Category And Homology

1 Category

1.1 Abstract and Concrete

What is category? Once god said there will be light, there will be. We aren′t god, but concept are material that could affect other concept by picturing on it. We gives a bunch of effect as $f, g, h...$, that′s means we have a indication of do-nothing effect, which means we have a effect as $1_{(..)}$, display itself which called **object**.

Summarize, we have **morphism**, each can be composite as effect chain $fgh...$. Then $f : X \to Y$ send effect one to another, as $1_Y f 1_X = f$. For simplicity, we write as $X \to Y$. Such system is nominated as **Category**.

Usually, we start a *concrete* objects constructed on sets.

Example.
- Group has groups has objects and homomorphisms as morphisms.
- Set has sets as object and functions as its morphisms

These objects are sets with specified mathematical structure. However, we can also define abstractly, we only consider object as relation of morphism.

Example.
- A poset $(P, \leq)$ whose elements of P are the objects, with a unique morphism $x \to y$ if and only if $x \leq y$. With identity picturing as $x = y$ if also $x \geq y$.
- A group G(or more generally, a monoid) defines a category BG as a single object, whose morphism is self-mapping, or endomorphism, as $g(h) = gh$, with $e \in G$ as identity as $efe = f$.

Russell′s paradox tells us that there′s no set whose elements are "all sets", for example, a "set" not in the set! That′s the reason we don′t lay out our basis on set. We distinguish "small" set and "large" set as set and class, with a abundant outcome from logic-theorist we won′t explore here.

Definition 1.1.1. *A category is **small** if morphisms formed a set.*

A small category is a category everything should be in **a set**, which is a global condition. That′s indeed means every do-nothing identity is **in** a set, which means object are also in a set.

However, Group or Set indeed are set of sets. So there aren′t small. However, we know each map on each objects is a set.

Definition 1.1.2. *A category is **locally small** if pick any pair of objects as morphisms form a set.*

We denote as $\mathrm{C}(X, Y)$ or $\mathrm{Hom}(X, Y)$ borrowed from group homomorphism. Notes, indeed, that's means for $\mathrm{C}(X, X)$ on the same object, formed a set, indicating a single object is a set. However, we can't guarantee everything as a whole is **a set**!

Now we evaluate morphisms. We can't rely on set concept, however we can rely on identity concept. We want to know which extent the effect retain informations on itself and the target.

The key is **natural**, a concept is valid or natural if it doens't depend on the selection of a way or object. Thus, if for any composite $fh = fg = f...$ for g, h and many as a selection, f doesn't care about selection would give us $g = h = ...$, evaluate that it retains the informations of every equal mapping from source, so we call it **monomorphism**.

Similarly, if $hf = gf = ...f$ giving us $h = g = ...$, it means f retains the informations of every equal mapping to target, so we call it **epimorphism**.

A morphism is a effect on other effect, as below, it means every morphism can be reformulate as a map on other map.

We can format that $f^* : \mathrm{C}(A, B) \to \mathrm{C}(A, C)$ as a post-composition or $f_* : \mathrm{C}(B, A) \to \mathrm{C}(C, A)$ as a pre-composition, giving us that $f^*(h) = f^*(g) \to h = g$ is a injection for monomorphism and epimorphism if such morphism set exist, as in locally small category.[1]

Now, what if morphism as composition is a surjection? Suppose for $\forall h \in C, \exists g \in B$ $s.t. f^*(g) = fg = h$. Indeed, it suggest that 1_C still in there with $f^*(g')h = 1_C h = h$ $\exists g' \in B$. Then $f^*(g) = h = f^*(g')h = f^*(g')f^*(g)$.

Now exploit that if $fg = 1_C$ always, f must map all information back to C integrally, we call it **split epimorphism**, also, reversely, we know g must map all information of C without loss, so we call it **split monomorphism**. Why? Because $h'f = hf \to h'fg = hfg \to h' = h$. That's what we seek, a stronger declaration.[2]

The key is due to the do-nothing effect of $1_{(...)}$, a looser condition is for any composition $(...)fh_1 = (...)fh_2$ as monomorphism, we can derive that:

$$h_1 = h_2 \leftarrow (...)fh_1 = (...)fh_2 \leftarrow fh_1 = fh_2 \tag{1.1}$$

With f as a monomorphism, and vice versa.

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[1] Some book denote locally small category as category.
[2] Note the definition is reverse in split version!

We delve further, a mirror copy or do-nothing which we call it **isomorphism**, can reverse back to original without loss. Then such morphism must, as a composition of morphism set, is a bijection.

Suppose f as composition f^* is a bijection, it means it's a monomorphism as injection and split epimorphism as surjection. Reversely, it must define a g with $fg = 1 \rightarrow fgf = f \rightarrow f(gf) = f \rightarrow f(gf) = f1_A \rightarrow gf = 1_A$. Therefore we define $g = f^{-1}$, the reverse of f uniquely as two side reverse.

