Review of Quantum

1

1.1

$g \in G$, given a action of group on a general space M equipped with function on it as $F(M)$. Thus we can apply g on M and lift upon $F(M)$ as well, i.e. $\varphi(g) \cdot x$ for $x \in M$ and $\rho(g) \cdot f$ for $f \in F(M)$ as a G-equivariant map.

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[d, "\varphi(g)"'] & & f(x) \arrow[d, "\rho(g)"'] \\ \varphi(g) \cdot x \arrow[r, "f"] & \rho(g) \cdot f(x) / f(\varphi(g) \cdot x) & \end{tikzcd}$$

Figure 1. Commutative diagram of group action on space and mapped space

Is commutative, i.e.

$$\rho(g) \cdot (\rho(h) \cdot f)(x) = \rho(h) \cdot f(\varphi(g) \cdot x) = f(\varphi(h) \cdot (\varphi(g) \cdot x)) = f(\varphi(h) \circ \varphi(g) \cdot x) \tag{1.1}$$

Is a anti-homomorphism, i.e. Such representation should be only with g element, thus a canonical choice is g^{-1} where below is given as a reduced notation:

$$g \cdot f(x) = f(g^{-1} \cdot x) \tag{1.2}$$

$V = W \oplus W' \to v = w + w'$, where the projection $q(v) = w$. Define $\overline{q} : v \to \frac{1}{|G|} \sum_{g \in G} \rho(g) q(\rho(g^{-1})v)$ which we drop ρ representation notation.

$$\overline{q}(w) = \frac{1}{|G|} \sum_{g \in G} g q(g^{-1} w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1} w) = w \tag{1.3}$$

$$h\overline{q}(v) = \frac{1}{|G|} \sum_{g \in G} hgq(g^{-1}h^{-1}hv) = \frac{1}{|G|} \sum_{g' \in G} g' q(g'^{-1}hv) = \overline{q}(hv) \tag{1.4}$$

If $v \in \operatorname{Ker}(\overline{q}), h \in G$ then $h\overline{q}(v) = 0 = \overline{q}(hv)$ that $hv \in \operatorname{Ker}(\overline{q})$ that we decompose by a G-invariant morphism based on solely vector space projection q.

How can we define a G-invariant operation on V? We should **always** remember that g act transitively on G itself that if $gG = G$. Then if we can put all elements of G on the operation, then it should be G-invariant. By the way, we can define the module of G-invariance which is $RG = \sum_{g_i \in G} r_i g_i \quad r_i \in R, g_i \in G$. The way is to average the operation by group action for all elements. Thus one has:

$$
\begin{aligned}
hf(v_1, ..., v_n) &= \frac{1}{|G|} \sum_{g \in G} hgf(g^{-1}h^{-1}hv_1, ..., g^{-1}h^{-1}hv_n) \\
&= \frac{1}{|G|} \sum_{g' \in G} g'f(g'^{-1}hv_1, ..., g'^{-1}hv_n) = f(hv_1, ..., hv_n)
\end{aligned}
\tag{1.5}
$$

$$
\begin{aligned}
f(hv_1, ..., hv_n) &= \frac{1}{|G|} \sum_{g \in G} gf(g^{-1}hv_1, ..., g^{-1}hv_n) \\
&= \frac{1}{|G|} \sum_{h^{-1}g \in G} h^{-1}gf\Big(\left(h^{-1}g\right)^{-1}v_1, ..., \left(h^{-1}g\right)^{-1}v_n\Big) = f(v_1, ..., v_n)
\end{aligned}
\tag{1.6}
$$

$$
\begin{aligned}
\langle v, u \rangle &= \langle u, v \rangle^* \\
\langle u, \alpha v + \beta w \rangle &= \alpha \langle u, v \rangle + \beta \langle u, w \rangle \\
\|u\|^2 &= \langle u, u \rangle
\end{aligned}
\tag{1.7}
$$

$$
\langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle \tag{1.8}
$$

If there’ s a invariant space that $\rho(g)(V_1) \subseteq V_1$, with the decomposition $V_2 = V_1^{\perp}$ that $V_2 = \{v \in V \mid (v, x) = 0 \ \forall x \in V_1\}$.

$$
\langle \rho(g)x, y \rangle = \langle \rho(g)^{-1}\rho(g)x, \rho(g)^{-1}y \rangle = \langle x, \rho(g)^{-1}y \rangle \quad \forall y \in V_1 \tag{1.9}
$$

Thus if $x \in V_2$, the inner product should be **zero**. Reversely, we deduce that $\rho(g)x \in V_2$ because $y \in V_1$ by first term. Therefore the representation can be decomposed if we seek a **sole** invariant space and repeat the procedure.

Thus given a inner product, which is also a bilinear-function, we can construct G-invariant form like above.

$$
\langle x, y \rangle := \sum_{h \in G} \langle \rho(h)x, \rho(h)y \rangle \tag{1.10}
$$

Which is *unitary* as above already proved. Same as integral form one has:

$$
\langle x, y \rangle := \int_G d\mu(h) \langle \rho(h)x, \rho(h)y \rangle \tag{1.11}
$$

Yields the same results. However, we can conclude the canonical G-invariance form for *finite* and *compact* group due to the convergence given by the integral only for compactness.

Given linear morphism for irreducible space, $\varphi_1 : V \to W$ and $\varphi_2 : V \to W$, one could inspect that due to $\ker(\varphi) = \{0\}$ or V due to **irreducible** property, one can acquire no smaller subspace. Thus, $\varphi = 0$ or isomorphism, by uniqueness, $\varphi_1 = \lambda\varphi_2$.

Now given a space transformation:

$$
\varphi(\rho(g)) = A^{-1}\rho(g)A = \rho'(g) \tag{1.12}
$$

By above suggestion on irreducible representation, one should have $A = 0$, or better, the irreducible representation should be isomorphism, thus $A = \lambda I$. However, for clarification, you should prefer $\mathbb{C}$ field that to fully decompose space rather the root problem would occur.

$$(v_1, v_2) \to v_1 \otimes v_2 \equiv v_1^i e_1^i \otimes v_2^j e_2^j = v_1^i \, v_2^i \, e_1^i \otimes e_2^j \tag{1.13}$$

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2) \tag{1.14}$$

$$\chi_\rho(g) = \operatorname{Tr}(\rho(g)) \tag{1.15}$$

$$\chi_{\rho'}(g) = \operatorname{Tr}(\rho'(g)) = \operatorname{Tr}(A^{-1}\rho(g)A) = \operatorname{Tr}(A^{-1}A\rho(g)) = \operatorname{Tr}(\rho(g)) = \chi_\rho(g) \tag{1.16}$$

$$\chi_{\rho_1 \oplus \rho_2}(g) = \operatorname{Tr}(\rho_1 \oplus \rho_2(g)) = \operatorname{Tr}(\rho_1(g)) + \operatorname{Tr}(\rho_2(g)) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g) \tag{1.17}$$

$$\begin{aligned} \chi_{\rho_1 \otimes \rho_2}(g) = \operatorname{Tr}(\rho_1 \otimes \rho_2(g)) = \operatorname{Tr}(\rho_1(g)) \otimes \operatorname{Tr}(\rho_2(g)) \\ = \chi_{\rho_1}(g) \otimes \chi_{\rho_2}(g) = \chi_{\rho_1}(g)\chi_{\rho_2}(g) \otimes 1 \cong \chi_{\rho_1}(g)\chi_{\rho_2}(g) \end{aligned} \tag{1.18}$$

$$\chi_\rho(e) = \operatorname{Tr}(\rho(e)) = \operatorname{Tr}(I_V) = \dim(V) \tag{1.19}$$

$$M_{jl}^{ik} = \int_G d\mu(g) \, \rho_1(g)_j^i \rho_2(g^{-1})_l^k \tag{1.20}$$

$$\rho_1(h)_a^i \rho_1(g)_j^a = \rho_1(hg)_j^i \tag{1.21}$$

$$\begin{aligned} \rho_1(h)_a^i M_{jl}^{ak} &= \int_G d\mu(g)\rho_1(h)_a^i \rho_1(g)_j^a \rho_2(g^{-1})_l^k \\ &= \int_G d\mu(hg)\rho_1(hg)_j^i \rho_2(g^{-1})_l^k \quad hg \to g' \\ &= \int_G d\mu(g')\rho_1(g')_j^i \rho_2(g'^{-1}h)_l^k \\ &= M_{jb}^{ik}\rho_2(h)_k^b \end{aligned} \tag{1.22}$$

By Schur' s lemma, one acquire a one for all relation that either isomorphism, or nothing for M as a transformation matrix. Thus, fix j, k as $M_{ik}^{jl} = \mathcal{M}_i^l$.

$$M_{ik}^{jl} = \alpha_k^j \delta_i^l \tag{1.23}$$

However, one can change the summation for matrix by apply direction as:

$$\rho_1(h)_j^a \rho_1(g)_a^i = \rho_1(gh)_j^i \tag{1.24}$$

Giving us:

$$M_{ik}^{jl} = \beta_i^l \delta_k^j = \alpha_k^j \delta_i^l \tag{1.25}$$

Thus it′ s actually a decomposition as $C\delta_k^j \delta_i^l$ for the same irreducible representation or else nothing. We then find it′ s actually a trace summation:

$$\int_G d\mu(g)\rho(g)_j^i \rho(g^{-1})_i^j = \int_G d\mu(g)\,\mathrm{Tr}(I_V) = n \int_G d\mu(g) = \sum_{i,j} M_{ji}^{ij} = Cn^2 \tag{1.26}$$

If we can normalize the integral:

$$\int_G d\mu(g) = 1 \to C = \frac{1}{n} = \frac{1}{\dim(V)} \tag{1.27}$$

$$\int_G d\mu(g)\rho(g)_i^i \rho(g^{-1})_j^j = \int_G d\mu(g)\chi_\rho(g)\chi_\rho(g^{-1}) = \sum_{ij} M_{ij}^{ij} = Cn^2 = n \tag{1.28}$$

Thus, one has:

$$\frac{1}{n}\int_G d\mu(g)\chi_{\rho_1}(g)\chi_{\rho_2}(g^{-1}) = 1_{\mathrm{id}}\delta_{\rho_1,\rho_2} \tag{1.29}$$

1.2

$$\mathbf{F}(q,p) = \begin{pmatrix} F_q(q,p) \\ F_p(q,p) \end{pmatrix} = \begin{pmatrix} \partial_q F(q,p) \\ \partial_p F(q,p) \end{pmatrix} \tag{1.30}$$

$$dH = H_q dq + H_p dp = -\frac{\mathrm{d}p}{\mathrm{d}t} dq + \frac{\mathrm{d}q}{\mathrm{d}t} dp \tag{1.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t}\partial_q + \frac{\mathrm{d}p}{\mathrm{d}t}\partial_p = H_p \partial_q - H_q \partial_p := \{\cdot, H\} := X_H \tag{1.32}$$

That′ s, Hamiltonian defines the evolution of the system.

$$\frac{\mathrm{d}F(q,p)}{\mathrm{d}t} = X_H F(q,p) = \{F(q,p), H\} \tag{1.33}$$

$$\begin{aligned} X_H(p\,dq) &= \dot{p}dq + pd\dot{q} = \dot{p}dq - \dot{q}dp + d(p\dot{q}) \\ &= -H_q dq + H_p dp + d(p\dot{q}) = d(-H + p\dot{q}) := dL(q,\dot{q}) \end{aligned} \tag{1.34}$$

$$\{F, H\} = X_H F = -X_F H \tag{1.35}$$

Where the $X_F = \frac{\partial F}{\partial p}\partial_q - \frac{\partial F}{\partial q}\partial_p$.

