side is called a **row-echelon form** of the augmented matrix, and the 1's at the pivotal positions are called the **leading** 1's. The process so far is called **Gaussian elimination.**

The last equation gives $z = -2$. Substituting $z = -2$ into the second equation gives $y = 5$. Now, putting these two values into the first equation, we get $x = -3$. This process is called **back substitution**. The computation is shown below: i.e., eliminating numbers above the leading 1's.

(5) Add -2 times the third row to the second and the first rows:

$$\begin{cases} x + 2y \qquad = 7 \\ \quad\;\; y \qquad = 5 \\ \qquad\quad\;\; z = -2 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(6) Add -2 times the second row to the first row:

$$\begin{cases} x \qquad\qquad = -3 \\ \quad\;\; y \qquad = 5 \\ \qquad\quad\;\; z = -2 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

This resulting matrix is called the **reduced row-echelon form** of the augmented matrix, which is row-equivalent to the original augmented matrix and gives the solution to the system. The whole process to obtain the reduced row-echelon form is called **Gauss–Jordan elimination.** □

In summary, by applying a finite sequence of elementary row operations, the augmented matrix for a system of linear equations can be transformed into its reduced row-echelon form, which is row equivalent to the original one. Hence the two corresponding systems have the same solutions. From the reduced row-echelon form, one can easily decide whether the system has a solution or not, and find the solution of the given system if it is consistent.

Definition 1.2 A **row-echelon form** of an augmented matrix is of the following form:

(1) The zero rows, if they exist, come last in the order of rows.
(2) The first nonzero entries in the nonzero rows are 1, called leading 1's.
(3) Below each leading 1 is a column of zeros. Thus, in any two consecutive nonzero rows, the leading 1 in the lower row appears farther to the right than the leading 1 in the upper row.

The **reduced row-echelon form** of an augmented matrix is of the form:

(4) Above each leading 1 is a column of zeros, in addition to a row-echelon form.

Example 1.4 The first three augmented matrices below are in reduced row-echelon form, and the last one is just in row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Recall that in an augmented matrix $[A\ \mathbf{b}]$, the last column $\mathbf{b}$ does not correspond to any variable. Thus, if the reduced row-echelon form of an augmented matrix for a nonhomogeneous system has a row of the form $[\,0\ 0\ \cdots\ 0\ b\,]$ with $b \neq 0$, then the associated equation is $0x_1 + 0x_2 + \cdots + 0x_n = b$ with $b \neq 0$, which means the system is inconsistent. If $b = 0$, then it has a row containing only 0's, which can be neglected and deleted. In this example, the third matrix shows the former case, and the first two matrices show the latter case. □

In the following example, we use Gauss–Jordan elimination again to solve a system which has infinitely many solutions.

Example 1.5 Solve the following system of linear equations by Gauss–Jordan elimination.

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 3 \\ 2x_1 + 6x_2 - 2x_3 + 4x_4 = 18 \\ \phantom{2x_1 + {}} x_2 + x_3 + 3x_4 = 10. \end{cases}$$

Solution: The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 2 & 6 & -2 & 4 & 18 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}.$$

The Gaussian elimination begins with:

(1) Adding -2 times the first row to the second produces

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}.$$

(2) Note that the coefficient of x_2 in the second equation is zero and that in the third equation is not. Thus, interchanging the second and the third rows produces

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix}.$$

(3) Dividing the third row by the pivot 2 produces a row-echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}.$$

We now continue the back-substitution:

(4) Adding -1 times the third row to the second, and 2 times the third row to the first produces

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 15 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}.$$

(5) Finally, adding -3 times the second row to the first produces the reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}.$$

The corresponding system of equations is

$$\left\{ \begin{array}{rrrrrr} x_1 & & & + & x_4 & = 3 \\ & x_2 & & + & x_4 & = 4 \\ & & x_3 & + & 2x_4 & = 6. \end{array} \right.$$

This system can be rewritten as follows:

$$\left\{ \begin{array}{rcrcr} x_1 & = & 3 & - & x_4 \\ x_2 & = & 4 & - & x_4 \\ x_3 & = & 6 & - & 2x_4. \end{array} \right.$$

Since there is no other condition on x_4, one can see that all the other variables x_1, x_2, and x_3 may be uniquely determined if an arbitrary real value $t \in \mathbb{R}$ is assigned to x_4 ($\mathbb{R}$ denotes the set of real numbers): thus the solutions can be written as

$$(x_1,\ x_2,\ x_3,\ x_4) = (3 - t,\ 4 - t,\ 6 - 2t,\ t),\ t \in \mathbb{R}. \qquad \square$$

Note that if we look at the reduced row-echelon form in Example 1.5, the variables x_1, x_2, and x_3 correspond to the columns containing leading 1's, while the column corresponding to x_4 contains no leading 1.

An augmented matrix of a system of linear equations may have more than one row-echelon form, but it has only one reduced row-echelon form (see Remark (2) on page 97 for a concrete proof). Thus the number of leading 1's in a system does not depend on the Gaussian elimination.

Definition 1.3 Among the variables in a system, the ones corresponding to the columns containing leading 1's are called the **basic variables**, and the ones corresponding to the columns without leading 1's, if there are any, are called the **free variables**.

Clearly the sum of the number of basic variables and that of free variables is equal to the total number of unknowns: the number of columns.

In Example 1.4, the first and the last augmented matrices have only basic variables but no free variables, while the second one has two basic variables x_1 and x_3, and two free variables x_2 and x_4. The third one has two basic variables x_1 and x_2, and only one free variable x_3.

In general, as we have seen in Example 1.5, a consistent system has infinitely many solutions if it has at least one free variable, and has a unique solution if it has no free variable. In fact, if a consistent system has a free variable (which always happens when the number of equations is less than that of unknowns), then by assigning arbitrary value to the free variable, one always obtains infinitely many solutions.

Theorem 1.2 *If a homogeneous system has more unknowns than equations, then it has infinitely many solutions.*

Problem 1.2 Suppose that the augmented matrices for some systems of linear equations have been reduced to reduced row-echelon forms as below by elementary row operations. Solve the systems:

$$(1)\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \qquad (2)\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}.$$

Problem 1.3 Solve the following systems of equations by Gaussian elimination. What are the pivots?

$$(1)\begin{cases} -x + y + 2z = 0 \\ 3x + 4y + z = 0 \\ 2x + 5y + 3z = 0. \end{cases} \qquad (2)\begin{cases} 2y - z = 1 \\ 4x - 10y + 3z = 5 \\ 3x - 3y = 6. \end{cases}$$

$$(3)\begin{cases} w + x + y = 3 \\ -3w - 17x + y + 2z = 1 \\ 4w - 17x + 8y - 5z = 1 \\ -5x - 2y + z = 1. \end{cases}$$

Problem 1.4 Determine the condition on b_i so that the following system has a solution.

$$(1)\begin{cases} x + 2y + 6z = b_1 \\ 2x - 3y - 2z = b_2 \\ 3x - y + 4z = b_3. \end{cases} \qquad (2)\begin{cases} x + 3y - 2z = b_1 \\ 2x - y + 3z = b_2 \\ 4x + 2y + z = b_3. \end{cases}$$

1.3 Sums and scalar multiplications of matrices

Rectangular arrays of real numbers arise in many real-world problems. Historically, it was an English mathematician J.J. Sylvester who first introduced the word "matrix" in the year 1848. It was the Latin word for womb, as a name for an array of numbers.

Definition 1.4 An m by n (written $m \times n$) **matrix** is a rectangular array of numbers arranged into m (horizontal) rows and n (vertical) columns. The **size** of a matrix is specified by the number m of the rows and the number n of the columns.

In general, a matrix is written in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n},$$

or just $A = [a_{ij}]$ if the size of the matrix is clear from the context. The number a_{ij} is called the (i, j)-**entry** of the matrix A, and is written as $a_{ij} = [A]_{ij}$.

An $m\times 1$ matrix is called a **column (matrix)** or sometimes a **column vector**, and a $1\times n$ matrix is called a **row (matrix)**, or a **row vector**. In general, we use capital letters like A, B, C for matrices and small boldface letters like $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ for column or row vectors.

Definition 1.5 Let $A = [a_{ij}]$ be an $m\times n$ matrix. The **transpose** of A is the $n\times m$ matrix, denoted by A^T, whose j-th column is taken from the j-th row of A: that is, $[A^T]_{ij} = [A]_{ji}$.

Example 1.6 **(1)** If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(2) The transpose of a column vector is a row vector and vice-versa:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \iff \mathbf{x}^T = [x_1\ x_2\ \cdots\ x_n].$$

□

Definition 1.6 A matrix $A = [a_{ij}]$ is called a **square matrix of order** n if the number of rows and the number of columns are both equal to n.

Definition 1.7 Let A be a square matrix of order n.

(1) The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **diagonal entries** of A.
(2) A is called a **diagonal** matrix if all the entries except for the diagonal entries are zero.
(3) A is called an **upper (lower) triangular** matrix if all the entries below (above, respectively) the diagonal are zero.

The following matrices U and L are the general forms of the upper triangular and lower triangular matrices, respectively:

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Note that a matrix which is both upper and lower triangular must be a diagonal matrix, and the transpose of an upper (lower, respectively) triangular matrix is lower (upper, respectively) triangular.

Definition 1.8 Two matrices A and B are said to be **equal**, written $A = B$, if their sizes are the same and their corresponding entries are equal: i.e., $[A]_{ij} = [B]_{ij}$ for all i and j.

This definition allows us to write a matrix equation. A simple example is $(A^T)^T = A$ by definition.

Let $M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with entries of real numbers. Among the elements of $M_{m\times n}(\mathbb{R})$, one can define two operations, called the scalar multiplication and the sum of matrices, as follows:

Definition 1.9 **(1)** (*Scalar multiplication*) For an $m\times n$ matrix $A = [a_{ij}] \in M_{m\times n}(\mathbb{R})$ and a *scalar* $k \in \mathbb{R}$ (which is simply a real number), the **scalar multiplication** of k and A is defined to be the matrix kA such that $[kA]_{ij} = k[A]_{ij}$ for all i and j: i.e., in an expanded form:

$$k\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{bmatrix}.$$

(2) (*Sum of matrices*) For two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_{m\times n}(\mathbb{R})$, the **sum** of A and B is defined to be the matrix $A+B$ such that $[A+B]_{ij} = [A]_{ij}+[B]_{ij}$ for all i and j: i.e., in an expanded form:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}.$$

The resulting matrices kA and $A + B$ from these two operations also belong to $M_{m\times n}(\mathbb{R})$. In this sense, we say $M_{m\times n}(\mathbb{R})$ is *closed* under the two operations. Note that matrices of different sizes cannot be added; for example, a sum

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

cannot be defined.

If B is any matrix, then $-B$ is by definition the multiplication $(-1)B$. Moreover, if A and B are two matrices of the same size, then the subtraction $A - B$ is by definition the sum $A + (-1)B$. A matrix whose entries are all zeros is called a **zero matrix**, denoted by the symbol $\mathbf{0}$ (or $\mathbf{0}_{m\times n}$ when the size is emphasized).

Clearly, matrix sum has the same properties as the sum of real numbers. The real numbers in the context here are traditionally called **scalars** even though "numbers" is a perfectly good name and "scalar" sounds more technical. The following theorem lists the basic arithmetic properties of the sum and scalar multiplication of matrices.

Theorem 1.3 *Suppose that the sizes of A, B and C are the same. Then the following arithmetic rules of matrices are valid:*

(1) $(A+B)+C = A+(B+C)$, (*written as* $A+B+C$) (Associativity),
(2) $A+\mathbf{0} = \mathbf{0}+A = A$,
(3) $A+(-A) = (-A)+A = \mathbf{0}$,
(4) $A+B = B+A$, (Commutativity),
(5) $k(A+B) = kA+kB$,
(6) $(k+\ell)A = kA+\ell A$,
(7) $(k\ell)A = k(\ell A)$.

Proof: We prove only the equality (5) and the remaining ones are left for exercises. For any (i, j),

$$\begin{aligned} ij = k[A+B]_{ij} &= k([A]_{ij}+[B]_{ij}) = [kA]_{ij}+[kB]_{ij} \\ &= [kA+kB]_{ij}. \end{aligned}$$

□

In particular, $A+A = 2A$, $A+(A+A) = 3A = (A+A)+A$, and inductively $nA = (n-1)A + A$ for any positive integer n.

Definition 1.10 A square matrix A is said to be **symmetric** if $A^T = A$, or **skew-symmetric** if $A^T = -A$.

For example, the matrices

$$A = \begin{bmatrix} 1 & a & b \\ a & 3 & c \\ b & c & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix},$$

are symmetric and skew-symmetric, respectively. Notice that all the diagonal entries of a skew-symmetric matrix must be zero, since $a_{ii} = -a_{ii}$.

By a direct computation, one can easily verify the following properties of the transpose of matrices:

Theorem 1.4 *Let A and B be $m \times n$ matrices. Then*

$$(kA)^T = kA^T \quad \textit{and} \quad (A+B)^T = A^T + B^T.$$

Problem 1.5 Prove the remaining parts of Theorem 1.3.

Problem 1.6 Find a matrix B such that $A + B^T = (A-B)^T$, where

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 4 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix}.$$

Problem 1.7 Find a, b, c and d such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2\begin{bmatrix} a & 3 \\ 2 & a+c \end{bmatrix} + \begin{bmatrix} 2+b & a+9 \\ c+d & b \end{bmatrix}.$$

1.4 Products of matrices

The sum and the scalar multiplication of matrices were introduced in Section 1.3. In this section, we introduce the product of matrices. Unlike the sum of two matrices, the product of matrices is a little bit more complicated, in the sense that it can be defined for two matrices of different sizes. The product of matrices will be defined in three steps:

Step (1) Product of vectors: For a $1 \times n$ row vector $\mathbf{a} = [a_1\ a_2\ \cdots\ a_n]$ and an $n \times 1$ column vector $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]^T$, the **product $\mathbf{ax}$** is a 1×1 matrix (i.e., just a number) defined by the rule

$$\mathbf{ax} = [a_1\ a_2\ \cdots\ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1x_1 + a_2x_2 + \cdots + a_nx_n] = \left[\sum_{i=1}^{n} a_i x_i \right].$$

Note that the number of entries of the first row vector is equal to the number of entries of the second column vector, so that an entry-wise multiplication is possible.

Step (2) Product of a matrix and a vector: For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}$$

with the row vectors $\mathbf{a}_i$'s and for an $n \times 1$ column vector $\mathbf{x} = [x_1\ \cdots\ x_n]^T$, the **product $A\mathbf{x}$** is by definition an $m \times 1$ matrix, whose m rows are computed according to Step (1):

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\mathbf{x} \\ \mathbf{a}_2\mathbf{x} \\ \vdots \\ \mathbf{a}_m\mathbf{x} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}x_i \\ \sum_{i=1}^{n} a_{2i}x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}x_i \end{bmatrix}.$$

Therefore, for a system of m linear equations in n unknowns, by writing the n unknowns as an $n \times 1$ column matrix $\mathbf{x}$ and the coefficients as an $m \times n$ matrix A, the system may be expressed as a matrix equation $A\mathbf{x} = \mathbf{b}$.

Step (3) Product of matrices: Let A be an $m \times n$ matrix and B an $n \times r$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r$, written as $B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_r]$. The **product AB** is defined to be an $m \times r$ matrix whose r columns are the products of A and the r columns of B, each computed according to Step (2) in corresponding order. That is,

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_r \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \cdots & \mathbf{a}_1\mathbf{b}_r \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \cdots & \mathbf{a}_2\mathbf{b}_r \\ \vdots & & \ddots & \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \cdots & \mathbf{a}_m\mathbf{b}_r \end{bmatrix},$$

which is an $m \times r$ matrix. Therefore, the (i, j)-entry $[AB]_{ij}$ of AB is:

$$\boxed{[AB]_{ij} = \mathbf{a}_i\mathbf{b}_j = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \textstyle\sum_{k=1}^{n} a_{ik}b_{kj}.}$$

This can be easily memorized as the sum of entry-wise multiplications of the boxed vectors in Figure 1.3.

$$AB \quad = \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots} & \boxed{a_{in}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & \boxed{b_{1j}} & \cdots & b_{1r} \\ b_{21} & \cdots & \boxed{b_{2j}} & & b_{2r} \\ \vdots & & \boxed{\vdots} & & \vdots \\ b_{n1} & \cdots & \boxed{b_{nj}} & \cdots & b_{nr} \end{bmatrix}$$

Figure 1.3. The entry $[AB]_{ij}$

Example 1.7 Consider the matrices

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix}.$$

The columns of AB are the product of A and each column of B:

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 3\cdot 5 \\ 4\cdot 1 + 0\cdot 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + 3\cdot(-1) \\ 4\cdot 2 + 0\cdot(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 0 + 3\cdot 0 \\ 4\cdot 0 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, AB is

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 1 & 0 \\ 4 & 8 & 0 \end{bmatrix}.$$

Since A is a 2×2 matrix and B is a 2×3 matrix, the product AB is a 2×3 matrix. If we concentrate, for example, on the (2, 1)-entry of AB, we single out the second row from A and the first column from B, and then we multiply corresponding entries together and add them up, i.e., $4 \cdot 1 + 0 \cdot 5 = 4$. □

Note that the product AB of A and B is not defined if the number of columns of A and the number of rows of B are not equal.

Remark: In step (2), instead of defining a product of a matrix and a vector, one can define alternatively the product of a $1 \times n$ row matrix $\mathbf{a}$ and an $n \times r$ matrix B using the same rule defined in step (1) to have a $1 \times r$ row matrix $\mathbf{a}B$. Accordingly, in step (3) an appropriate modification produces the same definition of the product of two matrices. We suggest that readers complete the details. (See Example 1.10.)

The **identity matrix** of order n, denoted by I_n (or I if the order is clear from the context), is a diagonal matrix whose diagonal entries are all 1, i.e.,

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

By a direct computation, one can easily see that $AI_n = A = I_nA$ for any $n \times n$ matrix A.

The operations of scalar multiplication, sum and product of matrices satisfy many, but not all, of the same arithmetic rules that real or complex numbers have. The matrix $\mathbf{0}_{m\times n}$ plays the role of the number 0, and I_n plays that of the number 1 in the set of usual numbers.

The rule that does not hold for matrices in general is commutativity $AB = BA$ of the product, while commutativity of the matrix sum $A+B = B+A$ always holds. The following example illustrates noncommutativity of the product of matrices.

Example 1.8 (*Noncommutativity of the matrix product*)
Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which shows $AB \neq BA$. □

The following theorem lists the basic arithmetic rules that hold in the matrix product.

Theorem 1.5 *Let A, B, C be arbitrary matrices for which the matrix operations below can be defined, and let k be an arbitrary scalar. Then*

(1) $A(BC) = (AB)C$, (*written as* ABC) (Associativity),
(2) $A(B+C) = AB + AC$, *and* $(A+B)C = AC + BC$, (Distributivity),
(3) $IA = A = AI$,
(4) $k(BC) = (kB)C = B(kC)$,
(5) $(AB)^T = B^TA^T$.

Proof: Each equality can be shown by direct computation of each entry on both sides of the equalities. We illustrate this by proving (1) only, and leave the others to the reader.

Assume that $A = [a_{ij}]$ is an $m \times n$ matrix, $B = [b_{k\ell}]$ is an $n \times p$ matrix, and $C = [c_{st}]$ is a $p \times r$ matrix. We now compute the (i, j)-entry of each side of the equation. Note that BC is an $n \times r$ matrix whose (i, j)-entry is $[BC]_{ij} = \sum_{\lambda=1}^{p} b_{i\lambda} c_{\lambda j}$. Thus

$$[A(BC)]_{ij} = \sum_{\mu=1}^{n} a_{i\mu}[BC]_{\mu j} = \sum_{\mu=1}^{n} a_{i\mu} \sum_{\lambda=1}^{p} b_{\mu\lambda} c_{\lambda j} = \sum_{\mu=1}^{n} \sum_{\lambda=1}^{p} a_{i\mu} b_{\mu\lambda} c_{\lambda j}.$$

Similarly, AB is an $m \times p$ matrix with the (i, j)-entry $[AB]_{ij} = \sum_{\mu=1}^{n} a_{i\mu} b_{\mu j}$, and

$$[(AB)C]_{ij} = \sum_{\lambda=1}^{p} [AB]_{i\lambda} c_{\lambda j} = \sum_{\lambda=1}^{p} \sum_{\mu=1}^{n} a_{i\mu} b_{\mu\lambda} c_{\lambda j} = \sum_{\mu=1}^{n} \sum_{\lambda=1}^{p} a_{i\mu} b_{\mu\lambda} c_{\lambda j}.$$

This clearly shows that $[A(BC)]_{ij} = [(AB)C]_{ij}$ for all i, j, and consequently $A(BC) = (AB)C$ as desired. □

Problem 1.8 Give an example of matrices A and B such that $(AB)^T \neq A^T B^T$.

Problem 1.9 Prove or disprove: If A is not a zero matrix and $AB = AC$, then $B = C$. Similarly, is it true or not that $AB = \mathbf{0}$ implies $A = \mathbf{0}$ or $B = \mathbf{0}$?

Problem 1.10 Show that any triangular matrix A satisfying $AA^T = A^T A$ is a diagonal matrix.

Problem 1.11 For a square matrix A, show that

(1) AA^T and $A + A^T$ are symmetric,
(2) $A - A^T$ is skew-symmetric, and
(3) A can be expressed as the sum of symmetric part $B = \frac{1}{2}(A + A^T)$ and skew-symmetric part $C = \frac{1}{2}(A - A^T)$, so that $A = B + C$.

As an application of our results on matrix operations, one can prove the following important theorem:

Theorem 1.6 *Any system of linear equations has either no solution, exactly one solution, or infinitely many solutions.*

Proof: We have seen that a system of linear equations may be written in matrix form as $A\mathbf{x} = \mathbf{b}$. This system may have either no solution or a solution. If it has only one solution, then there is nothing to prove. Suppose that the system has more than one solution and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two different solutions so that $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$. Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$. Then $\mathbf{x}_0 \neq \mathbf{0}$, and $A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$. Thus

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + kA\mathbf{x}_0 = \mathbf{b}.$$

This means that $\mathbf{x}_1 + k\mathbf{x}_0$ is also a solution of $A\mathbf{x} = \mathbf{b}$ for any k. Since there are infinitely many choices for k, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. $\square$

Problem 1.12 For which values of a does each of the following systems have no solution, exactly one solution, or infinitely many solutions?

$$(1) \begin{cases} x + 2y - 3z = 4 \\ 3x - y + 5z = 2 \\ 4x + y + (a^2 - 14)z = a + 2. \end{cases}$$

$$(2) \begin{cases} x - y + z = 1 \\ x + 3y + az = 2 \\ 2x + ay + 3z = 3. \end{cases}$$

1.5 Block matrices

In this section we introduce some techniques that may be helpful in manipulations of matrices. A **submatrix** of a matrix A is a matrix obtained from A by deleting certain rows and/or columns of A. Using some horizontal and vertical lines, one can partition a matrix A into submatrices, called **blocks**, of A as follows: Consider a matrix

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right],$$

divided up into four blocks by the dotted lines shown. Now, if we write

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} a_{34} \end{bmatrix},$$

then A can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

called a **block matrix**.

The product of matrices partitioned into blocks also follows the matrix product formula, as if the blocks A_{ij} were numbers: If

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

are block matrices and the number of columns in A_{ik} is equal to the number of rows in B_{kj}, then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

This will be true only if the columns of A are partitioned in the same way as the rows of B.

It is not hard to see that the matrix product by blocks is correct. Suppose, for example, that we have a 3×3 matrix A and partition it as

$$A = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

and a 3×2 matrix B which we partition as

$$B = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \hline b_{31} & b_{32} \end{array}\right] = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}.$$

Then the entries of $C = [c_{ij}] = AB$ are

$$c_{ij} = (a_{i1}b_{1j} + a_{i2}b_{2j}) + a_{i3}b_{3j}.$$

The quantity $a_{i1}b_{1j} + a_{i2}b_{2j}$ is simply the (i, j)-entry of $A_{11}B_{11}$ if $i \le 2$, and is the (i, j)-entry of $A_{21}B_{11}$ if $i = 3$. Similarly, $a_{i3}b_{3j}$ is the (i, j)-entry of $A_{12}B_{21}$ if $i \le 2$, and of $A_{22}B_{21}$ if $i = 3$. Thus AB can be written as

$$AB = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}.$$

Example 1.9 If an $m \times n$ matrix A is partitioned into blocks of column vectors: i.e., $A = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]$, where each block $\mathbf{c}_j$ is the j-th column, then the product $A\mathbf{x}$ with $\mathbf{x} = [x_1\ \cdots\ x_n]^T$ is the sum of the block matrices (or column vectors) with coefficients x_j's:

$$A\mathbf{x} = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n,$$

where $x_j\mathbf{c}_j = x_j[a_{1j}\ a_{2j}\ \cdots\ a_{nj}]^T$. Hence, a matrix equation $A\mathbf{x} = \mathbf{b}$ is nothing but the vector equation $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$. □

Example 1.10 Let A be an $m \times n$ matrix partitioned into the row vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots,$ $\mathbf{a}_n$ as its blocks, and let B be an $n \times r$ matrix so that their product AB is well defined. By considering the matrix B as a block, the product AB can be written

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \cdots & \mathbf{a}_1\mathbf{b}_r \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \cdots & \mathbf{a}_2\mathbf{b}_r \\ \vdots & & \ddots & \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \cdots & \mathbf{a}_m\mathbf{b}_r \end{bmatrix},$$

where $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r$ denote the columns of B. Hence, the row vectors of AB are the products of the row vectors of A and the column vectors of B. $\square$

Problem 1.13 Compute AB using block multiplication, where

$$A = \left[\begin{array}{rr|rr} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \\ \hline 0 & 0 & 2 & -1 \end{array}\right], \quad B = \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hline 2 & 3 & 4 \\ 3 & -2 & 1 \end{array}\right].$$

1.6 Inverse matrices

As shown in Section 1.4, a system of linear equations can be written as $A\mathbf{x} = \mathbf{b}$ in matrix form. This form resembles one of the simplest linear equations in one variable $ax = b$ whose solution is simply $x = a^{-1}b$ when $a \neq 0$. Thus it is tempting to write the solution of the system as $\mathbf{x} = A^{-1}\mathbf{b}$. However, in the case of matrices we first have to assign a meaning to A^{-1}. To discuss this we begin with the following definition.

Definition 1.11 For an $m \times n$ matrix A, an $n \times m$ matrix B is called a **left inverse** of A if $BA = I_n$, and an $n \times m$ matrix C is called a **right inverse** of A if $AC = I_m$.

Example 1.11 (*One-sided inverse*) From a direct calculation for two matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -3 \\ -1 & 5 \\ -2 & 7 \end{bmatrix},$$

we have $AB = I_2$, and $BA = \begin{bmatrix} -5 & 2 & -4 \\ 9 & -2 & 6 \\ 12 & -4 & 9 \end{bmatrix} \neq I_3$.

Thus, the matrix B is a right inverse but not a left inverse of A, while A is a left inverse but not a right inverse of B. $\square$

A matrix A has a right inverse if and only if A^T has a left inverse, since $(AB)^T = B^T A^T$ and $I^T = I$. In general, a matrix with a left (right) inverse need not have a right (left, respectively) inverse. However, the following lemma shows that if a matrix has both a left inverse and a right inverse, then they must be equal:

Lemma 1.7 *If an $n \times n$ square matrix A has a left inverse B and a right inverse C, then B and C are equal, i.e., $B = C$.*

Proof: A direct calculation shows that

$$B = BI_n = B(AC) = (BA)C = I_nC = C. \qquad \square$$

By Lemma 1.7, one can say that if a matrix A has both left and right inverses, then any two left inverses must be both equal to a right inverse C, and hence to each other. By the same reason, any two right inverses must be both equal to a left inverse B, and hence to each other. So there exists only one left and only one right inverse, which must be equal.

We will show later (Theorem 1.9) that if A is a square matrix and has a left inverse, then it has also a right inverse, and vice-versa. Moreover, Lemma 1.7 says that the left inverse and the right inverse must be equal. However, we shall also show in Chapter 3 that any non-square matrix A cannot have both a right inverse and a left inverse: that is, *a non-square matrix may have only a one-sided inverse.* The following example shows that such a matrix may have infinitely many one-sided inverses.

Example 1.12 (*Infinitely many one-sided inverses*) A non-square matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ can have more than one left inverse. In fact, for any $x, y \in \mathbb{R}$, the matrix $B = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}$ is a left inverse of A. □

Definition 1.12 An $n \times n$ square matrix A is said to be **invertible** (or **nonsingular**) if there exists a square matrix B of the same size such that

$$AB = I_n = BA.$$

Such a matrix B is called the **inverse** of A, and is denoted by A^{-1}. A matrix A is said to be **singular** if it is not invertible.

Lemma 1.7 implies that the inverse matrix of a square matrix is unique. That is why we call B 'the' inverse of A. For instance, consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then it is easy to verify that

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix},$$

since $AA^{-1} = I_2 = A^{-1}A$. Note that any zero matrix is singular.

Problem 1.14 Let A be an invertible matrix and k any nonzero scalar. Show that

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$;
(2) the matrix kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$;
(3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Theorem 1.8 *The product of invertible matrices is also invertible, whose inverse is the product of the individual inverses in reversed order:*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof: Suppose that A and B are invertible matrices of the same size. Then $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$, and similarly $(B^{-1}A^{-1})(AB) = I$. Thus, AB has the inverse $B^{-1}A^{-1}$. □

The inverse of A is written as 'A to the power -1', so one can give the meaning of A^k for any integer k: Let A be a square matrix. Define $A^0 = I$. Then, for any positive integer k, we define the power A^k of A inductively as

$$A^k = A(A^{k-1}).$$

Moreover, if A is invertible, then the negative integer power is defined as

$$A^{-k} = (A^{-1})^k \quad \text{for } k > 0.$$

It is easy to check that $A^{k+\ell} = A^k A^\ell$ whenever the right-hand side is defined. (If A is not invertible, $A^{3+(-1)}$ is defined but A^{-1} is not.)

Problem 1.15 Prove:

(1) If A has a zero row, so does AB.
(2) If B has a zero column, so does AB.
(3) Any matrix with a zero row or a zero column cannot be invertible.

Problem 1.16 Let A be an invertible matrix. Is it true that $(A^k)^T = (A^T)^k$ for any integer k? Justify your answer.

1.7 Elementary matrices and finding A^{-1}

We now return to the system of linear equations $A\mathbf{x} = \mathbf{b}$. If A has a right inverse B so that $AB = I_m$, then $\mathbf{x} = B\mathbf{b}$ is a solution of the system since

$$A\mathbf{x} = A(B\mathbf{b}) = (AB)\mathbf{b} = \mathbf{b}.$$

(Compare with Problem 1.23). In particular, if A is an invertible square matrix, then it has only one inverse A^{-1}, and $\mathbf{x} = A^{-1}\mathbf{b}$ is the only solution of the system. In this section, we discuss how to compute A^{-1} when A is invertible.

Recall that Gaussian elimination is a process in which the augmented matrix is transformed into its row-echelon form by a finite number of elementary row operations. In the following, one can see that each elementary row operation can be expressed as a nonsingular matrix, called an *elementary matrix*, so that the process of Gaussian elimination is the same as multiplying a finite number of corresponding elementary matrices to the augmented matrix.

Definition 1.13 An **elementary matrix** is a matrix obtained from the identity matrix I_n by executing only one elementary row operation.

For example, the following matrices are three elementary matrices corresponding to each type of the three elementary row operations:

(1st kind) $\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$: the second row of I_2 is multiplied by -5;

(2nd kind) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$: the second and the fourth rows of I_4 are interchanged;

(3rd kind) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: 3 times the third row is added to the first row of I_3.

It is an interesting fact that, if E is an elementary matrix obtained by executing a certain elementary row operation on the identity matrix I_m, then for any $m \times n$ matrix A, *the product EA is exactly the matrix that is obtained when the same elementary row operation in E is executed on A.*

The following example illustrates this argument. (Note that AE is not what we want. For this, see Problem 1.18.)

Example 1.13 (*Elementary operation by an elementary matrix*) Let $\mathbf{b} = [b_1\ b_2\ b_3]^T$ be a 3×1 column matrix. Suppose that we want to execute a third kind of elementary operation 'adding $(-2) \times$ the first row to the second row' on the matrix $\mathbf{b}$. First, we execute this operation on the identity matrix I_3 to get an elementary matrix E:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this elementary matrix E to $\mathbf{b}$ on the left produces the desired result:

$$E\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}.$$

Similarly, the second kind of elementary operation 'interchanging the first and the third rows' on the matrix $\mathbf{b}$ can be achieved by multiplying an elementary matrix P obtained from I_3 by interchanging the two rows, to $\mathbf{b}$ on the left:

$$P\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}. \qquad \square$$

Recall that each elementary row operation has an inverse operation, which is also an elementary operation, that brings the elementary matrix back to the original identity matrix. In other words, if E denotes an elementary matrix and if E' denotes the elementary matrix corresponding to the 'inverse' elementary row operation of E, then $E'E = I$, because

(1) if E multiplies a row by $c \neq 0$, then E' multiplies the same row by $\frac{1}{c}$;
(2) if E interchanges two rows, then E' interchanges them again;
(3) if E adds a multiple of one row to another, then E' subtracts it from the same row.

Furthermore, one can say that $E'E = I = EE'$: *every elementary matrix is invertible and inverse matrix $E^{-1} = E'$ is also an elementary matrix.*

Example 1.14 (*Inverse of an elementary matrix*) If

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \ E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \ E_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

Definition 1.14 A **permutation matrix** is a square matrix obtained from the identity matrix by permuting the rows.

In Example 1.14, E_3 is a permutation matrix, but E_2 is not.

Problem 1.17 Prove:

(1) A permutation matrix is the product of a finite number of elementary matrices each of which corresponds to the 'row-interchanging' elementary row operation.
(2) Every permutation matrix P is invertible and $P^{-1} = P^T$.
(3) The product of any two permutation matrices is a permutation matrix.
(4) The transpose of a permutation matrix is also a permutation matrix.

Problem 1.18 Define the **elementary column operations** for a matrix by just replacing 'row' by 'column' in the definition of the elementary row operations. Show that if A is an $m \times n$ matrix and if E is a matrix obtained by executing an elementary column operation on I_n, then AE is exactly the matrix that is obtained from A when the same column operation is executed on A. In particular, if D is an $n \times n$ diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$, then AD is obtained by multiplication by $d_1, d_2, \ldots, d_n$ of the columns of A, while DA is obtained by multiplication by $d_1, d_2, \ldots, d_n$ of the rows of A.

The next theorem establishes some fundamental relations between $n \times n$ square matrices and systems of n linear equations in n unknowns.

Theorem 1.9 *Let A be an $n \times n$ matrix. The following are equivalent:*

(1) *A has a left inverse;*
(2) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$;*

(3) *A is row-equivalent to I_n ;*
(4) *A is a product of elementary matrices;*
(5) *A is invertible;*
(6) *A has a right inverse.*

Proof: (1) $\Rightarrow$ (2): Let $\mathbf{x}$ be a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$, and let B be a left inverse of A. Then

$$\mathbf{x} = I_n\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B\mathbf{0} = \mathbf{0}.$$

(2) $\Rightarrow$ (3): Suppose that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$:

$$\left\{ \begin{array}{cccc} x_1 & & & = 0 \\ & x_2 & & = 0 \\ & & \ddots & \\ & & x_n & = 0. \end{array} \right.$$

This means that the augmented matrix $[A\ \mathbf{0}]$ of the system $A\mathbf{x} = \mathbf{0}$ is reduced to the system $[I_n\ \mathbf{0}]$ by Gauss–Jordan elimination. Hence, A is row-equivalent to I_n.

(3) $\Rightarrow$ (4): Assume A is row-equivalent to I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. Thus, one can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n.$$

By multiplying successively both sides of this equation by $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$ on the left, we obtain

$$A = E_1^{-1}E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1}E_2^{-1} \cdots E_k^{-1},$$

which expresses A as the product of elementary matrices.

(4) $\Rightarrow$ (5) is trivial, because any elementary matrix is invertible. In fact, $A^{-1} = E_k \cdots E_2 E_1$.

(5) $\Rightarrow$ (1) and (5) $\Rightarrow$ (6) are trivial.

(6) $\Rightarrow$ (5): If B is a right inverse of A, then A is a left inverse of B and one can apply (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) to B and conclude that B is invertible, with A as its unique inverse, by Lemma 1.7. That is, B is the inverse of A and so A is invertible. $\square$

If a triangular matrix A has a zero diagonal entry, then the system $A\mathbf{x} = \mathbf{0}$ has at least one free variable, so that it has infinitely many solutions. Hence, one can have the following corollary.

Corollary 1.10 *A triangular matrix is invertible if and only if it has no zero diagonal entry.*

From Theorem 1.9, one can see that a square matrix is invertible if it has a one-sided inverse. In particular, if a square matrix A is invertible, then $\mathbf{x} = A^{-1}\mathbf{b}$ is a unique solution to the system $A\mathbf{x} = \mathbf{b}$.

Problem 1.19 Find the inverse of the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As an application of Theorem 1.9, one can find a practical method for finding the inverse A^{-1} of an invertible $n\times n$ matrix A. If A is invertible, then A is row equivalent to I_n and so there are elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_n$. Hence,

$$A^{-1} = E_k \cdots E_2 E_1 = E_k \cdots E_2 E_1 I_n.$$

This means that A^{-1} *can be obtained by performing on* I_n *the same sequence of the elementary row operations that reduces* A *to* I_n. Practically, one first constructs an $n \times 2n$ augmented matrix $[A \mid I_n]$ and then performs a Gaussian–Jordan elimination that reduces A to I_n on $[A \mid I_n]$ to get $\left[I_n \mid A^{-1}\right]$: that is,

$$\begin{aligned} [A \mid I_n] &\rightarrow [E_\ell \cdots E_1 A \mid E_\ell \cdots E_1 I] = [U \mid K] \\ &\rightarrow [F_k \cdots F_1 U \mid F_k \cdots F_1 K] = [I \mid A^{-1}], \end{aligned}$$

where $E_\ell \cdots E_1$ represents a Gaussian elimination that reduces A to a row-echelon form U and $F_k \cdots F_1$ represents the back substitution. The following example illustrates the computation of an inverse matrix.

Example 1.15 (*Computing* A^{-1} *by Gauss–Jordan elimination*) Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

Solution: Apply Gauss–Jordan elimination to

$$\begin{aligned}
[A \mid I] &= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} (-2)\text{row } 1 + \text{row } 2 \\ (-1)\text{row } 1 + \text{row } 3 \end{array} \\
&\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \quad (-1)\text{row } 2 \\
&\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \quad (2)\text{row } 2 + \text{row } 3 \\
&\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right].
\end{aligned}$$

This is $[U \mid K]$ obtained by Gaussian elimination. Now continue the back substitution to reduce $[U \mid K]$ to $\left[I \mid A^{-1}\right]$.

$$[U \mid K] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \quad \begin{array}{l} (-1)\text{row } 3 + \text{row } 2 \\ (-3)\text{row } 3 + \text{row } 1 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -8 & 6 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \quad (-2)\text{row } 2 + \text{row } 1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] = \left[I \mid A^{-1}\right].$$

Thus, we get

$$A^{-1} = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}.$$

(The reader should verify that $AA^{-1} = I = A^{-1}A$.) □

Note that if A is not invertible, then, at some step in Gaussian elimination, a zero row will show up on the left-hand side in $[U \mid K]$. For example, the matrix $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 0 \end{bmatrix}$, which has a zero row and is not invertible.

By Theorem 1.9, a square matrix A is invertible if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. That is, a square matrix A is noninvertible if and only if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, say $\mathbf{x}_0$. Now, for any column vector $\mathbf{b} = [b_1 \cdots b_n]^T$, if $\mathbf{x}_1$ is a solution of $A\mathbf{x} = \mathbf{b}$ for a noninvertible matrix A, so is $k\mathbf{x}_0 + \mathbf{x}_1$ for any k, since

$$A(k\mathbf{x}_0 + \mathbf{x}_1) = k(A\mathbf{x}_0) + A\mathbf{x}_1 = k\mathbf{0} + \mathbf{b} = \mathbf{b}.$$

This argument strengthens Theorem 1.6 as follows when A is a square matrix:

Theorem 1.11 *If A is an invertible $n \times n$ matrix, then for any column vector $\mathbf{b} = [b_1 \cdots b_n]^T$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$. If A is not invertible, then the system has either no solution or infinitely many solutions according to the consistency of the system.*

Problem 1.20 Express A^{-1} as a product of elementary matrices for A given in Example 1.15.

Problem 1.21 When is a diagonal matrix $D = \begin{bmatrix} d_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & d_n \end{bmatrix}$ nonsingular, and what is D^{-1}?

Problem 1.22 Write the system of linear equations

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 10 \\ 2x & - & 2y & + & 3z & = & 1 \\ 4x & - & 3y & + & 5z & = & 4 \end{array}\right.$$

in matrix form $A\mathbf{x} = \mathbf{b}$ and solve it by finding $A^{-1}\mathbf{b}$.

Problem 1.23 True or false: If the matrix A has a left inverse C so that $CA = I_n$, then $\mathbf{x} = C\mathbf{b}$ is a solution of the system $A\mathbf{x} = \mathbf{b}$. Justify your answer.

1.8 *LDU* factorization

In this section, we show that the forward elimination for solving a system of linear equations $A\mathbf{x} = \mathbf{b}$ can be expressed by some invertible lower triangular matrix, so that the matrix A can be factored as a product of two or more triangular matrices.

We first assume that no permutations of rows (2^{nd} kind of operation) are necessary throughout the whole process of forward elimination on the augmented matrix $[A\ \mathbf{b}]$. Then the forward elimination is just multiplications of the augmented matrix $[A\ \mathbf{b}]$ by finitely many elementary matrices $E_k, \ldots, E_1$: that is,

$$[E_k \cdots E_1 A \quad E_k \cdots E_1 \mathbf{b}] = [U\ \mathbf{y}],$$

where each E_i is a lower triangular elementary matrix whose diagonal entries are all 1's and $[U\ \mathbf{y}]$ is a row-echelon form of $[A\ \mathbf{b}]$ without divisions of the rows by the pivots. (Note that if A is a square matrix, then U must be an upper triangular matrix). Therefore, if we set $L = (E_k \cdots E_1)^{-1} = E_1^{-1} \cdots E_k^{-1}$, then we have $A = LU$, where *L is a lower triangular matrix whose diagonal entries are all 1's*. (In fact, each E_i^{-1} is also a lower triangular matrix, and a product of lower triangular matrices is also lower triangular (see Problem 1.25). Such factorization $A = LU$ is called an LU **factorization** or an LU **decomposition** of A. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & d_2 & * & * & * \\ 0 & 0 & 0 & 0 & d_3 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = LU,$$

where d_i's are the pivots.

Now, let $A = LU$ be an LU factorization. Then the system $A\mathbf{x} = \mathbf{b}$ can be written as $LU\mathbf{x} = \mathbf{b}$. Let $U\mathbf{x} = \mathbf{y}$. Thus, the system

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$$

can be solved by the following two steps:

Step 1 Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.
Step 2 Solve $U\mathbf{x} = \mathbf{y}$ by back substitution.

The following example illustrates the convenience of an LU factorization of a matrix A for solving the system $A\mathbf{x} = \mathbf{b}$.

Example 1.16 Solve the system of linear equations

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = \mathbf{b}$$

by using an LU factorization of A.

Solution: The elementary matrices for the forward elimination on the augmented matrix $[A\ \mathbf{b}]$ are easily found to be

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix},$$

so that

$$E_3E_2E_1A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} = U.$$

Thus, if we set

$$L = E_1^{-1}\,E_2^{-1}\,E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix},$$

which is a lower triangular matrix with 1's on the diagonal, then

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}.$$

Now, the system

$$L\mathbf{y} = \mathbf{b}: \quad \left\{ \begin{array}{rcrcrcr} y_1 & & & & & = & 1 \\ 2y_1 & + & y_2 & & & = & -2 \\ -y_1 & - & 3y_2 & + & y_3 & = & 7 \end{array} \right.$$

can be easily solved inductively to get $\mathbf{y} = (1, -4, -4)$ and the system

$$Ux = y: \quad \left\{ \begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & x_3 & & & = & 1 \\ & - & x_2 & - & 2x_3 & + & x_4 & = & -4 \\ & & & - & 4x_3 & + & 4x_4 & = & -4 \end{array} \right.$$

also can be solved by back substitution to get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 & & \\ 2 & - & t \\ 1 & + & t \\ & & t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix},$$

for $t \in \mathbb{R}$, which is the solution for the original system. □

As shown in Example 1.16, it is a simple computation to solve the systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$, because the matrix L is lower triangular and the matrix U is the matrix obtained from A after forward elimination so that most entries of the lower-left side are zero.

Remark: For a system $A\mathbf{x} = \mathbf{b}$, the Gaussian elimination may be described as an LU factorization of a matrix A. Let us assume that one needs to solve several systems of linear equations $A\mathbf{x} = \mathbf{b}_i$ for $i = 1, 2, \ldots, \ell$ with the same coefficient matrix A. Instead of performing the Gaussian elimination process ℓ times to solve these systems, one can use an LU factorization of A and can solve first $L\mathbf{y} = \mathbf{b}_i$ for $i = 1, 2, \ldots, \ell$ to get solutions $\mathbf{y}_i$ and then the solutions of $A\mathbf{x} = \mathbf{b}_i$ are just those of $U\mathbf{x} = \mathbf{y}_i$. From an algorithmic point of view, the suggested method based on the LU factorization of A is much efficient than doing the Gaussian elimination repeatedly, in particular, when ℓ is large.

Problem 1.24 Determine an LU factorization of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

from which solve $A\mathbf{x} = \mathbf{b}$ for (1) $\mathbf{b} = [1\ 1\ 1]^T$ and (2) $\mathbf{b} = [2\ 0\ -1]^T$.

Problem 1.25 Let A and B be two lower triangular matrices. Prove that

(1) their product AB is also a lower triangular matrix;
(2) if A is invertible, then its inverse is also a lower triangular matrix;
(3) if the diagonal entries of A and B are all 1's, then the same holds for their product AB and their inverses.

Note that the same holds for upper triangular matrices, and for the product of more than two matrices.

The matrix U in the decomposition $A = LU$ of A can further be factored as the product $U = D\bar{U}$, where D is a diagonal matrix whose diagonal entries are the pivots of U or zeros and $\bar{U}$ is a row-echelon form of A with leading 1's, so that $A = LD\bar{U}$. For example,

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & d_2 & * & * & * \\ 0 & 0 & 0 & 0 & d_3 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = LU \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{*}{d_1} & * & * & * & \frac{*}{d_1} \\ 0 & 0 & 1 & \frac{*}{d_2} & * & \frac{*}{d_2} \\ 0 & 0 & 0 & 0 & 1 & \frac{*}{d_3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= LD\bar{U}.
\end{aligned}
$$

For notational convention, we replace $\bar{U}$ again by U and write $A = LDU$. This decomposition of A is called an *LDU* **factorization** or an *LDU* **decomposition** of A.

For example, the matrix A in Example 1.16 was factored as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} = LU.$$

It can be further factored as $A = LDU$ by taking

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = DU.$$

The LDU factorization of a matrix A is always possible when no row interchange is needed in the forward elimination process. In general, if a permutation matrix for a row interchange is necessary in the forward elimination process, then an LDU factorization may not be possible.

Example 1.17 (*The LDU factorization cannot exist*) Consider a matrix $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$. For forward elimination, it is necessary to interchange the first row with the second row. Without this interchange, A has no LU or LDU factorization. In fact, one can show that it cannot be expressed as a product of any lower triangular matrix L and any upper triangular matrix U. □

Suppose now that a row interchange is necessary during the forward elimination on the augmented matrix $[A\ \mathbf{b}]$. In this case, one can first do all the row interchanges before doing any other type of elementary row operations, since the interchange of rows can be done at any time, before or after the other elementary operations, with the same effect on the solution. Those 'row-interchanging' elementary matrices altogether form a permutation matrix P so that no more row interchanges are needed during the forward elimination on PA. Now, the matrix PA can have an LDU factorization.

Example 1.18 (*$PA = LDU$ factorization*) Now consider a square matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. For forward elimination, it is necessary to interchange the first row with the third row, that is, we need to multiply A by the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ so that

$$PA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = LDU.$$

Note that U is a row-echelon form of the matrix A. □

Of course, if we choose a different permutation P', then the LDU factorization of $P'A$ may be different from that of PA, even if there is another permutation matrix P'' that changes $P'A$ to PA. Moreover, as the following example shows, even if a permutation matrix is not necessary in the Gaussian elimination, the LDU factorization of A need not be unique.

Example 1.19 (*Infinitely many LDU factorizations*) The matrix

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has the LDU factorization

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = LDU$$

for any value x. It shows that a singular matrix B has infinitely many LDU factorizations. □

However, if the matrix A is invertible and if the permutation matrix P is fixed when it is necessary, then the matrix PA has a unique LDU factorization.

Theorem 1.12 *Let A be an invertible matrix. Then for a fixed suitable permutation matrix P the matrix PA has a unique LDU factorization.*

Proof: Suppose that $PA = L_1D_1U_1 = L_2D_2U_2$, where the L's are lower triangular, the U's are upper triangular whose diagonals are all 1's, and the D's are diagonal matrices with no zeros on the diagonal. One needs to show $L_1 = L_2$, $D_1 = D_2$, and $U_1 = U_2$ for the uniqueness.

Note that the inverse of a lower triangular matrix is also lower triangular, and the inverse of an upper triangular matrix is also upper triangular. And the inverse of a diagonal matrix is also diagonal. Therefore, by multiplying $(L_1D_1)^{-1} = D_1^{-1}L_1^{-1}$ on the left and U_2^{-1} on the right, the equation $L_1D_1U_1 = L_2D_2U_2$ becomes

$$U_1U_2^{-1} = D_1^{-1}L_1^{-1}L_2D_2.$$

The left-hand side is upper triangular, while the right-hand side is lower triangular. Hence, both sides must be diagonal. However, since the diagonal entries of the upper triangular matrix $U_1U_2^{-1}$ are all 1's, it must be the identity matrix I (see Problem 1.25). Thus $U_1U_2^{-1} = I$, i.e., $U_1 = U_2$. Similarly, $L_1^{-1}L_2 = D_1D_2^{-1}$ implies that $L_1 = L_2$ and $D_1 = D_2$. □

In particular, if an invertible matrix A is symmetric (i.e., $A = A^T$), and if it can be factored into $A = LDU$ without row interchanges, then we have

$$LDU = A = A^T = (LDU)^T = U^TD^TL^T = U^TDL^T,$$

and thus, by the uniqueness of factorizations, we have $U = L^T$ and $A = LDL^T$.

Problem 1.26 Find the factors L, D, and U for $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

What is the solution to $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = [1\ 0\ -1]^T$?

Problem 1.27 For all possible permutation matrices P, find the LDU factorization of PA for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

1.9 Applications

1.9.1 Cryptography

Cryptography is the study of sending messages in disguised form (secret codes) so that only the intended recipients can remove the disguise and read the message; modern cryptography uses advanced mathematics. As an application of invertible matrices, we introduce a simple coding. Suppose we associate a prescribed number with every letter in the alphabet; for example,

$$\begin{array}{cccccccccccc} A & B & C & D & \cdots & X & Y & Z & \text{Blank} & ? & ! \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 1 & 2 & 3 & \cdots & 23 & 24 & 25 & 26 & 27 & 28. \end{array}$$

Suppose that we want to send the message "GOOD LUCK." Replace this message by

$$6,\ 14,\ 14,\ 3,\ 26,\ 11,\ 20,\ 2,\ 10$$

according to the preceding substitution scheme. To use a matrix technique, we first break the message into three vectors in $\mathbb{R}^3$ each with three components, by adding extra blanks if necessary:

$$\begin{bmatrix} 6 \\ 14 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 26 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 2 \\ 10 \end{bmatrix}.$$

Next, choose a nonsingular 3×3 matrix A, say

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

which is supposed to be known to *both* sender and receiver. Then, as a matrix multiplication, A translates our message into

$$A\begin{bmatrix} 6 \\ 14 \\ 14 \end{bmatrix} = \begin{bmatrix} 6 \\ 26 \\ 34 \end{bmatrix}, \quad A\begin{bmatrix} 3 \\ 26 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 32 \\ 40 \end{bmatrix}, \quad A\begin{bmatrix} 20 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 42 \\ 32 \end{bmatrix}.$$

By putting the components of the resulting vectors consecutively, we transmit

$$6,\ 26,\ 34,\ 3,\ 32,\ 40,\ 20,\ 42,\ 32.$$

To decode this message, the receiver may follow the following process. Suppose that we received the following reply from our correspondent:

$$19,\ 45,\ 26,\ 13,\ 36,\ 41.$$

To decode it, first break the message into two vectors in $\mathbb{R}^3$ as before:

$$\begin{bmatrix} 19 \\ 45 \\ 26 \end{bmatrix}, \quad \begin{bmatrix} 13 \\ 36 \\ 41 \end{bmatrix}.$$

We want to find two vectors $\mathbf{x}_1$, $\mathbf{x}_2$ such that $A\mathbf{x}_i$ is the i-th vector of the above two vectors: i.e.,

$$A\mathbf{x}_1 = \begin{bmatrix} 19 \\ 45 \\ 26 \end{bmatrix}, \quad A\mathbf{x}_2 = \begin{bmatrix} 13 \\ 36 \\ 41 \end{bmatrix}.$$

Since A is invertible, the vectors $\mathbf{x}_1, \mathbf{x}_2$ can be found by multiplying the inverse of A to the two vectors given in the message. By an easy computation, one can find

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 19 \\ 45 \\ 26 \end{bmatrix} = \begin{bmatrix} 19 \\ 7 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 13 \\ 10 \\ 18 \end{bmatrix}.$$

The numbers one obtains are

$$19,\ 7,\ 0,\ 13,\ 10,\ 18.$$

Using our correspondence between letters and numbers, the message we have received is "THANKS."

Problem 1.28 Encode "TAKE UFO" using the same matrix A used in the above example.

1.9.2 Electrical network

In an electrical network, a simple current flow may be illustrated by a diagram like the one below. Such a network involves only voltage sources, like batteries, and resistors, like bulbs, motors, or refrigerators. The voltage is measured in *volts*, the resistance in *ohms*, and the current flow in amperes (*amps*, in short). For such an electrical network, current flow is governed by the following three laws:

- **Ohm's Law:** The voltage drop V across a resistor is the product of the current I and the resistance R: $V = IR$.
- **Kirchhoff's Current Law (KCL):** The current flow into a node equals the current flow out of the node.
- **Kirchhoff's Voltage Law (KVL):** The algebraic sum of the voltage drops around a closed loop equals the total voltage sources in the loop.

Example 1.20 Determine the currents in the network given in Figure 1.4.

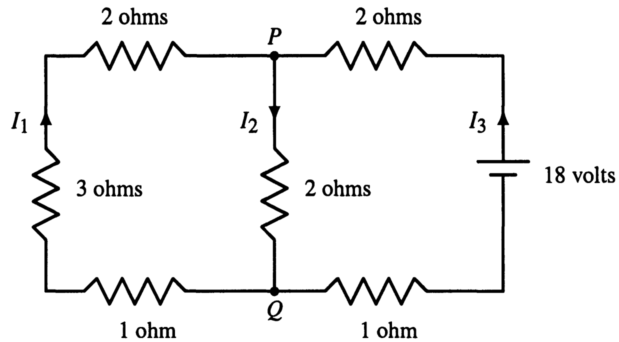

Figure 1.4. A circuit network

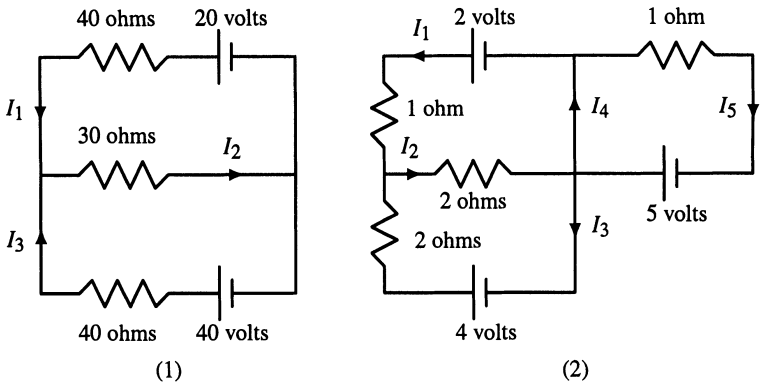

Figure 1.5. Two circuit networks

Solution: By applying KCL to nodes P and Q, we get equations

$$\begin{aligned} I_1 + I_3 &= I_2 \text{ at } P, \\ I_2 &= I_1 + I_3 \text{ at } Q. \end{aligned}$$

Observe that both equations are the same, and one of them is redundant. By applying KVL to each of the loops in the network clockwise direction, we get

$$\begin{aligned} 6I_1 + 2I_2 &= 0 \text{ from the left loop,} \\ 2I_2 + 3I_3 &= 18 \text{ from the right loop.} \end{aligned}$$

Collecting all the equations, we get a system of linear equations:

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 6I_1 + 2I_2 = 0 \\ 2I_2 + 3I_3 = 18. \end{cases}$$

By solving it, the currents are $I_1 = -1$ amp, $I_2 = 3$ amps and $I_3 = 4$ amps. The negative sign for I_1 means that the current I_1 flows in the direction opposite to that shown in the figure. □

Problem 1.29 Determine the currents in the networks given in Figure 1.5.

1.9.3 Leontief model

Another significant application of linear algebra is to a mathematical model in economics. In most nations, an economic society may be divided into many sectors that produce goods or services, such as the automobile industry, oil industry, steel industry, communication industry, and so on. Then a fundamental problem in economics is to find the *equilibrium* of the supply and the demand in the economy.

There are two kind of demands for the goods: the *intermediate demand* from the industries themselves (or the sectors) that are needed as inputs for their own production, and the *extra demand* from the consumer, the governmental use, surplus production, or exports. Practically, the interrelation between the sectors is very complicated, and the connection between the extra demand and the production is unclear. A natural question is *whether there is a production level such that the total amounts produced (or supply) will exactly balance the total demand for the production*, so that the equality

$$\begin{aligned}\{\text{Total output}\} &= \{\text{Total demand}\}\\ &= \{\text{Intermediate demand}\} + \{\text{Extra demand}\}\end{aligned}$$

holds. This problem can be described by a system of linear equations, which is called the *Leontief Input-Output Model*. To illustrate this, we show a simple example.

Suppose that a nation's economy consists of three sectors: I_1 = automobile industry, I_2 = steel industry, and I_3 = oil industry.

Let $\mathbf{x} = [x_1\, x_2\, x_3]^T$ denote the production vector (or production level) in $\mathbb{R}^3$, where each entry x_i denotes the total amount (in a common unit such as 'dollars' rather than quantities such as 'tons' or 'gallons') of the output that the industry I_i produces per year.

The intermediate demand may be explained as follows. Suppose that, for the total output x_2 units of the steel industry I_2, 20% is contributed by the output of I_1, 40% by that of I_2 and 20% by that of I_3. Then we can write this as a column vector, called a *unit consumption vector* of I_2:

$$\mathbf{c}_2 = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.2 \end{bmatrix}.$$

For example, if I_2 decides to produce 100 units per year, then it will order (or demand) 20 units from I_1, 40 units from I_2, and 20 units from I_3: i.e., the consumption vector of I_2 for the production $x_2 = 100$ units can be written as a column vector: $100\mathbf{c}_2 = [20\ 40\ 20]^T$. From the concept of the consumption vector, it is clear that the sum of decimal fractions in the column $\mathbf{c}_2$ must be ≤ 1.

In our example, suppose that the demands (inputs) of the outputs are given by the following matrix, called an *input-output matrix*:

$$A = \text{input}\ \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ \\ \\ \end{array} \begin{array}{c} \text{output} \\ \begin{array}{ccc} I_1 & I_2 & I_3 \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.3 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{array} \end{array}.$$

In this matrix, an industry looks down a column to see how much it needs from where to produce its total output, and it looks across a row to see how much of its output goes to where. For example, the second row says that, out of the total output x_2 units of the steel industry I_2, as the intermediate demand, the automobile industry I_1 demands 10% of the output x_1, the steel industry I_2 demands 40% of the output x_2 and the oil industry I_3 demands 10% of the output x_3. Therefore, it is now easy to see that the intermediate demand of the economy can be written as

$$A\mathbf{x} = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_1 + 0.2x_2 + 0.3x_3 \\ 0.1x_1 + 0.4x_2 + 0.1x_3 \\ 0.3x_1 + 0.2x_2 + 0.3x_3 \end{bmatrix}.$$

Suppose that the extra demand in our example is given by $\mathbf{d} = [d_1, d_2, d_3]^T = [30, 20, 10]^T$. Then the problem for this economy is to find the production vector $\mathbf{x}$ satisfying the following equation:

$$\mathbf{x} = A\mathbf{x} + \mathbf{d}.$$

Another form of the equation is $(I - A)\mathbf{x} = \mathbf{d}$, where the matrix $I - A$ is called the *Leontief matrix*. If $I - A$ is not invertible, then the equation may have no solution or infinitely many solutions depending on what $\mathbf{d}$ is. If $I - A$ is invertible, then the equation has the unique solution $\mathbf{x} = (I - A)^{-1}\mathbf{d}$. Now, our example can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}.$$

In this example, it turns out that the matrix $I - A$ is invertible and

$$(I - A)^{-1} = \begin{bmatrix} 2.0 & 1.0 & 1.0 \\ 0.5 & 2.0 & 0.5 \\ 1.0 & 1.0 & 2.0 \end{bmatrix}.$$

Therefore,

$$\mathbf{x} = (I - A)^{-1}\mathbf{d} = \begin{bmatrix} 90 \\ 60 \\ 70 \end{bmatrix},$$

which gives the total amount of product x_i of the industry I_i for one year to meet the required demand.

Remark: (1) Under the usual circumstances, the sum of the entries in a column of the consumption matrix A is less than one because a sector should require less than one units worth of inputs to produce one unit of output. This actually implies that $I - A$ is invertible and the production vector $\mathbf{x}$ is feasible in the sense that the entries in $\mathbf{x}$ are all nonnegative as the following argument shows.

(2) In general, by using induction one can easily verify that for any $k = 1, 2, \ldots,$

$$(I - A)(I + A + \cdots + A^{k-1}) = I - A^k.$$

If the sums of column entries of A are all strictly less than one, then $\lim_{k\to\infty} A^k = \mathbf{0}$ (see Section 6.4 for the limit of a sequence of matrices). Thus, we get $(I - A)(I + A + \cdots + A^k + \cdots) = I$, that is,

$$(I - A)^{-1} = I + A + \cdots + A^k + \cdots.$$

This also shows a practical way of computing $(I - A)^{-1}$ since by taking k sufficiently large the right-hand side may be made very close to $(I - A)^{-1}$. In Chapter 6, an easier method of computing A^k will be shown.

In summary, if A and $\mathbf{d}$ have nonnegative entries and if the sum of the entries of each column of A is less than one, then $I - A$ is invertible and the inverse is given as the above formula. Moreover, as the formula shows the entries of the inverse are all nonnegative, and so are those of the production vector $\mathbf{x} = (I - A)^{-1}\mathbf{d}$.

Problem 1.30 Determine the total demand for industries I_1, I_2 and I_3 for the input-output matrix A and the extra demand vector $\mathbf{d}$ given below:

$$A = \begin{bmatrix} 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \\ 0.4 & 0.2 & 0.2 \end{bmatrix} \text{ with } \mathbf{d} = \mathbf{0}.$$

Problem 1.31 Suppose that an economy is divided into three sectors: I_1 = services, I_2 = manufacturing industries, and I_3 = agriculture. For each unit of output, I_1 demands no services from I_1, 0.4 units from I_2, and 0.5 units from I_3. For each unit of output, I_2 requires 0.1 units from sector I_1 of services, 0.7 units from other parts in sector I_2, and no product from sector I_3. For each unit of output, I_3 demands 0.8 units of services I_1, 0.1 units of manufacturing products from I_2, and 0.1 units of its own output from I_3. Determine the production level to balance the economy when 90 units of services, 10 units of manufacturing, and 30 units of agriculture are required as the extra demand.

1.10 Exercises

1.1. Which of the following matrices are in row-echelon form or in reduced row-echelon form?

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & -2 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1.2. Find a row-echelon form of each matrix.

$$(1)\ \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \\ 3 & -9 & 10 & 2 & 9 \\ 2 & -6 & 4 & 2 & 4 \\ 2 & -6 & 8 & 1 & 7 \end{bmatrix}, \qquad (2)\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

1.3. Find the reduced row-echelon form of the matrices in Exercise **1.2**.

1.4. Solve the systems of equations by Gauss–Jordan elimination.

$$(1)\ \begin{cases} x_1 + x_2 + x_3 - x_4 = -2 \\ 2x_1 - x_2 + x_3 + x_4 = 0 \\ 3x_1 + 2x_2 - x_3 - x_4 = 1 \\ x_1 + x_2 + 3x_3 - 3x_4 = -8. \end{cases}$$

$$(2)\ \begin{cases} 2x - 3y = 8 \\ 4x - 5y + z = 15 \\ 2x + 4z = 1. \end{cases}$$

What are the pivots in each of 3^{rd} kind elementary operations?

1.5. Which of the following systems has a nontrivial solution?

$$(1)\ \begin{cases} x + 2y + 3z = 0 \\ 2y + 2z = 0 \\ x + 2y + 3z = 0. \end{cases} \qquad (2)\ \begin{cases} 2x + y - z = 0 \\ x - 2y - 3z = 0 \\ 3x + y - 2z = 0. \end{cases}$$

1.6. Determine all values of the b_i that make the following system consistent:

$$\begin{cases} x + y - z = b_1 \\ 2y + z = b_2 \\ y - z = b_3. \end{cases}$$

1.7. Determine the condition on b_i so that the following system has no solution:

$$\begin{cases} 2x + y + 7z = b_1 \\ 6x - 2y + 11z = b_2 \\ 2x - y + 3z = b_3. \end{cases}$$

1.8. Let A and B be matrices of the same size.

(1) Show that, if $A\mathbf{x} = 0$ for all $\mathbf{x}$, then A is the zero matrix.

(2) Show that, if $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$, then $A = B$.

1.9. Compute ABC and CAB for

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

1.10. Prove that if A is a 3×3 matrix such that $AB = BA$ for every 3×3 matrix B, then $A = cI_3$ for some constant c.

1.11. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. Find A^k for all integers k.

1.12. Compute $(2A - B)C$ and CC^T for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

1.13. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. For any square matrix A, a *matrix polynomial* $f(A)$ is defined as

$$f(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I.$$

For $f(x) = 3x^3 + x^2 - 2x + 3$, find $f(A)$ for

$$(1)\ A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

1.14. Find the symmetric part and the skew-symmetric part of each of the following matrices.

$$(1)\ A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 5 & 9 \\ -1 & 3 & 2 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

1.15. Find AA^T and A^TA for the matrix $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 3 & 1 \\ 2 & 8 & 4 & 0 \end{bmatrix}$.

1.16. Let $A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$.

(1) Find a matrix B such that $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$.

(2) Find a matrix C such that $AC = A^2 + A$.

1.17. Find all possible choices of a, b and c so that $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ has an inverse matrix such that $A^{-1} = A$.

1.18. Decide whether or not each of the following matrices is invertible. Find the inverses for invertible ones.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix}.$$

1.19. Find the inverse of each of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 2 \\ -6 & 4 & 11 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 2 & 4 & 8 \end{bmatrix}, C = \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix} (k \neq 0).$$

1.20. Suppose A is a 2×1 matrix and B is a 1×2 matrix. Prove that the product AB is not invertible.

1.21. Find three matrices which are row equivalent to $A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 5 & 2 & -3 & 4 \end{bmatrix}$.

1.22. Write the following systems of equations as matrix equations $A\mathbf{x} = \mathbf{b}$ and solve them by computing $A^{-1}\mathbf{b}$:

$$(1)\ \begin{cases} 2x_1 - x_2 + 3x_3 = 2 \\ \quad\ \ x_2 - 4x_3 = 5 \\ 2x_1 + x_2 - 2x_3 = 7, \end{cases} \quad (2)\ \begin{cases} x_1 - x_2 + x_3 = 5 \\ x_1 + x_2 - x_3 = -1 \\ 4x_1 - 3x_2 + 2x_3 = -3. \end{cases}$$

1.23. Find the LDU factorization for each of the following matrices:

$$(1)\ A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}.$$

1.24. Find the LDL^T factorization of the following symmetric matrices:

$$(1)\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 3 & 8 & 10 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

1.25. Solve $A\mathbf{x} = \mathbf{b}$ with $A = LU$, where L and U are given as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

Forward elimination is the same as $L\mathbf{c} = \mathbf{b}$, and back-substitution is $U\mathbf{x} = \mathbf{c}$.

1.26. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 5 \\ 1 & 4 & 7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

(1) Solve $A\mathbf{x} = \mathbf{b}$ by Gauss–Jordan elimination.

(2) Find the LDU factorization of A.

(3) Write A as a product of elementary matrices.

(4) Find the inverse of A.

1.27. A square matrix A is said to be *nilpotent* if $A^k = \mathbf{0}$ for a positive integer k.

(1) Show that any invertible matrix is not nilpotent.

(2) Show that any triangular matrix with zero diagonal is nilpotent.

(3) Show that if A is a nilpotent with $A^k = \mathbf{0}$, then $I - A$ is invertible with its inverse $I + A + \cdots + A^{k-1}$.

1.28. A square matrix A is said to be *idempotent* if $A^2 = A$.

(1) Find an example of an idempotent matrix other than $\mathbf{0}$ or I.

(2) Show that, if a matrix A is both idempotent and invertible, then $A = I$.

1.29. Determine whether the following statements are true or false, in general, and justify your answers.

(1) Let A and B be row-equivalent square matrices. Then A is invertible if and only if B is invertible.

(2) Let A be a square matrix such that $AA = A$. Then A is the identity.

(3) If A and B are invertible matrices such that $A^2 = I$ and $B^2 = I$, then $(AB)^{-1} = BA$.

(4) If A and B are invertible matrices, $A + B$ is also invertible.

(5) If A, B and AB are symmetric, then $AB = BA$.

(6) If A and B are symmetric and of the same size, then AB is also symmetric.

(7) If A is invertible and symmetric, then A^{-1} is also symmetric.

(8) Let $AB^T = I$. Then A is invertible if and only if B is invertible.

(9) If a square matrix A is not invertible, then neither is AB for any B.

(10) If E_1 and E_2 are elementary matrices, then $E_1E_2 = E_2E_1$.

(11) The inverse of an invertible upper triangular matrix is upper triangular.

(12) Any invertible matrix A can be written as $A = LU$, where L is lower triangular and U is upper triangular.

2

Determinants

2.1 Basic properties of the determinant

Our primary interest in Chapter 1 was in the solvability or finding solutions of a system $A\mathbf{x} = \mathbf{b}$ of linear equations. For an invertible matrix A, Theorem 1.9 shows that the system has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$.

Now the question is how to determine whether or not a square matrix A is invertible. In this chapter, we introduce the notion of *determinant* as a real-valued function of square matrices that satisfies certain axiomatic rules, and then show that *a square matrix A is invertible if and only if the determinant of A is not zero.* In fact, it was shown in Chapter 1 that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$. This number is called the determinant of A, written $\det A$, and is defined formally as follows:

Definition 2.1 For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$, the **determinant** of A is defined as $\det A = ad - bc$.

Geometrically, it turns out that the determinant of a 2×2 matrix A represents, up to sign, the area of a parallelogram in the xy-plane whose edges are constructed by the row vectors of A (see Theorem 2.10). Naturally, one can expect to define a determinant function on higher order square matrices so that it has a geometric interpretation similar to the 2×2 case. However, the formula itself in Definition 2.1 does not provide any clue of how to extend this idea of determinant to higher order matrices. Hence, we first examine some fundamental properties of the determinant function defined in Definition 2.1.

By a direct computation, one can easily verify that the function det in Definition 2.1 satisfies the following lemma.

Lemma 2.1 **(1)** $\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$

J H Kwak et al., *Linear Algebra*

(2) $\det\begin{bmatrix} c & d \\ a & b \end{bmatrix} = -\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(3) $\det\begin{bmatrix} ka+\ell a' & kb+\ell b' \\ c & d \end{bmatrix} = k\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \ell\det\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}$.

Proof: (2) $\det\begin{bmatrix} c & d \\ a & b \end{bmatrix} = bc - ad = -(ad - bc) = -\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(3)
$$\begin{aligned}\det\begin{bmatrix} ka+\ell a' & kb+\ell b' \\ c & d \end{bmatrix} &= (ka+\ell a')d - (kb+\ell b')c \\ &= k(ad-bc) + \ell(a'd - b'c) \\ &= k\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \ell\det\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}. \quad \square\end{aligned}$$

In Lemma 2.5, it will be shown that if a function $f : M_{2\times2}(\mathbb{R}) \to \mathbb{R}$ satisfies the properties (1)–(3) in Lemma 2.1, then it must be the function det defined in Definition 2.1, that is, $f(A) = ad - bc$. The properties (1)–(3) in Lemma 2.1 of the determinant on $M_{2\times2}(\mathbb{R})$ enable us to define the determinant function for any square matrix.

Definition 2.2 A real-valued function $f : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ of all $n\times n$ square matrices is called a **determinant** if it satisfies the following three rules:

(R_1) The value of f of the identity matrix is 1, i.e., $f(I_n) = 1$;
(R_2) the value of f changes sign if any two rows are interchanged;
(R_3) f is *linear* in the first row: that is, by definition,

$$f\begin{bmatrix} k\mathbf{r}_1 + \ell\mathbf{r}'_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = kf\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} + \ell f\begin{bmatrix} \mathbf{r}'_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix},$$

where $\mathbf{r}_i$'s denote the row vectors $[a_{i1} \ \cdots \ a_{in}]$ of a matrix.

Remark: (1) To be familiar with the linearity rule **(R_3)**, note that all row vectors of any $n \times n$ matrix belong to the set $M_{1\times n}(\mathbb{R})$, on which a matrix sum and a scalar multiplication are well defined. A real-valued function $f : M_{1\times n}(\mathbb{R}) \to \mathbb{R}$ is said to be **linear** if it preserves these two operations: that is, for any two vectors $\mathbf{x}, \ \mathbf{y} \in M_{1\times n}(\mathbb{R})$ and scalar k,

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) \quad \text{and} \quad f(k\mathbf{x}) = kf(\mathbf{x}),$$

or, equivalently $f(k\mathbf{x}+\ell\mathbf{y}) = kf(\mathbf{x}) + \ell f(\mathbf{y})$. Such a linear function will be discussed again in Chapter 4.
(2) The determinant is not defined for a non-square matrix.

It is already shown that the det on 2×2 matrices satisfies the rules **(R_1)–(R_3)**. In the next section, one can see that for each positive integer n there always exists a function

$f : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ satisfying the three rules $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and such a function is unique (*existence and uniqueness*). Therefore, we say 'the' determinant and designate it as 'det' in any order.

Let us first derive some direct consequences of the rules $(\mathbf{R}_1)$–$(\mathbf{R}_3)$.

Theorem 2.2 *The determinant satisfies the following properties.*

(1) *The determinant is linear in each row.*
(2) *If A has either a zero row or two identical rows, then* $\det A = 0$.
(3) *The elementary row operation that adds a constant multiple of one row to another row leaves the determinant unchanged.*

Proof: (1) Any row can be placed in the first row by interchanging rows with a change of sign in the determinant by the rule $(\mathbf{R}_2)$, and then using the linearity rule $(\mathbf{R}_3)$ and $(\mathbf{R}_2)$ again by interchanging the same rows.

(2) If A has a zero row, then this row is zero times the zero row so that $\det A = 0$ by (1). If A has two identical rows, then interchanging those two identical rows does not change the matrix itself but $\det A = -\det A$ by the rule $(\mathbf{R}_2)$, so that $\det A = 0$.

(3) By a direct computation using (1), one can get

$$\det \begin{bmatrix} \vdots \\ \mathbf{r}_i + k\mathbf{r}_j \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{bmatrix} + k \det \begin{bmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{bmatrix},$$

in which the second term on the right-hand side is zero by (2). □

The rule $(\mathbf{R}_2)$ of the determinant function is said to be the **alternating** property, and the property (1) in Theorem 2.2 is said to be **multilinearity**.

It is now easy to see the effect of elementary row operations on evaluations of the determinant. The first elementary row operation that 'multiplies a row by a constant k' changes the determinant to k times the determinant, by Theorem 2.2(1). The rule $(\mathbf{R}_2)$ explains the effect of the second elementary row operation that 'interchanges two rows'. The third elementary row operation that 'adds a constant multiple of a row to another' is explained in Theorem 2.2(3). In summary, one can see that

$$\det(EA) = \det E \det A \quad \text{for any elementary matrix } E.$$

For example, if E is the elementary matrix obtained from the identity matrix by 'multiplies a row by a constant k', then $\det(EA) = k\det A$ and $\det E = k$ by Theorem 2.2(1), so that $\det(EA) = \det E \det A$. As a consequence, if two matrices A and B are row-equivalent, then $\det A = k\det B$ for some nonzero number k.

Example 2.1 Consider a matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & b+a \end{bmatrix}.$$

If one adds the second row to the third, then the third row becomes

$$[a+b+c \quad a+b+c \quad a+b+c],$$

which is a scalar multiple of the first row. Thus, $\det A = 0$. □

Problem 2.1 Show that, for an $n \times n$ matrix A and $k \in \mathbb{R}$, $\det(kA) = k^n \det A$.

Problem 2.2 Explain why $\det A = 0$ for

$$(1)\, A = \begin{bmatrix} a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \\ a+3 & a+6 & a+9 \end{bmatrix}, \qquad (2)\, A = \begin{bmatrix} a & a^4 & a^7 \\ a^2 & a^5 & a^8 \\ a^3 & a^6 & a^9 \end{bmatrix}.$$

Recall that any square matrix can be transformed into an upper triangular matrix by forward elimination, possibly with row interchanges. Further properties of the determinant are obtained in the following theorem.

Theorem 2.3 *The determinant satisfies the following properties.*

(1) *The determinant of a triangular matrix is the product of the diagonal entries.*
(2) *The matrix A is invertible if and only if* $\det A \neq 0$.
(3) *For any two $n \times n$ matrices A and B,* $\det(AB) = \det A \det B$.
(4) $\det A^T = \det A$.

Proof: (1) If A is a diagonal matrix, then it is clear that $\det A = a_{11} \cdots a_{nn}$ by the multilinearity in Theorem 2.2(1) and rule $(\mathbf{R}_1)$. Suppose that A is a lower triangular matrix. If A has a zero diagonal entry, then a forward elimination, which does not change the determinant, produces a zero row, so that $\det A = 0$. If A does not have a zero diagonal entry, a forward elimination makes A row equivalent to the diagonal matrix D whose diagonal entries are exactly those of A, so that $\det A = \det D = a_{11} \cdots a_{nn}$. Similar arguments can be applied to an upper triangular matrix.

(2) A square matrix A is row equivalent to an upper triangular matrix U through a forward elimination possibly with row interchanges: that is, $A = PLU$ for some permutation matrix P and a lower triangular matrix L whose diagonal entries are all 1's. Thus $\det A = \pm \det U$, and the invertibility of U and A are equivalent. However, U is invertible if and only if U has no zero diagonal entry by Corollary 1.10, which is equivalent to $\det U \neq 0$ by (1).

(3) If A is not invertible, then neither is AB, and so $\det(AB) = 0 = \det A \det B$. If A is invertible, it can be written as a product of elementary matrices by Theorem 1.9, say $A = E_1 E_2 \cdots E_k$. Then by induction on k,

$$\begin{aligned}
\det(AB) &= \det(E_1 E_2 \cdots E_k B) \\
&= \det E_1 \ \det E_2 \ \cdots \ \det E_k \ \det B \\
&= \det(E_1 E_2 \cdots E_k) \det B \\
&= \det A \det B.
\end{aligned}$$

(4) Clearly, A is not invertible if and only if A^T is not. Thus, for a singular matrix A we have $\det A^T = 0 = \det A$. If A is invertible, then write it again as a product of elementary matrices, say $A = E_1 E_2 \cdots E_k$. But, $\det E = \det E^T$ for any elementary matrix E. In fact, if E is an elementary matrix obtained from the identity matrix by row interchange, then $\det E^T = -1 = \det E$ by $(\mathbf{R}_2)$, and all elementary matrices of other types are triangular, so that $\det E = \det E^T$. Hence, we have by (3)

$$\begin{aligned}
\det A^T &= \det(E_1 E_2 \cdots E_k)^T \\
&= \det(E_k^T \cdots E_2^T E_1^T) \\
&= \det E_k^T \cdots \det E_2^T \ \det E_1^T \\
&= \det E_k \cdots \det E_2 \ \det E_1 \\
&= \det A.
\end{aligned}$$

□

Remark: From the equality $\det A = \det A^T$, one could define the determinant in terms of columns instead of rows in Definition 2.2, and Theorem 2.2 is also true with 'columns' instead of 'rows'.

Example 2.2 (*Computing* $\det A$ *by a forward elimination*) Evaluate the determinant of

$$A = \begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & 0 & -1 & 2 \\ 3 & -4 & 3 & -1 \end{bmatrix}.$$

Solution: By using forward elimination, A can be transformed to an upper triangular matrix U. Since the forward elimination does not change the determinant, the determinant of A is simply the product of the diagonal entries of U:

$$\det A = \det U = \det \begin{bmatrix} 2 & -4 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 13 \end{bmatrix} = 2 \cdot (-1)^2 \cdot 13 = 26.$$

□

Problem 2.3 Prove that if A is invertible, then $\det A^{-1} = 1/\det A$.

Problem 2.4 Evaluate the determinant of each of the following matrices:

$$(1)\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 1 \\ -2 & 2 & 3 \end{bmatrix}, \quad (2)\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}, \quad (3)\begin{bmatrix} 1 & x & x^2 & x^3 \\ x^3 & 1 & x & x^2 \\ x^2 & x^3 & 1 & x \\ x & x^2 & x^3 & 1 \end{bmatrix}.$$

2.2 Existence and uniqueness of the determinant

Throughout this section, we prove the following fundamental theorem for the determinant.

Theorem 2.4 *For any natural number n,*

(1) **(Existence)** *there exists a real-valued function $f : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ which satisfies the three rules* $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ *in Definition 2.2.*
(2) **(Uniqueness)** *Such function is unique.*

Clearly, it is true when $n = 1$ with conclusion $\det[a] = a$.

For 2×2 **matrices:** When $n = 2$, the existence theorem comes from Lemma 2.1. The next lemma shows that any function $f : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ satisfying the three rules $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ must be the det in Definition 2.1, which implies the uniqueness of the determinant function on $M_{2\times 2}(\mathbb{R})$.

Lemma 2.5 *If a function $f : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ satisfies the rules* $(\mathbf{R}_1)$–$(\mathbf{R}_3)$, *then* $f\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. *That is,* $f(A) = \det A$.

Proof: First, note that $f\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$ by the rules $(\mathbf{R}_1)$ and $(\mathbf{R}_2)$.

$$\begin{aligned} f(A) &= f\begin{bmatrix} a & b \\ c & d \end{bmatrix} = f\begin{bmatrix} a+0 & 0+b \\ c & d \end{bmatrix} \\ &= f\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + f\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \\ &= f\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + f\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + f\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} + f\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \\ &= ad + 0 + 0 - bc = ad - bc, \end{aligned}$$

where the third and fourth equalities come from the multilinearity in Theorem 2.2(1). □

For 3×3 **matrices:** For $n = 3$, the same process as in the case of $n = 2$ can be applied. That is, by repeated use of the three rules $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ as in the proof of

Lemma 2.5, one can derive an *explicit formula* for $\det A$ of a matrix $A = [a_{ij}]$ in $M_{3\times 3}(\mathbb{R})$ as follows:

$$\begin{aligned}
&\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&= \det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} \\
&\quad + \det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \\
&= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.
\end{aligned}$$

The first equality is obtained by the multilinearity in Theorem 2.2(1): First, by applying it to the first row with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \end{bmatrix},$$

$\det A$ becomes the sum of determinants of three matrices. Observe that, in each of the three matrices, the first row has just one entry from A and all others zero. Subsequently, by applying the same multilinearity to the second and the third rows of each of the three matrices, one gets the sum of the determinants of $3^3 = 27$ matrices, each of which has exactly three entries from A, one in each of three rows, and all other entries zero. In each of those 27 matrices, if any two of the three entries of A are in the same column, then the matrix contains a zero column so that its determinant is zero. Consequently, the determinants of six matrices are left to get the first equality.

The second equality is just the computation of the six determinants by using the rules ($\mathbf{R}_2$) and Theorem 2.3(1). In fact, in each of those six matrices, no two entries from A are in the same row or in the same column, and thus one can take suitable 'column interchanges' to convert it to a diagonal matrix. Thus the determinant of each of them is just the product of the three entries with $\pm$ sign which is determined by the number of column interchanges.

Remark: The explicit formula for the determinant of a 3×3 matrix can easily be memorized by the following scheme. Copy the first two columns and put them on the right of the matrix, and compute the determinant by multiplying entries on six diagonals with a $+$ sign or a $-$ sign as in Figure 2.1. This is known as Sarrus's method for 3×3 matrices. It has no analogue for matrices of higher order $n \geq 4$.

The computation of the explicit formula for $\det A$ shows that, if any real-valued function $f : M_{3\times 3}(\mathbb{R}) \to \mathbb{R}$ satisfies the rules ($\mathbf{R}_1$)–($\mathbf{R}_3$), then $f(A) = \det A$ for any matrix $A = [a_{ij}] \in M_{3\times 3}(\mathbb{R})$. This proves the uniqueness theorem when $n = 3$. On the other hand, one can easily show that the given explicit formula for $\det A$ of a matrix $A \in M_{3\times 3}(\mathbb{R})$ satisfies the three rules, which proves the existence when

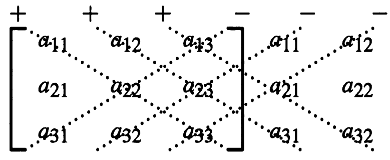

Figure 2.1. Sarrus's method

$n = 3$. Therefore, for $n = 3$, it shows both the uniqueness and the existence of the determinant function on $M_{3\times3}(\mathbb{R})$, which proves Theorem 2.4 when $n = 3$.

Problem 2.5 Show that the given explicit formula of the determinant for 3×3 matrices satisfies the three rules $(\mathbf{R}_1)$–$(\mathbf{R}_3)$.

Problem 2.6 Use Sarrus's method to evaluate the determinants of

$$(1)\ A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 1 \\ -2 & 2 & 3 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & -1 \\ -2 & 6 & 4 \end{bmatrix}.$$

Now, a reader might have an idea how to prove the uniqueness and the existence of the determinant function on $M_{n\times n}(\mathbb{R})$ for $n > 3$. If so, that reader may omit reading its continued proof below and rather concentrate on understanding the explicit formula of $\det A$ in Theorem 2.6.

For matrices of higher order $n > 3$: Again, we repeat the same procedure as for the 3×3 matrices to get an explicit formula for $\det A$ of any square matrix $A = [a_{ij}]$ of order n.

(Step 1) Just as the case for $n = 3$, use the multilinearity in each row of A to get $\det A$ as the sum of the determinants of n^n matrices. Notice that each one of n^n matrices has exactly n entries from A, one in each of n rows. However, if any two of the n entries from A are in the same column, then it must have a zero column, so that its determinant is zero and it can be neglected in the summation. Now, in each remaining matrix, the n entries from A must be in different columns: that is, no two of the n entries from A are in the same row or in the same column.

(Step 2) Now, we aim to estimate how many of them remain. From the observation in Step 1, in each of the remaining matrices, those n entries from A are of the form

$$a_{1i},\ a_{2j},\ a_{3k},\ \ldots,\ a_{n\ell}$$

with some column indices $i, j, k, \ldots, \ell$. Since no two of these n entries are in the same column, the column indices $i, j, k, \ldots, \ell$ are just a rearrangement of $1, 2, \ldots, n$ without repetition or omissions. It is not hard to see that there are just $n!$ ways of such rearrangements, so that $n!$ matrices remain for further consideration. (Here $n! = n(n-1)\cdots 2\cdot 1$, called n factorial.)

Remark: In fact, the $n!$ remaining matrices can be constructed from the matrix $A = [a_{ij}]$ as follows: First, choose any one entry from the first row of A, say a_{1i}, in the i-th column. Then all the other $n-1$ entries $a_{2j}, a_{3k}, \ldots, a_{n\ell}$ should be taken from the columns different from the i-th column. That is, they should be chosen from the submatrix of A obtained by deleting the row and the column containing a_{1i}. If the second entry a_{2j} is taken from the second row, then the third entry a_{3k} should be taken from the submatrix of A obtained by deleting the two rows and the two columns containing a_{1i} and a_{2j}, and so on. Finally, if the first $n-1$ entries

$$a_{1i},\ a_{2j},\ a_{3k},\ \ldots,\ a_{n-1\,p}$$

are chosen, then there is no alternative choice for the last one $a_{n\ell}$ since it is the one left after deleting $n-1$ rows and $n-1$ columns from A.

(Step 3) We now compute the determinant of each of the $n!$ remaining matrices. Since each of those matrices has just n entries from A so that no two of them are in the same row or in the same column, one can convert it into a diagonal matrix by 'suitable' column interchanges. Then the determinant will be just the product of the n entries from A with '$\pm$' sign, which will be determined by the number (actually the parity) of the column interchanges. To determine the sign, let us once again look back to the case of $n = 3$.

Example 2.3 (*Convert into a diagonal matrix by column interchanges*) Suppose that one of the six matrices is of the form:

$$\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}.$$

Then, one can convert this matrix into a diagonal matrix by interchanging the first and the third columns. That is,

$$\det \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} a_{13} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{31} \end{bmatrix} = -a_{13}a_{22}a_{31}.$$

Note that a column interchange is the same as an interchange of the corresponding column indices. Moreover, in each diagonal entry of a matrix, the row index must be the same as its column index. Hence, to convert such a matrix into a diagonal matrix, one has to convert the given arrangement of the column indices (in the example we have 3, 2, 1) to the standard order 1, 2, 3 to be matched with the arrangement 1, 2, 3 of the row indices.

In this case, there may be several ways of column interchanges to convert the given matrix to a diagonal matrix. For example, to convert the given arrangement 3, 2, 1 of the column indices to the standard order 1, 2, 3, one can take either just one interchanging of 3 and 1, or three interchanges: 3 and 2, 3 and 1, and then 2 and 1. In either case, the parity is odd so that the "$-$" sign in the computation of the determinant came from $(-1)^1 = (-1)^3$, where the exponents mean the numbers of interchanges of the column indices. $\square$

To formalize our discussions, we introduce a mathematical terminology for a rearrangement of n objects:

Definition 2.3 A **permutation** of n objects is a one-to-one function from the set of n objects onto itself.

In most cases, we use the set of integers $N_n = \{1, 2, \ldots, n\}$ for a set of n objects. A permutation σ of N_n assigns a number $\sigma(i)$ in N_n to each number i in N_n, and this permutation σ is usually denoted by

$$\sigma = \langle \sigma(1), \sigma(2), \ldots, \sigma(n) \rangle = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

Here, the first row is the usual lay-out of N_n as the domain set, and the second row is the image set showing an arrangement of the numbers in N_n in a certain order without repetitions or omissions. If S_n denotes the set of all permutations of N_n, then, as mentioned previously, one can see that S_n has exactly $n!$ permutations. For example, S_2 has $2 = 2!$, S_3 has $6 = 3!$, and S_4 has $24 = 4!$ permutations.

Definition 2.4 A permutation $\sigma = \langle j_1, j_2, \ldots, j_n \rangle$ is said to have an **inversion** if $j_s > j_t$ for $s < t$ (i.e., a larger number precedes a smaller number).

For example, the permutation $\sigma = \langle 3, 1, 5, 4, 2 \rangle$ has five inversions, since 3 precedes 1 and 2; 5 precedes 4 and 2; and 4 precedes 2. Note that the identity $\langle 1, 2, \ldots, n \rangle$ is the only one without inversions.

Definition 2.5 A permutation is said to be **even** if it has an even number of inversions, and it is said to be **odd** if it has an odd number of inversions. For a permutation σ in S_n, the **sign** of σ is defined as

$$\operatorname{sgn}(\sigma) = \left\{ \begin{array}{rl} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{array} \right\} = (-1)^k,$$

where k is the number of inversions of σ.

For example, when $n = 3$, the permutations $\langle 1, 2, 3 \rangle$, $\langle 2, 3, 1 \rangle$ and $\langle 3, 1, 2 \rangle$ are even, while the permutations $\langle 1, 3, 2 \rangle$, $\langle 2, 1, 3 \rangle$ and $\langle 3, 2, 1 \rangle$ are odd.

In general, one can convert a permutation $\sigma = \langle \sigma(1), \sigma(2), \ldots, \sigma(n) \rangle$ in S_n into the identity permutation $\langle 1, 2, \ldots, n \rangle$ by transposing each inversion of σ. However, the number of necessary transpositions to convert the given permutation into the identity permutation need not be unique as shown in Example 2.3. An interesting fact is that, *even though the number of necessary transpositions is not unique, the parity (even or odd) is always the same as that of the number of inversions.* (This may not be clear and the readers are suggested to convince themselves with a couple of examples.)

We now go back to Step 3 to compute the determinants of those remaining $n!$ matrices. Each of them has n entries of the form $a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)}$

for a permutation $\sigma \in S_n$. Moreover, this can be converted into a diagonal matrix by column interchanges corresponding to the inversions in the permutation $\sigma = \langle\sigma(1), \sigma(2), \ldots, \sigma(n)\rangle$. Hence, its determinant is equal to

$$\operatorname{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}.$$

This is called a **signed elementary product** of A.

Our discussions can be summarized as follows to get an explicit formula for $\det A$:

Theorem 2.6 *For an $n \times n$ matrix A,*

$$\boxed{\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}.}$$

That is, $\det A$ *is the sum of all signed elementary products of A.*

This shows that the determinant must be unique if it exists. On the other hand, one can show that the explicit formula for $\det A$ in Theorem 2.6 satisfies the three rules $(\mathbf{R}_1)$–$(\mathbf{R}_3)$. Therefore, we have both *existence* and *uniqueness* for the determinant function of square matrices of any order $n \geq 1$, which proves Theorem 2.4.

As the last part of this section, we add an example to demonstrate that any permutation σ can be converted into the identity permutation by the same number of transpositions as the number of inversions in σ.

Example 2.4 (*Convert into the identity permutation by transpositions*) Consider a permutation $\sigma = \langle 3, 1, 5, 4, 2\rangle$ in S_5. It has five inversions, and it can be converted to the identity permutation by composing five transpositions successively:

$$\begin{aligned} \sigma &= \langle 3,1,5,4,2\rangle \to \langle \mathbf{1},\mathbf{3},5,4,2\rangle \to \langle 1,3,5,\mathbf{2},\mathbf{4}\rangle \to \langle 1,3,\mathbf{2},\mathbf{5},4\rangle \\ &\to \langle 1,\mathbf{2},\mathbf{3},5,4\rangle \to \langle 1,2,3,\mathbf{4},\mathbf{5}\rangle. \end{aligned}$$

It was done by moving the number 1 to the first position, and then 2 to the second position by transpositions, and so on. In fact, the bold faced numbers are interchanged in each one of five steps, and the five transpositions used to convert σ into the identity permutation are shown below:

$$\sigma\langle \mathbf{2},\mathbf{1},3,4,5\rangle\langle 1,2,3,\mathbf{5},\mathbf{4}\rangle\langle 1,2,\mathbf{4},\mathbf{3},5\rangle\langle 1,\mathbf{3},\mathbf{2},4,5\rangle\langle 1,2,3,\mathbf{5},\mathbf{4}\rangle = \langle 1,2,3,4,5\rangle.$$

Here, two permutations are composed from right to left for notational convention: i.e., if we denote $\tau = \langle \mathbf{2},\mathbf{1},3,4,5\rangle$, then $\sigma\tau = \sigma \circ \tau$. For example, $\sigma\tau(2) = \sigma(1) = 3$.

Also, note that σ can be converted to the identity permutation by composing the following three transpositions successively:

$$\sigma\langle \mathbf{2},\mathbf{1},3,4,5\rangle\langle 1,\mathbf{5},3,4,\mathbf{2}\rangle\langle 1,2,\mathbf{5},4,\mathbf{3}\rangle = \langle 1,2,3,4,5\rangle. \qquad \square$$

It is not hard to see that the number of even permutations is equal to that of odd permutations, so it is $\frac{n!}{2}$. In the case $n = 3$, one can notice that there are three terms with $+$ sign and three terms with $-$ sign in $\det A$.

Problem 2.7 Show that the number of even permutations and the number of odd permutations in S_n are equal.

Problem 2.8 Let $A = [\mathbf{c}_1 \cdots \mathbf{c}_n]$ be an $n \times n$ matrix with the column vectors $\mathbf{c}_j$'s. Show that $\det[\mathbf{c}_j\, \mathbf{c}_1 \cdots \mathbf{c}_{j-1}\, \mathbf{c}_{j+1} \cdots \mathbf{c}_n] = (-1)^{j-1} \det[\mathbf{c}_1 \cdots \mathbf{c}_j \cdots \mathbf{c}_n]$. Note that the same kind of equality holds when A is written in row vectors.

2.3 Cofactor expansion

Even if one has found an explicit formula for the determinant as shown in Theorem 2.6, it is not much help in computing because one has to sum up $n!$ terms, which becomes a very large number as n gets large. Thus, we reformulate the formula by rewriting it in an inductive way, by which the summing time can be reduced.

The first factor $a_{1\sigma(1)}$ in each of the $n!$ terms is one of $a_{11}, a_{12}, \ldots, a_{1n}$ in the first row of A. Hence, one can divide the $n!$ terms of the expansion of $\det A$ into n groups according to the value of $\sigma(1)$: Say,

$$\begin{aligned}\det A &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \\ &= a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n},\end{aligned}$$

where, for $j = 1, 2, \ldots, n$, A_{1j} is defined as

$$A_{1j} = \sum_{\sigma \in S_n, \sigma(1) = j} \operatorname{sgn}(\sigma) a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

This number A_{1j} will turn out to be the determinant, up to a $\pm$ sign, of the submatrix of A obtained by deleting the 1-st row and j-th column. This submatrix will be denoted by M_{1j} and called the **minor** of the entry a_{1j}.

Remark: If we replace the entries $a_{11}, a_{12}, \ldots, a_{1n}$ in the first row of A as unknown variables $x_1, x_2, \ldots, x_n$, then $\det A$ is a polynomial of the variables x_i, and the number A_{1j} becomes the coefficient of the variable x_j in this polynomial.

We now aim to compute A_{1j} for $j = 1, 2, \ldots, n$. Clearly, when $j = 1$,

$$A_{11} = \sum_{\sigma \in S_n, \sigma(1) = 1} \operatorname{sgn}(\sigma) a_{2\sigma(2)} \cdots a_{n\sigma(n)} = \sum_{\tau} \operatorname{sgn}(\tau) a_{2\tau(2)} \cdots a_{n\tau(n)},$$

summing over all permutations τ of the numbers $2, 3, \ldots, n$. Note that each term in A_{11} contains no entries from the first row or from the first column of A. Hence, all the $(n-1)!$ terms in the sum of A_{11} are just the signed elementary products of the submatrix M_{11} of A obtained by deleting the first row and the first column of A, so that

$$A_{11} = \det M_{11}.$$

To compute the number A_{1j} for $j > 1$, let $A = [\mathbf{c}_1 \ \cdots \ \mathbf{c}_n]$ with the column vectors $\mathbf{c}_j$'s and let $B = [\mathbf{c}_j \, \mathbf{c}_1 \ \cdots \ \mathbf{c}_{j-1} \, \mathbf{c}_{j+1} \ \cdots \ \mathbf{c}_n]$ be the matrix obtained from A by interchanging the j-th column with its preceding $j-1$ columns one by one up to the first. Then, $\det A = (-1)^{j-1} \det B$ (see Problem 2.8). Write

$$\det B = b_{11}B_{11} + b_{12}B_{12} + \cdots + b_{1n}B_{1n}$$

as the expansion of $\det B$. Then, $a_{1j} = b_{11}$ and the number B_{11} is the coefficient of the entry b_{11} in the formula of $\det B$. By noting A_{1j} is the coefficient of the entry a_{1j} in the formula of $\det A$, one can have $A_{1j} = (-1)^{j-1}B_{11}$. Moreover, the minor M_{1j} of the entry a_{1j} is the same as the minor N_{11} of the entry b_{11}. Now, by applying the previous conclusion $A_{11} = \det M_{11}$ to the matrix B, one can obtain $B_{11} = \det N_{11}$ and then

$$A_{1j} = (-1)^{j-1}B_{11} = (-1)^{j-1} \det N_{11} = (-1)^{j-1} \det M_{1j}.$$

In summary, one can get an expansion of $\det A$ with respect to the first row:

$$\begin{aligned} \det A &= a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} \\ &= a_{11} \det M_{11} - a_{12} \det M_{12} + \cdots + (-1)^{1+n} a_{1n} \det M_{1n}. \end{aligned}$$

This is called the **cofactor expansion** of $\det A$ along the first row.

There is a similar expansion with respect to any other row, say the i-th row. To show this, first construct a new matrix C from A by moving the i-th row of A up to the first row by interchanges with its preceding $i-1$ rows one by one. Then $\det A = (-1)^{i-1} \det C$ as before. Now, the expansion of $\det C$ with respect to the first row $[a_{i1} \ \cdots \ a_{in}]$ is

$$\det C = a_{i1}C_{11} + a_{i2}C_{12} + \cdots + a_{in}C_{1n},$$

where $C_{1j} = (-1)^{j-1} \det \bar{M}_{1j}$ and $\bar{M}_{1j}$ denotes the minor of c_{1j} in the matrix C. Noting that $\bar{M}_{1j} = M_{ij}$ as minors, we have

$$A_{ij} = (-1)^{i-1}C_{1j} = (-1)^{i+j} \det M_{ij}$$

as before and then

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

The submatrix M_{ij} is called the **minor** of the entry a_{ij} and the number $A_{ij} = (-1)^{i+j} \det M_{ij}$ is called the **cofactor** of the entry a_{ij}.

Also, one can do the same with the column vectors because $\det A^T = \det A$. This gives the following theorem:

Theorem 2.7 *Let A be an $n \times n$ matrix and let A_{ij} be the cofactor of the entry a_{ij}. Then,*

(1) *for each $1 \leq i \leq n$,*

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in},$$

called the **cofactor expansion** *of* $\det A$ *along the i-th row.*

(2) *For each $1 \leq j \leq n$,*

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj},$$

called the **cofactor expansion** *of* $\det A$ *along the j-th column.*

This cofactor expansion gives an alternative way of defining the determinant inductively.

Remark: The sign $(-1)^{i+j}$ of the cofactor can be explained as follows:

$$\begin{bmatrix} + & - & + & \cdots & (-1)^{1+n} \\ - & + & - & \cdots & (-1)^{2+n} \\ + & - & + & \cdots & (-1)^{3+n} \\ \vdots & & & \ddots & \vdots \\ (-1)^{n+1} & (-1)^{n+2} & (-1)^{n+3} & \cdots & (-1)^{n+n} \end{bmatrix}.$$

Therefore, the determinant of an $n \times n$ matrix A is the sum of the products of the entries in any given row (or, any given column) with their cofactors.

Example 2.5 (*Computing* $\det A$ *by a cofactor expansion*) Let

$$A = [a_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Then the cofactors of a_{11}, a_{12} and a_{13} are

$$\begin{aligned} A_{11} &= (-1)^{1+1}\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} &&= 5\cdot 9 - 8\cdot 6 = -3, \\ A_{12} &= (-1)^{1+2}\det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} &&= (-1)(4\cdot 9 - 7\cdot 6) = 6, \\ A_{13} &= (-1)^{1+3}\det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} &&= 4\cdot 8 - 7\cdot 5 = -3, \end{aligned}$$

respectively. Hence the expansion of $\det A$ along the first row is

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1\cdot(-3) + 2\cdot 6 + 3\cdot(-3) = 0. \quad \square$$

The cofactor expansion formula of $\det A$ suggests that the evaluation of A_{ij} can be avoided whenever $a_{ij} = 0$, because the product $a_{ij}A_{ij}$ is zero regardless of the value of A_{ij}. Therefore, the computation of the determinant will be simplified by making the cofactor expansion along a row or a column that contains as many zero entries as possible. Moreover, by using the elementary row (or column) operations which do not alter the determinant, a matrix A may be simplified into another one having more zero entries in a row or in a column whose computation of the determinant may be simpler. For example, a forward elimination to a square matrix A will produce an upper triangular matrix U, and so the determinant of A will be just the product of the diagonal entries of U up to the sign caused by possible row interchanges. The next examples illustrate this method for an evaluation of the determinant.

Example 2.6 (*Computing* $\det A$ *by a forward elimination and a cofactor expansion*) Evaluate the determinant of

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{bmatrix}.$$

Solution: Apply the elementary operations:

$$\begin{aligned} 3 \times \text{row } 1 &+ \text{row } 2, \\ (-2) \times \text{row } 1 &+ \text{row } 3, \\ 2 \times \text{row } 1 &+ \text{row } 4 \end{aligned}$$

to A; then

$$\det A = \det \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 7 & -4 \\ 0 & -3 & -7 & 10 \\ 0 & 4 & 0 & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & 7 & -4 \\ -3 & -7 & 10 \\ 4 & 0 & -1 \end{bmatrix}.$$

Now apply the operation: $1 \times$ row $1 +$ row 2, to the matrix on the right-hand side, and take the cofactor expansion along the second column to get

$$\begin{aligned} \det \begin{bmatrix} 1 & 7 & -4 \\ -3 & -7 & 10 \\ 4 & 0 & -1 \end{bmatrix} &= \det \begin{bmatrix} 1 & 7 & -4 \\ -2 & 0 & 6 \\ 4 & 0 & -1 \end{bmatrix} \\ &= (-1)^{1+2} \cdot 7 \cdot \det \begin{bmatrix} -2 & 6 \\ 4 & -1 \end{bmatrix} \\ &= -7(2 - 24) = 154. \end{aligned}$$

Thus, $\det A = 154$. □

Example 2.7 Show that $\det A = (x-y)(x-z)(x-w)(y-z)(y-w)(z-w)$ for the **Vandermonde matrix** of order 4:

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{bmatrix}.$$

Solution: Use Gaussian elimination. To begin with, add $(-1)\times$ row 1 to rows 2, 3, and 4 of A:

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & y^2-x^2 & y^3-x^3 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & w-x & w^2-x^2 & w^3-x^3 \end{bmatrix} \\
&= \det \begin{bmatrix} y-x & y^2-x^2 & y^3-x^3 \\ z-x & z^2-x^2 & z^3-x^3 \\ w-x & w^2-x^2 & w^3-x^3 \end{bmatrix} \\
&= (y-x)(z-x)(w-x) \det \begin{bmatrix} 1 & y+x & y^2+xy+x^2 \\ 1 & z+x & z^2+xz+x^2 \\ 1 & w+x & w^2+xw+x^2 \end{bmatrix} \\
&= (x-y)(x-z)(w-x) \det \begin{bmatrix} 1 & y+x & y^2+xy+x^2 \\ 0 & z-y & (z-y)(z+y+x) \\ 0 & w-y & (w-y)(w+y+x) \end{bmatrix} \\
&= (x-y)(x-z)(w-x) \det \begin{bmatrix} z-y & (z-y)(z+y+x) \\ w-y & (w-y)(w+y+x) \end{bmatrix} \\
&= (x-y)(x-z)(x-w)(y-z)(w-y) \det \begin{bmatrix} 1 & z+y+x \\ 1 & w+y+x \end{bmatrix} \\
&= (x-y)(x-z)(x-w)(y-z)(y-w)(z-w).
\end{aligned}$$

□

Problem 2.9 Use cofactor expansions along a row or a column to evaluate the determinants of the following matrices:

$$(1)\ A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 0 \end{bmatrix}, \quad (2)\ A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 5 & 0 \end{bmatrix}.$$

Problem 2.10 Evaluate the determinant of

$$(1)\ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & y & z & w \\ x^2 & y^2 & z^2 & w^2 \\ x^4 & y^4 & z^4 & w^4 \end{bmatrix}, \qquad (2)\ B = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & c \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

2.4 Cramer's rule

In Chapter 1, we have studied two ways for solving a system of linear equations $A\mathbf{x} = \mathbf{b}$: (i) by Gauss–Jordan elimination (or by LDU factorization) or (ii) by using A^{-1} if A is invertible. In this section, we introduce another method for solving the system $A\mathbf{x} = \mathbf{b}$ for an invertible matrix A.

The cofactor expansion of the determinant gives a method for computing the inverse of an invertible matrix A. For $i \neq j$, let A^* be the matrix A with the j-th row replaced by the i-th row. Then the determinant of A^* must be zero, because the i-th and j-th rows are the same. Moreover, with respect to the j-th row , the cofactors of A^* are the same as those of A: that is, $A^*_{jk} = A_{jk}$ for all $k = 1, \ldots, n$. Therefore, we have

$$\begin{aligned} 0 = \det A^* &= a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn} \\ &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}. \end{aligned}$$

This proves the following lemma.

Lemma 2.8

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det A & \textit{if } i = j \\ 0 & \textit{if } i \neq j. \end{cases}$$

Definition 2.6 Let A be an $n \times n$ matrix and let A_{ij} denote the cofactor of a_{ij}. Then the new matrix

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

is called the **matrix of cofactors** of A. Its transpose is called the **adjugate** of A and is denoted by adj A.

It follows from Lemma 2.8 that

$$A \cdot \operatorname{adj} A = \begin{bmatrix} \det A & 0 & \cdots & 0 \\ 0 & \det A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det A \end{bmatrix} = (\det A)I.$$

If A is invertible, then $\det A \neq 0$ and we may write $A\left(\frac{1}{\det A} \operatorname{adj} A\right) = I$. Thus

$$\boxed{A^{-1} = \frac{1}{\det A}\,\mathrm{adj}\,A, \quad \text{and} \quad A = (\det A)\,\mathrm{adj}\,(A^{-1})}$$

by replacing A with A^{-1}.

Example 2.8 (*Computing* A^{-1} *with* adj A) For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, adj $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, and if $\det A = ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Problem 2.11 Compute adj A and A^{-1} for $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.

Problem 2.12 Show that A is invertible if and only if adj A is invertible, and that if A is invertible, then

$$(\mathrm{adj}\ A)^{-1} = \frac{A}{\det A} = \mathrm{adj}(A^{-1}).$$

Problem 2.13 Let A be an $n \times n$ invertible matrix with $n > 1$. Show that

(1) $\det(\mathrm{adj}\ A) = (\det A)^{n-1}$;
(2) $\mathrm{adj}(\mathrm{adj}\ A) = (\det A)^{n-2}A$.

Problem 2.14 For invertible matrices A and B, show that

(1) adj AB = adj $B \cdot$ adj A;
(2) adj $QAQ^{-1} = Q(\mathrm{adj}\ A)Q^{-1}$ for any invertible matrix Q;
(3) if $AB = BA$, then $(\mathrm{adj}\ A)B = B(\mathrm{adj}\ A)$.

In fact, these three properties are satisfied for any (invertible or not) two square matrices A and B. (See Exercise **6.5**.)

The next theorem establishes a formula for the solution of a system of n equations in n unknowns. It may not be useful as a practical method but can be used to study properties of the solution without solving the system.

Theorem 2.9 (Cramer's rule) *Let* $A\mathbf{x} = \mathbf{b}$ *be a system of* n *linear equations in* n *unknowns such that* $\det A \neq 0$. *Then the system has the unique solution given by*

$$x_j = \frac{\det C_j}{\det A}, \qquad j = 1, 2, \ldots, n,$$

where C_j *is the matrix obtained from* A *by replacing the* j*-th column with the column matrix* $\mathbf{b} = [b_1\ b_2\ \cdots\ b_n]^T$.

Proof: If $\det A \neq 0$, then A is invertible and $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$. Since

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det A}(\operatorname{adj} A)\mathbf{b},$$

it follows that

$$x_j = \frac{b_1 A_{1j} + b_2 A_{2j} + \cdots + b_n A_{nj}}{\det A} = \frac{\det C_j}{\det A}. \qquad \square$$

Example 2.9 Use Cramer's rule to solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 50 \\ 2x_1 + 2x_2 + x_3 = 60 \\ x_1 + 2x_2 + 3x_3 = 90. \end{cases}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 50 & 2 & 1 \\ 60 & 2 & 1 \\ 90 & 2 & 3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 50 & 1 \\ 2 & 60 & 1 \\ 1 & 90 & 3 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 2 & 50 \\ 2 & 2 & 60 \\ 1 & 2 & 90 \end{bmatrix}.$$

Therefore,

$$x_1 = \frac{\det C_1}{\det A} = 10, \quad x_2 = \frac{\det C_2}{\det A} = 10, \quad x_3 = \frac{\det C_3}{\det A} = 20. \qquad \square$$

Cramer's rule provides a convenient method for writing down the solution of a system of n linear equations in n unknowns in terms of determinants. To find the solution, however, one must evaluate $n + 1$ determinants of order n. Evaluating even two of these determinants generally involves more computations than solving the system by using Gauss–Jordan elimination.

Problem 2.15 Use Cramer's rule to solve the following systems.

$$(1) \quad \begin{cases} \quad\quad\ \ 4x_2 + 3x_3 = -2 \\ 3x_1 + 4x_2 + 5x_3 = 6 \\ -2x_1 + 5x_2 - 2x_3 = 1. \end{cases}$$

$$(2) \quad \begin{cases} \dfrac{2}{x} - \dfrac{3}{y} + \dfrac{5}{z} = 3 \\ -\dfrac{4}{x} + \dfrac{7}{y} + \dfrac{2}{z} = 0 \\ \quad\quad \dfrac{2}{y} - \dfrac{1}{z} = 2. \end{cases}$$

Problem 2.16 Let A be the matrix obtained from the identity matrix I_n with i-th column replaced by the column vector $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$. Compute $\det A$.

2.5 Applications

2.5.1 Miscellaneous examples for determinants

Example 2.10 Let A be the **Vandermonde matrix** of order n:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

Its determinant can be computed by the same method as in Example 2.7 as follows:

$$\det A = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Example 2.11 Let A_{ij} denote the cofactor of a_{ij} in a matrix $A = [a_{ij}]$. If $n > 1$, then

$$\det \begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_n \\ x_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ x_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_n & a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = -\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_i x_j.$$

Solution: First take the cofactor expansion along the first row, and then compute the cofactor expansion along the first column of each $n \times n$ submatrix. □

Example 2.12 Let $f_1, f_2, \ldots, f_n$ be n real-valued differentiable functions on $\mathbb{R}$. For the matrix

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix},$$

its determinant is called the **Wronskian** for $\{f_1(x), f_2(x), \ldots, f_n(x)\}$. For example,

$$\det \begin{bmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{bmatrix} = -x, \quad \text{but} \quad \det \begin{bmatrix} x & \sin x & x + \sin x \\ 1 & \cos x & 1 + \cos x \\ 0 & -\sin x & -\sin x \end{bmatrix} = 0.$$

In general, in $f_1, f_2, \ldots, f_n$, if one of them is a constant multiple of another or a sum of such multiples, then the Wronskian must be zero.

