# Review of Quantum

## 1

### 1.1

$g \in G$, given a action of group on a general space $M$ equipped with function on it as $F(M)$. Thus we can apply $g$ on $M$ and lift upon $F(M)$ as well, i.e. $\varphi(g) \cdot x$ for $x \in M$ and $\rho(g) \cdot f$ for $f \in F(M)$ as a $G$-equivariant map.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\varphi(g)"'] & f(x) \arrow[d, "\rho(g)"'] \\ \varphi(g) \cdot x \arrow[r, "f"] & \rho(g) \cdot f(x) / f(\varphi(g) \cdot x) \end{tikzcd}$$

Figure 1. Commutative diagram of group action on space and mapped space

Is commutative, i.e.

$$\rho(g) \cdot (\rho(h) \cdot f)(x) = \rho(h) \cdot f(\varphi(g) \cdot x) = f(\varphi(h) \cdot (\varphi(g) \cdot x)) = f(\varphi(h) \circ \varphi(g) \cdot x) \tag{1.1}$$

Is a anti-homomorphism, i.e. Such representation should be only with $g$ element, thus a canonical choice is $g^{-1}$ where below is given as a reduced notation:

$$g \cdot f(x) = f(g^{-1} \cdot x) \tag{1.2}$$

$V = W \oplus W' \to v = w + w'$, where the projection $q(v) = w$. Define $\overline{q} : v \to \frac{1}{|G|} \sum_{g \in G} \rho(g) q(\rho(g^{-1})v)$ which we drop $\rho$ representation notation.

$$\overline{q}(w) = \frac{1}{|G|} \sum_{g \in G} g q(g^{-1} w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1} w) = w \tag{1.3}$$

$$h\overline{q}(v) = \frac{1}{|G|} \sum_{g \in G} hgq(g^{-1}h^{-1}hv) = \frac{1}{|G|} \sum_{g' \in G} g' q(g'^{-1}hv) = \overline{q}(hv) \tag{1.4}$$

If $v \in \mathrm{Ker}(\overline{q}), h \in G$ then $h\overline{q}(v) = 0 = \overline{q}(hv)$ that $hv \in \mathrm{Ker}(\overline{q})$ that we decompose by a $G$-invariant morphism based on solely vector space projection $q$.

How can we define a $G$-invariant operation on $V$? We should **always** remember that $g$ act transitively on $G$ itself that if $gG = G$. Then if we can put all elements of $G$ on the operation, then it should be $G$-invariant. By the way, we can define the module of $G$-invariance which is $RG = \sum_{g_i \in G} r_i g_i \quad r_i \in R, g_i \in G$. The way is to average the operation by group action for all elements. Thus one has:

$$\begin{aligned} hf(v_1, ..., v_n) &= \frac{1}{|G|} \sum_{g \in G} hgf(g^{-1}h^{-1}hv_1, ..., g^{-1}h^{-1}hv_n) \\ &= \frac{1}{|G|} \sum_{g' \in G} g'f(g'^{-1}hv_1, ..., g'^{-1}hv_n) = f(hv_1, ..., hv_n) \end{aligned} \tag{1.5}$$

$$\begin{aligned} f(hv_1, ..., hv_n) &= \frac{1}{|G|} \sum_{g \in G} gf(g^{-1}hv_1, ..., g^{-1}hv_n) \\ &= \frac{1}{|G|} \sum_{h^{-1}g \in G} h^{-1}gf\left(\left(h^{-1}g\right)^{-1}v_1, ..., \left(h^{-1}g\right)^{-1}v_n\right) = f(v_1, ..., v_n) \end{aligned} \tag{1.6}$$

$$\begin{aligned} \langle v, u \rangle &= \langle u, v \rangle^* \\ \langle u, \alpha v + \beta w \rangle &= \alpha \langle u, v \rangle + \beta \langle u, w \rangle \\ \|u\|^2 &= \langle u, u \rangle \end{aligned} \tag{1.7}$$

$$\langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle \tag{1.8}$$

If there' s a invariant space that $\rho(g)(V_1) \subseteq V_1$, with the decomposition $V_2 = V_1^{\perp}$ that $V_2 = \{v \in V \mid (v, x) = 0 \ \forall x \in V_1\}$.

$$\langle \rho(g)x, y \rangle = \langle \rho(g)^{-1}\rho(g)x, \rho(g)^{-1}y \rangle = \langle x, \rho(g)^{-1}y \rangle \quad \forall y \in V_1 \tag{1.9}$$

Thus if $x \in V_2$, the inner product should be **zero**. Reversely, we deduce that $\rho(g)x \in V_2$ because $y \in V_1$ by first term. Therefore the representation can be decomposed if we seek a **sole** invariant space and repeat the procedure.

Thus given a inner product, which is also a bilinear-function, we can construct $G$-invariant form like above.

$$\langle x, y \rangle := \sum_{h \in G} \langle \rho(h)x, \rho(h)y \rangle \tag{1.10}$$

Which is *unitary* as above already proved. Same as integral form one has:

$$\langle x, y \rangle := \int_G d\mu(h) \langle \rho(h)x, \rho(h)y \rangle \tag{1.11}$$

Yields the same results. However, we can conclude the canonical $G$-invariance form for *finite* and *compact* group due to the convergence given by the integral only for compactness.

Given linear morphism for irreducible space, $\varphi_1 : V \to W$ and $\varphi_2 : V \to W$, one could inspect that due to $\ker(\varphi) = \{0\}$ or $V$ due to **irreducible** property, one can acquire no smaller subspace. Thus, $\varphi = 0$ or isomorphism, by uniqueness, $\varphi_1 = \lambda\varphi_2$.

Now given a space transformation:

$$\varphi(\rho(g)) = A^{-1}\rho(g)A = \rho'(g) \tag{1.12}$$

By above suggestion on irreducible representation, one should have $A = 0$, or better, the irreducible representation should be isomorphism, thus $A = \lambda I$. However, for clarification, you should prefer $\mathbb{C}$ field that to fully decompose space rather the root problem would occur.

$$(v_1, v_2) \to v_1 \otimes v_2 \equiv v_1^i e_1^i \otimes v_2^j e_2^j = v_1^i\, v_2^i\, e_1^i \otimes e_2^j \tag{1.13}$$

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2) \tag{1.14}$$

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)) \tag{1.15}$$

$$\chi_{\rho'}(g) = \mathrm{Tr}(\rho'(g)) = \mathrm{Tr}(A^{-1}\rho(g)A) = \mathrm{Tr}(A^{-1}A\rho(g)) = \mathrm{Tr}(\rho(g)) = \chi_\rho(g) \tag{1.16}$$

$$\chi_{\rho_1 \oplus \rho_2}(g) = \mathrm{Tr}(\rho_1 \oplus \rho_2(g)) = \mathrm{Tr}(\rho_1(g)) + \mathrm{Tr}(\rho_2(g)) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g) \tag{1.17}$$

$$\begin{aligned}\chi_{\rho_1 \otimes \rho_2}(g) = \mathrm{Tr}(\rho_1 \otimes \rho_2(g)) &= \mathrm{Tr}(\rho_1(g)) \otimes \mathrm{Tr}(\rho_2(g)) \\ &= \chi_{\rho_1}(g) \otimes \chi_{\rho_2}(g) = \chi_{\rho_1}(g)\chi_{\rho_2}(g) \otimes 1 \cong \chi_{\rho_1}(g)\chi_{\rho_2}(g)\end{aligned} \tag{1.18}$$

$$\chi_\rho(e) = \mathrm{Tr}(\rho(e)) = \mathrm{Tr}(I_V) = \dim(V) \tag{1.19}$$

$$M_{jl}^{ik} = \int_G d\mu(g)\, \rho_1(g)_j^i \rho_2(g^{-1})_l^k \tag{1.20}$$

$$\rho_1(h)_a^i \rho_1(g)_j^a = \rho_1(hg)_j^i \tag{1.21}$$

$$\begin{aligned}\rho_1(h)_a^i M_{jl}^{ak} &= \int_G d\mu(g)\rho_1(h)_a^i \rho_1(g)_j^a \rho_2(g^{-1})_l^k \\ &= \int_G d\mu(hg)\rho_1(hg)_j^i \rho_2(g^{-1})_l^k \quad hg \to g' \\ &= \int_G d\mu(g')\rho_1(g')_j^i \rho_2(g'^{-1}h)_l^k \\ &= M_{jb}^{ik}\rho_2(h)_k^b\end{aligned} \tag{1.22}$$

By Schur' s lemma, one acquire a one for all relation that either isomorphism, or nothing for $M$ as a transformation matrix. Thus, fix $j, k$ as $M_{ik}^{jl} = \mathcal{M}_i^l$.

$$M_{ik}^{jl} = \alpha_k^j \delta_i^l \tag{1.23}$$

However, one can change the summation for matrix by apply direction as:

$$\rho_1(h)_j^a \rho_1(g)_a^i = \rho_1(gh)_j^i \tag{1.24}$$

Giving us:

$$M_{ik}^{jl} = \beta_i^l \delta_k^j = \alpha_k^j \delta_i^l \tag{1.25}$$

Thus it' s actually a decomposition as $C\delta_k^j \delta_i^l$ for the same irreducible representation or else nothing. We then find it' s actually a trace summation:

$$\int_G d\mu(g)\rho(g)^i_j \rho(g^{-1})^j_i = \int_G d\mu(g)\,\mathrm{Tr}(I_V) = n\int_G d\mu(g) = \sum_{i,j} M_{ji}^{ij} = Cn^2 \tag{1.26}$$

If we can normalize the integral:

$$\int_G d\mu(g) = 1 \to C = \frac{1}{n} = \frac{1}{\dim(V)} \tag{1.27}$$

$$\int_G d\mu(g)\rho(g)^i_i \rho(g^{-1})^j_j = \int_G d\mu(g)\chi_\rho(g)\chi_\rho(g^{-1}) = \sum_{ij} M_{ij}^{ij} = Cn^2 = n \tag{1.28}$$

Thus, one has:

$$\frac{1}{n}\int_G d\mu(g)\chi_{\rho_1}(g)\chi_{\rho_2}(g^{-1}) = 1_{\mathrm{id}}\delta_{\rho_1,\rho_2} \tag{1.29}$$

## 1.2

$$\mathbf{F}(q,p) = \begin{pmatrix} F_q(q,p) \\ F_p(q,p) \end{pmatrix} = \begin{pmatrix} \partial_q F(q,p) \\ \partial_p F(q,p) \end{pmatrix} \tag{1.30}$$

$$dH = H_q dq + H_p dp = -\frac{\mathrm{d}p}{\mathrm{d}t} dq + \frac{\mathrm{d}q}{\mathrm{d}t} dp \tag{1.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t}\partial_q + \frac{\mathrm{d}p}{\mathrm{d}t}\partial_p = H_p\partial_q - H_q\partial_p \coloneqq \{\cdot, H\} \coloneqq X_H \tag{1.32}$$

That' s, Hamiltonian defines the evolution of the system.

$$\frac{\mathrm{d}F(q,p)}{\mathrm{d}t} = X_H F(q,p) = \{F(q,p), H\} \tag{1.33}$$

$$\begin{aligned} X_H(p\,dq) &= \dot{p}dq + pd\dot{q} = \dot{p}dq - \dot{q}dp + d(p\dot{q}) \\ &= -H_q dq + H_p dp + d(p\dot{q}) = d(-H + p\dot{q}) \coloneqq dL(q,\dot{q}) \end{aligned} \tag{1.34}$$

$$\{F, H\} = X_H F = -X_F H \tag{1.35}$$

Where the $X_F = \frac{\partial F}{\partial p}\partial_q - \frac{\partial F}{\partial q}\partial_p$.

Generally, one scalar function defined upon sympletic manifold can induce a vector field on it, with a general area preservation:

$$
\begin{gathered}
\omega = dq \wedge dp \\
\iota_{X_F}(\omega) = \iota_{\frac{\partial F}{\partial q}\partial_p - \frac{\partial F}{\partial p}\partial_q}(dq \wedge dp) = \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial p}dp = dF
\end{gathered}
\tag{1.36}
$$

$$
\iota_{X_H}\iota_{X_F}(\omega) = \iota_{X_H}(dF) = X_H F = \{F, H\} = \frac{\mathrm{d}F}{\mathrm{d}t} \tag{1.37}
$$

We thus induce a symmetric bilinear form:

$$
\omega(X_F, X_G) = \iota_{X_F}\iota_{X_G}(\omega) = \iota_{X_F}(dG) = X_F G = \{G, F\} = -\{F, G\} \tag{1.38}
$$

One can lift the Hamiltonian in exponential flow by dynamics of $X_H$.

$$
\frac{\mathrm{d}}{\mathrm{d}t}F(\exp(tX_H)) \mid_{t=0} = \{F(p,q), H(p,q)\} \tag{1.39}
$$

One thus can define lie group dynamics too just like above:

$$
\begin{gathered}
L \in \mathfrak{g} \to X_L \\
X_L F(p,q) = \frac{\mathrm{d}}{\mathrm{d}t}F\left(e^{tL} \cdot (p,q)\right)
\end{gathered}
\tag{1.40}
$$

However, due to the representation of any group should be a anti-homomorphism, we have:

$$
L \cdot F(p,q) = \frac{\mathrm{d}}{\mathrm{d}t}F\left(e^{-tL} \cdot (p,q)\right) = -X_L F(p,q) \tag{1.41}
$$

Now we can try translation group $T = \mathbb{R}^3$ first:

$$
a \cdot (q,p) = (q + a, p) \quad a \in T \leftrightarrow e^{-t\lambda} \in T \ \lambda \in \mathfrak{t} \tag{1.42}
$$

Here we **abuse** of notation because the lie algebra of translation induced by lie group is trivial.

$$
a \cdot F(p,q) = \frac{\mathrm{d}}{\mathrm{d}t}F\left(e^{-t\lambda} \cdot (p,q)\right) \mid_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}F(q - ta, p) \mid_{t=0} = -a\frac{\partial}{\partial q}F = -X_a F \tag{1.43}
$$

$$
X_a = a\frac{\partial}{\partial q} \tag{1.44}
$$

We know that $\iota_{X_F}(\omega)$ is a closed form that it corresponds to a certain function $F$ up to a constant, we can induce the scalar function of action by the lie group.

$$X_a(\omega) = ap \to \mu_a(p) = \mu_a(p, q) \tag{1.45}$$

We thus recover the momentum operator which configures the momentum for each translation element $a$. Such closed form check lift a function defined in the manifold which shared the same role as Hamiltonian, if $H$ is $G$-invariant, then $\mu_a$ is a conserved quantity, as the Hamiltonian version of Noether′ s theorem.

$$\iota_{X_H}(d\mu_L) = \{\mu_L, H\} = \frac{\mathrm{d}\mu_L}{\mathrm{d}t} = -\{H, \mu_L\} = -\iota_{X_L}(dH) = -X_L H = 0 \tag{1.46}$$