Generally, one scalar function defined upon sympletic manifold can induce a vector field on it, with a general area preservation:

$$\begin{gathered}\omega = dq \wedge dp \\ \iota_{X_F}(\omega) = \iota_{\frac{\partial F}{\partial q}\partial_p - \frac{\partial F}{\partial p}\partial_q}(dq \wedge dp) = \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial p}dp = dF\end{gathered} \tag{1.36}$$

$$\iota_{X_H}\iota_{X_F}(\omega) = \iota_{X_H}(dF) = X_H F = \{F, H\} = \frac{\mathrm{d}F}{\mathrm{d}t} \tag{1.37}$$

We thus induce a symmetric bilinear form:

$$\omega(X_F, X_G) = \iota_{X_F}\iota_{X_G}(\omega) = \iota_{X_F}(dG) = X_F G = \{G, F\} = -\{F, G\} \tag{1.38}$$

One can lift the Hamiltonian in exponential flow by dynamics of X_H.

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\exp(tX_H)) \mid_{t=0} = \{F(p, q), H(p, q)\} \tag{1.39}$$

One thus can define lie group dynamics too just like above:

$$\begin{gathered}L \in \mathfrak{g} \to X_L \\ X_L F(p, q) = \frac{\mathrm{d}}{\mathrm{d}t}F\left(e^{tL} \cdot (p, q)\right)\end{gathered} \tag{1.40}$$

However, due to the representation of any group should be a anti-homomorphism, we have:

$$L \cdot F(p, q) = \frac{\mathrm{d}}{\mathrm{d}t}F\left(e^{-tL} \cdot (p, q)\right) = -X_L F(p, q) \tag{1.41}$$

Now we can try translation group $T = \mathbb{R}^3$ first:

$$a \cdot (q, p) = (q + a, p) \quad a \in T \leftrightarrow e^{-t\lambda} \in T \ \lambda \in \mathfrak{t} \tag{1.42}$$

Here we **abuse** of notation because the lie algebra of translation induced by lie group is trivial.

$$a \cdot F(p, q) = \frac{\mathrm{d}}{\mathrm{d}t}F\left(e^{-t\lambda} \cdot (p, q)\right) \mid_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}F(q - ta, p) \mid_{t=0} = -a\frac{\partial}{\partial q}F = -X_a F \tag{1.43}$$

$$X_a = a\frac{\partial}{\partial q} \tag{1.44}$$

We know that $\iota_{X_F}(\omega)$ is a closed form that it corresponds to a certain function F up to a constant, we can induce the scalar function of action by the lie group.

$$X_a(\omega) = ap \to \mu_a(p) = \mu_a(p,q) \tag{1.45}$$

We thus recover the momentum operator which configures the momentum for each translation element a. Such closed form check lift a function defined in the manifold which shared the same role as Hamiltonian, if H is G-invariant, then μ_a is a conserved quantity, as the Hamiltonian version of Noether′ s theorem.

$$\iota_{X_H}(d\mu_L) = \{\mu_L, H\} = \frac{\mathrm{d}\mu_L}{\mathrm{d}t} = -\{H, \mu_L\} = -\iota_{X_L}(dH) = -X_L H = 0 \tag{1.46}$$

1.3

$$[\phi(X), \phi(Y)] = \phi([X,Y]) = \phi(X)\phi(Y) - (-1)^{\sigma(X)\sigma(Y)}\phi(Y)\phi(X) \tag{1.47}$$

$$\Omega(Jv_1, Jv_2) = \Omega(v_1, v_2) \to J \in \mathrm{Sp}(2n, \mathbb{R}) \tag{1.48}$$

$$\begin{gathered} A^T A = I \to A \in \mathrm{O}(n, \mathbb{R}) \\ A^T \Omega A = \Omega \to A \in \mathrm{Sp}(2n, \mathbb{R}) \\ A^T \Omega A = A^T A \Omega \to A^T(\Omega A - A\Omega) = 0 \to \Omega A = A\Omega \\ \Omega(Av, Aw) = g(A\Omega v, Aw) = g(\Omega Av, Aw) = \Omega(v, w) \\ A \in \mathrm{O}(n, \mathbb{R}) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{U}(n) \end{gathered} \tag{1.49}$$

$$z_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad \overline{z_j} = \frac{1}{\sqrt{2}}(q_j - ip_j) \tag{1.50}$$

$$\begin{gathered} L_1 = Q_2 P_3 - Q_3 P_2, \quad L_2 = Q_3 P_1 - Q_1 P_3, \quad L_3 = Q_1 P_2 - Q_2 P_1 \\ \Gamma(L_1) = a_3^\dagger a_2 - a_2^\dagger a_3, \quad \Gamma(L_2) = a_1^\dagger a_3 - a_3^\dagger a_1, \quad \Gamma(L_3) = a_2^\dagger a_1 - a_1^\dagger a_2 \end{gathered} \tag{1.51}$$

$$\begin{aligned} & F(\theta) = c_0 + c_1\theta + c_2\theta^2 + \dots + c_n\theta^n = c_0 + c_1\theta \\ & \frac{\partial F}{\partial \theta} = c_1 \\ & F(\theta_1, \dots, \theta_n) = F_A + \theta_j F_B \quad \exists F_A, F_B \in \bigwedge(\mathbb{R}^n) \\ & \frac{\partial FG}{\partial \theta_j} = \frac{\partial F}{\partial \theta_j} G + (-1)^{\sigma(F)} F \frac{\partial G}{\partial \theta_j} \quad \sigma(F) \text{ is the degree of } F \end{aligned} \tag{1.52}$$

Given a translation invariance property of integral:

$$\int f(\theta + \eta) d\theta = \int f(\theta) d\theta \tag{1.53}$$

$$\int (\theta + \eta) d\theta = \int \theta d\theta + \eta \int d\theta = \int \theta d\theta \to \int d\theta = 0 \tag{1.54}$$

Given with normalization convention:

$$\int \theta d\theta = 1 \rightarrow \int c_0 + c_1 \theta d\theta = c_1 \rightarrow \int f(\theta) d\theta = \frac{\mathrm{d}f}{\mathrm{d}\theta} \tag{1.55}$$

Thus for multi-variables calculus, one should be careful on applying order of partial derivative:

$$\int f(\theta_1, ..., \theta_n) d\theta_n ... d\theta_1 = \frac{\partial}{\partial \theta_1} ... \frac{\partial}{\partial \theta_n} f \tag{1.56}$$

Where swap of order of partial derivative will induce a minus sign.

$$\begin{aligned} \left[\theta_j, \theta_k\right]_+ &= \pm \delta_{jk} \\ \frac{\mathrm{d}}{\mathrm{d}t} \theta_j(t) &= \left[\theta_j(t), H\right]_+ \end{aligned} \tag{1.57}$$

Given H:

$$\begin{aligned} B &= \begin{pmatrix} 0 & B_{12} & B_{13} & ... & B_{1n} \\ -B_{12} & 0 & B_{23} & ... & B_{2n} \\ -B_{13} & -B_{23} & 0 & ... & B_{3n} \\ ... \\ -B_{1n} & -B_{2n} & -B_{3n} & ... & 0 \end{pmatrix} \\ H &= \frac{1}{2} \sum_{j,k=1}^{n} B_{jk} \theta_j \theta_k \end{aligned} \tag{1.58}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \theta_j(t) = \left[\theta_j(t), H\right]_+ = -\left[H, \theta_j(t)\right]_+ = \sum_{k=1}^{n} B_{jk} \theta_k(t) \tag{1.59}$$

The minus sign of commutator is due to $B_{jk} = -B_{kj}$. Which is highly similar to the bosonic case in evolution.

$$\Omega^+(Jv_1, Jv_2) = \Omega^+(v_1, v_2) \rightarrow J \in \mathrm{SO}(2n, \mathbb{R}) \tag{1.60}$$

$$\begin{aligned} U_A &= \sum_{jk} a_j^\dagger A_{jk} a_k \\ \left[U_A, a_j\right]_\pm &= \pm \sum_k A_{jk} a_k = \pm A \boldsymbol{a} \\ \left[U_A, a_j^\dagger\right]_\pm &= \sum_k a_k^\dagger A_{kj} = A^T \boldsymbol{a}^\dagger \end{aligned} \tag{1.61}$$

$$v \otimes w = \frac{1}{2}(v \otimes w - u \otimes v) + \frac{1}{2}(v \otimes w + w \otimes v) = v \wedge w + g(v, w) \tag{1.62}$$

Thus given a basis omit tensor notation, for example:

$$e_i e_j = e_i \wedge e_j \quad (i \neq j) \tag{1.63}$$

Or physics notation:

$$\boldsymbol{v} = (v_1, ..., v_n) \in \mathbb{R}^n \to \gamma(\boldsymbol{v}) = \not{v} = v_1\gamma_1 + ... + v_n\gamma_n \tag{1.64}$$

$$\begin{aligned} v' &= v - 2\frac{g(v,w)}{g(w,w)}w = \not{v} - \frac{\not{v}\not{w} - \not{w}\not{v}}{g(w,w)}\not{w} \\ &= -\frac{\not{w}\not{v}\not{w}}{g(w,w)} \end{aligned} \tag{1.65}$$

$$\begin{aligned} 2AB &= [A,B]_+ + [A,B]_- \\ 2BA &= [A,B]_+ - [A,B]_- \end{aligned} \tag{1.66}$$

Thus we can always decompose a product term into symmetric and antisymmetric parts.

$$ABC = \frac{1}{2}A[B,C]_+ + \frac{1}{2}A[B,C]_- = \frac{1}{2}[A,B]_+C - \frac{1}{2}[A,C]_+B + \frac{1}{2}[A,B]_-C - \frac{1}{2}[A,C]_-B \tag{1.67}$$

$$\begin{aligned} [AB,C]_- = ABC - CAB &= A\left(\frac{1}{2}[B,C]_+ + \frac{1}{2}[B,C]_-\right) - \left(\frac{1}{2}[A,C]_+ - \frac{1}{2}[A,C]_-\right)B \\ &= \frac{1}{2}(A[B,C]_+ - [A,C]_+B) + \frac{1}{2}(A[B,C]_- + [A,C]_-B) \end{aligned} \tag{1.68}$$

$$\begin{aligned} [AB,C]_+ = ABC + CAB &= A\left(\frac{1}{2}[B,C]_+ + \frac{1}{2}[B,C]_-\right) + \left(\frac{1}{2}[A,C]_+ - \frac{1}{2}[A,C]_-\right)B \\ &= \frac{1}{2}(A[B,C]_+ - [A,C]_-B) + \frac{1}{2}(A[B,C]_- + [A,C]_+B) \end{aligned} \tag{1.69}$$

Equal below:

$$[AB,C]_- = A[B,C]_- + [A,C]_-B = A[B,C]_+ - [A,C]_+B \tag{1.70}$$

$$[AB,C]_+ = A[B,C]_+ + [A,C]_-B = A[B,C]_- - [A,C]_+B \tag{1.71}$$

$$\begin{aligned} [AB,CD]_- &= A[B,CD]_- + [A,CD]_-B \\ &= A([B,C]_+D - C[B,D]_+) + ([A,C]_+D - C[A,D]_+)B \\ &= A[B,C]_+D - AC[B,D]_+ + [A,C]_+DB - C[A,D]_+B \end{aligned} \tag{1.72}$$

For Clifford algebra, symmetric part is given by metric, anti-symmetric part is given by wedge product.

$$\begin{aligned} M_{\mu\nu} &= \gamma_\mu \wedge \gamma_\nu \\ M_{\mu\nu} &= \gamma_\mu\gamma_\nu - g_{\mu\nu} \end{aligned} \tag{1.73}$$

$$M_{\mu\nu} \wedge \gamma_\rho = \gamma_\mu \gamma_\nu \wedge \gamma_\rho = \gamma_\mu g_{\nu\rho} - \gamma_\nu g_{\mu\rho} \tag{1.74}$$

$$M_{\mu\nu} \wedge M_{\rho\sigma} = \gamma_\mu \gamma_\nu \wedge \gamma_\rho \gamma_\sigma = \gamma_\mu \gamma_\sigma g_{\upsilon\rho} - \gamma_\mu \gamma_\rho g_{\nu\sigma} + \gamma_\nu \gamma_\rho g_{\mu\sigma} - \gamma_\nu \gamma_\sigma g_{\mu\rho} \tag{1.75}$$

Is same as above, however, I didn′ t found this is useful : (.