Example 2.13 An $n \times n$ matrix A is called a **circulant matrix** if the i-th row of A is obtained from the first row of A by a cyclic shift of the $i-1$ steps, i.e., the general form of the circulant matrix is

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix}.$$

If $n = 2$, $\det A = a_1^2 - a_2^2 = (a_1 + a_2)(a_1 - a_2)$. If $n = 3$,

$$\det A = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{bmatrix} = (a_1 + a_2 + a_3)(a_1 + a_2\omega + a_3\omega^2)(a_1 + a_2\omega^2 + a_3\omega),$$

where $\omega = e^{2\pi i/3}$ is the primitive root of unity. In general, for $n > 1$,

$$\det A = \prod_{j=0}^{n-1} \left(a_1 + a_2\omega_j + a_3\omega_j^2 + \cdots + a_n\omega_j^{n-1} \right),$$

where $\omega_j = e^{2\pi i j/n}$, $j = 0, 1, \ldots, n-1$, are the roots of unity. (See Example **8.18** for the proof.)

Example 2.14 A **tridiagonal** matrix is a square matrix of the form

$$T_n = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & a_3 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-2} & b_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & c_{n-1} & a_n \end{bmatrix}.$$

The determinant of this matrix can be computed by a recurrence relation: Set $D_0 = 1$ and $D_k = \det T_k$ for $k \geq 1$. By expanding T_k with respect to the k-th row, one can have a recurrence relation

$$D_k = a_k D_{k-1} - b_{k-1}c_{k-1}D_{k-2} \quad \text{for} \quad k \geq 2.$$

The following two special cases are interesting.

Case (1) Let all a_i, b_j, c_k be the same, say $a_i = b_j = c_k = b > 0$. Then, $D_1 = b$, $D_2 = 0$ and

$$D_n = bD_{n-1} - b^2 D_{n-2} \quad \text{for} \quad n \geq 3.$$

Successively, one can find $D_3 = -b^3$, $D_4 = -b^4$, $D_5 = 0$, $\ldots$. In general, the n-th term D_n of the recurrence relation is given by

$$D_n = b^n \left[\cos \frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3} \right].$$

Later in Section 6.3.1, it will be discussed how to find the n-th term of a given recurrence relation. (See Exercise **8.6**.)

Case (2) Let all $b_j = 1$ and all $c_k = -1$, and let us write

$$(a_1 \ldots a_n) = \det \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a_3 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-2} & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & a_n \end{bmatrix}.$$

Then,

$$\frac{(a_1 a_2 \ldots a_n)}{(a_2 a_3 \ldots a_n)} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}.$$

Proof: Let us prove it by induction on n. Clearly, $a_1 + \frac{1}{a_2} = \frac{(a_1 a_2)}{(a_2)}$. It remains to show that

$$a_1 + \cfrac{1}{\cfrac{(a_2 a_3 \ldots a_n)}{(a_3 a_4 \ldots a_n)}} = \frac{(a_1 a_2 \ldots a_n)}{(a_2 a_3 \ldots a_n)},$$

i.e., $a_1(a_2 \ldots a_n) + (a_3 \ldots a_n) = (a_1 a_2 \ldots a_n)$. But this identity follows from the previous recurrence relation, since $(a_1 a_2 \ldots a_n) = (a_n \ldots a_2 a_1)$. $\square$

Example 2.15 (Binet–Cauchy formula) Let A and B be matrices of size $n \times m$ and $m \times n$, respectively, and $n \leq m$. Then

$$\det(AB) = \sum_{1 \leq k_1 < k_2 < \cdots < k_n \leq m} A_{k_1 \ldots k_n} B^{k_1 \ldots k_n},$$

where $A_{k_1 \ldots k_n}$ is the minor obtained from the columns of A whose numbers are $k_1, \ldots, k_n$ and $B^{k_1 \ldots k_n}$ is the minor obtained from the rows of B whose numbers are $k_1, \ldots, k_n$. In other words, $\det(AB)$ is the sum of products of the corresponding majors of A and B, where a major of a matrix is, by definition, a determinant of maximal order minor in the matrix.

Proof: Let $C = AB$, $c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$. Then

$$\begin{aligned}
\det C &= \sum_{\sigma\in S_n}(-1)^{\sigma}\left(\sum_{k_1} a_{1k_1}b_{k_1\sigma(1)} \cdots \sum_{k_n} a_{nk_n}b_{k_n\sigma(n)}\right)\\
&= \sum_{k_1,\ldots,k_n=1}^{m} a_{1k_1}\cdots a_{nk_n}\sum_{\sigma\in S_n}(-1)^{\sigma} b_{k_1\sigma(1)}\cdots b_{k_n\sigma(n)}\\
&= \sum_{k_1,\ldots,k_n=1}^{m} a_{1k_1}\cdots a_{nk_n}B^{k_1\ldots k_n}.
\end{aligned}$$

The minor $B^{k_1\ldots k_n}$ is nonzero only if the numbers $k_1, \ldots, k_n$ are distinct. Thus, the summation can be performed over distinct numbers $k_1, \ldots, k_n$. Since $B^{\tau(k_1)\ldots\tau(k_n)} = (-1)^{\tau}B^{k_1\ldots k_n}$ for any permutation τ of the numbers $k_1, \ldots, k_n$, we have

$$\begin{aligned}
\textstyle\sum_{k_1,\ldots,k_n=1}^{m} a_{1k_1}\cdots a_{nk_n}B^{k_1\ldots k_n} &= \textstyle\sum_{k_1<k_2<\cdots<k_n}\sum_{\tau}(-1)^{\tau}a_{1\tau(1)}\cdots a_{n\tau(n)}B^{k_1\ldots k_n}\\
&= \textstyle\sum_{1\le k_1<k_2<\cdots<k_n\le m} A_{k_1\ldots k_n}B^{k_1\ldots k_n}.
\end{aligned}$$

□

For example, if $A = \begin{bmatrix} 1 & -1 & 3 & 3\\ 2 & 2 & -1 & 2\end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2\\ 2 & -1\\ -3 & 1\\ 1 & 2\end{bmatrix}$, then

$$\begin{aligned}
\det(AB) &= \det\begin{bmatrix} 1 & -1\\ 2 & 2\end{bmatrix}\det\begin{bmatrix} 1 & 2\\ 2 & -1\end{bmatrix} + \det\begin{bmatrix} 1 & 3\\ 2 & -1\end{bmatrix}\det\begin{bmatrix} 1 & 2\\ -3 & 1\end{bmatrix}\\
&+ \det\begin{bmatrix} 1 & 3\\ 2 & 2\end{bmatrix}\det\begin{bmatrix} 1 & 2\\ 1 & 2\end{bmatrix} + \det\begin{bmatrix} -1 & 3\\ 2 & -1\end{bmatrix}\det\begin{bmatrix} 2 & -1\\ -3 & 1\end{bmatrix}\\
&+ \det\begin{bmatrix} -1 & 3\\ 2 & 2\end{bmatrix}\det\begin{bmatrix} 2 & -1\\ 1 & 2\end{bmatrix} + \det\begin{bmatrix} 3 & 3\\ -1 & 2\end{bmatrix}\det\begin{bmatrix} -3 & 1\\ 1 & 2\end{bmatrix}\\
&= -167.
\end{aligned}$$

2.5.2 Area and volume

In this section, we demonstrate a geometrical interpretation of the determinant of a square matrix A as the volume (or area for $n = 2$) of the parallelepiped $\mathcal{P}(A)$ spanned by the row vectors of A. For this, we restrict our attention to the case of $n = 2$ or 3 for a visualization, even if a similar argument can be applied for $n > 3$.

For an $n \times n$ square matrix A, its row vectors $\mathbf{r}_i = [a_{i1} \ \cdots \ a_{in}]$ can be considered as elements in

$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \ : \ a_i \in \mathbb{R}, i = 1, 2, \ldots, n\}.$$

The set

$$\mathcal{P}(A) = \left\{ \sum_{i=1}^{n} t_i \mathbf{r}_i \ : \ 0 \le t_i \le 1, \ i = 1, 2, \ldots, n \right\}$$

is called a **parallelogram** if $n = 2$, or a **parallelepiped** if $n \ge 3$. Note that the row vectors of A form the edges of $\mathcal{P}(A)$, and changing the order of row vectors does not alter the shape of $\mathcal{P}(A)$.

Theorem 2.10 *The determinant* $\det A$ *of an* $n \times n$ *matrix* A *is the volume of* $\mathcal{P}(A)$ *up to sign. In fact, the volume of* $\mathcal{P}(A)$ *is equal to* $|\det A|$.

Proof: We give the proof for the case $n = 2$ only, and leave the case $n = 3$ to the readers. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix},$$

where $\mathbf{r}_1$, $\mathbf{r}_2$ are the row vectors of A. Let $Area(A)$ denote the area of the parallelogram $\mathcal{P}(A)$ (see Figure 2.3). Note that

$$Area \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = Area \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \end{bmatrix},$$

but

$$\det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = -\det \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \end{bmatrix}.$$

Thus, one can expect in general that

$$\det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = \pm Area \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix},$$

which explains why we say 'up to sign'. To determine the sign $\pm$, we first define the **orientation** of $A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$ to be

$$\rho(A) = \begin{cases} \frac{\det A}{|\det A|} = \pm 1 & \text{if } \det A \ne 0, \\ \quad 1 & \text{if } \det A = 0. \end{cases}$$

In general, $\rho(A) = 1$ if and only if $\det A > 0$: In this case, we say the ordered pair $(\mathbf{r}_1 \ \mathbf{r}_2)$ is **positively oriented**, while $\rho(A) = -1$ if and only if $\det A < 0$: In this case, α is **negatively oriented.** See Figure 2.2 (next page).

For example, $\rho\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$, while $\rho\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$.

To finish the proof, it is sufficient to show that the function $D(A) = \rho(A) Area(A)$ satisfies the rules $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ of the determinant, so that $\det = D = \pm Area$, or $Area(A) = |\det A|$. However,

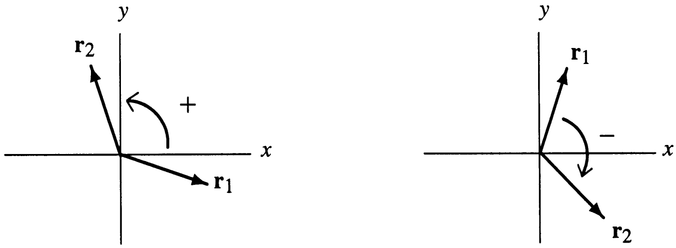

Figure 2.2. Orientation of vectors

(1) it is clear that $D\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$.

(2) $D\begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \end{bmatrix} = -D\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$, because $\rho\left(\begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \end{bmatrix}\right) = -\rho\left(\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\right)$ and $Area\begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \end{bmatrix} = Area\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$.

(3) $D\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = kD\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$ for any k. Indeed, if $k = 0$, it is clear. Suppose $k \neq 0$. Then, as illustrated in Figure 2.3, the bottom edge $\mathbf{r}_1$ of $\mathcal{P}(A)$ is elongated by $|k|$ while the height h remains unchanged. Thus

$$Area\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = |k|\; Area\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}.$$

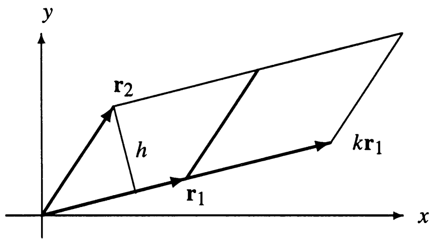

Figure 2.3. The parallelogram $\mathcal{P}\left(\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\right)$

On the other hand,

$$\rho\left(\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\right) = \frac{\det\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}}{\left|\det\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\right|} = \frac{k\det\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}}{|k|\left|\det\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\right|} = \frac{k}{|k|}\,\rho\left(\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\right).$$

Therefore, we have

$$\begin{aligned} D\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} &= \rho\left(\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\right) Area\begin{bmatrix} k\mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \\ &= \frac{k}{|k|}\rho\left(\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\right)|k| Area\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = kD\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}. \end{aligned}$$

(4) $D\begin{bmatrix} \mathbf{r}_1 + \mathbf{r}_2 \\ \mathbf{u} \end{bmatrix} = D\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{u} \end{bmatrix} + D\begin{bmatrix} \mathbf{r}_2 \\ \mathbf{u} \end{bmatrix}$ for any $\mathbf{u}$, $\mathbf{r}_1$ and $\mathbf{r}_2$ in $\mathbb{R}^2$. If $\mathbf{u} = \mathbf{0}$, there is nothing to prove. Assume that $\mathbf{u} \neq \mathbf{0}$. Choose any vector $\mathbf{v} \in \mathbb{R}^2$ such that $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$ and the pair $(\mathbf{u}, \mathbf{v})$ is positively oriented. Then $\mathbf{r}_i = a_i\mathbf{u} + b_i\mathbf{v}$, $i = 1, 2$, and

$$\begin{aligned} D\begin{bmatrix} \mathbf{r}_1 + \mathbf{r}_2 \\ \mathbf{u} \end{bmatrix} &= D\begin{bmatrix} (a_1 + a_2)\mathbf{u} + (b_1 + b_2)\mathbf{v} \\ \mathbf{u} \end{bmatrix} \\ &= D\begin{bmatrix} (b_1 + b_2)\mathbf{v} \\ \mathbf{u} \end{bmatrix} = (b_1 + b_2)D\begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} \\ &= D\begin{bmatrix} a_1\mathbf{u} + b_1\mathbf{v} \\ \mathbf{u} \end{bmatrix} + D\begin{bmatrix} a_2\mathbf{u} + b_2\mathbf{v} \\ \mathbf{u} \end{bmatrix} \\ &= D\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{u} \end{bmatrix} + D\begin{bmatrix} \mathbf{r}_2 \\ \mathbf{u} \end{bmatrix}. \end{aligned}$$

The second equality follows from (2) and Figure 2.4.

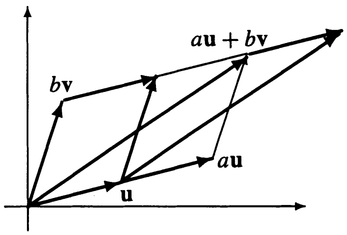

Figure 2.4. $D\begin{bmatrix} a\mathbf{u} + b\mathbf{v} \\ \mathbf{u} \end{bmatrix} = D\begin{bmatrix} b\mathbf{v} \\ \mathbf{u} \end{bmatrix}$.

□

Remark: **(1)** Note that if we have constructed the parallelepiped $\mathcal{P}(A)$ using the column vectors of A, then the shape of the parallelepiped is totally different from the one constructed using the row vectors. However, $\det A = \det A^T$ means their volumes are the same, which is a totally nontrivial fact.

(2) For $n \geq 3$, the volume of $\mathcal{P}(A)$ can be defined by induction on n, and exactly the same argument in the proof can be applied to show that the volume is the determinant. However, there is another way of looking at this fact. Let $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be n column vectors of an $m \times n$ matrix A. They constitute an n-dimensional parallelepiped in $\mathbb{R}^m$ such that

$$\mathcal{P}(A) = \left\{ \sum_{i=1}^{n} t_i \mathbf{c}_i \ : \ 0 \le t_i \le 1, \ i = 1, 2, \ldots, n \right\}.$$

A formula for the volume of this parallelepiped may be expressed as follows: We first consider a two-dimensional parallelepiped (a parallelogram) determined by two column vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ of $A = [\mathbf{c}_1 \ \mathbf{c}_2]$ in $\mathbb{R}^3$. The area of this parallelogram is

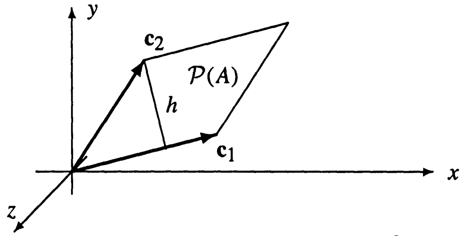

Figure 2.5. A parallelogram in $\mathbb{R}^3$

simply $Area(\mathcal{P}(A)) = \|\mathbf{c}_1\| h$, where $h = \|\mathbf{c}_2\| \sin\theta$ and θ is the angle between $\mathbf{c}_1$ and $\mathbf{c}_2$. Therefore, we have

$$\begin{aligned} Area(\mathcal{P}(A))^2 &= \|\mathbf{c}_1\|^2 \|\mathbf{c}_2\|^2 \sin^2\theta \ = \ \|\mathbf{c}_1\|^2 \|\mathbf{c}_2\|^2 (1 - \cos^2\theta) \\ &= (\mathbf{c}_1 \cdot \mathbf{c}_1)(\mathbf{c}_2 \cdot \mathbf{c}_2) \left(1 - \frac{(\mathbf{c}_1 \cdot \mathbf{c}_2)^2}{(\mathbf{c}_1 \cdot \mathbf{c}_1)(\mathbf{c}_2 \cdot \mathbf{c}_2)} \right) \\ &= (\mathbf{c}_1 \cdot \mathbf{c}_1)(\mathbf{c}_2 \cdot \mathbf{c}_2) - (\mathbf{c}_1 \cdot \mathbf{c}_2)^2 \\ &= \det \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{c}_1 & \mathbf{c}_1 \cdot \mathbf{c}_2 \\ \mathbf{c}_2 \cdot \mathbf{c}_1 & \mathbf{c}_2 \cdot \mathbf{c}_2 \end{bmatrix} \\ &= \det \left(\begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \right) \ = \ \det(A^T A), \end{aligned}$$

where "$\cdot$" denotes the dot product. In general, let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ be n column vectors of an $m \times n$ (not necessarily square) matrix A. Then one can show (for a proof see Exercise **5.17**) that the volume of the n-dimensional parallelepiped $\mathcal{P}(A)$ determined by those n column vectors $\mathbf{c}_j$'s in $\mathbb{R}^m$ is

$$\mathrm{vol}(\mathcal{P}(A)) = \sqrt{\det(A^T A)}.$$

In particular, if A is an $m \times m$ square matrix, then

$$\mathrm{vol}(\mathcal{P}(A)) = \sqrt{\det(A^T A)} = \sqrt{\det(A^T)\det(A)} = |\det A|,$$

as expected.

Problem 2.17 Show that the area of a triangle ABC in the plane $\mathbb{R}^2$, where $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$, is equal to the absolute value of

$$\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

2.6 Exercises

2.1. Determine the values of k for which $\det\begin{bmatrix} k & k \\ 4 & 2k \end{bmatrix} = 0$.

2.2. Evaluate $\det(A^2BA^{-1})$ and $\det(B^{-1}A^3)$ for the following matrices:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \\ 2 & 1 & 3 \end{bmatrix}.$$

2.3. Evaluate the determinant of

$$A = \begin{bmatrix} 3 & -2 & -5 & 4 \\ -5 & 2 & 8 & -5 \\ -3 & 4 & 7 & -3 \\ 2 & -3 & -5 & 8 \end{bmatrix}.$$

2.4. Evaluate $\det A$ for an $n \times n$ matrix $A = [a_{ij}]$ when

$$(1)\ a_{ij} = \begin{cases} 1 & i \neq j \\ 0 & i = j, \end{cases} \qquad (2)\ a_{ij} = i + j.$$

2.5. Find all solutions of the equation $\det(AB) = 0$ for

$$A = \begin{bmatrix} x+2 & 3x \\ 3 & x+2 \end{bmatrix}, \qquad B = \begin{bmatrix} x & 0 \\ 5 & x+2 \end{bmatrix}.$$

2.6. Prove that if A is an $n \times n$ skew-symmetric matrix and n is odd, then $\det A = 0$. Give an example of 4×4 skew-symmetric matrix A with $\det A \neq 0$.

2.7. Use the determinant function to find

(1) the area of the parallelogram with edges determined by $(4, 3)$ and $(7, 5)$,

(2) the volume of the parallelepiped with edges determined by the vectors $(1, 0, 4)$, $(0, -2, 2)$ and $(3, 1, -1)$.

2.8. Use Cramer's rule to solve each system.

$$(1)\ \begin{cases} x_1 + x_2 = 3 \\ x_1 - x_2 = -1. \end{cases}$$

$$(2)\ \begin{cases} x_1 + x_2 + x_3 = 2 \\ x_1 + 2x_2 + x_3 = 2 \\ x_1 + 3x_2 - x_3 = -4. \end{cases}$$

$$(3)\ \begin{cases} - x_2 + x_4 = -1 \\ x_1 + x_3 = 3 \\ x_1 - x_2 - x_3 - x_4 = 2 \\ x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

2.9. Use Cramer's rule to solve the given system:

$$(1)\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\mathbf{x}=\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad (2)\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix}\mathbf{x}=\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

2.10. Find a constant k so that the system of linear equations

$$\begin{cases} kx - 2y - z = 0 \\ (k+1)y + 4z = 0 \\ (k-1)z = 0 \end{cases}$$

has more than one solution. (Is it possible to apply Cramer's rule here?)

2.11. Solve the following system of linear equations by using Cramer's rule and by using Gaussian elimination:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}\mathbf{x}=\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

2.12. Solve the following system of equations by using Cramer's rule:

$$\begin{cases} 3x + 2y = 3z + 1 \\ 3x + 2z = 8 - 5y \\ 3z - 1 = x - 2y. \end{cases}$$

2.13. Calculate the cofactors A_{11}, A_{12}, A_{13} and A_{33} for the matrix A:

$$(1)\ A=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix},\ (2)\ A=\begin{bmatrix} 1 & 4 & 0 \\ 1 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix},\ (3)\ A=\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

2.14. Let A be the $n \times n$ matrix whose entries are all 1. Show that

(1) $\det(A - nI_n) = 0$.

(2) $(A - nI_n)_{ij} = (-1)^{n-1}n^{n-2}$ for all i, j, where $(A - nI_n)_{ij}$ denotes the cofactor of the (i, j)-entry of $A - nI_n$.

2.15. Show that if A is symmetric, so is adj A. Moreover, if it is invertible, then the inverse of A is also symmetric.

2.16. Use the adjugate formula to compute the inverses of the following matrices:

$$A=\begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}, \qquad B=\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

2.17. Compute adj A, $\det A$, $\det(\operatorname{adj} A)$, A^{-1}, and verify $A \cdot \operatorname{adj} A = (\det A)I$ for

$$(1)\ A=\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix},\ (2)\ A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}.$$

2.18. Let A, B be invertible matrices. Show that $\operatorname{adj}(AB) = \operatorname{adj} B \operatorname{adj} A$.
(The reader may also try to prove this equality for noninvertible matrices.)

2.19. For an $m \times n$ matrix A and $n \times m$ matrix B, show that

$$\det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \det(AB).$$

2.20. Find the area of the triangle with vertices at $(0, 0)$, $(1, 3)$ and $(3, 1)$ in $\mathbb{R}^2$.

2.21. Find the area of the triangle with vertices at $(0, 0, 0)$, $(1, 1, 2)$ and $(2, 2, 1)$ in $\mathbb{R}^3$.

2.22. For $A, B, C, D \in M_{n\times n}(\mathbb{R})$, show that $\det\begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix} = \det A \det D$. But, in general, $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq \det A \det D - \det B \det C$.

2.23. Determine whether or not the following statements are true in general, and justify your answers.

(1) For any square matrices A and B of the same size, $\det(A + B) = \det A + \det B$.

(2) For any square matrices A and B of the same size, $\det(AB) = \det(BA)$.

(3) If A is an $n \times n$ square matrix, then for any scalar c, $\det(cI_n - A) = c^n - \det A$.

(4) If A is an $n \times n$ square matrix, then for any scalar c, $\det(cI_n - A^T) = \det(cI_n - A)$.

(5) If E is an elementary matrix, then $\det E = \pm 1$.

(6) There is no matrix A of order 3 such that $A^2 = -I_3$.

(7) Let A be a nilpotent matrix, i.e., $A^k = \mathbf{0}$ for some natural number k. Then $\det A = 0$.

(8) $\det(kA) = k \det A$ for any square matrix A.

(9) The multilinearity holds in any two rows at the same time:

$$\det\begin{bmatrix} a+u & b+v & c+w \\ d+x & e+y & f+z \\ \ell & m & n \end{bmatrix} = \det\begin{bmatrix} a & b & c \\ d & e & f \\ \ell & m & n \end{bmatrix} + \det\begin{bmatrix} u & v & w \\ x & y & z \\ \ell & m & n \end{bmatrix}.$$

(10) Any system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\det A \neq 0$.

(11) For any $n \times 1$, $n \geq 2$, column vectors $\mathbf{u}$ and $\mathbf{v}$, $\det(\mathbf{u}\mathbf{v}^T) = 0$.

(12) If A is a square matrix with $\det A = 1$, then $\operatorname{adj}(\operatorname{adj} A) = A$.

(13) If the entries of A are all integers and $\det A = 1$ or -1, then the entries of A^{-1} are also integers.

(14) If the entries of A are 0's or 1's, then $\det A = 1, 0$, or -1.

(15) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

(16) If A is a permutation matrix, then $A^T = A$.

3

Vector Spaces

3.1 The n-space $\mathbb{R}^n$ and vector spaces

We have seen that the Gauss–Jordan elimination is the most basic technique for solving a system $A\mathbf{x} = \mathbf{b}$ of linear equations and it can be written in matrix notation as an LDU factorization. Moreover, the questions of the existence or the uniqueness of the solution are much easier to answer after the Gauss–Jordan elimination. In particular, if $\det A \neq 0$, $\mathbf{x} = \mathbf{0}$ is the unique solution $A\mathbf{x} = \mathbf{0}$. In general, the set of solutions of $A\mathbf{x} = \mathbf{0}$ has a kind of mathematical structure, called a *vector space*, and with this concept one can characterize the uniqueness of the solution of a system $A\mathbf{x} = \mathbf{b}$ of linear equations in a more systematic way.

In this chapter, we introduce the notion of a vector space, which is an abstraction of the usual algebraic structures of the 3-space $\mathbb{R}^3$ and then elaborate our study of a system of linear equations to this framework.

Usually, many physical quantities, such as length, area, mass, temperature are described by real numbers as magnitudes. Other physical quantities like force or velocity have directions as well as magnitudes. Such quantities with direction are called **vectors**, while the numbers are called **scalars**. For instance, a vector (or a point) $\mathbf{x}$ in the 3-space $\mathbb{R}^3$ is usually represented as a triple of real numbers:

$$\mathbf{x} = (x_1, x_2, x_3),$$

where $x_i \in \mathbb{R}$, $i = 1, 2, 3$, which are called the **coordinates** of $\mathbf{x}$. This expression provides a rectangular coordinate system in a natural way. On the other hand, pictorially such a point in the 3-space $\mathbb{R}^3$ can also be represented by an arrow from the origin to $\mathbf{x}$. In this way, a point in the 3-space $\mathbb{R}^3$ can be understood as a vector. The direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude.

In order to have a more general definition of vectors, we extract the most basic properties of those arrows in $\mathbb{R}^3$. Note that for all vectors (or points) in $\mathbb{R}^3$, there are two algebraic operations: the sum of two vectors and scalar multiplication of a vector by a scalar. That is, for two vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ and k a scalar, we define

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3),\\ k\mathbf{x} &= (kx_1,\ kx_2,\ kx_3).\end{aligned}$$

Then a vector $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ may be written as

$$\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ which were introduced as the rectangular coordinate system in vector calculus. The sum of vectors and the scalar multiplication of vectors in the 3-space $\mathbb{R}^3$ are illustrated in Figure 3.1:

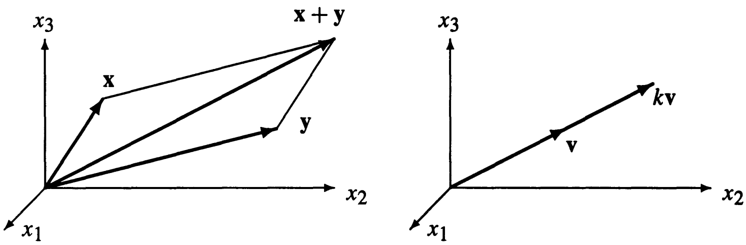

Figure 3.1. A vector sum and a scalar multiplication

Even though our geometric visualization of vectors does not go beyond the 3-space $\mathbb{R}^3$, it is possible to extend these algebraic operations of vectors in the 3-space $\mathbb{R}^3$ to the n**-space** $\mathbb{R}^n$ for any positive integer n. It is defined to be the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ of real numbers, called *vectors* : i.e.,

$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n)\ :\ a_i \in \mathbb{R},\ i = 1,\ 2,\ \ldots,\ n\}.$$

For any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in the n-space $\mathbb{R}^n$, and a scalar k, the *sum* $\mathbf{x} + \mathbf{y}$ and the *scalar multiplication* $k\mathbf{x}$ of them are vectors in $\mathbb{R}^n$ defined by

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= (x_1 + y_1,\ x_2 + y_2, \ldots, x_n + y_n),\\ k\mathbf{x} &= (kx_1,\ kx_2, \ldots, kx_n).\end{aligned}$$

It is easy to verify the following arithmetical rules of the operations:

Theorem 3.1 *For any scalars k and ℓ, and vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, and $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ in the n-space $\mathbb{R}^n$, the following rules hold:*

(1) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$,
(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$,
(3) $\mathbf{x}+\mathbf{0}=\mathbf{x}=\mathbf{0}+\mathbf{x}$,
(4) $\mathbf{x}+(-1)\mathbf{x}=\mathbf{0}$,
(5) $k(\mathbf{x}+\mathbf{y})=k\mathbf{x}+k\mathbf{y}$,
(6) $(k+\ell)\mathbf{x}=k\mathbf{x}+\ell\mathbf{x}$,
(7) $k(\ell\mathbf{x})=(k\ell)\mathbf{x}$,
(8) $1\mathbf{x}=\mathbf{x}$,

where $\mathbf{0}=(0,0,\ldots,0)$ *is the* **zero** *vector.*

We usually identify a vector $(a_1, a_2, \ldots, a_n)$ in the n-space $\mathbb{R}^n$ with an $n \times 1$ column vector

$$(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [a_1\ a_2\ \cdots\ a_n]^T.$$

Sometimes a vector in $\mathbb{R}^n$ is also identified with a $1 \times n$ row vector (see Section 3.5). Then, the two operations of the matrix sum and the scalar multiplication of column matrices coincide with those of vectors in $\mathbb{R}^n$, and Theorem 3.1 rephrases Theorem 1.3.

These rules of arithmetic of vectors are the most important ones because they are the only rules that we need to manipulate vectors in the n-space $\mathbb{R}^n$. Hence, an (abstract) vector space can be defined with respect to these rules of operations of vectors in the n-space $\mathbb{R}^n$ so that $\mathbb{R}^n$ itself becomes a vector space. In general, a *vector space* is defined to be a set with two operations: an addition and a scalar multiplication which satisfy the rules (1)–(8) in Theorem 3.1.

Definition 3.1 A (real) **vector space** is a nonempty set V of elements, called **vectors,** with two algebraic operations that satisfy the following rules.

(A) There is an operation called *vector addition* that associates to every pair $\mathbf{x}$ and $\mathbf{y}$ of vectors in V a unique vector $\mathbf{x}+\mathbf{y}$ in V, called the **sum** of $\mathbf{x}$ and $\mathbf{y}$, so that the following rules hold for all vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in V:

(1) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ (commutativity in addition),
(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}(=\mathbf{x}+\mathbf{y}+\mathbf{z})$ (associativity in addition),
(3) there is a unique vector $\mathbf{0}$ in V such that $\mathbf{x}+\mathbf{0}=\mathbf{x}=\mathbf{0}+\mathbf{x}$ for all $\mathbf{x} \in V$ (it is called the **zero vector**),
(4) for any $\mathbf{x} \in V$, there is a vector $-\mathbf{x} \in V$, called the **negative** of $\mathbf{x}$, such that $\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$.

(B) There is an operation called the **scalar multiplication** that associates to each vector $\mathbf{x}$ in V and each scalar k a unique vector $k\mathbf{x}$ in V so that the following rules hold for all vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in V and all scalars k, ℓ:

(5) $k(\mathbf{x}+\mathbf{y})=k\mathbf{x}+k\mathbf{y}$ (distributivity with respect to vector addition),
(6) $(k+\ell)\mathbf{x}=k\mathbf{x}+\ell\mathbf{x}$ (distributivity with respect to scalar addition),

(7) $k(\ell\mathbf{x}) = (k\ell)\mathbf{x}$ (associativity in scalar multiplication),
(8) $1\mathbf{x} = \mathbf{x}$.

Clearly, the n-space $\mathbb{R}^n$ is a vector space by Theorem 3.1. A **complex vector space** is obtained if, instead of real numbers, we take complex numbers for scalars. For example, the set $\mathbb{C}^n$ of all ordered n-tuples of complex numbers is a complex vector space. In Chapter 7 we shall discuss complex vector spaces, but until then we will discuss only real vector spaces unless otherwise stated.

Example 3.1 (*Miscellaneous examples of vector spaces*)

(1) For any two positive integers m and n, the set $M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices forms a vector space under the matrix sum and the scalar multiplication defined in Section 1.3. The zero vector in this space is the zero matrix $\mathbf{0}_{m\times n}$, and $-A$ is the negative of a matrix A.

(2) Let A be an $m \times n$ matrix. Then it is easy to show that the set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space (under the sum and the scalar multiplication of matrices).

(3) Let $C(\mathbb{R})$ denote the set of real-valued continuous functions defined on the real line $\mathbb{R}$. For two functions f and g, and a real number k, the sum $f + g$ and the scalar multiplication kf of them are defined by

$$\begin{aligned}(f+g)(x) &= f(x)+g(x),\\ (kf)(x) &= kf(x).\end{aligned}$$

Then the set $C(\mathbb{R})$ is a vector space under these operations. The zero vector in this space is the constant function whose value at each point is zero.

(4) Let $S(\mathbb{R})$ denote the set of real-valued functions defined on the set of integers. A function $f \in S(\mathbb{R})$ can be written as a doubly infinite sequence of real numbers:

$$\ldots,\ x_{-2},\ x_{-1},\ x_0,\ x_1,\ x_2,\ \ldots,$$

where $x_k = f(k)$ for each k. This kind of sequences appear frequently in engineering, and is called a discrete or a digital signal. One can define the sum of two functions and the scalar multiplication of a function with a scalar just as in $C(\mathbb{R})$ in (3) so that $S(\mathbb{R})$ becomes a vector space. □

Theorem 3.2 *Let V be a vector space and let $\mathbf{x}$, $\mathbf{y}$ be vectors in V. Then*

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}$ *implies* $\mathbf{x} = \mathbf{0}$,
(2) $0\mathbf{x} = \mathbf{0}$,
(3) $k\mathbf{0} = \mathbf{0}$ *for any* $k \in \mathbb{R}$,
(4) $-\mathbf{x}$ *is unique and* $-\mathbf{x} = (-1)\mathbf{x}$,
(5) *if* $k\mathbf{x} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{x} = \mathbf{0}$.

Proof: (1) By adding $-\mathbf{y}$ to both sides of $\mathbf{x}+\mathbf{y} = \mathbf{y}$, we have

$$\mathbf{x} = \mathbf{x}+\mathbf{0} = \mathbf{x}+\mathbf{y}+(-\mathbf{y}) = \mathbf{y}+(-\mathbf{y}) = \mathbf{0}.$$

(2) $0\mathbf{x} = (0+0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$ implies $0\mathbf{x} = \mathbf{0}$ by (1).

(3) This is an easy exercise.

(4) The uniqueness of the negative $-\mathbf{x}$ of $\mathbf{x}$ can be shown by a simple modification of Lemma 1.7. In fact, if $\bar{\mathbf{x}}$ is another negative of $\mathbf{x}$ such that $\mathbf{x} + \bar{\mathbf{x}} = \mathbf{0}$, then

$$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + (\mathbf{x} + \bar{\mathbf{x}}) = (-\mathbf{x} + \mathbf{x}) + \bar{\mathbf{x}} = \mathbf{0} + \bar{\mathbf{x}} = \bar{\mathbf{x}}.$$

On the other hand, the equation

$$\mathbf{x} + (-1)\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} = (1-1)\mathbf{x} = 0\mathbf{x} = \mathbf{0}$$

shows that $(-1)\mathbf{x}$ is another negative of $\mathbf{x}$, and hence $-\mathbf{x} = (-1)\mathbf{x}$ by the uniqueness of $-\mathbf{x}$.

(5) Suppose $k\mathbf{x} = \mathbf{0}$ and $k \neq 0$. Then $\mathbf{x} = 1\mathbf{x} = \frac{1}{k}(k\mathbf{x}) = \frac{1}{k}\mathbf{0} = \mathbf{0}$. □

Problem 3.1 Let V be the set of all pairs (x, y) of real numbers. Suppose that an addition and scalar multiplication of pairs are defined by

$$(x, y) + (u, v) = (x + 2u,\ y + 2v), \quad k(x, y) = (kx,\ ky).$$

Is the set V a vector space under those operations? Justify your answer.

3.2 Subspaces

Definition 3.2 A subset W of a vector space V is called a **subspace** of V if W itself is a vector space under the addition and the scalar multiplication defined in V.

In order to show that a subset W is a subspace of a vector space V, it is not necessary to verify all the arithmetic rules of the definition of a vector space. One only needs to check whether a given subset is closed under the same vector addition and scalar multiplication as in V. This is due to the fact that certain rules satisfied in the larger space are automatically satisfied in every subset.

Theorem 3.3 *A nonempty subset W of a vector space V is a subspace if and only if $\mathbf{x} + \mathbf{y}$ and $k\mathbf{x}$ are contained in W (or equivalently, $\mathbf{x} + k\mathbf{y} \in W$) for any vectors $\mathbf{x}$ and $\mathbf{y}$ in W and any scalar $k \in \mathbb{R}$.*

Proof: We need only to prove the sufficiency. Assume both conditions hold and let $\mathbf{x}$ be any vector in W. Since W is closed under scalar multiplication, $\mathbf{0} = 0\mathbf{x}$ and $-\mathbf{x} = (-1)\mathbf{x}$ are in W, so rules (3) and (4) for a vector space hold. All the other rules for a vector space are clear. □

A vector space V itself and the zero vector $\{\mathbf{0}\}$ are trivially subspaces. Some nontrivial subspaces are given in the following examples.

Example 3.2 (*Which planes in $\mathbb{R}^3$ can be a subspace?*) Let

$$W = \{(x, y, z) \in \mathbb{R}^3 \ : \ ax + by + cz = 0\},$$

where a, b, c are constants. If $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ are points in W, then clearly $\mathbf{x} + \mathbf{y} = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3)$ is also a point in W, because it satisfies the equation in W. Similarly, $k\mathbf{x}$ also lies in W for any scalar k. Hence, W is a subspace of $\mathbb{R}^3$, which is a plane passing through the origin in $\mathbb{R}^3$. □

Example 3.3 (*The solutions of* $A\mathbf{x} = \mathbf{0}$ *form a subspace*) Let A be an $m \times n$ matrix. Then, as shown in Example 3.1(2), the set

$$W = \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \mathbf{0}\}$$

of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space. Moreover, since the operations in W and in $\mathbb{R}^n$ coincide, W is a subspace of $\mathbb{R}^n$. □

Example 3.4 For a nonnegative integer n, let $P_n(\mathbb{R})$ denote the set of all real polynomials in x with degree $\leq n$. Then $P_n(\mathbb{R})$ is a subspace of the vector space $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$. □

Example 3.5 (*The space of symmetric or skew-symmetric matrices*) Let W be the set of all $n \times n$ real symmetric matrices. Then W is a subspace of the vector space $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices, because the sum of two symmetric matrices is symmetric and a scalar multiplication of a symmetric matrix is also symmetric. Similarly, the set of all $n \times n$ skew-symmetric matrices is also a subspace of $M_{n\times n}(\mathbb{R})$. □

Problem 3.2 Which of the following sets are subspaces of the 3-space $\mathbb{R}^3$? Justify your answer.

(1) $W = \{(x, y, z) \in \mathbb{R}^3 \ : \ xyz = 0\}$,
(2) $W = \{(2t,\ 3t,\ 4t) \in \mathbb{R}^3 \ : \ t \in \mathbb{R}\}$,
(3) $W = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 0\}$,
(4) $W = \{\mathbf{x} \in \mathbb{R}^3 \ : \ \mathbf{x}^T\mathbf{u} = \mathbf{0} = \mathbf{x}^T\mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are any two fixed nonzero vectors in $\mathbb{R}^3$.

Can you describe all subspaces of the 3-space $\mathbb{R}^3$?

Problem 3.3 Let $V = C(\mathbb{R})$ be the vector space of all continuous functions on $\mathbb{R}$. Which of the following sets W are subspaces of V? Justify your answer.

(1) W is the set of all differentiable functions on $\mathbb{R}$.
(2) W is the set of all bounded continuous functions on $\mathbb{R}$.
(3) W is the set of all continuous nonnegative-valued functions on $\mathbb{R}$, i.e., $f(x) \geq 0$ for any $x \in \mathbb{R}$.
(4) W is the set of all continuous odd functions on $\mathbb{R}$, i.e., $f(-x) = -f(x)$ for any $x \in \mathbb{R}$.
(5) W is the set of all polynomials with integer coefficients.

Definition 3.3 Let U and W be two subspaces of a vector space V.

(1) The **sum** of U and W is defined by

$$U + W = \{\mathbf{u} + \mathbf{w} \in V : \mathbf{u} \in U, \ \mathbf{w} \in W\}.$$

(2) A vector space V is called the **direct sum** of two subspaces U and W, written as $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{\mathbf{0}\}$.

It is easy to see that $U + W$ and $U \cap W$ are also subspaces of V. If $V = \mathbb{R}^2$ (xy-plane), $U = \{x\mathbf{i} : x \in \mathbb{R}\}$ (x-axis), and $W = \{y\mathbf{j} : y \in \mathbb{R}\}$ (y-axis), then it is easy to see that $\mathbb{R}^2 = U \oplus V = \mathbb{R} \oplus \mathbb{R}$, by considering the x-axis (and also the y-axis) as $\mathbb{R}$. Similarly, one can easily be convinced that $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}^1 = \mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1$.

Problem 3.4 Let U and W be subspaces of a vector space V.

(1) Suppose that Z is a subspace of V contained in both U and W. Show that Z is also contained in $U \cap W$.
(2) Suppose that Z is a subspace of V containing both U and W. Show that Z also contains $U + W$ as a subspace.

Theorem 3.4 *A vector space V is the direct sum of subspaces U and W, i.e., $V = U \oplus W$, if and only if for any $\mathbf{v} \in V$ there exist unique $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.*

Proof: ($\Rightarrow$) Suppose that $V = U \oplus W$. Then, for any $\mathbf{v} \in V$, there exist vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, since $V = U + W$. To show the uniqueness, suppose that $\mathbf{v}$ is also expressed as a sum $\mathbf{u}' + \mathbf{w}'$ for $\mathbf{u}' \in U$ and $\mathbf{w}' \in W$. Then $\mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}'$ implies

$$\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w} \in U \cap W = \{\mathbf{0}\}.$$

Hence, $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$.

($\Leftarrow$) Clearly, $V = U + W$. Suppose that there exists a nonzero vector $\mathbf{v}$ in $U \cap W$. Then $\mathbf{v}$ can be written as a sum of vectors in U and W in many different ways:

$$\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{v} = \frac{1}{3}\mathbf{v} + \frac{2}{3}\mathbf{v} \in U + W. \qquad \square$$

Example 3.6 (*Sum, but not direct sum*) In the 3-space $\mathbb{R}^3$, consider the three vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$. These three vectors are also well known as $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ respectively. Let $U = \{a\mathbf{i} + c\mathbf{k} : a, c \in \mathbb{R}\}$ be the xz-plane, and let $W = \{b\mathbf{j} + c\mathbf{k} : b, c \in \mathbb{R}\}$ be the yz-plane, which are both subspaces of $\mathbb{R}^3$. Then a vector in $U + W$ is of the form

$$\begin{aligned}(a\mathbf{i} + c_1\mathbf{k}) + (b\mathbf{j} + c_2\mathbf{k}) &= a\mathbf{i} + b\mathbf{j} + (c_1 + c_2)\mathbf{k} \\ &= a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \ = \ (a, b, c),\end{aligned}$$

where $c = c_1 + c_2$ and a, b, c can be arbitrary numbers. Thus $U + W = \mathbb{R}^3$. However, $\mathbb{R}^3 \neq U \oplus W$ since clearly $\mathbf{k} \in U \cap W \neq \{\mathbf{0}\}$. In fact, the vector $\mathbf{k} \in \mathbb{R}^3$ can be written as many linear combinations of vectors in U and W:

$$\mathbf{k} = \frac{1}{2}\mathbf{k} + \frac{1}{2}\mathbf{k} = \frac{1}{3}\mathbf{k} + \frac{2}{3}\mathbf{k} \in U + W.$$

Note that if we had taken $W = \{y\mathbf{j} : y \in \mathbb{R}\}$ to be the y-axis, then it would be easy to see that $\mathbb{R}^3 = U \oplus W$. Note also that there are many choices for W so that $\mathbb{R}^3 = U \oplus W$. □

Problem 3.5 Let U and W be the subspaces of the vector space $M_{n\times n}(\mathbb{R})$ consisting of all symmetric matrices and all skew-symmetric matrices, respectively. Show that $M_{n\times n}(\mathbb{R}) = U \oplus W$. Therefore, the decomposition of a square matrix A given in (3) of Problem 1.11 is unique.

3.3 Bases

As we know, a vector in the 3-space $\mathbb{R}^3$ is of the form (x_1, x_2, x_3), and also it can be written as

$$(x_1, x_2, x_3) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1).$$

That is, any vector in $\mathbb{R}^3$ can be expressed as a sum of scalar multiples of three vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$. The following definition gives a name to such an expression.

Definition 3.4 Let V be a vector space and let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a set of vectors in V. Then a vector $\mathbf{y}$ in V of the form

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m,$$

where $a_1, a_2, \ldots, a_m$ are scalars, is called a **linear combination** of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$.

The next theorem shows that the set of all linear combinations of a finite set of vectors in a vector space forms a subspace.

Theorem 3.5 *Let* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ *be vectors in a vector space* V. *Then the set* $W = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m : a_i \in \mathbb{R}\}$ *of all linear combinations of* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ *is a subspace of* V. *It is called the* **subspace of** V **spanned** *by* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$. *Or,* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ **span** *the subspace* W.

Proof: It is necessary to show that W is closed under the vector sum and the scalar multiplication. Let $\mathbf{u}$ and $\mathbf{w}$ be any two vectors in W. Then

$$\begin{array}{rcccccccc} \mathbf{u} & = & a_1\mathbf{x}_1 & + & a_2\mathbf{x}_2 & + & \cdots & + & a_m\mathbf{x}_m, \\ \mathbf{w} & = & b_1\mathbf{x}_1 & + & b_2\mathbf{x}_2 & + & \cdots & + & b_m\mathbf{x}_m \end{array}$$

for some scalars a_i's and b_i's. Therefore,

$$\mathbf{u} + \mathbf{w} = (a_1 + b_1)\mathbf{x}_1 + (a_2 + b_2)\mathbf{x}_2 + \cdots + (a_m + b_m)\mathbf{x}_m$$

and, for any scalar k,

$$k\mathbf{u} = (ka_1)\mathbf{x}_1 + (ka_2)\mathbf{x}_2 + \cdots + (ka_m)\mathbf{x}_m.$$

Thus, $\mathbf{u} + \mathbf{w}$ and $k\mathbf{u}$ are linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ and consequently contained in W. Therefore, W is a subspace of V. □

Example 3.7 (*A space can be spanned by many different sets*)

(1) For a nonzero vector $\mathbf{v}$ in a vector space V, a linear combination of $\mathbf{v}$ is simply a scalar multiple of $\mathbf{v}$. Thus the subspace W of V spanned by $\mathbf{v}$ is $W = \{k\mathbf{v} : k \in \mathbb{R}\}$. Note that this subspace W can be spanned by any $k\mathbf{v}$, $k \neq 0$.

(2) Consider three vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2 = (1, 1, 0)$ in $\mathbb{R}^3$. The subspace W_1 spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ is written as

$$W_1 = \{a_1\mathbf{e}_1 + a_2\mathbf{e}_2 = (a_1, a_2, 0) : a_i \in \mathbb{R}\},$$

and the subspace W_2 spanned by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{v}$ is written as

$$W_2 = \{a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{v} = (a_1 + a_3, a_2 + a_3, 0) : a_i \in \mathbb{R}\}.$$

Clearly, $W_1 \subseteq W_2$. On the other hand, since $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2 \in W_1$, $W_2 \subseteq W_1$. Thus $W_1 = W_2$ which is the xy-plane in $\mathbb{R}^3$. In general, a subspace in a vector space can have many different spanning sets. □

Example 3.8 (*For any $m \leq n$, $\mathbb{R}^m$ is a subspace of $\mathbb{R}^n$*) Let

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, 0, \ldots, 0), \\ \mathbf{e}_2 &= (0, 1, 0, \ldots, 0), \\ &\vdots \\ \mathbf{e}_n &= (0, 0, 0, \ldots, 1) \end{aligned}$$

be n vectors in the n-space $\mathbb{R}^n$ ($n \geq 3$). Then a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is of the form

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = (a_1, a_2, a_3, 0, \ldots, 0).$$

Hence, the set

$$W = \{(a_1, a_2, a_3, 0, \ldots, 0) \in \mathbb{R}^n \ : \ a_1, \ a_2, \ a_3 \in \mathbb{R}\}$$

is the subspace of the n-space $\mathbb{R}^n$ spanned by the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Note that the subspace W can be identified with the 3-space $\mathbb{R}^3$ through the identification

$$(a_1, a_2, a_3, 0, \ldots, 0) \equiv (a_1, a_2, a_3)$$

with $a_i \in \mathbb{R}$. In general, for $m \leq n$, the m-space $\mathbb{R}^m$ can be identified as a subspace of the n-space $\mathbb{R}^n$. □

Example 3.9 (*All* $A\mathbf{x}$*'s form the column space*) Let $A = [\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_n]$ be an $m \times n$ matrix with column $\mathbf{c}_i$'s. Then the column vectors $\mathbf{c}_i$ are in $\mathbb{R}^m$, and the matrix product $A\mathbf{x}$ represents the linear combination of the column vectors $\mathbf{c}_i$ whose coefficients are the components of $\mathbf{x} \in \mathbb{R}^n$, i.e., $A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$ (see Example 1.9). Therefore, the set

$$W = \{A\mathbf{x} \in \mathbb{R}^m \ :\ \mathbf{x} \in \mathbb{R}^n\}$$

of all linear combinations of the column vectors of A is a subspace of $\mathbb{R}^m$ called the **column space** of A. Consequently, $A\mathbf{x} = \mathbf{b}$ *has a solution* $(x_1, x_2, \ldots, x_n)$ *in* $\mathbb{R}^n$ *if and only if the vector* $\mathbf{b}$ *belongs to the column space of* A. □

Remark: One can take another point of view on the equation $A\mathbf{x} = \mathbf{b}$. For any vector $\mathbf{x} \in \mathbb{R}^n$, A assigns another vector $\mathbf{b}$ in $\mathbb{R}^m$ which is given as $A\mathbf{x}$. That is, A can be considered as a function from $\mathbb{R}^n$ into $\mathbb{R}^m$ and the column space of A is nothing but the image of this function.

Problem 3.6 Let $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ be vectors in a vector space V and let W be the subspace spanned by $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$. Show that W is the smallest subspace of V containing $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$. In other words, if U is a subspace of V containing $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$, then $W \subseteq U$.

As we saw in Theorem 3.5 and Example 3.7, any nonempty subset of a vector space V spans a subspace through the linear combinations of the vectors, and two different subsets may span the same subspace. This means that a vector can be written as linear combinations in various ways.

However, for some sets of vectors in a vector space V, any vector in V can be expressed uniquely as a linear combination of the set. Such a set of vectors is called a basis for V. In the following we will make this notion clear and show how to find such a basis.

Definition 3.5 A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in a vector space V is said to be **linearly independent** if the vector equation, called the **linear dependence** of $\mathbf{x}_i$'s,

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_m = 0$. Otherwise, it is said to be **linearly dependent**.

By definition, a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is linearly dependent if and only if the linear dependence

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$$

has a nontrivial solution $(c_1, c_2, \ldots, c_m)$. For example, if $c_m \neq 0$, the equation can be rewritten as

$$\mathbf{x}_m = -\frac{c_1}{c_m}\mathbf{x}_1 - \frac{c_2}{c_m}\mathbf{x}_2 - \cdots - \frac{c_{m-1}}{c_m}\mathbf{x}_{m-1}.$$

That is, *a set of vectors is linearly dependent if and only if at least one of the vectors in the set can be written as a linear combination of the others.* It means that at least one of the vectors can be expressed as a linear combination of the set in two different ways.

Example 3.10 (*Linear independence in* $\mathbb{R}^3$) Let $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (3, 2, 1)$ be two vectors in the 3-space $\mathbb{R}^3$. Then clearly $\mathbf{y} \neq \lambda\mathbf{x}$ for any $\lambda \in \mathbb{R}$ (or $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$ is possible only when $a = b = 0$). This means that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent in $\mathbb{R}^3$. If $\mathbf{w} = (3, 6, 9)$, then $\{\mathbf{x}, \mathbf{w}\}$ is linearly dependent since $\mathbf{w} - 3\mathbf{x} = \mathbf{0}$. In general, if $\mathbf{x}, \mathbf{y}$ are non-collinear vectors in the 3-space $\mathbb{R}^3$, the set of all linear combinations of $\mathbf{x}$ and $\mathbf{y}$ determines a plane W through the origin in $\mathbb{R}^3$, i.e., $W = \{a\mathbf{x} + b\mathbf{y} : a, b \in \mathbb{R}\}$. Let $\mathbf{z}$ be another nonzero vector in the 3-space $\mathbb{R}^3$. If $\mathbf{z} \in W$, then there are some scalars $a, b \in \mathbb{R}$, not all of them are zero, such that $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$, that is, the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent. If $\mathbf{z} \notin W$, then $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ is possible only when $a = b = c = 0$ (prove it). Therefore, the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent if and only if $\mathbf{z}$ does not lie in W. □

By abuse of language, it is sometimes convenient to say that "the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent," although this is really a property of a set.

Example 3.11 The columns of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 4 & 2 & 6 & 8 \\ 2 & -1 & 1 & 3 \end{bmatrix}$$

are linearly dependent in the 3-space $\mathbb{R}^3$, since the third column is the sum of the first and the second. □

As shown in Example 3.11, the concept of linear dependence can be applied to the row or column vectors of any matrix.

Example 3.12 Consider an upper triangular matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix}.$$

The linear dependence of the column vectors of A may be written as

$$c_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which, in matrix notation, may be written as a homogeneous system:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the third row, $c_3 = 0$, from the second row $c_2 = 0$, and substitution of them into the first row forces $c_1 = 0$, i.e., the homogeneous system has only the trivial solution, so that the column vectors are linearly independent. □

The following theorem can be proved by the same argument.

Theorem 3.6 *The nonzero rows of a matrix in row-echelon form are linearly independent, and so are the columns that contain leading 1's.*

In particular, the rows of any triangular matrix with nonzero diagonals are linearly independent, and so are the columns.

If $V = \mathbb{R}^m$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are n vectors in $\mathbb{R}^m$, then they form an $m \times n$ matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$. On the other hand, Example 3.9 shows that the linear dependence $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ of $\mathbf{v}_i$'s is nothing but the homogeneous equation $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (c_1, c_2, \ldots, c_n)$. Thus, one can say

(1) *the column vectors $\mathbf{v}_i$'s of A are linearly independent in $\mathbb{R}^m$ if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution,* and
(2) *they are linearly dependent if and only if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.*

If U is the reduced row-echelon form of A, then we know that $A\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{0}$ have the same set of solutions. Moreover, a homogeneous system $A\mathbf{x} = \mathbf{0}$ with more unknowns than equations always has a nontrivial solution by Theorem 1.2. This proves the following lemma.

Lemma 3.7 **(1)** *If $n > m$, any set of n vectors in the m-space $\mathbb{R}^m$ is linearly dependent.*
(2) *If U is the reduced row-echelon form of A, then the columns of U are linearly independent if and only if the columns of A are linearly independent.*

Example 3.13 Consider the vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ in the 3-space $\mathbb{R}^3$. The matrix $A = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3]$ is the identity matrix and so $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Thus, the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent and also spans $\mathbb{R}^3$. □

Example 3.14 (*The standard basis for* $\mathbb{R}^n$) The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are clearly linearly independent (see Theorem 3.6). Moreover, they span the n-space $\mathbb{R}^n$: In fact, a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a linear combination of the vector $\mathbf{e}_i$'s:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$ □

Definition 3.6 Let V be a vector space. A **basis** for V is a set of linearly independent vectors that spans V.

For example, as in Example 3.14, the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ forms a basis, called the **standard basis** for the n-space $\mathbb{R}^n$. Of course, there are many other bases for $\mathbb{R}^n$.

Example 3.15 (*A basis or not*)

(1) The set of vectors $(1, 1, 0)$, $(0, -1, 1)$, and $(1, 0, 1)$ is not a basis for the 3-space $\mathbb{R}^3$, since this set is linearly dependent (the third is the sum of the first two vectors), and cannot span $\mathbb{R}^3$. (The vector $(1, 0, 0)$ cannot be obtained as a linear combination of them (prove it).) This set does not have enough vectors spanning $\mathbb{R}^3$.

(2) The set of vectors $(1, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$ and $(0, 1, 0)$ is not a basis either, since they are not linearly independent (the sum of the first two minus the third makes the fourth) even though they span $\mathbb{R}^3$. This set of vectors has some redundant vectors spanning $\mathbb{R}^3$.

(3) The set of vectors $(1, 1, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$ is linearly independent and also spans $\mathbb{R}^3$. That is, it is a basis for $\mathbb{R}^3$, different from the standard basis. This set has the proper number of vectors spanning $\mathbb{R}^3$, since the set cannot be reduced to a smaller set nor does it need any additional vector to span $\mathbb{R}^3$. □

By definition, in order to show that a set of vectors in a vector space is a basis, one needs to show two things: *it is linearly independent, and it spans the whole space.* The following theorem shows that a basis for a vector space represents a coordinate system just like the rectangular coordinate system by the standard basis for $\mathbb{R}^n$.

Theorem 3.8 *Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Then each vector $\mathbf{x}$ in V can be uniquely expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, i.e., there are unique scalars a_i's, $i = 1, 2, \ldots, n$, such that*

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Proof: If $\mathbf{x}$ can also expressed as $\mathbf{x} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$, then we have $\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n$. By the linear independence of $\mathbf{v}_i$'s, $a_i = b_i$ for all $i = 1, 2, \ldots, n$. □

Example 3.16 (*Two different bases for* $\mathbb{R}^3$) Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1, 1, 1) = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_2 = (0, 1, 1) = \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_3 = (0, 0, 1) = \mathbf{e}_3$. Then, β is also a basis for $\mathbb{R}^3$ (see Example 3.15(3)). For any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, one can easily verify that

$$\begin{aligned} \mathbf{x} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \\ &= x_1\mathbf{v}_1 + (x_2 - x_1)\mathbf{v}_2 + (x_3 - x_2)\mathbf{v}_3. \end{aligned}$$ □

Problem 3.7 Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$ and $\mathbf{v}_3 = (3, 3, 4)$ in the 3-space $\mathbb{R}^3$ form a basis.

Problem 3.8 Show that the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(\mathbb{R})$, the vector space of all polynomials of degree $\leq n$ with real coefficients.

Problem 3.9 Let $\mathbf{x}_k$ denote the vector in $\mathbb{R}^n$ whose first $k - 1$ coordinates are zero and whose last $n - k + 1$ coordinates are 1. Show that the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$.

3.4 Dimensions

We often say that the line $\mathbb{R}^1$ is one-dimensional, the plane $\mathbb{R}^2$ is two-dimensional and the space $\mathbb{R}^3$ is three-dimensional, etc. This is mostly due to the fact that the freedom in choosing coordinates for each element in the space is 1, 2 or 3, respectively. This means that the concept of *dimension* is closely related to the concept of bases. Note that for a vector space in general there is no unique way in choosing a basis. However, there is something common to all bases, and this is related to the notion of dimension. We first need the following lemma from which one can define the dimension of a vector space.

Lemma 3.9 *Let V be a vector space and let $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a set of m vectors in V.*

(1) *If α spans V, then every set of vectors with more than m vectors cannot be linearly independent.*

(2) *If α is linearly independent, then any set of vectors with fewer than m vectors cannot span V.*

Proof: Since (2) follows from (1) directly, we prove only (1). Let $\beta = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be a set of n-vectors in V with $n > m$. We will show that β is linearly dependent. Indeed, since each vector $\mathbf{y}_j$ is a linear combination of the vectors in the spanning set α, i.e., for $j = 1, 2, \ldots, n$,

$$\mathbf{y}_j = a_{1j}\mathbf{x}_1 + a_{2j}\mathbf{x}_2 + \cdots + a_{mj}\mathbf{x}_m = \sum_{i=1}^{m} a_{ij}\mathbf{x}_i,$$

we have

$$\begin{aligned}
c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n &= c_1(a_{11}\mathbf{x}_1 + a_{21}\mathbf{x}_2 + \cdots + a_{m1}\mathbf{x}_m) \\
&\quad + c_2(a_{12}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{m2}\mathbf{x}_m) \\
&\qquad \vdots \\
&\quad + c_n(a_{1n}\mathbf{x}_1 + a_{2n}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_m) \\
&= (a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n)\mathbf{x}_1 \\
&\quad + (a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n)\mathbf{x}_2 \\
&\qquad \vdots \\
&\quad + (a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n)\mathbf{x}_m.
\end{aligned}$$

Thus, β is linearly dependent if and only if the system of linear equations

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{0}$$

has a nontrivial solution $(c_1, c_2, \ldots, c_n) \neq (0, 0, \ldots, 0)$. This is true if all the coefficients of $\mathbf{x}_i$'s are zero but not all of c_i's are zero. It means that the homogeneous system of linear equations in c_i's

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

must have a nontrivial solution. But it is guaranteed by Lemma 3.7, since $m < n$. □

It is clear by Lemma 3.9 that if a set $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of n vectors is a basis for a vector space V, then no other set $\beta = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ of r vectors can be a basis for V if $r \neq n$. This means that all bases for a vector space V have the same number of vectors, even if there are many different bases for a vector space. Therefore, we obtain the following important result:

Theorem 3.10 *If a basis for a vector space V consists of n vectors, so does every other basis.*

Definition 3.7 The **dimension** of a vector space V is the number, say n, of vectors in a basis for V, denoted by $\dim V = n$. When V has a basis of a finite number of vectors, V is said to be **finite dimensional.**

Example 3.17 (*Computing the dimension*) The following are trivial:

(1) If V has only the zero vector: $V = \{\mathbf{0}\}$, then $\dim V = 0$.
(2) If $V = \mathbb{R}^n$, then the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for V implies $\dim \mathbb{R}^n = n$.
(3) If $V = P_n(\mathbb{R})$ of all polynomials of degree less than or equal to n, then $\dim P_n(\mathbb{R}) = n + 1$ since $\{1, x, x^2, \ldots, x^n\}$ is a basis for V.
(4) If $V = M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices, then $\dim M_{m\times n}(\mathbb{R}) = mn$ since $\{E_{ij} : i = 1, \ldots, m,\ j = 1, \ldots, n\}$ is a basis for V, where E_{ij} is the $m \times n$ matrix whose (i, j)-th entry is 1 and all others are zero. □

If $V = C(\mathbb{R})$ of all real-valued continuous functions defined on the real line, then one can show that V is not finite dimensional. A vector space V is **infinite dimensional** if it is not finite dimensional. In this book, we are concerned only with finite-dimensional vector spaces unless otherwise stated.

Theorem 3.11 *Let V be a finite-dimensional vector space.*

(1) *Any linearly independent set in V can be extended to a basis by adding more vectors if necessary.*
(2) *Any set of vectors that spans V can be reduced to a basis by discarding vectors if necessary.*

Proof: We prove (1) and leave (2) as an exercise. Let $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a linearly independent set in V. If α spans V, then α is a basis. If α does not span V, then there exists a vector, say $\mathbf{x}_{k+1}$, in V that is not contained in the subspace spanned by the vectors in α. Now $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ is linearly independent (check why). If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ spans V, then this is a basis for V. If it does not span V, then the same procedure can be repeated, yielding a linearly independent set that spans V, i.e., a basis for V. This procedure must stop in a finite number of steps because of Lemma 3.9 for a finite-dimensional vector space V. $\square$

Theorem 3.11 shows that a basis for a vector space V is a set of vectors in V which is *maximally independent* and *minimally spanning* in the above sense. In particular, if W is a subspace of V, then any basis for W is linearly independent also in V, and can be extended to a basis for V. Thus $\dim W \leq \dim V$.

Corollary 3.12 *Let V be a vector space of dimension n. Then*

(1) *any set of n vectors that spans V is a basis for V, and*
(2) *any set of n linearly independent vectors is a basis for V.*

Proof: Again we prove (1) only. If a spanning set of n vectors were not linearly independent, then the set would be reduced to a basis that has a number of vectors smaller than n. $\square$

Corollary 3.12 means that if it is known that $\dim V = n$ and if a set of n vectors either is linearly independent or spans V, then it is already a basis for the space V.

Example 3.18 (*Constructing a basis*) Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{x}_1 = (1,\ -2,\ 5,\ -3),\quad \mathbf{x}_2 = (0,\ 1,\ 1,\ 4),\quad \mathbf{x}_3 = (1,\ 0,\ 1,\ 0).$$

Find a basis for W and extend it to a basis for $\mathbb{R}^4$.

Solution: Note that $\dim W \leq 3$ since W is spanned by three vectors $\mathbf{x}_i$'s. Let A be the 3×4 matrix whose rows are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Reduce A to a row-echelon form:

$$U = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & \frac{5}{6} \end{bmatrix}.$$

The three nonzero row vectors of U are clearly linearly independent, and they also span W because the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ can be expressed as a linear combination of these three nonzero row vectors of U. Hence, the three nonzero row vectors of U provides a basis for W. (Note that this implies $\dim W = 3$ and hence $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ is also a basis for W by Corollary 3.12. The linear independence of $\mathbf{x}_i$'s is a by-product of this fact).

To extend it to a basis for $\mathbb{R}^4$, just add any nonzero vector of the form $\mathbf{x}_4 = (0,\ 0,\ 0,\ t)$ to the rows of U. □

Problem 3.10 Let W be a subspace of a vector space V. Show that if $\dim W = \dim V$, then $W = V$.

Problem 3.11 Find a basis and the dimension of each of the following subspaces of $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices.

(1) The space of all $n \times n$ diagonal matrices whose traces are zero.
(2) The space of all $n \times n$ symmetric matrices.
(3) The space of all $n \times n$ skew-symmetric matrices.

As a direct consequence of Theorem 3.11 and the definition of the direct sum of subspaces, one can show the following corollary.

Corollary 3.13 *For any subspace U of V, there is a subspace W of V such that $V = U \oplus W$.*

Proof: Choose a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for U, and extend it to a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ for V. Then the subspace W spanned by $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ satisfies the requirement. □

Problem 3.12 Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V and let $W_i = \{r\mathbf{v}_i : r \in \mathbb{R}\}$ be the subspace of V spanned by $\mathbf{v}_i$. Show that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

3.5 Row and column spaces

In this section, we go back to systems of linear equations and study them in terms of the concepts introduced in the previous sections. Note that an $m \times n$ matrix A can be abbreviated by the row vectors or column vectors as follows:

$$
\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix},
\end{aligned}
$$

where $\mathbf{r}_i$ is the i-th row vectors of A in $\mathbb{R}^n$, and $\mathbf{c}_j$ is the j-th column vectors of A in $\mathbb{R}^m$.

Definition 3.8 Let A be an $m \times n$ matrix with row vectors $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$ and column vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$.

(1) The **row space** of A is the subspace in $\mathbb{R}^n$ spanned by the row vectors $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$, denoted by $\mathcal{R}(A)$.
(2) The **column space** of A is the subspace in $\mathbb{R}^m$ spanned by the column vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$, denoted by $\mathcal{C}(A)$.
(3) The solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is called the **null space** of A, denoted by $\mathcal{N}(A)$.

Note that the null space $\mathcal{N}(A)$ is a subspace of the n-space $\mathbb{R}^n$. Its dimension is called the **nullity** of A. Since the row vectors of A are just the column vectors of its transpose A^T and the column vectors of A are the row vectors of A^T, the row (column) space of A is just the column (row) space of A^T; that is,

$$\mathcal{R}(A) = \mathcal{C}(A^T) \quad \text{and} \quad \mathcal{C}(A) = \mathcal{R}(A^T).$$

Since $A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$ for any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we get

$$\mathcal{C}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}.$$

Thus, for a vector $\mathbf{b} \in \mathbb{R}^m$, *the system* $A\mathbf{x} = \mathbf{b}$ *has a solution if and only if* $\mathbf{b} \in \mathcal{C}(A) \subseteq \mathbb{R}^m$. In other words, the column space $\mathcal{C}(A)$ is the set of vectors $\mathbf{b} \in \mathbb{R}^m$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

It is quite natural to ask what the dimensions of those subspaces are, and how one can find bases for them. This will help us to understand the structure of all the solutions of the equation $A\mathbf{x} = \mathbf{b}$. Since the set of the row vectors and the set of the column vectors of A are spanning sets for the row space and the column space, respectively, a minimally spanning subset of each of them will be a basis for each of them. This is not a difficult problem for a matrix of a (reduced) row-echelon form.

Example 3.19 (*Find a basis for the null space*) Let U be in a reduced row-echelon form given as

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the first three nonzero row vectors containing leading 1's are linearly independent and they form a basis for the row space $\mathcal{R}(U)$, so that $\dim \mathcal{R}(U) = 3$. On the other hand, the first three columns containing leading 1's are also linearly independent (see Theorem 3.6), and the last two column vectors can be expressed as linear combinations of them. Hence, they form a basis for $\mathcal{C}(U)$, and $\dim \mathcal{C}(U) = 3$. To find a basis for the null space $\mathcal{N}(U)$, we first solve the system $U\mathbf{x} = \mathbf{0}$ to get the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 2t \\ s - 3t \\ -4s + t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = s\mathbf{n}_s + t\mathbf{n}_t,$$

where $\mathbf{n}_s = (-2,\ 1,\ -4,\ 1,\ 0)$, $\mathbf{n}_t = (-2,\ -3,\ 1,\ 0,\ 1)$, and s and t are arbitrary values for the free variables x_4 and x_5, respectively. It shows that these two vectors $\mathbf{n}_s$ and $\mathbf{n}_t$ span the null space $\mathcal{N}(U)$, and they are clearly linearly independent (see their last two entries). Hence, the set $\{\mathbf{n}_s, \mathbf{n}_t\}$ is a basis for the null space $\mathcal{N}(U)$. □

For any matrix A, we first investigate the row space $\mathcal{R}(A)$ and the null space $\mathcal{N}(A)$ of A by comparing them with those of the reduced row-echelon form U of A. Since $A\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{0}$ have the same solution set by Theorem 1.1, we clearly have $\mathcal{N}(A) = \mathcal{N}(U)$.

Let $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ be the row vectors of an $m \times n$ matrix A. The three elementary row operations change A into the matrices A_i of the following three types:

$$A_1 = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ k\mathbf{r}_i \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \text{ for } k \neq 0; \quad A_2 = \begin{bmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_i \\ \vdots \end{bmatrix} \text{ for } i < j; \quad A_3 = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i + k\mathbf{r}_j \\ \vdots \\ \mathbf{r}_m \end{bmatrix}.$$

It is clear that the row vectors of the three matrices A_1, A_2 and A_3 are linear combinations of the row vectors of A. On the other hand, by the inverse elementary row operations, these matrices can be changed into A. Thus, the row vectors of A can also be written as linear combinations of those of A_i's. This means that if matrices A and B are row equivalent, then their row spaces must be equal, i.e., $\mathcal{R}(A) = \mathcal{R}(B)$.

Now the nonzero row vectors in the reduced row-echelon form U are always linearly independent and span the row space of U (see Theorem 3.6). Thus they form a basis for the row space $\mathcal{R}(A)$ of A. It gives the following theorem.

Theorem 3.14 *Let U be a (reduced) row-echelon form of a matrix A. Then*

$$\mathcal{R}(A) = \mathcal{R}(U) \quad \textit{and} \quad \mathcal{N}(A) = \mathcal{N}(U).$$

Moreover, if U has r nonzero row vectors containing leading 1's, then they form a basis for the row space $\mathcal{R}(A)$, so that the dimension of $\mathcal{R}(A)$ is r.

The following example shows how to find bases for the row and the null spaces, and at the same time how to find a basis for the column space.

Example 3.20 (*Find bases for the row space and the column space of A*) Let A be a matrix given as

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{bmatrix}.$$

Find bases for the row space $\mathcal{R}(A)$, the null space $\mathcal{N}(A)$, and the column space $\mathcal{C}(A)$ of A.

Solution: (1) *Find a basis for* $\mathcal{R}(A)$: By Gauss–Jordan elimination on A, we get the reduced row-echelon form U:

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the three nonzero row vectors

$$\begin{aligned} \mathbf{v}_1 &= (1,\ 0,\ \ 2,\ 0,\ 1), \\ \mathbf{v}_2 &= (0,\ 1,\ -1,\ 0,\ 1), \\ \mathbf{v}_3 &= (0,\ 0,\ \ 0,\ 1,\ 1) \end{aligned}$$

of U are linearly independent, they form a basis for the row space $\mathcal{R}(U) = \mathcal{R}(A)$, so $\dim \mathcal{R}(A) = 3$. (Note that in the process of Gaussian elimination, we did not use a permutation matrix. This means that the three nonzero rows of U were obtained from the first three row vectors $\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3$ of A and the fourth row $\mathbf{r}_4$ of A turned out to be a linear combination of them. Thus the first three row vectors of A also form a basis for the row space.)

(2) *Find a basis for* $\mathcal{N}(A)$: It is enough to solve the homogeneous system $U\mathbf{x} = \mathbf{0}$, since $\mathcal{N}(A) = \mathcal{N}(U)$. That is, neglecting the fourth zero equation, the equation $U\mathbf{x} = \mathbf{0}$ takes the following system of equations:

$$\left\{ \begin{array}{rrrrrcl} x_1 & & +\ 2x_3 & & +\ x_5 & = & 0 \\ & x_2 & -\ \ x_3 & & +\ x_5 & = & 0 \\ & & & x_4 & +\ x_5 & = & 0. \end{array} \right.$$

Since the first, the second and the fourth columns of U contain the leading 1's, we see that the basic variables are $x_1,\ x_2,\ x_4$, and the free variables are $x_3,\ x_5$. As in Example 3.19 by assigning arbitrary values s and t to the free variables x_3 and x_5, one can find the solution $\mathbf{x}$ of $U\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = s\mathbf{n}_s + t\mathbf{n}_t,$$

where $\mathbf{n}_s = (-2,\ 1,\ 1,\ 0,\ 0)$ and $\mathbf{n}_t = (-1,\ -1,\ 0,\ -1,\ 1)$. In fact, the two vectors $\mathbf{n}_s$ and $\mathbf{n}_t$ are the solutions when $(x_3, x_5) = (s, t)$ is $(1, 0)$ and when $(x_3, x_5) = (s, t)$ is $(0, 1)$, respectively. They must be linearly independent, since $(1, 0)$ and $(0, 1)$, as the (x_3, x_5)-coordinates of $\mathbf{n}_s$ and $\mathbf{n}_t$ respectively, are linearly independent. Since any solution of $U\mathbf{x} = \mathbf{0}$ is a linear combination of them, the set $\{\mathbf{n}_s,\ \mathbf{n}_t\}$ is a basis for the null space $\mathcal{N}(U) = \mathcal{N}(A)$. Thus $\dim \mathcal{N}(A) = 2 =$ *the number of free variables in* $U\mathbf{x} = \mathbf{0}$.

(3) *Find a basis for* $\mathcal{C}(A)$: Let $\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3,\ \mathbf{c}_4,\ \mathbf{c}_5$ denote the column vectors of A in the given order. Since these column vectors of A span $\mathcal{C}(A)$, we only need to discard some of the columns that can be expressed as linear combinations of other column vectors. But, the linear dependence

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 + x_4\mathbf{c}_4 + x_5\mathbf{c}_5 = \mathbf{0}, \quad \text{i.e., } A\mathbf{x} = \mathbf{0},$$

holds if and only if $\mathbf{x} = (x_1, \ldots, x_5) \in \mathcal{N}(A)$. By taking $\mathbf{x} = \mathbf{n}_s = (-2,\ 1,\ 1,\ 0,\ 0)$ or $\mathbf{x} = \mathbf{n}_t = (-1,\ -1,\ 0,\ -1,\ 1)$, the basis vectors of $\mathcal{N}(A)$ given in (2), we obtain two nontrivial linear dependencies of $\mathbf{c}_i$'s:

$$\begin{array}{rcrcrcrcrcl} -2\mathbf{c}_1 & + & \mathbf{c}_2 & + & \mathbf{c}_3 & & & & & = & \mathbf{0}, \\ -\mathbf{c}_1 & - & \mathbf{c}_2 & & & - & \mathbf{c}_4 & + & \mathbf{c}_5 & = & \mathbf{0}, \end{array}$$

respectively. Hence, the column vectors $\mathbf{c}_3$ and $\mathbf{c}_5$ corresponding to the free variables in $A\mathbf{x} = \mathbf{0}$ can be written as

$$\begin{aligned} \mathbf{c}_3 &= 2\mathbf{c}_1 - \mathbf{c}_2, \\ \mathbf{c}_5 &= \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_4. \end{aligned}$$

That is, the column vectors $\mathbf{c}_3, \mathbf{c}_5$ of A are linear combinations of the column vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4$, which correspond to the basic variables in $A\mathbf{x} = \mathbf{0}$. Hence, $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ spans the column space $\mathcal{C}(A)$.

We claim that $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is linearly independent. Let $\tilde{A} = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_4]$ and $\tilde{U} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_4]$ be submatrices of A and U, respectively, where $\mathbf{u}_j$ is the j-th column vector of the reduced row-echelon form U of A obtained in (1):

$$\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and } \tilde{A} = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -1 \\ 0 & -3 & 4 \\ 3 & 6 & -7 \end{bmatrix}.$$

Then clearly $\tilde{U}$ is the reduced row-echelon form of $\tilde{A}$ so that $\mathcal{N}(\tilde{A}) = \mathcal{N}(\tilde{U})$. Since the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ are just the columns of U containing leading 1's, they are linearly independent, by Theorem 3.6, and $\tilde{U}\mathbf{x} = \mathbf{0}$ has only a trivial solution. This means that $\tilde{A}\mathbf{x} = \mathbf{0}$ has also only a trivial solution, so $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is linearly independent. Therefore, it is a basis for the column space $\mathcal{C}(A)$ and $\dim \mathcal{C}(A) = 3 =$ *the number of basic variables*. That is, *the column vectors of A corresponding to the basic variables in* $U\mathbf{x} = \mathbf{0}$ *form a basis for the column space* $\mathcal{C}(A)$. □

In summary, given a matrix A, we first find the (reduced) row-echelon form U of A by Gauss–Jordan elimination. Then

- a basis for $\mathcal{R}(A) = \mathcal{R}(U)$ is the set of nonzero row vectors of U,
- a basis for $\mathcal{N}(A) = \mathcal{N}(U)$ can be found by solving $U\mathbf{x} = \mathbf{0}$,
- for a basis for the column space $\mathcal{C}(A)$, one notices that $\mathcal{C}(U) \neq \mathcal{C}(A)$ in general, since the column space of A is not preserved by Gauss–Jordan elimination. (See Problem 3.16.) However, we have $\dim \mathcal{C}(A) = \dim \mathcal{C}(U)$, and a basis for $\mathcal{C}(A)$ can be formed by selecting the columns in A, not in U, which correspond to the basic variables (or the leading 1's in U).

Alternatively, a basis for the column space $\mathcal{C}(A)$ can also be found with the elementary column operations, which is the same as finding a basis for the row space $\mathcal{R}(A^T)$ of A^T.

Problem 3.13 Let A be the matrix given in Example 3.20. Find the conditions on a, b, c, d so that the vector $\mathbf{x} = (a, b, c, d)$ belongs to $\mathcal{C}(A)$.

Problem 3.14 Find bases for $\mathcal{R}(A)$ and $\mathcal{N}(A)$ of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}.$$

Also find a basis for $\mathcal{C}(A)$ by finding a basis for $\mathcal{R}(A^T)$.

Problem 3.15 Let A and B be two $n \times n$ matrices. Show that $AB = \mathbf{0}$ if and only if the column space of B is a subspace of the nullspace of A.

Problem 3.16 Find an example of a matrix A and its row-echelon form U such that $\mathcal{C}(A) \neq \mathcal{C}(U)$. What is wrong in $\mathcal{C}(A) = \mathcal{R}(A^T) = \mathcal{R}(U^T) = \mathcal{C}(U)$?

3.6 Rank and nullity

The argument in Example 3.20 is so general that it can be used to prove the following theorem, which is one of the most fundamental results in linear algebra. The proof given here is just a repetition of the argument in Example 3.20 in a general case, and so it may be skipped at the reader's discretion.

Theorem 3.15 (The fundamental theorem) *For any $m \times n$ matrix A, the row space and the column space of A have the same dimension; that is,* $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.

Proof: Let $\dim \mathcal{R}(A) = r$ and let U be the reduced row-echelon form of A. Then r is the number of the nonzero row (or column) vectors of U containing leading 1's, which is equal to the number of basic variables in $U\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \mathbf{0}$. We shall prove that the r columns of A corresponding to the leading 1's (or basic variables) form a basis for $\mathcal{C}(A)$, so that $\dim \mathcal{C}(A) = r = \dim \mathcal{R}(A)$.

(1) They are linearly independent: Let $\tilde{A}$ denote the submatrix of A whose columns are those of A corresponding to the r basic variables (or leading 1's) in U, and let $\tilde{U}$ denote the submatrix of U consisting of the r columns containing leading 1's. Then, it is clear that $\tilde{U}$ is the reduced row-echelon form of $\tilde{A}$, so that $\tilde{A}\mathbf{x} = \mathbf{0}$ if and only if $\tilde{U}\mathbf{x} = \mathbf{0}$. However, $\tilde{U}\mathbf{x} = \mathbf{0}$ has only a trivial solution since the columns

of U containing the leading 1's are linearly independent by Theorem 3.6. Therefore, $\tilde{A}\mathbf{x} = \mathbf{0}$ also has only the trivial solution, so the columns of $\tilde{A}$ are linearly independent.

(2) They span $\mathcal{C}(A)$: Note that the columns of A corresponding to the free variables are not contained in $\tilde{A}$, and each of these column vectors of A can be written as a linear combination of the column vectors of $\tilde{A}$ (see Example 3.20). To show this, let $\{\mathbf{c}_{i_1}, \mathbf{c}_{i_2}, \ldots, \mathbf{c}_{i_k}\}$ be the columns of A (not contained in $\tilde{A}$) corresponding to the free variables $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, and let x_{i_j} be any of these free variables. Then, by assigning the value 1 to x_{i_j} and 0 to all the other free variables, one can get a nontrivial solution of

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{0}.$$

When such a solution is substituted into this equation, one can see that the column $\mathbf{c}_{i_j}$ of A corresponding to $x_{i_j} = 1$ is written as a linear combination of the columns of $\tilde{A}$. This can be done for each free variable x_{i_j}, $j = 1, 2, \ldots, k$, so the columns of A corresponding to those free variables are redundant in the spanning set of $\mathcal{C}(A)$. □

Remark: (1) In the proof of Theorem 3.15, once we have shown that the columns in $\tilde{A}$ are linearly independent as in step (1), we may replace step (2) by the following argument: One can easily see that $\dim \mathcal{C}(A) \geq \dim \mathcal{R}(A)$ by Theorem 3.11. On the other hand, since this inequality holds for arbitrary matrices, applying to A^T particularly we get $\dim \mathcal{C}(A^T) \geq \dim \mathcal{R}(A^T)$. Moreover, $\mathcal{C}(A^T) = \mathcal{R}(A)$ and $\mathcal{R}(A^T) = \mathcal{C}(A)$ implies $\dim \mathcal{C}(A) \leq \dim \mathcal{R}(A)$, which means $\dim \mathcal{C}(A) = \dim \mathcal{R}(A)$. This also means that the column vectors of $\tilde{A}$ span $\mathcal{C}(A)$, and so form a basis.

(2) The proof (2) of Theorem 3.15 also shows that the reduced row-echelon form of a system is unique, which was stated on page 10. In fact, if U_1 and U_2 are two reduced row-echelon forms of an $m \times n$ matrix A, then the columns of U_1 and U_2 corresponding to the basic variables (i.e., containing leading 1's) must be the same and of the form $[0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]^T$ by the definition of the reduced row-echelon form. If there are no free variables, then it is quite clear that

$$U_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \\ & \cdots & & 1 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{bmatrix} = U_2.$$

Suppose that there is a free variable. Since $U_1\mathbf{x} = \mathbf{0}$ if and only if $U_2\mathbf{x} = \mathbf{0}$, one can easily check that the columns of U_1 and U_2 corresponding to each free variable must also be the same, so that $U_1 = U_2$.

In summary, the following equalities are now clear from Theorems 3.14 and 3.15:

$$
\begin{aligned}
\dim \mathcal{N}(A) &= \dim \mathcal{N}(U) \\
&= \text{the number of free variables in } U\mathbf{x} = \mathbf{0}. \\
\dim \mathcal{R}(A) &= \dim \mathcal{R}(U) \\
&= \text{the number of nonzero row vectors of } U \\
&= \text{the maximal number of linearly independent} \\
&\quad\ \text{row vectors of } A \\
&= \text{the number of basic variables in } U\mathbf{x} = \mathbf{0} \\
&= \text{the maximal number of linearly independent} \\
&\quad\ \text{column vectors of } A \\
&= \dim \mathcal{C}(A).
\end{aligned}
$$

Definition 3.9 For an $m \times n$ matrix A, the **rank** of A is defined to be the dimension of the row space (or the column space), denoted by rank A.

Clearly, rank $I_n = n$ and rank A = rank A^T. And for an $m \times n$ matrix A, since $\dim \mathcal{R}(A) \le m$ and $\dim \mathcal{C}(A) \le n$, we have the following corollary:

Corollary 3.16 *If A is an $m \times n$ matrix, then* rank $A \le \min\{m, n\}$.

Since $\dim \mathcal{R}(A) = \dim \mathcal{C}(A) =$ rank A is the number of basic variables in $A\mathbf{x} = \mathbf{0}$, and $\dim \mathcal{N}(A)$ = nullity of A is the number of free variables $A\mathbf{x} = \mathbf{0}$, we have the following theorem.

Theorem 3.17 (Rank Theorem) *For any $m \times n$ matrix A,*

$$
\begin{aligned}
\dim \mathcal{R}(A) + \dim \mathcal{N}(A) &= \text{rank } A + \text{nullity of } A &&= n, \\
\dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) &= \text{rank } A + \text{nullity of } A^T &&= m.
\end{aligned}
$$

If $\dim \mathcal{N}(A) = 0$ (or $\mathcal{N}(A) = \{\mathbf{0}\}$), then $\dim \mathcal{R}(A) = n$ (or $\mathcal{R}(A) = \mathbb{R}^n$), which means that A has exactly n linearly independent rows and n linearly independent columns. In particular, if A is a square matrix of order n, then the row vectors are linearly independent if and only if the column vectors are linearly independent. Therefore, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and by Theorem 1.9 we get the following corollary.

Corollary 3.18 *Let A be an $n \times n$ square matrix. Then A is invertible if and only if* rank $A = n$.

Example 3.21 (*Find the rank and the nullity*) For a 4×5 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix},$$

find the rank and the nullity of A.

Solution: Gaussian elimination gives

$$U = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first three nonzero rows containing leading 1's in U form a basis for $\mathcal{R}(U) = \mathcal{R}(A)$. Therefore, rank $A = \dim \mathcal{R}(A) = \dim \mathcal{C}(A) = 3$, the nullity of $A = \dim \mathcal{N}(A) = 5 - \dim \mathcal{R}(A) = 2$. □

Problem 3.17 Find the nullity and the rank of each of the following matrices:

$$(1)\ A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}.$$

For each of the matrices, show that $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$ directly by finding their bases.

Problem 3.18 Show that a system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if rank A = rank $[A\ \mathbf{b}]$, where $[A\ \mathbf{b}]$ denotes the augmented matrix for $A\mathbf{x} = \mathbf{b}$.

Theorem 3.19 *For any two matrices A and B for which AB can be defined,*

(1) $\mathcal{N}(AB) \supseteq \mathcal{N}(B)$,
(2) $\mathcal{N}((AB)^T) \supseteq \mathcal{N}(A^T)$,
(3) $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$,
(4) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$.

Proof: (1) and (2) are clear, since $B\mathbf{x} = \mathbf{0}$ implies $(AB)\mathbf{x} = A(B\mathbf{x}) = \mathbf{0}$.
(3) For an $m \times n$ matrix A and an $n \times p$ matrix B,

$$\begin{aligned} \mathcal{C}(AB) &= \{AB\mathbf{x} : \mathbf{x} \in \mathbb{R}^p\} \\ &\subseteq \{A\mathbf{y} : \mathbf{y} \in \mathbb{R}^n\} = \mathcal{C}(A), \end{aligned}$$

because $B\mathbf{x} \in \mathbb{R}^n$ for any $\mathbf{x} \in \mathbb{R}^p$. (See Example 3.9.)
(4) $\mathcal{R}(AB) = \mathcal{C}((AB)^T) = \mathcal{C}(B^TA^T) \subseteq \mathcal{C}(B^T) = \mathcal{R}(B)$. □

Corollary 3.20 $\mathrm{rank}(AB) \le \min\{\mathrm{rank}\, A,\ \mathrm{rank}\, B\}$.

In some particular cases, the equality holds. In fact, it will be shown later in Theorem 5.25 that for any square matrix A, $\mathrm{rank}(A^TA) = \mathrm{rank}\, A = \mathrm{rank}(AA^T)$. The following problem illustrates another such case.

Problem 3.19 Let A be an invertible square matrix. Show that, for any matrix B, $\mathrm{rank}(AB) = \mathrm{rank}\, B = \mathrm{rank}(BA)$.

Theorem 3.21 *Let A be an $m \times n$ matrix of rank r. Then*

(1) *for every submatrix C of A,* $\mathrm{rank}\, C \le r$, *and*
(2) *the matrix A has at least one $r \times r$ submatrix of rank r; that is, A has an invertible submatrix of order r.*

Proof: (1) Consider an intermediate matrix B which is obtained from A by removing the rows that are not wanted in C. Then clearly $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and hence $\mathrm{rank}\, B \le \mathrm{rank}\, A$. Moreover, since the columns of C are taken from those of B, $\mathcal{C}(C) \subseteq \mathcal{C}(B)$ and $\mathrm{rank}\, C \le \mathrm{rank}\, B$.

(2) Note that one can find r linearly independent row vectors of A, which form a basis for the row space of A. Let B be the matrix whose row vectors consist of these vectors. Then $\mathrm{rank}\, B = r$ and the column space of B must be of dimension r. By taking r linearly independent column vectors of B, one can find an $r \times r$ submatrix C of A with rank r. □

Problem 3.20 Prove that the rank of a matrix is equal to the largest order of its invertible submatrices.

Problem 3.21 For each of the matrices given in Problem 3.17, find an invertible submatrix of the largest order.

3.7 Bases for subspaces

In this section, we introduce two ways of finding bases for $V + W$ and $V \cap W$ of two subspaces V and W of the n-space $\mathbb{R}^n$, and then derive an important relationship between the dimensions of those subspaces in terms of the dimensions of V and W.

Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\ell\}$ be bases for V and W, respectively. Let Q be the $n \times (k + \ell)$ matrix whose columns are those basis vectors:

$$Q = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \, \mathbf{w}_1 \ \cdots \ \mathbf{w}_\ell]_{n\times(k+\ell)}.$$

Theorem 3.22 *Let V and W be two subspaces of $\mathbb{R}^n$, and Q be the matrix defined above.*

(1) $\mathcal{C}(Q) = V + W$, *so that a basis for the column space $\mathcal{C}(Q)$ is a basis for $V + W$.*
(2) $\mathcal{N}(Q)$ *can be identified with $V \cap W$ so that* $\dim(V \cap W) = \dim \mathcal{N}(Q)$.

Proof: (1) It is clear that $\mathcal{C}(Q) = V + W$.

(2) Let $\mathbf{x} = (a_1, \ldots, a_k, b_1, \ldots, b_\ell) \in \mathcal{N}(Q) \subseteq \mathbb{R}^{k+\ell}$. Then

$$Q\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell = \mathbf{0},$$

from which we get

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = -(b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell).$$

If we set

$$\begin{aligned} \mathbf{y} &= a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \\ &= -(b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell), \end{aligned}$$

then $\mathbf{y} \in V \cap W$ since the first right-hand side $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$ is in V as a linear combination of the basis vectors in α and the second right-hand side $-(b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell)$ is in W as a linear combination of the basis vectors in β. That is, to each $\mathbf{x} \in \mathcal{N}(Q)$, there corresponds a vector $\mathbf{y}$ in $V \cap W$.

On the other hand, if $\mathbf{y} \in V \cap W$, then $\mathbf{y}$ can be written in two linear combinations by the bases for V and W separately as

$$\begin{aligned} \mathbf{y} &= a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \in V, \\ \mathbf{y} &= b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell \in W, \end{aligned}$$

for some $a_1, \ldots, a_k$ and $b_1, \ldots, b_\ell$. Let $\mathbf{x} = (a_1, \ldots, a_k, -b_1, \ldots, -b_\ell) \in \mathbb{R}^{k+\ell}$. Then it is quite clear that $Q\mathbf{x} = \mathbf{0}$, i.e., $\mathbf{x} \in \mathcal{N}(Q)$. Therefore, the correspondence of $\mathbf{x}$ in $\mathcal{N}(Q) \subseteq \mathbb{R}^{k+\ell}$ to a vector $\mathbf{y}$ in $V \cap W \subseteq \mathbb{R}^n$ gives us a one-to-one correspondence between the sets $\mathcal{N}(Q)$ and $V \cap W$.

Moreover, if $\mathbf{x}_i$, $i = 1, 2$, correspond to $\mathbf{y}_i$, then one can easily check that $\mathbf{x}_1 + \mathbf{x}_2$ corresponds to $\mathbf{y}_1 + \mathbf{y}_2$, and $k\mathbf{x}_1$ corresponds to $k\mathbf{y}_1$. This means that the two vector spaces $\mathcal{N}(Q)$ and $V \cap W$ can be identified as vector spaces (see Section 4.2 for an exact meaning of this identification). In particular, for a basis for $\mathcal{N}(Q)$, the corresponding set in $V \cap W$ is a basis for $V \cap W$: that is, if the set of vectors

$$\left\{ \begin{aligned} \mathbf{x}_1 &= (a_{11}, \ldots, a_{1k}, b_{11}, \ldots, b_{1\ell}), \\ &\vdots \\ \mathbf{x}_s &= (a_{s1}, \ldots, a_{sk}, b_{s1}, \ldots, b_{s\ell}), \end{aligned} \right.$$

is a basis for $\mathcal{N}(Q)$, then the set of vectors

$$\left\{ \begin{aligned} \mathbf{y}_1 &= a_{11}\mathbf{v}_1 + \cdots + a_{1k}\mathbf{v}_k, \\ &\vdots \\ \mathbf{y}_s &= a_{s1}\mathbf{v}_1 + \cdots + a_{sk}\mathbf{v}_k, \end{aligned} \right. \quad \text{or} \quad \left\{ \begin{aligned} \mathbf{y}_1 &= -(b_{11}\mathbf{w}_1 + \cdots + b_{1\ell}\mathbf{w}_\ell), \\ &\vdots \\ \mathbf{y}_s &= -(b_{s1}\mathbf{w}_1 + \cdots + b_{s\ell}\mathbf{w}_\ell) \end{aligned} \right.$$

is a basis for $V \cap W$, and vice-versa. This implies that

$$\dim \mathcal{N}(Q) = \dim(V \cap W). \qquad \square$$

Note that $\dim(V + W) \neq \dim V + \dim W$, in general. The following theorem gives a relation between them.

Theorem 3.23 *For any subspaces V and W of the n-space $\mathbb{R}^n$,*

$$\dim(V+W)+\dim(V\cap W)=\dim V+\dim W.$$

Proof: Let $\alpha=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ and $\beta=\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_\ell\}$ be bases for V and W, respectively. Let Q be the $n\times(k+\ell)$ matrix whose columns are the previous basis vectors:

$$Q=[\mathbf{v}_1\ \cdots\ \mathbf{v}_k\ \mathbf{w}_1\ \cdots\ \mathbf{w}_\ell]_{n\times(k+\ell)}.$$

Then, by the Rank Theorem and Theorem 3.22, we have

$$k+\ell=\dim\mathcal{C}(Q)+\dim\mathcal{N}(Q)=\dim(V+W)+\dim(V\cap W). \qquad \square$$

In particular, $\dim(V+W)=\dim V+\dim W$ if and only if $V\cap W=\{\mathbf{0}\}$. In this case, $V+W=V\oplus W$.

Example 3.22 (*Find a basis for a subspace*) Let V and W be two subspaces of $\mathbb{R}^5$ with bases

$$\left\{\begin{array}{lcrrrrr} \mathbf{v}_1 &=& (1, & 3, & -2, & 2, & 3),\\ \mathbf{v}_2 &=& (1, & 4, & -3, & 4, & 2),\\ \mathbf{v}_3 &=& (1, & 3, & 0, & 2, & 3), \end{array}\right. \qquad \left\{\begin{array}{lcrrrrr} \mathbf{w}_1 &=& (2, & 3, & -1, & -2, & 9),\\ \mathbf{w}_2 &=& (1, & 5, & -6, & 6, & 1),\\ \mathbf{w}_3 &=& (2, & 4, & 4, & 2, & 8), \end{array}\right.$$

respectively. Find bases for $V+W$ and $V\cap W$.

Solution: The matrix Q takes the following form:

$$Q=[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{w}_1\ \mathbf{w}_2\ \mathbf{w}_3\,]=\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2\\ 3 & 4 & 3 & 3 & 5 & 4\\ -2 & -3 & 0 & -1 & -6 & 4\\ 2 & 4 & 2 & -2 & 6 & 2\\ 3 & 2 & 3 & 9 & 1 & 8 \end{bmatrix}.$$

The Gauss–Jordan elimination gives

$$U=\begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 0\\ 0 & 1 & 0 & -3 & 2 & 0\\ 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From this, one can directly see that $\dim(V+W)=4$, and the columns $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{w}_3$ corresponding to the basic variables in $Q\mathbf{x}=\mathbf{0}$ (or leading 1's in U) form a basis for $\mathcal{C}(Q)=V+W$. Moreover, $\dim\mathcal{N}(Q)=\dim(V\cap W)=2$, corresponding to two free variables x_4 and x_5 in $Q\mathbf{x}=\mathbf{0}$.

To find a basis for $V\cap W$, we solve $U\mathbf{x}=\mathbf{0}$ for $(x_4,x_5)=(1,0)$ and $(x_4,x_5)=(0,1)$ respectively to obtain a basis for $\mathcal{N}(Q)$:

$$\mathbf{x}_1 = (-5, 3, 0, 1, 0, 0) \text{ and } \mathbf{x}_2 = (0, -2, 1, 0, 1, 0).$$

From $Q\mathbf{x}_i = \mathbf{0}$, we obtain two equations:

$$\begin{array}{rcrcrcl} -5\mathbf{v}_1 & + & 3\mathbf{v}_2 & + & \mathbf{w}_1 & = & \mathbf{0}, \\ -2\mathbf{v}_2 & + & \mathbf{v}_3 & + & \mathbf{w}_2 & = & \mathbf{0}. \end{array}$$

Therefore, $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a basis for $V \cap W$, where

$$\mathbf{y}_1 = 5\mathbf{v}_1 - 3\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -2 \\ 9 \end{bmatrix} = \mathbf{w}_1, \quad \mathbf{y}_2 = 2\mathbf{v}_2 - \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ -6 \\ 6 \\ 1 \end{bmatrix} = \mathbf{w}_2.$$

Clearly, one can check

$$\dim(V + W) + \dim(V \cap W) = 4 + 2 = 3 + 3 = \dim V + \dim W. \qquad \square$$

Remark: (Another method for finding bases) Example 3.22 illustrates a method for finding bases for $V+W$ and $V \cap W$ for given subspaces V and W of $\mathbb{R}^n$ by constructing a matrix Q whose columns are basis vectors for V and basis vectors for W. There is another method for finding their bases by constructing a matrix Q whose rows are basis vectors for V and basis vectors for W. In this case, clearly $V + W = \mathcal{R}(Q)$. By finding a basis for the row space $\mathcal{R}(Q)$, one can get a basis for $V + W$.

On the other hand, a basis for $V \cap W$ can be found as follows: Let A be the $k \times n$ matrix whose rows are basis vectors for V, and B the $\ell \times n$ matrix whose rows are basis vectors for W. Then $V = \mathcal{R}(A)$ and $W = \mathcal{R}(B)$. Let $\bar{A}$ denote the matrix A with an additional unknown vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ attached as the bottom row, i.e.,

$$\bar{A} = \begin{bmatrix} A \\ \mathbf{x} \end{bmatrix},$$

and the matrix $\bar{B}$ is defined similarly. Then it is clear that $\mathcal{R}(A) = \mathcal{R}(\bar{A})$ and $\mathcal{R}(B) = \mathcal{R}(\bar{B})$ if and only if $\mathbf{x} \in V \cap W = \mathcal{R}(A) \cap \mathcal{R}(B)$. This means that the row-echelon form of A and that of $\bar{A}$ should be the same via the same Gaussian elimination. Thus, by comparing the row vectors of the row-echelon form of A with those of $\bar{A}$, one can obtain a system of linear equations for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. By the same argument applied to B and $\bar{B}$, one gets another system of linear equations for the same $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Common solutions of these two systems together will provide us a basis for $V \cap W$.

The following example illustrates how one can apply this argument to find bases for $V + W$ and $V \cap W$.

Example 3.23 (*Find a basis for a subspace*) Let V be the subspace of $\mathbb{R}^5$ spanned by

$$\begin{aligned}
\mathbf{v}_1 &= (1,\ 3,\ -2,\ 2,\ 3),\\
\mathbf{v}_2 &= (1,\ 4,\ -3,\ 4,\ 2),\\
\mathbf{v}_3 &= (2,\ 3,\ -1,\ -2,\ 10),
\end{aligned}$$

and W the subspace spanned by

$$\begin{aligned}
\mathbf{w}_1 &= (1,\ 3,\ 0,\ 2,\ 1),\\
\mathbf{w}_2 &= (1,\ 5,\ -6,\ 6,\ 3),\\
\mathbf{w}_3 &= (2,\ 5,\ 3,\ 2,\ 1).
\end{aligned}$$

Find a basis for $V + W$ and for $V \cap W$.

Solution: Note that the matrix A whose row vectors are $\mathbf{v}_i$'s is reduced to a row-echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

so that $\dim V = 3$. Similarly, the matrix B whose row vectors are $\mathbf{w}_j$'s is reduced to a row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so that $\dim W = 2$.

Now, if Q denotes the 6×5 matrix whose row vectors are $\mathbf{v}_i$'s and $\mathbf{w}_j$'s, then $V + W = \mathcal{R}(Q)$. By Gaussian elimination, Q is reduced to a row-echelon form, excluding zero rows:

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the four nonzero row vectors

$$(1,\ 3,\ -2,\ 2,\ 3),\ (0,\ 1,\ -1,\ 2,\ -1),\ (0,\ 0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 0,\ 1)$$

form a basis for $V + W$, so that $\dim(V + W) = 4$.

We now find a basis for $V \cap W$. A vector $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ is contained in $V \cap W$ if and only if $\mathbf{x}$ is contained in both the row space of A and that of B.

Let $\bar{A}$ be A with $\mathbf{x}$ attached as the last row:

$$\bar{A} = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 10 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}.$$

Then by the same Gaussian elimination to reduce A to its row-echelon form, $\bar{A}$ is reduced to

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -x_1+x_2+x_3 & 4x_1-2x_2+x_4 & 0 \end{bmatrix}.$$

Therefore, $\mathbf{x} \in \mathcal{R}(A) = V$ if and only if $\mathcal{R}(A) = \mathcal{R}(\bar{A})$. By comparing the row vectors of the row-echelon form of A with those of $\bar{A}$, one can say that $\mathbf{x} \in \mathcal{R}(A)$ if and only if the last row vector of the row-echelon form of $\bar{A}$ is the zero vector, that is, $\mathbf{x}$ is a solution of the homogeneous system of equations

$$\left\{ \begin{array}{rcrcrcrcl} -x_1 & + & x_2 & + & x_3 & & & = & 0 \\ 4x_1 & - & 2x_2 & & & + & x_4 & = & 0. \end{array} \right.$$

The same calculation with $\bar{B}$ gives another homogeneous system of linear equations for $\mathbf{x}$:

$$\left\{ \begin{array}{rcrcrcrcrcl} -9x_1 & + & 3x_2 & + & x_3 & & & & & = & 0 \\ 4x_1 & - & 2x_2 & & & + & x_4 & & & = & 0 \\ 2x_1 & - & x_2 & & & & & + & x_5 & = & 0. \end{array} \right.$$

Solving these two homogeneous systems together yields

$$V \cap W = \{t(1,\ 4,\ -3,\ 4,\ 2) : t \in \mathbb{R}\}.$$

Hence, $\{(1,\ 4,\ -3,\ 4,\ 2)\}$ is a basis for $V \cap W$ and $\dim(V \cap W) = 1$. $\square$

Problem 3.22 Let V and W be the subspaces of the vector space $P_3(\mathbb{R})$ spanned by

$$\left\{ \begin{array}{lcrcrcrcr} v_1(x) & = & 3 & - & x & + & 4x^2 & + & x^3 \\ v_2(x) & = & 5 & & & + & 5x^2 & + & x^3 \\ v_3(x) & = & 5 & - & 5x & + & 10x^2 & + & 3x^3 \end{array} \right.$$

and

$$\left\{ \begin{array}{lcrcrcrcr} w_1(x) & = & 9 & - & 3x & + & 3x^2 & + & 2x^3 \\ w_2(x) & = & 5 & - & x & + & 2x^2 & + & x^3 \\ w_3(x) & = & 6 & & & + & 4x^2 & + & x^3 \end{array} \right.$$

respectively. Find the dimensions and bases for $V + W$ and $V \cap W$.

Problem 3.23 Let

$$\begin{array}{lcl} V & = & \left\{(x, y, z, u) \in \mathbb{R}^4 \ : \ y + z + u = 0\right\}, \\ W & = & \left\{(x, y, z, u) \in \mathbb{R}^4 \ : \ x + y = 0,\ z = 2u\right\} \end{array}$$

be two subspaces of $\mathbb{R}^4$. Find bases for V, W, $V + W$, and $V \cap W$.

3.8 Invertibility

In Chapter 1, we have seen that a non-square matrix A may have only one-sided (right or left) inverses. In this section, it will be shown that the existence of a one-sided inverse (right or left) of A implies the existence or the uniqueness of the solutions of a system $A\mathbf{x} = \mathbf{b}$.

Theorem 3.24 (Existence) *Let A be an $m \times n$ matrix. Then the following statements are equivalent.*

(1) *For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has* at least *one solution $\mathbf{x}$ in $\mathbb{R}^n$.*
(2) *The column vectors of A span $\mathbb{R}^m$, i.e., $\mathcal{C}(A) = \mathbb{R}^m$.*
(3) rank $A = m$ *(hence $m \le n$).*
(4) *A has a right inverse (i.e., B such that $AB = I_m$).*

Proof: (1) $\Leftrightarrow$ (2): In general, $\mathcal{C}(A) \subseteq \mathbb{R}^m$. For any $\mathbf{b} \in \mathbb{R}^m$, there is a solution $\mathbf{x} \in \mathbb{R}^n$ of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b}$ is a linear combination of the column vectors of A, i.e., $\mathbf{b} \in \mathcal{C}(A)$. Thus $\mathbb{R}^m = \mathcal{C}(A)$.

(2) $\Leftrightarrow$ (3): $\mathcal{C}(A) = \mathbb{R}^m$ if and only if $\dim \mathcal{C}(A) = m \le n$ (see Problem 3.10). But $\dim \mathcal{C}(A) = \text{rank } A = \dim \mathcal{R}(A) \le \min\{m, n\}$,

(1) $\Rightarrow$ (4): Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ be the standard basis for $\mathbb{R}^m$. Then for each $\mathbf{e}_i$, one can find an $\mathbf{x}_i \in \mathbb{R}^n$ such that $A\mathbf{x}_i = \mathbf{e}_i$ by the hypothesis (1). If B is the $n \times m$ matrix whose columns are these $\mathbf{x}_i$'s: i.e., $B = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m]$, then, by matrix multiplication,

$$AB = A\,[\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m] = [\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_m] = I_m.$$

(4) $\Rightarrow$ (1): If B is a right inverse of A, then for any $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} = B\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. $\square$

Condition (2) means that A has m linearly independent column vectors, and condition (3) implies that there exist m linearly independent row vectors of A, since rank $A = m = \dim \mathcal{R}(A)$.

Note that if $\mathcal{C}(A) \subsetneq \mathbb{R}^m$, then $A\mathbf{x} = \mathbf{b}$ has no solution for $\mathbf{b} \notin \mathcal{C}(A)$.

Theorem 3.25 (Uniqueness) *Let A be an $m \times n$ matrix. Then the following statements are equivalent.*

(1) *For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has* at most *one solution $\mathbf{x}$ in $\mathbb{R}^n$.*
(2) *The column vectors of A are linearly independent.*
(3) $\dim \mathcal{C}(A) = \text{rank } A = n$ *(hence $n \le m$).*
(4) $\mathcal{R}(A) = \mathbb{R}^n$.
(5) $\mathcal{N}(A) = \{\mathbf{0}\}$.
(6) *A has a left inverse (i.e., C such that $CA = I_n$).*

Proof: (1) $\Rightarrow$ (2): Note that the column vectors of A are linearly independent if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. However,

$A\mathbf{x} = \mathbf{0}$ has always the trivial solution $\mathbf{x} = \mathbf{0}$, and the statement (1) implies that it is the only one.

(2) $\Leftrightarrow$ (3): Clear, because all the column vectors are linearly independent if and only if they form a basis for $\mathcal{C}(A)$, or $\dim \mathcal{C}(A) = n \leq m$.

(3) $\Leftrightarrow$ (4): Clear, because $\dim \mathcal{R}(A) = \text{rank } A = \dim \mathcal{C}(A) = n$ if and only if $\mathcal{R}(A) = \mathbb{R}^n$ (see Problem 3.10).

(4) $\Leftrightarrow$ (5): Clear, since $\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n$.

(2) $\Rightarrow$ (6): Suppose that the columns of A are linearly independent so that rank $A = n$. Extend these column vectors of A to a basis for $\mathbb{R}^m$ by adding $m-n$ additional independent vectors to them. Construct an $m \times m$ matrix S with those basis vectors in its columns. Then the matrix S has rank m, and hence it is invertible. Let C be the $n \times m$ matrix obtained from S^{-1} by throwing away the last $m - n$ rows. Since the first n columns of S constitute the matrix A, we have $CA = I_n$.

$$I_m = S^{-1}S = \begin{bmatrix} & C_{n\times m} & \\ \text{-} & \text{-----} & \text{-} \\ * & * & * \end{bmatrix} \begin{bmatrix} & \mid & * \\ A_{m\times n} & \mid & * \\ & \mid & * \end{bmatrix} = \begin{bmatrix} I_n & \mid & \mathbf{0} \\ - & - & - \\ \mathbf{0} & \mid & I \end{bmatrix}.$$

(6) $\Rightarrow$ (1): Let C be a left inverse of A. If $A\mathbf{x} = \mathbf{b}$ has no solution, then we are done. Suppose that $A\mathbf{x} = \mathbf{b}$ has two solutions, say $\mathbf{x}_1$ and $\mathbf{x}_2$. Then

$$\mathbf{x}_1 = CA\mathbf{x}_1 = C\mathbf{b} = CA\mathbf{x}_2 = \mathbf{x}_2.$$

Hence, the system can have at most one solution. □

Remark: **(1)** We have proved that an $m\times n$ matrix A has a right inverse if and only if rank $A = m$, while A has a left inverse if and only if rank $A = n$. Therefore, if $m \neq n$, A cannot have both left and right inverses.

(2) For a practical way of finding a right or a left inverse of an $m\times n$ matrix A, we will show later (see Remark (1) below Theorem 5.26) that if rank $A = m$, then $(AA^T)^{-1}$ exists and $A^T(AA^T)^{-1}$ is a right inverse of A, and if rank $A = n$, then $(A^TA)^{-1}$ exists and $(A^TA)^{-1}A^T$ is a left inverse of A (see Theorem 5.26).

(3) Note that if $m = n$ so that A is a square matrix, then A has a right inverse (and a left inverse) if and only if rank $A = m = n$. Moreover, in this case the inverses are the same (see Theorem 1.9). Therefore, a square matrix A has rank n if and only if A is invertible. This means that for a square matrix "Existence = Uniqueness," and the ten statements listed in Theorems 3.24–3.25 are all equivalent. In particular, for the invertibility of a square matrix it is enough to show the existence of a one-sided inverse.

Problem 3.24 For each of the following matrices, find all vectors $\mathbf{b}$ such that the system of linear equations $A\mathbf{x} = \mathbf{b}$ has at least one solution. Also, discuss the uniqueness of the solution.

$$(1)\ A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 2 & 3 \\ 3 & -7 \\ -6 & 1 \end{bmatrix},$$

$$(3)\ A = \begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}, \quad (4)\ A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 1 & 2 & -2 \end{bmatrix}.$$

Summarizing all the results obtained so far about solvability of a system, one can obtain several characterizations of the invertibility of a square matrix. The following theorem is a collection of the results proved in Theorems 1.9, 3.24, and 3.25.

Theorem 3.26 *For a square matrix A of order n, the following statements are equivalent.*

(1) *A is invertible.*
(2) $\det A \neq 0$.
(3) *A is row equivalent to I_n.*
(4) *A is a product of elementary matrices.*
(5) *Elimination can be completed: $PA = LDU$, with all $d_i \neq 0$.*
(6) *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^n$.*
(7) *$A\mathbf{x} = \mathbf{0}$ has only a trivial solution, i.e., $\mathcal{N}(A) = \{\mathbf{0}\}$.*
(8) *The columns of A are linearly independent.*
(9) *The columns of A span $\mathbb{R}^n$, i.e., $\mathcal{C}(A) = \mathbb{R}^n$.*
(10) *A has a left inverse.*
(11) $\operatorname{rank} A = n$.
(12) *The rows of A are linearly independent.*
(13) *The rows of A span $\mathbb{R}^n$, i.e., $\mathcal{R}(A) = \mathbb{R}^n$.*
(14) *A has a right inverse.*
(15)* *The linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ via $A(\mathbf{x}) = A\mathbf{x}$ is injective.*
(16)* *The linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is surjective.*
(17)* *Zero is not an eigenvalue of A.*

Proof: Exercise; where have we proved which claim? Prove any not covered. The numbers with asterisks will be explained in the following places: (15) and (16) in Remark on page 133 and (17) in Theorem 6.1. □

3.9 Applications

3.9.1 Interpolation

In many scientific experiments, a scientist wants to find the precise functional relationship between input data and output data. That is, in his experiment, he puts various input values into his experimental device and obtains output values corresponding to those input values. After his experiment, what he has is a table of inputs and outputs. The precise functional relationship might be very complicated, and sometimes it might be very hard or almost impossible to find the precise function. In this case, one thing he can do is to find a polynomial whose graph passes through each of the data points and comes very close to the function he wanted to find. That is, he is looking for a polynomial that approximates the precise function. Such a polynomial is called an **interpolating polynomial**. This problem is closely related to systems of linear equations.

Let us begin with a set of given data: Suppose that for $n+1$ distinct experimental input values $x_0, x_1, \ldots, x_n$, we obtained $n+1$ output values $y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)$. The output values are supposed to be related to the inputs by a certain (unknown) function f. We wish to construct a polynomial $p(x)$ of degree less than or equal to n which interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$: i.e., $p(x_i) = y_i = f(x_i)$ for $i = 0, 1, \ldots, n$.

