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This article generalizes the Cox, Ross, and Rubinstein (1979) binomial option-pricing model, and establishes a convergence from discrete-time multivariate multinomial models to continuous-time multidimensional diffusion models for contingent claims prices. The key to the approach is to approximate the N-dimensional diffusion price process by a sequence of N-variate, (N + 1)-nomial processes. It is shown that contingent claims prices and dynamic replicating portfolio strategies derived from the discrete time models converge to their corresponding continuous-time limits.

Over a decade ago, Cox, Ross, and Rubinstein (CRR) (1979) established a convergence of certain binomial processes to a lognormal process and showed that the Black–Scholes (1973) option-pricing formula is a limit of the discrete time binomial option-pricing formula.[1] The binomial approach provides an easy way of explaining (without using advanced mathematics) how uncertainties are resolved in the continuous-time model and how continuous trading in the stock and the bond can span infinitely many states of nature.

[1] The binomial or two-state variable approach was also developed independently in Sharpe (1978) and Rendleman and Bartter (1979).

More importantly, it provides an elegant numerical alternative to the partial differential equations (PDE) obtained in continuous-time models. The binomial option-pricing technique has now become an extremely powerful tool for valuing derivative securities that might be difficult to price under other alternative methods.

Recently, Evnine (1983), Boyle (1988), Cheyette (1988), Hull and White (HW) (1988), Boyle, Evnine, and Gibbs (BEG) (1989), and Madan, Milne, and Shefrin (MMS) (1989) have attempted to generalize the CRR method to approximate a multidimensional lognormal process. Although their approximations are adequate for purposes of valuation, none of these have provided economically satisfactory solutions. The paper by Cheyette is an exception. The discrete-time processes proposed by these authors converge to the corresponding continuous-time process under the risk-neutral probability measure. Although risk neutrality helps establish the convergence of contingent claims prices, the approximating discrete time models no longer have the usual Arrow–Debreu complete market property as is the case in the continuous-time counterpart. The model proposed by MMS has the complete market property but does not guarantee the convergence of multivariate contingent claims, such as options, on the maximum of two stocks. Cheyette's approximation does not rely on risk neutrality. However, his method applies only to lognormal processes.

In this article, we present a convergence from discrete-time multivariate multinomial models to a general continuous-time multidimensional diffusion model for contingent claims prices. The diffusion model consists of N risky stocks and one riskless bond, where the stocks and the bond form a dynamically complete securities market. We approximate the N-dimensional diffusion process for stock prices by a sequence of N-variate, $(N + 1)$-nomial processes. Thus, the stocks and the bond in the discrete-time multinomial models also form a dynamically complete securities market. We show that the contingent claims prices and the replicating portfolio strategies derived from the discrete-time models converge to the corresponding contingent claims prices and replicating portfolio strategies of the limiting continuous-time model.

The problem of approximating price processes when there are two stocks and one bond, where the stock prices follow two correlated lognormal processes, has long been of interest to financial economists. Intuitively, one would think that if one lognormal process can be approximated by one binomial process, then two lognormal processes should be approximated by two binomial processes. This would lead to a multinomial process with four uncertain states following each trading date. However, with four uncertain states and only two

stocks and one bond available for trading, markets cannot be completed by dynamic trading, and options cannot be priced by arbitrage. This is not the case in the continuous-time model, in which markets can be completed by continuous trading in the two stocks and the bond.

Evnine (1983) has proposed a "multiple" binomial model, which approximates the increments of two lognormal processes by three sequential moves. The approach he takes is first to let the price of the first stock move stochastically, while letting the price of the other stock grow at the riskless rate. He then lets the price of the second stock move stochastically, while letting the price of the first grow at the riskless rate. Finally, the prices of both stocks move together in order to capture the correlation. While Evnine manages to show that the discrete-price process matches the continuous-price process in distribution in the limit, the dynamic portfolio trading strategy implied by his model is always indeterminate, because the return on one of the two stocks is correlated perfectly with the return on the riskless bond. MMS (1988) construct an $(N + 1)$-nomial process for N stocks such that the distribution of the discrete-time price process for each individual stock converges to that of a one-dimensional lognormal process. Since they fail to specify the correlations among different assets and establish joint convergence for N stock prices, their model does not imply convergence for general multivariate claims prices.

In this article, we resolve this problem by showing that an N-dimensional diffusion process for stock prices can be approximated by an N-variate, $(N + 1)$-nomial process.[2] We utilize the fact that the increments of N independent Brownian motions can be approximated by N uncorrelated, *not necessarily independent,* random variables. Moreover, we construct N uncorrelated random variables, each of which takes only $N + 1$ possible values. As a result, an N-dimensional diffusion process can be approximated by an N-variate and $(N + 1)$-nomial process. For example, two correlated lognormal processes can be approximated by a trinomial model. A crucial distinction between our approximation and that of MMS is that the implicit Arrow–Debreu state price processes derived from the discrete-time models in ours converge to the corresponding continuous-time limit.

Computationally, our method may or may not perform better than those proposed by others. This is not surprising. It is well-known that, for numerical option prices, the trinomial approximation to a single lognormal process may perform better than the binomial approximation. In the case of two lognormal processes, although the

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[2] The number N is the martingale multiplicity discussed in Duffie and Huang (1985). When an $(N + 1)$-nomial process is used, the martingale multiplicities in both discrete- and continuous-time models are identical.

number of nodes in our models grows at a rate of n^2 (n is the number of time steps), which is significantly slower than the rate of growth reported in the models of HW and BEG, more time steps may be needed to reach the same level of approximation as in these articles.

Other related work has been done by Cheyette (1988), in which he approximates the returns of N assets by an $(N + 1)$-nomial process, where the asset prices follow an N-dimensional lognormal process. Similar to our model, the complete market property of the continuous-time model is preserved in the discrete-time models and the convergence of contingent claims prices is also achieved. While Cheyette's approach is similar to that here, we emphasize that his method applies only to the lognormal processes. The approximation procedure as well as the proof of convergence presented here are more general.

The rest of this article is organized as follows. In Section 1, the method is illustrated by some examples. In Section 2, a multivariate multinomial approximation to the multidimensional diffusion process is presented for stock prices and the bond price. In Section 3, the convergence of contingent claims prices and replicating portfolio strategies is dealt with. The article is concluded in Section 4.

1. Examples

In this section, we illustrate our method by some simple examples. Consider first the Black–Scholes economy with one stock and one bond. The movement of the stock price, S, and the bond price, B, can be described by the stochastic differential equations

$$dB_t = rB_t\, dt,$$
$$dS_t = \mu S_t\, dt + \sigma S_t\, dw_t,$$

where w is a one-dimensional Brownian motion. Suppose the time horizon is $[0, 1]$. CRR (1979) proposed the following binomial approximation:

$$S_{k+1}^n = \begin{cases} S_k^n e^{\sigma\sqrt{1/n}}, \\ S_k^n e^{-\sigma\sqrt{1/n}}, \end{cases}$$

where n denotes the number of time steps and S_k^n is the price of the stock at time k/n. The probability that the stock price will go up in the next period is equal to $\frac{1}{2}(1 + (\mu/\sigma)\sqrt{1/n})$. Although CRR's approximation serves our purposes well, we propose a more natural discretization scheme that focuses on approximating the increment of the Brownian motion. Intuitively, the increment of a Brownian motion can be approximated by the binomial random variable $\tilde{z}$ defined

by $P[\tilde{\epsilon} = 1] = P[\tilde{\epsilon} = -1] = \frac{1}{2}$, which has the required properties that $E(\tilde{\epsilon}) = 0$ and $\text{Var}(\tilde{\epsilon}) = 1$. Therefore, we propose the following binomial approximation with equally probable states:

$$S_{k+1}^n = \begin{cases} S_k^n + \dfrac{\mu S_k^n}{n} + \dfrac{\sigma S_k^n}{\sqrt{n}}, \\[2ex] S_k^n + \dfrac{\mu S_k^n}{n} - \dfrac{\sigma S_k^n}{\sqrt{n}}. \end{cases}$$