A rather hindsight on Set is indeed, every object is a set and also morphism betweens form a set, so morphism corresponds to set concept of injection, surjection, bijection. However, in element view, we can construct split epimorphism in epimorphism by choice axiom.

It's rather simple, $f(a) = b$ for all b, then we define a reverse $s.t.$ $g(b) = a$ for $\forall b \in B$. Choice axiom guarantee its existence, then $\forall b \in B$, we has $fg(b) = f(a) = b$, indeed, that's the definition of surjection actually. Same procedure can be done to derive for monomorphism as injection.

A caveat, in different category, monomorphism and epimorphism doesn't correspond to injection and surjection. For example, in Ring, ring category with all ring as objects and ring homomorphism as morphism, we has $\mathbb{Z} \rightarrow \mathbb{Q}$ as inclusion $s.t.$, every $f(p)/f(q) = f(p)f(q)^{-1} = f\left(\frac{p}{q}\right)$ Which means every element in $\mathbb{Q}$ is uniquely defined in such homomorphism which is a monic and epic. However, inclusion isn't surjection obviously. This means we can't find $\frac{p}{q} \rightarrow n \in \mathbb{Z}$ to reversely construct split epimorphism.

Above analysis indicates that a duality of morphism, suppose we have a operation which only reverse composition:

$$(fg)^{\mathrm{op}} = g^{\mathrm{op}} f^{\mathrm{op}} \tag{1.2}$$

Deduce that:

$$(...)^{\mathrm{op}} : Y \rightarrow X \leftrightarrow (...) : X \rightarrow Y \tag{1.3}$$

Indeed, this indicating that monomorphism becomes epimorphism and vice versa in composition meaning, with a invariant inclination on both-side invariance, for example, isomorphism.

In such view, we can evaluate on one side and reverse arrows to get a dual evaluation. In general, the concrete morphism in opposite meaning is vague for example, a set function $f : \{1, 2\} \rightarrow \{3\}$ is a epimorphism, but its reverse is vague to choose 1 or 2 as a concrete monomorphism, but indeed its a monomorphism.

1.2 Diagram and Functor

What's the relation between category? If such relation exsit, just like morphism, it must map each morphism and object, or just morphism, but globally rather pairwise! Into another category. Such map must be composable and maintain identity as $F(1_C) = 1_{F(C)}$ which is just object mapping.

We call above as **functor**. List example below:

Example.

- endofunctor, which is self mapping of category, $P : \mathsf{Set} \to \mathsf{Set}$ send a set to its power set and $f : A \to B$ to its $F(f) : PA \to PB$.

- **forgetful functor**, for every concrete category, define to forget the mathematical structure while maintaining as set function. $U : \mathsf{Group} \to \mathsf{Set}$ sends a group to underlying set.

This gives that a **concrete category** is a category equipped with forgetful functor.

How to acquire a measurement of functor? We thus need to divide concept locally and globally. Follow our routine for morphism and smallness, we could give a version of epimorphism and monomorphism in functor.

Definition 1.2.1.
- ***full*** *if for pair's morphisms* $\mathrm{C}(A, B) \to \mathrm{D}(F(A), F(B))$ *is surjection.*
- ***faithful*** *if for pair's morphisms* $\mathrm{C}(A, B) \to \mathrm{D}(F(A), F(B))$ *is injection.*

Indeed, we see it's a local condition, So above condition doesn't guarantee across pair's morphisms, for example, two morphisms in different source and target could be map to a same morphism set, which could also be **faithful** or **full**.

Example. Category $F : 2 \to 1$ with $F(1/2_2) = 1_1$. We analyze that:
- $2(1,1)/2(2,2) \to 1(1,1)$ is a $\left\{1_{1/2}\right\} \to \{1_1\}$ which is injection.
- $2(1,2) \to 1(1,1)$ is a $\{f\} \to \{1_1\}$ which is also a injection

Then we conclude above is a faithful and even full functor! The local condition doesn't ensure element shrinkage situation. At least, we know forgetful functor is **faithful**.

Thus, we introduce a global condition:

Definition 1.2.2.
- ***essentially full***[3] *if map of each morphisms is surjective.*
- ***embedding*** *if map of each morphisms is injective.*

Sorry to abuse terms here, we can formulate this as if $F(f_1) = F(f_2)$ then $f_1 = f_2$ globally for **embedding** and similarly, if $F(f) = g \quad \forall g \in \mathrm{D}$ for **essentially full**. That's
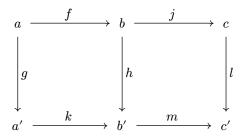
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[3]Or **essentially surjective on objects**, but that's lengthy actually

the violation of above example. We see it's not **embedding**. A refrain of above is to account of object rather only pair morphism, so we can say a functor is essentially full if $F(A) = B \quad B \in \mathrm{D}$.

Commonly, we call a **embedding** of a category to another is a **subcategory**, and a **full embedding** of a category is a **full subcategory**, which preserve all morphisms between objects.