Note that if there is such a polynomial, it must be unique. Indeed, if $q(x)$ is another such polynomial, then $h(x) = p(x) - q(x)$ is also a polynomial of degree less than or equal to n vanishing at $n+1$ distinct points $x_0, x_1, \ldots, x_n$. Hence $h(x)$ must be the identically zero polynomial so that $p(x) = q(x)$ for all $x \in \mathbb{R}$.

To find such a polynomial $p(x)$, let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

with $n+1$ unknowns a_i's. Then,

$$p(x_i) = a_0 + a_1 x_i + \cdots + a_n x_i^n = y_i = f(x_i),$$

for $i = 0, 1, \ldots, n$. In matrix notation,

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

The coefficient matrix A is a square matrix of order $n+1$, known as **Vandermonde's matrix** (see Example 2.10), whose determinant is

$$\det A = \prod_{0 \le i < j \le n} (x_j - x_i).$$

Since the x_i's are all distinct, $\det A \neq 0$. Hence, A is nonsingular, and $A\mathbf{x} = \mathbf{b}$ has a unique solution which determines the unique polynomial $p(x)$ of degree $\le n$ passing through the given $n+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ in the plane $\mathbb{R}^2$.

Example 3.24 (*Finding an interpolating polynomial*) Given four points

$$(0, 3),\ (1, 0),\ (-1, 2),\ (3, 6)$$

in the plane $\mathbb{R}^2$, let $p(x) = a_0+a_1x+a_2x^2+a_3x^3$ be the polynomial passing through the given four points. Then, we have a system of equations

$$\begin{cases} a_0 & & & & & & & = & 3 \\ a_0 & + & a_1 & + & a_2 & + & a_3 & = & 0 \\ a_0 & - & a_1 & + & a_2 & - & a_3 & = & 2 \\ a_0 & + & 3a_1 & + & 9a_2 & + & 27a_3 & = & 6. \end{cases}$$

Solving this system, one can get $a_0 = 3$, $a_1 = -2$, $a_2 = -2$, $a_3 = 1$, and the unique polynomial is $p(x) = 3 - 2x - 2x^2 + x^3$. $\square$

Problem 3.25 Let $f(x) = \sin x$. Then at $x = 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \pi$, the values of f are $y = 0, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 0$. Find the polynomial $p(x)$ of degree ≤ 4 that passes through these five points. (One may need to use a computer to avoid messy computation.)

Problem 3.26 Find a polynomial $p(x) = a + bx + cx^2 + dx^3$ that satisfies $p(0) = 1$, $p'(0) = 2$, $p(1) = 4$, $p'(1) = 4$.

Problem 3.27 Find the equation of a circle that passes through the three points $(2, -2)$, $(3, 5)$, and $(-4, 6)$ in the plane $\mathbb{R}^2$.

Remark: Note that the interpolating polynomial $p(x)$ of degree $\leq n$ is uniquely determined when we have the correct data, i.e., when we are given precisely $n + 1$ values of y at $n + 1$ distinct points $x_0, x_1, \ldots, x_n$.

However, if we are given fewer data, then the polynomial is under-determined: i.e., if we have m values of y with $m < n + 1$ at m distinct points $x_1, x_2, \ldots, x_m$, then there are as many interpolating polynomials as the null space of A since in this case A is an $m \times (n + 1)$ matrix with $m < n + 1$. (See the Existence Theorem 3.24.)

On the other hand, if we are given more than $n + 1$ data, then the polynomial is over-determined: i.e., if we have m values of y with $m > n + 1$ at m distinct points $x_1, x_2, \ldots, x_m$, then there may not exist an interpolating polynomial since the system could be inconsistent. (See the Uniqueness Theorem 3.25.) In this case, the best one can do is to find a polynomial of degree $\leq n$ to which the data is closest, called the *least square solution*. It will be reviewed again in Sections 5.9–5.9.2.

3.9.2 The Wronskian

Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ be n vectors in an m-dimensional vector space V. To check the independence of the vector $\mathbf{y}_i$'s, consider its linear dependence:

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{0}.$$

Let $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a basis for V. By expressing each $\mathbf{y}_i$ as a linear combination of the basis vectors $\mathbf{x}_i$'s: i.e., $\mathbf{y}_j = \sum_{i=1}^{n} a_{ij}\mathbf{x}_i$, the linear dependence of $\mathbf{y}_i$'s can be written as a linear combination of the basis vectors $\mathbf{x}_i$'s:

$$\begin{aligned}\mathbf{0} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n &= (a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n)\mathbf{x}_1 \\ &\quad + (a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n)\mathbf{x}_2 \\ &\quad \vdots \\ &\quad + (a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n)\mathbf{x}_m,\end{aligned}$$

so that all of the coefficients (which are also linear combinations of c_i's) must be zero. It gives a homogeneous system of linear equations in c_i's, say $A\mathbf{c} = \mathbf{0}$ with an $m \times n$ matrix A, as in the proof of Lemma 3.9:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Recall that the vectors $\mathbf{y}_i$'s are linearly independent if and only if the system $A\mathbf{c} = \mathbf{0}$ has only the trivial solution. Hence, the linear independence of a set of vectors in a finite dimensional vector space can be tested by solving a homogeneous system of linear equations.

If V is not finite dimensional, this test for the linear independence of a set of vectors cannot be applied. In this section, we introduce a test for the linear independence of a set of functions. For our purpose, let V be the vector space of all functions on $\mathbb{R}$ which are differentiable infinitely many times. Then one can easily see that V is an infinite dimensional vector space.

Let $f_1, f_2, \ldots, f_n$ be n functions in V. The n functions are *linearly independent* in V if

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = \mathbf{0}$$

implies that all $c_i = 0$. Note that the zero function $\mathbf{0}$ takes its value zero at all the points in the domain. Thus they are linearly independent if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in \mathbb{R}$ implies that all $c_i = 0$. By taking a differentiations $n-1$ times, one can obtain n equations:

$$c_1 f_1^{(i)}(x) + c_2 f_2^{(i)}(x) + \cdots + c_n f_n^{(i)}(x) = 0, \quad 0 \le i \le n-1,$$

for all $x \in \mathbb{R}$. Or, in matrix form:

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The determinant of the coefficient matrix is called the **Wronskian** for $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ and denoted by $W(x)$. Therefore, if there is a point $x_0 \in \mathbb{R}$ such that $W(x_0) \neq 0$, then the coefficient matrix is nonsingular at $x = x_0$, and so all $c_i = 0$. Therefore,

> *if the Wronskian $W(x) \neq 0$ for at least one $x \in \mathbb{R}$, then $f_1, f_2, \ldots, f_n$ are linearly independent.*

However, the Wronskian $W(x) = 0$ for all x does not imply linear dependence of the given functions f_i's. In fact, $W(x) = 0$ means that the functions are linearly dependent at each point $x \in \mathbb{R}$, but the constants c_i's giving nontrivial linear dependence may vary as x varies in the domain. (See Example 3.25 (2).)

Example 3.25 (*Test the linear independence of functions by Wronskian*)

(1) For the sets of functions $F_1 = \{x, \cos x, \sin x\}$ and $F_2 = \{x, e^x, e^{-x}\}$, the Wronskians are

$$W_1(x) = \det \begin{bmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} = x$$

and

$$W_2(x) = \det \begin{bmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{bmatrix} = 2x.$$

Since $W_i(x) \neq 0$ for $x \neq 0$, both F_1 and f_2 are linearly independent.

(2) For the set of functions $\{x|x|,\ x^2\}$ on $\mathbb{R}$, the Wronskian for them is

$$W(x) = \det \begin{bmatrix} x|x| & x^2 \\ 2|x| & 2x \end{bmatrix} = 0$$

for all x. These two functions are linearly dependent on each of $(-\infty, 0]$ and $[0, \infty)$, since $x|x| = -x^2$ on $(-\infty, 0]$ and $x|x| = x^2$ on $[0, \infty)$. But they are clearly linearly independent functions on $\mathbb{R}$. □

Problem 3.28 Show that $1, x, x^2, \ldots, x^n$ are linearly independent in the vector space $C(\mathbb{R})$ of continuous functions.

3.10 Exercises

3.1. Let V be the set of all pairs $(x,\ y)$ of real numbers. Define

$$\begin{aligned} (x,\ y) + (x_1,\ y_1) &= (x + x_1,\ y + y_1) \\ k(x,\ y) &= (kx,\ y). \end{aligned}$$

Is V a vector space with these operations?

3.2. For $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ and $k \in \mathbb{R}$, define two operations as

$$\mathbf{x} \oplus \mathbf{y} = \mathbf{x} - \mathbf{y}, \qquad k \cdot \mathbf{x} = -k\mathbf{x}.$$

The operations on the right-hand sides are the usual ones. Which of the rules in the definition of a vector space are satisfied for $(\mathbb{R}^n,\ \oplus,\ \cdot)$?

3.3. Determine whether the given set is a vector space with the usual addition and scalar multiplication of functions.

(1) The set of all continuous functions f defined on the interval $[-1,\ 1]$ such that $f(0) = 1$.

(2) The set of all continuous functions f defined on the real line $\mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$.

(3) The set of all twice differentiable functions f defined on $\mathbb{R}$ such that $f''(x) + f(x) = 0$.

3.4. Let $C^2[-1,\ 1]$ be the vector space of all functions with continuous second derivatives on the domain $[-1,\ 1]$. Which of the following subsets is a subspace of $C^2[-1,\ 1]$?

(1) $W = \{f(x) \in C^2[-1,\ 1] : f''(x) + f(x) = 0,\ -1 \le x \le 1\}$.

(2) $W = \{f(x) \in C^2[-1,\ 1] : f''(x) + f(x) = x^2,\ -1 \le x \le 1\}$.

3.5. Which of the following subsets of $C[-1,\ 1]$ is a subspace of the vector space $C[-1,\ 1]$ of continuous functions on $[-1,\ 1]$?

(1) $W = \{f(x) \in C[-1,\ 1] : f(-1) = -f(1)\}$.

(2) $W = \{f(x) \in C[-1,\ 1] : f(x) \ge 0$ for all x in $[-1,\ 1]\}$.

(3) $W = \{f(x) \in C[-1,\ 1] : f(-1) = -2$ and $f(1) = 2\}$.

(4) $W = \{f(x) \in C[-1,\ 1] : f(\frac{1}{2}) = 0\}$.

3.6. Show that the set of all matrices of the form $AB - BA$ cannot span the vector space $M_{n\times n}(\mathbb{R})$.

3.7. Does the vector $(3,\ -1,\ 0,\ -1)$ belong to the subspace of $\mathbb{R}^4$ spanned by the vectors $(2,\ -1,\ 3,\ 2)$, $(-1,\ 1,\ 1,\ -3)$ and $(1,\ 1,\ 9,\ -5)$?

3.8. Express the given function as a linear combination of functions in the given set Q.

(1) $p(x) = -1 - 3x + 3x^2$ and $Q = \{p_1(x),\ p_2(x),\ p_3(x)\}$, where $p_1(x) = 1 + 2x + x^2,\ p_2(x) = 2 + 5x,\ p_3(x) = 3 + 8x - 2x^2$.

(2) $p(x) = -2 - 4x + x^2$ and $Q = \{p_1(x),\ p_2(x),\ p_3(x),\ p_4(x)\}$, where $p_1(x) = 1 + 2x^2 + x^3,\ p_2(x) = 1 + x + 2x^3,\ p_3(x) = -1 - 3x - 4x^3,\ p_4(x) = 1 + 2x - x^2 + x^3$.

3.9. Is $\{\cos^2 x,\ \sin^2 x,\ 1,\ e^x\}$ linearly independent in the vector space $C(\mathbb{R})$?

3.10. In the n-space $\mathbb{R}^n$, determine whether or not the set

$$\{\mathbf{e}_1 - \mathbf{e}_2,\ \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n,\ \mathbf{e}_n - \mathbf{e}_1\}$$

is linearly dependent.

3.11. Show that the given sets of functions are linearly independent in the vector space $C[-\pi,\ \pi]$.

(1) $\{1,\ x,\ x^2,\ x^3,\ x^4\}$

(2) $\{1,\ e^x,\ e^{2x},\ e^{3x}\}$

(3) $\{1,\ \sin x,\ \cos x, \ldots, \sin kx,\ \cos kx\}$

3.12. Are the vectors

$$\mathbf{v}_1 = (1,\ 1,\ 2,\ 4), \qquad \mathbf{v}_2 = (2,\ -1,\ -5,\ 2),$$
$$\mathbf{v}_3 = (1,\ -1,\ -4,\ 0), \quad \mathbf{v}_4 = (2,\ 1,\ 1,\ 6)$$

linearly independent in the 4-space $\mathbb{R}^4$?

3.13. In the 3-space $\mathbb{R}^3$, let W be the set of all vectors (x_1, x_2, x_3) that satisfy the equation $x_1 - x_2 - x_3 = 0$. Prove that W is a subspace of $\mathbb{R}^3$. Find a basis for the subspace W.

3.14. Let W be the subspace of $C[-\pi,\ \pi]$ consisting of functions of the form $f(x) = a\sin x + b\cos x$. Determine the dimension of W.

3.15. Let V denote the set of all infinite sequences of real numbers:

$$V = \{\mathbf{x} : \mathbf{x} = \{x_i\}_{i=1}^{\infty}, x_i \in \mathbb{R}\}.$$

If $\mathbf{x} = \{x_i\}$ and $\mathbf{y} = \{y_i\}$ are in V, then $\mathbf{x} + \mathbf{y}$ is the sequence $\{x_i + y_i\}_{i=1}^{\infty}$. If c is a real number, then $c\mathbf{x}$ is the sequence $\{cx_i\}_{i=1}^{\infty}$.

(1) Prove that V is a vector space.

(2) Prove that V is not finite dimensional.

3.16. For two matrices A and B for which AB can be defined, prove the following statements:

(1) If both A and B have linearly independent column vectors, then the column vectors of AB are also linearly independent.

(2) If both A and B have linearly independent row vectors, then the row vectors of AB are also linearly independent.

(3) If the column vectors of B are linearly dependent, then the column vectors of AB are also linearly dependent.

(4) If the row vectors of A are linearly dependent, then the row vectors of AB are also linearly dependent.

3.17. Let $U = \{(x, y, z) \ : \ 2x + 3y + z = 0\}$ and $V = \{(x, y, z) \ : \ x + 2y - z = 0\}$ be subspaces of $\mathbb{R}^3$.

(1) Find a basis for $U \cap V$.

(2) Determine the dimension of $U + V$.

(3) Describe $U,\ V,\ U \cap V$ and $U + V$ geometrically.

3.18. How many 5×5 permutation matrices are there? Are they linearly independent? Do they span the vector space $M_{5\times 5}(\mathbb{R})$?

3.19. Find bases for the row space, the column space, and the null space for each of the following matrices.

$$(1)\ A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix}, \quad (2)\ B = \begin{bmatrix} 0 & 2 & 1 & -5 \\ 1 & 1 & -2 & 2 \\ 1 & 5 & 0 & 0 \end{bmatrix},$$

$$(3)\ C = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{bmatrix}, \quad (4)\ D = \begin{bmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{bmatrix}.$$

3.20. Find the rank of A as a function of x: $A = \begin{bmatrix} 2 & 2 & -6 & 8 \\ 3 & 3 & -9 & 8 \\ 1 & 1 & x & 4 \end{bmatrix}$.

3.21. Find the rank and the largest invertible submatrix of each of the following matrices.

$$(1) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2) \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 4 \end{bmatrix}, \quad (3) \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 4 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

3.22. For any nonzero column vectors $\mathbf{u}$ and $\mathbf{v}$, show that the matrix $A = \mathbf{u}\mathbf{v}^T$ has rank 1. Conversely, every matrix of rank 1 can be written as $\mathbf{u}\mathbf{v}^T$ for some vectors $\mathbf{u}$ and $\mathbf{v}$.

3.23. Determine whether the following statements are true or false, and justify your answers.

(1) The set of all $n \times n$ matrices A such that $A^T = A^{-1}$ is a subspace of the vector space $M_{n\times n}(\mathbb{R})$.

(2) If α and β are linearly independent subsets of a vector space V, so is their union $\alpha \cup \beta$.

(3) If U and W are subspaces of a vector space V with bases α and β respectively, then the intersection $\alpha \cap \beta$ is a basis for $U \cap W$.

(4) Let U be the row-echelon form of a square matrix A. If the first r columns of U are linearly independent, so are the first r columns of A.

(5) Any two row-equivalent matrices have the same column space.

(6) Let A be an $m \times n$ matrix with rank m. Then the column vectors of A span $\mathbb{R}^m$.

(7) Let A be an $m \times n$ matrix with rank n. Then $A\mathbf{x} = \mathbf{b}$ has at most one solution.

(8) If U is a subspace of V and $\mathbf{x}, \mathbf{y}$ are vectors in V such that $\mathbf{x} + \mathbf{y}$ is contained in U, then $\mathbf{x} \in U$ and $\mathbf{y} \in U$.

(9) Let U and V are vector spaces. Then U is a subspace of V if and only if $\dim U \leq \dim V$.

(10) For any $m \times n$ matrix A, $\dim \mathcal{C}(A^T) + \dim \mathcal{N}(A^T) = m$.

4

Linear Transformations

4.1 Basic properties of linear transformations

As shown in Chapter 3, there are many different vector spaces even with the same dimension. The question now is how one can determine whether or not two given vector spaces have the 'same' structure as vector spaces, or can be identified as the same vector space. To answer the question, one has to compare them first as sets, and then see whether their arithmetic rules are the same or not. A usual way of comparing two sets is to define a *function* between them. When a function f is given between the underlying sets of vector spaces, one can compare the arithmetic rules of the vector spaces by examining whether the function f preserves two algebraic operations: the vector addition and the scalar multiplication, that is, $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ and $f(k\mathbf{x}) = kf(\mathbf{x})$ for any vectors $\mathbf{x}, \mathbf{y}$ and any scalar k. In this chapter, we discuss this kind of functions between vector spaces.

Definition 4.1 Let V and W be vector spaces. A function $T : V \to W$ is called a **linear transformation** from V to W if for all $\mathbf{x},\ \mathbf{y} \in V$ and scalar k the following conditions hold:

(1) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$,
(2) $T(k\mathbf{x}) = kT(\mathbf{x})$.

We often call T simply **linear**. It is easy to see that the two conditions for a linear transformation can be combined into a single requirement

$$T(\mathbf{x}+k\mathbf{y}) = T(\mathbf{x}) + kT(\mathbf{y}).$$

Geometrically, the linearity is just the requirement for a straight line to be transformed to a straight line, since $\mathbf{x}+k\mathbf{y}$ represents a straight line through $\mathbf{x}$ in the direction $\mathbf{y}$ in V, and its image $T(\mathbf{x}) + kT(\mathbf{y})$ also represents a straight line through $T(\mathbf{x})$ in the direction of $T(\mathbf{y})$ in W.

Example 4.1 (*Linear or not*) Consider the following functions:

(1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$;
(2) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2 - x$;
(3) $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x,\ y) = (x - y,\ 2x)$;
(4) $k : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $k(x,\ y) = (xy,\ x^2 + 1)$.

One can easily see that g and k are not linear, while f and h are linear. Moreover, on the 1-space $\mathbb{R}$, all polynomials of degree greater than one are not linear. □

Example 4.2 (*A matrix A as a linear transformation*)

(1) For an $m \times n$ matrix A, the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by the matrix product

$$T(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation by the distributive law $A(\mathbf{x} + k\mathbf{y}) = A\mathbf{x} + kA\mathbf{y}$ for any $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ and for any scalar $k \in \mathbb{R}$. Therefore, a matrix A, identified with the transformation T, may be considered as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(2) For a vector space V, the **identity transformation** $id : V \to V$ is defined by $id(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$. If W is another vector space, the **zero transformation** $T_0 : V \to W$ is defined by $T_0(\mathbf{x}) = \mathbf{0}$ (the zero vector) for all $\mathbf{x} \in V$. Clearly, both transformations are linear. □

The following theorem is a direct consequence of the definition, and the proof is left for an exercise.

Theorem 4.1 *Let $T : V \to W$ be a linear transformation. Then*

(1) $T(\mathbf{0}) = \mathbf{0}$.
(2) *For any* $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n \in V$ *and scalars* $k_1,\ k_2,\ \ldots,\ k_n$,

$$T(k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n) = k_1T(\mathbf{x}_1) + k_2T(\mathbf{x}_2) + \cdots + k_nT(\mathbf{x}_n).$$

Nontrivial important examples of linear transformations are the rotations, reflections, and projections in a geometry.

Example 4.3 (*The rotations, reflections and projections in a geometry*)

(1) Let θ denote the angle between the x-axis and a fixed vector in $\mathbb{R}^2$. Then the matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

defines a linear transformation on $\mathbb{R}^2$ that rotates any vector in $\mathbb{R}^2$ through the angle θ about the origin. It is called a **rotation** by the angle θ.

(2) The **projection** on the x-axis is the linear transformation $P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by, for $\mathbf{x} = (x, y) \in \mathbb{R}^2$,

$$P(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

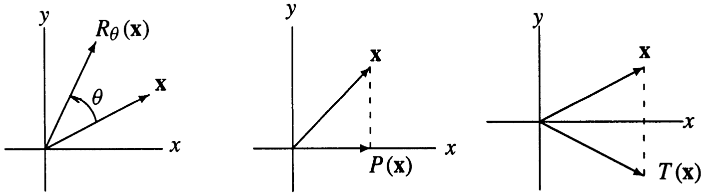

Figure 4.1. Three linear transformations on $\mathbb{R}^2$

(3) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by, for $\mathbf{x} = (x, y)$,

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

is the **reflection** about the x-axis. □

Problem 4.1 Find the matrix of the reflection about the line $y = x$ in the plane $\mathbb{R}^2$.

Example 4.4 (*Differentiations and integrations in calculus*) In calculus, it is well known that two transformations

$$D : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R}), \quad \mathcal{I} : P_{n-1}(\mathbb{R}) \to P_n(\mathbb{R})$$

defined by differentiation and integration,

$$D(f)(x) = f'(x), \quad \mathcal{I}(f)(x) = \int_0^x f(t)dt,$$

satisfy linearity, and so they are linear transformations. Many problems related with differential and integral equations may be reformulated in terms of linear transformations. □

Definition 4.2 Let V and W be two vector spaces, and let $T : V \to W$ be a linear transformation from V into W.

(1) $\text{Ker}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\} \subseteq V$ is called the **kernel** of T.
(2) $\text{Im}(T) = \{T(\mathbf{v}) \in W : \mathbf{v} \in V\} = T(V) \subseteq W$ is called the **image** of T.

Example 4.5 Let V and W be vector spaces and let $id : V \to V$ and $T_0 : V \to W$ be the identity and the zero transformations, respectively. Then it is easy to see that $\text{Ker}(id) = \{\mathbf{0}\}$, $\text{Im}(id) = V$, $\text{Ker}(T_0) = V$ and $\text{Im}(T_0) = \{\mathbf{0}\}$. □

Theorem 4.2 *Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Then the kernel* $\text{Ker}(T)$ *and the image* $\text{Im}(T)$ *are subspaces of V and W, respectively.*

Proof: Since $T(\mathbf{0}) = \mathbf{0}$, each of $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ is nonempty having $\mathbf{0}$.

(1) For any $\mathbf{x}, \mathbf{y} \in \mathrm{Ker}(T)$ and for any scalar k,

$$T(\mathbf{x} + k\mathbf{y}) = T(\mathbf{x}) + kT(\mathbf{y}) = \mathbf{0} + k\mathbf{0} = \mathbf{0}.$$

Hence $\mathbf{x} + k\mathbf{y} \in \mathrm{Ker}(T)$, so that $\mathrm{Ker}(T)$ is a subspace of V.

(2) If $\mathbf{v}, \mathbf{w} \in \mathrm{Im}(T)$, then there exist $\mathbf{x}$ and $\mathbf{y}$ in V such that $T(\mathbf{x}) = \mathbf{v}$ and $T(\mathbf{y}) = \mathbf{w}$. Thus, for any scalar k,

$$\mathbf{v} + k\mathbf{w} = T(\mathbf{x}) + kT(\mathbf{y}) = T(\mathbf{x} + k\mathbf{y}).$$

Thus $\mathbf{v} + k\mathbf{w} \in \mathrm{Im}(T)$, so that $\mathrm{Im}(T)$ is a subspace of W. □

Example 4.6 ($\mathrm{Ker}(A) = \mathcal{N}(A)$ *and* $\mathrm{Im}(A) = \mathcal{C}(A)$ *for any matrix* A) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by an $m \times n$ matrix A as in Example 4.2(1). The kernel $\mathrm{Ker}(A)$ of A consists of all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Hence, the kernel $\mathrm{Ker}(A)$ of A is nothing but the null space $\mathcal{N}(A)$ of the matrix A, and the image $\mathrm{Im}(A)$ of A is just the column space $\mathcal{C}(A) = \mathrm{Im}(A) \subseteq \mathbb{R}^m$ of the matrix A. Recall that $A\mathbf{x}$ is a linear combination of the column vectors of A. □

Example 4.7 (*The trace is linear*) A **trace** is a function $\mathrm{tr} : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined by the sum of diagonal entries

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

for $A = [a_{ij}] \in M_{n\times n}(\mathbb{R})$, and $\mathrm{tr}(A)$ is called the **trace** of the matrix A. It is easy to show that

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B) \quad \text{and} \quad \mathrm{tr}(kA) = k\,\mathrm{tr}(A)$$

for any two matrices A and B in $M_{n\times n}(\mathbb{R})$, which means that 'tr' is a linear transformation from $M_{n\times n}(\mathbb{R})$ to the 1-space $\mathbb{R}$. In addition, one can easily show that the set of all $n\times n$ matrices with trace 0 is a subspace of the vector space $M_{n\times n}(\mathbb{R})$. □

Problem 4.2 Let $W = \{A \in M_{n\times n}(\mathbb{R}) : \mathrm{tr}(A) = 0\}$. Show that W is a subspace, and find a basis for W.

Problem 4.3 Show that, for any matrices A and B in $M_{n\times n}(\mathbb{R})$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

One of the most important properties of linear transformations is that they are completely determined by their values on a basis.

Theorem 4.3 *Let V and W be vector spaces. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ be any vectors (possibly repeated) in W. Then there exists a unique linear transformation $T : V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \ldots, n$.*

Proof: Let $\mathbf{x} \in V$. Then it has a unique expression: $\mathbf{x} = \sum_{i=1}^{n} a_i\mathbf{v}_i$ for some scalars $a_1, a_2, \ldots, a_n$. Define

$$T : V \to W \quad \text{by} \quad T(\mathbf{x}) = \sum_{i=1}^{n} a_i \mathbf{w}_i.$$

In particular, $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \ldots, n$.

Linearity: For $\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{v}_i$, $\mathbf{y} = \sum_{i=1}^{n} b_i \mathbf{v}_i \in V$ and k a scalar, we have $\mathbf{x} + k\mathbf{y} = \sum_{i=1}^{n} (a_i + kb_i)\mathbf{v}_i$. Then

$$T(\mathbf{x} + k\mathbf{y}) = \sum_{i=1}^{n} (a_i + kb_i)\mathbf{w}_i = \sum_{i=1}^{n} a_i \mathbf{w}_i + k \sum_{i=1}^{n} b_i \mathbf{w}_i = T(\mathbf{x}) + kT(\mathbf{y}).$$

Uniqueness: Suppose that $S : V \to W$ is linear and $S(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \ldots, n$. Then for any $\mathbf{x} \in V$ with $\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{v}_i$, we have

$$S(\mathbf{x}) = \sum_{i=1}^{n} a_i S(\mathbf{v}_i) = \sum_{i=1}^{n} a_i \mathbf{w}_i = T(\mathbf{x}).$$

Hence, we have $S = T$. □

The uniqueness in Theorem 4.3 may be rephrased as the following corollary.

Corollary 4.4 *Let V and W be vector spaces and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. If S, $T : V \to W$ are linear transformations and $S(\mathbf{v}_i) = T(\mathbf{v}_i)$ for $i = 1, 2, \ldots, n$; then $S = T$, i.e., $S(\mathbf{x}) = T(\mathbf{x})$ for all $\mathbf{x} \in V$.*

Example 4.8 (*Linear extension of a transformation defined on a basis*) Let $\mathbf{w}_1 = (1,\ 0)$, $\mathbf{w}_2 = (2,\ -1)$, $\mathbf{w}_3 = (4,\ 3)$ be three vectors in $\mathbb{R}^2$.

(1) Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for the 3-space $\mathbb{R}^3$, and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(\mathbf{e}_1) = \mathbf{w}_1, \qquad T(\mathbf{e}_2) = \mathbf{w}_2, \qquad T(\mathbf{e}_3) = \mathbf{w}_3.$$

Find a formula for $T(x_1, x_2, x_3)$, and then use it to compute $T(2,\ -3,\ 5)$.

(2) Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be another basis for $\mathbb{R}^3$, where $\mathbf{v}_1 = (1,\ 1,\ 1)$, $\mathbf{v}_2 = (1,\ 1,\ 0)$, $\mathbf{v}_3 = (1,\ 0,\ 0)$, and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(\mathbf{v}_1) = \mathbf{w}_1, \qquad T(\mathbf{v}_2) = \mathbf{w}_2, \qquad T(\mathbf{v}_3) = \mathbf{w}_3.$$

Find a formula for $T(x_1, x_2, x_3)$, and then use it to compute $T(2,\ -3,\ 5)$.

Solution: (1) For $\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \mathbb{R}^3$,

$$\begin{aligned} T(\mathbf{x}) &= \sum_{i=1}^{3} x_i T(\mathbf{e}_i) = \sum_{i=1}^{3} x_i \mathbf{w}_i \\ &= x_1(1, 0) + x_2(2,\ -1) + x_3(4,\ 3) \\ &= (x_1 + 2x_2 + 4x_3,\ -x_2 + 3x_3). \end{aligned}$$

Thus, $T(2, -3, 5) = (16, 18)$. In matrix notation, this can be written as

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ -x_2 + 3x_3 \end{bmatrix}.$$

(2) In this case, we need to express $\mathbf{x} = (x_1, x_2, x_3)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, i.e.,

$$\begin{aligned} (x_1, x_2, x_3) = \sum_{i=1}^{3} k_i \mathbf{v}_i &= k_1(1, 1, 1) + k_2(1, 1, 0) + k_3(1, 0, 0) \\ &= (k_1 + k_2 + k_3)\mathbf{e}_1 + (k_1 + k_2)\mathbf{e}_2 + k_1 \mathbf{e}_3. \end{aligned}$$

By equating corresponding components we obtain a system of equations

$$\begin{cases} k_1 + k_2 + k_3 = x_1 \\ k_1 + k_2 = x_2 \\ k_1 = x_3. \end{cases}$$

The solution is $k_1 = x_3,\ k_2 = x_2 - x_3,\ k_3 = x_1 - x_2$. Therefore,

$$\begin{aligned} (x_1, x_2, x_3) &= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3, \quad \text{and} \\ T(x_1, x_2, x_3) &= x_3 T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\ &= x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) \\ &= (4x_1 - 2x_2 - x_3,\ 3x_1 - 4x_2 + x_3). \end{aligned}$$

From this formula, we obtain $T(2, -3, 5) = (9, 23)$. □

Problem 4.4 Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(3, 1, 0) = (1, 1)$ and $T(-6, -2, 0) = (2, 1)$? If yes, can you find an expression of $T(\mathbf{x})$ for $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$?

Problem 4.5 Let V and W be vector spaces and $T : V \to W$ be linear. Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be a linearly independent subset of the image $\mathrm{Im}(T) \subseteq W$. Suppose that $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is chosen so that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \ldots, k$. Prove that α is linearly independent.

4.2 Invertible linear transformations

A function f from a set X to a set Y is said to be **invertible** if there is a function g from Y to X such that their compositions satisfy $g \circ f = id$ and $f \circ g = id$. Such a function g is called the **inverse** function of f and denoted by $g = f^{-1}$. One can notice that if there exists an invertible function from a set X into another set Y, then it gives a one-to-one correspondence between these two sets so that they can be identified as sets. A useful criterion for a function between two given sets to be invertible is that it is one-to-one and onto. Recall that a function $f : X \to Y$ is **one-to-one** (or **injective**) if $f(u) = f(v)$ in Y implies $u = v$ in X, and is **onto** (or **surjective**) if for each element y in Y there exists an element x in X such that $f(x) = y$. A function is said to be **bijective** if it is both one-to-one and onto, that is, if for each element y in Y there is a *unique* element x in X such that $f(x) = y$.

Lemma 4.5 *A function $f : X \to Y$ is invertible if and only if it is bijective (or one-to-one and onto).*

Proof: Suppose $f : X \to Y$ is invertible, and let $g : Y \to X$ be its inverse. If $f(u) = f(v)$, then $u = g(f(u)) = g(f(v)) = v$. Thus f is one-to-one. For each $y \in Y$, let $g(y) = x$ in X. Then $f(x) = f(g(y)) = y$. Thus it is onto.

Conversely, suppose f is bijective. Then, for each $y \in Y$, there is a unique $x \in X$ such that $f(x) = y$. Now for each $y \in Y$ define $g(y) = x$. Then one can easily check that $g : Y \to X$ is well defined, and that $f \circ g = id$ and $g \circ f = id$, i.e., g is the inverse function of f. □

If $T : V \to W$ and $S : W \to Z$ are linear transformations, then it is easy to show that their composition $(S \circ T)(\mathbf{v}) = S(T(\mathbf{v}))$ is also a linear transformation. In particular, if two linear transformations are defined by matrices $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^m \to \mathbb{R}^k$ as in Example 4.2(1), then their composition is nothing but the matrix product BA of them, i.e., $(B \circ A)(\mathbf{x}) = B(A\mathbf{x}) = (BA)\mathbf{x}$.

The following lemma shows that if a given function is an invertible linear transformation from a vector space into another, then the linearity is preserved by the inversion.

Lemma 4.6 *Let V and W be vector spaces. If $T : V \to W$ is an invertible linear transformation, then its inverse $T^{-1} : W \to V$ is also linear.*

Proof: Let $\mathbf{w}_1, \mathbf{w}_2 \in W$, and let k be any scalar. Since T is invertible, there exist unique vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Then

$$\begin{aligned} T^{-1}(\mathbf{w}_1 + k\mathbf{w}_2) &= T^{-1}(T(\mathbf{v}_1) + kT(\mathbf{v}_2)) \\ &= T^{-1}(T(\mathbf{v}_1 + k\mathbf{v}_2)) \\ &= \mathbf{v}_1 + k\mathbf{v}_2 \\ &= T^{-1}(\mathbf{w}_1) + kT^{-1}(\mathbf{w}_2). \end{aligned}$$ □

Definition 4.3 A linear transformation $T : V \to W$ from a vector space V to another W is called an **isomorphism** if it is invertible (or one-to-one and onto). In this case, we say that V and W are **isomorphic** to each other.

Example 4.9 (*The vector space $P_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n+1}$*) Consider the vector space $P_2(\mathbb{R}) = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}$ of all polynomials of degree ≤ 2 with real coefficients. To each polynomial $a + bx + cx^2$ in the space $P_2(\mathbb{R})$, one can assign the column vector $[a\ b\ c]^T$ in $\mathbb{R}^3$. Then it is not hard to see that it is an isomorphism from the vector space $P_2(\mathbb{R})$ to the 3-space $\mathbb{R}^3$, by which one can *identify* the polynomial $a + bx + cx^2$ with the column vector $[a\ b\ c]^T$. It means that these two vector spaces can be considered as the same vector space through the isomorphism. In this sense, one often says that a vector space can be identified with another if they are isomorphic to each other. In general, the vector space $P_n(\mathbb{R})$ can be identified with the $(n+1)$-space $\mathbb{R}^{n+1}$. □

It is clear from Lemma 4.6 that if T is an isomorphism, then its inverse T^{-1} is also an isomorphism with $(T^{-1})^{-1} = T$. In particular, if a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is defined by an invertible $n \times n$ matrix A as in Example 4.2(1), then the inverse matrix A^{-1} plays the inverse linear transformation, so that it is also an isomorphism on $\mathbb{R}^n$. That is, a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ defined by an $n \times n$ square matrix A is an isomorphism if and only if A is invertible, that is, rank $A = n$.

Problem 4.6 Suppose that S and T are linear transformations whose composition $S \circ T$ is well defined. Prove that

(1) if $S \circ T$ is one-to-one, so is T,
(2) if $S \circ T$ is onto, so is S,
(3) if S and T are isomorphisms, so is $S \circ T$,
(4) if A and B are two $n \times n$ matrices of rank n, so is AB.

Problem 4.7 Let $T : V \to W$ be a linear transformation. Prove that

(1) T is one-to-one if and only if $\operatorname{Ker}(T) = \{\mathbf{0}\}$.
(2) If $V = W$, then T is one-to-one if and only if T is onto.

Theorem 4.7 *Two vector spaces V and W are isomorphic if and only if* $\dim V = \dim W$.

Proof: Let $T : V \to W$ be an isomorphism, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. Then we show that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is a basis for W so that $\dim W = n = \dim V$.

(1) *It is linearly independent:* Since T is one-to-one, the equation

$$\mathbf{0} = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n)$$

implies that $\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. Since the $\mathbf{v}_i$'s are linearly independent, we have $c_i = 0$ for all $i = 1, 2, \ldots, n$.

(2) *It spans W:* Since T is onto, for any $\mathbf{y} \in W$ there exists an $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{y}$. Write $\mathbf{x} = \sum_{i=1}^n a_i\mathbf{v}_i$. Then

$$\mathbf{y} = T(\mathbf{x}) = T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \cdots + a_nT(\mathbf{v}_n),$$

i.e., $\mathbf{y}$ is a linear combination of $T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \ldots,\ T(\mathbf{v}_n)$.

Conversely, if $\dim V = \dim W = n$, then one can choose bases $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ for V and W, respectively. By Theorem 4.3, there exist linear transformations $T : V \to W$ and $S : W \to V$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ and $S(\mathbf{w}_i) = \mathbf{v}_i$ for $i = 1, 2, \ldots, n$. Clearly, $(S \circ T)(\mathbf{v}_i) = \mathbf{v}_i$ and $(T \circ S)(\mathbf{w}_i) = \mathbf{w}_i$ for $i = 1, 2, \ldots, n$, which implies that $S \circ T$ and $T \circ S$ are the identity transformations on V and W, respectively, by the uniqueness in Corollary 4.4. Hence, T and S are isomorphisms, and consequently V and W are isomorphic. □

Corollary 4.8 *Let V and W be vector spaces.*

(1) *If* $\dim V = n$, *then V is isomorphic to the n-space $\mathbb{R}^n$.*
(2) *If* $\dim V = \dim W$, *any bijective function from a basis for V to a basis for W can be extended to an isomorphism from V to W.*

An isomorphism between a vector space V and $\mathbb{R}^n$ in Corollary 4.8 depends on the choices of bases for two spaces as shown in Theorem 4.7. However, an isomorphism is uniquely determined if we fix the bases in which the order of the vectors is also fixed.

An **ordered basis** for a vector space is a basis endowed with a specific order. For example, in the 3-space $\mathbb{R}^3$, two bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with the order $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ and $\{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$ with the order $\mathbf{e}_2,\ \mathbf{e}_1,\ \mathbf{e}_3$ are clearly different as ordered bases, but the same as unordered ones. The basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ with the order $\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n$ is called the **standard ordered basis** for $\mathbb{R}^n$. However, we often say simply a basis for an ordered basis if there is no ambiguity in the context.

Let V be a vector space of dimension n with an ordered basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, and let $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard ordered basis for $\mathbb{R}^n$. Then the isomorphism $\Phi : V \to \mathbb{R}^n$ defined by $\Phi(\mathbf{v}_i) = \mathbf{e}_i$ is called the **natural isomorphism** with respect to the basis α. By this isomorphism, a vector in V can be identified with a column vector in $\mathbb{R}^n$. In fact, for any $\mathbf{x} = \sum_{i=1}^n a_i \mathbf{v}_i \in V$, the image of $\mathbf{x}$ under this natural isomorphism is written as

$$\Phi(\mathbf{x}) = \sum_{i=1}^n a_i \Phi(\mathbf{v}_i) = \sum_{i=1}^n a_i \mathbf{e}_i = (a_1, \ldots, a_n) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n,$$

which is called the **coordinate vector** of $\mathbf{x}$ with respect to the ordered basis α, and it is denoted by $[\mathbf{x}]_\alpha$. Clearly $[\mathbf{v}_i]_\alpha = \mathbf{e}_i$.

Example 4.10 **(1)** Recall that, from Example 4.3, the rotation by the angle θ of $\mathbb{R}^2$ is given by the matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Clearly, it is invertible and hence is an isomorphism of $\mathbb{R}^2$. In fact, the inverse R_θ^{-1} is another rotation $R_{-\theta}$.

(2) Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard ordered basis, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_i = R_\theta \mathbf{e}_i$, $i = 1, 2$. Then β is also a basis for $\mathbb{R}^2$. The coordinate vectors of $\mathbf{v}_i$ with respect to α are

$$[\mathbf{v}_1]_\alpha = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad [\mathbf{v}_2]_\alpha = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix},$$

while

$$[\mathbf{v}_1]_\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [\mathbf{v}_2]_\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If we choose $\alpha' = \{\mathbf{e}_2, \mathbf{e}_1\}$ as a different ordered basis for $\mathbb{R}^2$, then the coordinate vectors of $\mathbf{v}_i$ with respect to α' are

$$[\mathbf{v}_1]_{\alpha'} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}, \quad [\mathbf{v}_2]_{\alpha'} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}.$$ □

Example 4.11 (*All reflections are of the form $R_\theta \circ T \circ R_{-\theta}$*) In the plane $\mathbb{R}^2$, the reflection about the line $y = x$ can be obtained by the compositions of the rotation by $-\frac{\pi}{4}$ of the plane, the reflection about the x-axis, and the rotation by $\frac{\pi}{4}$. Actually, it is a product of matrices given in (1) and (3) of Example 4.3 with $\theta = \frac{\pi}{4}$: Note that the rotation by $\frac{\pi}{4}$ is

$$R_{\frac{\pi}{4}} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

and the reflection about the x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{-\frac{\pi}{4}} = R_{\frac{\pi}{4}}^{-1}$. Hence, the matrix for the reflection about the line $y = x$ is

$$R_{\frac{\pi}{4}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{\frac{\pi}{4}}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In general, the reflection about a line ℓ in the plane can be expressed as the composition $R_\theta \circ T \circ R_{-\theta}$, where T is the reflection about the x-axis and θ is the angle between the x-axis and the line ℓ (see Figure 4.2). □

Problem 4.8 Find the matrix of reflection about the line $y = \sqrt{3}x$ in $\mathbb{R}^2$.

Problem 4.9 Find the coordinate vector of $5 + 2x + 3x^2$ with respect to the given ordered basis α for $P_2(\mathbb{R})$:
(1) $\alpha = \{1,\ x,\ x^2\}$; (2) $\alpha = \{1 + x,\ 1 + x^2,\ x + x^2\}$.

4.3 Matrices of linear transformations

We have seen that the product of an $m \times n$ matrix A and an $n \times 1$ column matrix $\mathbf{x}$ gives rise to a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Conversely, one can show that a linear transformation of a vector space into another can be represented by a matrix via

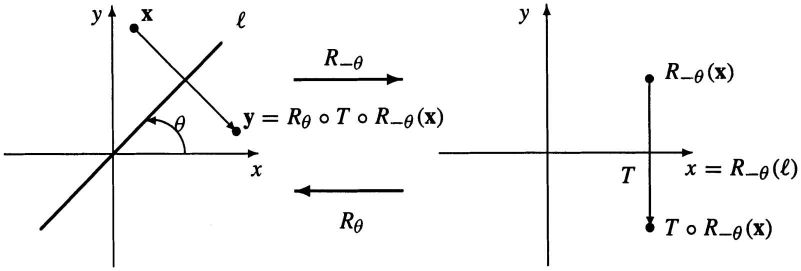

Figure 4.2. The reflection $R_\theta \circ T \circ R_{-\theta}$

the natural isomorphism between an n-dimensional vector space V and the n-space $\mathbb{R}^n$, which will be shown in this section.

Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W, and let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be any ordered bases for V and W, respectively, which will be fixed throughout this section. Then by Theorem 4.3 the linear transformation T is completely determined by its values on a basis α: Write them as

$$\left\{\begin{array}{rcccccccc} T(\mathbf{v}_1) & = & a_{11}\mathbf{w}_1 & + & a_{21}\mathbf{w}_2 & + & \cdots & + & a_{m1}\mathbf{w}_m \\ T(\mathbf{v}_2) & = & a_{12}\mathbf{w}_1 & + & a_{22}\mathbf{w}_2 & + & \cdots & + & a_{m2}\mathbf{w}_m \\ & \vdots & & & & & & & \vdots \\ T(\mathbf{v}_n) & = & a_{1n}\mathbf{w}_1 & + & a_{2n}\mathbf{w}_2 & + & \cdots & + & a_{mn}\mathbf{w}_m, \end{array}\right.$$

or, in a short form,

$$T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i \quad \text{for } 1 \le j \le n,$$

for some scalars a_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

Now, for any vector $\mathbf{x} = \sum_{j=1}^{n} x_j\mathbf{v}_j \in V$,

$$T(\mathbf{x}) = \sum_{j=1}^{n} x_j T(\mathbf{v}_j) = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij}\mathbf{w}_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)\mathbf{w}_i.$$

Equivalently, the coordinate vector of $T(\mathbf{x})$ with respect to the basis β in W is

$$[T(\mathbf{x})]_\beta = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A[\mathbf{x}]_\alpha.$$

That is, for any $\mathbf{x} \in V$ the coordinate vector $[T(\mathbf{x})]_\beta$ of $T(\mathbf{x})$ in W is just the product of a fixed matrix A and the coordinate vector $[\mathbf{x}]_\alpha$ of $\mathbf{x}$. This situation can be incorporated

$$\begin{tikzcd}
V \arrow[rrr, "T"] \arrow[ddd, "\Phi"'] & & & W \arrow[ddd, "\Psi"] \\
& \mathbf{x} \arrow[r, mapsto, squiggly] \arrow[d, mapsto] & T(\mathbf{x}) \arrow[d, mapsto] & \\
& {[\mathbf{x}]_\alpha} \arrow[r, mapsto, squiggly] & {[T(\mathbf{x})]_\beta} & \\
\mathbb{R}^n \arrow[rrr, "A = {[T]_\alpha^\beta}"'] & & & \mathbb{R}^m,
\end{tikzcd}$$

Figure 4.3. The associated matrix for T

in the commutative diagram in Figure 4.3 with the natural isomorphisms Φ and Ψ, defined in Section 4.2. Note that the commutativity of the diagram means that $A \circ \Phi = \Psi \circ T$.

Note that

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \big[[T(\mathbf{v}_1)]_\beta \ \cdots \ [T(\mathbf{v}_n)]_\beta\big]$$

is the matrix whose column vectors are just the coordinate vectors $[T(\mathbf{v}_j)]_\beta$ *of* $T(\mathbf{v}_j)$ *with respect to the basis* β. In fact, $A = [a_{ij}]$ is just the transpose of the coefficient matrix in the expression of $T(\mathbf{v}_i)$ with respect to the basis β in W. Note that this matrix $[T]_\alpha^\beta$ is unique since the coordinate expression of a vector with respect to a fixed basis is unique.

Definition 4.4 The matrix A is called the **associated matrix** for T (or the **matrix representation** of T) with respect to the ordered bases α and β, and denoted by $A = [T]_\alpha^\beta$. When $V = W$ and $\alpha = \beta$, we simply write $[T]_\alpha$ for $[T]_\alpha^\alpha$.

Now, the argument so far can be summarized in the following theorem.

Theorem 4.9 *Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. For fixed ordered bases $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for V and β for W, there corresponds a unique associated $m \times n$ matrix $[T]_\alpha^\beta$ for T such that for any vector $\mathbf{x} \in V$ the coordinate vector $[T(\mathbf{x})]_\beta$ of $T(\mathbf{x})$ with respect to β is given as a matrix product of the associated matrix $[T]_\alpha^\beta$ for T and the coordinate vector $[\mathbf{x}]_\alpha$, i.e.,*

$$\boxed{[T(\mathbf{x})]_\beta = [T]_\alpha^\beta [\mathbf{x}]_\alpha.}$$

The associated matrix $[T]_\alpha^\beta$ is given as

$$[T]_\alpha^\beta = \big[\, [T(\mathbf{v}_1)]_\beta \ [T(\mathbf{v}_2)]_\beta \ \cdots \ [T(\mathbf{v}_n)]_\beta \,\big].$$

The following examples illustrate the computation of the associated matrices for linear transformations.

Example 4.12 (*The associated matrix* $[id]_\alpha$) Let $id : V \to V$ be the identity transformation on a vector space V. Then for any ordered basis α for V, the matrix $[id]_\alpha = I$, the identity matrix, because if $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then

$$\left\{\begin{array}{rcrcrcccl} id(\mathbf{v}_1) & = & 1\mathbf{v}_1 & + & 0\mathbf{v}_2 & + & \cdots & + & 0\mathbf{v}_m \\ id(\mathbf{v}_2) & = & 0\mathbf{v}_1 & + & 1\mathbf{v}_2 & + & \cdots & + & 0\mathbf{v}_m \\ & \vdots & & & & & & & \vdots \\ id(\mathbf{v}_n) & = & 0\mathbf{v}_1 & + & 0\mathbf{v}_2 & + & \cdots & + & 1\mathbf{v}_m. \end{array}\right.$$

□

Example 4.13 (*The associated matrix* $[T]_\alpha^\beta$) Let $T : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$\big(T(p)\big)(x) = xp(x).$$

Find the associated matrix $[T]_\alpha^\beta$ with respected to ordered bases $\alpha = \{1,\ x\}$ and $\beta = \{1,\ x,\ x^2\}$ for $P_1(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively.

Solution: Clearly,

$$\left\{\begin{array}{rcccrcrcl} (T(1))(x) & = & x & = & 0\cdot 1 & + & 1x & + & 0x^2 \\ (T(x))(x) & = & x^2 & = & 0\cdot 1 & + & 0x & + & 1x^2. \end{array}\right.$$

Hence, the associated matrix for T is the transpose of the coefficient matrix in this expression, that is, $[T]_\alpha^\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. □

Example 4.14 (*The associated matrices* $[T]_\alpha^\beta$ *and* $[T]_\alpha^{\beta'}$) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T(x,\ y) = (x + 2y,\ 0,\ 2x + 3y)$ with respect to the standard bases α and β for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Then

$$\left\{\begin{array}{rcccccrcrcl} T(\mathbf{e}_1) & = & T(1,0) & = & (1,\ 0,\ 2) & = & 1\mathbf{e}_1 & + & 0\mathbf{e}_2 & + & 2\mathbf{e}_3 \\ T(\mathbf{e}_2) & = & T(0,1) & = & (2,\ 0,\ 3) & = & 2\mathbf{e}_1 & + & 0\mathbf{e}_2 & + & 3\mathbf{e}_3. \end{array}\right.$$

Hence, $[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}$. If $\beta' = \{\mathbf{e}_3,\ \mathbf{e}_2,\ \mathbf{e}_1\}$, then $[T]_\alpha^{\beta'} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}$. □

Example 4.15 (*The associated matrix* $[T]_\alpha$ *for the standard basis* α) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(1,\ 1) = (0,\ 1)$ and $T(-1,\ 1) = (2,\ 3)$. Find the matrix representation $[T]_\alpha$ of T with respect to the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$.

Solution: Note $(a,\ b) = a\mathbf{e}_1 + b\mathbf{e}_2$ for any $(a,\ b) \in \mathbb{R}^2$. Thus, the definition of T shows

$$\begin{array}{rcccccccl} T(\mathbf{e}_1) + T(\mathbf{e}_2) &=& T(\mathbf{e}_1 + \mathbf{e}_2) &=& T(1,\ 1) &=& (0, 1) &=& \mathbf{e}_2, \\ -T(\mathbf{e}_1) + T(\mathbf{e}_2) &=& T(-\mathbf{e}_1 + \mathbf{e}_2) &=& T(-1,\ 1) &=& (2,\ 3) &=& 2\mathbf{e}_1 + 3\mathbf{e}_2. \end{array}$$

By solving these equations, we obtain

$$\begin{array}{rcrcl} T(\mathbf{e}_1) &=& -\mathbf{e}_1 &-& \mathbf{e}_2, \\ T(\mathbf{e}_2) &=& \mathbf{e}_1 &+& 2\mathbf{e}_2. \end{array}$$

Therefore, $[T]_\alpha = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$. □

Example 4.16 (*The associated matrix $[T]_\beta$ for a non-standard basis β*) Let T be the linear transformation given in Example 4.15. Find $[T]_\beta$ for a basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = (0, 1)$ and $\mathbf{v}_2 = (2,\ 3)$.

Solution: From Example 4.15,

$$\begin{array}{rcl} T(\mathbf{v}_1) &=& \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [T(\mathbf{v}_1)]_\alpha, \\ T(\mathbf{v}_2) &=& \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = [T(\mathbf{v}_2)]_\alpha. \end{array}$$

To write these vectors as linear combinations of basis vectors in β, we put

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2 = \begin{bmatrix} 2b \\ a + 3b \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix} = c\mathbf{v}_1 + d\mathbf{v}_2 = \begin{bmatrix} 2d \\ c + 3d \end{bmatrix}.$$

Solving for $a,\ b,\ c$ and d, we obtain

$$[T(\mathbf{v}_1)]_\beta = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } [T(\mathbf{v}_2)]_\beta = \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Therefore, $[T]_\beta = \frac{1}{2}\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$. □

Remark: **(1)** Recall that any $m \times n$ matrix A can be considered as a linear transformation from the n-space $\mathbb{R}^n$ to the m-space $\mathbb{R}^m$ via $\mathbf{x} \mapsto A\mathbf{x}$. Clearly, its matrix representation with respect to the standard bases α for $\mathbb{R}^n$ and β for $\mathbb{R}^m$ is the matrix A itself, i.e., $A = [A]_\alpha^\beta$. (Note that $A\mathbf{e}_j$ is just the j-th column vector of A.) In particular, if A is an invertible $n \times n$ square matrix, then the column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ form another basis γ for $\mathbb{R}^n$. Thus, A is simply the linear transformation on $\mathbb{R}^n$ that takes the standard basis α to γ, in fact,

$$A\mathbf{e}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} = \mathbf{c}_j,$$

the j-th column of A, so that its matrix representation $[A]_\alpha^\gamma$ is the identity matrix.

(2) Let V and W be vector spaces with bases α and β, respectively, and let $T : V \to W$ be a linear transformation with the matrix representation $[T]_\alpha^\beta = A$. Then it is clear that $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are isomorphic to the null space $\mathcal{N}(A)$ and the column space $\mathcal{C}(A)$, respectively, via the natural isomorphisms. In particular, if $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ with the standard bases, then $\mathrm{Ker}(T) = \mathcal{N}(A)$ and $\mathrm{Im}(T) = \mathcal{C}(A)$. Therefore, from Theorem 3.17, we have

$$\boxed{\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim V.}$$

(3) Let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations with an $m \times n$ coefficient matrix A. By considering the matrix A as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, one can have other equivalent conditions to those mentioned in Theorems 3.24 and 3.25: The conditions in Theorem 3.24 (e.g., $\mathcal{C}(A) = \mathbb{R}^m$) are equivalent to the condition that A is *surjective*, and those in Theorem 3.25 (e.g., $\mathcal{N}(A) = \{\mathbf{0}\}$) are equivalent to the condition that A is *one-to-one*. This observation gives the proof of (15)–(16) in Theorem 3.26.

Problem 4.10 Find the matrix representations $[T]_\alpha$ and $[T]_\beta$ of each of the following linear transformations T on $\mathbb{R}^3$ with respect to the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and another $\beta = \{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1\}$:

(1) $T(x, y, z) = (2x - 3y + 4z,\ 5x - y + 2z,\ 4x + 7y)$,
(2) $T(x, y, z) = (2y + z,\ x - 4y,\ 3x)$.

Also, find the matrix representation $[T]_\alpha^\beta$ of each of the linear transformations T.

Problem 4.11 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x,\ y,\ z,\ u) = (x + 2y,\ x - 3z + u,\ 2y + 3z + 4u).$$

Let α and β be the standard bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Find $[T]_\alpha^\beta$.

Problem 4.12 Let $id : \mathbb{R}^n \to \mathbb{R}^n$ be the identity transformation. Let $\mathbf{x}_k$ denote the vector in $\mathbb{R}^n$ whose first $k - 1$ coordinates are zero and the last $n - k + 1$ coordinates are 1. Then clearly $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$ (see Problem 3.9). Let $\alpha = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Find the matrix representations $[id]_\alpha^\beta$ and $[id]_\beta^\alpha$.

4.4 Vector spaces of linear transformations

Let V and W be two vector spaces of dimensions n and m. Let $\mathcal{L}(V; W)$ denote the set of all linear transformations from V to W, i.e.,

$$\mathcal{L}(V; W) = \{T\ :\ T \text{ is a linear transformation from } V \text{ to } W\}.$$

For any two linear transformations S and T in $\mathcal{L}(V; W)$ and $\lambda \in \mathbb{R}$, we define the **sum** $S + T$ and the **scalar multiplication** λS by

$$(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v}) \quad \text{and} \quad (\lambda S)(\mathbf{v}) = \lambda(S(\mathbf{v}))$$

for any $\mathbf{v} \in V$. Clearly, the sum $S+T$ and the scalar multiplication λS are also linear and satisfy the operational rules of a vector space, so that $\mathcal{L}(V; W)$ becomes a vector space.

Let α and β be two ordered bases for V and W, respectively and let $T : V \to W$ be a linear transformation. Then the associated matrix $[T]_\alpha^\beta$ of T with respect to these bases is uniquely determined by Theorem 4.9. That is, the function $\phi : \mathcal{L}(V; W) \to M_{m\times n}(\mathbb{R})$ defined by

$$\phi(T) = [T]_\alpha^\beta \in M_{m\times n}(\mathbb{R})$$

for $T \in \mathcal{L}(V; W)$ is well defined (see Section 4.3).

Lemma 4.10 *The function $\phi : \mathcal{L}(V; W) \to M_{m\times n}(\mathbb{R})$ is a one-to-one correspondence between $\mathcal{L}(V; W)$ and $M_{m\times n}(\mathbb{R})$.*

Proof: (1) It is one-to-one: If $[S]_\alpha^\beta = [T]_\alpha^\beta$ for S and T in $\mathcal{L}(V; W)$, then we have $S = T$ by Corollary 4.4.

(2) It is onto: For any $m \times n$ matrix A (considered as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$), define a linear transformation $T : V \to W$ by $T = \Psi^{-1} \circ A \circ \Phi$ as the composition of A with the natural isomorphisms $\Phi : V \to \mathbb{R}^n$ and $\Psi : W \to \mathbb{R}^m$. Then clearly $[T]_\alpha^\beta = A$, i.e., ϕ is onto. □

Furthermore, the following lemma shows that ϕ is linear, so that it is in fact an isomorphism from $\mathcal{L}(V; W)$ to $M_{m\times n}(\mathbb{R})$.

Lemma 4.11 *Let V and W be vector spaces with ordered bases α and β, respectively, and let $S,\ T : V \to W$ be linear. Then we have*

$$[S+T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta \quad \textit{and} \quad [kS]_\alpha^\beta = k[S]_\alpha^\beta.$$

Proof: Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. Then we have unique expressions $S(\mathbf{v}_j) = \sum_{i=1}^m a_{ij}\mathbf{w}_i$ and $T(\mathbf{v}_j) = \sum_{i=1}^m b_{ij}\mathbf{w}_i$ for each $1 \le j \le n$, so that $[S]_\alpha^\beta = [a_{ij}]$ and $[T]_\alpha^\beta = [b_{ij}]$. Hence

$$(S+T)(\mathbf{v}_j) = \sum_{i=1}^m a_{ij}\mathbf{w}_i + \sum_{i=1}^m b_{ij}\mathbf{w}_i = \sum_{i=1}^m (a_{ij} + b_{ij})\mathbf{w}_i.$$

Thus

$$[S+T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta.$$

The proof of the second equality $[kS]_\alpha^\beta = k[S]_\alpha^\beta$ is similar and left as an exercise. □

In particular, if $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then the vector space $M_{m\times n}(\mathbb{R})$ of $m \times n$ matrices may be identified with the vector space $\mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$, since such a matrix A is a linear transformation and A itself is the matrix representation of itself with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

One can summarize our discussions in the following theorem:

Theorem 4.12 *For vector spaces V of dimension n and W of dimension m, the vector space $\mathcal{L}(V; W)$ of all linear transformations from V to W is isomorphic to the vector space $M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices, and*

$$\dim \mathcal{L}(V; W) = \dim M_{m\times n}(\mathbb{R}) = mn = \dim V \dim W.$$

Remark: With an isomorphism from the vector space $\mathcal{L}(V; W)$ to the vector space $M_{m\times n}(\mathbb{R})$ as mentioned in Theorem 4.12, one can prove that the following conditions for a linear transformation T on a vector space V are equivalent, as mentioned in Theorem 3.26:

(1) *T is an isomorphism,*
(2) *T is one-to-one,*
(3) *T is surjective.*

(One can also prove it directly by using the definition of a basis for V. See Problem 4.7.)

The next theorem shows that the one-to-one correspondence between $\mathcal{L}(V; W)$ and $M_{m\times n}(\mathbb{R})$ preserves not only the vector space structure but also the compositions of linear transformations. Let V, W and Z be vector spaces. Suppose that $S : V \to W$ and $T : W \to Z$ are linear transformations. Then the composition $T \circ S : V \to Z$ is also linear.

Theorem 4.13 *Let V, W and Z be vector spaces with ordered bases α, β, and γ, respectively. Suppose that $S : V \to W$ and $T : W \to Z$ are linear transformations. Then*

$$[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta.$$

Proof: Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ and $\gamma = \{\mathbf{z}_1, \ldots, \mathbf{z}_\ell\}$. Let $[T]_\beta^\gamma = [a_{ij}]$ and $[S]_\alpha^\beta = [b_{pq}]$. Then, for $1 \le i \le n$,

$$\begin{aligned}(T \circ S)(\mathbf{v}_i) &= T(S(\mathbf{v}_i)) = T\left(\sum_{k=1}^m b_{ki}\mathbf{w}_k\right) = \sum_{k=1}^m b_{ki} T(\mathbf{w}_k) \\ &= \sum_{k=1}^m b_{ki}\left(\sum_{j=1}^\ell a_{jk}\mathbf{z}_j\right) = \sum_{j=1}^\ell\left(\sum_{k=1}^m a_{jk}b_{ki}\right)\mathbf{z}_j.\end{aligned}$$

It shows that $[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$. $\square$

Problem 4.13 Let α be the standard basis for $\mathbb{R}^3$, and let $S,\ T : \mathbb{R}^3 \to \mathbb{R}^3$ be two linear transformations given by

$$\begin{array}{lll} S(\mathbf{e}_1) = (2,\ 2,\ 1), & S(\mathbf{e}_2) = (0,\ 1,\ 2), & S(\mathbf{e}_3) = (-1,\ 2,\ 1), \\ T(\mathbf{e}_1) = (1,\ 0,\ 1), & T(\mathbf{e}_2) = (0,\ 1,\ 1), & T(\mathbf{e}_3) = (1,\ 1,\ 2). \end{array}$$

Compute $[S+T]_\alpha$, $[2T-S]_\alpha$ and $[T \circ S]_\alpha$.

Problem 4.14 Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(f) = (3+x)f' + 2f$, and let $S : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the one defined by $S(a+bx+cx^2) = (a-b,\ a+b,\ c)$. For a basis $\alpha = \{1, x, x^2\}$ for $P_2(\mathbb{R})$ and the standard basis $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$, compute $[S]_\alpha^\beta$, $[T]_\alpha$ and $[S \circ T]_\alpha^\beta$.

Theorem 4.14 *Let V and W be vector spaces with ordered bases α and β, respectively, and let $T : V \to W$ be an isomorphism. Then*

$$[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}.$$

Proof: Since T is invertible, $\dim V = \dim W$, and the matrices $[T]_\alpha^\beta$ and $[T^{-1}]_\beta^\alpha$ are square and of the same size. Thus,

$$[T]_\alpha^\beta [T^{-1}]_\beta^\alpha = [T \circ T^{-1}]_\beta = [id]_\beta$$

is the identity matrix. Hence, $[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$. □

In particular, if a linear transformation $T : V \to W$ is an isomorphism, then $[T]_\alpha^\beta$ is an invertible matrix for any bases α for V and β for W.

Problem 4.15 For the vector spaces $P_1(\mathbb{R})$ and $\mathbb{R}^2$, choose the bases $\alpha = \{1, x\}$ for $P_1(\mathbb{R})$ and $\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$, respectively. Let $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation defined by $T(a+bx) = (a,\ a+b)$.

(1) Show that T is invertible.
(2) Find $[T]_\alpha^\beta$ and $[T^{-1}]_\beta^\alpha$.

4.5 Change of bases

In Section 4.2, we have seen that, in an n-dimensional vector space V with a fixed basis α, any vector $\mathbf{x}$ can be identified with a column vector $[\mathbf{x}]_\alpha$ in the n-space $\mathbb{R}^n$ via the natural isomorphism Φ. Of course, one may get a different column vector $[\mathbf{x}]_\beta$ if another basis β is given instead of α. Thus, one may naturally ask what the relation between $[\mathbf{x}]_\alpha$ and $[\mathbf{x}]_\beta$ is for two different bases α and β.

To answer this question, let us begin with an example in the plane $\mathbb{R}^2$. The coordinate expression of $\mathbf{x} = (x,\ y) \in \mathbb{R}^2$ with respect to the standard basis $\alpha = \{\mathbf{e}_1,\ \mathbf{e}_2\}$ is $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$, so that $[\mathbf{x}]_\alpha = \begin{bmatrix} x \\ y \end{bmatrix}$.

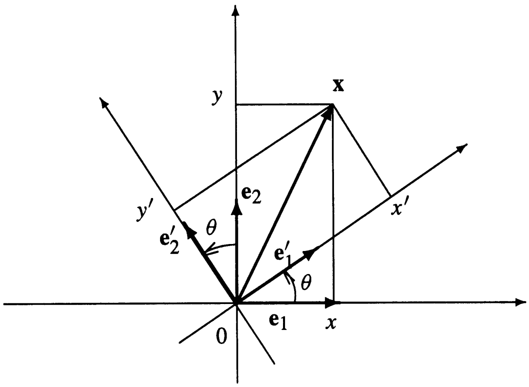

Figure 4.4. Coordinates $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2\}$

Now let $\beta = \{\mathbf{e}'_1, \mathbf{e}'_2\}$ be another basis for $\mathbb{R}^2$ obtained by rotating α counterclockwise through an angle θ as in Figure 4.4, and suppose that the coordinate expression of $\mathbf{x} \in \mathbb{R}^2$ with respect to β is written as $\mathbf{x} = x'\mathbf{e}'_1 + y'\mathbf{e}'_2$, or $[\mathbf{x}]_\beta = \begin{bmatrix} x' \\ y' \end{bmatrix}$. Then, the expressions of the vectors in β with respect to α are

$$\begin{cases} \mathbf{e}'_1 = id(\mathbf{e}'_1) = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2 \\ \mathbf{e}'_2 = id(\mathbf{e}'_2) = -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2, \end{cases}$$

so

$$[\mathbf{e}'_1]_\alpha = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad [\mathbf{e}'_2]_\alpha = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Therefore, from $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$ and

$$\mathbf{x} = x'\mathbf{e}'_1 + y'\mathbf{e}'_2 = (x'\cos\theta - y'\sin\theta)\mathbf{e}_1 + (x'\sin\theta + y'\cos\theta)\mathbf{e}_2,$$

one can have the matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \text{or} \quad [\mathbf{x}]_\alpha = [id]^\alpha_\beta [\mathbf{x}]_\beta,$$

where

$$[id]^\alpha_\beta = \big[[\mathbf{e}'_1]_\alpha \ [\mathbf{e}'_2]_\alpha\big] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

It means that two coordinate vectors $[\mathbf{x}]_\alpha$ and $[\mathbf{x}]_\beta$ in the 2-space $\mathbb{R}^2$ are related by the associated matrix $[id]^\alpha_\beta$ for the identity transformation id on $\mathbb{R}^2$. Note that $[id]^\beta_\alpha = ([id]^\alpha_\beta)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ by Theorem 4.14.

In general, if $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ are two ordered bases for an n-dimensional vector space V, then any vector $\mathbf{x} \in V$ has two expressions:

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{v}_i = \sum_{j=1}^{n} y_j \mathbf{w}_j.$$

In particular, each vector in β is expressed as a linear combination of the vectors in α: say $\mathbf{w}_j = id(\mathbf{w}_j) = \sum_{i=1}^{n} q_{ij}\mathbf{v}_i$ for $j = 1, 2, \ldots, n$, so that

$$[\mathbf{w}_j]_\alpha = [id(\mathbf{w}_j)]_\alpha = \begin{bmatrix} q_{1j} \\ \vdots \\ q_{nj} \end{bmatrix}.$$

Then for any $\mathbf{x} \in V$,

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{v}_i = \sum_{j=1}^{n} y_j \mathbf{w}_j = \sum_{j=1}^{n} y_j \sum_{i=1}^{n} q_{ij}\mathbf{v}_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} q_{ij} y_j \right) \mathbf{v}_i.$$

This is equivalent to the matrix equation

$$\boxed{[\mathbf{x}]_\alpha = \left[\sum_{j=1}^{n} q_{ij} y_j \right] = [id]_\beta^\alpha [\mathbf{x}]_\beta},$$

or

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ & \ddots & \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

where

$$[id]_\beta^\alpha = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ & \ddots & \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} = [\, [\mathbf{w}_1]_\alpha \ \cdots \ [\mathbf{w}_n]_\alpha].$$

This means that any two coordinate vectors of a vector in V with respect to two different ordered bases α and β are related by the matrix representation $[id]_\beta^\alpha$ of the identity transformation on V, and this can be incorporated in the commutative diagram in Figure 4.5 (next page).

Definition 4.5 The matrix representation $[id]_\beta^\alpha$ of the identity transformation $id : V \to V$ with respect to any two ordered bases α and β is called the **basis-change matrix** or the **coordinate-change matrix** from β to α.

Since the identity transformation $id : V \to V$ is invertible, the basis-change matrix $Q = [id]_\beta^\alpha$ is also invertible by Theorem 4.14. If we had taken the expressions of the vectors in the basis α with respect to the basis β: $\mathbf{v}_j = id(\mathbf{v}_j) = \sum_{i=1}^{n} p_{ij}\mathbf{w}_i$ for $j = 1, 2, \ldots, n$, then $[p_{ij}] = [id]_\alpha^\beta = Q^{-1}$ and

$$[\mathbf{x}]_\beta = [id]_\alpha^\beta [\mathbf{x}]_\alpha = ([id]_\beta^\alpha)^{-1} [\mathbf{x}]_\alpha.$$

$$\begin{tikzcd}
V \arrow[rrr, "id"] \arrow[ddd, "\Phi'"'] & & & V \arrow[ddd, "\Phi"] \\
& \mathbf{x} \arrow[r, mapsto, squiggly] \arrow[d, mapsto] & \mathbf{x} \arrow[d, mapsto] & \\
& {[\mathbf{x}]_\beta} \arrow[r, mapsto, squiggly] & {[\mathbf{x}]_\alpha} & \\
\mathbb{R}^n \arrow[rrr, "Q = [id]_\beta^\alpha"'] & & & \mathbb{R}^n,
\end{tikzcd}$$

Figure 4.5. The basis-change matrix $[id]_\beta^\alpha$

Example 4.17 (*Analytic interpretation of a basis-change matrix*) Consider a curve $xy = 1$ on the plane $\mathbb{R}^2$. Find the quadric equation of the curve which is obtained from the curve $xy = 1$ by rotating around the origin clockwise through an angle $\pi/4$.

Solution: Let $\beta = \{\mathbf{e}_1', \mathbf{e}_2'\}$ be the basis for $\mathbb{R}^2$ obtained by rotating the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$ counterclockwise through an angle $\pi/4$, and let $[\mathbf{x}]_\alpha = \begin{bmatrix} x \\ y \end{bmatrix}$ and $[\mathbf{x}]_\beta = \begin{bmatrix} x' \\ y' \end{bmatrix}$. Then,

$$\begin{bmatrix} x \\ y \end{bmatrix} = [id]_\beta^\alpha \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Hence, the equation $xy = 1$ is transformed to

$$1 = xy = \left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right)\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right) = \frac{(x')^2}{2} - \frac{(y')^2}{2},$$

which is a hyperbola. (See Figure 4.6.) $\square$

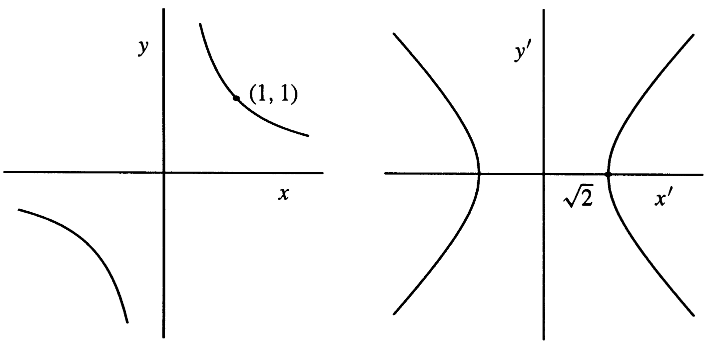

Figure 4.6. The graphs of $xy = 1$ and $\frac{(x')^2}{2} - \frac{(y')^2}{2} = 1$

Example 4.18 (*Computing a basis-change matrix*) Let the 3-space $\mathbb{R}^3$ be equipped with the standard xyz-coordinate system, i.e., with the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Take a new $x'y'z'$-coordinate system by rotating the xyz-system around its z-axis counterclockwise through an angle θ, i.e., we take a new basis $\beta = \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ by rotating the basis α about z axis through θ. Then we get

$$[\mathbf{e}'_1]_\alpha = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad [\mathbf{e}'_2]_\alpha = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \quad [\mathbf{e}'_3]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the basis-change matrix from β to α is

$$Q = [id]^\alpha_\beta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so

$$[\mathbf{x}]_\alpha = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = Q[\mathbf{x}]_\beta.$$

Moreover, $Q = [id]^\alpha_\beta$ is invertible and the basis-change matrix from α to β is

$$Q^{-1} = [id]^\beta_\alpha = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

□

Problem 4.16 Find the basis-change matrix from a basis α to another basis β for the 3-space $\mathbb{R}^3$, where $\alpha = \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$, $\beta = \{(2, 3, 1), (1, 2, 0), (2, 0, 3)\}$.

4.6 Similarity

The coordinate expression of a vector in a vector space V depends on the choice of an ordered basis. Hence, the matrix representation of a linear transformation is also dependent on the choice of bases.

Let V and W be two vector spaces of dimensions n and m with two ordered bases α and β, respectively, and let $T : V \to W$ be a linear transformation. In Section 4.3, we discussed how to find the associated matrix $[T]^\beta_\alpha$. If one takes different bases α' and β' for V and W, respectively, then one may get another associated matrix $[T]^{\beta'}_{\alpha'}$ of T. In fact, we have two different expressions

$$\begin{array}{lll} [\mathbf{x}]_\alpha \text{ and } [\mathbf{x}]_{\alpha'} \text{ in } \mathbb{R}^n & \text{for each } \mathbf{x} \in V, \\ [T(\mathbf{x})]_\beta \text{ and } [T(\mathbf{x})]_{\beta'} \text{ in } \mathbb{R}^m & \text{for } T(\mathbf{x}) \in W. \end{array}$$

They are related by the basis-change matrices as follows:

$$[\mathbf{x}]_{\alpha'} = [id_V]_\alpha^{\alpha'}[\mathbf{x}]_\alpha, \text{ and } [T(\mathbf{x})]_{\beta'} = [id_W]_\beta^{\beta'}[T(\mathbf{x})]_\beta.$$

On the other hand, by Theorem 4.9, we have

$$[T(\mathbf{x})]_\beta = [T]_\alpha^\beta[\mathbf{x}]_\alpha, \text{ and } [T(\mathbf{x})]_{\beta'} = [T]_{\alpha'}^{\beta'}[\mathbf{x}]_{\alpha'}.$$

Therefore, we get

$$\begin{aligned} [T]_{\alpha'}^{\beta'}[\mathbf{x}]_{\alpha'} &= [T(\mathbf{x})]_{\beta'} = [id_W]_\beta^{\beta'}[T(\mathbf{x})]_\beta = [id_W]_\beta^{\beta'}\,[T]_\alpha^\beta[\mathbf{x}]_\alpha \\ &= [id_W]_\beta^{\beta'}\,[T]_\alpha^\beta\,[id_V]_{\alpha'}^\alpha[\mathbf{x}]_{\alpha'}. \end{aligned}$$

This equation looks messy. However, by Theorem 4.13, this relation can be obtained directly from $T = id_W \circ T \circ id_V$ as

$$[T]_{\alpha'}^{\beta'} = [id_W \circ T \circ id_V]_{\alpha'}^{\beta'} = [id_W]_\beta^{\beta'}\,[T]_\alpha^\beta\,[id_V]_{\alpha'}^\alpha.$$

Note that $[T]_\alpha^\beta$ and $[T]_{\alpha'}^{\beta'}$ are $m \times n$ matrices, $[id_V]_{\alpha'}^\alpha$ is an $n \times n$ matrix and $[id_W]_\beta^{\beta'}$ is an $m \times m$ matrix.

The relation can also be incorporated in the diagram in Figure 4.7, in which all rectangles are commutative.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "{[T]_{\alpha'}^{\beta'}}"] \arrow[ddd, "{[id_V]_{\alpha'}^\alpha}"'] & & & \mathbb{R}^m \arrow[ddd, "{[id_W]_{\beta'}^\beta}"] \\
& (V,\alpha') \arrow[r, "T"] \arrow[ul, "\Phi'"] \arrow[d, "id_V"'] & (W,\beta') \arrow[ur, "\Psi'"'] \arrow[d, "id_W"] & \\
& (V,\alpha) \arrow[r, "T"] \arrow[dl, "\Phi"'] & (W,\beta) \arrow[dr, "\Psi"] & \\
\mathbb{R}^n \arrow[rrr, "{[T]_\alpha^\beta}"] & & & \mathbb{R}^m
\end{tikzcd}$$

Figure 4.7. Relating two associated matrices $[T]_\alpha^\beta$ and $[T]_{\alpha'}^{\beta'}$

Our discussion is summarized in the following theorem.

Theorem 4.15 *Let $T : V \to W$ be a linear transformation from a vector space V with bases α and α' to another vector space W with bases β and β'. Then*

$$[T]_{\alpha'}^{\beta'} = P^{-1}[T]_{\alpha}^{\beta} Q,$$

where $Q = [id_V]_{\alpha'}^{\alpha}$ and $P = [id_W]_{\beta'}^{\beta}$ are the basis-change matrices.

In particular, if we take $W = V$, $\alpha = \beta$ and $\alpha' = \beta'$, then $P = Q$ and we get to the following corollary.

Corollary 4.16 *Let $T : V \to V$ be a linear transformation on a vector space V and let α and β be ordered bases for V. Let $Q = [id]_{\beta}^{\alpha}$ be the basis-change matrix from β to α. Then*

(1) *Q is invertible, and $Q^{-1} = [id]_{\alpha}^{\beta}$.*
(2) *For any $\mathbf{x} \in V$, $[\mathbf{x}]_{\alpha} = Q[\mathbf{x}]_{\beta}$.*
(3) *$[T]_{\beta} = Q^{-1}[T]_{\alpha}Q$.*

The relation (3) of $[T]_{\beta}$ and $[T]_{\alpha}$ in Corollary 4.16 is called a *similarity*. In general, we have the following definition.

Definition 4.6 For any square matrices A and B, A is said to be **similar** to B if there exists a nonsingular matrix Q such that $B = Q^{-1}AQ$.

Note that if A is similar to B, then B is also similar to A. Thus we simply say that A and B are similar. We saw in Corollary 4.16 that if A and B are $n \times n$ matrices representing the same linear transformation T on a vector space V, then A and B are similar.

Example 4.19 (*Two similar associated matrices*) Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for the 3-space $\mathbb{R}^3$ consisting of $\mathbf{v}_1 = (1,\ 1,\ 0)$, $\mathbf{v}_2 = (1,\ 0,\ 1)$ and $\mathbf{v}_3 = (0,\ 1,\ 1)$. Let T be the linear transformation on $\mathbb{R}^3$ given by the matrix

$$[T]_{\beta} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}.$$

Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis. Find the basis-change matrix $[id]_{\alpha}^{\beta}$ and $[T]_{\alpha}$.

Solution: Since $\mathbf{v}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{v}_2 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{v}_3 = \mathbf{e}_2 + \mathbf{e}_3$, we have

$$[id]_{\beta}^{\alpha} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad [id]_{\alpha}^{\beta} = ([id]_{\beta}^{\alpha})^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Therefore,

$$[T]_{\alpha} = [id]_{\beta}^{\alpha}[T]_{\beta}[id]_{\alpha}^{\beta} = \frac{1}{2}\begin{bmatrix} 4 & 2 & 2 \\ 3 & -1 & 1 \\ -1 & 1 & 7 \end{bmatrix}.$$

□

Example 4.20 (*Computing an associated matrix*) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_2, x_1 + x_2 + 3x_3, -x_2).$$

Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard ordered basis for $\mathbb{R}^3$, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be another ordered basis consisting of $\mathbf{v}_1 = (-1, 0, 0)$, $\mathbf{v}_2 = (2, 1, 0)$, and $\mathbf{v}_3 = (1, 1, 1)$. Find the associated matrices $[T]_\alpha$ and $[T]_\beta$ for T. Also, show that $T(\mathbf{v}_j)$ is the linear combination of the basis vectors in β with the entries of the j-th column of $[T]_\beta$ as its coefficients for $j = 1, 2, 3$.

Solution: One can easily show that

$$[T]_\alpha = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad [id]^\alpha_\beta = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, with the inverse $[id]^\beta_\alpha = ([id]^\alpha_\beta)^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, it follows that

$$[T]_\beta = [id]^\beta_\alpha [T]_\alpha [id]^\alpha_\beta = \begin{bmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{bmatrix}.$$

To show the second statement, let $j = 2$. Then $T(\mathbf{v}_2) = T(2, 1, 0) = (5, 3, -1)$. On the other hand, the coefficients of $[T(\mathbf{v}_2)]_\beta$ are just the entries of the second column of $[T]_\beta$. Therefore,

$$\begin{aligned} T(\mathbf{v}_2) &= 2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3 \\ &= 2(-1, 0, 0) + 4(2, 1, 0) - (1, 1, 1) = (5, 3, -1), \end{aligned}$$

as expected. □

The next theorem shows that two similar matrices can be matrix representations of the same linear transformation.

Theorem 4.17 *Suppose that an $n \times n$ matrix A represents a linear transformation $T : V \to V$ on a vector space V with respect to an ordered basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, i.e., $[T]_\alpha = A$. If $B = Q^{-1}AQ$ for some nonsingular matrix Q, then there exists a basis β for V such that $B = [T]_\beta$ and $Q = [id]^\alpha_\beta$.*

Proof: Let $Q = [q_{ij}]$ and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ be the vectors in V defined by

$$\left\{ \begin{array}{lcl} \mathbf{w}_1 &=& q_{11}\mathbf{v}_1 + q_{21}\mathbf{v}_2 + \cdots + q_{n1}\mathbf{v}_n \\ \mathbf{w}_2 &=& q_{12}\mathbf{v}_1 + q_{22}\mathbf{v}_2 + \cdots + q_{n2}\mathbf{v}_n \\ &\vdots& \\ \mathbf{w}_n &=& q_{1n}\mathbf{v}_1 + q_{2n}\mathbf{v}_2 + \cdots + q_{nn}\mathbf{v}_n. \end{array} \right.$$

Then the nonsingularity of $Q = [q_{ij}]$ implies that $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an ordered basis for V, and Theorem 4.16(3) shows that $[T]_\beta = Q^{-1}[T]_\alpha Q = Q^{-1}AQ = B$ with $Q = [id]_\beta^\alpha$. □

Example 4.21 (*A matrix similar to an associated matrix is also an associated matrix*) Let D be the differential operator on the vector space $P_2(\mathbb{R})$. Given the ordered basis $\alpha = \{1,\ x,\ x^2\}$, first note that

$$\begin{array}{rcll} D(1) & = & 0 & = 0\cdot 1 + 0\cdot x + 0\cdot x^2 \\ D(x) & = & 1 & = 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ D(x^2) & = & 2x & = 0\cdot 1 + 2\cdot x + 0\cdot x^2. \end{array}$$

Hence, the matrix representation of D with respect to α is given by

$$[D]_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Choose a nonsingular matrix

$$Q = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad \text{with its inverse} \quad Q^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let

$$B = Q^{-1}[D]_\alpha Q = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, we are going to find a basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ so that $B = [D]_\beta$. But, if it is, the matrix Q must be the basis-change matrix $[id]_\beta^\alpha$, and then

$$\begin{array}{rcccl} \mathbf{v}_1 & = & 1\cdot 1 + 0\cdot x + 0\cdot x^2 & = & 1, \\ \mathbf{v}_2 & = & 0\cdot 1 + 2\cdot x + 0\cdot x^2 & = & 2x, \\ \mathbf{v}_3 & = & -2\cdot 1 + 0\cdot x + 4\cdot x^2 & = & -2 + 4x^2. \end{array}$$

Clearly, one can obtain

$$\begin{array}{rcccl} D(1) & = & 0 & = & 0\cdot 1 + 0\cdot 2x + 0\cdot(-2+4x^2), \\ D(2x) & = & 2 & = & 2\cdot 1 + 0\cdot 2x + 0\cdot(-2+4x^2), \\ D(-2+4x^2) & = & 8x & = & 0\cdot 1 + 4\cdot 2x + 0\cdot(-2+4x^2), \end{array}$$

and

$$[D]_\beta = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$ □

thus, as expected, that $[D]_\beta = B = Q^{-1}[D]_\alpha Q$.

Problem 4.17 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, -x_2, x_1 + 4x_3).$$

Let α be the standard basis, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be another ordered basis consisting of $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 1, 1)$ for $\mathbb{R}^3$. Find the associated matrix of T with respect to α and the associated matrix of T with respect to β. Are they similar?

Problem 4.18 Suppose that A and B are similar $n \times n$ matrices. Show that

(1) $\det A = \det B$,
(2) $\mathrm{tr}(A) = \mathrm{tr}(B)$,
(3) $\mathrm{rank}\, A = \mathrm{rank}\, B$.

Problem 4.19 Let A and B be $n \times n$ matrices. Show that if A is similar to B, then A^2 is similar to B^2. In general, A^n is similar to B^n for all $n \geq 2$.

4.7 Applications

4.7.1 Dual spaces and adjoint

Note that the space of all scalars is a one-dimensional vector space $\mathbb{R}$, and the set of all linear transformations from V to $\mathbb{R}$ is the vector space $\mathcal{L}(V; \mathbb{R})$ whose dimension is equal to the dimension of V (see Theorem 4.12), so that the two vector spaces $\mathcal{L}(V; \mathbb{R})$ and V are isomorphic. In this section, we are concerned exclusively with such linear transformations from V to the scalar space $\mathbb{R}$.

Definition 4.7 Let V be a vector space.

(1) The vector space $\mathcal{L}(V; \mathbb{R})$ of all linear transformations from V to $\mathbb{R}$ is called the **dual space** of V and denoted by V^*.
(2) An element (i.e., a linear transformation) in the dual space $\mathcal{L}(V; \mathbb{R})$ is called a **linear functional** of V.

From the definition, one can say that *any vector space is isomorphic to its dual space.*

Example 4.22 The trace function $\mathrm{tr} : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a linear functional of $M_{n\times n}(\mathbb{R})$.

The definite integral of continuous functions is one of the most important examples of linear functionals in mathematics.

Example 4.23 (*Fourier coefficients are linear functionals*) Let $C[a, b]$ be the vector space of all continuous real-valued functions on the interval $[a, b]$. The definite integral $\mathcal{I} : C[a, b] \to \mathbb{R}$ defined by

$$\mathcal{I}(f) = \int_a^b f(t)dt$$

is a linear functional of $C[a, b]$. In particular, if the interval is $[0, 2\pi]$ and n is an integer, then

$$\mathcal{A}_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\, dt \qquad \text{and} \quad \mathcal{B}_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\, dt$$

are linear functionals, called the n-th **Fourier coefficients** of f. □

For a matrix A regarded as a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$, the transpose A^T of A is another linear transformation $A^T : \mathbb{R}^m \to \mathbb{R}^n$. For a linear transformation $T : V \to W$ from a vector space V to W, one can naturally ask what its *transpose* is and what the definition is. In this section, we discuss this problem.

Recall that a linear functional $T : V \to \mathbb{R}$ is completely determined by the values on a basis for V. Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. For each $i = 1, 2, \ldots, n$, define a linear functional

$$\mathbf{v}_i^* : V \to \mathbb{R}$$

by $\mathbf{v}_i^*(\mathbf{v}_j) = \delta_{ij}$ for each $j = 1, 2, \ldots, n$. Then, for any $\mathbf{x} = \sum a_i\mathbf{v}_i \in V$, we have $\mathbf{v}_i^*(\mathbf{x}) = a_i$, which is the i-th coordinate of $\mathbf{x}$ with respect to α. Thus, the functional $\mathbf{v}_i^*$ is called the i-th **coordinate function** with respect to the basis α.

Theorem 4.18 *The set $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_n^*\}$ of coordinate functions forms a basis for the dual space V^*, and for any $T \in V^*$ we have*

$$T = \sum_{i=1}^{n} T(\mathbf{v}_i)\mathbf{v}_i^*.$$

Proof: Clearly, the set $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_n^*\}$ is linearly independent, since $\mathbf{0} = \sum_{i=1}^n c_i\mathbf{v}_i^*$ implies $0 = \sum_{i=1}^n c_i\mathbf{v}_i^*(\mathbf{v}_j) = c_j$ for each $j = 1, 2, \ldots, n$. Because $\dim V^* = \dim V = n$, these n linearly independent vectors in α^* must form a basis.

Now, for any $T \in V^*$, let $T = \sum_{i=1}^n c_i\mathbf{v}_i^*$. Then, $T(\mathbf{v}_j) = \sum_{i=1}^n c_i\mathbf{v}_i^*(\mathbf{v}_j) = c_j$. It gives $T = \sum_{i=1}^n T(\mathbf{v}_i)\mathbf{v}_i^*$. □

Definition 4.8 For a basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for a vector space V, the basis $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_n^*\}$ for V^* is called the **dual basis** of α.

Example 4.24 (*Computing a dual basis*) Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for $\mathbb{R}^2$, where $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (1, 3)$. To find its dual basis $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*\}$ of α, we consider the equations

$$\begin{aligned} 1 &= \mathbf{v}_1^*(\mathbf{v}_1) = \mathbf{v}_1^*(\mathbf{e}_1) + 2\mathbf{v}_1^*(\mathbf{e}_2), \\ 0 &= \mathbf{v}_1^*(\mathbf{v}_2) = \mathbf{v}_1^*(\mathbf{e}_1) + 3\mathbf{v}_1^*(\mathbf{e}_2). \end{aligned}$$

Solving these equations, we obtain that $\mathbf{v}_1^*(\mathbf{e}_1) = 3$ and $\mathbf{v}_1^*(\mathbf{e}_2) = -1$. Thus $\mathbf{v}_1^*(x, y) = 3x - y$. Similarly, it can be shown that $\mathbf{v}_2^*(x, y) = -2x + y$. □

The following example shows a natural isomorphism between the n-space $\mathbb{R}^n$ and its dual space $\mathbb{R}^{n*}$.

Example 4.25 (*The dual basis* $\mathbf{e}_i^*$ *is the coordinate function*) For the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for the n-space $\mathbb{R}^n$, its dual basis vector $\mathbf{e}_i^*$ is just the i-th coordinate function. In fact, for any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \in \mathbb{R}^n$, we have $\mathbf{e}_i^*(\mathbf{x}) = \mathbf{e}_i^*(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = x_i$ for all i.

On the other hand, when we write a vector in $\mathbb{R}^n$ as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with variables x_i, it means that for a given point $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, each x_i gives us the i-th coordinate of $\mathbf{a}$, that is, $x_i(\mathbf{a}) = a_i$ for all i. In this sense, one can identify $\mathbf{e}_i^* = x_i$ for $i = 1, 2, \ldots, n$, so that $\mathbb{R}^{n*} = \mathbb{R}^n$ and they are called coordinate functions. □

Problem 4.20 Let $\alpha = \{(1, 0, 1), (1, 2, 1), (0, 0, 1)\}$ be a basis for $\mathbb{R}^3$. Find the dual basis α^*.

For a given linear transformation $T : V \to W$, one can define $T^* : W^* \to V^*$ by $T^*(g) = g \circ T$ for any $g \in W^*$. In fact, for any linear functional $g \in W^*$, i.e., $g : W \to \mathbb{R}$, the composition $g \circ T : V \to \mathbb{R}$ given by $g \circ T(\mathbf{x}) = g(T(\mathbf{x}))$ for $\mathbf{x} \in V$ defines a linear functional on V, i.e., $T^*(g) = g \circ T \in V^*$.

Lemma 4.19 *The transformation* $T^* : W^* \to V^*$ *defined by* $T^*(g) = g \circ T$ *for* $g \in W^*$ *is a linear transformation. It is called the* **adjoint** (*or* **transpose**) *of* T. □

Proof: For any $f, g \in W^*$, $a, b \in \mathbb{R}$ and $\mathbf{x} \in V$,

$$T^*(af + bg)(\mathbf{x}) = af(T(\mathbf{x})) + bg(T(\mathbf{x})) = (aT^*(f) + bT^*(g))(\mathbf{x}).$$ □

Example 4.26 ($(id_V)^* = id_{V^*}$ and $(T \circ S)^* = S^* \circ T^*$)

(1) Let $id : V \to V$ be the identity transformation on a vector space V. Then for any $g \in V^*$, $id^*(g) = g \circ id = g$. Hence, the adjoint $id^* : V^* \to V^*$ is the identity transformation on V^*, i.e., $id^* = id$.

(2) Let $S : U \to V$ and $T : V \to W$ be two linear transformations. Then for any $g \in W^*$, we have

$$\begin{aligned}(T \circ S)^*(g) &= g \circ (T \circ S) = (g \circ T) \circ S \\ &= T^*(g) \circ S = S^*(T^*(g)) = (S^* \circ T^*)(g).\end{aligned}$$

It shows that $(T \circ S)^* = S^* \circ T^*$. □

Now, if $S : V \to W$ is an isomorphism, then $(S^{-1})^* \circ S^* = (S \circ S^{-1})^* = id^* = id$ shows that $S^* : W^* \to V^*$ is also an isomorphism.

Note that the linear transformation $* : V \to V^*$ defined by assigning a basis for V to its dual basis is an isomorphism, so that the composition $** : V \to V^{**}$ is also an isomorphism. However, an isomorphism between V and V^{**} can be defined without choosing a basis for V. In fact, for each $\mathbf{x} \in V$, one can first define $\tilde{\mathbf{x}} : V^* \to \mathbb{R}$ by

$\tilde{\mathbf{x}}(f) = f(\mathbf{x})$ for every $f \in V^*$. It is easy to verify that $\tilde{\mathbf{x}}$ is a linear functional on V^* so that $\tilde{\mathbf{x}} \in V^{**}$. The following theorem shows that the mapping $\Phi : V \to V^{**}$ defined by $\Phi(\mathbf{x}) = \tilde{\mathbf{x}}$ is an isomorphism and it is not dependent on the choice of basis for V.

Theorem 4.20 *The mapping $\Phi : V \to V^{**}$ defined by $\Phi(\mathbf{x}) = \tilde{\mathbf{x}}$ is an isomorphism from V to V^{**}.*

Proof: To show the linearity of Φ, let $\mathbf{x}, \mathbf{y} \in V$ and k a scalar. Then, for any $f \in V^*$,

$$\begin{aligned} \Phi(\mathbf{x} + k\mathbf{y})(f) &= \widetilde{(\mathbf{x} + k\mathbf{y})}(f) = f(\mathbf{x} + k\mathbf{y}) \\ &= f(\mathbf{x}) + kf(\mathbf{y}) = \tilde{\mathbf{x}}(f) + k\tilde{\mathbf{y}}(f) \\ &= (\tilde{\mathbf{x}} + k\tilde{\mathbf{y}})(f) = (\Phi(\mathbf{x}) + k\Phi(\mathbf{y}))\,(f). \end{aligned}$$

Hence, $\Phi(\mathbf{x} + k\mathbf{y}) = \Phi(\mathbf{x}) + k\Phi(\mathbf{y})$.

To show that Φ is injective, suppose $\mathbf{x} \in \mathrm{Ker}(\Phi)$. Then $\Phi(\mathbf{x}) = \tilde{\mathbf{x}} = \mathbf{0}$ in V^{**}, i.e., $\tilde{\mathbf{x}}(f) = 0$ for all $f \in V^*$. It implies that $\mathbf{x} = \mathbf{0}$: In fact, if $\mathbf{x} \neq \mathbf{0}$, one can choose a basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for V such that $\mathbf{v}_1 = \mathbf{x}$. Let $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_n^*\}$ be the dual basis of α. Then

$$0 = \tilde{\mathbf{x}}(\mathbf{v}_1^*) = \mathbf{v}_1^*(\mathbf{x}) = \mathbf{v}_1^*(\mathbf{v}_1) = 1,$$

which is a contradiction. Thus, $\mathbf{x} = \mathbf{0}$ and $\mathrm{Ker}(\Phi) = \{\mathbf{0}\}$.

Since $\dim V = \dim V^{**}$, Φ is an isomorphism. $\square$

Problem 4.21 Let $V = \mathbb{R}^3$ and define $f_i \in V^*$ as follows:

$$f_1(x, y, z) = x - 2y, \quad f_2(x, y, z) = x + y + z, \quad f_3(x, y, z) = y - 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual.

We now consider the matrix representation of the transpose $S^* : W^* \to V^*$ of a linear transformation $S : V \to W$. Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be bases for V and W with their dual bases $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_n^*\}$ and $\beta^* = \{\mathbf{w}_1^*, \mathbf{w}_2^*, \ldots, \mathbf{w}_m^*\}$, respectively.

Theorem 4.21 *The matrix representation of the transpose $S^* : W^* \to V^*$ is the transpose of the matrix representation of $S : V \to W$, that is,*

$$[S^*]_{\beta^*}^{\alpha^*} = \left([S]_\alpha^\beta\right)^T.$$

Proof: Let $S(\mathbf{v}_i) = \sum_{k=1}^m a_{ki}\mathbf{w}_k$, so that

$$[S]_\alpha^\beta = \left[[S(\mathbf{v}_1)]_\beta \;\cdots\; [S(\mathbf{v}_n)]_\beta\right] = [a_{ij}].$$

Then

$$[S^*]_{\beta^*}^{\alpha^*} = \left[[S^*(\mathbf{w}_1^*)]_{\alpha^*} \cdots [S^*(\mathbf{w}_m^*)]_{\alpha^*}\right].$$

Note that, for $1 \le j \le m$,

$$S^*(\mathbf{w}_j^*) = \sum_{i=1}^{n} S^*(\mathbf{w}_j^*)(\mathbf{v}_i)\mathbf{v}_i^* = \sum_{i=1}^{n} a_{ji}\mathbf{v}_i^*,$$

since

$$\begin{aligned} S^*(\mathbf{w}_j^*)(\mathbf{v}_i) &= (\mathbf{w}_j^* \circ S)(\mathbf{v}_i) = \mathbf{w}_j^*(S(\mathbf{v}_i)) \\ &= \mathbf{w}_j^*\left(\sum_{k=1}^{m} a_{ki}\mathbf{w}_k\right) = \sum_{k=1}^{m} a_{ki}\mathbf{w}_j^*(\mathbf{w}_k) = a_{ji}. \end{aligned}$$

Hence, we get $[S^*]_{\beta^*}^{\alpha^*} = \left([S]_\alpha^\beta\right)^T$. □

Example 4.27 (*The transpose A^T is the adjoint transformation of A*) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined by an $m \times n$ matrix A. Let α and β be the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then $[A]_\alpha^\beta = A$. By Theorem 4.21, we have $[A^*]_{\beta^*}^{\alpha^*} = ([A]_\alpha^\beta)^T$. Thus, with the identification $\mathbb{R}^{k^*} = \mathbb{R}^k$ via $\alpha^* = \alpha$ and $\beta^* = \beta$ as in Example 4.25, we have $[A^*]_{\beta^*}^{\alpha^*} = A^*$ and $A^* = A^T$. In this sense, we see that the transpose A^T is the adjoint transformation of A. □

As the final part of the section, we consider the dual space of a subspace. Let V be a vector space of dimension n, and let U be a subspace of V of dimension k. Then $U^* = \{T : U \to \mathbb{R} \ : \ T \text{ is linear on } U\}$ is not a subspace of V^*. However, one can extend each $T \in U^*$ to a linear functional on V as follows. Choose a basis $\alpha = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for U. Then by definition $\alpha^* = \{\mathbf{u}_1^*, \mathbf{u}_2^*, \ldots, \mathbf{u}_k^*\}$ is its dual basis for U^*. Now extend α to a basis $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ for V. For each $T \in U^*$, let $\overline{T} : V \to \mathbb{R}$ be the linear functional on V defined by

$$\overline{T}(\mathbf{u}_i) = \begin{cases} T(\mathbf{u}_i) & \text{if } i \le k, \\ \mathbf{0} & \text{if } k+1 \le i \le n. \end{cases}$$

Then clearly $\overline{T} \in V^*$ and the restriction $\overline{T}|_U$ of $\overline{T}$ on U is simply T: i.e., $\overline{T}|_U = T \in U^*$. It is easy to see that $\overline{(T + kS)} = \overline{T} + k\overline{S}$. In particular, it is also easy to see that $\{\overline{\mathbf{u}_1^*}, \overline{\mathbf{u}_2^*}, \ldots, \overline{\mathbf{u}_k^*}\}$ is linearly independent in V^* and $\overline{\mathbf{u}_i^*}|_U = \mathbf{u}_i^* \in U^*$, $i = 1, 2, \ldots, k$. Therefore, one obtains a one-to-one linear transformation

$$\varphi : U^* \to V^*$$

given by $\varphi(T) = \overline{T}$ for all $T \in U^*$. The image $\varphi(U^*)$ is now a subspace of V^*. By identifying U^* with the image $\varphi(U^*)$, one can say U^* is a subspace of V^*.

Problem 4.22 Let U and W be subspaces of a vector space V. Show that $U \subseteq W$ if and only if $W^* \subseteq U^*$.

Let S be an arbitrary subset of V, and let $\langle S\rangle$ denote the subspace of V spanned by the vectors in S. Let $S^\perp = \{f \in V^* : f(\mathbf{x}) = 0 \text{ for any } \mathbf{x} \in S\}$. Then it is easy to show that $S^\perp$ is a subspace of V^*, $S^\perp = \langle S\rangle^\perp$, and $\dim\langle S\rangle + \dim S^\perp = n$.

Let R be a subset of V^*. Then $R^\perp = \{\mathbf{x} \in V : f(\mathbf{x}) = 0 \text{ for any } f \in R\}$ is again a subspace of V such that $R^\perp = \langle R\rangle^\perp$ and $\dim R^\perp + \dim\langle R\rangle = n$.

Problem 4.23 For subspaces U and W of a vector space V, show that
(1) $(U + W)^\perp = U^\perp \cap W^\perp$ (2) $(U \cap W)^\perp = U^\perp + W^\perp$.

4.7.2 Computer graphics

One of the simple applications of a linear transformation is to animation or graphical display of pictures on a computer screen. For a simple display of the idea, let us consider a picture in the 2-plane $\mathbb{R}^2$. Note that a picture or an image on a screen usually consists of a number of points, lines or curves connecting some of them, and information about how to fill the regions bounded by the lines and curves. Assuming that the computer has information about how to connect the points and curves, a figure can be defined by a list of points.

For example, consider the capital letters 'LA' as in Figure 4.8. They can be represented by a matrix with coordinates of the vertices. For example, the coordinates of the 6 vertices of 'L' form a matrix:

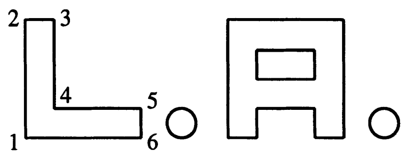

Figure 4.8. Letter L.A. on a screen

$$\begin{array}{c} \text{vertices} \\ \text{x-coordinate} \\ \text{y-coordinate} \end{array} \begin{array}{c} \begin{matrix} \;\;1 & \;\;\;2 & \;\;\;3 & \;\;\;4 & \;\;\;5 & \;\;\;6 \end{matrix} \\ \begin{bmatrix} 0.0 & 0.0 & 0.5 & 0.5 & 2.0 & 2.0 \\ 0.0 & 2.0 & 2.0 & 0.5 & 0.5 & 0.0 \end{bmatrix} \end{array} = A.$$

Of course, we assume that the computer knows which vertices are connected to which by lines via some algorithm. We know that line segments are transformed to other line segments by a matrix, considered as a linear transformation. Thus, by multiplying A by a matrix, the vertices are transformed to the other set of vertices, and the line segments connecting the vertices are preserved. For example, the matrix $B = \begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$ transforms the matrix A to the following form, which represents new coordinates of the vertices:

$$\begin{array}{cc} \text{vertices} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ BA = & \begin{bmatrix} 0.0 & 0.5 & 1.0 & 0.625 & 2.125 & 2.0 \\ 0.0 & 2.0 & 2.0 & 0.5 & 0.5 & 0.0 \end{bmatrix}. \end{array}$$

Now, the computer connects these vertices properly by lines according to the given algorithm and displays on the screen the changed figure as the left-hand side of the Figure 4.9. The multiplication of the matrix $C = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$ to BA shrinks the width

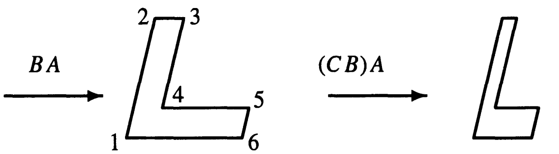

Figure 4.9. Tilting and Shrinking

of BA by half producing the right-hand side of Figure 4.9. Thus, changes in the shape of a figure may be obtained by compositions of appropriate linear transformations. Now, it is suggested that the readers try to find various matrices such as reflections, rotations, or any other linear transformations, and multiply A by them to see how the shape of the figure changes.

Problem 4.24 For the given matrices A and $B = \begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$ above, by the matrix $B^T A$ instead of BA, what kind of figure can you have ?

Remark: Incidentally, one can see that the composition of a rotation by π followed by a reflection about the x-axis is the same as the composition of the reflection followed by the rotation (see Figure 4.10). In general, a rotation and a reflection are not commutative, neither are a reflection and another reflection.

The above argument generally applies to a figure in any dimension. For instance, a 3×3 matrix may be used to convert a figure in $\mathbb{R}^3$ since each point has three components.

Example 4.28 (*Classifying all rotations in* $\mathbb{R}^3$) It is easy to see that the matrices

$$R_{(x,\alpha)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \quad R_{(y,\beta)} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$R_{(z,\gamma)} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the rotations about the x, y, z-axes by the angles α, β and γ, respectively.

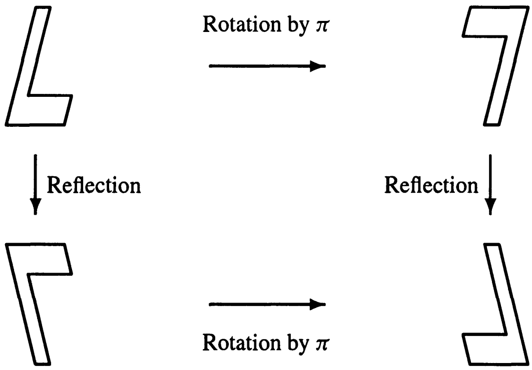

Figure 4.10. Commutativity of a rotation and a reflection

In general, the matrix that rotates $\mathbb{R}^3$ with respect to a given axis appears frequently in many applications. One can easily express such a general rotation as a composition of basic rotations such as $R_{(x,\alpha)}$, $R_{(y,\beta)}$ and $R_{(z,\gamma)}$:

First, note that by choosing a unit vector $\mathbf{u}$ in $\mathbb{R}^3$, one can determine an axis for a rotation by taking a line passing through $\mathbf{u}$ and the origin $\mathbf{0}$. (In fact, vectors $\mathbf{u}$ and $-\mathbf{u}$ determine the same line). Let $\mathbf{u} = (\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha)$, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta \leq 2\pi$ in the spherical coordinates. To find the matrix $R_{(\mathbf{u},\theta)}$ of the rotation about the $\mathbf{u}$-axis by θ, we first rotate the $\mathbf{u}$-axis about the z-axis into the xz-plane by $R_{(z,-\beta)}$ and then into the x-axis by the rotation $R_{(y,-\alpha)}$ about the y-axis. Then the rotation about the $\mathbf{u}$-axis is the same as the rotation about the x-axis followed by the inverses of the above rotations, i.e., take the rotation $R_{(x,\theta)}$ about the x-axis, and then get back to the rotation about the $\mathbf{u}$-axis via $R_{(y,\alpha)}$ and $R_{(z,\beta)}$. In summary,

$$R_{(\mathbf{u},\theta)} = R_{(z,\beta)}R_{(y,\alpha)}R_{(x,\theta)}R_{(y,-\alpha)}R_{(z,-\beta)}. \qquad \square$$

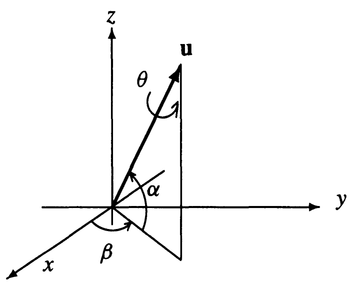

Figure 4.11. A rotation about the $\mathbf{u}$-axis

Problem 4.25 Find the matrix $R_{(\mathbf{u},\frac{\pi}{4})}$ for the rotation about the line determined by $\mathbf{u} = (1, 1, 1)$ by $\frac{\pi}{4}$.

So far, we have seen rotations, reflections, tilting (or say shear) or scaling (shrinking or enlargement) or their compositions as linear transformations on the plain $\mathbb{R}^2$ or the space $\mathbb{R}^3$ for computer graphics. However, another indispensable transformation for computer graphics is a translation: A **translation** is by definition a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$ for any $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{x}_0$ is a fixed vector in $\mathbb{R}^n$. Unfortunately, a translation is *not* linear if $\mathbf{x}_0 \neq \mathbf{0}$ and hence it cannot be represented by a matrix. To escape this disadvantage, we introduce a new coordinate system, called a **homogeneous coordinate**. For brevity, we will consider only the 3-space $\mathbb{R}^3$.

A point $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ in the rectangular coordinate can be viewed as the set of vectors $\mathbf{x} = (hx, hy, hz, h), h \neq 0$ in the 4-space $\mathbb{R}^4$ in a homogeneous coordinate. Most time, we use $(x, y, z, 1)$ as a representative of this set. Conversely, a point $(hx, hy, hz, h), (h \neq 0)$ in the 4-space $\mathbb{R}^4$ in a homogeneous coordinate corresponds to the point $(x/h, y/h, z/h) \in \mathbb{R}^3$ in the rectangular coordinate. Now, it is possible to represent all of our transformations including translations as 4×4 matrices by using the homogeneous coordinate and it will be shown case by case as follows.

(1) Translations: A translation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 = (x_0, y_0, z_0)$, can be represented by a matrix multiplication in the homogeneous coordinate as

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

(2) Rotations: With the notations in Example 4.28,

$$R_{(x,\alpha)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_{(y,\beta)} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_{(z,\gamma)} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are the rotations about the x, y, z-axes by the angles α, β and γ in the homogeneous coordinate, respectively.

(3) Reflections: An xy-reflection is represented by a matrix multiplication in the homogeneous coordinate as

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

Similarly, one can have an xz-reflection and a yz-reflection.

(4) Shear: An xy-shear can be represented by a matrix multiplication in the homogeneous coordinate as

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad a, b \text{ are positive.}$$

Similarly, one can have an xz-shear and a yz-shear with matrices of the form

$$\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively.

(5) Scaling: A scaling is represented by a matrix multiplication in the homogeneous coordinate as

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad a, b, c \text{ are positive.}$$

In summary, all of the transformations can be represented by matrix multiplications in the homogeneous coordinate and also their compositions can be done by their corresponding matrix multiplications.

4.8 Exercises

4.1. Which of the following functions T are linear transformations?

(1) $T(x,\ y) = (x^2 - y^2,\ x^2 + y^2)$.

(2) $T(x, y, z) = (x + y,\ 0,\ 2x + 4z)$.

(3) $T(x,\ y) = (\sin x,\ y)$.

(4) $T(x,\ y) = (x + 1,\ 2y,\ x + y)$.

(5) $T(x, y, z) = (|x|,\ 0)$.

4.2. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation such that $T(1) = 1$, $T(x) = x^2$ and $T(x^2) = x^3 + x$. Find $T(ax^2 + bx + c)$.

4.3. Find $S \circ T$ and/or $T \circ S$ whenever it is defined.

(1) $T(x, y, z) = (x - y + z,\ x + z)$, $S(x,\ y) = (x,\ x - y,\ y)$;

(2) $T(x,\ y) = (x,\ 3y + x,\ 2x - 4y,\ y)$, $S(x, y, z) = (2x,\ y)$.

4.4. Let $S : C(\mathbb{R}) \to C(\mathbb{R})$ be the function on the vector space $C(\mathbb{R})$ defined by, for $f \in C(\mathbb{R})$,

$$S(f)(x) = f(x) - \int_1^x uf(u)du.$$

Show that S is a linear transformation on the vector space $C(\mathbb{R})$.

4.5. Let T be a linear transformation on a vector space V such that $T^2 = id$ and $T \neq id$. Let $U = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{v}\}$ and $W = \{\mathbf{v} \in V : T(\mathbf{v}) = -\mathbf{v}\}$. Show that

(1) at least one of U and W is a nonzero subspace of V;

(2) $U \cap W = \{\mathbf{0}\}$;

(3) $V = U + W$.

4.6. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = (2x - z,\ 3x - 2y,\ x - 2y + z)$,

(1) determine the null space $\mathcal{N}(T)$ of T,

(2) determine whether T is one-to-one,

(3) find a basis for $\mathcal{N}(T)$.

4.7. Show that each of the following linear transformations T on $\mathbb{R}^3$ is invertible, and find a formula for T^{-1}:

(1) $T(x, y, z) = (3x,\ x - y,\ 2x + y + z)$.

(2) $T(x, y, z) = (2x,\ 4x - y,\ 2x + 3y - z)$.

4.8. Let $S, T : V \to V$ be linear transformations on a vector space V.

(1) Show that if $T \circ S$ is one-to-one, then T is an isomorphism.

(2) Show that if $T \circ S$ is onto, then T is an isomorphism.

(3) Show that if T^k is an isomorphism for some positive k, then T is an isomorphism.

4.9. Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$, and let S be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. Prove that the composition $S \circ T$ is not invertible.

4.10. Let T be a linear transformation on a vector space V satisfying $T - T^2 = id$. Show that T is invertible.

4.11. Let A be an $n \times n$ matrix, which is a linear transformation on the n-space $\mathbb{R}^n$ by the matrix multiplication $A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$. Suppose that $\mathbf{r}_1,\ \mathbf{r}_2,\ \ldots,\ \mathbf{r}_n$ are linearly independent vectors in $\mathbb{R}^n$ constituting a parallelepiped (see Remark (2) on page 70). Then A transforms this parallelepiped into another parallelepiped determined by $A\mathbf{r}_1,\ A\mathbf{r}_2,\ \ldots,\ A\mathbf{r}_n$. Suppose that we denote the $n \times n$ matrix whose j-th column is $\mathbf{r}_j$ by B, and the $n \times n$ matrix whose j-th column is $A\mathbf{r}_j$ by C. Prove that

$$\mathrm{vol}(\mathcal{P}(C)) = |\det\ A|\ \mathrm{vol}(\mathcal{P}(B)).$$

(This means that, for a square matrix A considered as a linear transformation, the absolute value of the determinant of A is the ratio between the volumes of a parallelepiped $\mathcal{P}(B)$ and its image parallelepiped $\mathcal{P}(C)$ under the transformation by A. If $\det A = 0$, then the image $\mathcal{P}(C)$ is a parallelepiped in a subspace of dimension less than n).

