Cat-Hom

1 Category

1.1 Additive/Abelian

Additive category is a category $\mathcal{C}$ for which that, given any pair A,B as object, the $\mathrm{Hom}(A, B)$ inherit abelian group law.

$$(f_1 + f_2)\circ g = f_1 \circ g + f_2 \circ g$$
$$f\circ(g_1 + g_2) = f\circ g_1 + f\circ g_2$$

It doesn′t mean category contains a native abelian operation $+$, rather, it′s for each object contained in the category. That is the reason of **additive** comes from. However, the operation without abelian isn′t well defined.

$$(f_1 \,\tilde{+}\, f_2)\circ(g_1 \,\tilde{+}\, g_2) = f_1 \circ g_1 \,\tilde{+}\, f_1 \circ g_2 \,\tilde{+}\, f_2 \circ g_1 \,\tilde{+}\, f_2 \circ g_2$$
$$\neq f_1 \circ g_1 \,\tilde{+}\, f_2 \circ g_1 \,\tilde{+}\, f_1 \circ g_2 \,\tilde{+}\, f_2 \circ g_2$$
$$= (f_1 \,\tilde{+}\, f_2)g_1 \,\tilde{+}\, (f_1 \,\tilde{+}\, f_2)g_2$$

That′s mean in such category, we has $f_1 - f_2 = 0$ if $f_1 = f_2$. The zero object of category which is the unique element(because the equivalence class of $[f_1 - f_2] \sim [0]$ deduced from the equivalence class of functions.) equipped with unique morphism to any other objects.

$(i_A, 0)$ could be the equivalence natural injection and projection between $A \times B$ and A. So in additive category, product and sum is a equal definition.

Kernel is the description of a morphism that send A to zero by f.

Definition 1.1.1. *A **Kernel** of a morphism $f : A \to B$ is a monomorphism $k : K \to A$ such that:*

1. *Composition as Zero: $f\circ k = 0$*
2. *Universal Property: For any $k' : C \to A$ such that $f\circ k' = 0$ will give us a unique morphism u, such that $k' = k\circ u$.*

The universal property doesn′t ensure its existence. So we just point it out rather prove it.

Usually, it does like zorn lemma in axiom choice, we can always select a general representation for a equivalence relation between possible underneath morphisms.

Thus, $\mathrm{Ker}(f)$, if exist, is the equalizer of every possible morphisms that composes f as zero.

We could define in same routine for Coker(f), which is the composition zero by $k \circ f$. That′s the representation of information in target object that doesn′t recorded by f.

Rather, the definition of Im(f) is second in category. We see by $c : B \to \mathrm{Coker}(f)$ $f : A \to B$. We revert the annihilation that $\mathrm{Im}(f) = \mathrm{Ker}(c)$. Which means the information that recorded by f.

We could define in same routine for Coim(f), which is $c : \mathrm{Ker}(f) \to A$ $f : A \to B$. $\mathrm{Coim}(f) = \mathrm{Coker}(c)$.

Then we factor out a morphism by f : $f = \mathrm{im}(f) \circ$ factor $\circ$ coim(f). Which means we decompose f as the information from source object A as coim(f) to target B as im(f). But what about "factor"? That′s the reason of definition **abelian category**, derived from the uncertainty of equivalence relation of coim(f) and im(f). We should ensure they are isomorphism but only in intuition.

Definition 1.1.2. *abelian category is an additive category with following additional conditions:*

    1. Any morphism admits a kernel and cokernel.
    2. $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for every f.

It also indicates a restriction that bijection is isomorphism because $\mathrm{Ker}(f) = 0$ and $\mathrm{Coker}(f) = 0$ indicates:

$$A \to \mathrm{Coim}(f) \to \mathrm{Im(f)} \to B$$

gives a isomorphism based on the bridge of condition (2).

We know we can factor out a object by a morphism that:

$$0 \to \mathrm{Ker}(f) \overset{\mathrm{ker}(f)}{\to} A \overset{\mathrm{im}(f)}{\to} \mathrm{Im}(f) \to 0$$
$$0 \to A \to B \to C \to 0$$

For which ker(f) is monomorphism and im(f) is epimorphism. Observe that $\mathrm{im}(f) \circ \mathrm{ker}(f) = 0$. Which means we fully factor out a object! We call this sequence of morphism is **exact**. We can define in same routine for left exact and right exact, which corresponds to partial factor by left monomorphism or right epimorphism.

If a functor apply to the sequence s.t.:

$$0 \to \mathrm{Hom}(X, A) \overset{u_*}{\to} \mathrm{Hom}(X, B) \overset{v_*}{\to} \mathrm{Hom}(X, C)$$

This functor maps every object in $\mathcal{C}$ to $\mathrm{Hom}(X, \mathcal{C})$ $X \in \mathcal{C}$. Each object is a morphism set and each morphism is a consecutive concatenation by $u_*(f) \sim u \circ f$ $u \in \mathrm{Hom}(A, B)$ in above map.