More generally, we consider the Black–Scholes economy with two stocks and one bond. The stock prices follow two correlated lognormal processes satisfying the stochastic differential equation:

$$dS_{t,1} = \mu_1 S_{t,1}\, dt + \sigma_1 S_{t,1}\, dw_{t,1},$$

$$dS_{t,2} = \mu_2 S_{t,2}\, dt + \sigma_2 \rho S_{t,2}\, dw_{t,1} + \sigma_2 \sqrt{1 - \rho_2} S_{t,2}\, dw_{t,2},$$

where w_1 and w_2 are two independent Brownian motions, σ_i^2 is the volatility of the instantaneous return on stock i, and ρ is the correlation coefficient between the instantaneous returns of the two stocks. Following the above argument, we approximate the increments of two independent Brownian motions by two random variables. We may use the two independent random variables, $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$, defined by

$$P[\tilde{\epsilon}_1 = 1, \tilde{\epsilon}_2 = 1] = 0.25, \qquad P[\tilde{\epsilon}_1 = 1, \tilde{\epsilon}_2 = -1] = 0.25,$$
$$P[\tilde{\epsilon}_1 = -1, \tilde{\epsilon}_2 = 1] = 0.25, \qquad P[\tilde{\epsilon}_1 = -1, \tilde{\epsilon}_2 = -1] = 0.25,$$

which have the required properties that $E(\tilde{\epsilon}_1) = E(\tilde{\epsilon}_2) = 0$, $\text{Var}(\tilde{\epsilon}_1) = \text{Var}(\tilde{\epsilon}_2) = 1$, and $\text{Cov}(\tilde{\epsilon}_1, \tilde{\epsilon}_2) = 0$. However, this approximation would result in an economically unsatisfactory multinomial process with four possible states following each trading date. If we allow ourselves to choose among uncorrelated random variables, then we can use the two random variables, $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$, defined by

$$P\left[\tilde{\epsilon}_1 = \frac{\sqrt{3}}{\sqrt{2}}, \tilde{\epsilon}_2 = \frac{1}{\sqrt{2}}\right] = \frac{1}{3}, \qquad P\left[\tilde{\epsilon}_1 = 0, \tilde{\epsilon}_2 = -\frac{2}{\sqrt{2}}\right] = \frac{1}{3},$$

$$P\left[\tilde{\epsilon}_1 = -\frac{\sqrt{3}}{\sqrt{2}}, \tilde{\epsilon}_2 = \frac{1}{\sqrt{2}}\right] = \frac{1}{3},$$

which also have the required properties $E(\tilde{\epsilon}_1) = E(\tilde{\epsilon}_2) = 0$, $\text{Var}(\tilde{\epsilon}_1) = \text{Var}(\tilde{\epsilon}_2) = 1$, and $\text{Cov}(\tilde{\epsilon}_1, \tilde{\epsilon}_2) = 0$. Thus, we propose the triominal approximation with equally probable states:

$$S^n_{k+1,1} = \begin{cases} S^n_{k,1} + \dfrac{\mu_1 S^n_{k,1}}{n} + \sigma_1 S^n_{k,1}\dfrac{\sqrt{3}}{\sqrt{2n}}, \\[2ex] S^n_{k,1} + \dfrac{\mu_1 S^n_{k,1}}{n}, \\[2ex] S^n_{k,1} + \dfrac{\mu_1 S^n_{k,1}}{n} - \sigma_1 S^n_{k,1}\dfrac{\sqrt{3}}{\sqrt{2n}}, \end{cases}$$

$$S^n_{k+1,2} = \begin{cases} S^n_{k,2} + \dfrac{\mu_2 S^n_{k,2}}{n} + \sigma_2\rho S^n_{k,2}\dfrac{\sqrt{3}}{\sqrt{2n}} + \sigma_2\sqrt{1-\rho^2}S^n_{k,2}\dfrac{1}{\sqrt{2n}}, \\[2ex] S^n_{k,2} + \dfrac{\mu_2 S^n_{k,2}}{n} - \sigma_2\sqrt{1-\rho^2}S^n_{k,2}\dfrac{2}{\sqrt{2n}}, \\[2ex] S^n_{k,2} + \dfrac{\mu_2 S^n_{k,2}}{n} - \sigma_2\rho S^n_{k,2}\dfrac{\sqrt{3}}{\sqrt{2n}} + \sigma_2\sqrt{1-\rho^2}S^n_{k,2}\dfrac{1}{\sqrt{2n}}. \end{cases}$$

We will show in the next section that the above trinomial processes converge weakly to the original lognormal processes.

We point out that the choice of such a pair of random variables is not unique. Let there be three equally probable states, and let the values of $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ be $(e_{1,1}, e_{2,1}, e_{3,1})$ and $(e_{1,2}, e_{2,2}, e_{3,2})$, respectively, then the restrictions on $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are that the two variables should have zero means, unit variances, and zero covariance, i.e.,

$$\tfrac{1}{3}(e_{1,1} + e_{2,1} + e_{3,1}) = 0,$$

$$\tfrac{1}{3}(e_{1,2} + e_{2,2} + e_{3,2}) = 0,$$

$$\tfrac{1}{3}(e_{1,1}e_{1,1} + e_{2,1}e_{2,1} + e_{3,1}e_{3,1}) = 1,$$

$$\tfrac{1}{3}(e_{1,2}e_{1,2} + e_{2,2}e_{2,2} + e_{3,2}e_{3,2}) = 1,$$

$$\tfrac{1}{3}(e_{1,1}e_{1,2} + e_{2,1}e_{2,2} + e_{3,1}e_{3,2}) = 0.$$

For example, $(1/\sqrt{8})\,(2, -1 + \sqrt{3}, -1 - \sqrt{3})$ and $(1/\sqrt{8})\,(2, -1 - \sqrt{3}, -1 + \sqrt{3})$ form one such pair. In fact, there exist infinitely many such pairs satisfying the above equations. Also, it is important to note that by collapsing those states that have the same levels of stock prices, the number of states in the trinomial model grows at a rate $(n + 1)\,(n + 2)/2$, which is significantly slower than the rate of growth in the models proposed by HW and BEG. Of course, more time steps are needed in order to be able to reach the same level of precision for option prices as that reported by HW and BEG. If the drift and diffusion terms were not linear in S, then the number of the

states in the trinomial model would grow at a rate 3^n [and $(N + 1)^n$ for $(N + 1)$-nomial processes].

2. Multinomial Approximation

In this section, we construct a sequence of N-variate, $(N + 1)$-nomial processes that converge weakly to the N-dimensional diffusion process for stock prices.[3] Moreover, we show that the sequence of the implicit Arrow–Debreu state price processes derived from the discrete-time model converges weakly to the corresponding continuous-time limit. This result plays an important role in establishing the weak convergence of contingent claims prices.

We consider a securities market consisting of N risky stocks and one locally riskless bond. The N-dimensional vector of stock prices, S, and the bond price, B, are described by the stochastic differential equations

$$dS_t = b(S_t) \ dt + \sigma(S_t) \ dw_t, \tag{1}$$

$$dB_t = B_t r(S_t) \ dt, \qquad B_0 = 1, \tag{2}$$

where w is an N-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$. We assume that $b: \Re^N \to \Re^N$, $\sigma: \Re^N \to \Re^{N \times N}$, and $r: \Re^N \to \Re$ are continuous and that σ is nonsingular. We assume further that b and σ satisfy the *uniform Lipschitz condition,* that is, there exists a constant $L > 0$, such that, for all $x, y \in \Re^N$,

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|. \tag{3}$$

We assume that r is nonnegative and that there exists a constant $K > 0$, such that, for all $x \in \Re^N$,

$$|x^2 r(x)| \leq K(1 + |x|^2). \tag{4}$$

Finally, the time horizon is assumed to be $[0, 1]$. It is easy to see that the regularity conditions imposed on the drift and the diffusion terms are readily satisfied for the Black–Scholes price system.