4.12. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T(x, y, z) = (x + y, y + z, x + z).$$

Let C denote the unit cube in $\mathbb{R}^3$ determined by the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Find the volume of the image parallelepiped $T(C)$ of C under T.

4.13. With respect to the ordered basis $\alpha = \{1,\ x,\ x^2\}$ for the vector space $P_2(\mathbb{R})$, find the coordinate vector of the following polynomials:

$$(1)\ f(x) = x^2 - x + 1, \quad (2)\ f(x) = x^2 + 4x - 1, \quad (3)\ f(x) = 2x + 5.$$

4.14. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by

$$Tf(x) = f''(x) - 4f'(x) + f(x).$$

Find the matrix $[T]_\alpha$ for the basis $\alpha = \{x,\ 1+x,\ x+x^2,\ x^3\}$.

4.15. Let T be the linear transformation on $\mathbb{R}^2$ defined by $T(x,\ y) = (-y,\ x)$.

(1) What is the matrix of T with respect to an ordered basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = (1,\ 2), \mathbf{v}_2 = (1,\ -1)$?

(2) Show that for every real number c the linear transformation $T - c\,id$ is invertible.

4.16. Find the matrix representation of each of the following linear transformations T on $\mathbb{R}^2$ with respect to the standard basis $\{\mathbf{e}_1,\ \mathbf{e}_2\}$.

(1) $T(x,\ y) = (2y,\ 3x - y)$.

(2) $T(x,\ y) = (3x - 4y,\ x + 5y)$.

4.17. Let $M = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

(1) Find the unique linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ so that M is the associated matrix of T with respect to the bases

$$\alpha_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \alpha_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(2) Find $T(x, y, z)$.

4.18. Find the matrix representation of each of the following linear transformations T on $P_2(\mathbb{R})$ with respect to the basis $\{1,\ x,\ x^2\}$.

(1) $T : p(x) \to p(x+1)$.

(2) $T : p(x) \to p'(x)$.

(3) $T : p(x) \to p(0)x$.

(4) $T : p(x) \to \dfrac{p(x) - p(0)}{x}$.

4.19. Consider the following ordered bases of $\mathbb{R}^3$: $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the standard basis and $\beta = \{\mathbf{u}_1 = (1,\ 1,\ 1),\ \mathbf{u}_2 = (1,\ 1,\ 0),\ \mathbf{u}_3 = (1, 0, 0)\}$.

(1) Find the basis-change matrix P from α to β.

(2) Find the basis-change matrix Q from β to α.

(3) Verify that $Q = P^{-1}$.

(4) Show that $[\mathbf{v}]_\beta = P[\mathbf{v}]_\alpha$ for any vector $\mathbf{v} \in \mathbb{R}^3$.

(5) Show that $[T]_\beta = Q^{-1}[T]_\alpha Q$ for the linear transformation T defined by $T(x, y, z) = (2y + x,\ x - 4y,\ 3x)$.

4.20. There are no matrices A and B in $M_{n \times n}(\mathbb{R})$ such that $AB - BA = I_n$.

4.21. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x, y, z) = (3x + 2y - 4z,\ x - 5y + 3z),$$

and let $\alpha = \{(1,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1, 0, 0)\}$ and $\beta = \{(1,\ 3),\ (2,\ 5)\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

(1) Find the associated matrix $[T]_\alpha^\beta$ for T.

(2) Verify $[T]_\alpha^\beta[\mathbf{v}]_\alpha = [T(\mathbf{v})]_\beta$ for any $\mathbf{v} \in \mathbb{R}^3$.

4.22. Find the basis-change matrix $[id]_\alpha^\beta$ from α to β, when

(1) $\alpha = \{(2, 3),\ (0, 1)\},\ \beta = \{(6, 4),\ (4, 8)\}$;

(2) $\alpha = \{(5, 1),\ (1, 2)\},\ \beta = \{(1, 0),\ (0, 1)\}$;

(3) $\alpha = \{(1, 1, 1),\ (1, 1, 0),\ (1, 0, 0)\},\ \beta = \{(2, 0, 3),\ (-1, 4, 1),\ (3, 2, 5)\}$;

(4) $\alpha = \{t,\ 1,\ t^2\},\ \beta = \{3 + 2t + t^2,\ t^2 - 4,\ 2 + t\}$.

4.23. Show that all matrices of the form $A_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ are similar.

4.24. Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ cannot be similar to a diagonal matrix.

4.25. Are the matrices $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 1 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ similar?

4.26. For a linear transformation T on a vector space V, show that T is one-to-one if and only if its transpose T^* is one-to-one.

4.27. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
$T(x, y, z) = (2y + z,\ -x + 4y + z,\ x + z)$.
Compute $[T]_\alpha$ and $[T^*]_{\alpha^*}$ for the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

4.28. Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by
$T(x_1, x_2, x_3) = (x_1 + x_2,\ 2x_3 - x_1)$.

(1) For the standard ordered bases α and β for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, find the associated matrix for T with respect to the bases α and β.

(2) Let $\alpha = \{\mathbf{x}_1, \mathbf{x}_2,\ \mathbf{x}_3\}$ and $\beta = \{\mathbf{y}_1,\ \mathbf{y}_2\}$, where $\mathbf{x}_1 = (1, 0, -1)$, $\mathbf{x}_2 = (1, 1, 1)$, $\mathbf{x}_3 = (1, 0, 0)$, and $\mathbf{y}_1 = (0, 1)$, $\mathbf{y}_2 = (1, 0)$. Find the associated matrices $[T]_\alpha^\beta$ and $[T^*]_{\beta^*}^{\alpha^*}$.

4.29. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ defined by
$T(x, y, z) = (2x + y + 4z,\ x + y + 2z,\ y + 2z,\ x + y + 3z)$.
Find the image and the kernel of T. What is the dimension of $\mathrm{Im}(T)$? Find $[T]_\alpha^\beta$ and $[T^*]_{\beta^*}^{\alpha^*}$, where
$\alpha = \{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$,
$\beta = \{(1, 0, 0, 0),\ (1, 1, 0, 0),\ (1, 1, 1, 0),\ (1, 1, 1, 1)\}$.

4.30. Let T be the linear transformation on $V = \mathbb{R}^3$, for which the associated matrix with respect to the standard ordered basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Find the bases for the kernel and the image of the transpose T^* on V^*.

4.31. Define three linear functionals on the vector space $V = P_2(\mathbb{R})$ by

$$f_1(p) = \int_0^1 p(x)dx, \quad f_2(p) = \int_0^2 p(x)dx, \quad f_3(p) = \int_0^{-1} p(x)dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by finding its dual basis for V.

4.32. Determine whether or not the following statements are true in general, and justify your answers.

(1) For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\mathrm{Ker}(T) = \{\mathbf{0}\}$ if $m > n$.

(2) For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\mathrm{Ker}(T) \neq \{\mathbf{0}\}$ if $m < n$.

(3) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if the nullspace of $[T]_\alpha^\beta$ is $\{\mathbf{0}\}$ for any basis α for $\mathbb{R}^n$ and any basis β for $\mathbb{R}^m$.

(4) For any linear transformation T on $\mathbb{R}^n$, the dimension of the image of T is equal to that of the row space of $[T]_\alpha$ for any basis α for $\mathbb{R}^n$.

(5) For any two linear transformations $T : V \to W$ and $S : W \to Z$, if $\mathrm{Ker}(S \circ T) = \mathbf{0}$, then $\mathrm{Ker}(T) = \mathbf{0}$.

(6) Any polynomial $p(x)$ is linear if and only if the degree of $p(x)$ is less than or equal to 1.

(7) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a function given as $T(\mathbf{x}) = (T_1(\mathbf{x}), T_2(\mathbf{x}))$ for any $\mathbf{x} \in \mathbb{R}^3$. Then T is linear if and only if their coordinate functions T_i, $i = 1, 2$, are linear.

(8) For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, if $[T]_\alpha^\beta = I_n$ for some bases α and β of $\mathbb{R}^n$, then T must be the identity transformation.

(9) If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one, then any matrix representation of T is nonsingular.

(10) Any $m \times n$ matrix A can be a matrix representation of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

(11) Every basis-change matrix is invertible.

(12) A matrix similar to a basis-change matrix is also a basis-change matrix.

(13) $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a linear functional.

(14) Every translation in $\mathbb{R}^n$ is a linear transformation.

5

Inner Product Spaces

5.1 Dot products and inner products

To study a geometry of a vector space, we go back to the case of the 3-space $\mathbb{R}^3$. The **dot** (or **Euclidean inner**) **product** of two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ is a number defined by the formula

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 = [x_1\ x_2\ x_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \mathbf{x}^T \mathbf{y},$$

where $\mathbf{x}^T\mathbf{y}$ is the matrix product of $\mathbf{x}^T$ and $\mathbf{y}$, which is also a number identified with the 1×1 matrix $\mathbf{x}^T\mathbf{y}$. Using the dot product, the **length** (or **magnitude**) of a vector $\mathbf{x} = (x_1, x_2, x_3)$ is defined by

$$\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2 + x_3^2}\,,$$

and the **Euclidean distance** between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

In this way, the dot product can be considered to be a ruler for measuring the length of a line segment in the 3-space $\mathbb{R}^3$. Furthermore, it can also be used to measure the angle between two nonzero vectors: in fact, the **angle** θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ is measured by the formula involving the dot product

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}, \quad 0 \le \theta \le \pi,$$

since the dot product satisfies the formula

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\| \cos\theta.$$

In particular, two vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** (i.e., they form a right angle $\theta = \pi/2$) if and only if the Pythagorean theorem holds:

J H Kwak et al., *Linear Algebra*

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2.$$

By rewriting this formula in terms of the dot product, we obtain another equivalent condition:

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 = 0.$$

In fact, this dot product is one of the most important structures with which $\mathbb{R}^3$ is equipped. Euclidean geometry begins with the vector space $\mathbb{R}^3$ together with the dot product, because the Euclidean distance can be defined by the dot product.

The dot product has a direct extension to the n-space $\mathbb{R}^n$ of any dimension n: for any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, their **dot product**, also called the **Euclidean inner product**, and the **length** (or **magnitude**) of a vector are defined similarly as

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= x_1y_1 + x_2y_2 + \cdots + x_ny_n = \mathbf{x}^T\mathbf{y}, \\ \|\mathbf{x}\| &= (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \end{aligned}$$

To extend this notion of the dot product to a (real) vector space, we extract the most essential properties that the dot product in $\mathbb{R}^n$ satisfies and take these properties as axioms for an inner product on a vector space V.

Definition 5.1 An **inner product** on a real vector space V is a function that associates a real number $\langle \mathbf{x}, \mathbf{y} \rangle$ to each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V in such a way that the following rules are satisfied: For any vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in V and any scalar k in $\mathbb{R}$,

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry),
(2) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (additivity),
(3) $\langle k\mathbf{x}, \mathbf{y} \rangle = k\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity),
(4) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$ (positive definiteness).

A pair $(V, \langle\ ,\ \rangle)$ of a (real) vector space V and an inner product $\langle\ ,\ \rangle$ is called a **(real) inner product space**. In particular, the pair $(\mathbb{R}^n, \cdot)$ is called the **Euclidean n-space**.

Note that by symmetry (1), additivity (2) and homogeneity (3) also hold for the second variable: i.e.,

(2′) $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$,
(3′) $\langle \mathbf{x}, k\mathbf{y} \rangle = k\langle \mathbf{x}, \mathbf{y} \rangle$.

It is easy to show that $\langle \mathbf{0}, \mathbf{y} \rangle = 0\langle \mathbf{0}, \mathbf{y} \rangle = 0$ and also $\langle \mathbf{x}, \mathbf{0} \rangle = 0$.

Remark: In Definition 5.1, the rules (2) and (3) mean that the inner product is linear for the first variable; and the rules (2′) and (3′) above mean that the inner product is also linear for the second variable. In this sense, the inner product is called **bilinear**.

Example 5.1 (*Non-Euclidean inner product on* $\mathbb{R}^2$) For any two vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$, define

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y}\rangle &= ax_1y_1 + c(x_1y_2 + x_2y_1) + bx_2y_2 \\ &= [x_1\ x_2]\begin{bmatrix} a & c \\ c & b\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} = \mathbf{x}^T A\mathbf{y},\end{aligned}$$

where a, b and c are arbitrary real numbers. Then, this function $\langle\ ,\ \rangle$ clearly satisfies the first three rules of the inner product, i.e., $\langle\ ,\ \rangle$ is symmetric and bilinear. Moreover, if $a > 0$ and $\det A = ab - c^2 > 0$ hold, then it also satisfies rule (4), the positive definiteness of the inner product. (Hint: The equation $\langle \mathbf{x}, \mathbf{x}\rangle = ax_1^2 + 2cx_1x_2 + bx_2^2 \geq 0$ if and only if either $x_2 = 0$ or the discriminant of $\langle \mathbf{x}, \mathbf{x}\rangle / x_2^2$ is nonpositive.) In the case of $c = 0$, this reduces to $\langle \mathbf{x}, \mathbf{y}\rangle = ax_1y_1 + bx_2y_2$. Notice also that $a = \langle \mathbf{e}_1, \mathbf{e}_1\rangle$, $b = \langle \mathbf{e}_2, \mathbf{e}_2\rangle$ and $c = \langle \mathbf{e}_1, \mathbf{e}_2\rangle = \langle \mathbf{e}_2, \mathbf{e}_1\rangle$. □

Problem 5.1 In Example 5.1, the converse is also true: Prove that if $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T A\mathbf{y}$ is an inner product in $\mathbb{R}^2$, then $a > 0$ and $ab - c^2 > 0$.

Example 5.2 (*Case of* $\mathbf{x} \neq \mathbf{0} \neq \mathbf{y}$ *but* $\langle \mathbf{x}, \mathbf{y}\rangle = 0$) Let $V = C[0,\ 1]$ be the vector space of all real-valued continuous functions on $[0,\ 1]$. For any two functions f and g in V, define

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx\,.$$

Then $\langle\ ,\ \rangle$ is an inner product on V (verify this). Let

$$f(x) = \begin{cases} 1 - 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then $f \neq \mathbf{0} \neq g$, but $\langle f, g\rangle = 0$. □

By a subspace W of an inner product space V, we mean a subspace of the vector space V together with the inner product that is the restriction of the inner product on V to W.

Example 5.3 (*A subspace as an inner product space*) The set $W = D^1[0,\ 1]$ of all real-valued *differentiable* functions on $[0,\ 1]$ is a subspace of $V = C[0,\ 1]$. The restriction to W of the inner product on V defined in Example 5.2 makes W an inner product subspace of V. However, one can define another inner product on W by the following formula: For any two functions $f(x)$ and $g(x)$ in W,

$$\langle\langle f, g\rangle\rangle = \int_0^1 f(x)g(x)dx + \int_0^1 f'(x)g'(x)dx\,.$$

Then $\langle\langle\ ,\ \rangle\rangle$ is also an inner product on W, which is different from the restriction to W of the inner product of V, and hence W with this new inner product is not a subspace of the inner product space V. □

Remark: From vector calculus, most readers might be already familiar with the dot product (or the inner product) and the cross product (or the outer product) in the 3-space $\mathbb{R}^3$. The concept of this dot product is extended to a higher dimensional Euclidean space $\mathbb{R}^n$ in this section. However, it is known in advanced mathematics that the cross product in the 3-space $\mathbb{R}^3$ cannot be extended to a higher dimensional Euclidean space $\mathbb{R}^n$. In fact, it is known that if there is a bilinear function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$; $f(\mathbf{x}, \mathbf{y}) = \mathbf{x} \times \mathbf{y}$ satisfying the property: $\mathbf{x} \times \mathbf{y}$ is perpendicular to $\mathbf{x}$ and $\mathbf{y}$, and $\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2$, then $n = 3$ or 7. Hence, the cross product or the outer product will not be introduced in linear algebra.

5.2 The lengths and angles of vectors

In this section, we study a geometry of an inner product space by introducing a length, an angle or a distance between two vectors. The following inequality will enable us to define an angle between two vectors in an inner product space V.

Theorem 5.1 (Cauchy–Schwarz inequality) *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in an inner product space* V*, then*

$$\langle \mathbf{x}, \mathbf{y} \rangle^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle.$$

Proof: If $\mathbf{x} = \mathbf{0}$, it is clear. Assume $\mathbf{x} \neq \mathbf{0}$. For any scalar t, we have

$$0 \leq \langle t\mathbf{x} + \mathbf{y}, t\mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle t^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle t + \langle \mathbf{y}, \mathbf{y} \rangle.$$

This inequality implies that the polynomial $\langle \mathbf{x}, \mathbf{x} \rangle t^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle t + \langle \mathbf{y}, \mathbf{y} \rangle$ in t has either no real roots or a repeated real root. Therefore, its discriminant must be nonpositive:

$$\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \leq 0,$$

which implies the inequality. □

Problem 5.2 Prove that the equality in the Cauchy–Schwarz inequality holds if and only if the vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

The lengths of vectors and angles between two vectors in an inner product space are defined in a similar way to the case of the Euclidean n-space.

Definition 5.2 Let V be an inner product space.

(1) The **magnitude** $\|\mathbf{x}\|$ (or the **length**) of a vector $\mathbf{x}$ is defined by

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

(2) The **distance** $d(\mathbf{x}, \mathbf{y})$ between two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

(3) From the Cauchy–Schwarz inequality, we have $-1 \le \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} \le 1$ for any two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$. Hence, there is a unique number $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} \quad \text{or} \quad \langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos\theta.$$

Such a number θ is called the **angle** between $\mathbf{x}$ and $\mathbf{y}$.

For example, the dot product in the Euclidean 3-space $\mathbb{R}^3$ defines the Euclidean distance in $\mathbb{R}^3$. However, one can define infinitely many non-Euclidean distances in $\mathbb{R}^3$ as shown in the following example.

Example 5.4 (*Infinitely many different inner products on* $\mathbb{R}^2$ *or* $\mathbb{R}^3$)

(1) In $\mathbb{R}^2$ equipped with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 3x_2y_2$, the angle between $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (1, 0)$ is computed as

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} = \frac{2}{\sqrt{5} \cdot 2} = 0.6324 \cdots .$$

Thus, $\theta = \cos^{-1}(\frac{2}{\sqrt{10}})$. Notice that in the Euclidean 2-space $\mathbb{R}^2$ with the dot product, the angle between $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (1, 0)$ is clearly $\frac{\pi}{4}$ and $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.7071 \cdots$. It shows that the angle between two vectors depends actually on an inner product on a vector space.

(2) For any diagonal matrix $A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ with all $d_i > 0$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y} = d_1x_1y_1 + d_2x_2y_2 + d_3x_3y_3$$

defines an inner product on $\mathbb{R}^3$. Thus, there are infinitely many different inner products on $\mathbb{R}^3$. Moreover, an inner product in the 3-space $\mathbb{R}^3$ may play the roles of a ruler and a protractor in our physical world $\mathbb{R}^3$. $\square$

Problem 5.3 In Example 5.4(2), show that $\mathbf{x}^T A\mathbf{y}$ cannot be an inner product if A has a negative diagonal entry $d_i < 0$.

Problem 5.4 Prove the following properties of length in an inner product space V: For any vectors $\mathbf{x}$, $\mathbf{y} \in V$,

(1) $\|\mathbf{x}\| \ge 0$,
(2) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,
(3) $\|k\mathbf{x}\| = |k| \, \|\mathbf{x}\|$,
(4) $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality).

Problem 5.5 Let V be an inner product space. Show that for any vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in V,

(1) $d(\mathbf{x}, \mathbf{y}) \geq 0$,
(2) $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$,
(3) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$,
(4) $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ (triangle inequality).

Definition 5.3 Two vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space are said to be **orthogonal** (or **perpendicular**) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Note that for nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ if and only if $\theta = \pi/2$.

Lemma 5.2 *Let V be an inner product space and let $\mathbf{x} \in V$. Then, the vector $\mathbf{x}$ is orthogonal to every vector $\mathbf{y}$ in V (i.e., $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y}$ in V) if and only if $\mathbf{x} = \mathbf{0}$.*

Proof: If $\mathbf{x} = \mathbf{0}$, clearly $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y}$ in V. Suppose that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y}$ in V. Then $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, implying $\mathbf{x} = \mathbf{0}$ by positive definiteness. □

Corollary 5.3 *Let V be an inner product space, and let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Then, a vector $\mathbf{x}$ in V is orthogonal to every basis vector $\mathbf{v}_i$ in α if and only if $\mathbf{x} = \mathbf{0}$.*

Proof: If $\langle \mathbf{x}, \mathbf{v}_i \rangle = 0$ for $i = 1, 2, \ldots, n$, then $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} y_i \langle \mathbf{x}, \mathbf{v}_i \rangle = 0$ for any $\mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{v}_i \in V$. □

Example 5.5 (Pythagorean theorem) Let V be an inner product space, and let $\mathbf{x}$ and $\mathbf{y}$ be any two nonzero vectors in V with the angle θ. Then, $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos\theta$ gives the equality

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\| \|\mathbf{y}\| \cos\theta.$$

Moreover, it deduces the Pythagorean theorem: $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ for any orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$. □

Theorem 5.4 *If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are nonzero mutually orthogonal vectors in an inner product space V (i.e., each vector is orthogonal to every other vector), then they are linearly independent.*

Proof: Suppose $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$. Then for each $i = 1, 2, \ldots, k$,

$$\begin{aligned} 0 &= \langle \mathbf{0}, \mathbf{x}_i \rangle = \langle c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k, \mathbf{x}_i \rangle \\ &= c_1\langle \mathbf{x}_1, \mathbf{x}_i \rangle + \cdots + c_i\langle \mathbf{x}_i, \mathbf{x}_i \rangle + \cdots + c_k\langle \mathbf{x}_k, \mathbf{x}_i \rangle \\ &= c_i \|\mathbf{x}_i\|^2, \end{aligned}$$

because $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are mutually orthogonal. Since each $\mathbf{x}_i$ is not the zero vector, $\|\mathbf{x}_i\| \neq 0$; so $c_i = 0$ for $i = 1, 2, \ldots, k$. □

Problem 5.6 Let $f(x)$ and $g(x)$ be continuous real-valued functions on $[0,\ 1]$. Prove

(1) $\left[\int_0^1 f(x)g(x)dx\right]^2 \le \left[\int_0^1 f^2(x)dx\right]\left[\int_0^1 g^2(x)dx\right]$,

(2) $\left[\int_0^1 (f(x)+g(x))^2 dx\right]^{\frac{1}{2}} \le \left[\int_0^1 f^2(x)dx\right]^{\frac{1}{2}} + \left[\int_0^1 g^2(x)dx\right]^{\frac{1}{2}}$.

Problem 5.7 Let $V = C[0,\ 1]$ be the inner product space of all real-valued continuous functions on $[0,\ 1]$ equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx \quad \text{for any } f \text{ and } g \text{ in } V.$$

For the following two functions f and g in V, compute the angle between them: For any natural numbers $k,\ \ell$,

(1) $f(x) = kx$ and $g(x) = \ell x$,
(2) $f(x) = \sin 2\pi kx$ and $g(x) = \sin 2\pi \ell x$,
(3) $f(x) = \cos 2\pi kx$ and $g(x) = \cos 2\pi \ell x$.

5.3 Matrix representations of inner products

Let A be an $n \times n$ diagonal matrix with positive diagonal entries. Then one can show that $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T A\mathbf{y}$ defines an inner product on the n-space $\mathbb{R}^n$ as shown in Example 5.4(2). The converse is also true: every inner product on a vector space can be expressed in such a matrix product form. Let $(V,\ \langle\ ,\ \rangle)$ be an inner product space, and let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a fixed ordered basis for V. Then for any $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{v}_i$ and $\mathbf{y} = \sum_{j=1}^n y_j\mathbf{v}_j$ in V,

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle \mathbf{v}_i, \mathbf{v}_j\rangle$$

holds. If we set $a_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j\rangle$ for $i, j = 1, 2,\ \ldots,\ n$, then these numbers constitute a symmetric matrix $A = [a_{ij}]$, since $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \langle \mathbf{v}_j, \mathbf{v}_i\rangle$. Thus, in matrix notation, the inner product may be written as

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j a_{ij} = [\mathbf{x}]_\alpha^T A[\mathbf{y}]_\alpha.$$

The matrix A is called the **matrix representation** of the inner product $\langle\ ,\ \rangle$ with respect to the basis α.

Example 5.6 (*Matrix representation of an inner product*)

(1) With respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n\}$ for the Euclidean n-space $\mathbb{R}^n$, *the matrix representation of the dot product is the identity matrix,* since

$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Thus, for $\mathbf{x} = \sum_i x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_j y_j \mathbf{e}_j \in \mathbb{R}^n$, the dot product is just the matrix product $\mathbf{x}^T \mathbf{y}$:

$$\mathbf{x} \cdot \mathbf{y} = [x_1 \cdots x_n] \begin{bmatrix} 1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \mathbf{x}^T \mathbf{y}.$$

(2) On $V = P_2([0, 1])$, we define an inner product on V as

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Then for a basis $\alpha = \{f_1(x) = 1,\ f_2(x) = x,\ f_3(x) = x^2\}$ for V, one can easily find its matrix representation $A = [a_{ij}]$: For instance,

$$a_{23} = \langle f_2, f_3 \rangle = \int_0^1 f_2(x) f_3(x) dx = \int_0^1 x \cdot x^2 dx = \frac{1}{4}. \qquad \square$$

The expression of the dot product as a matrix product is very useful in stating or proving theorems in the Euclidean space.

For any symmetric matrix A and for a fixed basis α, the formula $\langle \mathbf{x}, \mathbf{y} \rangle = [\mathbf{x}]_\alpha^T A [\mathbf{y}]_\alpha$ seems to give rise to an inner product on V. In fact, the formula clearly is symmetric and bilinear, but does not necessarily satisfy the fourth rule, *positive definiteness*. The following theorem gives a necessary condition for a symmetric matrix A to give rise to an inner product. Some necessary and sufficient conditions will be discussed in Chapter 8.

Theorem 5.5 *The matrix representation A of an inner product (with respect to any basis) on a vector space V is invertible. That is,* $\det A \neq 0$.

Proof: Let $\langle \mathbf{x}, \mathbf{y} \rangle = [\mathbf{x}]_\alpha^T A [\mathbf{y}]_\alpha$ be an inner product in a vector space V with respect to a basis α, and let $A[\mathbf{y}]_\alpha = \mathbf{0}$ as a homogeneous system of linear equations. Then

$$\langle \mathbf{y}, \mathbf{y} \rangle = [\mathbf{y}]_\alpha^T A [\mathbf{y}]_\alpha = 0.$$

It implies that $A[\mathbf{y}]_\alpha = \mathbf{0}$ has only the trivial solution $\mathbf{y} = \mathbf{0}$, or equivalently A is invertible by Theorem 1.9. $\square$

Recall that the conditions $a > 0$ and $\det A = ab - c^2 > 0$ in Example 5.1 are sufficient for A to give rise to an inner product on $\mathbb{R}^2$.

5.4 Gram–Schmidt orthogonalization

The standard basis for the Euclidean n-space $\mathbb{R}^n$ has a special property: The basis vectors are mutually orthogonal and are of length 1. In this sense, it is called the

rectangular coordinate system for $\mathbb{R}^n$. In an inner product space, a vector with length 1 is called a **unit vector**. If $\mathbf{x}$ is a nonzero vector in an inner product space V, the vector $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a unit vector. The process of obtaining a unit vector from a nonzero vector by multiplying the reciprocal of its length is called a **normalization**. Thus, if there is a set of mutually orthogonal vectors (or a basis) in an inner product space, then the vectors can be converted to unit vectors by normalizing them without losing their mutual orthogonality.

Definition 5.4 A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ in an inner product space V is said to be **orthonormal** if

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonality)}, \\ 1 & \text{if } i = j \quad \text{(normality)}. \end{cases}$$

A set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of vectors is called an **orthonormal basis** for V if it is a basis and orthonormal.

Problem 5.8 Determine whether each of the following sets of vectors in $\mathbb{R}^2$ is orthogonal, orthonormal, or neither with respect to the Euclidean inner product.

$$\text{(1)} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\} \quad \text{(2)} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$$

$$\text{(3)} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\} \quad \text{(4)} \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

It will be shown later in Theorem 5.6 that every inner product space has an orthonormal basis, just like the standard basis for the Euclidean n-space $\mathbb{R}^n$.

The following example illustrates how to construct such an orthonormal basis.

Example 5.7 (*How to construct an orthonormal basis?*) For a matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix},$$

find an orthonormal basis for the column space $\mathcal{C}(A)$ of A.

Solution: Let $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ be the column vectors of A in the order from left to right. It is easily verified that they are linearly independent, so they form a basis for the column space $\mathcal{C}(A)$ of dimension 3 in $\mathbb{R}^4$. For notational convention, we denote by $\mathbf{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ the subspace spanned by $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

(1) First normalize $\mathbf{c}_1$ to get

$$\mathbf{u}_1 = \frac{\mathbf{c}_1}{\|\mathbf{c}_1\|} = \frac{\mathbf{c}_1}{2} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right),$$

which is a unit vector. Then $\mathbf{Span}\{\mathbf{u}_1\}$ =$\mathbf{Span}\{\mathbf{c}_1\}$, because one is a scalar multiple of the other.

(2) Noting that the vector $\mathbf{c}_2 - \langle \mathbf{u}_1, \mathbf{c}_2\rangle\mathbf{u}_1 = \mathbf{c}_2 - 2\mathbf{u}_1 = (0,\ 1,\ -1,\ 0)$ is a nonzero vector orthogonal to $\mathbf{u}_1$, we set

$$\mathbf{u}_2 = \frac{\mathbf{c}_2 - \langle \mathbf{u}_1, \mathbf{c}_2\rangle\mathbf{u}_1}{\|\mathbf{c}_2 - \langle \mathbf{u}_1, \mathbf{c}_2\rangle\mathbf{u}_1\|} = \frac{1}{\sqrt{2}}(0,\ 1,\ -1,\ 0) = \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right).$$

Then, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is orthonormal and $\mathbf{Span}\{\mathbf{u}_1, \mathbf{u}_2\} = \mathbf{Span}\{\mathbf{c}_1, \mathbf{c}_2\}$, because each $\mathbf{u}_i$ is a linear combination of $\mathbf{c}_1$ and $\mathbf{c}_2$, and the converse is also true.

(3) Finally, note that $\mathbf{c}_3 - \langle \mathbf{u}_1, \mathbf{c}_3\rangle\mathbf{u}_1 - \langle \mathbf{u}_2, \mathbf{c}_3\rangle\mathbf{u}_2 = \mathbf{c}_3 - 4\mathbf{u}_1 + \sqrt{2}\mathbf{u}_2 = (0,\ 1,\ 1,\ -2)$ is also a nonzero vector orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. In fact,

$$\begin{aligned} \langle \mathbf{u}_1,\ \mathbf{c}_3 - 4\mathbf{u}_1 + \sqrt{2}\,\mathbf{u}_2\rangle &= \langle \mathbf{u}_1,\ \mathbf{c}_3\rangle - 4\langle \mathbf{u}_1, \mathbf{u}_1\rangle + \sqrt{2}\,\langle \mathbf{u}_1,\ \mathbf{u}_2\rangle = 0, \\ \langle \mathbf{u}_2,\ \mathbf{c}_3 - 4\mathbf{u}_1 + \sqrt{2}\,\mathbf{u}_2\rangle &= \langle \mathbf{u}_2,\ \mathbf{c}_3\rangle - 4\langle \mathbf{u}_2,\ \mathbf{u}_1\rangle + \sqrt{2}\,\langle \mathbf{u}_2, \mathbf{u}_2\rangle = 0. \end{aligned}$$

By the normalization, the vector

$$\mathbf{u}_3 = \frac{\mathbf{c}_3 - \langle \mathbf{u}_1, \mathbf{c}_3\rangle\mathbf{u}_1 - \langle \mathbf{u}_2, \mathbf{c}_3\rangle\mathbf{u}_2}{\|\mathbf{c}_3 - \langle \mathbf{u}_1, \mathbf{c}_3\rangle\mathbf{u}_1 - \langle \mathbf{u}_2, \mathbf{c}_3\rangle\mathbf{u}_2\|} = \frac{1}{\sqrt{6}}(0,\ 1,\ 1,\ -2)$$

is a unit vector, and one can also show that $\mathbf{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbf{Span}\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\} = \mathcal{C}(A)$. Consequently, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathcal{C}(A)$. $\square$

The orthonormalization process in Example 5.7 indicates how to prove the following general case, called the **Gram–Schmidt orthogonalization.**

Theorem 5.6 *Every inner product space has an orthonormal basis.*

Proof: [Gram–Schmidt orthogonalization process] Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for an n-dimensional inner product space V. Let

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}, \qquad \mathbf{u}_2 = \frac{\mathbf{x}_2 - \langle \mathbf{u}_1, \mathbf{x}_2\rangle\mathbf{u}_1}{\|\mathbf{x}_2 - \langle \mathbf{u}_1, \mathbf{x}_2\rangle\mathbf{u}_1\|}.$$

Of course, $\mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{u}_1\rangle\mathbf{u}_1 \neq \mathbf{0}$, because $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent. Generally, one can define by induction on $k = 1, 2, \ldots, n$,

$$\mathbf{u}_k = \frac{\mathbf{x}_k - \langle \mathbf{u}_1, \mathbf{x}_k\rangle\mathbf{u}_1 - \langle \mathbf{u}_2, \mathbf{x}_k\rangle\mathbf{u}_2 - \cdots - \langle \mathbf{u}_{k-1}, \mathbf{x}_k\rangle\mathbf{u}_{k-1}}{\|\mathbf{x}_k - \langle \mathbf{u}_1, \mathbf{x}_k\rangle\mathbf{u}_1 - \langle \mathbf{u}_2, \mathbf{x}_k\rangle\mathbf{u}_2 - \cdots - \langle \mathbf{u}_{k-1}, \mathbf{x}_k\rangle\mathbf{u}_{k-1}\|}.$$

Then, as Example 5.7 shows, the vectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \ldots,\ \mathbf{u}_n$ are orthonormal in the n-dimensional vector space V. Since every orthonormal set is linearly independent, it is an orthonormal basis for V. $\square$

Problem 5.9 Use the Gram–Schmidt orthogonalization on the Euclidean space $\mathbb{R}^4$ to transform the basis

$$\{(0,\ 1,\ 1,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (1,\ 2,\ 0,\ -1),\ (-1,\ 0,\ 0,\ -1)\}$$

into an orthonormal basis.

Problem 5.10 Find an orthonormal basis for the subspace W of the Euclidean space $\mathbb{R}^3$ given by $x + 2y - z = 0$.

Problem 5.11 Let $V = C[0, 1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx \quad \text{for any } f \text{ and } g \text{ in } V.$$

Find an orthonormal basis for the subspace spanned by $1, x$ and x^2.

The next theorem shows that an orthornormal basis acts just like the standard basis for the Euclidean n-space $\mathbb{R}^n$.

Theorem 5.7 *Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for a subspace U in an inner product space V. Then, for any vector $\mathbf{x}$ in U,*

$$\mathbf{x} = \langle \mathbf{u}_1, \mathbf{x}\rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{x}\rangle \mathbf{u}_2 + \cdots + \langle \mathbf{u}_k, \mathbf{x}\rangle \mathbf{u}_k.$$

Proof: For any vector $\mathbf{x} \in U$, one can write $\mathbf{x} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_k\mathbf{u}_k$, as a linear combination of the basis vectors. However, for each $i = 1, \ldots, n$,

$$\begin{aligned} \langle \mathbf{u}_i, \mathbf{x}\rangle &= \langle \mathbf{u}_i,\ x_1\mathbf{u}_1 + \cdots + x_k\mathbf{u}_k\rangle \\ &= x_1\langle \mathbf{u}_i, \mathbf{u}_1\rangle + \cdots + x_i\langle \mathbf{u}_i, \mathbf{u}_i\rangle + \cdots + x_k\langle \mathbf{u}_i, \mathbf{u}_k\rangle \\ &= x_i, \end{aligned}$$

because $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is orthonormal. □

In particular, if $\alpha = \{\mathbf{v}_1, \mathbf{v}_2,\ \ldots,\ \mathbf{v}_n\}$ is an orthonormal basis for V, then any vector $\mathbf{x}$ in V can be written uniquely as

$$\mathbf{x} = \langle \mathbf{v}_1, \mathbf{x}\rangle \mathbf{v}_1 + \langle \mathbf{v}_2, \mathbf{x}\rangle \mathbf{v}_2 + \cdots + \langle \mathbf{v}_n, \mathbf{x}\rangle \mathbf{v}_n.$$

Moreover, one can identify an n-dimensional inner product space V with the Euclidean n-space $\mathbb{R}^n$. Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2,\ \ldots,\ \mathbf{v}_n\}$ be an orthonormal basis for the space V. With this orthonormal basis α, the natural isomorphism $\Phi : V \to \mathbb{R}^n$ given by $\Phi(\mathbf{v}_i) = [\mathbf{v}_i]_\alpha = \mathbf{e}_i, i = 1, 2,\ \ldots,\ n$ preserves the inner product on vectors: For a vector $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{v}_i$ in V with $x_i = \langle \mathbf{x}, \mathbf{v}_i\rangle$, the coordinate vector of $\mathbf{x}$ with respect to α is a column matrix

$$\Phi(\mathbf{x}) = [\mathbf{x}]_\alpha = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Moreover, for another vector $\mathbf{y} = \sum_{i=1}^n y_i\mathbf{v}_i$ in V,

$$\langle \mathbf{x}, \mathbf{y}\rangle = \left\langle \sum_{i=1}^n x_i\mathbf{v}_i, \sum_{j=1}^n y_j\mathbf{v}_j \right\rangle = \sum_{i=1}^n x_i y_i = [\mathbf{x}]_\alpha^T[\mathbf{y}]_\alpha.$$

The right-hand side of this equation is just the dot product of vectors in the Euclidean space $\mathbb{R}^n$. That is,

$$\langle \mathbf{x}, \mathbf{y}\rangle = [\mathbf{x}]_\alpha^T[\mathbf{y}]_\alpha = \Phi(\mathbf{x}) \cdot \Phi(\mathbf{y})$$

for any $\mathbf{x}$, $\mathbf{y} \in V$. Hence, *the natural isomorphism Φ preserves the inner product,* and we have the following theorem (compare with Corollary 4.8(1)).

Theorem 5.8 *Any n-dimension inner product space V with an inner product $\langle\ ,\ \rangle$ is isomorphic to the Euclidean n-space $\mathbb{R}^n$ with the dot product $\cdot$, which means that an isomorphism preserves the inner product.*

In this sense, someone likes to restrict the study of an inner product space to the case of the Euclidean n-space $\mathbb{R}^n$ with the dot product.

A special kind of linear transformation that preserves the inner product such as the natural isomorphism from V to $\mathbb{R}^n$ plays an important role in linear algebra, and it will be studied in detail in Section 5.8.

5.5 Projections

Let U be a subspace of a vector space V. Then, by Corollary 3.13 there is another subspace W of V such that $V = U \oplus W$, so that any $\mathbf{x} \in V$ has a unique expression as $\mathbf{x} = \mathbf{u} + \mathbf{w}$ for $\mathbf{u} \in U$ and $\mathbf{w} \in W$. As an easy exercise, one can show that a function $T : V \to V$ defined by $T(\mathbf{x}) = T(\mathbf{u} + \mathbf{w}) = \mathbf{u}$ is a linear transformation, whose image $\mathrm{Im}(T) = T(V)$ is the subspace U and kernel $\mathrm{Ker}(T)$ is the subspace W.

Definition 5.5 Let U and W be subspaces of a vector space V. A linear transformation $T : V \to V$ is called the **projection** of V onto the subspace U along W if $V = U \oplus W$ and $T(\mathbf{x}) = \mathbf{u}$ for $\mathbf{x} = \mathbf{u} + \mathbf{w} \in U \oplus W$.

Example 5.8 (*Infinitely many different projections of $\mathbb{R}^2$ onto the x-axis*) Let X, Y and Z be the 1-dimensional subspaces of the Euclidean 2-space $\mathbb{R}^2$ spanned by the vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2 = (1, 1)$, respectively:

$$\begin{aligned} X &= \{r\mathbf{e}_1 : r \in \mathbb{R}\} = x\text{-axis}, \\ Y &= \{r\mathbf{e}_2 : r \in \mathbb{R}\} = y\text{-axis}, \\ Z &= \{r(\mathbf{e}_1 + \mathbf{e}_2) : r \in \mathbb{R}\}. \end{aligned}$$

Since the pairs $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}_1, \mathbf{v}\}$ are linearly independent, the space $\mathbb{R}^2$ can be expressed as the direct sum in two ways: $\mathbb{R}^2 = X \oplus Y = X \oplus Z$.

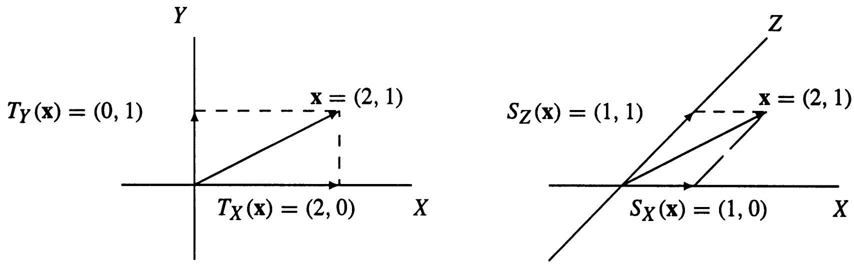

Figure 5.1. Two decompositions of $\mathbb{R}^2$

Thus, a vector $\mathbf{x} = (2, 1) \in \mathbb{R}^2$ may be written in two ways:

$$\mathbf{x} = (2, 1) = \begin{cases} 2(1,0) + (0,1) & \in \; X \oplus Y \; = \; \mathbb{R}^2, \quad \text{or} \\ \;\;(1,0) + (1,1) & \in \; X \oplus Z \; = \; \mathbb{R}^2. \end{cases}$$

Let T_X and S_X denote the projections of $\mathbb{R}^2$ onto X along Y and Z, respectively. Then

$$\begin{aligned} T_X(\mathbf{x}) &= 2(1,0) \in X, \qquad T_Y(\mathbf{x}) = (0,1) \in Y, \;\text{ and} \\ S_X(\mathbf{x}) &= \;\;(1,0) \in X, \qquad S_Z(\mathbf{x}) = (1,1) \in Z. \end{aligned}$$

This shows that a projection of $\mathbb{R}^2$ onto the subspace X depends on a choice of complementary subspace of X. For example, by choosing $Z_n = \{r(n, 1) : r \in \mathbb{R}\}$ for any integer n as a complementary subspace, one can construct infinitely many different projections of $\mathbb{R}^2$ onto the x-axis. □

Note that for a given subspace U of V, a projection T of V onto U depends on the choice of a complementary subspace W of U as shown in Example 5.8. However, by definition, $T(\mathbf{u}) = \mathbf{u}$ for any $\mathbf{u} \in U$ and for any choice of W. That is, $T \circ T = T$ for every projection T of V.

The following theorem shows an algebraic characterization of a linear transformation to be a projection.

Theorem 5.9 *A linear transformation $T : V \to V$ is a projection if and only if $T = T^2$ ($= T \circ T$ by definition).*

Proof: The necessity is clear, because $T \circ T = T$ for any projection T.

For the sufficiency, suppose $T^2 = T$. It suffices to show that $V = \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$ and $T(\mathbf{u} + \mathbf{w}) = \mathbf{u}$ for any $\mathbf{u} + \mathbf{w} \in \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$. First, one needs to prove $\mathrm{Im}(T) \cap \mathrm{Ker}(T) = \{\mathbf{0}\}$ and $V = \mathrm{Im}(T) + \mathrm{Ker}(T)$. Indeed, if $\mathbf{y} \in \mathrm{Im}(T) \cap \mathrm{Ker}(T)$, then there exists $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{y}$ and $T(\mathbf{y}) = \mathbf{0}$. It implies

$$\mathbf{y} = T(\mathbf{x}) = T^2(\mathbf{x}) = T(T(\mathbf{x})) = T(\mathbf{y}) = \mathbf{0}.$$

The hypothesis $T^2 = T$ also shows that $T(\mathbf{v}) \in \mathrm{Im}(T)$ and $\mathbf{v} - T(\mathbf{v}) \in \mathrm{Ker}(T)$ for any $\mathbf{v} \in V$. It implies $V = \mathrm{Im}(T) + \mathrm{Ker}(T)$. Finally, note that $T(\mathbf{u} + \mathbf{w}) = T(\mathbf{u}) + T(\mathbf{w}) = T(\mathbf{u}) = \mathbf{u}$ for any $\mathbf{u} + \mathbf{w} \in \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$. $\square$

Let $T : V \to V$ be a projection, so that $V = \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$. It is not difficult to show that $\mathrm{Im}(id_V - T) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(id_V - T) = \mathrm{Im}(T)$ for the identity transformation id_V on V.

Corollary 5.10 *A linear transformation $T : V \to V$ is a projection if and only if $id_V - T$ is a projection. Moreover, if T is the projection of V onto a subspace U along W, then $id_V - T$ is the projection of V onto W along U.* $\square$

Proof: It is enough to show that $(id_V - T) \circ (id_V - T) = id_V - T$. But

$$(id_V - T) \circ (id_V - T) = (id_V - T) - (T - T^2) = id_V - T. \qquad \square$$

Problem 5.12 For $V = U \oplus W$, let T_U denote the projection of V onto U along W, and let T_W denote the projection of V onto W along U. Prove the following.

(1) For any $\mathbf{x} \in V$, $\mathbf{x} = T_U(\mathbf{x}) + T_W(\mathbf{x})$.
(2) $T_U \circ (id_V - T_U) = \mathbf{0}$.
(3) $T_U \circ T_W = T_W \circ T_U = \mathbf{0}$.
(4) For any projection $T : V \to V$, $\mathrm{Im}(id_V - T) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(id_V - T) = \mathrm{Im}(T)$.

5.6 Orthogonal projections

Let U be a subspace of a vector space V. As shown in Example 5.8, we learn that there are infinitely many projections of V onto U which depend on a choice of complementary subspace W of U. However, if V is an inner product space, there is a particular choice of complementary subspace W, called the *orthogonal complement* of U, along which the projection onto U is called the *orthogonal projection,* defined below. To show this, we first extend the orthogonality of two vectors to an orthogonality of two subspaces.

Definition 5.6 Let U and W be subspaces of an inner product space V.

(1) Two subspaces U and W are said to be **orthogonal**, written $U \perp W$, if $\langle \mathbf{u}, \mathbf{w} \rangle = 0$ for each $\mathbf{u} \in U$ and $\mathbf{w} \in W$.
(2) The set of all vectors in V that are orthogonal to every vector in U is called the **orthogonal complement** of U, denoted by $U^{\perp}$, i.e.,

$$U^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{u} \rangle = 0 \text{ for all } \mathbf{u} \in U\}.$$

One can easily show that $U^\perp$ is a subspace of V, and $\mathbf{v} \in U^\perp$ if and only if $\langle \mathbf{v}, \mathbf{u} \rangle = 0$ for every $\mathbf{u} \in \beta$, where β is a basis for U. Moreover, $W \perp U$ if and only if $W \subseteq U^\perp$.

Problem 5.13 Let U and W be subspaces of an inner product space V. Show that

(1) If $U \perp W$, $U \cap W = \{\mathbf{0}\}$. (2) $U \subseteq W$ if and only if $W^\perp \subseteq U^\perp$.

Theorem 5.11 *Let U be a subspace of an inner product space V. Then*

(1) $(U^\perp)^\perp = U$.
(2) $V = U \oplus U^\perp$: *that is, for each* $\mathbf{x} \in V$, *there exist unique vectors* $\mathbf{x}_U \in U$ *and* $\mathbf{x}_{U^\perp} \in U^\perp$ *such that* $\mathbf{x} = \mathbf{x}_U + \mathbf{x}_{U^\perp}$. *This is called the* **orthogonal decomposition** *of V (or of $\mathbf{x}$) by U.*

Proof: Let $\dim U = k$. To show $(U^\perp)^\perp = U$, take an orthonormal basis for U, say $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, by the Gram–Schmidt orthogonalization, and then extend it to an orthonormal basis for V, say $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$, which is always possible. Then, clearly $\gamma = \{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ forms an (orthonormal) basis for $U^\perp$, which means that $(U^\perp)^\perp = U$ and $V = U \oplus U^\perp$. □

Definition 5.7 Let U be a subspace of an inner product space V, and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be an orthonormal basis for U. The **orthogonal projection** Proj_U from V onto the subspace U is defined by

$$\boxed{\mathrm{Proj}_U(\mathbf{x}) = \langle \mathbf{u}_1, \mathbf{x} \rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{x} \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{u}_m, \mathbf{x} \rangle \mathbf{u}_m}$$

for any $\mathbf{x} \in V$.

Clearly, Proj_U is linear and a projection, because $\mathrm{Proj}_U \circ \mathrm{Proj}_U = \mathrm{Proj}_U$. Moreover, $\mathrm{Proj}_U(\mathbf{x}) \in U$ and $\mathbf{x} - \mathrm{Proj}_U(\mathbf{x}) \in U^\perp$, because

$$\langle \mathbf{x} - \mathrm{Proj}_U(\mathbf{x}), \mathbf{u}_i \rangle = \langle \mathbf{x}, \mathbf{u}_i \rangle - \langle \mathrm{Proj}_U(\mathbf{x}), \mathbf{u}_i \rangle = \langle \mathbf{x}, \mathbf{u}_i \rangle - \langle \mathbf{u}_i, \mathbf{x} \rangle = 0$$

for every basis vector $\mathbf{u}_i$. Hence, by Theorem 5.7, we have

Corollary 5.12 *The orthogonal projection* Proj_U *is the projection of V onto a subspace U along its orthogonal complement $U^\perp$.*

Therefore, in Definition 5.7, the projection $\mathrm{Proj}_U(\mathbf{x})$ is independent of the choice of an orthonormal basis for the subspace U. In this sense, it is called *the* orthogonal projection from the inner product space V onto the subspace U.

Almost all projections used in linear algebra are orthogonal projections.

Example 5.9 (*The orthogonal projection from $\mathbb{R}^3$ onto the xy-plane*) In the Euclidean 3-space $\mathbb{R}^3$, let U be the xy-plane with the orthonormal basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$. Then, the orthogonal projection $\mathrm{Proj}_U(\mathbf{x}) = \langle \mathbf{e}_1, \mathbf{x}\rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{x}\rangle \mathbf{e}_2$ is the orthogonal projection onto the xy-plane in a usual sense in geometry, and $\mathbf{x} - \mathrm{Proj}_U(\mathbf{x}) \in U^\perp$, which is the z-axis. It actually means that $\mathrm{Proj}_U(x_1, x_2, x_3) = (x_1, x_2, 0)$ for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. □

Example 5.10 (*The orthogonal projection from $\mathbb{R}^2$ onto the x-axis*) As in Example 5.8, let X, Y and Z be the 1-dimensional subspaces of the Euclidean 2-space $\mathbb{R}^2$ spanned by the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2 = (1, 1)$, respectively. Then clearly $Y = X^\perp$ and $V \neq X^\perp$. And, for the projections T_X and S_X of $\mathbb{R}^2$ given in Example 5.8, T_X is the orthogonal projection, but S_X is not, so that $T_X = \mathrm{Proj}_X$ and $S_X \neq \mathrm{Proj}_X$. □

Theorem 5.13 *Let U be a subspace of an inner product space V, and let $\mathbf{x} \in V$. Then, the orthogonal projection $\mathrm{Proj}_U(\mathbf{x})$ of $\mathbf{x}$ satisfies*

$$\|\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})\| \leq \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{y} \in U$. The equality holds if and only if $\mathbf{y} = \mathrm{Proj}_U(\mathbf{x})$.

Proof: First, note that for any vector $\mathbf{x} \in V$, we have $\mathrm{Proj}_U(\mathbf{x}) \in U$ and $\mathbf{x} - \mathrm{Proj}_U(\mathbf{x}) \in U^\perp$. Thus, for all $\mathbf{y} \in U$,

$$\begin{aligned}
\|\mathbf{x} - \mathbf{y}\|^2 &= \|(\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})) + (\mathrm{Proj}_U(\mathbf{x}) - \mathbf{y})\|^2 \\
&= \|\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})\|^2 + \|\mathrm{Proj}_U(\mathbf{x}) - \mathbf{y}\|^2 \\
&\geq \|\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})\|^2,
\end{aligned}$$

where the second equality comes from the Pythagorean theorem for the orthogonality $(\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})) \perp (\mathrm{Proj}_U(\mathbf{x}) - \mathbf{y})$. (See Figure 5.2.) □

It follows from Theorem 5.13 that *the orthogonal projection* $\mathrm{Proj}_U(\mathbf{x})$ *of* $\mathbf{x}$ *is the unique vector in U that is closest to* $\mathbf{x}$ in the sense that it minimizes the distance from $\mathbf{x}$ to the vectors in U. It also shows that in Definition 5.7, the vector $\mathrm{Proj}_U(\mathbf{x})$ is independent of the choice of an orthonormal basis for the subspace U. Geometrically, Figure 5.2 depicts the vector $\mathrm{Proj}_U(\mathbf{x})$.

Problem 5.14 Find the point on the plane $x - y - z = 0$ that is closest to $\mathbf{p} = (1, 2, 0)$.

Problem 5.15 Let $U \subset \mathbb{R}^4$ be the subspace of the Euclidean 4-space $\mathbb{R}^4$ spanned by $(1, 1, 0, 0)$ and $(1, 0, 1, 0)$, and let $W \subset \mathbb{R}^4$ be the subspace spanned by $(0, 1, 0, 1)$ and $(0, 0, 1, 1)$. Find a basis for and the dimension of each of the following subspaces:
(1) $U + W$, (2) $U^\perp$, (3) $U^\perp + W^\perp$, (4) $U \cap W$.

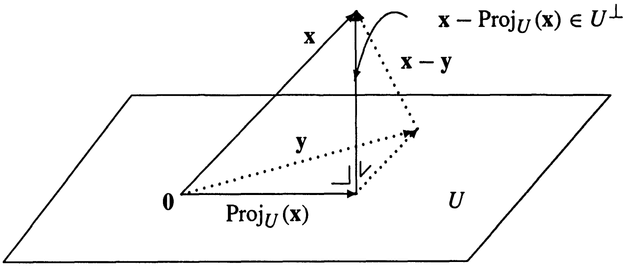

Figure 5.2. Orthogonal projection Proj_U

Problem 5.16 Let U and W be subspaces of an inner product space V. Show that

(1) $(U+W)^{\perp} = U^{\perp} \cap W^{\perp}$. (2) $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$.

As a particular case, let $V = \mathbb{R}^n$ be the Euclidean n-space with the dot product and let $U = \{r\mathbf{u} : r \in \mathbb{R}\}$ be a 1-dimensional subspace determined by a unit vector $\mathbf{u}$. Then for a vector $\mathbf{x}$ in $\mathbb{R}^n$, the orthogonal projection of $\mathbf{x}$ into U is

$$\mathrm{Proj}_U(\mathbf{x}) = (\mathbf{u} \cdot \mathbf{x})\mathbf{u} = (\mathbf{u}^T\mathbf{x})\mathbf{u} = \mathbf{u}(\mathbf{u}^T\mathbf{x}) = (\mathbf{u}\mathbf{u}^T)\mathbf{x}.$$

(Here, the last two equalities come from the facts that $\mathbf{u} \cdot \mathbf{x} = \mathbf{u}^T\mathbf{x}$ is a scalar and the associativity of a matrix product $\mathbf{u}\mathbf{u}^T\mathbf{x}$, respectively.) This equation shows that the matrix representation of the orthogonal projection Proj_U with respect to the standard basis α is

$$[\mathrm{Proj}_U]_\alpha = \mathbf{u}\mathbf{u}^T.$$

If U is an m-dimensional subspace of $\mathbb{R}^n$ with an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, then for any $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}\mathrm{Proj}_U(\mathbf{x}) &= (\mathbf{u}_1 \cdot \mathbf{x})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{x})\mathbf{u}_2 + \cdots + (\mathbf{u}_m \cdot \mathbf{x})\mathbf{u}_m \\ &= \mathbf{u}_1(\mathbf{u}_1^T\mathbf{x}) + \mathbf{u}_2(\mathbf{u}_2^T\mathbf{x}) + \cdots + \mathbf{u}_m(\mathbf{u}_m^T\mathbf{x}) \\ &= (\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_m\mathbf{u}_m^T)\mathbf{x}.\end{aligned}$$

Thus, the matrix representation of the orthogonal projection Proj_U with respect to the standard basis α is

$$[\mathrm{Proj}_U]_\alpha = \mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_m\mathbf{u}_m^T.$$

Definition 5.8 The matrix representation $[\mathrm{Proj}_U]_\alpha$ of the orthogonal projection $\mathrm{Proj}_U : \mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ onto a subspace U with respect to the standard basis α is called the (**orthogonal**) **projection matrix** on U.

Further discussions about the orthogonal projection matrices will be continued in Section 5.9.3.

Example 5.11 (*Distance from a point to a line*) Let $ax+by+c=0$ be a line L in the plane $\mathbb{R}^2$. (Note that the line L cannot be a subspace of $\mathbb{R}^2$ if $c \neq 0$.) For any two points $Q=(x_1, y_1)$ and $R=(x_2, y_2)$ on the line, the equality $a(x_2-x_1)+b(y_2-y_1)=0$ implies that the nonzero vector $\mathbf{n}=(a, b)$ is perpendicular to the line L, that is, $\overrightarrow{QR} \perp \mathbf{n}$.

Let $P=(x_0, y_0)$ be any point in the plane $\mathbb{R}^2$. Then the distance d between the point P and the line L is simply the length of the orthogonal projection of $\overrightarrow{QP}$ into $\mathbf{n}$, for any point $Q=(x_1, y_1)$ in the line. Thus,

$$\begin{aligned} d &= \|\mathrm{Proj}_{\mathbf{n}}(\overrightarrow{QP})\| \\ &= \left|\left\langle \overrightarrow{QP}, \frac{\mathbf{n}}{\|\mathbf{n}\|} \right\rangle\right| \qquad \text{(the dot product)} \\ &= \frac{|a(x_0-x_1)+b(y_0-y_1)|}{\sqrt{a^2+b^2}} \\ &= \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}. \end{aligned}$$

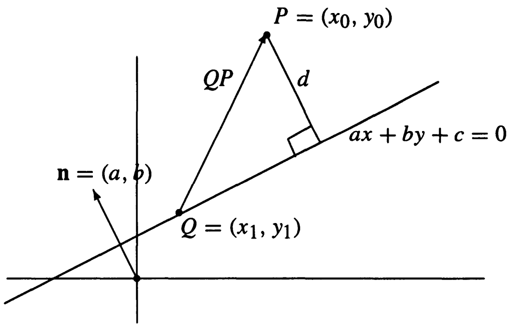

Figure 5.3. Distance from a point to a line

Note that the last equality is due to the fact that the point Q is on the line (i.e., $ax_1+by_1+c=0$).

To find the orthogonal projection matrix, let $\mathbf{u}=\frac{\mathbf{n}}{\|\mathbf{n}\|}=\frac{1}{\sqrt{a^2+b^2}}(a, b)$. Then the orthogonal projection matrix onto $U=\{r\mathbf{u} \,:\, r \in \mathbb{R}\}$ is

$$\mathbf{u}\mathbf{u}^T = \frac{1}{a^2+b^2}\begin{bmatrix} a \\ b \end{bmatrix}[a\ b] = \frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}.$$

Thus, if $\mathbf{x}=(1,1) \in \mathbb{R}^2$, then

$$\mathrm{Proj}_U(\mathbf{x}) = (\mathbf{u}\mathbf{u}^T)\mathbf{x} = \frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} a^2+ab \\ b^2+ab \end{bmatrix}. \qquad \square$$

Problem 5.17 Let $V = P_3(\mathbb{R})$ be the vector space of polynomials of degree ≤ 3 equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx \quad \text{for any } f \text{ and } g \text{ in } V.$$

Let W be the subspace of V spanned by $\{1,\ x\}$, and define $f(x) = x^2$. Find the orthogonal projection $\mathrm{Proj}_W(f)$ of f onto W.

5.7 Relations of fundamental subspaces

We now go back to the study of a system $A\mathbf{x} = \mathbf{b}$ of linear equations with an $m \times n$ matrix A. One of the most important applications of the orthogonal projection of vectors onto a subspace is the decompositions of the domain space and the image space of A by the four fundamental subspaces $\mathcal{N}(A)$, $\mathcal{R}(A)$ in $\mathbb{R}^n$ and $\mathcal{C}(A)$, $\mathcal{N}(A^T)$ in $\mathbb{R}^m$ (see Theorem 5.16). From these decompositions, one can completely determine the solution set of a consistent system $A\mathbf{x} = \mathbf{b}$.

Lemma 5.14 *For an $m \times n$ matrix A, the null space $\mathcal{N}(A)$ and the row space $\mathcal{R}(A)$ are orthogonal: i.e., $\mathcal{N}(A) \perp \mathcal{R}(A)$ in $\mathbb{R}^n$. Similarly, $\mathcal{N}(A^T) \perp \mathcal{C}(A)$ in $\mathbb{R}^m$.*

Proof: Note that $\mathbf{w} \in \mathcal{N}(A)$ if and only if $A\mathbf{w} = \mathbf{0}$, i.e., for every row vector $\mathbf{r}$ in A, $\mathbf{r} \cdot \mathbf{w} = 0$. For the second statement, do the same with A^T. □

From Lemma 5.14, it is clear that

$$\begin{array}{rcll} \mathcal{N}(A) & \subseteq & \mathcal{R}(A)^{\perp} & (\text{or } \mathcal{R}(A) \subseteq \mathcal{N}(A)^{\perp}), \text{ and} \\ \mathcal{N}(A^T) & \subseteq & \mathcal{C}(A)^{\perp} & (\text{or } \mathcal{C}(A) \subseteq \mathcal{N}(A^T)^{\perp}). \end{array}$$

Moreover, by comparing the dimensions of these subspaces and by using Theorem 5.11 and Rank Theorem 3.17, we have

$$\begin{array}{rcccl} \dim \mathcal{R}(A) + \dim \mathcal{N}(A) & = & n & = & \dim \mathcal{R}(A) + \dim \mathcal{R}(A)^{\perp}, \\ \dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) & = & m & = & \dim \mathcal{C}(A) + \dim \mathcal{C}(A)^{\perp}. \end{array}$$

This means that the inclusions are actually equalities.

Lemma 5.15 **(1)** *$\mathcal{N}(A) = \mathcal{R}(A)^{\perp}$ (or $\mathcal{R}(A) = \mathcal{N}(A)^{\perp}$).*
(2) *$\mathcal{N}(A^T) = \mathcal{C}(A)^{\perp}$ (or $\mathcal{C}(A) = \mathcal{N}(A^T)^{\perp}$).*

We show that the row space $\mathcal{R}(A)$ is the orthogonal complement of the null space $\mathcal{N}(A)$ in $\mathbb{R}^n$, and vice-versa. Similarly, the same thing happens for the column space $\mathcal{C}(A)$ and the null space $\mathcal{N}(A^T)$ of A^T in $\mathbb{R}^m$. Hence, by Theorem 5.11, we have the following orthogonal decomposition.

Theorem 5.16 *For any* $m \times n$ *matrix* A,

$$\boxed{\begin{array}{l}(1)\ \mathcal{N}(A) \oplus \mathcal{R}(A) = \mathbb{R}^n, \\ (2)\ \mathcal{N}(A^T) \oplus \mathcal{C}(A) = \mathbb{R}^m.\end{array}}$$

Note that if rank $A = r$ so that $\dim \mathcal{R}(A) = r = \dim \mathcal{C}(A)$, then $\dim \mathcal{N}(A) = n - r$ and $\dim \mathcal{N}(A^T) = m - r$. Considering the matrix A as a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$, Figure 5.4 depicts Theorem 5.16.

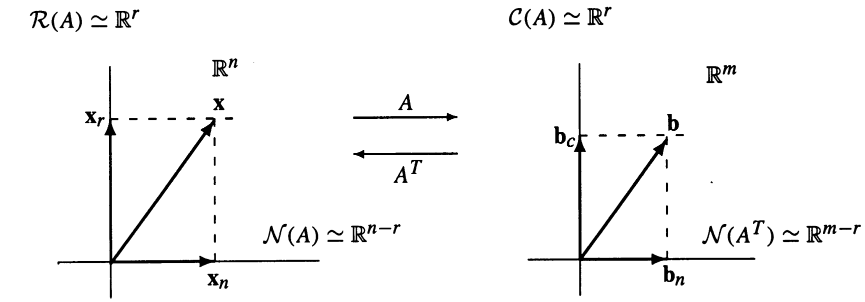

Figure 5.4. Relations of four fundamental subspaces

Corollary 5.17 *The set of solutions of a consistent system* $A\mathbf{x} = \mathbf{b}$ *is precisely* $\mathbf{x}_0 + \mathcal{N}(A)$, *where* $\mathbf{x}_0$ *is any solution of* $A\mathbf{x} = \mathbf{b}$.

Proof: Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a solution of a system $A\mathbf{x} = \mathbf{b}$. Now consider the set $\mathbf{x}_0 + \mathcal{N}(A)$, which is just a translation of $\mathcal{N}(A)$ by $\mathbf{x}_0$.

(1) For any vector $\mathbf{x}_0 + \mathbf{n}$ in $\mathbf{x}_0 + \mathcal{N}(A)$, it is also a solution because $A(\mathbf{x}_0 + \mathbf{n}) = A\mathbf{x}_0 = \mathbf{b}$.

(2) If $\mathbf{x}$ is another solution, then clearly $\mathbf{x} - \mathbf{x}_0$ is in the null space $\mathcal{N}(A)$ so that $\mathbf{x} = \mathbf{x}_0 + \mathbf{n}$ for some $\mathbf{n} \in \mathcal{N}(A)$, i.e., $\mathbf{x} \in \mathbf{x}_0 + \mathcal{N}(A)$. □

In particular, if rank $A = m$ (so that $m \leq n$), then $\mathcal{C}(A) = \mathbb{R}^m$. Thus, for any $\mathbf{b} \in \mathbb{R}^m$, the system $A\mathbf{x} = \mathbf{b}$ has a solution in $\mathbb{R}^n$. (This is the case of the existence Theorem 3.24).

On the other hand, if rank $A = n$ (so that $n \leq m$), then $\mathcal{N}(A) = \{\mathbf{0}\}$ and $\mathcal{R}(A) = \mathbb{R}^n$. Therefore, the system $A\mathbf{x} = \mathbf{b}$ has at most one solution, that is, it has a unique solution $\mathbf{x}$ in $\mathcal{R}(A)$ if $\mathbf{b} \in \mathcal{C}(A)$, and has no solution if $\mathbf{b} \notin \mathcal{C}(A)$. (This is the case of the uniqueness Theorem 3.25). The latter case may occur when $m > r = \operatorname{rank} A$; that is, $\mathcal{N}(A^T)$ is a nontrivial subspace of $\mathbb{R}^m$, and will be discussed later in Section 5.9.1.

Problem 5.18 Prove the following statements.

(1) If $A\mathbf{x} = \mathbf{b}$ and $A^T\mathbf{y} = \mathbf{0}$, then $\mathbf{y}^T\mathbf{b} = 0$, i.e., $\mathbf{y} \perp \mathbf{b}$.
(2) If $A\mathbf{x} = \mathbf{0}$ and $A^T\mathbf{y} = \mathbf{c}$, then $\mathbf{x}^T\mathbf{c} = 0$, i.e., $\mathbf{x} \perp \mathbf{c}$.

Problem 5.19 Given two vectors $(1, 2, 1, 2)$ and $(0, -1, -1, 1)$ in $\mathbb{R}^4$, find all vectors in $\mathbb{R}^4$ that are perpendicular to them.

Problem 5.20 Find a basis for the orthogonal complement of the row space of A:

$$(1)\ A = \begin{bmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ 3 & 0 & 6 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

5.8 Orthogonal matrices and isometries

In Chapter 4, we saw that a linear transformation can be associated with a matrix, and vice-versa. In this section, we are mainly interested in those linear transformations (or matrices) that preserve the length of a vector in an inner product space.

Let $A = [\mathbf{c}_1 \cdots \mathbf{c}_n]$ be an $n \times n$ square matrix, with columns $\mathbf{c}_1, \ldots, \mathbf{c}_n$. Then, a simple computation shows that

$$A^T A = \begin{bmatrix} -- & \mathbf{c}_1^T & -- \\ & \vdots & \\ -- & \mathbf{c}_n^T & -- \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{c}_1 & \cdots & \mathbf{c}_n \\ | & & | \end{bmatrix} = [\mathbf{c}_i^T \mathbf{c}_j] = [\mathbf{c}_i \cdot \mathbf{c}_j].$$

Hence, if the column vectors are orthonormal, $\mathbf{c}_i^T \mathbf{c}_j = \delta_{ij}$, then $A^T A = I_n$, that is, A^T is a left inverse of A, and vice-versa. Since A is a square matrix, this left inverse must be the right inverse of A, i.e., $AA^T = I_n$. Equivalently, the row vectors of A are also orthonormal. This argument can be summarized as follows.

Lemma 5.18 *For an $n \times n$ matrix A, the following are equivalent.*

(1) *The column vectors of A are orthonormal.*
(2) $A^T A = I_n$.
(3) $A^T = A^{-1}$.
(4) $AA^T = I_n$.
(5) *The row vectors of A are orthonormal.*

Definition 5.9 A square matrix A is called an **orthogonal** matrix if A satisfies one (and hence all) of the statements in Lemma 5.18.

Clearly, A is orthogonal if and only if A^T is orthogonal.

Example 5.12 (*Rotations and reflections are orthogonal*) The matrices

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

are orthogonal, and satisfy

$$A^{-1} = A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad B^{-1} = B^T = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Note that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a rotation through the angle θ, while $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(\mathbf{x}) = B\mathbf{x}$ is the reflection about the line passing through the origin that forms an angle $\theta/2$ with the positive x-axis.□

Example 5.13 (*All* 2×2 *orthogonal matrices*) Show that every 2×2 orthogonal matrix must be one of the forms

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Solution: Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an orthogonal matrix, so that $AA^T = I_2 = A^TA$. The first equality gives $a^2 + b^2 = 1$, $ac + bd = 0$, and $c^2 + d^2 = 1$. The second equality gives $a^2 + c^2 = 1$, $ab + cd = 0$, and $b^2 + d^2 = 1$. Thus, $b = \pm c$. If $b = -c$, then we get $a = d$. If $b = c$, then we get $a = -d$. Now, choose θ so that $a = \cos\theta$ and $b = \sin\theta$. □

Problem 5.21 Find the inverse of each of the following matrices.

$$(1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad (2) \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What are they as linear transformations on $\mathbb{R}^3$: rotations, reflections, or other?

Problem 5.22 Find eight 2×2 orthogonal matrices which transform the square $-1 \le x, y \le 1$ onto itself.

As shown in Examples 5.12 and 5.13, all rotations and reflections on the Euclidean 2-space $\mathbb{R}^2$ are orthogonal and preserve intuitively both the lengths of vectors and the angle between two vectors. In fact, every orthogonal matrix A preserves the lengths of vectors:

$$\|A\mathbf{x}\|^2 = A\mathbf{x} \cdot A\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2.$$

Definition 5.10 Let V and W be two inner product spaces. A linear transformation $T : V \to W$ is called an **isometry**, or an **orthogonal transformation**, if it preserves the lengths of vectors, that is, for every vector $\mathbf{x} \in V$

$$\|T(\mathbf{x})\| = \|\mathbf{x}\|.$$

Clearly, any orthogonal matrix is an isometry as a linear transformation. If $T : V \to W$ is an isometry, then T is one-to-one, since the kernel of T is trivial: $T(\mathbf{x}) = \mathbf{0}$ implies $\|\mathbf{x}\| = \|T(\mathbf{x})\| = 0$. Thus, if $\dim V = \dim W$, then an isometry is also an isomorphism.

The following theorem gives an interesting characterization of an isometry.

Theorem 5.19 *Let $T : V \to W$ be a linear transformation from an inner product space V to another W. Then, T is an isometry if and only if T preserves inner products, that is,*

$$\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$$

for any vectors $\mathbf{x}$, $\mathbf{y}$ in V.

Proof: Let T be an isometry. Then $\|T(\mathbf{x})\|^2 = \|\mathbf{x}\|^2$ for any $\mathbf{x} \in V$. Hence,

$$\langle T(\mathbf{x}+\mathbf{y}), T(\mathbf{x}+\mathbf{y})\rangle = \|T(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle$$

for any $\mathbf{x}, \mathbf{y} \in V$. On the other hand,

$$\begin{aligned}\langle T(\mathbf{x}+\mathbf{y}), T(\mathbf{x}+\mathbf{y})\rangle &= \langle T(\mathbf{x}), T(\mathbf{x})\rangle + 2\langle T(\mathbf{x}), T(\mathbf{y})\rangle + \langle T(\mathbf{y}), T(\mathbf{y})\rangle, \\ \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle &= \langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle,\end{aligned}$$

from which we get $\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$.

The converse is quite clear by choosing $\mathbf{y} = \mathbf{x}$. □

Theorem 5.20 *Let A be an $n \times n$ matrix. Then, A is an orthogonal matrix if and only if $A : \mathbb{R}^n \to \mathbb{R}^n$, as a linear transformation, preserves the dot product. That is, for any vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$,*

$$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

Proof: The necessity is clear. For the sufficiency, suppose that A preserves the dot product. Then for any vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$,

$$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x}^T A^T A\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

Take $\mathbf{x} = \mathbf{e}_i$ and $\mathbf{y} = \mathbf{e}_j$. Then, this equation is just $[A^T A]_{ij} = \delta_{ij}$. □

Since $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ for any $\mathbf{x}$ and $\mathbf{y}$ in V, one can easily derive the following corollary.

Corollary 5.21 *A linear transformation $T : V \to W$ is an isometry if and only if*

$$d(T(\mathbf{x}), T(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$$

for any $\mathbf{x}$ *and* $\mathbf{y}$ *in* V.

Recall that if θ is the angle between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V, then for any isometry $T : V \to V$,

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{\langle T\mathbf{x}, T\mathbf{y}\rangle}{\|T\mathbf{x}\|\|T\mathbf{y}\|}.$$

Hence, we have

Corollary 5.22 *An isometry preserves the angle.*

The converse of Corollary 5.22 is not true in general. A linear transformation $T(\mathbf{x}) = 2\mathbf{x}$ on the Euclidean space $\mathbb{R}^n$ preserves the angle but not the lengths of vectors (i.e., not an isometry). Such a linear transformation is called a **dilation.**

We have seen that any orthogonal matrix is an isometry as the linear transformation $T(\mathbf{x}) = A\mathbf{x}$. The following theorem says that the converse is also true, that is, the matrix representation of an isometry with respect to an orthonormal basis is an orthogonal matrix.

Theorem 5.23 *Let $T : V \to W$ be an isometry from an inner product space V to another W of the same dimension. Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be orthonormal bases for V and W, respectively. Then, the matrix $[T]_\alpha^\beta$ for T with respect to the bases α and β is an orthogonal matrix.*

Proof: Note that the k-th column vector of the matrix $[T]_\alpha^\beta$ is just $[T(\mathbf{v}_k)]_\beta$. Since T preserves inner products and α, β are orthonormal, we get

$$[T(\mathbf{v}_k)]_\beta \cdot [T(\mathbf{v}_\ell)]_\beta = \langle T(\mathbf{v}_k), T(\mathbf{v}_\ell)\rangle = \langle \mathbf{v}_k, \mathbf{v}_\ell\rangle = \delta_{k\ell},$$

which shows that the column vectors of $[T]_\alpha^\beta$ are orthonormal. □

Remark: In summary, for a linear transformation $T : V \to W$, the following are equivalent:

(1) T is an isometry: that is, T preserves the lengths of vectors.
(2) T preserves the inner product.
(3) T preserves the distance.
(4) $[T]_\alpha^\beta$ with respect to orthonormal bases α and β is an orthogonal matrix.

Any one (hence all) of these conditions implies that T preserves the angle, but the converse is not true.

Problem 5.23 Find values $r > 0,\ s > 0,\ a > 0,\ b$ and c such that matrix Q is orthogonal.

$$(1)\ Q = \begin{bmatrix} r & s & a \\ 0 & 2s & b \\ r & -s & c \end{bmatrix}, \quad (2)\ Q = \begin{bmatrix} r & -s & a \\ r & 3s & b \\ r & -2s & c \end{bmatrix}.$$

Problem 5.24 **(Bessel's Inequality)** Let V be an inner product space, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a set of orthonormal vectors in V (not necessarily a basis for V). Prove that for any $\mathbf{x}$ in V, $\|\mathbf{x}\|^2 \geq \sum_{i=1}^{m} |\langle \mathbf{x}, \mathbf{v}_i \rangle|^2$.

Problem 5.25 Determine whether the following linear transformations on the Euclidean space $\mathbb{R}^3$ are orthogonal.

(1) $T(x, y, z) = (z,\ \frac{\sqrt{3}}{2}x + \frac{1}{2}y,\ \frac{x}{2} - \frac{\sqrt{3}}{2}y)$.
(2) $T(x, y, z) = (\frac{5}{13}x + \frac{11}{13}z,\ \frac{12}{13}y - \frac{5}{13}z,\ x)$.

5.9 Applications

5.9.1 Least squares solutions

In the previous section, we have completely determined the solution set for a system $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b} \in \mathcal{C}(A)$. In this section, we discuss what we can do when the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, that is, when $\mathbf{b} \notin \mathcal{C}(A) \subseteq \mathbb{R}^m$. Certainly, there exists no solution in this case, but one can find a 'pseudo'-solution in the following sense.

Note that for any vector $\mathbf{x}$ in $\mathbb{R}^n$, $A\mathbf{x} \in \mathcal{C}(A)$. Hence, the best we can do is to find a vector $\mathbf{x}_0 \in \mathbb{R}^n$ so that $A\mathbf{x}_0$ is the *closest* to the given vector $\mathbf{b} \in \mathbb{R}^m$: i.e., $\|A\mathbf{x}_0 - \mathbf{b}\|$ is as small as possible. Such a vector $\mathbf{x}_0$ will give us the best approximation $A\mathbf{x}$ to $\mathbf{b}$ for all vectors $\mathbf{x}$ in $\mathbb{R}^n$, and it is called a **least squares solution** of $A\mathbf{x} = \mathbf{b}$.

To find a least squares solution, we first need to find a vector in $\mathcal{C}(A)$ that is closest to $\mathbf{b}$. However, from the orthogonal decomposition $\mathbb{R}^m = \mathcal{C}(A) \oplus \mathcal{N}(A^T)$, any $\mathbf{b} \in \mathbb{R}^m$ has the unique orthogonal decomposition as

$$\mathbf{b} = \mathbf{b}_c + \mathbf{b}_n \in \mathcal{C}(A) \oplus \mathcal{N}(A^T) = \mathbb{R}^m,$$

where $\mathbf{b}_c = \mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b}) \in \mathcal{C}(A)$ and $\mathbf{b}_n = \mathbf{b} - \mathbf{b}_c \in \mathcal{N}(A^T)$. Here, the vector $\mathbf{b}_c = \mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b}) \in \mathcal{C}(A)$ has two basic properties:

(1) There always exists a solution $\mathbf{x}_0 \in \mathbb{R}^n$ of $A\mathbf{x} = \mathbf{b}_c$, since $\mathbf{b}_c \in \mathcal{C}(A)$,
(2) $\mathbf{b}_c$ is the closest vector to $\mathbf{b}$ among the vectors in $\mathcal{C}(A)$ (see Theorem 5.13).

Therefore, a least squares solution $\mathbf{x}_0 \in \mathbb{R}^n$ of $A\mathbf{x} = \mathbf{b}$ is just a solution of $A\mathbf{x} = \mathbf{b}_c$. Furthermore, if $\mathbf{x}_0 \in \mathbb{R}^n$ is a least squares solution, then the set of all least squares solutions is $\mathbf{x}_0 + \mathcal{N}(A)$ by Corollary 5.17.

In particular, if $\mathbf{b} \in \mathcal{C}(A)$, then $\mathbf{b} = \mathbf{b}_c$, so that the least squares solutions are just the 'true' solutions of $A\mathbf{x} = \mathbf{b}$. The second property of $\mathbf{b}_c$ means that a least squares solution $\mathbf{x}_0 \in \mathbb{R}^n$ of $A\mathbf{x} = \mathbf{b}$ gives the best approximation $A\mathbf{x}_0 = \mathbf{b}_c$ to $\mathbf{b}$: i.e., for any vector $\mathbf{x}$ in $\mathbb{R}^n$,

$$\|A\mathbf{x}_0 - \mathbf{b}\| \leq \|A\mathbf{x} - \mathbf{b}\|.$$

In summary, to have a least squares solution of $A\mathbf{x} = \mathbf{b}$, the first step is to find the orthogonal projection $\mathbf{b}_c = \mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b}) \in \mathcal{C}(A)$ of $\mathbf{b}$, and then solve $A\mathbf{x} = \mathbf{b}_c$ as usual.

One can find $\mathbf{b}_c$ from $\mathbf{b} \in \mathbb{R}^m$ by using the orthogonal projection if we have an orthonormal basis for $\mathcal{C}(A)$. But, such a computation of $\mathbf{b}_c$ could be uncomfortable, because the only way we know so far to find an orthonormal basis for $\mathcal{C}(A)$ is the Gram–Schmidt orthogonalization (whose computation may be cumbersome).

However, there is a bypass to avoid the Gram–Schmidt orthogonalization. For this, let us examine a least squares solution once again. If $\mathbf{x}_0 \in \mathbb{R}^n$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$, then

$$A\mathbf{x}_0 - \mathbf{b} = A\mathbf{x}_0 - (\mathbf{b}_c + \mathbf{b}_n) = -\mathbf{b}_n \in \mathcal{N}(A^T)$$

holds since $A\mathbf{x}_0 = \mathbf{b}_c$. Thus, $A^T(A\mathbf{x}_0 - \mathbf{b}) = A^T(-\mathbf{b}_n) = \mathbf{0}$ or equivalently $A^T A\mathbf{x}_0 = A^T\mathbf{b}$, that is, $\mathbf{x}_0$ is a solution of the equation

$$\boxed{A^T A\mathbf{x} = A^T\mathbf{b}.}$$

This equation is very interesting because it is also a sufficient condition for a least squares solution as the next theorem shows, and it is called the **normal equation** of $A\mathbf{x} = \mathbf{b}$.

Theorem 5.24 *Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$ be any vector. Then, a vector $\mathbf{x}_0 \in \mathbb{R}^n$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}_0$ is a solution of the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$.*

Proof: We only need to show the sufficiency. Let $\mathbf{x}_0$ be a solution of the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$. Then, $A^T(A\mathbf{x}_0 - \mathbf{b}) = \mathbf{0}$, so $A\mathbf{x}_0 - \mathbf{b} \in \mathcal{N}(A^T)$. Say, $A\mathbf{x}_0 - \mathbf{b} = \mathbf{n}$ in $\mathcal{N}(A^T)$, and let $\mathbf{b} = \mathbf{b}_c + \mathbf{b}_n \in \mathcal{C}(A) \oplus \mathcal{N}(A^T)$. Then, $A\mathbf{x}_0 - \mathbf{b}_c = \mathbf{n} + \mathbf{b}_n \in \mathcal{N}(A^T)$. Since $A\mathbf{x}_0 - \mathbf{b}_c$ is also contained in $\mathcal{C}(A)$ and $\mathcal{N}(A^T) \cap \mathcal{C}(A) = \{\mathbf{0}\}$, $A\mathbf{x}_0 = \mathbf{b}_c = \mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b})$, i.e., $\mathbf{x}_0$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$. □

Example 5.14 (*The best approximated solution of an inconsistent system* $A\mathbf{x} = \mathbf{b}$) Find all the least squares solutions of $A\mathbf{x} = \mathbf{b}$, and then determine the orthogonal projection $\mathbf{b}_c$ of $\mathbf{b}$ into the column space $\mathcal{C}(A)$, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: (The reader may check that $A\mathbf{x} = \mathbf{b}$ has no solutions).

$$A^T A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}.$$

From the normal equation, a least squares solution of $A\mathbf{x} = \mathbf{b}$ is a solution of $A^T A\mathbf{x} = A^T\mathbf{b}$, i.e.,

$$\begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}.$$

By solving this system of equations (left for an exercise), one can obtain all the least squares solutions, which are of the form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -8 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

for any number $t \in \mathbb{R}$. Moreover,

$$\mathbf{b}_c = A\mathbf{x} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix} \in \mathcal{C}(A).$$

Note that the set of least squares solutions is $\mathbf{x}_0 + \mathcal{N}(A)$, where $\mathbf{x}_0 = [-8/3 \quad -5/3 \quad 0]^T$ and $\mathcal{N}(A) = \{t[5 \;\; 3 \;\; 1]^T : t \in \mathbb{R}\}$. □

Problem 5.26 Find all least squares solutions $\mathbf{x}$ in $\mathbb{R}^3$ of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix}.$$

Note that the normal equation is always consistent by the construction, and, as Example 5.14 shows, a least squares solution can be found by the Gauss–Jordan elimination even though $A^T A$ is not invertible. If $\mathbf{b} \in \mathcal{C}(A)$ or, even better, if the rows of A are linearly independent (thus, rank $A = m$ and $\mathcal{C}(A) = \mathbb{R}^m$), then $\mathbf{b}_c = \mathbf{b}$ so that the system $A\mathbf{x} = \mathbf{b}$ is always consistent and the least squares solutions coincide with the true solutions. Therefore, for any given system $A\mathbf{x} = \mathbf{b}$, consistent or inconsistent, by solving the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$, one can obtain either the true solutions or the least squares solutions.

If the square matrix $A^T A$ is invertible, then the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$ of the system $A\mathbf{x} = \mathbf{b}$ resolves to $\mathbf{x} = (A^T A)^{-1}A^T\mathbf{b}$, which is a least squares solution. In particular, if $A^T A = I_n$, or equivalently the columns of A are orthonormal (see Lemma 5.18), then the normal equation reduces to the least squares solution $\mathbf{x} = A^T\mathbf{b}$.

The following theorem gives a condition for $A^T A$ to be invertible.

Theorem 5.25 *For any $m \times n$ matrix A, $A^T A$ is a symmetric $n \times n$ square matrix and* $\operatorname{rank}(A^T A) = \operatorname{rank} A$.

Proof: Clearly, $A^T A$ is square and symmetric. Since the number of columns of A and $A^T A$ are both n, we have

$$\operatorname{rank} A + \dim \mathcal{N}(A) \;=\; n \;=\; \operatorname{rank}\,(A^T A) + \dim \mathcal{N}(A^T A).$$

Hence, it suffices to show that $\mathcal{N}(A) = \mathcal{N}(A^T A)$ so that $\dim \mathcal{N}(A) = \dim \mathcal{N}(A^T A)$. It is trivial to see that $\mathcal{N}(A) \subseteq \mathcal{N}(A^T A)$, since $A\mathbf{x} = \mathbf{0}$ implies $A^T A\mathbf{x} = \mathbf{0}$. Conversely, suppose that $A^T A\mathbf{x} = \mathbf{0}$. Then

$$A\mathbf{x} \cdot A\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T(A^T A\mathbf{x}) = \mathbf{x}^T\mathbf{0} = 0.$$

Hence $A\mathbf{x} = \mathbf{0}$, and $\mathbf{x} \in \mathcal{N}(A)$. □

It follows from Theorem 5.25 that $A^T A$ is invertible if and only if $\operatorname{rank} A = n$: that is, the columns of A are linearly independent. In this case, $\mathcal{N}(A) = \{\mathbf{0}\}$ and so the system $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution $\mathbf{x}_0$ in $\mathcal{R}(A) = \mathbb{R}^n$, which is

$$\mathbf{x}_0 = (A^T A)^{-1}A^T\mathbf{b}.$$

This can be summarized in the following theorem:

Theorem 5.26 *Let A be an $m \times n$ matrix. If* $\operatorname{rank} A = n$*, or equivalently the columns of A are linearly independent, then*

(1) *$A^T A$ is invertible so that $(A^T A)^{-1}A^T$ is a left inverse of A,*
(2) *the vector $\mathbf{x}_0 = (A^T A)^{-1}A^T\mathbf{b}$ is the unique least squares solution of a system $A\mathbf{x} = \mathbf{b}$, and*
(3) *$A\mathbf{x}_0 = A(A^T A)^{-1}A^T\mathbf{b} = \mathbf{b}_c = \operatorname{Proj}_{\mathcal{C}(A)}(\mathbf{b})$, that is, the orthogonal projection of $\mathbb{R}^m$ onto $\mathcal{C}(A)$ is $\operatorname{Proj}_{\mathcal{C}(A)} = A(A^T A)^{-1}A^T$.*

Remark: (1) For an $m \times n$ matrix A, by applying Theorem 5.26 to A^T, one can say that $\operatorname{rank} A = m$ if and only if AA^T is invertible. In this case $A^T(AA^T)^{-1}$ is a right inverse of A (cf. Remark after Theorem 3.25). Moreover, AA^T is invertible if and only if the rows of A are linearly independent by Theorem 5.25.

(2) If the matrix A is orthogonal, then the columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of A form an orthonormal basis for the column space $\mathcal{C}(A)$, so that for any $\mathbf{b} \in \mathbb{R}^m$,

$$\mathbf{b}_c = (\mathbf{u}_1 \cdot \mathbf{b})\mathbf{u}_1 + \cdots + (\mathbf{u}_n \cdot \mathbf{b})\mathbf{u}_n = (\mathbf{u}_1\mathbf{u}_1^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T)\mathbf{b}$$

and the projection matrix is

$$\mathrm{Proj}_{\mathcal{C}(A)} = \mathbf{u}_1\mathbf{u}_1^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T.$$

In fact, this result coincides with Theorem 5.26: If A is orthogonal so that $A^T A = I_n$, then

$$\begin{aligned} \mathrm{Proj}_{\mathcal{C}(A)} &= A(A^TA)^{-1}A^T = AA^T \\ &= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \\ &= \mathbf{u}_1\mathbf{u}_1^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T, \end{aligned}$$

and the least squares solution is

$$\mathbf{x}_0 = (A^TA)^{-1}A^T\mathbf{b} = A^T\mathbf{b} = \begin{bmatrix} -- & \mathbf{u}_1^T & -- \\ & \vdots & \\ -- & \mathbf{u}_n^T & -- \end{bmatrix} \mathbf{b} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{b} \\ \vdots \\ \mathbf{u}_n \cdot \mathbf{b} \end{bmatrix},$$

which is the coordinate expression of $A\mathbf{x}_0 = \mathbf{b}_c = \mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b})$ with respect to the orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for $\mathcal{C}(A)$.

In general, the columns of A need not be orthonormal, in which case the above formula is not possible. In Section 5.9.3, we will discuss more about this general case.

(3) If rank $A = r < n$, one can reduce the columns of A to a basis for the column space and work with this reduced matrix $\bar{A}$ (thus, $\bar{A}^T\bar{A}$ is invertible) to find the orthogonal projection $\mathrm{Proj}_{\mathcal{C}(A)} = \bar{A}(\bar{A}^T\bar{A})^{-1}\bar{A}^T$ of $\mathbb{R}^m$ onto the column space $\mathcal{C}(A)$. However, the least squares solutions of $A\mathbf{x} = \mathbf{b}$ should be found from the original normal equation directly, since the least squares solution $\bar{\mathbf{x}}_0 = (\bar{A}^T\bar{A})^{-1}\bar{A}^T\mathbf{b}$ of $\bar{A}\mathbf{x} = \mathbf{b}$ has only r components so that it cannot be a solution of $A\mathbf{x} = \mathbf{b}_c$.

Example 5.15 (*Solving an inconsistent system* $A\mathbf{x} = \mathbf{b}$ *by the normal equation*) Find the least squares solutions of the system:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix} = \mathbf{b}.$$

Determine also the orthogonal projection $\mathbf{b}_c$ of $\mathbf{b}$ in the column space $\mathcal{C}(A)$.

Solution: Clearly, the two columns of A are linearly independent and $\mathcal{C}(A)$ is the xy-plane. Thus, $\mathbf{b} \notin \mathcal{C}(A)$. Note that

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 7 & 29 \end{bmatrix},$$

which is invertible. By a simple computation one can obtain

$$(A^TA)^{-1} = \frac{1}{9} \begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix}.$$

Hence,

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{9} \begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 23 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 42 \\ -3 \end{bmatrix} = \begin{bmatrix} 14/3 \\ -1/3 \end{bmatrix}$$

is a least squares solution, which is unique since $\mathcal{N}(A) = \mathbf{0}$. The orthogonal projection of $\mathbf{b}$ in $\mathcal{C}(A)$ is

$$\mathbf{b}_c = A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 14/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}. \qquad \square$$

Problem 5.27 Find all the least squares solutions of the following inconsistent system of linear equations:

$$\begin{cases} x + y &= 1 \\ 2x + 2y &= 3 \\ 3x + 4y &= 4. \end{cases}$$

5.9.2 Polynomial approximations

In this section, one can find a reason for the name of the "*least squares*" solutions, and the following example illustrates an application of the least squares solution to the determination of the spring constants in physics.

Example 5.16 *Hooke's law* for springs in physics says that for a uniform spring, *the length stretched or compressed is a linear function of the force applied*, that is, the force F applied to the spring is related to the length x stretched or compressed by the equation

$$F = a + kx,$$

where a and k are some constants determined by the spring.

Suppose now that, given a spring of length 6.1 inches, we want to determine the constants a and k under the experimental data: The lengths are measured to be 7.6, 8.7 and 10.4 inches when forces of 2, 4 and 6 kilograms, respectively, are applied to the spring. However, by plotting these data

$$(x, F) = (6.1,\ 0),\ (7.6,\ 2),\ (8.7,\ 4),\ (10.4,\ 6),$$

in the xF-plane, one can easily recognize that they are not on a straight line of the form $F = a + kx$ in the xF-plane, which may be caused by experimental errors. This means that the system of linear equations

$$\begin{cases} F_1 &= a + 6.1k &= 0 \\ F_2 &= a + 7.6k &= 2 \\ F_3 &= a + 8.7k &= 4 \\ F_4 &= a + 10.4k &= 6 \end{cases}$$

is inconsistent (i.e., has no solutions so the second equality in each equation may not be a true equality). It means that if we put $\mathbf{b} = (0, 2, 4, 6)$ and $\mathbf{F} = (F_1, F_2, F_3, F_4)$ as vectors in $\mathbb{R}^4$ representing the data and the points on the line at x_i's, respectively, then $\|\mathbf{b} - \mathbf{F}\|$ is not zero. Thus, the best thing one can do is to determine the straight line $a + kx = F$ that 'fits' the data best: that is, to minimize the sum of the squares of the vertical distances from the line to the data (x_i, y_i) for $i = 1, 2, 3, 4$ (See Figure 5.5) (this is the reason why we say *least squares*):

$$(0 - F_1)^2 + (2 - F_2)^2 + (4 - F_3)^2 + (6 - F_4)^2 = \|\mathbf{b} - \mathbf{F}\|^2.$$

Thus, for the original inconsistent system

$$A\mathbf{x} = \begin{bmatrix} 1 & 6.1 \\ 1 & 7.6 \\ 1 & 8.7 \\ 1 & 10.4 \end{bmatrix} \begin{bmatrix} a \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} = \mathbf{b} \notin \mathcal{C}(A),$$

we are looking for $\mathbf{F} \in \mathcal{C}(A)$, which is the projection of $\mathbf{b}$ onto the column space $\mathcal{C}(A)$ of A, and the least squares solution $\mathbf{x}_0$, which satisfies $A\mathbf{x}_0 = \mathbf{F}$.

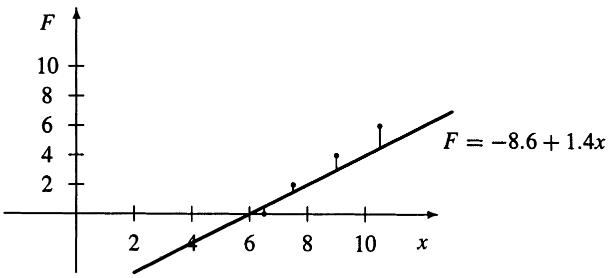

Figure 5.5. Least squares fitting

It is now easily computed as (by solving the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$)

$$\begin{bmatrix} a \\ k \end{bmatrix} = \mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} -8.6 \\ 1.4 \end{bmatrix}.$$

It gives $F = -8.6 + 1.4x$. □

In general, a common problem in experimental work is to obtain a polynomial $y = f(x)$ in two variables x and y that best 'fits' the data of various values of y determined experimentally for inputs x, say

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

plotted in the xy-plane. Some possible fitting polynomials are

(1) by a straight line: $y = a + bx$,
(2) by a quadratic polynomial: $y = a + bx + cx^2$, or
(3) by a polynomial of degree k: $y = a_0 + a_1x + \cdots + a_kx^k$, etc.

As a general case, suppose that we are looking for a polynomial $y = f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ of degree k that passes through the given data. Then we obtain a system of linear equations,

$$\begin{cases} f(x_1) = a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_kx_1^k &= y_1 \\ f(x_2) = a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_kx_2^k &= y_2 \\ \qquad\vdots & \\ f(x_n) = a_0 + a_1x_n + a_2x_n^2 + \cdots + a_kx_n^k &= y_n, \end{cases}$$

or, in matrix form, the system may be written as $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The left-hand side $A\mathbf{x}$ represents the values of the polynomial at x_i's and the right-hand side represents the data obtained from the inputs x_i's in the experiment.

If $n \leq k + 1$, then the cases have already been discussed in Section 3.9.1. If $n > k + 1$, this kind of system may be inconsistent. Therefore, the best thing one can do is to find the polynomial $f(x)$ that minimizes the sum of the squares of the vertical distances between the graph of the polynomial and the data. But, it is equivalent to find the least squares solution of the system $A\mathbf{x} = \mathbf{b}$, because for any $\mathbf{c} \in \mathcal{C}(A)$ of the form

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} a_0 + a_1x_1 + \cdots + a_kx_1^k \\ a_0 + a_1x_2 + \cdots + a_kx_2^k \\ \vdots \\ a_0 + a_1x_n + \cdots + a_kx_n^k \end{bmatrix} = \mathbf{c},$$

we have

$$\begin{aligned} \|\mathbf{b} - \mathbf{c}\|^2 = &\ (y_1 - a_0 - a_1x_1 - \cdots - a_kx_1^k)^2 + \cdots \\ &+ (y_n - a_0 - a_1x_n - \cdots - a_kx_n^k)^2. \end{aligned}$$

The previous theory says that the orthogonal projection $\mathbf{b}_c$ of $\mathbf{b}$ into the column space of A minimizes this quantity and shows how to find $\mathbf{b}_c$ and a least squares solution $\mathbf{x}_0$.

Example 5.17 Find a straight line $y = a + bx$ that fits best the given experimental data, $(1, 0)$, $(2, 3)$, $(3, 4)$ and $(4, 4)$.

Solution: We are looking for a line $y = a + bx$ that minimizes the sum of squares of the vertical distances $|y_i - a - bx_i|$'s from the line $y = a + bx$ to the data (x_i, y_i). By adapting matrix notation

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 4 \end{bmatrix},$$

we have $A\mathbf{x} = \mathbf{b}$ and want to find a least squares solution of $A\mathbf{x} = \mathbf{b}$. But the columns of A are linearly independent, and the least squares solution is $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$. Now,

$$A^TA = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}, \quad (A^TA)^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} 11 \\ 34 \end{bmatrix}.$$

Hence, we have

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} -\frac{1}{2} \\ \frac{13}{10} \end{bmatrix},$$

and $y = -\frac{1}{2} + \frac{13}{10}x$ is the desired line. □

Problem 5.28 From Newton's second law of motion, a body near the surface of the earth falls vertically downward according to the equation

$$s(t) = s_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2,$$

where $s(t)$ is the distance that the body travelled in time t, and s_0, $\mathbf{v}_0$ are the initial displacement and velocity, respectively, of the body, and $\mathbf{g}$ is the gravitational acceleration at the earth's surface. Suppose a weight is released, and the distances that the body has fallen from some reference point were measured to be $s = -0.18,\ 0.31,\ 1.03,\ 2.48,\ 3.73$ feet at times $t = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$ seconds, respectively. Determine approximate values of s_0, $\mathbf{v}_0$, $\mathbf{g}$ using these data.

5.9.3 Orthogonal projection matrices

In Section 5.9.1, we have seen that the orthogonal projection $\mathrm{Proj}_{\mathcal{C}(A)}$ of the Euclidean space $\mathbb{R}^m$ on the column space $\mathcal{C}(A)$ of an $m \times n$ matrix A plays an important role in finding a least squares solution of an inconsistent system $A\mathbf{x} = \mathbf{b}$. Also, the orthogonal projection $\mathrm{Proj}_{\mathcal{C}(A)}$ is the main tool in the Gram–Schmidt orthogonalization.

In general, for a given subspace U of $\mathbb{R}^m$, the computation of the orthogonal projection Proj_U of $\mathbb{R}^m$ onto U appears quite often in applied science and engineering problems. The least squares solution method can also be used to find the orthogonal projection: Indeed, by taking first a basis for U and then making an $m \times n$ matrix A with these basis vectors as columns, one clearly gets $U = \mathcal{C}(A)$, and so by Theorem 5.26

$$\boxed{\mathrm{Proj}_U = \mathrm{Proj}_{\mathcal{C}(A)} = A(A^TA)^{-1}A^T.}$$

In fact, this projection itself is the **orthogonal projection matrix**, that is, the matrix representation of Proj_U with respect to the standard basis α,

$$\mathrm{Proj}_U = [\mathrm{Proj}_U]_\alpha.$$

Note that this projection matrix Proj_U is independent of the choice of a basis for U due to the uniqueness of the matrix representation of a linear transformation with respect to a fixed basis.

Example 5.18 Find the projection matrix P on the plane $2x - y - 3z = 0$ in the space $\mathbb{R}^3$ and calculate $P\mathbf{b}$ for $\mathbf{b} = (1,\ 0,\ 1)$.

Solution: Choose any basis for the plane $2x - y - 3z = 0$, say,

$$\mathbf{v}_1 = (0,\ 3,\ -1) \quad \text{and} \quad \mathbf{v}_2 = (1,\ 2,\ 0).$$

Let $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}$ be the matrix with $\mathbf{v}_1$ and $\mathbf{v}_2$ as columns. Then

$$(A^TA)^{-1} = \begin{bmatrix} 10 & 6 \\ 6 & 5 \end{bmatrix}^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}.$$

The orthogonal projection matrix $P = \mathrm{Proj}_{\mathcal{C}(A)}$ is

$$\begin{aligned} P &= A(A^TA)^{-1}A^T \\ &= \frac{1}{14}\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}\begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{bmatrix}, \end{aligned}$$

and

$$P\mathbf{b} = \frac{1}{14}\begin{bmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 16 \\ -1 \\ 11 \end{bmatrix}. \qquad \square$$

If an orthonormal basis for U is known, then the computation of $\mathrm{Proj}_U = A(A^TA)^{-1}A^T$ is easy, as shown in Remark (2) on page 185: For an orthonormal basis $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for U, the orthogonal projection matrix onto the subspace U is given as

$$\boxed{\mathrm{Proj}_U = AA^T = \mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T.}$$

Example 5.19 If $A = [\mathbf{c}_1\ \mathbf{c}_2]$, where $\mathbf{c}_1 = (1, 0, 0)$, $\mathbf{c}_2 = (0, 1, 0)$, then the column vectors of A are orthonormal, $\mathcal{C}(A)$ is the xy-plane, and the projection of $\mathbf{b} = (x, y, z) \in \mathbb{R}^3$ onto $\mathcal{C}(A)$ is $\mathbf{b}_c = (x, y, 0)$. In fact,

$$P = AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix},$$

which is equal to $\mathbf{c}_1\mathbf{c}_1^T + \mathbf{c}_2\mathbf{c}_2^T$. $\square$

Note that, if we denote by $\mathrm{Proj}_{\mathbf{u}_i}$ the orthogonal projection of $\mathbb{R}^m$ on the subspace spanned by the basis vector $\mathbf{u}_i$ for each i, then its projection matrix is $\mathbf{u}_i\mathbf{u}_i^T$, and so

$$\mathrm{Proj}_W = \mathrm{Proj}_{\mathbf{u}_1} + \mathrm{Proj}_{\mathbf{u}_2} + \cdots + \mathrm{Proj}_{\mathbf{u}_n},$$

and

$$\mathrm{Proj}_{\mathbf{u}_j} \circ \mathrm{Proj}_{\mathbf{u}_i} = \mathbf{u}_j\mathbf{u}_j^T\mathbf{u}_i\mathbf{u}_i^T = \mathbf{u}_j\delta_{ji}\mathbf{u}_i^T = \begin{cases} \mathbf{0} & \text{if } i \neq j \\ \mathrm{Proj}_{\mathbf{u}_j} & \text{if } i = j. \end{cases}$$

Problem 5.29 Let $\mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ be a vector in $\mathbb{R}^2$ which determines 1-dimensional subspace $U = \{a\mathbf{u} = (\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}) : a \in \mathbb{R}\}$. Show that the matrix

$$A = \mathbf{u}\mathbf{u}^T = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}[1\ 1] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

considered as a linear transformation on $\mathbb{R}^2$, is an orthogonal projection onto the subspace U.

Problem 5.30 Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is an orthonormal basis for $\mathbb{R}^m$, then $\mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T + \cdots + \mathbf{v}_m\mathbf{v}_m^T = I_m$.

In general, if a basis $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ for U is given, but not orthonormal, then one has to directly compute

$$\mathrm{Proj}_U = A(A^T A)^{-1} A^T,$$

where $A = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]$ is an $m \times n$ matrix whose columns are the given basis vectors $\mathbf{c}_i$'s.

Sometimes it is necessary to compute an orthonormal basis from the given basis for U by the Gram–Schmidt orthogonalization. Its computation gives us a decomposition of the matrix A into an orthogonal part and an upper triangular part, from which the computation of the projection matrix might be easier.

QR **decomposition method:** Let $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be an arbitrary basis for a subspace U. The Gram–Schmidt orthogonalization process to this basis may be written as the following steps:

(1) From the basis $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$, find an orthogonal basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ for U by

$$\begin{aligned}
\mathbf{q}_1 &= \mathbf{c}_1 \\
\mathbf{q}_2 &= \mathbf{c}_2 - \frac{\langle \mathbf{q}_1, \mathbf{c}_2 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\
&\ \ \vdots \\
\mathbf{q}_n &= \mathbf{c}_n - \frac{\langle \mathbf{q}_{n-1}, \mathbf{c}_n \rangle}{\langle \mathbf{q}_{n-1}, \mathbf{q}_{n-1} \rangle} \mathbf{q}_{n-1} - \cdots - \frac{\langle \mathbf{q}_1, \mathbf{c}_n \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1.
\end{aligned}$$

(2) By normalization of these vectors: $\mathbf{u}_i = \mathbf{q}_i / \|\mathbf{q}_i\|$, one obtains an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for U.

(3) By rewriting those equations in (1), one gets

$$\begin{aligned}
\mathbf{c}_1 &= \mathbf{q}_1 && = b_{11}\mathbf{u}_1 \\
\mathbf{c}_2 &= a_{12}\mathbf{q}_1 + \mathbf{q}_2 && = b_{12}\mathbf{u}_1 + b_{22}\mathbf{u}_2 \\
&\ \ \vdots \\
\mathbf{c}_n &= a_{1n}\mathbf{q}_1 + \cdots + a_{n-1n}\mathbf{q}_{n-1} + \mathbf{q}_n && = b_{1n}\mathbf{u}_1 + \cdots + b_{nn}\mathbf{u}_n,
\end{aligned}$$

where $a_{ij} = \frac{\langle \mathbf{q}_i, \mathbf{c}_j \rangle}{\langle \mathbf{q}_i, \mathbf{q}_i \rangle}$ for $i < j$, $a_{ii} = 1$, and

$$b_{ij} = a_{ij}\|\mathbf{q}_i\| = \frac{\langle \mathbf{q}_i, \mathbf{c}_j \rangle}{\langle \mathbf{q}_i, \mathbf{q}_i \rangle}\|\mathbf{q}_i\| = \langle \mathbf{u}_i, \mathbf{c}_j \rangle,$$

for $i \le j$, which is just the component of $\mathbf{c}_j$ in $\mathbf{u}_i$ direction.

(4) Let $A = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]$. Then, the equations in (3) can be written in matrix notation as

$$A = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n] = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n] \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & b_{nn} \end{bmatrix} = QR,$$

where $Q = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n]$ is the $m \times n$ matrix whose orthonormal columns are obtained from $\mathbf{c}_i$'s by the Gram–Schmidt orthogonalization, and R is the $n \times n$ upper triangular matrix.

(5) Note that rank A = rank $Q = n \le m$, and $\mathcal{C}(Q) = U = \mathcal{C}(A)$ which is of dimension n in $\mathbb{R}^m$. Moreover, the matrix R is an invertible $n \times n$ matrix, since each $b_{jj} = \langle \mathbf{u}_j, \mathbf{c}_j \rangle$ is equal to $\|\mathbf{c}_j - \mathrm{Proj}_{U_{j-1}}(\mathbf{c}_j)\|$, where U_{j-1} is the subspace of $\mathbb{R}^m$ spanned by $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{j-1}\}$ (or equivalently, by $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{j-1}\}$), and so $b_{jj} \neq 0$ for all j because $\mathbf{c}_j \notin U_{j-1}$.

One of the byproducts of this computation is the following theorem.

Theorem 5.27 *Any $m \times n$ matrix of rank n can be factored into a product QR, where Q is an $m \times n$ matrix with orthonormal columns and R is an $n \times n$ invertible upper triangular matrix.*

Definition 5.11 The decomposition $A = QR$ is called the QR **factorization** or the QR **decomposition** of an $m \times n$ matrix A, (rank $A = n$), where the matrix $Q = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n]$ is called the **orthogonal part** of A, and the matrix $R = [b_{ij}]$ is called the **upper triangular part** of A.

Remark: In the QR factorization of $A = QR$, the orthonormality of the column vectors of Q means $Q^T Q = I_n$, and the j-th column of the matrix R is simply the coordinate vector of $\mathbf{c}_j$ with respect to the orthonormal basis $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for U: i.e.,

$$\mathbf{c}_j = \langle \mathbf{u}_1, \mathbf{c}_j \rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{c}_j \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{u}_j, \mathbf{c}_j \rangle \mathbf{u}_j,$$

and so

$$[\mathbf{c}_j]_\beta = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{c}_j \rangle \\ \vdots \\ \langle \mathbf{u}_j, \mathbf{c}_j \rangle \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{jj} \\ \vdots \\ 0 \end{bmatrix}.$$

With this QR decomposition of A, the projection matrix and the least squares solution of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} \in \mathbb{R}^m$ can be calculated easily as

$$\begin{aligned} P &= A(A^TA)^{-1}A^T = QR(R^TQ^TQR)^{-1}R^TQ^T = QQ^T, \\ \mathbf{x}_0 &= (A^TA)^{-1}A^T\mathbf{b} = (R^TQ^TQR)^{-1}R^TQ^T\mathbf{b} = R^{-1}Q^T\mathbf{b}. \end{aligned}$$

Corollary 5.28 *Let A be an $m \times n$ matrix of rank n and let $A = QR$ be its QR factorization. Then,*

(1) *the projection matrix on the column space of A is $[\mathrm{Proj}_{\mathcal{C}(A)}]_\alpha = QQ^T$.*

(2) *The least squares solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}_0 = R^{-1}Q^T\mathbf{b}$, which can be solved by using back substitution to the system $R\mathbf{x} = Q^T\mathbf{b}$.*

Example 5.20 (*QR decomposition of A*) Find the QR factorization $A = QR$ and the orthogonal projection matrix $P = [\mathrm{Proj}_{\mathcal{C}(A)}]_\alpha$ for

$$A = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: We first find the decomposition of A into Q and R, the orthogonal part and the upper triangular part. Use the Gram–Schmidt orthogonalization to get the column vectors of Q:

$$\begin{aligned} \mathbf{q}_1 &= \mathbf{c}_1 = (1,\ 1,\ 0,\ 0) \\ \mathbf{q}_2 &= \mathbf{c}_2 - \frac{\mathbf{c}_2 \cdot \mathbf{q}_1}{\mathbf{q}_1 \cdot \mathbf{q}_1}\,\mathbf{q}_1 = \left(\frac{1}{2},\ -\frac{1}{2},\ 1,\ 0\right) \\ \mathbf{q}_3 &= \mathbf{c}_3 - \frac{\mathbf{c}_3 \cdot \mathbf{q}_2}{\mathbf{q}_2 \cdot \mathbf{q}_2}\,\mathbf{q}_2 - \frac{\mathbf{c}_3 \cdot \mathbf{q}_1}{\mathbf{q}_1 \cdot \mathbf{q}_1}\,\mathbf{q}_1 = \left(-\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3},\ 1\right), \end{aligned}$$

and $\|\mathbf{q}_1\| = \sqrt{2}$, $\|\mathbf{q}_2\| = \sqrt{3/2}$, $\|\mathbf{q}_3\| = \sqrt{7/3}$. Hence,

$$\begin{aligned} \mathbf{u}_1 &= \frac{\mathbf{q}_1}{\|\mathbf{q}_1\|} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right) \\ \mathbf{u}_2 &= \frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} = \left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{\sqrt{2}}{\sqrt{3}},\ 0\right) \\ \mathbf{u}_3 &= \frac{\mathbf{q}_3}{\|\mathbf{q}_3\|} = \left(-\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{\sqrt{3}}{\sqrt{7}}\right). \end{aligned}$$

Then, $\mathbf{c}_1 = \sqrt{2}\mathbf{u}_1$, $\mathbf{c}_2 = \frac{1}{\sqrt{2}}\mathbf{u}_1 + \sqrt{\frac{3}{2}}\mathbf{u}_2$, $\mathbf{c}_3 = \frac{1}{\sqrt{2}}\mathbf{u}_1 + \frac{1}{\sqrt{6}}\mathbf{u}_2 + \sqrt{\frac{7}{3}}\mathbf{u}_3$. In fact, these equations can also be derived from $A = QR$ with an upper triangular matrix R. (It gives that the (i, j)-entry of R is $b_{ij} = \langle \mathbf{u}_i, \mathbf{c}_j \rangle$.) Therefore,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{7}/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -2/\sqrt{21} \\ 1/\sqrt{2} & -1/\sqrt{6} & 2/\sqrt{21} \\ 0 & \sqrt{2}/\sqrt{3} & 2/\sqrt{21} \\ 0 & 0 & \sqrt{3}/\sqrt{7} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{7}/\sqrt{3} \end{bmatrix} = QR, \end{aligned}$$

and

$$P = QQ^T = \begin{bmatrix} 6/7 & 1/7 & 1/7 & -2/7 \\ 1/7 & 6/7 & -1/7 & 2/7 \\ -1/7 & -1/7 & 6/7 & 2/7 \\ -2/7 & 2/7 & 2/7 & 3/7 \end{bmatrix}. \qquad \square$$

Problem 5.31 Find the 2×2 matrix P that projects the xy-plane onto the line $y = x$.

