

Review of Quantum

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$g \in G$, given a action of group on a general space M equipped with function on it as $F(M)$. Thus we can apply g on M and lift upon $F(M)$ as well, i.e. $\varphi(g) \cdot x$ for $x \in M$ and $\rho(g) \cdot f$ for $f \in F(M)$ as a G -equivariant map.

$$\begin{array}{ccc} x & \xrightarrow{f} & f(x) \\ \varphi(g) \downarrow & & \downarrow \rho(g) \\ \varphi(g) \cdot x & \xrightarrow{f} & \rho(g) \cdot f(x) / f(\varphi(g) \cdot x) \end{array}$$

Figure 1. Commutative diagram of group action on space and mapped space

Is commutative, i.e.

$$\rho(g) \cdot (\rho(h) \cdot f)(x) = \rho(h) \cdot f(\varphi(g) \cdot x) = f(\varphi(h) \cdot (\varphi(g) \cdot x)) = f(\varphi(h) \circ \varphi(g) \cdot x) \quad (1.1)$$

Is a anti-homomorphism, i.e. Such representation should be only with g element, thus a canonical choice is g^{-1} where below is given as a reduced notation:

$$g \cdot f(x) = f(g^{-1} \cdot x) \quad (1.2)$$

$V = W \oplus W' \rightarrow v = w + w'$, where the projection $q(v) = w$. Define $\bar{q} : v \rightarrow \frac{1}{|G|} \sum_{g \in G} \rho(g) q(\rho(g^{-1})v)$ which we drop ρ representation notation.

$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} g q(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = w \quad (1.3)$$

$$h\bar{q}(v) = \frac{1}{|G|} \sum_{g \in G} h g q(g^{-1}h^{-1}hv) = \frac{1}{|G|} \sum_{g' \in G} g' q(g'^{-1}hv) = \bar{q}(hv) \quad (1.4)$$

If $v \in \text{Ker}(\bar{q})$, $h \in G$ then $h\bar{q}(v) = 0 = \bar{q}(hv)$ that $hv \in \text{Ker}(\bar{q})$ that we decompose by a G -invariant morphism based on solely vector space projection q .

How can we define a G -invariant operation on V ? We should **always** remember that g act transitively on G itself that if $gG = G$. Then if we can put all elements of G on the operation, then it should be G -invariant. By the way, we can define the module of G -invariance which is $RG = \sum_{g_i \in G} r_i g_i$ $r_i \in R, g_i \in G$. The way is to average the operation by group action for all elements. Thus one has:

$$\begin{aligned}
hf(v_1, \dots, v_n) &= \frac{1}{|G|} \sum_{g \in G} hgf(g^{-1}h^{-1}hv_1, \dots, g^{-1}h^{-1}hv_n) \\
&= \frac{1}{|G|} \sum_{g' \in G} g'f(g'^{-1}hv_1, \dots, g'^{-1}hv_n) = f(hv_1, \dots, hv_n)
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
f(hv_1, \dots, hv_n) &= \frac{1}{|G|} \sum_{g \in G} gf(g^{-1}hv_1, \dots, g^{-1}hv_n) \\
&= \frac{1}{|G|} \sum_{h^{-1}g \in G} h^{-1}gf\left((h^{-1}g)^{-1}v_1, \dots, (h^{-1}g)^{-1}v_n\right) = f(v_1, \dots, v_n)
\end{aligned} \tag{1.6}$$

$$\begin{aligned}
\langle v, u \rangle &= \langle u, v \rangle^* \\
\langle u, \alpha v + \beta w \rangle &= \alpha \langle u, v \rangle + \beta \langle u, w \rangle \\
\|u\|^2 &= \langle u, u \rangle
\end{aligned} \tag{1.7}$$

$$\langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle \tag{1.8}$$

If there' s a invariant space that $\rho(g)(V_1) \subseteq V_1$, with the decomposition $V_2 = V_1^\perp$ that $V_2 = \{v \in V \mid (v, x) = 0 \ \forall x \in V_1\}$.

$$\langle \rho(g)x, y \rangle = \langle \rho(g)^{-1}\rho(g)x, \rho(g)^{-1}y \rangle = \langle x, \rho(g)^{-1}y \rangle \quad \forall y \in V_1 \tag{1.9}$$

Thus if $x \in V_2$, the inner product should be **zero**. Reversely, we deduce that $\rho(g)x \in V_2$ because $y \in V_1$ by first term. Therefore the representation can be decomposed if we seek a **sole** invariant space and repeat the procedure.

Thus given a inner product, which is also a bilinear-function, we can construct G -invariant form like above.

$$\langle x, y \rangle := \sum_{h \in G} \langle \rho(h)x, \rho(h)y \rangle \tag{1.10}$$

Which is *unitary* as above already proved. Same as integral form one has:

$$\langle x, y \rangle := \int_G d\mu(h) \langle \rho(h)x, \rho(h)y \rangle \tag{1.11}$$

Yields the same results. However, we can conclude the canonical G -invariance form for *finite* and *compact* group due to the convergence given by the integral only for compactness.

Given linear morphism for irreducible space, $\varphi_1 : V \rightarrow W$ and $\varphi_2 : V \rightarrow W$, one could inspect that due to $\ker(\varphi) = \{0\}$ or V due to **irreducible** property, one can acquire no smaller subspace. Thus, $\varphi = 0$ or isomorphism, by uniqueness, $\varphi_1 = \lambda\varphi_2$.

Now given a space transformation:

$$\varphi(\rho(g)) = A^{-1}\rho(g)A = \rho'(g) \tag{1.12}$$

By above suggestion on irreducible representation, one should have $A = 0$, or better, the irreducible representation should be isomorphism, thus $A = \lambda I$. However, for clarification, you should prefer \mathbb{C} field that to fully decompose space rather the root problem would occur.

$$(v_1, v_2) \rightarrow v_1 \otimes v_2 \equiv v_1^i e_1^i \otimes v_2^j e_2^j = v_1^i v_2^j e_1^i \otimes e_2^j \quad (1.13)$$

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2) \quad (1.14)$$

$$\chi_\rho(g) = \text{Tr}(\rho(g)) \quad (1.15)$$

$$\chi_{\rho'}(g) = \text{Tr}(\rho'(g)) = \text{Tr}(A^{-1}\rho(g)A) = \text{Tr}(A^{-1}A\rho(g)) = \text{Tr}(\rho(g)) = \chi_\rho(g) \quad (1.16)$$

$$\chi_{\rho_1 \oplus \rho_2}(g) = \text{Tr}(\rho_1 \oplus \rho_2(g)) = \text{Tr}(\rho_1(g)) + \text{Tr}(\rho_2(g)) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g) \quad (1.17)$$

$$\begin{aligned} \chi_{\rho_1 \otimes \rho_2}(g) &= \text{Tr}(\rho_1 \otimes \rho_2(g)) = \text{Tr}(\rho_1(g)) \otimes \text{Tr}(\rho_2(g)) \\ &= \chi_{\rho_1}(g) \otimes \chi_{\rho_2}(g) = \chi_{\rho_1}(g) \chi_{\rho_2}(g) \otimes 1 \cong \chi_{\rho_1}(g) \chi_{\rho_2}(g) \end{aligned} \quad (1.18)$$

$$\chi_\rho(e) = \text{Tr}(\rho(e)) = \text{Tr}(I_V) = \dim(V) \quad (1.19)$$

$$M_{jl}^{ik} = \int_G d\mu(g) \rho_1(g)_j^i \rho_2(g^{-1})_l^k \quad (1.20)$$

$$\rho_1(h)_a^i \rho_1(g)_j^a = \rho_1(hg)_j^i \quad (1.21)$$

$$\begin{aligned} \rho_1(h)_a^i M_{jl}^{ak} &= \int_G d\mu(g) \rho_1(h)_a^i \rho_1(g)_j^a \rho_2(g^{-1})_l^k \\ &= \int_G d\mu(hg) \rho_1(hg)_j^i \rho_2(g^{-1})_l^k \quad hg \rightarrow g' \\ &= \int_G d\mu(g') \rho_1(g')_j^i \rho_2(g'^{-1}h)_l^k \\ &= M_{jb}^{ik} \rho_2(h)_k^b \end{aligned} \quad (1.22)$$

By Schur's lemma, one acquire a one for all relation that either isomorphism, or nothing for M as a transformation matrix. Thus, fix j, k as $M_{ik}^{jl} = \mathcal{M}_i^l$.

$$M_{ik}^{jl} = \alpha_k^j \delta_i^l \quad (1.23)$$

However, one can change the summation for matrix by apply direction as:

$$\rho_1(h)_j^a \rho_1(g)_a^i = \rho_1(hg)_j^i \quad (1.24)$$

Giving us:

$$M_{ik}^{jl} = \beta_i^l \delta_k^j = \alpha_k^j \delta_i^l \quad (1.25)$$

Thus it' s actually a decomposition as $C\delta_k^j\delta_i^l$ for the same irreducible representation or else nothing. We then find it' s actually a trace summation:

$$\int_G d\mu(g) \rho(g)_j^i \rho(g^{-1})_i^j = \int_G d\mu(g) \text{Tr}(I_V) = n \int_G d\mu(g) = \sum_{i,j} M_{ji}^{ij} = Cn^2 \quad (1.26)$$

If we can normalize the integral:

$$\int_G d\mu(g) = 1 \rightarrow C = \frac{1}{n} = \frac{1}{\dim(V)} \quad (1.27)$$

$$\int_G d\mu(g) \rho(g)_i^i \rho(g^{-1})_j^j = \int_G d\mu(g) \chi_\rho(g) \chi_\rho(g^{-1}) = \sum_{ij} M_{ij}^{ij} = Cn^2 = n \quad (1.28)$$

Thus, one has:

$$\frac{1}{n} \int_G d\mu(g) \chi_{\rho_1}(g) \chi_{\rho_2}(g^{-1}) = 1_{\text{id}} \delta_{\rho_1, \rho_2} \quad (1.29)$$

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$$\mathbf{F}(q, p) = \begin{pmatrix} F_q(q, p) \\ F_p(q, p) \end{pmatrix} = \begin{pmatrix} \partial_q F(q, p) \\ \partial_p F(q, p) \end{pmatrix} \quad (1.30)$$

$$dH = H_q dq + H_p dp = -\frac{dp}{dt} dq + \frac{dq}{dt} dp \quad (1.31)$$

$$\frac{d}{dt} = \frac{dq}{dt} \partial_q + \frac{dp}{dt} \partial_p = H_p \partial_q - H_q \partial_p := \{\cdot, H\} := X_H \quad (1.32)$$

That' s, Hamiltonian defines the evolution of the system.

$$\frac{dF(q, p)}{dt} = X_H F(q, p) = \{F(q, p), H\} \quad (1.33)$$

$$\begin{aligned} X_H(p dq) &= \dot{p} dq + p d\dot{q} = \dot{p} dq - \dot{q} dp + d(p\dot{q}) \\ &= -H_q dq + H_p dp + d(p\dot{q}) = d(-H + p\dot{q}) := dL(q, \dot{q}) \end{aligned} \quad (1.34)$$

$$\{F, H\} = X_H F = -X_F H \quad (1.35)$$

Where the $X_F = \frac{\partial F}{\partial p} \partial_q - \frac{\partial F}{\partial q} \partial_p$.

Generally, one scalar function defined upon symplectic manifold can induce a vector field on it, with a general area preservation:

$$\begin{aligned}\omega &= dq \wedge dp \\ \iota_{X_F}(\omega) &= \iota_{\frac{\partial F}{\partial q} \partial_p - \frac{\partial F}{\partial p} \partial_q}(dq \wedge dp) = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial p} dp = dF\end{aligned}\tag{1.36}$$

$$\iota_{X_H} \iota_{X_F}(\omega) = \iota_{X_H}(dF) = X_H F = \{F, H\} = \frac{dF}{dt}\tag{1.37}$$

We thus induce a symmetric bilinear form:

$$\omega(X_F, X_G) = \iota_{X_F} \iota_{X_G}(\omega) = \iota_{X_F}(dG) = X_F G = \{G, F\} = -\{F, G\}\tag{1.38}$$

One can lift the Hamiltonian in exponential flow by dynamics of X_H .

$$\frac{d}{dt} F(\exp(tX_H)) \big|_{t=0} = \{F(p, q), H(p, q)\}\tag{1.39}$$

One thus can define lie group dynamics too just like above:

$$\begin{aligned}L &\in \mathfrak{g} \rightarrow X_L \\ X_L F(p, q) &= \frac{d}{dt} F(e^{tL} \cdot (p, q))\end{aligned}\tag{1.40}$$

However, due to the representation of any group should be a anti-homomorphism, we have:

$$L \cdot F(p, q) = \frac{d}{dt} F(e^{-tL} \cdot (p, q)) = -X_L F(p, q)\tag{1.41}$$

Now we can try translation group $T = \mathbb{R}^3$ first:

$$a \cdot (q, p) = (q + a, p) \quad a \in T \leftrightarrow e^{-t\lambda} \in T \quad \lambda \in \mathfrak{t}\tag{1.42}$$

Here we **abuse** of notation because the lie algebra of translation induced by lie group is trivial.

$$a \cdot F(p, q) = \frac{d}{dt} F(e^{-t\lambda} \cdot (p, q)) \big|_{t=0} = \frac{d}{dt} F(q - ta, p) \big|_{t=0} = -a \frac{\partial}{\partial q} F = -X_a F\tag{1.43}$$

$$X_a = a \frac{\partial}{\partial q}\tag{1.44}$$

We know that $\iota_{X_F}(\omega)$ is a closed form that it corresponds to a certain function F up to a constant, we can induce the scalar function of action by the lie group.

$$X_a(\omega) = ap \rightarrow \mu_a(p) = \mu_a(p, q)\tag{1.45}$$

We thus recover the momentum operator which configures the momentum for each translation element a . Such closed form check lift a function defined in the manifold which shared the same role as Hamiltonian, if H is G -invariant, then μ_a is a conserved quantity, as the Hamiltonian version of Noether's theorem.

$$\iota_{X_H}(d\mu_L) = \{\mu_L, H\} = \frac{d\mu_L}{dt} = -\{H, \mu_L\} = -\iota_{X_L}(dH) = -X_L H = 0 \quad (1.46)$$

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$$[\phi(X), \phi(Y)] = \phi([X, Y]) = \phi(X)\phi(Y) - (-1)^{\sigma(X)\sigma(Y)}\phi(Y)\phi(X) \quad (1.47)$$

$$\Omega(Jv_1, Jv_2) = \Omega(v_1, v_2) \rightarrow J \in \text{Sp}(2n, \mathbb{R}) \quad (1.48)$$

$$\begin{aligned} A^T A &= I \rightarrow A \in \text{O}(n, \mathbb{R}) \\ A^T \Omega A &= \Omega \rightarrow A \in \text{Sp}(2n, \mathbb{R}) \\ A^T \Omega A &= A^T A \Omega \rightarrow A^T (\Omega A - A \Omega) = 0 \rightarrow \Omega A = A \Omega \\ \Omega(Av, Aw) &= g(A \Omega v, Aw) = g(\Omega Av, Aw) = \Omega(v, w) \\ A &\in \text{O}(n, \mathbb{R}) \cap \text{Sp}(2n, \mathbb{R}) = \text{U}(n) \end{aligned} \quad (1.49)$$

$$z_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j - ip_j) \quad (1.50)$$

$$\begin{aligned} L_1 &= Q_2 P_3 - Q_3 P_2, \quad L_2 = Q_3 P_1 - Q_1 P_3, \quad L_3 = Q_1 P_2 - Q_2 P_1 \\ \Gamma(L_1) &= a_3^\dagger a_2 - a_2^\dagger a_3, \quad \Gamma(L_2) = a_1^\dagger a_3 - a_3^\dagger a_1, \quad \Gamma(L_3) = a_2^\dagger a_1 - a_1^\dagger a_2 \end{aligned} \quad (1.51)$$

$$\begin{aligned} F(\theta) &= c_0 + c_1 \theta + c_2 \theta^2 + \dots + c_n \theta^n = c_0 + c_1 \theta \\ \frac{\partial F}{\partial \theta} &= c_1 \\ F(\theta_1, \dots, \theta_n) &= F_A + \theta_j F_B \quad \exists F_A, F_B \in \bigwedge(\mathbb{R}^n) \\ \frac{\partial FG}{\partial \theta_j} &= \frac{\partial F}{\partial \theta_j} G + (-1)^{\sigma(F)} F \frac{\partial G}{\partial \theta_j} \quad \sigma(F) \text{ is the degree of } F \end{aligned} \quad (1.52)$$

Given a translation invariance property of integral:

$$\int f(\theta + \eta) d\theta = \int f(\theta) d\theta \quad (1.53)$$

$$\int (\theta + \eta) d\theta = \int \theta d\theta + \eta \int d\theta = \int \theta d\theta \rightarrow \int d\theta = 0 \quad (1.54)$$

Given with normalization convention:

$$\int \theta d\theta = 1 \rightarrow \int c_0 + c_1 \theta d\theta = c_1 \rightarrow \int f(\theta) d\theta = \frac{df}{d\theta} \quad (1.55)$$

Thus for multi-variables calculus, one should be careful on applying order of partial derivative:

$$\int f(\theta_1, \dots, \theta_n) d\theta_n \dots d\theta_1 = \frac{\partial}{\partial \theta_1} \dots \frac{\partial}{\partial \theta_n} f \quad (1.56)$$

Where swap of order of partial derivative will induce a minus sign.

$$\begin{aligned} [\theta_j, \theta_k]_+ &= \pm \delta_{jk} \\ \frac{d}{dt} \theta_j(t) &= [\theta_j(t), H]_+ \end{aligned} \quad (1.57)$$