As in the Black–Scholes economy, we assume that markets are dynamically complete.[4] That is, contingent claims, such as options written on the stocks, can be spanned by dynamic trading in the stocks and the bond. To rule out arbitrage opportunities, we assume that there exists a unique equivalent martingale measure or a risk-neutral

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[3] Let X^n and X be the stochastic processes defined on the probability spaces $(\Omega_n, \mathscr{F}_n, P_n)$ and $(\Omega, \mathscr{F}, P)$ with values in $D^M[0, 1]$, the space of functions from $[0, 1]$ to $\Re^M$ which are right-continuous with left limits. The sequence of processes X^n is said to converge weakly to X if, for any bounded continuous mapping, b, from $D^M[0, 1]$ to $\Re$, we have $E_n[b(X^n)] \to E[b(X)]$. See Billingsley (1968, chapter 1) for details.

[4] Rigorously speaking, dynamic completeness requires the existence of a unique equivalent martingale measure and a proper choice of the space of feasible trading strategies (Cox and Huang, 1989).

probability measure for the price system defined by (1) and (2) (i.e., the stock prices discounted by the bond price become martingale under this measure). Letting

$$\kappa(x) \equiv -\sigma(x)^{-1}(b(x) - r(x)x)$$

and

$$\xi_t \equiv \exp\left(\int_0^t \kappa(S_s) \, dw_s - \tfrac{1}{2} \int_0^t |\kappa(S_s)|^2 \, ds\right),$$

then the equivalent martingale measure, denoted by Q, has the following form:

$$Q(A) \equiv \int_A \xi_1(\omega) P \, (d\omega).$$

We assume throughout that κ is continuous in x. One can easily verify that the stochastic process $\{\xi_t\}$ satisfies the stochastic differential equation:

$$d\xi_t = \kappa(S_t)\xi_t \, dw_t, \qquad \xi_0 = 1. \tag{5}$$

In the literature $\{\xi_t(\omega)\}$ is usually interpreted as the implicit system of Arrow–Debreu state prices for a security that pays off one dollar at time t, state ω and nothing otherwise (Cox and Huang, 1989). More specifically, the price of a contingent claim that pays off X_t dollars (in flow) at time $t \in [0, 1]$ and Y dollars (in lump sum) at time 1 can be calculated as

$$E\left[\int_0^1 \xi_t(\omega)\frac{X_t(\omega)}{B_t(\omega)} \, dt + \xi_1(\omega)\frac{Y(\omega)}{B_1(\omega)}\right].$$

Hence, to be more precise, $\{\xi_t\}$ is the price of the discounted dollar payoff per unit of probability. With a minor abuse of terminology, we call ξ the implicit Arrow–Debreu state price process.

We now proceed with the construction of a sequence of $(N + 1)$-nomial processes for the N stock prices and the bond price. As we have illustrated earlier, the basic idea here is to approximte the increments of N independent Brownian motions by N uncorrelated random variables.

We first construct N uncorrelated random variables. Since we need to approximate the increments of the Brownian motions on all of the subintervals, we will in fact construct a sequence of independent, identically distributed, N-dimensional random vectors, $\tilde{\epsilon}^k = (\tilde{\epsilon}_1^k, \ldots, \tilde{\epsilon}_N^k)^\top$, $k = 1, 2, \ldots, n$, where n is the number of time steps. Let A be

an $(N + 1)$, real, and orthogonal matrix, such that the last column of A is $(1/\sqrt{N + 1}, \ldots, 1/\sqrt{N + 1})^{\mathsf{T}}$.[5]

For example, when $N = 1$, we can choose

$$
A = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix}.
$$

When $N = 2$, we can choose

$$
A = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\ 0 & -\dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \end{pmatrix}.
$$

Define $e_{s,j} \equiv a_{sj}\sqrt{N + 1}$, where a_{sj} is the sth element in jth column of A. Let $\Omega = \{\omega_1, \ldots, \omega_{N+1}\}$ be the sample space on which $\tilde{\epsilon}^k$ is defined. We assign an equal probability to each of the states ω_s, that is, $P[\omega_s] = 1/(N + 1)$, for $s = 1, \ldots, N + 1$. We now define $\tilde{\epsilon}_j^k(\omega_s) = e_{s,j}$, for $s = 1, \ldots, N + 1$, where k is the time step. Thus $e_{s,j}$ is the realization of the jth element of the random vector in state ω_s. Note also that $\tilde{\epsilon}^k$ is stationary, since $e_{s,j}$ is independent of k. It is easy to verify that, for a fixed k, $\tilde{\epsilon}_1^k, \ldots, \tilde{\epsilon}_N^k$ are uncorrelated with each other, and have mean 0 and variance 1. Now, let $\Omega_n = \Omega \times \cdots \times \Omega$ (n times) be the natural product space and P_n be the natural product measure on Ω_n, then $\{\tilde{\epsilon}^1, \tilde{\epsilon}^2, \ldots, \tilde{\epsilon}^n\}$ can be treated as a sequence of independent and identically distributed random vectors defined on (Ω_n, P_n). We now define the N-variate, $(N + 1)$-nomial process for stock prices, S^n, as the solution to the stochastic difference equation:

$$
S_{k+1}^n = S_k^n + \frac{b(S_k^n)}{n} + \sigma(S_k^n)\frac{\tilde{\epsilon}^k}{\sqrt{n}}, \tag{6}
$$

where S_k^n denotes the vector of stock prices at time k/n and $S_0^n = S_0$. More explicitly, we have

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[5] Such a matrix always exists! In fact, let D be the linear subspace (in $\mathfrak{R}^{N+1}$) generated by the vector $(1, \ldots, 1)^{\mathsf{T}}$, then the first N columns of A can be formed by any orthonormal basis of $D^{\perp}$, the largest linear subspace that is orthogonal to D. A general procedure for finding an orthonormal basis for a subspace can be found in Anton (1981).

$$
S_{k+1,\,j}^{n} = \begin{cases}
S_{k,j}^{n} + \dfrac{b_j(S_k^n)}{n} + \sigma_{j,1}(S_k^n)\dfrac{e_{1,1}}{\sqrt{n}} + \cdots + \sigma_{j,N}(S_k^n)\dfrac{e_{1,N}}{\sqrt{n}}, \\[2ex]
S_{k,j}^{n} + \dfrac{b_j(S_k^n)}{n} + \sigma_{j,1}(S_k^n)\dfrac{e_{2,1}}{\sqrt{n}} + \cdots + \sigma_{j,N}(S_k^n)\dfrac{e_{2,N}}{\sqrt{n}}, \\[2ex]
\qquad\qquad\qquad\vdots \\[1ex]
S_{k,j}^{n} + \dfrac{b_j(S_k^n)}{n} + \sigma_{j,1}(S_k^n)\dfrac{e_{N+1,1}}{\sqrt{n}} + \cdots + \sigma_{j,N}(S_k^n)\dfrac{e_{N+1,N}}{\sqrt{n}},
\end{cases}
$$

where $S_{k,j}^n$ denotes the price of stock j at time k/n, b_j and $\sigma_{j,i}$ are the jth elements of b and the ith column of σ, respectively. From (6), it is clear that the kth random vector, $\tilde{e}^k$, is used to approximate the random increments of the Brownian motions from time k/n to time $(k + 1)/n$. Equation (6) can be viewed as the finite difference approximation to the stochastic differential equation (1). Similarly, we define the bond price process, B^n, by the following equation:

$$
B_{k+1}^n = B_k^n(1 + r(S_k^n)/n), \tag{7}
$$

with $B_0^n = 1$. Note that the bond price is locally riskless.

