Review of Quantum

1

1.1

$g \in G$, given a action of group on a general space M equipped with function on it as $F(M)$. Thus we can apply g on M and lift upon $F(M)$ as well, i.e. $\varphi(g) \cdot x$ for $x \in M$ and $\rho(g) \cdot f$ for $f \in F(M)$ as a G-equivariant map.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\varphi(g)"'] & f(x) \arrow[d, "\rho(g)"'] \\ \varphi(g) \cdot x \arrow[r, "f"] & \rho(g) \cdot f(x)/f(\varphi(g) \cdot x) \end{tikzcd} \tag{1.1}$$

Is commutative, i.e.

$$\rho(g) \cdot (\rho(h) \cdot f)(x) = \rho(h) \cdot f(\varphi(g) \cdot x) = f(\varphi(h) \cdot (\varphi(g) \cdot x)) = f(\varphi(h) \circ \varphi(g) \cdot x) \tag{1.2}$$

Is a anti-homomorphism, i.e. Such representation should be only with g element, thus a canonical choice is g^{-1} where below is given as a reduced notation:

$$g \cdot f(x) = f(g^{-1} \cdot x) \tag{1.3}$$

$V = W \oplus W' \to v = w + w'$, where the projection $q(v) = w$. Define $\overline{q} : v \to \frac{1}{|G|} \sum_{g \in G} \rho(g) q(\rho(g^{-1})v)$ which we drop ρ representation notation.

$$\overline{q}(w) = \frac{1}{|G|} \sum_{g \in G} g q(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = w \tag{1.4}$$

$$h\overline{q}(v) = \frac{1}{|G|} \sum_{g \in G} hgq(g^{-1}h^{-1}hv) = \frac{1}{|G|} \sum_{g' \in G} g' q(g'^{-1}hv) = \overline{q}(hv) \tag{1.5}$$

If $v \in \mathrm{Ker}(\overline{q}), h \in G$ then $h\overline{q}(v) = 0 = \overline{q}(hv)$ that $hv \in \mathrm{Ker}(\overline{q})$ that we decompose by a G-invariant morphism based on solely vector space projection q.

How can we define a G-invariant operation on V? We should **always** remember that g act transitively on G itself that if $gG = G$. Then if we can put all elements of G on the operation, then it should be G-invariant. By the way, we can define the module of G-invariance which is $RG = \sum_{g_i \in G} r_i g_i \quad r_i \in R, g_i \in G$. The way is to average the operation by group action for all elements. Thus one has:

$$
\begin{aligned}
hf(v_1, ..., v_n) &= \frac{1}{|G|} \sum_{g \in G} hgf(g^{-1}h^{-1}hv_1, ..., g^{-1}h^{-1}hv_n) \\
&= \frac{1}{|G|} \sum_{g' \in G} g'f(g'^{-1}hv_1, ..., g'^{-1}hv_n) = f(hv_1, ..., hv_n)
\end{aligned} \tag{1.6}
$$

$$
\begin{aligned}
f(hv_1, ..., hv_n) &= \frac{1}{|G|} \sum_{g \in G} gf(g^{-1}hv_1, ..., g^{-1}hv_n) \\
&= \frac{1}{|G|} \sum_{h^{-1}g \in G} h^{-1}gf\left(\left(h^{-1}g\right)^{-1}v_1, ..., \left(h^{-1}g\right)^{-1}v_n\right) = f(v_1, ..., v_n)
\end{aligned} \tag{1.7}
$$

$$
\begin{aligned}
\langle v, u \rangle &= \langle u, v \rangle^* \\
\langle u, \alpha v + \beta w \rangle &= \alpha \langle u, v \rangle + \beta \langle u, w \rangle \\
\|u\|^2 &= \langle u, u \rangle
\end{aligned} \tag{1.8}
$$

$$
\langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle \tag{1.9}
$$

If there' s a invariant space that $\rho(g)(V_1) \subseteq V_1$, with the decomposition $V_2 = V_1^{\perp}$ that $V_2 = \{v \in V \mid (v, x) = 0 \ \forall x \in V_1\}$.

$$
\langle \rho(g)x, y \rangle = \langle \rho(g)^{-1}\rho(g)x, \rho(g)^{-1}y \rangle = \langle x, \rho(g)^{-1}y \rangle \quad \forall y \in V_1 \tag{1.10}
$$

Thus if $x \in V_2$, the inner product should be **zero**. Reversely, we deduce that $\rho(g)x \in V_2$ because $y \in V_1$ by first term. Therefore the representation can be decomposed if we seek a **sole** invariant space and repeat the procedure.