## 1.3

$$[\phi(X), \phi(Y)] = \phi([X, Y]) = \phi(X)\phi(Y) - (-1)^{\sigma(X)\sigma(Y)}\phi(Y)\phi(X) \tag{1.47}$$

$$\Omega(Jv_1, Jv_2) = \Omega(v_1, v_2) \to J \in \mathrm{Sp}(2n, \mathbb{R}) \tag{1.48}$$

$$\begin{gathered} A^T A = I \to A \in \mathrm{O}(n, \mathbb{R}) \\ A^T \Omega A = \Omega \to A \in \mathrm{Sp}(2n, \mathbb{R}) \\ A^T \Omega A = A^T A \Omega \to A^T(\Omega A - A\Omega) = 0 \to \Omega A = A\Omega \\ \Omega(Av, Aw) = g(A\Omega v, Aw) = g(\Omega A v, Aw) = \Omega(v, w) \\ A \in \mathrm{O}(n, \mathbb{R}) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{U}(n) \end{gathered} \tag{1.49}$$

$$z_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad \overline{z_j} = \frac{1}{\sqrt{2}}(q_j - ip_j) \tag{1.50}$$

$$\begin{gathered} L_1 = Q_2 P_3 - Q_3 P_2, \quad L_2 = Q_3 P_1 - Q_1 P_3, \quad L_3 = Q_1 P_2 - Q_2 P_1 \\ \Gamma(L_1) = a_3^\dagger a_2 - a_2^\dagger a_3, \quad \Gamma(L_2) = a_1^\dagger a_3 - a_3^\dagger a_1, \quad \Gamma(L_3) = a_2^\dagger a_1 - a_1^\dagger a_2 \end{gathered} \tag{1.51}$$

$$\begin{aligned} & F(\theta) = c_0 + c_1\theta + c_2\theta^2 + \ldots + c_n\theta^n = c_0 + c_1\theta \\ & \frac{\partial F}{\partial \theta} = c_1 \\ & F(\theta_1, \ldots, \theta_n) = F_A + \theta_j F_B \quad \exists F_A, F_B \in \bigwedge(\mathbb{R}^n) \\ & \frac{\partial FG}{\partial \theta_j} = \frac{\partial F}{\partial \theta_j} G + (-1)^{\sigma(F)} F \frac{\partial G}{\partial \theta_j} \quad \sigma(F) \text{ is the degree of } F \end{aligned} \tag{1.52}$$

Given a translation invariance property of integral:

$$\int f(\theta + \eta) d\theta = \int f(\theta) d\theta \tag{1.53}$$

$$\int (\theta + \eta) d\theta = \int \theta d\theta + \eta \int d\theta = \int \theta d\theta \to \int d\theta = 0 \tag{1.54}$$

Given with normalization convention:

$$\int \theta d\theta = 1 \rightarrow \int c_0 + c_1 \theta d\theta = c_1 \rightarrow \int f(\theta) d\theta = \frac{\mathrm{d}f}{\mathrm{d}\theta} \tag{1.55}$$

Thus for multi-variables calculus, one should be careful on applying order of partial derivative:

$$\int f(\theta_1, ..., \theta_n) d\theta_n ... d\theta_1 = \frac{\partial}{\partial \theta_1} ... \frac{\partial}{\partial \theta_n} f \tag{1.56}$$

Where swap of order of partial derivative will induce a minus sign.

$$\begin{gathered} [\theta_j, \theta_k]_+ = \pm \delta_{jk} \\ \frac{\mathrm{d}}{\mathrm{d}t} \theta_j(t) = [\theta_j(t), H]_+ \end{gathered} \tag{1.57}$$

Given $H$:

$$\begin{gathered} B = \begin{pmatrix} 0 & B_{12} & B_{13} & ... & B_{1n} \\ -B_{12} & 0 & B_{23} & ... & B_{2n} \\ -B_{13} & -B_{23} & 0 & ... & B_{3n} \\ ... & & & & \\ -B_{1n} & -B_{2n} & -B_{3n} & ... & 0 \end{pmatrix} \\ H = \frac{1}{2} \sum_{j,k=1}^{n} B_{jk} \theta_j \theta_k \end{gathered} \tag{1.58}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \theta_j(t) = [\theta_j(t), H]_+ = -[H, \theta_j(t)]_+ = \sum_{k=1}^{n} B_{jk} \theta_k(t) \tag{1.59}$$

The minus sign of commutator is due to $B_{jk} = -B_{kj}$. Which is highly similar to the bosonic case in evolution.

$$\Omega^+(Jv_1, Jv_2) = \Omega^+(v_1, v_2) \rightarrow J \in \mathrm{SO}(2n, \mathbb{R}) \tag{1.60}$$

$$\begin{gathered} U_A = \sum_{jk} a_j^\dagger A_{jk} a_k \\ [U_A, a_j]_\pm = \pm \sum_k A_{jk} a_k = \pm A \boldsymbol{a} \\ [U_A, a_j^\dagger]_\pm = \sum_k a_k^\dagger A_{kj} = A^T \boldsymbol{a}^\dagger \end{gathered} \tag{1.61}$$

$$v \otimes w = \frac{1}{2}(v \otimes w - u \otimes v) + \frac{1}{2}(v \otimes w + w \otimes v) = v \wedge w + g(v, w) \tag{1.62}$$

Thus given a basis omit tensor notation, for example:

$$e_i e_j = e_i \wedge e_j \quad (i \neq j) \tag{1.63}$$

Or physics notation:

$$\boldsymbol{v} = (v_1, ..., v_n) \in \mathbb{R}^n \to \gamma(\boldsymbol{v}) = \not{v} = v_1 \gamma_1 + ... + v_n \gamma_n \tag{1.64}$$

$$\begin{aligned} v' &= v - 2\frac{g(v,w)}{g(w,w)} w = \not{v} - \frac{\not{v}\not{w} - \not{w}\not{v}}{g(w,w)} \not{w} \\ &= -\frac{\not{w}\not{v}\not{w}}{g(w,w)} \end{aligned} \tag{1.65}$$

$$\begin{aligned} 2AB &= [A,B]_+ + [A,B]_- \\ 2BA &= [A,B]_+ - [A,B]_- \end{aligned} \tag{1.66}$$

Thus we can always decompose a product term into symmetric and antisymmetric parts.

$$ABC = \frac{1}{2}A[B,C]_+ + \frac{1}{2}A[B,C]_- = \frac{1}{2}[A,B]_+ C - \frac{1}{2}[A,C]_+ B + \frac{1}{2}[A,B]_- C - \frac{1}{2}[A,C]_- B \tag{1.67}$$

$$\begin{aligned} [AB,C]_- = ABC - CAB &= A\left(\frac{1}{2}[B,C]_+ + \frac{1}{2}[B,C]_-\right) - \left(\frac{1}{2}[A,C]_+ - \frac{1}{2}[A,C]_-\right)B \\ &= \frac{1}{2}(A[B,C]_+ - [A,C]_+ B) + \frac{1}{2}(A[B,C]_- + [A,C]_- B) \end{aligned} \tag{1.68}$$

$$\begin{aligned} [AB,C]_+ = ABC + CAB &= A\left(\frac{1}{2}[B,C]_+ + \frac{1}{2}[B,C]_-\right) + \left(\frac{1}{2}[A,C]_+ - \frac{1}{2}[A,C]_-\right)B \\ &= \frac{1}{2}(A[B,C]_+ - [A,C]_- B) + \frac{1}{2}(A[B,C]_- + [A,C]_+ B) \end{aligned} \tag{1.69}$$

Equal below:

$$[AB,C]_- = A[B,C]_- + [A,C]_- B = A[B,C]_+ - [A,C]_+ B \tag{1.70}$$

$$[AB,C]_+ = A[B,C]_+ + [A,C]_- B = A[B,C]_- - [A,C]_+ B \tag{1.71}$$

$$\begin{aligned} [AB,CD]_- &= A[B,CD]_- + [A,CD]_- B \\ &= A([B,C]_+ D - C[B,D]_+) + ([A,C]_+ D - C[A,D]_+)B \\ &= A[B,C]_+ D - AC[B,D]_+ + [A,C]_+ DB - C[A,D]_+ B \end{aligned} \tag{1.72}$$

For Clifford algebra, symmetric part is given by metric, anti-symmetric part is given by wedge product.

$$\begin{aligned} M_{\mu\nu} &= \gamma_\mu \wedge \gamma_\nu \\ M_{\mu\nu} &= \gamma_\mu \gamma_\nu - g_{\mu\nu} \end{aligned} \tag{1.73}$$

$$M_{\mu\nu} \wedge \gamma_\rho = \gamma_\mu \gamma_\nu \wedge \gamma_\rho = \gamma_\mu g_{\nu\rho} - \gamma_\nu g_{\mu\rho} \tag{1.74}$$

$$M_{\mu\nu} \wedge M_{\rho\sigma} = \gamma_\mu \gamma_\nu \wedge \gamma_\rho \gamma_\sigma = \gamma_\mu \gamma_\sigma g_{\upsilon\rho} - \gamma_\mu \gamma_\rho g_{\nu\sigma} + \gamma_\nu \gamma_\rho g_{\mu\sigma} - \gamma_\nu \gamma_\sigma g_{\mu\rho} \tag{1.75}$$

Is same as above, however, I didn′ t found this is useful : (.

## 1.4

$$\frac{L}{[L, L]} \text{ is abelian that } \forall a, b \in L, [a, b] = 0 \tag{1.76}$$

Thus given a central series:

$$[L^n, L^n] = L^{n+1} \rightarrow \frac{L^n}{[L^n, L^n]} = \frac{L^n}{L^{n+1}} \tag{1.77}$$

Factor through, for all possible elements, otherwise $[L, L] = L$ is irreducible. Thus induce a maximal nilpotent ideal of $L$ which is the maximal solvable ideal of $L$, else it must be a **semi-simple** structure without additional smaller space. Where nilpotent is a structure that can be shared with a basis with commutative eigenvectors, that′ s $XYv = YXv + [X, Y]v = \lambda(X)Yv + \lambda([X, Y])v$ where $[X, Y]$ in some smaller space by deduction of nilpotent ideal.

Given a basis $\{X_1, ..., X_n\}$ with structure constants for a semi-simple lie algebra:

$$\left[X_i, X_j\right] = \sum_k {c_{ij}}^k X_k \tag{1.78}$$

Given the commutation relation, one has:

$$\begin{aligned} {c_{ij}}^k &= -{c_{ji}}^k \\ {c_{ij}}^k + {c_{jk}}^i + {c_{ki}}^j &= 0 \end{aligned} \tag{1.79}$$

Moreover, we can define the representation based on the structure constants:

$$\begin{aligned} &(T_i)^k_j = \left(D_{X_i}\right)^k_j = {c_{ij}}^k \\ &D_{X_i} : \mathfrak{g} \rightarrow \operatorname{ad}(\mathfrak{g}) \end{aligned} \tag{1.80}$$

$$D_{X_i} X_j = \left[X_i, X_j\right] = \sum_k {c_{ij}}^k X_k \tag{1.81}$$

$$\begin{aligned} \left[D_{X_i}, D_{X_j}\right] &= D_{[X_i, X_j]} \\ \left[D_{X_i}, D_{X_j}\right] X_k &= D_{[X_i, X_j]} X_k \\ \left[X_i \left[X_j X_k\right]\right] - \left[X_j [X_i X_k]\right] &= \left[\left[X_i, X_j\right] X_k\right] \end{aligned} \tag{1.82}$$

Which adjoin representation with **homomorphism consistency** should restrict all lie algebras to contain Jacobi identity.

$$\begin{aligned} D_{X_i} D_{X_j} X_l = D_{X_i} c^k{}_{jl} X_k = c_{ik}{}^m c_{jl}{}^k X_m \\ \left(D_{X_i} D_{X_j}\right)^m_l = c_{ik}{}^m c_{jl}{}^k \\ \operatorname{Tr}\left(D_{X_i} D_{X_j}\right) = \left(D_{X_i} D_{X_j}\right)^m_m = c_{ik}{}^m c_{jm}{}^k = g_{ij} = c_{jm}{}^k c_{ik}{}^m = g_{ji} \end{aligned} \tag{1.83}$$

Where:

$$g_{ij} g^{jl} = c_{ik}{}^m c_{jm}{}^k c^{jn}{}_f c^{lf}{}_n = c_{ik}{}^m c^{lf}{}_n \delta_m{}^n \delta^k{}_f = \delta_i{}^l \tag{1.84}$$

$$\langle v, w \rangle = v^i \left(D_{X_i}\right)^l_k v^j \left(D_{X_j}\right)^k_l = v^i v^j g_{ij} \tag{1.85}$$

Cyclic property:

$$0 = \operatorname{Tr}([A, BC]) = \operatorname{Tr}(B[A, C]) + \operatorname{Tr}([A, B]C) \tag{1.86}$$

$$\begin{aligned} \operatorname{Tr}([A, B]C) &= \operatorname{Tr}((AB - BA)C) \\ &= \operatorname{Tr}(ABC) - \operatorname{Tr}(BAC) \\ &= \operatorname{Tr}(ABC) - Tr(ACB) = \operatorname{Tr}(A[B, C]) \end{aligned} \tag{1.87}$$

$$\begin{aligned} 0 = \operatorname{Tr}\left(\left[\rho(X_i)\rho\left(X_j\right), \rho(X_k)\right]\right) &= \operatorname{Tr}\left(\rho(X_i)\left[\rho\left(X_j\right), \rho(X_k)\right]\right) + \operatorname{Tr}\left([\rho(X_i), \rho(X_k)]\rho\left(X_j\right)\right) \\ &= \operatorname{Tr}\left(\rho(X_i) f^d_{jk} \rho(X_d)\right) + \operatorname{Tr}\left(f^d_{ik} \rho(X_d) \rho\left(X_j\right)\right) \\ &= f^d_{jk} g_{id} + f^d_{ik} g_{dj} \\ &= f_{jki} + f_{ikj} \end{aligned} \tag{1.88}$$

Given with $f_{jki} = -f_{kji}$, one has $f_{jki} = -f_{kji} = f_{ijk} = -f_{jik}$. It' s totally anti-symmetric tensor.

Fix indices as $f^k_{ij} = (\not{f}_i)^k_j$:

$$\begin{aligned} \not{f}_j g^{(\rho)} + \left(\not{f}_i g^{(\rho)}\right)^T &= 0 \\ g^{(-1)^{(\rho)}} \not{f}_j g^{(\rho)} &= -\not{f}^T_i \quad \text{by } g = g^T \\ \not{f}_j g^{(\rho)} &= g^{(\rho)} g^{(-1)^{(\rho)}} \not{f}_j g^{(\rho)} \\ \not{f}_j g^{(\rho)} g^{(-1)^{(\rho)}} &= g^{(\rho)} g^{(-1)^{(\rho)}} \not{f}_j \end{aligned} \tag{1.89}$$

Thus by Schur' s lemma, for *simple* lie algebra, it should be up to constant for the representation of the metric form compared with the canonical metric form.

However, it doesn' t mean that **semi-simple** structure contains no ideal. We try to diagonalize as much as possible even we can' t achieve for all like solvable case. That' s $\mathfrak{t} \subset \mathfrak{g}$ which all elements are commutative. We call such maximal commutative subalgebra as **Cartan subalgebra**, which can be diagonalized simultaneously. Thus if one choose a vector space it acts on, we can decompose the space into eigen-space for all:

$$V = \bigoplus_{\lambda(\mathfrak{t})} V_\lambda \quad V_\lambda = \{v \in V \mid \forall H \in \mathfrak{t}, Hv = \lambda(H)v\} \tag{1.90}$$

Given a adjoin representation or homomorphism as $X_i \to D_{X_i}$, we can then apply the Cartan subalgebra to decompose the lie algebra itself:

$$\mathfrak{g} = \bigoplus_{\alpha(\mathfrak{t})} \mathfrak{g}_\alpha \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{t}, [H, X] = D_H(X) = \alpha(H)X\} \tag{1.91}$$

Which is called **root space decomposition**, where $\alpha \in \mathfrak{t}^*$ is the eigenvalue on $\mathfrak{t}$.