1.4

$$\frac{L}{[L,L]} \text{ is abelian that } \forall a, b \in L, [a,b] = 0 \tag{1.76}$$

Thus given a central series:

$$[L^n, L^n] = L^{n+1} \to \frac{L^n}{[L^n, L^n]} = \frac{L^n}{L^{n+1}} \tag{1.77}$$

Factor through, for all possible elements, otherwise $[L, L] = L$ is irreducible. Thus induce a maximal nilpotent ideal of L which is the maximal solvable ideal of L, else it must be a **semi-simple** structure without additional smaller space. Where nilpotent is a structure that can be shared with a basis with commutative eigenvectors, that′ s $XYv = YXv + [X,Y]v = \lambda(X)Yv + \lambda([X,Y])v$ where $[X,Y]$ in some smaller space by deduction of nilpotent ideal.

Given a basis $\{X_1, ..., X_n\}$ with structure constants for a semi-simple lie algebra:

$$[X_i, X_j] = \sum_k {c_{ij}}^k X_k \tag{1.78}$$

Given the commutation relation, one has:

$$\begin{aligned} {c_{ij}}^k &= -{c_{ji}}^k \\ {c_{ij}}^k + {c_{jk}}^i + {c_{ki}}^j &= 0 \end{aligned} \tag{1.79}$$

Moreover, we can define the representation based on the structure constants:

$$\begin{aligned} &(T_i)^k_j = \left(D_{X_i}\right)^k_j = {c_{ij}}^k \\ &D_{X_i} : \mathfrak{g} \to \mathrm{ad}(\mathfrak{g}) \end{aligned} \tag{1.80}$$

$$D_{X_i} X_j = [X_i, X_j] = \sum_k {c_{ij}}^k X_k \tag{1.81}$$

$$\begin{aligned} \left[D_{X_i}, D_{X_j}\right] &= D_{[X_i, X_j]} \\ \left[D_{X_i}, D_{X_j}\right] X_k &= D_{[X_i, X_j]} X_k \\ [X_i [X_j X_k]] - [X_j [X_i X_k]] &= [[X_i, X_j] X_k] \end{aligned} \tag{1.82}$$

Which adjoin representation with **homomorphism consistency** should restrict all lie algebras to contain Jacobi identity.

$$\begin{aligned} D_{X_i} D_{X_j} X_l = D_{X_i} {c^k}_{jl} X_k = {c_{ik}}^m {c_{jl}}^k X_m \\ \left(D_{X_i} D_{X_j}\right)^m_l = {c_{ik}}^m {c_{jl}}^k \\ \operatorname{Tr}\left(D_{X_i} D_{X_j}\right) = \left(D_{X_i} D_{X_j}\right)^m_m = {c_{ik}}^m {c_{jm}}^k = g_{ij} = {c_{jm}}^k {c_{ik}}^m = g_{ji} \end{aligned} \tag{1.83}$$

Where:

$$g_{ij} g^{jl} = {c_{ik}}^m {c_{jm}}^k {c^{jn}}_f {c^{lf}}_n = {c_{ik}}^m {c^{lf}}_n {\delta_m}^n {\delta^k}_f = {\delta_i}^l \tag{1.84}$$

$$\langle v, w \rangle = v^i \left(D_{X_i}\right)^l_k v^j \left(D_{X_j}\right)^k_l = v^i v^j g_{ij} \tag{1.85}$$

Cyclic property:

$$0 = \operatorname{Tr}([A, BC]) = \operatorname{Tr}(B[A, C]) + \operatorname{Tr}([A, B]C) \tag{1.86}$$

$$\begin{aligned} \operatorname{Tr}([A, B]C) &= \operatorname{Tr}((AB - BA)C) \\ &= \operatorname{Tr}(ABC) - \operatorname{Tr}(BAC) \\ &= \operatorname{Tr}(ABC) - Tr(ACB) = \operatorname{Tr}(A[B, C]) \end{aligned} \tag{1.87}$$

$$\begin{aligned} 0 = \operatorname{Tr}\left(\left[\rho(X_i)\rho\left(X_j\right), \rho(X_k)\right]\right) &= \operatorname{Tr}\left(\rho(X_i)\left[\rho\left(X_j\right), \rho(X_k)\right]\right) + \operatorname{Tr}\left(\left[\rho(X_i), \rho(X_k)\right]\rho\left(X_j\right)\right) \\ &= \operatorname{Tr}\left(\rho(X_i) f^d_{jk} \rho(X_d)\right) + \operatorname{Tr}\left(f^d_{ik} \rho(X_d) \rho\left(X_j\right)\right) \\ &= f^d_{jk} g_{id} + f^d_{ik} g_{dj} \\ &= f_{jki} + f_{ikj} \end{aligned} \tag{1.88}$$

Given with $f_{jki} = -f_{kji}$, one has $f_{jki} = -f_{kji} = f_{ijk} = -f_{jik}$. It' s totally anti-symmetric tensor.

Fix indices as $f^k_{ij} = (\ell_i)^k_j$:

$$\begin{aligned} \ell_j g^{(\rho)} + \left(\ell_i g^{(\rho)}\right)^T &= 0 \\ g^{(-1)^{(\rho)}} \ell_j g^{(\rho)} &= -\ell_i^T \quad \text{by } g = g^T \\ \ell_j g^{(\rho)} &= g^{(\rho)} g^{(-1)^{(\rho)}} \ell_j g^{(\rho)} \\ \ell_j g^{(\rho)} g^{(-1)^{(\rho)}} &= g^{(\rho)} g^{(-1)^{(\rho)}} \ell_j \end{aligned} \tag{1.89}$$

Thus by Schur' s lemma, for *simple* lie algebra, it should be up to constant for the representation of the metric form compared with the canonical metric form.

However, it doesn' t mean that **semi-simple** structure contains no ideal. We try to diagonalize as much as possible even we can' t achieve for all like solvable case. That' s $\mathfrak{t} \subset \mathfrak{g}$ which all elements are commutative. We call such maximal commutative subalgebra as **Cartan subalgebra**, which can be diagonalized simultaneously. Thus if one choose a vector space it acts on, we can decompose the space into eigen-space for all:

$$V = \bigoplus_{\lambda(\mathfrak{t})} V_\lambda \quad V_\lambda = \{v \in V \mid \forall H \in \mathfrak{t}, Hv = \lambda(H)v\} \tag{1.90}$$

Given a adjoin representation or homomorphism as $X_i \to D_{X_i}$, we can then apply the Cartan subalgebra to decompose the lie algebra itself:

$$\mathfrak{g} = \bigoplus_{\alpha(\mathfrak{t})} \mathfrak{g}_\alpha \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{t}, [H, X] = D_H(X) = \alpha(H)X\} \tag{1.91}$$

Which is called **root space decomposition**, where $\alpha \in \mathfrak{t}^*$ is the eigenvalue on $\mathfrak{t}$.

We often see the ladder operator in physics that a state shared by all other raising and lowering operators, which is the eigenvector of Cartan subalgebra.

$$\begin{gathered} \boldsymbol{H} = (H_1, ..., H_r) \\ \left[H_i, H_j\right] = 0 \quad \forall i, j = 1, ..., r \to \left[\rho(H_i), \rho\left(H_j\right)\right] = 0 \quad \text{for finite representation} \end{gathered} \tag{1.92}$$

Due to mutually commutation, each H_i can share the same eigenstate with weights $\boldsymbol{m}$.

$$\rho(H_i)|\boldsymbol{m}\rangle = m_i|\boldsymbol{m}\rangle \quad \boldsymbol{m} = (m_1, ..., m_r) \tag{1.93}$$

Plus, given by the adjoint representation, the same routine is applied for remained decomposition.

$$D_{H_i}(E_{\boldsymbol{\alpha}}) = [H_i, E_{\boldsymbol{\alpha}}] = \boldsymbol{\alpha}(H_i)E_{\boldsymbol{\alpha}} = \alpha_i E_{\boldsymbol{\alpha}} \quad \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r) \tag{1.94}$$

$$\begin{aligned} D_{H_i}\left(\left[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}\right]\right) = \left[H_i, \left[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}\right]\right] &= \left[[H_i, E_{\boldsymbol{\alpha}}], E_{\boldsymbol{\beta}}\right] + \left[E_{\boldsymbol{\alpha}}, \left[H_i, E_{\boldsymbol{\beta}}\right]\right] \\ &= (\alpha_i + \beta_i)\left[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}\right] \sim H_i E_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \leftrightarrow \left[\mathfrak{g}_\alpha, \mathfrak{g}_\beta\right] \subseteq \mathfrak{g}_{\alpha+\beta} \end{aligned} \tag{1.95}$$

Just like you already have seen in generator of $\mathfrak{su}(2)$ that $[\xi_3, \xi^\pm] = \pm\xi^\pm$.

We know if the ladder operator act too many times in finite representation, it will reach zero, that' s D_X is nilpotent for $X \in \mathfrak{g}_\alpha, \alpha \neq 0$, or we simply call it nilpotent that $\mathfrak{g}_{\beta+n\alpha} = 0$.

For clarity, $\mathfrak{g}_0$ means $[\mathfrak{t}, \mathfrak{g}_0] = 0 \cdot \mathfrak{g}_0 = 0$ thus $\mathfrak{t} \subset \mathfrak{g}_0$, however, due to maximality of Cartan subalgebra, we have $\mathfrak{g}_0 = \mathfrak{t}$.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{t} \tag{1.96}$$

Given a unitary representation, with the observable eigenvalue should be *real*, thus, we restrict $\boldsymbol{m}, \boldsymbol{\alpha}$ to real value, or $\boldsymbol{H}^\dagger = \boldsymbol{H}$.

$$[H_i, E_{\boldsymbol{\alpha}}]^\dagger = (\alpha_i E_{\boldsymbol{\alpha}})^\dagger \to \left[E_{\boldsymbol{\alpha}}^\dagger, H_i\right] = \alpha_i^* E_{\boldsymbol{\alpha}}^\dagger = -\left[H_i, E_{\boldsymbol{\alpha}}^\dagger\right] \to E_{\boldsymbol{\alpha}}^\dagger \in \mathfrak{g}_{-\alpha} \tag{1.97}$$

Which is justify the existence of $\mathfrak{g}_{-\alpha}$

$$[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \sum_i^r b^i H_i \tag{1.98}$$

Restrict to $\mathfrak{t}$, we acquire its metric as $\mathrm{Tr}\left(H_i H_j\right) = g_{ij}$.

$$\sum_j g_{ij} b^j = \mathrm{Tr}(H_i [E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}]) = \mathrm{Tr}([H_i, E_{\boldsymbol{\alpha}}] E_{-\boldsymbol{\alpha}}) = \alpha_i \, \mathrm{Tr}(E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}) \tag{1.99}$$

If we normalize that $\mathrm{Tr}(E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}) = 1$.

$$\sum_j g_{ij} b^j = \alpha_i \to b^j = \sum_i g^{ij} \alpha_i \tag{1.100}$$

We thus could define $H_{\boldsymbol{\alpha}} \coloneqq [E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \sum_j b_j H_j$.

$$\left[H_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}\right] = \sum_j b_j \left[H_j, E_{\boldsymbol{\beta}}\right] = (\boldsymbol{b} \cdot \beta) E_{\boldsymbol{\beta}} \to [H_{\boldsymbol{\alpha}}, E_{\pm\boldsymbol{\alpha}}] = \pm(\boldsymbol{b} \cdot \boldsymbol{\alpha}) E_{\pm\boldsymbol{\alpha}} \tag{1.101}$$

However:

$$\boldsymbol{b} \cdot \boldsymbol{\alpha} = \sum_{ij} \alpha_i \alpha_j g^{ij} = \langle \alpha, \alpha \rangle \tag{1.102}$$

And we actually acquire a more ease representation as $H_{\boldsymbol{\alpha}}$.