What property could be preserved? We see in original exact sequence that u is monomorphism s.t. $u_*(f) = u \circ f$ implies that if $u \circ f = u \circ 0$, then $f = 0$. Which still preserves the

property(Same operation could be reversed to prove original category left exactness). We can do the same to prove $\text{Ker}(v_*) = \text{Im}(u_*)$. However, we can see directly that left operation can′t ensure right property which is epimorphism, resulting only left exactness.

Finally, there are also **half exact** functor which only preserves $\text{Ker}(F(v)) = \text{Im}(F(v))$. Everything can also be reversed as in $\mathcal{C}^{\text{op}}$ as a reversed factor(epi to mono, mono to epi, then reverse the arrow).

1.2 Generators and Cogenerators

These elements aren′t the smallest elements, but rather abundant elements capturing enough information.

Definition 1.2.1. *There exist a family of objects* $(U_i)_{i \in I}$ *is a family of generators that for any object A and its suboject B, there always exist* $u_i : U_i \to A$ *such that the image isn′t contained within B.*

The above proposition can be also explained as u_i can′t be factored as a smaller morphism through B. However, it also indicating that we can always determines B by morphisms from other possible set of U_i, or we can always have enough informations to determine a larger subobject through out any object.

We can reversely defining cogenerators by declaring that every $A \to U_i$ must goes through subobject B.

Above definition is already in abelian category, so actually projection and inclusion would be same, gives a isomorphism between generator and cogenerator.

In abelian category, we say that $\pi : \bigoplus U_i = U \to A$ as a quotient of possible index set of U_i.

1.3 Injective and Projective Objects

Injective/Projective objects is the special objects that having enough spaces to contains or gives out information.

As a symmetry, we choose injective:

Definition 1.3.1. *An object M is injective if any morphism into M from subobject* $B \subseteq A$ *can be extended to morphism from larger object A*

We can reverse the definition that projective object would extended to smaller object.

However, it gives a deeper relation to exact sequence by Hom(-,M) contravariant functor.

$$0 \to B \xrightarrow{u} A \xrightarrow{v} A/B \to 0$$

Where $\text{Coker}(u)$ is B. Apply functor that:

$$0 \to \mathrm{Hom}(A/B, M) \overset{v_*}{\to} \mathrm{Hom}(A, M) \overset{u_*}{\to} \mathrm{Hom}(B, M) \to 0?$$

We don't know wether the final arrow is possible to construct. If M is injective, then $u^*(h) = h \circ u$ for $h : A \to M$, would always possible to extended to larger object by the monomorphism u from B s.t. $u^*(h) = h \circ u = g$ for $g : B \to M$. Where we recover the definition of injective object.

We can reverse the definition that projective object applied as $\mathrm{Hom}(P, \text{-})$ preserving right exactness by $v_*(k) = v \circ k = g$ for $\forall k : P \to A$ with $g : P \to A/B$.

The useful technique could be derived from generators. Suppose $i_u : V_i \to U_i$ as a inclusion in subobject of U_i generator, $i_b : B \to B'$ as subojects inclusion of A, given a relation of $U_i \overset{j}{\to} B$.

$$V \overset{\phi'}{\to} U \times B \overset{\phi}{\to} B' \to 0$$

with $\phi'(v) = (i_u(v), -j \circ i_u(v))$ and $\phi(u, b) = i_b \circ j(u) + i_b(b)$.

If the sequence is exact, i.e. $\phi \circ \phi'(v) = i_b \circ j \circ i_u(v) - i_b \circ j \circ i_u(v) = 0$, indicating that the V properly included in B, that's the preimage, i.e $j^{-1}(U_i) = V_i$. We thus construct B' as $\mathrm{Coker}(\phi')$, which is $(U \times B)/V$, modulus already known information of U already in B, gives us B'.

The greatest thing is if a category has generator, we can inductively apply this procedure, while due to zorn lemma, every object violating such property would be contradicted to our proposition of such building procedure, giving us a maximal element.

This proof can be stronger that: i_b as inclusion can be proved rather imposed.

For every possible morphism $Z \overset{m}{\to} B \overset{f}{\to} U \times B \to B' \to 0$, s.t. $i_b \circ m = 0$, which means the image of morphism m is contained in V, s.t. $i_b \circ m = \phi' \circ v$ for $v : Z \to V_i$. We know the inclusion $f \circ m = (0, m) = \sigma \circ v$.

From left component: $(\sigma \text{ to left}) \circ v = 0 \sim v = 0$ because σ is inclusion as injection. Then $m = 0$ from right component, therefore $i_b \circ m = 0 \sim m = 0$, gives i_b is injection.

That's reasonable because V_i is the information both from B and U_i, so if the morphism m image is only contained B is absurd, rather, the gained information must be comes from U_i as a extension.

Above observation implies that rather for object and its suboject relation, every object can be given a injection object, resulting injective property in such category.

1.4 δ-Functor

Commonly, a functor isn't exact, it may be half exact, left or right exact, which means the sequence isn't fully decomposed by morphism.