We often see the ladder operator in physics that a state shared by all other raising and lowering operators, which is the eigenvector of Cartan subalgebra.

$$\begin{gathered} \boldsymbol{H} = (H_1, ..., H_r) \\ [H_i, H_j] = 0 \quad \forall i, j = 1, ..., r \to [\rho(H_i), \rho(H_j)] = 0 \quad \text{for finite representation} \end{gathered} \tag{1.92}$$

Due to mutually commutation, each $H_i$ can share the same eigenstate with weights $\boldsymbol{m}$.

$$\rho(H_i)|\boldsymbol{m}\rangle = m_i|\boldsymbol{m}\rangle \quad \boldsymbol{m} = (m_1, ..., m_r) \tag{1.93}$$

Plus, given by the adjoint representation, the same routine is applied for remained decomposition.

$$D_{H_i}(E_{\boldsymbol{\alpha}}) = [H_i, E_{\boldsymbol{\alpha}}] = \boldsymbol{\alpha}(H_i)E_{\boldsymbol{\alpha}} = \alpha_i E_{\boldsymbol{\alpha}} \quad \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r) \tag{1.94}$$

$$\begin{aligned} D_{H_i}([E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}]) = [H_i, [E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}]] &= [[H_i, E_{\boldsymbol{\alpha}}], E_{\boldsymbol{\beta}}] + [E_{\boldsymbol{\alpha}}, [H_i, E_{\boldsymbol{\beta}}]] \\ &= (\alpha_i + \beta_i)[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] \sim H_i E_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \leftrightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \end{aligned} \tag{1.95}$$

Just like you already have seen in generator of $\mathfrak{su}(2)$ that $[\xi_3, \xi^\pm] = \pm\xi^\pm$.

We know if the ladder operator act too many times in finite representation, it will reach zero, that' s $D_X$ is nilpotent for $X \in \mathfrak{g}_\alpha, \alpha \neq 0$, or we simply call it nilpotent that $\mathfrak{g}_{\beta+n\alpha} = 0$.

For clarity, $\mathfrak{g}_0$ means $[\mathfrak{t}, \mathfrak{g}_0] = 0 \cdot \mathfrak{g}_0 = 0$ thus $\mathfrak{t} \subset \mathfrak{g}_0$, however, due to maximality of Cartan subalgebra, we have $\mathfrak{g}_0 = \mathfrak{t}$.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{t} \tag{1.96}$$

Given a unitary representation, with the observable eigenvalue should be *real*, thus, we restrict $\boldsymbol{m}, \boldsymbol{\alpha}$ to real value, or $\boldsymbol{H}^\dagger = \boldsymbol{H}$.

$$[H_i, E_{\boldsymbol{\alpha}}]^\dagger = (\alpha_i E_{\boldsymbol{\alpha}})^\dagger \to \left[E^\dagger_{\boldsymbol{\alpha}}, H_i\right] = \alpha_i^* E^\dagger_{\boldsymbol{\alpha}} = -\left[H_i, E^\dagger_{\boldsymbol{\alpha}}\right] \to E^\dagger_{\boldsymbol{\alpha}} \in \mathfrak{g}_{-\alpha} \tag{1.97}$$

Which is justify the existence of $\mathfrak{g}_{-\alpha}$

$$[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \sum_i^r b^i H_i \tag{1.98}$$

Restrict to $\mathfrak{t}$, we acquire its metric as $\mathrm{Tr}(H_i H_j) = g_{ij}$.

$$\sum_j g_{ij} b^j = \mathrm{Tr}(H_i [E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}]) = \mathrm{Tr}([H_i, E_{\boldsymbol{\alpha}}] E_{-\boldsymbol{\alpha}}) = \alpha_i \, \mathrm{Tr}(E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}) \tag{1.99}$$

If we normalize that $\mathrm{Tr}(E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}) = 1$.

$$\sum_j g_{ij} b^j = \alpha_i \to b^j = \sum_i g^{ij} \alpha_i \tag{1.100}$$

We thus could define $H_{\boldsymbol{\alpha}} \coloneqq [E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \sum_j b_j H_j$.

$$\left[H_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}\right] = \sum_j b_j \left[H_j, E_{\boldsymbol{\beta}}\right] = (\boldsymbol{b} \cdot \beta) E_{\boldsymbol{\beta}} \to [H_{\boldsymbol{\alpha}}, E_{\pm\boldsymbol{\alpha}}] = \pm(\boldsymbol{b} \cdot \boldsymbol{\alpha}) E_{\pm\boldsymbol{\alpha}} \tag{1.101}$$

However:

$$\boldsymbol{b} \cdot \boldsymbol{\alpha} = \sum_{ij} \alpha_i \alpha_j g^{ij} = \langle \alpha, \alpha \rangle \tag{1.102}$$

And we actually acquire a more ease representation as $H_{\boldsymbol{\alpha}}$.

### 1.4.1

$$\rho(H_i)\rho(E_{\boldsymbol{\alpha}})|\rho; \boldsymbol{m}\rangle = ([\rho(H_i), \rho(E_{\boldsymbol{\alpha}})] + \rho(E_{\boldsymbol{\alpha}})\rho(H_i))|\rho; \boldsymbol{m}\rangle = (\boldsymbol{\alpha}_i + m_i) E_{\boldsymbol{\alpha}} |\rho; \boldsymbol{m}\rangle \tag{1.103}$$

$$\rho(E_{\boldsymbol{\alpha}})|\rho; \boldsymbol{m}\rangle = N_\alpha(\boldsymbol{m})|\rho; \boldsymbol{m} + \boldsymbol{\alpha}\rangle \tag{1.104}$$

$$\begin{aligned} &[\rho(E_{\boldsymbol{\alpha}}), \rho(E_{-\boldsymbol{\alpha}})] = \rho(H_{\boldsymbol{\alpha}}) \\ \to\ & N_\alpha(\boldsymbol{m} - \boldsymbol{\alpha}) N_{-\alpha}(\boldsymbol{m}) - N_{-\alpha}(\boldsymbol{m} + \boldsymbol{\alpha}) N_\alpha(\boldsymbol{m}) = (\boldsymbol{\alpha}, \boldsymbol{m}) \end{aligned} \tag{1.105}$$

$$E_{\boldsymbol{\alpha}} |\rho; \boldsymbol{m} + p\boldsymbol{\alpha}\rangle = 0 \quad E_{-\boldsymbol{\alpha}} |\rho; \boldsymbol{m} - q\boldsymbol{\alpha}\rangle = 0 \tag{1.106}$$

$$N_\alpha(\boldsymbol{m} + (k-1)\boldsymbol{\alpha}) N_{-\alpha}(\boldsymbol{m} + k\boldsymbol{\alpha}) - N_{-\alpha}(\boldsymbol{m} + (k+1)\boldsymbol{\alpha}) N_\alpha(\boldsymbol{m} + k\boldsymbol{\alpha}) = \langle \boldsymbol{\alpha}, \boldsymbol{m} + k\boldsymbol{\alpha} \rangle \tag{1.107}$$

$$
\begin{aligned}
&\sum_{k=-q}^{p} N_\alpha(\boldsymbol{m}+(k-1)\boldsymbol{\alpha})N_{-\alpha}(\boldsymbol{m}+k\boldsymbol{\alpha}) \\
&\qquad -N_{-\alpha}(\boldsymbol{m}+(k+1)\boldsymbol{\alpha})N_\alpha(\boldsymbol{m}+k\boldsymbol{\alpha}) = \sum_{k=-q}^{p}\langle\boldsymbol{\alpha},\boldsymbol{m}+k\boldsymbol{\alpha}\rangle
\end{aligned}
\tag{1.108}
$$

Then, for second term, change the summation from $k \to k-1$, we cancel each term except the first and last.

$$
\begin{aligned}
&N_\alpha(\boldsymbol{m}-(q+1)\boldsymbol{\alpha})N_{-\alpha}(\boldsymbol{m}-q\boldsymbol{\alpha}) \\
&-N_{-\alpha}(\boldsymbol{m}+(p+1)\boldsymbol{\alpha})N_\alpha(\boldsymbol{m}+p\boldsymbol{\alpha}) = \sum_{k=-q}^{p}\langle\boldsymbol{\alpha},\boldsymbol{m}+k\boldsymbol{\alpha}\rangle
\end{aligned}
\tag{1.109}
$$

However, due to the finite order restriction, the left terms all vanish.

$$
\begin{aligned}
(p+q+1)\langle\boldsymbol{\alpha},\boldsymbol{m}\rangle + \left(\frac{1}{2}p(p+1) - \frac{1}{2}q(q+1)\right)\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle &= 0 \\
2\frac{\langle\boldsymbol{\alpha},\boldsymbol{m}\rangle}{\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle} &= q-p
\end{aligned}
\tag{1.110}
$$

So what′ s the procedure to investigate the roots of a lie algebra? First, given a canonical matrix representation, one should find the Cartan subalgebra, how? We seek the maximal independent *diagonal* matrix set. Then seek the relation of $\boldsymbol{H}$ and remaining elements, you may not pretty lucky to find direct scalar relation rather mixed terms. Then, you should make a combination like $v = a^i M_i$, and try to build by $\left[H_i, a^j M_j\right] = \lambda_i a^j M_j$ exactly. Given each coefficient with normalization, you may find a set of $\delta_{i,j} + \delta_{i+1,j} + \ldots$ for possible slots. Summarize, it can be evaluated for $\boldsymbol{\alpha}(H_i)$ for $i$ index of this vector, it indicates to extract the $i$ set to form a basis combination by $\delta_{ij} + \delta_{i+1,j} \sim e_j$ etc, which is the root.

### 1.4.2 $A_n = \mathfrak{sl}(n+1)$

Given $\mathbf{e}_i$ for $i = 1, \ldots, n+1$ in $\mathbb{R}^{n+1}$ parametrize the lie algebra $\mathfrak{sl}(n+1)$. Thus the matrix is represented by $\left(M_{ij}\right)_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$ which Kronecker index connects the matrix slot and generator index.

It′ s more natural to just seek a matrix summation to yield results, rather a direct index summation is false. Looking at the $jk$ summation yielding:

$$
M_{i,j}M_{k,l} = \delta_{j,k}M_{i,l} \tag{1.111}
$$

$$
\left[M_{i,j}, M_{k,l}\right] = \delta_{j,k}M_{i,l} - \delta_{i,l}M_{j,k} \tag{1.112}
$$

Given $\mathrm{Tr}(\mathfrak{sl}(n+1)) = 0$, we should take the maximal diagonal and traceless basis set, then we choose:

$$
\begin{aligned}
H_i &= M_{i,i} - M_{i+1,i+1} \\
\left[H_i, M_{k,l}\right] &= \delta_{i,k}M_{i,l} - \delta_{i+1,k}M_{i+1,l} + \delta_{i,l}M_{k,i} - \delta_{i+1,l}M_{k,i+1} \\
&= \left(\delta_{i,k} - \delta_{i,l} - \delta_{i+1,k} + \delta_{i+1,l}\right)M_{k,l}
\end{aligned}
\tag{1.113}
$$

Now, be clever to extract the basis of $i$ index to form simple root set.

$$\boldsymbol{\alpha} = \mathbf{e}_k - \mathbf{e}_l \quad E_{\boldsymbol{\alpha}} = M_{k,l} \tag{1.114}$$

$$\begin{aligned} [H_i, E_{\boldsymbol{\alpha}}] &= \boldsymbol{\alpha}(H_i) E_{\boldsymbol{\alpha}} \\ \boldsymbol{\alpha}(H_i) &= \left(\delta_{i,k} - \delta_{i,l}\right) - \left(\delta_{i+1,k} - \delta_{i+1,l}\right) \end{aligned} \tag{1.115}$$

Where we choose root as $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} \quad i = 1, ..., n$ to expand into more clarified basis set.

$$\mathbf{e}_k - \mathbf{e}_l = \sum_{k}^{l-1} \left(\mathbf{e}_i - \mathbf{e}_{i+1}\right) \tag{1.116}$$

$$\left\langle \alpha_i, \alpha_j \right\rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| \geq -2 \end{cases} \tag{1.117}$$

We only care about the normalization relation upon the inner product, which build the relation between each basis. How to express this clearly? One named *Dynkin*, creates the diagram as follows:

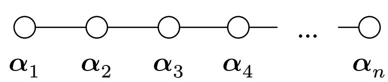


Figure 2. Dynkin diagram of $A_n = \mathfrak{sl}(n+1)$

Where the solid line connects node indicates $-\frac{1}{2} = \cos\frac{\pi}{3}$ by normalization of simple roots in $i, j$ where $|i-j| = 1$. Thus, given a relation between nodes, we connects nodes by single or multiple solid lines to indicates different normalization angles. We could also express in matrix form by $C_{ij} = \left\langle \alpha_i, \alpha_j \right\rangle$ called *Cartan* matrix which is too sparse in expression compared to Dynkin diagram.

### 1.4.3 $D_n = \mathfrak{so}(2n)$

As for algebra $\mathfrak{so}(2n)$, the antisymmetric matrix should be denoted as $M_{ij} = -M_{ji} = -M_{ij}^T$ for $i, j = 1, ..., 2n$.

$$\left(M_{ij}\right)_{\alpha\beta} = \delta_{a,\alpha}\delta_{b,\beta} - \delta_{a,\beta}\delta_{b,\alpha} \tag{1.118}$$

We should notie that one should transpose the index to sum again in second term, thus given $(i,j)(k,l)$, we should merge $j, k$ first and then $i, k$.

$$\left[M_{i,j}, M_{k,l}\right] = \delta_{j,k} M_{i,l} - \delta_{i,k} M_{j,l} + \delta_{l,i} M_{k,j} - \delta_{k,i} M_{l,j} \tag{1.119}$$

We define the Cartan subalgebra as, for maximal $n$ ranks for $\mathfrak{so}(2n)$:

$$H_i = M_{2i-1,2i} \quad i = 1, 2, ..., n \tag{1.120}$$

Which is **not** diagonalizable actually, that it must mix four elements by previous calculation. If one need to diagonalize such basis, we should take in $\mathbb{C}$ with $i$ factor appended. Rank $n$ is the consequence derived from maximally $n$ independent rotation.

We aren' t lucky enough to extract the scalar for a single basis, rather those are scalars distributing around different ones. Thus, we choose a vector format to calculate by choose a various $j, k = 1, 2, ..., n$ basis:

$$\begin{gathered} \boldsymbol{M} = aM_{2j-1,2k-1} + bM_{2j,2k-1} + cM_{2j-1,2k} + dM_{2j,2k} \\ [H_i, \boldsymbol{M}] = a\big(\delta_{2i,2j-1}M_{2i-1,2k-1} - \delta_{2i-1,2j-1}M_{2i,2k-1} + \delta_{2k-1,2i-1}M_{2i,2j-1} - \delta_{2k-1,2i}M_{2i-1,2j-1}\big) + ... \end{gathered} \tag{1.121}$$

We suddenly found that $2i \neq 2j - 1$ ever and ever because one is odd and one is even! We immediately reduce the index into $2i - 1, 2j - 1$ or $2k - 1, 2i - 1$. Then, we note that $\delta_{i,j}M_{2i,2k-1} = M_{2j,2k-1}$, $\delta_{k,i}M_{2i,2j-1} = -M_{2j-1,2k}$.