In order to define the discrete time stock and bond prices on the entire time horizon $[0, 1]$, we set $\tilde{B}_t^n \equiv \tilde{B}_{[nt]}^n$ and $\tilde{S}_t^n \equiv S_{[nt]}^n$, where $[\ ]$ denotes the largest integer that is less than or equal to nt. The sample paths of $\tilde{S}^n$ and $\tilde{B}^n$ are piecewise constant and have jumps only at $t = k/n$.[6]

Since there are $N + 1$ securities traded and $N + 1$ possible uncertain states following each trading date, and since σ and A are invertible, it can be verified that markets are dynamically complete.[7] The unique Arrow–Debreu state price $\pi(\omega_s; S_k^n)$ at time k/n, conditional on the stock price at time k/n being S_k^n, for a security that pays off one dollar at time $(k + 1)/n$, state ω_s and nothing otherwise must satisfy the following conditions:[8]

$$
\sum_{s=1}^{N+1} \pi(\omega_s;\, S_k^n) S_{k+1}^n(\omega_s) = S_k^n, \tag{8}
$$

$$
\sum_{s=1}^{N+1} \pi(\omega_s;\, S_k^n) B_{k+1}^n(\omega_s) = B_k^n. \tag{9}
$$

Solving (8) and (9) by substituting (6) and (7) into these equations, we obtain

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[6] Let $\mathscr{F}_n = \{F_k, \ k \le n\}$, where $F_k = \sigma(\tilde{e}^1, \ldots, \tilde{e}^k)$. Then $(\tilde{S}^n, \tilde{B}^n)$ is a process defined on the probability space $(\Omega_n, \mathscr{F}_n, P_n)$ with values in $D^{N+1}[0, 1]$.

[7] If σ and A are invertible, the $(N + 1) \times (N + 1)$ matrix formed by the prices of the N stocks and the bond at the $N + 1$ states for any subtree is also invertible.

[8] Rigorously speaking, we should write $S_{k+1}^n(\omega_s; S_k^n)$ for S_{k+1}^n, since S^n is a Markov chain and its value at time $(k + 1)/n$ depends upon its value at time k/n.

$$\pi(\omega_s; S_k^n) = \frac{1}{N+1}\left(1 + \frac{\kappa(S_k^n)^{\mathsf{T}}}{\sqrt{n}}\tilde{\epsilon}^k(\omega_s)\right)\left(1 + \frac{r(S_k^n)}{n}\right)^{-1}.$$

Note that $\pi(\ ; S_k^n)$ is the one-period Arrow–Debreu state price [from time k/n to time $(k+1)/n$]. To obtain the Arrow–Debreu state price at time 0 for a security that pays off one dollar at time k/n, we need to multiply together all of the one-period Arrow–Debreu state prices from period 1 to period k, where period i is from time $(i-1)/n$ to time i/n. Let π_k^n denote this state price. Then

$$\pi_k^n = \pi(\ ; S_{k-1}^n)\pi(\ ; S_{k-2}^n)\cdots\pi(\ ; S_0^n), \qquad k = 1, \ldots, n,$$

with $\pi_0^n = 1$. Moreover, π_k^n satisfies the stochastic difference equation:

$$\pi_{k+1}^n = \frac{\pi_k^n}{N+1}\left(1 + \frac{\kappa(S_k^n)^{\mathsf{T}}}{\sqrt{n}}\tilde{\epsilon}^k\right)\left(1 + \frac{r(S_k^n)}{n}\right)^{-1}.$$

We assume that all of the π's are nonnegative for sufficiently large n. A sufficient condition for this to be true is that κ be bounded.

In order to relate π^n to ξ, we introduce a new process ξ^n, defined by

$$\xi_k^n \equiv \pi_k^n B_k^n (N+1)^k.$$

Using the fact that

$$\sum_{s=1}^{N+1}\pi(\omega_s; S_k^n) = \left(1 + \frac{r(S_k^n)}{n}\right)^{-1},$$

one can verify that $E_n[\xi_n^n] = 1$, where E_n denotes the expectation under the probability measure P_n on Ω_n. Now defining

$$Q_n(A) = \int_A \xi_n^n(\omega)P_n(d\omega),$$

then (8) and (9) imply that Q_n is an equivalent martingale measure (i.e., the discounted stock price processes become martingales under this measure). Defining $\tilde{\xi}_t^n = \xi_{[nt]}^n$, for $t \in [0, 1]$, we call $\tilde{\xi}^n$ the implicit Arrow–Debreu state price process, similar to the convention we adopted in the continuous-time case. Analogous to (5), ξ^n can be represented by the stochastic difference equation:

$$\xi_{k+1}^n = \xi_k^n\left(1 + \kappa(S_k^n)^{\mathsf{T}}\frac{\tilde{\epsilon}^k}{\sqrt{n}}\right). \tag{10}$$

We have the following convergence theorem for the proposed $(N + 1)$-nomial approximation. (All of the proofs can be found in the Appendix.)

Theorem 1. *Let $\tilde{X}^n = (\tilde{S}^n, \tilde{B}^n, \tilde{\xi}^n)$ and $X = (S, B, \xi)$. Then $\tilde{X}^n$ converges weakly to X.*

Remark 1. *(a) The proof of this theorem employs the Martingale Central Limit Theorem developed by Ethier and Kurtz (1986, p. 354). Theorem 1 shows that the local movements of N-independent Brownian motions can be approximated by N uncorrelated but possibly dependent random variables $\{\tilde{\epsilon}_j^k, j = 1, \ldots, N\}$.*
(b) $N + 1$ is the smallest number of the states one can allow in order to keep $\{\tilde{\epsilon}_j^k\}, j = 1, \ldots, N$, mutually uncorrelated.

3. Convergence of Contingent Claims Prices

In this section, we demonstrate that contingent claims price processes, obtained from the discrete-time models based on a no-arbitrage argument, converge weakly to their continuous-time counterpart. Moreover, the dynamic portfolio strategies that replicate the payoff on the underlying contingent claims also converge weakly. It then follows that the contingent claims prices and the replicating portfolio strategies, computed at time 0, converge (numerically) to the corresponding continuous-time limits.

We begin with a definition of a contingent claim. Let g be a measurable function mapping from $\mathfrak{R}^N$ to $\mathfrak{R}$. A contingent claim on the stocks is defined to be a security that pays $g(S_1)$ dollars at the final date (i.e., $t = 1$).[9] This formulation subsumes all of the usual examples. For example, it includes as special cases an option on one stock, $\max(S_1^{(1)} - K, 0)$, and an option on the maximum of the two stocks, $\max(\max(S_1^{(1)}, S_1^{(2)}) - K, 0)$, where $S_1^{(i)}$ denotes the price of the ith security at time 1.

Following Harrison and Kreps (1979), the price of a contingent claim at time t can be evaluated by taking the conditional expectation of the discounted final payoff under the equivalent martingale measure Q, that is,

$$V(S, t) = E_Q\left[\exp\left(-\int_t^1 r(S_r, \tau)\, d\tau\right) g(S_1) \,\middle|\, S_t = S\right]$$

$$= E\left[\frac{\xi_1}{\xi_t} \exp\left(-\int_t^1 r(S_r, \tau)\, d\tau\right) g(S_1) \,\middle|\, S_t = S\right]. \quad (11)$$

---

[9] We ignore the possibility of cash payout before the final date or path-dependent final payoffs, although conceptually these pose no difficulty at all.

