Math Physics

1 Number System

The evolution of cognition of mathematics stem from **natural number**. What′s the natural number? It describe a infinite process starts from a base. The base is **empty**. The negation of everything denoted as $\emptyset$. From a primitive view, the higher process includes the lower process which suggests that $? \to S(?)$ is a inclusion morphism. Such view is **ordinal** view. A child will imagine things in ordinal view, 3 apples is 1 apples more on 2 apples, etc. That′s, inclusion process, a ordered structure. Such inclusion indicates that $?$ as a ordinal is a subset of its successor.

We can safely process $\emptyset$ in mind, as a subset of a larger set. $S(\emptyset) = \{\emptyset\}$. The successor must be a larger set contains both empty and successor of empty, a well-inclusion process suggests $\{\emptyset, \{\emptyset\}\} = S(\{\emptyset\})$. Arabs number system spread as our todays number system. However we immediately found that such inclusion process itself is a number system. $S(*) \to S(S(*))$, it said, object is nothing but its relation, and base isn′t such important, because select any number as base is reasonable. Such process generally and widely used in mathematics.

$S(b) = b + 1$ is not a equation but a interpretation which we translate function notation to Arabs number notation. But that′s much more than this. First, we have no $+$, second, 1 is a instance, but we haven′t expanded such structure that 1 is special to any others as such notation.

From 0 to 1, We has such equivalence that $(+1) \to S(*)$. Because a larger set of a set is a inclusion of elements of the set and the set itself, and the empty set as **nothing**, we acquire 1 as a instance. However, from morphism view, we see $0 = \emptyset \sim \mathrm{id}_{0/\emptyset}$, and $1 \sim S(\mathrm{id}_0(*)) = S \cdot \mathrm{id}_0 = S(*)$. Emphasis again, the instance is nothing but its relations, natural number as instance is actually the intepretation of infinite process.

That′s, $(+1) \sim S(*)$. Arabians already knows addition which is multiple inclusion process. That′s $S \to S \cdot S \to S \cdot S \cdot S(*) \to \dots$ We thus immediately acquire addition as infinite process of such inclusion process itself.

So $(+n) \sim \overbrace{S \cdot S\dots}^{n \text{ times}}$ gives us the operation. The commutativity is trivial because we apply the same induction process as S, there′s no permutation at all. It′s useful to use our infinite process again, it giving us multiplication as multiple inclusion of $(+n) \to (+n)(+n)(*) \to \dots$. Usually, Arabians use an to denote a times addtition of n. But it implicate a vagueness in addition and multiplication process priority. So in such number system, we should use bracket or restrict a multiplication priority:

$$nx + b \neq n(x + b) \tag{1.1}$$

Where we fix x as applied element. The distribution property is a corollary of commutativity of $+$.

$$n(a + b) = \overbrace{(a + b) + (a + b) + ...}^{n \text{ times}} = \overbrace{a + b + a + b + ...}^{n \text{ times}}$$
$$= \overbrace{a + a + ...}^{n \text{ times}} + \overbrace{b + b + ...}^{n \text{ times}} = na + nb \tag{1.2}$$

We remove bracket is because there's no operation priority for $+$ as a repeated operation, we can place $+$ to anywhere with a element beside of it.

Now, we have a simpler way to deduce commutativity.

$$na(*) = n \left(\overbrace{1 + 1 + ...}^{a \text{ times}} \right)(*) = \overbrace{n + n + ...}^{a \text{ times}}(*) = an(*) \tag{1.3}$$

Finally, the relation of multiplication and successor:

$$S(n)(*) = (n + 1)(*) = n(*) + (*) \tag{1.4}$$

We apply induction process again to deduce power, $(* n) \to (* n)(* n) \to ... \sim n^x$ for x times induction. We has the distribution, commutativity, and relation to addition(lower second order) in power too, with its multiplication as element:

$$(ab)^x = \overbrace{(ab)(ab)...}^{x \text{ times}} = a^x b^x$$

$$(n^x)^y = \left(\overbrace{n...}^{x \text{ times}} \right)^y = \overbrace{n^y...}^{x \text{ times}} = (n^y)^x \tag{1.5}$$

$$(n)^{x+y} = \overbrace{n...}^{x+y \text{ times}} = \overbrace{n...}^{x \text{ times}} \overbrace{n...}^{y \text{ times}} = n^x n^y$$

You see that the operation commutativity and distribution is based on the lower order but not all order. You can apply it to addition too.

Now, what if we deduce again? $(n)^n \to ((n)^n)^n \to (((n)^n)^n)^n \to ... \sim \overbrace{n^{n^{n\cdots}}}^{x \text{ times}} = a \uparrow\uparrow b$ where we has a thing much like power. Such induction process called **Hyperoperation**.

We need to acquire a inverse operation of such system. This is, the extension of number system. Suppose addition first, $a + x = b$. If such reverse operation is always defined, we should has $x = (a +)^{-1}(b)$ to give a pleasant result. The partial order indicates a violation if $a > b$. Where there shouldn't exist a number in successor logic. But if it's defined, the new number is coming out as a extension of original number system. We test that:

$$x = (a +)^{-1}(0) = (a +)^{-1}(\mathrm{id}_0) = (a +)^{-1}$$
$$(a +)^{-1}(a +)x = x \tag{1.6}$$
$$(a +)^{-1}(a +) = 0$$

If it's applicable, and indeed so, called negative number corresponding to positive number. $\mathbb{N} \to \mathbb{Z}$ as the answer. But what's there order? It should be a reverse successor that, if original is greater, this should be smaller to retains the correct order. That's, $S^{-1}(a) < a$, $S^{-1}(0) < 0$. Now, there's no lower bound.

We try multiplication too, $ax = b$ is also in such corner. Define $x = (a *)^{-1}b$ reveals a huger number system. The reverse operation should has the same property as the original to maintain consistency. Test that:

$$(a *)^{-1}(b) + (a *)^{-1}(c) = (a *)^{-1}(b + c) \tag{1.7}$$

$$(a *)^{-1}b + (c *)^{-1}d = (a *)^{-1}(c *)(c *)^{-1}b + ...$$
$$= (a *)^{-1}(c *)^{-1}(bc + ad) \tag{1.8}$$

Such extension is $\mathbb{Z} \to \mathbb{Q}$, integer to rational number. It shows that given a rational number the order is determined by $bc - ad$. Where $0 < (n *)^{-1}1 < 1$ is a results.