Thus we choose:

$$\begin{aligned} &\boldsymbol{\alpha} = \eta\mathbf{e}_j + \eta'\mathbf{e}_k \quad j \neq k \quad \eta, \eta' = 1, -1 \\ &E_{\boldsymbol{\alpha}} = \frac{1}{2}\big(M_{2j-1,2k-1} + \eta M_{2j,2k-1} + \eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}\big) \\ &\boldsymbol{\alpha}(H_i) = \eta\delta_{i,j} + \eta'\delta_{i,k} \\ &[H_i, E_{\boldsymbol{\alpha}}] = \big(\eta\delta_{i,j} + \eta'\delta_{i,k}\big)E_{\boldsymbol{\alpha}} \end{aligned} \tag{1.122}$$

By counting numbers of generators, we found that they are $2n(n-1)$ for $j, k$ root generators.

$$\begin{aligned} &\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} \quad i = 1, ..., n-1 \\ &\alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n \end{aligned} \tag{1.123}$$

Actually, choose arbitrary sign is acceptable, but such is preferable.

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{if } |i - j| = 0 \\ 0 & \text{if } i = n - 1 \; j = n \end{cases} \tag{1.124}$$

We draw the *Dynkin* diagram as below:

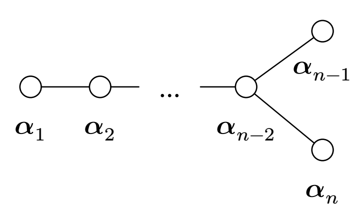


Figure 3. Dynkin diagram of $D_n = \mathfrak{so}(2n)$


Figure 4. Dynkin diagram of $B_n = \mathfrak{so}(2n+1)$

### 1.4.4 Coincidence

$$
\begin{aligned}
\mathfrak{so}(3;\mathbb{C}) &= \mathfrak{sl}(2;\mathbb{C}) = \mathfrak{sp}(2;\mathbb{C}) \\
\mathfrak{so}(4;\mathbb{C}) &= \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}) \\
\mathfrak{so}(5;\mathbb{C}) &= \mathfrak{sp}(4;\mathbb{C}) \\
\mathfrak{so}(6;\mathbb{C}) &= \mathfrak{sl}(4;\mathbb{C})
\end{aligned}
\tag{1.125}
$$

### 1.4.5

$\mathfrak{su}(2) = \left\{X \in \mathfrak{gl}\ (2,\mathbb{C}) \mid \overline{X}^t + X = 0, \mathrm{Tr}(X) = 0\right\}$ and $\mathfrak{so}(3)$ are all real lie algebras of dimension 3.

(a) The basis:

$$
\xi_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \xi_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\tag{1.126}
$$

With $[\xi_k, \xi_l] = \varepsilon_{klm}\xi_m$.

$$
\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2
\tag{1.127}
$$

$$
[\xi^2, \xi_k] = 0
\tag{1.128}
$$

We construct a commute element which the maximal diagonalized state is $|b, m\rangle$.

$$
\begin{aligned}
\xi^2|b, m\rangle &= b|b, m\rangle \\
\xi_3|b, m\rangle &= m|b, m\rangle
\end{aligned}
\tag{1.129}
$$

$$
\xi^+ = \xi_1 + i\xi_2, \quad \xi^- = \xi_1 - i\xi_2
\tag{1.130}
$$

$$
[\xi_3, \xi^+] = \xi^+, \quad [\xi_3, \xi^-] = -\xi^-
\tag{1.131}
$$

$$
[\xi^+, \xi^-] = 2\xi_3
\tag{1.132}
$$

$$
\xi_3\xi^\pm|b, m\rangle = [\xi_3, \xi^\pm]|b, m\rangle + \xi^\pm\xi_3|b, m\rangle = (m \pm 1)\xi^\pm|b, m\rangle
\tag{1.133}
$$

Thus we conclude that $\xi^\pm|b, m\rangle = C|b, m \pm 1\rangle, \quad C \in \mathbb{C}$, and $\xi_3|b, m \pm 1\rangle = (m \pm 1)|b, m \pm 1\rangle$.

We call such operators ladder operators or raising and lowering operators.

$$
\xi^+|b, j\rangle = 0 \quad \exists j \in ?
\tag{1.134}
$$

$$
\xi^-\xi^+ = (\xi_1 - i\xi_2)(\xi_1 + i\xi_2) = \xi_1^2 + \xi_2^2 + i[\xi_1, \xi_2] = \xi^2 - \xi_3^2 + \xi_3
\tag{1.135}
$$

$$0 = \xi^- \xi^+ |b, j\rangle = (b - j^2 + j)|b, j\rangle \to b = j(j+1) \tag{1.136}$$

Same for $\xi^+ \xi^-$, one has $b - j'^2 + j' = 0$

$$\begin{aligned} b = j'(j'-1) = -j'(-j'+1) \\ \to j' = -j \end{aligned} \tag{1.137}$$

We conclude that for $m$ can be $-j, -j+1, ..., j-1, j$, thus $2j \in \mathbb{N}$ is a integer for finite dimension.

$$\begin{aligned} & j = 0, \quad m = 0, \quad j(j+1) = 0 \\ & j = \frac{1}{2}, \quad m = \frac{1}{2}, -\frac{1}{2}, \quad j(j+1) = \frac{3}{4} \\ & j = 1, \quad m = 1, 0, -1, \quad j(j+1) = 2 \\ & \dots \end{aligned} \tag{1.138}$$

From the definition of $\xi^+ = \xi^{-^\dagger}$ by its expansion, we have such normalization factor:

$$\begin{aligned} \left\langle b, m \middle| \xi^{+^\dagger} \xi^+ \middle| b, m \right\rangle &= \langle b, m | \xi^- \xi^+ | b, m \rangle \\ &= \langle b, m | \xi^2 - \xi_3^2 - \xi_3 | b, m \rangle \\ &= b - m^2 - m = j(j+1) - m^2 - m = C^2 \end{aligned} \tag{1.139}$$

$$C = \sqrt{j(j+1) - m(m+1)} \tag{1.140}$$

Same, one has for $\xi^-$ that $\tilde{C} = \sqrt{j(j+1) - m(m-1)}$.

### 1.4.6

Take the transformation $SO(3)$ as $GL(3; \mathbb{R})$ matrix representation act naturally on $\mathbb{R}^3$ equipped with function valued on it as a scalar form.

$$\begin{aligned} l_1(f)(\boldsymbol{x}) = f(l_1^{-1} x) &= \frac{d}{dt} f \left( \exp \left( t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) |_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) |_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f \left( \begin{pmatrix} x_1 \\ x_2 \cos t + x_3 \sin t \\ -x_2 \sin t + x_3 \cos t \end{pmatrix} \right) \\ &= \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) \cdot \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix} \\ &= x_3 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_3} \end{aligned} \tag{1.141}$$

If we have $x_1 = r\sin\theta\cos\phi, x_2 = r\sin\theta\sin\phi, x_3 = r\cos\theta$, we can construct the transformation upon $\theta, \phi$ too.

$$\frac{\partial}{\partial x_i} = \frac{\partial r_i}{\partial x_i}\frac{\partial}{\partial r_i} \tag{1.142}$$

In differential geometry as a basis vector transformation, or familiar Jacobi.

$$\begin{aligned} l_1 &= x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \\ l_2 &= -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi} \\ l_3 &= -\frac{\partial}{\partial\phi} \end{aligned} \tag{1.143}$$

$$\begin{aligned} l^+ &= e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) \\ l^- &= e^{i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) \end{aligned} \tag{1.144}$$

Given the representation $F(\theta,\phi)(m)$ with the weight $m$ chosen by $l_3$, with the highest weight denoted as $l$:

$$l_3 F(\theta,\phi)(m) = -\frac{\partial}{\partial\phi}F(\theta,\phi)(m) = mF(\theta,\phi)(m) \tag{1.145}$$

Thus we have $F(\theta,\phi) \sim e^{m\phi}G(\theta)$, however, we have such restriction $\phi + 2\pi \sim \phi$, thus we must have $m \to im$ to match such period.

We can immediately decompose the representation scalar function by raising operator for the highest weight:

$$\begin{aligned} \left(\frac{\partial}{\partial\theta} - l\cot\theta\right)G(\theta)(l) = 0 \\ \ln G(\theta)(l) = l\ln\sin\theta \\ G(\theta)(l) = C_l\sin^l\theta \end{aligned} \tag{1.146}$$

Where the constant is, for the representation of $2l+1$ dimension in $l$ weight. Apply lowering operator would giving us:

$$\begin{aligned} F(\theta,\phi)(m) &= C_m(l^-)^{l-m}F(\theta,\phi)(l) \\ &= C_m\left(e^{-i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right)\right)^{l-m} e^{-il\phi}\sin^l\theta \end{aligned} \tag{1.147}$$

However, one usually encompass the whole $2l+1$ representation for arbitrary $l$ as:

$$Y_l^m(\theta,\phi) = C_{lm}(\ldots) \text{ as representation for } 2l+1 \text{ dimension with weight } m \tag{1.148}$$

Casimir operator with commutative property acting on will give us the eigenvalue.

$$l^2 = l_1^2 + l_2^2 + l_3^2 \tag{1.149}$$

$$l^2 Y_l^m(\theta, \phi) = l(l+1) Y_l^m(\theta, \phi) \tag{1.150}$$

$$\begin{aligned} l^2 &= l^- l^+ + i l_3 + l_3^2 \\ &= e^{i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right)e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) - i\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} \\ &= \frac{\partial^2}{\partial\theta^2} + i\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\phi}\right) + e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\left(-i\frac{\partial}{\partial\theta}\right)\right) + e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}\right) + (...) \end{aligned} \tag{1.151}$$

Evaluate that:

$$\begin{aligned} i\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\phi}\right) &= \frac{i}{\sin^2\theta}\frac{\partial}{\partial\phi} + i\cot\theta\frac{\partial^2}{\partial\theta\partial\phi} \\ e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\left(-i\frac{\partial}{\partial\theta}\right)\right) &= -\cot\theta\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial^2}{\partial\theta\partial\phi} \\ e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}\right) &= -i\cot^2\theta\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} \end{aligned} \tag{1.152}$$

The first term evaluate as:

$$\begin{aligned} \text{First term} &= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + i\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi} - \cot^2\theta\frac{\partial}{\partial\phi}\right) + \cot^2\theta\frac{\partial^2}{\partial\phi^2} \\ &= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} \end{aligned} \tag{1.153}$$

$$\begin{aligned} \text{LHS} &= \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\phi^2} \\ &= \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right)\sin\theta + \frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\ &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} = \Delta_\Omega \end{aligned} \tag{1.154}$$

Which is the **Spherical Laplacian**[1].

This, actually can be formalize in such insight, that Laplacian as:

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \Delta_\Omega \tag{1.155}$$

[1]Actually, calculate by $l_1^2 + l_2^2 + l_3^2$ is more simpler, or use some notation reduction would be easier in burden.

Is rotation invariant given by $[\Delta_\Omega, l_i] = 0$. Which is same as Casimir operator by its **uniqueness** up to constant. Thus we decompose $L^2(S^2)$ into basis equipped with $Y_l^m(\theta, \phi)$.

### 1.4.7

Given $SO(2)$, it contains one generator which rotates the plane:

$$l = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim l = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial \phi} \tag{1.156}$$

It' s simple that only one element share the eigenvalue solely.

$$l|m\rangle = m|m\rangle \sim |m\rangle = F(\phi)(m) = e^{-im\phi} \tag{1.157}$$

With period condition induced.

$SO(2) \rtimes \mathbb{R}^+ = E(2)$

$$\begin{pmatrix} R(\theta) & \boldsymbol{t} \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \tag{1.158}$$

$\mathfrak{e}(2)$

$$l \to \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad p_x \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad p_y \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.159}$$

$$[l, p_x] = p_y \quad [l, p_y] = -p_x \quad [p_x, p_y] = 0 \tag{1.160}$$

$$\begin{aligned} l &\to x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial \phi} \\ p_x &\to \frac{\partial}{\partial x} \\ p_y &\to \frac{\partial}{\partial y} \end{aligned} \tag{1.161}$$

Given a Cartan subalgebra, we choose $l$, because the Cartan subalgebra must be a semi-simple decomposition thus we shouldn' t take a nilpotent ideal which would **vanish in commutation**.

$$p^\pm := p_x \pm i p_y = \frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} = e^{\pm i\phi} \left( \frac{\partial}{\partial r} \pm \frac{i}{r} \frac{\partial}{\partial \phi} \right) \tag{1.162}$$

$$[l, p^\pm] = \pm p^\pm \quad [p^+, p^-] = 0 \tag{1.163}$$

Casimir

$$
\begin{aligned}
p^+p^- - p^+p^- &= 0 \\
[l, p^+]p^- + p^+[l, p^-] &= 0 \\
[l, p^+p^-] &= 0 \\
\left[l, p_x^2 + p_y^2\right] &= 0 \\
\left[l, p^2\right] &= 0
\end{aligned}
\tag{1.164}
$$

Thus, it' s the Casimir operator, why there' s no $l^2$? Because there' s no element can generate $l$ to absorb in. Thus we clarify such operator should be unique up to scalar, which is $\Delta_\perp$ in representation.

$$
p^+p^- = e^{i\phi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \phi}\right)(e^{-i\phi})\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \phi}\right)
\tag{1.165}
$$

$$
\begin{aligned}
&e^{i\phi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \phi}\right)\left(e^{-i\phi}B\right) \\
&= \frac{\partial}{\partial r}B + \frac{1}{r}B + \frac{i}{r}\frac{\partial}{\partial \phi}B
\end{aligned}
\tag{1.166}
$$

$$
\begin{aligned}
p^+p^- &= \frac{\partial^2}{\partial r^2} - \frac{i}{r}\frac{\partial^2}{\partial r \partial \phi} + \frac{i}{r^2}\frac{\partial}{\partial \phi} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{i}{r^2}\frac{\partial}{\partial \phi} + \frac{i}{r}\frac{\partial^2}{\partial \phi \partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}
\end{aligned}
\tag{1.167}
$$

We inspect that the commutation relation differs from the case in semi-simple algebra like $SU(2)$, which we can' t generate the angular part from the translation. If you review the proof of weight boundedness of $SU(2)$ representation that Casimir operator build the connection with maximal and minimal weight, however, we can' t reach the results in this one, indicating *unboundedness* of the weight.

$$
p^{\pm}F(r, \phi, z)(m, k_z) = e^{\pm i\phi}\left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial \phi}\right)F(r, \phi, z)
\tag{1.168}
$$

$$
\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} = p^2
\tag{1.169}
$$

Given by weight $m$, we extract terms like $e^{im\phi}$:

$$
\begin{aligned}
(\Delta_\perp + \partial_z^2)F(r, \phi, z)(m, k_z) &= \lambda F(r, \phi, z)(m, k_z) \\
\Delta_\perp\left(e^{im\phi}R(r)(m)\right) &= \left(\lambda - k_z^2\right)\left(e^{im\phi}R(r)(m)\right) \\
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \left(k_z^2 - \lambda\right)\right)R(r)(m) &= 0
\end{aligned}
\tag{1.170}
$$

Which is called *Bessel Equation*, the solution is called *Bessel Function* $J_m(r)$ with $m$ as weight notation. Notice that $k_z^2 - \lambda$ is a isolated scalar generated from $z$ translation which could be reduced, we denote as $r \to \sqrt{k_z^2 - \lambda} r := kr$.