Alternatively, if V is continuously differentiable with respect to t and twice continuously differentiable with respect to S, then V can be determined as a solution to the following partial differential equation [Cox, Ingersoll, and Ross (1985)]:

$$\tfrac{1}{2}\text{trace}[\sigma^{\mathsf{T}} V_{SS}\sigma] + rS^{\mathsf{T}} V_S + V_t - rV = 0,$$

$$V(S, 1) = g(S), \tag{12}$$

where

$$V_S(S, t) = \left[\frac{\partial V(S, t)}{\partial S_1}, \cdots, \frac{\partial V(S, t)}{\partial S_N}\right]^{\mathsf{T}},$$

$$V_{SS}(S, t) = \left[\frac{\partial V(S, t)}{\partial S_i \partial S_j}\right]_{N \times N}.$$

Equation (12) is usually called the *fundamental partial differential equation* in the option-pricing literature. The dynamic portfolio strategy that replicates the final payoff of this claim is given by

$$\theta(S_t, t) = V_S(S_t, t),$$
$$\alpha(S_t, t) = (1/B_t)(V(S_t, t) - \theta(S_t, t)^{\mathsf{T}} S_t),$$

where θ_i, the ith element of θ, and α denote the number of shares held in the ith stock and in the bond, respectively.

This valuation technique can easily be applied to the discrete-time model. The price of the contingent claim at any time k/n, denoted by V^n, can be evaluated by taking the conditional expectation under the equivalent martingale measure Q_n, that is,

$$V^n\left(S, \frac{k}{n}\right) = E_{Q_n}\left[g(S_n^n)\frac{B_k^n}{B_n^n}\;\middle|\; S_k^n = S\right]$$

$$= E_n\left[\frac{\xi_n^n}{\xi_k^n}g(S_n^n)\frac{B_k^n}{B_n^n}\;\middle|\; S_k^n = S\right]. \tag{13}$$

Alternately, V^n can be determined as a solution to the recurrent equation:

$$V^n\left(S_k^n, \frac{k}{n}\right) = \sum_{s=1}^{n+1} \pi(\omega_s; S_k^n) V^n\left(S_{k+1}^n(\omega_s), \frac{k+1}{n}\right),$$

$$V^n(S_n^n, 1) = g(S_n^n). \tag{14}$$

The dynamic portfolio strategy that replicates the final payoff of this claim is determined by the following system of linear equations:

$$\alpha_k^n B_{k+1}^n(\omega_s) + \theta_{k,1}^n S_{k+1,1}^n(\omega_s) + \cdots + \theta_{k,N}^n S_{k+1,N}^n(\omega_s)$$

$$= V\left(S_{k+1}^n(\omega_s), \frac{k+1}{n}\right),$$

for $s = 1, \ldots, N + 1$. Since there are $N + 1$ equations and $N + 1$ unknowns, and since σ and A are invertible, the solution for (α_k^n, θ_k^n) is uniquely determined. Morever, α_k^n and θ_k^n are functions of the stock prices S_k^n and time k/n. The following theorem establishes the convergence of contingent claim prices from discrete-time models to the continuous-time diffusion model. We start with a definition.

Definition 1. *A function $f(x, t): \Re^N \times \Re_+ \to \Re^M$ is said to satisfy a polynomial growth condition if there exists a constant $\gamma > 0$ and a positive integer q, such that*

$$|f(x, t)| \le \gamma(1 + |x|^{2q}), \qquad \forall \ (x, t) \in \Re^N \times \Re_+.$$

Theorem 2. *Suppose that V is continuously differentiable up to the third order and that V and all of its derivatives up to the third order satisfy a polynomial growth condition. Then,*

(1) letting $\tilde{V}_t = V(S_t, t)$, $V_k^n = V^n(S_k^n, k/n)$, and $\tilde{V}_t^n = V_{[nt]}^n$, we have that $\tilde{V}^n$ converges weakly to $\tilde{V}$;

(2) letting $\tilde{\alpha}_t = \alpha(S_t, t)$, $\tilde{\theta}_t = \theta(S_t, t)$, $\tilde{\alpha}_t^n = \alpha_{[nt]}^n$, and $\tilde{\theta}_t^n = \theta_{[nt]}^n$, we have that $\tilde{\alpha}^n$ and $\tilde{\theta}^n$ converge weakly to $\tilde{\alpha}$ and $\tilde{\theta}$, respectively. As a result, we have that $\tilde{V}_0^n \to \tilde{V}_0$, and $(\tilde{\alpha}_0^n, \tilde{\theta}_0^n) \to (\tilde{\alpha}_0 \tilde{\theta}_0)$, as $n \to \infty$.

The basic technique of the proof is to write the discrete-time contingent claim price as follows:

$$V^n\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) = V\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - \left[V\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - V^n\left(\tilde{S}_t^n, \frac{[nt]}{n}\right)\right]$$

$$= V\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - \tilde{e}_t^n.$$

Since V is continuous and S^n converges weakly to S, the stochastic process defined by the first term on the right-hand side of the above equation converges weakly to $\tilde{V}$ by the Continuous Mapping Theorem (theorem 1.5.1 of Billingsley, 1968, p. 30). We show in the Appendix that the stochastic process defined by the second term on the right-hand side of the above equation converges weakly to zero.

We emphasize that the convergence of contingent claims prices can be well-anticipated from Theorem 1 and Equations (11) and (13).

Clearly, convergence of the implicit Arrow–Debreu state price processes is crucial for Theorem 2 to be true.

The smoothness condition for V and the polynomial growth condition for V and its derivatives can be guaranteed if the following two conditions are satisfied:

1. $g(x)$ is piecewise and continuously differentiable up to the sixth order, with all of these derivatives satisfying a polynomial growth condition;

2. $r(x)$ and $xr(x)$ satisfy a uniform Lipschitz condition for $x \in \Re^N$, and $r(x)$ and $\sigma(x)$ are continuously differentiable with respect to x up to the sixth order with all of these derivatives satisfying a polynomial growth condition.

For example, $g(x) = \max(0, x - K)$ satisfies the first condition, although it is not differentiable at $x = K$. We refer the reader to He (1989, p. 68) for details. The second condition is readily satisfied for a price system in which the stock prices follow a multidimensional lognormal process and the bond price grows at a constant rate of interest.

4. Concluding Remarks

We have generalized the Cox, Ross, and Rubinstein (1979) binomial option-pricing model, and developed a convergence from discrete-time multivariate multinomial models to continuous-time multidimensional diffusion models for contingent claims prices. We approximated the N-dimensional diffusion price process by a sequence of N-variate, $(N + 1)$-nomial processes. The contingent claims prices, as well as dynamic replicating portfolio strategies derived from the discrete-time models, converge to their corresponding continuous-time limits.

Appendix: Proofs

Proof of Theorem 1
We need to apply the Martingale Central Limit Theorem (theorem 7.4.1 of Ethier and Kurtz, 1986, p. 354) to prove Theorem 1.

Lemma 1 (Martingale Central Limit Theorem). *Let $b(x)$ be an $M \times 1$ vector and $\sigma(x)$ be an $M \times N$ matrix for all $x \in \Re^M$. Assume that b and σ are continuous such that the stochastic differential equation*

$$dX_t = b(X_t)\ dt + \sigma(X_t)\ dw_t \qquad \text{(A1)}$$

admits a unique weak solution,[10] where w is an N-dimensional Brownian motion. Let X be the weak solution to (A1) with $X_0 = x_0$. Suppose X^n is a sequence of Markov processes with sample paths in $D^M[0, 1]$, where $D^M[0, 1]$ is the space of functions from [0, 1] to $\Re^M$ that are right-continuous with left limits. Let L^n and A^n be $N \times 1$ and (symmetric) $N \times N$ matrix-valued processes, respectively, such that each of their elements has sample path in $D^1[0, 1]$ and $A_t^n - A_s^n$ is nonnegative definite for $t > s \geq 0$. Define $\tau_n^q = \inf\{t \leq T: |X_t^n| \geq q$ or $|X_{t-}^n| \geq q\}$, and suppose further that

(a) X_0^n converges to x_0 in distribution;
(b) $M^n \equiv X^n - L^n$ and $M^n M^{nT} - A^n$ are martingales;
(c) for all $q > 0$,

$$\lim_{n \to \infty} E_n \left[\sup_{t \leq \tau_n^q} |X_t^n - X_{t-}^n|^2 \right] = 0,$$

$$\lim_{n \to \infty} E_n \left[\sup_{t \leq \tau_n^q} |L_t^n - L_{t-}^n|^2 \right] = 0,$$

$$\lim_{n \to \infty} E_n \left[\sup_{t \leq \tau_n^q} |A_t^n - A_{t-}^n| \right] = 0,$$

(d)

$$\sup_{t \leq \tau_n^q} \left| L_t^n - \int_0^t b(X_s^n) \, ds \right| \to 0$$

in probability for all $q > 0$, as $n \to \infty$, where $a = \sigma\sigma^T$. Then X^n converges weakly to X.