Thus given a inner product, which is also a bilinear-function, we can construct G-invariant form like above.

$$
\langle x, y \rangle := \sum_{h \in G} \langle \rho(h)x, \rho(h)y \rangle \tag{1.11}
$$

Which is *unitary* as above already proved. Same as integral form one has:

$$
\langle x, y \rangle := \int_G d\mu(h) \langle \rho(h)x, \rho(h)y \rangle \tag{1.12}
$$

Yields the same results. However, we can conclude the canonical G-invariance form for *finite* and *compact* group due to the convergence given by the integral only for compactness.

1.2

$$
(v_1, v_2) \to v_1 \otimes v_2 \equiv v_1^i e_1^i \otimes v_2^j e_2^j = v_1^i \, v_2^i \, e_1^i \otimes e_2^j \tag{1.13}
$$

$$
(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2) \tag{1.14}
$$

1.3

$$\mathbf{F}(q,p) = \begin{pmatrix} F_q(q,p) \\ F_p(q,p) \end{pmatrix} = \begin{pmatrix} \partial_q F(q,p) \\ \partial_p F(q,p) \end{pmatrix} \tag{1.15}$$

$$dH = H_q dq + H_p dp = -\frac{\mathrm{d}p}{\mathrm{d}t} dq + \frac{\mathrm{d}q}{\mathrm{d}t} dp \tag{1.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t}\partial_q + \frac{\mathrm{d}p}{\mathrm{d}t}\partial_p = H_p\partial_q - H_q\partial_p := \{\cdot, H\} := X_H \tag{1.17}$$

That' s, Hamiltonian defines the evolution of the system.

$$\frac{\mathrm{d}F(q,p)}{\mathrm{d}t} = X_H F(q,p) = \{F(q,p), H\} \tag{1.18}$$

$$\begin{aligned} X_H(p\,dq) &= \dot{p}dq + pd\dot{q} = \dot{p}dq - \dot{q}dp + d(p\dot{q}) \\ &= -H_q dq + H_p dp + d(p\dot{q}) = d(-H + p\dot{q}) := dL(q,\dot{q}) \end{aligned} \tag{1.19}$$

$$\{F, H\} = X_H F = -X_F H \tag{1.20}$$

Where the $X_F = \frac{\partial F}{\partial p}\partial_q - \frac{\partial F}{\partial q}\partial_p$.

Generally, one scalar function defined upon sympletic manifold can induce a vector field on it, with a general area preservation:

$$\begin{gathered} \omega = dq \wedge dp \\ \iota_{X_F}(\omega) = \iota_{\frac{\partial F}{\partial q}\partial_p - \frac{\partial F}{\partial p}\partial_q}(dq \wedge dp) = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial p} dp = dF \end{gathered} \tag{1.21}$$

$$\iota_{X_H}\iota_{X_F}(\omega) = \iota_{X_H}(dF) = X_H F = \{F, H\} = \frac{\mathrm{d}F}{\mathrm{d}t} \tag{1.22}$$

We thus induce a symmetric bilinear form:

$$\omega(X_F, X_G) = \iota_{X_F}\iota_{X_G}(\omega) = \iota_{X_F}(dG) = X_F G = \{G, F\} = -\{F, G\} \tag{1.23}$$

One can lift the Hamiltonian in exponential flow by dynamics of X_H.

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\exp(tX_H)) \mid_{t=0} = \{F(p,q), H(p,q)\} \tag{1.24}$$

One thus can define lie group dynamics too just like above:

$$
\begin{aligned}
L \in \mathfrak{g} &\to X_L \\
X_L F(p, q) &= \frac{\mathrm{d}}{\mathrm{d}t} F\left(e^{tL} \cdot (p, q)\right)
\end{aligned}
\tag{1.25}
$$

However, due to the representation of any group should be a anti-homomorphism, we have:

$$
L \cdot F(p, q) = \frac{\mathrm{d}}{\mathrm{d}t} F\left(e^{-tL} \cdot (p, q)\right) = -X_L F(p, q) \tag{1.26}
$$

Now we can try translation group $T = \mathbb{R}^3$ first:

$$
a \cdot (q, p) = (q + a, p) \quad a \in T \leftrightarrow e^{-t\lambda} \in T \ \lambda \in \mathfrak{t} \tag{1.27}
$$

Here we **abuse** of notation because the lie algebra of translation induced by lie group is trivial.

$$
a \cdot F(p, q) = \frac{\mathrm{d}}{\mathrm{d}t} F\left(e^{-t\lambda} \cdot (p, q)\right) \mid_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} F(q - ta, p) \mid_{t=0} = -a \frac{\partial}{\partial q} F = -X_a F \tag{1.28}
$$

$$
X_a = a \frac{\partial}{\partial q} \tag{1.29}
$$

We know that $\iota_{X_F}(\omega)$ is a closed form that it corresponds to a certain function F up to a constant, we can induce the scalar function of action by the lie group.