Given H :

$$B = \begin{pmatrix} 0 & B_{12} & B_{13} & \dots & B_{1n} \\ -B_{12} & 0 & B_{23} & \dots & B_{2n} \\ -B_{13} & -B_{23} & 0 & \dots & B_{3n} \\ \dots & & & & \\ -B_{1n} & -B_{2n} & -B_{3n} & \dots & 0 \end{pmatrix} \quad (1.58)$$

$$H = \frac{1}{2} \sum_{j,k=1}^n B_{jk} \theta_j \theta_k$$

$$\frac{d}{dt} \theta_j(t) = [\theta_j(t), H]_+ = -[H, \theta_j(t)]_+ = \sum_{k=1}^n B_{jk} \theta_k(t) \quad (1.59)$$

The minus sign of commutator is due to $B_{jk} = -B_{kj}$. Which is highly similar to the bosonic case in evolution.

$$\Omega^+(Jv_1, Jv_2) = \Omega^+(v_1, v_2) \rightarrow J \in \text{SO}(2n, \mathbb{R}) \quad (1.60)$$

$$\begin{aligned} U_A &= \sum_{jk} a_j^\dagger A_{jk} a_k \\ [U_A, a_j]_\pm &= \pm \sum_k A_{jk} a_k = \pm A \mathbf{a} \\ [U_A, a_j^\dagger]_\pm &= \sum_k a_k^\dagger A_{kj} = A^T \mathbf{a}^\dagger \end{aligned} \quad (1.61)$$

$$v \otimes w = \frac{1}{2}(v \otimes w - u \otimes v) + \frac{1}{2}(v \otimes w + w \otimes v) = v \wedge w + g(v, w) \quad (1.62)$$

Thus given a basis omit tensor notation, for example:

$$e_i e_j = e_i \wedge e_j \quad (i \neq j) \quad (1.63)$$

Or physics notation:

$$\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n \rightarrow \gamma(\mathbf{v}) = \mathcal{V} = v_1 \gamma_1 + \dots + v_n \gamma_n \quad (1.64)$$

$$\begin{aligned} v' &= v - 2 \frac{g(v, w)}{g(w, w)} w = \mathcal{V} - \frac{\mathcal{V}\mathcal{W} - \mathcal{W}\mathcal{V}}{g(w, w)} \mathcal{W} \\ &= - \frac{\mathcal{W}\mathcal{W}\mathcal{W}}{g(w, w)} \end{aligned} \quad (1.65)$$

$$\begin{aligned} 2AB &= [A, B]_+ + [A, B]_- \\ 2BA &= [A, B]_+ - [A, B]_- \end{aligned} \quad (1.66)$$

Thus we can always decompose a product term into symmetric and antisymmetric parts.

$$ABC = \frac{1}{2}A[B, C]_+ + \frac{1}{2}A[B, C]_- = \frac{1}{2}[A, B]_+C - \frac{1}{2}[A, C]_+B + \frac{1}{2}[A, B]_-C - \frac{1}{2}[A, C]_-B \quad (1.67)$$

$$\begin{aligned} [AB, C]_- &= ABC - CAB = A\left(\frac{1}{2}[B, C]_+ + \frac{1}{2}[B, C]_-\right) - \left(\frac{1}{2}[A, C]_+ - \frac{1}{2}[A, C]_-\right)B \\ &= \frac{1}{2}(A[B, C]_+ - [A, C]_+B) + \frac{1}{2}(A[B, C]_- + [A, C]_-B) \end{aligned} \quad (1.68)$$

$$\begin{aligned} [AB, C]_+ &= ABC + CAB = A\left(\frac{1}{2}[B, C]_+ + \frac{1}{2}[B, C]_-\right) + \left(\frac{1}{2}[A, C]_+ - \frac{1}{2}[A, C]_-\right)B \\ &= \frac{1}{2}(A[B, C]_+ - [A, C]_-B) + \frac{1}{2}(A[B, C]_- + [A, C]_+B) \end{aligned} \quad (1.69)$$

Equal below:

$$[AB, C]_- = A[B, C]_- + [A, C]_-B = A[B, C]_+ - [A, C]_+B \quad (1.70)$$

$$[AB, C]_+ = A[B, C]_+ + [A, C]_-B = A[B, C]_- - [A, C]_+B \quad (1.71)$$

$$\begin{aligned} [AB, CD]_- &= A[B, CD]_- + [A, CD]_-B \\ &= A([B, C]_+D - C[B, D]_+) + ([A, C]_+D - C[A, D]_+)B \\ &= A[B, C]_+D - AC[B, D]_+ + [A, C]_+DB - C[A, D]_+B \end{aligned} \quad (1.72)$$

For Clifford algebra, symmetric part is given by metric, anti-symmetric part is given by wedge product.

$$\begin{aligned} M_{\mu\nu} &= \gamma_\mu \wedge \gamma_\nu \\ M_{\mu\nu} &= \gamma_\mu \gamma_\nu - g_{\mu\nu} \end{aligned} \quad (1.73)$$

$$M_{\mu\nu} \wedge \gamma_\rho = \gamma_\mu \gamma_\nu \wedge \gamma_\rho = \gamma_\mu g_{\nu\rho} - \gamma_\nu g_{\mu\rho} \quad (1.74)$$

$$M_{\mu\nu} \wedge M_{\rho\sigma} = \gamma_\mu \gamma_\nu \wedge \gamma_\rho \gamma_\sigma = \gamma_\mu \gamma_\sigma g_{\nu\rho} - \gamma_\mu \gamma_\rho g_{\nu\sigma} + \gamma_\nu \gamma_\rho g_{\mu\sigma} - \gamma_\nu \gamma_\sigma g_{\mu\rho} \quad (1.75)$$

Is same as above, however, I didn't find this is useful : (

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$$\frac{L}{[L, L]} \text{ is abelian that } \forall a, b \in L, [a, b] = 0 \quad (1.76)$$

Thus given a central series:

$$[L^n, L^n] = L^{n+1} \rightarrow \frac{L^n}{[L^n, L^n]} = \frac{L^n}{L^{n+1}} \quad (1.77)$$

Factor through, for all possible elements, otherwise $[L, L] = L$ is irreducible. Thus induce a maximal nilpotent ideal of L which is the maximal solvable ideal of L , else it must be a **semi-simple** structure without additional smaller space. Where nilpotent is a structure that can be shared with a basis with commutative eigenvectors, that's $XYv = YXv + [X, Y]v = \lambda(X)Yv + \lambda([X, Y])v$ where $[X, Y]$ in some smaller space by deduction of nilpotent ideal.

Given a basis $\{X_1, \dots, X_n\}$ with structure constants for a semi-simple lie algebra:

$$[X_i, X_j] = \sum_k c_{ij}^k X_k \quad (1.78)$$

Given the commutation relation, one has:

$$\begin{aligned} c_{ij}^k &= -c_{ji}^k \\ c_{ij}^k + c_{jk}^i + c_{ki}^j &= 0 \end{aligned} \quad (1.79)$$

Moreover, we can define the representation based on the structure constants:

$$(T_i)_j^k = (D_{X_i})_j^k = c_{ij}^k \quad (1.80)$$

$$D_{X_i} : \mathfrak{g} \rightarrow \text{ad}(\mathfrak{g})$$

$$D_{X_i} X_j = [X_i, X_j] = \sum_k c_{ij}^k X_k \quad (1.81)$$

$$\begin{aligned} [D_{X_i}, D_{X_j}] &= D_{[X_i, X_j]} \\ [D_{X_i}, D_{X_j}] X_k &= D_{[X_i, X_j]} X_k \\ [X_i [X_j X_k]] - [X_j [X_i X_k]] &= [[X_i, X_j] X_k] \end{aligned} \quad (1.82)$$

Which adjoin representation with **homomorphism consistency** should restrict all lie algebras to contain Jacobi identity.

$$\begin{aligned}
D_{X_i} D_{X_j} X_l &= D_{X_i} c_{jl}^k X_k = c_{ik}^m c_{jl}^k X_m \\
(D_{X_i} D_{X_j})_l^m &= c_{ik}^m c_{jl}^k \\
\text{Tr}(D_{X_i} D_{X_j}) &= (D_{X_i} D_{X_j})_m^m = c_{ik}^m c_{jm}^k = g_{ij} = c_{jm}^k c_{ik}^m = g_{ji}
\end{aligned} \tag{1.83}$$

Where:

$$g_{ij} g^{jl} = c_{ik}^m c_{jm}^k c_{fn}^{jn} c_n^{lf} = c_{ik}^m c_n^{lf} \delta_m^n \delta_f^k = \delta_i^l \tag{1.84}$$

$$\langle v, w \rangle = v^i (D_{X_i})_k^l v^j (D_{X_j})_l^k = v^i v^j g_{ij} \tag{1.85}$$

Cyclic property:

$$0 = \text{Tr}([A, BC]) = \text{Tr}(B[A, C]) + \text{Tr}([A, B]C) \tag{1.86}$$

$$\begin{aligned}
\text{Tr}([A, B]C) &= \text{Tr}((AB - BA)C) \\
&= \text{Tr}(ABC) - \text{Tr}(BAC) \\
&= \text{Tr}(ABC) - \text{Tr}(ACB) = \text{Tr}(A[B, C])
\end{aligned} \tag{1.87}$$

$$\begin{aligned}
0 &= \text{Tr}([\rho(X_i)\rho(X_j), \rho(X_k)]) = \text{Tr}(\rho(X_i)[\rho(X_j), \rho(X_k)]) + \text{Tr}([\rho(X_i), \rho(X_k)]\rho(X_j)) \\
&= \text{Tr}(\rho(X_i)f_{jk}^d \rho(X_d)) + \text{Tr}(f_{ik}^d \rho(X_d)\rho(X_j)) \\
&= f_{jk}^d g_{id} + f_{ik}^d g_{dj} \\
&= f_{jki} + f_{ikj}
\end{aligned} \tag{1.88}$$

Given with $f_{jki} = -f_{kji}$, one has $f_{jki} = -f_{kji} = f_{ijk} = -f_{jik}$. It's totally anti-symmetric tensor.

Fix indices as $f_{ij}^k = (\ell_i)_j^k$:

$$\begin{aligned}
\ell_j g^{(\rho)} + (\ell_i g^{(\rho)})^T &= 0 \\
g^{(-1)^{(\rho)}} \ell_j g^{(\rho)} &= -\ell_i^T \quad \text{by } g = g^T \\
\ell_j g^{(\rho)} &= g^{(\rho)} g^{(-1)^{(\rho)}} \ell_j g^{(\rho)} \\
\ell_j g^{(\rho)} g^{(-1)^{(\rho)}} &= g^{(\rho)} g^{(-1)^{(\rho)}} \ell_j
\end{aligned} \tag{1.89}$$

Thus by Schur's lemma, for *simple* lie algebra, it should be up to constant for the representation of the metric form compared with the canonical metric form.

However, it doesn't mean that **semi-simple** structure contains no ideal. We try to diagonalize as much as possible even we can't achieve for all like solvable case. That's $\mathfrak{t} \subset \mathfrak{g}$ which all elements are commutative. We call such maximal commutative subalgebra as **Cartan subalgebra**, which can be diagonalized simultaneously. Thus if one choose a vector space it acts on, we can decompose the space into eigen-space for all:

$$V = \bigoplus_{\lambda(\mathfrak{t})} V_\lambda \quad V_\lambda = \{v \in V \mid \forall H \in \mathfrak{t}, Hv = \lambda(H)v\} \quad (1.90)$$

Given a adjoint representation or homomorphism as $X_i \rightarrow D_{X_i}$, we can then apply the Cartan subalgebra to decompose the lie algebra itself:

$$\mathfrak{g} = \bigoplus_{\alpha(\mathfrak{t})} \mathfrak{g}_\alpha \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{t}, [H, X] = D_H(X) = \alpha(H)X\} \quad (1.91)$$

Which is called **root space decomposition**, where $\alpha \in \mathfrak{t}^*$ is the eigenvalue on \mathfrak{t} .

We often see the ladder operator in physics that a state shared by all other raising and lowering operators, which is the eigenvector of Cartan subalgebra.

$$\begin{aligned} \mathbf{H} &= (H_1, \dots, H_r) \\ [H_i, H_j] &= 0 \quad \forall i, j = 1, \dots, r \rightarrow [\rho(H_i), \rho(H_j)] = 0 \quad \text{for finite representation} \end{aligned} \quad (1.92)$$

Due to mutually commutation, each H_i can share the same eigenstate with weights \mathbf{m} .

$$\rho(H_i)|\mathbf{m}\rangle = m_i|\mathbf{m}\rangle \quad \mathbf{m} = (m_1, \dots, m_r) \quad (1.93)$$

Plus, given by the adjoint representation, the same routine is applied for remained decomposition.

$$D_{H_i}(E_\alpha) = [H_i, E_\alpha] = \alpha(H_i)E_\alpha = \alpha_i E_\alpha \quad \alpha = (\alpha_1, \dots, \alpha_r) \quad (1.94)$$

$$\begin{aligned} D_{H_i}([E_\alpha, E_\beta]) &= [H_i, [E_\alpha, E_\beta]] = [[H_i, E_\alpha], E_\beta] + [E_\alpha, [H_i, E_\beta]] \\ &= (\alpha_i + \beta_i)[E_\alpha, E_\beta] \sim H_i E_{\alpha+\beta} \leftrightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \end{aligned} \quad (1.95)$$

Just like you already have seen in generator of $\mathfrak{su}(2)$ that $[\xi_3, \xi^\pm] = \pm \xi^\pm$.

We know if the ladder operator act too many times in finite representation, it will reach zero, that's D_X is nilpotent for $X \in \mathfrak{g}_\alpha, \alpha \neq 0$, or we simply call it nilpotent that $\mathfrak{g}_{\beta+n\alpha} = 0$.

For clarity, \mathfrak{g}_0 means $[\mathfrak{t}, \mathfrak{g}_0] = 0 \cdot \mathfrak{g}_0 = 0$ thus $\mathfrak{t} \subset \mathfrak{g}_0$, however, due to maximality of Cartan subalgebra, we have $\mathfrak{g}_0 = \mathfrak{t}$.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{t} \quad (1.96)$$

Given a unitary representation, with the observable eigenvalue should be *real*, thus, we restrict \mathbf{m}, α to real value, or $\mathbf{H}^\dagger = \mathbf{H}$.

$$[H_i, E_\alpha]^\dagger = (\alpha_i E_\alpha)^\dagger \rightarrow [E_\alpha^\dagger, H_i] = \alpha_i^* E_\alpha^\dagger = -[H_i, E_\alpha^\dagger] \rightarrow E_\alpha^\dagger \in \mathfrak{g}_{-\alpha} \quad (1.97)$$

Which is justify the existence of $\mathfrak{g}_{-\alpha}$

$$[E_{\alpha}, E_{-\alpha}] = \sum_i^r b^i H_i \quad (1.98)$$

Restrict to \mathfrak{t} , we acquire its metric as $\text{Tr}(H_i H_j) = g_{ij}$.

$$\sum_j g_{ij} b^j = \text{Tr}(H_i [E_{\alpha}, E_{-\alpha}]) = \text{Tr}([H_i, E_{\alpha}] E_{-\alpha}) = \alpha_i \text{Tr}(E_{\alpha}, E_{-\alpha}) \quad (1.99)$$

If we normalize that $\text{Tr}(E_{\alpha}, E_{-\alpha}) = 1$.

$$\sum_j g_{ij} b^j = \alpha_i \rightarrow b^j = \sum_i g^{ij} \alpha_i \quad (1.100)$$

We thus could define $H_{\alpha} := [E_{\alpha}, E_{-\alpha}] = \sum_j b_j H_j$.

$$[H_{\alpha}, E_{\beta}] = \sum_j b_j [H_j, E_{\beta}] = (\mathbf{b} \cdot \beta) E_{\beta} \rightarrow [H_{\alpha}, E_{\pm\alpha}] = \pm(\mathbf{b} \cdot \alpha) E_{\pm\alpha} \quad (1.101)$$

However:

$$\mathbf{b} \cdot \alpha = \sum_{ij} \alpha_i \alpha_j g^{ij} = \langle \alpha, \alpha \rangle \quad (1.102)$$

And we actually acquire a more ease representation as H_{α} .

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$$\rho(H_i) \rho(E_{\alpha}) |\rho; \mathbf{m}\rangle = ([\rho(H_i), \rho(E_{\alpha})] + \rho(E_{\alpha}) \rho(H_i)) |\rho; \mathbf{m}\rangle = (\alpha_i + m_i) E_{\alpha} |\rho; \mathbf{m}\rangle \quad (1.103)$$

$$\rho(E_{\alpha}) |\rho; \mathbf{m}\rangle = N_{\alpha}(\mathbf{m}) |\rho; \mathbf{m} + \alpha\rangle \quad (1.104)$$

$$\begin{aligned} [\rho(E_{\alpha}), \rho(E_{-\alpha})] &= \rho(H_{\alpha}) \\ \rightarrow N_{\alpha}(\mathbf{m} - \alpha) N_{-\alpha}(\mathbf{m}) - N_{-\alpha}(\mathbf{m} + \alpha) N_{\alpha}(\mathbf{m}) &= (\alpha, \mathbf{m}) \end{aligned} \quad (1.105)$$

$$E_{\alpha} |\rho; \mathbf{m} + p\alpha\rangle = 0 \quad E_{-\alpha} |\rho; \mathbf{m} - q\alpha\rangle = 0 \quad (1.106)$$

$$N_{\alpha}(\mathbf{m} + (k-1)\alpha) N_{-\alpha}(\mathbf{m} + k\alpha) - N_{-\alpha}(\mathbf{m} + (k+1)\alpha) N_{\alpha}(\mathbf{m} + k\alpha) = \langle \alpha, \mathbf{m} + k\alpha \rangle \quad (1.107)$$

$$\begin{aligned} \sum_{k=-q}^p N_{\alpha}(\mathbf{m} + (k-1)\alpha) N_{-\alpha}(\mathbf{m} + k\alpha) \\ - N_{-\alpha}(\mathbf{m} + (k+1)\alpha) N_{\alpha}(\mathbf{m} + k\alpha) &= \sum_{k=-q}^p \langle \alpha, \mathbf{m} + k\alpha \rangle \end{aligned} \quad (1.108)$$

Then, for second term, change the summation from $k \rightarrow k-1$, we cancel each term except the first and last.

$$\begin{aligned}
& N_{\alpha}(\mathbf{m} - (q+1)\alpha)N_{-\alpha}(\mathbf{m} - q\alpha) \\
& - N_{-\alpha}(\mathbf{m} + (p+1)\alpha)N_{\alpha}(\mathbf{m} + p\alpha) = \sum_{k=-q}^p \langle \alpha, \mathbf{m} + k\alpha \rangle
\end{aligned} \tag{1.109}$$

However, due to the finite order restriction, the left terms all vanish.

$$\begin{aligned}
(p+q+1)\langle \alpha, \mathbf{m} \rangle + \left(\frac{1}{2}p(p+1) - \frac{1}{2}q(q+1) \right) \langle \alpha, \alpha \rangle &= 0 \\
2 \frac{\langle \alpha, \mathbf{m} \rangle}{\langle \alpha, \alpha \rangle} &= q - p
\end{aligned} \tag{1.110}$$

So what' s the procedure to investigate the roots of a lie algebra? First, given a canonical matrix representation, one should find the Cartan subalgebra, how? We seek the maximal independent *diagonal* matrix set. Then seek the relation of \mathbf{H} and remaining elements, you may not pretty lucky to find direct scalar relation rather mixed terms. Then, you should make a combination like $v = a^i M_i$, and try to build by $[H_i, a^j M_j] = \lambda_i a^j M_j$ exactly. Given each coefficient with normalization, you may find a set of $\delta_{i,j} + \delta_{i+1,j} + \dots$ for possible slots. Summarize, it can be evaluated for $\alpha(H_i)$ for i index of this vector, it indicates to extract the i set to form a basis combination by $\delta_{ij} + \delta_{i+1,j} \sim e_j$ etc, which is the root.