We summarize current cardinality where $|\mathbb{Q}| = |\mathbb{Z} \times \mathbb{Z}^*|$ Where we remove 0, but reasonably, there could be a $\{\pm\infty\}$ as a additional number, but it will break the order. We can check any infinity has same order. But negative infinity is lower than any number, positive infinity is higher than any number, forming a circle order.[1]

Now we proceed on power, or exponential. Because it's not commutative now. Such astonishing consequence derived from the canonical morphism that $n \to S^n$. $S^m S^n \sim S^n S^m$. But such distribution failed in exponential for S that $S(n)m \neq S(mn)$ as we already shown in multiplication non-commutativity with successor.

Such problem immediately cause a infinite sequence which we call **polynomial**.

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[1] It's undefined and rather prohibited as we usually learned, but limit haven't been "created" now.

$$\sum_{\mathbb{Q}} a_\sigma x^\sigma = 0 \qquad (1.9)$$

Where we use σ as a general notation of infinite iteration on all possible number of $\mathbb{Q}$. We can, thus deduce that the new number system must be $\prod_{|\mathbb{Q}|} \mathbb{Q}$. However, $\mathbb{Q}$ as a product, can be drawn as a grid and iterate in anyway to fill it. Which states that $f(n) \to \mathbb{Q}$ is injective and surjective, for example $\frac{m}{n} \leftrightarrow 2^m 3^n \in \mathbb{N}$. Thus $\prod_{|\mathbb{Q}|} \mathbb{Q} = \prod_{|\mathbb{N}|} \mathbb{Q}$. Which is a infinite sequence. How we express such system? The structure should inherit from $\mathbb{Q}$, we can define it element-wise that each element endure the regulation of order of $\mathbb{Q}$. But it doe not induce a native order like its base number system. That's, we need to induce a morphism into a order set or canonical number system to construct closeness.

Such infinite sequence can be expressed in many ways. galois extension states such extension as a vector space s.t. $Q\left[\sqrt{2}, \sqrt{3}, i, \pi...\right]$[2], or well known 10 digit expression s.t. $(..., 0, 1, 2, 3, ...) \leftrightarrow ...0.123...$ etc. The new number system is larger than we expected, as $|\mathbb{N}|^{|\mathbb{N}|}$, higher than $|\mathbb{N}|$.

We apply induction of finite sequence with element of $\mathbb{Q}$, the new number system must be a infinite sequence with a **nearness** no matter how we express it. Such process we call it **limit**. $\lim_{n\to\infty}$ express the higher cardinality level as $\lim_{n\to\infty}(..., q_n) \to r$. Or, you can express it in anyway you want, for example, notorious $0.9999... = 1$ is the same logic here, also $\lim_{n\to\infty} q_n = r$ for rational q_n. The latter is **Cauchy Sequence** common in analysis. All above follows a order in $\mathbb{Q}$ to construct $\mathbb{R}$.

A additional structure should be established. A usual procedure to reach a real number by infinitesimal is that, $\lim_{n\to\infty}(r - q_n) = 0$, where $r - q_n < \varepsilon, \forall \varepsilon \exists n > N \in \mathbb{N}$. This is a standard analysis that ε is a residual number exist in Real which can be arbitrary small. Presume for most of elements in sequence they are **same**. The two real numbers should be infinitesimally close, construct a equality. Such sequence procedure with a **nearness**, notates as ${}^*\mathbb{Q} \cong \mathbb{R}$ could be a intermediate procedure during extension. We can apply it to every that $\mathbb{N}^{|\mathbb{N}|}$ to construct ${}^*\mathbb{N}$ etc, with same cardinality as $\mathbb{R}$.[3]

In order to create such **nearness**(or convergence), suggest a filter on infinite elements in canonical order[4], against finite, the case for ${}^*\mathbb{R}$ as a intermediate number system could be constructed. Ultra filter said that we can select collections of subsets of $\mathbb{N}$, where its infinite, so it contains $\mathbb{N}$, but also a consistent selection for intersections or superset other

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[2]It should be clarified that the extension of all finite polynomial is called algebraic number $\mathbb{A}$, which is a subfield of $\mathbb{C}$. If you consider a infinite polynomial, for example $\sin(x) = \sum_i a_i x^i$, we can get π from $\sin(x) = 0$, absolutely not in $\mathbb{A}$.

[3]Such construction is called **hyper**, which is a nonstandard extension. The tedious comes from a order field embedding and the equivalence proposition of $\mathbb{R}$ and ${}^*\mathbb{R}$

[4]Indeed if we use digit expression, we should compare the higher digit first. The filter on sequence should be restricted to the canonical order to avoid prohibited expression of sequence.

than finite sets. Thus every sequence could be compared through a equivalence class of such filter ensured by axiom choice although the conrete form is unknown.[5]

Thus for positive sequence that smaller than any number of $\mathbb{R}^{>0}$, gives a infinitesimal(hypersmall) class $[\varepsilon]$, for positive sequence that greater than any number of $\mathbb{R}^{>0}$, gives a infinite(hyperlarge) class $[\infty]$.

Now, we try to formalize continuity as a attempt. $\forall \delta \in {}^*\mathbb{R}, \delta \sim [\varepsilon] : {}^*f(c + \delta) - f(c) \sim [\varepsilon]$, where c is a standard number in $\mathbb{R}$.[6] Also the monotone boundary sequence has limit as $\exists N \sim [\infty] \in {}^*\mathbb{N}$[7] , $\forall n \in N \ s_N \geq s_n$ is a element of upper bound, we also know it's a element of sequence, which suggest it's a least upper bound element, as $s_N = \lim_{n \to \infty} s_n$ in standard analysis.

The final problem here is that the number system can be inductively constructed higher than $\mathbb{R}$ by operation completion? Currently, I don't know such operation as extension can be proceed further. But the cardinality, suggest a map from $\mathbb{R}$ to $\mathbb{R}$, which is $|\mathbb{R}|^{|\mathbb{R}|}$. That's where function(s) comes in.

2 Function Space

In function, we can consider each real number in source as the ordinal of sequence $\mathbb{N}$, and each element in the sequence as domain. Thus natively, we inherit the property of every operation in number like addition, multiplication, exponential etc. $(f + g)(x), f(x)g(x), f(x)^{g(x)}....$ A structure has a basic additional **completion**(Or induction process completion if you want) is called **vector space**[8]. It's usual, even it's not trivial to define a ultra filter to measure how close between functions.

But wait here, such system seems, if acquire a total order by ultra filter, losing too much as we already seen in $\mathbb{R}$. Where function space becomes a monotone space by a specific ultra filter. But if a ultra filter restrict too much, a less strict design should be considered. First we could define limit as usual.