$$\begin{gathered} G(r, \phi)(m) = e^{im\phi} R(r)(m) = J_m(kr) e^{im\phi} \\ p^+(G(r, \phi)(m)) = G(r, \phi)(m+1) \end{gathered} \tag{1.171}$$

$$J_{m+1}(r) = \frac{m}{x} J_m(r) - J_m'(r) \tag{1.172}$$

However, we have no choice like $SU(2)$ to denote the minimal or maximal basis function and iterates for else. Thus, we can only acquire a iteration relation like above. You may expect the different solution or *differential seed* will inherit the same iteration property like above.

$$\begin{gathered} [l, p^\pm] = \pm p^\pm \\ e^{tl} p^\pm e^{-tl} = e^{tD_l}(p^\pm) = \sum_n^\infty \frac{t^n}{n!} (\pm 1)^n p^\pm = e^{\pm t} p^\pm \end{gathered} \tag{1.173}$$

We then set $t \to i\phi$ for period restriction.

$$\begin{gathered} U(\phi) := e^{i\phi l} \\ U(\phi)|m\rangle = e^{i\phi l}|m\rangle = e^{im\phi}|m\rangle \end{gathered} \tag{1.174}$$

We thus immediately decompose it to the $m$-weight space, recall the representation decomposition, one has $\chi(g) = e^{im\phi}$:

$$\begin{gathered} \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} e^{im\phi} d\phi = 1_m \\ \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} U(\phi)|m\rangle d\phi = 1_{\mathrm{id}} \\ \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} U(\phi) d\phi = \Pi_m \end{gathered} \tag{1.175}$$

$$\mathrm{Tr}(U(\phi)) = \langle m|U(\phi)|m\rangle = \sum_{m \in \mathbb{Z}} e^{im\phi} = 2\pi \sum_{m \in \mathbb{Z}} \delta(\phi - 2\pi n) = \sum_m \Pi_m = 1_{\mathrm{id}} \tag{1.176}$$

$$\sum_m \Pi_m\left(e^{ikr\cos\phi}\right) = \sum_{m \in \mathbb{Z}} J_m(kr) e^{im\phi} \tag{1.177}$$

## 1.5

$$S[*] : \mathcal{F} \to \mathbb{R}; \mathcal{F} = \{\boldsymbol{q}(t) : t \in [t_0, t_1] \subset \mathbb{R} \to \mathbb{R}^M\} \tag{1.178}$$

With differential set if applicable:

$$\mathcal{F}_\varepsilon = \{\delta \boldsymbol{q}(t); |\delta \boldsymbol{q}(t)| < \varepsilon; |\delta \dot{\boldsymbol{q}}(t)| < \varepsilon; \forall t \in [t_0, t_1] \subset \mathbb{R}\} \tag{1.179}$$

$$\delta S(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \to \delta \boldsymbol{q}(t_0) = \delta \boldsymbol{q}(t_1) = 0 \tag{1.180}$$

Given by a certain basis product form:

$$\begin{gathered} |x\rangle \text{ or } |x+\delta\rangle? \to \text{continuous basis} \\ \langle x|\psi\rangle = \langle x|x'\rangle\langle x'|\psi\rangle \end{gathered} \tag{1.181}$$

Thus $\varphi$ should be a continuous basis expansion for the representation of coordinates:

$$\begin{gathered} \int d\mu\, \varphi(x|x') \leftrightarrow \langle x|x'\rangle \\ \int d\mu\, \varphi(x|x')\varphi(x'|x_0) \leftrightarrow \langle x|x'\rangle\langle x'|x_0\rangle = \langle x|x_0\rangle \leftrightarrow \delta(x'|x_0) \end{gathered} \tag{1.182}$$

$$\begin{gathered} \varphi(x|x_0) = \varphi(x|x_{N-1})\varphi(x_{N-1}|x_{N-2})...\varphi(x_1|x_0) \\ \varphi(x|x_0) = \lim_{N\to\infty} \prod^{\infty} T(\varphi(x_\varepsilon)) \end{gathered} \tag{1.183}$$

Denotes $M(V;\mathbb{C})$ as set of non-degenerate complex-valued symmetric bilinear forms on $V$ with non-negative definite real part, with dense set $M^\circ(V;\mathbb{C})$. Now fix a translation invariant volume form $dx$ on $V$, thus we can define the determinant $\det(B)$ $B \in M^\circ(V;\mathbb{C})$.

Let $\mathcal{S}(V)$ be the *Schwartz space* of $V$ where the smooth functions on $V$ whose all derivatives are rapidly decaying at $\infty$ faster than any power of $|x|$, equipped with the dual space $\mathcal{S}^*(V)$ in natural inclusions $\mathcal{S}(V) \subset L^2(V) \subset \mathcal{S}'(V)$.

$$\begin{gathered} \mathcal{F} : \mathcal{S}(V) \to \mathcal{S}(V^*) \\ \mathcal{F}[g](p) = (2\pi)^{-\frac{d}{2}} \int_V g(x) e^{-i\langle p,x\rangle} dx \end{gathered} \tag{1.184}$$

Isometry is given by $\mathcal{F}^2[g](x) = g(-x)$. By duality, such operator could be extended to $\mathcal{S}^*(V) \to \mathcal{S}^*(V^*)$ for tempered distribution. The function $\exp\left(-\frac{1}{2}B(x,x)\right)$ for matrix $B \in M^\circ(V)$ belongs to $\mathcal{S}^*(V)$ if $\operatorname{Re} B \geq 0$ by its decaying behavoir.

For dimension 1, where a diagonalizable $B$:

$$
\begin{aligned}
\mathcal{F}\left(\exp\left(-\frac{1}{2}B(x,x)\right)\right) &= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left(-ipx-\frac{1}{2}ax^2\right)dx \\
&= (2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}a^{-1}p^2\right)\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}a\left(x+ia^{-1}p\right)^2\right)dx \\
&= (2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}a^{-1}p^2\right)\int_{\mathbb{R}+ia^{-1}p}\exp\left(-\frac{1}{2}a\left(x+ia^{-1}p\right)^2\right)dx \\
&= a^{-\frac{1}{2}}\exp\left(-\frac{1}{2}a^{-1}p^2\right)
\end{aligned}
\tag{1.185}
$$

We therefore could extend such matrix to diagonalizable by dense property that for arbitrary countable dimensions where a general results is given by:

$$
\mathcal{F}\left(\exp\left(-\frac{1}{2}B(x,x)\right)\right) = (\det B)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}B^{-1}(p,p)\right)
\tag{1.186}
$$

That the $\det(B)^{-\frac{1}{2}} = \prod_i \lambda_i$ by eigenvalues which is trivial in dimension 1.

$$
\begin{aligned}
I[g;B](\hbar) &= \frac{1}{\hbar^{\frac{1}{2}}}\int_V g(x)e^{-\frac{1}{2\hbar}B(x,x)}dx \quad \hbar \geq 0 \\
&= \int_V g\left(\hbar^{\frac{1}{2}}x\right)e^{-\frac{1}{2}B(x,x)}dx \quad x \to \frac{x}{\hbar^{\frac{1}{2}}}
\end{aligned}
\tag{1.187}
$$

$$
I[g;B](0) = \int_V (\ldots)dx\ g(0) = (2\pi)^{\frac{d}{2}}(\det B)^{-\frac{1}{2}}g(0)
\tag{1.188}
$$

$$
\begin{aligned}
I[g;B](\hbar) &= \left\langle g\left(\hbar^{\frac{1}{2}}x\right), \exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar^{-\frac{d}{2}}(\det B)^{-\frac{1}{2}}\left\langle \mathcal{F}[g]\left(\hbar^{-\frac{1}{2}}p\right), \exp\left(-\frac{1}{2}B^{-1}(p,p)\right)\right\rangle \\
&= (\det B)^{-\frac{1}{2}}\left\langle \mathcal{F}[g](p), \exp\left(-\frac{\hbar}{2}B^{-1}(p,p)\right)\right\rangle
\end{aligned}
\tag{1.189}
$$

Where $\hbar \to 0 \rightsquigarrow \exp\left(-\frac{\hbar}{2}B^{-1}(p,p)\right) = 1$.

If $l \in V^*$ and $g \in \mathcal{S}(V)$, then:

$$
\begin{aligned}
I[lg;B](\hbar) &= \left\langle l\left(\hbar^{\frac{1}{2}}x\right)g\left(\hbar^{\frac{1}{2}}x\right), \exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar^{\frac{1}{2}}\left\langle g\left(\hbar^{\frac{1}{2}}x\right), l(x)\exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= -\hbar^{\frac{1}{2}}\left\langle g\left(\hbar^{\frac{1}{2}}x\right), \partial_{B^{-1}l}\exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar^{\frac{1}{2}}\left\langle \partial_{B^{-1}l}g\left(\hbar^{\frac{1}{2}}x\right), \exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar\left\langle (\partial_{B^{-1}l}g)\left(\hbar^{\frac{1}{2}}x\right), \exp\left(-\frac{1}{2}B(x,x)\right)\right\rangle \\
&= \hbar I[\partial_{B^{-1}l}g;B](\hbar)
\end{aligned} \tag{1.190}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\hbar}\left(g\left(\hbar^{\frac{1}{2}}x\right)\right) = \frac{1}{2}\hbar^{-\frac{1}{2}}\left(x^* \cdot \partial_{\hbar^{\frac{1}{2}}x}g\right)\left(\hbar^{\frac{1}{2}}x\right) = \frac{1}{2}\hbar^{-\frac{1}{2}}\hbar^{-\frac{1}{2}}(x^* \cdot \partial_x g)\left(\hbar^{\frac{1}{2}}x\right) = \frac{1}{2}\hbar^{-1}(x^* \cdot \partial_x g)\left(\hbar^{\frac{1}{2}}x\right) \tag{1.191}
$$

Where $x^*$ is the covector compared to vector $x$. Compose two results we get a Laplacian operator:

$$
I'[g;B](\hbar) = \frac{1}{2}\hbar^{-1}I[x^* \cdot \partial_x g](\hbar) = \frac{1}{2}\hbar^{-1}\hbar I[\partial_{B^{-1}x^*} \cdot \partial_x g)](\hbar) = I\left[\frac{1}{2}\Delta_B g;B\right](\hbar) \tag{1.192}
$$

By deduction, we can apply as much as possible that:

$$
I^{(n)}[g;B](\hbar) = I\left[\left(\frac{1}{2}\Delta_B\right)^n;B\right](\hbar) \tag{1.193}
$$

If moreover $g$ vanishes at the origin to order $2n+1$, then for every differential operator of order $\leq 2n$ annihilates $g$ at 0, hence:

$$
I_g(0) = I_g'(0) = \ldots = I_g^{(n)}(0) = 0 \tag{1.194}
$$

Assume $f$ attains a global minimum at a unique point $c \in [a,b]$, s.t. $a < c < b$ and $f''(c) > 0$. To acquire more information, we could expand the function as:

$$
f(x) = f(c) + \frac{1}{2}f''(c)(x-c)^2 + \ldots \tag{1.195}
$$

Or even better to take a variable transformation with equality by rescale and shift:

$$
c \to 0 \quad f(c) = 0 \rightsquigarrow f(x) = \frac{M}{2}x^2 \tag{1.196}
$$

If given a compact support that $\{c\} \in U$ where divide the function as $g = g_1 + g_2 \quad g_1 \subset U, g_2 \subset [a,b] \setminus U$. Hence the integral of second term will rapidly decaying if we choose a enough huge neighborhood support around $c$. We left with the first term as:

$$I[g](\hbar) = \int_{-\infty}^{\infty} g\left(\hbar^{\frac{1}{2}} y\right) e^{-\frac{M}{2} y^2} dy \tag{1.197}$$

$$\hbar^{\frac{1}{2}} I[g; f] = \int_a^b g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} I[\tilde{g}; B = (\cdot)^2] \tag{1.198}$$

It actually show a general strategy to expand the integral by:

$$\begin{aligned} c = 0 \quad f(x) = \frac{1}{2} p(x)^2 \\ p'(0) = \sqrt{f''(0)} > 0 \end{aligned} \tag{1.199}$$

$$\int_{-\infty}^{\infty} g(x) e^{-\frac{p(x)^2}{2\hbar}} dx \sim \hbar^{\frac{1}{2}} \int_{-\infty}^{\infty} \tilde{g}\left(\hbar^{\frac{1}{2}} y\right) e^{-\frac{y^2}{2}} dy \tag{1.200}$$

$$\begin{aligned} &\frac{p(x)}{\hbar^{\frac{1}{2}}} = y \\ &dx = dp^{-1}\left(\hbar^{\frac{1}{2}} y\right) = \left(p^{-1}\right)'\left(\hbar^{\frac{1}{2}} y\right) \hbar^{\frac{1}{2}} dy \end{aligned} \tag{1.201}$$

$$\begin{aligned} p(p^{-1}(x)) &= x \\ p'\left(p^{-1}(x)\right) \cdot \left(p^{-1}\right)'(x) &= 1 \\ \left(p^{-1}\right)'(x) &= \frac{1}{p'\left(p^{-1}(x)\right)} \end{aligned} \tag{1.202}$$

$$\tilde{g}(y) := g(p^{-1}(y)) \left(p^{-1}\right)'(y) = \frac{g(p^{-1}(y))}{p'\left(p^{-1}(y)\right)} \tag{1.203}$$

We could expand $\tilde{g}$ by Taylor polynomials to acquire a series of summation.