142 $A_n = \mathfrak{sl}(n+1)$

Given \mathbf{e}_i for $i = 1, \dots, n+1$ in \mathbb{R}^{n+1} parametrize the lie algebra $\mathfrak{sl}(n+1)$. Thus the matrix is represented by $(M_{ij})_{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta}$ which Kronecker index connects the matrix slot and generator index.

It' s more natural to just seek a matrix summation to yield results, rather a direct index summation is false. Looking at the jk summation yielding:

$$M_{i,j} M_{k,l} = \delta_{j,k} M_{i,l} \tag{1.111}$$

$$[M_{i,j}, M_{k,l}] = \delta_{j,k} M_{i,l} - \delta_{i,l} M_{j,k} \tag{1.112}$$

Given $\text{Tr}(\mathfrak{sl}(n+1)) = 0$, we should take the maximal diagonal and traceless basis set, then we choose:

$$\begin{aligned}
H_i &= M_{i,i} - M_{i+1,i+1} \\
[H_i, M_{k,l}] &= \delta_{i,k} M_{i,l} - \delta_{i+1,k} M_{i+1,l} + \delta_{i,l} M_{k,i} - \delta_{i+1,l} M_{k,i+1} \\
&= (\delta_{i,k} - \delta_{i,l} - \delta_{i+1,k} + \delta_{i+1,l}) M_{k,l}
\end{aligned} \tag{1.113}$$

Now, be clever to extract the basis of i index to form simple root set.

$$\alpha = \mathbf{e}_k - \mathbf{e}_l \quad E_{\alpha} = M_{k,l} \tag{1.114}$$

$$\begin{aligned}
[H_i, E_{\alpha}] &= \alpha(H_i) E_{\alpha} \\
\alpha(H_i) &= (\delta_{i,k} - \delta_{i,l}) - (\delta_{i+1,k} - \delta_{i+1,l})
\end{aligned} \tag{1.115}$$

Where we choose root as $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ $i = 1, \dots, n$ to expand into more clarified basis set.

$$\mathbf{e}_k - \mathbf{e}_l = \sum_k^{l-1} (\mathbf{e}_i - \mathbf{e}_{i+1}) \quad (1.116)$$

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{if } |i - j| \geq 2 \end{cases} \quad (1.117)$$

We only care about the normalization relation upon the inner product, which build the relation between each basis. How to express this clearly? One named *Dynkin*, creates the diagram as follows:

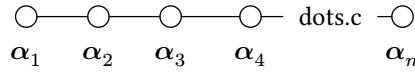


Figure 2. Dynkin diagram of $A_n = \mathfrak{sl}(n+1)$

Where the solid line connects node indicates $-\frac{1}{2} = \cos \frac{\pi}{3}$ by normalization of simple roots in i, j where $|i - j| = 1$. Thus, given a relation between nodes, we connects nodes by single or multiple solid lines to indicates different normalization angles. We could also express in matrix form by $C_{ij} = \langle \alpha_i, \alpha_j \rangle$ called *Cartan* matrix which is too sparse in expression compared to Dynkin diagram.

143 $D_n = \mathfrak{so}(2n)$

As for algebra $\mathfrak{so}(2n)$, the antisymmetric matrix should be denoted as $M_{ij} = -M_{ji} = -M_{ij}^T$ for $i, j = 1, \dots, 2n$.

$$(M_{ij})_{\alpha\beta} = \delta_{a,\alpha} \delta_{b,\beta} - \delta_{a,\beta} \delta_{b,\alpha} \quad (1.118)$$

We should notice that one should transpose the index to sum again in second term, thus given $(i, j)(k, l)$, we should merge j, k first and then i, l .

$$[M_{i,j}, M_{k,l}] = \delta_{j,k} M_{i,l} - \delta_{i,k} M_{j,l} + \delta_{l,i} M_{k,j} - \delta_{k,i} M_{l,j} \quad (1.119)$$

We define the Cartan subalgebra as, for maximal n ranks for $\mathfrak{so}(2n)$:

$$H_i = M_{2i-1, 2i} \quad i = 1, 2, \dots, n \quad (1.120)$$

Which is **not** diagonalizable actually, that it must mix four elements by previous calculation. If one need to diagonalize such basis, we should take in \mathbb{C} with i factor appended. Rank n is the consequence derived from maximally n independent rotation.

We aren't lucky enough to extract the scalar for a single basis, rather those are scalars distributing around different ones. Thus, we choose a vector format to calculate by choose a various $j, k = 1, 2, \dots, n$ basis:

$$\begin{aligned} \mathbf{M} &= aM_{2j-1, 2k-1} + bM_{2j, 2k-1} + cM_{2j-1, 2k} + dM_{2j, 2k} \\ [H_i, \mathbf{M}] &= a(\delta_{2i, 2j-1} M_{2i-1, 2k-1} - \delta_{2i-1, 2j-1} M_{2i, 2k-1} + \delta_{2k-1, 2i-1} M_{2i, 2j-1} - \delta_{2k-1, 2i} M_{2i-1, 2j-1}) + \dots \end{aligned} \quad (1.121)$$

We suddenly found that $2i \neq 2j - 1$ ever and ever because one is odd and one is even! We immediately reduce the index into $2i - 1, 2j - 1$ or $2k - 1, 2i - 1$. Then, we note that $\delta_{i,j} M_{2i,2k-1} = M_{2j,2k-1}, \delta_{k,i} M_{2i,2j-1} = -M_{2j-1,2k}$.

Thus we choose:

$$\begin{aligned}\alpha &= \eta \mathbf{e}_j + \eta' \mathbf{e}_k \quad j \neq k \quad \eta, \eta' = 1, -1 \\ E_\alpha &= \frac{1}{2} (M_{2j-1,2k-1} + \eta M_{2j,2k-1} + \eta' M_{2j-1,2k} - \eta \eta' M_{2j,2k}) \\ \alpha(H_i) &= \eta \delta_{i,j} + \eta' \delta_{i,k} \\ [H_i, E_\alpha] &= (\eta \delta_{i,j} + \eta' \delta_{i,k}) E_\alpha\end{aligned}\tag{1.122}$$

By counting numbers of generators, we found that they are $2n(n-1)$ for j, k root generators.

$$\begin{aligned}\alpha_i &= \mathbf{e}_i - \mathbf{e}_{i+1} \quad i = 1, \dots, n-1 \\ \alpha_n &= \mathbf{e}_{n-1} + \mathbf{e}_n\end{aligned}\tag{1.123}$$

Actually, choose arbitrary sign is acceptable, but such is preferable.

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{if } |i - j| = 0 \\ 0 & \text{if } i = n-1, j = n \end{cases}\tag{1.124}$$

We draw the *Dynkin* diagram as below:

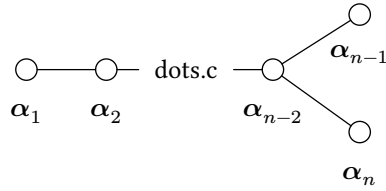


Figure 3. Dynkin diagram of $D_n = \mathfrak{so}(2n)$

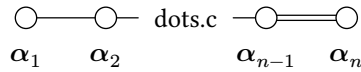


Figure 4. Dynkin diagram of $B_n = \mathfrak{so}(2n+1)$

144 Coincidence

$$\begin{aligned}\mathfrak{so}(3; \mathbb{C}) &= \mathfrak{sl}(2; \mathbb{C}) = \mathfrak{sp}(2; \mathbb{C}) \\ \mathfrak{so}(4; \mathbb{C}) &= \mathfrak{sl}(2; \mathbb{C}) \oplus \mathfrak{sl}(2; \mathbb{C}) \\ \mathfrak{so}(5; \mathbb{C}) &= \mathfrak{sp}(4; \mathbb{C}) \\ \mathfrak{so}(6; \mathbb{C}) &= \mathfrak{sl}(4; \mathbb{C})\end{aligned}\tag{1.125}$$

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$\mathfrak{su}(2) = \{X \in \mathfrak{gl}(2, \mathbb{C}) \mid \overline{X}^t + X = 0, \text{Tr}(X) = 0\}$ and $\mathfrak{so}(3)$ are all real lie algebras of dimension 3.

(a) The basis:

$$\xi_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \xi_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (1.126)$$

With $[\xi_k, \xi_l] = \varepsilon_{klm} \xi_m$.

$$\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \quad (1.127)$$

$$[\xi^2, \xi_k] = 0 \quad (1.128)$$

We construct a commute element which the maximal diagonalized state is $|b, m\rangle$.

$$\begin{aligned} \xi^2 |b, m\rangle &= b |b, m\rangle \\ \xi_3 |b, m\rangle &= m |b, m\rangle \end{aligned} \quad (1.129)$$

$$\xi^+ = \xi_1 + i\xi_2, \quad \xi^- = \xi_1 - i\xi_2 \quad (1.130)$$

$$[\xi_3, \xi^+] = \xi^+, \quad [\xi_3, \xi^-] = -\xi^- \quad (1.131)$$

$$[\xi^+, \xi^-] = 2\xi_3 \quad (1.132)$$

$$\xi_3 \xi^\pm |b, m\rangle = [\xi_3, \xi^\pm] |b, m\rangle + \xi^\pm \xi_3 |b, m\rangle = (m \pm 1) \xi^\pm |b, m\rangle \quad (1.133)$$

Thus we conclude that $\xi^\pm |b, m\rangle = C |b, m \pm 1\rangle$, $C \in \mathbb{C}$, and $\xi_3 |b, m \pm 1\rangle = (m \pm 1) |b, m \pm 1\rangle$.

We call such operators ladder operators or raising and lowering operators.

$$\xi^+ |b, j\rangle = 0 \quad \exists j \in ? \quad (1.134)$$

$$\xi^- \xi^+ = (\xi_1 - i\xi_2)(\xi_1 + i\xi_2) = \xi_1^2 + \xi_2^2 + i[\xi_1, \xi_2] = \xi^2 - \xi_3^2 + \xi_3 \quad (1.135)$$

$$0 = \xi^- \xi^+ |b, j\rangle = (b - j^2 + j) |b, j\rangle \rightarrow b = j(j+1) \quad (1.136)$$

Same for $\xi^+ \xi^-$, one has $b - j'^2 + j' = 0$

$$\begin{aligned} b &= j'(j' - 1) = -j'(-j' + 1) \\ &\rightarrow j' = -j \end{aligned} \quad (1.137)$$

We conclude that for m can be $-j, -j+1, \dots, j-1, j$, thus $2j \in \mathbb{N}$ is a integer for finite dimension.

$$\begin{aligned} j &= 0, \quad m = 0, \quad j(j+1) = 0 \\ j &= \frac{1}{2}, \quad m = \frac{1}{2}, -\frac{1}{2}, \quad j(j+1) = \frac{3}{4} \\ j &= 1, \quad m = 1, 0, -1, \quad j(j+1) = 2 \\ &\dots \end{aligned} \quad (1.138)$$

From the definition of $\xi^+ = \xi^{-\dagger}$ by its expansion, we have such normalization factor:

$$\begin{aligned}\langle b, m | \xi^{+\dagger} \xi^+ | b, m \rangle &= \langle b, m | \xi^- \xi^+ | b, m \rangle \\ &= \langle b, m | \xi^2 - \xi_3^2 - \xi_3 | b, m \rangle \\ &= b - m^2 - m = j(j+1) - m^2 - m = C^2\end{aligned}\tag{1.139}$$

$$C = \sqrt{j(j+1) - m(m+1)}\tag{1.140}$$

Same, one has for ξ^- that $\tilde{C} = \sqrt{j(j+1) - m(m-1)}$.

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Take the transformation $SO(3)$ as $GL(3; \mathbb{R})$ matrix representation act naturally on \mathbb{R}^3 equipped with function valued on it as a scalar form.

$$\begin{aligned}l_1(f)(x) &= f(l_1^{-1}x) = \frac{d}{dt} f \left(\exp \left(t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \Big|_{t=0} \\ &= \frac{d}{dt} f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \Big|_{t=0} \\ &= \frac{d}{dt} f \left(\begin{pmatrix} x_1 \\ x_2 \cos t + x_3 \sin t \\ -x_2 \sin t + x_3 \cos t \end{pmatrix} \right) \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) \cdot \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix} \\ &= x_3 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_3}\end{aligned}\tag{1.141}$$

If we have $x_1 = r \sin \theta \cos \phi, x_2 = r \sin \theta \sin \phi, x_3 = r \cos \theta$, we can construct the transformation upon θ, ϕ too.

$$\frac{\partial}{\partial x_i} = \frac{\partial r_i}{\partial x_i} \frac{\partial}{\partial r_i}\tag{1.142}$$

In differential geometry as a basis vector transformation, or familiar Jacobi.

$$\begin{aligned}l_1 &= x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \\ l_2 &= -\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \\ l_3 &= -\frac{\partial}{\partial \phi}\end{aligned}\tag{1.143}$$

$$\begin{aligned}
l^+ &= e^{-i\phi} \left(-i \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \phi} \right) \\
l^- &= e^{i\phi} \left(i \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \phi} \right)
\end{aligned} \tag{1.144}$$

Given the representation $F(\theta, \phi)(m)$ with the weight m chosen by l_3 , with the highest weight denoted as l :

$$l_3 F(\theta, \phi)(m) = -\frac{\partial}{\partial \phi} F(\theta, \phi)(m) = m F(\theta, \phi)(m) \tag{1.145}$$

Thus we have $F(\theta, \phi) \sim e^{m\phi} G(\theta)$, however, we have such restriction $\phi + 2\pi \sim \phi$, thus we must have $m \rightarrow im$ to match such period.

We can immediately decompose the representation scalar function by raising operator for the highest weight:

$$\begin{aligned}
\left(\frac{\partial}{\partial \theta} - l \cot \theta \right) G(\theta)(l) &= 0 \\
\ln G(\theta)(l) &= l \ln \sin \theta \\
G(\theta)(l) &= C_l \sin^l \theta
\end{aligned} \tag{1.146}$$

Where the constant is, for the representation of $2l + 1$ dimension in l weight. Apply lowering operator would giving us:

$$\begin{aligned}
F(\theta, \phi)(m) &= C_m (l^-)^{l-m} F(\theta, \phi)(l) \\
&= C_m \left(e^{-i\phi} \left(i \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \phi} \right) \right)^{l-m} e^{-il\phi} \sin^l \theta
\end{aligned} \tag{1.147}$$

However, one usually encompass the whole $2l + 1$ representation for arbitrary l as:

$$Y_l^m(\theta, \phi) = C_{lm}(\dots) \text{ as representation for } 2l + 1 \text{ dimension with weight } m \tag{1.148}$$

Casimir operator with commutative property acting on will give us the eigenvalue.

$$l^2 = l_1^2 + l_2^2 + l_3^2 \tag{1.149}$$

$$l^2 Y_l^m(\theta, \phi) = l(l+1) Y_l^m(\theta, \phi) \tag{1.150}$$

$$\begin{aligned}
l^2 &= l^- l^+ + i l_3 + l_3^2 \\
&= e^{i\phi} \left(i \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \phi} \right) e^{-i\phi} \left(-i \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \phi} \right) - i \frac{\partial}{\partial \phi} + \frac{\partial^2}{\partial \phi^2} \\
&= \frac{\partial^2}{\partial \theta^2} + i \frac{\partial}{\partial \theta} \left(\cot \theta \frac{\partial}{\partial \phi} \right) + e^{i\phi} \cot \theta \frac{\partial}{\partial \phi} \left(e^{-i\phi} \left(-i \frac{\partial}{\partial \theta} \right) \right) + e^{i\phi} \cot \theta \frac{\partial}{\partial \phi} \left(e^{-i\phi} \cot \theta \frac{\partial}{\partial \phi} \right) + (\dots)
\end{aligned} \tag{1.151}$$

Evaluate that:

$$\begin{aligned}
i \frac{\partial}{\partial \theta} \left(\cot \theta \frac{\partial}{\partial \phi} \right) &= \frac{i}{\sin^2 \theta} \frac{\partial}{\partial \phi} + i \cot \theta \frac{\partial^2}{\partial \theta \partial \phi} \\
e^{i\phi} \cot \theta \frac{\partial}{\partial \phi} \left(e^{-i\phi} \left(-i \frac{\partial}{\partial \theta} \right) \right) &= -\cot \theta \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial^2}{\partial \theta \partial \phi} \\
e^{i\phi} \cot \theta \frac{\partial}{\partial \phi} \left(e^{-i\phi} \cot \theta \frac{\partial}{\partial \phi} \right) &= -i \cot^2 \theta \frac{\partial}{\partial \phi} + \cot^2 \theta \frac{\partial^2}{\partial \phi^2}
\end{aligned} \tag{1.152}$$

The first term evaluate as:

$$\begin{aligned}
\text{First term} &= \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} + i \left(\frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi} - \cot^2 \theta \frac{\partial}{\partial \phi} \right) + \cot^2 \theta \frac{\partial^2}{\partial \phi^2} \\
&= \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} + i \frac{\partial}{\partial \phi} + \cot^2 \theta \frac{\partial^2}{\partial \phi^2}
\end{aligned} \tag{1.153}$$

$$\begin{aligned}
\text{LHS} &= \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} + \cot^2 \theta \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \phi^2} \\
&= \frac{1}{\sin \theta} \left(\frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial \theta} \right) \sin \theta + \frac{\partial}{\partial \theta} (\sin \theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \\
&= \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} = \Delta_{\Omega}
\end{aligned} \tag{1.154}$$

Which is the **Spherical Laplacian**¹.