1.4.1

$$\rho(H_i)\rho(E_{\boldsymbol{\alpha}})|\rho; \boldsymbol{m}\rangle = ([\rho(H_i), \rho(E_{\boldsymbol{\alpha}})] + \rho(E_{\boldsymbol{\alpha}})\rho(H_i))|\rho; \boldsymbol{m}\rangle = (\boldsymbol{\alpha}_i + m_i) E_{\boldsymbol{\alpha}} |\rho; \boldsymbol{m}\rangle \tag{1.103}$$

$$\rho(E_{\boldsymbol{\alpha}})|\rho; \boldsymbol{m}\rangle = N_\alpha(\boldsymbol{m})|\rho; \boldsymbol{m} + \boldsymbol{\alpha}\rangle \tag{1.104}$$

$$\begin{aligned} [\rho(E_{\boldsymbol{\alpha}}), \rho(E_{-\boldsymbol{\alpha}})] &= \rho(H_{\boldsymbol{\alpha}}) \\ \to N_\alpha(\boldsymbol{m} - \boldsymbol{\alpha}) N_{-\alpha}(\boldsymbol{m}) - N_{-\alpha}(\boldsymbol{m} + \boldsymbol{\alpha}) N_\alpha(\boldsymbol{m}) &= (\boldsymbol{\alpha}, \boldsymbol{m}) \end{aligned} \tag{1.105}$$

$$E_{\boldsymbol{\alpha}} |\rho; \boldsymbol{m} + p\boldsymbol{\alpha}\rangle = 0 \quad E_{-\boldsymbol{\alpha}} |\rho; \boldsymbol{m} - q\boldsymbol{\alpha}\rangle = 0 \tag{1.106}$$

$$N_\alpha(\boldsymbol{m} + (k-1)\boldsymbol{\alpha}) N_{-\alpha}(\boldsymbol{m} + k\boldsymbol{\alpha}) - N_{-\alpha}(\boldsymbol{m} + (k+1)\boldsymbol{\alpha}) N_\alpha(\boldsymbol{m} + k\boldsymbol{\alpha}) = \langle \boldsymbol{\alpha}, \boldsymbol{m} + k\boldsymbol{\alpha} \rangle \tag{1.107}$$

$$\begin{aligned}\sum_{k=-q}^{p} N_\alpha(\boldsymbol{m}+(k-1)\boldsymbol{\alpha})N_{-\alpha}(\boldsymbol{m}+k\boldsymbol{\alpha})\\ -N_{-\alpha}(\boldsymbol{m}+(k+1)\boldsymbol{\alpha})N_\alpha(\boldsymbol{m}+k\boldsymbol{\alpha}) = \sum_{k=-q}^{p}\langle\boldsymbol{\alpha},\boldsymbol{m}+k\boldsymbol{\alpha}\rangle\end{aligned} \tag{1.108}$$

Then, for second term, change the summation from $k \to k-1$, we cancel each term except the first and last.

$$\begin{aligned}&N_\alpha(\boldsymbol{m}-(q+1)\boldsymbol{\alpha})N_{-\alpha}(\boldsymbol{m}-q\boldsymbol{\alpha})\\ &-N_{-\alpha}(\boldsymbol{m}+(p+1)\boldsymbol{\alpha})N_\alpha(\boldsymbol{m}+p\boldsymbol{\alpha}) = \sum_{k=-q}^{p}\langle\boldsymbol{\alpha},\boldsymbol{m}+k\boldsymbol{\alpha}\rangle\end{aligned} \tag{1.109}$$

However, due to the finite order restriction, the left terms all vanish.

$$\begin{aligned}(p+q+1)\langle\boldsymbol{\alpha},\boldsymbol{m}\rangle + \left(\frac{1}{2}p(p+1)-\frac{1}{2}q(q+1)\right)\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle = 0\\ 2\frac{\langle\boldsymbol{\alpha},\boldsymbol{m}\rangle}{\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle} = q-p\end{aligned} \tag{1.110}$$

So what′ s the procedure to investigate the roots of a lie algebra? First, given a canonical matrix representation, one should find the Cartan subalgebra, how? We seek the maximal independent *diagonal* matrix set. Then seek the relation of $\boldsymbol{H}$ and remaining elements, you may not pretty lucky to find direct scalar relation rather mixed terms. Then, you should make a combination like $v = a^i M_i$, and try to build by $\left[H_i, a^j M_j\right] = \lambda_i a^j M_j$ exactly. Given each coefficient with normalization, you may find a set of $\delta_{i,j} + \delta_{i+1,j} + \dots$ for possible slots. Summarize, it can be evaluated for $\boldsymbol{\alpha}(H_i)$ for i index of this vector, it indicates to extract the i set to form a basis combination by $\delta_{ij} + \delta_{i+1,j} \sim e_j$ etc, which is the root.

1.4.2 $A_n = \mathfrak{sl}(n+1)$

Given $\mathbf{e}_i$ for $i = 1, \dots, n+1$ in $\mathbb{R}^{n+1}$ parametrize the lie algebra $\mathfrak{sl}(n+1)$. Thus the matrix is represented by $\left(M_{ij}\right)_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$ which Kronecker index connects the matrix slot and generator index.

It′ s more natural to just seek a matrix summation to yield results, rather a direct index summation is false. Looking at the jk summation yielding:

$$M_{i,j}M_{k,l} = \delta_{j,k}M_{i,l} \tag{1.111}$$

$$\left[M_{i,j}, M_{k,l}\right] = \delta_{j,k}M_{i,l} - \delta_{i,l}M_{j,k} \tag{1.112}$$

Given $\mathrm{Tr}(\mathfrak{sl}(n+1)) = 0$, we should take the maximal diagonal and traceless basis set, then we choose:

$$\begin{aligned}H_i &= M_{i,i} - M_{i+1,i+1}\\ \left[H_i, M_{k,l}\right] &= \delta_{i,k}M_{i,l} - \delta_{i+1,k}M_{i+1,l} + \delta_{i,l}M_{k,i} - \delta_{i+1,l}M_{k,i+1}\\ &= \left(\delta_{i,k} - \delta_{i,l} - \delta_{i+1,k} + \delta_{i+1,l}\right)M_{k,l}\end{aligned} \tag{1.113}$$

Now, be clever to extract the basis of i index to form simple root set.

$$\boldsymbol{\alpha} = \mathbf{e}_k - \mathbf{e}_l \quad E_{\boldsymbol{\alpha}} = M_{k,l} \tag{1.114}$$

$$\begin{aligned} [H_i, E_{\boldsymbol{\alpha}}] &= \boldsymbol{\alpha}(H_i) E_{\boldsymbol{\alpha}} \\ \boldsymbol{\alpha}(H_i) &= \left(\delta_{i,k} - \delta_{i,l}\right) - \left(\delta_{i+1,k} - \delta_{i+1,l}\right) \end{aligned} \tag{1.115}$$

Where we choose root as $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} \quad i = 1, ..., n$ to expand into more clarified basis set.

$$\mathbf{e}_k - \mathbf{e}_l = \sum_k^{l-1} (\mathbf{e}_i - \mathbf{e}_{i+1}) \tag{1.116}$$

$$\left\langle \alpha_i, \alpha_j \right\rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{if } |i - j| \geq -2 \end{cases} \tag{1.117}$$

We only care about the normalization relation upon the inner product, which build the relation between each basis. How to express this clearly? One named *Dynkin*, creates the diagram as follows:

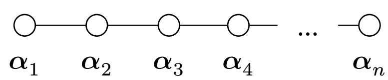

Figure 2. Dynkin diagram of $A_n = \mathfrak{sl}(n+1)$

Where the solid line connects node indicates $-\frac{1}{2} = \cos\frac{\pi}{3}$ by normalization of simple roots in i, j where $|i - j| = 1$. Thus, given a relation between nodes, we connects nodes by single or multiple solid lines to indicates different normalization angles. We could also express in matrix form by $C_{ij} = \left\langle \alpha_i, \alpha_j \right\rangle$ called *Cartan* matrix which is too sparse in expression compared to Dynkin diagram.

1.4.3 $D_n = \mathfrak{so}(2n)$

As for algebra $\mathfrak{so}(2n)$, the antisymmetric matrix should be denoted as $M_{ij} = -M_{ji} = -M_{ij}^T$ for $i, j = 1, ..., 2n$.

$$\left(M_{ij}\right)_{\alpha\beta} = \delta_{a,\alpha}\delta_{b,\beta} - \delta_{a,\beta}\delta_{b,\alpha} \tag{1.118}$$

We should notie that one should transpose the index to sum again in second term, thus given $(i, j)(k, l)$, we should merge j, k first and then i, k.

$$\left[M_{i,j}, M_{k,l}\right] = \delta_{j,k} M_{i,l} - \delta_{i,k} M_{j,l} + \delta_{l,i} M_{k,j} - \delta_{k,i} M_{l,j} \tag{1.119}$$

We define the Cartan subalgebra as, for maximal n ranks for $\mathfrak{so}(2n)$:

$$H_i = M_{2i-1,2i} \quad i = 1, 2, ..., n \tag{1.120}$$

Which is **not** diagonalizable actually, that it must mix four elements by previous calculation. If one need to diagonalize such basis, we should take in $\mathbb{C}$ with i factor appended. Rank n is the consequence derived from maximally n independent rotation.

We aren' t lucky enough to extract the scalar for a single basis, rather those are scalars distributing around different ones. Thus, we choose a vector format to calculate by choose a various $j, k = 1, 2, ..., n$ basis:

$$\begin{gathered} \boldsymbol{M} = aM_{2j-1,2k-1} + bM_{2j,2k-1} + cM_{2j-1,2k} + dM_{2j,2k} \\ [H_i, \boldsymbol{M}] = a\left(\delta_{2i,2j-1}M_{2i-1,2k-1} - \delta_{2i-1,2j-1}M_{2i,2k-1} + \delta_{2k-1,2i-1}M_{2i,2j-1} - \delta_{2k-1,2i}M_{2i-1,2j-1}\right) + ... \end{gathered} \tag{1.121}$$

We suddenly found that $2i \neq 2j-1$ ever and ever because one is odd and one is even! We immediately reduce the index into $2i-1, 2j-1$ or $2k-1, 2i-1$. Then, we note that $\delta_{i,j}M_{2i,2k-1} = M_{2j,2k-1}$, $\delta_{k,i}M_{2i,2j-1} = -M_{2j-1,2k}$.