Now given a special functor, which called **diagram**, or we want to make it a functor. It maps a index category which has bunch of vertices in a set as objects and its corresponding morphism as edges, which are also in a set. Thus it's a small category. What's the role? Suppose the diagram functor $F : \mathrm{J} \to \mathrm{C}$ maps into a category to mimic the relation of such category, we realize that index category equipped with a preorder which means any composite of morphism as routes or paths from same source to same target give a "same" morphism. Rephrase, it means any routes as composition is unique because of preorder property. In its codomain or C, we acquire that any morphisms composition would yield the same results. We call such property is **commutative**, and diagram as **commutative diagram**.

In general, morphisms isn't commutative, but a few emerged if we seek:

$$
\begin{array}{ccccc}
a & \xrightarrow{\ f\ } & b & \xrightarrow{\ j\ } & c \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} \\
a' & \xrightarrow{\ k\ } & b' & \xrightarrow{\ m\ } & c'
\end{array}
$$

If we know outer rectangle is commutative, how about inner? Indeed, a whole path is commutative can't ensure inner path is too unless:

$$gh_n...h_1f = gk_m...k_1f \tag{1.4}$$

We know if g is monomorphism and f is epimorphism giving the **cancellable** to infer inner path is commutative.

Then a given selection of inner rectangle could infer commutative? Suppose we know left or right rectangle is commutative, and also the outer:

$$\text{outer} : ljf = mkg$$
$$\text{left} : hf = kg \tag{1.5}$$
$$\rightarrow ljf = mhf$$

Now we know if f is epimorphism yielding the right square commutativity. Reverse arrow to deduce that m must be monomorphism to yield the left square commutativity.

A other useful consequence is that if commutativity is just a equality between different morphisms composition, then injection suggests that if image is same, preimage must be also same too. It shows that a faithful functor always preserve commutativity if codomain is, then domain is too.[4]

1.3 Naturality

For category $\mathcal{C}_1$ and $\mathcal{C}_2$, we define $\mathrm{Fun}(\mathcal{C}_1, \mathcal{C}_2)$ as the category of functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. The objects are functors and morphisms are natural transformations. Is that well defined? Usually, we should ensure that morphisms of this category is a set. This can be achieved if $\mathcal{C}_1$ and $\mathcal{C}_2$ is small, a.k.a the objects are in a set. Then for $F_1(\mathcal{C}_1) \xrightarrow{\eta} F_2(\mathcal{C}_1)$, we know $\mathcal{C}_2^{\mathcal{C}_1}$ defines the maximal set of possible functors between them. Then the power set of $\mathcal{C}_2^{\mathcal{C}_1}$ is a set, which is the set of all possible morphisms between them, gives a set of natural transformations. So indeed, $\mathrm{Fun}(\mathcal{C}_1, \mathcal{C}_2)$ is a category.

We can also define a functor $\mathrm{Env}(\mathcal{C}_1, \mathrm{Fun}(\mathcal{C}_1, \mathcal{C}_2))$, given by $\mathrm{Env}(A, F) \rightarrow F(A)$ which is called **evaluation functor**.

Given a pair that $(A, F) \in \mathcal{C}_1 \times \mathrm{Fun}(\mathcal{C}_1, \mathrm{Set})$, we give a correspondence $\mathrm{Hom}_{\mathcal{C}_1}(A, A) \rightarrow F(A)$. Naturally, we can define generally $\mathrm{Hom}_{\mathcal{C}_1}(A, -) \rightarrow F(-)$. Which usually, the left term is also a object in $\mathrm{Fun}(\mathcal{C}_1, \mathrm{Set})$ because $\mathrm{Hom}_{\mathcal{C}_1}(A, -)$ is a functor from $\mathcal{C}_1$ to Set of morphisms, which is a functor. We can see indeed, the map that $\mathrm{Hom}(I_A, f)\big(\mathrm{Hom}_{\mathcal{C}_1}(A, A)\big) \rightarrow \mathrm{Hom}_{\mathcal{C}_1}(A, B)$ while $F(f)(F(A)) \rightarrow F(B)$. Now suppose various $\mathrm{Hom}(I_A, f)(g) \rightarrow \mathrm{Hom}(I_A, f{\circ}g)$ corresponds to $F(g)F(f)$. And finally $\mathrm{Hom}(I_A, I_A) \rightarrow F(I_A)$ which is identity. It seems that we construct a relation between these two objects in $\mathrm{Fun}(\mathcal{C}_1, \mathrm{Set})$.

Now a deep sight is every map of object $B, C...$ is a morphism from $A \xrightarrow{f} B$ or $A \xrightarrow{g} C$, which is just a **representation** by $f, g...$ from $\mathrm{Hom}_{\mathcal{C}_1}(A, -)$ mapping to $F(B) = F(f(A)), F(C) = F(g(A))...$, we define the morphism or the natural transformation as $\Theta_{A,B,C...} = \Theta$. For every $x, y... \in A$, we can define each different $\Theta', \Theta''...$ given by $\Theta(f) \rightarrow F(f)(x)$ which repeat above process.