This, actually can be formalize in such insight, that Laplacian as:

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \Delta_{\Omega} \tag{1.155}$$

Is rotation invariant given by $[\Delta_{\Omega}, l_i] = 0$. Which is same as Casimir operator by its **uniqueness** up to constant. Thus we decompose $L^2(S^2)$ into basis equipped with $Y_l^m(\theta, \phi)$.

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Given $SO(2)$, it contains one generator which rotates the plane:

$$l = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim l = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial \phi} \tag{1.156}$$

It's simple that only one element share the eigenvalue solely.

$$l|m\rangle = m|m\rangle \sim |m\rangle = F(\phi)(m) = e^{-im\phi} \tag{1.157}$$

With period condition induced.

$$SO(2) \rtimes \mathbb{R}^+ = E(2)$$

¹Actually, calculate by $l_1^2 + l_2^2 + l_3^2$ is more simpler, or use some notation reduction would be easier in burden.

$$\begin{pmatrix} R(\theta) & \mathbf{t} \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix} \quad (1.158)$$

$\mathfrak{e}(2)$

$$l \rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad p_x \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad p_y \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.159)$$

$$[l, p_x] = p_y \quad [l, p_y] = -p_x \quad [p_x, p_y] = 0 \quad (1.160)$$

$$\begin{aligned} l &\rightarrow x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial \phi} \\ p_x &\rightarrow \frac{\partial}{\partial x} \\ p_y &\rightarrow \frac{\partial}{\partial y} \end{aligned} \quad (1.161)$$

Given a Cartan subalgebra, we choose l , because the Cartan subalgebra must be a semi-simple decomposition thus we shouldn't take a nilpotent ideal which would **vanish in commutation**.

$$p^\pm := p_x \pm i p_y = \frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} = e^{\pm i\phi} \left(\frac{\partial}{\partial r} \pm \frac{i}{r} \frac{\partial}{\partial \phi} \right) \quad (1.162)$$

$$[l, p^\pm] = \pm p^\pm \quad [p^+, p^-] = 0 \quad (1.163)$$

Casimir

$$\begin{aligned} p^+ p^- - p^- p^+ &= 0 \\ [l, p^+] p^- + p^+ [l, p^-] &= 0 \\ [l, p^+ p^-] &= 0 \\ [l, p_x^2 + p_y^2] &= 0 \\ [l, p^2] &= 0 \end{aligned} \quad (1.164)$$

Thus, it's the Casimir operator, why there's no l^2 ? Because there's no element can generate l to absorb in. Thus we clarify such operator should be unique up to scalar, which is Δ_\perp in representation.

$$p^+ p^- = e^{i\phi} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right) (e^{-i\phi}) \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \phi} \right) \quad (1.165)$$

$$\begin{aligned}
& e^{i\phi} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right) (e^{-i\phi} B) \\
&= \frac{\partial}{\partial r} B + \frac{1}{r} B + \frac{i}{r} \frac{\partial}{\partial \phi} B
\end{aligned} \tag{1.166}$$

$$\begin{aligned}
p^+ p^- &= \frac{\partial^2}{\partial r^2} - \frac{i}{r} \frac{\partial^2}{\partial r \partial \phi} + \frac{i}{r^2} \frac{\partial}{\partial \phi} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{i}{r^2} \frac{\partial}{\partial \phi} + \frac{i}{r} \frac{\partial^2}{\partial \phi \partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}
\end{aligned} \tag{1.167}$$

We inspect that the commutation relation differs from the case in semi-simple algebra like $SU(2)$, which we can't generate the angular part from the translation. If you review the proof of weight boundedness of $SU(2)$ representation that Casimir operator build the connection with maximal and minimal weight, however, we can't reach the results in this one, indicating *unboundedness* of the weight.

$$p^\pm F(r, \phi, z)(m, k_z) = e^{\pm i\phi} \left(\frac{\partial}{\partial r} \pm \frac{i}{r} \frac{\partial}{\partial \phi} \right) F(r, \phi, z) \tag{1.168}$$

$$\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} = p^2 \tag{1.169}$$

Given by weight m , we extract terms like $e^{im\phi}$:

$$\begin{aligned}
(\Delta_\perp + \partial_z^2) F(r, \phi, z)(m, k_z) &= \lambda F(r, \phi, z)(m, k_z) \\
\Delta_\perp (e^{im\phi} R(r)(m)) &= (\lambda - k_z^2) (e^{im\phi} R(r)(m)) \\
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} + (k_z^2 - \lambda) \right) R(r)(m) &= 0
\end{aligned} \tag{1.170}$$

Which is called *Bessel Equation*, the solution is called *Bessel Function* $J_m(r)$ with m as weight notation. Notice that $k_z^2 - \lambda$ is a isolated scalar generated from z translation which could be reduced, we denote as $r \rightarrow \sqrt{k_z^2 - \lambda} r := kr$.

$$\begin{aligned}
G(r, \phi)(m) &= e^{im\phi} R(r)(m) = J_m(kr) e^{im\phi} \\
p^+(G(r, \phi)(m)) &= G(r, \phi)(m+1)
\end{aligned} \tag{1.171}$$

$$J_{m+1}(r) = \frac{m}{x} J_m(r) - J'_m(r) \tag{1.172}$$

However, we have no choice like $SU(2)$ to denote the minimal or maximal basis function and iterates for else. Thus, we can only acquire a iteration relation like above. You may expect the different solution or *differential seed* will inherit the same iteration property like above.

$$\begin{aligned}
[l, p^\pm] &= \pm p^\pm \\
e^{tl} p^\pm e^{-tl} &= e^{tD_l}(p^\pm) = \sum_n \frac{t^n}{n!} (\pm 1)^n p^\pm = e^{\pm t} p^\pm
\end{aligned} \tag{1.173}$$

We then set $t \rightarrow i\phi$ for period restriction.

$$\begin{aligned}
U(\phi) &:= e^{i\phi l} \\
U(\phi)|m\rangle &= e^{i\phi l}|m\rangle = e^{im\phi}|m\rangle
\end{aligned} \tag{1.174}$$

We thus immediately decompose it to the m -weight space, recall the representation decomposition, one has $\chi(g) = e^{im\phi}$:

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} e^{im\phi} d\phi &= 1_m \\
\frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} U(\phi)|m\rangle d\phi &= 1_{\text{id}} \\
\frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} U(\phi) d\phi &= \Pi_m
\end{aligned} \tag{1.175}$$

$$\text{Tr}(U(\phi)) = \langle m|U(\phi)|m\rangle = \sum_{m \in \mathbb{Z}} e^{im\phi} = 2\pi \sum_{m \in \mathbb{Z}} \delta(\phi - 2\pi n) = \sum_m \Pi_m = 1_{\text{id}} \tag{1.176}$$

$$\sum_m \Pi_m(e^{ikr \cos \phi}) = \sum_{m \in \mathbb{Z}} J_m(kr) e^{im\phi} \tag{1.177}$$

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$$S[*] : \mathcal{F} \rightarrow \mathbb{R}; \mathcal{F} = \{\mathbf{q}(t) : t \in [t_0, t_1] \subset \mathbb{R} \rightarrow \mathbb{R}^M\} \tag{1.178}$$

With differential set if applicable:

$$\mathcal{F}_\varepsilon = \{\delta \mathbf{q}(t); |\delta \mathbf{q}(t)| < \varepsilon; |\delta \dot{\mathbf{q}}(t)| < \varepsilon; \forall t \in [t_0, t_1] \subset \mathbb{R}\} \tag{1.179}$$

$$\delta S(\mathbf{q}, \dot{\mathbf{q}}) = 0 \rightarrow \delta \mathbf{q}(t_0) = \delta \mathbf{q}(t_1) = 0 \tag{1.180}$$

Given by a certain basis product form:

$$\begin{aligned}
|x\rangle \text{ or } |x + \delta\rangle? &\rightarrow \text{continuous basis} \\
\langle x|\psi\rangle &= \langle x|x'\rangle \langle x'|\psi\rangle
\end{aligned} \tag{1.181}$$

Thus φ should be a continuous basis expansion for the representation of coordinates:

$$\int d\mu \varphi(x|x') \leftrightarrow \langle x|x' \rangle$$

$$\int d\mu \varphi(x|x') \varphi(x'|x_0) \leftrightarrow \langle x|x' \rangle \langle x'|x_0 \rangle = \langle x|x_0 \rangle \leftrightarrow \delta(x'|x_0)$$
(1.182)

$$\varphi(x|x_0) = \varphi(x|x_{N-1}) \varphi(x_{N-1}|x_{N-2}) \cdots \varphi(x_1|x_0)$$

$$\varphi(x|x_0) = \lim_{N \rightarrow \infty} \prod_{\varepsilon}^{\infty} T(\varphi(x_{\varepsilon}))$$
(1.183)

Denotes $M(V; \mathbb{C})$ as set of non-degenerate complex-valued symmetric bilinear forms on V with non-negative definite real part, with dense set $M^\circ(V; \mathbb{C})$. Now fix a translation invariant volume form dx on V , thus we can define the determinant $\det(B)$ $B \in M^\circ(V; \mathbb{C})$.

Let $\mathcal{S}(V)$ be the *Schwartz space* of V where the smooth functions on V whose all derivatives are rapidly decaying at ∞ faster than any power of $|x|$, equipped with the dual space $\mathcal{S}^*(V)$ in natural inclusions $\mathcal{S}(V) \subset L^2(V) \subset \mathcal{S}'(V)$.

$$\mathcal{F} : \mathcal{S}(V) \rightarrow \mathcal{S}(V^*)$$

$$\mathcal{F}[g](p) = (2\pi)^{-\frac{d}{2}} \int_V g(x) e^{-i\langle p, x \rangle} dx$$
(1.184)

Isometry is given by $\mathcal{F}^2[g](x) = g(-x)$. By duality, such operator could be extended to $\mathcal{S}^*(V) \rightarrow \mathcal{S}^*(V^*)$ for tempered distribution. The function $\exp(-\frac{1}{2}B(x, x))$ for matrix $B \in M^\circ(V)$ belongs to $\mathcal{S}^*(V)$ if $\text{Re } B \geq 0$ by its decaying behavior.

For dimension 1, where a diagonalizable B :

$$\begin{aligned} \mathcal{F}\left(\exp\left(-\frac{1}{2}B(x, x)\right)\right) &= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-ipx - \frac{1}{2}ax^2\right) dx \\ &= (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}a^{-1}p^2\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}a(x + ia^{-1}p)^2\right) dx \\ &= (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}a^{-1}p^2\right) \int_{\mathbb{R} + ia^{-1}p} \exp\left(-\frac{1}{2}a(x + ia^{-1}p)^2\right) dx \\ &= a^{-\frac{1}{2}} \exp\left(-\frac{1}{2}a^{-1}p^2\right) \end{aligned}$$
(1.185)

We therefore could extend such matrix to diagonalizable by dense property that for arbitrary countable dimensions where a general results is given by:

$$\mathcal{F}\left(\exp\left(-\frac{1}{2}B(x, x)\right)\right) = (\det B)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}B^{-1}(p, p)\right)$$
(1.186)

That the $\det(B)^{-\frac{1}{2}} = \prod_i \lambda_i$ by eigenvalues which is trivial in dimension 1.

$$\begin{aligned}
I[g; B](\hbar) &= \frac{1}{\hbar^{\frac{1}{2}}} \int_V g(x) e^{-\frac{1}{2\hbar} B(x, x)} dx \quad \hbar \geq 0 \\
&= \int_V g(\hbar^{\frac{1}{2}} x) e^{-\frac{1}{2} B(x, x)} dx \quad x \rightarrow \frac{x}{\hbar^{\frac{1}{2}}}
\end{aligned} \tag{1.187}$$

$$I[g; B](0) = \int_V (\cdots) dx g(0) = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} g(0) \tag{1.188}$$

$$\begin{aligned}
I[g; B](\hbar) &= \left\langle g(\hbar^{\frac{1}{2}} x), \exp\left(-\frac{1}{2} B(x, x)\right) \right\rangle \\
&= \hbar^{-\frac{d}{2}} (\det B)^{-\frac{1}{2}} \left\langle \mathcal{F}[g](\hbar^{-\frac{1}{2}} p), \exp\left(-\frac{1}{2} B^{-1}(p, p)\right) \right\rangle \\
&= (\det B)^{-\frac{1}{2}} \left\langle \mathcal{F}[g](p), \exp\left(-\frac{\hbar}{2} B^{-1}(p, p)\right) \right\rangle
\end{aligned} \tag{1.189}$$

Where $\hbar \rightarrow 0 \rightsquigarrow \exp\left(-\frac{\hbar}{2} B^{-1}(p, p)\right) = 1$.

If $l \in V^*$ and $g \in \mathcal{S}(V)$, then:

$$\begin{aligned}
I[lg; B](\hbar) &= \left\langle l(\hbar^{\frac{1}{2}} x) g(\hbar^{\frac{1}{2}} x), \exp\left(-\frac{1}{2} B(x, x)\right) \right\rangle \\
&= \hbar^{\frac{1}{2}} \left\langle g(\hbar^{\frac{1}{2}} x), l(x) \exp\left(-\frac{1}{2} B(x, x)\right) \right\rangle \\
&= -\hbar^{\frac{1}{2}} \left\langle g(\hbar^{\frac{1}{2}} x), \partial_{B^{-1}l} \exp\left(-\frac{1}{2} B(x, x)\right) \right\rangle \\
&= \hbar^{\frac{1}{2}} \left\langle \partial_{B^{-1}l} g(\hbar^{\frac{1}{2}} x), \exp\left(-\frac{1}{2} B(x, x)\right) \right\rangle \\
&= \hbar \left\langle (\partial_{B^{-1}l} g)(\hbar^{\frac{1}{2}} x), \exp\left(-\frac{1}{2} B(x, x)\right) \right\rangle \\
&= \hbar I[\partial_{B^{-1}l} g; B](\hbar)
\end{aligned} \tag{1.190}$$

$$\frac{d}{d\hbar} (g(\hbar^{\frac{1}{2}} x)) = \frac{1}{2} \hbar^{-\frac{1}{2}} (x^* \cdot \partial_{\hbar^{\frac{1}{2}} x} g)(\hbar^{\frac{1}{2}} x) = \frac{1}{2} \hbar^{-\frac{1}{2}} \hbar^{-\frac{1}{2}} (x^* \cdot \partial_x g)(\hbar^{\frac{1}{2}} x) = \frac{1}{2} \hbar^{-1} (x^* \cdot \partial_x g)(\hbar^{\frac{1}{2}} x) \tag{1.191}$$

Where x^* is the covector compared to vector x . Compose two results we get a Laplacian operator:

$$I'[g; B](\hbar) = \frac{1}{2} \hbar^{-1} I[x^* \cdot \partial_x g](\hbar) = \frac{1}{2} \hbar^{-1} \hbar I[\partial_{B^{-1}x^*} \cdot \partial_x g](\hbar) = I\left[\frac{1}{2} \Delta_B g; B\right](\hbar) \tag{1.192}$$

By deduction, we can apply as much as possible that:

$$I^{(n)}[g; B](\hbar) = I\left[\left(\frac{1}{2} \Delta_B\right)^n; B\right](\hbar) \tag{1.193}$$

If moreover g vanishes at the origin to order $2n + 1$, then for every differential operator of order $\leq 2n$ annihilates g at 0, hence:

$$I_g(0) = I'_g(0) = \dots = I_g^{(n)}(0) = 0 \quad (1.194)$$

Assume f attains a global minimum at a unique point $c \in [a, b]$, s.t. $a < c < b$ and $f''(c) > 0$. To acquire more information, we could expand the function as:

$$f(x) = f(c) + \frac{1}{2}f''(c)(x - c)^2 + \dots \quad (1.195)$$

Or even better to take a variable transformation with equality by rescale and shift:

$$c \rightarrow 0 \quad f(c) = 0 \rightsquigarrow f(x) = \frac{M}{2}x^2 \quad (1.196)$$

If given a compact support that $\{c\} \in U$ where divide the function as $g = g_1 + g_2$ $g_1 \subset U, g_2 \subset [a, b] \setminus U$. Hence the integral of second term will rapidly decaying if we choose a enough huge neighborhood support around c . We left with the first term as:

$$I[g](\hbar) = \int_{-\infty}^{\infty} g\left(\hbar^{\frac{1}{2}}y\right)e^{-\frac{M}{2}y^2}dy \quad (1.197)$$

$$\hbar^{\frac{1}{2}}I[g; f] = \int_a^b g(x)e^{-\frac{f(x)}{\hbar}}dx = \hbar^{\frac{1}{2}}e^{-\frac{f(c)}{\hbar}}I[\tilde{g}; B = (\cdot)^2] \quad (1.198)$$

It actually show a general strategy to expand the integral by:

$$\begin{aligned} c = 0 \quad f(x) &= \frac{1}{2}p(x)^2 \\ p'(0) &= \sqrt{f''(0)} > 0 \end{aligned} \quad (1.199)$$

$$\int_{-\infty}^{\infty} g(x)e^{-\frac{p(x)^2}{2\hbar}}dx \sim \hbar^{\frac{1}{2}} \int_{-\infty}^{\infty} \tilde{g}\left(\hbar^{\frac{1}{2}}y\right)e^{-\frac{y^2}{2}}dy \quad (1.200)$$

$$\begin{aligned} \frac{p(x)}{\hbar^{\frac{1}{2}}} &= y \\ dx &= dp^{-1}\left(\hbar^{\frac{1}{2}}y\right) = (p^{-1})'\left(\hbar^{\frac{1}{2}}y\right)\hbar^{\frac{1}{2}}dy \end{aligned} \quad (1.201)$$

$$\begin{aligned} p(p^{-1}(x)) &= x \\ p'(p^{-1}(x)) \cdot (p^{-1})'(x) &= 1 \\ (p^{-1})'(x) &= \frac{1}{p'(p^{-1}(x))} \end{aligned} \quad (1.202)$$

$$\tilde{g}(y) := g(p^{-1}(y)) (p^{-1})'(y) = \frac{g(p^{-1}(y))}{p'(p^{-1}(y))} \quad (1.203)$$

We could expand \tilde{g} by Taylor polynomials to acquire a series of summation.