Problem 5.32 Find the projection matrix P of the Euclidean 3-space $\mathbb{R}^3$ onto the column space $\mathcal{C}(A)$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem 5.33 Find the projection matrix P on the x_1, x_2, x_4 coordinate subspace of the Euclidean 4-space $\mathbb{R}^4$.

Problem 5.34 Find the QR factorization of the matrix $\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & 0 \end{bmatrix}$.

As the last part of this section, we introduce a characterization of the orthogonal projection matrices.

Theorem 5.29 *A square matrix P is an orthogonal projection matrix if and only if it is symmetric and idempotent, i.e., $P^T = P$ and $P^2 = P$.*

Proof: Let P be an orthogonal projection matrix. Then, the matrix P can be written as $P = A(A^TA)^{-1}A^T$ for a matrix A whose column vectors form a basis for the column space of P. It gives

$$\begin{aligned} P^T &= \left(A(A^TA)^{-1}A^T\right)^T = A(A^TA)^{-1^T}A^T = A(A^TA)^{-1}A^T = P, \\ P^2 &= PP = \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right) = A(A^TA)^{-1}A^T = P. \end{aligned}$$

In fact, this second equation was already shown in Theorem 5.9.

Conversely, by Theorem 5.16, one has the orthogonal decomposition $\mathbb{R}^m = \mathcal{C}(P) \oplus \mathcal{N}(P^T)$. But, $\mathcal{N}(P^T) = \mathcal{N}(P)$ since $P^T = P$. Thus, for any $\mathbf{u} + \mathbf{n} \in \mathcal{C}(P) \oplus \mathcal{N}(P^T) = \mathbb{R}^m$, $P(\mathbf{u}+\mathbf{n}) = P\mathbf{u} + P\mathbf{n} = P\mathbf{u} = \mathbf{u}$, because $P^2 = P$ implies $P\mathbf{u} = \mathbf{u}$ for $\mathbf{u} \in \mathcal{C}(P)$. It shows that P is an orthogonal projection matrix. (Alternatively, one can use directly Theorem 5.9). $\square$

From Corollary 5.10, if P is a projection matrix on $\mathcal{C}(P)$, then $I - P$ is a projection matrix on the null space $\mathcal{N}(P)$ $(= \mathcal{C}(I - P))$, which is orthogonal to $\mathcal{C}(P)$ $(= \mathcal{N}(I - P))$.

Example 5.21 Let $P_i : \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$P_i(x_1, \ldots, x_m) = (0, \ldots, 0,\ x_i,\ 0, \ldots, 0),$$

for $i = 1, 2, \ldots, m$. Then, each P_i is the projection of $\mathbb{R}^m$ onto the i-th axis, whose matrix form looks like

$$P_i = \begin{bmatrix} \ddots & & & & 0 \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \end{bmatrix}, \quad I - P_i = \begin{bmatrix} \ddots & & & & 0 \\ & 1 & & & \\ & & 0 & & \\ & & & 1 & \\ 0 & & & & \ddots \end{bmatrix}.$$

When we restrict the image to $\mathbb{R}$, P_i is an element in the dual space $\mathbb{R}^{n*}$, and usually denoted by x_i as the i-th coordinate function (see Example 4.25). □

Problem 5.35 Show that any square matrix P that satisfies $P^T P = P$ is a projection matrix.

5.10 Exercises

5.1. Decide which of the following functions on $\mathbb{R}^2$ are inner products and which are not. For $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 x_2 y_2$,

(2) $\langle \mathbf{x}, \mathbf{y} \rangle = 4x_1 y_1 + 4x_2 y_2 - x_1 y_2 - x_2 y_1$,

(3) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 - x_2 y_1$,

(4) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 3x_2 y_2$,

(5) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2$.

5.2. Show that the function $\langle A, B \rangle = \operatorname{tr}(A^T B)$ for $A, B \in M_{n\times n}(\mathbb{R})$ defines an inner product on $M_{n\times n}(\mathbb{R})$.

5.3. Find the angle between the vectors $(4, 7, 9, 1, 3)$ and $(2, 1, 1, 6, 8)$ in $\mathbb{R}^5$.

5.4. Determine the values of k so that the given vectors are orthogonal with respect to the Euclidean inner product in $\mathbb{R}^4$.

$$(1) \left\{ \begin{bmatrix} 2 \\ 3 \\ k \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ 3 \\ -5 \end{bmatrix} \right\}, \quad (2) \left\{ \begin{bmatrix} 2 \\ 8 \\ 4 \\ k \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 2 \\ k \end{bmatrix} \right\}.$$

5.5. Consider the space $C[0, 1]$ with the inner product defined by

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx.$$

Compute the length of each vector and the cosine of the angle between each pair of vectors in each of the following:

(1) $f(x) = 1$, $g(x) = x$;

(2) $f(x) = x^m$, $g(x) = x^n$, where m, n are positive integers;

(3) $f(x) = \sin m\pi x$, $g(x) = \cos n\pi x$, where m, n are positive integers.

5.6. Prove that

$$(a_1 + \cdots + a_n)^2 \le n(a_1^2 + \cdots + a_n^2)$$

for any real numbers $a_1, a_2, \ldots, a_n$. When does equality hold?

5.7. Let $V = P_2([0, 1])$ be the vector space of polynomials of degree ≤ 2 on $[0, 1]$ equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

(1) Compute $\langle f, g\rangle$ and $\|f\|$ for $f(x) = x + 2$ and $g(x) = x^2 - 2x - 3$.

(2) Find the orthogonal complement of the subspace of scalar polynomials.

5.8. Find an orthonormal basis for the Euclidean 3-space $\mathbb{R}^3$ by applying the Gram–Schmidt orthogonalization to the three vectors $\mathbf{x} = (1, 0, 1), \mathbf{x}_2 = (1, 0, -1), \mathbf{x}_3 = (0, 3, 4)$.

5.9. Let W be the subspace of the Euclidean space $\mathbb{R}^3$ spanned by the vectors $\mathbf{v}_1 = (1, 1, 2)$ and $\mathbf{v}_2 = (1, 1, -1)$. Find $\mathrm{Proj}_W(\mathbf{b})$ for $\mathbf{b} = (1, 3, -2)$.

5.10. Show that if $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then every scalar multiple of $\mathbf{u}$ is also orthogonal to $\mathbf{v}$. Find a unit vector orthogonal to $\mathbf{v}_1 = (1, 1, 2)$ and $\mathbf{v}_2 = (0, 1, 3)$ in the Euclidean 3-space $\mathbb{R}^3$.

5.11. Determine the orthogonal projection of $\mathbf{v}_1$ onto $\mathbf{v}_2$ for the following vectors in the n-space $\mathbb{R}^n$ with the Euclidean inner product.

(1) $\mathbf{v}_1 = (1, 2, 3), \ \mathbf{v}_2 = (1, 1, 2)$,

(2) $\mathbf{v}_1 = (1, 2, 1), \ \mathbf{v}_2 = (2, 1, -1)$,

(3) $\mathbf{v}_1 = (1, 0, 1, 0), \ \mathbf{v}_2 = (0, 2, 2, 0)$.

5.12. Let $S = \{\mathbf{v}_i\}$, where $\mathbf{v}_i$'s are given below. For each S, find a basis for $S^\perp$ with respect to the Euclidean inner product on $\mathbb{R}^n$.

(1) $\mathbf{v}_1 = (0, 1, 0), \ \mathbf{v}_2 = (0, 0, 1)$,

(2) $\mathbf{v}_1 = (1, 1, 0), \ \mathbf{v}_2 = (1, 1, 1)$,

(3) $\mathbf{v}_1 = (1, 0, 1, 2), \ \mathbf{v}_2 = (1, 1, 1, 1), \ \mathbf{v}_3 = (2, 2, 0, 1)$.

5.13. Which of the following matrices are orthogonal?

$$(1)\begin{bmatrix} 1/2 & -1/3 \\ -1/2 & 1/3 \end{bmatrix}, \qquad (2)\begin{bmatrix} 4/5 & -3/5 \\ -3/5 & 4/5 \end{bmatrix},$$

$$(3)\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}, \quad (4)\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}.$$

5.14. Let W be the subspace of the Euclidean 4-space $\mathbb{R}^4$ consisting of all vectors that are orthogonal to both $\mathbf{x} = (1, 0, -1, 1)$ and $\mathbf{y} = (2, 3, -1, 2)$. Find a basis for the subspace W.

5.15. Let V be an inner product space. For vectors $\mathbf{x}$ and $\mathbf{y}$ in V, establish the following identities:

(1) $\langle \mathbf{x}, \mathbf{y}\rangle = \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2$ (polarization identity),

(2) $\langle \mathbf{x}, \mathbf{y}\rangle = \frac{1}{2}\left(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\right)$ (polarization identity),

(3) $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$ (parallelogram equality).

5.16. Show that $\mathbf{x} + \mathbf{y}$ is perpendicular to $\mathbf{x} - \mathbf{y}$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.

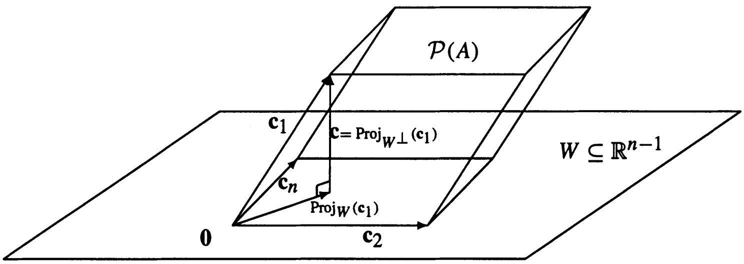

Figure 5.6. n-dimensional parallelepiped $\mathcal{P}(A)$

5.17. Let A be the $m \times n$ matrix whose columns are $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ in the Euclidean m-space $\mathbb{R}^m$. Prove that the volume of the n-dimensional parallelepiped $\mathcal{P}(A)$ determined by those vectors $\mathbf{c}_j$'s in $\mathbb{R}^m$ is given by

$$\mathrm{vol}(A) = \sqrt{\det(A^T A)}\,.$$

(Note that the volume of the n-dimensional parallelepiped determined by the vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ in $\mathbb{R}^m$ is by definition the multiplication of the volume of the $(n-1)$-dimensional parallelepiped (base) determined by $\mathbf{c}_2, \ldots, \mathbf{c}_n$ and the height of $\mathbf{c}_1$ from the plane W which is spanned by $\mathbf{c}_2, \ldots, \mathbf{c}_n$. Here, the height is the length of the vector $\mathbf{c} = \mathbf{c}_1 - \mathrm{Proj}_W(\mathbf{c}_1)$, which is orthogonal to W. (See Figure 5.6.) If the vectors are linearly dependent, then the parallelepiped is degenerate, i.e., it is contained in a subspace of dimension less than n.)

5.18. Find the volume of the three-dimensional tetrahedron in the Euclidean 4-space $\mathbb{R}^4$ whose vertices are at $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,2,2)$ and $(0,0,1,2)$.

5.19. For an orthogonal matrix A, show that $\det A = \pm 1$. Give an example of an orthogonal matrix A for which $\det A = -1$.

5.20. Find orthonormal bases for the row space and the null space of each of the following matrices.

$$(1)\begin{bmatrix} 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad (2)\begin{bmatrix} 1 & 4 & 0 \\ -2 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad (3)\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

5.21. Let A be an $m \times n$ matrix of rank r. Find a relation among m, n and r so that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every $\mathbf{b} \in \mathbb{R}^m$.

5.22. Find the equation of the straight line that fits best the data of the four points $(0, 1)$, $(1, 3)$, $(2, 4)$, and $(3, 4)$.

5.23. Find the cubic polynomial that fits best the data of the five points
$(-1, -14)$, $(0, -5)$, $(1, -4)$, $(2, 1)$, and $(3, 22)$.

5.24. Let W be the subspace of the Euclidean 4-space $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_i$'s given in each of the following problems. Find the projection matrix P for the subspace W and the null space $\mathcal{N}(P)$ of P. Compute $P\mathbf{b}$ for $\mathbf{b}$ given in each problem.

(1) $\mathbf{x}_1 = (1, 1, 1, 1), \mathbf{x}_2 = (1, -1, 1, -1), \mathbf{x}_3 = (-1, 1, 1, 0)$, and $\mathbf{b} = (1, 2, 1, 1)$.

(2) $\mathbf{x}_1 = (0, -2, 2, 1), \mathbf{x}_2 = (2, 0, -1, 2)$, and $\mathbf{b} = (1, 1, 1, 1)$.

(3) $\mathbf{x}_1 = (2, 0, 3, -6), \mathbf{x}_2 = (-3, 6, 8, 0)$, and $\mathbf{b} = (-1, 2, -1, 1)$.

5.25. Find the matrix for orthogonal projection from the Euclidean 3-space $\mathbb{R}^3$ to the plane spanned by the vectors $(1, 1, 1)$ and $(1, 0, 2)$.

5.26. Find the projection matrix for the row space and the null space of each of the following matrices:

$$(1) \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}, \quad (2) \begin{bmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad (3) \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 2 \\ 2 & 3 & -1 \end{bmatrix}.$$

5.27. Consider the space $C[-1, 1]$ with the inner product defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

A function $f \in C[-1, 1]$ is *even* if $f(-x) = f(x)$, or *odd* if $f(-x) = -f(x)$. Let U and V be the sets of all even functions and odd functions in $C[-1, 1]$, respectively.

(1) Prove that U and V are subspaces and $C[-1, 1] = U + V$.

(2) Prove that $U \perp V$.

(3) Prove that for any $f \in C[-1, 1]$, $\|f\|^2 = \|h\|^2 + \|g\|^2$ where $f = h + g \in U \oplus V$.

5.28. Determine whether the following statements are true or false, in general, and justify your answers.

(1) An inner product can be defined on any vector space.

(2) Two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space are linearly independent if and only if the angle between $\mathbf{x}$ and $\mathbf{y}$ is not zero.

(3) If V is perpendicular to W, then $V^\perp$ is perpendicular to $W^\perp$.

(4) Let V be an inner product space. Then $\|\mathbf{x} - \mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|$ for any vectors $\mathbf{x}$ and $\mathbf{y}$ in V.

(5) Every permutation matrix is an orthogonal matrix.

(6) For any $n \times n$ symmetric matrix A, $\mathbf{x}^T A\mathbf{y}$ defines an inner product on $\mathbb{R}^n$.

(7) A square matrix A is a projection matrix if and only if $A^2 = I$.

(8) For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, T is an orthogonal projection if and only if $id_{\mathbb{R}^n} - T$ is an orthogonal projection.

(9) For any $m \times n$ matrix A, the row space $\mathcal{R}(A)$ and the column space $\mathcal{C}(A)$ are orthogonal.

(10) A linear transformation T is an isomorphism if and only if it is an isometry.

(11) For any $m \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^m$, $A^T A\mathbf{x} = A^T\mathbf{b}$ always has a solution.

(12) The least squares solution of $A\mathbf{x} = \mathbf{b}$ is unique for any matrix A.

(13) The least squares solution of $A\mathbf{x} = \mathbf{b}$ is the orthogonal projection of $\mathbf{b}$ on the column space of A.

6

Diagonalization

6.1 Eigenvalues and eigenvectors

Gaussian elimination plays a fundamental role in solving a system $A\mathbf{x} = \mathbf{b}$ of linear equations. In general, instead of solving the given system, one could try to solve the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$, whose solutions are the true solutions or the least squares solutions depending on whether or not the given system is consistent. Note that the matrix $A^T A$ is a symmetric square matrix, and so one may assume that the matrix in the system is a square matrix. For this kind of reason, we focus on a diagonal matrix or a linear transformation from a vector space to itself throughout this chapter.

Recall that a square matrix A, as a linear transformation on $\mathbb{R}^n$, may have various matrix representations depending on the choice of the bases, which are all in similar relations. In particular, A itself is the matrix representation with respect to the standard basis. One may now ask whether there exists a basis β with respect to which the matrix representation $[A]_\beta$ of A is diagonal or not. But then A and a diagonal matrix $D = [A]_\beta$ are similar: i.e., there is an invertible matrix Q such that $D = Q^{-1}AQ$.

In this chapter, we will see which matrices can have diagonal matrix representations and how one can find such representations. For this we introduce *eigenvalues* and *eigenvectors*, which play important roles in their own right in mathematics and have far-reaching applications not only in mathematics, but also in other fields of science and engineering. Some specific applications of diagonalization of a square matrix A are to

(1) solving a system $A\mathbf{x} = \mathbf{b}$ of linear equations,
(2) checking the invertibility of A or estimation of $\det A$,
(3) calculating a power A^n or the limit of a matrix series $\sum_{n=1}^{\infty} A^n$,
(4) solving systems of linear differential equations or difference equations,
(5) finding a simple form of the matrix representation of a linear transformation, etc.

One might notice that some of these problems are easy if A is diagonal.

Definition 6.1 Let A be an $n \times n$ square matrix. A nonzero vector $\mathbf{x}$ in the n-space $\mathbb{R}^n$ is called an **eigenvector** (or **characteristic vector**) of A if there is a scalar λ in $\mathbb{R}$

J H Kwak et al., *Linear Algebra*

such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The scalar λ is called an **eigenvalue** (or **characteristic value**) of A, and we say $\mathbf{x}$ **belongs** to λ.

Geometrically, an eigenvector of a matrix A is a nonzero vector $\mathbf{x}$ in the n-space $\mathbb{R}^n$ such that the vectors $\mathbf{x}$ and $A\mathbf{x}$ are parallel. In other words, the subspace W spanned by $\mathbf{x}$ is *invariant* under the linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ in the sense $A(W) \subset W$. Algebraically, an eigenvector $\mathbf{x}$ is a nontrivial solution of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ of linear equations, that is, an eigenvector $\mathbf{x}$ is a nonzero vector in the null space $\mathcal{N}(\lambda I - A)$.

There are two unknowns in the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$: an eigenvalue λ and an eigenvector $\mathbf{x}$. To find those unknowns, first we should determine an eigenvalue λ by using the fact that the equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$ if and only if λ satisfies the equation

$$\det(\lambda I - A) = 0,$$

called the **characteristic equation** of A. Note that $\det(\lambda I - A)$ is a polynomial of degree n in λ and it will be called the **characteristic polynomial** of A. Thus, the eigenvalues are just the roots of the characteristic equation $\det(\lambda I - A) = 0$.

Next, the eigenvectors of A can be determined by solving the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ for each eigenvalue λ. In summary, by referring to Theorem 3.26 we have the following theorem.

Theorem 6.1 *For any square matrix A, the following are equivalent:*

(1) *λ is an eigenvalue of A;*
(2) $\det(\lambda I - A) = 0$ *(or* $\det(A - \lambda I) = 0$*);*
(3) *$\lambda I - A$ is singular;*
(4) *the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.*

Recall that the eigenvectors of A belonging to an eigenvalue λ are just the nonzero vectors $\mathbf{x}$ in the null space $\mathcal{N}(\lambda I - A)$. This null space is called the **eigenspace** of A belonging to λ, and denoted by $E(\lambda)$.

Example 6.1 (*Matrix having distinct eigenvalues*) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & -\sqrt{2} \\ -\sqrt{2} & \lambda - 1 \end{bmatrix} = \lambda^2 - 3\lambda = \lambda(\lambda - 3).$$

Thus the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$. To determine the eigenvectors belonging to λ_i's, we should solve the homogeneous system of equations $(\lambda_i I - A)\mathbf{x} = \mathbf{0}$. Let us take $\lambda_1 = 0$ first; then the system of equations $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ becomes

$$\begin{cases} -2x_1 - \sqrt{2}x_2 = 0, \\ -\sqrt{2}x_1 - x_2 = 0, \end{cases} \quad \text{or } x_2 = -\sqrt{2}x_1.$$

Hence, $\mathbf{x}_1 = (x_1, x_2) = (-1, \sqrt{2})$ is an eigenvector belonging to $\lambda_1 = 0$, and $E(0) = \{t\mathbf{x}_1 : t \in \mathbb{R}\}$. (Here, one can take any nonzero solution (x_1, x_2) as an eigenvector $\mathbf{x}_1$ belonging to $\lambda_1 = 0$.)

For $\lambda_2 = 3$, the system of equations $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$ becomes

$$\begin{cases} x_1 - \sqrt{2}x_2 = 0, \\ -\sqrt{2}x_1 + 2x_2 = 0, \end{cases} \quad \text{or } x_1 = \sqrt{2}x_2.$$

Thus, by a similar calculation, $\mathbf{x}_2 = (\sqrt{2}, 1)$ is one of the eigenvectors belonging to $\lambda_2 = 3$ and $E(3) = \{t\mathbf{x}_2 : t \in \mathbb{R}\}$. Note that the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ belonging to the eigenvalues λ_1 and λ_2 respectively are linearly independent. □

Example 6.2 (*Matrix having a repeated eigenvalue but full eigenvectors*) Find a basis for the eigenspaces of

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Solution: The characteristic polynomial of A is $(\lambda-1)(\lambda-5)^2$, so that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$ with multiplicity 2. Thus, there are two eigenspaces of A. By definition, $\mathbf{x} = (x_1, x_2, x_3)$ is an eigenvector of A belonging to λ if and only if $\mathbf{x}$ is a nontrivial solution of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} \lambda-3 & 2 & 0 \\ 2 & \lambda-3 & 0 \\ 0 & 0 & \lambda-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If $\lambda_1 = 1$, then the system becomes

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving this system yields $x_1 = t$, $x_2 = t$, $x_3 = 0$ for $t \in \mathbb{R}$. Thus, the eigenvectors belonging to $\lambda_1 = 1$ are nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R},$$

so that $(1, 1, 0)$ is a basis for the eigenspace $E(\lambda_1)$ belonging to $\lambda_1 = 1$.

If $\lambda_2 = 5$, then the system becomes

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving this system yields $x_1 = -s$, $x_2 = s$, $x_3 = t$ for $s,\ t \in \mathbb{R}$. Thus, the eigenvectors of A belonging to $\lambda_2 = 5$ are nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} -s \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $s,\ t \in \mathbb{R}$. Since $(-1,\ 1,\ 0)$ and $(0,\ 0,\ 1)$ are linearly independent, they form a basis for the eigenspace $E(\lambda_2)$ belonging to $\lambda_2 = 5$. □

For each eigenvalue λ of A in Examples 6.1 and 6.2, one can see that the dimension of the eigenspace $E(\lambda)$ is equal to the multiplicity of λ as a root of the equation $\det(\lambda I - A) = 0$. But, in general it is not true as the next example shows.

Example 6.3 (*Matrix having a repeated eigenvalue with insufficient eigenvectors*) Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

A simple computation shows that the characteristic polynomial of the matrix A is $(\lambda - 2)^5$ so that the eigenvalue $\lambda = 2$ is of multiplicity 5. However, there is only one linearly independent eigenvector $\mathbf{e}_1 = (1,\ 0,\ 0,\ 0,\ 0)$ belonging to $\lambda = 2$ because $\operatorname{rank}(2I - A) = 4$, which shows that $\dim E(\lambda) = \dim \mathcal{N}(2I - A) = 1$ is less than the multiplicity of λ. This kind of matrix will be discussed later in Chapter 8. □

Note that the equation $\det(\lambda I - A) = 0$ may have complex roots, which are called **complex eigenvalues**. However, the complex numbers are not scalars of the real vector space. In many cases, it is necessary to deal with those complex numbers, that is, we need to expand the set of scalars to the set of complex numbers. This expansion of the set of scalars to the set of complex numbers leads us to work with complex vector spaces, which will be treated in Chapter 7. In this chapter, we restrict our discussion to the case of real eigenvalues, even though the entire discussion in this chapter applies in the same way to the complex vector spaces.

Example 6.4 (*Matrix having complex eigenvalues*) The characteristic polynomial of the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is $\lambda^2 - 2\cos\theta\lambda + (\cos^2\theta + \sin^2\theta)$. Thus, the eigenvalues are $\lambda = \cos\theta \pm i\sin\theta$, which are complex numbers, so this matrix as a rotation of $\mathbb{R}^2$ has no real eigenvalues unless $\theta = n\pi$, $n = 0,\ \pm 1,\ \pm 2,\ \ldots$. □

Problem 6.1 Let λ be an eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Use mathematical induction to show that λ^m is an eigenvalue of A^m and $\mathbf{x}$ is an eigenvector of A^m belonging to λ^m for each $m = 1, 2, \ldots$.

In the following, we derive some basic properties of the eigenvalues and eigenvectors.

Lemma 6.2 **(1)** *If A is a triangular matrix, then the diagonal entries are exactly the eigenvalues of A.*
(2) *If A and B are square matrices similar to each other, then they have the same characteristic polynomial.*

Proof: (1) The characteristic equation of an upper triangular matrix is

$$\begin{aligned}\det(\lambda I - A) &= \det\begin{bmatrix} \lambda - a_{11} & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ \mathbf{0} & & & \lambda - a_{nn} \end{bmatrix} \\ &= (\lambda - a_{11}) \cdots (\lambda - a_{nn}) = 0.\end{aligned}$$

(2) Since there exists a nonsingular matrix Q such that $B = Q^{-1}AQ$,

$$\begin{aligned}\det(\lambda I - B) &= \det\left(Q^{-1}(\lambda I)Q - Q^{-1}AQ\right) \\ &= \det\left(Q^{-1}(\lambda I - A)Q\right) \\ &= \det Q^{-1} \det(\lambda I - A) \det Q \\ &= \det(\lambda I - A).\end{aligned}$$

□

Lemma 6.2(2) says that similar matrices have the same eigenvalues, i.e., *the eigenvalues are invariant under the similarity.* However, their eigenvectors might be different: in fact, $\mathbf{x}$ is an eigenvector of B belonging to λ if and only if $Q\mathbf{x}$ is an eigenvector of A belonging to λ, since $AQ = QB$, and $AQ\mathbf{x} = QB\mathbf{x} = \lambda Q\mathbf{x}$.

Theorem 6.3 *Let an $n \times n$ matrix A have n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ possibly with repetition. Then,*

(1) $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$, *(the product of the n eigenvalues),*
(2) $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, *(the sum of the n eigenvalues).*

Proof: (1) Since eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the zeros of the characteristic polynomial of A, we have

$$\det(\lambda I - A) = (\lambda - \lambda_1)\,(\lambda - \lambda_2)\,\cdots\,(\lambda - \lambda_n).$$

If we take $\lambda = 0$ in both sides, then we get

$$(-1)^n \det A = (-1)^n \lambda_1\,\lambda_2\,\cdots\,\lambda_n.$$

(2) On the other hand,

$$\begin{aligned}(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) &= \det(\lambda I - A)\\ &= \det\begin{bmatrix}\lambda - a_{11} & -a_{12} & \cdots & -a_{1n}\\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n}\\ \vdots & \vdots & & \vdots\\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn}\end{bmatrix},\end{aligned}$$

which is a polynomial of the form $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ in λ. One can compute the coefficient c_{n-1} of λ^{n-1} in two ways by expanding both sides, and get $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn} = \mathrm{tr}(A)$. □

Problem 6.2 Show that

(1) for any 2×2 matrix A, $\det(\lambda I - A) = \lambda^2 + \mathrm{tr}(A)\lambda + \det A$;
(2) for any 3×3 matrix A,

$$\det(\lambda I - A) = -\lambda^3 + \mathrm{tr}(A)\lambda^2 + \frac{1}{2}\sum_{i \neq j}(a_{ij}a_{ji} - a_{ii}a_{jj})\lambda + \det A.$$

In Theorem 6.3, we assume that the matrix A has n (real) eigenvalues counting multiplicities. But, by allowing the scalars to be complex numbers, which will be done in the next chapter, every $n \times n$ matrix has n eigenvalues counting multiplicities, so that Theorem 6.3 remains true for any square matrix.

Corollary 6.4 *The determinant and the trace of A are invariant under similarity.*

Recall that any square matrix A is singular if and only if $\det A = 0$. However, $\det A$ is the product of its n eigenvalues. Thus a square matrix A is singular if and only if zero is an eigenvalue of A, or A *is invertible if and only if zero is not an eigenvalue of* A.

The following corollaries are easy consequences of this fact.

Corollary 6.5 *For any $n \times n$ matrices A and B, the following are equivalent.*

(1) *Zero is an eigenvalue of AB.*
(2) *A or B is singular.*
(3) *Zero is an eigenvalue of BA.*

Corollary 6.6 *For any $n \times n$ matrices A and B, the matrices AB and BA have the same eigenvalues.*

Proof: By Corollary 6.5, zero is an eigenvalue of AB if and only if it is an eigenvalue of BA. Let λ be a nonzero eigenvalue of AB with $(AB)\mathbf{x} = \lambda\mathbf{x}$ for a nonzero vector $\mathbf{x}$. Then the vector $B\mathbf{x}$ is not zero, since $\lambda \neq 0$, but

$$(BA)(B\mathbf{x}) = B(\lambda\mathbf{x}) = \lambda(B\mathbf{x}).$$

This means that $B\mathbf{x}$ is an eigenvector of BA belonging to the eigenvalue λ, and λ is an eigenvalue of BA. Similarly, any nonzero eigenvalue of BA is also an eigenvalue of AB. □

Problem 6.3 Find the matrices A and B such that $\det A = \det B$, $\mathrm{tr}(A) = \mathrm{tr}(B)$, but A is not similar to B.

Problem 6.4 Show that A and A^T have the same eigenvalues. Do they necessarily have the same eigenvectors?

Problem 6.5 Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix A. Then

(1) A is invertible if and only if $\lambda_i \neq 0$ for all $i = 1, 2, \ldots, n$.

(2) If A is invertible, then the inverse A^{-1} has eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

Problem 6.6 For any $n \times n$ matrices A and B, show that AB and BA are similar if A or B is nonsingular. Is it true for two singular matrices A and B?

6.2 Diagonalization of matrices

In this section, we are going to show what kinds of square matrices are similar to diagonal matrices. That is, given a square matrix A, we want to know whether there exists an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix, and if so, how one can find such a matrix Q.

Definition 6.2 A square matrix A is said to be **diagonalizable** if there exists an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix (i.e., A is similar to a diagonal matrix).

If a square matrix A is diagonalizable, then the similarity $D = Q^{-1}AQ$ gives an easy way to solve some problems related to the matrix A, like (1)–(5) on page 201. For instance, let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations with a square matrix A, and suppose that there is an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix D. Then the system $A\mathbf{x} = \mathbf{b}$ can be written as $QDQ^{-1}\mathbf{x} = \mathbf{b}$, or equivalently $DQ^{-1}\mathbf{x} = Q^{-1}\mathbf{b}$. Hence, for $\mathbf{c} = Q^{-1}\mathbf{b}$ the solution $\mathbf{y}$ of $D\mathbf{y} = \mathbf{c}$ yields the solution $\mathbf{x} = Q\mathbf{y}$ of the system $A\mathbf{x} = \mathbf{b}$. Note that $D\mathbf{y} = \mathbf{c}$ can be solved easily.

The next theorem characterizes a diagonalizable matrix, and the proof shows a practical way of diagonalizing a matrix.

Theorem 6.7 *Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.*

Proof: ($\Rightarrow$) Suppose A is diagonalizable. Then there is an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix D, say

$$Q^{-1}AQ = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

or, equivalently, $AQ = QD$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ denote the column vectors of Q. Since

$$\begin{aligned} AQ &= [A\mathbf{x}_1\ A\mathbf{x}_2\ \cdots\ A\mathbf{x}_n], \\ QD &= [\lambda_1\mathbf{x}_1\ \lambda_2\mathbf{x}_2\ \cdots\ \lambda_n\mathbf{x}_n], \end{aligned}$$

the matrix equation $AQ = QD$ implies $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for $i = 1, \ldots, n$. Moreover, since Q is invertible, their column vectors are nonzero and are linearly independent, that is, the $\mathbf{x}_i$'s are n linearly independent eigenvectors of A.

($\Leftarrow$) Assume that A has n linearly independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ belonging to the eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, so that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for $i = 1, \ldots, n$. If we define a matrix Q as

$$Q = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n]$$

with $\mathbf{x}_j$ as the j-th column vector, then the same equation shows $AQ = QD$, where D is the diagonal matrix having the eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal. Since the column vectors of Q are assumed to be linearly independent, Q is invertible, so $Q^{-1}AQ = D$. □

Remark: **(1)** The proof of Theorem 6.7 reveals how to diagonalize an $n \times n$ matrix A.

Step 1	Find n linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of A.
Step 2	Form the matrix $Q = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n]$.
Step 3	The matrix $Q^{-1}AQ$ will be a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ as its successive diagonal entries, where λ_j is the eigenvalue associated with the eigenvector $\mathbf{x}_j$, $j = 1, 2, \ldots, n$.

(2) Let α denote the standard basis for $\mathbb{R}^n$ and let $\beta = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be the basis for $\mathbb{R}^n$ consisting of n linearly independent eigenvectors of A. Then the matrix

$$Q = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n] = [[\mathbf{x}_1]_\alpha\ [\mathbf{x}_2]_\alpha\ \cdots\ [\mathbf{x}_n]_\alpha] = [id]_\beta^\alpha$$

is the basis-change matrix from β to α, and the matrix representation of A, as a linear transformation, with respect to β, is

$$[A]_\beta = [id]^\beta_\alpha [A]_\alpha [id]^\alpha_\beta = Q^{-1}AQ = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix}.$$

Note that the diagonal entries λ_i's are the eigenvalues of A.

(3) *Not all matrices are diagonalizable.* A standard example is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Its eigenvalues are $\lambda_1 = \lambda_2 = 0$. Hence, if A is diagonalizable, then

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \mathbf{0}$$

for some invertible matrix Q, and then A must be the zero matrix. Since A is not the zero matrix, no invertible matrix Q can be obtained so that $Q^{-1}AQ$ is diagonal.

Example 6.5 (*Several different types of a diagonalization*) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}.$$

Solution: A direct calculation gives that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$ and $\lambda_3 = -2$, and their associated eigenvectors are

$$\mathbf{x}_1 = (1,\ 0,\ 0),\ \mathbf{x}_2 = (0,\ 1,\ 1) \quad \text{and} \quad \mathbf{x}_3 = (1,\ 2,\ 1),$$

respectively. They are linearly independent, and the first two vectors $\mathbf{x}_1, \mathbf{x}_2$ form a basis for the eigenspace $E(1)$ belonging to $\lambda_1 = \lambda_2 = 1$, and $\mathbf{x}_3$ forms a basis for the eigenspace $E(-2)$ belonging to $\lambda_3 = -2$. Thus, the matrix

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

diagonalizes A. In fact, one can verify that

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned}$$

What would happen if one chose different eigenvectors belonging to the eigenvalues 1 and -2? According to the proof of Theorem 6.7, nothing would happen: Any matrix whose columns are linearly independent eigenvectors will diagonalize A. For example, $\{(-1, 0, 0), (0, -1, -1)\}$ is another basis for $E(1)$, and $\{(2, 4, 2)\}$ is also a basis for $E(-2)$. The matrix

$$Q = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 4 \\ 0 & -1 & 2 \end{bmatrix} \text{ also diagonalizes } A \text{ as } Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

A change in the order of the eigenvectors in constructing a basis-change matrix Q does not change the diagonalizability of A, but the eigenvalues appearing on the main diagonal of the resulting diagonal matrix would appear in accordance with the order of the eigenvectors in the basis-change matrix. For example, let

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then, S will diagonalize A, because it has linearly independent eigenvectors as columns. In fact, one can show that

$$S^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

Problem 6.7 Show that the following matrices are not diagonalizable.

$$(1)\ A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad (2)\ B = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}, \quad \lambda \text{ is any scalar.}$$

Problem 6.8 Construct a 2×2 matrix A whose eigenvalues are 2 and 3, and whose eigenvectors are $(2, 1)$ and $(3, 2)$, respectively.

From Theorem 6.7, we learn how to diagonalize a matrix and what the diagonal matrix is when the matrix has a full set of linearly independent eigenvectors. The next question is when a square matrix A can have a full set of linearly independent eigenvectors. The following theorem shows that it can happen if an $n \times n$ matrix has n distinct (real) eigenvalues.

Theorem 6.8 *Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of a matrix A and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ eigenvectors belonging to them, respectively. Then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is linearly independent.*

Proof: Let r be the largest integer such that $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is linearly independent. If $r = k$, then there is nothing to prove. Suppose not, i.e., $1 \leq r < k$. Then

$\{\mathbf{x}_1, \ldots, \mathbf{x}_{r+1}\}$ is linearly dependent. Thus, there exist scalars $c_1, c_2, \ldots, c_{r+1}$ with $c_{r+1} \neq 0$ such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{r+1}\mathbf{x}_{r+1} = \mathbf{0}. \tag{1}$$

Multiplying both sides by A and using

$$A\mathbf{x}_1 = \lambda_1\mathbf{x}_1, \ A\mathbf{x}_2 = \lambda_2\mathbf{x}_2, \ \ldots, \ A\mathbf{x}_{r+1} = \lambda_{r+1}\mathbf{x}_{r+1},$$

one can get

$$c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = \mathbf{0}. \tag{2}$$

Multiplying both sides of (1) by λ_{r+1} and subtracting the resulting equation from (2) yields

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + c_2(\lambda_2 - \lambda_{r+1})\mathbf{x}_2 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = \mathbf{0}.$$

Since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ is linearly independent and $\lambda_1, \lambda_2, \ldots, \lambda_{r+1}$ are all distinct, it follows that $c_1 = c_2 = \cdots = c_r = 0$. Substituting these values in (1) yields $c_{r+1} = 0$, which is a contradiction to the assumption. □

As a consequence of Theorems 6.7 and 6.8, we obtain the following.

Theorem 6.9 *If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.*

It follows from Theorem 6.9 that if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are eigenvectors of an $n \times n$ matrix A belonging to n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, then they form a basis for $\mathbb{R}^n$ and the matrix representation of A with respect to this basis should be a diagonal matrix as shown in Remark (2) on page 208.

Of course, some matrices can have eigenvalues with multiplicities > 1 so that the number of distinct eigenvalues is strictly less than n. In this case, if such a matrix still has n linearly independent eigenvectors, then it is also diagonalizable, because for a diagonalization all we need is n linearly independent eigenvectors (see Example 6.2 or try with the matrix λI_n). In some cases, such a matrix does not have n linearly independent eigenvectors (see Example 6.3), so a diagonalization is impossible. This case will be discussed in Chapter 8.

The next example shows a simple application of the diagonalization to the computation of the power A^n of a matrix A.

Example 6.6 Compute A^{100} for $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

Solution: Its eigenvalues are 5 and -2 with associated eigenvectors $(1, 1)$ and $(-4, 3)$, respectively. Hence $Q = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}$ diagonalizes A, i.e.,

$$Q^{-1}AQ = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{or} \quad A = Q\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}Q^{-1}.$$

Therefore,

$$\begin{aligned} A^{100} &= Q\begin{bmatrix} 5^{100} & 0 \\ 0 & (-2)^{100} \end{bmatrix} Q^{-1} \\ &= \frac{1}{7}\begin{bmatrix} 3\cdot 5^{100}+4\cdot 2^{100} & 4\cdot 5^{100}-4\cdot 2^{100} \\ 3\cdot 5^{100}-3\cdot 2^{100} & 4\cdot 5^{100}+3\cdot 2^{100} \end{bmatrix}. \end{aligned}$$ □

Problem 6.9 For the matrix $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$,

(1) diagonalize the matrix A; and (2) find the eigenvalues of $A^{10} + A^7 + 5A$.

6.3 Applications

6.3.1 Linear recurrence relations

Early in the thirteenth century, Fibonacci posed the following problem: "Suppose that a newly born pair of rabbits produces no offspring during the first month of their lives, but each pair gives birth to a new pair once a month from the second month onward. Starting with one ($= x_1$) newly born pair in the first month, how many pairs of rabbits can be bred in a given time, assuming no rabbit dies?"

Initially, there is one pair. After one month there is still one pair, but two months later it gives a birth, so there are two pairs. If at the end of n months there are x_n pairs, then after $n+1$ months the number will be the x_n pairs plus the number of offspring of the x_{n-1} pairs who were alive at $n-1$ months. Therefore, we have for $n \geq 2$,

$$x_{n+1} = x_n + x_{n-1}.$$

Here, if we assume $x_0 = 0$ and $x_1 = 1$, then the first several terms of the sequence become

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \ldots .$$

This sequence is called the **Fibonacci sequence** and each term is called a **Fibonacci number**.

Example 6.7 Find the 2000$^{\text{th}}$ Fibonacci number.

Solution: A standard trick is to consider a trivial extra equation $x_n = x_n$ together with the given equation:

$$\begin{cases} x_{n+1} = x_n + x_{n-1} \\ \quad\; x_n = x_n. \end{cases}$$

Equivalently in matrix notation,

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix},$$

which is of the form

$$\mathbf{x}_n = A\mathbf{x}_{n-1} = A^n\mathbf{x}_0, \quad n = 1, 2, \ldots,$$

where $\mathbf{x}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Thus, the problem is reduced to computing A^n. However, a simple computation gives the eigenvalues $\lambda_1 = \frac{1}{2}(1+\sqrt{5})$, $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$ of A and their associated eigenvectors $\mathbf{v}_1 = (\lambda_1, 1)$, $\mathbf{v}_2 = (\lambda_2, 1)$, respectively. Moreover, the basis-change matrix and its inverse are found to be

$$Q = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}, \quad Q^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}.$$

With $D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$,

$$A^n = QD^nQ^{-1} = Q \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} Q^{-1}.$$

For instance, if $n = 2000$, then

$$\begin{aligned} \begin{bmatrix} x_{2001} \\ x_{2000} \end{bmatrix} = \mathbf{x}_{2000} &= A^{2000} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = QD^{2000}Q^{-1}\mathbf{x}_0 \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{2001} - \lambda_2^{2001} \\ \lambda_1^{2000} - \lambda_2^{2000} \end{bmatrix}. \end{aligned}$$

It gives

$$x_{2000} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{2000} - \left(\frac{1-\sqrt{5}}{2} \right)^{2000} \right).$$

In general, the Fibonacci numbers satisfy

$$x_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n} - \left(\frac{1-\sqrt{5}}{2} \right)^{n} \right)$$

for $n \geq 0$.

Note that since x_{2000} must be an integer, we look for the nearest integer to the huge number $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{2000}$, because $\left(\frac{1-\sqrt{5}}{2}\right)^k$ is actually very small for large k.

Historically, the number $\frac{1+\sqrt{5}}{2}$, which is very close to the ratio $\frac{x_{2001}}{x_{2000}}$, is called the **golden mean.** $\square$

Remark: The golden mean is one of the mysterious naturally occurring numbers, like $e = 2.71828182\cdots$ or $\pi = 3.14159265\cdots$ and it is denoted by ϕ. Its decimal representation is $\phi = 1.61803398\cdots$. It is also described as $\phi = \frac{\ell}{s}$ for $0 < s < \ell$ satisfying $\frac{\ell}{s} = \frac{\ell+s}{\ell}$.

Definition 6.3 A sequence $\{x_n : n \geq 0\}$ of numbers is said to satisfy a **linear recurrence relation of order** k if there exist k constants a_i, $i = 1, \ldots, k$ with a_1 and a_k nonzero such that

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k} \quad \text{for all } n \geq k.$$

For example, the relation $x_n = ax_{n-1}$ of order 1 for $n = 1, 2, \ldots$ gives a geometric sequence $x_n = a^n x_0$, and the relation $x_{n+1} = x_n + x_{n-1}$ of order 2 with $x_0 = 0$, $x_1 = 1$ for $n = 1, 2, \ldots$ gives the Fibonacci sequence. A **solution** to the recurrence relation is any sequence $\{x_n : n \geq 0\}$ of numbers that satisfies the equation. Of course, a solution can be found by simply writing out enough terms of the sequence if k beginning values $x_0, x_1, \ldots, x_{k-1}$, called the **initial values**, are given.

As in the case of the Fibonacci sequence, one can write the recurrence relation

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k} \quad \text{for all } n \geq k,$$

or equivalently,

$$x_{n+k-1} = a_1x_{n+k-2} + a_2x_{n+k-3} + \cdots + a_kx_{n-1} \quad \text{for all } n \geq 1.$$

Its matrix form with some trivial extra equations is

$$\mathbf{x}_n = \begin{bmatrix} x_{n+k-1} \\ x_{n+k-2} \\ \vdots \\ x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{k-1} & a_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+k-2} \\ x_{n+k-3} \\ \vdots \\ x_n \\ x_{n-1} \end{bmatrix} = A\mathbf{x}_{n-1}$$

for $n \geq 1$, or simply $\mathbf{x}_n = A\mathbf{x}_{n-1}$. The matrix A is called the **companion matrix** of a recurrence relation $\mathbf{x}_n = A\mathbf{x}_{n-1}$.

To solve a recurrence relation $\mathbf{x}_n = A\mathbf{x}_{n-1}$, we first compute the characteristic polynomial of a companion matrix A.

Lemma 6.10 *For a companion matrix*

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{k-1} & a_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad \textit{with } a_1 \textit{ and } a_k \textit{ nonzero},$$

(1) *the characteristic polynomial of* A *is* $\lambda^k - a_1\lambda^{k-1} - \cdots - a_{k-1}\lambda - a_k$.

(2) *All eigenvalues of* A *are nonzero and for any eigenvalue* λ *of* A, $x_n = \lambda^n$ *is a solution of the recurrence relation* $\mathbf{x}_n = A\mathbf{x}_{n-1}$.

Proof: (1) Use induction on k. Clearly true for $k = 1$. Assume the equality for $k = n-1$. Let $k = n$. By taking the cofactor expansion of $\det(\lambda I - A)$ along the last column, the induction hypothesis gives

$$\begin{aligned}\det(\lambda I - A) &= \lambda(\lambda^{n-1} - a_1\lambda^{n-2} - \cdots - a_{n-1}) + (-1)^{2n-1}a_n \\ &= \lambda^n - a_1\lambda^{n-1} - \cdots - a_{n-1}\lambda - a_n.\end{aligned}$$

(2) Clearly all eigenvalues are not zero, because $a_k \neq 0$. It follows from (1) that for any eigenvalue λ of A, $x_n = \lambda^n$ satisfies the recurrence relation

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}.$$

□

Remark: (1) By Lemma 6.10(1), every *monic* polynomial, a polynomial whose coefficient of the highest degree is 1, can be expressed as the characteristic polynomial of some matrix A. This matrix A is also called the companion matrix of the monic polynomial $p(\lambda) = \lambda^k - a_1\lambda^{k-1} - \cdots - a_{k-1}\lambda - a_k$.

(2) From Lemma 6.10(1), one can see that if a recurrence relation

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}$$

is given, then the characteristic equation of the associated companion matrix A can be obtained from the recurrence relation by replacing x_i with λ^i and dividing the resulting equation by λ^{n-k}. This relation between the recurrence relation and the characteristic equation of the matrix A can be a reason why $\{\lambda^n : n \geq 0\}$ is a solution for each eigenvalue λ.

Lemma 6.11 *If* λ_0 *is an eigenvalue of the companion matrix* A *of a linear difference equation of order* k, *then the eigenspace* $E(\lambda_0)$ *is a 1-dimensional subspace and contains* $[\lambda_0^{k-1} \;\cdots\; \lambda_0 \; 1]^T$.

Proof: An entry-wise comparison of $A\mathbf{x} = \lambda_0\mathbf{x}$,

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\begin{bmatrix} x_k \\ \vdots \\ x_2 \\ x_1 \end{bmatrix} = \lambda_0\begin{bmatrix} x_k \\ \vdots \\ x_2 \\ x_1 \end{bmatrix},$$

shows that $x_i = \lambda_0 x_{i-1} = \lambda_0^{i-1}x_1$ for $i = 2, \ldots, k$. □

The recurrence relation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ can be solved if the companion matrix A is diagonalizable.

Example 6.8 (*Recurrence relation* $\mathbf{x}_n = A\mathbf{x}_{n-1}$ *with diagonalizable* A) Solve the recurrence relation

$$x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3} \quad \text{for } n \geq 3$$

with initial values $x_0 = 0,\ x_1 = 1,\ x_2 = -1$.

Solution: In matrix form, it is

$$\mathbf{x}_n = \begin{bmatrix} x_{n+2} \\ x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{bmatrix} = A\mathbf{x}_{n-1} \quad \text{for } n \geq 1.$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3),$$

by Lemma 6.10. Hence, the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$ and their associated eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix},$$

respectively, by Lemma 6.11. Moreover, the basis-change matrix and its inverse can be found to be

$$Q = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad Q^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -5 & 6 \\ -2 & 8 & -6 \\ 1 & -3 & 2 \end{bmatrix}.$$

With

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad Q^{-1}\mathbf{x}_0 = Q^{-1}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix},$$

one can get

$$\begin{aligned} \mathbf{x}_n &= A^n\mathbf{x}_0 = QD^nQ^{-1}\mathbf{x}_0 \\ &= \begin{bmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix} \\ \text{or,} \quad \begin{bmatrix} x_{n+2} \\ x_{n+1} \\ x_n \end{bmatrix} &= \begin{bmatrix} -3 + 5 \times 2^{n+2} - 2 \times 3^{n+2} \\ -3 + 5 \times 2^{n+1} - 2 \times 3^{n+1} \\ -3 + 5 \times 2^n - 2 \times 3^n \end{bmatrix}. \end{aligned}$$

It implies that the solution is $x_n = -3 + 5 \times 2^n - 2 \times 3^n$. □

As a generalization of a recurrence relation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ with a companion matrix A, let us consider a sequence $\{\mathbf{x}_n\}$ of vectors in $\mathbb{R}^k$ defined by a matrix equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ with arbitrary $k \times k$ square matrix A (not necessarily a companion matrix). Such an equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is called a **linear difference equation.**

A **solution** to the linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is any sequence $\{\mathbf{x}_n \in \mathbb{R}^k : n \geq 0\}$ of vectors that satisfies the equation. In fact, a solution of a linear difference equation is reduced to a simple computation of A^n if the starting vector $\mathbf{x}_0$, called the **initial value**, is given.

We first examine the set of its solutions.

Theorem 6.12 *For any $k \times k$ matrix A, the set of all solutions of the linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is a k-dimensional vector space.*

In particular, the set of solutions of the recurrence relation of order k,

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}, \quad n \geq k$$

with nonzero a_1 and a_k, is a k-dimensional vector space.

Proof: Since the proofs are similar, we prove this only for the recurrence relation. Let W be the set of solutions $\{x_n\}$ of the recurrence relation. Clearly, a sum of two solutions and any scalar multiplication of a solution are also solutions. Hence, the solutions form a vector space as shown in Example 3.1(4). One can show that the function $f : W \to \mathbb{R}^k$ defined by $f(\{x_n\}) = (x_{k-1}, \ldots, x_1, x_0)$ is a linear transformation. Clearly, it is bijective, because any given initial k values of $x_0, x_1, \ldots, x_{k-1}$ generate recursively a unique sequence $\{x_n : n \geq 0\}$ of numbers that satisfies the equation. Hence, $\dim W = \dim \mathbb{R}^k = k$. (For the linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$, see Problem 6.10). $\square$

Problem 6.10 Let $\mathbf{x}_n = A\mathbf{x}_{n-1}$ be any linear difference equation with a $k \times k$ matrix A. For each basis vector $\mathbf{e}_i$ in $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ in $\mathbb{R}^k$, there is a unique solution of $\mathbf{x}_n = A\mathbf{x}_{n-1}$ with initial value $\mathbf{e}_i$. Show that such k solutions form a basis for the solution space of $\mathbf{x}_n = A\mathbf{x}_{n-1}$.

Definition 6.4 A basis for the space of solutions of a linear difference equation or a recurrence relation is called a **fundamental set of solutions**, and a **general solution** is described as its linear combination. If the initial value is specified, the solution is uniquely determined and it is called a **particular solution.**

By Theorem 6.12, it is enough to find k linearly independent solutions in order to solve a given linear difference equation or a recurrence relation of order k, and its general solution is just a linear combination of those linearly independent solutions.

First, we assume that the square matrix A is diagonalizable with k linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ belonging to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for $\mathbb{R}^k$, any initial vector $\mathbf{x}_0$ can be written as

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Since $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$, we have

$$\mathbf{x}_1 = A\mathbf{x}_0 = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_k\lambda_k\mathbf{v}_k,$$

and, in general for all $n = 1, 2, \ldots$,

$$\mathbf{x}_n = A^n\mathbf{x}_0 = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + \cdots + c_k\lambda_k^n\mathbf{v}_k.$$

In particular, if the companion matrix A of the recurrence relation $x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}$ of order k has k *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then its solution x_n is (as the $(k, 1)$-entry of the vector $\mathbf{x}_n$) a linear combination of $\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n$. In fact, for each $1 \le j \le k$, $x_n = \lambda_j^n$ is a solution of the recurrence relation, and these k solutions are linearly independent, so that it forms a fundamental set of solutions by Theorem 6.12. (One can also directly show that $x_n = \lambda_j^n$ satisfies the recurrence relation). Note that all these solutions are geometric sequences.

We can summarize as follows.

Theorem 6.13 *Let A be a $k \times k$ diagonalizable matrix with k linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ belonging to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively. Then, a general solution of a linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ can be written as*

$$\mathbf{x}_n = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + \cdots + c_k\lambda_k^n\mathbf{v}_k$$

for some constants $c_1, c_2, \ldots, c_k$.

In particular, for the recurrence relation

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}, \quad n \ge k$$

with nonzero a_1 and a_k, if the associated companion matrix A has k distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then its general solution is

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_k\lambda_k^n \quad \textit{with constants } c_i\textit{'s}.$$

Example 6.9 (*Recurrence relation with distinct eigenvalues*) Solve the recurrence relation

$$x_n = x_{n-1} + 7x_{n-2} - x_{n-3} - 6x_{n-4} \quad \text{for } n \ge 4,$$

and also find its particular solution satisfying the initial conditions $x_0 = 0$, $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

Solution: By Lemma 6.10, the characteristic polynomial of the companion matrix A associated with the given recurrence relation is

$$\det(\lambda I - A) = \lambda^4 - \lambda^3 - 7\lambda^2 + \lambda + 6 = (\lambda - 1)(\lambda + 1)(\lambda + 2)(\lambda - 3),$$

so that A has four distinct eigenvalues $\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = -2,\ \lambda_4 = 3$. Hence, the geometric sequences $\{1^n\},\ \{(-1)^n\},\ \{(-2)^n\},\ \{3^n\}$ are linearly independent and a general solution is a linear combination of them by Theorem 6.13:

$$x_n = c_1 1^n + c_2(-1)^n + c_3(-2)^n + c_4 3^n$$

with constants $c_1,\ c_2,\ c_3,\ c_4$. And the initial values give

$$\begin{array}{lcrcrcrcrcr} \text{if } n=0 & : & c_1 & + & c_2 & + & c_3 & + & c_4 & = & 0, \\ \text{if } n=1 & : & c_1 & + & (-1)^1 c_2 & + & (-2)^1 c_3 & + & 3^1 c_4 & = & 1, \\ \text{if } n=2 & : & c_1 & + & (-1)^2 c_2 & + & (-2)^2 c_3 & + & 3^2 c_4 & = & -1, \\ \text{if } n=3 & : & c_1 & + & (-1)^3 c_2 & + & (-2)^3 c_3 & + & 3^3 c_4 & = & 2, \end{array}$$

which is a system of linear equations with a 4×4 Vandermonde matrix as its coefficient matrix. Hence, one can solve it to have $c_1 = \frac{5}{12},\ c_2 = -\frac{1}{8},\ c_3 = -\frac{4}{15},\ c_4 = -\frac{1}{40}$ and then its particular solution is

$$x_n = \frac{5}{12}1^n - \frac{1}{8}(-1)^n - \frac{4}{15}(-2)^n - \frac{1}{40}3^n. \qquad \square$$

Problem 6.11 Let $\{a_n\}$ be a sequence with $a_0 = 1, a_1 = 2, a_2 = 0$, and the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$. Find the n-th term a_n.

The next example illustrates how to solve a recurrence relation when its associated companion matrix A has a repeated eigenvalue.

Example 6.10 (*Recurrence relation with a repeated eigenvalue*) Solve the recurrence relation

$$x_n = -2x_{n-1} - x_{n-2} \quad \text{for } n \geq 2,$$

and also find its particular solution satisfying the initial conditions $x_0 = 1,\ x_1 = 2$.

Solution: Its characteristic polynomial is

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2,$$

and $\lambda = -1$ is an eigenvalue of multiplicity 2. Hence, the geometric sequence $\{x_n\} = \{(-1)^n\}$ is a solution of the recurrence relation. Since its solution space is of dimension 2 by Theorem 6.12, we should find one more solution which is independent of $\{x_n\} = \{(-1)^n\}$. But, in this case $\{x_n\} = \{n(-1)^n\}$ is also a solution of the recurrence relation. In fact, for $n \geq 2$,

$$-2x_{n-1} - x_{n-2} = -2(n-1)(-1)^{n-1} - (n-2)(-1)^{n-2} = n(-1)^n = x_n.$$

Clearly, two solutions $\{(-1)^n\}$ and $\{n(-1)^n\}$ are linearly independent, and so $x_n = c_1(-1)^n + c_2 n(-1)^n$ is a general solution. The initial condition gives $c_1 = 1, c_2 = -3$ and $x_n = (-1)^n - 3n(-1)^n$ is the particular solution of the recurrence relation. □

In Example 6.10, we show that $\lambda = -1$ is an eigenvalue of multiplicity 2, and two geometric sequences $\{(-1)^n\}$ and $\{n(-1)^n\}$ are linearly independent solutions of the recurrence relation.

As a general case, let us consider a recurrence relation

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}, \quad n \geq k$$

with nonzero a_1 and a_k, and let the associated companion matrix A have s distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ with multiplicity $m_1, m_2, \ldots, m_s$, respectively. For each eigenvalue λ_i with multiplicity $m_i > 1$, we have their derivatives $f(\lambda_i) = f'(\lambda_i) = \cdots = f^{(m_i-1)}(\lambda_i) = 0$, where $f(\lambda) = \lambda^k - a_1\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_{k-1}\lambda - a_k$ is the characteristic polynomial of A. Hence, for a new function $F_1(\lambda)$ defined by

$$\begin{aligned} F_1(\lambda) &= \lambda^{n-k} f(\lambda) \\ &= \lambda^n - a_1\lambda^{n-1} - \cdots - a_{k-1}\lambda^{n-k+1} - a_k\lambda^{n-k}, \end{aligned}$$

one can see that the derivative $F_1'(\lambda) = 0$ at $\lambda = \lambda_i$ and then $F_2(\lambda) = \lambda F_1'(\lambda) = 0$ at $\lambda = \lambda_i$. That is,

$$n\lambda^n - a_1(n-1)\lambda^{n-1} - \cdots - a_{k-1}(n-k+1)\lambda^{n-k+1} - a_k(n-k)\lambda^{n-k} = 0.$$

It shows that $x_n = n\lambda_i^n$ is also a solution of the recurrence relation. Inductively, $F_j(\lambda) = \lambda F_{j-1}'(\lambda) = 0$ at $\lambda = \lambda_i$ shows that $x_n = n^{j-1}\lambda_i^n$ is also a solution for $j = 1, \ldots, m_i$. Therefore, one can conclude that $x_n = \lambda_i^n, n\lambda_i^n, \ldots, n^{m_i-1}\lambda_i^n$ are m_i linearly independent solutions of the recurrence relation. Getting together such m_i linearly independent solutions for each eigenvalue λ_i, one can get a fundamental set of solutions of the recurrence relation.

In summary, we have the following theorem.

Theorem 6.14 *For any given recurrence relation*

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}, \quad n \geq k$$

with nonzero a_1 and a_k, let the associated companion matrix A have s distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ with multiplicity $m_1, m_2, \ldots, m_s$, respectively. Then

$$\left\{\{\lambda_i^n\}, \{n\lambda_i^n\}, \ldots, \{n^{m_i-1}\lambda_i^n\} \mid i = 1, 2, \ldots, s\right\}$$

forms a fundamental set of solutions, and a general solution is a linear combination of them.

Problem 6.12 Prove that if $\lambda = q$ is an eigenvalue of a recurrence relation with multiplicity m, then the m solutions $\{q^n\}$, $\{nq^n\}$, $\ldots$, $\{n^{m-1}q^n\}$ are linearly independent.

Problem 6.13 Solve the recurrence relation $x_n = 3x_{n-1} - 4x_{n-3}$ for $n \geq 3$. What is it if $x_0 = 1, x_1 = x_2 = 1$?

6.3.2 Linear difference equations

A linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ represents sometimes mathematical models of dynamic processes that change over time and are widely used in such areas as economics, electrical engineering, and ecology. In this case, vectors $\mathbf{x}_n$ give information about a dynamic process when time n passes.

In this concept, a linear difference equation

$$\mathbf{x}_n = A\mathbf{x}_{n-1}, \; n = 1, 2, \ldots$$

with a square matrix A is also called a **discrete dynamical system.** If the matrix A is a companion matrix, then it is nothing but a recurrence relation.

If the matrix A is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_k$, then, by Theorem 6.13, a general solution of $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is

$$\mathbf{x}_n = c_1\lambda_1^n\mathbf{e}_1 + c_2\lambda_2^n\mathbf{e}_2 + \cdots + c_k\lambda_k^n\mathbf{e}_k = (c_1\lambda_1^n, c_2\lambda_2^n, \ldots, c_k\lambda_k^n)$$

with some constants $c_1, c_2, \ldots, c_k$.

Throughout this section, we are concerned with only a linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}, \; n = 1, 2, \ldots$ for a diagonalizable matrix A, because if A is not diagonalizable, it is not easy in general to solve it. However, it can be done after reducing A to a simpler form called the Jordan canonical form and this case will be discussed again in Chapter 8.

Let A be a $k \times k$ diagonalizable matrix with k linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ belonging to the eigenvalues $\lambda_1, \ldots, \lambda_k$, respectively. Then, by Theorem 6.13 again, a general solution of $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is

$$\mathbf{x}_n = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + \cdots + c_k\lambda_k^n\mathbf{v}_k$$

with some constants $c_1, c_2, \ldots, c_k$. Hence, if $|\lambda_i| < 1$ for all i, then the vector $\mathbf{x}_n$ must approach the zero vector as n increases. On the other hand, if there exists an eigenvalue λ_i with $|\lambda_i| > 1$, this vector $\mathbf{x}_n$ may grow exponentially in magnitude.

Therefore, we have three possible cases for a dynamic process given by $\mathbf{x}_n = A\mathbf{x}_{n-1}, n = 1, 2, \ldots$. The process is said to be

(1) **unstable** if A has an eigenvalue λ with $|\lambda| > 1$,
(2) **stable** if $|\lambda| < 1$ for all eigenvalues of A,
(3) **neutrally stable** if the maximum value of the eigenvalues of A is 1.

To determine the stability of a dynamic process, it is often necessary to estimate the (upper) bound for the absolute values of the eigenvalues of a square matrix A. To do this, for any square matrix $A = [a_{ij}]$ of order k, let

$$\begin{aligned} \mathrm{R}(A) &= \max\{\mathrm{R}_i(A) = \sum_{j=1}^{k} |a_{ij}| \ : \ 1 \le i \le k\}, \\ \mathrm{C}(A) &= \max\{\mathrm{C}_j(A) = \sum_{i=1}^{k} |a_{ij}| \ : \ 1 \le j \le k\}, \\ s_i &= \mathrm{R}_i(A) - |a_{ii}|. \end{aligned}$$

Theorem 6.15 (Gerschgorin's Theorem) *For any square matrix A of order k, every eigenvalue λ of A satisfies $|\lambda - a_{\ell\ell}| \le s_\ell$ for some $1 \le \ell \le k$.*

Proof: Let λ be an eigenvalue with eigenvector $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_k]^T$. Then $\sum_{j=1}^{k} a_{ij}x_j = \lambda x_i$ for $i = 1, \ldots, k$. Take a coordinate x_ℓ of $\mathbf{x}$ with the largest absolute value. Then clearly $x_\ell \ne 0$, and

$$|\lambda - a_{\ell\ell}||x_\ell| = |\lambda x_\ell - a_{\ell\ell}x_\ell| = \left|\sum_{j\ne\ell} a_{\ell j}x_j\right| \le \sum_{j\ne\ell} |a_{\ell j}||x_\ell| = s_\ell |x_\ell|.$$

Since $|x_\ell| > 0$, $|\lambda - a_{\ell\ell}| \le s_\ell$. □

Corollary 6.16 *For any square matrix A of order k, every eigenvalue λ of A satisfies $|\lambda| \le \min\{\mathrm{R}(A), \mathrm{C}(A)\}$.*

Proof: Note that $|\lambda| \le |\lambda - a_{\ell\ell}| + |a_{\ell\ell}| \le s_\ell + |a_{\ell\ell}| = \mathrm{R}_\ell(A) \le \mathrm{R}(A)$. Moreover, since λ is also an eigenvalue of A^T, $|\lambda| \le \mathrm{R}(A^T) = \mathrm{C}(A)$. □

Example 6.11 (*Stable or unstable*) Solve a discrete dynamical system $\mathbf{x}_n = A\mathbf{x}_{n-1}$, $n = 1, 2, \ldots$, where

$$(1)\ A = \begin{bmatrix} 0.8 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}, \quad (2)\ A = \begin{bmatrix} 1.2 & 0.0 \\ 0.0 & 0.6 \end{bmatrix}, \quad (3)\ A = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Solution: (1) Clearly, the eigenvalues of A are 0.8 and 0.5 with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. Hence, its general solution is

$$\mathbf{x}_n = c_1(0.8)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.5)^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(0.8)^n \\ c_2(0.5)^n \end{bmatrix}.$$

It concludes that the system $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is stable. (See Figure 6.1.)

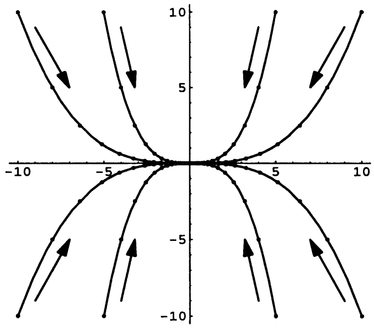

Figure 6.1. A stable dynamical system

(2) Similarly, one can show that

$$\mathbf{x}_n = c_1(1.2)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.6)^n \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

in which the system is unstable if $c_1 \neq 0$. (See Figure 6.2.)

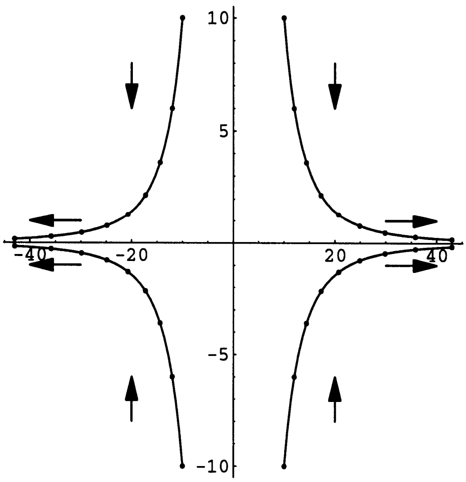

Figure 6.2. An unstable dynamical system

(3) The eigenvalues of A are 1 and 2 with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively. Hence, a general solution of $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is

$$\mathbf{x}_n = c_1\, 1^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\, 2^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is unstable if $c_2 \neq 0$: For example, if $c_1 = -1$, $c_2 = 1$, then

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 9 \\ 7 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 17 \\ 15 \end{bmatrix}, \ldots. \quad \square$$

The following example is a special type of a discrete dynamical system, called a *Markov process.*

Example 6.12 (*Markov process with distinct eigenvalues*) Suppose that the population of a certain metropolitan area starts with x_0 people outside a big city and y_0 people inside the city. Suppose that each year 20% of the people outside the city move in, and 10% of the people inside move out. What is the 'eventual' distribution of the population?

Solution: At the end of the first year, the distribution of the population will be

$$\begin{cases} x_1 &= 0.8\,x_0 + 0.1\,y_0 \\ y_1 &= 0.2\,x_0 + 0.9\,y_0. \end{cases}$$

Or, in matrix form,

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = A\mathbf{x}_0.$$

Thus if $\mathbf{x}_n = (x_n, y_n)$ denotes the distribution in the metropolitan area of the population after n years, we get $\mathbf{x}_n = A^n\mathbf{x}_0$.

In this formulation, the problem can be summarized as follows:

(1) The entries of A are all nonnegative because the entries of each column of A represent the probabilities of residing in one of the two locations in the next year,
(2) the entries of each column of A add up to 1 because the total population of the metropolitan area remains constant.

Now, to solve the problem, we first find the eigenvalues and eigenvectors of A. They are $\lambda_1 = 1$, $\lambda_2 = 0.7$ and $\mathbf{v}_1 = (1, 2)$, $\mathbf{v}_2 = (-1, 1)$, respectively, so that its general solution is

$$\mathbf{x}_n = c_1(1)^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2(0.7)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(1)^n - c_2(0.7)^n \\ 2c_1(1)^n + c_2(0.7)^n \end{bmatrix}.$$

But, the initial condition $\mathbf{x}_0 = (x_0, y_0)$ gives $c_1 = \frac{x_0}{3} + \frac{y_0}{3}$ and $c_2 = \frac{-2x_0}{3} + \frac{y_0}{3}$, so that

$$\begin{aligned} \mathbf{x}_n &= \left(\frac{x_0}{3} + \frac{y_0}{3}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(\frac{-2x_0}{3} + \frac{y_0}{3}\right)(0.7)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &\to \left(\frac{x_0}{3} + \frac{y_0}{3}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{as } n \to \infty. \end{aligned}$$

Note that, since $x_n + y_n = a$ is fixed for all n, the process in time remains on the straight line $x + y = a$. Thus, for a given initial total population a, the eventual ratio $x_n : y_n$ of the populations tends to $1 : 2$ which is independent of the initial distribution. For initial populations of $a = 3, 4, 5, 6$ million people, the processes are shown in Figure 6.3. □

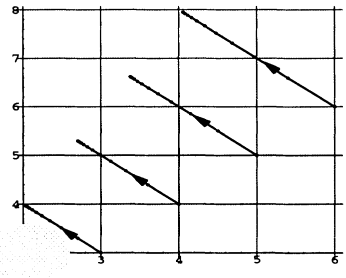

Figure 6.3. A Markov process $A^n\mathbf{x}_0$

Recall that the matrix A in Example 6.12 satisfies the following two conditions:

(1) all entries of A are nonnegative,
(2) the entries of each column of A add up to 1.

Such a matrix A is called a **Markov matrix** (or, a **stochastic matrix**). In general, a dynamical system $\mathbf{x}_n = A\mathbf{x}_{n-1}$ with a Markov matrix A is called a **Markov process**.

The next theorem follows directly from Gerschgorin's Theorem 6.15.

Theorem 6.17 *If λ is an eigenvalue of any Markov matrix A, then $|\lambda| \le 1$.*

In fact, every Markov matrix has an eigenvalue $\lambda = 1$. To show this, let A be any Markov matrix. Then the entries of each column of A add up to 1. It means that the sum of each column of $A - I$ is 0, or equivalently $\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_n = \mathbf{0}$ for the row vectors $\mathbf{r}_i$ of $A - I$. This is a nontrivial linear combination of the row vectors, and so these row vectors are linearly dependent, so that $\det(A - I) = 0$. Consequently, $\lambda = 1$ is an eigenvalue of A. If $\mathbf{x}$ is an eigenvector of A belonging to $\lambda = 1$, then $A\mathbf{x} = \mathbf{x}$. This is called the **equilibrium state**.

Theorem 6.18 *If A is a stochastic matrix, then*

(1) $\lambda = 1$ *is an eigenvalue of A,*
(2) *there exists an equilibrium state $\mathbf{x}$ that remains fixed by the Markov process.*

Problem 6.14 Suppose that a land use in a city in 2000 is

Residential	$x_0 = 30\%$,
Commercial	$y_0 = 20\%$,
Industrial	$z_0 = 50\%$.

Denote by x_k, y_k, z_k the percentage of residential, commercial, and industrial, respectively, after k years, and assume that the stochastic matrix is given as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 & 0.0 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}.$$

Find the land use in the city after 50 years.

Problem 6.15 A car rental company has three branch offices in different cities. When a car is rented at one of the offices, it may be returned to any of three offices. This company started business with 900 cars, and initially an equal number of cars was distributed to each office. When the week-by-week distribution of cars is governed by a stochastic matrix

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.7 & 0.6 \end{bmatrix},$$

determine the number of cars at each office in the k-th week. Also, find $\lim_{k\to\infty} A^k$.

6.3.3 Linear differential equations I

A first-order differential equation is a relation between a real-valued differentiable function $y(t)$ of time t and its first derivative $y'(t)$, and it can be written in the form

$$y'(t) = \frac{df(t)}{dt} = f(t, y(t)).$$

As a special case, if it can be written as $y'(t) = g(t)$ for an integrable function $g(t)$, then its solution is $y(t) = \int g(t)dt + c$ for a constant c. However, it is difficult to solve it in most other cases such as $y'(t) = \sin(ty^2)$.

As another case, if $y'(t) = 5y(t)$, then it has a *general solution* $y = ce^{5t}$, where c is an arbitrary constant. If an additional condition $y(0) = 3$, called an *initial condition*, is given, then its solution is $y = 3e^{5t}$, called a *particular solution*.

The second case can be generalized to a system of n *linear differential equations* with constant coefficients, which is by definition of the form

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ \quad\vdots \\ y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n, \end{cases}$$

where $y_i = f_i(t)$ for $i = 1, 2, \ldots, n$ are real-valued differentiable functions on an interval $I = (a, b)$. In most cases, one may assume that the interval I contains 0, and some initial conditions are given as $f_i(0) = d_i$ at $0 \in I$.

Let $\mathbf{y} = [f_1\ f_2\ \cdots\ f_n]^T$ denote the vector whose entries are the differentiable functions $y_i = f_i$'s defined on an interval $I = (a, b)$: thus, for each $t \in I$, $\mathbf{y}(t) = [f_1(t)\ f_2(t)\ \cdots\ f_n(t)]^T$ is a vector in $\mathbb{R}^n$. Its derivative is defined by

$$\mathbf{y}' = \begin{bmatrix} f_1' \\ f_2' \\ \vdots \\ f_n' \end{bmatrix} \quad \text{or} \quad \mathbf{y}'(t) = \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix}.$$

If A denotes the coefficient matrix of the system of linear differential equations, the matrix form of the system can be written as

$$\mathbf{y}' = A\mathbf{y}, \text{ or } \mathbf{y}'(t) = A\mathbf{y}(t) \text{ for all } t \in I.$$

An initial condition is given by $\mathbf{y}_0 = \mathbf{y}(0) = (d_1, \ldots, d_n) \in \mathbb{R}^n$.

A differentiable vector function $\mathbf{y}(t)$ is called a **solution** of the system $\mathbf{y}'(t) = A\mathbf{y}(t)$ if it satisfies the equation. In general, the entries of the coefficient matrix A could be functions. However, in this book, we restrict our attention to the systems with constant coefficients.

Example 6.13 Consider the following three systems:

$$\begin{cases} y_1' &= 2y_1 - 3y_2 \\ y_2' &= 2y_1 + \ y_2 \end{cases}, \quad \begin{cases} y_1' &= ty_1 + t^2 y_2 \\ y_2' &= 2y_1 + t^3 y_2 \end{cases}, \quad \begin{cases} y_1' &= 2y_1 - 3y_2^2 \\ y_2' &= \sin y_1 + 5y_2 \end{cases}.$$

The first two systems are linear, but the coefficients of the second are functions of t. The third is not linear because of the terms y_2^2 and $\sin y_1$. □

Example 6.14 (*Population model*) Let $p(t)$ denote the population of a given species like bacteria at time t and let $r(t, p)$ denote the difference between its birth rate and its death rate at time t. If $r(t, p)$ is independent of time t, i.e., it is a constant, then $\frac{dp(t)}{dt} = rp(t)$ is the rate of change of the population and its general solution is $p(t) = p(0)e^{rt}$. □

Some basic facts about a system $\mathbf{y}'(t) = A\mathbf{y}(t)$, A is any $n \times n$ matrix, of linear differential equations defined on $I = (a, b)$ are listed below.

(I) (*The fundamental theorem for a system of linear differential equations*) The system $\mathbf{y}'(t) = A\mathbf{y}(t)$ always has a solution. In addition, if an initial condition $\mathbf{y}_0$ is given, then there is a unique solution $\mathbf{y}(t)$ on I which satisfies the initial condition. If $\mathbf{y} = [y_1 \ y_2 \ \cdots \ y_n]^T$ is a solution on I, then it draws a curve in $\mathbb{R}^n$ passing through the initial vector $\mathbf{y}_0 = \mathbf{y}(0) = (d_1, \ldots, d_n)$ as t varies in the interval I.

(II) (*Linear independence of solutions*) Let $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ be a set of n solutions of the system $\mathbf{y}' = A\mathbf{y}$ on I. The linear independence of the solutions $\mathbf{y}_1, \ldots, \mathbf{y}_n$ on I is defined as usual: if $c_1\mathbf{y}_1 + \cdots + c_n\mathbf{y}_n = \mathbf{0}$ implies $c_1 = \cdots = c_n = 0$. Or, equivalently, they are linearly dependent if and only if one of them can be written as a linear combination of all the others. Define

$$Y(t) = [\mathbf{y}_1(t) \ \cdots \ \mathbf{y}_n(t)] = \begin{bmatrix} y_{11}(t) & y_{12}(t) & \cdots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \cdots & y_{2n}(t) \\ \vdots & \vdots & & \vdots \\ y_{n1}(t) & y_{n2}(t) & \cdots & y_{nn}(t) \end{bmatrix} \quad \text{for } t \in I.$$

If the n solutions are linearly dependent, then $\det Y(t) = 0$ for all $t \in I$. Or, equivalently, if $\det Y(t) \neq 0$ for at least one point $t \in I$, then the solutions are linearly independent. However, the next lemma says that

$$\det Y(t) \neq 0 \textit{ for all } t \in I \textit{ if and only if } \det Y(t) \neq 0 \textit{ at one point } t \in I.$$

The determinant of $Y(t)$ is called the **Wronskian** of the solutions, denoted by $W(t) = \det Y(t)$ for $t \in I$. Note that the Wronskian $W(t)$ is a real-valued differentiable function on I.

Lemma 6.19 $W'(t) = \operatorname{tr}(A)W(t)$. □

Proof:

$$\begin{aligned}
W'(t) &= (\det Y(t))' = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(y_{1\sigma(1)} \cdots y_{n\sigma(n)})' \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) y'_{1\sigma(1)} \cdots y_{n\sigma(n)} + \cdots + \sum_{\sigma} \operatorname{sgn}(\sigma) y_{1\sigma(1)} \cdots y'_{n\sigma(n)} \\
&= (y'_{11}Y_{11} + \cdots + y'_{1n}Y_{1n}) + \cdots + (y'_{n1}Y_{n1} + \cdots + y'_{nn}Y_{nn}) \\
&= \sum_i^n \sum_j^n y'_{ij}Y_{ij} = \sum_i^n \left(\sum_j^n y'_{ij}[\operatorname{adj} Y]_{ji} \right) = \sum_i^n [Y' \cdot \operatorname{adj} Y]_{ii} \\
&= \operatorname{tr}(Y' \cdot \operatorname{adj} Y) = \operatorname{tr}(A \cdot (Y \cdot \operatorname{adj} Y)) \\
&= \operatorname{tr}(\det Y(t) A) = \operatorname{tr}(A)W(t),
\end{aligned}$$

where $Y_{ij}(t)$ is the cofactor of y_{ij}, and the equalities in the last two lines are due to the fact that

$$\begin{aligned}
Y'(t) &= [\mathbf{y}'_1(t) \cdots \mathbf{y}'_n(t)] = A[\mathbf{y}_1(t) \cdots \mathbf{y}_n(t)] = AY(t), \\
Y(t) \operatorname{adj} Y(t) &= \det Y(t) I_n = W(t) I_n.
\end{aligned}$$ □

From Lemma 6.19, it is clear that the Wronskian $W(t)$ is an exponential function of the form $W(t) = ce^{\operatorname{tr}(A)t}$ with an initial condition $W(0) = c$. It implies that the value of $W(t)$ is zero for all t or never zero on I depending on whether or not $c = 0$. Thus, we have the following lemma.

Lemma 6.20 *Let* $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ *be a set of n solutions of the system* $\mathbf{y}' = A\mathbf{y}$ *on I, where A is any $n \times n$ matrix. Then the following are equivalent.*

(1) *The vectors* $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ *are linearly independent.*
(2) $W(t) \neq 0$ *for some t, that is,* $\mathbf{y}_1(t), \mathbf{y}_2(t), \ldots, \mathbf{y}_n(t)$ *are linearly independent in* $\mathbb{R}^n$ *for some t.*
(3) $W(t) \neq 0$ *for all t, that is,* $\mathbf{y}_1(t), \mathbf{y}_2(t), \ldots, \mathbf{y}_n(t)$ *are linearly independent in* $\mathbb{R}^n$ *for all t.*

(III) (*Dimension of the solution space*) Clearly, the set of all solutions of $\mathbf{y}'(t) = A\mathbf{y}(t)$ is a vector space. In fact, for any two solutions $\mathbf{y}_1, \mathbf{y}_2$ of the system $\mathbf{y}'(t) = A\mathbf{y}(t)$, we have

$$(c_1\mathbf{y}_1 + c_2\mathbf{y}_2)' = c_1\mathbf{y}_1' + c_2\mathbf{y}_2' = c_1A\mathbf{y}_1 + c_2A\mathbf{y}_2 = A(c_1\mathbf{y}_1 + c_2\mathbf{y}_2).$$

Thus, $c_1\mathbf{y}_1 + c_2\mathbf{y}_2$ is also a solution for any constants c_i's.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. For each $\mathbf{e}_i$, there exists a unique solution $\mathbf{y}_i$ of $\mathbf{y}'(t) = A\mathbf{y}(t)$ such that $\mathbf{y}_i(0) = \mathbf{e}_i$, by (I). All such solutions $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are linearly independent by Lemma 6.20. Moreover, they generate the vector space of solutions. To show this, let $\mathbf{y}$ be any solution. Then the vector $\mathbf{y}(0)$ can be written as a linear combination of the standard basis vectors: say, $\mathbf{y}_0 = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n$. Then, by the uniqueness of the solution in (I), we have

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t).$$

This proves the following theorem.

Theorem 6.21 *For any $n \times n$ matrix A, the set of solutions of a system $\mathbf{y}' = A\mathbf{y}$ on I is an n-dimensional vector space.*

Definition 6.5 A basis for the solution space is called a **fundamental set** of solutions. The solution expressed as a linear combination of a fundamental set is called a **general solution** of the system. The solution determined by a given initial condition is called a **particular solution**.

By Theorem 6.21, it is enough to find n linearly independent solutions to solve a system $\mathbf{y}' = A\mathbf{y}$ on I, and then its general solution is just a linear combination of those linearly independent solutions. This may be considered in three steps: (1) A is diagonal, (2) A is diagonalizable, and finally (3) A is any square matrix.

(1) First suppose that A is a diagonal matrix D. Then $\mathbf{y}'(t) = A\mathbf{y}(t)$ is

$$\begin{bmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1(t) \\ \vdots \\ \lambda_n y_n(t) \end{bmatrix}.$$

This system is just n simple linear differential equations of the first order:

$$y_i'(t) = \lambda_i y_i(t), \quad i = 1, 2, \ldots, n,$$

and their solutions are trivial: $y_i(t) = c_i e^{\lambda_i t}$ with a constant c_i for $i = 1, 2, \ldots, n$.

On the other hand, the diagonal matrix A has n linearly independent eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ belonging to the eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively. One can see that $\mathbf{y}_i(t) = e^{\lambda_i t}\mathbf{e}_i$ is a solution of the system $\mathbf{y}'(t) = A\mathbf{y}(t)$ for $i = 1, 2, \ldots, n$. Moreover, at $t = 0$, the solution set $\{\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)\} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is linearly independent. Hence, by Lemma 6.20, a general solution of the system $\mathbf{y}'(t) = A\mathbf{y}(t)$ is

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t) = c_1 e^{\lambda_1 t}\mathbf{e}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{e}_n$$

with constants c_i's. Or in matrix notation,

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = e^{tD}\mathbf{y}_0,$$

where e^{tD} is by definition,

$$e^{tD} = \begin{bmatrix} e^{\lambda_1 t} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{\lambda_n t} \end{bmatrix}. \qquad \square$$

Example 6.15 (*A* **predator-prey** *problem as a model of a system of differential equations*) One of the fundamental problems of mathematical ecology is the predator-prey problem. Let $x(t)$ and $y(t)$ denote the populations at time t of two species in a specified region, one of which x preys upon the other y. For example, $x(t)$ and $y(t)$ may be the number of sharks and small fishes, respectively, in a restricted region of the ocean. Without the small fishes (preys) the population of the sharks (predators) will decrease, and without the sharks the population of the fishes will increase. A mathematical model showing their interactions and whether an ecological balance exists can be written as the following system of differential equations:

$$\begin{cases} x'(t) = a\,x(t) - b\,x(t)y(t) \\ y'(t) = -c\,y(t) + d\,x(t)y(t). \end{cases}$$

In this equation, the coefficients a and c are the birth rate of x and the death rate of y, respectively. The nonlinear $x(t)y(t)$ terms in the two equations mean the interaction of the two species such as the number of contacts per unit time between predators and prey, so the coefficients b and d are the measures of the effect of the interaction between them. A study of this general system of differential equations leads to very interesting developments in the theory of dynamical systems and can be found in any book on ordinary differential equations. Here, we restrict our study to the case of x and y very small, i.e., near the origin in the plane. In this case, one can neglect the nonlinear terms in the equations, so the system is assumed to be given as follows:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Thus, the eigenvalues are $\lambda_1 = a$ and $\lambda_2 = -c$ with their associated eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Therefore, its general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{at} \\ c_2 e^{-ct} \end{bmatrix} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{-ct} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 e^{at}\mathbf{e}_1 + c_2 e^{-ct}\mathbf{e}_2. \qquad \square$$

(2) We next assume that a matrix A in the system $\mathbf{y}'(t) = A\mathbf{y}(t)$ is diagonalizable, that is, it has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ belonging to the eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively. Then the basis-change matrix $Q = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$ diagonalizes A and

$$A = QDQ^{-1} = Q \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix} Q^{-1}.$$

Thus the system becomes $Q^{-1}\mathbf{y}' = DQ^{-1}\mathbf{y}$. If we take a change of variables by the new vector $\mathbf{x} = Q^{-1}\mathbf{y}$ (or $\mathbf{y} = Q\mathbf{x}$), then we obtain a new system

$$\mathbf{x}' = D\mathbf{x},$$

with an initial condition $\mathbf{x}_0 = Q^{-1}\mathbf{y}_0 = (c_1, \ldots, c_n)$. Since D is diagonal, its general solution is

$$\mathbf{x} = e^{tD}\mathbf{x}_0 = c_1 e^{\lambda_1 t}\mathbf{e}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{e}_n.$$

Now, a general solution of the original system $\mathbf{y}' = A\mathbf{y}$ is

$$\begin{aligned} \mathbf{y} = Q\mathbf{x} &= Qe^{tD}Q^{-1}\mathbf{y}_0 \\ &= [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n] \begin{bmatrix} e^{\lambda_1 t} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n. \end{aligned}$$

Remark: One can check directly that each vector function $\mathbf{y}_i(t) = e^{\lambda_i t}\mathbf{v}_i$ is the particular solution of the system with the initial condition $\mathbf{y}_i(0) = \mathbf{v}_i$ for $i = 1, \ldots, n$. Since $\mathbf{y}_i(0) = \mathbf{v}_i$ for $i = 1, \ldots, n$ are linearly independent, $e^{\lambda_i t}\mathbf{v}_i$ for $i = 1, \ldots, n$ form a fundamental set of solutions.

Thus, we have obtained the following theorem:

Theorem 6.22 *Let A be a diagonalizable $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ belonging to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Then, a general solution of the system of linear differential equations $\mathbf{y}'(t) = A\mathbf{y}(t)$ is*

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$$

with constants $c_1, c_2, \ldots, c_n$.

Note that a particular solution can be obtained from a general solution by determining the coefficients depending on the given initial condition.

Example 6.16 ($\mathbf{y}' = A\mathbf{y}$ *with a diagonalizable matrix A*) Solve the system of linear differential equations

$$\begin{cases} y_1' = 5y_1 - 4y_2 + 4y_3 \\ y_2' = 12y_1 - 11y_2 + 12y_3 \\ y_3' = 4y_1 - 4y_2 + 5y_3. \end{cases}$$

Solution: In matrix form, the system may be written as $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = -3$, and their associated eigenvectors are $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$ and $\mathbf{v}_3 = (1, 3, 1)$, respectively, which are linearly independent (see Problem 6.9). Hence, by Theorem 6.22, its general solution is

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= c_1e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3e^{-3t} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\ &= c_1e^t\mathbf{v}_1 + c_2e^t\mathbf{v}_2 + c_3e^{-3t}\mathbf{v}_3. \end{aligned}$$

□

(3) A system $\mathbf{y}' = A\mathbf{y}$ of linear differential equations with a non-diagonalizable matrix A will be discussed in Section 6.5.

6.4 Exponential matrices

Just like the Maclaurin series of the exponential function e^x, we define the exponential of a matrix.

Definition 6.6 For any square matrix A, the **exponential matrix** of A is defined as the series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots.$$

That is, the exponential matrix e^A is defined to be the (entry-wise) limit of the sequence: $\left[e^A\right]_{ij} = \lim_{m\to\infty} \left[\sum_{k=0}^{m} \frac{A^k}{k!}\right]_{ij}$ for all i, j.

Example 6.17 If

$$D = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix}, \quad \text{then} \quad D^k = \begin{bmatrix} \lambda_1^k & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n^k \end{bmatrix} \quad \text{for any } k \geq 0.$$

Thus, the exponential matrix e^D is

$$e^D = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{\lambda_n} \end{bmatrix},$$

which coincides with the definition given on page 230. □

Practically, the computation of e^A involves the computation of the powers A^k for all $k \geq 1$, and hence it is not easy in general. Nevertheless, one can show that the limit e^A exists for any square matrix A.

Theorem 6.23 *For any square matrix A, the matrix e^A exists. In other words, each (i, j)-entry of e^A is convergent.*

Proof: Since A has only n^2 entries, there is a number M such that $|a_{ij}| \leq M$ for all (i, j)-entries a_{ij} of A. Then one can easily show that $[A^k]_{ij} \leq n^{k-1}M^k$ for all k and i, j. Thus

$$[e^A]_{ij} \leq \sum_{k=0}^{\infty} \frac{1}{k!} n^{k-1} M^k = \frac{1}{n} e^{nM},$$

so by the comparison test, each entry of $e^A = \sum_{n=0}^{\infty} \frac{A^k}{k!}$ is absolutely convergent for any square matrix A. □

Example 6.18 If $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$, then

$$\begin{aligned} e^A &= I + A + \frac{1}{2}A^2 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 1+3 \\ 0 & 3^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^1 & * \\ 0 & e^3 \end{bmatrix}. \end{aligned}$$

It is a good exercise to calculate the missing entry $*$ directly from the definition. □

Problem 6.16 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $\lim_{k\to\infty} A^k$ if it exists. (Note that the matrix A is not diagonalizable.)

The following theorem is sometimes helpful to compute e^A.

Theorem 6.24 *Let $A_1, A_2, A_3, \ldots$ be a sequence of $m \times n$ matrices such that $\lim_{k\to\infty} A_k = L$. Then*

$$\lim_{k\to\infty} BA_k = BL \quad \textit{and} \quad \lim_{k\to\infty} A_k C = LC$$

for any matrices B and C for which the products can be defined.

Proof: By comparing the (i, j)-entries of both sides

$$\begin{aligned}\lim_{k\to\infty}[BA_k]_{ij} &= \lim_{k\to\infty}\left(\sum_{\ell=1}^{m}[B]_{i\ell}[A_k]_{\ell j}\right) = \sum_{\ell=1}^{m}[B]_{i\ell}\lim_{k\to\infty}[A_k]_{\ell j}\\ &= \sum_{\ell=1}^{m}[B]_{i\ell}[L]_{\ell j} = [BL]_{ij},\end{aligned}$$

we get $\lim_{k\to\infty} BA_k = BL$. Similarly $\lim_{k\to\infty} A_kC = LC$. □

For example, if A is a diagonalizable matrix and $Q^{-1}AQ = D$ is diagonal for some invertible matrix Q, then, for each integer $k \geq 0$, $A^k = QD^kQ^{-1}$ and

$$\lim_{k\to\infty} A^k = Q\left(\lim_{k\to\infty} D^k\right)Q^{-1} = Q\begin{bmatrix}\lim_{k\to\infty}\lambda_1^k & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \lim_{k\to\infty}\lambda_n^k\end{bmatrix}Q^{-1}.$$

Thus, $\lim_{k\to\infty} A^k$ exists if and only if $\lim_{k\to\infty}\lambda_i^k$ exists for $i = 1, 2, \ldots, n$.

Also, by Theorem 6.24,

$$\begin{aligned}e^A = e^{QDQ^{-1}} &= I + QDQ^{-1} + \frac{(QDQ^{-1})^2}{2!} + \frac{(QDQ^{-1})^3}{3!} + \cdots\\ &= Q\left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots\right)Q^{-1}\\ &= Qe^DQ^{-1},\end{aligned}$$

whose computation is easy.

Example 6.19 (*Computing e^A for a diagonalizable matrix A*) Let $A = \frac{1}{2}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$. Then its eigenvalues are 1 and 2 with associated eigenvectors $\mathbf{u}_1 = \begin{bmatrix}-1\\ 1\end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$, respectively. Thus $A = QDQ^{-1}$ with $D = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ and $Q = \begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}$. Therefore,

$$e^A = \frac{1}{2}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^1 & 0\\ 0 & e^2\end{bmatrix}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}e+e^2 & -e+e^2\\ -e+e^2 & e+e^2\end{bmatrix}.$$ □

The following theorem shows some basic properties of the exponential matrices, whose proofs are easy, and are left for exercises.

Theorem 6.25 **(1)** $e^{A+B} = e^Ae^B$ *provided that* $AB = BA$.

(2) *e^A is invertible for any square matrix A, and $(e^A)^{-1} = e^{-A}$.*
(3) *$e^{Q^{-1}AQ} = Q^{-1}e^A Q$ for any invertible matrix Q.*
(4) *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix A with their associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then e^{λ_i}'s are the eigenvalues of e^A with the same associated eigenvectors $\mathbf{v}_i$'s for $i = 1, 2, \ldots, n$. Moreover,* $\det e^A = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\operatorname{tr}(A)} \neq 0$ *for any square matrix A.*

Problem 6.17 Prove Theorem 6.25.

Problem 6.18 Finish the computation of e^A for the matrix A in Example 6.18.

Problem 6.19 Prove that if A is skew-symmetric, then e^A is orthogonal.

In general, the computation of e^A is not easy at all if A is not diagonalizable. However, if A is a triangular matrix, it is relatively easy as shown in the following example.

Example 6.20 (*Computing e^A for a triangular matrix of the form $A = \lambda I + N$*)
For $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, compute e^A.

Solution: Write $A = 2I + N$ with $N = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$. Since $(2I)N = N(2I)$, by Theorem 6.25(1), $e^A = e^{2I}e^N$. From the direct computation of the series expansion, we get $e^{2I} = e^2 I$. Moreover, since $N^k = \mathbf{0}$ for $k \geq 2$, $e^N = I + N + \frac{N^2}{2!} + \cdots = I + N = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. Thus,

$$e^A = e^2(I + N) = e^2 \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^2 & 3e^2 \\ 0 & e^2 \end{bmatrix}. \qquad \square$$

Problem 6.20 Compute e^A for $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

6.5 Applications continued

6.5.1 Linear differential equations II

One of the most prominent applications of exponential matrices is to the theory of linear differential equations. In this section, we show that a general solution of $\mathbf{y}'(t) = A\mathbf{y}(t)$ is of the form $\mathbf{y}(t) = e^{tA}\mathbf{y}_0$.

Lemma 6.26 *For any $t \in \mathbb{R}$ and any square matrix A, the exponential matrix*

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

is a differentiable function of t, and $\frac{d}{dt}e^{tA} = Ae^{tA}$.

Proof: By absolute convergence of the series expansion of e^{tA} one can use term by term differentiation, i.e.,

$$\begin{aligned} \frac{d}{dt}e^{tA} &= \frac{d}{dt}(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots) \\ &= A + tA^2 + \frac{t^2}{2!}A^3 + \cdots = Ae^{tA}. \end{aligned}$$ □

As a direct consequence of Lemma 6.26, one can see that $\mathbf{y}(t) = e^{tA}\mathbf{y}_0$ is a solution of the linear differential equation $\mathbf{y}' = A\mathbf{y}$. In fact, by taking the initial vector $\mathbf{y}_0$ as the standard basis vector $\mathbf{e}_i$, $0 \le i \le n$, the n columns of e^{tA} become solutions and they are clearly linearly independent, so they form a fundamental set of solutions. Hence, we have the following.

Theorem 6.27 *For any $n \times n$ matrix A, the linear differential equation $\mathbf{y}' = A\mathbf{y}$ has a general solution*

$$\mathbf{y}(t) = e^{tA}\mathbf{y}_0.$$

In particular, if A is diagonalizable, say $Q^{-1}AQ = D$ is diagonal with a basis-change matrix $Q = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ consisting of n linearly independent eigenvectors of A belonging to the eigenvalues λ_i's, then a general solution of a system $\mathbf{y}' = A\mathbf{y}$ is

$$\begin{aligned} \mathbf{y}(t) = e^{tA}\mathbf{y}_0 &= e^{tQDQ^{-1}}\mathbf{y}_0 = Qe^{tD}Q^{-1}\mathbf{y}_0 \\ &= [\mathbf{v}_1 \cdots \mathbf{v}_n] \begin{bmatrix} e^{\lambda_1 t} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n. \end{aligned}$$

In fact, $\{\mathbf{y}_i(t) = e^{\lambda_i t}\mathbf{v}_i : i = 1, \ldots, n\}$ forms a fundamental set of solutions, and the constants $(c_1, \ldots, c_n) = Q^{-1}\mathbf{y}_0$ can be determined if an initial condition is given. Note that this just rephrases Theorem 6.22.

Example 6.21 ($\mathbf{y}' = A\mathbf{y}$ *for a diagonalizable matrix A*) Solve the system

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \text{or} \quad \begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y_1(t) \end{cases}$$

with initial conditions $y_1(0) = 1$, $y_2(0) = 0$.

Solution: (1) The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = -1$ with associated eigenvectors $\mathbf{v}_1 = [1\ 1]^T$ and $\mathbf{v}_2 = [1\ -1]^T$, respectively.

(2) By setting $Q = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$.

(3) A general solution $\mathbf{y}(t) = e^{tA}\mathbf{y}_0$ is

$$\begin{aligned} e^{tA}\mathbf{y}_0 &= e^{tQDQ^{-1}}\mathbf{y}_0 = Qe^{tD}Q^{-1}\mathbf{y}_0 \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} Q^{-1}\mathbf{y}_0 = c_1e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

with constants c_1, c_2. The initial conditions $y_1(0) = 1$, $y_2(0) = 0$ determine $c_1 = c_2 = \frac{1}{2}$, so that

$$\mathbf{y}(t) = e^{tA}\mathbf{y}_0 = \frac{1}{2}\begin{bmatrix} e^t \\ e^t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}. \qquad \square$$

Problem 6.21 Solve the system $\begin{cases} y_1' = y_1 + y_2 \\ y_2' = 4y_1 - 2y_2. \end{cases}$

Problem 6.22 Solve the system $\begin{cases} y_1' = 4y_1 + y_3 \\ y_2' = -2y_1 + y_2 \\ y_3' = -2y_1 + y_3, \end{cases}$
and find the particular solution of the system satisfying the initial conditions $y_1(0) = -1$, $y_2(0) = 1$, $y_3(0) = 0$.

If A is not diagonalizable, then it is not easy in general to compute e^{tA} directly. However, one can still reduce A to a simpler form called the Jordan canonical form, which will be introduced in Chapter 8, and then the computation of e^{tA} is made relatively easy. The following example shows that the computation of e^A is possible for some triangular matrices A even if they are not diagonalizable. A general case will be treated again in Chapter 8.

Example 6.22 ($\mathbf{y}' = A\mathbf{y}$ *for a triangular matrix* $A = \lambda I + N$) Solve the system $\mathbf{y}' = A\mathbf{y}$ of linear differential equations with initial condition $\mathbf{y}(0) = \mathbf{y}_0$, where

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad \mathbf{y}_0 = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Solution: First note that A has an eigenvalue λ of multiplicity 2 and is not diagonalizable. One can rewrite A as

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \lambda I + N.$$

Then, by the same argument as in Example 6.20,

$$e^{tA} = e^{t(\lambda I+N)} = e^{\lambda t}e^{tN} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Therefore, the solution is

$$\mathbf{y} = e^{tA}\mathbf{y}_0 = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (a+bt)e^{\lambda t} \\ be^{\lambda t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} + te^{\lambda t}\begin{bmatrix} b \\ 0 \end{bmatrix}.$$

In terms of components, $y_1 = (a+bt)e^{\lambda t}$, $y_2 = be^{\lambda t}$. □

Example 6.23 ($\mathbf{y}' = A\mathbf{y}$ *with A having complex eigenvalues*) Find a general solution of the system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad b \neq 0.$$

Solution: Note that the eigenvalues of A are $a \pm ib$, which are not real. However, one can compute e^{tA} directly without using diagonalization. We first write A as

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = aI + bJ.$$

Then clearly $IJ = JI$ and $e^{tA} = e^{atI+btJ} = e^{at}e^{btJ}$. Since

$$J^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I, \quad J^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -J, \quad J^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

one can deduce $J^k = J^{k+4}$ for all $k = 1, 2, \ldots$, and

$$\begin{aligned} e^{btJ} &= I + \frac{btJ}{1!} + \frac{(bt)^2J^2}{2!} + \frac{(bt)^3J^3}{3!} + \frac{(bt)^4J^4}{4!} + \cdots \\ &= \begin{bmatrix} 1 - \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} - \cdots & -(bt) + \frac{(bt)^3}{3!} - \frac{(bt)^5}{5!} + \cdots \\ (bt) - \frac{(bt)^3}{3!} + \frac{(bt)^5}{5!} - \cdots & 1 - \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} - \cdots \end{bmatrix} \\ &= \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \end{aligned}$$

for any constant b and t. Thus, a general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y} = e^{tA}\mathbf{c} = e^{at}e^{btJ}\mathbf{c} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

In terms of components,

$$\begin{cases} y_1 = e^{at}(c_1 \cos bt - c_2 \sin bt) \\ y_2 = e^{at}(c_1 \sin bt + c_2 \cos bt). \end{cases}$$

□

Problem 6.23 Solve the system $\mathbf{y}' = A\mathbf{y}$ with initial condition $\mathbf{y}(0) = \mathbf{y}_0$ by computing $e^{tA}\mathbf{y}_0$ for

$$(1)\ A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix},\ \mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (2)\ A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix},\ \mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Remark: Consider the n-th order homogeneous linear differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + a_2 \frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_n y = 0,$$

where a_i are constants and $y(t)$ is a differentiable function on an interval $I = (a, b)$. A fundamental theorem of differential equations says that such a differential equation has a unique solution $y(t)$ on I satisfying the given initial condition: For a point t_0 in I and arbitrary constants $c_0, \ldots, c_{n-1}$, there is a unique solution $y = y(t)$ of the equation such that $y(t_0) = c_0$, $y'(t_0) = c_1$, $\ldots$, $y^{(n-1)}(t_0) = c_{n-1}$. This can be confirmed as follows: Let

$$\begin{aligned} y_1 &= y, \\ y_2 &= y' = \frac{dy_1}{dt}, \\ y_3 &= y'' = \frac{dy_2}{dt}, \\ &\ \vdots \\ y_n &= y^{(n-1)} = \frac{dy_{n-1}}{dt}. \end{aligned}$$

Then the original homogeneous linear differential equation is nothing but

$$\frac{dy_n}{dt} = \frac{d^n y}{dt^n} = -a_1 y_n - a_2 y_{n-1} - \cdots - a_{n-1} y_2 - a_n y_1.$$

In matrix notation,

$$\mathbf{y}'(t) = \begin{bmatrix} y_n' \\ \vdots \\ y_2' \\ y_1' \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ \vdots \\ y_2 \\ y_1 \end{bmatrix} = A\mathbf{y}(t),$$

which is just a system of linear differential equations with a companion matrix A. It is treated in Section 6.3.3 (see Theorem 6.22). Therefore, the solution of the original differential equation is just the solution of $\mathbf{y}'(t) = A\mathbf{y}(t)$, which is of the form:

$$y(t) = c_1 e^{\lambda_1 t} + \cdots + c_n e^{\lambda_n t}$$

if A has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. In Chapter 8, we will discuss the case of eigenvalues with multiplicity.

6.6 Diagonalization of linear transformations

Recall that two matrices are similar if and only if they can be the matrix representations of the same linear transformation, and similar matrices have the same eigenvalues. In this section, we aim to find a basis α so that the matrix representation of a linear transformation with respect to α is a diagonal matrix. First, we start with the eigenvalues and the eigenvectors of a linear transformation.

Definition 6.7 Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear transformation on V. Then the **eigenvalues** and **eigenvectors** of T can be defined by the same equation, $T\mathbf{x} = \lambda\mathbf{x}$, with a nonzero vector $\mathbf{x} \in V$.

Practically, the eigenvalues and eigenvectors of T can be computed as follows: Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. Then the natural isomorphism $\Phi : V \to \mathbb{R}^n$ identifies the associated matrix $A = [T]_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ with the linear transformation $T : V \to V$ via the following commutative diagram.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi"', "\cong"] & V \arrow[d, "\Phi"', "\cong"] \\ \mathbb{R}^n \arrow[r, "A = [T]_\alpha"'] & \mathbb{R}^n . \end{tikzcd}$$

Now, the eigenvalues of T are those of its matrix representation $A = [T]_\alpha$ because if $[T]_\alpha$ is similar to $[T]_\beta$ for any other basis β for V, then their eigenvalues are the same by Theorem 6.3. For eigenvectors of T, note that $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is an eigenvector of A belonging to λ ($A\mathbf{x} = \lambda\mathbf{x}$) if and only if $\Phi^{-1}(\mathbf{x}) = \mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \in V$ is an eigenvector of T ($T(\mathbf{v}) = \lambda\mathbf{v}$), because the commutativity of the diagram shows

$$[T(\mathbf{v})]_\alpha = [T]_\alpha[\mathbf{v}]_\alpha = A\mathbf{x} = \lambda\mathbf{x} = [\lambda\mathbf{v}]_\alpha.$$

Therefore, if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent eigenvectors of $A = [T]_\alpha$, then $\Phi^{-1}(\mathbf{x}_1), \Phi^{-1}(\mathbf{x}_2), \ldots, \Phi^{-1}(\mathbf{x}_k)$ are linearly independent eigenvectors of T. Hence, the linear transformation T has a diagonal matrix representation if and only if it has n linearly independent eigenvectors, by Theorem 6.7.

The following example illustrates how to find a diagonal matrix representation of a linear transformation on a vector space.

Example 6.24 Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$(Tf)(x) = f(x) + xf'(x) + f'(x).$$

Find a basis for $P_2(\mathbb{R})$ with respect to which the matrix of T is diagonal.

Solution: First of all, we find the eigenvalues and the eigenvectors of T. Take a basis for the vector space $P_2(\mathbb{R})$, say $\alpha = \{1,\ x,\ x^2\}$. Then the matrix of T with respect to α is

$$[T]_\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

which is upper triangular. Hence, the eigenvalues of T are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. By a simple computation, one can verify that the vectors $\mathbf{x}_1 = (1,\ 0,\ 0)$, $\mathbf{x}_2 = (1,\ 1,\ 0)$ and $\mathbf{x}_3 = (1,\ 2,\ 1)$ are eigenvectors of $[T]_\alpha$ in $\mathbb{R}^3$ belonging to eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$, respectively. Their associated eigenvectors of T in $P_2(\mathbb{R})$ are $f_1(x) = 1$, $f_2(x) = 1 + x$, $f_3(x) = 1 + 2x + x^2$, respectively. Since the eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$ are all distinct, the eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2,\ \mathbf{x}_3\}$ of $[T]_\alpha$ are linearly independent and so are $\beta = \{f_1,\ f_2,\ f_3\}$ in $P_2(\mathbb{R})$. Thus, each f_i is a basis for the eigenspace $E(\lambda_i)$ of T belonging to λ_i for $i = 1,\ 2,\ 3$, and the basis-change matrix is

$$Q = [id]^\alpha_\beta = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = [\,[f_1]_\alpha\ [f_2]_\alpha\ [f_3]_\alpha\,] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, by changing the basis α to β, the matrix representation of T is a diagonal matrix:

$$[T]_\beta = [id]^\beta_\alpha [T]_\alpha [id]^\alpha_\beta = Q^{-1}[T]_\alpha Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D. \qquad \square$$

Note that, if $T = A$ is an $n \times n$ square matrix written in column vectors, $A = [\mathbf{c}_1\ \cdots\ \mathbf{c}_n]$, then the linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is given by $A(\mathbf{e}_i) = \mathbf{c}_i$, $i = 1,\ \ldots,\ n$, so that A itself is just the matrix representation with respect to the standard basis $\alpha = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$, say $A = [A]_\alpha$. Now if there is a basis $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of n linearly independent eigenvectors of A, then the natural isomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\Phi(\mathbf{x}_i) = \mathbf{e}_i$ is simply a change of basis by the basis-change matrix $Q = [id]^\alpha_\beta$ and the matrix representation of A with respect to β is a diagonal matrix:

$$D = [A]_\beta = Q^{-1}[A]_\alpha Q = Q^{-1}AQ.$$

Problem 6.24 Let T be the linear transformation on $\mathbb{R}^3$ defined by

$$T(x, y, z) = (4x + z,\ 2x + 3y + 2z,\ x + 4z).$$

Find all the eigenvalues and their eigenvectors of T and diagonalize T.

Problem 6.25 Let $M_{2\times2}(\mathbb{R})$ be the vector space of all real 2×2 matrices and let T be the linear transformation on $M_{2\times2}(\mathbb{R})$ defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b+d & a+b+c \\ b+c+d & a+c+d \end{bmatrix}.$$

Find the eigenvalues and basis for each of the eigenspaces of T, and diagonalize T.

Problem 6.26 Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f(x) + xf'(x)$. Find all the eigenvalues of T and find a basis α for $P_2(\mathbb{R})$ so that $[T]_\alpha$ is a diagonal matrix.

6.7 Exercises

6.1. Find the eigenvalues and eigenvectors for the given matrix, if they exist.

$$(1) \begin{bmatrix} 6 & 0 \\ -2 & 2 \end{bmatrix}, \qquad (2) \begin{bmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix},$$

$$(3) \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad (4) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$(5) \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \qquad (6) \begin{bmatrix} 1 & -5 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

6.2. Find the characteristic polynomial, eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 3 & 2 & 3 \\ 4 & -1 & 6 \end{bmatrix}.$$

6.3. Show that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has

(1) two distinct real eigenvalues if $(a-d)^2 + 4bc > 0$,

(2) one eigenvalue if $(a-d)^2 + 4bc = 0$,

(3) no real eigenvalues if $(a-d)^2 + 4bc < 0$,

(4) only real eigenvalues if it is symmetric (i.e., $b = c$).

6.4. Suppose that a 3×3 matrix A has eigenvalues $-1,\ 0,\ 1$ with eigenvectors $\mathbf{u},\ \mathbf{v},\ \mathbf{w}$, respectively. Describe the null space $\mathcal{N}(A)$, and the column space $\mathcal{C}(A)$.

6.5. For any two matrices A and B, show that

(1) adj AB = adj $B \cdot$ adj A;

(2) adj $QAQ^{-1} = Q(\text{adj } A)Q^{-1}$ for any invertible matrix Q;

(3) if $AB = BA$, then $(\text{adj } A)B = B(\text{adj } A)$.

(Hint: It was mentioned for any two invertible matrices A and B in Problem 2.14)

6.6. If a 3×3 matrix A has eigenvalues 1, 2, 3, what are the eigenvectors of $B = (A - I)(A - 2I)(A - 3I)$?

6.7. Show that any 2×2 skew-symmetric nonzero matrix has no real eigenvalue.

6.8. Find a 3×3 matrix that has the eigenvalues $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$ with the associated eigenvectors $\mathbf{x}_1 = (2,\ -1,\ 0),\ \mathbf{x}_2 = (-1,\ 2,\ -1),\ \mathbf{x}_3 = (0,\ -1,\ 2)$.

6.9. Let P be the projection matrix that projects $\mathbb{R}^n$ onto a subspace W. Find the eigenvalues and the eigenspaces for P.

6.10. Let $\mathbf{u},\ \mathbf{v}$ be $n \times 1$ column vectors, and let $A = \mathbf{u}\mathbf{v}^T$. Show that $\mathbf{u}$ is an eigenvector of A, and find the eigenvalues and the eigenvectors of A.

6.11. Show that if λ is an eigenvalue of an idempotent $n \times n$ matrix A (i.e., $A^2 = A$), then λ must be either 0 or 1.

6.12. Prove that if A is an idempotent matrix, then $\operatorname{tr}(A) = \operatorname{rank} A$.

6.13. Let $A = [a_{ij}]$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that

$$\lambda_j = a_{jj} + \sum_{i \neq j} (a_{ii} - \lambda_i) \qquad \text{for } j = 1, \ldots, n.$$

6.14. Prove that if two diagonalizable matrices A and B have the same eigenvectors (i.e., there exists an invertible matrix Q such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal; such matrices A and B are said to be **simultaneously diagonalizable**), then $AB = BA$. In fact, the converse is also true. (See Exercise **7.17**.) Prove the converse with an assumption that the eigenvalues of A are all distinct.

6.15. Let $D : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the differentiation defined by $Df(x) = f'(x)$ for $f \in P_3(\mathbb{R})$. Find all eigenvalues and eigenvectors of D and of D^2.

6.16. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$T(a_2x^2 + a_1x + a_0) = (a_0 + a_1)x^2 + (a_1 + a_2)x + (a_0 + a_2).$$

Find a basis for $P_2(\mathbb{R})$ with respect to which the matrix representation for T is diagonal.

6.17. Determine whether or not each of the following matrices is diagonalizable.

$$(1) \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 2 & 3 \end{bmatrix}, \quad (2) \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad (3) \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & -1 \end{bmatrix}.$$

6.18. Find an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$ for

$$(1)\ A = \begin{bmatrix} -3 & 2 & 4 \\ 2 & -6 & 2 \\ 4 & 2 & -3 \end{bmatrix}, \quad (2)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}, \quad (3)\ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

6.19. Calculate $A^{10}\mathbf{x}$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -2 \\ 0 & 6 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}.$

6.20. For $n \geq 1$, let a_n denote the number of subsets of $\{1, 2, \ldots, n\}$ that contain no consecutive integers. Find the number a_n for all $n \geq 1$.

6.21. Find a general solution of each of the following recurrence relations.

(1) $x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3}, \quad n \geq 3,$

(2) $x_n = 3x_{n-1} - 4x_{n-2} + 2x_{n-3}, \quad n \geq 3,$

(3) $x_n = 4x_{n-1} - 6x_{n-2} + 4x_{n-3} - x_{n-4}, \quad n \geq 4.$

6.22. Let $A = \begin{bmatrix} 0 & 0.3 \\ 0.6 & x \end{bmatrix}$. Find a value x so that A has an eigenvalue $\lambda = 1$. For $\mathbf{x}_0 = (1,\ 1)$, calculate $\lim_{k\to\infty} \mathbf{x}_k$, where $\mathbf{x}_k = A\mathbf{x}_{k-1}$, $k = 1, 2, \ldots$.