$$
X_a(\omega) = ap \to \mu_a(p) = \mu_a(p, q) \tag{1.30}
$$

We thus recover the momentum operator which configures the momentum for each translation element a. Such closed form check lift a function defined in the manifold which shared the same role as Hamiltonian, if H is G-invariant, then μ_a is a conserved quantity, as the Hamiltonian version of Noether′ s theorem.

$$
\iota_{X_H}(d\mu_L) = \{\mu_L, H\} = \frac{\mathrm{d}\mu_L}{\mathrm{d}t} = -\{H, \mu_L\} = -\iota_{X_L}(dH) = -X_L H = 0 \tag{1.31}
$$

1.4

$$
[\phi(X), \phi(Y)] = \phi([X, Y]) = \phi(X)\phi(Y) - (-1)^{\sigma(X)\sigma(Y)} \phi(Y)\phi(X) \tag{1.32}
$$

$$
\Omega(Jv_1, Jv_2) = \Omega(v_1, v_2) \to J \in \mathrm{Sp}(2n, \mathbb{R}) \tag{1.33}
$$

$$
\begin{gathered}
A^T A = I \to A \in \mathrm{O}(n, \mathbb{R}) \\
A^T \Omega A = \Omega \to A \in \mathrm{Sp}(2n, \mathbb{R}) \\
A^T \Omega A = A^T A \Omega \to A^T (\Omega A - A\Omega) = 0 \to \Omega A = A \Omega \\
\Omega(Av, Aw) = g(A\Omega v, Aw) = g(\Omega A v, Aw) = \Omega(v, w) \\
A \in \mathrm{O}(n, \mathbb{R}) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{U}(n)
\end{gathered} \tag{1.34}
$$

$$
z_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad \overline{z_j} = \frac{1}{\sqrt{2}}(q_j - ip_j) \tag{1.35}
$$

$$
\begin{gathered}
L_1 = Q_2 P_3 - Q_3 P_2, \quad L_2 = Q_3 P_1 - Q_1 P_3, \quad L_3 = Q_1 P_2 - Q_2 P_1 \\
\Gamma(L_1) = a_3^\dagger a_2 - a_2^\dagger a_3, \quad \Gamma(L_2) = a_1^\dagger a_3 - a_3^\dagger a_1, \quad \Gamma(L_3) = a_2^\dagger a_1 - a_1^\dagger a_2
\end{gathered} \tag{1.36}
$$

$$
\begin{aligned}
& F(\theta) = c_0 + c_1\theta + c_2\theta^2 + ... + c_n\theta^n = c_0 + c_1\theta \\
& \frac{\partial F}{\partial \theta} = c_1 \\
& F(\theta_1, ..., \theta_n) = F_A + \theta_j F_B \quad \exists F_A, F_B \in \bigwedge(\mathbb{R}^n) \\
& \frac{\partial FG}{\partial \theta_j} = \frac{\partial F}{\partial \theta_j} G + (-1)^{\sigma(F)} F \frac{\partial G}{\partial \theta_j} \quad \sigma(F) \text{ is the degree of } F
\end{aligned} \tag{1.37}
$$

Given a translation invariance property of integral:

$$
\int f(\theta + \eta) d\theta = \int f(\theta) d\theta \tag{1.38}
$$

$$
\int (\theta + \eta) d\theta = \int \theta d\theta + \eta \int d\theta = \int \theta d\theta \to \int d\theta = 0 \tag{1.39}
$$

Given with normalization convention:

$$
\int \theta d\theta = 1 \to \int c_0 + c_1 \theta d\theta = c_1 \to \int f(\theta) d\theta = \frac{\mathrm{d}f}{\mathrm{d}\theta} \tag{1.40}
$$

Thus for multi-variables calculus, one should be careful on applying order of partial derivative:

$$
\int f(\theta_1, ..., \theta_n) d\theta_n ... d\theta_1 = \frac{\partial}{\partial \theta_1} ... \frac{\partial}{\partial \theta_n} f \tag{1.41}
$$

Where swap of order of partial derivative will induce a minus sign.

$$
\begin{gathered}
[\theta_j, \theta_k]_+ = \pm \delta_{jk} \\
\frac{\mathrm{d}}{\mathrm{d}t} \theta_j(t) = [\theta_j(t), H]_+
\end{gathered} \tag{1.42}
$$