Notice that any odd order polynomials expansion will vanish due to the whole space integral. Thus we can freely expand in $O(\hbar^n)$ order as:

$$\hbar^{\frac{1}{2}} I[g; f] = \hbar^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} I[\tilde{g}; B] = (2\pi\hbar)^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} \sum_{n \geq 0} a_n \hbar^n \quad (1.204)$$

The Gaussian integral can be expanded as $\int_{-\infty}^{\infty} y^{2m} e^{-\frac{y^2}{2}} dy = (2\pi)^{\frac{1}{2}} (2m-1)!!$, So we can exploit the constant that $\tilde{g}(y) = \sum_{n \geq 0} b_n y^n \rightarrow a_n = b_{2n} (2m-1)!!$.

$$\int_a^b g(x) e^{\frac{if(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{\frac{if(c)}{\hbar}} e^{\frac{\pm \pi i}{4}} I[g](\hbar) \quad (1.205)$$

$$\begin{aligned} I(\hbar) &= \sum_{n \geq 0} a_n \hbar^n = \sum_{n \geq 0} a_n \frac{\hbar^n}{n!} n! \\ &= \sum_{n \geq 0} a_n \frac{\hbar^n}{n!} \int_0^\infty u^n e^{-u} du \\ &= \int_0^\infty g(\hbar u) e^{-u} du \quad g(\hbar) = \sum_{n \geq 0} a_n \frac{\hbar^n}{n!} \\ &= \int_{-\infty}^\infty |v| g(\hbar v^2) e^{-v^2} dv \quad u \rightarrow v^2 \\ &= \hbar^{-\frac{1}{2}} \int_{-\infty}^\infty \tilde{g}(\hbar^{\frac{1}{2}} v) e^{-v^2} dv \quad \tilde{g}(v) = |v| g(v^2) \end{aligned} \quad (1.206)$$

$$\begin{aligned} \Gamma(s+1) &= \int_0^\infty t^s e^{-t} dt \quad s > 0 \\ &= s^{s+1} \int_0^\infty x^s e^{-sx} dx \quad t \rightarrow sx \\ &= s^{s+1} \int_0^\infty e^{-s(x - \ln x)} dx \quad \hbar \rightarrow \frac{1}{s}, f(x) \rightarrow x - \ln x, g(x) \rightarrow 1 \\ &= s^{s+\frac{1}{2}} e^{-s} (2\pi)^{\frac{1}{2}} \sum_{n \geq 0} a_n \left(\frac{1}{s}\right)^n \quad f(c) = f(1) = 1 \end{aligned} \quad (1.207)$$

$$\begin{aligned} p(x) &= \sqrt{2(x - \ln x)} = \sqrt{2 \left(x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots \right) \right)} \\ &= x \sqrt{1 - \frac{2x}{3} + \frac{x^2}{2} + \dots} = x - \frac{x^2}{3} + \frac{7x^3}{36} + \dots \end{aligned} \quad (1.208)$$

$$\begin{aligned}
p^{-1}(x) &= x + \frac{x^2}{3} + \frac{x^3}{36} + \dots \\
(p^{-1})'(x) &= 1 + \frac{2x}{3} + \frac{x^2}{12} + \dots
\end{aligned} \tag{1.209}$$

$$\tilde{g}(y) = p'^{-1}(y) \rightarrow a_0 = 1, a_1 = \frac{1}{12}(2-1)!! = \frac{1}{12} \tag{1.210}$$

Multidimensional steepest descent formula:

$$\hbar^{\frac{d}{2}} I^D[g; f] = \int_D g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{-\frac{f(c)}{\hbar}} I[g; B] \tag{1.211}$$

$$\begin{aligned}
T &\rightarrow \Pi(T) \\
|\Pi_k| &= \frac{(2k)!}{2^k k!} = \frac{(2k)!}{(2k)!!} = (2k-1)!!
\end{aligned} \tag{1.212}$$

$$\langle l_1 \dots l_N \rangle := \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D l_1(x) \dots l_N(x) e^{-\frac{S(x)}{\hbar}} dx \tag{1.213}$$

$$S(x) = \frac{B(x, x)}{2} + \tilde{S}(x) = \frac{B(x, x)}{2} - \sum_{i \geq 0} g_i \frac{B_i(x, \dots, x)}{i!} \tag{1.214}$$

$$Z = I[1; S(x)] = \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx \tag{1.215}$$

$$\begin{aligned}
\mathbf{n} &= (n_0, n_1, n_2, \dots) \\
F_\Gamma &= \prod_i g_i^{n_i} \cdot \tilde{F}_\Gamma
\end{aligned} \tag{1.216}$$

$$\begin{aligned}
x &\rightarrow y = \hbar^{-\frac{1}{2}} x \\
B_i(x, \dots, x) &\rightarrow \hbar^{\frac{i}{2}} B_i(x, \dots, x)
\end{aligned} \tag{1.217}$$

$$Z_{\mathbf{n}} = \int_V e^{-\frac{B(y, y)}{2}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \left(\hbar^{\frac{i}{2}-1} B_i(y, \dots, y) \right)^{n_i} dy$$

Contract each vertex.

$$-\chi(\Gamma) = E(\Gamma) - V(\Gamma) = \frac{1}{2} \sum_{i \geq 0} n_i i - \sum_{i \geq 0} n_i = \sum_{i \geq 0} n_i \left(\frac{i}{2} - 1 \right) \tag{1.218}$$

$$Z_n = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \sum_{\sigma \in \Pi(T_n)} \tilde{F}(\sigma) \tag{1.219}$$

1. permutation of "flowers" of given valency: S_{n_i}
2. permutation of the i th edges inside each i -valent "flower" : $S_i^{n_i}$

$$\begin{aligned}\mathbb{G}_{\mathbf{n}} &= \prod_i (S_{n_i} \ltimes S_i^{n_i}) \\ |\mathbb{G}_{\mathbf{n}}| &= \prod_i (n_i!)(i!)^{n_i}\end{aligned}\tag{1.220}$$

The group $\mathbb{G}_{\mathbf{n}}$ acts on the set $\Pi(T_{\mathbf{n}})$, which the stabilizer of a given matching is $\text{Aut}(\Gamma)$.

$$N_{\Gamma} = \frac{\prod_i (n_i!)(i!)^{n_i}}{|\text{Aut}(\Gamma)|}\tag{1.221}$$

$$\sum_{\sigma \in \Pi(T_{\mathbf{n}})} \tilde{F}(\sigma) = \sum_{\Gamma} \frac{\prod_i (n_i!)(i!)^{n_i}}{|\text{Aut}(\Gamma)|} F_{\Gamma} = \sum_{\Gamma} N_{\Gamma} \tilde{F}_{\Gamma}\tag{1.222}$$

$$\chi(\Gamma) = \beta_0(\Gamma) - \beta_1(\Gamma) = V - E\tag{1.223}$$

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}} = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G(\mathbf{n})} \frac{\tilde{F}_{\Gamma}}{|\text{Aut}(\Gamma)|}\tag{1.224}$$

We know that such exponential can be expanded as infinite sum of expectation of products.

$$\langle e^l \rangle := \sum_{n \geq 0} \frac{1}{n!} \langle l^n \rangle = \hbar^{\frac{d}{2}} \int_V e^{l(x) - \frac{S(x)}{\hbar}} dx\tag{1.225}$$

We immediately inspect that the infinite sum of $Z_{\mathbf{n}}$ is exact the same as the exponential expansion of Z , a specific case for $B_1(\cdot) = l(\cdot)$ as the external vertices plus a arbitrary action of $\frac{S(x)}{\hbar}$ to induce the Feynman amplitude for $F_{\Gamma}(l_1, \dots, l_N)$ $\Gamma \in G(\mathbf{n})$.

Given below diagram, one have two external vertices and a loop, also with a 3-valent flower and 4-valent flower too.

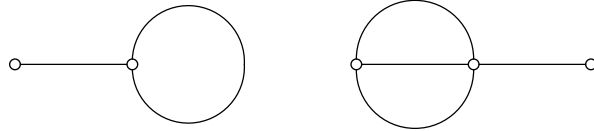


Figure 5. A graph with separated components.

$$\begin{aligned}B_3 &= \sum_i b_i^1 \otimes b_i^2 \otimes b_i^3 \\ B_4 &= \sum_j c_j^1 \otimes c_j^2 \otimes c_j^3 \otimes c_j^4\end{aligned}\tag{1.226}$$

In expansion of B_3 and B_4 exponential will generate the 3-valent and 4-valent flowers, plus two external vertices l_1, l_2 , all which will be contracted by the edges B_2 .

$$F_\Gamma(l_1, l_2) = \sum_i B^{-1}(l_1, b_i^1) B^{-1}(b_i^2, b_i^3) \sum_{ij} B^{-1}(b_i^1, c_j^1) B^{-1}(b_i^2, c_j^2) B^{-1}(b_i^3, c_j^3) B^{-1}(c_j^4, l_2) \quad (1.227)$$

Suppose the coefficients $g_i = 1$, we acquire the general expression for expectation as sum of all possible graphs:

$$\langle l_1 \dots l_N \rangle = \sum_{\Gamma \in G(N)} \frac{\hbar^{-\chi(\Gamma)}}{|\text{Aut}(\Gamma)|} \tilde{F}_\Gamma(l_1, \dots, l_N) \quad (1.228)$$

Where we reduce the product of $\prod_i (g_i \hbar^{\frac{i}{2}-1})$ in $\hbar^{-\chi(\Gamma)}$ by the Euler characteristic.

Now, suppose in 1-dimension, given a examples as simple as possible that $B(y, y) = y^2$, $B_i = z^i$, $g_i = g$, $\hbar = 1$:

$$Z = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + g \sum_{i \geq 0} \frac{(zx)^i}{i!}\right) dy = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + ge^{zx}\right) dy \quad (1.229)$$

It actually contains infinite many vertices with arbitrary **valency**, we expand the second term ge^{zx} :

$$Z = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) \sum_{n \geq 0} \frac{(ge^{zy})^n}{n!} dy = (2\pi)^{-\frac{1}{2}} \sum_{n \geq 0} \frac{g^n}{n!} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + nzy\right) dy \quad (1.230)$$

We can explain that $z(\cdot)$ pair a vertex, thus n vertices will contribute n factors of z as $z(\cdot) + z(\cdot) + \dots$:

$$\int \exp\left(-\frac{y^2}{2} + nzy\right) dy = \exp\left(\frac{n^2 z^2}{2}\right) \int \exp\left(-\frac{1}{2}(y - nzy)^2\right) dy = (2\pi)^{\frac{1}{2}} \exp\left(\frac{n^2 z^2}{2}\right) \quad (1.231)$$

$$Z = \sum_{n \geq 0} \frac{g^n}{n!} \exp\left(\frac{n^2 z^2}{2}\right) = \sum_{n \geq 0} \frac{g^n}{n!} \sum_{k \geq 0} \frac{(n^2 \frac{z^2}{2})^k}{k!} = \sum_{n, k \geq 0} \frac{g^n n^{2k}}{2^k k! n!} z^{2k} = \sum_{n \geq 0} g^n \sum_{\Gamma \in G(n)} \frac{\tilde{F}_\Gamma}{|\text{Aut}(\Gamma)|} \quad (1.232)$$

Given a number of vertices n with contribution zy to each $B^{-1}(zy, zy) = z^2 B^{-1}(y, y) = z^2$. Thus a general graph with k edges contribute $F_\Gamma = z^{2k} \times (B^{-1})^k = z^{2k}$. It can also be evaluated by expanding $\exp(nzy)$ then integrating. Therefore, we can identify the $|\text{Aut}(\Gamma)|$ too:

$$\sum_{\Gamma \in G(n)} (\dots) = \sum_k \sum_{\Gamma \in G(n; k)} \frac{1}{|\text{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k! n!} \quad (1.233)$$

To scrutinize our answer in combinatorics, we pick out two vertices from n vertices with n^2 choice with order. For list of k edges, we have $n^2 \cdot n^2 \cdot \dots = n^{2k}$ choices. To cancel out the order, we first remove orientation of edges

²It should also be considered as the contraction order from 1 to k , so if we pick out $(a_1, b_1), (a_2, b_2)$, it's also reasonable to pick out $(a_2, b_2), (a_1, b_1)$ which remove the contribution of edges order.

which contribute $2 \cdot 2 \cdot 2 \cdots = 2^k$ choices. Then we remove the permutation of edges² which contribute $k!$ choices. We still treat the vertices as labelled, so remove the permutation of vertices which contribute $n!$ choices.

Try to decompose the graph to connected components, we denote $\Gamma = \bigsqcup_{j=1}^r \Gamma_j^{k_j}$ with \mathbf{k} as tuple of number of copies of each connected components.

$$\begin{aligned} F_{\Gamma_1 \sqcup \Gamma_2} &= F_{\Gamma_1} \times F_{\Gamma_2} \\ \chi(\Gamma_1 \sqcup \Gamma_2) &= \chi(\Gamma_1) + \chi(\Gamma_2) \\ |\text{Aut}(\Gamma_1^{k_1} \sqcup \Gamma_2^{k_2} \cdots \sqcup \Gamma_r^{k_r})| &= \prod_{j=1}^r |\text{Aut}(\Gamma_j)|^{k_j} k_j! \end{aligned} \quad (1.234)$$

Here $k_j!$ of the third equality comes from the permutation of identical connected components.

Given a connected graph set $G_c(*)$, $\Gamma = \bigsqcup_{\gamma \in G_c(*)} \gamma^{k_\gamma}$ where the index is same as the graph itself for simplicity.

$$w(\Gamma) := \frac{F_\Gamma}{|\text{Aut}(\Gamma)|} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \quad (1.235)$$

$$\frac{\tilde{F}_\Gamma}{|\text{Aut}(\Gamma)|} = \prod_{\gamma \in G_c(*)} \left(\frac{\tilde{F}_\gamma}{|\text{Aut}(\gamma)|} \right)^{k_\gamma} \frac{1}{k_\gamma!} \quad (1.236)$$

We decompose the second product $\prod_i (\cdots)$ too, therefore we reduce the terms as:

$$w(\Gamma) = \prod_{\gamma \in G_c(*)} \frac{w(\gamma)^{k_\gamma}}{k_\gamma!} \quad (1.237)$$

$$Z = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \sum_{\mathbf{n}} \sum_{\Gamma \in G(\mathbf{n})} w(\Gamma) = Z_0 \sum_{\Gamma \in G(*)} w(\Gamma) \quad Z_0 = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \quad (1.238)$$

Where we just reduce our previous expansion of Z in a more concise expression.

$$\frac{Z}{Z_0} = \sum_{\Gamma \in G(*)} w(\Gamma) = \sum_{k_\gamma} \prod_{\gamma \in G_c(*)} \frac{w(\gamma)^{k_\gamma}}{k_\gamma!} = \exp \left(\sum_{\gamma \in G_c(*)} w(\gamma) \right) \quad (1.239)$$

For any summation of arbitrary graphs in $G(*)$, we can decompose it to connected components with reordering $\mathbf{n} \rightarrow \mathbf{k} \sim \sum_{\mathbf{n}} \prod_i \rightarrow \sum_{\mathbf{k}} \prod_\gamma$ by counting the flowers into counting the connected components.

$$\ln \frac{Z}{Z_0} = \sum_{\gamma \in G_c(*)} w(\gamma) = \sum_{\gamma \in G_c(*)} \frac{\tilde{F}_\gamma}{|\text{Aut}(\gamma)|} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i(\gamma)} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1}) \sum_{\gamma \in G_c(\mathbf{n})} \frac{\tilde{F}_\gamma}{|\text{Aut}(\gamma)|} \quad (1.240)$$

A spanning-tree $T \subset \Gamma$ contains all V vertices and without any cycles, thus a spanning-tree on V vertices contains exactly $V - 1$ edges. Thus the extra edges will contribute exact the same number of loops.

$$\beta_1(\Gamma) = 1 - \chi(\Gamma) = E - V + 1 \quad (1.241)$$

$$\begin{aligned}\beta_0(\Gamma) &= 1 \text{ for connected graph} \\ \beta_1(\Gamma) &= E - V + 1\end{aligned}\tag{1.242}$$