The introduction of Lebesgue integral is highly resemble to filter[9] where we need a basic order topology which usually contains all interval of $\mathbb{R}$ which called Borel σ-algebra. Where is closed under countable unions and intersections. We can consider it

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[5]Even sequence like $(1, 2, 3, 1, 2, 3, ...)$ is not a canonical number in $\mathbb{R}$, it could be assigned a determined order in ${}^*\mathbb{R}$. Notably, we only compare the number in $\mathbb{R}$, that's why it's not standard.

[6]It should be clear that if in continuous process both number could be non-standard, it's called uniform continuity, for example, $\frac{1}{N}, \frac{1}{N+1} \sim [\varepsilon]$ for $\frac{1}{x}$ is $N + 1 \not\approx N$ is not uniform continuous.

[7]Where we give the same sequence procedure on $\mathbb{N}$, in which $[\infty] \sim [1, 2, 3, ...]$ could be a instance of such equivalence class.

[8]I choose such because a mathematical structure must stem from canonical number system, multiplication could be derived as a induction of addition therefore construct a endomorphism as $v + v + v... \sim nv, n \in \mathbb{Z}$ and contrive a extension if we could.

[9]Rather ultra filter because for example, $[0, 0.5]$ is a set where itself or its complement either is measurable, which doesn't satisfy **in or not** property of ultra filter.

as a application of filter language. The measure, is actually full of tendency to create a consistent additive morphisms into $\mathbb{R}^{\geq 0}$ where built upon characteristic function.

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \in \mathbb{R}/E \end{cases} \tag{2.10}$$

Where $f(x)$ is a fine addition of χ_E. But if we consider it as a sequence that such integration is actually natural and the limit $f(x) - f_N(x)\, N \simeq [\infty] \in {}^*\mathbb{N}$ can be reached if, the filter exclude all countable set which has smaller cardinality than $\mathbb{R}$, constructing **almost all** property.[10]

However, it there′s no order, we should has a weaker order logic. We try to induce a morphism $d(*, *) \to S$ for a order set S(Or any given canonical number system).

The history of topology is based on selection of set. A finer selection means a huger control on element. Given a ordered set, we can always construct a open set which has $\{x \in X | a < x < b\}$ stem from canonical order of number system. For a finite set, which has a cardinality equal to a subset of $\mathbb{N}$, can be equipped with a order in $\mathbb{N}$ as a topology space, it can be finer that for every element discretely separated, define every open set also as closed set as $\{x\}$.

Given a morphism $d(*, *) : X \times X \to S$ where S a order set, it must has property to reflect the order of S. X has no necessity to deduce a **total order** where every element can be smaller or greater, that′s, we will encounter a violation if $d(a, c) < 0, d(b, c) > 0$ but $d(a, b) < 0$. But we indeed has $d(x, x) = d(x, x) + d(x, x)$ as a identity element of such addition structure, $d(x, x) = 0$. The continuity given by distance $d(x, B(x, \varepsilon)) = d(B(x, \varepsilon), x) \to (x, y) = d(y, x)$ for the open ball defined as $\{y \in X \mid d(x, y)/d(y, x) < \varepsilon\}$. Finally, suppose we want to find a distance in minimal lower bound by a selection of element $d(x, y_m)$, with a limit, $d(y_m, y_n) > d(x, y_m) + d(x, y_n) \simeq [\varepsilon]$ cause a violation. Thus $d(y_m, y_n) \leq (\ldots)$ forms a triangle inequality.

Thus, $\mathbb{Q}^{|\mathbb{N}|}$ can also be a vector space equipped with $d(*, *)$. But is this necessary, the answer is no. It′s at least indicates a Hausdorff topology. A weaker property is acceptable but the order induced by canonical number system is indeed Hausdorff, so we will take all this as a canonical way.

The distance morphism could retains vector structure in domain. First, it should be translation invariant which $d(x, y) \simeq d(x + \alpha, y + \alpha)$ independent of reference base. Second, it must retain scalar as linearity, $f(ax) = af(x) \to d(ax, ay) = ad(x, y)$. Thus, we has $d(x, y) = d(x - y, 0) \simeq \|x - y\|$.

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[10]The basic consequence is that the integral of $\xi_{\mathbb{Q}}$ is 0.

The highly non trivial property of inequality ensure the limit existence. Which is another name of convexity. Suppose a homotopy $F(\lambda)$ $\lambda \in [0, 1]$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \tag{2.11}$$

Where we recover back to triangle inequality if $\lambda = \frac{1}{2}$.

Now we are already into the language of function space. The basic analysis tells that $(*)^p$ is convex if $p \geq 1$. With a pointwise inequality:

$$|\lambda f(t) + (1 - \lambda)g(t)|^p \leq \lambda |f(t)|^p + (1 - \lambda)|g(t)|^p \tag{2.12}$$

Now, we apply integral which is addition of sequence or the given limit of characteristic function.

$$\int |\lambda f(t) + (1 - \lambda)g(t)|^p d\mu \leq \lambda \int |f(t)|^p d\mu + (1 - \lambda) \int |g(t)|^p d\mu \tag{2.13}$$

Now a selection of λ is the trick to solve here. Suppose for $\|x + y\| \leq \|x\| + \|y\|$:

$$\left\| \frac{x + y}{\|x\| + \|y\|} \right\| \leq 1 \tag{2.14}$$

Thus take $\lambda = \frac{\|x\|}{\|x\| + \|y\|}$, $u = \frac{x}{\|x\|}$ and $v = \frac{x}{\|y\|}$, it becomes:

$$\|\lambda u + (1 - \lambda)v\| \leq 1 \tag{2.15}$$

then take equivalence selection on above integration, it suggests that the norm should be $\left(\int |f(t)|^p \, d\mu \right)^{\frac{1}{p}}$. Then the inequality in integral called **Minkowski inequality** in L_p, where L_p is the integrable sequence space with the given norm.

So, what about $p \in [\infty]$? We could take a insight from basic analysis that $c^p \sim [\varepsilon]$ if $c \in (0, 1)$. Thus $\left\| \frac{u}{\sup(|u|)} \right\|_\infty = 1$ Where all element are suppressed only if the supreme is retained. That′s, $\|f\|_\infty = \inf_{\mu(A)=0} \sup_{t \in \Omega \setminus A} |f|$ excluding all measure 0.

The completeness is a verification of efficiency of approximation. That′s, the near point as $d(a, b) < \varepsilon$ for $\varepsilon \in \mathbb{R}^{>0}$ and a or b is standard(Not the infinitesimal here!), is a same set as $a \approx b$ or $a = b + [\varepsilon]$, For all nonstandard number here. Which states that the metric is effective reflective of point nearness.