6.23. Compute e^A for

(1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, (2) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

6.24. In 2000, the initial status of the car owners in a city was reported as follows: 40% of the car owners drove large cars, 20% drove medium-sized cars, and 40% drove small cars. In 2005, 70% of the large-car owners in 2000 still owned large cars, but 30% had changed to a medium-sized car. Of those who owned medium-sized cars in 2000, 10% had changed to large cars, 70% continued to drive medium-sized cars, and 20% had changed to small cars. Finally, of those who owned the small cars in 2000, 10% had changed to medium-sized cars and 90% still owned small cars in 2005. Assuming that these trends continue, and that no car owners are born, die or otherwise add realism to the problem, determine the percentage of car owners who will own cars of each size in 2035.

6.25. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(1) Compute e^A directly from the expansion.

(2) Compute e^A by diagonalizing A.

6.26. Let $A(t)$ be a matrix whose entries are all differentiable functions in t and invertible for all t. Compute the following:

(1) $\frac{d}{dt}(A(t)^3)$, (2) $\frac{d}{dt}(A(t)^{-1})$.

6.27. Solve $\mathbf{y}' = A\mathbf{y}$, where

(1) $A = \begin{bmatrix} -6 & 24 & 8 \\ -1 & 8 & 4 \\ 2 & -12 & -6 \end{bmatrix}$ and $\mathbf{y}(1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(2) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

6.28. Solve the system $\begin{cases} y_1' = y_1 - y_2 + 2y_3 \\ y_2' = 3y_1 + 4y_3 \\ y_3' = 2y_1 + y_2 \end{cases}$

with initial conditions $y_1(0) = 0$, $y_2(0) = 2$, $y_3(0) = 1$.

6.29. Let $f(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of A. Evaluate $f(A)$ for

(1) $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$, (2) $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$.

In fact, $f(A) = \mathbf{0}$ for any square matrix A and its characteristic polynomial $f(\lambda)$. (This is the Cayley–Hamilton theorem).

6.30. Determine whether the following statements are true or false, in general, and justify your answers.

(1) If B is obtained from A by interchanging two rows, then B is similar to A.

(2) If λ is an eigenvalue of A of multiplicity k, then there exist k linearly independent eigenvectors belonging to λ.

(3) If A and B are diagonalizable, so is AB.

(4) Every invertible matrix is diagonalizable.

(5) Every diagonalizable matrix is invertible.

(6) Interchanging the rows of a 2×2 matrix reverses the signs of its eigenvalues.

(7) A matrix A cannot be similar to $A + I$.

(8) Each eigenvalue of $A + B$ is a sum of an eigenvalue of A and one of B.

(9) The total sum of eigenvalues of $A + B$ equals the sum of all the eigenvalues of A and of those of B.

(10) A sum of two eigenvectors of A is also an eigenvector of A.

(11) Any two similar matrices have the same eigenvectors.

(12) For any square matrix A, $\det e^A = e^{\det A}$.

7

Complex Vector Spaces

7.1 The n-space $\mathbb{C}^n$ and complex vector spaces

So far, we have been dealing with matrices having only real entries and vector spaces with real scalars. Also, in any system of linear (difference or differential) equations, we assumed that the coefficients of an equation are all real. However, for many applications of linear algebra, it is desirable to extend the scalars to complex numbers. For example, by allowing complex scalars, any polynomial of degree n (even with complex coefficients) has n complex roots counting multiplicity. (This is well known as the fundamental theorem of algebra). By applying it to a characteristic polynomial of a matrix, one can say that all the square matrices of order n will have n eigenvalues counting multiplicity. For instance, the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ has no real eigenvalues, but it has two complex eigenvalues $\lambda = 1 \pm i$. Thus, it is indispensable to work with complex numbers to find the full set of eigenvalues and eigenvectors. Therefore, it is natural to extend the concept of real vector spaces to that of complex vector spaces, and develop the basic properties of complex vector spaces.

The complex n-space $\mathbb{C}^n$ is the set of all ordered n-tuples $(z_1, z_2, \dots, z_n)$ of complex numbers:

$$\mathbb{C}^n = \{(z_1, z_2, \dots, z_n) : z_i \in \mathbb{C},\ i = 1, 2, \dots, n\},$$

and it is clearly a complex vector space with addition and scalar multiplication defined as follows:

$$\begin{aligned} (z_1, z_2, \dots, z_n) + (z'_1, z'_2, \dots, z'_n) &= (z_1 + z'_1, z_2 + z'_2, \dots, z_n + z'_n) \\ k(z_1, z_2, \dots, z_n) &= (kz_1, kz_2, \dots, kz_n) \ \text{ for } k \in \mathbb{C}. \end{aligned}$$

The standard basis for the space $\mathbb{C}^n$ is again $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ as the real case, but the scalars are now complex numbers so that any vector $\mathbf{z}$ in $\mathbb{C}^n$ is of the form $\mathbf{z} = \sum_{k=1}^{n} z_k \mathbf{e}_k$ with $z_k = x_k + iy_k \in \mathbb{C}$, i.e., $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ with $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$.

In a complex vector space, linear combinations are defined in the same way as the real case except that scalars are allowed to be complex numbers. Thus the same is true for linear independence, spanning spaces, basis, dimension, and subspace. For complex matrices, whose entries are complex numbers, the matrix sum and product follow the same rules as real matrices. The same is true for the concept of a linear transformation $T : V \to W$ from a complex vector space V to a complex vector space W. The definitions of the kernel and the image of a linear transformation remain the same as those in the real case, as well as the facts about null spaces, column spaces, matrix representations of linear transformations, similarity, and so on.

However, if we are concerned about the inner product, there should be a modification from the real case. Note that the *absolute value* (or *modulus*) of a complex number $z = x+iy$ is defined as the nonnegative real number $|z| = (\bar{z}z)^{\frac{1}{2}} = \sqrt{x^2 + y^2}$, where $\bar{z}$ is the complex conjugate of z. Accordingly, the length of a vector $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ in the n-space $\mathbb{C}^n$ with $z_k = x_k + iy_k \in \mathbb{C}$ has to be modified: if one would take an inner product in $\mathbb{C}^n$ as $\|\mathbf{z}\|^2 = z_1^2 + \cdots + z_n^2$, then a nonzero vector $(1, i)$ in $\mathbb{C}^2$ would have zero length: $1^2 + i^2 = 0$. In any case, a modified definition should coincide with the old definition, when the vectors and matrices were real. The following is the definition of a usual inner product on the n-space $\mathbb{C}^n$.

Definition 7.1 For two vectors $\mathbf{u} = [u_1\ u_2\ \cdots\ u_n]^T$ and $\mathbf{v} = [v_1\ v_2\ \cdots\ v_n]^T$ in $\mathbb{C}^n$, $u_k, v_k \in \mathbb{C}$, the **dot** (or **Euclidean inner**) **product** $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \overline{u_1}v_1 + \overline{u_2}v_2 + \cdots + \overline{u_n}v_n = \overline{\mathbf{u}}^T \mathbf{v},$$

where $\overline{\mathbf{u}} = [\overline{u}_1\ \overline{u}_2\ \cdots\ \overline{u}_n]^T$, the conjugate of $\mathbf{u}$. The **Euclidean length** (or **magnitude**) of a vector $\mathbf{u}$ in $\mathbb{C}^n$ is defined by

$$\|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{\frac{1}{2}} = \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2},$$

where $|u_k|^2 = \overline{u}_k u_k$, and the **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{C}^n$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

In an (abstract) complex vector space, one can also define an inner product by adopting the basic properties of the Euclidean inner product on $\mathbb{C}^n$ as axioms.

Definition 7.2 A (complex) **inner product** (or **Hermitian inner product**) on a complex vector space V is a function that associates a complex number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V in such a way that the following rules are satisfied: For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and all scalars k in $\mathbb{C}$,

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$,
(2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ (additivity),
(3) $\langle k\mathbf{u}, \mathbf{v} \rangle = \bar{k}\langle \mathbf{u}, \mathbf{v} \rangle$ (antilinear),
(4) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positive definiteness).

A complex vector space together with an inner product is called a **complex inner product space** or a **unitary space**. In particular, the n-space $\mathbb{C}^n$ with the dot product is called the **Euclidean** (complex) **n-space**.

The following properties are immediate from the definition of an inner product:

(5) $\langle \mathbf{0}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$,
(6) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$,
(7) $\langle \mathbf{u}, k\mathbf{v}\rangle = k\langle \mathbf{u}, \mathbf{v}\rangle$.

Remark: There is another way to define an inner product on a complex vector space. If we redefine the dot product $\mathbf{u}\cdot\mathbf{v}$ on the n-space $\mathbb{C}^n$ by

$$\mathbf{u}\cdot\mathbf{v} = u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n},$$

then the third rule in Definition 7.2 should be modified to be

(3′) $\langle \mathbf{u}, k\mathbf{v}\rangle = \bar{k}\langle \mathbf{u}, \mathbf{v}\rangle$, so that $\langle k\mathbf{u}, \mathbf{v}\rangle = k\langle \mathbf{u}, \mathbf{v}\rangle$.

But these two different definitions do not induce any essential difference in a complex vector space.

In a complex inner product space, as the real case, the **length** (or **magnitude**) of a vector $\mathbf{u}$ and the **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ are defined by

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{\frac{1}{2}}, \quad d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|,$$

respectively.

Example 7.1 (*A complex inner product on a function space*) Let $C_{\mathbb{C}}[a, b]$ denote the set of all complex-valued continuous functions defined on $[a, b]$. Thus an element in $C_{\mathbb{C}}[a, b]$ is of the form $\mathbf{f}(x) = f_1(x)+if_2(x)$, where $f_1(x)$ and $f_2(x)$ are real-valued and continuous on $[a, b]$. Note that $\mathbf{f}$ is continuous if and only if each component function f_i is continuous. Clearly, the set $C_{\mathbb{C}}[a, b]$ is a complex vector space under the sum and scalar multiplication of functions. For a vector $\mathbf{f}(x) = f_1(x)+if_2(x)$ in $C_{\mathbb{C}}[a, b]$, its integral is defined as follows:

$$\int_a^b \mathbf{f}(x)dx = \int_a^b [f_1(x)+if_2(x)]\,dx = \int_a^b f_1(x)dx + i\int_a^b f_2(x)dx.$$

It is an elementary exercise to show that, for vectors $\mathbf{f}(x) = f_1(x) + if_2(x)$ and $\mathbf{g}(x) = g_1(x)+ig_2(x)$ in the complex vector space $C_{\mathbb{C}}[a, b]$, the following formula defines an inner product on $C_{\mathbb{C}}[a, b]$:

$$\begin{aligned}
\langle \mathbf{f}, \mathbf{g}\rangle &= \int_a^b \overline{\mathbf{f}(x)}\mathbf{g}(x)dx \\
&= \int_a^b [f_1(x) - if_2(x)]\,[g_1(x)+ig_2(x)]\,dx \\
&= \int_a^b [f_1(x)g_1(x) + f_2(x)g_2(x)]\,dx \\
&\quad + i\int_a^b [f_1(x)g_2(x) - f_2(x)g_1(x)]\,dx.
\end{aligned}$$

□

Problem 7.1 Show that the Euclidean inner product on $\mathbb{C}^n$ satisfies all the inner product axioms.

The definitions of such terms as orthogonal sets, orthogonal complements, orthonormal sets, and orthonormal basis remain the same in complex inner product spaces as in real inner product spaces. Moreover, the Gram–Schmidt orthogonalization is still valid in complex inner product spaces, and can be used to convert an arbitrary basis into an orthonormal basis. If V is an n-dimensional complex vector space, then by taking an orthonormal basis for V, there is a natural isometry from V to $\mathbb{C}^n$ that preserves the inner product as in the real case. Hence, without loss of generality, one may work only in $\mathbb{C}^n$ with the Euclidean inner product, and we use $\cdot$ and $\langle\ ,\ \rangle$ interchangeably.

On the other hand, one may consider the set $\mathbb{C}^n$ as a real vector space by defining addition and scalar multiplication as

$$\begin{aligned}(z_1, z_2, \ldots, z_n) + (z_1', z_2', \ldots, z_n') &= (z_1 + z_1', z_2 + z_2', \ldots, z_n + z_n')\\ r(z_1, z_2, \ldots, z_n) &= (rz_1, rz_2, \ldots, rz_n) \text{ for } r \in \mathbb{R}.\end{aligned}$$

Two vectors $\mathbf{e}_1 = (1, 0, \ldots, 0)$ and $i\mathbf{e}_1 = (i, 0, \ldots, 0)$ are linearly dependent when the space $\mathbb{C}^n$ is considered as a complex vector space. However, they are linearly independent if $\mathbb{C}^n$ is considered as a real vector space. In general,

$$\{\mathbf{e}_1, \ldots, \mathbf{e}_n,\ i\mathbf{e}_1, \ldots, i\mathbf{e}_n\}$$

forms a basis for $\mathbb{C}^n$ considered as a real vector space. In this way, $\mathbb{C}^n$ is naturally identified with the $2n$-dimensional real vector space $\mathbb{R}^{2n}$. That is, $\dim \mathbb{C}^n = n$ when $\mathbb{C}^n$ is considered as a complex vector space, but $\dim \mathbb{C}^n = 2n$ when $\mathbb{C}^n$ is considered as a real vector space.

Note that when $\mathbb{C}^n$ is considered as a $2n$-dimensional real vector space, the space $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$ is a subspace of $\mathbb{C}^n$, but not when $\mathbb{C}^n$ is considered as an n-dimensional complex vector space.

Example 7.2 (*Gram–Schmidt orthogonalization on a complex vector space*) Consider the complex vector space $\mathbb{C}^3$ with the Euclidean inner product. Apply the Gram–Schmidt orthogonalization to convert the basis $\mathbf{x}_1 = (i,\ i,\ i)$, $\mathbf{x}_2 = (0,\ i,\ i)$, $\mathbf{x}_3 = (0,\ 0,\ i)$ into an orthonormal basis.

Solution: Step 1: Set

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{(i,\ i,\ i)}{\sqrt{3}} = \left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right).$$

Step 2: Let W_1 denote the subspace spanned by $\mathbf{u}_1$. Then

$$\begin{aligned}\mathbf{x}_2 - \mathrm{Proj}_{W_1}\mathbf{x}_2 &= \mathbf{x}_2 - \langle \mathbf{u}_1, \mathbf{x}_2\rangle \mathbf{u}_1\\ &= (0,\ i,\ i) - \frac{2}{\sqrt{3}}\left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)\\ &= \left(-\frac{2i}{3}, \frac{i}{3}, \frac{i}{3}\right).\end{aligned}$$

Therefore,

$$\mathbf{u}_2 = \frac{\mathbf{x}_2 - \mathrm{Proj}_{W_1}\mathbf{x}_2}{\|\mathbf{x}_2 - \mathrm{Proj}_{W_1}\mathbf{x}_2\|} = \frac{3}{\sqrt{6}}\left(-\frac{2i}{3}, \frac{i}{3}, \frac{i}{3}\right) = \left(-\frac{2i}{\sqrt{6}}, \frac{i}{\sqrt{6}}, \frac{i}{\sqrt{6}}\right).$$

Step 3: Let W_2 denote the subspace spanned by $\{\mathbf{u}_1, \mathbf{u}_2\}$. Then

$$\begin{aligned}
&\mathbf{x}_3 - \mathrm{Proj}_{W_2}\mathbf{x}_3 \\
&= \mathbf{x}_3 - \langle \mathbf{u}_1, \mathbf{x}_3\rangle \mathbf{u}_1 - \langle \mathbf{u}_2, \mathbf{x}_3\rangle \mathbf{u}_2 \\
&= (0,\ 0,\ i) - \frac{1}{\sqrt{3}}\left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{2i}{\sqrt{6}}, \frac{i}{\sqrt{6}}, \frac{i}{\sqrt{6}}\right) \\
&= \left(0, -\frac{i}{2}, \frac{i}{2}\right).
\end{aligned}$$

Therefore,

$$\mathbf{u}_3 = \frac{\mathbf{x}_3 - \mathrm{Proj}_{W_2}\mathbf{x}_3}{\|\mathbf{x}_3 - \mathrm{Proj}_{W_2}\mathbf{x}_3\|} = \sqrt{2}\left(0, -\frac{i}{2}, \frac{i}{2}\right) = \left(0, -\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right).$$

Thus,

$$\mathbf{u}_1 = \left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right), \quad \mathbf{u}_2 = \left(-\frac{2i}{\sqrt{6}}, \frac{i}{\sqrt{6}}, \frac{i}{\sqrt{6}}\right), \quad \mathbf{u}_3 = \left(0, -\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)$$

form an orthonormal basis for $\mathbb{C}^3$. □

Example 7.3 (*An orthonormal set in the complex-valued function space* $C_{\mathbb{C}}[0,\ 2\pi]$) Let $C_{\mathbb{C}}[0,\ 2\pi]$ be the complex vector space with the inner product given in Example 7.1, and let W be the set of vectors in $C_{\mathbb{C}}[0,\ 2\pi]$ of the form

$$e^{ikx} = \cos kx + i \sin kx,$$

where k is an integer. The set W is orthogonal. In fact, if

$$\mathbf{g}_k(x) = e^{ikx} \quad \text{and} \quad \mathbf{g}_\ell(x) = e^{i\ell x}$$

are vectors in W, then

$$\begin{aligned}
\langle \mathbf{g}_k, \mathbf{g}_\ell \rangle &= \int_0^{2\pi} \overline{e^{ikx}} e^{i\ell x} dx = \int_0^{2\pi} e^{-ikx} e^{i\ell x} dx = \int_0^{2\pi} e^{i(\ell - k)x} dx \\
&= \int_0^{2\pi} \cos(\ell - k)x dx + i \int_0^{2\pi} \sin(\ell - k)x dx \\
&= \begin{cases} \left[\frac{1}{\ell - k}\sin(\ell - k)x\right]_0^{2\pi} + i\left[\frac{-1}{\ell - k}\cos(\ell - k)x\right]_0^{2\pi} & \text{if } k \neq \ell, \\ \int_0^{2\pi} dx & \text{if } k = \ell. \end{cases} \\
&= \begin{cases} 0 & \text{if } k \neq \ell, \\ 2\pi & \text{if } k = \ell. \end{cases}
\end{aligned}$$

Thus, the vectors in W are orthogonal and each vector has length $\sqrt{2\pi}$. By normalizing each vector in the orthogonal set W, one can get an orthonormal set. Therefore, the vectors

$$\mathbf{f}_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}, \quad k = 0,\ \pm 1,\ \pm 2,\ \ldots$$

form an orthonormal set in the complex vector space $C_{\mathbb{C}}[0,\ 2\pi]$. □

Problem 7.2 Prove that in a complex inner product space V,

(1) $|\langle \mathbf{x}, \mathbf{y}\rangle|^2 \le \langle \mathbf{x}, \mathbf{x}\rangle\langle \mathbf{y}, \mathbf{y}\rangle$ (Cauchy–Schwarz inequality),
(2) $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality),
(3) $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal (Pythagorean theorem).

The definitions of eigenvalues and eigenvectors in a complex vector space are the same as the real case, but the eigenvalues can now be complex numbers. Hence, for any $n \times n$ (real or complex) matrix A, the characteristic polynomial $\det(\lambda I - A)$ has always n complex roots (i.e., eigenvalues) counting multiplicities.

For example, consider a rotation matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

with real entries. This matrix has two complex eigenvalues for any $\theta \in \mathbb{R}$, but no real eigenvalues unless $\theta = k\pi$ for an integer k.

Therefore, all theorems and corollaries in Chapter 6 regarding eigenvalues and eigenvectors remain true without requiring the existence of n eigenvalues explicitly, and exactly the same proofs as the real case are valid since the arguments in the proofs are not concerned with what the scalars are. For example, one can have a theorem like *'for an $n \times n$ matrix A, the eigenvectors belonging to distinct eigenvalues are linearly independent'*, and *'if the n eigenvalues of A are distinct, then the eigenvectors belonging to them form a basis for $\mathbb{C}^n$ so that A is diagonalizable'*.

An $n \times n$ real matrix A can be considered as a linear transformation on both $\mathbb{R}^n$ and $\mathbb{C}^n$:

$$\begin{aligned} T &: \mathbb{R}^n \to \mathbb{R}^n \quad \text{defined by} \quad T(\mathbf{x}) = A\mathbf{x}, \\ S &: \mathbb{C}^n \to \mathbb{C}^n \quad \text{defined by} \quad S(\mathbf{x}) = A\mathbf{x}. \end{aligned}$$

Since the entries are all real, the coefficients of the characteristic polynomial $f(\lambda) = \det(\lambda I - A)$ of A are all real. Thus, if λ is a root of $f(\lambda) = 0$, then its conjugate $\bar{\lambda}$ is also a root because $f(\bar{\lambda}) = \overline{f(\lambda)} = 0$. In other words, *if λ is an eigenvalue of a real matrix A, then $\bar{\lambda}$ is also an eigenvalue.* In particular, any $n \times n$ real matrix A has at least one real eigenvalue if n is odd.

Moreover, if $\mathbf{x}$ is an eigenvector belonging to a complex eigenvalue λ, then the complex conjugate $\bar{\mathbf{x}}$ is an eigenvector belonging to $\bar{\lambda}$. In fact, if $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$, then

$$A\bar{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}},$$

where $\bar{\mathbf{x}}$ denotes the vector whose entries are the complex conjugates of the corresponding entries of $\mathbf{x}$.

Using this fact, the following example shows that any 2×2 matrix with no real eigenvalues can be written as a scalar multiple of a rotation.

Example 7.4 Show that if A is a 2×2 real matrix having no real eigenvalues, then A is similar to a matrix of the form

$$\begin{bmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}.$$

Solution: Let A be a 2×2 real matrix having no real eigenvalues, and let $\lambda = a + ib$ and $\bar{\lambda} = a - ib$ with $a,\ b \in \mathbb{R}$ and $b \neq 0$ be two complex eigenvalues of A with associated eigenvectors $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ and $\bar{\mathbf{x}} = \mathbf{u} - i\mathbf{v}$ with $\mathbf{u},\ \mathbf{v} \in \mathbb{R}^2$, respectively. It follows immediately that

$$\begin{aligned} \mathbf{u} &= \tfrac{1}{2}(\mathbf{x} + \bar{\mathbf{x}}), & \mathbf{v} &= -\tfrac{i}{2}(\mathbf{x} - \bar{\mathbf{x}}), \\ a &= \frac{1}{2}(\lambda + \bar{\lambda}), & b &= -\frac{i}{2}(\lambda - \bar{\lambda}). \end{aligned}$$

Since $\lambda \neq \bar{\lambda}$, the eigenvectors $\mathbf{x}$ and $\bar{\mathbf{x}}$ are linearly independent in the complex vector space $\mathbb{C}^2$, as they are when $\mathbb{C}^2$ is considered as a real vector space. It implies that the vectors $\mathbf{x} + \bar{\mathbf{x}}$ and $\mathbf{x} - \bar{\mathbf{x}}$ are linearly independent in the real vector space $\mathbb{C}^2$, (see Problem 7.3 below), so that the real vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent in the subspace $\mathbb{R}^2$ of the real vector space $\mathbb{C}^2$. Thus $\alpha = \{\mathbf{u},\ \mathbf{v}\}$ is a basis for the real vector space $\mathbb{R}^2$, and

$$\begin{aligned} A\mathbf{u} &= \frac{1}{2}(A\mathbf{x} + A\bar{\mathbf{x}}) = \frac{1}{2}(\lambda\mathbf{x} + \bar{\lambda}\bar{\mathbf{x}}) \\ &= \lambda\left(\frac{\mathbf{u} + i\mathbf{v}}{2}\right) + \bar{\lambda}\left(\frac{\mathbf{u} - i\mathbf{v}}{2}\right) = a\mathbf{u} - b\mathbf{v}. \end{aligned}$$

Similarly, one can get $A\mathbf{v} = b\mathbf{u} + a\mathbf{v}$, implying that the matrix representation of the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the basis α is

$$[A]_\alpha = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

That is, any 2×2 matrix that has no real eigenvalues is similar to a matrix of such form. Now, by setting $r = \sqrt{a^2 + b^2} > 0$, one can get $a = r\cos\theta$ and $b = r\sin\theta$ for some $\theta \in \mathbb{R}$, so

$$[A]_\alpha = \begin{bmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}. \qquad \square$$

Problem 7.3 Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in a vector space V. Show that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent if and only if $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are linearly independent.

Problem 7.4 Find the eigenvalues and the eigenvectors of

$$(1)\ \begin{bmatrix} i & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -i \end{bmatrix}, \qquad (2)\ \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 0 \\ 1-i & 0 & 1 \end{bmatrix}.$$

Problem 7.5 Prove that an $n \times n$ complex matrix A is diagonalizable if and only if A has n linearly independent eigenvectors in the complex vector space $\mathbb{C}^n$.

7.2 Hermitian and unitary matrices

Recall that the dot product of real vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is given by $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$ in matrix form. For complex vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, the Euclidean inner product is defined by $\mathbf{u} \cdot \mathbf{v} = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n = \bar{\mathbf{u}}^T\mathbf{v}$, which involves the conjugate transpose, not just the transpose.

Definition 7.3 For a complex matrix A, its complex conjugate transpose, $A^H = \overline{A}^T$, is called the **adjoint** of A.

Note that $\overline{A}$ is the matrix whose entries are the complex conjugates of the corresponding entries in A. Thus, $[a_{ij}]^H = [\overline{a_{ji}}]$. With this notation, the Euclidean inner product on $\mathbb{C}^n$ can be written as

$$\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{u}}^T\mathbf{v} = \mathbf{u}^H\mathbf{v}.$$

Problem 7.6 Show that $(AB)^H = B^H A^H$ when AB can be defined.

Problem 7.7 Prove that if A is invertible, so is A^H, and $(A^H)^{-1} = (A^{-1})^H$.

For complex matrices, the notion of symmetry and skew-symmetry real matrices are replaced by Hermitian and skew-Hermitian matrices, respectively.

Definition 7.4 A complex square matrix A is said to be **Hermitian** (or **self-adjoint**) if $A^H = A$, or **skew-Hermitian** if $A^H = -A$.

For matrices

$$A = \begin{bmatrix} 2 & 4+i \\ 4-i & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} i & 1+i \\ -1+i & -i \end{bmatrix},$$

one can see that A is Hermitian and B is skew-Hermitian.

A Hermitian matrix with real entries is just a real symmetric matrix, and conversely, any real symmetric matrix is Hermitian.

Like real matrices, any $m \times n$ (complex) matrix A can be considered as a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$, and

$$(A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x})^H\mathbf{y} = \mathbf{x}^H A^H \mathbf{y} = \mathbf{x} \cdot (A^H\mathbf{y})$$

for any $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$. The following theorem lists some important properties of Hermitian matrices.

Theorem 7.1 *Let A be a Hermitian matrix.*

(1) *For any (complex) vector $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^H A\mathbf{x}$ is real.*
(2) *All (complex) eigenvalues of A are real. In particular, an $n \times n$ real symmetric matrix has precisely n real eigenvalues.*
(3) *The eigenvectors of A belonging to distinct eigenvalues are mutually orthogonal.*

Proof: (1) Since $\mathbf{x}^H A\mathbf{x}$ is a 1×1 matrix, $\overline{(\mathbf{x}^H A\mathbf{x})} = (\mathbf{x}^H A\mathbf{x})^H = \mathbf{x}^H A\mathbf{x}$.

(2) If $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}^H A\mathbf{x} = \mathbf{x}^H \lambda\mathbf{x} = \lambda\mathbf{x}^H\mathbf{x} = \lambda\|\mathbf{x}\|^2$. The left-hand side is real and $\|\mathbf{x}\|^2$ is real and positive, because $\mathbf{x} \neq \mathbf{0}$. Therefore, λ must be real.

(3) Let $\mathbf{x}$ and $\mathbf{y}$ be eigenvectors of A belonging to eigenvalues λ and μ, respectively. Let $\lambda \neq \mu$. Because $A = A^H$ and λ is real, it follows that

$$\lambda(\mathbf{x}\cdot\mathbf{y}) = (\lambda\mathbf{x})\cdot\mathbf{y} = A\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot A\mathbf{y} = \mu(\mathbf{x}\cdot\mathbf{y}).$$

Since $\lambda \neq \mu$, it gives that $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^H\mathbf{y} = 0$, i.e., $\mathbf{x}$ is orthogonal to $\mathbf{y}$. □

In particular, eigenvectors belonging to distinct eigenvalues of a real symmetric matrix are orthogonal.

Remark: Condition (1) in Theorem 7.1 (i.e., $\mathbf{x}^H A\mathbf{x}$ is real for any complex vector $\mathbf{x} \in \mathbb{C}^n$) is equivalent to saying that the diagonals of A are real:

$$\begin{aligned} \mathbf{x}^H A\mathbf{x} &= [\bar{x}_1 \ \cdots \ \bar{x}_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= \sum_{i,j} a_{ij}\bar{x}_i x_j \\ &= \sum_i a_{ii}|x_i|^2 + C + \bar{C}, \end{aligned}$$

where $C = \sum_{i<j} a_{ij}\bar{x}_i x_j$. Since $C + \bar{C}$ is real, all $a_{ii} \in \mathbb{R}$ if and only if $\mathbf{x}^H A\mathbf{x} \in \mathbb{R}$ for any $\mathbf{x} \in \mathbb{C}^n$.

Problem 7.8 Prove that the determinant of any Hermitian matrix is real.

Problem 7.9 Let $\mathbf{x}$ be a nonzero vector in the complex vector space $\mathbb{C}^n$, and $A = \mathbf{x}\mathbf{x}^H$. Show that A is Hermitian, and find all the eigenvalues and their eigenspaces for A.

It is easy to see that if A is Hermitian, then the matrix iA is skew-Hermitian; similarly, if A is skew-Hermitian, then iA is Hermitian. Therefore, the following theorem is a direct consequence of this fact and Theorem 7.1. The proof is left for an exercise.

Theorem 7.2 *Let A be a skew-Hermitian matrix.*

(1) *For any complex vector $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^H A\mathbf{x}$ is purely imaginary, and the diagonal entries of A are purely imaginary.*
(2) *All eigenvalues of A are purely imaginary. In particular, a real skew-symmetric matrix has purely imaginary n eigenvalues.*
(3) *The eigenvectors of A belonging to distinct eigenvalues are mutually orthogonal.*

Problem 7.10 Prove Theorem 7.2 by using Theorem 7.1, and prove (3) directly.

Problem 7.11 Show that $A = B + iC$ (B and C real matrices) is skew-Hermitian if and only if B is skew-symmetric and C is symmetric.

Problem 7.12 Let A and B be either both Hermitian or both skew-Hermitian.

(1) AB is Hermitian if and only if $AB = BA$.
(2) AB is skew-Hermitian if and only if $AB = -BA$.

Recall that a square matrix Q with real entries is orthogonal if their column vectors are orthonormal (i.e., $Q^T Q = I$). The same is true for complex matrices (compare with Lemma 5.18).

Lemma 7.3 *For a complex square matrix U, the following are equivalent:*

(1) *the column vectors of U are orthonormal;*
(2) $U^H U = I$;
(3) $U^{-1} = U^H$;
(4) $UU^H = I$;
(5) *the row vectors of U are orthonormal.*

The complex analogue to an orthogonal matrix is a unitary matrix.

Definition 7.5 A complex square matrix U is said to be **unitary** if it satisfies any one (and hence, all) of the conditions in Lemma 7.3.

Like a real orthogonal matrix, any unitary matrix preserves the lengths of vectors.

Theorem 7.4 *Let U be an $n \times n$ unitary matrix.*

(1) *U preserves the dot product on $\mathbb{C}^n$: i.e., for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}^n$,*

$$(U\mathbf{x})^H (U\mathbf{y}) = \mathbf{x}^H \mathbf{y}.$$

(2) *If λ is an eigenvalue of U, then $|\lambda| = 1$.*
(3) *The eigenvectors of U belonging to distinct eigenvalues are mutually orthogonal.*

Proof: (1) $(U\mathbf{x})^H (U\mathbf{y}) = \mathbf{x}^H U^H U\mathbf{y} = \mathbf{x}^H \mathbf{y}$.

(2) For $U\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x}^H\mathbf{x} = (U\mathbf{x})^H (U\mathbf{x}) = |\lambda|^2 \mathbf{x}^H \mathbf{x}$.

(3) Let $U\mathbf{x} = \lambda\mathbf{x}$, $U\mathbf{y} = \mu\mathbf{y}$, and $\lambda \neq \mu$. Since U is unitary, we have $\lambda\bar{\lambda} = 1 = \mu\bar{\mu}$, and $U^{-1}\mathbf{y} = \mu^{-1}\mathbf{y} = \bar{\mu}\mathbf{y}$. Therefore,

$$\bar{\lambda}\mathbf{x}^H\mathbf{y} = (\lambda\mathbf{x})^H\mathbf{y} = (U\mathbf{x})^H\mathbf{y} = \mathbf{x}^H U^{-1}\mathbf{y} = \mathbf{x}^H(\bar{\mu}\mathbf{y}) = \bar{\mu}\mathbf{x}^H\mathbf{y}$$

holds, and $\lambda \neq \mu$ implies $\mathbf{x}^H\mathbf{y} = 0$. □

From the same argument as in the proof of Theorem 5.19, U preserves the dot product if and only if it preserves the lengths of vectors: $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{C}^n$. Thus, a unitary matrix is an isometry.

Theorem 7.5 *A basis-change matrix from one orthonormal basis to another in a complex vector space is unitary.*

Proof: Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be two orthonormal bases, and let $Q = [q_{ij}]$ be the basis-change matrix from the basis β to the basis α. By definition,

$$\mathbf{w}_j = \sum_{\ell=1}^{n} q_{\ell j}\mathbf{v}_\ell.$$

Thus,

$$\begin{aligned} \delta_{ij} = \langle \mathbf{w}_i, \mathbf{w}_j \rangle &= \left\langle \sum_{k=1}^{n} q_{ki}\mathbf{v}_k, \sum_{\ell=1}^{n} q_{\ell j}\mathbf{v}_\ell \right\rangle \\ &= \sum_{k=1}^{n} \overline{q_{ki}} \sum_{\ell=1}^{n} q_{\ell j}\langle \mathbf{v}_k, \mathbf{v}_\ell \rangle \\ &= \sum_{k=1}^{n} \overline{q_{ki}} q_{kj} = \sum_{k=1}^{n} [Q^H]_{ik}[Q]_{kj}. \end{aligned}$$

This means that the columns of Q are orthonormal and Q is unitary. □

Just as in the real case, it is true that two matrices representing the same linear transformation on a complex vector space with respect to different bases are similar. If the two bases are both orthonormal, then the basis-change matrix is unitary (or orthogonal).

Problem 7.13 Show that $|\det U| = 1$ for any unitary matrix U.

Problem 7.14 Show that

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{bmatrix}$$

is unitary but neither Hermitian nor skew-Hermitian.

Problem 7.15 Show that the adjoint of a unitary matrix is unitary, and the product of two unitary matrices is unitary.

Problem 7.16 Describe all 3×3 matrices that are simultaneously Hermitian, unitary, and diagonal. How many are there?

7.3 Unitarily diagonalizable matrices

In the previous section, it was shown that if an $n \times n$ square matrix A is Hermitian, skew-Hermitian or unitary, then the eigenvectors belonging to distinct eigenvalues are mutually orthogonal. Hence, if such a matrix A has n distinct eigenvalues, then there exists an orthonormal basis α for $\mathbb{C}^n$ consisting of eigenvectors of A so that the matrix representation $[A]_\alpha$ is diagonal, i.e., A is diagonalizable by a unitary matrix. In this section, it will be shown that any Hermitian, skew-Hermitian or unitary matrix has n orthonormal eigenvectors even if the eigenvalues are not all distinct. In particular, it is always diagonalizable by a unitary matrix.

Definition 7.6 **(1)** Two real matrices A and B are **orthogonally similar** if there exists an orthogonal matrix P such that $P^{-1}AP = B$. A matrix is **orthogonally diagonalizable** if it is orthogonally similar to a diagonal matrix.

(2) Two complex matrices A and B are **unitarily similar** if there exists a unitary matrix U such that $U^{-1}AU = B$. A matrix is **unitarily diagonalizable** if it is unitarily similar to a diagonal matrix.

We begin with a classical theorem due to Schur (1909) concerning orthogonal and unitary similarity.

Lemma 7.6 (Schur's Lemma) **(1)** *If an $n \times n$ real matrix A has only real eigenvalues, then A is orthogonally similar to an upper triangular matrix.*

(2) *Every $n \times n$ complex matrix is unitarily similar to an upper triangular matrix.*

Proof: We prove only the second assertion (2) by mathematical induction on n, because (1) can be done in a similar way. Clearly, it is true for $n = 1$. Assume now that the assertion (2) holds for $n = r - 1$. Let A be any $r \times r$ complex matrix and let λ_1 be an eigenvalue of A with a normalized eigenvector $\mathbf{x}$. Extend it to an orthonormal basis by the Gram–Schmidt orthogonalization, say $\{\mathbf{x}, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ for $\mathbb{C}^r$. Set a unitary matrix $U_1 = [\mathbf{x}\ \mathbf{u}_2 \cdots\ \mathbf{u}_r]$ with these basis vectors as its columns. A direct computation of the product $U_1^{-1}AU_1$ shows

$$\begin{aligned}
U_1^{-1}AU_1 &= U_1^H AU_1 = U_1^H[A\mathbf{x}\ A\mathbf{u}_2\ \cdots\ A\mathbf{u}_r] \\
&= \begin{bmatrix} -- & \bar{\mathbf{x}}^T & -- \\ -- & \bar{\mathbf{u}}_2^T & -- \\ & \vdots & \\ -- & \bar{\mathbf{u}}_r^T & -- \end{bmatrix} \begin{bmatrix} | & | & & | \\ \lambda_1\mathbf{x} & A\mathbf{u}_2 & \cdots & A\mathbf{u}_r \\ | & | & & | \end{bmatrix} \\
&= \left[\begin{array}{c|ccc} \lambda_1 & & * & \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right],
\end{aligned}$$

where B is an $(r-1) \times (r-1)$ matrix. By the induction hypothesis there exists an $(r-1) \times (r-1)$ unitary matrix U_2 such that $U_2^{-1} B U_2$ is an upper triangular matrix with diagonal entries $\lambda_2, \lambda_3, \ldots, \lambda_r$. Define

$$U = U_1 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{bmatrix}.$$

Then it is easy to check that U is also a unitary matrix, and

$$U^{-1}AU = U^H AU = \begin{bmatrix} \lambda_1 & * \\ \mathbf{0} & U_2^H B U_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & * \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \lambda_r \end{bmatrix}. \qquad \square$$

Schur's lemma is a cornerstone in the study of complex matrices.

Theorem 7.7 *If A is either a Hermitian, a skew-Hermitian or a unitary matrix, then it is unitarily diagonalizable.*

Proof: By Schur's lemma, $U^H AU = B$ is an upper triangular matrix for some unitary matrix U. However,

$$B^H = U^H A^H U = \begin{cases} U^H(\pm A)U = \pm B & \text{if } A \text{ is (skew-) Hermitian} \\ U^{-1}A^{-1}U = B^{-1} & \text{if } A \text{ is unitary,} \end{cases}$$

where the right-hand sides of the equalities depend on whether A is either a Hermitian, a skew-Hermitian or a unitary matrix. This means that the upper triangular matrix B takes the same type; a Hermitian, a skew-Hermitian or unitary, as A.

Note that B^H is a lower triangular matrix and B^{-1} is an upper triangular matrix because B is upper triangular. Therefore, the upper triangular matrix B must be a diagonal matrix in each case of Hermitian, skew-Hermitian or unitary. $\square$

Note that, in the similarity condition $U^{-1}AU (= U^H AU) = D$ of A to a diagonal matrix D through a unitary matrix U, the equation $AU = UD$ shows that the column vectors of U constitute a set of n orthonormal eigenvectors of A while the diagonal entries of D are eigenvalues of A as shown in Theorem 6.7. Therefore, by Theorems 7.1, 7.2 and 7.4, all the diagonal entries of D are real, purely imaginary or of unit length depending on the types (Hermitian, skew-Hermitian or unitary, respectively) of the matrix A.

Example 7.5 (*A Hermitian matrix is unitarily diagonalizable*) Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$$

by a unitary matrix.

Solution: Since A is Hermitian, it is unitarily diagonalizable. One can show that A has the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 0$ with associated eigenvectors $\mathbf{x}_1 = (1-i,\ 1)$ and $\mathbf{x}_2 = (-1,\ 1+i)$, respectively. Let

$$\begin{aligned} \mathbf{u}_1 &= \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{1}{\sqrt{3}}(1-i,\ 1), \\ \mathbf{u}_2 &= \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{1}{\sqrt{3}}(-1,\ 1+i), \end{aligned}$$

and let

$$U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1-i & -1 \\ 1 & 1+i \end{bmatrix}.$$

Then, U is a unitary matrix and diagonalizes A:

$$\begin{aligned} U^H AU &= \frac{1}{3}\begin{bmatrix} 1+i & 1 \\ -1 & 1-i \end{bmatrix}\begin{bmatrix} 2 & 1-i \\ 1+1 & 1 \end{bmatrix}\begin{bmatrix} 1-i & -1 \\ 1 & 1+i \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$ □

Since all the real symmetric matrices are Hermitian matrices, they are unitarily diagonalizable by Theorem 7.7. However, the following theorem says more than that.

Theorem 7.8 *For any $n \times n$ real matrix A, the following are equivalent.*

(1) *A is symmetric.*
(2) *A is orthogonally diagonalizable.*
(3) *A has a full set of n orthonormal eigenvectors.*

Proof: (1) $\Rightarrow$ (2): If A is real and symmetric, then it is a Hermitian matrix, so it has only real eigenvalues. By Schur's lemma 7.6, A is orthogonally similar to an upper triangular matrix, which must be already diagonal. Hence it is orthogonally diagonalizable.

(2) $\Rightarrow$ (3): If A is diagonalized by an orthogonal matrix Q, then the column vectors of Q are eigenvectors of A. Hence A has a full set of n orthonormal eigenvectors.

(3) $\Rightarrow$ (1): If A has a full set of n orthonormal eigenvectors, then these eigenvectors form an orthogonal basis-change matrix Q such that $AQ = QD$. It is now trivial to show that $A = QDQ^{-1} = QDQ^T$ is symmetric. □

Corollary 7.9 *Let A be a real symmetric matrix, and let λ be an eigenvalue of A of multiplicity m_λ. Then*

$$\dim E(\lambda) = \dim \mathcal{N}(\lambda I - A) = m_\lambda.$$

By Theorem 7.8, all real symmetric matrices are always diagonalizable, even more, orthogonally. Moreover, they are all that can be "orthogonally" diagonalized. Even though not all matrices are diagonalizable, certain non-symmetric matrices may still have a full set of linearly independent eigenvectors so that they are diagonalizable, but in this case the eigenvectors cannot be orthogonal. That is, the basis-change matrix Q cannot be an orthogonal matrix. For example, any triangular matrix having all distinct diagonal entries is diagonalizable because their eigenvalues are all distinct, but cannot be orthogonally diagonalizable if it is not diagonal (i.e., not symmetric).

Problem 7.17 Show that the non-symmetric matrices

$$(1)\ A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad (2)\ A = \begin{bmatrix} x & 0 & 1 \\ 0 & x+1 & 0 \\ 0 & 0 & x+2 \end{bmatrix},\ x \in \mathbb{R}$$

are diagonalizable, but not orthogonally.

Remark: The procedure for orthogonal diagonalization of a symmetric matrix A can be summarized as follows.

Step 1 Find a basis for each eigenspace of A.
Step 2 Apply the Gram–Schmidt orthogonalization to each of these bases to obtain an orthonormal basis for each eigenspace.
Step 3 Form the matrix Q whose columns are the basis vectors constructed in Step 2; this matrix orthogonally diagonalizes A.

The justification of this procedure should be clear, because eigenvectors belonging to *distinct* eigenvalues are orthogonal, while an application of the Gram–Schmidt orthogonalization assures that the eigenvectors obtained within the *same* eigenspace are orthonormal. Thus, the entire set of eigenvectors obtained by this procedure is orthonormal.

Example 7.6 (*A symmetric matrix is orthogonally diagonalizable*) Find an orthogonal matrix Q that diagonalizes the symmetric matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

Solution: The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{bmatrix} = (\lambda-2)^2(\lambda-8).$$

Thus, the eigenvalues of A are $\lambda = 2$ and $\lambda = 8$. By the method used in Example 6.2, it can be shown that

$$\mathbf{x}_1 = (-1,\ 1,\ 0) \quad \text{and} \quad \mathbf{x}_2 = (-1,\ 0,\ 1)$$

form a basis for the eigenspace belonging to $\lambda = 2$. Applying the Gram–Schmidt orthogonalization to $\{\mathbf{x}_1, \mathbf{x}_2\}$ yields the following orthonormal eigenvectors (verify):

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(-1,\ 1,\ 0) \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{6}}(-1,\ -1,\ 2).$$

The eigenspace belonging to $\lambda = 8$ has $\mathbf{x}_3 = (1,\ 1,\ 1)$ as a basis. The normalization of $\mathbf{x}_3$ yields $\mathbf{u}_3 = \frac{1}{\sqrt{3}}(1,\ 1,\ 1)$. Finally, using $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ as column vectors, one can obtain

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

which orthogonally diagonalizes A. (It is suggested that readers verify that $Q^T AQ$ is actually a diagonal matrix.) □

Example 7.7 (*Diagonal, but neither Hermitian, skew-Hermitian, nor unitary*) A matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & x \end{bmatrix}, \quad x \in \mathbb{R} \text{ with } |x| \neq 1,$$

is neither Hermitian, skew-Hermitian, nor unitary, but is a diagonal matrix. Hence, there are infinitely many unitarily diagonalizable matrices which are neither Hermitian, skew-Hermitian, nor unitary. □

Problem 7.18 For each of matrices

$$(1)\begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix} \qquad (2)\begin{bmatrix} 2i & 0 & 0 \\ i & -1 & -i \\ -1 & 0 & 2i \end{bmatrix}$$

find a unitary matrix U such that $U^{-1}AU$ is an upper triangular matrix.

7.4 Normal matrices

We have seen that Hermitian, skew-Hermitian and unitary matrices are all unitarily diagonalizable. However, it turns out that they do not constitute the entire class of unitarily diagonalizable matrices, whereas in the class of real matrices the real symmetric matrices are the only matrices with real entries that are orthogonally diagonalizable. That is, there are infinitely many unitarily diagonalizable matrices that are neither one of the above-mentioned classes of matrices, (see Example 7.7). Actually, all unitarily diagonalizable matrices belong to the following class of matrices, called *normal* matrices.

Definition 7.7 A complex square matrix A is called **normal** if

$$AA^H = A^H A.$$

Note that all the Hermitian, skew-Hermitian and unitary matrices are normal. But, there are infinitely many matrices that are normal but are none of these, as shown in Example 7.7. Moreover, there exists an example of such matrices which are not diagonal.

Example 7.8 (*Normal, but neither Hermitian, skew-Hermitian, unitary, nor diagonal*) The 2×2 matrix

$$A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

is normal, but is neither Hermitian, skew-Hermitian, unitary, nor diagonal. However, one can easily check that this matrix is unitarily diagonalizable. In fact,

$$U^{-1}AU = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} = D. \quad \square$$

Problem 7.19 Which of following matrices are Hermitian, skew-Hermitian, unitary or normal?

$$(1) \begin{bmatrix} 1 & 1+i \\ 1-i & 0 \end{bmatrix}, \quad (2) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad (3) \begin{bmatrix} i & 1 & -i \\ -1 & 2i & 0 \\ -i & 0 & -i \end{bmatrix},$$

$$(4) \begin{bmatrix} -i & 3 \\ 3 & 2 \end{bmatrix}, \quad (5) \begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{bmatrix}, \quad (6) \begin{bmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{bmatrix}.$$

As a matter of fact, it will be shown that the normal matrices are all classified as the unitarily diagonalizable matrices. We begin with a lemma.

Lemma 7.10 *If an upper triangular matrix T is normal, then it must be a diagonal matrix.*

Proof: Use induction on k in comparing the diagonal (k, k)-entry of both sides of $TT^H = T^HT$:

$$\begin{bmatrix} t_{11} & & t_{1n} \\ & \ddots & \\ 0 & & t_{nn} \end{bmatrix}\begin{bmatrix} \overline{t_{11}} & & 0 \\ & \ddots & \\ \overline{t_{1n}} & & \overline{t_{nn}} \end{bmatrix} = \begin{bmatrix} \overline{t_{11}} & & 0 \\ & \ddots & \\ \overline{t_{1n}} & & \overline{t_{nn}} \end{bmatrix}\begin{bmatrix} t_{11} & & t_{1n} \\ & \ddots & \\ 0 & & t_{nn} \end{bmatrix}.$$

If $k = 1$, the equality

$$[TT^H]_{11} = |t_{11}|^2 + \cdots + |t_{1n}|^2, \text{ and } [T^HT]_{11} = |t_{11}|^2$$

implies $t_{12} = \cdots = t_{1n} = 0$. Inductively, assume that $t_{i-1i} = \cdots = t_{i-1n} = 0$ has been shown for $i = 1, \ldots, k$. Then

$$[TT^H]_{kk} = |t_{kk}|^2 + \cdots + |t_{kn}|^2$$

and

$$[T^H T]_{kk} = |t_{1k}|^2 + \cdots + |t_{k-1k}|^2 + |t_{kk}|^2 = |t_{kk}|^2$$

because $t_{1k} = \cdots = t_{k-1k} = 0$ by an induction hypothesis. But $TT^H = T^H T$ yields $t_{kk+1} = \cdots = t_{kn} = 0$. It concludes that $t_{kk+1} = \cdots = t_{kn} = 0$ for all $k = 1, \ldots, n$, which shows that all the entries of T off the diagonal are zero, i.e., T is diagonal. □

Theorem 7.11 *For any $n \times n$ complex matrix A, the following are equivalent:*

(1) *A is normal;*
(2) *A is unitarily diagonalizable;*
(3) *A has a full set of n orthonormal eigenvectors.*

Proof: (1) $\Rightarrow$ (2): Suppose that A is normal. By Schur's lemma, there exists a unitary matrix U such that $T = U^H AU$ is an upper triangular matrix. Then T is also normal, since

$$\begin{aligned} TT^H &= U^H AUU^H A^H U = U^H AA^H U = U^H A^H AU \\ &= U^H A^H UU^H AU = T^H T. \end{aligned}$$

Thus, by Lemma 7.10, T is already diagonal so that A is unitarily diagonalizable.

(2) $\Rightarrow$ (3): It is clear that the columns of the basis-change matrix U are n orthonormal eigenvectors of A.

(3) $\Rightarrow$ (1): Let U be the unitary matrix whose columns are the n orthonormal eigenvectors . Then $AU = UD$ or $A = UDU^H$, and

$$\begin{aligned} AA^H &= UDU^H UD^H U^H = UDD^H U^H = UD^H DU^H \\ &= UD^H U^H UDU^H = A^H A. \end{aligned}$$

That is, A is normal. □

Note that there exist infinitely many non-normal complex matrices that are still diagonalizable, but of course not unitarily. One can find such examples among the triangle matrices having distinct diagonal entries.

Recall that any $n \times n$ real matrix A can be written as the sum $S + T$ of a symmetric matrix $S = \frac{1}{2}(A + A^T)$ and a skew-symmetric matrix $T = \frac{1}{2}(A - A^T)$. (See Problem 1.11.) The same kind of expression is also possible for a complex matrix. A complex matrix A can be written as the sum $A = H_1 + iH_2$, where

$$H_1 = \frac{1}{2}(A + A^H), \quad H_2 = -\frac{i}{2}(A - A^H); \text{ or } iH_2 = \frac{1}{2}(A - A^H).$$

Clearly both H_1 and H_2 are Hermitian, and iH_2 is skew-Hermitian.

Problem 7.20 Show that the matrix $A = \begin{bmatrix} x & 0 \\ 1 & i \end{bmatrix}$, $x \in \mathbb{R}$, is not normal, so it cannot be unitarily diagonalizable. But it is diagonalizable.

Problem 7.21 Determine whether or not the matrix

$$A = \begin{bmatrix} 1 & i & i \\ i & 1 & i \\ i & i & 1 \end{bmatrix}$$

is unitarily diagonalizable. If it is, find a unitary matrix U that diagonalizes A.

Problem 7.22 Let H_1 and H_2 be two Hermitian matrices. Show that $A = H_1 + iH_2$ is normal if and only if $H_1H_2 = H_2H_1$.

Problem 7.23 For any unitarily diagonalizable matrix A, prove that

(1) A is Hermitian if and only if A has only real eigenvalues;
(2) A is skew-Hermitian if and only if A has only purely imaginary eigenvalues;
(3) A is unitary if and only if $|\lambda| = 1$ for any eigenvalue λ of A.

7.5 Application

7.5.1 The spectral theorem

As shown in the previous section, the normal matrices are the only matrices that can be unitarily diagonalized. That is, A is normal if and only if there exists a basis α for $\mathbb{C}^n$ consisting of orthonormal eigenvectors of A such that the matrix representation $[A]_\alpha$ of A with respect to α is diagonal.

Theorem 7.12 (Spectral theorem) *Let A be a normal matrix, and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a set of orthonormal eigenvectors belonging to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, respectively. Then A can be written as*

$$\boxed{A = UDU^H = \lambda_1\mathbf{u}_1\mathbf{u}_1^H + \lambda_2\mathbf{u}_2\mathbf{u}_2^H + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^H}$$

and $\mathbf{u}_i\mathbf{u}_i^H$ is the orthogonal projection matrix onto the subspace spanned by the eigenvector $\mathbf{u}_i$ for $i = 1, \ldots, n$.

Proof: Note that $U = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n]$ is a unitary matrix that transforms A into a diagonal matrix D, i.e., $U^{-1}AU = U^HAU = D$. Then

$$\begin{aligned} A &= UDU^H = [\lambda_1\mathbf{u}_1\ \lambda_2\mathbf{u}_2\ \cdots\ \lambda_n\mathbf{u}_n]\begin{bmatrix} \mathbf{u}_1^H \\ \mathbf{u}_2^H \\ \vdots \\ \mathbf{u}_n^H \end{bmatrix} \\ &= \lambda_1\mathbf{u}_1\mathbf{u}_1^H + \lambda_2\mathbf{u}_2\mathbf{u}_2^H + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^H \\ &= \lambda_1P_1 + \lambda_2P_2 + \cdots + \lambda_nP_n, \end{aligned}$$

where

$$P_i = \mathbf{u}_i\mathbf{u}_i^H = \begin{bmatrix} u_{1i} \\ \vdots \\ u_{ni} \end{bmatrix} [\bar{u}_{1i} \cdots \bar{u}_{ni}] = \begin{bmatrix} |u_{1i}|^2 & \cdots & u_{1i}\bar{u}_{ni} \\ \vdots & \ddots & \vdots \\ u_{ni}\bar{u}_{1i} & \cdots & |u_{ni}|^2 \end{bmatrix},$$

which is a Hermitian matrix. Now, for any $\mathbf{x} \in \mathbb{C}^n$ and $i, j = 1, \ldots, n$,

$$\begin{aligned} P_i\mathbf{x} &= \mathbf{u}_i\mathbf{u}_i^H\mathbf{x} = \langle \mathbf{u}_i, \mathbf{x}\rangle\mathbf{u}_i, \\ P_iP_j &= \mathbf{u}_i\mathbf{u}_i^H\mathbf{u}_j\mathbf{u}_j^H = \langle \mathbf{u}_i, \mathbf{u}_j\rangle\mathbf{u}_i\mathbf{u}_j^H \\ &= \begin{cases} 1\mathbf{u}_i\mathbf{u}_i^H = P_i & \text{if } i = j, \\ 0\mathbf{u}_i\mathbf{u}_j^H = \mathbf{0} & \text{if } i \neq j, \end{cases} \\ (P_1 + \cdots + P_n)\mathbf{x} &= P_1\mathbf{x} + \cdots + P_n\mathbf{x} = \sum_{i=1}^n \langle \mathbf{u}_i, \mathbf{x}\rangle\mathbf{u}_i = \mathbf{x} = id(\mathbf{x}). \end{aligned}$$

Therefore, each P_i is nothing but the orthogonal projection onto the subspace spanned by the eigenvector $\mathbf{u}_i$. □

Note that the equation $P_1 + \cdots + P_n = id$ means that if one restricts the image of the P_i to be the subspace spanned by $\mathbf{u}_i$ which is isomorphic to $\mathbb{C}$, then $(P_1, \ldots, P_n)$ defines another orthogonal coordinate system with respect to the orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ just like $(z_1, \ldots, z_n)$ of the $\mathbb{C}^n$ (see Sections 5.6 and 5.9.3).

Note that any $\mathbf{x} \in \mathbb{C}^n$ has the unique expression $\mathbf{x} = \sum\langle \mathbf{u}_i, \mathbf{x}\rangle\mathbf{u}_i$ as a linear combination of the orthonormal basis vectors $\mathbf{u}_i$, and by the spectral theorem,

$$\begin{aligned} A\mathbf{x} &= \lambda_1P_1\mathbf{x} + \lambda_2P_2\mathbf{x} + \cdots + \lambda_nP_n\mathbf{x} \\ &= \lambda_1\mathbf{u}_1(\mathbf{u}_1^H\mathbf{x}) + \cdots + \lambda_n\mathbf{u}_n(\mathbf{u}_n^H\mathbf{x}) \\ &= \lambda_1\langle \mathbf{u}_1, \mathbf{x}\rangle\mathbf{u}_1 + \cdots + \lambda_n\langle \mathbf{u}_n, \mathbf{x}\rangle\mathbf{u}_n. \end{aligned}$$

If an eigenvalue λ has multiplicity ℓ, i.e., $\lambda = \lambda_{i_1} = \cdots = \lambda_{i_\ell}$, with a set of ℓ orthonormal eigenvectors $\mathbf{u}_{i_1}, \ldots, \mathbf{u}_{i_\ell}$, then they form an orthonormal basis for the eigenspace $E(\lambda)$, and

$$P_\lambda = P_{i_1} + \cdots + P_{i_\ell} = \mathbf{u}_{i_1}\mathbf{u}_{i_1}^H + \cdots + \mathbf{u}_{i_\ell}\mathbf{u}_{i_\ell}^H$$

is the orthogonal projection matrix onto $E(\lambda) = \mathcal{N}(\lambda I - A)$. Therefore, counting the multiplicity of each eigenvalue, every normal matrix A has the unique **spectral decomposition** into the projections

$$A = \lambda_1P_{\lambda_1} + \cdots + \lambda_kP_{\lambda_k},$$

for $k \leq n$, where λ_i's are all distinct.

Corollary 7.13 *Let A be a normal matrix.*

(1) *The eigenvectors of A belonging to distinct eigenvalues are mutually orthogonal.*
(2) *If an eigenvalue λ of A has multiplicity k, then the eigenspace $\mathcal{N}(A - \lambda I)$ belonging to λ is of dimension k.*

Corollary 7.14 *Let A be a normal matrix with the spectral decomposition $A = \lambda_1 P_{\lambda_1} + \cdots + \lambda_k P_{\lambda_k}$. Then, for any positive integer m,*

$$A^m = \lambda_1^m P_{\lambda_1} + \cdots + \lambda_k^m P_{\lambda_k}.$$

Moreover, if A is invertible, then for any positive integer ℓ,

$$A^{-\ell} = \frac{1}{\lambda_1^\ell} P_{\lambda_1} + \cdots + \frac{1}{\lambda_k^\ell} P_{\lambda_k}.$$

Example 7.9 (*Spectral decomposition of a symmetric matrix*) Find the spectral decomposition of

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

Solution: From Example 7.6, the spectral decomposition is

$$A = 2(P_1 + P_2) + 8P_3,$$

where the projections are

$$\begin{aligned}
P_1 &= \mathbf{u}_1\mathbf{u}_1^H = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}[-1\ 1\ 0] = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
P_2 &= \mathbf{u}_2\mathbf{u}_2^H = \frac{1}{6}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}[-1\ -1\ 2] = \frac{1}{6}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}, \\
P_3 &= \mathbf{u}_3\mathbf{u}_3^H = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}[1\ 1\ 1] = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.
\end{aligned}$$

Hence,

$$P_1' = P_1 + P_2 = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

is the projection onto the eigenspace $E(2)$ belonging to $\lambda = 2$, P_3 is the projection onto the eigenspace $E(8)$ belonging to $\lambda = 8$, and

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} + \frac{8}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \qquad \square$$

Problem 7.24 Given $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$, find an orthogonal matrix Q that diagonalizes A, and find the spectral decomposition of A.

Example 7.10 (*Spectral decomposition of a normal matrix*) Find the spectral decomposition of a normal matrix

$$A = \begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{bmatrix}.$$

Solution: Since A is normal ($AA^H = A^H A$), it is unitarily diagonalizable. The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & -i \\ 0 & \lambda - i & 0 \\ -i & 0 & \lambda \end{bmatrix} = (\lambda - i)^2(\lambda + i).$$

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = i$ of multiplicity 2 and $\lambda_3 = -i$ of multiplicity 1. By a simple computation using the Gram–Schmidt orthogonalization, one can find that

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are orthonormal eigenvectors of A belonging to the eigenvalues λ_1, λ_2 and λ_3, respectively. Now, the spectral decomposition is $A = i(P_1 + P_2) - iP_3$, where the projection matrices are

$$P_1 = \mathbf{u}_1\mathbf{u}_1^H = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [1\ 0\ 1] = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$P_2 = \mathbf{u}_2\mathbf{u}_2^H = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [0\ 1\ 0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_3 = \mathbf{u}_3\mathbf{u}_3^H = \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} [-1\ 0\ 1] = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Hence,

$$A = i(P_1 + P_2) - iP_3 = \frac{i}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \frac{i}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \qquad \square$$

Problem 7.25 Find the spectral decomposition of each of the following matrices:

$$(1)\ A=\begin{bmatrix}2&1\\1&2\end{bmatrix},\qquad (2)\ B=\begin{bmatrix}1&2+i\\2-i&3\end{bmatrix},$$

$$(3)\ C=\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix},\qquad (4)\ D=\begin{bmatrix}1&1&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\end{bmatrix}.$$

7.6 Exercises

7.1. Calculate $\|\mathbf{x}\|$ for

$$(1)\ \mathbf{x}=\begin{bmatrix}1+i\\2\end{bmatrix},\quad (2)\ \mathbf{x}=\begin{bmatrix}1-2i\\i\\3+i\end{bmatrix}.$$

7.2. Construct an orthonormal basis for $\mathbb{C}^2$ from $\{(i,\ 4+2i),\ (5+6i,\ 1)\}$ by applying the Gram–Schmidt orthogonalization.

7.3. Find the rank of the matrix $A=\begin{bmatrix}i&1&1-i&1+i\\1-i&1+i&1&2+i\\1+3i&1-i&2-i&1+4i\end{bmatrix}$.

7.4. Find the eigenvalues and eigenvectors for each of the following matrices:

$$(1)\ \begin{bmatrix}-2&-1\\5&2\end{bmatrix},\quad (2)\ \begin{bmatrix}0&i\\-i&0\end{bmatrix},$$

$$(3)\ \begin{bmatrix}5&-5&-5\\-1&4&2\\3&-5&-3\end{bmatrix},\quad (4)\ \begin{bmatrix}0&-i&0\\i&1&i\\0&-i&0\end{bmatrix}.$$

7.5. Find the third column vector $\mathbf{v}$ so that $U=\begin{bmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{2}}&|\\\frac{1}{\sqrt{3}}&0&\mathbf{v}\\\frac{1}{\sqrt{3}}&-\frac{1}{\sqrt{2}}&|\end{bmatrix}$ is unitary. How much freedom is there in this choice?

7.6. Find a real matrix A such that $A+rI$ is invertible for all $r\in\mathbb{R}$. Does there exist a square matrix A such that $A+cI$ is invertible for all $c\in\mathbb{C}$?

7.7. Find a unitary matrix whose first row is

(1) $k(1,\ 1-i)$ where k is a number, (2) $\left(\frac{1}{2},\ \frac{i}{2},\ \frac{1-i}{2}\right)$.

7.8. Let $V=\mathbb{C}^2$ with the Euclidean inner product. Let T be the linear transformation on V with the matrix representation $A=\begin{bmatrix}1&i\\1&1\end{bmatrix}$ with respect to the standard basis. Show that T is normal and find a set of orthonormal eigenvectors of T.

7.9. Prove that the following matrices are unitarily similar:

$$\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix},\quad \begin{bmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{bmatrix},\ \text{where } \theta \text{ is a real number.}$$

7.10. For each of the following real symmetric matrices A, find a real orthogonal matrix Q such that $Q^T AQ$ is diagonal:

(1) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, (2) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

7.11. For each of the following Hermitian matrices A, find a unitary matrix U such that $U^H AU$ is diagonal.

(1) $\begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$, (2) $\begin{bmatrix} 1 & 2+3i \\ 2-3i & -1 \end{bmatrix}$, (3) $\begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$.

7.12. Find the diagonal matrices to which the following matrices are unitarily similar. Determine whether each of them is Hermitian, unitary or orthogonal.

(1) $\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$, (2) $\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$, (3) $\begin{bmatrix} 1 & i & 0 \\ -i & 1 & i \\ 0 & -i & 1 \end{bmatrix}$.

7.13. For a skew-Hermitian matrix A, show that

(1) $A - I$ is invertible, (2) e^A is unitary.

7.14. Let U be a unitary matrix. Prove that U and U^T have the same set of eigenvalues.

7.15. Verify that $A = \begin{bmatrix} 2 & i \\ i & 2 \end{bmatrix}$ is normal. Diagonalize A by a unitary matrix U.

7.16. Show that the non-symmetric real matrix

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 3 \\ -2 & -4 & -5 \end{bmatrix}$ can be diagonalized.

7.17. Suppose that A, B are diagonalizable $n \times n$ matrices. Prove that $AB = BA$ if and only if A and B can be diagonalized simultaneously by the same matrix Q, i.e., $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

7.18. Find the spectral decomposition of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

7.19. Let A and B be 2×2 symmetric matrices. Prove that A and B are similar if and only if $\det A = \det B$ and $\operatorname{tr}(A) = \operatorname{tr}(B)$.

7.20. Let A be a real symmetric $n \times n$ matrix and λ an eigenvalue of A with multiplicity m. Show that $\dim \mathcal{N}(A - \lambda I) = m$.

7.21. Show that a matrix A is nilpotent, i.e., $A^n = \mathbf{0}$ for some integer $n \geq 1$, if and only if its eigenvalues are all zero.

7.22. Determine whether the following statements are true or false, in general, and justify your answers.

(1) Every Hermitian matrix is unitarily similar to a diagonal matrix.

(2) An orthogonal matrix is always unitarily similar to a real diagonal matrix.

(3) For any square matrix A, AA^H and $A^H A$ have the same eigenvalues.

(4) If a triangular matrix is similar to a diagonal matrix, it is already diagonal.

(5) If all the columns of a square matrix A are orthonormal, then A is diagonalizable.

(6) Every permutation matrix is diagonalizable.

(7) Every permutation matrix is Hermitian.

(8) A nonzero nilpotent matrix cannot be Hermitian.

(9) Every square matrix is similar to a triangular matrix.

(10) If A is a Hermitian matrix, then $A + iI$ is invertible.

(11) If A is a real matrix, then $A + iI$ is invertible.

(12) If A is an orthogonal matrix, then $A + \frac{1}{2}I$ is invertible.

(13) Every unitarily diagonalizable matrix with real eigenvalues is Hermitian.

(14) Every diagonalizable matrix is normal.

(15) Every invertible matrix is similar to a unitary matrix.

8

Jordan Canonical Forms

8.1 Basic properties of Jordan canonical forms

Most problems related to a (complex) matrix A can be easily solved if the matrix is diagonalizable, as shown in previous chapters. For example, this is true in computing the power A^n, in solving a linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ or a linear differential equation $\mathbf{y}'(t) = A\mathbf{y}(t)$. In this chapter, we discuss how to solve the same problems for a non-diagonalizable matrix A by introducing the Jordan canonical form of a square matrix.

Recall that an $n \times n$ matrix A is diagonalizable if and only if A has a full set of n linearly independent eigenvectors, or equivalently, the dimension of each eigenspace $E(\lambda) = \mathcal{N}(\lambda I - A)$ is equal to the multiplicity of the eigenvalue λ. Hence, if $\lambda_1, \ldots, \lambda_t$ are distinct eigenvalues of A with multiplicities $m_{\lambda_1}, \ldots, m_{\lambda_t}$, respectively, then

$$\dim E(\lambda_1) + \cdots + \dim E(\lambda_t) = m_{\lambda_1} + \cdots + m_{\lambda_t} = n$$

and

$$\mathbb{C}^n = E(\lambda_1) \oplus \cdots \oplus E(\lambda_t).$$

On the other hand, a matrix A is *not* diagonalizable if and only if A has an eigenvalue λ with multiplicity $m_\lambda > 1$ such that

$$1 \le \dim E(\lambda) < m_\lambda,$$

so that the number of linearly independent eigenvectors belonging to λ must be less than m_λ.

However, even if a matrix A is not diagonalizable, one may try to find a matrix similar to A which has as many zero entries as possible except diagonals. Schur's lemma says that any square matrix is (unitarily) similar to an upper triangular matrix. But, it is a fact that any square matrix A can be similar to a matrix much "closer" to a diagonal matrix, called a *Jordan canonical form*. Its diagonal entries are the eigenvalues of A, the entry just above each diagonal entry is 0 or 1, and all other

J H Kwak et al., *Linear Algebra*

entries are 0. In this case, the columns of a basis-change matrix Q are something like eigenvectors, but not the same in general. They are called *generalized eigenvectors.*

Theorem 8.1 *For any square matrix A, A is similar to a matrix J of the following form, called the* **Jordan canonical form** *of A or a* **Jordan canonical matrix,**

$$J = Q^{-1}AQ = \begin{bmatrix} J_1 & & & \mathbf{0} \\ & J_2 & & \\ & & \ddots & \\ \mathbf{0} & & & J_s \end{bmatrix},$$

in which

(1) *s is the number of linearly independent eigenvectors of A, and*
(2) *each J_i is an upper triangular matrix of the form*

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda_i \end{bmatrix},$$

where λ_i is a single eigenvalue of J_i with only one linearly independent associated eigenvector. Such J_i is called a **Jordan block** *belonging to the eigenvalue λ_i.*

The proof of Theorem 8.1 may be beyond a beginning linear algebra course. Hence, we leave it to some advanced books and we are only concerned with how to find the Jordan canonical form J of A and a basis-change matrix Q in this book.

First, note that if an $n \times n$ matrix A has a full set of n linearly independent eigenvectors (that is $s = n$), then there have to be n Jordan blocks so that each Jordan block is just a 1×1 matrix, and an eigenvalue λ appears as many times as its multiplicity. In this case, the Jordan canonical form J of A is just the diagonal matrix with eigenvalues on the diagonal and a basis-change matrix Q is defined by taking the n linearly independent eigenvectors as its columns. Hence, a diagonal matrix is a particular case of the Jordan canonical form.

Remark: (1) For a given Jordan canonical form J of A, one can get another one by permuting the Jordan blocks of J, and this new one is also similar to A. It will be shown later that any two Jordan canonical forms of A cannot be similar except this possibility. In other words, any (complex) square matrix A is similar to only one Jordan canonical matrix up to permutations of the Jordan blocks. In this sense, it is called *the* Jordan canonical form of a matrix A.
(2) As an alternative way to define the Jordan canonical form of A, one can take the transpose J^T of the Jordan canonical matrix J given in Theorem 8.1. In this case, each Jordan block becomes a lower triangular matrix with a single eigenvalue. But this alternative definition does not induce any essential difference from the original one.

If J is the Jordan canonical form of a matrix A, then they have the same eigenvalues and the same number of linearly independent eigenvectors, but not the same set of them in general. (Note that $\mathbf{x}$ is an eigenvector of $J = Q^{-1}AQ$ if and only if $Q\mathbf{x}$ is an eigenvector of A).

Actually, for a given matrix A, its Jordan canonical form J is completely determined by the number s of linearly independent eigenvectors of A and their *ranks* (which will be defined in Section 8.2): each eigenvector corresponds to a Jordan block in J, and the *rank* of an eigenvector determines the size of the corresponding block.

The following example illustrates which matrices A have the Jordan canonical form J and how the s linearly independent eigenvectors of A (or J) correspond to the Jordan blocks in J.

Example 8.1 (*Each Jordan block to each eigenvector*) Let J be a Jordan canonical matrix of the form:

$$J = \begin{bmatrix} \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix} & & \\ & \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} & \\ & & [2] \end{bmatrix} = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix}.$$

Find the number of linearly independent eigenvectors of J and determine all matrices whose Jordan canonical forms are J.

Solution: Since J is a triangular matrix, the eigenvalues of J are the diagonal entries 6 and 2 with multiplicities 2 and 3, respectively. The eigenspace $E(6)$ has a single linearly independent eigenvector because $\dim E(6) = \dim \mathcal{N}(J-6I) = 5-\text{rank}(J-6I) = 1$, by the Rank Theorem 3.17. In fact, $\mathbf{e}_1 = (1, 0, 0, 0, 0)$ is such a vector and $\lambda = 6$ appears only in a single block J_1. Similarly, one can see that the eigenspace $E(2)$ has two linearly independent eigenvectors $\mathbf{e}_3$ and $\mathbf{e}_5$ with $\dim E(2) = 5 - \text{rank}(J - 2I) = 2$, and $\lambda = 2$ appears in two blocks J_2 and J_3.

Hence, one can conclude that if a matrix A is similar to J, then A is a 5×5 matrix whose eigenvalues are 6 and 2 with multiplicity 2 and 3 respectively, but there is only one linearly independent eigenvector belonging to 6, (i.e., $\dim E(6) = 1$) and there are only two linearly independent eigenvectors belonging to 2, (i.e., $\dim E(2) = 2$). Moreover, the converse is also true by Theorem 8.1. In general, one can say that *if a matrix A is similar to a Jordan canonical matrix J, then both matrices have the same eigenvalues of the same multiplicities and* $\dim \mathcal{N}(\lambda I - A) = \dim \mathcal{N}(\lambda I - J)$ *for each eigenvalue λ.* □

As shown in Example 8.1, the standard basis vectors $\mathbf{e}_j$'s associated with the first column vectors of the Jordan blocks J_i's of J form linearly independent eigenvectors of the matrix J, and then the vectors $Q\mathbf{e}_j$'s form linearly independent eigenvectors of the matrix A, where $Q^{-1}AQ = J$.

Problem 8.1 Note that the matrix

$$A = \begin{bmatrix} 6 & -1 & -1 & 0 & 1 \\ 0 & 6 & 4 & 1 & -4 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

has the eigenvalues 6 and 2 with multiplicities 2 and 3, respectively. Moreover, there are two linearly independent eigenvectors $\mathbf{u}_1 = (0, -1, 1, 0, 0)$ and $\mathbf{u}_2 = (0, 1, 0, 0, 1)$ belonging to $\lambda = 2$, and $\mathbf{v}_1 = (-1, 0, 0, 0, 0)$ is an eigenvector belonging to $\lambda = 6$. Show that the Jordan canonical form of A is the matrix J given in Example 8.1 by showing $Q^{-1}AQ = J$ with an invertible matrix

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 8.2 Show that for any Jordan block J, the order of J is the smallest positive integer k such that $(J - \lambda I)^k = \mathbf{0}$, where λ is an eigenvalue of J.

An eigenvalue λ may appear in several blocks. In fact, the number of Jordan blocks belonging to an eigenvalue λ is equal to the number of linearly independent eigenvectors of A belonging to λ, which is the dimension of the eigenspace $E(\lambda) = \mathcal{N}(\lambda I - A)$. Moreover, the sum of the orders of all Jordan blocks belonging to an eigenvalue λ is equal to the multiplicity m_λ of λ.

Next, one might ask how to determine the Jordan blocks belonging to a given eigenvalue λ. The following example shows all possible cases of Jordan blocks belonging to an eigenvalue λ when its multiplicity m_λ is fixed.

Example 8.2 (*Classifying Jordan canonical matrices having a single eigenvalue*) Classify all possible Jordan canonical forms of a 5×5 matrix A that has a single eigenvalue λ of multiplicity 5 (up to permutations of the Jordan blocks).

Solution: There are seven possible Jordan canonical forms as follows.

(1) Suppose A has only one linearly independent eigenvector belonging to λ. Then the Jordan canonical form of A is of the form

$$J^{(1)} = Q^{-1}AQ = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix},$$

which consists of only one Jordan block with eigenvalue λ on the diagonal. And, both A and $J^{(1)}$ have only one linearly independent eigenvector belonging to λ. (Note that $\operatorname{rank}(J^{(1)} - \lambda I) = 4$.)

(2) Suppose it has two linearly independent eigenvectors belonging to λ. Then the Jordan canonical form of A is either one of the forms

$$J^{(2)} = \begin{bmatrix} \lambda & 1 & & & \\ 0 & \lambda & & & \\ & & \lambda & 1 & 0 \\ & & 0 & \lambda & 1 \\ & & 0 & 0 & \lambda \end{bmatrix} \quad \text{or} \quad J^{(3)} = \begin{bmatrix} \lambda & & & & \\ & \lambda & 1 & 0 & 0 \\ & 0 & \lambda & 1 & 0 \\ & 0 & 0 & \lambda & 1 \\ & 0 & 0 & 0 & \lambda \end{bmatrix},$$

each of which consists of two Jordan blocks belonging to the eigenvalue λ. Note that $J^{(2)}$ has two linearly independent eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_3$, while $J^{(3)}$ has $\mathbf{e}_1$ and $\mathbf{e}_2$. These two matrices $J^{(2)}$ and $J^{(3)}$ cannot be similar, because $(J^{(2)} - \lambda I)^3 = \mathbf{0}$, but $(J^{(3)} - \lambda I)^3 \neq \mathbf{0}$. (One can justify it by a direct computation.)

(3) Suppose it has three linearly independent eigenvectors belonging to λ. Then the Jordan canonical form of A is either one of the forms

$$J^{(4)} = \begin{bmatrix} \lambda & & & & \\ & \lambda & 1 & & \\ & 0 & \lambda & & \\ & & & \lambda & 1 \\ & & & 0 & \lambda \end{bmatrix} \quad \text{or} \quad J^{(5)} = \begin{bmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda & 1 & 0 \\ & & 0 & \lambda & 1 \\ & & 0 & 0 & \lambda \end{bmatrix},$$

each of which consists of three Jordan blocks belonging to the eigenvalue λ. Note that $J^{(4)}$ has three linearly independent eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_4$, while $J^{(5)}$ has $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. These two matrices $J^{(4)}$ and $J^{(5)}$ are not similar, because $(J^{(4)} - \lambda I)^2 = \mathbf{0}$, but $(J^{(5)} - \lambda I)^2 \neq \mathbf{0}$.

(4) Suppose it has four linearly independent eigenvectors belonging to λ. Then the Jordan canonical form of A is of the form

$$J^{(6)} = \begin{bmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \lambda & 1 \\ & & & 0 & \lambda \end{bmatrix},$$

which consists of four Jordan blocks with eigenvalue λ.

(5) Suppose it has five linearly independent eigenvectors belonging to λ. Then the Jordan canonical form of A is the diagonal matrix $J^{(7)}$ with diagonal entries λ.

Note that all of these seven possible Jordan canonical matrices have the same trace, determinant and the same characteristic polynomial, but any two of them are not similar to each other. □

As shown in the case (2) (also in (3)) of Example 8.2, two Jordan canonical matrices $J^{(2)}$ and $J^{(3)}$ have the same eigenvalue of the same multiplicity and they also have the same number of linearly independent eigenvectors, but they are not similar. The problem of choosing one of the two possible Jordan canonical forms which is similar to the given matrix A depends on the sequence of $\text{rank}(A - \lambda I)^{\ell}$ for $\ell = 1, 2, \ldots, n$.

The next example illustrates how to determine the orders of the Jordan blocks belonging to the same eigenvalue λ.

Example 8.3 (*Determine Jordan blocks belonging to the same eigenvalue*) Let J be a Jordan canonical matrix with a single eigenvalue λ:

$$J = \begin{bmatrix} J_1 & & & \mathbf{0} \\ & J_2 & & \\ & & J_3 & \\ \mathbf{0} & & & J_4 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} & & & \\ & \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} & & \\ & & \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} & \\ & & & [\lambda] \end{bmatrix}.$$

Then,

$$(J - \lambda I)^2 = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & & & \\ & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & & \\ & & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \\ & & & [0] \end{bmatrix} \text{ and } (J - \lambda I)^3 = \mathbf{0}. \quad \square$$

Thus, one can get $\text{rank}(J - \lambda I) = 4$, $\text{rank}(J - \lambda I)^2 = 1$ and $\text{rank}(J - \lambda I)^3 = 0$. This sequence of ranks, $\text{rank}(J - \lambda I)^{\ell}$ for $\ell = 1, 2, 3$, determines completely the orders of blocks of J. In fact, one can notice that

(i) the fact that $(J - \lambda I)^3 = \mathbf{0}$ but $(J - \lambda I)^2 \neq \mathbf{0}$ implies that the largest block has order 3,

(ii) $\text{rank}(J - \lambda I)^2 = 1$ is equal to the number of blocks of order 3,

(iii) $\text{rank}(J - \lambda I) = 4$ is equal to twice the number of blocks of order 3 plus the number of blocks of order 2, so there are two of them,

(iv) the number of blocks of order 1 is $8 - (2 \times 2) - (3 \times 1) = 1$.

Recall that two similar matrices have the same rank. Hence, if J is the Jordan canonical form of A, then for any eigenvalue λ and for any positive integer ℓ, we have $\text{rank}(A - \lambda I)^{\ell} = \text{rank}\ (J - \lambda I)^{\ell}$. Furthermore, in a sequence of matrices $J - \lambda I$, $(J - \lambda I)^2$, $(J - \lambda I)^3$, $\ldots$, all Jordan blocks belonging to the eigenvalue λ will terminate to a zero matrix but all other blocks (belonging to an eigenvalue different from λ) remain as upper triangular matrices with nonzero diagonal entries. Hence, the sequence of $\text{rank}(J - \lambda I)^k$ must stop decreasing at $n - m_\lambda$. (Note: $\text{rank}(J - \lambda I)^k = n - m_\lambda$, when $k = m_\lambda$.)

Let $c_\lambda = n - m_\lambda$ for simplicity. Then, the decreasing sequence

$$\{\mathrm{rank}(A - \lambda I)^k - c_\lambda \ : \ k = 1, \ldots, m_\lambda\}$$

determines completely the orders of the blocks of J belonging to λ as shown in Example 8.3:

(i) The order of the largest block belonging to λ is the smallest positive integer k such that $\mathrm{rank}(A - \lambda I)^k - c_\lambda = 0$. And the number of such largest blocks is equal to $\mathrm{rank}(A - \lambda I)^{k-1} - c_\lambda$, (say $= \ell_1$).

(ii) The number of blocks of order $k - 1$ is equal to $\mathrm{rank}(A - \lambda I)^{k-2} - c_\lambda - 2\ell_1$, (say $= \ell_2$).

(iii) The number of blocks of order $k - 2$ is equal to $\mathrm{rank}(A - \lambda I)^{k-3} - c_\lambda - 3\ell_1 - 2\ell_2$, (say $= \ell_3$), and so on.

In general, if $\ell_1, \ell_2, \ldots, \ell_j$ are given, one can determine ℓ_{j+1} inductively as follows:

(iv) The number of blocks of order $k - j$ is equal to $\mathrm{rank}(A - \lambda I)^{k-(j+1)} - c_\lambda - (j+1)\ell_1 - j\ell_2 - \cdots - 2\ell_j$, (say $= \ell_{j+1}$) with $\ell_0 = 0$ for $j = 0, \ldots, k - 1$.

In summary, one can determine the Jordan canonical form J of an $n \times n$ matrix A by the following procedure.

Step 1 Find all distinct eigenvalues $\lambda_1, \ \ldots, \ \lambda_t$ of A. Let their multiplicities be $m_{\lambda_1}, \ \ldots, \ m_{\lambda_t}$, respectively, so that $m_{\lambda_1} + \cdots + m_{\lambda_\ell} = n$, and let $c_{\lambda_i} = n - m_{\lambda_i}$.

Step 2 For each eigenvalue λ, the Jordan blocks belonging to the eigenvalue λ are determined by the following criteria:
(i) The order of the largest block belonging to λ is the smallest positive integer k such that $\mathrm{rank}(A - \lambda I)^k - c_\lambda = 0$, and
(ii) the number of blocks of order $k - j$ is inductively determined as $\mathrm{rank}(A - \lambda I)^{k-(j+1)} - c_\lambda - (j+1)\ell_1 - j\ell_2 - \cdots - 2\ell_j$, (say $= \ell_{j+1}$) with $\ell_0 = 0$ for $j = 0, \ldots, k - 1$.

This is a general guide to determine the Jordan canonical form of a matrix. However, for a matrix of large order, the evaluation of $\mathrm{rank}(A - \lambda I)^k$ might not be easy at all, while, for matrices of lower order or relatively simple matrices, the computations may be accessible.

Example 8.4 (*Jordan canonical form of a triangular matrix*) Find the Jordan canonical form J of the matrix

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution: Since A is triangular, the eigenvalues of A are the diagonal entries $\lambda_1 = \lambda_2 = 2, \ \lambda_3 = 3$. Hence, there are two possibilities of the Jordan canonical form of A:

$$J^{(1)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{or} \quad J^{(2)} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

But, one can see that $\text{rank}(A - 2I) = 2$. It implies that $\text{rank}(J - 2I) = 2$ and so the Jordan canonical form of A must be $J^{(2)}$. □

Example 8.5 (*Jordan canonical form of a companion matrix*) Find the Jordan canonical form J of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix}.$$

Solution: The characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = (\lambda - 1)^4.$$

The eigenvalue of A is $\lambda = 1$ of multiplicity 4. Note that the rank of the matrix

$$A - I = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 4 & -6 & 3 \end{bmatrix}$$

is 3; by noting that the first three rows are linearly independent, or the determinant of the 3×3 principal submatrix of the upper left part is not zero. Hence, the rank of $J - I$ is also 3 and so $J - I$ must have three 1's beyond diagonal entries. It means that

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Also, one can check the following equations:

$$(A - I)^3 = \begin{bmatrix} -1 & 3 & -3 & 1 \\ -1 & 3 & -3 & 1 \\ -1 & 3 & -3 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \neq \mathbf{0}, \quad \text{but} \quad (A - I)^4 = \mathbf{0},$$

which says that the order of the largest Jordan block is 4. □

Problem 8.3 Let A be a 5×5 matrix with two distinct eigenvalues: λ of multiplicity 3 and μ of multiplicity 2. Find all possible Jordan canonical forms of A up to permutations of the Jordan blocks.

Problem 8.4 Find the Jordan canonical form for each of the following matrices:

$$(1) \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}, \quad (2) \begin{bmatrix} 4 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad (3) \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

8.2 Generalized eigenvectors

In Section 8.1, assuming Theorem 8.1, we have shown how to determine the Jordan canonical form J of a matrix A. In this section, we discuss how to determine a basis-change matrix Q so that $Q^{-1}AQ = J$ is the Jordan canonical form of A. In fact, if J is given, then a basis-change matrix Q is a nonsingular solution of a matrix equation $AQ = QJ$.

The following example illustrates how to determine a basis-change matrix Q when a matrix A and its Jordan canonical form J are given.

Example 8.6 (*Each Jordan block to each chain of generalized eigenvectors*) Let A be a 5×5 matrix similar to a Jordan block of the form

$$Q^{-1}AQ = J = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Determine a basis-change matrix $Q = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4\ \mathbf{x}_5]$.

Solution: Clearly, two similar matrices A and J have the same eigenvalue λ of multiplicity 5. Since $\operatorname{rank}(J - \lambda I) = 4$, $\dim \mathcal{N}(J - \lambda I) = \dim E(\lambda) = 1$. In fact, J has only one linearly independent eigenvector, which is $\mathbf{e}_1 = (1, 0, 0, 0, 0)$. Thus $Q\mathbf{e}_1 = \mathbf{x}_1$ is a linearly independent eigenvector of A. Also, note that the smallest positive integer k such that $(A - \lambda I)^k = (J - \lambda I)^k = \mathbf{0}$ is 5 which is the order of the block J.

To see what the other columns of Q are, we expand $AQ = QJ$ as

$$[A\mathbf{x}_1\ A\mathbf{x}_2\ A\mathbf{x}_3\ A\mathbf{x}_4\ A\mathbf{x}_5] = [\lambda\mathbf{x}_1\ \ \mathbf{x}_1 + \lambda\mathbf{x}_2\ \ \mathbf{x}_2 + \lambda\mathbf{x}_3\ \ \mathbf{x}_3 + \lambda\mathbf{x}_4\ \ \mathbf{x}_4 + \lambda\mathbf{x}_5].$$

By comparing the column vectors, we have

$$\begin{array}{lll} A\mathbf{x}_5 = \mathbf{x}_4 + \lambda\mathbf{x}_5, & \text{or} & (A - \lambda I)\mathbf{x}_5 = (A - \lambda I)^1\mathbf{x}_5 = \mathbf{x}_4, \\ A\mathbf{x}_4 = \mathbf{x}_3 + \lambda\mathbf{x}_4, & \text{or} & (A - \lambda I)\mathbf{x}_4 = (A - \lambda I)^2\mathbf{x}_5 = \mathbf{x}_3, \\ A\mathbf{x}_3 = \mathbf{x}_2 + \lambda\mathbf{x}_3, & \text{or} & (A - \lambda I)\mathbf{x}_3 = (A - \lambda I)^3\mathbf{x}_5 = \mathbf{x}_2, \\ A\mathbf{x}_2 = \mathbf{x}_1 + \lambda\mathbf{x}_2, & \text{or} & (A - \lambda I)\mathbf{x}_2 = (A - \lambda I)^4\mathbf{x}_5 = \mathbf{x}_1, \\ A\mathbf{x}_1 = \lambda\mathbf{x}_1, & \text{or} & (A - \lambda I)\mathbf{x}_1 = (A - \lambda I)^5\mathbf{x}_5 = \mathbf{0}. \end{array}$$

Note that the vector $\mathbf{x}_5$ satisfies $(A-\lambda I)^5\mathbf{x}_5 = \mathbf{0}$ but $(A-\lambda I)^4\mathbf{x}_5 = \mathbf{x}_1 \neq \mathbf{0}$. However, $(A-\lambda I)^5 = (J-\lambda I)^5 = \mathbf{0}$. Hence, if one gets $\mathbf{x}_5$ as a solution of $(A-\lambda I)^4\mathbf{x} \neq \mathbf{0}$, then all other $\mathbf{x}_i$'s can be obtained by $\mathbf{x}_4 = (A-\lambda I)\mathbf{x}_5$, $\mathbf{x}_3 = (A-\lambda I)\mathbf{x}_4$, $\mathbf{x}_2 = (A-\lambda I)\mathbf{x}_3$, etc. Such a vector $\mathbf{x}_5$ is called a *generalized eigenvector* of rank 5, and the (ordered) set $\{\mathbf{x}_1, \ldots, \mathbf{x}_5\}$ is called a *chain of generalized eigenvectors* belonging to λ. Therefore, the columns of the basis-change matrix Q form a chain of generalized eigenvectors. □

In general, by expanding $AQ = QJ$, one can see that the columns of Q corresponding to the first columns of Jordan blocks of J form a maximal set of linearly independent eigenvectors of A, and remaining columns of Q are generalized eigenvectors.

Definition 8.1 A nonzero vector $\mathbf{x}$ is said to be a **generalized eigenvector** of A of **rank** k belonging to an eigenvalue λ if

$$(A-\lambda I)^k\mathbf{x} = \mathbf{0} \quad \text{and} \quad (A-\lambda I)^{k-1}\mathbf{x} \neq \mathbf{0}.$$

Note that if $k = 1$, this is the usual definition of an eigenvector. For a generalized eigenvector $\mathbf{x}$ of rank $k \geq 1$ belonging to an eigenvalue λ, define

$$\boxed{\begin{array}{lllll}
\mathbf{x}_k & = & \mathbf{x}, & & \\
\mathbf{x}_{k-1} & = & (A-\lambda I)\mathbf{x}_k & = & (A-\lambda I)\mathbf{x}, \\
\mathbf{x}_{k-2} & = & (A-\lambda I)\mathbf{x}_{k-1} & = & (A-\lambda I)^2\mathbf{x}, \\
\vdots & & & & \\
\mathbf{x}_2 & = & (A-\lambda I)\mathbf{x}_3 & = & (A-\lambda I)^{k-2}\mathbf{x}, \\
\mathbf{x}_1 & = & (A-\lambda I)\mathbf{x}_2 & = & (A-\lambda I)^{k-1}\mathbf{x}.
\end{array}}$$

Thus, for each ℓ, $1 < \ell \leq k$, we have $(A-\lambda I)^{\ell}\mathbf{x}_\ell = (A-\lambda I)^k\mathbf{x} = \mathbf{0}$ and $(A-\lambda I)^{\ell-1}\mathbf{x}_\ell = \mathbf{x}_1 \neq \mathbf{0}$. Note also that $(A-\lambda I)^{\ell}\mathbf{x}_i = \mathbf{0}$ for $\ell \geq i$. Hence, the vector $\mathbf{x}_\ell = (A-\lambda I)^{k-\ell}\mathbf{x}$ is a generalized eigenvector of A of rank ℓ. See Figure 8.1.

Definition 8.2 The set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is called a **chain of generalized eigenvectors** belonging to the eigenvalue λ. The eigenvector $\mathbf{x}_1$ is called the **initial eigenvector** of the chain.

The following successive three theorems show that a basis-change matrix Q can be constructed from the chains of generalized eigenvectors initiated from s linearly independent eigenvectors of A, and also justify the invertibility of Q. (A reader may omit reading their proofs below if not interested in details.)

Theorem 8.2 *A chain of generalized eigenvectors* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ *belonging to an eigenvalue* λ *is linearly independent.*

Proof: Let $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$ with constants c_i, $i = 1, \ldots, k$. If we multiply (on the left) both sides of this equation by $(A-\lambda I)^{k-1}$, then for $i = 1, \ldots, k-1$,

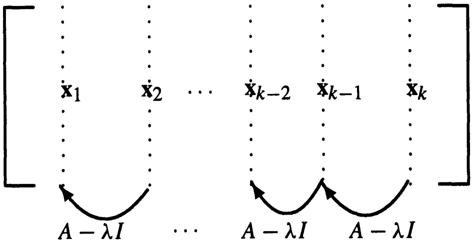

Figure 8.1. A chain of generalized eigenvectors

$$(A-\lambda I)^{k-1}\mathbf{x}_i = (A-\lambda I)^{k-(i+1)}(A-\lambda I)^i\mathbf{x}_i = \mathbf{0}.$$

Thus, $c_k(A-\lambda I)^{k-1}\mathbf{x}_k = \mathbf{0}$, and, hence, $c_k = 0$.

Do the same to the equation $c_1\mathbf{x}_1 + \cdots + c_{k-1}\mathbf{x}_{k-1} = \mathbf{0}$ with $(A-\lambda I)^{k-2}$ and get $c_{k-1} = 0$. Proceeding successively, one can show that $c_i = 0$ for all $i = 1, \ldots, k$. □

Theorem 8.3 *The union of chains of generalized eigenvectors of a square matrix A belonging to distinct eigenvalues is linearly independent.*

Proof: For brevity, we prove this theorem for only two distinct eigenvalues. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_\ell\}$ be the chains of generalized eigenvectors of A belonging to the eigenvalues λ and μ, respectively, and let $\lambda \neq \mu$. In order to show that the set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}_1, \ldots, \mathbf{y}_\ell\}$ is linearly independent, let

$$c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k + d_1\mathbf{y}_1 + \cdots + d_\ell\mathbf{y}_\ell = \mathbf{0}$$

with constants c_i's and d_j's. We multiply both sides of the equation by $(A-\lambda I)^k$ and note that $(A-\lambda I)^k\mathbf{x}_i = \mathbf{0}$ for all $i = 1, \ldots, k$. Thus we have

$$(A-\lambda I)^k(d_1\mathbf{y}_1 + d_2\mathbf{y}_2 + \cdots + d_\ell\mathbf{y}_\ell) = \mathbf{0}.$$

Again, multiply this equation by $(A-\mu I)^{\ell-1}$ and note that

$$\begin{aligned} (A-\mu I)^{\ell-1}(A-\lambda I)^k &= (A-\lambda I)^k(A-\mu I)^{\ell-1}, \\ (A-\mu I)^{\ell-1}\mathbf{y}_\ell &= \mathbf{y}_1, \\ (A-\mu I)^{\ell-1}\mathbf{y}_i &= \mathbf{0} \end{aligned}$$

for $i = 1, \ldots, \ell-1$. Thus we obtain

$$\mathbf{0} = d_\ell(A-\lambda I)^k\mathbf{y}_1.$$

Because $(A-\mu I)\mathbf{y}_1 = \mathbf{0}$ (or $A\mathbf{y}_1 = \mu\mathbf{y}_1$), this reduces to

$$d_\ell(\mu-\lambda)^k\mathbf{y}_1 = \mathbf{0},$$

which implies that $d_\ell = 0$ by the assumption $\lambda \neq \mu$ and $\mathbf{y}_1 \neq \mathbf{0}$. Proceeding successively, one can show that $d_i = 0,\ i = \ell, \ell - 1, \ldots, 2, 1$, so we are left with

$$c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k = \mathbf{0}.$$

Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is already linearly independent by Theorem 8.2, $c_i = 0$ for all $i = 1, \ldots, k$. Thus the set of generalized eigenvectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}_1, \ldots, \mathbf{y}_\ell\}$ is linearly independent. □

The next step to determine Q such that $AQ = QJ$ is how to choose chains of generalized eigenvectors from a generalized eigenspace, which is defined below, so that the union of the chains is linearly independent.

Definition 8.3 Let λ be an eigenvalue of A. The **generalized eigenspace** of A belonging to λ, denoted by K_λ, is the set

$$K_\lambda = \{\mathbf{x} \in \mathbb{C}^n \ :\ (A - \lambda I)^p\mathbf{x} = \mathbf{0} \quad \text{for some positive integer } p\}.$$

It turns out that dim K_λ is the multiplicity of λ, and it contains the usual eigenspace $\mathcal{N}(A - \lambda I)$. The following theorem enables us to choose a basis for K_λ, but we omit the proof even though it can be proved by induction on the number of vectors in $S \cup T$.

Theorem 8.4 *Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ and $T = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_\ell\}$ be two chains of generalized eigenvectors of A belonging to the same eigenvalue λ. If the initial vectors $\mathbf{x}_1$ and $\mathbf{y}_1$ are linearly independent, then the union $S \cup T$ is also linearly independent.*

Note that Theorem 8.4 extends easily to a finite number of chains of generalized eigenvectors of A belonging to an eigenvalue λ, and the union of such chains will form a basis for K_λ so that the matrix Q may be constructed from these bases for each eigenvalue as usual.

Example 8.7 (*Basis-change matrix for a triangular matrix*) For a matrix

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix},$$

find a basis-change matrix Q so that $Q^{-1}AQ$ is the Jordan canonical matrix.

Solution: *Method 1*: In Example 8.4, the Jordan canonical form J of A is determined as

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

By comparing the column vectors of $AQ = QJ$ with $Q = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$, one can get

$$A\mathbf{x}_1 = 2\mathbf{x}_1,\ \ A\mathbf{x}_2 = 2\mathbf{x}_2 + \mathbf{x}_1,\ \ A\mathbf{x}_3 = 3\mathbf{x}_3.$$

Since $\mathbf{x}_1$ and $\mathbf{x}_3$ are eigenvectors of A belonging to $\lambda = 2$ and $\lambda = 3$, one can take $\mathbf{x}_1 = (1, 0, 0)$ and $\mathbf{x}_3 = (3, -1, 1)$. Also, from the equation $A\mathbf{x}_2 = 2\mathbf{x}_2 + \mathbf{x}_1$, one can conclude $\mathbf{x}_2 = (a, 1, 0)$ with any constant a, so that

$$Q = \begin{bmatrix} 1 & a & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

One may check directly the equality $AQ = QJ$ by a direct computation.

Method 2: This is a direct method to compute Q without using the Jordan canonical form J. Clearly, the eigenvalues of A are $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 3$. Since $\text{rank}(A - \lambda_1 I) = 2$, the dimension of the eigenspace $\mathcal{N}(A - \lambda_1 I)$ is 1. Thus there is only one linearly independent eigenvector belonging to $\lambda_1 = \lambda_2 = 2$, and an eigenvector belonging to $\lambda_3 = 3$ is found to be $\mathbf{x}_3 = (3, -1, 1)$. We need to find a generalized eigenvector $\mathbf{x}_2$ of rank 2 belonging to $\lambda = 2$, which is a solution of the following systems:

$$\begin{aligned} (A - 2I)\mathbf{x} &= \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \neq \mathbf{0}, \\ (A - 2I)^2\mathbf{x} &= \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}. \end{aligned}$$

From the second equation, $\mathbf{x}_2$ has to be of the form $(a, b, 0)$, and from the first equation we must have $b \neq 0$. Let us take $\mathbf{x}_2 = (0, 1, 0)$ as a generalized eigenvector of rank 2. Then, we have $\mathbf{x}_1 = (A - 2I)\mathbf{x}_2 = (1, 0, 0)$. Now, one can set

$$Q = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The reader may check by a direct computation

$$Q^{-1}AQ = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} \quad \text{with } Q^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

where $J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $J_2 = [3]$. □

Example 8.8 (*Basis-change matrix for a companion matrix*) Find a basis-change matrix Q so that $Q^{-1}AQ = J$ is the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix}.$$

Solution: *Method 1*: In Example 8.5, we computed

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now, one can find a basis-change matrix $Q = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4]$ by comparing the column vectors of $AQ = QJ$:

$$A\mathbf{x}_1 = \mathbf{x}_1,\ A\mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1,\ A\mathbf{x}_3 = \mathbf{x}_3 + \mathbf{x}_2,\ A\mathbf{x}_4 = \mathbf{x}_4 + \mathbf{x}_3.$$

By computing an eigenvector of A belonging to $\lambda = 1$, one can get $\mathbf{x}_1 = (1,\ 1,\ 1,\ 1)$. The equation $A\mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1$ gives $\mathbf{x}_2 = (a,\ a+1,\ a+2,\ a+3)$ for any a. Take $\mathbf{x}_2 = (0,\ 1,\ 2,\ 3)$. Similarly, from equations $A\mathbf{x}_3 = \mathbf{x}_3 + \mathbf{x}_2$ and $A\mathbf{x}_4 = \mathbf{x}_4 + \mathbf{x}_3$, one can get $\mathbf{x}_3 = (b,\ b,\ b+1,\ b+3)$ for any b and set $\mathbf{x}_3 = (0,\ 0,\ 1,\ 3)$, and successively one can take $\mathbf{x}_4 = (0,\ 0,\ 0,\ 1)$. We conclude that

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

One may check $AQ = QJ$ by a direct matrix multiplication.

Method 2: The characteristic polynomial of the matrix A is

$$\det(A - \lambda I) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = (\lambda - 1)^4.$$

The only eigenvalue of A is $\lambda = 1$ of multiplicity 4. Note that $\dim \mathcal{N}(A - I) = 1$ because the rank of the matrix

$$A - I = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 4 & -6 & 3 \end{bmatrix}$$

is 3. Thus, a basis-change matrix $Q = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4]$ consists of a chain of generalized eigenvectors. First, we find a generalized eigenvector $\mathbf{x}_4$ of rank 4, which is a solution $\mathbf{x}$ of the following equations:

$$\begin{aligned} (A - I)^3\mathbf{x} &= \begin{bmatrix} -1 & 3 & -3 & 1 \\ -1 & 3 & -3 & 1 \\ -1 & 3 & -3 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \mathbf{x} \neq \mathbf{0}, \\ (A - I)^4\mathbf{x} &= \mathbf{0}. \end{aligned}$$

But, a direct computation shows that the matrix $(A - I)^4 = \mathbf{0}$. Hence, one can take any vector that satisfies the first equation as a generalized eigenvector of rank 4: Take $\mathbf{x}_4 = (-1,\ 0,\ 0,\ 0)$. Then,

$$
\begin{aligned}
\mathbf{x}_3 &= (A-I)\mathbf{x}_4 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 4 & -6 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\
\mathbf{x}_2 &= (A-I)\mathbf{x}_3 = (-1,\ 0,\ 1,\ 2), \\
\mathbf{x}_1 &= (A-I)\mathbf{x}_2 = (1,\ 1,\ 1,\ 1).
\end{aligned}
$$

Now, one can set

$$
Q = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}.
$$

Then,

$$
Q^{-1}AQ = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = J \quad \text{with } Q^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix}. \qquad \square
$$

Remark: In Examples 8.7–8.8, we use two different methods to determine a basis-change matrix. In Method 1, we first find an initial eigenvector $\mathbf{x}_1$ in order to get a chain $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ of generalized eigenvectors belonging to λ. After that, we find $\mathbf{x}_2$ as a solution of $(A-\lambda I)\mathbf{x}_2 = \mathbf{x}_1$, and $\mathbf{x}_3$ as a solution of $(A-\lambda I)\mathbf{x}_3 = \mathbf{x}_2$, and so on. In this method, we don't need to compute the power matrix $(A-\lambda I)^2$ and $(A-\lambda I)^3$, etc. But, this method may not work sometimes, when the matrix $A-\lambda I$ is not invertible. See the next Example 8.9. In Method 2, we first find a generalized eigenvector $\mathbf{x}_k$ of rank k as a solution of $(A-\lambda I)^k\mathbf{x} = \mathbf{0}$ but $(A-\lambda I)^{k-1}\mathbf{x} \neq \mathbf{0}$. With this $\mathbf{x}_k$, one can get $\mathbf{x}_{k-1}$ as $\mathbf{x}_{k-1} = (A-\lambda I)\mathbf{x}_k$, and $\mathbf{x}_{k-2} = (A-\lambda I)\mathbf{x}_{k-1}$, and so on. This method works always, but we need to compute a power $(A-\lambda I)^k$.

The next example shows that a chain of generalized eigenvectors may sometimes not be obtained from an initial eigenvector of the chain.

Example 8.9 For a matrix

$$
A = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix},
$$

find a basis-change matrix Q so that $Q^{-1}AQ$ is the Jordan canonical matrix.

Solution: *Method 1*: The eigenvalue of A is $\lambda = 1$ of multiplicity 3, and the rank of the matrix

$$
A - I = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}
$$

is 1. (Note that the second and the third rows are scalar multiples of the first row.) Hence, there are two linearly independent eigenvectors belonging to $\lambda = 1$, and the Jordan canonical form J of A is determined as

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, one may find a basis-change matrix $Q = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$ by comparing the column vectors of $AQ = QJ$:

$$A\mathbf{x}_1 = \mathbf{x}_1, \quad A\mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1, \quad A\mathbf{x}_3 = \mathbf{x}_3.$$

By computing an eigenvector of A belonging to $\lambda = 1$, one may get two linearly independent eigenvectors: take $\mathbf{u}_1 = (1,\ 0,\ 2)$ and $\mathbf{u}_2 = (0,\ 2,\ -3)$. If we take the eigenvector $\mathbf{x}_1$ as $\mathbf{u}_1$ or $\mathbf{u}_2$, then a generalized eigenvector $\mathbf{x}_2$ must be a solution of

$$(A - I)\mathbf{x} = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{u}_1 \text{ or } \mathbf{u}_2.$$

But this system is inconsistent and one cannot get a generalized eigenvector $\mathbf{x}_2$ in this way. It means that we are supposed to take an eigenvector $\mathbf{x}_1 \in E(1)$ carefully in order to get $\mathbf{x}_2$ as a solution of $(A - I)\mathbf{x} = \mathbf{x}_1$.

Method 2: First, note that A has an eigenvalue $\lambda = 1$ of multiplicity 3 and there are two linearly independent eigenvectors belonging to $\lambda = 1$. Hence, we need to find a generalized eigenvector of rank 2, which is a solution $\mathbf{x}$ of the following equations:

$$\begin{aligned} (A - I)\mathbf{x} &= \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \mathbf{0}, \\ (A - I)^2\mathbf{x} &= \mathbf{0}. \end{aligned}$$

But, a direct computation shows that the matrix $(A - I)^2 = \mathbf{0}$. Hence, one can take any vector that satisfies the first equation as a generalized eigenvector of rank 2: take $\mathbf{x}_2 = (0,\ 0,\ -1)$. Then,

$$\mathbf{x}_1 = (A - I)\mathbf{x}_2 = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}.$$

Now, by taking another eigenvector $\mathbf{x}_3 = (1,\ 0,\ 2)$, so that $\mathbf{x}_1$ and $\mathbf{x}_3$ are linearly independent, one can get

$$Q = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 0 \\ -2 & -1 & 2 \end{bmatrix}.$$

One may check by a direct computation

$$Q^{-1}AQ = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = J \quad \text{with } Q^{-1} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 2 & -\frac{3}{2} & -1 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix}. \qquad \square$$

Problem 8.5 (From Problem 8.4) Find a full set of generalized eigenvectors of the following matrices:

$$(1) \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}, \quad (2) \begin{bmatrix} 4 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad (3) \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

8.3 The power A^k and the exponential e^A

In this section, we discuss how to compute the power A^k and the exponential matrix e^A, which are necessary for solving linear difference or differential equations as mentioned in Sections 6.3 and 6.5. It can be dealt with in two ways. Firstly, it can be done by computing the power J^k and the exponential matrix e^J for the Jordan canonical form J of A, as shown in this section. The second method is based on the Cayley–Hamilton theorem (or the minimal polynomial) and it will be discussed later in Sections 8.6.1–8.6.2.

Let J be the Jordan canonical form of a square matrix A and let

$$Q^{-1}AQ = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$$

with a basis-change matrix Q. Since

$$A^k = QJ^kQ^{-1} = Q \begin{bmatrix} J_1^k & & \\ & \ddots & \\ & & J_s^k \end{bmatrix} Q^{-1}$$

for $k = 1, 2, \ldots,$ it is enough to compute J^k for a single Jordan block J. Now an $n \times n$ Jordan block J belonging to an eigenvalue λ of A can be written as

$$J = \begin{bmatrix} \lambda & 1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$= \lambda I + N.$$

Since I is the identity matrix, clearly $IN = NI$ and

$$J^k = (\lambda I + N)^k = \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} N^j.$$

Note that $N^k = \mathbf{0}$ for $k \geq n$. Thus, by assuming $\binom{k}{\ell} = 0$ if $k < \ell$,

$$\begin{aligned}
J^k &= \sum_{j=0}^{n-1} \binom{k}{j} \lambda^{k-j} N^j \\
&= \lambda^k I + \binom{k}{1} \lambda^{k-1} N + \cdots + \binom{k}{n-1} \lambda^{k-(n-1)} N^{n-1} \\
&= \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{n-1}\lambda^{k-n+1} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{n-2}\lambda^{k-n+2} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & \cdots & \cdots & 0 & \lambda^k \end{bmatrix}.
\end{aligned}$$

Next, to compute the exponential matrix e^A, we first note that

$$e^A = e^{QJQ^{-1}} = Qe^J Q^{-1} = Q \begin{bmatrix} e^{J_1} & & & \mathbf{0} \\ & e^{J_2} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{J_s} \end{bmatrix} Q^{-1}$$

Thus, it is enough to compute e^J for a single Jordan block J. Let $J = \lambda I + N$, as before. Then, $N^k = \mathbf{0}$ for $k \geq n$ and

$$e^J = e^{\lambda I} e^N = e^{\lambda} \sum_{k=0}^{n-1} \frac{N^k}{k!} = e^{\lambda} \begin{bmatrix} 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(n-2)!} & \frac{1}{(n-1)!} \\ 0 & 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(n-2)!} \\ \vdots & \ddots & 1 & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 & \frac{1}{2!} \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Example 8.10 (*Computing A^k and e^A by using the Jordan canonical form*) Compute the power A^k and the exponential matrix e^A by using the Jordan canonical form of A for

$$(1)\ A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \qquad (2)\ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix}.$$

Solution: (1) From Examples 8.4–8.7, one can see that

$$A = QJQ^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$J^k = \begin{bmatrix} 2^k & \binom{k}{1}2^{k-1} & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \quad \text{and} \quad e^J = \begin{bmatrix} e^2 & e^2 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix}.$$

Hence,

$$\begin{aligned} A^k = QJ^kQ^{-1} &= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^k & \binom{k}{1}2^{k-1} & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^k & k2^{k-1} & -3\cdot 2^k + k2^{k-1} + 3\cdot 3^k \\ 0 & 2^k & 2^k - 3^k \\ 0 & 0 & 3^k \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} e^A = Qe^JQ^{-1} &= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^2 & e^2 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^2 & e^2 & -2e^2 + 3e^3 \\ 0 & e^2 & e^2 - e^3 \\ 0 & 0 & e^3 \end{bmatrix}. \end{aligned}$$

(2) Do the same process as (1). From Examples 8.5–8.8,

$$A = QJQ^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix},$$

and

$$J^k = \begin{bmatrix} 1 & \binom{k}{1} & \binom{k}{2} & \binom{k}{3} \\ 0 & 1 & \binom{k}{1} & \binom{k}{2} \\ 0 & 0 & 1 & \binom{k}{1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad e^J = e\begin{bmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 1 & 1 & \frac{1}{2!} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now, one can compute $A^k = QJ^kQ^{-1}$ and $e^A = Qe^JQ^{-1}$. For example,

$$\begin{aligned} e^A &= e\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 1 & 1 & \frac{1}{2!} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \\ &= e\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{6} \\ -\frac{1}{6} & 1 & -\frac{1}{2} & \frac{2}{3} \\ -\frac{2}{3} & \frac{5}{2} & -3 & \frac{13}{6} \\ -\frac{13}{6} & 8 & -\frac{21}{2} & \frac{17}{3} \end{bmatrix}. \end{aligned}$$

□

Example 8.11 (*Computing A^k and e^A by using the Jordan canonical form*) Compute the power A^k and the exponential matrix e^A by using the Jordan canonical form of A for

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}.$$

Solution: (1) The characteristic polynomial of A is $\det(\lambda I - A) = \lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 16 = (\lambda - 2)^4$, and $\lambda = 2$ is an eigenvalue of multiplicity 4. Since

$$\operatorname{rank}(A - 2I) = \operatorname{rank}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} = 2$$

and

$$\operatorname{rank}(A - 2I)^2 = \operatorname{rank}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = 1,$$

the Jordan canonical form J of A must be of the form

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

and a basis-change matrix Q such that $Q^{-1}AQ = J$ is

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad \text{and then} \quad Q^{-1} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & -2 \end{bmatrix}.$$

Therefore,

$$A^k = QJ^kQ^{-1} = Q \begin{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}^k & \mathbf{0} \\ \mathbf{0} & [\,2\,]^k \end{bmatrix} Q^{-1},$$

and

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}^k = \begin{bmatrix} 2^k & \binom{k}{1}2^{k-1} & \binom{k}{2}2^{k-2} \\ 0 & 2^k & \binom{k}{1}2^{k-1} \\ 0 & 0 & 2^k \end{bmatrix}.$$

Hence,

$$\begin{aligned} A^k &= QJ^kQ^{-1} = Q \begin{bmatrix} \begin{bmatrix} 2^k & \binom{k}{1}2^{k-1} & \binom{k}{2}2^{k-2} \\ 0 & 2^k & \binom{k}{1}2^{k-1} \\ 0 & 0 & 2^k \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & 2^k \end{bmatrix} Q^{-1} \\ &= \begin{bmatrix} 2^k - k2^{k-1} & k2^{k-1} & \frac{k(k-1)}{2}2^{k-2} + k2^{k-1} & k2^{k-1} \\ 0 & 2^k & 2k2^{k-1} & 0 \\ 0 & 0 & 2^k & 0 \\ -k2^{k-1} & k2^{k-1} & \frac{k(k-1)}{2}2^{k-2} & k2^{k-1} + 2^k \end{bmatrix}. \end{aligned}$$

(2) With the same notation,

$$e^A = e^{QJQ^{-1}} = Qe^JQ^{-1} = Q \begin{bmatrix} e^{J_1} & \mathbf{0} \\ \mathbf{0} & e^{J_2} \end{bmatrix} Q^{-1},$$

where

$$J_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = 2I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 2I + N \quad \text{and} \quad J_2 = [2].$$

Hence,

$$e^{J_1} = e^{2I}e^N = e^2\sum_{k=0}^{2}\frac{N^k}{k!} = e^2\begin{bmatrix} 1 & 1 & \frac{1}{2!} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus we have

$$e^A = Qe^JQ^{-1} = e^2Q\begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}Q^{-1} = \begin{bmatrix} 0 & e^2 & \frac{3}{2}e^2 & e^2 \\ 0 & e^2 & 2e^2 & 0 \\ 0 & 0 & e^2 & 0 \\ -e^2 & e^2 & \frac{1}{2}e^2 & 2e^2 \end{bmatrix}. \qquad \square$$

Problem 8.6 Compute A^k and e^A by using the Jordan canonical form for

$$(1)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix}.$$

8.4 Cayley–Hamilton theorem

As we saw in earlier chapters, the association of the characteristic polynomial with a matrix is very useful in studying matrices. In this section, using this association of polynomials with matrices, we prove one more useful theorem, called the *Cayley–Hamilton theorem*, which makes the calculation of matrix polynomials simple, and has many applications to real problems.