It suggests $\hbar^{-\chi(\gamma)}$ will contribute in different order of $O(\hbar)$. For **tree**, $\beta_1 = 0$, thus a connected graph contribute $-\chi(\gamma) = -(V - E) = -(\beta_0 - \beta_1) = -1 \sim O(\hbar^{-1})$ order. For **1-loop** graph, $\beta_1 = 1$, thus contribute $O(\hbar)$ order, and so forth. Given a set of classes of graphs with j loops:

$$\hbar^{j-1} \left(\ln \frac{Z}{Z_0} \right)_j := \sum_{\mathbf{n}} \prod_i (g_i^{n_i(\gamma)} \hbar^{j-1} \sum_{\gamma \in G_c(\mathbf{n})} \frac{\tilde{F}_\gamma}{|\text{Aut}(\gamma)|})\tag{1.243}$$

$$\ln \frac{Z}{Z_0} = \sum_{j \geq 0} \left(\ln \frac{Z}{Z_0} \right)_j \hbar^{j-1}\tag{1.244}$$

$$Z := \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx = e^{-\frac{S(x_0)}{\hbar}} I[1; B = B_2] = e^{-\frac{S(x_0)}{\hbar}} (\det B)^{\frac{1}{2}} \det(S''(x_0))^{-\frac{1}{2}} \sum_{i \geq 0} a_i \hbar^i\tag{1.245}$$

By steepest descent formula in the neighborhood of critical point x_0 where $S'(x_0) = 0$.

$$\log \frac{Z}{Z_0} = -\frac{S(x_0)}{\hbar} + \frac{1}{2} \log \frac{\det(B)}{\det(S''(x_0))} + \log \left(\sum_{i \geq 0} a_i \hbar^i \right)\tag{1.246}$$

Often, the first term is called *classical approximation* by physicists, the second term is called *1-loop approximation* etc.

$$\begin{aligned}S'(x) &= 0 \\ \left(\frac{1}{2} B(x, x) - \sum_{i \geq 3} g_i B_i \frac{x, \dots, x}{i!} \right)' &= B(x, \cdot) - \sum_{i \geq 3} g_i \frac{B_i(x, \dots, x, \cdot)}{(i-1)!} = 0 \\ \sum_{i \geq 3} g_i B^{-1} \frac{B_i(x, \dots, x, \cdot)}{(i-1)!} &:= \beta(x) = x \quad B^{-1} : V^* \rightarrow V\end{aligned}\tag{1.247}$$

In the sense of power series norm, β is a contraction mapping in a neighborhood of 0.

$$x_0 = \lim_{n \rightarrow \infty} \beta^n(x)\tag{1.248}$$

- Each application of β produces a new **vertex** with $i - 1$ incoming arguments and one outgoing edge.
- B^{-1} contracts the open leg (argument of the slot $B(x, \dots, -)$) with the upper vertex.

$$x^2 = \beta(x^1) = \sum_i g_i B^{-1} \frac{B_i(x^1, \dots, x^1)}{(i-1)!}\tag{1.249}$$

$$x_0 = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G^0(\mathbf{n})} \frac{\tilde{F}_T}{|\text{Aut}(T)|}\tag{1.250}$$

Where the graph Γ is a *tree* with one external vertex. Take the convergence results x_0 back to the action $S(x) \rightarrow S(x_0)$. To explain this, $\frac{B(x_0, x_0)}{2}$ corresponds to gluing two trees with both single external vertices; $\frac{1}{i!} B_i(x_0, \dots, x_0)$ corresponds to gluing i trees with external vertices together into a i -valent flower. We can think of the first term as counting tree E times (once per edge), and rest of terms as counting each tree V times (once per vertex), therefore $-S(x_0) = V - E = 1$ (per tree). So the tree summation contribution is $-S(x_0)$.

$$S(x) := \frac{x^2}{2} - gh(x) \quad h(x) = \sum_{n \geq 0} c_n x^n \text{ with } c_1 \neq 0 \quad g \text{ is constant} \quad (1.251)$$

$$\begin{aligned} S'(x) &= 0 \\ x_0 &= gh'(x_0) \\ \frac{x_0}{h'(x_0)} &= g \\ \left(\frac{x_0}{h'(x_0)} \right)^{-1} \frac{x_0}{h'(x_0)} &= \left(\frac{x_0}{h'(x_0)} \right)^{-1} (g) \\ x_0 &= f(g) \quad \left(\frac{x_0}{h'(x_0)} \right)^{-1} = f \end{aligned} \quad (1.252)$$

$$-S(x_0) = F(g) = -\frac{f(g)^2}{2} + gh(f(g)) \quad (1.253)$$

$$F'(g) = -f(g)f'(g) + h(f(g)) + gh'(f(g))f'(g) \quad (1.254)$$

But $x_0 = gh'(x_0) \rightarrow f(g) = gh'(f(g)) \rightarrow h'(f(g)) = \frac{f(g)}{g}$, first and third terms cancel each other.

$$F'(g) = h(f(g)) \rightarrow -S(x_0) = \int_0^g h(f(a)) da \quad (1.255)$$

Consider $B_i = 1, g_i = g$ with $S(x) = \frac{x^2}{2} - ge^x$, expanding e^x will give the all flowers or external edges for us to compute the number of trees.

$$\sum_{n \geq 0} g^n \sum_{\gamma \in G_c^0(n)} \frac{1}{|\text{Aut}(\gamma)|} = -S(x_0) \quad (1.256)$$

$$S'(x) = 0 \rightarrow x = ge^x \rightarrow x_0 = f(g) = \left(\frac{x}{e^x} \right)^{-1} (g) \quad (1.257)$$

$$\frac{f(x)}{e^{f(x)}} = x \rightarrow e^{f(x)} = \frac{f(x)}{x} \quad (1.258)$$

$$-S(x_0) = \int_0^g h(f(a)) da \rightarrow \int e^{f(a)} da \rightarrow \int \frac{f(a)}{a} da \quad (1.259)$$

Now we need to evaluate the coefficients of Taylor series of $f(x)$ to acquire the count of summation of automorphism class of graphs.

$$f(g) = \sum_{n \geq 1} a_n g^n$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(g)}{g^{n+1}} dg \xrightarrow{g \rightarrow f^{-1}(x)} \frac{1}{2\pi i} \oint \frac{x}{(xe^{-x})^{n+1}} d(xe^{-x}) = \frac{1}{2\pi i} \oint e^{nx} \frac{1-x}{x^n} dx \quad (1.260)$$

We take residue in a simpler way, expand $e^{nx} = \sum_{i \geq 0} \frac{(nx)^i}{i!}$, we immediately see the residue equal to $\frac{n^i}{i!} x^{i-k}$ $i - k = -1$. Therefore $k = n$ or $n - 1$, we get:

$$a_n = \frac{n^{n-1}}{(n-1)!} - \frac{n^{n-2}}{(n-2)!} = \frac{n^{n-2}}{(n-1)!} \quad (1.261)$$

$$-S(x_0) = \int_0^a \frac{f(a)}{a} da = \sum_{n \geq 1} \frac{n^{n-2}}{(n-1)!} \int_0^g a^{n-1} da = \sum_{n \geq 1} \frac{n^{n-2}}{n!} g^n \quad (1.262)$$

$$-S(x_0) = \sum_{n \geq 1} \frac{n^{n-2}}{n!} g^n = \sum_{n \geq 0} g^n \sum_{\gamma \in G_c^0(n)} \frac{1}{|\text{Aut}(\gamma)|}$$

$$\frac{n^{n-2}}{n!} = \sum_{\gamma \in G_c^0(n)} \frac{1}{|\text{Aut}(\gamma)|} \quad (1.263)$$

We conclude that the number of labelled tree of n vertices is n^{n-2} as $\frac{n!}{|\text{Aut}(\gamma)|}$ non-isomorphic labellings, The theorem is called *Cayley* theorem.

Consider now $B_i = 1, g_i = g$ with $S(x) = \frac{x^2}{2} - g\left(x + \frac{x^3}{6}\right)$ which is the set of graphs composed by 1-valent and 3-valent flowers. However, we can directly calculate the fixed point by solving algebraically.

$$S'(x) = 0 \rightarrow x = g \left(1 + \frac{x^2}{2} \right)$$

$$x_0 = \frac{1 - (1 - 2g^2)^{\frac{1}{2}}}{2} \quad (1.264)$$

$$\begin{aligned}
-S(x_0) &= \int_0^g \frac{1 - (1 - 2a^2)^{\frac{1}{2}}}{a} + \frac{\left(1 - (1 - 2a^2)^{\frac{1}{2}}\right)^3}{6a^3} da \\
&= \int_0^g \frac{1}{6a^3} \left(6a^2 - 6a^2(1 - 2a^2)^{\frac{1}{2}} + 1 + 3(1 - 2a^2) - 3(1 - 2a^2)^{\frac{1}{2}} - (1 - 2a^2)^{\frac{3}{2}}\right) da \\
&= \int_0^g \frac{1}{6a^3} \left(4 - (6a^2 + 3 + 1 - 2a^2)(1 - 2a^2)^{\frac{1}{2}}\right) da \\
&= \frac{2}{3} \int_0^g \frac{1}{a^3} \left(1 - (1 + a^2)(1 - 2a^2)^{\frac{1}{2}}\right) da \quad \text{integrate by part } dv = \frac{1}{a^3} \\
&= \frac{2}{3} \left(-\frac{1}{2a^2}\right) \Big|_0^g - \frac{2}{3} \left(1 + a^2\right)(1 - 2a^2)^{\frac{1}{2}} \left(-\frac{1}{2a^2}\right) \Big|_0^g \\
&\quad - \frac{2}{3} \int_0^g -\frac{1}{2a^2} \left(2a(1 - 2a^2)^{\frac{1}{2}} + (1 + a^2) \cdot \frac{1}{2}(1 - 2a^2)^{-\frac{1}{2}} \cdot (-4a)\right) da
\end{aligned} \tag{1.265}$$

We focus on third term in:

$$\begin{aligned}
du &= 2a(1 - 2a^2)^{\frac{1}{2}} - 2a(1 + a^2)(1 - 2a^2)^{-\frac{1}{2}} \\
&= \frac{2a(1 - 2a^2) - 2a(1 + a^2)}{(1 - 2a^2)^{\frac{1}{2}}} \\
&= \frac{6a^3}{(1 - 2a^2)^{\frac{1}{2}}}
\end{aligned} \tag{1.266}$$

$$\text{third term} = \int_0^g \frac{2a}{(1 - 2a^2)^{\frac{1}{2}}} da = \int_0^g \frac{1}{(1 - 2a^2)^{\frac{1}{2}}} da^2 = -(1 - 2a^2)^{\frac{1}{2}} \Big|_0^g \tag{1.267}$$

Combine two terms together:

$$\begin{aligned}
I &= \frac{1}{3a^2} \left((1 + a^2)(1 - 2a^2)^{\frac{1}{2}} - 1 - 3a^2(1 - 2a^2)^{\frac{1}{2}}\right) \\
&= \frac{1}{3a^2} \left((1 - 2a^2)(1 - 2a^2)^{\frac{1}{2}} - 1\right) \\
&= \frac{1}{3a^2} \left((1 - 2a^2)^{\frac{3}{2}} - 1\right)
\end{aligned} \tag{1.268}$$

$$\begin{aligned}
I|_0^g &= \frac{1}{3g^2} \left((1 - 2g^2)^{\frac{3}{2}} - 1\right) - \lim_{a \rightarrow 0} \frac{1}{3a^2} (-3a^2) \quad \text{L'Hôpital's Rule} \\
&= \frac{1}{3g^2} \left((1 - 2g^2)^{\frac{3}{2}} - 1\right) + 1
\end{aligned} \tag{1.269}$$

$$\begin{aligned}
-S(x_0) &= \frac{1}{3g^2} \left((1-2g^2)^{\frac{3}{2}} - 1 \right) + 1 \\
&= \frac{1}{3g^2} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{3}{2} \cdot \frac{1}{2} \cdot \dots \cdot \frac{3-2n}{2} (-2)^{n+1} g^{2(n+1)} + \frac{1}{3g^2} - \frac{1}{3g^2} + 1 \text{ by shift 1 index} \\
&= \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \frac{1}{2} \cdot \dots \cdot \frac{2n-3}{2} (-1)^n (-2)^n g^{2n} - 1 + 1 \text{ by extract first term} \\
&= \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \dots \cdot (2n-3)}{(n+1)!} g^{2n}
\end{aligned} \tag{1.270}$$

Now we see each coefficients corresponds to counts of tree with $m = 2n$ vertices, if we multiply $2n$ then corresponds to the one with vertices labelled.

Feynman calculus can be used to count not only oriented but also oriented graphs. Suppose we want to count not labelled oriented trees, how to represent this? Given two labelled point x, y rather x^2 , the bilinear form is thus xy . Identify x as the source of the edge and y as the sink of the edge, we extend all flowers same as before but summation for both: $be^x + ae^y$. Combine together: $S(x, y) = xy - be^x - ae^y$. The critical point is the zeros of partial derivative in such 2-dimension manifold.

$$\begin{aligned}
\frac{\partial S(x, y)}{\partial x} &= y - be^x = 0 \rightarrow be^x = y \\
\frac{\partial S(x, y)}{\partial y} &= x - ae^y = 0 \rightarrow ae^y = x
\end{aligned} \tag{1.271}$$

Such transcendental function can't be solved directly, we expand each coefficients:

$$\begin{aligned}
x &= a \left(\sum_{i \geq 0} \frac{y^i}{i!} \right) = a + \sum_{p \geq 1, q \geq 1} c_{p,q} a^p b^q \\
y &= b \left(\sum_{i \geq 0} \frac{x^i}{i!} \right) = b + \sum_{p \geq 1, q \geq 1} d_{p,q} a^p b^q
\end{aligned} \tag{1.272}$$

Evaluate the coefficient of each order by Cauchy's theorem:

$$c_{p,q} = \frac{1}{(2\pi i)^2} \oint \oint x a^{-(p+1)} b^{-(q+1)} da \wedge db \tag{1.273}$$

$$\begin{aligned}
da &= e^{-y}(dx - xdy) \quad db = e^{-x}(dy - ydx) \\
da \wedge db &= e^{-x-y}(dx - xdy) \wedge (dy - ydx) = e^{-x-y}(1 - xy)dx \wedge dy
\end{aligned} \tag{1.274}$$

$$a^{-(p+1)} = x^{-(p+1)} e^{(p+1)y}, b^{-(q+1)} = y^{-(q+1)} e^{(q+1)x} \tag{1.275}$$

$$\begin{aligned}
c_{p,q} &= \frac{1}{(2\pi i)^2} \oint \oint x x^{-(p+1)} e^{(p+1)y} y^{-(q+1)} e^{(q+1)x} e^{-x-y} (1-xy) dx \wedge dy \\
&= \frac{1}{(2\pi i)^2} \oint \oint (1-xy) x^{-p} y^{-(q+1)} e^{qx+py} dx \wedge dy
\end{aligned} \tag{1.276}$$

We evaluate $x^{-k} y^{-l} e^{qx+py}$ to tackle general case:

$$x^{-k} y^{-l} \sum_{i,j \geq 0} \frac{(qx)^i}{i!} \frac{(py)^j}{j!} \rightarrow i-k = -1, j-l = -1 \rightarrow I = \frac{q^{k-1}}{(k-1)!} \frac{p^{l-1}}{(l-1)!} \tag{1.277}$$

$$\begin{aligned}
c_{p,q} &= \frac{q^{p-1}}{(p-1)!} \frac{p^q}{q!} - \frac{q^{p-2}}{(p-2)!} \frac{p^{q-1}}{(q-1)!} = \frac{q^{p-1} p^q - (p-1) q^{p-2} p^{q-1}}{(p-1)! q!} \\
&= \frac{(p - (p-1)) q^{p-1} p^{q-1}}{(p-1)! q!} \\
&= \frac{q^{p-1} p^{q-1}}{(p-1)! q!}
\end{aligned} \tag{1.278}$$

Similarly, $d_{p,q}$ is solved by same strategy while $x \leftrightarrow y$, thus $p \leftrightarrow q$, is $\frac{q^{p-1} p^{q-1}}{(q-1)! p!}$. Use integral trick to compute $S(x_0, y_0)$, but before on the hand, we decompose:

$$\frac{\partial S(x_0, y_0)}{\partial a} = -e^{y_0} = -\frac{x_0}{a}, \quad \frac{\partial S(x_0, y_0)}{\partial a} = -e^{x_0} = -\frac{y_0}{b} \tag{1.279}$$

Thus we can evaluate the integral of the curve by the total derivative of the action:

$$\begin{aligned}
-S(x_0, y_0)(a, b) &= \int_{a,0}^{a,b} \int_{0,0}^{a,0} \frac{x_0}{u} du + \frac{y_0}{v} dv \\
&= \int_{0,0}^{a,0} \frac{x_0(u, 0)}{u} du + \int_{a,0}^{a,b} \frac{y_0(a, v)}{v} dv \\
&= a + \int_0^b \left(1 + \sum_{p \geq 1, q \geq 1} d_{p,q} a^p v^{q-1} \right) dv \\
&= a + b + \sum_{p \geq 1, q \geq 1} \frac{q^{p-1} p^{q-1}}{q! p!} a^p b^q
\end{aligned} \tag{1.280}$$

Now we see that the number of trees with p sources and q sinks is $\frac{q^{p-1} p^{q-1}}{q! p!}$ labelled for source and sink one, or $q^{p-1} p^{q-1} \frac{(p+q)!}{q! p!}$ unlabelled for which is source or sink, or $p^{q-1} p^{q-1}$ if the vertices in source and sink are also labelled.