Notice that any odd order polynomials expansion will vanish due to the whole space integral. Thus we can freely expand in $O(\hbar^n)$ order as:

$$\hbar^{\frac{1}{2}} I[g; f] = \hbar^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} I[\tilde{g}; B] = (2\pi\hbar)^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} \sum_{n \geq 0} a_n \hbar^n \tag{1.204}$$

The Gaussian integral can be expanded as $\int_{-\infty}^{\infty} y^{2m} e^{-\frac{y^2}{2}} dy = (2\pi)^{\frac{1}{2}} (2m-1)!!$, So we can exploit the constant that $\tilde{g}(y) = \sum_{n \geq 0}^{\infty} b_n y^n \to a_n = b_{2n} (2m-1)!!$.

$$\int_a^b g(x) e^{\frac{i f(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{\frac{i f(c)}{\hbar}} e^{\frac{\pm \pi i}{4}} I[g](\hbar) \tag{1.205}$$

$$
\begin{aligned}
I(\hbar) = \sum_{n\geq 0} a_n \hbar^n &= \sum_{n\geq 0} a_n \frac{\hbar^n}{n!} n! \\
&= \sum_{n\geq 0} a_n \frac{\hbar^n}{n!} \int_0^\infty u^n e^{-u} du \\
&= \int_0^\infty g(\hbar u) e^{-u} du \quad g(\hbar) = \sum_{n\geq 0} a_n \frac{\hbar^n}{n!} \\
&= \int_{-\infty}^\infty |v| g(\hbar v^2) e^{-v^2} dv \quad u \to v^2 \\
&= \hbar^{-\frac{1}{2}} \int_{-\infty}^\infty \tilde{g}\left(\hbar^{\frac{1}{2}} v\right) e^{-v^2} dv \quad \tilde{g}(v) = |v| g(v^2)
\end{aligned}
\tag{1.206}
$$

$$
\begin{aligned}
\Gamma(s+1) &= \int_0^\infty t^s e^{-t} dt \quad s > 0 \\
&= s^{s+1} \int_0^\infty x^s e^{-sx} dx \quad t \to sx \\
&= s^{s+1} \int_0^\infty e^{-s(x-\ln x)} dx \quad \hbar \to \frac{1}{s}, f(x) \to x - \ln x, g(x) \to 1 \\
&= s^{s+\frac{1}{2}} e^{-s} (2\pi)^{\frac{1}{2}} \sum_{n\geq 0} a_n \left(\frac{1}{s}\right)^n \quad f(c) = f(1) = 1
\end{aligned}
\tag{1.207}
$$

$$
\begin{aligned}
p(x) = \sqrt{2(x - \ln x)} &= \sqrt{2\left(x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)\right)} \\
&= x\sqrt{1 - \frac{2x}{3} + \frac{x^2}{2} + \ldots} = x - \frac{x^2}{3} + \frac{7x^3}{36} + \ldots
\end{aligned}
\tag{1.208}
$$

$$
\begin{aligned}
p^{-1}(x) &= x + \frac{x^2}{3} + \frac{x^3}{36} + \ldots \\
\left(p^{-1}\right)'(x) &= 1 + \frac{2x}{3} + \frac{x^2}{12} + \ldots
\end{aligned}
\tag{1.209}
$$

$$
\tilde{g}(y) = p'^{-1}(y) \to a_0 = 1, a_1 = \frac{1}{12}(2-1)!! = \frac{1}{12}
\tag{1.210}
$$

Multidimensional steepest descent formula:

$$
\hbar^{\frac{d}{2}} I^D[g; f] = \int_D g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{-\frac{f(c)}{\hbar}} I[g; B]
\tag{1.211}
$$

$$
\begin{aligned}
T &\to \Pi(T) \\
|\Pi_k| &= \frac{(2k)!}{2^k k!} = \frac{(2k)!}{(2k)!!} = (2k-1)!!
\end{aligned}
\tag{1.212}
$$

$$\langle l_1 ... l_N \rangle := \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D l_1(x) ... l_N(x) e^{-\frac{S(x)}{\hbar}} dx \tag{1.213}$$

$$S(x) = \frac{B(x,x)}{2} + \tilde{S}(x) = \frac{B(x,x)}{2} - \sum_{i \geq 0} g_i \frac{B_i(x, ..., x)}{i!} \tag{1.214}$$

$$Z = I[1; S(x)] = \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx \tag{1.215}$$

$$\begin{aligned} \mathbf{n} &= (n_0, n_1, n_2, ...) \\ F_\Gamma &= \prod_i g_i^{n_i} \cdot \tilde{F}_\Gamma \end{aligned} \tag{1.216}$$

$$\begin{aligned} x \to y &= \hbar^{-\frac{1}{2}} x \\ B_i(x, ..., x) &\to \hbar^{\frac{i}{2}} B_i(x, ..., x) \\ Z_\mathbf{n} &= \int_V e^{-\frac{B(y,y)}{2}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \left(\hbar^{\frac{i}{2}-1} B_i(y, ..., y)\right)^{n_i} dy \end{aligned} \tag{1.217}$$

Contract each vertex.

$$-\chi(\Gamma) = E(\Gamma) - V(\Gamma) = \frac{1}{2} \sum_{i \geq 0} n_i i - \sum_{i \geq 0} n_i = \sum_{i \geq 0} n_i \left(\frac{i}{2} - 1\right) \tag{1.218}$$

$$Z_n = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \sum_{\sigma \in \prod(T_\mathbf{n})} \tilde{F}(\sigma) \tag{1.219}$$

1. permutation of "flowers" of given valency: $S_{n_i}$
2. permutation of the $i$th edges inside each $i$-valent "flower" : $S_i^{n_i}$

$$\begin{aligned} \mathbb{G}_\mathbf{n} &= \prod_i \left(S_{n_i} \ltimes S_i^{n_i}\right) \\ |\mathbb{G}_\mathbf{n}| &= \prod_i (n_i!)(i!)^{n_i} \end{aligned} \tag{1.220}$$

The group $\mathbb{G}_\mathbf{n}$ acts on the set $\Pi(T_\mathbf{n})$, which the stabilizer of a given matching is $\mathrm{Aut}(\Gamma)$.

$$N_\Gamma = \frac{\prod_i (n_i!)(i!)^{n_i}}{|\mathrm{Aut}(\Gamma)|} \tag{1.221}$$

$$\sum_{\sigma \in \Pi(T_\mathbf{n})} \tilde{F}(\sigma) = \sum_\Gamma \frac{\prod_i (n_i!)(i!)^{n_i}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma = \sum_\Gamma N_\Gamma \tilde{F}_\Gamma \tag{1.222}$$

$$\chi(\Gamma) = \beta_0(\Gamma) - \beta_1(\Gamma) = V - E \tag{1.223}$$

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}} = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\Gamma \in G(\mathbf{n})} \frac{\tilde{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|} \tag{1.224}$$

We know that such exponential can be expanded as infinite sum of expectation of products.

$$\langle e^l \rangle := \sum_{n \geq 0} \frac{1}{n!} \langle l^n \rangle = \hbar^{\frac{d}{2}} \int_V e^{l(x) - \frac{S(x)}{\hbar}} dx \tag{1.225}$$

We immediately inspect that the infinite sum of $Z_{\mathbf{n}}$ is exact the same as the exponential expansion of $Z$, a specific case for $B_1(\cdot) = l(\cdot)$ as the external vertices plus a arbitrary action of $\frac{S(x)}{\hbar}$ to induce the Feynman amplitude for $F_\Gamma(l_1, ..., l_N)$ $\Gamma \in G(\mathbf{n})$.

Given below diagram, one have two external vertices and a loop, also with a 3-valent flower and 4-valent flower too.

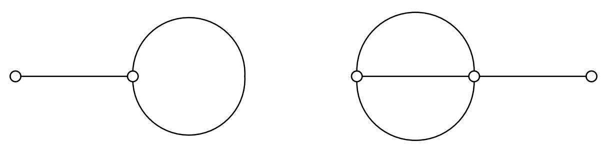

Figure 5. A graph with separated components.

$$\begin{aligned} B_3 &= \sum_i b_i^1 \otimes b_i^2 \otimes b_i^3 \\ B_4 &= \sum_j c_j^1 \otimes c_j^2 \otimes c_j^3 \otimes c_j^4 \end{aligned} \tag{1.226}$$

In expansion of $B_3$ and $B_4$ exponential will generate the 3-valent and 4-valent flowers, plus two external vertices $l_1, l_2$, all which will be contracted by the edges $B_2$.

$$F_\Gamma(l_1, l_2) = \sum_i B^{-1}(l_1, b_i^1) B^{-1}(b_i^2, b_i^3) \sum_{ij} B^{-1}(b_i^1, c_j^1) B^{-1}(b_i^2, c_j^2) B^{-1}(b_i^3, c_j^3) B^{-1}(c_j^4, l_2) \tag{1.227}$$

Suppose the coefficients $g_i = 1$, we acquire the general expression for expectation as sum of all possible graphs:

$$\langle l_1 ... l_N \rangle = \sum_{\Gamma \in G(N)} \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \tilde{F}_\Gamma(l_1, ..., l_N) \tag{1.228}$$

Where we reduce the product of $\prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)$ in $\hbar^{-\chi(\Gamma)}$ by the Euler characteristic.

Now, suppose in 1-dimension, given a examples as simple as possible that $B(y, y) = y^2$, $B_i = z^i$, $g_i = g$, $\hbar = 1$:

$$Z = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + g \sum_{i \geq 0} \frac{(zx)^i}{i!}\right) dy = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + g e^{zx}\right) dy \tag{1.229}$$

It actually contains infinite many vertices with arbitrary **valency**, we expand the second term $ge^{zx}$:

$$Z = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) \sum_{n\geq 0} \frac{(ge^{zy})^n}{n!} dy = (2\pi)^{-\frac{1}{2}} \sum_{n\geq 0} \frac{g^n}{n!} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2} + nzy\right) dy \tag{1.230}$$

We can explain that $z(\cdot)$ pair a vertex, thus $n$ vertices will contribute $n$ factors of $z$ as $z(\cdot) + z(\cdot) + ...$:

$$\int \exp\left(-\frac{y^2}{2} + nzy\right) dy = \exp\left(\frac{n^2 z^2}{2}\right) \int \exp\left(-\frac{1}{2}(y - nz^2)\right) dy = (2\pi)^{\frac{1}{2}} \exp\left(\frac{n^2 z^2}{2}\right) \tag{1.231}$$

$$Z = \sum_{n\geq 0} \frac{g^n}{n!} \exp\left(\frac{n^2 z^2}{2}\right) = \sum_{n\geq 0} \frac{g^n}{n!} \sum_{k\geq 0} \frac{\left(n^2 \frac{z^2}{2}\right)^k}{k!} = \sum_{n,k\geq 0} \frac{g^n n^{2k}}{2^k k! n!} z^{2k} = \sum_{n\geq 0} g^n \sum_{\Gamma\in G(n)} \frac{\tilde{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|} \tag{1.232}$$

Given a number of vertices $n$ with contribution $zy$ to each $B^{-1}(zy, zy) = z^2 B^{-1}(y, y) = z^2$. Thus a general graph with $k$ edges contribute $F_\Gamma = z^{2k} \times \left(B^{-1}\right)^k = z^{2k}$. It can also be evaluated by expanding $\exp(nzy)$ then integrating. Therefore, we can identify the $|\mathrm{Aut}(\Gamma)|$ too:

$$\sum_{\Gamma\in G(n)} (...) = \sum_k \sum_{\Gamma\in G(n;k)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k! n!} \tag{1.233}$$

To scrutinize our answer in combinatorics, we pick out two vertices from $n$ vertices with $n^2$ choice with order. For list of $k$ edges, we have $n^2 \cdot n^2 \cdot ... = n^{2k}$ choices. To cancel out the order, we first remove orientation of edges which contribute $2 \cdot 2 \cdot 2... = 2^k$ choices. Then we reomove the permutation of edges[2] which contribute $k!$ choices. We still treats the vertices as labelled, so remove the permutation of vertices which contribute $n!$ choices.

Try to decompose the graph to connected components, we denote $\Gamma = \bigsqcup_{j=1}^{r} \Gamma_j^{k_j}$ with $\mathbf{k}$ as tuple of number of copies of each connected components.

$$\begin{gathered} F_{\Gamma_1 \sqcup \Gamma_2} = F_{\Gamma_1} \times F_{\Gamma_2} \\ \chi(\Gamma_1 \sqcup \Gamma_2) = \chi(\Gamma_1) + \chi(\Gamma_2) \\ \left|\mathrm{Aut}\left(\Gamma_1^{k_1} \sqcup \Gamma_2^{k_2} ... \sqcup \Gamma_r^{k_r}\right)\right| = \prod_{j=1}^{r} \left|\mathrm{Aut}\left(\Gamma_j\right)\right|^{k_j} k_j! \end{gathered} \tag{1.234}$$

Here $k_{j!}$ of the third equality comes from the permutation of identical connected components.

Given a connected graph set $G_c(*)$, $\Gamma = \bigsqcup_{\gamma\in G_c(*)} \gamma^{k_\gamma}$ where the index is same as the graph itself for simplicity.

$$w(\Gamma) := \frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \tag{1.235}$$

[2] It should also be considered as the contraction order from 1 to $k$, so if we pick out $(a_1, b_1), (a_2, b_2)$, it′ s also reasonable to pick out $(a_2, b_2), (a_1, b_1)$ which remove the contribution of edges order.

$$\frac{\tilde{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|} = \prod_{\gamma \in G_c(*)} \left( \frac{\tilde{F}_\gamma}{|\mathrm{Aut}(\gamma)|} \right)^{k_\gamma} \frac{1}{k_\gamma !} \tag{1.236}$$

We decompose the second product $\prod_i (\ldots)$ too, therefore we reduce the terms as:

$$w(\Gamma) = \prod_{\gamma \in G_c(*)} \frac{w(\gamma)^{k_\gamma}}{k_\gamma !} \tag{1.237}$$

$$Z = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \sum_{\mathbf{n}} \sum_{\Gamma \in G(n)} w(\Gamma) = Z_0 \sum_{\Gamma \in G(*)} w(\Gamma) \quad Z_0 = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} \tag{1.238}$$

Where we just reduce our previous expansion of $Z$ in a more concise expression.

$$\frac{Z}{Z_0} = \sum_{\Gamma \in G(*)} w(\Gamma) = \sum_{k_\gamma} \prod_{\gamma \in G_c(*)} \frac{w(\gamma)^{k_\gamma}}{k_\gamma !} = \exp\left( \sum_{\gamma \in G_c(*)} w(\gamma) \right) \tag{1.239}$$

For any summation of arbitrary graphs in $G(*)$, we can decompose it to connected components with reordering $\mathbf{n} \to \mathbf{k} \sim \sum_{\mathbf{n}} \prod_i \to \sum_{\mathbf{k}} \prod_\gamma$ by counting the flowers into counting the connected components.

$$\ln \frac{Z}{Z_0} = \sum_{\gamma \in G_c(*)} w(\gamma) = \sum_{\gamma \in G_c(*)} \frac{\tilde{F}_\gamma}{|\mathrm{Aut}(\gamma)|} \prod_i \left( g_i \hbar^{\frac{i}{2}-1} \right)^{n_i(\gamma)} = \sum_{\mathbf{n}} \prod_i \left( g_i \hbar^{\frac{i}{2}-1} \right) \sum_{\gamma \in G_c(\mathbf{n})} \frac{\tilde{F}_\gamma}{|\mathrm{Aut}(\gamma)|} \tag{1.240}$$

A spanning-tree $T \subset \Gamma$ contains all $V$ vertices and without any cycles, thus a spanning-tree on $V$ vertices contains exactly $V-1$ edges. Thus the extra edges will contribute exact the same number of loops.

$$\beta_1(\Gamma) = 1 - \chi(\Gamma) = E - V + 1 \tag{1.241}$$

$$\begin{aligned} &\beta_0(\Gamma) = 1 \text{ for connected graph} \\ &\beta_1(\Gamma) = E - V + 1 \end{aligned} \tag{1.242}$$

It suggests $\hbar^{-\chi(\gamma)}$ will contribute in different order of $O(\hbar)$. For **tree**, $\beta_1 = 0$, thus a connected graph contribute $-\chi(\gamma) = -(V - E) = -(\beta_0 - \beta_1) = -1 \sim O(\hbar^{-1})$ order. For **1-loop** graph, $\beta_1 = 1$, thus contribute $O(\hbar)$ order, and so forth. Given a set of classes of graphs with $\boldsymbol{j}$ loops:

$$\hbar^{j-1} \left( \ln \frac{Z}{Z_0} \right)_j := \sum_{\mathbf{n}} \prod_i (g_i^{n_i(\gamma)} \hbar^{j-1} \sum_{\gamma \in G_c(\mathbf{n})} \frac{\tilde{F}_\gamma}{|\mathrm{Aut}(\gamma)|} \tag{1.243}$$

$$\ln \frac{Z}{Z_0} = \sum_{j \geq 0} \left( \ln \frac{Z}{Z_0} \right)_j \hbar^{j-1} \tag{1.244}$$

$$Z := \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx = e^{-\frac{S(x_0)}{\hbar}} I[1; B = B_2] = e^{-\frac{S(x_0)}{\hbar}} (\det B)^{\frac{1}{2}} \det(S''(x_0))^{-\frac{1}{2}} \sum_{i \geq 0} a_i \hbar^i \tag{1.245}$$