Thus we choose:

$$\begin{aligned} &\boldsymbol{\alpha} = \eta\mathbf{e}_j + \eta'\mathbf{e}_k \quad j \neq k \quad \eta, \eta' = 1, -1 \\ &E_{\boldsymbol{\alpha}} = \frac{1}{2}\left(M_{2j-1,2k-1} + \eta M_{2j,2k-1} + \eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}\right) \\ &\boldsymbol{\alpha}(H_i) = \eta\delta_{i,j} + \eta'\delta_{i,k} \\ &[H_i, E_{\boldsymbol{\alpha}}] = \left(\eta\delta_{i,j} + \eta'\delta_{i,k}\right)E_{\boldsymbol{\alpha}} \end{aligned} \tag{1.122}$$

By counting numbers of generators, we found that they are $2n(n-1)$ for j, k root generators.

$$\begin{aligned} &\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} \quad i = 1, ..., n-1 \\ &\alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n \end{aligned} \tag{1.123}$$

Actually, choose arbitrary sign is acceptable, but such is preferable.

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| = 0 \\ 0 & \text{if } i = n-1 \ j = n \end{cases} \tag{1.124}$$

We draw the *Dynkin* diagram as below:

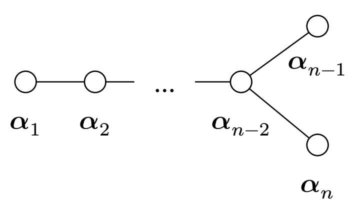

Figure 3. Dynkin diagram of $D_n = \mathfrak{so}(2n)$

Figure 4. Dynkin diagram of $B_n = \mathfrak{so}(2n+1)$

1.4.4 Coincidence

$$\begin{aligned}
\mathfrak{so}(3;\mathbb{C}) &= \mathfrak{sl}(2;\mathbb{C}) = \mathfrak{sp}(2;\mathbb{C}) \\
\mathfrak{so}(4;\mathbb{C}) &= \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}) \\
\mathfrak{so}(5;\mathbb{C}) &= \mathfrak{sp}(4;\mathbb{C}) \\
\mathfrak{so}(6;\mathbb{C}) &= \mathfrak{sl}(4;\mathbb{C})
\end{aligned} \tag{1.125}$$

1.4.5

$\mathfrak{su}(2) = \left\{ X \in \mathfrak{gl}\ (2, \mathbb{C}) \mid \overline{X}^t + X = 0, \mathrm{Tr}(X) = 0 \right\}$ and $\mathfrak{so}(3)$ are all real lie algebras of dimension 3.

(a) The basis:

$$\xi_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \xi_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{1.126}$$

With $[\xi_k, \xi_l] = \varepsilon_{klm}\xi_m$.

$$\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \tag{1.127}$$

$$[\xi^2, \xi_k] = 0 \tag{1.128}$$

We construct a commute element which the maximal diagonalized state is $|b, m\rangle$.

$$\begin{aligned}
\xi^2 |b, m\rangle &= b|b, m\rangle \\
\xi_3 |b, m\rangle &= m|b, m\rangle
\end{aligned} \tag{1.129}$$

$$\xi^+ = \xi_1 + i\xi_2, \quad \xi^- = \xi_1 - i\xi_2 \tag{1.130}$$

$$[\xi_3, \xi^+] = \xi^+, \quad [\xi_3, \xi^-] = -\xi^- \tag{1.131}$$

$$[\xi^+, \xi^-] = 2\xi_3 \tag{1.132}$$

$$\xi_3\xi^\pm |b, m\rangle = [\xi_3, \xi^\pm]|b, m\rangle + \xi^\pm\xi_3|b, m\rangle = (m \pm 1)\xi^\pm|b, m\rangle \tag{1.133}$$

Thus we conclude that $\xi^\pm|b, m\rangle = C|b, m \pm 1\rangle, \quad C \in \mathbb{C}$, and $\xi_3|b, m \pm 1\rangle = (m \pm 1)|b, m \pm 1\rangle$.

We call such operators ladder operators or raising and lowering operators.

$$\xi^+|b, j\rangle = 0 \quad \exists j \in ? \tag{1.134}$$

$$\xi^-\xi^+ = (\xi_1 - i\xi_2)(\xi_1 + i\xi_2) = \xi_1^2 + \xi_2^2 + i[\xi_1, \xi_2] = \xi^2 - \xi_3^2 + \xi_3 \tag{1.135}$$

$$0 = \xi^- \xi^+ |b, j\rangle = (b - j^2 + j)|b, j\rangle \to b = j(j+1) \tag{1.136}$$

Same for $\xi^+ \xi^-$, one has $b - j'^2 + j' = 0$

$$\begin{aligned} b = j'(j'-1) = -j'(-j'+1) \\ \to j' = -j \end{aligned} \tag{1.137}$$

We conclude that for m can be $-j, -j+1, ..., j-1, j$, thus $2j \in \mathbb{N}$ is a integer for finite dimension.

$$\begin{aligned} & j = 0, \quad m = 0, \quad j(j+1) = 0 \\ & j = \frac{1}{2}, \quad m = \frac{1}{2}, -\frac{1}{2}, \quad j(j+1) = \frac{3}{4} \\ & j = 1, \quad m = 1, 0, -1, \quad j(j+1) = 2 \\ & \dots \end{aligned} \tag{1.138}$$

From the definition of $\xi^+ = \xi^{-\dagger}$ by its expansion, we have such normalization factor:

$$\begin{aligned} \left\langle b, m \middle| \xi^{+\dagger} \xi^+ \middle| b, m \right\rangle &= \langle b, m | \xi^- \xi^+ | b, m \rangle \\ &= \langle b, m | \xi^2 - \xi_3^2 - \xi_3 | b, m \rangle \\ &= b - m^2 - m = j(j+1) - m^2 - m = C^2 \end{aligned} \tag{1.139}$$

$$C = \sqrt{j(j+1) - m(m+1)} \tag{1.140}$$

Same, one has for ξ^- that $\tilde{C} = \sqrt{j(j+1) - m(m-1)}$.

1.4.6

Take the transformation $SO(3)$ as $GL(3;\mathbb{R})$ matrix representation act naturally on $\mathbb{R}^3$ equipped with function valued on it as a scalar form.

$$\begin{aligned} l_1(f)(\boldsymbol{x}) = f(l_1^{-1} x) &= \frac{d}{dt} f\left(\exp\left(t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f\left(\begin{pmatrix} x_1 \\ x_2 \cos t + x_3 \sin t \\ -x_2 \sin t + x_3 \cos t \end{pmatrix}\right) \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \cdot \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix} \\ &= x_3 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_3} \end{aligned} \tag{1.141}$$

If we have $x_1 = r \sin\theta \cos\phi, x_2 = r \sin\theta \sin\phi, x_3 = r\cos\theta$, we can construct the transformation upon θ, ϕ too.

$$\frac{\partial}{\partial x_i} = \frac{\partial r_i}{\partial x_i}\frac{\partial}{\partial r_i} \tag{1.142}$$

In differential geometry as a basis vector transformation, or familiar Jacobi.

$$\begin{aligned} l_1 &= x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} = \sin\phi \frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi} \\ l_2 &= -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi} \\ l_3 &= -\frac{\partial}{\partial\phi} \end{aligned} \tag{1.143}$$

$$\begin{aligned} l^+ &= e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) \\ l^- &= e^{i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) \end{aligned} \tag{1.144}$$

Given the representation $F(\theta,\phi)(m)$ with the weight m chosen by l_3, with the highest weight denoted as l:

$$l_3 F(\theta,\phi)(m) = -\frac{\partial}{\partial\phi}F(\theta,\phi)(m) = mF(\theta,\phi)(m) \tag{1.145}$$

Thus we have $F(\theta,\phi) \sim e^{m\phi}G(\theta)$, however, we have such restriction $\phi + 2\pi \sim \phi$, thus we must have $m \to im$ to match such period.

We can immediately decompose the representation scalar function by raising operator for the highest weight:

$$\begin{aligned} \left(\frac{\partial}{\partial\theta} - l\cot\theta\right)G(\theta)(l) &= 0 \\ \ln G(\theta)(l) &= l\ln\sin\theta \\ G(\theta)(l) &= C_l \sin^l\theta \end{aligned} \tag{1.146}$$

Where the constant is, for the representation of $2l+1$ dimension in l weight. Apply lowering operator would giving us:

$$\begin{aligned} F(\theta,\phi)(m) &= C_m (l^-)^{l-m} F(\theta,\phi)(l) \\ &= C_m\left(e^{-i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right)\right)^{l-m} e^{-il\phi}\sin^l\theta \end{aligned} \tag{1.147}$$

However, one usually encompass the whole $2l+1$ representation for arbitrary l as:

$$Y_l^m(\theta,\phi) = C_{lm}(\ldots) \text{ as representation for } 2l+1 \text{ dimension with weight } m \tag{1.148}$$

Casimir operator with commutative property acting on will give us the eigenvalue.

$$l^2 = l_1^2 + l_2^2 + l_3^2 \tag{1.149}$$

$$l^2 Y_l^m(\theta,\phi) = l(l+1) Y_l^m(\theta,\phi) \tag{1.150}$$

$$\begin{aligned} l^2 &= l^- l^+ + i l_3 + l_3^2 \\ &= e^{i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) - i\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} \\ &= \frac{\partial^2}{\partial\theta^2} + i\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\phi}\right) + e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\left(-i\frac{\partial}{\partial\theta}\right)\right) + e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}\right) + (...) \end{aligned} \tag{1.151}$$

Evaluate that:

$$\begin{aligned} i\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\phi}\right) &= \frac{i}{\sin^2\theta}\frac{\partial}{\partial\phi} + i\cot\theta\frac{\partial^2}{\partial\theta\partial\phi} \\ e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\left(-i\frac{\partial}{\partial\theta}\right)\right) &= -\cot\theta\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial^2}{\partial\theta\partial\phi} \\ e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}\right) &= -i\cot^2\theta\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} \end{aligned} \tag{1.152}$$

The first term evaluate as:

$$\begin{aligned} \text{First term} &= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + i\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi} - \cot^2\theta\frac{\partial}{\partial\phi}\right) + \cot^2\theta\frac{\partial^2}{\partial\phi^2} \\ &= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} \end{aligned} \tag{1.153}$$

$$\begin{aligned} \text{LHS} &= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\phi^2} \\ &= \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right)\sin\theta + \frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\ &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} = \Delta_\Omega \end{aligned} \tag{1.154}$$

Which is the **Spherical Laplacian**[1].

This, actually can be formalize in such insight, that Laplacian as:

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \Delta_\Omega \tag{1.155}$$

[1]Actually, calculate by $l_1^2 + l_2^2 + l_3^2$ is more simpler, or use some notation reduction would be easier in burden.

Is rotation invariant given by $[\Delta_\Omega, l_i] = 0$. Which is same as Casimir operator by its **uniqueness** up to constant. Thus we decompose $L^2(S^2)$ into basis equipped with $Y_l^m(\theta, \phi)$.

1.4.7

Given $SO(2)$, it contains one generator which rotates the plane:

$$l = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim l = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial \phi} \tag{1.156}$$

It' s simple that only one element share the eigenvalue solely.

$$l|m\rangle = m|m\rangle \sim |m\rangle = F(\phi)(m) = e^{-im\phi} \tag{1.157}$$

With period condition induced.