Given H:

$$
\begin{aligned}
B &= \begin{pmatrix} 0 & B_{12} & B_{13} & \ldots & B_{1n} \\ -B_{12} & 0 & B_{23} & \ldots & B_{2n} \\ -B_{13} & -B_{23} & 0 & \ldots & B_{3n} \\ \ldots & & & & \\ -B_{1n} & -B_{2n} & -B_{3n} & \ldots & 0 \end{pmatrix} \\
H &= \frac{1}{2} \sum_{j,k=1}^{n} B_{jk} \theta_j \theta_k
\end{aligned}
\tag{1.43}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} \theta_j(t) = \left[\theta_j(t), H\right]_+ = -\left[H, \theta_j(t)\right]_+ = \sum_{k=1}^{n} B_{jk} \theta_k(t)
\tag{1.44}
$$

The minus sign of commutator is due to $B_{jk} = -B_{kj}$. Which is highly similar to the bosonic case in evolution.

$$
\Omega^+(Jv_1, Jv_2) = \Omega^+(v_1, v_2) \to J \in \mathrm{SO}(2n, \mathbb{R})
\tag{1.45}
$$

$$
\begin{aligned}
U_A &= \sum_{jk} a_j^\dagger A_{jk} a_k \\
\left[U_A, a_j\right]_\pm &= \pm \sum_k A_{jk} a_k = \pm A \boldsymbol{a} \\
\left[U_A, a_j^\dagger\right]_\pm &= \sum_k a_k^\dagger A_{kj} = A^T \boldsymbol{a}^\dagger
\end{aligned}
\tag{1.46}
$$

$$
v \otimes w = \frac{1}{2}(v \otimes w - u \otimes v) + \frac{1}{2}(v \otimes w + w \otimes v) = v \wedge w + g(v, w)
\tag{1.47}
$$

Thus given a basis omit tensor notation, for example:

$$
e_i e_j = e_i \wedge e_j \quad (i \neq j)
\tag{1.48}
$$

Or physics notation:

$$
\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \to \gamma(\boldsymbol{v}) = \not{\boldsymbol{v}} = v_1 \gamma_1 + \ldots + v_n \gamma_n
\tag{1.49}
$$

$$
\begin{aligned}
v' &= v - 2 \frac{g(v, w)}{g(w, w)} w = \not{v} - \frac{\not{v}\not{w} - \not{w}\not{v}}{g(w, w)} \not{w} \\
&= -\frac{\not{w}\not{v}\not{w}}{g(w, w)}
\end{aligned}
\tag{1.50}
$$

$$
\begin{aligned}
2AB &= [A, B]_+ + [A, B]_- \\
2BA &= [A, B]_+ - [A, B]_-
\end{aligned}
\tag{1.51}
$$

Thus we can always decompose a product term into symmetric and antisymmetric parts.

$$ABC = \frac{1}{2}A[B,C]_+ + \frac{1}{2}A[B,C]_- = \frac{1}{2}[A,B]_+C - \frac{1}{2}[A,C]_+B + \frac{1}{2}[A,B]_-C - \frac{1}{2}[A,C]_-B \tag{1.52}$$

$$\begin{aligned}[AB,C]_- = ABC - CAB &= A\left(\frac{1}{2}[B,C]_+ + \frac{1}{2}[B,C]_-\right) - \left(\frac{1}{2}[A,C]_+ - \frac{1}{2}[A,C]_-\right)B \\ &= \frac{1}{2}(A[B,C]_+ - [A,C]_+B) + \frac{1}{2}(A[B,C]_- + [A,C]_-B)\end{aligned} \tag{1.53}$$

$$\begin{aligned}[AB,C]_+ = ABC + CAB &= A\left(\frac{1}{2}[B,C]_+ + \frac{1}{2}[B,C]_-\right) + \left(\frac{1}{2}[A,C]_+ - \frac{1}{2}[A,C]_-\right)B \\ &= \frac{1}{2}(A[B,C]_+ - [A,C]_-B) + \frac{1}{2}(A[B,C]_- + [A,C]_+B)\end{aligned} \tag{1.54}$$

Equal below:

$$[AB,C]_- = A[B,C]_- + [A,C]_-B = A[B,C]_+ - [A,C]_+B \tag{1.55}$$

$$[AB,C]_+ = A[B,C]_+ + [A,C]_-B = A[B,C]_- - [A,C]_+B \tag{1.56}$$

$$\begin{aligned}[AB,CD]_- &= A[B,CD]_- + [A,CD]_-B \\ &= A([B,C]_+D - C[B,D]_+) + ([A,C]_+D - C[A,D]_+)B \\ &= A[B,C]_+D - AC[B,D]_+ + [A,C]_+DB - C[A,D]_+B\end{aligned} \tag{1.57}$$

For Clifford algebra, symmetric part is given by metric, anti-symmetric part is given by wedge product.