Let $f(x) = a_mx^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$ be a polynomial, and let A be an $n \times n$ square matrix. The matrix defined by

$$f(A) = a_mA^m + a_{m-1}A^{m-1} + \cdots + a_1A + a_0I_n$$

is called a **matrix polynomial** of A.

For example, if $f(x) = x^2 - 2x + 2$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then

$$\begin{aligned} f(A) &= A^2 - 2A + 2I_2 \\ &= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}. \end{aligned}$$

Problem 8.7 Let λ be an eigenvalue of a matrix A. For any polynomial $f(x)$, show that $f(\lambda)$ is an eigenvalue of the matrix polynomial $f(A)$.

Theorem 8.5 (Cayley–Hamilton) *For any $n \times n$ matrix A, if $f(\lambda) = \det(\lambda I - A)$ is the characteristic polynomial of A, then $f(A) = \mathbf{0}$.*

Proof: If A is diagonal, its proof is an easy exercise. For an arbitrary square matrix A, let its Jordan canonical form be

$$J = \begin{bmatrix} J_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & J_s \end{bmatrix} = Q^{-1}AQ,$$

so that $f(A) = Qf(J)Q^{-1}$. Since

$$J^k = \begin{bmatrix} J_1^k & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & J_s^k \end{bmatrix} \quad \text{and} \quad f(J) = \begin{bmatrix} f(J_1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & f(J_s) \end{bmatrix},$$

it is sufficient to show that $f(J_i) = \mathbf{0}$ for each Jordan block J_i. Let $J_i = \lambda_0 I + N$ with eigenvalue λ_0 of multiplicity m, in which $N^m = \mathbf{0}$. Since $f(\lambda) = \det(\lambda I - A) = \det(\lambda I - J) = (\lambda - \lambda_0)^m g(\lambda)$ for some $g(\lambda)$, we have

$$f(J_i) = (J_i - \lambda_0 I)^m g(J_i) = (\lambda_0 I + N - \lambda_0 I)^m g(J_i) = N^m g(J_i) = \mathbf{0}\, g(J_i) = \mathbf{0}.$$

□

Remark: For a Jordan block

$$J = \begin{bmatrix} \lambda_0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda_0 \end{bmatrix}$$

with a single eigenvalue λ_0 of multiplicity m, we have

$$J^2 = \begin{bmatrix} \lambda_0^2 & 2\lambda_0 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 2\lambda_0 \\ \mathbf{0} & & & \lambda_0^2 \end{bmatrix} \quad \text{and } J^k = \begin{bmatrix} \lambda_0^k & \binom{k}{1}\lambda_0^{k-1} & \cdots & \binom{k}{m-1}\lambda_0^{k-m+1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \binom{k}{1}\lambda_0^{k-1} \\ \mathbf{0} & & & \lambda_0^k \end{bmatrix}.$$

Hence, for any polynomial $p(\lambda)$,

$$p(J) = \begin{bmatrix} p(\lambda_0) & \partial_\lambda p(\lambda_0) & \cdots & \frac{\partial_\lambda^{(m-1)} p(\lambda_0)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \partial_\lambda p(\lambda_0) \\ \mathbf{0} & & & p(\lambda_0) \end{bmatrix},$$

where ∂_λ denotes the derivative with respect to λ. In particular, for the characteristic polynomial $f(\lambda) = \det(\lambda I - J) = (\lambda - \lambda_0)^m$, we have $f(\lambda_0) = \partial_\lambda f(\lambda_0) = \cdots = \partial_\lambda^{(m-1)} f(\lambda_0) = 0$ and hence $f(J) = \mathbf{0}$.

Example 8.12 The characteristic polynomial of

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 0 & 2 & 0 \\ -3 & -12 & -6 \end{bmatrix}$$

is $f(\lambda) = \det(\lambda I - A) = \lambda^3 + \lambda^2 - 6\lambda$, and

$$\begin{aligned} f(A) &= A^3 + A^2 - 6A \\ &= \begin{bmatrix} 27 & 78 & 54 \\ 0 & 8 & 0 \\ -27 & -102 & -54 \end{bmatrix} + \begin{bmatrix} -9 & -42 & -18 \\ 0 & 4 & 0 \\ 9 & 30 & 18 \end{bmatrix} \\ &\quad - 6\begin{bmatrix} 3 & 6 & 6 \\ 0 & 2 & 0 \\ -3 & -12 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$ □

Problem 8.8 Let us prove the Cayley–Hamilton Theorem 8.5 as follows: by setting $\lambda = A$, $f(A) = \det(AI - A) = \det \mathbf{0} = 0$. Is it correct or not? If not, what is a wrong step?

Problem 8.9 Prove the Cayley–Hamilton theorem for a diagonal matrix A by computing $f(A)$ directly. By using this, do the same for a diagonalizable matrix A.

The Cayley–Hamilton theorem can be used to find the inverse of a nonsingular matrix. If $f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ is the characteristic polynomial of a matrix A, then

$$\begin{aligned} \mathbf{0} = f(A) &= A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I, \\ \text{or} \qquad -a_0 I &= (A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I)A. \end{aligned}$$

Since $a_0 = f(0) = \det(0I - A) = \det(-A) = (-1)^n \det A$, A is nonsingular if and only if $a_0 = (-1)^n \det A \neq 0$. Therefore, if A is nonsingular,

$$A^{-1} = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I).$$

Example 8.13 (*Compute A^{-1} by the Cayley–Hamilton theorem*) The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

is $f(\lambda) = \det(\lambda I_3 - A) = \lambda^3 - 8\lambda^2 + 17\lambda - 10$, and the Cayley–Hamilton theorem yields

$$A^3 - 8A^2 + 17A = 10I_3.$$

Hence

$$\begin{aligned} A^{-1} &= \frac{1}{10}(A^2 - 8A + 17I_3) \\ &= \frac{1}{10}\begin{bmatrix} 10 & 6 & -6 \\ -39 & 7 & 18 \\ -30 & 12 & 13 \end{bmatrix} - \frac{8}{10}\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} + \frac{17}{10}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} -5 & -10 & 10 \\ 1 & 0 & 2 \\ -14 & -20 & 22 \end{bmatrix}. \end{aligned}$$

□

Problem 8.10 Let A and B be square matrices, not necessarily of the same order, and let $f(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of A. Show that $f(B)$ is invertible if and only if A has no eigenvalue in common with B.

The Cayley–Hamilton theorem can also be used to simplify the calculation of matrix polynomials. Let $p(\lambda)$ be any polynomial and let $f(\lambda)$ be the characteristic polynomial of a square matrix A. A theorem of algebra tells us that there are polynomials $q(\lambda)$ and $r(\lambda)$ such that

$$p(\lambda) = q(\lambda)f(\lambda) + r(\lambda),$$

where the degree of $r(\lambda)$ is less than the degree of $f(\lambda)$. Then

$$p(A) = q(A)f(A) + r(A).$$

By the Cayley–Hamilton theorem, $f(A) = \mathbf{0}$ and

$$p(A) = r(A).$$

Thus, the problem of evaluating a polynomial of an $n \times n$ matrix A or in particular a power A^k can be reduced to the problem of evaluating a matrix polynomial of degree less than n.

Example 8.14 The characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is $f(\lambda) = \lambda^2 - 2\lambda - 3$. Let $p(\lambda) = \lambda^4 - 7\lambda^3 - 3\lambda^2 + \lambda + 4$ be a polynomial. A division by $f(\lambda)$ gives that

$$p(\lambda) = (\lambda^2 - 5\lambda - 10)f(\lambda) - 34\lambda - 26.$$

Therefore

$$\begin{aligned} p(A) &= (A^2 - 5A - 10)f(A) - 34A - 26I \\ &= -34A - 26I \\ &= -34\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 26\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -60 & -68 \\ -68 & -60 \end{bmatrix}. \end{aligned}$$

□

Example 8.15 (*Computing A^k by the Cayley–Hamilton theorem*) Compute the power A^{10} by using the Cayley–Hamilton theorem for

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}.$$

Solution: The characteristic polynomial of A is $f(\lambda) = \det(\lambda I - A) = \lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 16 = (\lambda - 2)^4$, see Example 8.11. A division by $f(\lambda)$ gives

$$\lambda^{10} = q(\lambda)f(\lambda) + r(\lambda)$$

with a quotient polynomial $q(\lambda) = \lambda^6 + 8\lambda^5 + 40\lambda^4 + 160\lambda^3 + 560\lambda^2 + 1792\lambda + 5376$ and a remainder polynomial $r(\lambda) = 15360\lambda^3 - 80640\lambda^2 + 143360\lambda - 86016$. Hence,

$$\begin{aligned} A^{10} &= r(A) = 15360A^3 - 80640A^2 + 143360A - 86016I \\ &= \begin{bmatrix} -4096 & 5120 & 16640 & 5120 \\ 0 & 1024 & 10240 & 0 \\ 0 & 0 & 1024 & 0 \\ -5120 & 5120 & 11520 & 6144 \end{bmatrix}. \end{aligned}$$

(Compare this result with A^k given in Example 8.11). One may notice that this computational method for A^n will become increasingly complicated if n becomes bigger and bigger. A simpler method will be shown later in Example 8.22. □

Problem 8.11 For the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, (1) evaluate the power matrix A^{10} and the inverse A^{-1}; (2) evaluate the matrix polynomial $A^5 + 3A^4 + A^3 - A^2 + 4A + 6I$.

8.5 The minimal polynomial of a matrix

Let A be a square matrix of order n and let

$$f(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = \prod_{i=1}^{t}(\lambda - \lambda_i)^{m_{\lambda_i}}$$

be the characteristic polynomial of A, where m_{λ_i} is the multiplicity of the eigenvalue λ_i. Then, the Cayley–Hamilton theorem says that

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = \prod_{i=1}^{t}(A - \lambda_i I)^{m_{\lambda_i}} = \mathbf{0}.$$

The **minimal polynomial** of a matrix A is the monic polynomial $m(\lambda) = \sum_{i=0}^{m} c_i\lambda^i$ of smallest degree m such that

$$m(A) = \sum_{i=0}^{m} c_i A^i = \mathbf{0}.$$

Clearly, the minimal polynomial $m(\lambda)$ divides any polynomial $p(\lambda)$ satisfying $p(A) = \mathbf{0}$. In fact, if $p(\lambda) = q(\lambda)m(\lambda) + r(\lambda)$ as on page 297, where the degree of $r(\lambda)$ is less than the degree of $m(\lambda)$, then $\mathbf{0} = p(A) = q(A)m(A) + r(A) = r(A)$ and the minimality of $m(\lambda)$ implies that $r(\lambda) = 0$. In particular, the minimal polynomial $m(\lambda)$ divides the characteristic polynomial so that

$$m(\lambda) = \prod_{i=1}^{t}(\lambda - \lambda_i)^{k_i}$$

with $k_i \le m_i$. For example, the characteristic polynomial of the $n \times n$ zero matrix is λ^n and its minimal polynomial is just λ.

Clearly, any two similar matrices have the same minimal polynomial because

$$\prod_{i=1}^{t}\left(Q^{-1}AQ - \lambda_i I\right)^{k_i} = Q^{-1}\left(\prod_{i=1}^{t}(A - \lambda_i I)^{k_i}\right)Q$$

for any invertible matrix Q.

Example 8.16 For a diagonal matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix},$$

its characteristic polynomial is $f(\lambda) = (\lambda-2)^3(\lambda-5)^2$. However, for the polynomial $m(\lambda) = (\lambda - 2)(\lambda - 5)$, the matrix $m(A)$ is

$$m(A) = (A - 2I)(A - 5I) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Hence, the minimal polynomial of A is $m(\lambda) = (\lambda - 2)(\lambda - 5)$. □

Example 8.17 (*The minimal polynomial of a diagonal matrix*)

(1) Any diagonal matrix of the form $\lambda_0 I$ has the minimal polynomial $\lambda - \lambda_0$. In particular, the minimal polynomial of the zero matrix is the monomial λ, and the minimal polynomial of the identity matrix is the monomial $\lambda - 1$.

(2) If an $n \times n$ matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then its minimal polynomial coincides with the characteristic polynomial $f(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$. In fact, for the diagonal matrix D having distinct diagonal entries $\lambda_1, \ldots, \lambda_n$ successively and for any given j, the (j, j)-entry of $\prod_{i\neq j}(D - \lambda_i I)$ is equal to $\prod_{i\neq j}(\lambda_j - \lambda_i)$, which is not zero. □

Example 8.18 (*The minimal polynomial of a Jordan canonical matrix J having one or two blocks*)

(1) For any 5×5 matrix A similar to a Jordan block of the form

$$Q^{-1}AQ = J = \begin{bmatrix} \lambda_0 & 1 & 0 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & 0 & \lambda_0 \end{bmatrix},$$

its minimal polynomial is equal to the characteristic polynomial $f(\lambda) = (\lambda - \lambda_0)^5$, because $(J - \lambda_0 I)^4 \neq \mathbf{0}$ but $(J - \lambda_0 I)^5 = \mathbf{0}$.

(2) For a matrix J having two Jordan blocks belonging to a single eigenvalue λ, say

$$J = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & \lambda_0 \end{bmatrix} & \\ & \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix} \end{bmatrix},$$

the minimal polynomial of the smaller block J_2 is a divisor of the minimal polynomial of the larger block J_1. In general, *if a matrix A or its Jordan canonical form J has a single eigenvalue λ_0, then its minimal polynomial is $(\lambda - \lambda_0)^k$, where k is the smallest positive integer ℓ such that $(A - \lambda I)^\ell = \mathbf{0}$*. In fact, such number k is known as the order of the largest Jordan block belonging to λ.

(3) For a matrix J having two Jordan blocks belonging to two different eigenvalues $\lambda_0 \neq \lambda_1$ respectively, say

$$J = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \lambda_0 & 1 & 0 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & 0 & \lambda_0 \end{bmatrix} & \\ & \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \end{bmatrix},$$

the minimal polynomial of J is a product of the minimal polynomials of J_1 and J_2, which is $(\lambda - \lambda_0)^5(\lambda - \lambda_1)^3$.

In general, for a Jordan canonical matrix J, let J_λ denote the direct sum of Jordan blocks belonging to the eigenvalue λ. Then, $J = \bigoplus_{i=1}^{t} J_{\lambda_i}$, where t is the number of the distinct eigenvalues of J. In this case, *the minimal polynomial of J is the product of the minimal polynomials of the summands J_{λ_i}'s.* □

By a method similar to Example 8.18 (2) and (3) with Step 2(i) on page 279, one can have the following theorem.

Theorem 8.6 *For any $n \times n$ matrix A or its Jordan canonical matrix J, its minimal polynomial is $\prod_{i=1}^{t}(\lambda - \lambda_i)^{k_i}$, where $\lambda_1, \ldots, \lambda_t$ are the distinct eigenvalues of A and k_i is the order of the largest Jordan block in J belonging to the eigenvalue λ_i. Or equivalently, k_i is the smallest positive integer ℓ such that* $\operatorname{rank}(A - \lambda_i I)^\ell + m_{\lambda_i} = n$, *where m_{λ_i} is the multiplicity of the eigenvalue λ_i.*

Corollary 8.7 *A matrix A is diagonalizable if and only if its minimal polynomial is equal to $\prod_{i=1}^{t}(\lambda - \lambda_i)$, where $\lambda_1, \ldots, \lambda_t$ are the distinct eigenvalues of A.*

Example 8.19 (*Computing the minimal polynomial*) Compute the minimal polynomial of A for

$$(1)\ A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \quad (2)\ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix}.$$

Solution: (1) Since A is triangular, its eigenvalues are $\lambda_1 = \lambda_2 = 2,\ \lambda_3 = 3$. But, $\operatorname{rank}(A - 2I) = 2$ and so A is not diagonalizable. Hence, its minimal polynomial is $m(A) = (\lambda - 2)^2(\lambda - 3)$.

(2) Recalling Example 8.5, we know that the eigenvalue of A is $\lambda = 1$ of multiplicity 4, and $\operatorname{rank}(A - I) = 3$. It implies that the Jordan canonical form of A is

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the minimal polynomial of A is $m(A) = (\lambda - 1)^4$, by Theorem 8.6. □

Example 8.20 Compute the minimal polynomial of A for

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}.$$

Solution: In Example 8.11, we show that the characteristic polynomial of A is $f(\lambda) = (\lambda - 2)^4$, $\operatorname{rank}(A - 2I) = 2$, $\operatorname{rank}(A - 2I)^2 = 1$ and the Jordan canonical form J of A is

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

So, the minimal polynomial of A is $m(A) = (\lambda - 2)^3$, by Theorem 8.6. □

Problem 8.12 In Example 8.2, we have seen that there are seven possible (nonsimilar) Jordan canonical matrices of order 5 that have a single eigenvalue. Compute the minimal polynomial of each of them.

8.6 Applications

8.6.1 The power matrix A^k again

We already know how to compute a power matrix A^k in two ways: One is by using the Jordan canonical form J of A with a basis-change matrix Q as shown in Section 8.3; and the other is by using the Cayley–Hamilton theorem with the characteristic polynomial of A as shown in Section 8.4. In this section, we introduce the third method by using the minimal polynomial, as a possibly simpler method.

The following example demonstrates how to compute a power A^k and A^{-1} for a diagonalizable matrix A by using the minimal polynomial instead of the characteristic polynomial and also without using the Jordan canonical form J of A.

Example 8.21 (*Computing A^k by the minimal polynomial when A is diagonalizable*) Compute the power A^k and A^{-1} by using the minimal polynomial for a symmetric matrix

$$A = \begin{bmatrix} 4 & 0 & 1 & -1 \\ 0 & 4 & 1 & 1 \\ 1 & 1 & 5 & 0 \\ -1 & 1 & 0 & 5 \end{bmatrix}.$$

Solution: Its characteristic polynomial is $f(\lambda) = (\lambda - 3)^2(\lambda - 6)^2$. Since A is symmetric and so diagonalizable, its minimal polynomial is $m(\lambda) = (\lambda - 3)(\lambda - 6)$, or equivalently $m(A) = A^2 - 9A + 18I = \mathbf{0}$. Hence,

$$A^{-1} = -\frac{1}{18}(A - 9I) = -\frac{1}{18}\begin{bmatrix} -5 & 0 & 1 & -1 \\ 0 & -5 & 1 & 1 \\ 1 & 1 & -4 & 0 \\ -1 & 1 & 0 & -4 \end{bmatrix}.$$

To compute A^k for any natural number k, first note that the power A^k with $k \geq 2$ can be written as a linear combination of I and A, because $A^2 = 9A - 18I$. (See page 297.) Hence, one can write

$$A^k = x_0 I + x_1 A$$

with unknown coefficients x_i's. Now, by multiplying an eigenvector $\mathbf{x}$ of A belonging to each eigenvalue λ in this equation, that is, by computing $A^k\mathbf{x} = (x_0 I + x_1 A)\mathbf{x}$, we have a system of equations

$$\begin{array}{lrcrcl} \text{as } \lambda = 3, & 3^k & = & x_0 & + & 3x_1, \\ \text{as } \lambda = 6, & 6^k & = & x_0 & + & 6x_1. \end{array}$$

Its solution is $x_0 = 2 \cdot 3^k - 6^k$ and $x_1 = \frac{1}{3}(6^k - 3^k)$. Hence,

$$A^k = x_0 I + x_1 A = \frac{1}{3}\begin{bmatrix} 2 \cdot 3^k + 6^k & 0 & 6^k - 3^k & -6^k + 3^k \\ 0 & 2 \cdot 3^k + 6^k & 6^k - 3^k & 6^k - 3^k \\ 6^k - 3^k & 6^k - 3^k & 3^k + 2 \cdot 6^k & 0 \\ -6^k + 3^k & 6^k - 3^k & 0 & 3^k + 2 \cdot 6^k \end{bmatrix}. \qquad \square$$

Now, we discuss how to compute a power A^k in two separate cases depending on the diagonalizability of A.

(1) Firstly, suppose that A is diagonalizable. Then its minimal polynomial is equal to $m(\lambda) = \prod_{i=1}^{t}(\lambda - \lambda_i)$, where $\lambda_1, \ldots, \lambda_t$ are the distinct eigenvalues of A. Now, by using $m(A) = \prod_{i=1}^{t}(A - \lambda_i I) = \mathbf{0}$ as in Example 8.21, one can write

$$A^k = x_0 I + x_1 A + \cdots + x_{t-1}A^{t-1}$$

with unknowns x_i's. Now, by multiplying an eigenvector $\mathbf{x}_i$ of A belonging to each eigenvalue λ_i in this equation, one can have a system of equations

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{t-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{t-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_t & \cdots & \lambda_t^{t-1} \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} {\lambda_1}^k \\ {\lambda_2}^k \\ \vdots \\ {\lambda_t}^k \end{bmatrix}.$$

Its coefficient matrix $[\lambda_i^j]$ is an invertible Vandermonde matrix of order k because the eigenvalues λ_i's are all distinct. Hence, the system is consistent and its unique solution x_i's determines the power A^k completely.

(2) Secondly, let A be any $n \times n$, may not be diagonalizable. By Theorem 8.6, the minimal polynomial of A is $m(\lambda) = \prod_{i=1}^t (\lambda - \lambda_i)^{k_i}$, where $\lambda_1, \ldots, \lambda_t$ are the distinct eigenvalues of A and k_i is the smallest positive integer ℓ such that rank$(A - \lambda_i I)^\ell + m_{\lambda_i} = n$. Let $s = \sum_{i=1}^t k_i$ be the degree of the minimal polynomial $m(\lambda)$ and let

$$A^k = x_0 I + x_1 A + \cdots + x_{s-1} A^{s-1}$$

with unknown x_i's as before. Now, by multiplying an eigenvector $\mathbf{x}$ of A belonging to each eigenvalue λ_i in this equation, we have

$$\lambda_i^k = x_0 + \lambda_i x_1 + \cdots + \lambda_i^{s-1} x_{s-1}.$$

For the eigenvalue λ_i, let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k_i}\}$ be a chain of generalized eigenvectors belonging to λ_i. Then these vectors satisfy

$$\begin{array}{rcl} A\mathbf{x}_1 &=& \lambda_i \mathbf{x}_1, \\ A\mathbf{x}_2 &=& \lambda_i \mathbf{x}_2 + \mathbf{x}_1, \\ &\vdots& \\ A\mathbf{x}_{k_i} &=& \lambda_i \mathbf{x}_{k_i} + \mathbf{x}_{k_i-1}. \end{array}$$

By using the first two equations repeatedly, one can get

$$A^k \mathbf{x}_2 = \lambda_i^k \mathbf{x}_2 + k\lambda_i^{k-1} \mathbf{x}_1.$$

On the other hand,

$$\begin{array}{rcl} A^k \mathbf{x}_2 &=& \left(x_0 I + x_1 A + \cdots + x_{s-1} A^{s-1}\right) \mathbf{x}_2 \\ &=& x_0 \mathbf{x}_2 + x_1(\lambda_i \mathbf{x}_2 + \mathbf{x}_1) + \cdots + x_{s-1}\left(\lambda_i^{s-1}\mathbf{x}_2 + (s-1)\lambda_i^{s-2}\mathbf{x}_1\right) \\ &=& \left(x_0 + \lambda_i x_1 + \cdots + \lambda_i^{s-1} x_{s-1}\right)\mathbf{x}_2 \\ && \quad + \left(x_1 + 2\lambda_i x_2 + \cdots + (s-1)\lambda_i^{s-2} x_{s-1}\right)\mathbf{x}_1. \end{array}$$

In these two equations of $A^k \mathbf{x}_2$, since the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, their coefficients must be the same. It means that

$$\begin{array}{rcccccccccc} \lambda_i{}^k &=& x_0 &+& \lambda_i x_1 &+& \lambda_i^2 x_2 &+& \cdots &+& \lambda_i^{s-1} x_{s-1}, \\ k\lambda_i{}^{k-1} &=& && x_1 &+& 2\lambda_i x_2 &+& \cdots &+& (s-1)\lambda_i^{s-2} x_{s-1}. \end{array}$$

Here, the second equation is the derivative of the first equation with respect to λ_i.

Similarly, one can write $A^k \mathbf{x}_{k_i}$ as a linear combination of linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k_i}$ in two different ways and then a comparison of their coefficients gives the following k_i equations:

$$\begin{array}{rcllll}
\lambda_i{}^k & = & x_0 + & \lambda_i x_1 & + \cdots + & \lambda_i^{s-1} x_{s-1}, \\
\binom{k}{1}\lambda_i{}^{k-1} & = & \binom{1}{1}x_1 + & \binom{2}{1}\lambda_i x_2 & + \cdots + & \binom{s-1}{1}\lambda_i^{s-2} x_{s-1}, \\
\binom{k}{2}\lambda_i{}^{k-2} & = & \binom{2}{2}x_2 + & \binom{3}{2}\lambda_i x_3 & + \cdots + & \binom{s-1}{2}\lambda_i^{s-3} x_{s-1}, \\
 & \vdots & & & & \\
\binom{k}{k_i-1}\lambda_i{}^{k-k_i+1} & = & \binom{k_i-1}{k_i-1}x_{k_i-1} & + \binom{k_i}{k_i-1}\lambda_i x_{k_i} & + \cdots + & \binom{s-1}{k_i-1}\lambda_i^{s-k_i} x_{s-1}.
\end{array}$$

Note that the last $k_i - 1$ equations are equivalent to the consecutive derivatives of the first equation with respect to λ_i. For example, the two times of the third equation is just the second derivative of the first equation, and the $(k_i - 1)!$ times of the last equation is the $(k_i - 1)$-th derivative of the first equation. Getting together all of such equations for each eigenvalue λ_i, $i = 1, \ldots, s$, one can get a system of s equations with s unknowns x_j's with an invertible coefficient matrix. Therefore, the unknowns $x_0, \ldots, x_{s-1}$ can be uniquely determined and so can the power A^k.

Example 8.22 (*Computing A^k by the minimal polynomial when A is not diagonalizable*) (Example 8.11 again) Compute the power A^k and A^{-1} by using the minimal polynomial for

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}.$$

Solution: In Example 8.20, the minimal polynomial of A is determined as $m(\lambda) = (\lambda - 2)^3$. Hence, one can write

$$A^k = x_0 I + x_1 A + x_2 A^2$$

with unknown coefficients x_i's and

$$A^2 = \begin{bmatrix} 0 & 4 & 5 & 4 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 4 & 0 \\ -4 & 4 & 1 & 8 \end{bmatrix}.$$

Now, at the eigenvalue $\lambda = 2$, one can have

$$\begin{array}{llcllll}
\text{as } \lambda = 2, & 2^k & = & x_0 & + \ 2x_1 & + & 2^2 x_2, \\
\text{take } \partial_\lambda, & k2^{k-1} & = & & \quad x_1 & + & 2 \cdot 2x_2, \\
\text{take } \partial_\lambda^2, & k(k-1)2^{k-2} & = & & & & 2x_2.
\end{array}$$

Its solution is

$$x_2 = k(k-1)2^{k-3}; \; x_1 = k(2-k)2^{k-1} \text{ and } x_0 = 2^k\left(\frac{1}{2}k^2 - \frac{3}{2}k + 1\right).$$

Thus, one can have (the same power as in Example 8.11)

$$A^k = x_0 I + x_1 A + x_2 A^2$$
$$= \begin{bmatrix} 2^k - k2^{k-1} & k2^{k-1} & 2^{k-3}(3k+k^2) & k2^{k-1} \\ 0 & 2^k & k2^k & 0 \\ 0 & 0 & 2^k & 0 \\ -k2^{k-1} & k2^{k-1} & k(k-1)2^{k-3} & k2^{k-1}+2^k \end{bmatrix}.$$

To find A^{-1}, first note that $m(A) = A^3 - 6A^2 + 12A - 8I = \mathbf{0}$. Hence,

$$A^{-1} = \frac{1}{8}\left(A^2 - 6A + 12I\right) = \frac{1}{8}\begin{bmatrix} 6 & -2 & -1 & -2 \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 4 & 0 \\ 2 & -2 & 1 & 2 \end{bmatrix}. \qquad \square$$

Problem 8.13 (Problem 8.11 again) For the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, evaluate the power matrix A^{10} and the inverse A^{-1}.

Problem 8.14 (Problem 8.6 again) Compute A^k by using the minimal polynomial for

$$(1)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix}.$$

8.6.2 The exponential matrix e^A again

As a continuation of computing the exponential matrix e^A in Section 8.3 in which we use the Jordan canonical form J of A and a basis-change matrix Q, we introduce another method with its minimal polynomial. This method is quite similar to computing a power A^k discussed in Section 8.6.1 (see also Example 8.22). It means that we don't need the Jordan canonical form J of A and a basis-change matrix Q. Because of a similarity to computing A^k, we state only the difference in computing the exponential matrix e^A in this section.

We also discuss how to compute e^A in two cases depending on the diagonalizability of A.

(1) First, let A be diagonalizable. Then its minimal polynomial is equal to $m(\lambda) = \prod_{i=1}^{t}(\lambda - \lambda_i)$, where $\lambda_1, \ldots, \lambda_t$ are the distinct eigenvalues of A. Now, as before, one can set

$$e^A = x_0 I + x_1 A + \cdots + x_{t-1} A^{t-1}$$

with unknowns x_i's and by multiplying an eigenvector $\mathbf{x}$ of A belonging to each eigenvalue λ_i in this equation, one can get

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{t-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{t-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_t & \cdots & \lambda_t^{t-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} \\ e^{\lambda_2} \\ \vdots \\ e^{\lambda_t} \end{bmatrix}.$$

The coefficient matrix is invertible and the unique solution x_i's determines the matrix e^A.

(2) Next, let A be any $n \times n$, may not be diagonalizable. Then, the minimal polynomial of A is $m(\lambda) = \prod_{i=1}^{t}(\lambda - \lambda_i)^{k_i}$, where $\lambda_1, \ldots, \lambda_t$ are the distinct eigenvalues of A and k_i is the smallest positive integer ℓ such that $\operatorname{rank}(A - \lambda_i I)^{\ell} + m_{\lambda_i} = n$. Let $s = \sum_{i=1}^{t} k_i$ be the degree of $m(\lambda)$ and let

$$e^A = x_0 I + x_1 A + \cdots + x_{s-1} A^{s-1}$$

with unknown x_i's as before.

For each eigenvalue λ_i and a chain of generalized eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k_i}\}$ belonging to λ_i, a parallel procedure in computing $e^A\mathbf{x}_{k_i}$ to that for $A^k\mathbf{x}_{k_i}$ gives the following k_i equations:

$$\begin{array}{rcl} e^{\lambda_i} & = & x_0 + \lambda_i x_1 + \cdots + \lambda_i^{s-1} x_{s-1}, \\ \frac{1}{1!}e^{\lambda_i} & = & \binom{1}{1}x_1 + \binom{2}{1}\lambda_i x_2 + \cdots + \binom{s-1}{1}\lambda_i^{s-2} x_{s-1}, \\ \frac{1}{2!}e^{\lambda_i} & = & \binom{2}{2}x_2 + \binom{3}{2}\lambda_i x_3 + \cdots + \binom{s-1}{2}\lambda_i^{s-3} x_{s-1}, \\ & \vdots & \\ \frac{1}{(k_i-1)!}e^{\lambda_i} & = & \binom{k_i-1}{k_i-1}x_{k_i-1} + \binom{k_i}{k_i-1}\lambda_i x_{k_i} + \cdots + \binom{s-1}{k_i-1}\lambda_i^{s-k_i} x_{s-1}. \end{array}$$

Note that the last $k_i - 1$ equations are equivalent to the consecutive derivatives of the first equation with respect to λ_i. For example, the 2! times of the third equation is just the second derivative of the first equation, and the $(k_i - 1)!$ times of the last equation is the $(k_i - 1)$-th derivative of the first equation. Getting together all of such equations for each eigenvalue λ_i, $i = 1, \ldots, t$, one can get a system of s equations with s unknowns x_j's with an invertible coefficient matrix. Therefore, the unknowns $x_0, \ldots, x_{s-1}$ can be uniquely determined and so can the exponential e^A.

Example 8.23 (*Computing e^A by the minimal polynomial when A is diagonalizable*) Compute the exponential matrix e^A by using the minimal polynomial for

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}.$$

Solution: First, recall that the matrix A is the coefficient matrix of the system of linear differential equations given in Example 6.16:

$$\begin{cases} y_1' = 5y_1 - 4y_2 + 4y_3 \\ y_2' = 12y_1 - 11y_2 + 12y_3 \\ y_3' = 4y_1 - 4y_2 + 5y_3. \end{cases}$$

It was known that A is diagonalizable with the eigenvalues $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = -3$. Hence, the minimal polynomial of A is $m(\lambda) = (\lambda - 1)(\lambda + 3)$, and one can write $e^A = x_0 I + x_1 A$ with unknowns x_i's. Then

$$\begin{array}{rrcrcr} \text{as } \lambda = 1, & e & = & x_0 & + & x_1, \\ \text{as } \lambda = -3, & e^{-3} & = & x_0 & - & 3x_1. \end{array}$$

By solving it, one can get $x_0 = \frac{1}{4}(3e + e^{-3})$, $x_1 = \frac{1}{4}(e - e^{-3})$ and

$$e^A = x_0 I + x_1 A = \begin{bmatrix} 2e - e^{-3} & -e + e^{-3} & e - e^{-3} \\ 3e - 3e^{-3} & -2e + 3e^{-3} & 3e - 3e^{-3} \\ e - e^{-3} & -e + e^{-3} & 2e - e^{-3} \end{bmatrix}. \qquad \square$$

Example 8.24 (Example 8.11 again) (*Computing e^A by the minimal polynomial when A is not diagonalizable*) Compute the exponential matrix e^A by using the minimal polynomial for

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}.$$

Solution: In Example 8.20, the minimal polynomial of A is computed as $m(\lambda) = (\lambda - 2)^3$. Hence, one can set as in Example 8.22,

$$e^A = x_0 I + x_1 A + x_2 A^2$$

with unknown coefficients x_i's and

$$A^2 = \begin{bmatrix} 0 & 4 & 5 & 4 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 4 & 0 \\ -4 & 4 & 1 & 8 \end{bmatrix}.$$

Now, at the eigenvalue $\lambda = 2$, one can have

$$\begin{array}{rrcrcrcr} \text{as } \lambda = 2, & e^2 & = & x_0 & + & 2x_1 & + & 2^2 x_2, \\ \text{take } \partial_\lambda, & e^2 & = & & & x_1 & + & 2 \cdot 2x_2, \\ \text{take } (\partial_\lambda)^2, & e^2 & = & & & & & 2x_2. \end{array}$$

Its solution is $x_2 = \frac{1}{2}e^2$; $x_1 = -e^2$ and $x_0 = e^2$. Hence,

$$e^A = x_0 I + x_1 A + x_2 A^2 = e^2\left(I - A + \frac{1}{2}A^2\right) = \begin{bmatrix} 0 & e^2 & \frac{3}{2}e^2 & e^2 \\ 0 & e^2 & 2e^2 & 0 \\ 0 & 0 & e^2 & 0 \\ -e^2 & e^2 & \frac{1}{2}e^2 & 2e^2 \end{bmatrix},$$

as shown in Example 8.11. $\square$

Problem 8.15 (Problem 8.6 again) Compute e^A by using the minimal polynomial for

$$(1)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2)\ A = \begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix}.$$

8.6.3 Linear difference equations again

A linear difference equation is a matrix equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ with a $k \times k$ matrix A and if an initial vector $\mathbf{x}_0$ is given, then $\mathbf{x}_n = A^n\mathbf{x}_0$ for all n.

If the matrix A is diagonalizable with k linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ belonging to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively, a general solution of a linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is known as

$$\mathbf{x}_n = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + \cdots + c_k\lambda_k^n\mathbf{v}_k$$

with constants $c_1, c_2, \ldots, c_k$. (See Theorem 6.13.)

On the other hand, a linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ can be solved for any square matrix A (not necessarily diagonalizable) by using the power A^n, whose computation was discussed in Sections 8.3 and 8.5.

Example 8.25 Solve a linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}.$$

Solution: In Example 8.11, it was shown that the matrix A has an eigenvalue 2 with multiplicity 4, and A is not diagonalizable. However, the solution of $\mathbf{x}_n = A\mathbf{x}_{n-1}$ is $\mathbf{x}_n = A^n\mathbf{x}_0$, where

$$A^n = \begin{bmatrix} 2^n - n2^{n-1} & n2^{n-1} & 2^{n-3}(3n + n^2) & n2^{n-1} \\ 0 & 2^n & n2^n & 0 \\ 0 & 0 & 2^n & 0 \\ -n2^{n-1} & n2^{n-1} & n(n-1)2^{n-3} & n2^{n-1} + 2^n \end{bmatrix},$$

which is given in Examples 8.11 and 8.22 □

Problem 8.16 Solve the linear difference equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ for

$$(1)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \ (2)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \ (3)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

8.6.4 Linear differential equations again

Now, we go back to a system of linear differential equations

$$\mathbf{y}' = A\mathbf{y} \quad \text{with initial condition} \quad \mathbf{y}(0) = \mathbf{y}_0.$$

Its solution is known as $\mathbf{y}(t) = e^{tA}\mathbf{y}_0$. (See Theorem 6.27.) In particular, if A is diagonalizable, a general solution of $\mathbf{y}' = A\mathbf{y}$ is known as

$$\mathbf{y}(t) = e^{tA}\mathbf{y}_0 = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n,$$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the eigenvectors belonging to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, respectively.

For any square matrix A (not necessarily diagonalizable), the matrix e^{tA} can be computed in two different ways. Firstly, let $Q^{-1}AQ = J$ be the Jordan canonical form of A. Then, the solution $\mathbf{y}(t) = e^{tA}\mathbf{y}_0$ is

$$e^{tA}\mathbf{y}_0 = Qe^{tJ}Q^{-1}\mathbf{y}_0 = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n]\begin{bmatrix} e^{tJ_1} & 0 & \cdots & 0 \\ 0 & e^{tJ_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{tJ_s} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where $Q^{-1}\mathbf{y}_0 = (c_1, \ldots, c_n)$ and the $\mathbf{u}_i$'s are generalized eigenvectors of A. In particular, if $Q^{-1}AQ = J$ is a single Jordan block with corresponding generalized eigenvectors $\mathbf{u}_i$ of order k, then the solution becomes

$$\begin{aligned} e^{tA}\mathbf{y}_0 &= e^{\lambda t}Qe^{tN}Q^{-1}\mathbf{y}_0 \\ &= e^{\lambda t}[\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n] \\ &\quad \times \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & t & \frac{t^2}{2!} \\ \vdots & & & \ddots & 1 & t \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= e^{\lambda t}\left(\left(\sum_{k=0}^{n-1} c_{k+1}\frac{t^k}{k!}\right)\mathbf{u}_1 + \left(\sum_{k=0}^{n-2} c_{k+2}\frac{t^k}{k!}\right)\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n\right). \end{aligned}$$

As a simpler method to compute e^{tA}, one can use the minimal polynomial of tA as discussed in Section 8.6.2. First, note that if $m(\lambda)$ is the minimal polynomial of A, then $m(t\lambda)$ is that of the matrix tA.

Example 8.26 Solve the linear differential equation $\mathbf{y}' = A\mathbf{y}$ with initial condition $\mathbf{y}(0) = \mathbf{y}_0$, where

$$A = \begin{bmatrix} 4 & -3 & -1 \\ 1 & 0 & -1 \\ -1 & 2 & 3 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Solution: *Method 1*: (i) Note that the characteristic polynomial of A is $\det(\lambda I - A) = \lambda^3 - 7\lambda^2 + 16\lambda - 12 = (\lambda - 3)(\lambda - 2)^2$ and A is not diagonalizable (because $\operatorname{rank}(A - 2I) = 2$). By taking $\mathbf{x}_1 = (-1, -1, 1)$ and $\mathbf{x}_3 = (2, 1, -1)$ as eigenvectors belonging to $\lambda = 2$ and $\lambda = 3$, respectively, one can compute the Jordan canonical form of A as follows:

$$J = Q^{-1}AQ = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix},$$

where

$$J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad J_2 = [3], \quad \text{and } Q = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

(ii) Let $\mathbf{y} = Q\mathbf{x}$. Then the given system changes to $\mathbf{x}' = J\mathbf{x}$ with

$$\mathbf{x}(0) = Q^{-1}\mathbf{y}(0) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

and its solution is

$$\mathbf{x}(t) = e^{tJ}\mathbf{x}(0) = \begin{bmatrix} e^{tJ_1} & \mathbf{0} \\ \mathbf{0} & e^{tJ_2} \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix},$$

since

$$e^{tJ_1} = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad e^{tJ_2} = e^{3t}.$$

(iii) Thus, we get

$$\begin{aligned} \mathbf{y}(t) &= Q\mathbf{x}(t) = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} \\ &= e^{2t}\left((5 + 5t) \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) + e^{3t} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Method 2: To use the minimal polynomial of the matrix tA, first recall that the characteristic polynomial of A is $f(\lambda) = (\lambda - 3)(\lambda - 2)^2$ and A is not diagonalizable. Hence, its minimal polynomial coincides with the characteristic polynomial. Therefore, the minimal polynomial of tA is the polynomial $f(t\lambda)$, and one can write

$$e^{tA} = x_0(t)I + x_1(t)A + x_2(t)A^2$$

with unknown coefficient functions $x_i(t)$'s and

$$A^2 = \begin{bmatrix} 14 & -14 & -4 \\ 5 & -5 & -4 \\ -5 & 9 & 8 \end{bmatrix}.$$

Now, at each eigenvalue λ, one can have

$$\begin{array}{lrcrcrcr} \text{as } \lambda = 2, & e^{2t} & = & x_0(t) & + & 2x_1(t) & + & 2^2x_2(t), \\ \text{take } \partial_\lambda, & te^{2t} & = & & & x_1(t) & + & 2\cdot 2x_2(t), \\ \text{as } \lambda = 3, & e^{3t} & = & x_0(t) & + & 3x_1(t) & + & 3^2x_2(t). \end{array}$$

Its solution is $x_0(t) = -3e^{2t} - 6te^{2t} + 4e^{3t}$; $x_1(t) = 5te^{2t} - 4e^{3t} + 4e^{2t}$ and $x_2(t) = e^{3t} - e^{2t} - te^{2t}$. Hence,

$$\begin{aligned} e^{tA} &= x_0(t)I + x_1(t)A + x_2(t)A^2 \\ &= \begin{bmatrix} -e^{2t} + 2e^{3t} & -te^{2t} - 2e^{3t} + 2e^{2t} & -te^{2t} \\ e^{3t} - e^{2t} & 2e^{2t} - te^{2t} - e^{3t} & -te^{2t} \\ -e^{3t} + e^{2t} & te^{2t} + e^{3t} - e^{2t} & e^{2t} + te^{2t} \end{bmatrix}. \end{aligned}$$

Now, one might compare the value $\mathbf{y}(t) = e^{tA}\mathbf{y}(0)$ with the solution obtained by Method 1. The reader can easily notice that Method 2 is simpler than Method 1. □

Example 8.27 (*Computing e^{tA} when A is diagonalizable*) (Example 6.16 again) Solve the system of linear differential equations

$$\begin{cases} y_1' = 5y_1 - 4y_2 + 4y_3 \\ y_2' = 12y_1 - 11y_2 + 12y_3 \\ y_3' = 4y_1 - 4y_2 + 5y_3, \end{cases}$$

and also find its particular solution satisfying the initial conditions $y_1(0) = 0$, $y_2(0) = 3$ and $y_3(0) = 2$.

Solution: The matrix form of the system is $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix},$$

and its general solution is $\mathbf{y} = e^{tA}\mathbf{y}_0$. It was known that A is diagonalizable and the minimal polynomial of A is $m(\lambda) = (\lambda-1)(\lambda+3)$. If we write $e^{tA} = x_0(t)I + x_1(t)A$ with unknown functions $x_i(t)$'s, then

$$\begin{array}{rcrcrcr} \text{as } \lambda = 1, & e^{t} & = & x_0(t) & + & x_1(t), \\ \text{as } \lambda = -3, & e^{-3t} & = & x_0(t) & - & 3x_1(t). \end{array}$$

(Compare with Example 8.23). By solving it, we have $x_0(t) = \frac{1}{4}(3e^t + e^{-3t})$, $x_1(t) = \frac{1}{4}(e^t - e^{-3t})$ and

$$e^{tA} = x_0(t)I + x_1(t)A = \begin{bmatrix} 2e^t - e^{-3t} & -e^t + e^{-3t} & e^t - e^{-3t} \\ 3e^t - 3e^{-3t} & -2e^t + 3e^{-3t} & 3e^t - 3e^{-3t} \\ e^t - e^{-3t} & -e^t + e^{-3t} & 2e^t - e^{-3t} \end{bmatrix}.$$

Moreover, with the initial conditions $y_1(0) = 0$, $y_2(0) = 3$ and $y_3(0) = 2$, the particular solution is

$$\begin{aligned} \mathbf{y}(t) = e^{tA}\mathbf{y}_0 &= \begin{bmatrix} 2e^t - e^{-3t} & -e^t + e^{-3t} & e^t - e^{-3t} \\ 3e^t - 3e^{-3t} & -2e^t + 3e^{-3t} & 3e^t - 3e^{-3t} \\ e^t - e^{-3t} & -e^t + e^{-3t} & 2e^t - e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -e^t + e^{-3t} \\ 3e^{-3t} \\ e^t + e^{-3t} \end{bmatrix}. \end{aligned}$$

One might compare this method with that given in Example 6.16. □

Note: In Example 8.27, we compute e^{tA} by finding the unknown functions $x_i(t)$'s in $e^{tA} = x_0(t)I + x_1(t)A$. However, in Example 8.23, we determined $e^A = x_0 I + x_1 A$ with $x_0 = \frac{1}{4}(3e + e^{-3})$ and $x_1 = \frac{1}{4}(e - e^{-3})$. Hence, it looks true that $e^{tA} = x_0 I + x_1(tA)$ with the same $x_0 = \frac{1}{4}(3e + e^{-3})$ and $x_1 = \frac{1}{4}(e - e^{-3})$. But, it is *not* a fact, because if we put $e^{tA} = x_0 I + x_1 A$, then x_0 and x_1 must be functions of t.

Example 8.28 (*Computing e^{tA} when A is not diagonalizable*) Solve the system of linear differential equations $\mathbf{y}'(t) = A\mathbf{y}(t)$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}.$$

Solution: In Example 8.20, the minimal polynomial of A is computed as $m(\lambda) = (\lambda - 2)^3$. Hence, as in Example 8.24, one can set

$$e^{tA} = x_0(t)I + x_1(t)A + x_2(t)A^2$$

with unknown coefficient functions $x_i(t)$'s. Now, at the eigenvalue $\lambda = 2$, one can have

$$\begin{array}{rrcrcrcr} \text{as } \lambda = 2, & e^{2t} & = & x_0(t) & + & 2x_1(t) & + & 2^2x_2(t), \\ \text{take } \partial_\lambda, & te^{2t} & = & & & x_1(t) & + & 2\cdot 2x_2(t), \\ \text{take } \partial_\lambda^2, & t^2e^{2t} & = & & & & & 2x_2(t). \end{array}$$

Its solution is $x_2(t) = \frac{1}{2}t^2e^{2t}$; $x_1(t) = te^{2t} - 2t^2e^{2t}$ and $x_0(t) = e^{2t} - 2te^{2t} + 2t^2e^{2t}$. Hence,

$$\begin{aligned} e^{tA} &= x_0(t)I + x_1(t)A + x_2(t)A^2 \\ &= \begin{bmatrix} e^{2t} - te^{2t} & te^{2t} & te^{2t} + \frac{1}{2}t^2e^{2t} & te^{2t} \\ 0 & e^{2t} & 2te^{2t} & 0 \\ 0 & 0 & e^{2t} & 0 \\ -te^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} & e^{2t} + te^{2t} \end{bmatrix}, \end{aligned}$$

and the solution of $\mathbf{y}'(t) = A\mathbf{y}(t)$ is given by $\mathbf{y}(t) = e^{tA}\mathbf{y}_0$. □

Example 8.29 (*Solving* $\mathbf{y}'(t) = A\mathbf{y}(t)$ *by the minimal polynomial*) Solve the system of linear differential equations $\mathbf{y}'(t) = A\mathbf{y}(t)$, where

$$A = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix}.$$

Solution: (1) The characteristic polynomial of A is $\det(\lambda I - A) = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$, so that the eigenvalue of A is $\lambda = 1$ of multiplicity 3.

(2) In the matrix

$$A - I = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix},$$

one can see that the second and the third rows are constant multiples of the first row, and hence $\operatorname{rank}(A - I) = 1$. Hence, A is not diagonalizable, but $(A - I)^2 = \mathbf{0}$. It means that the minimal polynomial of A is $m(\lambda) = (\lambda - 1)^2$. Therefore, one can write

$$e^{tA} = x_0(t)I + x_1(t)A$$

with unknown coefficient functions $x_i(t)$'s, and one can have

$$\begin{array}{rrcrcr} \text{as } \lambda = 1, & e^{t} & = & x_0(t) & + & 1\cdot x_1(t), \\ \text{take } \partial_\lambda, & te^{t} & = & & & x_1(t). \end{array}$$

Its solution is $x_0(t) = e^t(1-t)$; $x_1(t) = te^t$. Hence,

$$e^{tA} = x_0(t)I + x_1(t)A = \begin{bmatrix} (1+4t)e^t & -3te^t & -2te^t \\ 8te^t & (1-6t)e^t & -4te^t \\ -4te^t & 3te^t & (1+2t)e^t \end{bmatrix},$$

and a general solution of $\mathbf{y}'(t) = A\mathbf{y}(t)$ is given by $\mathbf{y}(t) = e^{tA}\mathbf{y}_0$. □

Problem 8.17 Solve the system of linear differential equations $\mathbf{y}' = A\mathbf{y}$ with the initial condition $\mathbf{y}(0) = \mathbf{y}_0$, where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Problem 8.18 Solve the system of linear differential equations $\mathbf{y}' = A\mathbf{y}$ for

$$(1)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad (2)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad (3)\ A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

8.7 Exercises

8.1. For $A=\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$ $(\lambda \neq 0)$, find A^{-1} and its Jordan canonical form J.

8.2. Show that if A is nonsingular, then A^{-1} has the same block structure in its Jordan canonical form as A does.

8.3. Find the number of linearly independent eigenvectors for each of the following matrices:

$$(1) \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad (2) \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \quad (3) \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

8.4. Find the Jordan canonical form for each of the following matrices:

$$(1) \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}, \quad (2) \begin{bmatrix} -2 & 0 & -2 \\ -1 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}, \quad (3) \begin{bmatrix} -6 & 31 & -14 \\ -1 & 6 & -2 \\ 0 & 2 & 1 \end{bmatrix}.$$

Also, find a full set of generalized eigenvectors of each of them.

8.5. Show that a Jordan block J is similar to its transpose, $J^T = P^{-1}JP$, by the permutation matrix $P = [\mathbf{e}_n \ \cdots \ \mathbf{e}_1]$. Deduce that every matrix is similar to its transpose.

8.6. Evaluate $\det A_n$ for a tridiagonal matrix

$$A_n = \begin{bmatrix} b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b \end{bmatrix}, \quad b > 0.$$

8.7. Solve the system of linear equations

$$\begin{cases} (1-i)x + (1+i)y = 2-i \\ (1+i)x + (1+i)y = 1+3i. \end{cases}$$

8.8. Solve the system of three difference equations:

$$\begin{cases} x_{n+1} = 3x_n + 5y_n + 2z_n \\ y_{n+1} = x_n - y_n + z_n \\ z_{n+1} = 2x_n + y_n + 3z_n \end{cases}$$

for $n = 0, 1, 2, \ldots$.

8.9. Solve $\mathbf{y}_n = A\mathbf{y}_{n-1}$ for $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ with $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

8.10. Solve $\mathbf{y}_n = A\mathbf{y}_{n-1}$ for $A = \begin{bmatrix} -6 & 24 & 8 \\ -1 & 8 & 4 \\ 2 & -12 & -6 \end{bmatrix}$ with $\mathbf{y}_0 = (2, 1, 0)$.

8.11. Solve $\mathbf{y}' = A\mathbf{y}$ for $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ with $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

8.12. Solve $\mathbf{y}' = A\mathbf{y}$ for $A = \begin{bmatrix} -6 & 24 & 8 \\ -1 & 8 & 4 \\ 2 & -12 & -6 \end{bmatrix}$ with $\mathbf{y}(1) = (2, 1, 0)$.

8.13. Solve the initial value problem

$$\begin{cases} y_1' = -y_1 + 2y_3, & y_1(0) = -2 \\ y_2' = 2y_1 + y_2 - 2y_3, & y_2(0) = 0 \\ y_3' = -2y_1 + 3y_3, & y_3(0) = -1. \end{cases}$$

8.14. Consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(1) Find a necessary and sufficient condition for A to be diagonalizable.

(2) The characteristic polynomial for A is $f(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$. Show that $f(A) = 0$.

8.15. For each of the following matrices, find its Jordan canonical form and the minimal polynomial.

$$(1)\ \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix},\quad (2)\ \begin{bmatrix} 2 & -3 \\ 7 & -4 \end{bmatrix},\quad (3)\ \begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$

8.16. Compute the minimal polynomial of each of the following matrices and from these results compute A^{-1}, if it exists.

$$(1)\ \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix},\quad (2)\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},\quad (3)\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

8.17. Compute A^{-1}, A^n and e^A for

$$(1)\ \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix},\quad (2)\ \begin{bmatrix} 4 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix},\quad (3)\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(4)\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix},\quad (5)\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

8.18. An $n \times n$ matrix A is called a **circulant matrix** if the i-th row of A is obtained from the first row of A by a cyclic shift of the $i - 1$ steps, i.e., the general form of the circulant matrix is

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix}.$$

(1) Show that any circulant matrix is normal.

(2) Find all eigenvalues of the $n \times n$ circulant matrix

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(3) Find all eigenvalues of the circulant matrix A by showing that

$$A = \sum_{i=1}^{n} a_i W^{i-1}.$$

(4) Compute $\det A$. (Hint: It is the product of all eigenvalues.)

(5) Use your answer to find the eigenvalues of

$$B = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

8.19. Determine whether the following statements are true or false, in general, and justify your answers.

(1) Any square matrix is similar to a triangular matrix.

(2) If a matrix A has exactly k linearly independent eigenvectors, then the Jordan canonical form of A has k Jordan blocks.

(3) If a matrix A has k distinct eigenvalues, then the Jordan canonical form of A has k Jordan blocks.

(4) If two square matrices A and B have the same characteristic polynomial $\det(\lambda I - A) = \det(\lambda I - B)$ and for each eigenvalue λ the dimensions of their eigenspaces $\mathcal{N}(\lambda I - A)$ and $\mathcal{N}(\lambda I - B)$ are the same, then A and B are similar.

(5) If a 4×4 matrix A has eigenvalues 1 and 2, each of multiplicity 2, such that $\dim E(1) = 2$ and $\dim E(2) = 1$, then the Jordan canonical form of A has three Jordan blocks.

(6) If there is an eigenvalue λ of A with multiplicity m_λ and $\dim E(\lambda_i) \neq m_\lambda$, then A is not diagonalizable.

(7) For any Jordan block J with eigenvalue λ, $\det e^J = e^\lambda$.

(8) For any square matrix A, A and A^T have the same Jordan canonical form.

(9) If $f(x)$ is a polynomial and A is a square matrix such that $f(A) = \mathbf{0}$, then $f(x)$ is a multiple of the characteristic polynomial of A.

(10) The minimal polynomial of a Jordan canonical matrix J is the product of the minimal polynomials of its Jordan blocks J_i.

(11) If the degree of the minimal polynomial of A is equal to the number of the distinct eigenvalues of A, then A is diagonalizable.

9

Quadratic Forms

9.1 Basic properties of quadratic forms

In the beginning of this book, we started with systems of linear equations, one of which can be written as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

The left-hand side $a_1x_1 + a_2x_2 + \cdots + a_nx_n = \mathbf{a}^T\mathbf{x}$ of the equation is a (homogeneous) polynomial of degree 1 in n real variables. In this chapter, we study a (homogeneous) polynomial of degree 2 in several variables, called a *quadratic form*, and show that matrices also play an important role in the study of a quadratic form. Quadratic forms arise in a variety of applications, including geometry, number theory, vibrations of mechanical systems, statistics, electrical engineering, etc. A more general type of a quadratic form is a *bilinear form* which will be described in Section 9.6. As a matter of fact, a quadratic form (or bilinear form) can be associated with a real symmetric matrix, and vice-versa.

A **quadratic equation** in two variables x and y is an equation of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0,$$

in which the left-hand side consists of a constant term f, a linear form $dx + ey$, and a quadratic form $ax^2 + 2bxy + cy^2$. Note that this quadratic form may be written in matrix notation as

$$ax^2 + 2bxy + cy^2 = [\, x \quad y \,] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x}^T A \mathbf{x},$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Note also that the matrix A is taken to be a (real) symmetric matrix.

Geometrically, the solution set of a quadratic equation in x and y usually represents a *conic section*, such as an ellipse, a parabola or a hyperbola in the xy-plane. (See Figure 9.1.)

J H Kwak et al., *Linear Algebra*

Definition 9.1 (1) A **linear form** on the Euclidean space $\mathbb{R}^n$ is a polynomial of degree 1 in n variables $x_1, x_2, \ldots, x_n$ of the form

$$\mathbf{b}^T\mathbf{x} = \sum_{i=1}^{n} b_i x_i,$$

where $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$ and $\mathbf{b} = [b_1 \ \cdots \ b_n]^T$ in $\mathbb{R}^n$.

(2) A **quadratic equation** on $\mathbb{R}^n$ is an equation in n variables $x_1, x_2, \ldots, x_n$ of the form

$$f(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j + \sum_{i=1}^{n} b_i x_i + c = 0,$$

where a_{ij}, b_j and c are real constants. In matrix form, it can be written as

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = 0,$$

where $A = [a_{ij}]$, $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$ and $\mathbf{b} = [b_1 \ \cdots \ b_n]^T$ in $\mathbb{R}^n$.

(3) A **quadratic form** on $\mathbb{R}^n$ is a (homogeneous) polynomial of degree 2 in n variables $x_1, x_2, \ldots, x_n$ of the form

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n][a_{ij}]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j,$$

where $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]^T \in \mathbb{R}^n$ and $A = [a_{ij}]$ is a real $n \times n$ symmetric matrix.

It is also possible to define a linear form and a quadratic form on the Euclidean complex n-space $\mathbb{C}^n$ instead of the Euclidean real n-space $\mathbb{R}^n$.

Definition 9.2 (1) A **linear form** on the complex n-space $\mathbb{C}^n$ is a polynomial of degree 1 in n complex variables $x_1, x_2, \ldots, x_n$ of the form

$$\mathbf{b}^H\mathbf{x} = \sum_{i=1}^{n} \overline{b_i} x_i,$$

where $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$ and $\mathbf{b} = [b_1 \ \cdots \ b_n]^T$ in $\mathbb{C}^n$.

(2) A **complex quadratic form** on $\mathbb{C}^n$ is a polynomial of degree 2 in n complex variables $x_1, x_2, \ldots, x_n$ of the form

$$q(\mathbf{x}) = \mathbf{x}^H A\mathbf{x} = [\overline{x_1} \ \overline{x_2} \ \cdots \ \overline{x_n}][a_{ij}]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\overline{x_i} x_j,$$

where $\mathbf{x} \in \mathbb{C}^n$ and $A = [a_{ij}]$ is an $n \times n$ Hermitian matrix.

The real quadratic form on $\mathbb{R}^n$ and the complex quadratic form on $\mathbb{C}^n$ can be denoted simultaneously as $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$ by using the dot product on the real n-space $\mathbb{R}^n$ or on the complex n-space $\mathbb{C}^n$.

Remark: (1) A quadratic equation $f(\mathbf{x})$ is said to be **consistent** if it has a solution, i.e., there is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $f(\mathbf{x}) = 0$. Otherwise, it is said to be **inconsistent**. For instance, the equation $2x^2 + 3y^2 = -1$ in $\mathbb{R}^2$ is inconsistent. In the following, we will consider only consistent equations.

(2) A linear form is simply the dot product on the real n-space $\mathbb{R}^n$ or on the complex n-space $\mathbb{C}^n$ with a fixed vector $\mathbf{b}$.

(3) The matrix A in the definition of a real quadratic form can be any square matrix. In fact, a square matrix A can be expressed as the sum of a symmetric part B and a skew-symmetric part C, say

$$A = B + C, \quad \text{where } B = \frac{1}{2}(A + A^T) \text{ and } C = \frac{1}{2}(A - A^T).$$

For the skew-symmetric matrix C, we have

$$\mathbf{x}^T C\mathbf{x} = \left(\mathbf{x}^T C\mathbf{x}\right)^T = \mathbf{x}^T C^T \mathbf{x} = -\mathbf{x}^T C\mathbf{x}.$$

Hence, as a real number, $\mathbf{x}^T C\mathbf{x} = 0$. Therefore,

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T (B + C)\mathbf{x} = \mathbf{x}^T B\mathbf{x}.$$

This means that, without loss of generality, one may assume that the matrix A in the definition of a real quadratic form is a symmetric matrix.

(4) For the definition of a complex quadratic form, let A be any $n \times n$ complex matrix. Then, for any $\mathbf{x} \in \mathbb{C}^n$, the matrix product $\mathbf{x}^H A\mathbf{x}$ is a complex number. But, for the matrix A, it is known that there are Hermitian matrices B and C such that $A = B + iC$. (See page 264.) Hence,

$$\mathbf{x}^H A\mathbf{x} = \mathbf{x}^H (B + iC)\mathbf{x} = \mathbf{x}^H B\mathbf{x} + i\mathbf{x}^H C\mathbf{x},$$

in which $\mathbf{x}^H B\mathbf{x}$ and $\mathbf{x}^H C\mathbf{x}$ are real numbers. Hence, for a complex quadratic form on $\mathbb{C}^n$, we are only concerned with a Hermitian matrix A so that $\mathbf{x}^H A\mathbf{x}$ is a real number for any $\mathbf{x} \in \mathbb{C}^n$.

The solution set of a consistent quadratic equation $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = 0$ is a level surface in $\mathbb{R}^n$, that is, a curved surface that can be parameterized in $n - 1$ variables. In particular, if $n = 2$, the solution set of a quadratic equation is called a **quadratic curve**, or more commonly a **conic section**. When $n = 3$, it is called a **quadratic surface**, which is an ellipsoid, a paraboloid or a hyperboloid.

Example 9.1 (*The standard three types of conic sections*)

(1) (circle or ellipse) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $A = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix}$.

(2) (hyperbola) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ with

$$A = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} -\frac{1}{b^2} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}.$$

(3) (parabola) $x^2 = ay$ or $y^2 = bx$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

All of these cases are illustrated in Figures 9.1 as conic sections. □

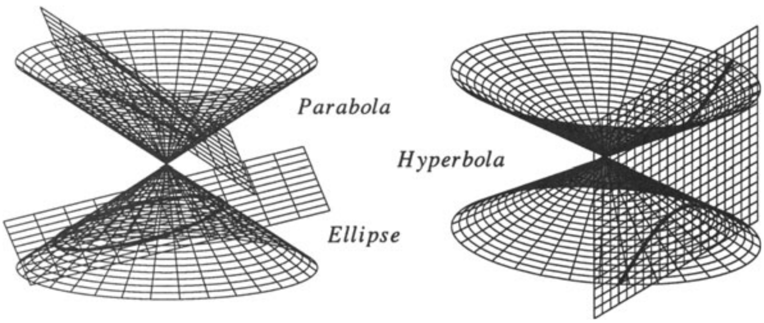

Figure 9.1. Conic sections

Example 9.2 (*The standard four types of quadratic surfaces*)

(1) (ellipsoids) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with $A = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{bmatrix}$.

(2) (hyperboloids of one or two sheets) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (of one sheet) or $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (of two sheets) with

$$A = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & -\frac{1}{c^2} \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} -\frac{1}{a^2} & 0 & 0 \\ 0 & -\frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{bmatrix}.$$

(3) (cones) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ with $A = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & -\frac{1}{c^2} \end{bmatrix}$.

(4) (paraboloids; elliptic or hyperbolic) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ (elliptic) or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$, $c > 0$ (hyperbolic) with

$$A = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & -\frac{1}{b^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

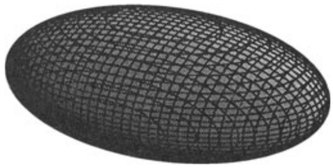

Figure 9.2. Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

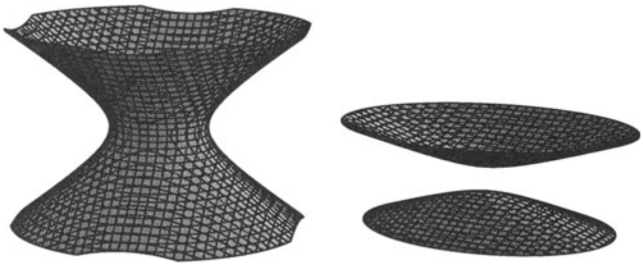

Figure 9.3. Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; and of two sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

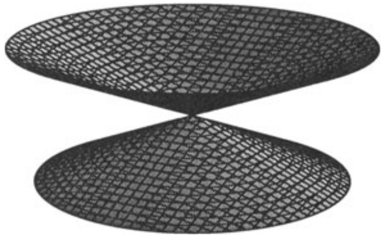

Figure 9.4. Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

All of these cases are illustrated in Figures 9.2–9.5. $\square$

Problem 9.1 Find the symmetric matrices representing the quadratic forms

(1) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + 2x_2x_3$,
(2) $x_1x_2 + x_1x_3 + x_2x_3$,
(3) $x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2x_1x_2 - 10x_1x_4 + 4x_3x_4$.

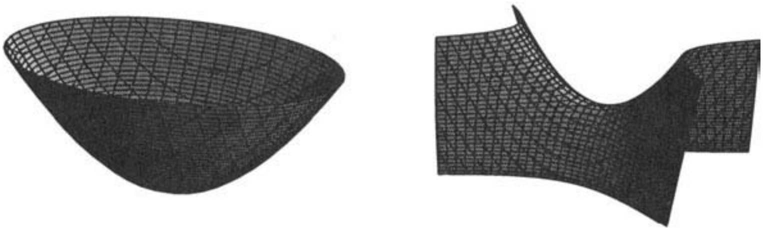

Figure 9.5. Elliptic paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ and Hyperbolic paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$, $c > 0$

9.2 Diagonalization of quadratic forms

In this section, we discuss how to sketch the level surface of a quadratic equation on $\mathbb{R}^n$. To do this for a quadratic equation $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = 0$, we first consider a special case of the type $\mathbf{x}^T A\mathbf{x} = c$ without a linear form.

A quadratic form on $\mathbb{R}^n$ without a linear form may be written as the sum of two parts:

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n} a_{ii}x_i^2 + 2\sum_{i<j} a_{ij}x_ix_j,$$

in which the first part $\sum_{i=1}^{n} a_{ii}x_i^2$ is called the **(perfect) square terms** and the second part $\sum_{i\neq j} a_{ij}x_ix_j$ is called the **cross-product terms**. Actually, what makes it hard to sketch the level surface of a quadratic equation is the cross-product terms. However, the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ can be transformed into a new quadratic form without the cross-product terms by a suitable change of variables. It can be done by computing the eigenvalues of A and their associated eigenvectors. In fact, the symmetric matrix A can be orthogonally diagonalized, i.e., there exists an orthogonal matrix P such that

$$P^T AP = P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix}.$$

Here, the diagonal entries λ_i's are the eigenvalues of A and the column vectors of P are their associated eigenvectors of A. Now, by setting $\mathbf{x} = P\mathbf{y}$, (that is, by a change of variables), we have

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2,$$

which is a quadratic form without the cross-product terms.

It is also true for a complex quadratic form $q(\mathbf{x}) = \mathbf{x}^H A\mathbf{x}$ with a Hermitian matrix A. Since every Hermitian matrix is unitarily diagonalizable, there exists a unitary matrix U such that $U^H AU = D$ is a diagonal matrix. Hence, by a change of variables $\mathbf{x} = U\mathbf{y}$, the quadratic form $q(\mathbf{x}) = \mathbf{y}^H D\mathbf{y}$ has only square terms.

In either case of real or complex, we consequently have the following theorem.

Theorem 9.1 (The principal axes theorem)

(1) *Let* $\mathbf{x}^T A\mathbf{x}$ *be a quadratic form in* $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]^T \in \mathbb{R}^n$ *for a symmetric matrix* A. *Then, there is a change of coordinates of* $\mathbf{x}$ *into* $\mathbf{y} = P^T\mathbf{x} = [y_1\ y_2\ \cdots\ y_n]^T$ *such that*

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2,$$

where P *is an orthogonal matrix and* $P^T AP = D$ *is diagonal.*

(2) *Let* $\mathbf{x}^H A\mathbf{x}$ *be a complex quadratic form on* $\mathbb{C}^n$ *with a Hermitian matrix* A. *Then, there is a change of coordinates of* $\mathbf{x}$ *into* $\mathbf{y} = U^H\mathbf{x} = [y_1\ y_2\ \cdots\ y_n]^T$ *such that*

$$\mathbf{x}^H A\mathbf{x} = \mathbf{y}^H D\mathbf{y} = \lambda_1|y_1|^2 + \lambda_2|y_2|^2 + \cdots + \lambda_n|y_n|^2,$$

where U *is a unitary matrix and* $U^H AU = D$ *is diagonal.*

Clearly, the columns of the matrix P (U, respectively) in Theorem 9.1 form an orthonormal basis for $\mathbb{R}^n$ (for $\mathbb{C}^n$, respectively) and it is called the **principal axes** of the quadratic form. The vector $\mathbf{y}$ is just the coordinate expression of $\mathbf{x}$ with respect to the principal axes.

Example 9.3 (*Via a diagonalization of a quadratic form*) Determine the conic section $3x^2 + 2xy + 3y^2 - 8 = 0$ on $\mathbb{R}^2$.

Solution: In matrix form, it is

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 8.$$

The matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$ with associated unit eigenvectors

$$\mathbf{v}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \mathbf{v}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

respectively, which form an orthonormal basis β. If α denotes the standard basis, then the basis-change matrix

$$P = [id]_\beta^\alpha = [\mathbf{v}_1\ \ \mathbf{v}_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix},$$

which is a rotation through 45° in the clockwise direction such that $P^T = P^{-1}$. It gives a change of coordinates, $\mathbf{x} = P\mathbf{y}$, i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \\ -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \end{bmatrix}.$$

It implies that

$$\begin{aligned} 3x^2 + 2xy + 3y^2 &= \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T P^T A P\mathbf{y} \\ &= \mathbf{y}^T \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{y} = 2(x')^2 + 4(y')^2 = 8, \end{aligned}$$

or

$$\frac{(x')^2}{4} + \frac{(y')^2}{2} = 1,$$

which is an ellipse with the principal axes $\mathbf{v}_1 = P^T\mathbf{e}_1$ and $\mathbf{v}_2 = P^T\mathbf{e}_2$. □

To determine the type of a quadratic form, we introduce the following definition for a symmetric matrix A or a quadratic form $\mathbf{x}^T A\mathbf{x}$.

Definition 9.3 Let $A = [a_{ij}] \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix and let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then, the matrix A, or a quadratic form $\mathbf{x}^T A\mathbf{x}$, is said to be

(1) positive definite if $\mathbf{x}^T A\mathbf{x} = \sum_{i,j} a_{ij}x_i x_j > 0$ for all nonzero $\mathbf{x}$,
(2) positive semidefinite if $\mathbf{x}^T A\mathbf{x} = \sum_{i,j} a_{ij}x_i x_j \geq 0$ for all $\mathbf{x}$,
(3) negative definite if $\mathbf{x}^T A\mathbf{x} = \sum_{i,j} a_{ij}x_i x_j < 0$ for all nonzero $\mathbf{x}$,
(4) negative semidefinite if $\mathbf{x}^T A\mathbf{x} = \sum_{i,j} a_{ij}x_i x_j \leq 0$ for all $\mathbf{x}$,
(5) indefinite if $\mathbf{x}^T A\mathbf{x}$ takes both positive and negative values.

Similarly, one can define the same terminologies for a Hermitian matrix or for a complex quadratic form $\mathbf{x}^H A\mathbf{x}$ on $\mathbb{C}^n$.

For example, the real symmetric matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite, because the quadratic form satisfies

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} &= [x_1\ x_2\ x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= [x_1\ x_2\ x_3] \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix} \\ &= x_1(2x_1 - x_2) + x_2(-x_1 + 2x_2 - x_3) + x_3(-x_2 + 2x_3) \\ &= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \\ &= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0 \end{aligned}$$

unless $x_1 = x_2 = x_3 = 0$.

The following characterizations follow from the principal axes theorem.

Corollary 9.2 *A real symmetric or a Hermitian matrix A is*

(1) *positive definite if and only if all the eigenvalues of A are positive,*
(2) *positive semidefinite if and only if all the eigenvalues of A are nonnegative,*
(3) *negative definite if and only if all the eigenvalues of A are negative,*
(4) *negative semidefinite if and only if all the eigenvalues of A are nonpositive,*
(5) *indefinite if and only if A takes both positive and negative eigenvalues.*

Note that if A is positive definite, $\det A > 0$ (as the product of all eigenvalues). If the eigenvalues of A are all negative, then $-A$ must be positive definite and consequently A must be negative definite. If A has eigenvalues that differ in sign, then A is indefinite. Indeed, if λ_1 is a positive eigenvalue of A and $\mathbf{x}_1$ is an eigenvector belonging to λ_1, then

$$q(\mathbf{x}_1) = \langle \mathbf{x}_1, A\mathbf{x}_1 \rangle = \lambda_1 \langle \mathbf{x}_1, \mathbf{x}_1 \rangle = \lambda_1 \|\mathbf{x}_1\|^2 > 0,$$

and if λ_2 is a negative eigenvalue with eigenvector $\mathbf{x}_2$, then

$$q(\mathbf{x}_2) = \langle \mathbf{x}_2, A\mathbf{x}_2 \rangle = \lambda_2 \langle \mathbf{x}_2, \mathbf{x}_2 \rangle = \lambda_2 \|\mathbf{x}_2\|^2 < 0.$$

If A is definite, then $\mathbf{0}$ is the only critical point of a quadratic form $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$, and $q(\mathbf{0}) = 0$ is the *global minimum* if A is positive definite and the *global maximum* if A is negative definite. If A is indefinite, then $\mathbf{0}$ is a *saddle point.*

9.3 A classification of level surfaces

We have seen already in Section 9.1 that the geometric type of the level surface of a quadratic equation $\mathbf{x}^T A\mathbf{x} = 0$ depends on the signs of the eigenvalues of A. In fact, it is completely determined by the numbers of positive, negative and zero eigenvalues of A.

Definition 9.4 The **inertia** of a Hermitian (or a symmetric) matrix A is a triple of integers denoted by $\mathrm{In}(A) = (p, q, k)$, where p, q and k are the numbers of positive, negative and zero eigenvalues of A, respectively.

The inertia $\mathrm{In}(A)$ determines the geometric type of the quadratic surface $\mathbf{x}^T A\mathbf{x} = 0$ on $\mathbb{R}^n$ in the following sense. Since $\mathrm{In}(-A) = (q, p, k)$ if $\mathrm{In}(A) = (p, q, k)$ and the equation $\mathbf{x}^T A\mathbf{x} = c$ is inconsistent if $p = 0$ and $c > 0$, it suffices to consider the cases of $c \geq 0$ and $p > 0$. Excluding those inconsistent cases, we have the following characterization of the solution sets for $n = 2$ and 3:

For $n = 2$, there are only three possible cases for $\text{In}(A)$:

$\text{In}(A)$	The solution of $\mathbf{x}^T A\mathbf{x} = c$	
(p, q, k)	$c > 0$	$c = 0$
$(2, 0, 0)$	ellipse	a point
$(1, 1, 0)$	hyperbola	two lines crossing at $\mathbf{0}$
$(1, 0, 1)$	two parallel lines	a line

For $n = 3$, there are six possibilities:

$\text{In}(A)$	The solution of $\mathbf{x}^T A\mathbf{x} = c$	
(p, q, k)	$c > 0$	$c = 0$
$(3, 0, 0)$	ellipsoid	a point
$(2, 1, 0)$	one-sheeted hyperboloid	elliptic cone
$(2, 0, 1)$	elliptic cylinder	a line
$(1, 2, 0)$	two-sheeted hyperboloid	elliptic cone
$(1, 1, 1)$	hyperbolic cylinder	two planes crossing in a line
$(1, 0, 2)$	two parallel planes	a plane

In general, for an $n \times n$ symmetric matrix A, $\text{In}(A)$ will have $n(n+1)/2$ possibilities, each characterizing a different geometric type of a quadratic form. For example, if $\text{In}(A) = (n, 0, 0)$ and $c > 0$, i.e., the eigenvalues of A are all positive, then the quadratic form describes an ellipsoid in $\mathbb{R}^n$, etc.

Example 9.4 (*The inertia determines the geometric type of the quadratic form*) Determine the quadratic surface for $2xy + 2xz = 1$ on $\mathbb{R}^3$.

Solution: The matrix for the given quadratic form is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and the eigenvalues of A can be found to be $\lambda_1 = \sqrt{2}$, $\lambda_2 = -\sqrt{2}$, $\lambda_3 = 0$, with associated orthonormal eigenvectors

$$\mathbf{v}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right), \quad \mathbf{v}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

respectively. Hence, an orthogonal matrix P that diagonalizes A is

$$P = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \end{bmatrix},$$

and with the change of coordinates $\mathbf{x} = P\mathbf{y}$, that is,

$$x = \frac{1}{\sqrt{2}}(x' - y'), \quad y = \frac{1}{2}(x' + y' - \sqrt{2}z'), \quad z = \frac{1}{2}(x' + y' + \sqrt{2}z'),$$

the equation is transformed to $\sqrt{2}(x')^2 - \sqrt{2}(y')^2 = 1$, which is a hyperbolic cylinder as shown in Figure 9.6. Note that $\mathrm{In}(A) = (1, 1, 1)$. □

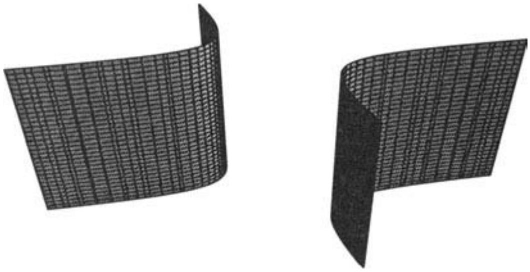

Figure 9.6. Hyperbolic cylinder: $2xy + 2xz = 1$

Now, consider a general form of a quadratic equation on $\mathbb{R}^n$

$$\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} = c.$$

(1) If it does not have a linear form, i.e., $\mathbf{b} = \mathbf{0}$, then, as shown already, a *parabolic* level surface does not appear as a solution of the quadratic equation.

(2) Suppose that it has a nonzero linear form, i.e., $\mathbf{b} \neq \mathbf{0}$. If the matrix A is invertible, then, by taking a change of variables as $\mathbf{y} = \mathbf{x} + \frac{1}{2}A^{-1}\mathbf{b}$ (or $\mathbf{x} = \mathbf{y} - \frac{1}{2}A^{-1}\mathbf{b}$), (it is a translation) the given quadratic equation is transformed into a new quadratic equation $\mathbf{y}^T A\mathbf{y} = d$ without a linear form, where $d = c + \frac{1}{4}\mathbf{b}^T A^{-1}\mathbf{b}$. However, if A is not invertible, the solution of the quadratic equation depends not only on the inertia of A, but also on the type of linear form, and a parabolic level surface appears as the solution of the quadratic equation with a nonzero linear form. For example, the equation $x^2 - z = c$ has a singular quadratic form for which $\mathrm{In}(A) = (1, 0, 2)$ and also has a nonzero linear form that cannot be removed by any change of variables. The solution of this equation is a **parabolic cylinder** when $n = 3$.

Example 9.5 (*A quadratic equation having a linear form*) Determine the conic section for $3x^2 - 6xy + 4y^2 + 2x - 2y = 0$.

Solution: The matrix for the quadratic form $3x^2 - 6xy + 4y^2$ is

$$A = \begin{bmatrix} 3 & -3 \\ -3 & 4 \end{bmatrix}.$$

Its inverse is

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

With the change of variables $\mathbf{y} = \mathbf{x} + \frac{1}{2}A^{-1}\mathbf{b}$, that is

$$x' = x + \frac{1}{3}, \quad y' = y,$$

the equation is transformed to a new equation $3(x')^2 - 6x'y' + 4(y')^2 = \frac{1}{3}$. Clearly, the matrix representation of the new quadratic form is also A, and its eigenvalues are $\frac{1}{2}(7 \pm \sqrt{37})$. Therefore, $\text{In}(A) = (2, 0, 0)$ and the solution of the equation is an ellipse. □

The following is another view of the classifying of the conic sections in $\mathbb{R}^2$ and it can be skipped at the reader's discretion.

Example 9.6 (*The classification of the conic sections in* $\mathbb{R}^2$) Consider a quadratic equation in two variables on $\mathbb{R}^2$:

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0.$$

Or, in matrix form

$$\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + f = 0,$$

with the symmetric matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $\mathbf{b} = [d\ e]^T$ and $\mathbf{x} = [x\ y]^T$ in $\mathbb{R}^2$. We present here the classification of the conic sections according to the coefficients.

(1) If $b = 0$, then A is already a diagonal matrix with the eigenvalues a and c, and the equation becomes

$$ax^2 + cy^2 + dx + ey + f = 0.$$

(i) If $a = 0 = c$, then the conic section is a line in the plane.
(ii) If $a \neq 0 = c$, then it is a parabola when $e \neq 0$, or one or two lines when $e = 0$.
(iii) If $a \neq 0 \neq c$, then the quadratic equation becomes

$$ax^2 + cy^2 + dx + ey + f = a(x-p)^2 + c(y-q)^2 + h = 0$$

for some constants p, q, and h. If $h = 0$, the cases are easily classified (try). Suppose $h \neq 0$. Then, the conic section is a circle if $a = c$, an ellipse if $ac > 0$, or a hyperbola if $ac < 0$.

(2) Suppose that $b \neq 0$. Since A is symmetric, it can be diagonalized by an orthogonal matrix P whose columns are orthonormal eigenvectors, and the diagonal matrix has eigenvalues λ_1 and λ_2 on the diagonal. By a basis-change by P, the quadratic equation becomes

$$ax^2 + 2bxy + cy^2 + dx + ey + f = \lambda_1 u^2 + \lambda_2 v^2 + d'u + e'v + f = 0$$

for some constants d' and e'. Hence, the classification of the conic sections is reduced to the case (1) according to the various possible cases of the eigenvalues of A. However, the eigenvalues are given as

$$\lambda_1 = \frac{(a+c)+\sqrt{(a-c)^2+4b^2}}{2}, \quad \lambda_2 = \frac{(a+c)-\sqrt{(a-c)^2+4b^2}}{2},$$

which are determined by the coefficients a, b, and c. Hence one can classify the conic section according to the various possible cases a, b, and c (see Exercise **9.4**).

(3) The axes of the conic section are the directions of the eigenvectors, which are orthogonal to each other. Since we only need to find axis lines, but not the direction vectors, we may choose them to be the rotation of the standard coordinate x- and y-axes which are determined by $\mathbf{e}_1, \mathbf{e}_2$. Now, a pair of orthogonal eigenvectors are found to be

$$\mathbf{v}_i = \begin{bmatrix} v_{i1} \\ v_{i2} \end{bmatrix} = \begin{bmatrix} b \\ -(a-\lambda_i) \end{bmatrix}$$

for $i = 1, 2$. The slope of $\mathbf{v}_1$ from the x-axis is

$$\begin{aligned} \frac{-(a-\lambda_1)}{b} &= -\frac{a-c}{2b} + \sqrt{\left(\frac{a-c}{2b}\right)^2 + 1} \\ &= -\cot 2\theta + \operatorname{cosec} 2\theta \;=\; \tan\theta, \end{aligned}$$

where $\cot 2\theta = \frac{a-c}{2b}$ for some θ. Since $b \neq 0$ and $a - c > 0$, one may assume that $0 < \theta < \pi$. This means that if we set $\tan 2\theta = \frac{2b}{a-c}$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then θ is the rotation angle we were looking for. Therefore, the orthonormal eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of A may be chosen as the rotation of the standard basis through the angle θ. The basis-change matrix is now

$$P = [\mathbf{u}_1 \;\; \mathbf{u}_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \text{and} \quad P^TAP = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

By a change of coordinates $\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix}$, the quadratic equation becomes

$$ax^2 + 2bxy + cy^2 + dx + ey + f \;=\; \lambda_1 x'^2 + \lambda_2 y'^2 + d'x' + e'y' + f \;=\; 0,$$

where $d' = d\,\cos\theta + e\,\sin\theta$ and $e' = -d\,\sin\theta + e\,\cos\theta$. □

Problem 9.2 Sketch the level surface of each of the following quadratic equations:

(1) $2x^2 + 2y^2 + 6yz + 10z^2 = 9$;
(2) $x^2 - 8xy + 16y^2 - 3z^2 = 8$;
(3) $4x^2 + 12xy + 9y^2 + 3x - 4 = 0$.

9.4 Characterizations of definite forms

In the previous section, we have seen that the geometric type of a quadratic equation $\langle \mathbf{x}, A\mathbf{x} \rangle = 0$ depends on the inertia of the matrix A. Hence, it is important to determine whether or not a Hermitian (or symmetric) matrix A is positive definite or negative definite. In most cases, the definition does not help much for such criteria. But we have seen that Corollary 9.2 gives us a practical characterization of positive definite matrices: *A is positive definite if and only if all eigenvalues of A are positive.* We will find some other practical criteria in terms of the determinant of the matrix. For this, we again look at the quadratic form in two real variables, $q(x, y) = ax^2 + 2bxy + cy^2$, which may be rewritten in a complete square form as

$$q(\mathbf{x}) = ax^2 + 2bxy + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \left(c - \frac{b^2}{a}\right)y^2.$$

We see that q is positive definite, i.e., $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} > 0$ for any nonzero vector $\mathbf{x} = (x, y) \in \mathbb{R}^2$, if and only if $a > 0$ and $ac > b^2$, or equivalently, the determinants of

$$[a] \quad \text{and} \quad \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

are positive.

A generalization of these conditions will involve all n submatrices of A, called the **principal submatrices** of A, which are defined as the upper left square submatrices

$$A_1 = \begin{bmatrix} a_{11} \end{bmatrix}, \; A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \; \ldots, \; A_n = A.$$

With this construction, we have the following characterization of positive definite matrices.

Theorem 9.3 *The following are equivalent for a Hermitian (or a real symmetric) matrix A:*

(1) *A is positive definite, i.e., $\langle \mathbf{x}, A\mathbf{x} \rangle > 0$ for all nonzero vectors $\mathbf{x}$;*
(2) *all the eigenvalues of A are positive;*
(3) *all the principal submatrices A_k's have positive determinants;*
(4) *A can be reduced to an upper triangular matrix by using only the elementary operation of "adding a constant multiple of a row to another," (without row interchanges), and all the pivots are positive;*
(5) *there exists a lower triangular matrix L with positive diagonal entries such that $A = LL^H$, $(= LL^T$ if A is real symmetric) (called a* **Cholesky decomposition** *or a* **Cholesky factorization***);*
(6) *there exists a nonsingular matrix W such that $A = W^H W$, $(= W^T W$ if A is real symmetric).*

Proof: (1) $\Leftrightarrow$ (2) was shown.

(2) $\Rightarrow$ (3) First, we prove it for the real case with a symmetric matrix A. If A has positive eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\det A = \lambda_1\lambda_2\cdots\lambda_n > 0$. To prove the same result for all the submatrices A_k, we claim that if A is positive definite, so is every A_k. For each $k = 1, \ldots, n$, consider all the vectors whose last $n-k$ components are zero, say $\mathbf{x} = [x_1 \ \cdots \ x_k \ 0 \ \cdots \ 0]^T = [\mathbf{x}_k \ \mathbf{0}]^T$, where $\mathbf{x}_k^T$ is any vector in $\mathbb{R}^k$. Then

$$\mathbf{x}^T A\mathbf{x} = [\mathbf{x}_k \ \mathbf{0}]^T \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{0} \end{bmatrix} = \mathbf{x}_k^T A_k \mathbf{x}_k.$$

Since $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero $\mathbf{x}$, $\mathbf{x}_k^T A_k \mathbf{x}_k > 0$ for all nonzero $\mathbf{x}_k \in \mathbb{R}^k$; that is, A_k's are positive definite, all eigenvalues of A_k are positive, and their determinants are positive. The complex case with a Hermitian matrix A can be proved by the same argument except for using ^{H} instead of ^{T}.

(3) $\Rightarrow$ (4) Let $A = [a_{ij}]$. Then the first principal submatrix $A_1 = [a_{11}]$ has positive determinant, i.e., $a_{11} > 0$. Thus, a_{11} can be used as a pivot for forward elimination to make all other first column entries below a_{11} zero. Let $a_{22}^{(1)}$ denote the (2, 2)-entry of the resulting matrix, so that the principal submatrix A_2 has been transformed into a matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}^{(1)} \end{bmatrix}.$$

Since the elementary operation of "adding a constant multiple of a row to another" does not change the determinant, we have

$$\det A_2 = a_{11}a_{22}^{(1)} \quad \text{and} \quad a_{22}^{(1)} = \frac{\det A_2}{a_{11}} = \frac{\det A_2}{\det A_1} > 0.$$

Since $a_{22}^{(1)} \neq 0$, it can be used as a pivot in the second forward elimination, to transform the principal submatrix A_3 into

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}.$$

Also, one can show that

$$\det A_3 = a_{11}a_{22}^{(1)}a_{33}^{(2)} \quad \text{and} \quad a_{33}^{(2)} = \frac{\det A_3}{a_{11}a_{22}^{(1)}} = \frac{\det A_3}{\det A_2} > 0.$$

A similar process can be repeated to get an upper triangular matrix as a row-echelon form of A with the k-th pivot $a_{kk}^{(k-1)}$, which is exactly the ratio of $\det A_k$ to $\det A_{k-1}$. Hence, all pivots are positive.

(4) $\Rightarrow$ (5) First, consider the real case. By the hypothesis (4) and the uniqueness of the LDU factorization of a symmetric matrix, the matrix A can be factored as a product LDL^T with

$$D = \begin{bmatrix} d_1 & & & \mathbf{0} \\ & d_2 & & \\ & & \ddots & \\ \mathbf{0} & & & d_n \end{bmatrix}, \quad d_i > 0.$$

Define

$$\sqrt{D} = \begin{bmatrix} \sqrt{d_1} & & & \mathbf{0} \\ & \sqrt{d_2} & & \\ & & \ddots & \\ \mathbf{0} & & & \sqrt{d_n} \end{bmatrix}.$$

Then, clearly $\det(\sqrt{D}) > 0$, $D = \sqrt{D}\sqrt{D}$ and $(\sqrt{D})^T = \sqrt{D}$. Hence,

$$A = LDL^T = (L\sqrt{D})(\sqrt{D}L^T) = (L\sqrt{D})(L\sqrt{D})^T,$$

as desired. A similar process can be repeated for a complex case with ^{H} instead of ^{T}.

(5) $\Rightarrow$ (6) is easy.

(6) $\Rightarrow$ (1) Let $A = W^T W$ for a nonsingular matrix W. Then, for $\mathbf{x} \neq \mathbf{0}$,

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T (W^T W)\mathbf{x} = (W\mathbf{x})^T (W\mathbf{x}) = \|W\mathbf{x}\|^2 > 0,$$

because $W\mathbf{x} \neq \mathbf{0}$. Similarly for a complex case. □

Problem 9.3 Determine which one of the following matrices A and B is positive definite. For the positive definite one, find a nonsingular matrix W such that it is $W^T W$.

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Problem 9.4 Let A be a positive definite matrix. Prove that $C^T AC$ (or $C^H AC$ for a complex case) is also positive definite for any nonsingular matrix C.

Since a Hermitian matrix A is negative definite if and only if $-A$ is positive definite, one can get the following theorem from Theorem 9.3.

Theorem 9.4 *The following statements are equivalent for a Hermitian (or a real symmetric) matrix A:*

(1) *A is negative definite, i.e., $\langle \mathbf{x}, A\mathbf{x}\rangle < 0$ for all nonzero vectors $\mathbf{x}$;*

(2) *all the eigenvalues of A are negative;*

(3) *the determinants of the principal submatrices A_k's alternate in sign: i.e.,* $\det A_1 < 0$, $\det A_2 > 0$, $\det A_3 < 0$, *and so on;*

(4) *A can be reduced to an upper triangular matrix by using only the elementary operation of "adding a constant multiple of a row to another," (without row interchanges), and all the pivots are negative;*
(5) *there exists a lower triangular matrix L with positive diagonal entries such that* $A = -LL^H$, $(= -LL^T$ *if A is real symmetric);*
(6) *there exists a nonsingular matrix W such that* $A = -W^H W$, $(= -W^T W$ *if A is real symmetric).*

Problem 9.5 Show that the determinant of a negative definite $n \times n$ symmetric matrix is positive if n is even and negative if n is odd.

One can easily establish the following analogous theorem for semidefinite matrices.

Theorem 9.5 *The following statements are equivalent for a Hermitian (or a real symmetric) matrix A:*

(1) *A is positive semidefinite, i.e.,* $\langle \mathbf{x}, A\mathbf{x} \rangle \geq 0$ *for all nonzero vectors* $\mathbf{x}$*;*
(2) *all the eigenvalues of A are nonnegative;*
(3) *A can be reduced to an upper triangular matrix by using only the elementary operation of "adding a constant multiple of a row to another," (without row interchanges), and all the pivots are nonnegative;*
(4) *there exists a matrix W, possibly singular, such that* $A = W^H W$.
$(= W^T W$ *if A is real symmetric).*

Problem 9.6 Determine whether the following statement is true or false: A Hermitian matrix A is positive semidefinite if and only if all the principal submatrices A_k's have nonnegative determinants.

Problem 9.7 State the corresponding conditions to the ones in Theorem 9.5 for the negative semidefinite forms.

Problem 9.8 Which of the following matrices are positive definite? negative definite? indefinite?

$$(1) \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad (2) \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 3 \\ 0 & 3 & 5 \end{bmatrix}, \quad (3) \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

9.5 Congruence relation

As we have seen already, in a quadratic equation $\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = 0$ on $\mathbb{R}^n$, the linear form may be eliminated by a change of variables when A is invertible, and then by the principal axis theorem the equation can be transformed into a simple form $\mathbf{y}^T A\mathbf{y} = c$ having only square terms. Hence, the geometric type of the quadratic equation may be easily classified. However, these changes of variables contain basis changes by some invertible matrices.

Let us now consider a change of basis (or variables) and a relation between two different matrix representations of a quadratic form. Usually a real or complex quadratic form $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$ is expressed in the coordinates of $\mathbf{x}$ with respect to the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$ or for $\mathbb{C}^n$ depending on a real or complex case. Let $\beta = \{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ be another basis. Then, any vector $\mathbf{x}$ has two coordinate representations $[\mathbf{x}]_\alpha$ and $[\mathbf{x}]_\beta$ through the equations

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = \mathbf{x} = y_1\mathbf{e}'_1 + y_2\mathbf{e}'_2 + \cdots + y_n\mathbf{e}'_n.$$

They are related as $[\mathbf{x}]_\alpha = P[\mathbf{x}]_\beta$, where $P = [id]^\alpha_\beta$ is the basis-change matrix from β to α. This is just a change of variables. If we set notations $\mathbf{x} = [\mathbf{x}]_\alpha$ and $\mathbf{y} = [\mathbf{x}]_\beta$, then the quadratic form can be written as

$$q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle = \langle P\mathbf{y}, AP\mathbf{y} \rangle = \langle \mathbf{y}, P^H AP\mathbf{y} \rangle = \langle \mathbf{y}, B\mathbf{y} \rangle,$$

where $B = P^H AP$ and $\langle \mathbf{y}, B\mathbf{y} \rangle$ is the expression of $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$ in a new basis (or a new coordinate system) β.

Definition 9.5 (1) Two real $n \times n$ matrices A and B are said to be **congruent** if there exists an invertible real matrix P such that $P^T AP = B$.

(2) Two complex $n \times n$ matrices A and B are said to be **Hermitian congruent** if there exists an invertible complex matrix P such that $P^H AP = B$.

It is easily seen that the congruence relation is an equivalence relation in the vector space $M_{n\times n}(\mathbb{R})$, and *any two matrix representations of a quadratic form on* $\mathbb{R}^n$ *with respect to different bases are congruent.* A similar statement also holds for a Hermitian congruence and a complex quadratic form.

Remark: **(1)** *Two orthogonally similar real matrices are clearly congruent, but the converse is not true* in general. Clearly, a real symmetric matrix A is congruent to a diagonal matrix D by an orthogonal matrix P. However, it can be congruent to infinitely many different diagonal matrices (not necessarily by orthogonal matrices). In fact, if $P^T AP = D$ by an orthogonal matrix P, then the matrix $Q = kP$, $k \neq 0$, also diagonalizes A to a different diagonal matrix via a congruence relation:

$$Q^T AQ = (kP)^T A(kP) = k^2 P^T AP = k^2 D,$$

which is also diagonal with diagonal entries $k^2\lambda_1, k^2\lambda_2, \ldots, k^2\lambda_n$. In this case, if $k \neq \pm 1$, Q is not an orthogonal matrix and the resulting diagonal entries are not the eigenvalues of A anymore.

(2) Sylvester's law of inertia (Theorem 9.10 in Section 9.6) says that even though a real symmetric matrix A may be congruent to various diagonal matrices, the numbers of positive, negative and zero diagonal entries are invariant under the congruence relation. That is, *any two symmetric matrices which are congruent have the same inertia.* A similarity holds for Hermitian matrices: *any two Hermitian matrices which are Hermitian congruent have the same inertia.* (See Corollary 9.11.)

Certainly, the inertia of a real symmetric matrix (a Hermitian matrix in a complex case) can be found by computing the eigenvalues. However, there is another practical method of diagonalizing it through the congruence (the Hermitian congruence in a complex case) relation by using the elementary row operation of adding a constant multiple of a row (or a column) to another row (or a column).

First, suppose that a real symmetric matrix A is diagonalized by an invertible matrix P through the congruence relation $P^TAP = D$. Since both P and P^T are invertible matrices, P^T can be written as a product of elementary matrices, say $P^T = E_k \cdots E_2E_1$. Then we have

$$D = P^TAP = E_k \cdots E_2E_1AE_1^TE_2^T \cdots E_k^T.$$

Recall that for any elementary matrix E, the product EA is exactly the matrix that is obtained from A when the same elementary row operation is executed on A. Clearly, if E is an elementary matrix, so is E^T. Moreover, if an elementary matrix E is obtained by executing an elementary operation on the i-th row, then the product EAE^T is just the matrix that is obtained from A when the same elementary operation is executed both on the i-th row and on the i-th column. Since A is symmetric, the operation EAE^T will have the same effect on the diagonally opposite entries of A simultaneously. For instance, if

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 6 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is an elementary matrix adding -1 times the first row to the second row, then EA is the matrix obtained from A by replacing the second row $[1\ \ 0\ \ 3]$ by $[0\ \ -1\ \ 1]$. Now, the matrix EAE^T is obtained from the matrix EA by replacing the second column by $[0\ \ -1\ \ 1]^T$ for the symmetry of the matrix EAE^T. In fact,

$$EAE^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & 6 \end{bmatrix}.$$

It implies that the operations performed from the left of A (i.e., the product of $E_k \cdots E_2E_1$) are nothing but a forward elimination on A to get an upper triangular matrix P^TA and those from the right (i.e., the product of $E_1^TE_2^T \cdots E_k^T$) are the corresponding column operations to yield a diagonal matrix D. In summary, *if we take a forward elimination on A to get an upper triangular matrix by the elementary matrices $E_1, \ldots, E_k$, then $E_k \cdots E_1AE_1^T \cdots E_k^T = D$ is diagonal and $P^T = E_k \cdots E_1$.* It gives

$$\begin{aligned} [A \mid I] &\to [E_1AE_1^T \mid E_1I] \to [E_2E_1AE_1^TE_2^T \mid E_2E_1I] \to \cdots \\ &\to [E_k \cdots E_1AE_1^T \cdots E_k^T \mid E_k \cdots E_1I] = [D \mid P^T]. \end{aligned}$$

Remark: **(1)** In the conjugate relation $P^TAP = D$, the matrix P need not be an orthogonal matrix, and the diagonal entries of D need not be eigenvalues of A.

(2) Be careful not to apply the same argument for the diagonalization of symmetric matrices through the similarity $P^{-1}AP = D$, because multiplying E^{-1} on the right of A is not the same column operation as E^T, so that the operations EAE^T do not work for the diagonalization of A.

To diagonalize a Hermitian matrix A through the Hermitian congruence as a complex case, a similar argument as the real case can be applied in parallel with ^{H} instead of ^{T}.

The following example shows how to determine the inertia of a real symmetric matrix A through the congruence relation (instead of computing the eigenvalues).

Example 9.7 (*Computing In*(A) *through the congruence relation*) Determine the inertia of the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 6 \end{bmatrix}.$$

Is it positive definite, negative definite, or indefinite?

Solution: The preceding method produces

$$\begin{aligned}
[A \mid I] &= \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \\
\longrightarrow [E_2E_1AE_1^TE_2^T \mid E_2E_1I] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{array}\right] \\
\longrightarrow [E_3E_2E_1AE_1^TE_2^TE_3^T \mid E_3E_2E_1I] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 1 & 1 \end{array}\right] \\
&= [D \mid P^T],
\end{aligned}$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since the diagonal entries of D are $1, -1$ and 3, we get $\text{In}(A) = (2, 1, 0)$ and A is indefinite. One can check that $P^TAP = D$ by a direct computation. $\square$

Problem 9.9 Find an invertible matrix P such that $P^T AP$ is diagonal for each of the following symmetric matrices:

$$(1)\ A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix},\ (2)\ A = \begin{bmatrix} 1 & -3 & 1 \\ -3 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix},\ (3)\ A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Problem 9.10 For each A of the following Hermitian matrices, find an invertible matrix P such that $P^H AP$ is diagonal and determine $\mathrm{In}(A)$:

$$(1)\ A = \begin{bmatrix} 0 & 1 & i \\ 1 & 1 & 0 \\ -i & 0 & 2 \end{bmatrix},\ (2)\ A = \begin{bmatrix} 1 & 1-3i & 1 \\ 1+3i & 4 & 2i \\ 1 & -2i & 5 \end{bmatrix},\ (3)\ A = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}.$$

9.6 Bilinear and Hermitian forms

In this section, we are concerned with two new forms, *bilinear* and *Hermitian*, to have a little deep insight into a real or complex quadratic form, and prove Sylvester's law of inertia as one of the main results.

Definition 9.6 A **bilinear form** on a pair of real vector spaces V and W is a real-valued function $b : V \times W \to \mathbb{R}$ on $V \times W$ satisfying

(1) $b(k\mathbf{x} + \ell\mathbf{x}',\ \mathbf{y}) = k\, b(\mathbf{x},\ \mathbf{y}) + \ell\, b(\mathbf{x}',\ \mathbf{y})$,
(2) $b(\mathbf{x},\ k\mathbf{y} + \ell\mathbf{y}') = k\, b(\mathbf{x},\ \mathbf{y}) + \ell\, b(\mathbf{x},\ \mathbf{y}')$

for any $\mathbf{x},\ \mathbf{x}'$ in V, $\mathbf{y},\ \mathbf{y}'$ in W and any scalars $k,\ \ell$. In particular, if $V = W$, $b : V \times V \to \mathbb{R}$ is called a bilinear form on V.

The conditions (1) and (2) say that b is linear in the first variable and also in the second variable. In this sense, the function $b : V \times W \to \mathbb{R}$ is said to be *bilinear.*

Example 9.8 (*Every inner product is a bilinear form*) Let A be an $m \times n$ real matrix and let $b : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be defined by $b(\mathbf{x},\ \mathbf{y}) = \mathbf{x}^T A\mathbf{y}$ for $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$. Then b is clearly a bilinear form. In particular, if $m = n$ and $A = I_n$, the identity matrix, then it shows

(1) the dot product on $\mathbb{R}^n$ is a bilinear form. In general,
(2) any inner product $b(\mathbf{x},\ \mathbf{y}) = \langle\mathbf{x},\ \mathbf{y}\rangle$ on a real vector space is a bilinear form. □

Example 9.9 Let V be a vector space and V^* its dual vector space, that is, $V^* = \mathcal{L}(V; \mathbb{R})$. Let $b : V \times V^* \to \mathbb{R}$ be defined by

$$b(\mathbf{v},\ \mathbf{v}^*) = \mathbf{v}^*(\mathbf{v}) \quad \text{for any } \mathbf{v} \in V,\ \mathbf{v}^* \in V^*.$$

Then, b is clearly a bilinear form on the pair of vector spaces V and V^*. □

Definition 9.7 A bilinear form b on a vector space V is said to be **symmetric** if $b(\mathbf{x},\ \mathbf{y}) = b(\mathbf{y},\ \mathbf{x})$ for any $\mathbf{x},\ \mathbf{y} \in V$, and is **skew-symmetric** (or **alternating**) if $b(\mathbf{x},\ \mathbf{y}) = -b(\mathbf{y},\ \mathbf{x})$ for any $\mathbf{x},\ \mathbf{y} \in V$.

For example, the bilinear form $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $b(\mathbf{x},\ \mathbf{y}) = \mathbf{x}^T A\mathbf{y}$ is symmetric (skew-symmetric, respectively) if and only if the matrix A is symmetric (skew-symmetric, respectively). Clearly, the dot product and a real quadratic form on $\mathbb{R}^n$ are symmetric bilinear forms.

Problem 9.11 Show that a bilinear form b on $\mathbb{R}^n$ is skew-symmetric if and only if $b(\mathbf{x},\ \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

Definition 9.8 A **sesquilinear form** on a complex vector space V is a complex-valued function $b : V \times V \to \mathbb{C}$ satisfying

(1) $b(k\mathbf{x} + \ell\mathbf{x}',\ \mathbf{y}) = \bar{k}\, b(\mathbf{x},\ \mathbf{y}) + \bar{\ell}\, b(\mathbf{x}',\ \mathbf{y})$ (semilinear in 1^{st} variable),
(2) $b(\mathbf{x},\ k\mathbf{y} + \ell\mathbf{y}') = k\, b(\mathbf{x},\ \mathbf{y}) + \ell\, b(\mathbf{x},\ \mathbf{y}')$ (linear in 2^{nd} variable)

for any $\mathbf{x},\ \mathbf{x}',\ \mathbf{y},\ \mathbf{y}'$ in V and any complex scalars $k,\ \ell$.

A sesquilinear form is called **Hermitian** if it satisfies

(3) $b(\mathbf{x},\ \mathbf{y}) = \overline{b(\mathbf{y},\ \mathbf{x})}$ for any $\mathbf{x},\ \mathbf{y}$ in V.

Example 9.10 (*Every complex inner product is a Hermitian form*) For any $n \times n$ complex matrix A, the function $b : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by $b(\mathbf{x},\ \mathbf{y}) = \mathbf{x}^H A\mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ is a Hermitian form if and only if A is a Hermitian matrix. In fact, $b(\mathbf{x},\ \mathbf{y}) = \mathbf{x}^H A\mathbf{y}$ is certainly semilinear in the first variable, and $\mathbf{y}^H A\mathbf{x} = \overline{\mathbf{x}^H A\mathbf{y}}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ if and only if the matrix A is Hermitian. As a special case, if one takes $A = I_n$, the identity matrix, then it shows

(1) the dot product on $\mathbb{C}^n$ is a Hermitian form. In general,
(2) any complex inner product $b(\mathbf{x},\ \mathbf{y}) = \langle \mathbf{x},\ \mathbf{y} \rangle$ on a complex vector space is a Hermitian form. □

Let $b : V \times V \to \mathbb{R}$ be a bilinear form on a real vector space V, and let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. Such a bilinear form is completely determined by the values $b(\mathbf{v}_i,\ \mathbf{v}_j)$ of the vectors $\mathbf{v}_i, \mathbf{v}_j$ in the basis α because of the bilinearity. In fact, if

$$\begin{array}{rcl} \mathbf{x} &=& x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n, \\ \mathbf{y} &=& y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n \end{array}$$

are vectors in V, then

$$b(\mathbf{x},\ \mathbf{y}) = \sum_{i,j=1}^{n} x_i y_j b(\mathbf{v}_i,\ \mathbf{v}_j) = [\mathbf{x}]_\alpha^T A[\mathbf{y}]_\alpha,$$

where $A = [a_{ij}]$, $a_{ij} = b(\mathbf{v}_i,\ \mathbf{v}_j)$. It is called the **matrix representation** of b with respect to the basis α and denoted by $[b]_\alpha$. Let β be another basis for the vector space V and let $P = [id]_\beta^\alpha$ be the basis-change matrix from β to α. Then we get

$$P[\mathbf{x}]_\beta = [id]_\beta^\alpha[\mathbf{x}]_\beta = [\mathbf{x}]_\alpha$$

for any $\mathbf{x}$ in V, and

$$b(\mathbf{x},\ \mathbf{y}) = [\mathbf{x}]_\alpha^T A[\mathbf{y}]_\alpha = [\mathbf{x}]_\beta^T(P^T AP)[\mathbf{y}]_\beta$$

for any $\mathbf{x}$ and $\mathbf{y}$ in V. Thus, *two matrix representations of a bilinear form b with respect to different bases are congruent, and conversely any two congruent matrices can be matrix representations of the same bilinear form* (verify it). Moreover, a bilinear form is symmetric (or skew-symmetric) if and only if its matrix representation is symmetric (or skew-symmetric) for any basis.

A similar process works in a complex case with ^{H} instead of ^{T} in order to have a **matrix representation** $[b]_\alpha$ of the sesquilinear form b with respect to the basis α. As the real case, one can show that *two matrix representations of a sesquilinear form b with respect to different bases are Hermitian congruent, and conversely any two Hermitian congruent matrices can be matrix representations of the same sesquilinear form.* Moreover, a sesquilinear form b is Hermitian if and only if its matrix representation is Hermitian for any basis.

Problem 9.12 Prove:

(1) A bilinear form b is symmetric (or skew-symmetric, resp.) if and only if $b(\mathbf{v}_i,\ \mathbf{v}_j) = b(\mathbf{v}_j,\ \mathbf{v}_i)$ (or $b(\mathbf{v}_i,\ \mathbf{v}_j) = -b(\mathbf{v}_j,\ \mathbf{v}_i)$, resp.) for any vectors $\mathbf{v}_i,\ \mathbf{v}_j$ in a basis α, or equivalently, the matrix representation $[b]_\alpha$ is symmetric (or skew-symmetric, resp.) for some basis α.

(2) A sesquilinear form is Hermitian if and only if the matrix representation $[b]_\alpha$ is Hermitian for some basis α.

(3) A sesquilinear form on a complex vector space V is called **skew-Hermitian** if it satisfies $b(\mathbf{x},\ \mathbf{y}) = -\overline{b(\mathbf{y},\ \mathbf{x})}$ for any $\mathbf{x},\ \mathbf{y}$ in V. Show that a sesquilinear form b is skew-Hermitian if and only if its matrix representation $[b]_\alpha$ is skew-Hermitian for some basis α.

Note that congruent or Hermitian congruent matrices have the same rank because the basis-change matrix P is nonsingular and so is P^T (or P^H).

Definition 9.9 The **rank** of a bilinear or a sesquilinear form b on a vector space V, written rank(b), is defined as the rank of any matrix representation of b.

Example 9.11 (*Computing* rank(b)) Let $b : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $b(\mathbf{x},\ \mathbf{y}) = x_1y_1 + 3x_1y_2 + 2x_2y_1 - x_2y_2$ with respect to the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$. Then, b is clearly a bilinear form but not symmetric, and the matrix representation of b with respect to α is

$$[b]_\alpha = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}.$$

If $\beta = \{\mathbf{v}_1,\ \mathbf{v}_2\}$ with $\mathbf{v}_1 = (1, 0),\ \mathbf{v}_2 = (1,\ 1)$ is another basis for $\mathbb{R}^2$, then the matrix representation of b with respect to β becomes

$$[b]_\beta = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix},$$

because $b(\mathbf{v}_1, \mathbf{v}_1) = 1$, $b(\mathbf{v}_1, \mathbf{v}_2) = 4$, $b(\mathbf{v}_2, \mathbf{v}_1) = 3$ and $b(\mathbf{v}_2, \mathbf{v}_2) = 5$. Hence, $\operatorname{rank}[b]_\alpha = \operatorname{rank}[b]_\beta = 2$, and the rank of b is also 2. □

Problem 9.13 (1) Let $b : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by $b(\mathbf{x}, \mathbf{y}) = x_1y_1 - 2x_1y_2 + x_2y_1 - x_3y_3$ with respect to the standard basis. Is this a bilinear form? If so, find the matrix representation of b with respect to the basis

$$\alpha = \{\mathbf{v}_1 = (1, 0, 1), \mathbf{v}_2 = (1, 0, -1), \mathbf{v}_3 = (0, 1, 0)\}.$$

Find its rank.

(2) Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices, and let $b : V \times V \to \mathbb{R}$ be defined by $b(A, B) = \operatorname{tr}(A) \cdot \operatorname{tr}(B)$. Is this a bilinear form? If so, find the matrix representation of b with respect to the basis

$$\alpha = \left\{E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

Find its rank.

9.7 Diagonalization of bilinear or Hermitian forms

Every inner product $\langle \mathbf{x}, A\mathbf{x}\rangle$ on a real vector space can be represented by a symmetric matrix A, which is diagonalizable. However, it is not true for a bilinear form.

Definition 9.10 A bilinear (or sesquilinear) form b on V is **diagonalizable** if there exists a basis α for V such that the matrix representation $[b]_\alpha$ of b with respect to α is diagonal.