$SO(2) \rtimes \mathbb{R}^+ = E(2)$

$$\begin{pmatrix} R(\theta) & \boldsymbol{t} \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \tag{1.158}$$

$\mathfrak{e}(2)$

$$l \to \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad p_x \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad p_y \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.159}$$

$$[l, p_x] = p_y \quad [l, p_y] = -p_x \quad [p_x, p_y] = 0 \tag{1.160}$$

$$\begin{aligned} l &\to x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial \phi} \\ p_x &\to \frac{\partial}{\partial x} \\ p_y &\to \frac{\partial}{\partial y} \end{aligned} \tag{1.161}$$

Given a Cartan subalgebra, we choose l, because the Cartan subalgebra must be a semi-simple decomposition thus we shouldn' t take a nilpotent ideal which would **vanish in commutation**.

$$p^\pm := p_x \pm i p_y = \frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} = e^{\pm i\phi} \left(\frac{\partial}{\partial r} \pm \frac{i}{r} \frac{\partial}{\partial \phi} \right) \tag{1.162}$$

$$[l, p^\pm] = \pm p^\pm \quad [p^+, p^-] = 0 \tag{1.163}$$

Casimir

$$\begin{aligned} p^+p^- - p^+p^- &= 0 \\ [l, p^+]p^- + p^+[l, p^-] &= 0 \\ [l, p^+p^-] &= 0 \\ \left[l, p_x^2 + p_y^2\right] &= 0 \\ \left[l, p^2\right] &= 0 \end{aligned} \tag{1.164}$$

Thus, it' s the Casimir operator, why there' s no l^2? Because there' s no element can generate l to absorb in. Thus we clarify such operator should be unique up to scalar, which is $\Delta_\perp$ in representation.

$$p^+p^- = e^{i\phi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \phi}\right)\left(e^{-i\phi}\right)\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \phi}\right) \tag{1.165}$$

$$\begin{aligned} &e^{i\phi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \phi}\right)\left(e^{-i\phi}B\right) \\ &= \frac{\partial}{\partial r}B + \frac{1}{r}B + \frac{i}{r}\frac{\partial}{\partial \phi}B \end{aligned} \tag{1.166}$$

$$\begin{aligned} p^+p^- &= \frac{\partial^2}{\partial r^2} - \frac{i}{r}\frac{\partial^2}{\partial r \partial \phi} + \frac{i}{r^2}\frac{\partial}{\partial \phi} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{i}{r^2}\frac{\partial}{\partial \phi} + \frac{i}{r}\frac{\partial^2}{\partial \phi \partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \end{aligned} \tag{1.167}$$

We inspect that the commutation relation differs from the case in semi-simple algebra like $SU(2)$, which we can' t generate the angular part from the translation. If you review the proof of weight boundedness of $SU(2)$ representation that Casimir operator build the connection with maximal and minimal weight, however, we can' t reach the results in this one, indicating *unboundedness* of the weight.

$$p^{\pm}F(r, \phi, z)(m, k_z) = e^{\pm i\phi}\left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial \phi}\right)F(r, \phi, z) \tag{1.168}$$

$$\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} = p^2 \tag{1.169}$$

Given by weight m, we extract terms like $e^{im\phi}$:

$$\begin{aligned} (\Delta_\perp + \partial_z^2)F(r, \phi, z)(m, k_z) &= \lambda F(r, \phi, z)(m, k_z) \\ \Delta_\perp\left(e^{im\phi}R(r)(m)\right) &= \left(\lambda - k_z^2\right)\left(e^{im\phi}R(r)(m)\right) \\ \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \left(k_z^2 - \lambda\right)\right)R(r)(m) &= 0 \end{aligned} \tag{1.170}$$

Which is called *Bessel Equation*, the solution is called *Bessel Function* $J_m(r)$ with m as weight notation. Notice that $k_z^2 - \lambda$ is a isolated scalar generated from z translation which could be reduced, we denote as $r \to \sqrt{k_z^2 - \lambda} r := kr$.

$$\begin{gathered} G(r, \phi)(m) = e^{im\phi} R(r)(m) = J_m(kr) e^{im\phi} \\ p^+(G(r, \phi)(m)) = G(r, \phi)(m + 1) \end{gathered} \tag{1.171}$$

$$J_{m+1}(r) = \frac{m}{x} J_m(r) - J'_m(r) \tag{1.172}$$

However, we have no choice like $SU(2)$ to denote the minimal or maximal basis function and iterates for else. Thus, we can only acquire a iteration relation like above. You may expect the different solution or *differential seed* will inherit the same iteration property like above.

$$\begin{gathered} [l, p^\pm] = \pm p^\pm \\ e^{tl} p^\pm e^{-tl} = e^{tD_l}(p^\pm) = \sum_n^\infty \frac{t^n}{n!} (\pm 1)^n p^\pm = e^{\pm t} p^\pm \end{gathered} \tag{1.173}$$

We then set $t \to i\phi$ for period restriction.

$$\begin{gathered} U(\phi) := e^{i\phi l} \\ U(\phi)|m\rangle = e^{i\phi l}|m\rangle = e^{im\phi}|m\rangle \end{gathered} \tag{1.174}$$

We thus immediately decompose it to the m-weight space, recall the representation decomposition, one has $\chi(g) = e^{im\phi}$:

$$\begin{gathered} \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} e^{im\phi} d\phi = 1_m \\ \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} U(\phi)|m\rangle d\phi = 1_{\text{id}} \\ \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} U(\phi) d\phi = \Pi_m \end{gathered} \tag{1.175}$$

$$\mathrm{Tr}(U(\phi)) = \langle m|U(\phi)|m\rangle = \sum_{m \in \mathbb{Z}} e^{im\phi} = 2\pi \sum_{m \in \mathbb{Z}} \delta(\phi - 2\pi n) = \sum_m \Pi_m = 1_{\text{id}} \tag{1.176}$$

$$\sum_m \Pi_m\left(e^{ikr\cos\phi}\right) = \sum_{m \in \mathbb{Z}} J_m(kr) e^{im\phi} \tag{1.177}$$

1.5

$$S[*] : \mathcal{F} \to \mathbb{R}; \mathcal{F} = \left\{ \boldsymbol{q}(t) : t \in [t_0, t_1] \subset \mathbb{R} \to \mathbb{R}^M \right\} \tag{1.178}$$

With differential set if applicable:

$$\mathcal{F}_{\varepsilon} = \{\delta \boldsymbol{q}(t); |\delta \boldsymbol{q}(t)| < \varepsilon; |\delta \dot{\boldsymbol{q}}(t)| < \varepsilon; \forall t \in [t_0, t_1] \subset \mathbb{R}\} \tag{1.179}$$

$$\delta S(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \to \delta \boldsymbol{q}(t_0) = \delta \boldsymbol{q}(t_1) = 0 \tag{1.180}$$

Given by a certain basis product form:

$$\begin{gathered} |x\rangle \text{ or } |x+\delta\rangle ? \to \text{continuous basis} \\ \langle x|\psi\rangle = \langle x|x'\rangle\langle x'|\psi\rangle \end{gathered} \tag{1.181}$$

Thus φ should be a continuous basis expansion for the representation of coordinates:

$$\begin{gathered} \int d\mu\, \varphi(x|x') \leftrightarrow \langle x|x'\rangle \\ \int d\mu\, \varphi(x|x')\varphi(x'|x_0) \leftrightarrow \langle x|x'\rangle\langle x'|x_0\rangle = \langle x|x_0\rangle \leftrightarrow \delta(x'|x_0) \end{gathered} \tag{1.182}$$

$$\begin{gathered} \varphi(x|x_0) = \varphi(x|x_{N-1})\varphi(x_{N-1}|x_{N-2}) ... \varphi(x_1|x_0) \\ \varphi(x|x_0) = \lim_{N\to\infty} \prod^{\infty} T(\varphi(x_\varepsilon)) \end{gathered} \tag{1.183}$$

Denotes $M(V;\mathbb{C})$ as set of non-degenerate complex-valued symmetric bilinear forms on V with non-negative definite real part, with dense set $M^{\circ}(V;\mathbb{C})$. Now fix a translation invariant volume form dx on V, thus we can define the determinant $\det(B)$ $B \in M^{\circ}(V;\mathbb{C})$.

Let $\mathcal{S}(V)$ be the *Schwartz space* of V where the smooth functions on V whose all derivatives are rapidly decaying at ∞ faster than any power of $|x|$, equipped with the dual space $\mathcal{S}^*(V)$ in natural inclusions $\mathcal{S}(V) \subset L^2(V) \subset \mathcal{S}'(V)$.

$$\begin{gathered} \mathcal{F} : \mathcal{S}(V) \to \mathcal{S}(V^*) \\ \mathcal{F}[g](p) = (2\pi)^{-\frac{d}{2}} \int_V g(x) e^{-i\langle p,x\rangle} dx \end{gathered} \tag{1.184}$$

Isometry is given by $\mathcal{F}^2[g](x) = g(-x)$. By duality, such operator could be extended to $\mathcal{S}^*(V) \to \mathcal{S}^*(V^*)$ for tempered distribution. The function $\exp\left(-\frac{1}{2}B(x,x)\right)$ for matrix $B \in M^{\circ}(V)$ belongs to $\mathcal{S}^*(V)$ if $\operatorname{Re} B \geq 0$ by its decaying behavoir.

For dimension 1, where a diagonalizable B:

$$
\begin{aligned}
\mathcal{F}\left(\exp\left(-\frac{1}{2}B(x,x)\right)\right) &= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left(-ipx-\frac{1}{2}ax^2\right)dx \\
&= (2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}a^{-1}p^2\right)\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}a\left(x+ia^{-1}p\right)^2\right)dx \\
&= (2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}a^{-1}p^2\right)\int_{\mathbb{R}+ia^{-1}p}\exp\left(-\frac{1}{2}a\left(x+ia^{-1}p\right)^2\right)dx \\
&= a^{-\frac{1}{2}}\exp\left(-\frac{1}{2}a^{-1}p^2\right)
\end{aligned}
\tag{1.185}
$$

We therefore could extend such matrix to diagonalizable by dense property that for arbitrary countable dimensions where a general results is given by:

$$
\mathcal{F}\left(\exp\left(-\frac{1}{2}B(x,x)\right)\right) = (\det B)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}B^{-1}(p,p)\right)
\tag{1.186}
$$

That the $\det(B)^{-\frac{1}{2}} = \prod_i \lambda_i$ by eigenvalues which is trivial in dimension 1.

$$
\begin{aligned}
I[g;B](\hbar) &= \frac{1}{\hbar^{\frac{1}{2}}}\int_V g(x)e^{-\frac{1}{2\hbar}B(x,x)}dx \quad \hbar \geq 0 \\
&= \int_V g\left(\hbar^{\frac{1}{2}}x\right)e^{-\frac{1}{2}B(x,x)}dx \quad x \to \frac{x}{\hbar^{\frac{1}{2}}}
\end{aligned}
\tag{1.187}
$$

$$
I[g;B](0) = \int_V (...)dx\ g(0) = (2\pi)^{\frac{d}{2}}(\det B)^{-\frac{1}{2}}g(0)
\tag{1.188}
$$

$$
\begin{aligned}
I[g;B](\hbar) &= \left\langle g\left(\hbar^{\frac{1}{2}}x\right), \exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar^{-\frac{d}{2}}(\det B)^{-\frac{1}{2}}\left\langle \mathcal{F}[g]\left(\hbar^{-\frac{1}{2}}p\right), \exp\left(-\frac{1}{2}B^{-1}(p,p)\right)\right\rangle \\
&= (\det B)^{-\frac{1}{2}}\left\langle \mathcal{F}[g](p), \exp\left(-\frac{\hbar}{2}B^{-1}(p,p)\right)\right\rangle
\end{aligned}
\tag{1.189}
$$

Where $\hbar \to 0 \rightsquigarrow \exp\left(-\frac{\hbar}{2}B^{-1}(p,p)\right) = 1$.