The completeness criterion could be transferred from Cauchy in sequence into Cauchy in series. Suppose a absolute convergent $w_n = \sum_{i=1}^n s_i$ that $\|w_n - w_m\| = \left\| \sum_m^n s_i \right\| \leq \sum_m^n \|s_i\|$ where $n, m \sim [\infty]$, then $\sum_m^n \|s_i\| \sim [\varepsilon]$, that′s smaller than any $\mathbb{R}^{>0}$, the

completeness shows its converges. Reversely, a absolute convergent series converges, can be formulated as positive sequence $w_n = \sum_i^n (x_{i+1} - x_i)$. Which states that every sequence converges too. But what if the sequence contains negative? Where we separate here all $x_k = x_k^+ - x_k^-$ where positive part and negative is indeed converges $0 \leq x_k^+ \leq |x_k|$ suggests whole sequence converges, otherwise either part didn't converge cause a contradiction.

The same logic applied to L_p here, that, every bounded or absolute convergent series of positive functions as:

$$\left\| \sum_k^n f_k \right\|_p \leq \sum_k^n \|f_k\|_p \leq M \tag{2.16}$$

A limit of infinity is bounded too. Where the tails must also be converges to zero too. And apply a decomposition on positive and negative yielding a general results.

Such complete norm space called **Banach Space**.

3 Functional Space

Now we try to proceed further, that with cardinality as $\mathbb{R}^{|\mathbb{R}|^{|\mathbb{R}|}}$ as a functional space, evaluated each as a map from function to $\mathbb{R}$(Or $\mathbb{C}$). With a linearity based on base number system.

For $|T(f_n - f)| \sim [\varepsilon]$, for $n \sim [\infty]$, That's, $C[\varepsilon] \sim [\varepsilon]$ for any standard real number C. So the functional is bounded if:

$$|T(f)| \leq C\|f\| \tag{3.17}$$

Proceed further that if $\|f\| = 1$, we acquire:

$$|T(f)| \leq C \tag{3.18}$$

Which only depended on the bounded constant, which should be: $\sup_{\|f\|=1}(|T(f)|) = C$, we also establish another definition that:

$$\|T\| = \sup\left(\frac{T(f)}{\|f\|}\right) = \sup_{\|f\|=1}(T(f))$$
$$|T(f)| \leq \|T\|\|f\| \tag{3.19}$$

Where we just change name from C to $\|T\|$.

We know that, based on our fundamental spirit, a mathematical object can be represented as the relation upon that. Now we see that the **bounded** functional[11] can be formulated as the relation upon its working space. Where we call it **Dual Space** upon certain Banach space X, equipped with the $\|*\|_{\mathrm{op}}$ functional norms, is a Banach space by the applied element on X itself.

The problem is, is this always a natural corresponding relation between X and its dual space X^*? We can formulates X as double dual or dual of dual as X^{**}. That's if $x^{**}(\theta) = y^{**}(\theta)$ for all $\theta \in X^*$, then $x = y$. However, we should notice that we haven't prove this, it say that $()^{**} : X \to X^{**}$ is injective by $x^{**} = y^{**} \to x = y$ if above is valid.

In finite dimension, it's always workable by finite selection extension, however, we can apply Zorn lemma(ensured by axiom choice) for infinite case. Suppose we already has a corresponding f on given element or a subspace like w, then we want to do the same on $x = w + \alpha z$ or $x \in Z := W + \mathrm{span}(z)$.

$$g(w + \alpha z) = f(w) + \alpha c \quad c \in \mathbb{R} \tag{3.20}$$

Which should be **bounded**. But how? We can select suitable c to nudge over possible restriction where $g(w + \alpha z) \leq k\|w + \alpha z\| \to c \leq \frac{1}{\alpha}(k\|w + \alpha z\| - f(w))$.

Indeed α can be negative or positive so we turn around for all positive here and indeed smaller than the bound.

$$\left(-\frac{1}{\alpha_1}\right)(\|v - \alpha_1 z\| + f(v)) \leq c \leq (v \leftrightarrow w, \alpha_1 \leftrightarrow -\alpha_2) \quad \forall v, w \in W \tag{3.21}$$

We make a mess with algebra to yield:

$$f(\alpha_1 v + \alpha_2 w) \leq k\|\alpha_2 v + \alpha_1 w\| \leq k(\|\alpha_1 v - \alpha_1 \alpha_2 z\| + (v \leftrightarrow w, \alpha_1 \leftrightarrow -\alpha_2)) \tag{3.22}$$

Which is reasonable. That's, we can always extend the **bounded** functional element to the maximal element as the union of them. Such theorem called **Hahn-Banach** theorem.

If there's a $f(x) \neq f(y)$ in a subspace, then it can be extended the whole space which yield the contradiction.

Such selection gives another topology, even the convergence of norm fails, the convergence of $\theta(x_k), \forall \theta \in X^*$ is possible. We can thus extend it into the normed space where x_k is bounded and $\theta(x_k), \forall \theta \in A$. for the dense set $A \in X^*$. It shows that the object of X can converges in **element** that each sequence element will converge into the desired object, but the whole sum in norm is not. That's, **pointwise**, in Lebesgue

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[11]If not it's not continuous and no limit at all.

integral sense(delta function if you like) for L_p, or pointwise convergence sense for $C(K)$ where K is a compact topology space. The topology is indeed weaker than norm topology, called **Weak Topology**, with $\tau_w \subseteq \tau_s$ compared to **Strong(norm) Topology**, where the open set or reversely, closed set cardinality indeed lesser. But we can prove that they are the same in certain condition, that if we say $f(x_k) = gT(x_k)$ for $g \in Y^*$, $T \in \text{Hom}(X, Y)$, the key is T, from geometric level, if such sum or norm is restrict to compact space, where finite cover is enough, there restriction should be same. Indeed, for a compact space, we can always has a convergence sub-sequence, if not, we can extend the ball union infinitely where the finite cover doesn't work. Thus if T is **compact**(Not only bound, but the domain is compact), we has the convergence limit as $T(x_k) \to T(x)$.