By steepest descent formula in the neighborhood of critical point $x_0$ where $S'(x_0) = 0$.

$$\log \frac{Z}{Z_0} = -\frac{S(x_0)}{\hbar} + \frac{1}{2} \log \frac{\det(B)}{\det(S''(x_0))} + \log \left( \sum_{i \geq 0} a_i \hbar^i \right) \tag{1.246}$$

Often, the first term is called *classical approximation* by physicists, the second term is called *1-loop approximation* etc.

$$\begin{gathered} S'(x) = 0 \\ \left( \frac{1}{2} B(x,x) - \sum_{i \geq 3} g_i B_i \frac{x, \dots, x}{i!} \right)' = B(x, \cdot) - \sum_{i \geq 3} g_i \frac{B_i(x, \dots, x, \cdot)}{(i-1)!} = 0 \\ \sum_{i \geq 3} g_i B^{-1} \frac{B_i(x, \dots, x, \cdot)}{(i-1)!} := \beta(x) = x \quad B^{-1} : V^* \to V \end{gathered} \tag{1.247}$$

In the sense of power series norm, $\beta$ is a contraction mapping in a neighborhood of 0.

$$x_0 = \lim_{n \to \infty} \beta^n(x) \tag{1.248}$$

- Each application of $\beta$ produces a new **vertex** with $i-1$ incoming arguments and one outgoing edge.
- $B^{-1}$ contracts the open leg (argument of the slot $B(x, \dots, -)$) with the upper vertex.

$$x^2 = \beta(x^1) = \sum_i g_i B^{-1} \frac{B_i(x^1, \dots, x^1)}{(i-1)!} \tag{1.249}$$

$$x_0 = \sum_{\mathbf{n}} \prod_i \left( g_i \hbar^{\frac{i}{2}-1} \right)^{n_i} \sum_{\Gamma \in G^0(\mathbf{n})} \frac{\tilde{F}_T}{|\mathrm{Aut}(T)|} \tag{1.250}$$

Where the graph $\Gamma$ is a *tree* with one external vertex. Take the convergence results $x_0$ back to the action $S(x) \to S(x_0)$. To explain this, $\frac{B(x_0,x_0)}{2}$ corresponds to gluing two trees with both single external vertices; $\frac{1}{i!} B_i(x_0, \dots, x_0)$ corresponds to gluing $i$ trees with external vertices together into a $i$-valent flower. We can think of the first term as counting tree $E$ times (once per edge), and rest of terms as counting each tree $V$ times (once per vertex), therefore $-S(x_0) = V - E = 1$ (per tree). So the tree summation contribution is $-S(x_0)$.

$$S(x) := \frac{x^2}{2} - gh(x) \quad h(x) = \sum_{n \geq 0} c_n x^n \text{ with } c_1 \neq 0 \quad g \text{ is constant} \tag{1.251}$$

$$
\begin{aligned}
S'(x) &= 0 \\
x_0 &= gh'(x_0) \\
\frac{x_0}{h'(x_0)} &= g \\
\left(\frac{x_0}{h'(x_0)}\right)^{-1} \frac{x_0}{h'(x_0)} &= \left(\frac{x_0}{h'}(x_0)\right)^{-1}(g) \\
x_0 &= f(g) \quad \left(\frac{x_0}{h'(x_0)}\right)^{-1} = f
\end{aligned}
\tag{1.252}
$$

$$
-S(x_0) = F(g) = -\frac{f(g)^2}{2} + gh(f(g)) \tag{1.253}
$$

$$
F'(g) = -f(g)f'(g) + h(f(g)) + gh'(f(g))f'(g) \tag{1.254}
$$

But $x_0 = gh'(x_0) \to f(g) = gh'(f(g)) \to h'(f(g)) = \frac{f(g)}{g}$, first and third terms cancel each other.

$$
F'(g) = h(f(g)) \to -S(x_0) = \int_0^g h(f(a))da \tag{1.255}
$$

Consider $B_i = 1, g_i = g$ with $S(x) = \frac{x^2}{2} - ge^x$, expanding $e^x$ will give the all flowers or external edges for us to compute the number of trees.

$$
\sum_{n \geq 0} g^n \sum_{\gamma \in G_c^0(n)} \frac{1}{|\mathrm{Aut}(\gamma)|} = -S(x_0) \tag{1.256}
$$

$$
S'(x) = 0 \to x = ge^x \to x_0 = f(g) = \left(\frac{x}{e^x}\right)^{-1}(g) \tag{1.257}
$$

$$
\frac{f(x)}{e^{f(x)}} = x \to e^{f(x)} = \frac{f(x)}{x} \tag{1.258}
$$

$$
-S(x_0) = \int_0^g h(f(a))da \to \int e^{f(a)}da \to \int \frac{f(a)}{a}da \tag{1.259}
$$

Now we need to evaluate the coefficients of Taylor series of $f(x)$ to acquire the count of summation of automorphism class of graphs.

$$
\begin{gathered}
f(g) = \sum_{n \geq 1} a_n g^n \\
a_n = \frac{1}{2\pi i} \oint \frac{f(g)}{g^{n+1}} dg \overset{g \to f^{-1}(x)}{\to} \frac{1}{2\pi i} \oint \frac{x}{(xe^{-x})^{n+1}} d(xe^{-x}) = \frac{1}{2\pi i} \oint e^{nx} \frac{1-x}{x^n} dx
\end{gathered}
\tag{1.260}
$$

We take residue in a simpler way, expand $e^{nx} = \sum_{i\geq 0} \frac{(nx)^i}{i!}$, we immediately see the residue equal to $\frac{n^i}{i!}x^{i-k} \quad i-k=-1$. Therefore $k=n$ or $n-1$, we get:

$$a_n = \frac{n^{n-1}}{(n-1)!} - \frac{n^{n-2}}{(n-2)!} = \frac{n^{n-2}}{(n-1)!} \tag{1.261}$$

$$-S(x_0) = \int_0^a \frac{f(a)}{a} da = \sum_{n\geq 1} \frac{n^{n-2}}{(n-1)!} \int_0^g a^{n-1} da = \sum_{n\geq 1} \frac{n^{n-2}}{n!} g^n \tag{1.262}$$

$$\begin{gathered} -S(x_0) = \sum_{n\geq 1} \frac{n^{n-2}}{n!} g^n = \sum_{n\geq 0} g^n \sum_{\gamma\in G_c^0(n)} \frac{1}{|\mathrm{Aut}(\gamma)|} \\ \frac{n^{n-2}}{n!} = \sum_{\gamma\in G_c^0(n)} \frac{1}{|\mathrm{Aut}(\gamma)|} \end{gathered} \tag{1.263}$$

We conclude that the number of labelled tree of $n$ vertices is $n^{n-2}$ as $\frac{n!}{|\mathrm{Aut}(\gamma)|}$ non-isomorphic labellings, The theorem is called *Cayley* theorem.

Consider now $B_i = 1, g_i = g$ with $S(x) = \frac{x^2}{2} - g\left(x + \frac{x^3}{6}\right)$ which is the set of graphs composed by 1-valent and 3-valent flowers. However, we can directly calculate the fixed point by solving algebraically.

$$\begin{gathered} S'(x) = 0 \to x = g\left(1 + \frac{x^2}{2}\right) \\ x_0 = \frac{1 - (1-2g^2)^{\frac{1}{2}}}{2} \end{gathered} \tag{1.264}$$

$$\begin{aligned} -S(x_0) &= \int_0^g \frac{1-(1-2a^2)^{\frac{1}{2}}}{a} + \frac{\left(1-(1-2a^2)^{\frac{1}{2}}\right)^3}{6a^3} da \\ &= \int_0^g \frac{1}{6a^3}\left(6a^2 - 6a^2(1-2a^2)^{\frac{1}{2}} + 1 + 3(1-2a^2) - 3(1-2a^2)^{\frac{1}{2}} - (1-2a^2)^{\frac{3}{2}}\right) da \\ &= \int_0^g \frac{1}{6a^3}\left(4 - (6a^2+3+1-2a^2)(1-2a^2)^{\frac{1}{2}}\right) da \\ &= \frac{2}{3}\int_0^g \frac{1}{a^3}\left(1-(1+a^2)(1-2a^2)^{\frac{1}{2}}\right) da \quad \text{integrate by part } dv = \frac{1}{a^3} \\ &= \frac{2}{3}\left(-\frac{1}{2a^2}\right)\Bigg|_0^g - \frac{2}{3}\left(1+a^2)(1-2a^2)^{\frac{1}{2}}\left(-\frac{1}{2a^2}\right)\right|_0^g \\ &\quad -\frac{2}{3}\int_0^g -\frac{1}{2a^2}\left(2a(1-2a^2)^{\frac{1}{2}} + (1+a^2)\cdot\frac{1}{2}(1-2a^2)^{-\frac{1}{2}}\cdot(-4a)\right) da \end{aligned} \tag{1.265}$$

We focus on third term in:

$$\begin{aligned} du &= 2a(1-2a^2)^{\frac{1}{2}} - 2a(1+a^2)(1-2a^2)^{-\frac{1}{2}} \\ &= \frac{2a(1-2a^2) - 2a(1+a^2)}{(1-2a^2)^{\frac{1}{2}}} \\ &= \frac{6a^3}{(1-2a^2)^{\frac{1}{2}}} \end{aligned} \tag{1.266}$$

$$\text{third term} = \int_0^g \frac{2a}{(1-2a^2)^{\frac{1}{2}}} da = \int_0^g \frac{1}{(1-2a^2)^{\frac{1}{2}}} da^2 = -(1-2a^2)^{\frac{1}{2}}\Big|_0^g \tag{1.267}$$

Combine two terms together:

$$\begin{aligned} I &= \frac{1}{3a^2}\left((1+a^2)(1-2a^2)^{\frac{1}{2}} - 1 - 3a^2(1-2a^2)^{\frac{1}{2}}\right) \\ &= \frac{1}{3a^2}\left((1-2a^2)(1-2a^2)^{\frac{1}{2}} - 1\right) \\ &= \frac{1}{3a^2}\left((1-2a^2)^{\frac{3}{2}} - 1\right) \end{aligned} \tag{1.268}$$

$$\begin{aligned} I|_0^g &= \frac{1}{3g^2}\left((1-2g^2)^{\frac{3}{2}} - 1\right) - \lim_{a\to 0}\frac{1}{3a^2}(-3a^2) \quad \text{L'Hôpital's Rule} \\ &= \frac{1}{3g^2}\left((1-2g^2)^{\frac{3}{2}} - 1\right) + 1 \end{aligned} \tag{1.269}$$

$$\begin{aligned} -S(x_0) &= \frac{1}{3g^2}\left((1-2g^2)^{\frac{3}{2}} - 1\right) + 1 \\ &= \frac{1}{3g^2}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{3}{2}\cdot\frac{1}{2}\cdot\ldots\cdot\frac{3-2n}{2}(-2)^{n+1}g^{2(n+1)} + \frac{1}{3g^2} - \frac{1}{3g^2} + 1 \text{ by shift 1 index} \\ &= \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\frac{1}{2}\cdot\ldots\cdot\frac{2n-3}{2}(-1)^n(-2)^n g^{2n} - 1 + 1 \text{ by extract first term} \\ &= \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{(n+1)!}g^{2n} \end{aligned} \tag{1.270}$$

Now we see each coefficients corresponds to counts of tree with $m = 2n$ vertices, if we multiply $2n$ then corresponds to the one with vertices labelled.

Feynman calculus can be used to count not only oriented but also oriented graphs. Suppose we want to count not labelled oriented trees, how to represent this? Given two labelled point $x, y$ rather $x^2$, the bilinear form is thus $xy$. Identify $x$ as the source of the edge and $y$ as the sink of the edge, we extend all flowers same as before but summation for both: $be^x + ae^y$. Combine together: $S(x, y) = xy - be^x - ae^y$. The critical point is the zeros of partial derivative in such 2-dimension manifold.

$$\begin{aligned}\frac{\partial S(x,y)}{\partial x} &= y - be^x = 0 \to be^x = y \\ \frac{\partial S(x,y)}{\partial y} &= x - ae^y = 0 \to ae^y = x\end{aligned} \tag{1.271}$$

Such transcendental function can' t be solved directly, we expand each coefficients:

$$\begin{aligned}x &= a\left(\sum_{i\geq 0}\frac{y^i}{i!}\right) = a + \sum_{p\geq 1, q\geq 1} c_{p,q}a^p b^q \\ y &= b\left(\sum_{i\geq 0}\frac{x^i}{i!}\right) = b + \sum_{p\geq 1, q\geq 1} d_{p,q}a^p b^q\end{aligned} \tag{1.272}$$

Evaluate the coefficient of each order by Cauchy' s theorem:

$$c_{p,q} = \frac{1}{(2\pi i)^2}\oint\oint xa^{-(p+1)}b^{-(q+1)}da \wedge db \tag{1.273}$$

$$\begin{gathered}da = e^{-y}(dx - xdy) \quad db = e^{-x}(dy - ydx) \\ da \wedge db = e^{-x-y}(dx - xdy) \wedge (dy - ydx) = e^{-x-y}(1 - xy)dx \wedge dy\end{gathered} \tag{1.274}$$

$$a^{-(p+1)} = x^{-(p+1)}e^{(p+1)y}, b^{-(q+1)} = y^{-(q+1)}e^{(q+1)x} \tag{1.275}$$

$$\begin{aligned}c_{p,q} &= \frac{1}{(2\pi i)^2}\oint\oint xx^{-(p+1)}e^{(p+1)y}y^{-(q+1)}e^{(q+1)x}e^{-x-y}(1 - xy)dx \wedge dy \\ &= \frac{1}{(2\pi i)^2}\oint\oint(1 - xy)x^{-p}y^{-(q+1)}e^{qx+py}dx \wedge dy\end{aligned} \tag{1.276}$$

We evaluate $x^{-k}y^{-l}e^{qx+py}$ to tackle general case:

$$x^{-k}y^{-l}\sum_{i,j\geq 0}\frac{(qx)^i}{i!}\frac{(py)^j}{j!} \to i - k = -1, j - l = -1 \to I = \frac{q^{k-1}}{(k-1)!}\frac{p^{l-1}}{(l-1)!} \tag{1.277}$$

$$\begin{aligned}c_{p,q} = \frac{q^{p-1}}{(p-1)!}\frac{p^q}{q!} - \frac{q^{p-2}}{(p-2)!}\frac{p^{q-1}}{(q-1)!} &= \frac{q^{p-1}p^q - (p-1)q^{p-2}qp^{q-1}}{(p-1)!q!} \\ &= \frac{(p-(p-1))q^{p-1}p^{q-1}}{(p-1)!q!} \\ &= \frac{q^{p-1}p^{q-1}}{(p-1)!q!}\end{aligned} \tag{1.278}$$

Similarly, $d_{p,q}$ is solved by same strategy while $x \leftrightarrow y$, thus $p \leftrightarrow q$, is $\frac{q^{p-1}p^{q-1}}{(q-1)!p!}$. Use integral trick to compute $S(x_0, y_0)$, but before on the hand, we decompose:

$$\frac{\partial S(x_0, y_0)}{\partial a} = -e^{y_0} = -\frac{x_0}{a}, \quad \frac{\partial S(x_0, y_0)}{\partial a} = -e^{x_0} = -\frac{y_0}{b} \tag{1.279}$$

Thus we can evaluate the integral of the curve by the total derivative of the action:

$$\begin{aligned} -S(x_0, y_0)(a, b) &= \int_{a,0}^{a,b} \int_{0,0}^{a,0} \frac{x_0}{u} du + \frac{y_0}{v} dv \\ &= \int_{0,0}^{a,0} \frac{x_0(u,0)}{u} du + \int_{a,0}^{a,b} \frac{y_0(a,v)}{v} dv \\ &= a + \int_0^b \left(1 + \sum_{p\geq 1, q\geq 1} d_{p,q} a^p v^{q-1}\right) dv \\ &= a + b + \sum_{p\geq 1, q\geq 1} \frac{q^{p-1} p^{q-1}}{q!p!} a^p b^q \end{aligned} \tag{1.280}$$

Now we see that the number of trees with $p$ sources and $q$ sinks is $\frac{q^{p-1}p^{q-1}}{q!p!}$ labelled for source and sink one, or $q^{p-1}p^{q-1}\frac{(p+q)!}{q!p!}$ unlabelled for which is source or sink, or $p^{q-1}p^{q-1}$ if the vertices **in** source and sink are also labelled.