Proof of Theorem 1. Following remark 16.4 of Rogers and Williams (1987, p. 150), the uniform Lipschitz condition for b and σ guarantees that (1) has a unique weak solution. As a result, (2) and (5) also have unique solutions. Next, letting

---

[10] Equation (A1) is said to have a weak solution X with initial distribution μ if there exists $(\Omega, \{\mathscr{F}_t\}, P)$, such that w is an $\{\mathscr{F}_t\}$-Brownian motion, X_0 has distribution μ, and X satisfies

$$X_t = X_0 + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dw_s,$$

where all of the stochastic integrals are well-defined. The weak solution to (A1) is said to be unique if, whenever $\{X_t, t \geq 0\}$ and $\{X_t', t \geq 0\}$ are two solutions such that the distributions of X_0 and X_0' are the same, the distributions of X_t and X_t' are the same. See Rogers and Williams (1987, chapter 5).

$$L_t^n \equiv \begin{pmatrix} \int_0^{[nt]/n} b(\tilde{S}_s^n) \ ds \\ \int_0^{[nt]/n} r(\tilde{S}_s^n)\tilde{B}_s^n \ ds \\ 0 \end{pmatrix},$$

$$A_t^n \equiv \begin{pmatrix} \int_0^{[nt]/n} \sigma(\tilde{S}_s^n)\sigma^{\top}(\tilde{S}_s^n) \ ds & 0 & -\int_0^{[nt]/n} \sigma(\tilde{S}_s^n)\kappa(\tilde{S}_s^n)\tilde{\xi}_s^n \ ds \\ 0 & 0 & 0 \\ -\int_0^{[nt]/n} \kappa(\tilde{S}_s^n)^{\top}\sigma(\tilde{S}_s^n)^{\top} \ \tilde{\xi}_s^n \ ds & 0 & \int_0^{[nt]/n} |\kappa(\tilde{S}_s^n)\tilde{\xi}_s^n|^2 \ ds \end{pmatrix},$$

then the matrix $A_t^n - A_s^n$ is nonnegative definite for $t > s$, and $M^n \equiv \tilde{X}^n - L^n$, $M^n M^{n\top} - A^n$ are martingales. If we can verify that conditions (c) and (d) of Lemma 1 are satisfied, we can conclude that $\tilde{X}^n$ converges weakly to X.

To verify (c), we observe that with the control of the stopping time τ_n^q and the continuity assumption on r, b, σ, and κ, $|X_t^n - X_{t-}^n|$ is of order $n^{-1/2}$, $|L_t^n - L_{t-}^n|$ is of order n^{-1}, and $|A_t^n - A_{t-}^n|$ is of order n^{-1}. Thus, (c) is satisfied. A similar argument applies to (d). This completes the proof. ∎

Proof of Theorem 2
We need a lemma to proceed with the proof.

Lemma 2. *For any integers m, l, k ≥ 0, where l ≤ k ≤ n, there exists a constant A > 0, depending only upon m, L, and K, such that*

$$\hat{E}_n[S_k^n]^{2m} \le A(1 + \hat{E}_n[S_l^n]^{2m}),$$

where L and K are constants, such that the regularity conditions in (3) and (4) are satisfied for b(x), σ(x), and x²r(x), and $\hat{E}_n$ denotes the expectation under Q_n.

Proof. We demonstrate this inequality for $N = 1$ (the proof for $N > 1$ is analogous). Our proof follows closely the proof of theorem 2.3 of Friedman (1975, p. 107), where he obtained this result for multidimensional diffusion processes.

Given the regularity conditions on b, σ, and r, we can find some $K' > 0$, depending only upon L and K, such that, for any $x \in \Re$,

$$|b(x)| \le K'(1 + |x|), \qquad |b(x)|^2 \le K'(1 + |x|^2),$$

$$|\sigma(x)| \le K'(1 + |x|),$$

$$|\sigma(x)|^2 \le K'(1 + |x|^2), \qquad |xb(x)| \le K'(1 + |x|^2),$$

$$|b(x)\sigma(x)| \le K'(1 + |x|^2), \qquad |x^2 r(x)| \le K'(1 + |x|^2).$$

Applying a Taylor expansion to the function x^{2m}, we obtain

$$
\begin{aligned}
\left[S_{k+1}^n\right]^{2m} &= \left[S_k^n\right]^{2m} + 2m\left[S_k^n\right]^{2m-1}(S_{k+1}^n - S_k^n) \\
&\quad + m(2m - 1)\left[\bar{S}_k^n\right]^{2m-2}(S_{k+1}^n - S_k^n)^2 \\
&= \left[S_k^n\right]^{2m} + 2m\left[S_k^n\right]^{2m-1}\left(\frac{b}{n} + \frac{\sigma}{\sqrt{n}}\tilde{\epsilon}^k\right) \\
&\quad + m(2m - 1)\left[\bar{S}_k^n\right]^{2m-2}\left(\frac{b^2}{n^2} + \frac{2b\sigma}{n\sqrt{n}}\tilde{\epsilon}^k + \frac{\sigma^2}{n}|\tilde{\epsilon}^k|^2\right),
\end{aligned}
$$

where

$$\bar{S}_k^n = S_k^n + \beta\left(\frac{b}{n} + \frac{\sigma}{\sqrt{n}}\tilde{\epsilon}^k\right),$$

for some $\beta \in [0, 1]$. Taking expectation $\hat{E}_n$ (under Q_n) on both sides and noticing that $|\bar{S}_k^n| \le |S_k^n| + |b| + |\sigma|$ and $|\tilde{\epsilon}^k| = 1$, we obtain

$$
\begin{aligned}
\hat{E}_n&\left[S_{k+1}^n\right]^{2m} \\
&\le \hat{E}_n[S_k^n]^{2m} + \frac{2mK'}{n}\hat{E}_n\left(\left[S_k^n\right]^{2m-2} + \left[S_k^n\right]^{2m}\right) + \frac{m(2m - 1)}{n} \\
&\quad \cdot \tilde{E}_n\left((|S_k^n| + |b| + |\sigma|)^{2m-2}(b^2 + 2|b\sigma| + \sigma^2)\right),
\end{aligned}
$$

where we have used the fact that $|x^2 r(x)| \le K'(1 + x^2)$ and $\hat{E}_n[\tilde{\epsilon}^k] = \kappa(S_k^n)/\sqrt{n}$. Noticing further that $x^{2m-2} \le 1 + x^{2m}$ and $(x + y)^m \le 2^m(x^m + y^m)$ when $x, y > 0$, we obtain

$$
\begin{aligned}
\hat{E}_n\left[S_{k+1}^n\right]^{2m} &\le \hat{E}_n[S_k^n]^{2m} + \frac{2mK'}{n}\hat{E}_n[1 + 2[S_k^n]^{2m}] \\
&\quad + \frac{4K'm(2m - 1)}{n} \\
&\quad \cdot \hat{E}_n\left([2K' + (1 + 2K')|S_k^n|]^{2m-2}(1 + [S_k^n]^2)\right) \\
&\le \hat{E}_n[S_k^n]^{2m} + \frac{2mK'}{n}\hat{E}_n[1 + 2[S_k^n]^{2m}] \\
&\quad + \frac{4K'm(2m - 1)}{n}2^{2m-2}\left((2K')^{2m-2}(1 + \hat{E}_n[S_k^n]^2)\right. \\
&\quad \left. + (1 + 2K')^{2m-2}\hat{E}_n\left[[S_k^n]^{2m-2} + [S_k^n]^{2m}\right]\right).
\end{aligned}
$$