So for every possible natural transformations, we assign it to a unique element $x \in A$ by $\Theta(I_A) = F(x)$. Gives a bijection.

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[4]If domain is, then codomian is. Such proposition is trivial indeed.

$$\text{Hom}_{\text{Fun}(\mathcal{C}_1, \text{Set})}\Big(\text{Hom}_{\mathcal{C}_1}(A, -), F(-)\Big) = \Theta_{(-)} \to F(A) \tag{1.6}$$

is a bijection.

Comment: Reader may confuse about Θ as many components as Θ_A, Θ_B..., they are the same thing. Rather, the different thing is Θ, Θ' assign to different $x, y \in F(A)$.

A useful application or corollary is Given:

$$\text{Hom}\Big(\text{Hom}_{\mathcal{C}_1}(A, -), \text{Hom}_{\mathcal{C}_1}(B, -)\Big) \to \text{Hom}_{\mathcal{C}_1}(B, A) \tag{1.7}$$

Is a bijection. That's a indication that a object is nothing but the relation of other objects. The formally statement should be the relations of two objects are just the possible connections of their relations to other objects.

1.4 Additive/Abelian

Additive category is a category $\mathcal{C}$ for which that, given any pair A,B as object, the $\text{Hom}(A, B)$ inherit abelian group law.

$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$$
$$f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2 \tag{1.8}$$

It doesn't mean category contains a native abelian operation $+$, rather, it's for each object contained in the category. That is the reason of **additive** comes from. However, the operation without abelian isn't well defined.

$$(f_1 \tilde{+} f_2) \circ (g_1 \tilde{+} g_2) = f_1 \circ g_1 \tilde{+} f_1 \circ g_2 \tilde{+} f_2 \circ g_1 \tilde{+} f_2 \circ g_2$$
$$\neq f_1 \circ g_1 \tilde{+} f_2 \circ g_1 \tilde{+} f_1 \circ g_2 \tilde{+} f_2 \circ g_2 \tag{1.9}$$
$$= (f_1 \tilde{+} f_2) g_1 \tilde{+} (f_1 \tilde{+} f_2) g_2$$

That's mean in such category, we has $f_1 - f_2 = 0$ if $f_1 = f_2$. The zero object of category which is the unique element(because the equivalence class of $[f_1 - f_2] \sim [0]$ deduced from the equivalence class of functions.) equipped with unique morphism to any other objects.

$(i_A, 0)$ could be the equivalence natural injection and projection between $A \times B$ and A. So in additive category, product and sum is a equal definition.

Kernel is the description of a morphism that send A to zero by f.

Definition 1.4.1. *A **Kernel** of a morphism $f : A \to B$ is a monomorphism $k : K \to A$ such that:*

1. *Composition as Zero:* $f \circ k = 0$
2. *Universal Property: For any $k' : C \to A$ such that $f \circ k' = 0$ will give us a unique morphism u, such that $k' = k \circ u$.*

The universal property doesn' t ensure its existence. So we just point it out rather prove it.

Usually, it does like zorn lemma in axiom choice, we can always select a general representation for a equivalence relation between possible underneath morphisms.

Thus, Ker(f), if exist, is the equalizer of every possible morphisms that composes f as zero.

We could define in same routine for Coker(f), which is the composition zero by $k \circ f$. That' s the representation of information in target object that doesn' t recorded by f.

Rather, the definition of Im(f) is second in category. We see by $c : B \to \text{Coker}(f)$ $f : A \to B$. We revert the annihilation that Im(f) = Ker(c). Which means the information that recorded by f.

We could define in same routine for Coim(f), which is $c : \text{Ker}(f) \to A$ $f : A \to B$. Coim(f) = Coker(c).

Then we factor out a morphism by f : $f = \text{im}(f) \circ$ factor $\circ$ coim(f). Which means we decompose f as the information from source object A as coim(f) to target B as im(f). But what about "factor" ? That' s the reason of definition **abelian category**, derived from the uncertainty of equivalence relation of coim(f) and im(f). We want to ensure they are isomorphism but only left in intuition.

Definition 1.4.2. *abelian category is an additive category with following additional conditions:*

1. *Any morphism admits a kernel and cokernel.*
2. $\text{Coim}(f) \to \text{Im}(f)$ *is an isomorphism for every f.*

It also indicates a restriction that bijection is isomorphism because Ker(f) = 0 and Coker(f) = 0 indicates:

$$A \to \text{Coim}(f) \to \text{Im(f)} \to B \tag{1.10}$$

gives a isomorphism based on the bridge of condition (2).