Then we apply inductively again on $x^{**}(f_k)$, where the canonical injection say that $f_k(x)$ converges, such dual of weak topology as **Weak* Topology**. So what's matter? Suppose we separate $\prod_{x \in X}\{-\|x\|, \|x\|\} = \{f : X \to \mathbb{R} \mid |f(x)| \leq \|x\| \forall x \in X\} \sim \{\|f\| \leq 1\}$ as the bounded set of each sequence element. Then a ball, defines a set can be modulo the unsatisfied element not in X^*. Such product topology identify each f to the evaluation $(f(x))_{x \in X}$, is a product of compact(bounded) interval in $\mathbb{R}$. Thus such embed of ball of X^* into the product space assign $f_k \to f \sim (f_k(x_n)) \to f(x_n)$ as the support of f. Which is compact.[12] It said that B_{X^*} is weak* compact in the sense of evaluation on X. Where we can take $x^{**}(f) = f(x)$ embedding the $x \in X$ into f where $f \in B_{X^*}$ $\|f\| \leq 1$ to redefine x as $\sup_f |f(x)| = \|x\|_X$ in B_{X^*}. Which called **universal** embedding as $X \to C(B_{X^*})$

We reverse the theorem that for the dual space and ball $B_{X^{**}}$ in weak topology, identify the double dual with dual, if $X^{**} \cong X$, then the dual theorem exist and vice versa. We call it **reflexive** under canonical embedding.

A non-trivial application is to consider a concave inequality:

$$\ln(\lambda a + (1 - \lambda)b) \geq \lambda \ln a + (1 - \lambda) \ln b \tag{3.23}$$

Choose $\lambda = p, 1 - \lambda = q$.

$$\ln(pa + qb) \geq \ln a^p + \ln b^q = \ln a^p b^q \tag{3.24}$$

Choose $a = x^{\frac{1}{p}}, b = y^{\frac{1}{q}}$:

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[12] We use the Tychonoff theorem where the product of compact space is compact. You can expand the detail that the f_k bounded for certain x_n must also has $f_k(x_n) \leftrightarrow f_l(x_n)$ for a bounded sequence thus convergent(in $\mathbb{R}$) with $k, l \simeq [\infty] \in \mathbb{N}^*$. Thus $f_k \leftrightarrow f_l$.

$$\ln\left(px^{\frac{1}{p}} + qy^{\frac{1}{q}}\right) \geq \ln xy$$

$$e^{\ln\cdots} \geq e^{\ln\cdots} \tag{3.25}$$

$$px^{\frac{1}{p}} + qy^{\frac{1}{q}} \geq xy$$

Such inequality is called **Young's inequality**.

Take function evaluation pointwisely on x, y, we should take $\frac{1}{p} \sim p$ and $\frac{1}{q} \sim q$, rescaling $\|f\|_p = \|g\|_q = 1$ for simplicity.

$$|f(t)g(t)| \leq \frac{|f(t)|^p}{p} + \frac{|g(t)|^q}{q}$$

$$\int |fg| d\mu = \|fg\|_1 \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q \tag{3.26}$$

Called **Holder's inequality**. Rescaling on a different factor which is $s \geq r$ that for bounded measure $\mu(\Omega)$:[13]

$$\|f\|_r^r \leq \||f|^r\|_{\frac{s}{r}} \|1\|_q \quad \frac{1}{q} + \frac{1}{\frac{s}{r}} = 1$$

$$= \|f\|_s \mu(\Omega) \tag{3.27}$$

It means $L_r \subseteq L_s$ for $1 \leq r \leq s \leq [\infty]$.

The greatness comes from a minimal insight that:

$$f(x) = f\left(x_{\mu(A)} e_{\mu(A)}\right) = x_{\mu(A)} f\left(e_{\mu(A)}\right) \tag{3.28}$$

Define:

$$f\left(e_{\mu(A)}\right) := y_{\mu(A)} \tag{3.29}$$

Do I abuse notation? I hope not[14], even I don't suppose separable or countable basis but use the sequence view as before. Then for all $y_{\mu(A)}$, there exist a x s.t. Holder inequality is a equality, namely, choose $x := |y_{\mu(A)}|^{q-1} e_{\mu(A)}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

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[13]You may seek a counter example that $\frac{1}{x}$ on $L_2 \not\subset L_1$, That, L_2 is bounded integral, isn't the case in L_1.

[14]It's called **Radon-Nikodym Theorem** where the all $f(1_A)$ is new mesure can be represented as $\int g d\mu$ for a measurable function g which is proved by arrange function of $\mu + \nu$.

$$|f(x)| = \left|x_{\mu(A)}y_{\mu(A)}\right| = \left|y^q_{\mu(A)}\right| \leq \left\|x_{\mu(A)}\right\|_p \left\|y_{\mu(A)}\right\|_q$$

$$= \left|y^{\frac{q}{p}p}_{\mu(A)}\right|^{\frac{1}{p}}\left|y^q_{\mu(A)}\right|^{\frac{1}{q}} = \left|y^q_{\mu(A)}\right| = |f(x)| \tag{3.30}$$

With $|f(x)| \leq \|f\|\|x\|_p$, so $\left\|y_{\mu(A)}\right\|_q \leq \|f\| \leq \left\|y_{\mu(A)}\right\|_q$. Giving us the results.[15]

That's, L_p^* can be represented as a weight function y in integral form of L_q where $\frac{1}{p} + \frac{1}{q} = 1$. So we see that indeed, $L_p^{**} = L_p$ where in weak topology sense that converges element-wise. Generally, for a compact topology space K, the dual $C(K)^*$ is:

$$f(x) := \int f d\mu$$

$$\|f\| = |\mu|(K) \tag{3.31}$$

Where $|\mu|(K)$ is the total variation on the measured space, actually you are may acquainte like $\int d|\mu| = \int |f| dx$ for some function but we shouldn't constraint ourselves at such concrete form. However, in $p = 1, L_1^* = L_\infty$ by a bounded function as a compact space support, doesn't yield the same results on L_∞^* where the bounded restriction is lessen on the supreme support, that's we only need finite additive rather σ-additive or countable additive, that bounded finite additive measure space is the L_∞^*.

A notice should be mentioned that in descrete space, the inclusion is reverse that, for $1 \leq r \leq s \leq [\infty]$, suppose that $\|x\|_r = 1$:

$$x_n^r < 1$$

$$x_n^s \leq x_n^r$$

$$\|x\|_s \leq 1 = \|x\|_r \tag{3.32}$$

Is a reverse inclusion compared to $\mathbb{R}$ cardinality integral space.