Since $\hat{E}_n[S_k^n]^2 \leq 1 + \hat{E}_n[S_k^n]^{2m}$, for $m \geq 1$, and $\hat{E}_n[S_k^n]^{2m-2} \leq 1 + \hat{E}_n[S_k^n]^{2m}$, we can find a constant $C > 0$, depending only upon m and K', such that

$$\hat{E}_n[S_{k+1}^n]^{2m} \leq C/n + (1 + C/n)\hat{E}_n[S_k^n]^{2m}.$$

This implies that

$$\hat{E}_n[S_k^n]^{2m} \leq (1 + C/n)^{k-l}(1 + \hat{E}_n[S_l^k]^{2m}) \leq A(1 + \hat{E}_n[S_l^n]^{2m}),$$

where $A = \sup_n(1 + C/n)^n < \infty$. ∎

Proof of Theorem 2. We first prove this theorem for $N = 1$. We write the discrete-time contingent claim price as follows:

$$V^n\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) = V\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - \left[V\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - V^n\left(\tilde{S}_t^n, \frac{[nt]}{n}\right)\right]$$

$$= V\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - \tilde{e}_t^n.$$

Since $\tilde{S}^n$ converges weakly to S, and V is continuous, the stochastic process defined by the first term on the right-hand side of the above equation converges weakly to $\tilde{V}$ by the Continuous Mapping Theorem, theorem 1.5.1 of Billingsley (1968, p. 30). We show below that the stochastic process defined by the second term, the truncation errors, on the right-hand side of the above equation converges weakly to zero.

The basic idea of the proof is to substitute the value function V into the recurrent equation that defines V^n to get an estimate for the truncation errors:

$$e_k^n = V\left(S_k^n, \frac{k}{n}\right) - V^n\left(S_k^n, \frac{k}{n}\right).$$

Let us use $+$ and $-$ to denote the states $\{\tilde{\epsilon}^k = 1\}$ and $\{\tilde{\epsilon}^k = -1\}$, respectively, and define

$$S_{k+1}^{n+} = S_k^n + \frac{b(S_k^n)}{n} + \frac{\sigma(S_k^n)}{\sqrt{n}},$$

$$S_{k+1}^{n-} = S_k^n + \frac{b(S_k^n)}{n} - \frac{\sigma(S_k^n)}{\sqrt{n}},$$

where S_{k+1}^{n+} and S_{k+1}^{n-} denote the prices at time $(k+1)/n$ when $\tilde{\epsilon}^k = 1$ and $\tilde{\epsilon}^k = -1$, respectively. Similarly, we define two functions $f_+^{k,n}$ and $f_-^{k,n}$ as follows:

$$f_+^{k,n}(\tau) = V(S_k^n + \tau(S_{k+1}^{n+} - S_k^n), t_k^n + \tau(t_{k+1}^n - t_k^n)),$$
$$f_-^{k,n}(\tau) = V(S_k^n + \tau(S_{k+1}^{n-} - S_k^n), t_k^n + \tau(t_{k+1}^n - t_k^n)),$$

where $t_k^n = k/n$. Since V is differentiable up to the third order, $f_+^{k,n}$ and $f_-^{k,n}$ admit the following Taylor expansions (we omit the superscripts for f):

$$f_+(1) = f_+(0) + f_+'(0) + \tfrac{1}{2}f_+''(0) + R_k^n,$$

$$f_-(1) = f_-(0) + f_-'(0) + \tfrac{1}{2}f_-''(0) + Q_k^n,$$

where

$$R_k^n = \tfrac{1}{6} \int_0^1 (1 - s)^3 f_+^{(3)}(s) \, ds,$$

$$Q_k^n = \tfrac{1}{6} \int_0^1 (1 - s)^3 f_-^{(3)}(s) \, ds.$$

We have

$$f_+'(0) = V_s(S_{k+1}^{n+} - S_k^n) + \frac{1}{n}V_t,$$

$$f_-'(0) = V_s(S_{k+1}^{n-} - S_k^n) + \frac{1}{n}V_t,$$

$$f_+''(0) = V_{ss}(S_{k+1}^{n+} - S_k^n)^2 + V_{tt}\frac{1}{n^2} + 2V_{st}\frac{1}{n}(S_{k+1}^{n+} - S_k^n),$$

$$f_-''(0) = V_{ss}(S_{k+1}^{n-} - S_k^n)^2 + V_{tt}\frac{1}{n^2} + 2V_{st}\frac{1}{n}(S_{k+1}^{n-} - S_k^n),$$

where V and the partial derivatives of V are evaluated at (S_k^n, t_k^n). Now, to obtain an estimate for e_k^n, we substitute the expressions for $f_+^{k,n}(1)$ and $f_-^{k,n}(1)$ into the recurrent equation. Using the fact that V satisfies (12), we obtain

$$\pi(+; S_k^n)f_+(1) + \pi(-; S_k^n)f_-(1)$$

$$= V(S_k^n) + \frac{1}{2}\left(V_{ss}\left(\frac{b^2}{n^2} + \frac{2b\sigma\kappa}{n^2}\right) + 2V_{st}\frac{rS_k^n}{n^2}\right)\left(1 + \frac{r}{n}\right)^{-1}$$

$$+ \frac{1}{2}V_{tt}\frac{1}{n^2}\left(1 + \frac{r}{n}\right)^{-1} - \gamma_k^n\left(1 + \frac{r}{n}\right)^{-1},$$

where $\gamma_k^n = -(\pi(+; S_k^n)R_k^n + \pi(-; S_k^n)Q_k^n)$. Letting $b = -\tfrac{1}{2}V_{ss}(b^2 + 2b\sigma\kappa) - V_{st}rS - \tfrac{1}{2}V_{tt}$, we obtain the following recurrent equation for e_k^n:

$$e_k^n = \pi(+; S_k^n)e_{k+1}^{n+} + \pi(-; S_k^n)e_{k+1}^{n-}$$

$$+ \frac{1}{n^2}b\left(S_k^n, \frac{k}{n}\right)\left(1 + \frac{r}{n}\right)^{-1} + \gamma_k^n\left(1 + \frac{r}{n}\right)^{-1}.$$

We want to show that there exists a constant $C > 0$ and an integer $q > 0$, independent of k and n, such that

$$|e_k^n| \leq (C/\sqrt{n})(1 + |S_k^n|^{2q}). \tag{A2}$$

To prove that, we express e_k^n as follows:

$$e_n^n = 0,$$

$$e_k^n = \hat{E}_n \left[\sum_{m=k}^{n-1} \frac{1}{n^2} b\left(S_m^n, \frac{m}{n}\right) \frac{B_k^n}{B_{m+1}^n} + \gamma_m^n \frac{B_k^n}{B_{m+1}^n} \,\middle|\, S_k^n \right]. \tag{A3}$$

Since, by assumption, b satisfies a polynomial growth condition, we can find a constant $C' > 0$ and an integer $q > 0$, such that

$$|b(x,t)| \leq C'(1 + |x|^{2q}).$$

Applying Lemma 2, we obtain

$$\hat{E}_n \left[b\left(S_m^n, \frac{m}{n}\right) \,\middle|\, \middle|\, S_k^n \right] \leq C' \left(1 + A(1 + |S_k^n|^{2q})\right),$$

for all $m \geq k$. Since $B_k^n \leq B_{m+1}^n$, we can find a constant $D > 0$, such that

$$\hat{E}_n \left[\sum_{m=k}^{n-1} \frac{1}{n^2} \middle| b\left(S_m^n, \frac{m}{n}\right) \middle| \frac{B_k^n}{B_{m+1}^n} \,\middle|\, S_k^n \right] \leq \frac{D}{n}(1 + |S_k^n|^{2q}). \tag{A4}$$