We know we can factor out a object by a morphism that:

$$0 \to \text{Ker}(f) \overset{\text{ker}(f)}{\to} A \overset{\text{im}(f)}{\to} \text{Im}(f) \to 0$$
$$0 \to A \to B \to C \to 0 \tag{1.11}$$

For which $\ker(f)$ is monomorphism and $\mathrm{im}(f)$ is epimorphism. Observe that $\mathrm{im}(f)\circ\ker(f) = 0$. Which means we fully factor out a object! We call this sequence of morphism is **exact**. We can define in same routine for left exact and right exact, which corresponds to partial factor by left monomorphism or right epimorphism.

If a functor apply to the sequence s.t.:

$$0 \to \mathrm{Hom}(X, A) \xrightarrow{u_*} \mathrm{Hom}(X, B) \xrightarrow{v_*} \mathrm{Hom}(X, C) \tag{1.12}$$

This functor maps every object in $\mathcal{C}$ to $\mathrm{Hom}(X, \mathcal{C})$ $X \in \mathcal{C}$. Each object is a morphism set and each morphism is a consecutive concatenation by $u_*(f) \sim u\circ f$ $u \in \mathrm{Hom}(A, B)$ in above map.

What property could be preserved? We see in original exact sequence that u is monomorphism s.t. $u_*(f) = u\circ f$ implies that if $u\circ f = u\circ 0$, then $f = 0$. Which still preserves the property(Same operation could be reversed to prove original category left exactness). We can do the same to prove $\mathrm{Ker}(v_*) = \mathrm{Im}(u_*)$. However, we can see directly that left operation can't ensure right property which is epimorphism, resulting only left exactness.

Finally, there are also **half exact** functor which only preserves $\mathrm{Ker}(F(v)) = \mathrm{Im}(F(v))$. Everything can also be reversed as in $\mathcal{C}^{\mathrm{op}}$ as a reversed factor(epi to mono, mono to epi, then reverse the arrow).

$$\mathrm{Hom}(A, X) \xleftarrow{u_*} \mathrm{Hom}(B, X) \xleftarrow{v_*} \mathrm{Hom}(C, X) \leftarrow 0 \tag{1.13}$$

Which is still left **monomorphism** exact. We notice $v_*(f) = f\circ v$, giving us the map from B to X. If v is epimorphism, we know $f\circ v = 0$ indicating $f = 0$, resulting the monomorphism of $v_*(-)$. How can we ensure u_* as epimorphism, we gains nothing unless additional property is given.

1.5 Generators and Cogenerators

These elements aren't the smallest elements, but rather abundant elements capturing enough information.

Definition 1.5.1. *There exist a family of objects $(U_i)_{i\in I}$ is a family of generators that for any object A and its suboject B, there always exist $u_i : U_i \to A$ such that the image isn't contained within B.*

The above proposition can be also explained as u_i can't be factored as a smaller morphism through B. However, it also indicating that we can always determines B by

morphisms from other possible set of U_i, or we can always have enough informations to determine a larger subobject through out any object.

We can reversely defining cogenerators by declaring that every $A \to U_i$ must goes through subobject B.

Above definition is already in abelian category, so actually projection and inclusion would be same, gives a isomorphism between generator and cogenerator.

In abelian category, we say that $\pi : \bigoplus U_i = U \to A$ as a quotient of possible index set of U_i.

1.6 Injective and Projective Objects

Injective/Projective objects is the special objects that having enough spaces to contains or gives out information.

As a symmetry, we choose injective:

Definition 1.6.1. *An object M is injective if any morphism into M from subobject $B \subseteq A$ can be extended to morphism from larger object A*

We can reverse the definition that projective object would extended to smaller object.

However, it gives a deeper relation to exact sequence by Hom(-,M) contravariant functor.

$$0 \to B \xrightarrow{u} A \xrightarrow{v} A/B \to 0 \tag{1.14}$$

Where Coker(u) is B. Apply functor that:

$$0 \to \mathrm{Hom}(A/B, M) \xrightarrow{v_*} \mathrm{Hom}(A, M) \xrightarrow{u_*} \mathrm{Hom}(B, M) \to 0? \tag{1.15}$$

We don′t know wether the final arrow is possible to construct. If M is injective, then $u^*(h) = h \circ u$ for $h : A \to M$, would always possible to extended to larger object by the monomorphism u from B s.t. $u^*(h) = h \circ u = g$ for $g : B \to M$. Where we recover the definition of injective object.

We can reverse the definition that projective object applied as Hom(P, -) preserving right exactness by $v_*(k) = v \circ k = g$ for $\forall k : P \to A$ with $g : P \to A/B$.

The useful technique could be derived from generators. Suppose $i_u : V_i \to U_i$ as a inclusion in subobject of U_i generator, $i_b : B \to B'$ as subojects inclusion of A, given a relation of $U_i \xrightarrow{j} B$.

$$V \xrightarrow{\phi'} U \times B \xrightarrow{\phi} B' \to 0 \tag{1.16}$$

with $\phi'(v) = (i_u(v), -j \circ i_u(v))$ and $\phi(u, b) = i_b \circ j(u) + i_b(b)$.