$$\begin{aligned}M_{\mu\nu} &= \gamma_\mu \wedge \gamma_\nu \\ M_{\mu\nu} &= \gamma_\mu\gamma_\nu - g_{\mu\nu}\end{aligned} \tag{1.58}$$

$$M_{\mu\nu} \wedge \gamma_\rho = \gamma_\mu\gamma_\nu \wedge \gamma_\rho = \gamma_\mu g_{\nu\rho} - \gamma_\nu g_{\mu\rho} \tag{1.59}$$

$$M_{\mu\nu} \wedge M_{\rho\sigma} = \gamma_\mu\gamma_\nu \wedge \gamma_\rho\gamma_\sigma = \gamma_\mu\gamma_\sigma g_{v\rho} - \gamma_\mu\gamma_\rho g_{\nu\sigma} + \gamma_\nu\gamma_\rho g_{\mu\sigma} - \gamma_\nu\gamma_\sigma g_{\mu\rho} \tag{1.60}$$

Is same as above, however, I didn′ t found this is useful : (.

1.5

$$\frac{L}{[L,L]} \text{ is abelian that } \forall a,b \in L, [a,b] = 0 \tag{1.61}$$

Thus given a central series:

$$[L^n, L^n] = L^{n+1} \to \frac{L^n}{[L^n,L^n]} = \frac{L^n}{L^{n+1}} \tag{1.62}$$

Factor through, for all possible elements, otherwise $[L, L] = L$ is irreducible. Thus induce a maximal nilpotent ideal of L which is the maximal solvable ideal of L, else it must be a **semi-simple** structure without additional smaller space. Where nilpotent is a structure that can be shared with a basis with commutative eigenvectors, that' s $XYv = YXv + [X, Y]v = \lambda(X)Yv + \lambda([X, Y])v$ where $[X, Y]$ in some smaller space by deduction of nilpotent ideal.

However, it doesn' t mean that **semi-simple** structure contains no ideal. We try to diagonalize as much as possible even we can' t achieve for all like solvable case. That' s $\mathfrak{t} \subset \mathfrak{g}$ which all elements are commutative. We call such maximal commutative subalgebra as **Cartan subalgebra**, which can be diagonalized simultaneously. Thus if one choose a vector space it acts on, we can decompose the space into eigen-space for all:

$$V = \bigoplus_{\lambda(\mathfrak{t})} V_\lambda \quad V_\lambda = \{v \in V \mid \forall H \in \mathfrak{t}, Hv = \lambda(H)v\} \tag{1.63}$$

Given a adjoin representation or homomorphism as $X_i \to D_{X_i}$, we can then apply the Cartan subalgebra to decompose the lie algebra itself:

$$\mathfrak{g} = \bigoplus_{\alpha(\mathfrak{t})} \mathfrak{g}_\alpha \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{t}, [H, X] = D_H(X) = \alpha(H)X\} \tag{1.64}$$

Which is called **root space decomposition**, where $\alpha \in \mathfrak{t}^*$ is the eigenvalue on $\mathfrak{t}$.

We often see the ladder operator in physics that a state shared by all other raising and lowering operators, which is the eigenvector of Cartan subalgebra.

$$D_t([X, Y]) = [t, [X, Y]] = [[t, X], Y] + [X, [t, Y]] = (\alpha + \beta)(t)[X, Y] \to [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \tag{1.65}$$

Plus, we know if the ladder operator act too many times, it will reach zero, that' s D_X is nilpotent for $X \in \mathfrak{g}_\alpha, \alpha \neq 0$, or we simply call it nilpotent that $\mathfrak{g}_{\beta+n\alpha} = 0$. Another thing is, for clarity, $\mathfrak{g}_0$ means $[\mathfrak{t}, \mathfrak{g}_0] = 0 \cdot \mathfrak{g}_0 = 0$ thus $\mathfrak{t} \subset \mathfrak{g}_0$, however, due to maximality, we have $\mathfrak{g}_0 = \mathfrak{t}$.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{t} \tag{1.66}$$

Given each $\boldsymbol{H} = (H_1, ..., H_r)$, one has the evaluation on g_α which gives $[H_i, X_\alpha] = \alpha_i X_\alpha$. Thus we normalize:

$$E_\pm \equiv \frac{X_{\pm\alpha}}{|\alpha|} \tag{1.67}$$

$$[\rho(H_i), \rho(E_\alpha)] = \alpha_i \rho(E_\alpha) \tag{1.68}$$

$$\begin{gathered} \boldsymbol{\mu} = (m_1, ..., m_r) \\ \rho(H_i)\rho(E_\alpha)|\rho; \boldsymbol{\mu}\rangle = ([\rho(H_i), \rho(E_\alpha)] + \rho(E_\alpha)\rho(H_i))|\rho; \boldsymbol{\mu}\rangle = (\alpha_i + m_i)E_\alpha|\rho; \boldsymbol{\mu}\rangle \end{gathered} \tag{1.69}$$