For the second summation, we argue that it is of order $1/\sqrt{n}$. To see this, one can express R_k^n and Q_k^n explicitly by writing down $f_+^{(3)}(s)$ and $f_-^{(3)}(s)$, and argue that they are the sum of terms, of order $n^{-3/2}$ or higher. For example, a typical term of R_k^n has the following form:

$$\int_0^1 (1 - s)^3 V_{sss}\left((1 - s)S_k^n + sS_{k+1}^{n+}, \frac{k+1}{n}\right)(S_{k+1}^{n+} - S_k^n)^3 \, ds$$

$$= \int_0^1 (1 - s)^3 V_{sss}\left((1 - s)S_k^n + sS_{k+1}^{n+}, \frac{k+1}{n}\right)\frac{\sigma^3(S_k^n)}{n^{3/2}} ds + \cdots.$$

Since V_{sss} satisfies a polynomial growth condition, so does the integral. We can choose q to be larger enough so that all of the polynomial growth conditions have the same power $2q$. Now applying the same procedure, we can get an inequality for γ_m^n, which is of order $n^{-3/2}$. Hence, we can find a constant $D' > 0$, such that

$$\hat{E}_n \left[\sum_{m=k}^{n-1} |\gamma_m^n| \frac{B_k^n}{B_{m+1}^n} \,\middle|\, S_k^n \right] \leq \frac{D'}{\sqrt{n}}(1 + |S_k^n|^{2q}). \tag{A5}$$

Combining (A4) and (A5), we conclude that (A2) is true. Letting $\tilde{e}_t^n \equiv e_{nt}^n$, we have $\tilde{e}^n$ converges weakly to 0. We conclude that $\tilde{V}^n$ converges weakly to $\tilde{V}$.

We now substitute f_- and f_+ into the equation that defines θ_k^n. This yields

$$
\theta_k^n = \frac{F^n(S_{k+1}^{n+}, (k+1)/n) - F^n(S_{k+1}^{n-}, (k+1)/n)}{S_{k+1}^{n+} - S_{k+1}^{n-}}
$$

$$
= \frac{f_+^{k,n}(1) - f_-^{k,n}(1)}{S_{k+1}^{n+} - S_{k+1}^{n-}} - \frac{e_{k+1}^{n+} - e_{k+1}^{n-}}{S_{k+1}^{n+} - S_{k+1}^{n-}}
$$

$$
= V_s + \frac{2b}{n} V_{ss} + \frac{2V_{st}}{n} + \frac{2\sqrt{n}}{\sigma}(R_k^n - Q_k^n)
$$

$$
+ \frac{2\sqrt{n}}{\sigma}(e_{k+1}^{n+} - e_{k+1}^{n-}). \tag{A6}
$$

Since R_k^n and Q_k^n are of order $n^{-3/2}$ or higher, we conclude that

$$
\frac{2b}{n} V_{ss} + \frac{2V_{st}}{n} + \frac{2\sqrt{n}}{\sigma}(R_k^n - Q_k^n)
$$

converges weakly to zero. We further argue that $(2\sqrt{n}/\sigma)(e_{k+1}^{n+} - e_{k+1}^{n-})$ converges weakly to zero as well. For simplicity, we assume $r = 0$. Letting $t = k/n$, we have from (A3) that

$$
\sqrt{n}\,\tilde{e}_t^n = \hat{E}_n\left[\int_t^1 \left(\int_0^1 V_{sss}((1-u)\tilde{S}_s^n + u\tilde{S}_s^{n+}, s)\, du\right)\right.
$$

$$
\left. \cdot \sigma^3(\tilde{S}_s^n)\, ds \,\middle|\, \tilde{S}_s^n\right] + O\left(\frac{1}{\sqrt{n}}\right)
$$

$$
\Rightarrow E\left[\int_t^1 V_{sss}(S_s, s)\sigma^3(S_s)\, ds \,\middle|\, S_t\right],
$$

where the convergence follows from Aldous (1981, theorem 21.2 and lemma 16.2). Hence, $\{\sqrt{n}(e_{k+1}^{n+} - e_{k+1}^{n-})\}$ converges weakly to zero, and $\tilde{\theta}^n$ converges weakly to $\{V_s(S_t, t)\}$. Finally, since $\tilde{V}^n$ converges weakly to $\tilde{V}$ and $\tilde{V}_t^n = \tilde{\alpha}_t^n \tilde{B}_t^n + \tilde{\theta}_t^{nT} \tilde{S}_t^n$, we conclude that $\tilde{\alpha}^n$ converges weakly to $\tilde{\alpha}$. This proves part 2. The claims that $\tilde{V}_0^n \to \tilde{V}_0$, $\tilde{\alpha}_0^n \to \tilde{\alpha}_0$, and $\tilde{\theta}_0^n \to \tilde{\theta}_0$ follow directly from the definition of the Skorokhod topology (Billingsley, 1968, p. 121) and from the fact that they are deterministic.

We now consider the case when $N > 1$. For simplicity, we assume $N = 2$. The proof can be carried out in exactly the same way as we did for $N = 1$. Using a similar procedure, we can get a recurrent equation for the truncation error e_k^n:

$$e_k^n = \pi(\omega_1; S_k^n) e_{k+1}^n(\omega_1) + \pi(\omega_2; S_k^n) e_{k+1}^n(\omega_2)$$
$$+ \pi(\omega_3; S_k^n) e_{k+1}^n(\omega_3) + O\left(\frac{1}{n^{3/2}}\right).$$

This allows us to argue that $\tilde{e}^n$ converges weakly to zero in the order of $1/\sqrt{n}$. Next, we have

$$\alpha_k^n B_{k+1}^n(\omega_s) + \theta_{k,1}^n S_{k+1,1}^n(\omega_s) + \theta_{k,2}^n S_{k+1,2}^n(\omega_s) = F^n\left(S_{k+1}^n(\omega_s), \frac{k+1}{n}\right),$$

for $s = 1, 2, 3$. This implies that

$$\begin{pmatrix} S_{k+1,1}^n(\omega_1) - S_{k+1,1}^n(\omega_3) & S_{k+1,2}^n(\omega_1) - S_{k+1,2}^n(\omega_3) \\ S_{k+1,1}^n(\omega_2) - S_{k+1,1}^n(\omega_3) & S_{k+1,2}^n(\omega_2) - S_{k+1,2}^n(\omega_3) \end{pmatrix} \begin{pmatrix} \theta_{k,1}^n \\ \theta_{k,2}^n \end{pmatrix}$$

$$= \begin{pmatrix} f_{\omega_1}^{n,k}(1) - f_{\omega_3}^{n,k}(1) \\ f_{\omega_2}^{n,k}(1) - f_{\omega_3}^{n,k}(1) \end{pmatrix} - \begin{pmatrix} e_{k+1}^{n\omega_1} - e_{k+1}^{n\omega_3} \\ e_{k+1}^{n\omega_2} - e_{k+1}^{n\omega_3} \end{pmatrix}.$$

Applying Taylor expansion to $f_{\omega_s}^{n,k}$, and then solving for θ_k^n, we obtain

$$\begin{pmatrix} \theta_{k,1}^n \\ \theta_{k,2}^n \end{pmatrix} = V_s\left(S_k^n, \frac{k}{n}\right) + o(1).$$

Hence, $\tilde{\theta}^n$ converges weakly to $\{V_s(S,t)\}$. The proof is now completed.

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