If the sequence is exact, i.e. $\phi \circ \phi'(v) = i_b \circ j \circ i_u(v) - i_b \circ j \circ i_u(v) = 0$, indicating that the V properly included in B, that's the preimage, i.e $j^{-1}(U_i) = V_i$. We thus construct B' as $\mathrm{Coker}(\phi')$, which is $(U \times B)/V$, modulus already known information of U already in B, gives us B'.

The greatest thing is if a category has generator, we can inductively apply this procedure, while due to zorn lemma, every object violating such property would be contradicted to our proposition of such building procedure, giving us a maximal element.

This proof can be stronger that: i_b as inclusion can be proved rather imposed.

For every possible morphism $Z \xrightarrow{m} B \xrightarrow{f} U \times B \to B' \to 0$, s.t. $i_b \circ m = 0$, which means the image of morphism m is contained in V, s.t. $i_b \circ m = \phi' \circ v$ for $v : Z \to V_i$. We know the inclusion $f \circ m = (0, m) = \sigma \circ v$.

From left component: $(\sigma$ to left$) \circ v = 0 \sim v = 0$ because σ is inclusion as injection. Then $m = 0$ from right component, therefore $i_b \circ m = 0 \sim m = 0$, gives i_b is injection.

That's reasonable because V_i is the information both from B and U_i, so if the morphism m image is only contained B is absurd, rather, the gained information must be comes from U_i as a extension.

Above observation implies that rather for object and its suboject relation, every object can be given a injection object, resulting injective property in such category.

1.7 δ-Functor

Commonly, a functor isn't exact, it may be half exact, left or right exact, which means the sequence isn't fully decomposed by morphism.

Consider a functor F from an abelian category to a additive category and at least half exact, apply to a short exact sequence:

$$? \to F(A) \xrightarrow{F(u)} F(M) \xrightarrow{F(v)} F(A') \to ? \tag{1.17}$$

$F(v)$ isn't surjective now, if we want to measure the failure of surjectivity, we can define $\frac{F(A')}{\mathrm{Im}(F(v))}$. Rather, a general question is whether or not the definition is canonical? If M is injective, the original exact sequence exactness indicates the definition of A' is always defined by M itself. So $\frac{F(A')}{\mathrm{Im}((F(v)))} \to \frac{F(A')}{\mathrm{Coker}(F(u))}$ is well defined. Then we can seek a general definition by choose many possible M as injective object, information of A can always be preserved by various M. Indicating that $\frac{F(A')}{\mathrm{Coker}(F(u))}$ is invariant under the choice of M, we denote it as $SF(A)$. If the category has enough injective objects, we can

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always define such object for every possible A, giving a canonical functor definition as $SF(\alpha) : SF(A) \to SF(B)$. We are close to a general definition...

We denote a system of functors $T = (T_i)$ from an abelian category to a addtive category. Where $T_i : \mathcal{C} \to \mathcal{C}'$, together with special connecting morphsims named as boundary maps $\delta/\partial : T_i(A'') \to T_{i+1}(A')$ for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in each relevant degree i.

To maintain consistency of a functor, if T applies to sequence-A, it can applies to sequence-B, while maintaining commutative property.

The second property is special to δ-functor itself, which is any two morphisms composition is zero, therefore a weaker condition than exactness, at least we won't lose it.

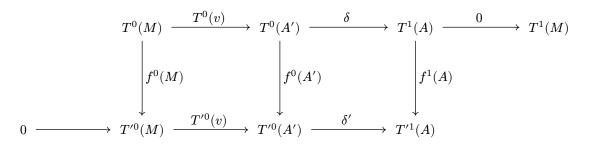An exact δ-functor contains a stronger second condition which is indeed the exactness.

We could reverse arrow to get contravariant δ-functor, or decrease the degree to get δ^*-functor, however, they are all symmetric representation.

If a category has enough injective objects, we can always construct a exact sequence by $0 \to A \overset{u}{\to} M \to \mathrm{Coker}(u) \to 0$. Where M is a injective object. Even, this could be applied to M itself that $0 \to M_0 \to M_1 \to M_2 \to ...$ or simpler $0 \to M \to M \to 0$ as a self injection. This indicates that all rest of M_i will be a trivial object for self injection of M. Take a δ-functor, we know $F(M_i) \overset{F(0)}{\to} F(M_{i+1})$, which means $T^i(M) = 0$ for $i > 0$ as **effacable**. We call such object is **acyclic** stemming from the zero map.

Use a simpler notation that $0 \to A \overset{u}{\to} M \overset{v}{\to} A' \to 0$ for a injection.

We apply exact δ-functor to the sequence, we know $... \to T^0(M) \overset{T^0(v)}{\to} T^0(A') \overset{\delta}{\to} T^1(A) \overset{T^1(u)}{\to} T^1(M) \to ....$ By exactness, we know $T^i(M) = 0$, so δ is surjective as $\mathrm{Im}(\delta) = \mathrm{Ker}(0)$.