We denote a system of functors $T = (T_i)$ from an abelian category to a addtive category. Where $T_i : \mathcal{C} \to \mathcal{C}'$, together with special connecting morphsims named as boundary maps $\delta/\partial : T_i(A'') \to T_{i+1}(A')$ for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in each relevant degree i.

To maintain consistency of a functor, if T applies to sequence-A, it can applies to sequence-B, while maintaining commutative property.

The second property is special to δ-functor itself, which is any two morphisms composition is zero, therefore a weaker condition than exactness, at least we won't lose it.

An exact δ-functor contains a stronger second condition which is indeed the exactness.

We could reverse arrow to get contravariant δ-functor, or decrease the degree to get δ^*-functor, however, they are all symmetric representation.

If a category has enough injective objects, we can always construct a exact sequence by $0 \to A \xrightarrow{u} M \to \mathrm{Coker}(u) \to 0$. Where M is a injective object. Even, this could be applied to M itself that $0 \to M_0 \to M_1 \to M_2 \to ...$ or simpler $0 \to M \to M \to 0$ as a self injection. This indicates that all rest of M_i will be a trivial object for self injection of M. Take a δ-functor, we know $F(M_i) \xrightarrow{F(0)} F(M_{i+1})$, which means $T^i(M) = 0$ for $i > 0$. We call such object is **acyclic** stemming from the zero map.

Use a simpler notation that $0 \to A \xrightarrow{u} M \xrightarrow{v} A' \to 0$ for a injection.

Now, we apply exact δ-functor to the sequence, we know $... \to T^0(M) \xrightarrow{T^0(v)} T^0(A') \xrightarrow{\delta} T^1(A) \xrightarrow{T^1(u)} T^1(M) \to ....$ By exactness, we know $T^i(M) = 0$, so δ is surjective as $\mathrm{Im}(\delta) = \mathrm{Ker}(0)$.

Given another T'^i, we could construct a sequence morphism:

Among all δ-functors, universal δ-functor are particularly important, which means for any δ-functor $T' = (T'^i)$ and any natural transformation $f^0 : T^0 \to T'^0$, there exist **unique** morphism of δ-functors $f = (f^i) : T \to T'$ given by f^0. It means the whole behavoir is determined by its behavoir at degree 0!(Apply identity natural transformation to itself.)

Therefore, we define universal δ-functor by $F = T^0$ and right satellite functors of F by $S^i F = T^i$. We can apply same routine to δ^* as decreasing degree, or reverse arrow to define in contravariant δ-functors.

2 Simplex

How to make a general topological basic element? We now that 1-dim would be (x), 2-dim would be (x, y) etc... But there aren't bounded! If we want to make a bounded element. For example, a point, we restrict $x = c$ as a constant. How about 2-dim? We could let $ax + by = c$, oh! Still unbounded! Let $x > 0 \quad y = 0$, everything works. Why we don't

make a point as nothing? Like (0) and make a line as (x) with $0 < x < 1$? Actually, that's still a reasonable definition.

Definition 2.1. *The geometric n-dimensional* **simplex** *is the topological space:*

$$\triangle^n = \left\{(x_0, ...x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 \quad x_i \geq 0\right\}$$

We omit coefficients because those are flexible for topology.

We denote $\sigma = (e_0, ..., e_n)$ as a instance of simplex, with

$$C_n(x) = \bigoplus_{\substack{\sigma_i \in X \\ \dim(\sigma_i)=n}} \mathbb{Z}\sigma_i$$

Now we want to get the differential of the simplex, which is its **boundary**. Suppose we trace a simplex from point e_1 to e_n, we know that a small hyper-surface is its all but omit certain e_n. For example (e_1, e_2, e_3), omit e_1, we get (e_2, e_3) which is a boundary line.

Naively, we could think of that d_n as a differential of n-dim simplex would be

$$d_n : C_n \to C_{n-1}, \sigma_i \to \sum_{k=0}^{n} \sigma_i^{\text{omit kth } e_k}$$

However, the bounded simplex has direction in boundary, same as previous example, if we omit e_2, (e_1, e_3) isn't t the correct direction. The correct direction in number-wise would be $(e_1, e_2), (e_2, e_3), (e_3, e_1)$, so if the whole number permutation in right direction, the left part points sequence will be counted as final part rather initial part. The omitted point will cause a $(-1)^k$ to give a correct answer.

Apply twice first by k then by l:

$$d_n \circ d_{n-1} = \sum_{k=0}^{n}\sum_{l=0}^{n-1} (-1)^{k+l}\sigma_i^{\text{omit k,lth}} = \sum_{k>l}^{...} (-1)^{k+l}\sigma_i^{\text{omit k,lth}} + \sum_{k<l}^{...} (-1)^{k+l-1}\sigma_i^{\text{omit l,kth}} = 0$$

The 3rd step is because if $k < l$, first omit kth will results lth index becomes (l-1)th. Then each sum is symmetric, we get zero in end. It's like a discrete style of total derivative, which is also zero due to symmetry of partial derivative.

However, could a topological space be decomposed by these minimal element? Actually, Partially is. Could the result be independent of triangulation? Yes with a lengthy theorem.