If $l \in V^*$ and $g \in \mathcal{S}(V)$, then:

$$
\begin{aligned}
I[lg;B](\hbar) &= \left\langle l\left(\hbar^{\frac{1}{2}}x\right)g\left(\hbar^{\frac{1}{2}}x\right), \exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar^{\frac{1}{2}}\left\langle g\left(\hbar^{\frac{1}{2}}x\right), l(x)\exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= -\hbar^{\frac{1}{2}}\left\langle g\left(\hbar^{\frac{1}{2}}x\right), \partial_{B^{-1}l}\exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar^{\frac{1}{2}}\left\langle \partial_{B^{-1}l}g\left(\hbar^{\frac{1}{2}}x\right), \exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar\left\langle (\partial_{B^{-1}l}g)\left(\hbar^{\frac{1}{2}}x\right), \exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar I[\partial_{B^{-1}l}g;B](\hbar)
\end{aligned}
\tag{1.190}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\hbar}\left(g\left(\hbar^{\frac{1}{2}}x\right)\right) = \frac{1}{2}\hbar^{-\frac{1}{2}}\left(x^*\cdot\partial_{\hbar^{\frac{1}{2}}x}g\right)\left(\hbar^{\frac{1}{2}}x\right) = \frac{1}{2}\hbar^{-\frac{1}{2}}\hbar^{-\frac{1}{2}}(x^*\cdot\partial_x g)\left(\hbar^{\frac{1}{2}}x\right) = \frac{1}{2}\hbar^{-1}(x^*\cdot\partial_x g)\left(\hbar^{\frac{1}{2}}x\right) \tag{1.191}
$$

Where x^* is the covector compared to vector x. Compose two results we get a Laplacian operator:

$$
I'[g;B](\hbar) = \frac{1}{2}\hbar^{-1}I[x^*\cdot\partial_x g](\hbar) = \frac{1}{2}\hbar^{-1}\hbar I[\partial_{B^{-1}x^*}\cdot\partial_x g)](\hbar) = I\left[\frac{1}{2}\Delta_B g;B\right](\hbar) \tag{1.192}
$$

By deduction, we can apply as much as possible that:

$$
I^{(n)}[g;B](\hbar) = I\left[\left(\frac{1}{2}\Delta_B\right)^n;B\right](\hbar) \tag{1.193}
$$

If moreover g vanishes at the origin to order $2n+1$, then for every differential operator of order $\leq 2n$ annihilates g at 0, hence:

$$
I_g(0) = I_g'(0) = \ldots = I_g^{(n)}(0) = 0 \tag{1.194}
$$

Assume f attains a global minimum at a unique point $c \in [a,b]$, s.t. $a < c < b$ and $f''(c) > 0$. To acquire more information, we could expand the function as:

$$
f(x) = f(c) + \frac{1}{2}f''(c)(x-c)^2 + \ldots \tag{1.195}
$$

Or even better to take a variable transformation with equality by rescale and shift:

$$
c \to 0 \quad f(c) = 0 \rightsquigarrow f(x) = \frac{M}{2}x^2 \tag{1.196}
$$

If given a compact support that $\{c\} \in U$ where divide the function as $g = g_1 + g_2 \quad g_1 \subset U, g_2 \subset [a,b] \setminus U$. Hence the integral of second term will rapidly decaying if we choose a enough huge neighborhood support around c. We left with the first term as:

$$I[g](\hbar) = \int_{-\infty}^{\infty} g\left(\hbar^{\frac{1}{2}} y\right) e^{-\frac{M}{2} y^2} dy \tag{1.197}$$

$$\hbar^{\frac{1}{2}} I[g; f] = \int_a^b g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} I\left[\tilde{g}; B = (\cdot)^2\right] \tag{1.198}$$

It actually show a general strategy to expand the integral by:

$$\begin{aligned} c = 0 \quad f(x) = \frac{1}{2} p(x)^2 \\ p'(0) = \sqrt{f''(0)} > 0 \end{aligned} \tag{1.199}$$

$$\int_{-\infty}^{\infty} g(x) e^{-\frac{p(x)^2}{2\hbar}} dx \sim \hbar^{\frac{1}{2}} \int_{-\infty}^{\infty} \tilde{g}\left(\hbar^{\frac{1}{2}} y\right) e^{-\frac{y^2}{2}} dy \tag{1.200}$$

$$\begin{aligned} &\frac{p(x)}{\hbar^{\frac{1}{2}}} = y \\ &dx = dp^{-1}\left(\hbar^{\frac{1}{2}} y\right) = \left(p^{-1}\right)'\left(\hbar^{\frac{1}{2}} y\right) \hbar^{\frac{1}{2}} dy \end{aligned} \tag{1.201}$$

$$\begin{aligned} p\left(p^{-1}(x)\right) &= x \\ p'\left(p^{-1}(x)\right) \cdot \left(p^{-1}\right)'(x) &= 1 \\ \left(p^{-1}\right)'(x) &= \frac{1}{p'\left(p^{-1}(x)\right)} \end{aligned} \tag{1.202}$$

$$\tilde{g}(y) \coloneqq g\left(p^{-1}(y)\right) \left(p^{-1}\right)'(y) = \frac{g\left(p^{-1}(y)\right)}{p'\left(p^{-1}(y)\right)} \tag{1.203}$$

We could expand $\tilde{g}$ by Taylor polynomials to acquire a series of summation.

Notice that any odd order polynomials expansion will vanish due to the whole space integral. Thus we can freely expand in $O(\hbar^n)$ order as:

$$\hbar^{\frac{1}{2}} I[g; f] = \hbar^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} I[\tilde{g}; B] = (2\pi\hbar)^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} \sum_{n \geq 0} a_n \hbar^n \tag{1.204}$$

The Gaussian integral can be expanded as $\int_{-\infty}^{\infty} y^{2m} e^{-\frac{y^2}{2}} dy = (2\pi)^{\frac{1}{2}} (2m-1)!!$, So we can exploit the constant that $\tilde{g}(y) = \sum_{n \geq 0}^{\infty} b_n y^n \to a_n = b_{2n} (2m-1)!!$.

$$\int_a^b g(x) e^{\frac{i f(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{\frac{i f(c)}{\hbar}} e^{\frac{\pm \pi i}{4}} I[g](\hbar) \tag{1.205}$$

$$
\begin{aligned}
I(\hbar) &= \sum_{n\geq 0} a_n \hbar^n = \sum_{n\geq 0} a_n \frac{\hbar^n}{n!} n! \\
&= \sum_{n\geq 0} a_n \frac{\hbar^n}{n!} \int_0^\infty u^n e^{-u} du \\
&= \int_0^\infty g(\hbar u) e^{-u} du \quad g(\hbar) = \sum_{n\geq 0} a_n \frac{\hbar^n}{n!} \\
&= \int_{-\infty}^{\infty} |v| g(\hbar v^2) e^{-v^2} dv \quad u \to v^2 \\
&= \hbar^{-\frac{1}{2}} \int_{-\infty}^{\infty} \tilde{g}\left(\hbar^{\frac{1}{2}} v\right) e^{-v^2} dv \quad \tilde{g}(v) = |v| g(v^2)
\end{aligned}
\tag{1.206}
$$

$$
\begin{aligned}
\Gamma(s+1) &= \int_0^\infty t^s e^{-t} dt \quad s > 0 \\
&= s^{s+1} \int_0^\infty x^s e^{-sx} dx \quad t \to sx \\
&= s^{s+1} \int_0^\infty e^{-s(x - \ln x)} dx \quad \hbar \to \frac{1}{s}, f(x) \to x - \ln x, g(x) \to 1 \\
&= s^{s+\frac{1}{2}} e^{-s} (2\pi)^{\frac{1}{2}} \sum_{n\geq 0} a_n \left(\frac{1}{s}\right)^n \quad f(c) = f(1) = 1
\end{aligned}
\tag{1.207}
$$

$$
\begin{aligned}
p(x) = \sqrt{2(x - \ln x)} &= \sqrt{2\left(x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)\right)} \\
&= x\sqrt{1 - \frac{2x}{3} + \frac{x^2}{2} + \ldots} = x - \frac{x^2}{3} + \frac{7x^3}{36} + \ldots
\end{aligned}
\tag{1.208}
$$

$$
\begin{aligned}
p^{-1}(x) &= x + \frac{x^2}{3} + \frac{x^3}{36} + \ldots \\
\left(p^{-1}\right)'(x) &= 1 + \frac{2x}{3} + \frac{x^2}{12} + \ldots
\end{aligned}
\tag{1.209}
$$

$$
\tilde{g}(y) = p'^{-1}(y) \to a_0 = 1, a_1 = \frac{1}{12}(2-1)!! = \frac{1}{12}
\tag{1.210}
$$

Multidimensional steepest descent formula:

$$
\hbar^{\frac{d}{2}} I^D[g; f] = \int_D g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{-\frac{f(c)}{\hbar}} I[g; B]
\tag{1.211}
$$

$$
\begin{aligned}
T &\to \Pi(T) \\
|\Pi_k| &= \frac{(2k)!}{2^k k!} = \frac{(2k)!}{(2k)!!} = (2k-1)!!
\end{aligned}
\tag{1.212}
$$

$$\langle l_1 ... l_N \rangle := \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D l_1(x)...l_N(x) e^{-\frac{S(x)}{\hbar}} dx \tag{1.213}$$

$$S(x) = \frac{B(x,x)}{2} + \tilde{S}(x) = \frac{B(x,x)}{2} - \sum_{i \geq 0} g_i \frac{B_i(x,...,x)}{i!} \tag{1.214}$$

$$Z = I[1; S(x)] = \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx \tag{1.215}$$

$$\begin{aligned} \mathbf{n} &= (n_0, n_1, n_2, ...) \\ F_\Gamma &= \prod_i g_i^{n_i} \cdot \tilde{F}_\Gamma \end{aligned} \tag{1.216}$$

$$\begin{aligned} x \to y &= \hbar^{-\frac{1}{2}} x \\ B_i(x,...,x) &\to \hbar^{\frac{i}{2}} B_i(x,...,x) \\ Z_{\mathbf{n}} &= \int_V e^{-\frac{B(y,y)}{2}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \left(\hbar^{\frac{i}{2}-1} B_i(y,...,y)\right)^{n_i} dy \end{aligned} \tag{1.217}$$

Contract each vertex.

$$-\chi(\Gamma) = E(\Gamma) - V(\Gamma) = \frac{1}{2} \sum_{i \geq 0} n_i i - \sum_{i \geq 0} n_i = \sum_{i \geq 0} n_i \left(\frac{i}{2} - 1\right) \tag{1.218}$$

$$Z_n = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \sum_{\sigma \in \prod(T_{\mathbf{n}})} \tilde{F}(\sigma) \tag{1.219}$$