Given another T'^i, we could construct a sequence morphism:

$$
\begin{array}{ccccccc}
T^0(M) & \overset{T^0(v)}{\longrightarrow} & T^0(A') & \overset{\delta}{\longrightarrow} & T^1(A) & \overset{0}{\longrightarrow} & T^1(M) \\
\downarrow {\scriptstyle f^0(M)} & & \downarrow {\scriptstyle f^0(A')} & & \downarrow {\scriptstyle f^1(A)} & & \\
0 \longrightarrow T'^0(M) & \overset{T'^0(v)}{\longrightarrow} & T'^0(A') & \overset{\delta'}{\longrightarrow} & T'^1(A) & &
\end{array}
$$

Based on a usual design of commutative diagram, we want to exploit the benefit of M for each T^i and T'^i. If we already know $f^0(...)$ is a natural transformation, which is commutative, exploiting that $\delta' \circ f^0(A') \circ T^0(v) = \delta' \circ T'^0(v) \circ f^0(M) = 0 \circ f^0(M) = 0$, with $\mathrm{Ker}(\delta) = \mathrm{Im}(T^0(v))$, we can get $\mathrm{Ker}(\delta' \circ f^0(A')) \supseteq \mathrm{Ker}(\delta)$. Thus we can factor

$\delta' \circ f^0(A') \circ \ker(\delta) = 0$ by Coker universal property, which means $\text{Im}(\delta' \circ f^0(A')) \subseteq \text{Coker}(\ker(\delta)) = \text{Im}(\delta)$, giving $\delta' \circ f^0(A') = g \circ \text{coker}(\ker(\delta)) = g \circ \text{im}(\delta)$ for a unique morphism g. Recall that δ is surjective, a.k.a $\text{im}(\delta) = \text{coker}(\ker(\delta)) = \delta$. Thus $g = f^1(A)$ is the unique morphism we want to find, making the diagram commutative.

The final technique we use is often common to construct a unique morphism making such diagram commutative. The dual technique is also useful, which just assert that if $\text{coker}(f) \circ h = 0$, a.k.a $h = \ker(\text{coker}(f)) \circ g \to \text{Coim}(h) \subseteq \text{Coim}(f)$, we can decompose $h = \text{coim}(f) \circ g = \text{im}(f) \circ g$ for a unique morphism g. If f is also a monomorphism, we get $h = f \circ g$.

Then we can also denote $f^{-1}(C)$ for subobject C of A and morphism $f : B \to A$ as **preimage**, That's $f^{-1}(A) = \text{im}(f) = \text{coim}(f)$ is always well defined even f^{-1} itself doesn't exist. We know that $B \xrightarrow{f} A \xrightarrow{\pi} C$, given $f^{-1}(\pi^{-1}(C)) = (\pi \circ f)^{-1}(C)$ is also well defined.

A rather instructive insight comes from diagram chasing, which is called **snake lemma**. Based on our previous analysis on right square, we know $\text{Ker}(f^1(A))$ must also the appended Ker upon $\text{Ker}(\delta)$. Based on epimorphism, gives a preimage as $\delta^{-1}(\text{Ker}(f^1))$, then $\delta' f^0(A')(\delta^{-1}(\text{Ker}(f^1)) = 0$, by exactness of lower sequence, we know that $(T'^0(v))^{-1}(f^0(A')\delta^{-1}(\text{Ker}(f^1)))$ is the preimage. Another route would be exploit the fact that $\delta^{-1}(\text{Ker}(f^1)) \subseteq \text{Coker}(T^0(v))$, due to upper sequence exactness.

Due to lower sequence, we know we can map $\text{Ker}(f^1(A)) \to T'^0(M)$. Further, a stronger declaration can be achieved, with commutativity $T^0(v) \circ f^0(A') = T'0(v) \circ f^0(M)$. By previous preimage of $(T'^0(v))^{-1}(f^0(A')\delta^{-1}(\text{Ker}(f^1)))$ denoted as $(...)$, trace up $f^0(A') \circ T^0(v)\left((f^0(M))^{-1}(..)\right) = 0 = T'^0(v) \circ f^0(M)\left((f^0(M))^{-1}(..)\right)$, indicating that $f^0(M)^{-1}(..) = 0$. Which means, indeed $T^0(M) \xrightarrow{f^0(M)} T'^0(M) \to \text{Coker}(f^0(M)) \supseteq (..)$. That's indicating a morphism, if we can construct $d : \text{Ker}(f^1(A)) \to \text{Coker}(f^0(M))$, notice the preimage $(..)$ isn't injective and surjective corresponds to $\text{Ker}(f^1(A))$, so we can only achieve a morphism. But it's a great progress!