For the special case where $X \cong X^*$ as its compact support, that $\frac{1}{p} + \frac{1}{p} = 1$, $p = 2$ give us a symmetry space that the product representation of f is in itself. Such product symmetry representation called **Inner Product**. That $X = X^*$ space called **Hilbert Space** denoted as H, where the famous **Cauchy-Schwartz inequality** is the special case of Holder inequality. With a elegant projection property that the minimal product form:

$$D(t) := \|v + ty\|^2 = \|v\|^2 + 2t(v, y) + t^2\|y\|^2 \tag{3.33}$$

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[15]Note we choose a specific x for arbitrary y to acquire the supreme as equality. Not reversely!

Where $t = 0$ is the projection for minimal distance, we can always decompose the space into a subspace and its orthogonal complement as $H = M \oplus M^\perp$.

We summarize above. Throughout the whole context, you won't see physics but it's appearance. The truth is, the sole appearance. Evaluate the number system is the quantification of the metric which represents the world, for example, time-space. A function is a state of the system which represents a evaluation of background for potential world. Therefore functional is the evaluation of the state, and operator is a transformation of the state, thus the system.

4 ?

The cardinality start from $\mathbb{R} \uparrow\uparrow 4$. However, the space is restricted to a symmetry in inner product evaluation. That's, $\mathrm{Hom}(X \to X)$ for linear operator. Thus, given a base number system, we can construct the algebra indeed with addition, multiplication and norm inequality.

We denote the operator domain as $D(T)$ for source space E, where the graph $(x, Tx) \in \Gamma(T)$ for morphism system $(T, D(T))$, is a closed space if, given a canonical norm by $\|x\|_E + \|Tx\|_F$ has limit. We know that bounded space will faithfully inject the whole space E without lossing information, $\|T\|_F \le M$. Thus the **closed** graph suggests a bounded operator if $D(T) = E$. A operator can be extended to a closed space if the uniqueness is ensured by original operator that $u \leftrightarrow Tu$ is unique, a.k.a injection, for example $u_n \to 0 \nrightarrow Tu_n \to 0$ by linearity. That's, $\overline{\Gamma(T)} \sim \Gamma(\overline{T})$ as the smallest close extension indeed.

Actually a simple operator is not closable, for example $T(\varphi) = \varphi(0)$ where $\varphi(0) = 1$ is reasonable even $\varphi_n = \varphi(nx) \to \|\varphi_n\| = n^{-\frac{d}{2}}\|\varphi\| \to 0$ for $L_2(\mathbb{R}^d)$ space, you may guess it's a delta-like function.

We restrict ourselves to Hilbert space where the inner product representation is a canonical form. That $((*)w, v) : \mathrm{Hom}(X \to X) \to \mathbb{R} \; \forall w, v \in H$ is a injection, it's true that $(Aw, v) = 0$ for all v, w and its coefficients which indeed zero. That, due to the domain problem, unbounded operator on inner product should consider two condition, first, if the inner product is same for all $v, w \in D(T)$, then we call it **Symmetric**, indicates that $D(T) \in D(T^*)$ for T can take any role of T^* can do. If there $D(T) = D(T^*)$, we call it **Self-Adjoint** plus the symmetric condition.

The uniqueness comes from the test elements on whole H. Where, if $D(T)$ is dense in H, the T^* has a unique corresponding element by the uniqueness of Hilbert space.

Such pair $J : (x, y) \to (-y, x)$ that $(Tx, v) \leftrightarrow (x, T^*v)$ is a $(x, Tx) \to (-Tx, x) \to (vTx) - (T^*(v)x) = 0 = (-Tx, x) \cdot (v, T^*(v))$. Where we can say that $\gamma(T^*) = J(\Gamma(T))^\perp$. Thus, $\Gamma(T^*)^\perp = (...)^{(\perp)^\perp} = \overline{J(\Gamma(T))}$, indicates T^* is closed. You can achieve the same by calling convergence on $(Tx, v_n) = (x, T^*v_n)$. If $D(T^*)$ is dense, the T^{**} is

well defined, $J \cdot J = -I, \Gamma(T^{**}) = (\text{twice}) = \overline{\Gamma(T)}$ suggests $\overline{T} = T^{**}$, which means T is closable.

$(x, T^*v) = (Tx, v) = (T^{**}x, v)$ for all $x \in D(T^{**})$ suggests that $T \subset T^{**}$. For dense T, $\overline{T}^* = (T^{**})^* = (T^*)^{**} = \overline{T}^* = T^*$. The symmetric T suggests that $T \subset T^*$ is also dense too. Then $T \subset \overline{T} \subset \overline{T}^* = T^*$. We know this means T is closable, so extend T into T^{**} giving us equality, T is self-adjoint by its symmetric property.

Suppose a morphism $l : \mathcal{B}(\mathbb{C}, \mathbb{C}) \to \mathbb{C}$ from a measurable function $\mathbb{C} \to \mathbb{C}$ to $\mathbb{C}$. We apply it: $l(f) = (f(T)v, v)$. The key is $(\|f(T)\|^2 v, v) = 0 \to \|f(T)\| = 0 \to f(T) = 0$. That's, the kernel of l. Due to injectivity of norm, we decompose it to $\varphi_T : C(\mathbb{C}) \to \text{End}(H)$ with $f \to f(T)$ for linearity operator between H. The kernel $f(T) = 0$ must be well defined. Suppose a point valued function f that $T - \lambda_{\{x\}}$ that $f(T)|_{\{x\}} = 0$ must be well defined, thus for all possible, may not countable λ_μ for certain measure μ must be defined upon it to depict the whole kernel, for example the product form like $\prod_\mu (T - \lambda_\mu)$. Now we said that such λ called **spectrum** of T. We can immediately inspect that there must countable or isolated points set and a continuous part as $\mu_s + \mu_c$ if you want. Thus we see $f(T) = 0 \Leftrightarrow f|_{\sigma(T)} = 0$ giving us $\mathcal{B}/\left(f|_{\sigma(T)} = 0\right) \to \mathcal{B}(\sigma(T))$ where $\varphi_T(f) = 0$ if $f|_{\sigma(T)} = 0$.