1. permutation of "flowers" of given valency: S_{n_i}
2. permutation of the ith edges inside each i-valent "flower" : $S_i^{n_i}$

$$\begin{aligned} \mathbb{G}_{\mathbf{n}} &= \prod_i \left(S_{n_i} \ltimes S_i^{n_i}\right) \\ |\mathbb{G}_{\mathbf{n}}| &= \prod_i (n_i!)(i!)^{n_i} \end{aligned} \tag{1.220}$$

The group $\mathbb{G}_{\mathbf{n}}$ acts on the set $\Pi(T_{\mathbf{n}})$, which the stabilizer of a given matching is $\mathrm{Aut}(\Gamma)$.

$$N_\Gamma = \frac{\prod_i (n_i!)(i!)^{n_i}}{|\mathrm{Aut}(\Gamma)|} \tag{1.221}$$

$$\sum_{\sigma \in \Pi(T_{\mathbf{n}})} \tilde{F}(\sigma) = \sum_\Gamma \frac{\prod_i (n_i!)(i!)^{n_i}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma = \sum_\Gamma N_\Gamma \tilde{F}_\Gamma \tag{1.222}$$

$$\chi(\Gamma) = \beta_0(\Gamma) - \beta_1(\Gamma) = V - E \tag{1.223}$$

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}} = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\Gamma \in G(\mathbf{n})} \frac{\tilde{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|} \tag{1.224}$$

We know that such exponential can be expanded as infinite sum of expectation of products.

$$\langle e^l \rangle := \sum_{n \geq 0} \frac{1}{n!} \langle l^n \rangle = \hbar^{\frac{d}{2}} \int_V e^{l(x) - \frac{S(x)}{\hbar}} dx \tag{1.225}$$

We immediately inspect that the infinite sum of $Z_{\mathbf{n}}$ is exact the same as the exponential expansion of Z, a specific case for $B_1(\cdot) = l(\cdot)$ as the external vertices plus a arbitrary action of $\frac{S(x)}{\hbar}$ to induce the Feynman amplitude for $F_\Gamma(l_1, ..., l_N)$ $\Gamma \in G(\mathbf{n})$.

Given below diagram, one have two external vertices and a loop, also with a 3-valent flower and 4-valent flower too.

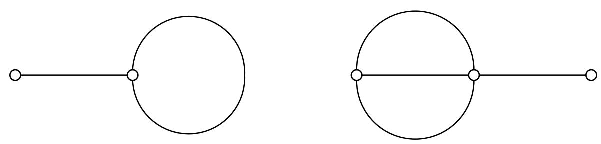

Figure 5. A graph with separated components.

$$\begin{aligned} B_3 &= \sum_i b_i^1 \otimes b_i^2 \otimes b_i^3 \\ B_4 &= \sum_j c_j^1 \otimes c_j^2 \otimes c_j^3 \otimes c_j^4 \end{aligned} \tag{1.226}$$

In expansion of B_3 and B_4 exponential will generate the 3-valent and 4-valent flowers, plus two external vertices l_1, l_2, all which will be contracted by the edges B_2.

$$F_\Gamma(l_1, l_2) = \sum_i B^{-1}(l_1, b_i^1) B^{-1}(b_i^2, b_i^3) \sum_{ij} B^{-1}(b_i^1, c_j^1) B^{-1}(b_i^2, c_j^2) B^{-1}\left(b_i^3, c_j^3\right) B^{-1}(c_j^4, l_2) \tag{1.227}$$

Suppose the coefficients $g_i = 1$, we acquire the general expression for expectation as sum of all possible graphs:

$$\langle l_1 ... l_N \rangle = \sum_{\Gamma \in G(N)} \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \tilde{F}_\Gamma(l_1, ..., l_N) \tag{1.228}$$

Where we reduce the product of $\prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)$ in $\hbar^{-\chi(\Gamma)}$ by the Euler characteristic.

Now, suppose in 1-dimension, given a examples as simple as possible that $B(y, y) = y^2$, $B_i = z^i$, $g_i = g$, $\hbar = 1$:

$$Z = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + g \sum_{i \geq 0} \frac{(zx)^i}{i!}\right) dy = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + g e^{zx}\right) dy \tag{1.229}$$

It actually contains infinite many vertices with arbitrary **valency**, we expand the second term ge^{zx}:

$$Z = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) \sum_{n\geq 0} \frac{\left(ge^{zy}\right)^n}{n!} dy = (2\pi)^{-\frac{1}{2}} \sum_{n\geq 0} \frac{g^n}{n!} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + nzy\right) dy \tag{1.230}$$

We can explain that $z(\cdot)$ pair a vertex, thus n vertices will contribute n factors of z as $z(\cdot) + z(\cdot) + \ldots$:

$$\int \exp\left(-\frac{y^2}{2} + nzy\right) dy = \exp\left(\frac{n^2 z^2}{2}\right) \int \exp\left(-\frac{1}{2}\left(y - nz^2\right)\right) dy = (2\pi)^{\frac{1}{2}} \exp\left(\frac{n^2 z^2}{2}\right) \tag{1.231}$$

$$Z = \sum_{n\geq 0} \frac{g^n}{n!} \exp\left(\frac{n^2 z^2}{2}\right) = \sum_{n\geq 0} \frac{g^n}{n!} \sum_{k\geq 0} \frac{\left(n^2 \frac{z^2}{2}\right)^k}{k!} = \sum_{n,k\geq 0} \frac{g^n n^{2k}}{2^k k! n!} z^{2k} = \sum_{n\geq 0} g^n \sum_{\Gamma \in G(n)} \frac{\tilde{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|} \tag{1.232}$$

Given a number of vertices n with contribution zy to each $B^{-1}(zy, zy) = z^2 B^{-1}(y, y) = z^2$. Thus a general graph with k edges contribute $F_\Gamma = z^{2k} \times \left(B^{-1}\right)^k = z^{2k}$. It can also be evaluated by expanding $\exp(nzy)$ then integrating. Therefore, we can identify the $|\mathrm{Aut}(\Gamma)|$ too:

$$\sum_{\Gamma \in G(n)} (\ldots) = \sum_k \sum_{\Gamma \in G(n;k)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k! n!} \tag{1.233}$$

To scrutinize our answer in combinatorics, we pick out two vertices from n vertices with n^2 choice with order. For list of k edges, we have $n^2 \cdot n^2 \cdot \ldots = n^{2k}$ choices. To cancel out the order, we first remove orientation of edges which contribute $2 \cdot 2 \cdot 2 \ldots = 2^k$ choices. Then we reomove the permutation of edges[2] which contribute $k!$ choices. We still treats the vertices as labelled, so remove the permutation of vertices which contribute $n!$ choices.

Try to decompose the graph to connected components, we denote $\Gamma = \bigsqcup_{j=1}^{r} \Gamma_j^{k_j}$ with $\mathbf{k}$ as tuple of number of copies of each connected components.

$$\begin{gathered} F_{\Gamma_1 \sqcup \Gamma_2} = F_{\Gamma_1} \times F_{\Gamma_2} \\ \chi(\Gamma_1 \sqcup \Gamma_2) = \chi(\Gamma_1) + \chi(\Gamma_2) \\ \left|\mathrm{Aut}\left(\Gamma_1^{k_1} \sqcup \Gamma_2^{k_2} \ldots \sqcup \Gamma_r^{k_r}\right)\right| = \prod_{j=1}^{r} \left|\mathrm{Aut}\left(\Gamma_j\right)\right|^{k_j} k_j! \end{gathered} \tag{1.234}$$

Here $k_{j!}$ of the third equality comes from the permutation of identical connected components.

Given a connected graph set $G_c(*)$, $\Gamma = \bigsqcup_{\gamma \in G_c(*)} \gamma^{k_\gamma}$ where the index is same as the graph itself for simplicity.

$$w(\Gamma) := \frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \tag{1.235}$$

[2] It should also be considered as the contraction order from 1 to k, so if we pick out $(a_1, b_1), (a_2, b_2)$, it′ s also reasonable to pick out $(a_2, b_2), (a_1, b_1)$ which remove the contribution of edges order.

$$\frac{\tilde{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|} = \prod_{\gamma \in G_c(*)} \left(\frac{\tilde{F}_\gamma}{|\mathrm{Aut}(\gamma)|} \right)^{k_\gamma} \frac{1}{k_\gamma!} \tag{1.236}$$

We decompose the second product $\prod_i (\ldots)$ too, therefore we reduce the terms as:

$$w(\Gamma) = \prod_{\gamma \in G_c(*)} \frac{w(\gamma)^{k_\gamma}}{k_\gamma!} \tag{1.237}$$

$$Z = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \sum_{\mathbf{n}} \sum_{\Gamma \in G(n)} w(\Gamma) = Z_0 \sum_{\Gamma \in G(*)} w(\Gamma) \quad Z_0 = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \tag{1.238}$$

Where we just reduce our previous expansion of Z in a more concise expression.

$$\frac{Z}{Z_0} = \sum_{\Gamma \in G(*)} w(\Gamma) = \sum_{k_\gamma} \prod_{\gamma \in G_c(*)} \frac{w(\gamma)^{k_\gamma}}{k_\gamma!} = \exp\left(\sum_{\gamma \in G_c(*)} w(\gamma) \right) \tag{1.239}$$

For any summation of arbitrary graphs in $G(*)$, we can decompose it to connected components with reordering $\mathbf{n} \to \mathbf{k} \sim \sum_{\mathbf{n}} \prod_i \to \sum_{\mathbf{k}} \prod_\gamma$ by counting the flowers into counting the connected components.

$$\ln \frac{Z}{Z_0} = \sum_{\gamma \in G_c(*)} w(\gamma) = \sum_{\gamma \in G_c(*)} \frac{\tilde{F}_\gamma}{|\mathrm{Aut}(\gamma)|} \prod_i \left(g_i \hbar^{\frac{i}{2}-1} \right)^{n_i(\gamma)} = \sum_{\mathbf{n}} \prod_i (\ldots) \tag{1.240}$$

A spanning-tree $T \subset \Gamma$ contains all V vertices and without any cycles, thus a spanning-tree on V vertices contains exactly $V-1$ edges. Thus the extra edges will contribute exact the same number of loops.

$$\beta_1(\Gamma) = 1 - \chi(\Gamma) = E - V + 1 \tag{1.241}$$

$$\begin{aligned} &\beta_0(\Gamma) = 1 \text{ for connected graph} \\ &\beta_1(\Gamma) = E - V + 1 \end{aligned} \tag{1.242}$$

It suggests $\hbar^{-\chi(\gamma)}$ will contribute in different order of $O(\hbar)$. For **tree**, $\beta_1 = 0$, thus a connected graph contribute $-\chi(\gamma) = -(V - E) = -(\beta_0 - \beta_1) = -1 \sim O(\hbar^{-1})$ order. For **1-loop** graph, $\beta_1 = 1$, thus contribute $O(\hbar)$ order, and so forth.

$$Z := \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx = e^{-\frac{S(x_0)}{\hbar}} I[1; B = B_2] = e^{-\frac{S(x_0)}{\hbar}} (\det B)^{\frac{1}{2}} \det(S''(x_0))^{-\frac{1}{2}} \sum_{i \geq 0} a_i \hbar^i \tag{1.243}$$

By steepest descent formula in the neighborhood of critical point x_0 where $S'(x_0) = 0$.

$$\log \frac{Z}{Z_0} = -\frac{S(x_0)}{\hbar} + \frac{1}{2} \log \frac{\det(B)}{\det(S''(x_0))} + \log \left(\sum_{i \geq 0} a_i \hbar^i \right) \tag{1.244}$$

Often, the first term is called *classical approximation* by physicists, the second term is called *1-loop approximation* etc.