Theorem 9.6 **(1)** *A bilinear form b on a real vector space V is symmetric if and only if it is diagonalizable.*

(2) *A sesquilinear form b on a complex vector space V is Hermitian if and only if it is diagonalizable in which all diagonal entries are real.*

Proof: We prove only (1) and leave (2) as an exercise. Since every symmetric matrix is orthogonally diagonalizable, we only need to prove the sufficiency. Let a bilinear form b be diagonalizable so that the matrix representation $[b]_\alpha$ is diagonal for some basis α. Then, for any vectors $\mathbf{v}_i, \mathbf{v}_j$ in a basis $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, we have $b(\mathbf{v}_i, \mathbf{v}_j) = b(\mathbf{v}_j, \mathbf{v}_i)$. Now, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in V, let $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{v}_i$ and $\mathbf{y} = \sum_{j=1}^n y_j\mathbf{v}_j$. Then,

$$b(\mathbf{x}, \mathbf{y}) = \sum_{i,j=1}^{n} x_iy_jb(\mathbf{v}_i, \mathbf{v}_j) = \sum_{i,j=1}^{n} y_jx_ib(\mathbf{v}_j, \mathbf{v}_i) = b(\mathbf{y}, \mathbf{x}).$$

Hence, b is symmetric. (See also Problem 9.12(1).) □

Problem 9.14 Prove Theorem 9.6(2): a sesquilinear form b on a complex vector space V is Hermitian if and only if it is diagonalizable in which all diagonal entries are real.

Example 9.12 (*Diagonalizing a symmetric bilinear form*) Let $b : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form defined by

$$b(\mathbf{x},\ \mathbf{y}) = x_1y_3 - 2x_2y_2 + 2x_2y_3 + x_3y_1 + 2x_3y_2 - x_3y_3.$$

Clearly, $b(\mathbf{x},\ \mathbf{y}) = b(\mathbf{y},\ \mathbf{x})$, and the matrix representation of b with respect to the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2,\ \mathbf{e}_3\}$ is

$$[b]_\alpha = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix},$$

which is symmetric. Hence, the bilinear form b is symmetric. By Theorem 9.6, it is diagonalizable through the congruence. In fact,

$$\begin{aligned}
[[b]_\alpha \mid I] &= \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \\
\longrightarrow\ [E_1[b]_\alpha E_1^T \mid E_1 I] &= \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \\
\longrightarrow\ [E_2E_1[b]_\alpha E_1^T E_2^T \mid E_2E_1 I] &= \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & -1 & -1 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \\
&= [D \mid P^T],
\end{aligned}$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\quad E_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By a direct computation, one can show that $P^T[b]_\alpha P = D$.

Moreover, if we take another basis $\beta = \{\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3\}$ consisting of the column vectors of the matrix P, then $P = [id]_\beta^\alpha$ and

$$b(\mathbf{x},\ \mathbf{y}) = [\mathbf{x}]_\alpha^T[b]_\alpha[\mathbf{y}]_\alpha = ([id]_\beta^\alpha[\mathbf{x}]_\beta)^T[b]_\alpha([id]_\beta^\alpha[\mathbf{y}]_\beta) = [\mathbf{x}]_\beta^T D[\mathbf{y}]_\beta.$$

Hence, if we write $[\mathbf{x}]_\beta = (x_1', x_2', x_3')$ and $[\mathbf{y}]_\beta = (y_1', y_2', y_3')$ as new variables, then the bilinear form b becomes

$$b(\mathbf{x},\ \mathbf{y}) = -x_1'y_1' - 2x_2'y_2' + x_3'y_3'.$$

□

A skew-symmetric matrix is not diagonalizable in general, but the following theorem shows the structure of a skew-symmetric bilinear form. Note that a bilinear form b is skew-symmetric if and only if $b(\mathbf{x}, \mathbf{x}) = 0$ for any $\mathbf{x}$ in V.

Theorem 9.7 *Let $b : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear form. Then there exists a basis α for V with respect to which the matrix representation $[b]_\alpha$ is of the form*

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & & & & \mathbf{0} \\ & \ddots & & & & \\ & & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & \mathbf{0} & & & & 0 \end{bmatrix}.$$

Proof: If $b = 0$, then $[b]_\alpha$ is the zero matrix. Also if $\dim V = 1$, then $b(\mathbf{x}, \mathbf{x}) = 0$ for any basis vector $\mathbf{x}$ in V, so $b = 0$.

Now, we assume that $b \neq 0$ and prove it by induction on $\dim V$. Since $b \neq 0$, there exist nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in V such that $b(\mathbf{x}, \mathbf{y}) \neq 0$. By the bilinearity of b, one can assume that $b(\mathbf{x}, \mathbf{y}) = 1$. Such vectors $\mathbf{x}$ and $\mathbf{y}$ must be linearly independent, because if $\mathbf{y} = k\mathbf{x}$, then $b(\mathbf{x}, \mathbf{y}) = kb(\mathbf{x}, \mathbf{x}) = 0$. Let U be the subspace of V spanned by $\mathbf{x}$ and $\mathbf{y}$, and let

$$W = \{\mathbf{v} \in V : b(\mathbf{v}, \mathbf{u}) = 0 \text{ for any } \mathbf{u} \in U\}.$$

Then, one can easily show that W is also a subspace of V and $U \cap W = \{\mathbf{0}\}$. Moreover, $U + W = V$. In fact for a given vector $\mathbf{v} \in V$, let $\mathbf{u} = b(\mathbf{v}, \mathbf{y})\mathbf{x} - b(\mathbf{v}, \mathbf{x})\mathbf{y}$. It is easy to show that $\mathbf{u} \in U$ and $\mathbf{v} - \mathbf{u} \in W$. Thus $V = U \oplus W$, where $\dim W = n-2$. Clearly, the matrix representation of the restriction of b to U with respect to the basis $\{\mathbf{x}, \mathbf{y}\}$ is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and the restriction of b to W is also skew-symmetric. The induction hypothesis can be applied to W, and then one can finish the proof. $\square$

Problem 9.15 Prove that $U \cap W = \{\mathbf{0}\}$ in the proof of Theorem 9.7.

Example 9.13 (*Block diagonalizing a skew-symmetric bilinear form*) Let $b : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form defined by

$$b(\mathbf{x}, \mathbf{y}) = x_1y_2 - x_2y_1 + x_3y_1 - x_1y_3 + x_2y_3 - x_3y_2.$$

Clearly, $b(\mathbf{x}, \mathbf{y}) = -b(\mathbf{y}, \mathbf{x})$, and the matrix representation of b with respect to the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is

$$[b]_\alpha = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

which is skew-symmetric. By a simple computation, $b(\mathbf{e}_1, \mathbf{e}_2) = 1 = -b(\mathbf{e}_2, \mathbf{e}_1)$. Let U be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, i.e., the xy-plane. If we set $W = \{\mathbf{v} \in V : b(\mathbf{v}, \mathbf{u}) = 0 \text{ for any } \mathbf{u} \in U\}$, then $W = \{\lambda\mathbf{z} : \lambda \in \mathbb{R}\}$, where $\mathbf{z} = (1, 1, 1)$. Clearly, $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{z}\}$ is a basis for $\mathbb{R}^3$ and $b(\mathbf{z}, \mathbf{z}) = 0$ so that

$$[b]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \square$$

Problem 9.16 Show that any bilinear form b on a vector space V is the sum of a symmetric bilinear form and a skew-symmetric bilinear form.

The following theorem shows how quadratic forms and symmetric bilinear forms are related.

Theorem 9.8 *If b is a symmetric bilinear form on $\mathbb{R}^n$, then the function $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$ is a quadratic form.*

Conversely, for every quadratic form q, there is a unique symmetric bilinear form b such that $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Proof: If $b(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{y}$ is a symmetric bilinear form, then $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is clearly a quadratic form.

Conversely, if b is a symmetric bilinear form, then

$$b(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) = b(\mathbf{x}, \mathbf{x}) + 2b(\mathbf{x}, \mathbf{y}) + b(\mathbf{y}, \mathbf{y}),$$

which is called the *polar form* of b. Hence, for any given quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ with a symmetric matrix A, a bilinear form b can be defined by

$$b(\mathbf{x}, \mathbf{y}) = \frac{1}{2}[q(\mathbf{x}+\mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y})].$$

This form b is clearly symmetric, bilinear and $b(\mathbf{x}, \mathbf{x}) = q(\mathbf{x})$. The uniqueness also comes from this relation. $\square$

The following theorem shows how complex quadratic forms and Hermitian forms are related.

Theorem 9.9 *If b is a Hermitian form on $\mathbb{C}^n$, then the function $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$ for $\mathbf{x} \in \mathbb{C}^n$ is a complex quadratic form.*

Conversely, for every complex quadratic form q, there is a unique Hermitian form b such that $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{C}^n$.

Proof: If $b(\mathbf{x}, \mathbf{y}) = \mathbf{x}^H A\mathbf{y}$ is a Hermitian form, then $q(\mathbf{x}) = b(\mathbf{x}, \mathbf{x}) = \mathbf{x}^H A\mathbf{x}$ is clearly a complex quadratic form.

Conversely, if b is a Hermitian form, then

$$\begin{aligned} b(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) &= b(\mathbf{x}, \mathbf{x}) + b(\mathbf{y}, \mathbf{y}) + b(\mathbf{x}, \mathbf{y}) + \overline{b(\mathbf{x}, \mathbf{y})}, \\ b(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}) &= b(\mathbf{x}, \mathbf{x}) + b(\mathbf{y}, \mathbf{y}) - b(\mathbf{x}, \mathbf{y}) - \overline{b(\mathbf{x}, \mathbf{y})}. \end{aligned}$$

Hence, for any given complex quadratic form $q(\mathbf{x}) = \mathbf{x}^H A\mathbf{x}$ with a Hermitian matrix A, a Hermitian form b can be defined by

$$b(\mathbf{x}, \mathbf{y}) = \frac{1}{4}[q(\mathbf{x}+\mathbf{y}) - q(\mathbf{x}-\mathbf{y})] + \frac{i}{4}[q(i\mathbf{x}+\mathbf{y}) - q(i\mathbf{x}-\mathbf{y})],$$

which is called the *polar form* of a Hermitian form b. This form b is clearly Hermitian and $b(\mathbf{x}, \mathbf{x}) = q(\mathbf{x})$, which implies the uniqueness of b. □

Now, we prove Sylvester's law of inertia.

Theorem 9.10 (Sylvester's law of inertia) *Let b be a symmetric bilinear or a Hermitian form on a vector space V. Then, the number of positive diagonal entries and the number of negative diagonal entries of any diagonal representation of b are both independent of the diagonal representation.*

Proof: We only prove it for a symmetric bilinear form, because the other case can be proved by a similar method. Let b be a symmetric bilinear form on a vector space V and let $\alpha = \{\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1}, \ldots, \mathbf{x}_n\}$ be an ordered basis for V in which

$$\begin{aligned} b(\mathbf{x}_i, \mathbf{x}_i) > 0 &\quad \text{for } i = 1, 2, \ldots, p, \text{ and} \\ b(\mathbf{x}_i, \mathbf{x}_i) \le 0 &\quad \text{for } i = p+1, \ldots, n, \end{aligned}$$

and let $\beta = \{\mathbf{y}_1, \ldots, \mathbf{y}_{p'}, \mathbf{y}_{p'+1}, \ldots, \mathbf{y}_n\}$ be another ordered basis for V in which

$$\begin{aligned} b(\mathbf{y}_i, \mathbf{y}_i) > 0 &\quad \text{for } i = 1, 2, \ldots, p', \text{ and} \\ b(\mathbf{y}_i, \mathbf{y}_i) \le 0 &\quad \text{for } i = p'+1, \ldots, n. \end{aligned}$$

To show $p = p'$, let U and W be subspaces of V spanned by $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ and $\{\mathbf{y}_{p'+1}, \ldots, \mathbf{y}_n\}$, respectively. Then, $b(\mathbf{u}, \mathbf{u}) > 0$ for any nonzero vector $\mathbf{u} \in U$ and $b(\mathbf{w}, \mathbf{w}) \le 0$ for any nonzero vector $\mathbf{w} \in W$ by the bilinearity of b. Thus, $U \cap W = \{\mathbf{0}\}$, and

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = p + (n - p') \le n,$$

or $p \le p'$. Similarly, one can show $p' \le p$ to conclude $p = p'$. Therefore, any two diagonal matrix representations of b have the same number of positive diagonal entries. By considering the bilinear form $-b$ instead of b, one can also have that any two diagonal matrix representations of b have the same number of negative diagonal entries. □

Corollary 9.11 (1) *Any two symmetric matrices which are congruent have the same inertia.*
(2) *Any two Hermitian matrices which are Hermitian congruent have the same inertia.*

Definition 9.11 Let A be a real symmetric or a Hermitian matrix. The number of positive eigenvalues of A is called the **index** of A. The difference between the number of positive eigenvalues and the number of negative eigenvalues of A is called the **signature** of A.

Hence, the index and signature together with the rank of a symmetric or a Hermitian matrix are invariants under the congruence relation, and any two of these invariants determine the third: that is,

$$\boxed{\begin{aligned} \text{the index } &= \text{ the number of positive eigenvalues,} \\ \text{the rank } &= \text{ the index } + \text{ the number of negative eigenvalues,} \\ \text{the signature } &= \text{ the index } - \text{ the number of negative eigenvalues.} \end{aligned}}$$

We have shown the necessary condition of the following corollary.

Corollary 9.12 (1) *Two symmetric matrices are congruent if and only if they have the same invariants; index, signature and rank.*
(2) *Two Hermitian matrices are Hermitian congruent if and only if they have the same invariants.*

Proof: We only prove (1). Suppose that two symmetric matrices A and B have the same invariants, and let D and E be diagonal matrices congruent to A and B, respectively. Without loss of generality, one may choose D and E so that the diagonal entries are in the order of positive, negative and zero. Let p and r denote the index and the rank, respectively, of both D and E. Let d_i denote the i-th diagonal entry of D. Define the diagonal matrix Q whose i-th diagonal entry q_i is given by

$$q_i = \begin{cases} 1/\sqrt{d_i} & \text{if } 1 \le i \le p \\ 1/\sqrt{-d_i} & \text{if } p < i \le r \\ 1 & \text{if } r < i \le n. \end{cases}$$

Then,

$$Q^T D Q = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_{r-p} & 0 \\ 0 & 0 & 0 \end{bmatrix} = J_{pr}.$$

Hence, A is congruent to J_{pr}, and similarly so is B. It concludes that A is congruent to B. $\square$

Example 9.14 Determine the index, the signature and the rank for each of the following matrices.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Which are congruent to each other?

Solution: In Example 9.7, we saw that the matrix A is congruent to the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Therefore, A has rank 3, index 2 and signature 1. The matrix B is already diagonal, and has rank 3, index 3 and signature 3. Using the method of Example 9.7, one can show that C is congruent to the diagonal matrix with diagonal entries $1, 1, -4$. Therefore, C has rank 3, index 2 and signature 1. (Note that it is not necessary to find the eigenvalue of C to determine its invariants.) We conclude that A and C are congruent and B is congruent to neither A nor C by Corollary 9.12. □

Problem 9.17 Prove that if the diagonal entries of a diagonal matrix are permuted, then the resulting diagonal matrix is congruent to the original one.

Problem 9.18 Prove that the total number of distinct equivalence classes of congruent $n \times n$ real symmetric matrices is equal to $\frac{1}{2}(n+1)(n+2)$.

Problem 9.19 Find the signature, the index and the rank of each of the following matrices.

$$(1) \begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, \quad (2) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad (3) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Which are congruent to each other?

9.8 Applications

9.8.1 Extrema of real-valued functions on $\mathbb{R}^n$

In calculus, one uses the second derivative test to see whether a given function $y = f(x)$ takes a local maximum or a local minimum at a critical point. In this section, we show a similar test for a function of more than one variable and also show how quadratic forms arise and how they can be used in this context.

Let $f(\mathbf{x})$ be a real-valued function (not necessarily a quadratic equation) on $\mathbb{R}^n$. A point $\mathbf{x}_0$ in $\mathbb{R}^n$ at which either a first partial derivative of f fails to exist or the first partial derivatives of f are all zero is called a **critical point** of f. If $f(\mathbf{x})$ has either a local maximum or a local minimum at a point $\mathbf{x}_0$ and all the first partial derivatives of f exist at $\mathbf{x}_0$, then all of them must be zero, i.e., $f_{x_i}(\mathbf{x}_0) = 0$ for all $i = 1, 2, \ldots, n$. Thus, if $f(\mathbf{x})$ has first partial derivatives everywhere, its local maxima and minima will occur at critical points.

Let us first consider a function of two variables: $f(\mathbf{x})$, $\mathbf{x} = (x, y) \in \mathbb{R}^2$, which has a critical point $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$. If f has continuous third partial derivatives in a neighborhood of $\mathbf{x}_0$, it can be expanded in a Taylor series about that point: For $\mathbf{x} = (x_0 + h, y_0 + k)$,

$$\begin{aligned} f(\mathbf{x}) &= f(x_0 + h, y_0 + k) = f(\mathbf{x}_0) + \big(hf_x(\mathbf{x}_0) + kf_y(\mathbf{x}_0)\big) \\ &\quad + \frac{1}{2}\Big(h^2 f_{xx}(\mathbf{x}_0) + 2hkf_{xy}(\mathbf{x}_0) + k^2 f_{yy}(\mathbf{x}_0)\Big) + R \\ &= f(\mathbf{x}_0) + \frac{1}{2}(ah^2 + 2bhk + ck^2) + R, \end{aligned}$$

where

$$a = f_{xx}(\mathbf{x}_0), \qquad b = f_{xy}(\mathbf{x}_0), \qquad c = f_{yy}(\mathbf{x}_0),$$

and the remainder R is given by

$$R = \frac{1}{6}\Big(h^3 f_{xxx}(\mathbf{z}) + 3h^2 k f_{xxy}(\mathbf{z}) + 3hk^2 f_{xyy}(\mathbf{z}) + k^3 f_{yyy}(\mathbf{z})\Big),$$

with $\mathbf{z} = (x_0 + \theta h, y_0 + \theta k)$ for some $0 < \theta < 1$.

If h and k are sufficiently small, $|R|$ will be smaller than the absolute value of $\frac{1}{2}(ah^2 + 2bhk + ck^2)$, and hence $f(\mathbf{x}) - f(\mathbf{x}_0)$ and $ah^2 + 2bhk + ck^2$ will have the same sign. Note that the expression

$$q(h, k) = ah^2 + 2bhk + ck^2 = [h\ k]H\begin{bmatrix} h \\ k \end{bmatrix}$$

is a quadratic form in the variables h and k, where

$$H = H(\mathbf{x}_0) = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} f_{xx}(\mathbf{x}_0) & f_{xy}(\mathbf{x}_0) \\ f_{xy}(\mathbf{x}_0) & f_{yy}(\mathbf{x}_0) \end{bmatrix}$$

is a symmetric matrix, called the **Hessian** of f at $\mathbf{x}_0 = (x_0, y_0)$. Hence, $f(x, y)$ has a local minimum (or maximum) at $\mathbf{x}_0$ if the quadratic form $q(h, k)$ is positive (or negative, respectively) for all sufficiently small (h, k). The critical point $\mathbf{x}_0$ is called a **saddle point** if $q(h, k)$ takes both positive and negative values. Thus, at this point $f(x, y)$ has neither a local minimum nor a local maximum. (This is *the second derivative test* for a local extrema of $f(x, y)$.)

In particular, a quadratic form

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = [x\ y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$$

for $\mathbf{x} = [x\ y]^T \in \mathbb{R}^2$ is itself a function of two variables, and its first partial derivatives are

$$\begin{aligned} q_x &= 2ax + 2by, \\ q_y &= 2bx + 2cy. \end{aligned}$$

By setting these equal to zero, we see that $\mathbf{0} = (0,\ 0)$ is a critical point of q. If $ac - b^2 \neq 0$, this will be the only critical point of q. Note the Hessian of q is

$$H = 2\begin{bmatrix} a & b \\ b & c \end{bmatrix} = 2A.$$

Thus, H is nonsingular if and only if $ac - b^2 \neq 0$.

Since $q(\mathbf{0}) = 0$, it follows that the quadratic form q takes the *global minimum* at $\mathbf{0}$ if and only if

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} > 0 \qquad \text{for all} \qquad \mathbf{x} \neq \mathbf{0},$$

and q takes the *global maximum* at $\mathbf{0}$ if and only if

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} < 0 \qquad \text{for all} \qquad \mathbf{x} \neq \mathbf{0}.$$

If $\mathbf{x}^T A\mathbf{x}$ takes both positive and negative values, then $\mathbf{0}$ is a saddle point. Thus, if A is nonsingular, the quadratic form q will have either the global minimum, the global maximum or a saddle point at $\mathbf{0}$.

In general, if a function f of two variables has a nonsingular Hessian H at a critical point $\mathbf{x}_0 = (x_0,\ y_0)$ which has nonzero eigenvalues λ_1 and λ_2, then the *second derivative test* for $f(\mathbf{x})$ says

(1) f has a minimum at $\mathbf{x}_0$ if both λ_1 and λ_2 are positive,
(2) f has a maximum at $\mathbf{x}_0$ if both λ_1 and λ_2 are negative,
(3) f has a saddle point at $\mathbf{x}_0$ if λ_1 and λ_2 have different signs.

Example 9.15 (*The extrema of a quadratic form $f(x, y) = \mathbf{x}^T A\mathbf{x}$ can be determined by the inertia of A*) For $q(x, y) = 2x^2 - 4xy + 5y^2$, determine the nature of the critical point $(0,\ 0)$.

Solution: The matrix of the quadratic form is

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}.$$

There are two methods:

(1) *Similarity method*: Solve $\det(\lambda I - A)$ to get eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$. Since both eigenvalues are positive, A is positive definite and hence $(0,\ 0)$ is a global minimum.

(2) *Congruence method*: Diagonalize the matrix A through the congruence relation to get

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \quad \longrightarrow \quad EAE^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

where $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. It shows that $\text{In}(A) = (2, 0, 0)$ and A is positive definite and hence $(0,\ 0)$ is a global minimum. $\square$

Example 9.16 (*The inertia of the Hessian determines the local extrema of any (non-quadratic) $f(x, y)$ at critical points*) Find and describe all critical points of the function

$$f(x,\ y) = \frac{1}{3}x^3 + xy^2 - 4xy + 1.$$

Solution: The first partial derivatives of f are

$$f_x = x^2 + y^2 - 4y, \quad f_y = 2xy - 4x \ = \ 2x(y-2).$$

Setting $f_y = 0$, we get $x = 0$ or $y = 2$. Setting $f_x = 0$, we see that if $x = 0$, then y must be either 0 or 4, and if $y = 2$, then $x = \pm 2$. Thus, $(0,\ 0)$, $(0,\ 4)$, $(2,\ 2)$, $(-2,\ 2)$ are the critical points of f. To classify these critical points, we compute the second partial derivatives:

$$f_{xx} = 2x, \quad f_{xy} = 2y - 4, \quad f_{yy} = 2x.$$

For each critical point $(x_0,\ y_0)$, one can determine the eigenvalues λ_1 and λ_2 of the Hessian

$$H = \begin{bmatrix} 2x_0 & 2y_0 - 4 \\ 2y_0 - 4 & 2x_0 \end{bmatrix}.$$

These values are summarized in the following table:

Critical Point $(x_0,\ y_0)$	λ_1	λ_2	Description
$(0,\ 0)$	4	-4	saddle point
$(0,\ 4)$	4	-4	saddle point
$(2,\ 2)$	4	4	local minimum
$(-2,\ 2)$	-4	-4	local maximum

As an alternative method, one can compute the inertia of the Hessian at each critical point by a congruence relation and get the same description of the nature of critical points. $\square$

Beyond the functions of two variables, the same argument of the second derivative test for functions of two variables can be justified for functions of more than two variables with a Tayler series about critical points: Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ be a real-valued function whose third partial derivatives are all continuous. If $\mathbf{x}_0$ is a critical point of f, the **Hessian** of f at $\mathbf{x}_0$ is the $n \times n$ symmetric matrix $H = H(\mathbf{x}_0) = [h_{ij}]$ given by

$$h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0).$$

The critical point can be classified as follows:

(1) f has a local minimum at $\mathbf{x}_0$ if $H(\mathbf{x}_0)$ is positive definite,
(2) f has a local maximum at $\mathbf{x}_0$ if $H(\mathbf{x}_0)$ is negative definite,
(3) $\mathbf{x}_0$ is a saddle point of f if $H(\mathbf{x}_0)$ is indefinite.

Example 9.17 (*The inertia of the Hessian determines the extrema of any $f(x, y, z)$ at critical points*) Find the local extrema of the function

$$f(x, y, z) = x^2 + xz - 3\cos y + z^2.$$

Solution: The first partial derivatives of f are

$$f_x = 2x + z, \quad f_y = 3\sin y, \quad f_z = x + 2z.$$

It follows that (x, y, z) is a critical point of f if and only if $x = z = 0$ and $y = n\pi$, where n is an integer. Let $\mathbf{x}_0 = (0,\ 2k\pi,\ 0)$. The Hessian of f at $\mathbf{x}_0$ is given by

$$H(\mathbf{x}_0) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

It can be diagonalized through the congruence relation to get

$$H(\mathbf{x}_0) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}.$$

It shows that $\mathrm{In}(H(\mathbf{x}_0)) = (3, 0, 0)$ and $H(\mathbf{x}_0)$ is positive definite and hence f has a local minimum at $\mathbf{x}_0$. (Alternatively, one can compute the eigenvalues of $H(\mathbf{x}_0)$ which are 3, 3, and 1, which implies that $H(\mathbf{x}_0)$ is positive definite.)

On the other hand, at a critical point of the form $\mathbf{x}_1 = (0,\ (2k - 1)\pi,\ 0)$, the Hessian will be

$$H(\mathbf{x}_1) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

One can show either that $\mathrm{In}(H(\mathbf{x}_1)) = (2, 1, 0)$ by using a congruence relation or that the eigenvalues of $H(\mathbf{x}_1)$ are -3, 3, and 1. Either one shows that $H(\mathbf{x}_1)$ is indefinite and hence $\mathbf{x}_1$ is a saddle point of f. □

Problem 9.20 For each of the following functions, determine whether the given critical point corresponds to a local minimum, local maximum, or saddle point:

(1) $f(x,\ y) = 3x^2 - xy + y^2$ at $(0,\ 0)$;

(2) $f(x, y, z) = x^3 + xyz + y^2 - 3x$ at $(1, 0, 0)$.

Problem 9.21 Show that for a continuous function $f(x, y)$ on $\mathbb{R}^2$ which has continuous third partial derivatives, a critical point $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ is a saddle point if and only if $\det H(\mathbf{x}_0) < 0$.

Is it also true for such a function $f(x, y, z)$ on $\mathbb{R}^3$?

9.8.2 Constrained quadratic optimization

One of the most important problems in applied mathematics is the optimization (minimization or maximization) of a real-valued function f of n variables subject to constraints on the variables. For example, when the function f is a linear form subject to constraints in the form of linear equalities and/or inequalities, the optimization problem is known as linear programming. Those optimization problems are extensively used in the military, industrial, governmental planning fields, among others.

In this section, we consider an optimization problem of a quadratic form in n variables. If there are no constraints on the variables, then such an optimization problem was discussed in Section 9.8.1.

As a quadratic optimization problem with constraints, we consider a very special one: Find the maximum and minimum values of a (real or complex) quadratic form $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$ subject to the constraint $\|\mathbf{x}\| = 1$. Advanced calculus tells us that such constraint extrema of $q(\mathbf{x})$ always exists.

Theorem 9.13 *Let A be a symmetric or a Hermitian matrix, and let the eigenvalues of A be $\lambda_{\min} = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n = \lambda_{\max}$ in increasing order. Then,*

(1) $\lambda_{\min} \|\mathbf{x}\|^2 \le \langle \mathbf{x}, A\mathbf{x} \rangle \le \lambda_{\max} \|\mathbf{x}\|^2$ *for all* $\mathbf{x}$.

(2) $\langle \mathbf{x}, A\mathbf{x} \rangle = \lambda \|\mathbf{x}\|^2$ *if $\mathbf{x}$ is an eigenvector of A belonging to an eigenvalue λ.*

(3) $\lambda_{\max} = \max\limits_{\mathbf{x} \ne \mathbf{0}} \dfrac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} = \max\limits_{\|\mathbf{x}\|=1} \langle \mathbf{x}, A\mathbf{x} \rangle$, *and for a unit vector $\mathbf{x}$, $\lambda_{\max} = \langle \mathbf{x}, A\mathbf{x} \rangle$ if and only if $\mathbf{x}$ is an eigenvector belonging to the eigenvalue $\lambda_{\max}$.*

(4) $\lambda_{\min} = \min\limits_{\mathbf{x} \ne \mathbf{0}} \dfrac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} = \min\limits_{\|\mathbf{x}\|=1} \langle \mathbf{x}, A\mathbf{x} \rangle$, *and for a unit vector $\mathbf{x}$, $\lambda_{\min} = \langle \mathbf{x}, A\mathbf{x} \rangle$ if and only if $\mathbf{x}$ is an eigenvector belonging to the eigenvalue $\lambda_{\min}$.*

In particular, the maximum and minimum values of a (real or complex) quadratic form $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$ subject to the constraint $\|\mathbf{x}\| = 1$ is the largest and the smallest eigenvalues of A, respectively.

Proof: We prove it for only a Hermitian matrix A and the other case of a real symmetric matrix is left as an exercise. If A is Hermitian, there is a unitary matrix U such that $U^H AU = D$ is a diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its diagonal entries. Moreover, with a change of coordinates $\mathbf{y} = U^H\mathbf{x} = [y_1\ y_2\ \cdots\ y_n]^T$ we have

$$\mathbf{x}^H A\mathbf{x} = \mathbf{y}^H D\mathbf{y} = \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2 + \cdots + \lambda_n |y_n|^2,$$

by the principal axes theorem, and $||\mathbf{x}|| = ||\mathbf{y}||$ because U is unitary. It implies that

$$\begin{aligned}\langle \mathbf{x}, A\mathbf{x}\rangle = \mathbf{x}^H A\mathbf{x} &= \lambda_1|y_1|^2 + \lambda_2|y_2|^2 + \cdots + \lambda_n|y_n|^2 \\ &\le \lambda_n|y_1|^2 + \lambda_n|y_2|^2 + \cdots + \lambda_n|y_n|^2 \\ &= \lambda_n(|y_1|^2 + |y_2|^2 + \cdots + |y_n|^2) \\ &= \lambda_n||\mathbf{y}||^2 = \lambda_{\max}\ ||\mathbf{x}||^2\end{aligned}$$

since $\lambda_n = \lambda_{\max}$ is the largest eigenvalue. Similarly, one can show $\lambda_{\min} \le \langle \mathbf{x}, A\mathbf{x}\rangle$ for all $\mathbf{x}$. It proves (1).

(2) If $\mathbf{x}$ is an eigenvector of A belonging to λ, then

$$\langle \mathbf{x}, A\mathbf{x}\rangle = \langle \mathbf{x}, \lambda\mathbf{x}\rangle = \lambda\langle \mathbf{x}, \mathbf{x}\rangle = \lambda||\mathbf{x}||^2.$$

In particular, if $\mathbf{x}$ is an eigenvector of A belonging to $\lambda_{\max}$ ($\lambda_{\min}$, respectively) and $||\mathbf{x}|| = 1$, then $\langle \mathbf{x}, A\mathbf{x}\rangle = \lambda_{\max}$ ($\lambda_{\min}$, respectively).

(3) We only prove the necessity part of the second assertion, because all other parts are clear from (1) and (2). To show this, suppose that $\lambda_{\max} = \langle \mathbf{x}, A\mathbf{x}\rangle$ for a Hermitian matrix A and a unit vector $\mathbf{x}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A with associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, respectively. One can assume that $\lambda_{\max} = \lambda_1$ and the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthonormal since A is Hermitian. Let $\mathbf{x} = \sum_j a_j\mathbf{v}_j$. Then, we have

$$\begin{aligned}\lambda_1 = \langle \mathbf{x}, A\mathbf{x}\rangle &= \left\langle \sum_{j=1}^n a_j\mathbf{v}_j, A\left(\sum_{j=1}^n a_j\mathbf{v}_j\right)\right\rangle = \left\langle \sum_{j=1}^n a_j\mathbf{v}_j, \sum_{j=1}^n a_j\lambda_j\mathbf{v}_j\right\rangle \\ &= \sum_{j=1}^n |a_j|^2\lambda_j \le \sum_{j=1}^n |a_j|^2\lambda_1 = \lambda_1\end{aligned}$$

since $\lambda_{\max} = \lambda_1$. Hence, it should be $a_j = 0$ whenever $\lambda_j \ne \lambda_1$, which implies that $\mathbf{x}$ is an eigenvector belonging to λ_1.

(4) can be proved in a similar way to (3). □

Definition 9.12 The **Rayleigh quotient** of a symmetric or Hermitian matrix A is the function R_A defined for $\mathbf{x} \ne \mathbf{0}$ by

$$R_A(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} \quad \text{for } \mathbf{x} \ne \mathbf{0}.$$

It follows from Theorem 9.13 that, subject to the constraint $||\mathbf{x}|| = 1$, the quadratic form $\langle \mathbf{x}, A\mathbf{x}\rangle$ has the maximum value $\lambda_{\max}$ and the minimum value $\lambda_{\min}$. It means that the smallest and the largest eigenvalues of a Hermitian matrix are characterized as the solutions of a constrained minimum and maximum problem of the Rayleigh quotient. This is very important in vibration problems ranging from aerodynamics to particle physics.

Example 9.18 (*The extreme values of a constrained quadratic form*) Find the maximum and minimum values of the quadratic form

$$x_1^2 + x_2^2 + 4x_1x_2$$

subject to the constraint $x_1^2 + x_2^2 = 1$, and determine values of x_1 and x_2 at which the maximum and minimum occur.

Solution: The quadratic form can be written as

$$x_1^2 + x_2^2 + 4x_1x_2 = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda = 3$ and $\lambda = -1$, which are the largest and smallest eigenvalues, respectively. Their associated unit eigenvectors are

$$\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

respectively. Note that those extreme values of the quadratic form occur at those unit eigenvectors. Thus, subject to the constraint $x_1^2 + x_2^2 = 1$, the maximum value of the quadratic form is $\lambda = 3$, which occurs at $\mathbf{x} = \pm(1/\sqrt{2},\ 1/\sqrt{2})$, and the minimum value is $\lambda = -1$, which occurs at $\mathbf{x} = \pm(1/\sqrt{2},\ -1/\sqrt{2})$. Clearly, the quadratic equation $\mathbf{x}^T A\mathbf{x} = c$ is a hyperbola and the extreme values occur when a hyperbola and the unit circle intersect as in Figure 9.7. □

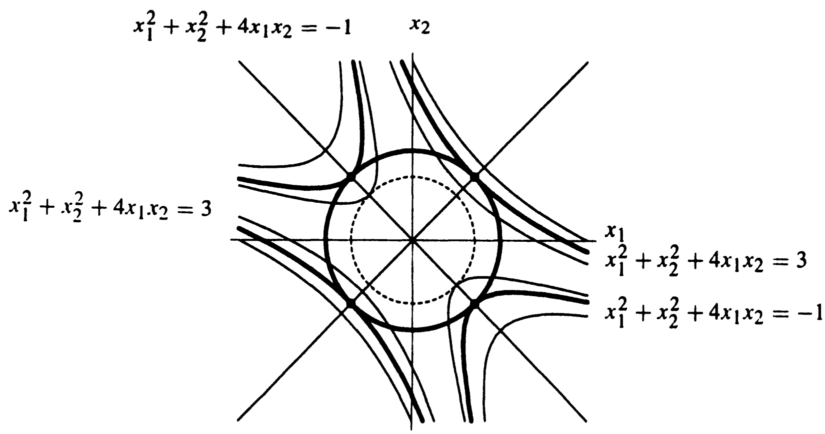

Figure 9.7. Extreme values of the constraint quadratic form

Remark: In Example 9.18, it is shown that the maximum value of the quadratic form $x_1^2+x_2^2+4x_1x_2$ subject to the constraint $x_1^2+x_2^2=1$ is 3. By examining Figure 9.7, one can also see the following *dual* constraint optimization: the minimum value of the quadratic form $x_1^2+x_2^2$ subject to the constraint $x_1^2+x_2^2+4x_1x_2=3$ is 1.

Problem 9.22 Find the maximum and the minimum values of the Rayleigh quotient of each of

$$(1)\ A=\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2)\ A=\begin{bmatrix} 2 & -3 & 0 \\ -3 & 0 & i \\ 0 & -i & 1 \end{bmatrix}.$$

Problem 9.23 Find the maximum and minimum values of the quadratic form

$$2x_1^2+2x_2^2+3x_1x_2$$

subject to the constraint $x_1^2+x_2^2=1$, and determine values of x_1 and x_2 at which the maximum and minimum occur.

Problem 9.24 Find the maximum and minimum of the following quadratic forms subject to the constraint $x_1^2+x_2^2+x_3^2=1$ and determine the values of x_1, x_2, and x_3 at which the maximum and minimum occur:

(1) $x_1^2+x_2^2+2x_3^2-2x_1x_2+4x_1x_3+4x_2x_3$,
(2) $2x_1^2+x_2^2+x_3^2+2x_1x_3+2x_1x_2$.

We have seen that the Rayleigh quotient characterizes the largest and the smallest eigenvalues and their associated eigenvectors of a real symmetric or a Hermitian matrix A in terms of a constrained optimization problem. But all other eigenvalues and their associated eigenvectors can be characterized in a similar way. For example, the second largest eigenvalue can be characterized as the maximum value of the quadratic form $\langle \mathbf{x}, A\mathbf{x}\rangle$ subject to the constraint $\mathbf{x}^H\mathbf{v}_n=0$, where $\mathbf{v}_n$ is an eigenvector belonging to the largest eigenvalue $\lambda_{\max}$. For a future discussion, one can refer to some advanced linear algebra books.

9.9 Exercises

9.1. Find the matrix representing each of the following quadratic forms:
(1) $x_1^2+4x_1x_2+3x_2^2$,
(2) $x_1^2-x_2^2+x_3^2+4x_1x_3-5x_2x_3$,
(3) $x_1^2-2x_2^2-3x_3^2+4x_1x_2+6x_1x_3-8x_2x_3$,
(4) $3x_1y_1-2x_1y_2+5x_2y_1+7x_2y_2-8x_2y_3+4x_3y_2-x_3y_3$,
(5) $[x_1\ x_2]\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

9.2. Sketch the level surface of each of the following quadratic equations:
(1) $xy=2$,

(2) $53x^2 - 72xy + 32y^2 = 80$,

(3) $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$.

9.3. Let q be a quadratic form on $\mathbb{R}^3$ and let $A = \begin{bmatrix} 7 & 4 & -5 \\ 4 & -2 & 4 \\ -5 & 4 & 7 \end{bmatrix}$ be the matrix representing q with respect to the basis

$$\alpha = \{(1,\ 0,\ 1),\ (1,\ 1,\ 0),\ (0, 0, 1)\}.$$

(1) Diagonalize A, i.e., find an orthogonal matrix P so that $P^T AP$ is a diagonal matrix.

(2) Construct a basis β for $\mathbb{R}^3$ such that the elements of β are the principal axes of the quadratic surface $q(\mathbf{x}) = 0$.

9.4. For a given quadratic equation $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ with $b \neq 0$, classify the conic section according to the various possible cases of a, b, and c (see Example 9.6).

9.5. For a positive definite quadratic form $q(\mathbf{x}) = ax^2 + 2bxy + cy^2$, the curve $q(\mathbf{x}) = 1$ is an ellipse. When $a = c = 2$ and $b = -1$, sketch the ellipse.

9.6. Show that if A and B are both positive definite, so are A^2, A^{-1} and $A + B$.

9.7. Prove that if A and B are symmetric and positive definite, so is $A^2 + B^{-1}$.

9.8. Find a substitution $\mathbf{x} = Q\mathbf{y}$ that diagonalizes each of the following quadratic forms, where Q is orthogonal. Also, classify the form as positive definite, positive semidefinite, and so on.

(1) $q(\mathbf{x}) = 2x^2 + 6xy + 2y^2$.

(2) $q(\mathbf{x}) = x^2 + y^2 + z^2 + 2(xy + xz + yz)$.

9.9. Determine whether or not each of the following matrices is positive definite:

$$(1)\ A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad (2)\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Use the decomposition $A = LDL^T$ to write $\mathbf{x}^T A\mathbf{x}$ as the sum of squares.

9.10. Let b be a bilinear form on $\mathbb{R}^2$ defined by

$$b(\mathbf{x},\ \mathbf{y}) = 2x_1y_1 - 3x_1y_2 + x_2y_2.$$

(1) Find the matrix A of b with respect to the basis $\alpha = \{(1, 0),\ (1,\ 1)\}$.

(2) Find the matrix B of b with respect to the basis $\beta = \{(2,\ 1),\ (1,\ -1)\}$.

(3) Find the basis-change matrix Q from the basis β to the basis α and verify that $B = Q^T AQ$.

9.11. Find the signature, index and rank of each of the following symmetric matrices:

$$(1) \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 4 \end{bmatrix}, \quad (2) \begin{bmatrix} 2 & 3 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 0 \end{bmatrix}, \quad (3) \begin{bmatrix} 4 & -3 & 5 \\ -3 & 2 & 1 \\ 5 & 1 & -6 \end{bmatrix}.$$

9.12. Which of the following functions b on $\mathbb{R}^2$ are of bilinear form?

(1) $b(\mathbf{x},\ \mathbf{y}) = 1$

(2) $b(\mathbf{x},\ \mathbf{y}) = (x_1 - y_1)^2 + x_2y_2$

(3) $b(\mathbf{x},\ \mathbf{y}) = (x_1 + y_1)^2 - (x_1 - y_1)^2$

(4) $b(\mathbf{x}, \mathbf{y}) = x_1y_2 - x_2y_1$

9.13. For a bilinear form on $\mathbb{R}^2$ defined by $b(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2$, find the matrix representation of b with respect to each of the following bases:

$$\alpha = \{(1,0), (0,1)\}, \quad \beta = \{(1, -1), (1, 1)\}, \quad \gamma = \{(1, 2), (3, 4)\}.$$

9.14. Which one of the following bilinear forms on $\mathbb{R}^3$ are symmetric or skew-symmetric? For each symmetric one, find its matrix representation of the diagonal form, and for each skew-symmetric one, find its matrix representation of the block form in Theorem 9.7.

(1) $b(\mathbf{x}, \mathbf{y}) = x_1y_3 + x_3y_1$

(2) $b(\mathbf{x}, \mathbf{y}) = x_1y_1 + 2x_1y_3 + 2x_3y_1 - x_2y_2$

(3) $b(\mathbf{x}, \mathbf{y}) = x_1y_2 + 2x_1y_3 - x_2y_3 - x_2y_1 - 2x_3y_1 + x_3y_2$

(4) $b(\mathbf{x}, \mathbf{y}) = \sum_{i,j=1}^{3}(i - j)x_iy_j$

9.15. Determine whether each of the following matrices takes a local minimum, local maximum or saddle point at the given point:

(1) $f(x, y) = -1 + 4(e^x - x) - 5x \sin y + 6y^2$ at the point $(x, y) = (0, 0)$;

(2) $f(x, y) = (x^2 - 2x)\cos y$ at $(x, y) = (1, \pi)$.

9.16. Show that the quadratic form $q(\mathbf{x}) = 2x^2 + 4xy + y^2$ has a saddle point at the origin, despite the fact that its coefficients are positive. Show that q can be written as the difference of two perfect squares.

9.17. Find the eigenvalues of the following matrices and the maximum value of the associated quadratic forms on the unit sphere.

$$(1)\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad (2)\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \quad (3)\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

9.18. A bilinear form $b : V \times W \to \mathbb{R}$ on vector spaces V and W is said to be **nondegenerate** if it satisfies

$$\begin{array}{llll} b(\mathbf{v}, \mathbf{w}) = 0 & \text{for all } \mathbf{w} \in W & \text{implies } \mathbf{v} = \mathbf{0}, & \text{and} \\ b(\mathbf{v}, \mathbf{w}) = 0 & \text{for all } \mathbf{v} \in V & \text{implies } \mathbf{w} = \mathbf{0}. & \end{array}$$

As an example, an inner product on a vector space V is just a symmetric, nondegenerate bilinear form on V. Let $b : V \times W \to \mathbb{R}$ be a nondegenerate bilinear form. For a fixed $\mathbf{w} \in W$, we define $\varphi_{\mathbf{w}} : V \to \mathbb{R}$ by

$$\varphi_{\mathbf{w}}(\mathbf{v}) = b(\mathbf{v}, \mathbf{w}) \quad \text{for } \mathbf{v} \in V.$$

Then, the bilinearity of b proves that $\varphi_{\mathbf{w}} \in V^*$, from which we obtain a linear transformation

$$\varphi : W \to V^* \quad \text{defined by} \quad \varphi(\mathbf{w}) = \varphi_{\mathbf{w}}.$$

Similarly, we can have a linear transformation $\psi : V \to W^*$ defined by

$$\psi(\mathbf{v})(\mathbf{w}) = b(\mathbf{v}, \mathbf{w}) \quad \text{for } \mathbf{v} \in V \text{ and } \mathbf{w} \in W.$$

Prove the following statements:

(1) If $b : V \times W \to \mathbb{R}$ is a nondegenerate bilinear form, then the linear transformations $\varphi : W \to V^*$ and $\psi : V \to W^*$ are isomorphisms.

(2) If there exists a nondegenerate bilinear form $b : V \times W \to \mathbb{R}$, then $\dim V = \dim W$.

9.19. Determine whether the following statements are true or false, in general, and justify your answers.

(1) For any quadratic form q on $\mathbb{R}^n$, there exists a basis α for $\mathbb{R}^n$ with respect to which the matrix representation of q is diagonal.

(2) Any two matrix representations of a quadratic form have the same inertia.

(3) If A is positive definite symmetric matrix, then every square submatrix of A has positive determinant.

(4) If A is negative definite, $\det A < 0$.

(5) The sum of a positive definite quadratic form and a negative definite quadratic form is indefinite.

(6) If A is a real symmetric positive definite matrix, then the solution set of $\mathbf{x}^T A\mathbf{x} = 1$ is an ellipsoid.

(7) For any nontrivial bilinear form $b \neq 0$ on a vector space V, if $b(\mathbf{v}, \mathbf{v}) = 0$, then $\mathbf{v} = \mathbf{0}$.

(8) Any symmetric matrix is congruent to a diagonal matrix.

(9) Any two congruent matrices have the same eigenvalues.

(10) Any two congruent matrices have the same determinant.

(11) The sum of two bilinear forms on V is also a bilinear form.

(12) Any matrix representation of a bilinear form is diagonalizable.

(13) If a real symmetric matrix A is both positive semidefinite and negative semidefinite, then A must be the zero matrix.

(14) Any two similar real symmetric matrices have the same signature.

Selected Answers and Hints

Chapter 1

Problems

1.2 (1) Inconsistent.
(2) $(x_1, x_2, x_3, x_4) = (-1-4t,\ 6-2t,\ 2-3t,\ t)$ for any $t \in \mathbb{R}$.

1.3 (1) $(x, y, z) = (t, -t, t)$. (3) $(w, x, y, z) = (2, 0, 1, 3)$

1.4 (1) $b_1 + b_2 - b_3 = 0$. (2) For any b_i's.

1.7 $a = -\frac{17}{2}$, $b = \frac{13}{2}$, $c = \frac{13}{4}$, $d = -4$.

1.9 Consider the matrices: $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix}$.

1.10 Compare the diagonal entries of AA^T and A^TA.

1.12 (1) Infinitely many for $a = 4$, exactly one for $a \neq \pm 4$, and none for $a = -4$.
(2) Infinitely many for $a = 2$, none for $a = -3$, and exactly one otherwise.

1.14 (3) $I = I^T = (AA^{-1})^T = (A^{-1})^TA^T$ means by definition $(A^T)^{-1} = (A^{-1})^T$.

1.16 Use Problem 1.14(3).

1.17 Any permutation on n objects can be obtained by taking a finite number of interchangings of two objects.

1.21 Consider the case that some d_i is zero.

1.22 $x = 2, y = 3, z = 1$.

1.23 True if $A\mathbf{x} = \mathbf{b}$ is consistent, but not true in general.

1.24 $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

1.25 (1) Consider (i, j)-entries of AB for $i < j$.
(2) A can be written as a product of lower triangular elementary matrices.

1.26 $L=\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$, $D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$, $U=\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$.

1.27 There are four possibilities for P.

1.29 (1) $I_1 = 0.5$, $I_2 = 6$, $I_3 = 0.55$. (2) $I_1 = 0$, $I_2 = I_3 = 1$, $I_4 = I_5 = 5$.

1.30 $\mathbf{x} = k\begin{bmatrix} 0.35 \\ 0.40 \\ 0.25 \end{bmatrix}$ for $k > 0$.

1.31 $A = \begin{bmatrix} 0.0 & 0.1 & 0.8 \\ 0.4 & 0.7 & 0.1 \\ 0.5 & 0.0 & 0.1 \end{bmatrix}$ with $\mathbf{d} = \begin{bmatrix} 90 \\ 10 \\ 30 \end{bmatrix}$.

Exercises

1.1 Row-echelon forms are A, B, D, F. Reduced row-echelon forms are A, B, F.

1.2 (1) $\begin{bmatrix} 1 & -3 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

1.3 (1) $\begin{bmatrix} 1 & -3 & 0 & 3/2 & 1/2 \\ 0 & 0 & 1 & -1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

1.4 (1) $x_1 = 0$, $x_2 = 1$, $x_3 = -1$, $x_4 = 2$. (2) $x = 17/2$, $y = 3$, $z = -4$.

1.5 (1) and (2).

1.6 For any b_i's.

1.7 $b_1 - 2b_2 + 5b_3 \neq 0$.

1.8 (1) Take x the transpose of each row vector of A.

1.10 Try it with several kinds of diagonal matrices for B.

1.11 $A^k = \begin{bmatrix} 1 & 2k & 3k(k-1) \\ 0 & 1 & 3k \\ 0 & 0 & 1 \end{bmatrix}$.

1.13 (2) $\begin{bmatrix} 5 & -22 & 101 \\ 0 & 27 & -60 \\ 0 & 0 & 87 \end{bmatrix}$.

1.14 See Problem 1.9.

1.16 (1) $A^{-1}AB = B$. (2) $A^{-1}AC = C = A + I$.

1.17 $a = 0$, $c^{-1} = b \neq 0$.

1.18 $A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{bmatrix}$.

1.19 $A^{-1} = \frac{1}{15}\begin{bmatrix} 8 & -19 & 2 \\ 1 & -23 & 4 \\ 4 & -2 & 1 \end{bmatrix}$.

1.22 (1) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1/3 & 1/6 & 1/6 \\ -4/3 & -5/3 & 4/3 \\ -1/3 & -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 8/3 \\ -5/3 \\ -5/3 \end{bmatrix}$.

1.23 (1) $A = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} = LDU$, (2) $L = A$, $D = U = I$.

1.24 (1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,

(2) $\begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d - b^2/a \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$.

1.25 $\mathbf{c} = [2\ -1\ 3]^T, \mathbf{x} = [4\ 2\ 3]^T$.

1.26 (2) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}$.

1.27 (1) $(A^k)^{-1} = (A^{-1})^k$. (2) $A^{n-1} = \mathbf{0}$ if $A \in M_{n\times n}$.
(3) $(I - A)(I + A + \cdots + A^{k-1}) = I - A^k$.

1.28 (1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. (2) $A = A^{-1}A^2 = A^{-1}A = I$.

1.29 Exactly seven of them are true.
(1) See Theorem 1.9. (2) Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.
(3) $A^2 = I \Rightarrow A = A^{-1}$. (4) If $A = -B$? (5) $AB = (AB)^T = B^TA^T = BA$.
(6) Consider $A = \begin{bmatrix} 0 & 2 & 5 \\ 2 & 1 & 3 \\ 5 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 & 1 \\ 7 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.
(7) $(A^{-1})^T = (A^T)^{-1} = A^{-1}$. (8) If A^{-1} exists, $A^{-1} = A^{-1}(AB^T) = B^T$.
(9) If AB has the (right) inverse C, then $A^{-1} = BC$.
(10) Consider $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
(12) Consider a permutation matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Chapter 2

Problems

2.2 (1) Note: 2^{nd} column $-$ 1^{st} column $=$ 3^{rd} column $-$ 2^{nd} column.

2.4 (1) -27, (2) 0, (3) $(1 - x^4)^3$.

2.7 Let σ be a transposition in S_n. Then the composition of σ with an even (odd) permutation in S_n is an odd (even, respectively) permutation.

2.9 (1) -14. (2) 0.

2.10 (1) $(y - x)(-x + z)(z - y)(w - x)(w - y)(w - z)(w + y + x + z)$.
(2) $(fa - bc + cd)(fa + cd - eb)$.

2.11 $A^{-1} = \begin{bmatrix} -1 & \frac{5}{3} & -\frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 2 & -\frac{8}{3} & \frac{5}{3} \end{bmatrix}$; $\mathrm{adj}A = \begin{bmatrix} 3 & -5 & 2 \\ 0 & -1 & 1 \\ -6 & 8 & -5 \end{bmatrix}$.

2.12 If $A = \mathbf{0}$, then clearly adj $A = \mathbf{0}$. Otherwise, use $A \cdot \text{adj } A = (\det A)I$.

2.13 Use adj $A \cdot \text{adj}(\text{adj } A) = \det(\text{adj } A)\, I$.

2.14 If A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. Since for any invertible matrix A we have adj $A = (\det A)A^{-1}$, (1) and (2) are obvious. To show (3), let $AB = BA$ and A be invertible. Then $A^{-1}B = A^{-1}(BA)A^{-1} = A^{-1}(AB)A^{-1} = BA^{-1}$, which gives (3).

2.15 (1) $x_1 = 4,\ x_2 = 1,\ x_3 = -2$.
(2) $x = \dfrac{10}{23},\ y = \dfrac{5}{6},\ z = \dfrac{5}{2}$.

2.16 The solution of the system $id(\mathbf{x}) = \mathbf{x}$ is $x_i = \frac{\det C_i}{\det I} = \det A$.

Exercises

2.1 $k = 0$ or 2.

2.2 It is not necessary to compute A^2 or A^3.

2.3 -37.

2.4 (1) $\det A = (-1)^{n-1}(n-1)$. (2) 0.

2.5 $-2, 0, 1, 4$.

2.6 Consider $\sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$.

2.7 (1) 1, (2) 24.

2.8 (3) $x_1 = 1,\ x_2 = -1,\ x_3 = 2,\ x_4 = -2$.

2.9 (2) $\mathbf{x} = (3, 0, 4/11)^T$.

2.10 $k = 0$ or ± 1.

2.11 $\mathbf{x} = (-5, 1, 2, 3)^T$.

2.12 $x = 3,\ y = -1,\ z = 2$.

2.13 (3) $A_{11} = -2, A_{12} = 7, A_{13} = -8, A_{33} = 3$.

2.16 $A^{-1} = \frac{1}{72}\begin{bmatrix} -3 & 5 & 9 \\ 18 & -6 & 18 \\ 6 & 14 & -18 \end{bmatrix}$.

2.17 (1) adj $A = \begin{bmatrix} 2 & -7 & -6 \\ 1 & -7 & -3 \\ -4 & 7 & 5 \end{bmatrix}$, $\det A = -7$, $\det(\text{adj } A) = 49$,
$A^{-1} = -\frac{1}{7}\text{adj } A$. (2) adj $A = \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$,
$\det A = 2$, $\det(\text{adj } A) = 4$, $A^{-1} = \frac{1}{2}\text{adj } A$.

2.19 Multiply $\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$.

2.20 If we set $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, then the area is $\frac{1}{2}|\det A| = 4$.

2.21 If we set $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$, then the area is $\frac{1}{2}\sqrt{|\det(A^TA)|} = \frac{3\sqrt{2}}{2}$.

2.22 Use $\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$.

2.23 Exactly seven of them are true.

(1) Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$.

(2) $\det(AB) = \det A \det B = \det B \det A$.

(3) Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $c = 3$. (4) $(cI_n - A)^T = cI_n - A^T$.

(5) Consider $E = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. (6) and (7) Compare their determinants.

(8) If $A = I_2$? (9) Find its counter example.

(10) What happened for $A = \mathbf{0}$?

(11) $\mathbf{u}\mathbf{v}^T = \mathbf{u}[v_1 \cdots v_n] = [v_1\mathbf{u} \cdots v_n\mathbf{u}]$;
$\det(\mathbf{u}\mathbf{v}^T) = v_1 \cdots v_n \det([\mathbf{u} \cdots \mathbf{u}]) = 0$.

(12) and (13) Use $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

(14) Consider $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. (15) If $\det A = 0$?

(16) $A^t = A^{-1}$ for any permutation matrix A.

Chapter 3

Problems

3.1 Check the commutativity in addition.

3.2 (2), (4).

3.3 (1), (2), (4).

3.5 See Problem 1.11.

3.6 Note that any vector $\mathbf{v}$ in W is of the form $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m$ which is a vector in U.

3.7 Use Lemma 3.7(2).

3.9 Use Theorem 3.6

3.10 Any basis for W must be a basis for V already, by Corollary 3.12.

3.11 (1) dim$= n-1$, (2) dim$= \frac{n(n+1)}{2}$, (3) dim$= \frac{n(n-1)}{2}$.

3.13 $63a + 39b - 13c + 5d = 0$.

3.15 If $\mathbf{b}_1, \ldots, \mathbf{b}_n$ denote the column vectors of B, then $AB = [A\mathbf{b}_1 \cdots A\mathbf{b}_n]$.

3.16 Consider the matrix A from Example 3.20.

3.17 (1) rank $= 3$, nullity $= 1$. (2) rank $= 2$, nullity $= 2$.

3.18 $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \mathcal{C}(A)$.

3.19 $A^{-1}(AB) = B$ implies rank $B = \operatorname{rank} A^{-1}(AB) \le \operatorname{rank}(AB)$.

3.20 By (2) of Theorem 3.21 and Corollary 3.18, a matrix A of rank r must have an invertible submatrix C of rank r. By (1) of the same theorem, the rank of C must be the largest.

3.22 $\dim(V + W) = 4$ and $\dim(V \cap W) = 1$.

3.23 A basis for V is $\{(1, 0, 0, 0), (0, -1, 1, 0), (0, -1, 0, 1)\}$, for W: $\{(-1, 1, 0, 0), (0, 0, 2, 1)\}$, and for $V \cap W$: $\{(3, -3, 2, 1)\}$. Thus, $\dim(V + W) = 4$ means $V + W = \mathbb{R}^4$ and any basis for $\mathbb{R}^4$ works for $V + W$.

3.26

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \text{ and } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

Exercises

3.1 Consider $0(1, 1)$.

3.2 (5).

3.3 (2), (3). For (1), if $f(0) = 1, 2f(0) = 2$.

3.4 (1).

3.5 (1), (4).

3.6 $\operatorname{tr}(AB - BA) = 0$.

3.7 No.

3.8 (1) $p(x) = -p_1(x) + 3p_2(x) - 2p_3(x)$.

3.9 No.

3.10 Linearly dependent.

3.12 No.

3.13 $\{(1, 1, 0), (1, 0, 1)\}$.

3.14 2.

3.15 Consider $\{\mathbf{e}_j = \{a_i\}_{i=1}^{\infty}\}$ where $a_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$

3.16 (1) $\mathbf{0} = c_1 A\mathbf{b}^1 + \cdots + c_p A\mathbf{b}^s = A(c_1\mathbf{b}^1 + \cdots + c_p\mathbf{b}^s)$ implies $c_1\mathbf{b}^1 + \cdots + c_p\mathbf{b}^p = \mathbf{0}$ since $\mathcal{N}(A) = \mathbf{0}$, and this also implies $c_i = 0$ for all $i = 1, \ldots, p$ since columns of B are linearly independent.
(2) B has a right inverse. (3) and (4): Look at (1) and (2) above.

3.17 (1) $\{(-5, 3, 1)\}$. (2) 3.

3.18 $5!$, and dependent.

3.19 (1) $\mathcal{R}(A) = \langle(1, 2, 0, 3), (0, 0, 1, 2)\rangle$, $\mathcal{C}(A) = \langle(5, 0, 1), (0, 5, 2)\rangle$, $\mathcal{N}(A) = \langle(-2, 1, 0, 0), (-3, 0, -2, 1)\rangle$.
(2) $\mathcal{R}(B) = \langle(1, 1, -2, 2), (0, 2, 1, -5), (0, 0, 0, 1)\rangle$, $\mathcal{C}(B) = \langle(1, -2, 0), (0, 1, 1), (0, 0, 1)\rangle$, $\mathcal{N}(B) = \langle(5, -1, 2, 0)\rangle$.

3.20 rank $= 2$ when $x = -3$, rank $= 3$ when $x \neq -3$.

3.22 See Exercise **2.23**: Each column vector of $\mathbf{u}\mathbf{v}^T$ is of the form $v_i\mathbf{u}$, that is, $\mathbf{u}$ spans the column space. Conversely, if A is of rank 1, then the column space is spanned by any one column of A, say the first column $\mathbf{u}$ of A, and the remaining columns are of the form $v_i\mathbf{u}$, $i = 2, \ldots, n$. Take $\mathbf{v} = [1 \ v_2 \ \cdots \ v_n]^T$. Then one can easily see that $A = \mathbf{u}\mathbf{v}^T$.

3.23 Four of them are true.
(1) $A - A =?$ or $2A =?$
(2) In $\mathbb{R}^2$, let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\beta = \{\mathbf{e}_1, -\mathbf{e}_2\}$.

(3) Even $U = W$, $\alpha \cap \beta$ can be an empty set.
(4) How can you find a basis for $\mathcal{C}(A)$. See Example 3.20.
(5) Consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
(6) See Theorem 3.24. (7) See Theorem 3.25. (8) If $\mathbf{x} = -\mathbf{y}$,
(9) In $\mathbb{R}^2$, Consider $U = \mathbb{R}^2 \times \mathbf{0}$ and $V = \mathbf{0} \times \mathbb{R}^2$.
(10) Note $\dim \mathcal{C}(A^T) = \dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.
(11) By the fundamental Theorem and the Rank Theorem.

Chapter 4

Problems

4.1 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since it is simply the change of coordinates x and y.

4.2 To show W is a subspace, see Theorem 4.2. Let E_{ij} be the matrix with 1 at the (i, j)-th position and 0 at others. Let F_k be the matrix with 1 at the (k, k)-th position, -1 at the (n, n)-th position and 0 at others. Then the set $\{E_{ij}, F_k : 1 \le i \ne j \le n,\ k = 1, \ldots, n-1\}$ is a basis for W. Thus $\dim W = n^2 - 1$.

4.3 $\mathrm{tr}(AB) = \sum_{i=1}^m \sum_{k=1}^n a_{ik}b_{ki} = \sum_{k=1}^n \sum_{i=1}^m b_{ki}a_{ik} = \mathrm{tr}(BA)$.

4.4 If yes, $(2,\ 1) = T(-6,\ -2,\ 0) = -2T(3,\ 1,\ 0) = (-2,\ -2)$.

4.5 If $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$, then
$\mathbf{0} = T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k) = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k$ implies $a_i = 0$ for $i = 1, \ldots, k$.

4.6 (1) If $T(\mathbf{x}) = T(\mathbf{y})$, then $S \circ T(\mathbf{x}) = S \circ T(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$. (4) They are invertible.

4.7 (1) $T(\mathbf{x}) = T(\mathbf{y})$ if and only if $T(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, i.e., $\mathbf{x} - \mathbf{y} \in \mathrm{Ker}(T)$.
(2) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. If T is one-to-one, then the set $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly independent as the proof of Theorem 4.7 shows. Corollary 3.12 shows it is a basis for V. Thus, for any $\mathbf{y} \in V$, we can write it as $\mathbf{y} = \sum_{i=1}^n a_i T(\mathbf{v}_i) = T(\sum_{i=1}^n a_i\mathbf{v}_i)$. Set $\mathbf{x} = \sum_{i=1}^n a_i\mathbf{v}_i \in V$. Then clearly $T(\mathbf{x}) = \mathbf{y}$ so that T is onto. If T is onto, then for each $i = 1, \ldots, n$ there exists $\mathbf{x}_i \in V$ such that $T(\mathbf{x}_i) = \mathbf{v}_i$. Then the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is linearly independent in V, since, if $\sum_{i=1}^n a_i\mathbf{x}_i = \mathbf{0}$, then $\mathbf{0} = T(\sum_{i=1}^n a_i\mathbf{x}_i) = \sum_{i=1}^n a_i T(\mathbf{x}_i) = \sum_{i=1}^n a_i\mathbf{v}_i$ implies $a_i = 0$ for all $i = 1, \ldots, n$. Thus it is a basis by Corollary 3.12 again. If $T(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} = \sum_{i=1}^n a_i\mathbf{x}_i \in V$, then $\mathbf{0} = T(\mathbf{x}) = \sum_{i=1}^n a_i T(\mathbf{x}_i) = \sum_{i=1}^n a_i\mathbf{v}_i$ implies $a_i = 0$ for all $i = 1, \ldots, n$, that is $\mathbf{x} = \mathbf{0}$. Thus Ker $(T) = \{\mathbf{0}\}$.

4.8 Use rotation $R_{\frac{\pi}{3}}$ and reflection $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ about the x-axis.

4.9 (1) (5, 2, 3). (2) (2, 3, 0).

4.10 (1) $[T]_\alpha = \begin{bmatrix} 2 & -3 & 4 \\ 5 & -1 & 2 \\ 4 & 7 & 0 \end{bmatrix}$, $[T]_\beta = \begin{bmatrix} 0 & 7 & 4 \\ 2 & -1 & 5 \\ 4 & -3 & 2 \end{bmatrix}$.

4.11 $[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & -3 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix}$.

4.13 $[S+T]_\alpha = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}, [T \circ S]_\alpha = \begin{bmatrix} 3 & 2 & 0 \\ 3 & 3 & 3 \\ 6 & 5 & 3 \end{bmatrix}.$

4.14 $[S]_\alpha^\beta = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [T]_\alpha = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}.$

4.15 (2) $[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} [T^{-1}]_\beta^\alpha = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$

4.16 $[id]_\beta^\alpha = \frac{1}{2}\begin{bmatrix} 0 & -1 & 5 \\ 4 & 3 & -1 \\ 2 & 1 & 1 \end{bmatrix}, [id]_\alpha^\beta = \begin{bmatrix} -2 & -3 & 7 \\ 3 & 5 & -10 \\ 1 & 1 & -2 \end{bmatrix}.$

4.17 $[T]_\alpha = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}, [T]_\beta = \begin{bmatrix} 1 & 4 & 5 \\ -1 & -2 & -6 \\ 1 & 1 & 5 \end{bmatrix}.$

4.18 Write $B = Q^{-1}AQ$ with some invertible matrix Q.
(1) $\det B = \det(Q^{-1}AQ) = \det Q^{-1} \det A \det Q = \det A$. (2) $\operatorname{tr}(B) = \operatorname{tr}(Q^{-1}AQ) = \operatorname{tr}(QQ^{-1}A) = \operatorname{tr}(A)$ (see Problem 4.3). (3) Use Problem 3.19.

4.20 $\alpha^* = \{f_1(x, y, z) = x - \frac{1}{2}y,\ f_2(x, y, z) = \frac{1}{2}y,\ f_3(x, y, z) = -x + z\}$.

4.24 By BA, we get a tilting along the x-axis; while by B^TA, one can get a tilting along the y-axis.

Exercises

4.1 (2).

4.2 $ax^3 + bx^2 + ax + c$.

4.4 S is linear because the integration satisfies the linearity.

4.5 (1) Consider the decomposition of $\mathbf{v} = \frac{\mathbf{v}+T(\mathbf{v})}{2} + \frac{\mathbf{v}-T(\mathbf{v})}{2}$.

4.6 (1) $\{(x, \frac{3}{2}x, 2x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$.

4.7 (2) $T^{-1}(r, s, t) = (\frac{1}{2}r,\ 2r - s,\ 7r - 3s - t)$.

4.8 (1) Since $T \circ S$ is one-to-one from V into V, $T \circ S$ is also onto and so T is onto. Moreover, if $S(\mathbf{u}) = S(\mathbf{v})$, then $T \circ S(\mathbf{u}) = T \circ S(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$. Thus, S is one-to-one, and so onto. This implies T is one-to-one. In fact, if $T(\mathbf{u}) = T(\mathbf{v})$, then there exist $\mathbf{x}$ and $\mathbf{y}$ such that $S(\mathbf{x}) = \mathbf{u}$ and $S(\mathbf{y}) = \mathbf{v}$. Thus $T \circ S(\mathbf{x}) = T \circ S(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$ and so $\mathbf{u} = T(\mathbf{x}) = T(\mathbf{y}) = \mathbf{v}$.

4.9 Note that T cannot be one-to-one and S cannot be onto.

4.12 $\operatorname{vol}(T(C)) = |\det(A)|\operatorname{vol}(C)$, for the matrix representation A of T.

4.13 (3) $(5, 2, 0)$.

4.14 $\begin{bmatrix} 5 & 4 & -6 & 18 \\ -4 & -3 & -2 & 0 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

4.15 (1) $\begin{bmatrix} -1/3 & 2/3 \\ -5/3 & 1/3 \end{bmatrix}.$

4.16 (1) $\begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$, (2) $\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}$.

4.17 (1) $T(1,0,0) = (4,\ 0)$, $T(1,\ 1,\ 0) = (1,\ 3)$, $T(1,\ 1,\ 1) = (4,\ 3)$.
(2) $T(x, y, z) = (4x - 2y + z,\ y + 2z)$.

4.18 (1) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, (4) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

4.19 (1) $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$, (2) $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = P^{-1}$.

4.20 Compute the trace of $AB - BA$.

4.21 (1) $\begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$.

4.22 (2) $\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$, (4) $\begin{bmatrix} -2/3 & 1/3 & 4/3 \\ 2/3 & -1/3 & -1/3 \\ 7/3 & -2/3 & -8/3 \end{bmatrix}$.

4.23 ' A_θ represents reflection in a line at angle $\theta/2$ to the x-axis. And, any two such reflections are similar (by a rotation).

4.25 Compute their determinants.

4.27 $[T]_\alpha = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 4 & 1 \\ 1 & 0 & 1 \end{bmatrix} = ([T^*]_{\alpha^*})^T$.

4.28 (1) $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$. (2) $[T]_\alpha^\beta = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

4.29 $\mathcal{N}(T) = \{\mathbf{0}\}$,

$$\mathcal{C}(T) = \langle\ (2,1,0,1),\ (1,1,1,1),\ (4,2,2,3)\ \rangle,\ [T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix}.$$

4.31 $p_1(x) = 1 + x - \frac{3}{2}x^2$, $p_2(x) = -\frac{1}{6} + \frac{1}{2}x^2$, $p_3(x) = -\frac{1}{3} + x - \frac{1}{2}x^2$.

4.32 Five of them are false.
(1) Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, 0, 0)$.
(2) Note $\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim \mathbb{R}^n = n$.
(3) $\dim \mathrm{Ker}(T) = \dim \mathbb{N}([T]_\alpha^\beta)$.
(4) $\dim \mathrm{Im}(T) = \dim \mathbb{C}([T]_\alpha) = \dim \mathbb{R}([T]_\alpha)$.
(5) $\mathrm{Ker}(T) \subset \mathrm{Ker}(S \circ T)$.
(6) $p(x) = 2$ is not linear.
(7) Use the definition of a linear transformation.
(8) and (10) See Remark (1) in Section 4.3.
(9) $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one iff T is an isomorphism. See Remark in Section 4.4 and Theorem 4.14.
(11) By Definition 4.5 and Theorem 4.14.
(12) Cf. Theorem 4.17. (13) $\det(A + B) \neq \det(A) + \det(B)$ in general.
(14) $T(\mathbf{0}) \neq (\mathbf{0})$ in general for a translation T.

Chapter 5

Problems

5.1 Let $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$ be an inner product. Then, for $\mathbf{x} = (1, 0)$, $\langle \mathbf{x}, \mathbf{x} \rangle = a x_1 y_1 + c(x_1 y_2 + x_2 y_1) + b x_2 y_2 > 0$ implies $a > 0$. Similarly, for any $\mathbf{x} = (x, 1)$, $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ implies $ab - c^2 > 0$.

5.2 $\langle \mathbf{x}, \mathbf{y} \rangle^2 = \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle$ if and only if $\|t\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle t^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle t + \langle \mathbf{y}, \mathbf{y} \rangle = 0$ has a repeated real root t_0.

5.3 If $d_1 < 0$, then for $\mathbf{x} = (1,\ 0,\ 0)$, $\langle \mathbf{x}, \mathbf{x} \rangle = d_1 < 0$: Impossible.

5.4 (4) Compute the square of both sides and then use Cauchy–Schwarz inequality.

5.5 (4) Use Problem 5.4(4): triangle inequality in the length.

5.6 $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ defines an inner product on $C\,[0,\ 1]$. Use Cauchy–Schwarz inequality.

5.7 (2)-(3): Use $\int_0^1 f(x)g(x)\,dx = 0$ if $k \neq \ell$; and $= \frac{1}{2}$ if $k = \ell$.

5.8 (1): Orthogonal, (2) and (3): None, (4): Orthonormal.

5.10 Clearly, $\mathbf{x}_1 = (1, 0, 1), \mathbf{x}_2 = (0, 1, 2)$ are in W. The Gram–Schmidt orthogonalization gives $\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = (1, 0, 1)$, $\mathbf{u}_2 = \frac{1}{\sqrt{3}}(-1, 1, 1)$ which form a basis.

5.11 $\left\{1,\ \sqrt{3}(2x - 1),\ \sqrt{5}(6x^2 - 6x + 1)\right\}$.

5.12 (4) $\mathrm{Im}(id_V - T) \subseteq \mathrm{Ker}(T)$ because $T(id_V - T)(\mathbf{x}) = T(\mathbf{x}) - T^2(\mathbf{x}) = \mathbf{0}$. $\mathrm{Im}(id_V - T) \supseteq \mathrm{Ker}(T)$ because if $T(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} = \mathbf{x} - \mathbf{0} = \mathbf{x} - T(\mathbf{x}) = (id_V - T)\mathbf{x}$.

5.13 (1) $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ for only $\mathbf{x} = \mathbf{0}$. (2) Use Definition 5.6(2).

5.16 1) is just the definition, and use (1) to prove (2).

5.17 $-\frac{1}{6} + x$.

5.18 (1) $\mathbf{b} \in \mathcal{C}(A)$ and $\mathbf{y} \in \mathcal{N}(A^T)$.

5.19 The null space of the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & -1 & 1 \end{bmatrix}$ is $\mathbf{x} = t[1\ -1\ 1\ 0]^T + s[-4\ 1\ 0\ 1]^T$ for $t, s \in \mathbb{R}$.

5.20 Note: $\mathcal{R}(A)^{\perp} = \mathcal{N}(A)$.

5.22 There are 4 rotations and 4 reflections.

5.23 (1) $r = \frac{1}{\sqrt{2}},\ s = \frac{1}{\sqrt{6}},\ a = \frac{1}{\sqrt{3}},\ b = -\frac{1}{\sqrt{3}},\ c = -\frac{1}{\sqrt{3}}$.

5.24 Extend $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ to an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \ldots,\ \mathbf{v}_n\}$. Then $\|\mathbf{x}\|^2 = \sum_{i=1}^m |\langle \mathbf{x}, \mathbf{v}_i \rangle|^2 + \sum_{j=m+1}^n |\langle \mathbf{x}, \mathbf{v}_j \rangle|^2$.

5.25 (1) orthogonal. (2) not orthogonal.

5.26 $\mathbf{x} = (1,\ -1,\ 0) + t(2,\ 1,\ -1)$ for any number t.

5.28 $$\begin{bmatrix} s_0 \\ \mathbf{v}_0 \\ \frac{1}{2}\mathbf{g} \end{bmatrix} = \mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} -0.4 \\ 0.35 \\ 16.1 \end{bmatrix}.$$

5.30 For $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} = \langle \mathbf{v}_1, \mathbf{x} \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{v}_m, \mathbf{x} \rangle \mathbf{v}_m = (\mathbf{v}_1 \mathbf{v}_1^T)\mathbf{x} + \cdots + (\mathbf{v}_m \mathbf{v}_m^T)\mathbf{x}$.

5.31 The line is a subspace with an orthonormal basis $\frac{1}{\sqrt{2}}(1, 1)$, or is the column space of $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

5.32 $P = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

5.33 Note that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$ is an orthonormal basis for the subspace.

5.35 Hint: First, show that P is symmetric.

Exercises

5.1 Inner products are (2), (4), (5).

5.2 For the last condition of the definition, note that $\langle A, A\rangle = \mathrm{tr}(A^T A) = \sum_{i,j} a_{ij}^2 = 0$ if and only if $a_{ij} = 0$ for all i, j.

5.4 (1) $k = 3$.

5.5 (3) $\|f\| = \|g\| = \sqrt{1/2}$, The angle is 0 if $n = m$, $\frac{\pi}{2}$ if $n \neq m$.

5.6 Use the Cauchy–Schwarz inequality and Problem 5.2 with $\mathbf{x} = (a_1, \ldots, a_n)$ and $\mathbf{y} = (1, \ldots, 1)$ in $(\mathbb{R}^n, \cdot)$.

5.7 (1) $37/4$, $\sqrt{19/3}$.
(2) If $\langle h, g\rangle = h(\frac{a}{3} + \frac{b}{2} + c) = 0$ with $h \neq 0$ a constant and $g(x) = ax^2 + bx + c$, then (a, b, c) is on the plane $\frac{a}{3} + \frac{b}{2} + c = 0$ in $\mathbb{R}^3$.

5.9 Hint: For $A = [\mathbf{v}_1\ \mathbf{v}_2]$, two columns are linearly independent, and its column space is W.

5.11 (1) $\frac{3}{2}\mathbf{v}_2$, (2) $\frac{1}{2}\mathbf{v}_2$.

5.13 Orthogonal: (4). Nonorthogonal: (1), (2), (3).

5.17 Use induction on n. Let B be the matrix A with the first column $\mathbf{c}_1$ replaced by $\mathbf{c} = \mathbf{c}_1 - \mathrm{Proj}_W(\mathbf{c}_1)$, and write $\mathrm{Proj}_W(\mathbf{c}_1) = a_2\mathbf{c}_2 + \cdots + a_n\mathbf{c}_n$ for some a_i's. Show that $\sqrt{\det(A^T A)} = \sqrt{\det(B^T B)} = \|\mathbf{c}\|\mathrm{vol}(\mathbf{c}_2, \ldots, \mathbf{c}_n) = \mathrm{vol}(\mathcal{P}(A))$.

5.18 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Then the volume of the tetrahedron is $\frac{1}{3}\sqrt{\det(A^T A)} = 1$.

5.19 $A^T A = I$ and $\det A^T = \det A$ imply $\det A = \pm 1$.
The matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ is orthogonal with $\det A = -1$.

5.21 $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if $r = m$. It has infinitely many solutions if $nullity = n - r = n - m > 0$.

5.22 Find a least squares solution of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$ for (a, b) in $y = a + bx$. Then $y = x + \frac{3}{2}$.

5.23 Follow Exercise **5.22** with $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}$. Then $y = 2x^3 - 4x^2 + 3x - 5$.

5.27 (1) Let $h(x) = \frac{1}{2}(f(x) + f(-x))$ and $g(x) = \frac{1}{2}(f(x) - f(-x))$. Then $f = h + g$.
(2) For $f \in U$ and $g \in V$, $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx = -\int_{1}^{-1} f(-t)g(-t)dt$
$= -\int_{-1}^{1} f(t)g(t)dt = -\langle f, g\rangle$, by change of variable $x = -t$.
(3) Expand the length in the inner product.

5.28 Five of them are true.
(1) Possible via a natural isomorphism. See Theorem 5.8.
(2) Consider $(1, 0)$ and $(-1, 0)$.
(3) Consider two subspaces U and W of $\mathbb{R}^3$ spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively.
(4) $\|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\| \geq \|\mathbf{x}\|$ by the triangle inequality.
(5) The columns of any permutation matrix P are $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in some order.
(6) See Theorem 5.5.
(7) A is a projection iff $A^2 = A$. (See Theorem 5.9.)
(8) By Corollaries 5.10 and 5.12.
(9) $\mathcal{R}(A)$ and $\mathcal{C}(A)$ are not subspaces of the same vector space.
(10) A dilation is an isomorphism.
(11) $A^T\mathbf{b} \in \mathcal{C}(A^TA)$ always.
(12) The solution set is $\mathbf{x}_0 + \mathcal{N}(A)$ by Corollary 5.17.

Chapter 6

Problems

6.3 Consider the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

6.4 Check with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

6.5 (1) Use $\det A = \lambda_1 \cdots \lambda_n$. (2) $A\mathbf{x} = \lambda\mathbf{x}$ if and only if $\mathbf{x} = \lambda A^{-1}\mathbf{x}$.

6.6 If A is invertible, then $AB = A(BA)A^{-1}$. For two singular matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, AB and BA are not similar, but they have the same eigenvalues.

6.7 (1) If $Q = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$ diagonalizes A, then the diagonal matrix must be λI and $AQ = \lambda QI$. Expand this equation and compare the corresponding columns of the equation to find a contradiction on the invertibility of Q.

6.8 $Q = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $A = QDQ^{-1} = \begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$.

6.9 (1) The eigenvalues of A are $1, 1, -3$, and their associated eigenvectors are $(1, 1, 0)$, $(-1, 0, 1)$ and $(1, 3, 1)$, respectively.
(2) If $f(x) = x^{10} + x^7 + 5x$, then $f(1)$, $f(1)$ and $f(-3)$ are the eigenvalues of $A^{10} + A^7 + 5A$.

6.11 Note that $\begin{bmatrix} a_{n+1} \\ a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix}$. The eigenvalues are $1, 2, -1$ and eigenvectors are $(1, 1, 1)$, $(4, 2, 1)$ and $(1, -1, 1)$, respectively. It turns out that $a_n = 2 - (-1)^n\frac{2}{3} - \frac{2^n}{3}$.

6.13 Its characteristic polynomial is $f(\lambda) = (\lambda + 1)(\lambda - 2)^2$; so $(-1)^n, 2^n, n2^n$ form a fundamental set.

6.14 The eigenvalues are 0.6, 0.8, and 1.

6.15 The eigenvalues are 0, 0.4, and 1, and their eigenvectors are $(1, 4, -5)$, $(1, 0, -1)$ and $(3, 2, 5)$, respectively.

6.17 For (1), use $(A + B)^k = \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i}$ if $AB = BA$. For (2) and (3), use the definition of e^A. Use (1) for (4).

6.19 Note that $e^{(A^T)} = (e^A)^T$ by definition (thus, if A is symmetric, so is e^A), and use (4).

6.20 $A = 2I + N$ with $N = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Then $N^3 = \mathbf{0}$. $e^A = e^2 \begin{bmatrix} 1 & 3 & \frac{3^2}{2!} \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

6.21 $y_1 = c_1 e^{2x} - \frac{1}{4} c_2 e^{-3x}; \quad y_2 = c_1 e^{2x} + c_2 e^{-3x}$.

6.22 $\begin{cases} y_1 = - c_2 e^{2x} + c_3 e^{3x} \\ y_2 = c_1 e^x + 2c_2 e^{2x} - c_3 e^{3x} \\ y_3 = 2c_2 e^{2x} - c_3 e^{3x} \end{cases}, \quad \begin{cases} y_1 = e^{2x} - 2e^{3x} \\ y_2 = e^x - 2e^{2x} + 2e^{3x} \\ y_3 = - 2e^{2x} + 2e^{3x}. \end{cases}$

6.23 (1) $\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$, (2) $\begin{bmatrix} 3e^t - 2 \\ 2 - e^{-t} \\ e^{-t} \end{bmatrix}$.

6.24 With respect to the standard basis α, $[T]_\alpha = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$ with eigenvalues 3, 3, 5 and eigenvectors $(0, 1, 0)$, $(-1, 0, 1)$ and $(1, 2, 1)$, respectively.

6.25 With the standard basis for $M_{2\times2}(\mathbb{R})$:

$$\alpha = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

$[T]_\alpha = A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$. The eigenvalues are 3, 1, 1, -1, and their associated eigenvectors are $(1, 1, 1, 1)$, $(-1, 0, 1, 0)$, $(0, -1, 0, 1)$, and $(-1, 1, -1, 1)$, respectively.

6.26 With the basis $\alpha = \{1, x, x^2\}$, $[T]_\alpha = A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Exercises

6.1 (4) 0 of multiplicity 3, 4 of multiplicity 1. Eigenvectors are $\mathbf{e}_i - \mathbf{e}_{i+1}$ for $1 \le i \le 3$ and $\sum_{i=1}^{4} \mathbf{e}_i$.

6.2 $f(\lambda) = (\lambda + 2)(\lambda^2 - 8\lambda + 15)$, $\lambda_1 = -2$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\mathbf{x}_1 = (-35,\ 12,\ 19)$, $\mathbf{x}_2 = (0,\ 3,\ 1)$, $\mathbf{x}_3 = (0,\ 1,\ 1)$.

6.4 $\{\mathbf{v}\}$ is a basis for $\mathcal{N}(A)$, and $\{\mathbf{u},\ \mathbf{w}\}$ is a basis for $\mathcal{C}(A)$.

6.5 Assume that it is true for invertible matrices. In each of the equations (1)–(3) both sides continuously depend on the elements of A and B. Any matrix A can be approximated by matrices of the form $A_\varepsilon = A + \varepsilon I$ which are invertible for sufficiently small nonzero ε. (Actually, if $\lambda_1, \ldots, \lambda_n$ is the whole set of eigenvalues of A, then A_ε is invertible for all $\varepsilon \neq -\lambda_i$.) Besides, if $AB = BA$, then $A_\varepsilon B = BA_\varepsilon$.

6.6 Note that the order in the product doesn't matter, and any eigenvector of A is killed by B. Since the eigenvalues are all different, the eigenvectors belonging to 1, 2, 3 form a basis. Thus $B = 0$, that is, B has only the zero eigenvalue, so all vectors are eigenvectors of B.

6.8 $A = QDQ^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & -2 & -1 \\ 1 & 4 & -1 \\ 1 & 2 & 7 \end{bmatrix}$.

6.9 Note that $\mathbb{R}^n = W \oplus W^\perp$ and $P(\mathbf{w}) = \mathbf{w}$ for $\mathbf{w} \in W$ and $P(\mathbf{v}) = \mathbf{0}$ for $\mathbf{v} \in W^\perp$. Thus, the eigenspace belonging to $\lambda = 1$ is W, and that to $\lambda = 0$ is $W^\perp$.

6.10 For any $\mathbf{w} \in \mathbb{R}^n$, $A\mathbf{w} = \mathbf{u}(\mathbf{v}^T\mathbf{w}) = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$. Thus $A\mathbf{u} = (\mathbf{v}\cdot\mathbf{u})\mathbf{u}$, so $\mathbf{u}$ is an eigenvector belonging to the eigenvalue $\lambda = \mathbf{v}\cdot\mathbf{u}$. The other eigenvectors are those in $\mathbf{v}^\perp$ with eigenvalue zero. Thus, A has either two eigenspaces $E(\lambda)$ that are 1-dimensional spanned by $\mathbf{u}$ and $E(0) = \mathbf{v}^\perp$ if $\mathbf{v}\cdot\mathbf{u} \neq 0$, or just one eigenspace $E(0) = \mathbb{R}^n$ if $\mathbf{v}\cdot\mathbf{u} = 0$.

6.11 $\lambda\mathbf{v} = A\mathbf{v} = A^2\mathbf{v} = \lambda^2\mathbf{v}$ implies $\lambda(\lambda - 1) = 0$.

6.13 Use $\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}$.

6.14 $A = QD_1Q^{-1}$ and $B = QD_2Q^{-1}$ imply $AB = BA$ since $D_1D_2 = D_2D_1$. Conversely, Suppose $AB = BA$ and all eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are distinct. Then the eigenspaces $E(\lambda_i)$ are all 1-dimensional for $i = 1, \ldots, n$. But if $A\mathbf{x} = \lambda_i\mathbf{x}$, then $AB\mathbf{x} = BA\mathbf{x} = \lambda B\mathbf{x}$ implies $B\mathbf{x} \in E(\lambda_i)$. Thus $B\mathbf{x} = \mu\mathbf{x}$ means $\mathbf{x}$ is also an eigenvector of B. By the same reason, any eigenvector of B is also an eigenvector of A. Choose a set of linearly independent eigenvectors of A, which form an invertible matrix Q such that $Q^{-1}AQ = D_1$ and $Q^{-1}BQ = D_2$.

6.16 With respect to the basis $\alpha = \{1, x, x^2\}$, $[T]_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The eigenvalues are 2, 1, -1 and the eigenvectors are $(1, 1, 1)$, $(-1, 1, 0)$ and $(1, 1, -2)$, respectively.

6.17 None is diagonalizable.

6.18 (1) $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$ (2) $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ (3) $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

6.19 Eigenvalues are 1, 1, 2 and eigenvectors are $(1, 0, 0)$, $(0, 1, 2)$ and $(1, 2, 3)$. $A^{10}\mathbf{x} = (1025,\ 2050,\ 3076)$.

6.20 Fibonacci sequence: $a_{n+1} = a_n + a_{n-1}$ with $a_1 = 2$ and $a_2 = 3$.

6.22 The characteristic equation is $\lambda^2 - x\lambda - 0.18 = 0$. Since $\lambda = 1$ is a solution, $x = 0.82$. The eigenvalues are now 1, -0.18 and the eigenvectors are $(-0.3, -1)$ and $(1, -0.6)$.

6.23 (1) $e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$.

6.24 The initial status in 1985 is $\mathbf{x}_0 = (x_0, y_0, z_0) = (0.4, 0.2, 0.4)$, where x, y, z represent the percentage of large, medium, and small car owners. In 1995, the status is $\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.3 & 0.7 & 0.1 \\ 0 & 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix} = A\mathbf{x}_0$. Thus, in 2025, the status is $\mathbf{x}_4 = A^4\mathbf{x}_0$. The eigenvalues are 0.5, 0.8, and 1, whose eigenvectors are $(-0.41, 0.82, -0.41)$, $(0.47, 0.47, -0.94)$, and $(-0.17, -0.52, -1.04)$, respectively.

6.27 (1) $\begin{cases} y_1(x) &= -2e^{2(1-x)} + 4e^{2(x-1)} \\ y_2(x) &= -e^{2(1-x)} + 2e^{2(x-1)} \\ y_3(x) &= 2e^{2(1-x)} - 2e^{2(x-1)}. \end{cases}$ (2) $\begin{cases} y_1(x) &= e^{2x}(\cos x - \sin x) \\ y_2(x) &= 2e^{2x}\sin x. \end{cases}$

6.28 $y_1 = 0,\ y_2 = 2e^{2t},\ y_3 = e^{2t}$.

6.29 (1) $f(\lambda) = \lambda^3 - 10\lambda^2 + 28\lambda - 24$, eigenvalues are 6, 2, 2, and eigenvectors are $(1, 2, 1)$, $(-1, 1, 0)$ and $(-1, 0, 1)$.

(2) $f(\lambda) = (\lambda - 1)(\lambda^2 - 6\lambda + 9)$, eigenvalues are 1, 3, 3, and eigenvectors are $(2, -1, 1)$, $(1, 1, 0)$ and $(1, 0, 1)$.

6.30 Two of them are true:

(1) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, if $B = Q^{-1}AQ$ then B must be the identity.

Or, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ have a different eigenvalue.

(2) See Example 6.3.

(3) Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

(4) Consider $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. (5) Consider $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. (6) Consider $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(7) For any eigenvalue λ of A, $\lambda + 1$ is an eigenvalue of $A + I$.

(8) Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(9) $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$. See Theorem 6.3.

(10) If both belong to the same eigenvalue.

(11) In Example 6.6, $Q^{-1}AQ$ is diagonal and its two linearly independent eigenvectors are $\mathbf{e}_1$ and $\mathbf{e}_2$.

(12) Use Theorem 6.25 with $A = I_n$.

Chapter 7

Problems

7.1 (1) $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{u}}^T\mathbf{v} = \sum_i \overline{u_i}v_i = \overline{\sum_i \overline{v_i}u_i} = \overline{\mathbf{v} \cdot \mathbf{u}}$.

(3) $(k\mathbf{u}) \cdot \mathbf{v} = \sum_i \overline{ku_i}v_i = \overline{k}\sum_i \overline{u_i}v_i = \overline{k}(\mathbf{u} \cdot \mathbf{v})$.

(4) $\mathbf{u} \cdot \mathbf{u} = \sum_i |u_i|^2 \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $u_i = 0$ for all i.

7.2 (1) If $\mathbf{x} = \mathbf{0}$, clear. Suppose $\mathbf{x} \neq \mathbf{0} \neq \mathbf{y}$. For any scalar k,
$0 \leq \langle \mathbf{x} - k\mathbf{y}, \mathbf{x} - k\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - k\langle \mathbf{x}, \mathbf{y} \rangle - \overline{k}\langle \mathbf{y}, \mathbf{x} \rangle + k\overline{k}\langle \mathbf{y}, \mathbf{y} \rangle$. Let $k = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$ to obtain $|\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle - |\langle \mathbf{x}, \mathbf{y} \rangle|^2 \geq 0$. Note that equality holds if and only if $\mathbf{x} = k\mathbf{y}$ for some scalar k.

(2) Expand $||\mathbf{x} + \mathbf{y}||^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle$ and use (1).

7.3 Suppose that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and consider the linear dependence $a(\mathbf{x}+\mathbf{y})+b(\mathbf{x}-\mathbf{y})=\mathbf{0}$ of $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$. Then $\mathbf{0}=(a+b)\mathbf{x}+(a-b)\mathbf{y}$. Since $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, we have $a+b=0$ and $a-b=0$ which are possible only for $a=0=b$. Thus $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are linearly independent. Conversely, if $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are linearly independent, then the linear dependence $a\mathbf{x}+b\mathbf{y}=\mathbf{0}$ of $\mathbf{x}$ and $\mathbf{y}$ gives $\frac{1}{2}(a+b)(\mathbf{x}+\mathbf{y})+\frac{1}{2}(a-b)(\mathbf{x}-\mathbf{y})=\mathbf{0}$. Thus we get $a=0=b$. Thus $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

7.4 (1) Eigenvalues are $0, 0, 2$ and their eigenvectors are $(1, 0, -i)$ and $(0, 1, 0)$, respectively. (2) Eigenvalues are 3, $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$, and their eigenvectors are $(1, -i, \frac{1-i}{2})$, $(\frac{\sqrt{5}-3}{2}i, 1, \frac{1-\sqrt{5}}{2}(1+i))$, and $(-\frac{\sqrt{5}+3}{2}i, 1, \frac{1+\sqrt{5}}{2}(1+i))$, respectively.

7.5 Refer to the real case.

7.6 $(AB)^H=(\overline{AB})^T=\overline{B}^T\overline{A}^T=B^HA^H$.

7.7 $(A^H)(A^{-1})^H=(A^{-1}A)^H=I$.

7.8 The determinant is just the product of the eigenvalues and a Hermitian matrix has only real eigenvalues.

7.9 See Exercise **6.10**.

7.10 To prove (3) directly, show that $\overline{\lambda}(\mathbf{x}\cdot\mathbf{y})=\overline{\mu}(\mathbf{x}\cdot\mathbf{y})$ by using the fact that $A^H\mathbf{x}=-\mu\mathbf{x}$ when $A\mathbf{x}=\mu\mathbf{x}$.

7.11 $A^H=B^H+(iC)^H=B^T-iC^T=-B-iC=-A$.

7.12 $\pm AB=(AB)^H=B^HA^H=(\pm B)(\pm A)=BA$, $+$ if they are Hermitian, $-$ if they are skew-Hermitian.

7.13 Note that $\det U^H=\overline{\det U}$, and $1=\det I=\det(U^HU)=|\det U|^2$.

7.15 If $A^{-1}=A^H$ and $B^{-1}=B^H$, then $(AB)^HAB=I$.

7.16 Hermitian means the diagonal entries are real, and diagonality implies off-diagonal entries are zero. Unitary means the diagonal entries must be ± 1.

7.18 (1) If $U=\begin{bmatrix}\frac{1}{6}i\sqrt{3}+\frac{1}{2} & -\frac{1}{6}i\sqrt{3}+\frac{1}{2}\\ -\frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3}\end{bmatrix}$, $U^{-1}AU=\begin{bmatrix}\frac{1}{2}-\frac{1}{2}i\sqrt{3} & 0\\ 0 & \frac{1}{2}+\frac{1}{2}i\sqrt{3}\end{bmatrix}$

(2) If $U=\begin{bmatrix}0 & 0 & 1\\ -\frac{6}{25}-\frac{8}{25}i & \frac{2}{5}+\frac{1}{5}i & \frac{6}{25}+\frac{8}{25}i\\ 0 & -1 & 0\end{bmatrix}$, $U^{-1}AU=\begin{bmatrix}-1 & 0 & 0\\ 0 & 2i & 1\\ 0 & 0 & 2i\end{bmatrix}$

7.20 Note that A has two distinct eigenvalues.

7.21 This is a normal matrix. From a direct computation, one can find the eigenvalues, $1-i$, $1-i$ and $1+2i$, and the associated eigenvectors: $(-1,0,1)$, $(-1,1,0)$ and $(1,1,1)$, respectively, which are not orthogonal. But, by an orthonormalization, one can obtain a unitary basis-change matrix so that A is unitarily diagonalizable.

7.22 $A^HA=(H_1-H_2)(H_1+H_2)=(H_1+H_2)(H_1-H_2)=AA^H$ if and only if $H_1H_2-H_2H_1=0$.

7.23 In one direction these are all already proven in the theorems. Suppose that $U^HAU=D$ for a unitary matrix U and a diagonal matrix D.
(1) and (2). If all the eigenvalues of A are real (or purely imaginary), then the diagonal entries of D are all real (or purely imaginary). Thus $D^H=\pm D$, so that A is Hermitian (or skew-Hermitian).
(3) The diagonal entries of D satisfy $|\lambda|=1$. Thus, $D^H=D^{-1}$, and
$A^H=UD^{-1}U^{-1}=A^{-1}$.

7.24 $Q = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{bmatrix}$.

7.25 (1) $A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

(2) $B = \frac{3+2\sqrt{6}}{6}\begin{bmatrix} 1 & \frac{(1+\sqrt{6})(2+i)}{5} \\ \frac{(1+\sqrt{6})(2-i)}{5} & \frac{7+2\sqrt{6}}{5} \end{bmatrix}$

$+ \frac{3-2\sqrt{6}}{6}\begin{bmatrix} 1 & \frac{(1-\sqrt{6})(2+i)}{5} \\ \frac{(1-\sqrt{6})(2-i)}{5} & \frac{7-2\sqrt{6}}{5} \end{bmatrix}$.

Exercises

7.1 (1) $\sqrt{6}$, (2) 4.

7.4 (1) $\lambda = i, \mathbf{x} = t(1,\ -2-i), \lambda = -i, \mathbf{x} = t(1,\ -2+i)$.
(2) $\lambda = 1, \mathbf{x} = t(i,\ 1), \lambda = -1, \mathbf{x} = t(-i,\ 1)$.
(3) Eigenvalues are $2, 2+i, 2-i$, and eigenvectors are $(0, -1, 1))$, $(1, -\frac{1}{5}(2+i), 1), (1, -\frac{1}{5}(2-i), 1)$.
(4) Eigenvalues are $0, -1, 2$, and eigenvectors are $(1, 0, -1)), (1, -i, 1), (1, 2i, 1)$.

7.6 $A + cI$ is invertible if $\det(A + cI) \neq 0$. However, for any matrix A, $\det(A + cI) = 0$ as a complex polynomial has always a (complex) solution. For the real matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $A + rI$ is invertible for every real number r since A has no real eigenvalues.

7.7 (1) $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$, (2) $\frac{1}{2}\begin{bmatrix} 1 & i & 1-i \\ \sqrt{2}i & \sqrt{2} & 0 \\ 1 & i & -1+i \end{bmatrix}$.

7.10 (2) $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

7.12 (1) Unitary; diagonal entries are $\{1,\ i\}$. (2) Orthogonal; $\{\cos\theta + i\sin\theta, \cos\theta - i\sin\theta\}$, where $\theta = \cos^{-1}(0.6)$. (3) Hermitian; $\{1,\ 1+\sqrt{2},\ 1-\sqrt{2}\}$.

7.13 (1) Since the eigenvalues of a skew-Hermitian matrix must always be purely imaginary, 1 cannot be an eigenvalue.
(2) Note that, for any invertible matrix A, $(e^A)^H = e^{A^H} = e^{-A} = (e^A)^{-1}$.

7.14 $\det(U - \lambda I) = \det(U - \lambda I)^T = \det(U^T - \lambda I)$.

7.15 $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = U^H A U = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$.

7.17 See Exercise **6.14**.

7.18 The eigenvalues are 1, 1, 4, and the orthonormal eigenvectors are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Therefore,

$$A = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

7.20 See Theorem 7.8.

7.21 If λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n. Thus, if $A^n = \mathbf{0}$, then $\lambda^n = 0$ or $\lambda = 0$. Conversely, by Schur's lemma, A is similar to an upper triangular matrix, whose diagonals are eigenvalues that are supposed to be zero. Then it is easy to conclude A is nilpotent.

7.22 Ten of them are true.

(1) See Theorem 7.7. (2) Consider $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta \neq k\pi$.

(3) True: See Corollary 6.6. (4) Consider $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(5) Such a matrix A is unitary.

(6) and (7) A permutation matrix is an orthogonal matrix, but need not be symmetric.

(8) True: If A is Hermitian, by Schur's lemma, A is orthogonally similar to an upper triangular matrix T. If A is nilpotent, the eigenvalues of A, as the diagonal entries of T, are all zero. By showing $T^H = T$, one can conclude that such A must be the zero matrix.

(9) Schur's lemma.

(10) For a Hermitian matrix A, $-i$ cannot be an eigenvalue of A. Hence, $\det(A+iI) \neq 0$.

(11) Consider $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

(12) Modify (10).

(13) If $U^H AU = D$ with real diagonal entries, then $A^H = A$.

(15) $|\det U| = 1$ for any unitary matrix U.

Chapter 8

Problems

8.2 Hint: Let

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Then,

$$J - \lambda I = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (J - \lambda I)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and that $(J - \lambda I)^4 \neq \mathbf{0}$ but $(J - \lambda I)^5 = \mathbf{0}$.

8.3 Six different possibilities.

8.4 (1) $\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$, (2) $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$, (3) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

8.5 (1) $Q=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, (2) $Q=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, (3) $Q=\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

8.7 See Problem 6.1.

8.8 $f(A) \neq \det(AI - A)$ in general.

8.9 For a diagonal D, all diagonal entries of $f(D)$ are zero. For a diagonalizable $A = Q^{-1}DQ$, $f(A) = Q^{-1}f(D)Q$.

8.10 Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then

$$f(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n).$$

Thus, $f(B) = (B - \lambda_1 I_m) \cdots (B - \lambda_n I_m)$ is nonsingular if and only if $B - \lambda_i I_m$, $i = 1, \ldots, n$, are all nonsingular. That is, none of the λ_i's is an eigenvalue of B.

8.11 (1) $A^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ and $A^{10} = \begin{bmatrix} 1 & 0 & 1023 \\ 0 & 1024 & 0 \\ 0 & 0 & 1024 \end{bmatrix}$.

(2) The characteristic polynomial of A is $f(\lambda) = (\lambda - 1)(\lambda - 2)^2$, and the remainder is $104A^2 - 228A + 138I = \begin{bmatrix} 14 & 0 & 84 \\ 0 & 98 & 0 \\ 0 & 0 & 98 \end{bmatrix}$.

8.12 For $J^{(2)}$, $m(x) = (x - \lambda)^3$. For $J^{(3)}$, $m(x) = (x - \lambda)^4$.

8.14 (1) $m(\lambda) = (\lambda - 1)(\lambda - 2)^3$.

(2) $m(\lambda) = \lambda^2(\lambda - 2)$. $A^n = \begin{bmatrix} 0 & -2^{n+1} & 2^n & 2^n \\ -2^n & 0 & 0 & 2^n \\ -2^n & 0 & 0 & 2^n \\ -2^n & -2^{n+1} & 2^n & 2^{n+1} \end{bmatrix}$ $n \geq 2$.

8.15 (2) $e^A = \begin{bmatrix} 1 & 3-2e^2 & -2+e^2 & -1+e^2 \\ 1-e^2 & 2 & -1 & -1+e^2 \\ 1-e^2 & 1 & 0 & -1+e^2 \\ 1-e^2 & 3-2e^2 & -2+e^2 & -1+2e^2 \end{bmatrix}$.

8.17 The eigenvalue is -1 of multiplicity 3 and has only one linearly independent eigenvector $(1, 0, 3)$. The solution is

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = e^{-t} \begin{bmatrix} -1 - 5t + 2t^2 \\ -1 + 4t \\ 1 - 15t + 6t^2 \end{bmatrix}.$$

Exercises

8.1 $A^{-1} = \begin{bmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda^2} & \frac{1}{\lambda^3} & -\frac{1}{\lambda^4} \\ 0 & \frac{1}{\lambda} & -\frac{1}{\lambda^2} & \frac{1}{\lambda^3} \\ 0 & 0 & \frac{1}{\lambda} & -\frac{1}{\lambda^2} \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix}$ and $J = \begin{bmatrix} \frac{1}{\lambda} & 1 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} & 1 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix}$.

8.2 Find the Jordan canonical form of A as $Q^{-1}AQ = J$. Since A is nonsingular, all the diagonal entries λ_i of J, as the eigenvalues of A, are nonzero. Hence, each Jordan blocks J_j of J is invertible. Now one can easily show that $(Q^{-1}AQ)^{-1} = Q^{-1}A^{-1}Q = J^{-1}$ which is the Jordan form of A^{-1}, whose Jordan blocks are of the form J_j^{-1}.

8.3 (1) $\{\mathbf{e}_1, \mathbf{e}_4\}$; (3) $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$.

8.4 (1) For $\lambda = i-1$, $\mathbf{x}_1 = (1,\ -1)$; for $\lambda = i+1$, $\mathbf{x}_2 = (1,\ 1)$. (2) For $\lambda = -1$, $\mathbf{x}_1 = (-2,\ 0,\ 1)$, $\mathbf{x}_2 = (0,\ 1,\ 1)$, and for $\lambda = 0$, $\mathbf{x}_1 = (-1,\ 1,\ 1)$. (3) For $\lambda = 1$, $\mathbf{x}_1 = (2,\ 0,\ -1)$, $\mathbf{x}_2 = (-\frac{7}{2},\ -\frac{1}{2},\ \frac{1}{2})$, and for $\lambda = -1$, $\mathbf{x}_1 = (9,\ 1,\ -1)$.

8.6 Solve the recurrence relation in Example 2.14.

8.7 $(x, y) = \frac{1}{2}(4+i, i)$.

8.9 Use $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

8.10 Use $A = QJQ^{-1} = \begin{bmatrix} -6 & 2 & 8 \\ -3 & 1 & 2 \\ 6 & -1 & -4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{3} \\ 0 & 2 & 1 \\ \frac{1}{4} & -\frac{1}{2} & 0 \end{bmatrix}$.

8.11 $\mathbf{y}(t) = \sqrt{2}e^{4t}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \sqrt{2}e^{2t}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

8.12 $\begin{cases} y_1(t) = -2e^{2(1-t)} + 4e^{2(t-1)} \\ y_2(t) = -e^{2(1-t)} + 2e^{2(t-1)} \\ y_3(t) = 2e^{2(1-t)} - 2e^{2(t-1)} \end{cases}$

8.13 $\begin{cases} y_1(t) = 2(t-1)e^t \\ y_2(t) = -2te^t \\ y_3(t) = (2t-1)e^t \end{cases}$

8.14 (1) $(a-d)^2 + 4bc \neq 0$ or $A = aI$.

8.15 (1) $t^2 + t - 11$, (2) $t^2 + 2t + 13$, (3) $(t-1)(t^2 - 2t - 5)$.

8.17 (3)

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \quad A^n = \begin{bmatrix} 1 & 0 & n \\ 0 & 2^n & 2^n - 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(5)

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad A^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ n & 1 & 0 & 0 \\ \frac{n(n-1)}{2} & n & 1 & 0 \\ -n & 0 & 0 & 1 \end{bmatrix}.$$

8.18 (2) The characteristic polynomial of W is $f(\lambda) = \lambda^n - 1$. So, its eigenvalues are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = e^{2\pi i/n}$.
(3) Eigenvalues of A are $\lambda_k = \sum_{i=1}^{n} a_i\omega^{(i-1)k} = a_1 + a_2\omega^k + a_3\omega^{2k} + \cdots + a_n\omega^{(n-1)k}$, $k = 0,\ 1,\ \ldots,\ n-1$.
(4) $\det A = \prod_{k=0}^{n-1}(a_1 + a_2\omega_k + a_3\omega_k^2 + \cdots + a_n\omega_k^{n-1})$, where $\omega_k = e^{2\pi ik/n}$.
(5) The characteristic polynomial of B is $f(\lambda) = (\lambda - n + 1)(\lambda + 1)^{n-1}$.

8.19 Six of them are true.
(1) and (2) See Theorem 8.1.
(3) and (4): Check it with Example 8.2.
(5) and (6) See Examples 8.2 and 8.3.
(7) $\det e^{I_n} = e^n$.
(8) For any Jordan matrix J, J and J^T are similar.
(9) What is the minimal polynomial of I_n? (10) See Example 8.18(2).
(11) See Corollary 8.7.

Chapter 9

Problems

9.1

$$(1) \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & 1 \\ -4 & 1 & 4 \end{bmatrix}, \quad (2)\ \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad (3) \begin{bmatrix} 1 & 1 & 0 & -5 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ -5 & 0 & 2 & -1 \end{bmatrix}.$$

9.2 (1) The eigenvalues of A are 1, 2, 11. (2) The eigenvalues are 17, 0, −3, and so it is a hyperbolic cylinder. (3) A is singular and the linear form is present, thus the graph is a parabola.

9.3 B with the eigenvalues 2, $2+\sqrt{2}$ and $2-\sqrt{2}$.

9.5 The determinant is the product of the eigenvalues.

9.6 False with a counter-example $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

9.8 (1) is indefinite. (2) and (3) are positive definite.

9.13 (2) $b_{11} = b_{14} = b_{41} = b_{44} = 1$, all others are zero.

9.15 If $\mathbf{u} \in U \cap W$, then $\mathbf{u} = \alpha\mathbf{x} + \beta\mathbf{y} \in W$ for some scalars α and β. Since $\mathbf{x}, \mathbf{y} \in U$, $b(\mathbf{u}, \mathbf{x}) = b(\mathbf{u}, \mathbf{y}) = 0$. But $b(\mathbf{u}, \mathbf{x}) = \beta b(\mathbf{y}, \mathbf{x}) = -\beta$ and $b(\mathbf{u}, \mathbf{y}) = \alpha b(\mathbf{x}, \mathbf{y}) = \alpha$.

9.16 Let $c(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(b(\mathbf{x}, \mathbf{y}) + b(\mathbf{y}, \mathbf{x}))$ and $d(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(b(\mathbf{x}, \mathbf{y}) - b(\mathbf{y}, \mathbf{x}))$. Then $b = c + d$.

9.17 Let D be a diagonal matrix, and let D' be obtained from D by interchanging two diagonal entries d_{ii} and d_{jj}, $i \neq j$. Let P be the permutation matrix interchanging i-th and j-th rows. Then $PDP^T = D'$.

9.18 Count the number of distinct inertia (p, q, k). For n, the number of inertia with $p = i$ is $n - i + 1$.

9.19 (3) index = 2, signature = 1, and rank = 3.

9.20 (1) local minimum, (2) saddle point.

9.21 Check it with $f(x, y, z) = x^2 - y^2 - z^2$.

9.22 Note that the maximum value of $R(\mathbf{x})$ is the maximum eigenvalue of A, and similarly for the minimum value.

9.23 $\max = \frac{7}{2}$ at $\pm(1/\sqrt{2},\ 1/\sqrt{2})$, $\min = \frac{1}{2}$ at $\pm(1/\sqrt{2},\ -1/\sqrt{2})$.

9.24 (1) $\max = 4$ at $\pm\frac{1}{\sqrt{6}}(1,\ 1,\ 2)$, $\min = -2$ at $\pm\frac{1}{\sqrt{3}}(-1,\ -1,\ 1)$;
(2) $\max = 3$ at $\pm\frac{1}{\sqrt{6}}(2,\ 1,\ 1)$, $\min = 0$ at $\pm\frac{1}{\sqrt{3}}(1,\ -1,\ -1)$.

Exercises

9.1 (1) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, (3) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix}$, (4) $\begin{bmatrix} 3 & -2 & 0 \\ 5 & 7 & -8 \\ 0 & 4 & -1 \end{bmatrix}$.

9.3 (2) $\{(2,\ 1,\ 2),\ (-1,\ -2,\ 2),\ (1,0,0)\}$.

9.4 (i) If $a = 0 = c$, then $\lambda_i = \pm b$. Thus the conic section is a hyperbola.
(ii) Since we assumed that $b \neq 0$, the discriminant $(a-c)^2 + 4b^2 > 0$. By the symmetry of the equation in x and y, we may assume that $a - c \geq 0$.
If $a - c = 0$, then $\lambda_i = a \pm b$. Thus, the conic section is an ellipse if $\lambda_1\lambda_2 = a^2 - b^2 > 0$, or a hyperbola if $a^2 - b^2 < 0$. If $\lambda_1\lambda_2 = a^2 - b^2 = 0$, then it is a parabola when $\lambda_1 \neq 0$ and $e' \neq 0$, or a line or two lines for the other cases.
If $a - c > 0$. Let $r^2 = (a-c)^2 + 4b^2 > 0$. Then $\lambda_i = \frac{(a+c)\pm r}{2}$ for $i = 1, 2$. Hence, $4\lambda_1\lambda_2 = (a+c)^2 - r^2 = 4(ac - b^2)$. Thus, the conic section is an ellipse if $\det A = ac - b^2 > 0$, or a hyperbola if $\det A = ac - b^2 < 0$. If $\det A = ac - b^2 = 0$, it is a parabola, or a line or two lines depending on some possible values of d', e' and the eigenvalues.

9.6 If λ is an eigenvalue of A, then λ^2 and $\frac{1}{\lambda}$ are eigenvalues of A^2 and A^{-1}, respectively. Note $\mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x}$.

9.8 (1) $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. The form is indefinite with eigenvalues $\lambda = 5$ and $\lambda = -1$.

9.10 (1) $A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$, (2) $B = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix}$, (3) $Q = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$.

9.11 (2) The signature is 1, the index is 2, and the rank is 3.

9.15 (2) The point $(1, \pi)$ is a critical point, and the Hessian is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Hence, $f(1, \pi)$ is a local maximum.

9.18 (1) Suppose that $\varphi_{\mathbf{w}} = \varphi_{\mathbf{w}'}$. Then, for all $\mathbf{v} \in V$,

$$b(\mathbf{v},\ \mathbf{w}) = \varphi(\mathbf{w})(\mathbf{v}) = \varphi(\mathbf{w}')(\mathbf{v}) = b(\mathbf{v},\ \mathbf{w}') \quad \text{or} \quad b(\mathbf{v},\ \mathbf{w} - \mathbf{w}') = 0.$$

The non-degeneracy of b implies that $\mathbf{w} = \mathbf{w}'$, that is, φ is one-to-one. This also implies that $\dim W \leq \dim V^*$. A similar argument shows that the linear transformation $\psi : V \to W^*$ is also one-to-one, and therefore $\dim V \leq \dim W^*$. Since $\dim V = \dim V^*$ and $\dim W = \dim W^*$ from Theorem 4.18, we have $\dim V \leq \dim W^* = \dim W \leq \dim V^* = \dim V$. Therefore, φ and ψ are surjective, and so are isomorphisms.
(2) comes from (1).

9.19 Exactly seven of them are true.
(1) See Theorem 9.1 (The principal axes theorem) (1)
(2) Any two congruent matrices have the same inertia.
(3) Consider $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. (4) Consider $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
(7) Consider a bilinear form $b(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_2y_2$ on $\mathbb{R}^2$.
(9) The identity I is congruent to $k^2 I$ for all $k \in \mathbb{R}$.
(10) See (9).
(12) Consider a bilinear form $b(\mathbf{x}, \mathbf{y}) = x_1y_2$. Its matrix $Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

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Index