Place all thing $\text{Ker}(T^0(M)) \to \text{Ker}(T^0(A')) \to \text{Ker}(T^1(A))$ is exact because they are just part of the above upper sequence, we can give a restriction by π on every component, which gives the same results. Second, $\text{Coker}(T'^0(M)) \to \text{Coker}(T'^0(A')) \to \text{Coker}(T'^1(A))$ is same.

$$\text{Ker}(T^0(M)) \to \text{Ker}(T^0(A')) \to \text{Ker}(T^1(A)) \xrightarrow{d}$$
$$\text{Coker}(T'^0(M)) \to \text{Coker}(T'^0(A')) \to \text{Coker}(T'^1(A))$$

(1.18)

Is a exact sequence, also, if $T^0(v)$ is monomorphism, then $\mathrm{Ker}(T^0(M)) \to \mathrm{Ker}(T^0(A'))$ is monomorphism for the same reason above. If δ' is a epimorphism, yielding $\mathrm{Coker}(T'^0(A')) \to \mathrm{Coker}(T'^1(A))$ is epimorphism.

However, such astonishing consequence is we can link each short exact sequence to a larger sequence, why bother that? Actually we can chain each object vertically to denote each apply functor component in each degree, and horizon is each short exact sequence, then we can see indeed, we chain up them degree by degree!

Now cease the stray, we see that by the above construction, we can always form a sequence natural transformations from the given condition T^i and f_0, continuing building the sequence of T^i and T'^i by f^i if and only if T^i is effaable and with enough injective objects. Thus we can define universal δ-functor, which means for any δ-functor $T' = (T'^i)$ and any natural transformation $f^0 : T^0 \to T'^0$, there exist **unique** morphism of δ-functors $f = (f^i) : T \to T'$ given by f^0. It means the whole behavoir is determined by its behavoir at degree 0!(Apply identity natural transformation to itself.)

Therefore, we define universal δ-functor by $F = T^0$ and right satellite functors of F by $S^i F = T^i$. We can apply same routine to δ^* as decreasing degree, or reverse arrow to define in contravariant δ-functors.

Given this, we chain each exact sequence by higher degree functors.

$$0 \to F(A) \to F(M) \to F(A') \xrightarrow{\delta_0} S^1 F(A) \to S^1 F(M) \to \ldots \qquad (1.19)$$

For acyclic object M, this gives $S^i F(A) = S^{i-1} F(A')$ due to $S^i F(M) = 0$. Then we know $S^1 F(A) =$

2 Simplex

How to make a general topological basic element? We now that 1-dim would be (x), 2-dim would be (x, y) etc... But there aren′t bounded! If we want to make a bounded element. For example, a point, we restrict $x = c$ as a constant. How about 2-dim? We could let $ax + by = c$, oh! Still unbounded! Let $x > 0 \quad y = 0$, everything works. Why we don′t make a point as nothing? Like (0) and make a line as (x) with $0 < x < 1$? Actually, that′s still a reasonable definition.

Definition 2.1. *The geometric n-dimensional **simplex** is the topological space:*

$$\triangle^n = \left\{ (x_0, \ldots x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1 \quad x_i \geq 0 \right\}$$

We omit coefficients because those are flexible for topology.

We denote $\sigma = (e_0, \ldots, e_n)$ as a instance of simplex, with

$$C_n(x) = \bigoplus_{\substack{\sigma_i \in X \\ \dim(\sigma_i) = n}} \mathbb{Z}\sigma_i \tag{2.20}$$

Now we want to get the differential of the simplex, which is its **boundary**. Suppose we trace a simplex from point e_1 to e_n, we know that a small hyper-surface is its all but omit certain e_n. For example (e_1, e_2, e_3), omit e_1, we get (e_2, e_3) which is a boundary line.

Naively, we could think of that d_n as a differential of n-dim simplex would be

$$d_n : C_n \to C_{n-1}, \sigma_i \to \sum_{k=0}^{n} \sigma_i^{\text{omit kth } e_k} \tag{2.21}$$

However, the bounded simplex has direction in boundary, same as previous example, if we omit e_2, (e_1, e_3) isn' t the correct direction. The correct direction in number-wise would be $(e_1, e_2), (e_2, e_3), (e_3, e_1)$, so if the whole number permutation in right direction, the left part points sequence will be counted as final part rather initial part. The omitted point will cause a $(-1)^k$ to give a correct answer.

Apply twice first by k then by l:

$$d_n \circ d_{n-1} = \sum_{k=0}^{n} \sum_{l=0}^{n-1} (-1)^{k+l} \sigma_i^{\text{omit k,lth}} = \sum_{k>l}^{\dots} (-1)^{k+l} \sigma_i^{\text{omit k,lth}} + \sum_{k<l}^{\dots} (-1)^{k+l-1} \sigma_i^{\text{omit l,kth}} = 0 \tag{2.22}$$

The 3rd step is because if $k < l$, first omit kth will results lth index becomes (l-1)th. Then each sum is symmetric, we get zero in end. It' s like a discrete style of total derivative, which is also zero due to symmetry of partial derivative.

However, could a topological space be decomposed by these minimal element? Actually, Partially is. Could the result be independent of triangulation? Yes with a lengthy theorem.