Can we extend the vertex to arbitrary finite labelled choices or colored? For example, the previous case has two color choices for each vertex, then the choice for different color can be considered as source and sink. Therefore we have a tuple of vertex color as (x_1, \dots, x_n) and their connection edges by $\frac{1}{2} \mathbf{x}^T B \mathbf{x}$ bilinear form while the previous case is $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Different flowers for each color can be constructed by $\sum_j a_j e^{x_j}$, reaching the final actions:

$$S(\mathbf{x}) = \frac{1}{2} \sum_{ij} B_{ij} x_i x_j - \sum_j^m a_j e^{x_j} \quad (1.281)$$

Differentiate to get fixed points tuple:

$$\frac{\partial S}{\partial x_i} = 0 \rightarrow \sum_j B_{ij} x_j = a_i e^{x_i} \quad (1.282)$$

Define $X_i := \sum_j B_{ij} x_j$ as the co-vector, the critical equations are:

$$X_i = a_i e^{x_i} \rightarrow a_i = X_i e^{-x_i}, \quad x_i := \sum_j (B^{-1})_{ij} X_j \quad (1.283)$$

We already know the final form of the action on critical points is restricted by:

$$\begin{aligned} \frac{\partial S(\mathbf{x}_0)(\mathbf{a})}{\partial a_i} &= e^{x_i} = \frac{X_i}{a_i} \\ -S(\mathbf{x}_0) &= \sum_i \int_0^{a_i} \frac{X_i(a_1, \dots, a_i = u, 0, \dots)}{u} du \end{aligned} \quad (1.284)$$

So the only problem is to solve the coefficients of expansion of each critical points.

$$X_i = a_i + \sum_{p_j \geq 1} c_{\mathbf{p},i} \prod_j^m a_j^{p_j} \quad (1.285)$$

Where the coefficient $c_{\mathbf{p},i}$ indicates the power counts tuple p_j for each $a_j^{p_j}$ while the i indicates the X_i co-vector.

Induce that any $X_i(a_1, \dots, u, 0, \dots) = a_i$ because all product term vanishes, thus we only need to know the coefficient and integrate to get the result: $\frac{c_{\mathbf{p},i}}{p_i}$.

$$\begin{aligned} c(\mathbf{p}, i) &= \frac{1}{(2\pi i)^m} \oint X_i \left(\prod_k a_k^{-p_k-1} \right) d\mathbf{a} \\ &= \frac{1}{(2\pi i)^m} \oint X_i \left(\prod_k (X_k e^{-x_k})^{-p_k-1} \right) d(X_1 e^{-x_1}) \wedge \dots \wedge d(X_m e^{-x_m}) \end{aligned} \quad (1.286)$$

We first calculate the single term, transforming all into X_i :

$$\frac{\partial X_i e^{-x_i}}{\partial X_j} = e^{-x_i} \delta_{ij} - X_i e^{-x_i} B_{ij}^{-1} = e^{-x_i} (\delta_{ij} - B_{ij}^{-1} X_i) \quad (1.287)$$

Thus the Jacobian matrix $J_{ij} = e^{-x_i} (\delta_{ij} - B_{ij}^{-1} X_i)$. We can inspect in matrix form:

$$J = \Lambda(e^{-x_1}, \dots, e^{-x_m}) \cdot (I - \Lambda(X_1, \dots, X_m) B^{-1}) \quad \Lambda(\dots) \text{ is diagonal matrix} \quad (1.288)$$

$$\det(J) = \prod_{i=1}^m e^{-x_i} \det(I - \Lambda(X_1, \dots, X_m) B^{-1}) = e^{-\sum_{i=1}^m x_i} \det(I - \Lambda(X_1, \dots, X_m) B^{-1}) \quad (1.289)$$

Generally, we should evaluate the concrete form of $\det(\lambda I - A)$ for certain matrix A . To calculate the determinant, recall the linearity of determinant for row and column:

$$\begin{aligned} \det(\lambda I - A) &= \det(\lambda_1 - a^{(1)}, \dots, \lambda_n - a^{(n)}) \quad a^{(i)} \text{ is column } (i) \\ &= \det(\lambda_1, \dots, \lambda_n) + \sum_{\{1, \dots, n\} \supset T, |T|=1} \det(\lambda_1, \dots, \lambda_{n-1}, -a^{(n)}) + \dots \end{aligned} \quad (1.290)$$

The dumb expansion could be summarized in more terse form, scrutinize that we can pick arbitrary columns $a^{(i)}$ from the tuple, which **count** as a subset $T \subset \{1, \dots, n\}$. For example, we pick two columns $\{1, 2\} \in \{1, 2, 3\}$ for a 3-dimension matrix resulting $\det(-a^{(1)}, -a^{(2)}, \lambda_3) = \lambda_3 \det(a^{(1)}, a^{(2)}, 1) := \lambda_3 D_{\{1,2\}}(A)$ where we use D to represent the residual or *minor* determinant of the matrix A . Thus the determinant is the summation of all subsets:

$$\det(\lambda I - A) = \sum_{T \subset \{1, \dots, n\}} (-1)^{|T|} D_T(A) \prod_{r \in \{1, \dots, n\} \setminus T} \lambda_r \quad (1.291)$$

You can see the product of λ_r comes from the remained index subtracted from T .

$$\begin{aligned} \det(J) &= e^{-\sum_{i=1}^m x_i} \sum_{T \subset \{1, \dots, m\}} (-1)^{|T|} D_T(\Lambda(X_1, \dots, X_m) B^{-1}) \\ &= (\dots) \prod_{r \in T} X_r D_T(B^{-1}) = (\dots) \prod_{i=1}^m X_i \prod_{r \in \{1, \dots, n\} \setminus T} X_r^{-1} D_T(B^{-1}) \end{aligned} \quad (1.292)$$

Where the final term is just reverse the product in set T to the product of whole dividing the product **not in** the T .

$$\begin{aligned} c(\mathbf{p}, i) &= \frac{1}{(2\pi i)^m} \oint \sum_{T \subset \{1, \dots, m\}} (-1)^{|T|} D_T(B^{-1}) X_i \prod_{r \in \{1, \dots, n\} \setminus T} X_r^{-1} \\ &\quad \times \prod_{k=1}^m X_k^{-p_k-1} \prod_{h=1}^m X_h \exp\left(\sum_{k=1}^m (p_k + 1) x_k - \sum_{h=1}^m x_h\right) d\mathbf{X} \end{aligned} \quad (1.293)$$

We see a lot of dumb index repeated count, reduce them:

$$\begin{aligned} c(\mathbf{p}, i) &= \frac{1}{(2\pi i)^m} \oint \sum_{T \subset \{1, \dots, m\}} (-1)^{|T|} D_T(B^{-1}) X_i \prod_{r \in \{1, \dots, n\} \setminus T} X_r^{-1} \\ &\quad \times \prod_{k=1}^m X_k^{-p_k} \exp\left(\sum_{k,l} p_k B_{k,l}^{-1} X_l\right) d\mathbf{X} \end{aligned} \quad (1.294)$$

The residue power counting is same as before, to simplify notation, identify $T^c = \{1, \dots, n\} \setminus T$. Focus on the certain dumb index l :

$$\begin{aligned}
n_l - p_l - \delta(l \in T^c) + \delta(i = l) &= -1 \\
n_l = p_l + \delta(l \in T^c) - \delta(i = l) - 1 &\rightarrow \prod_l^m \frac{\left(\sum_k^m p_k B_{k,l}^{-1}\right)^{p_l + \delta(l \in T^c) - \delta(i=l) - 1}}{(p_l + \delta(l \in T^c) - \delta(i = l) - 1)!}
\end{aligned} \tag{1.295}$$

The delta notation impede us reduce further, we reformat the first delta $\delta(l \in T^c) = 1 - \delta(l \in T)$, second, we see that $\delta(i = l)$ can be decomposed into $\delta(i \in T^c = l) + \delta(i \in T = l)$, but it only effect X_i , so extract out i index: $p_i - \delta(i \in T) - \delta(i \in T) - \delta(i \in T^c) = p_i - 1 - \delta(i \in T)$. Focus the i index term, we balance the power:

$$\frac{\left(\sum_k^m p_k B_{k,i}^{-1}\right)^{p_i - \delta(i \in T) - 1}}{(p_i - \delta(i \in T) - 1)!} = \frac{p_i - \delta(i \in T)}{\sum_k^m p_k B_{k,i}^{-1}} \frac{\left(\sum_k^m p_k B_{k,i}^{-1}\right)^{p_i - \delta(i \in T)}}{(p_i - \delta(i \in T))!} \tag{1.296}$$

To think the i index term into a whole product. Now we combine all into:

$$c(\mathbf{p}, i) = \sum_{T \subset \{1, \dots, m\}} (-1)^{|T|} D_T(B^{-1}) \frac{p_i - \delta(i \in T)}{\sum_k^m p_k B_{k,i}^{-1}} \prod_l^m \frac{\left(\sum_k^m p_k B_{k,l}^{-1}\right)^{p_l - \delta(l \in T)}}{(p_l - \delta(l \in T))!} \tag{1.297}$$

$$\prod_l^m (\dots) = \prod_{l \in T}^m \frac{\left(\sum_k^m p_k B_{k,l}^{-1}\right)^{p_l}}{p_l!} \prod_{l \notin T}^m \frac{\left(\sum_k^m p_k B_{k,l}^{-1}\right)^{p_l - 1}}{(p_l - 1)!} = \prod_l^m \frac{C_l^{p_l}}{p_l!} \prod_{l \notin T}^m \left(\frac{p_l}{C_l}\right) \tag{1.298}$$

$$C_{k,l} = p_k B_{k,l}^{-1} \quad C_l = \sum_k^m p_k B_{k,l}^{-1}$$

Where we extract by $\prod_{l \notin T}^m \frac{C_l^{p_l - 1}}{(p_l - 1)!} = \prod_{l \notin T}^m \frac{C_l^{p_l}}{p_l!} \frac{p_l}{C_l}$.

$$c(\mathbf{p}, i) = \prod_i^m \frac{C_i^{p_i}}{p_i!} \sum_{T \subset \{1, \dots, m\}} (-1)^{|T|} \frac{p_i - 1 + \delta(i \in T^c)}{C_i} D_T(B^{-1}) \prod_{l \notin T}^m \left(\frac{p_l}{C_l}\right) \tag{1.299}$$

Here the key, we decompose by $(p_i - 1) + \delta(i \in T^c)$ into two parts:

$$\begin{aligned}
\det M &:= \det(C_i \delta_{il} - p_i B_{il}^{-1}) = \prod_k^m C_k \det\left(\delta_{il} - \frac{p_i}{C_i} B_{il}^{-1}\right) \\
&= \prod_k^m C_k \sum_{T \subset \{1, \dots, m\}} (-1)^{|T|} D_T(B^{-1}) \prod_{l \notin T}^m \left(\frac{p_l}{C_l}\right)
\end{aligned} \tag{1.300}$$

Where each column or row summation is zero, thus, we can find a vector that C_i as columns with $C_i - \sum_l C_{il} = 0$ which is the linear dependence.

$$\det M := \det(C_i \delta_{il} - C_{il}) := 0 \tag{1.301}$$

$$\text{term 1} = \frac{p_i - 1}{C_i} \det\left(\delta_{il} - \frac{p_i}{C_i} B_{il}^{-1}\right) = (p_i - 1) \frac{1}{C_i} \prod_k^m \frac{1}{C_k} \det M = 0 \tag{1.302}$$

$$\sum_{T \subset \{1, \dots, m\}} \delta(i \in T^c) = \sum_{T \subset \{1, \dots, m\} \setminus i} \rightarrow \det D_{\{1, \dots, m\} \setminus i}(M) := (\det M)_{(i)} \quad (1.303)$$

Where we can see due to the delta notation, the summation must be restricted to index without i . Thus the determinant should be ignore the whole j column and row, which is the minor determinant without j index.

$$\text{term } 2 = \frac{1}{C_i} \det \left(\delta_{il} - \frac{p_i}{C_i} B_{il}^{-1} \right)_{(i)} = \frac{1}{C_i} \prod_{k \neq i}^m \frac{1}{C_k} (\det M)_{(i)} = \prod_k^m \frac{1}{C_k} (\det M)_{(i)} \quad (1.304)$$

$$c(\mathbf{p}, i) = \prod_l^m \frac{C_l^{p_l-1}}{p_l!} (\det M)_{(i)} \quad (1.305)$$

Is the final answer, notice the $p_l - 1$ power counts rather p_l due to the factor division from $\prod_k^m \frac{1}{C_k}$. It can also be formatted as:

$$\begin{aligned} \det M &:= \prod_k^m \frac{1}{p_k} \det(C_i p_l \delta_{i,l} - C_{i,l} p_l) := \prod_k^m \frac{1}{p_k} \det L \\ &\rightarrow N_\Gamma(\mathbf{p}) := \frac{c(\mathbf{p}, i)}{p_i} = \prod_k^m \frac{1}{p_k} \prod_l^m \frac{C_l^{p_l-1}}{p_l!} (\det(L))_i \end{aligned} \quad (1.306)$$

Be careful! The **left** $\frac{1}{p_i}$ pair with $\prod_{k \neq i}^m \frac{1}{p_k}$ to forming the whole product. The theorem generalize into a weighted version of *Kirchnoff's matrix tree theorem*, a highly non-trivial extension of *Cayley's theorem*, where we have p_i numbers of vertex a_i of Γ . Suppose only a_1 type of vertex, one has $C_l = \sum_k^m p_k B_{k,l}^{-1} = p_1$. Because such special case, the $(\det(L))_i$ is a zero matrix which is defined as 1. Thus, $N_\Gamma(p_1) = \frac{p_1^{p-2}}{p!}$, which is exactly the answer of *Cayley's theorem*. We then apply it to the case containing vertices type a_1 and a_2 or source and sinks:

$$\begin{aligned} B &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \mathbf{p} &= (p, q) \end{aligned} \quad (1.307)$$

$$\begin{aligned} C_l &= \sum_{k=1}^2 p_k B_{k,l}^{-1} = (q, p) \\ \det(L) &= C_i p_l \delta_{i,l} - C_{i,l} p_l = \begin{pmatrix} pq & -pq \\ -pq & pq \end{pmatrix} \rightarrow (\det(L))_i = pq \end{aligned} \quad (1.308)$$

$$N_\Gamma((p, q)) = \frac{1}{pq} \frac{p^{q-1} q^{p-1}}{p! q!} \cdot pq = \frac{p^{q-1} q^{p-1}}{p! q!} \quad (1.309)$$

Which is same as the results of previous calculation.

$$\begin{aligned} \mathcal{A} &: \text{set of combinatorial objects or simply objects} \\ \omega(\mathbf{s})(a) &: \mathcal{A} \rightarrow \mathbb{N} : a \rightarrow \text{counts of } \mathbf{s} - \text{subobjects in } a \end{aligned} \quad (1.310)$$

$$[\mathcal{A} : \omega(\mathbf{s})](x) := \begin{cases} \sum_{a \in \mathcal{A}} x^{\omega(\mathbf{s})(a)} & \text{labelled} \\ \sum_{a \in \mathcal{A}} \frac{x^{\omega(\mathbf{s})(a)}}{\omega(\mathbf{s})(a)!} & \text{unlabelled} \end{cases} \quad (1.311)$$

We can say that we represent any potential a by $[\mathcal{A} : \omega(\mathbf{s})](x)$.

For representation of set $\mathcal{U} := \{\emptyset, \{1\}, \{1, 2\}, \dots\}$ of unlabelled combinatorial objects, one get:

$$[\mathcal{U} : \omega(\mathbf{s})](x) = \sum_{k \geq 0} \frac{x^k}{k!} = e^x \quad (1.312)$$

$$\begin{aligned} [\mathcal{A} \cup \mathcal{B} : \omega(\mathbf{s})] &= [\mathcal{A} : \omega(\mathbf{s})] + [\mathcal{B} : \omega(\mathbf{s})] \\ [\mathcal{A} \setminus \mathcal{B} : \omega(\mathbf{s})] &= [\mathcal{A} : \omega(\mathbf{s})] - [\mathcal{B} : \omega(\mathbf{s})] \\ [\mathcal{A} \times \mathcal{B} : \omega(\mathbf{s}) \oplus \omega(\mathbf{r})] &= [\mathcal{A} : \omega(\mathbf{s})] \cdot [\mathcal{B} : \omega(\mathbf{r})] \\ [\mathcal{A} \circ \mathcal{B} : \omega(\mathbf{s})](y) &= ([\mathcal{A} : \omega(\mathbf{s})] \circ [\mathcal{B} : \omega(\mathbf{r})])(y) \\ [\partial_{\mathbf{s}} \mathcal{A} : \omega(\mathbf{s})](x) &= \frac{d}{dx} [\mathcal{A} : \omega(\mathbf{s})](x) \end{aligned} \quad (1.313)$$

Where deletion operator will remove all single \mathbf{s} –object for possible $a \in \mathcal{A}$. Therefore, we can also distinguish the object for all possible ways.

$$[\mathbf{s} \partial_{\mathbf{s}} \mathcal{A} : \omega(\mathbf{s})](x) = x \frac{d}{dx} [\mathcal{A} : \omega(\mathbf{s})](x) \quad (1.314)$$

Thus a multi-variate series can represent multiple type of objects.

$$[\mathcal{A} : \omega(\mathbf{s}_1) \otimes \omega(\mathbf{s}_2)](x, y) = \sum_{a \in \mathcal{A}} x^{\omega(\mathbf{s}_1)} y^{\omega(\mathbf{s}_2)} \quad (1.315)$$

$$G(y) := \left[\mathcal{G} : \bigotimes_{k \geq 1} \omega(\mathbf{v}_k) \otimes \omega(\mathbf{e}) \right] (y; \lambda_k, u) \quad (1.316)$$

One can construct a tree by sum its all possible ways: $\mathcal{T} = \bigcup \{\text{possible way to recover trees}\}$. We should investigate the ways by distinguishing method. First, for any k –valent flower λ_k or closed 1 –valent y vertex, we can attach it by deleting each closed vertex y of k trees and attach the flowers to connect k trees or just amend attaching closed 1 –valent vertex for our deleted 1 –valent vertex. However, edge is more special, to attach an edge to recover a tree, we need to delete each vertex and its edge already in two trees and attach one new.³ To summarize above operations, we should notice that edge operations are smaller one count than the vertex operations because a tree always has $V - E = 1$. The attaching ways of vertex are always one plus than the attaching ways of edge, yielding the below results.