Given a basis $\{X_1, ..., X_n\}$ with structure constants:

$$[X_i, X_j] = \sum_k c_{ij}{}^k X_k \tag{1.70}$$

Given the commutation relation, one has:

$$\begin{aligned} c_{ij}{}^k &= -c_{ji}{}^k \\ c_{ij}{}^k + c_{jk}{}^i + c_{ki}{}^j &= 0 \end{aligned} \tag{1.71}$$

Moreover, we can define the representation based on the structure constants:

$$\begin{aligned} (T_i)_{jk} &= \left(D_{X_i}\right)_{jk} = c_{ij}{}^k \\ D_{X_i} &: \mathfrak{g} \to \mathrm{ad}(\mathfrak{g}) \end{aligned} \tag{1.72}$$

$$D_{X_i} X_j = [X_i, X_j] = \sum_k c_{ij}{}^k X_k \tag{1.73}$$

$$\begin{aligned} \left[D_{X_i}, D_{X_j}\right] &= D_{[X_i, X_j]} \\ \left[D_{X_i}, D_{X_j}\right] X_k &= D_{[X_i, X_j]} X_k \\ [X_i [X_j X_k]] - [X_j [X_i X_k]] &= [[X_i, X_j] X_k] \end{aligned} \tag{1.74}$$

Which adjoin representation with **homomorphism consistency** should restrict all lie algebras to contain Jacobi identity.

$$\begin{aligned} D_{X_i} D_{X_j} X_l &= D_{X_i} c^k{}_{jl} X_k = c_{ik}{}^m c_{jl}{}^k X_m \\ \left(D_{X_i} D_{X_j}\right)_{ml} &= c_{ik}{}^m c_{jl}{}^k \\ \mathrm{Tr}\left(D_{X_i}\right) D_{X_j}) &= \left(D_{X_i} D_{X_j}\right)_{mm} = c_{ik}{}^m c_{jm}{}^k = g_{ij} \end{aligned} \tag{1.75}$$

Where:

$$g_{ij} g^{jl} = c_{ik}{}^m c_{jm}{}^k c^{jn}{}_f c^{lf}{}_n = c_{ik}{}^m c^{lf}{}_n \delta_m{}^n \delta^k{}_f = \delta_i{}^l \tag{1.76}$$

$$\left[D_{X_i}, D_{X_j}\right]_{kl} = \sum_m \left(c_{ij}{}^m c_{mk}{}^l - c_{ji}{}^m c_{mk}{}^l\right) = \sum_m c_{ij}{}^m c_{mk}{}^l \tag{1.77}$$

1.5.1

$\mathfrak{su}(2) = \left\{X \in \mathfrak{gl}\ (2, \mathbb{C}) \mid \overline{X}^t + X = 0, \mathrm{Tr}(X) = 0\right\}$ and $\mathfrak{so}(3)$ are all real lie algebras of dimension 3.

(a) The basis:

$$\xi_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \xi_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{1.78}$$

With $[\xi_k, \xi_l] = \varepsilon_{klm}\xi_m$.

$$\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \tag{1.79}$$

$$[\xi^2, \xi_k] = 0 \tag{1.80}$$

We construct a commute element which the maximal diagonalized state is $|b, m\rangle$.

$$\begin{aligned} \xi^2|b, m\rangle &= b|b, m\rangle \\ \xi_3|b, m\rangle &= m|b, m\rangle \end{aligned} \tag{1.81}$$

$$\xi^+ = \xi_1 + i\xi_2, \quad \xi^- = \xi_1 - i\xi_2 \tag{1.82}$$

$$[\xi_3, \xi^+] = \xi^+, \quad [\xi_3, \xi^-] = -\xi^- \tag{1.83}$$

$$[\xi^+, \xi^-] = 2\xi_3 \tag{1.84}$$

$$\xi_3\xi^\pm|b, m\rangle = [\xi_3, \xi^\pm]|b, m\rangle + \xi^\pm\xi_3|b, m\rangle = (m \pm 1)\xi^\pm|b, m\rangle \tag{1.85}$$

Thus we conclude that $\xi^\pm|b, m\rangle = C|b, m \pm 1\rangle, \quad C \in \mathbb{C}$, and $\xi_3|b, m \pm 1\rangle = (m \pm 1)|b, m \pm 1\rangle$.