Measure given on $(f(T)v, v) = \int_\mathbb{R} f(x)d\mu(x)$ for non vanishing function f on spectrum with $\mu(x) = \mu_{T,v}(x)$ upon specific T and v. The transformation giving by $U : L_2(\sigma(T), \mu) \to H$ is isometry which retains the norm. So that's the light comes from:

$$T(U(f))(v) = T(f(T))(v) = (xf)(T)(v) = U(xf)v \qquad (4.34)$$

$S = U^{-1}TU = x(*)$, is a multiplication operator on $L_2(\sigma(T), \mu_v)$. It should be clarified that we don't construct somehow isomorphism, such map between them is not injective, Therefore the x corresponds to a unknown measurable function g in original space.[16]

We formalize a new measure on the inner product where the orthogonal projection $\mu_v(A) = (\Pi_A(v), v) \in \mathbb{R}$ induce the Π_A as the idempotent measure as $\Pi_{T,\mathbb{R}} = \Pi_{T,[-r,r]} = I$ $\Pi^2 = \Pi = P$ as the project operator, thus $(T(v), v) = \left(\int_\mathbb{R} fd\Pi(v), v\right) = \int_\mathbb{R} f(\lambda)d\mu_v(\lambda)$ should be uniquely determined by f. We decompose $f = \sum_i \alpha_i \chi_A$ for specific $A_i \in \mathbb{R}$. Such measure is finite that $\left\|\int f(\lambda)d\mu_{v(\lambda)}(v)\right\| \leq \|f\|_\infty \|v\|$. Thus $\left\|\int f(\lambda)d\mu_v\right\| \leq \|f(\lambda)\|_\infty$. The identity operator is $I = \int_\mathbb{R} d\mu_v(\lambda)$. You may say bad luck that we can't deduce $f(\lambda)$ ever and ever! But the trick is to change variable:[17]

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[16]Such technique corresponding to descrete space is also related to polynomial, that defines a $\mathbb{F}[x]$-module with multiplication as endomorphism $x \cdot v = Av$ which has a non-zero kernel. Then $\mathbb{F}^n[x]$ should construct a homomorphism $\mathbb{F}^n[x] \to V$ to deduce the diagonal or rational form.

[17]It suggests that first is to quote T can be transformed into the multiplicative function g, then we use projection measure to transform onto λ variable by χ_g. But I guess jumping to the projection measure is a fine idea.

$$\int_{\mathbb{R}} f(\lambda) d\mu_v(\lambda) = \int_{\mathbb{R}} \lambda d\nu_v(\lambda) \tag{4.35}$$

$$f(\lambda) = \lambda'$$

If there' s a $h \in H$ that $\left(h, \int_{\mathbb{R}} f(\lambda) d\Pi(\lambda)(h)\right) \leq \|f(T)h\|\|h\| = 0 \to f(T)h = 0$. That' s, decompose the whole space except $\ker(T)$ into its orthogonal projection. In practise, that' s where $Lv = \lambda v$ comes from.

All above is just a rephrase of $T = \lambda_\mu v_\mu \otimes w_\mu$ for certain measure. Thus we has $(T - \lambda)^{-1} \sim \frac{1}{\lambda - \lambda_\mu} w_\mu \otimes v_\mu$, thus $\frac{1}{2\pi i} \oint_C (...) = w_\mu \otimes v_\mu$, take product on arbitrary function inducing $(x_\mu, w_\mu) \otimes (v_\mu, y_\mu) = \delta(x - y)$ called **Green Function** or **Resolvent** of T as $G(x, y, \lambda)$. Usually we take $\lambda = 0 \to G(x, y)$, where $TG = w_\mu \otimes v_\mu$, thus, $G(x, x') = \frac{1}{\lambda_\mu}(x_\mu, w_\mu) \otimes (v_\mu, x'_\mu)$.

We has the below form for spectrum where the $\sigma(T)$ is the set of all λ support that $T - \lambda$ is not injective on $D(T)$ where the support x losing information. Suppose $v \in \text{Ker}(T - \lambda_0)$:

$$0 = ((T - \lambda_0)v, w) = \left(v, T^* - \overline{\lambda_0}w\right) = \left(v, \left(T - \overline{\lambda_0}\right)w\right) \tag{4.36}$$

If $\lambda_0 = \overline{\lambda_0}$, a real number, for all $w \in D(T)$, then $T - \lambda_0 = 0$ yields a contradiction. Thus, λ_0 must be a complex number and $v = 0$. Thus, the spectrum is real subset.[18])

It' s rather tiring to prove the unbounded case which I would rather quote the support of the space if could. You will find that we choose $T - i$ as a cheat, because it' s not in the spectrum thus it' s injective and dense for bijection, therefore it' s continuous and then bounded. We apply again to prove that.

5 Royal Road

A partial derivative operator:

$$i\partial_x : D(i\partial_x) = \{f \in H \mid f \in C^1\} \to H \tag{5.37}$$

If we apply inner product on H:

$$\int f i\partial_x g d\mu = \int i\partial_x f g d\mu \tag{5.38}$$

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[18]It' s not a proof, seriously you need to prove $T + i$ for its kernel and image that $(Tv, v) = (v, Tv) = \pm i\|v\|^2 \to v = 0$, thus i shouldn' t be in the spectrum. The same for $-i$. And therefore the $\text{Ker}(T - i) = \text{Im}(T - i)^\perp = 0$ for arbitrary complex number that not a real.

Wait, are we missing something? We force a vanishing upon the boundary condition. Suppose for $H(\mathbb{R})$, for a compact support that such f, g always vanishing on the boundary is called distributional derivative. If the derivative is in the H, such space called **Sobolev Space** $H^1(\mathbb{R})$ with a finite condition on $\|\partial_x f\|$. We can define a higher order derivative for higher order Sobolev space $H^k(\mathbb{R})$.

Dual of Sobolev space $H^{-k}(\mathbb{R})$, can be represented as a sum of L_2 functions actings on derivatives of test functions as $\sum_\alpha \int f_\alpha D^\alpha u$ with $f_\alpha \in L_2$ upon u. For example, the delta function is in H^{-k} but not in the H^k space, where $H^{-k} \supset L_2 \supset H^k$.

It should be noted that $\delta(x)$ upon 1_μ is 1_μ.

Given a spectrum of $i\partial_x$:

$$i\partial_x f = \lambda f$$
$$f(x) = e^{i\lambda x} \quad \lambda \in \mathbb{R}$$

(5.39)

Represent the diagonal transformation. For any function f in H^1 can be densely inserted into L_2 space. We thus decompose a function into its spectrum by:

$$\left(f, e^{i\lambda x}\right) = \int f(x) e^{-i\lambda x} d\mu(x) = \mathcal{F}(f(x))(\lambda)$$

(5.40)

Now we can say that $e^{i\lambda x}$ is the general eigenfunction of operator $i\partial_x$, with the orthogonal relation:

$$\int e^{i\lambda x} e^{-i\lambda' x} d\mu(x) = \frac{e^{i(\lambda - \lambda')x}}{i(\lambda - \lambda')}\Big|_{-\infty}^{+\infty} = 2\pi\delta(\lambda - \lambda')$$

(5.41)

By contour integral.