$$\mathcal{T} \simeq \bigcup_{k \geq 1} \mathbf{v}_k \partial_{\mathbf{v}_k} \mathcal{T} \setminus \mathbf{e} \cdot \mathcal{U}_2 \circ (\mathbf{e}^{-1} \partial_{\mathbf{v}_1} \mathcal{T}) \quad (1.317)$$

³Or better, we remove each vertex of tree and one edge of **either** tree, then connect them together. The result is same: $\frac{1}{2u} \frac{\partial \mathcal{T}^2}{\partial y}$

$$y \frac{\partial T}{\partial y} + \sum_{k \geq 3} \frac{\lambda_k}{k!} \left(\frac{\partial T}{\partial y} \right)^k - \frac{u}{2} \left(\frac{1}{u} \frac{\partial T}{\partial y} \right)^2 = T \quad (1.318)$$

$$y = u^{-1} \left(\frac{\partial}{\partial x} \frac{x^2}{2} \right) \circ_y (y) = u^{-1} \left(\frac{\partial}{\partial x} \frac{x^2}{2} \right) \circ_y \left(\frac{\partial}{\partial y} \frac{uy^2}{2} \right) \quad (1.319)$$

$$\mathbf{e}^{-1} \partial_{\mathbf{v}_1} \leftrightarrow u^{-1} \frac{\partial}{\partial y}.$$

$$-u^{-1} \left(\frac{\partial}{\partial x} \frac{x^2}{2} \right) \circ \frac{\partial}{\partial y} T + \left(\sum_{k \geq 3} \lambda_k \frac{x^k}{k!} \right) \circ \frac{\partial}{\partial y} T \quad (1.320)$$

$$y \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial y} + \frac{\partial}{\partial \frac{\partial T}{\partial y}} \left(\sum_{k \geq 3} \frac{\lambda_k}{k!} \left(\frac{\partial T}{\partial y} \right)^k - \frac{1}{2u} \left(\frac{\partial T}{\partial y} \right)^2 \right) \frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial y}$$

$$\frac{\partial^2 T}{\partial y^2} \left(y + \frac{\partial}{\partial \frac{\partial T}{\partial y}} (\dots) \right) = 0 \quad (1.321)$$

$$\frac{\partial}{\partial \frac{\partial T}{\partial y}} \left(\frac{1}{2u} \left(\frac{\partial T}{\partial y} \right)^2 - \sum_{k \geq 3} \frac{1}{k!} \left(\frac{\partial T}{\partial y} \right)^k \right) = y$$

$$y = \frac{\partial}{\partial x} \left(\frac{x^2}{2u} - \sum_{k \geq 3} \frac{x^k}{k!} \right) \circ_{x:y} \left(\frac{\partial T}{\partial y} \right) \quad (1.322)$$

We recover the vertex expression by tree in a astonishing way. Therefore establishes a bijection between T and $F = \frac{x^2}{2u} - \sum_{k \geq 3} \frac{x^k}{k!}$.

$$L[T] \left(x = \frac{\partial T}{\partial y} \right) = y \frac{\partial T}{\partial y} - T = \frac{x^2}{2u} - \sum_{k \geq 3} \frac{x^k}{k!} = F(x) \quad (1.323)$$

$$L^2[T] \left(x = \frac{\partial T}{\partial y} \right) = L[F] \left(z = \frac{\partial F}{\partial x} \right)$$

$$= x \frac{\partial F}{\partial x} - F \quad (1.324)$$

$$= \left(\frac{z^2}{2u^{-1}} - \sum_{k \geq 3} \frac{z^k}{k!} \right) \circ_{z:x} \left(\frac{\partial F}{\partial x} \right) \quad \left(\frac{\partial F}{\partial x} = y \right)$$

$$= (\dots) \circ_{z:x} (y) = T(z)$$

Imply a original tree. It suggests that if we consider a general graph as a tree of 1-irreducible components or 2-connected components, taking Legendre transformation yielding a set of 1-irreducible components. It' s especially useful that we could investigate irreducible components only.

A supervector space V over \mathbb{R} or any \mathbb{k} field of characteristic 0 is just a \mathbb{Z}_2 -graded vector space:

$$\text{gr}_I(V) = \bigoplus_I V^I \quad (1.325)$$

$$\text{fil}(V) = 0 \subset F^1(V) \subset F^2(V) \subset \dots \subset F^k(V) = V \quad (1.326)$$

$$\text{agr}(V) = \bigoplus_I F^{i+1}(V)/F_i(V) \neq V \quad (1.327)$$

$$V = V_0 \oplus V_1 = \mathbb{k}^n \oplus \mathbb{k}^m := \mathbb{k}^{n|m} \quad \eta : V_{0/1} \rightarrow 0/1 \quad (1.328)$$

A tensor algebra is:

$$T(V) = \bigoplus_{n=0}^{\infty} T^n(V) = \bigoplus_{n=0}^{\infty} \underbrace{V \otimes V \dots \otimes V}_n = \bigoplus_{n=0}^{\infty} V^{\otimes n} \quad (1.329)$$

$$\begin{aligned} S(V) &= T(V)/(v \otimes w - w \otimes v) \rightarrow T(V)/(v \otimes w - (-1)^{\eta(w)\eta(v)} w \otimes v) \\ \eta : \underbrace{\xi_{i_1} \otimes \xi_{i_2} \dots \otimes \xi_{i_n}}_n &= n \bmod 2 \end{aligned} \quad (1.330)$$

$$\Lambda(V) = T(V)/(v \otimes w + w \otimes v) \rightarrow T(V)/(v \otimes w + (-1)^{\eta(w)\eta(v)} w \otimes v) \quad (1.331)$$

$$S(V_0) \overset{\Pi}{\leftrightarrow} \Lambda(V_1), \quad S(V_1) \overset{\Pi}{\leftrightarrow} \Lambda(V_0) \quad (1.332)$$

$$\begin{aligned} S(V) &\cong S(V_0) \otimes \Lambda(V_1) \cong \bigoplus S^i(V_0) \otimes \bigoplus \Lambda^i(V_1) : v \otimes w = (-1)^{\eta(v)\eta(w)} w \otimes v \\ \Lambda(V) &\cong \Lambda(V_0) \otimes S(V_1) \cong \bigoplus \Lambda^i(V_0) \otimes \bigoplus S^i(V_1) : v \wedge w = -(-1)^{\eta(v)\eta(w)} w \wedge v \end{aligned} \quad (1.333)$$

$$\begin{aligned} \mathcal{O}(V) &= \mathbb{k}[x_1, \dots, x_n; \xi_1, \dots, \xi_m] = S(V_0^*) \otimes \Lambda(V_1^*) \text{ in linear coordinates} \\ x_i x_j &= x_j x_i, x_i \xi_j = \xi_j x_i, \xi_i \xi_j = -\xi_j \xi_i \end{aligned} \quad (1.334)$$

$C^\infty(V_1)$ have no topology defined upon due to $\xi^2 = 0$. Thus there' s no smoothness. But we can still define the polynomial algebra with a correspondence by $f(\xi_1, \dots, \xi_n) = \sum_{k \geq 0} \frac{f^{(k)}}{k!} \xi_{k_1} \xi_{k_2} \dots \xi_{k_n}$, explicitly, one define $e^\xi = \sum_{k \geq 0} \frac{\xi^k}{k!} = 1 + \xi$.

$$C_s^\infty(V_1) \text{ doesn't exist!} \rightarrow C_s^\infty := C^\infty(V_0) \otimes \Lambda(V_1^*) \quad (1.335)$$

$$\begin{aligned} (M, C_s^\infty(M)) &\supset (U := U_0 \times V_1, C_s^\infty(U_0) := C^\infty(U_0) \otimes \Lambda(V_1^*)) \\ F : M(x_1, \dots, x_n; \xi_1, \dots, \xi_m) &\rightarrow N(y_1, \dots, y_p; \zeta_1, \dots, \zeta_q) \end{aligned} \quad (1.336)$$

$$\begin{aligned}
y_i &= f_i(x_1, \dots, x_n) + f_i^{j_1, j_2}(x_1, \dots, x_n) \xi_{j_1} \xi_{j_2} + \dots \text{ with } \eta(y_i) = 0 \\
\zeta_i &= g_i^{j_1}(x_1, \dots, x_n) \xi_{j_1} + g_i^{j_1, j_2, j_3}(x_1, \dots, x_n) \xi_{j_1} \xi_{j_2} \xi_{j_3} + \dots \text{ with } \eta(\zeta_i) = 1
\end{aligned} \tag{1.337}$$

Therefore for any scalar function defined upon manifold, one has such pullback:

$$\begin{aligned}
F^* : C_s^\infty(M) &\rightarrow C_s^\infty(N) \\
F^*(f)(x_1, \dots, x_n; \xi_1, \dots, \xi_m) &= f \circ F(y_1, \dots, y_p; \zeta_1, \dots, \zeta_q)
\end{aligned} \tag{1.338}$$

$$\begin{aligned}
F : M = \mathbb{R}^{1|2} &\rightarrow N = \mathbb{R}^{1|2} = (x_1, \xi_1, \xi_2) \rightarrow (x + \xi_1 \xi_2, \xi_1, \xi_2) \\
F^*(f(x_1)) &= f(x_1 + \xi_1 \xi_2) = f(x) + f'(x) \xi_1 \xi_2
\end{aligned} \tag{1.339}$$

$$\begin{aligned}
R &:= R_0 \oplus R_1 \quad R_i R_j \subseteq R_{i+j \bmod 2} \\
xy &= (-1)^{\eta(x)\eta(y)} yx
\end{aligned} \tag{1.340}$$

$$T : V \rightarrow V \text{ is even if } T(V_i) \subseteq V_i \quad \text{is odd if } T(V_i) \subseteq V_{i+1 \bmod 2} \tag{1.341}$$

$$\begin{aligned}
T &\text{ is even if } \eta(T) = 1 \\
T &\text{ is odd if } \eta(T) = -1
\end{aligned} \tag{1.342}$$

$$[A, B] := AB - (-1)^{\eta(A)\eta(B)} BA \tag{1.343}$$

Thus if $T \rightarrow \text{Mat}_{n|m}(R)(e_1, \dots, e_n; f_1, \dots, f_m)$ with basis e_i of V_0 and f_i of V_1 , we immediately see that:

$$GL_{n|m}(R) \cong \text{Mat}_{n|m}(R) = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \tag{1.344}$$

Where A_{00} is n by n , A_{11} is m by m is *even* mapping to the same space while $A_{10/01}$ is $m \times n$ or $n \times m$ is *odd* mapping to another space changing the parity.

$$\begin{aligned}
\text{sTr} : \text{Mat}_{n|m}(R) &\rightarrow R \\
\text{sTr} \left(\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \right) &\rightarrow \text{Tr}(A_{00}) + \varepsilon \text{Tr}(A_{11})
\end{aligned} \tag{1.345}$$

$$\begin{aligned}
A &= \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \\
AB &= \begin{pmatrix} XY & 0 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 0 \\ 0 & YX \end{pmatrix}
\end{aligned} \tag{1.346}$$

$$\begin{aligned}
\text{sTr}(AB) &= \text{Tr}(XY) + \varepsilon \cdot 0 = \text{Tr}(XY) \\
\text{sTr}(BA) &= \varepsilon \text{Tr}(YX)
\end{aligned} \tag{1.347}$$

$$\text{sTr}(AB) = \text{sTr}(BA) \rightarrow \text{Tr}(XY) = \varepsilon \text{Tr}(YX) \quad (1.348)$$

$$\begin{aligned} \text{Tr}(YX) &= \sum_{i,j}^{m,n} Y_{ji} X_{ij} = - \sum_{i,j}^{m,n} X_{ij} Y_{ji} = -\text{Tr}(XY) \\ &\rightarrow \text{Tr}(XY) = \varepsilon(-\text{Tr}(XY)) \\ &\rightarrow \varepsilon = -1 \end{aligned} \quad (1.349)$$

$$\text{sTr}(A) := \text{Tr}(A_{00}) - \text{Tr}(A_{11}) \quad (1.350)$$

For block-diagonal matrix $A = A_{00} \otimes A_{11}$:

$$\text{sDet}(e^A) = e^{\text{sTr}(A)} = e^{\text{Tr}(A_{00}) - \text{Tr}(A_{11})} = \frac{\text{Det}(A_{00})}{\text{Det}(A_{11})} \quad (1.351)$$

Generally, apply Gauss elimination in block-level. To achieve this, we have to suppose the block is invertible.

$$\begin{aligned} \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} &= \begin{pmatrix} I & 0 \\ A_{10}A_{00}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{00} & A_{01} \\ 0 & A_{11} - A_{10}A_{00}^{-1}A_{01} \end{pmatrix} \\ &= (\dots) \begin{pmatrix} A_{00} & 0 \\ 0 & A_{11} - A_{10}A_{00}^{-1}A_{01} \end{pmatrix} \begin{pmatrix} I & A_{00}^{-1}A_{01} \\ 0 & I \end{pmatrix} \end{aligned} \quad (1.352)$$

$$\begin{aligned} \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} &= \begin{pmatrix} I & A_{01}A_{11}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{00} - A_{01}A_{11}^{-1}A_{10} & 0 \\ A_{10} & A_{11} \end{pmatrix} \\ &= (\dots) \begin{pmatrix} A_{00} - A_{01}A_{11}^{-1}A_{10} & 0 \\ 0 & A_{11} \end{pmatrix} \begin{pmatrix} I & 0 \\ A_{11}^{-1}A_{10} & I \end{pmatrix} \end{aligned} \quad (1.353)$$

$$\text{sDet}\left(\begin{pmatrix} I & A_{10} \\ A_{10} & I \end{pmatrix}\right) = \exp(\text{sTr}(\dots)) = \exp(0) = 1 \quad (1.354)$$

$$\text{sDet}(AB) = \text{sDet}(A) \text{sDet}(B)$$

$$\text{sDet}(A) = \frac{\text{Det}(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\text{Det}(A_{11})} \text{ or } \frac{\text{Det}(A_{00})}{\text{Det}(A_{11} - A_{10}A_{00}^{-1}A_{01})} \quad (1.355)$$

$$\begin{aligned} n=1, m=0; x' = \lambda x : \int f(x')dx' &= \int f(\lambda x)|\lambda|dx \\ n=0, m=1; \xi' = \mu\xi : \int f(\xi')d\xi' &= \int f(\mu\xi)|\mu|^{-1}d\xi \end{aligned} \quad (1.356)$$

$$\begin{aligned} T_h : x \rightarrow x + h &\Rightarrow T_h(I) = I \text{ by translation invariance} \\ \int f(\mathbf{x}, \boldsymbol{\xi})dx^I d\xi^I &= \int f(\mathbf{x} + \boldsymbol{\lambda}, \boldsymbol{\xi} + \boldsymbol{\mu})dx^I d\xi^I \end{aligned} \quad (1.357)$$

Differentiate in coordinates zero, we immediately found that any partial derivatives in any directions is zero:

$$\int \partial_{x_i/\xi_i} f(\mathbf{x}, \xi) dx^I d\xi^I = 0 \quad (1.358)$$

$$\int \partial_\xi \xi d\xi = \int d\xi = 0 \quad (1.359)$$

$$\int \xi d\xi = \text{constant} \xrightarrow{\text{normalization}} \int \xi d\xi = 1 \quad (1.360)$$

$$B(\xi, \xi) = \sum_{i,j} b_{ij} \xi_i \xi_j \quad (1.361)$$

It's easy to see that $b_{ij} = -b_{ji}$ to make sense, otherwise it must be zero due to anti-commutation of odd variables.

$$B(\xi, \xi) = \sum_{i,j} b_{ij} \xi_i \xi_j = - \sum_{i,j} b_{ij} \xi_j \xi_i = \sum_{i,j} b_{ji} \xi_j \xi_i = B(\xi, \xi) \quad (1.362)$$

$$e^{\frac{1}{2}B(\xi, \xi)} = \sum_{k \geq 0} \frac{1}{2^k k!} (B(\xi, \xi))^k \quad (1.363)$$

We must have even numbers of odd variables due to unmatched integral for certain odd variable ξ contributing zero. For each k -degree term, the only nonzero contribution must be indices distinction due to $\xi^2 = 0$. Thus:

$$\begin{aligned} (B(\xi, \xi))^k &= \left(2 \sum_{i < j} b_{ij} \xi_i \xi_j \right)^k = 2^k \sum_{\sigma \in \Pi_k} b_{\sigma(i_1)(j_1)} \cdots b_{\sigma(i_k)(j_k)} \xi_{\sigma(i_1)} \xi_{\sigma(j_1)} \cdots \xi_{\sigma(i_k)} \xi_{\sigma(j_k)} \\ &= 2^k k! \sum_{\sigma \in \Pi_k} \prod_m b_{\sigma(i_m)(j_m)} \xi_{i_1} \xi_{j_1} \cdots \xi_{i_k} \xi_{j_k} \\ &= 2^k k! \sum_{\sigma \in \Pi_k} \prod_m b_{\sigma(i_m)\sigma(j_m)} \eta(\sigma) \xi_1 \xi_2 \cdots \xi_{2m} \end{aligned} \quad (1.364)$$

$$\begin{aligned} \int e^{\frac{1}{2}B(\xi, \xi)} d\xi &= \int \sum_{\sigma \in \Pi_k} \prod_m b_{\sigma(i_m)\sigma(j_m)} \eta(\sigma) \xi_1 \cdots \xi_{2m} d\xi_{2m} \cdots d\xi_1 \\ &= \sum_{\sigma \in \Pi_k} \eta(\sigma) \prod_m b_{\sigma(i)\sigma(j)} \end{aligned} \quad (1.365)$$

Where the pairing sign is depended on the order. We can see by swapping times, one swap corresponds to one minus sign. Quicker, we can see if each pairing number crossing each other. $\{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$, we see by $\{1, 4\}$ across else two terms, and $\{2, 5\}$ cross $\{3, 6\}$, thus $(-1)^3 = -1$, is minus sign. Thus the maximal minus sign must be $m - 1 + m - 2 + \dots = \frac{m(m-1)}{2}$ times.