We call such operators ladder operators or raising and lowering operators.

$$\xi^+|b, j\rangle = 0 \quad \exists j \in\ ? \tag{1.86}$$

$$\xi^-\xi^+ = (\xi_1 - i\xi_2)(\xi_1 + i\xi_2) = \xi_1^2 + \xi_2^2 + i[\xi_1, \xi_2] = \xi^2 - \xi_3^2 + \xi_3 \tag{1.87}$$

$$0 = \xi^-\xi^+|b, j\rangle = (b - j^2 + j)|b, j\rangle \to b = j(j+1) \tag{1.88}$$

Same for $\xi^+\xi^-$, one has $b - j'^2 + j' = 0$

$$\begin{aligned} b &= j'(j' - 1) = -j'(-j' + 1) \\ &\to j' = -j \end{aligned} \tag{1.89}$$

We conclude that for m can be $-j, -j+1, ..., j-1, j$, thus $2j \in \mathbb{N}$ is a integer for finite dimension.

$$\begin{aligned} &j = 0, \quad m = 0, \quad j(j+1) = 0 \\ &j = \frac{1}{2}, \quad m = \frac{1}{2}, -\frac{1}{2}, \quad j(j+1) = \frac{3}{4} \\ &j = 1, \quad m = 1, 0, -1, \quad j(j+1) = 2 \\ &\dots \end{aligned} \tag{1.90}$$

From the definition of $\xi^+ = \xi^{-\dagger}$ by its expansion, we have such normalization factor:

$$\begin{aligned}\left\langle b,m\middle|{\xi^+}^\dagger\xi^+\middle|b,m\right\rangle &= \langle b,m|\xi^-\xi^+|b,m\rangle \\ &= \langle b,m|\xi^2-\xi_3^2-\xi_3|b,m\rangle \\ &= b-m^2-m = j(j+1)-m^2-m = C^2\end{aligned} \tag{1.91}$$

$$C = \sqrt{j(j+1)-m(m+1)} \tag{1.92}$$

Same, one has for ξ^- that $\tilde{C} = \sqrt{j(j+1)-m(m-1)}$.

1.5.2

Take the transformation $SO(3)$ as $GL(3;\mathbb{R})$ matrix representation act naturally on $\mathbb{R}^3$ equipped with function valued on it as a scalar form.

$$\begin{aligned}l_1(f)(\boldsymbol{x}) = f(l_1^{-1}x) &= \frac{d}{dt}f\left(\exp\left(t\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\right)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}f\left(\begin{pmatrix}1&0&0\\0&\cos t&\sin t\\0&-\sin t&\cos t\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}f\left(\begin{pmatrix}x_1\\x_2\cos t+x_3\sin t\\-x_2\sin t+x_3\cos t\end{pmatrix}\right) \\ &= \left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3}\right)\cdot\begin{pmatrix}0\\x_3\\-x_2\end{pmatrix} \\ &= x_3\frac{\partial f}{\partial x_2}-x_2\frac{\partial f}{\partial x_3}\end{aligned} \tag{1.93}$$

If we have $x_1 = r\sin\theta\cos\phi, x_2 = r\sin\theta\sin\phi, x_3 = r\cos\theta$, we can construct the transformation upon θ,ϕ too.

$$\frac{\partial}{\partial x_i} = \frac{\partial r_i}{\partial x_i}\frac{\partial}{\partial r_i} \tag{1.94}$$

In differential geometry as a basis vector transformation, or familiar Jacobi.

$$\begin{aligned}l_1 &= x_3\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_3} = \sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi} \\ l_2 &= -\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi} \\ l_3 &= -\frac{\partial}{\partial\phi}\end{aligned} \tag{1.95}$$

$$
\begin{aligned}
l^{+} &= e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) \\
l^{-} &= e^{i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right)
\end{aligned}
\tag{1.96}
$$

Given the representation $F(\theta, \phi)(m)$ with the weight m chosen by l_3, with the highest weight denoted as l:

$$
l_3 F(\theta, \phi)(m) = -\frac{\partial}{\partial\phi}F(\theta, \phi)(m) = mF(\theta, \phi)(m) \tag{1.97}
$$

Thus we have $F(\theta, \phi) \sim e^{m\phi}G(\theta)$, however, we have such restriction $\phi + 2\pi \sim \phi$, thus we must have $m \to im$ to match such period.