With the summation of spectrum:

$$f(x) = \frac{1}{2\pi} \int \left(f, e^{i\lambda x}\right) e^{i\lambda x} d\mu(\lambda)$$

(5.42)

As the so called inverse Fourier transform.

Now we impose the operator on $H[a, b]$ with periodic boundary condition that $f(a) = f(b)$, thus is self-adjoint for $i\partial_x$:

$$i\partial_x f = \lambda f$$
$$f(a) = f(b)$$

(5.43)

$$e^{i\lambda a} = e^{i\lambda b}$$

$$\lambda = \frac{2\pi n}{b-a} = \frac{2\pi n}{L} \tag{5.44}$$

$$f(x) = e^{x\frac{2\pi i n}{L}}$$

With a orthogonal relation:

$$\int e^{i\lambda x} e^{-i\lambda' x} d\mu(x) = (...)|_a^b = L\delta(\lambda - \lambda') \tag{5.45}$$

Take the decomposition of f:

$$\left(f, e^{i\lambda x}\right) = \int f(x) e^{-i\lambda x} d\mu(x) = \mathcal{F}(f(\lambda)) \quad \lambda \in \frac{2\pi n}{L} \tag{5.46}$$

With the summation:

$$f(x) = \frac{1}{L} \int \left(f, e^{i\lambda x}\right) e^{i\lambda x} d\mu(\lambda) \tag{5.47}$$

Usually the terms L or 2π will be placed on one side like inverse Fourier transform or evenly placed as $\frac{1}{\sqrt{*}}$.[19]

Actually, because $i\partial_x \leftrightarrow \partial_x$, the ∂_x all spectrum should be $\mathbb{C}/\mathbb{R}$. Thus $(A - \lambda I)^{-1}$ is well defined as $G(\lambda), \lambda \in \mathbb{R}$:

$$(\lambda - \partial_{x'})G(x, x') = \delta(x - x')$$
$$G(x, x') = e^{-\lambda(x-x')}\Theta(x - x') \tag{5.48}$$

Where $Tu = f$ gives:

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[19]You can choose many conditions as long as it forms a self-adjoint operator. Usually, they are Dirichlet or periodic boundary condition, or Neumann boundary condition that $\partial_x f(a) = \partial_x f(b)$, or Robin boundary condition that $\alpha_1 f(a) + \alpha_2 \partial_x f(a) = \beta_1 f(b) + \beta_2 \partial_x f(b)$ for some constants α_i, β_i where the operator is self-adjoint. The key is to ensure the operator is self-adjoint, then the spectrum is real and the eigenfunction is orthogonal.

$$u(x) = \int uT(x')G(x, x')d\mu(x')$$

$$= \int ((\lambda + \partial_{x'})uG d\mu(x') + uG|_?^?$$

$$= \int fG d\mu + uG|_?^?$$

$$= \int fe^{-\lambda(x-x')}d\mu(x') + (ue^{-\lambda(x-x')}|_?^?$$

(5.49)

Given a function where $x > x'$, or we change variable to $x - x' = t$, which called **Resolvent** of ∂_x. Even through the operator is not equal to its dual. If you choose $t \in L_1[0, \infty]$ that the boundary term only left with $-u(0)$. Now given a arbitrary λ as $R(\partial_x)(\lambda)$:

$$u(x) = \int_{-\infty}^{x} f(x')e^{-\lambda(x-x')}d\mu(x') = \int_0^{+\infty} f(x+t)e^{-\lambda t}dt \qquad (5.50)$$

$$R(\partial_x)(\lambda) = \int_0^{+\infty} f(t)e^{-\lambda t}dt = u(0)(\lambda) \qquad (5.51)$$

Is the boundary term, which is **Laplace Transform**. Evaluate further that $f(x+t) = D(t)(f(x)) = e^{t\partial_x}(f(x))$. Generally, ∂_x is the generator of the translation operator. Where $D(t) = e^{t\partial_x}$ is the continuous semi-group of the generator. We know that the contour of green function would be:

$$\frac{1}{2\pi i} \oint_C (\lambda I - T)^{-1}d\mu(\lambda) = I \qquad (5.52)$$

For possible measure, thus we has below:

$$\frac{1}{2\pi i} \oint_C e^{t\lambda}(\lambda I - T)^{-1}d\mu(\lambda) = e^{tT} = D(t) \qquad (5.53)$$

Where given ∂_x we has:

$$\frac{1}{2\pi i} \oint_C e^{t\partial_\lambda}(\lambda I - \partial_x)^{-1}(f(x))d\mu(\lambda) = D(t)(f(x)) = f(x+t) \qquad (5.54)$$

Choose $x = 0$ gives us **Inverse Laplace Transform**.

$$(L_1(r_i, ..., r_n) + L_2(r_i, ..., r_{n'}))\psi = 0 \tag{5.55}$$

If there's a separation indicates:

$$L_1(\{r_i\})\psi = -L_2(\{r_{i'}\})\psi$$
$$L_1(\{r_i\})\psi = \lambda\psi \tag{5.56}$$
$$\psi(\{r_i, r_{i'}\}) = C(\{r_{i'}\})\psi(\{r_i\})$$

Where $r_i, r_{i'}$ are distinct, and ψ is a eigenfunction of the operator L_2, then it must give us a eigenfunction on L_1 which is a smaller decomposition. We may repeat again and again until the set of coordinates are exhausted and sole as $L(r_i)\psi = \lambda\psi$ for a specific r_i where the coordinates are independent. It should be considered that ψ must be automatically a general composition of all eigenfunctions.

For example:

$$\frac{\partial\psi}{\partial t} = \kappa\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\frac{\partial\psi}{\partial\varphi}\right) \tag{5.57}$$

In order to derive r single coordinate, we already see that we already separate t, so it's a first constant. Second, we multiply r^2 and we inspect that all terms are separated from r, so we acquire:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \left(\frac{\lambda_1}{\kappa}r^2 + \lambda_3 + \lambda_4\right)\psi = 0 \tag{5.58}$$

As for θ, we multiply r^2 and separate all unrelated terms, we leave only θ and φ. Then, multiply $\sin^2(\theta)$ to yield a independent operator on θ:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \left(\left(\frac{\lambda_1}{\kappa} + \lambda_2\right)\sin^2\theta + \lambda_4\right)\psi = 0 \tag{5.59}$$

Where we should know that the equality will reduce the number of known, so we choose $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{\lambda_1}{\kappa} + \lambda_2 + \lambda_3 + \lambda_4 = 0$, or other way you want to reduce the number of knowns.