We can immediately decompose the representation scalar function by raising operator for the highest weight:

$$
\begin{aligned}
\left(\frac{\partial}{\partial\theta} - l\cot\theta\right)G(\theta)(l) &= 0 \\
\ln G(\theta)(l) &= l\ln\sin\theta \\
G(\theta)(l) &= C_l \sin^l\theta
\end{aligned}
\tag{1.98}
$$

Where the constant is, for the representation of $2l + 1$ dimension in l weight. Apply lowering operator would giving us:

$$
\begin{aligned}
F(\theta, \phi)(m) &= C_m (l^{-})^{l-m} F(\theta, \phi)(l) \\
&= C_m \left(e^{-i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right)\right)^{l-m} e^{-il\phi}\sin^l\theta
\end{aligned}
\tag{1.99}
$$

However, one usually encompass the whole $2l + 1$ representation for arbitrary l as:

$$
Y_l^m(\theta, \phi) = C_{lm}(\ldots) \text{ as representation for } 2l + 1 \text{ dimension with weight } m \tag{1.100}
$$

Casimir operator with commutative property acting on will give us the eigenvalue.

$$
l^2 = l_1^2 + l_2^2 + l_3^2 \tag{1.101}
$$

$$
l^2 Y_l^m(\theta, \phi) = l(l+1)Y_l^m(\theta, \phi) \tag{1.102}
$$

$$
\begin{aligned}
l^2 &= l^{-}l^{+} + il_3 + l_3^2 \\
&= e^{i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right)e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) - i\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} \\
&= \frac{\partial^2}{\partial\theta^2} + i\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\phi}\right) + e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\left(-i\frac{\partial}{\partial\theta}\right)\right) + e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}\right) + (\ldots)
\end{aligned}
\tag{1.103}
$$

Evaluate that:

$$
\begin{aligned}
i\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\phi}\right) &= \frac{i}{\sin^2\theta}\frac{\partial}{\partial\phi} + i\cot\theta\frac{\partial^2}{\partial\theta\partial\phi} \\
e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\left(-i\frac{\partial}{\partial\theta}\right)\right) &= -\cot\theta\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial^2}{\partial\theta\partial\phi} \\
e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}\right) &= -i\cot^2\theta\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2}
\end{aligned}
\tag{1.104}
$$

The first term evaluate as:

$$
\begin{aligned}
\text{First term} &= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + i\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi} - \cot^2\theta\frac{\partial}{\partial\phi}\right) + \cot^2\theta\frac{\partial^2}{\partial\phi^2} \\
&= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2}
\end{aligned}
\tag{1.105}
$$

$$
\begin{aligned}
\text{LHS} &= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\phi^2} \\
&= \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right)\sin\theta + \frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\
&= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} = \Delta_\Omega
\end{aligned}
\tag{1.106}
$$

Which is the **Spherical Laplacian**[1].

This, actually can be formalize in such insight, that Laplacian as:

$$
\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \Delta_\Omega \tag{1.107}
$$

Is rotation invariant given by $[\Delta_\Omega, l_i] = 0$. Which is same as Casimir operator by its **uniqueness** up to constant. Thus we decompose $L^2(S^2)$ into basis equipped with $Y_l^m(\theta,\phi)$.

1.6

$$
S[*] : \mathcal{F} \to \mathbb{R}; \mathcal{F} = \{\boldsymbol{q}(t) : t \in [t_0, t_1] \subset \mathbb{R} \to \mathbb{R}^M\} \tag{1.108}
$$

With differential set if applicable:

$$
\mathcal{F}_\varepsilon = \{\delta\boldsymbol{q}(t); |\delta\boldsymbol{q}(t)| < \varepsilon; |\delta\dot{\boldsymbol{q}}(t)| < \varepsilon; \forall t \in [t_0, t_1] \subset \mathbb{R}\} \tag{1.109}
$$

$$
\delta S(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \to \delta\boldsymbol{q}(t_0) = \delta\boldsymbol{q}(t_1) = 0 \tag{1.110}
$$

[1] Actually, calculate by $l_1^2 + l_2^2 + l_3^2$ is more simpler, or use some notation reduction would be easier in burden.

Given by a certain basis product form:

$$
\begin{gathered}
|x\rangle \text{ or } |x+\delta\rangle ? \to \text{continuous basis} \\
\langle x|\psi\rangle = \langle x|x'\rangle\langle x'|\psi\rangle
\end{gathered}
\tag{1.111}
$$

Thus φ should be a continuous basis expansion for the representation of coordinates:

$$
\begin{gathered}
\int d\mu\, \varphi(x|x') \leftrightarrow \langle x|x'\rangle \\
\int d\mu\, \varphi(x|x')\varphi(x'|x_0) \leftrightarrow \langle x|x'\rangle\langle x'|x_0\rangle = \langle x|x_0\rangle \leftrightarrow \delta(x'|x_0)
\end{gathered}
\tag{1.112}
$$

$$
\begin{gathered}
\varphi(x|x_0) = \varphi(x|x_{N-1})\varphi(x_{N-1}|x_{N-2})...\varphi(x_1|x_0) \\
\varphi(x|x_0) = \lim_{N\to\infty} \prod^{\infty} T(\varphi(x_\varepsilon))
\end{gathered}
\tag{1.113}
$$