Can we extend the vertex to arbitrary finite labelled choices or colored? For example, the previous case has two color choices for each vertex, then the choice for different color can be considered as source and sink. Therefore we have a tuple of vertex color as $(x_1, ..., x_n)$ and their connection edges by $\frac{1}{2}\mathbf{x}^T B \mathbf{x}$ bilinear form while the previous case is $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Different flowers for each color can be constructed by $\sum_j^m a_j e^{x_j}$, reaching the final actions:

$$S(\mathbf{x}) = \frac{1}{2} \sum_{ij} B_{ij} x_i x_j - \sum_j^m a_j e^{x_j} \tag{1.281}$$

Differentiate to get fixed points tuple:

$$\frac{\partial S}{\partial x_i} = 0 \rightarrow \sum_j B_{ij} x_j = a_i e^{x_i} \tag{1.282}$$

Define $X_i := \sum_j B_{ij} x_j$ as the co-vector, the critical equations are:

$$X_i = a_i e^{x_i} \rightarrow a_i = X_i e^{-x_i}, \quad x_i := \sum_i (B^{-1})_{ij} X_j \tag{1.283}$$

We already know the final form of the action on critical points is restricted by:

$$\begin{aligned} \frac{\partial S(\mathbf{x_0})(\mathbf{a})}{\partial a_i} &= e^{x_i} = \frac{X_i}{a_i} \\ -S(\mathbf{x}_0) &= \sum_i \int_0^{a_i} \frac{X_i(a_1, ..., a_i = u, 0, ...)}{u} du \end{aligned} \tag{1.284}$$

So the only problem is to solve the coefficients of expansion of each critical points.

$$X_i = a_i + \sum_{p_j \geq 1} c_{\mathbf{p},i} \prod_j^m a_j^{p_j} \tag{1.285}$$

Where the coefficient $c_{\mathbf{p},i}$ indicates the power counts tuple $p_j$ for each $a_j^{p_j}$ while the $i$ indicates the $X_i$ co-vector.

Induce that any $X_i(a_1, .., u, 0, ...) = a_i$ because all product term vanishes, thus we only need to know the coefficient and integrate to get the result: $\frac{c_{\mathbf{p},i}}{p_i}$.

$$\begin{aligned} c(\mathbf{p}, i) &= \frac{1}{(2\pi i)^m} \oint X_i \left( \prod_k a_k^{-p_k - 1} \right) d\mathbf{a} \\ &= \frac{1}{(2\pi i)^m} \oint X_i \left( \prod_k (X_k e^{-x_k})^{-p_k - 1} \right) d(X_1 e^{-x_1}) \wedge ... \wedge d(X_m e^{-x_m}) \end{aligned} \tag{1.286}$$

We first calculate the single term, transforming all into $X_i$:

$$\frac{\partial X_i e^{-x_i}}{\partial X_j} = e^{-x_i} \delta_{ij} - X_i e^{-x_i} B_{ij}^{-1} = e^{-x_i} \left( \delta_{ij} - B_{ij}^{-1} X_i \right) \tag{1.287}$$

Thus the Jacobian matrix $J_{ij} = e^{-x_i} \left( \delta_{ij} - B_{ij}^{-1} X_i \right)$. We can inspect in matrix form:

$$J = \Lambda(e^{-x_1}, ..., e^{-x_m}) \cdot \left( I - \Lambda(X_1, ..., X_m) B^{-1} \right) \quad \Lambda(...) \text{ is diagonal matrix} \tag{1.288}$$

$$\det(J) = \prod_{i=1}^m e^{-x_i} \det\left(I - \Lambda(X_1, ..., X_m) B^{-1}\right) = e^{-\sum_{i=1}^m x_i} \det\left(I - \Lambda(X_1, ..., X_m) B^{-1}\right) \tag{1.289}$$

Generally, we should evaluate the concrete form of $\det(\lambda I - A)$ for certain matrix $A$. To calculate the determinant, recall the linearity of determinant for row and column:

$$\begin{aligned} \det(\lambda I - A) &= \det\left(\lambda_1 - a^{(1)}, ..., \lambda_n - a^{(n)}\right) \quad a^{(i)} \text{ is column } (i) \\ &= \det(\lambda_1, ..., \lambda_n) + \sum_{\{1,...,n\} \supset T, |T| = 1} \det\left(\lambda_1, ..., \lambda_{n-1}, -a^{(n)}\right) + ... \end{aligned} \tag{1.290}$$

The dumb expansion could be summarized in more terse form, scrutinize that we can pick arbitrary columns $a^{(i)}$ from the tuple, which **count** as a subset $T \subset \{1, ..., n\}$. For example, we pick two columns $\{1, 2\} \in \{1, 2, 3\}$ for a 3-dimension matrix resulting $\det\left(-a^{(1)}, -a^{(2)}, \lambda_3\right) = \lambda_3 \det\left(a^{(1)}, a^{(2)}, 1\right) := \lambda_3 D_{\{1,2\}}(A)$ where we use $D$ to represent the residual or *minor* determinant of the matrix $A$. Thus the determinant is the summation of all subsets:

$$\det(\lambda I - A) = \sum_{T \subset \{1,...,n\}} (-1)^{|T|} D_T(A) \prod_{r \in \{1,...,n\} \setminus T} \lambda_r \tag{1.291}$$

You can see the product of $\lambda_r$ comes from the remained index subtracted from $T$.

$$
\begin{aligned}
\det(J) &= e^{-\sum_{i=1}^{m} x_i} \sum_{T\subset\{1,\ldots,m\}} (-1)^{|T|} D_T\big(\Lambda(X_1,\ldots,X_m)B^{-1}\big) \\
&= (\ldots) \prod_{r\in T} X_r D_T(B^{-1}) = (\ldots) \prod_{i=1}^{m} X_i \prod_{r\in\{1,\ldots,n\}\setminus T} X_r^{-1} D_T(B^{-1})
\end{aligned}
\tag{1.292}
$$

Where the final term is just reverse the product in set $T$ to the product of whole dividing the product **not in** the $T$.

$$
\begin{aligned}
c(\mathbf{p}, i) = \frac{1}{(2\pi i)^m} \oint &\sum_{T\subset\{1,\ldots,m\}} (-1)^{|T|} D_T(B^{-1}) X_i \prod_{r\in\{1,\ldots,n\}\setminus T} X_r^{-1} \\
&\times \prod_{k=1}^{m} X_k^{-p_k-1} \prod_{h=1}^{m} X_h \exp\left(\sum_{k=1}^{m} (p_k+1)x_k - \sum_{h=1}^{m} x_i\right) d\mathbf{X}
\end{aligned}
\tag{1.293}
$$

We see a lot of dumb index repeated count, reduce them:

$$
\begin{aligned}
c(\mathbf{p}, i) = \frac{1}{(2\pi i)^m} \oint &\sum_{T\subset\{1,\ldots,m\}} (-1)^{|T|} D_T(B^{-1}) X_i \prod_{r\in\{1,\ldots,n\}\setminus T} X_r^{-1} \\
&\times \prod_{k=1}^{m} X_k^{-p_k} \exp\left(\sum_{k,l}^{m} p_k B_{k,l}^{-1} X_l\right) d\mathbf{X}
\end{aligned}
\tag{1.294}
$$

The residue power counting is same as before, to simplify notation, identify $T^c = \{1, \ldots, n\} \setminus T$. Focus on the certain dumb index $l$:

$$
\begin{aligned}
& n_l - p_l - \delta(l \in T^c) + \delta(i = l) = -1 \\
& n_l = p_l + \delta(l \in T^c) - \delta(i = l) - 1 \to \prod_{l}^{m} \frac{\left(\sum_k^m p_k B_{k,l}^{-1}\right)^{p_l + \delta(l\in T^c) - \delta(i=l) - 1}}{(p_l + \delta(l \in T^c) - \delta(i = l) - 1)!}
\end{aligned}
\tag{1.295}
$$

The delta notation impede us reduce further, we reformat the first delta $\delta(l \in T^c) = 1 - \delta(l \in T)$, second, we see that $\delta(i = l)$ can be decomposed into $\delta(i \in T^c = l) + \delta(i \in T = l)$, but it only effect $X_i$, so extract out $i$ index: $p_i - \delta(i \in T) - \delta(i \in T) - \delta(i \in T^c) = p_i - 1 - \delta(i \in T)$. Focus the $i$ index term, we balance the power:

$$
\frac{\left(\sum_k^m p_k B_{k,i}^{-1}\right)^{p_l - \delta(i\in T) - 1}}{(p_i - \delta(i \in T) - 1)!} = \frac{p_i - \delta(i \in T)}{\sum_k^m p_k B_{k,i}^{-1}} \frac{\left(\sum_k^m p_k B_{k,i}^{-1}\right)^{p_l - \delta(i\in T)}}{(p_i - \delta(i \in T))!}
\tag{1.296}
$$

To thunk the $i$ index term into a whole product. Now we combine all into:

$$
c(\mathbf{p}, i) = \sum_{T\subset\{1,\ldots,m\}} (-1)^{|T|} D_T(B^{-1}) \frac{p_i - \delta(i \in T)}{\sum_k^m p_k B_{k,l}^{-1}} \prod_{l}^{m} \frac{\left(\sum_k^m p_k B_{k,l}^{-1}\right)^{p_l - \delta(l\in T)}}{(p_l - \delta(l \in T))!}
\tag{1.297}
$$

$$\prod_l^m(\ldots) = \prod_{l\in T}^m \frac{\left(\sum_k^m p_k B_{k,l}^{-1}\right)^{p_l}}{p_l!} \prod_{l\notin T}^m \frac{\left(\sum_k^m p_k B_{k,l}^{-1}\right)^{p_l-1}}{(p_l-1)!} = \prod_l^m \frac{C_l^{p_l}}{p_l!} \prod_{l\notin T}^m \left(\frac{p_l}{C_l}\right)$$

$$C_{k,l} = p_k B_{k,l}^{-1} \quad C_l = \sum_k^m p_k B_{k,l}^{-1} \tag{1.298}$$

Where we extract by $\prod_{l\notin T}^m \frac{C_l^{p_l-1}}{(p_l-1)!} = \prod_{l\notin T}^m \frac{C_l^{p_l}}{p_l!}\frac{p_l}{C_l}$.

$$c(\mathbf{p}, i) = \prod_i^m \frac{C_l^{p_l}}{p_l!} \sum_{T\subset\{1,\ldots,m\}} (-1)^{|T|} \frac{p_i - 1 + \delta(i\in T^c)}{C_i} D_T(B^{-1}) \prod_{l\notin T}^m \left(\frac{p_l}{C_l}\right) \tag{1.299}$$

Here the key, we decompose by $(p_i - 1) + \delta(i \in T^c)$ into two parts:

$$\det M := \det(C_i\delta_{il} - p_i B_{il}^{-1}) = \prod_k^m C_k \det\left(\delta_{il} - \frac{p_i}{C_i} B_{il}^{-1}\right)$$
$$= \prod_k^m C_k \sum_{T\subset\{1,..,m\}} (-1)^{|T|} D_T(B^{-1}) \prod_{l\notin T}^m \left(\frac{p_l}{C_l}\right) \tag{1.300}$$

Where each column or row summation is zero, thus, we can find a vector that $C_i$ as columns with $C_i - \sum_l C_{il} = 0$ which is the linear dependence.

$$\det M := \det(C_i\delta_{il} - C_{il}) := 0 \tag{1.301}$$

$$\text{term } 1 = \frac{p_i - 1}{C_i} \det\left(\delta_{il} - \frac{p_i}{C_i} B_{il}^{-1}\right) = (p_i - 1)\frac{1}{C_i} \prod_k^m \frac{1}{C_k} \det M = 0 \tag{1.302}$$

$$\sum_{T\subset\{1,\ldots,m\}} \delta(i \in T^c) = \sum_{T\subset\{1,\ldots,m\}\setminus i} \to \det D_{\{1,\ldots,m\}\setminus i}(M) := (\det M)_{(i)} \tag{1.303}$$

Where we can see due to the delta notation, the summation must be restricted to index without $i$. Thus the determinant should be ignore the whole $j$ column and row, which is the minor determinant without $j$ index.

$$\text{term } 2 = \frac{1}{C_i} \det\left(\delta_{il} - \frac{p_i}{C_i} B_{il}^{-1}\right)_{(i)} = \frac{1}{C_i} \prod_{k\neq i}^m \frac{1}{C_k} (\det M)_{(i)} = \prod_k \frac{1}{C_k} (\det M)_{(i)} \tag{1.304}$$

$$c(\mathbf{p}, i) = \prod_l^m \frac{C_l^{p_l-1}}{p_l!} (\det M)_{(i)} \tag{1.305}$$

Is the final answer, notice the $p_l - 1$ power counts rather $p_l$ due to the factor division from $\prod_k^m \frac{1}{C_k}$. It can also be formatted as:

$$
\begin{aligned}
\det M &:= \prod_k^m \frac{1}{p_k} \det\left(C_i p_l \delta_{i,l} - C_{i,l} p_l\right) := \prod_k^m \frac{1}{p_k} \det L \\
\to N_\Gamma(\mathbf{p}) &:= \frac{c(\mathbf{p}, i)}{p_i} = \prod_k^m \frac{1}{p_k} \prod_l^m \frac{C_l^{p_l - 1}}{p_l!} (\det(L))_i
\end{aligned}
\tag{1.306}
$$

Be careful! The **left** $\frac{1}{p_i}$ pair with $\prod_{k\neq i}^m \frac{1}{p_k}$ to forming the whole product. The theorem generalize into a weighted version of *Kirchnoff′ s matrix tree theorem*, a highly non-trivial extension of *Cayley′ s theorem*, where we have $p_i$ numbers of vertex $a_i$ of $\Gamma$. Suppose only $a_1$ type of vertex, one has $C_l = \sum_k^m p_k B_{k,l}^{-1} = p_1$. Because such special case, the $(\det(L))_i$ is a zero matrix which is defined as 1. Thus, $N_\Gamma(p_1) = \frac{p^{p-2}}{p!}$, which is exactly the answer of *Cayley′ s theorem*. We then apply it to the case containing vertices type $a_1$ and $a_2$ or source and sinks:

$$
\begin{aligned}
B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\to B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\mathbf{p} &= (p, q)
\end{aligned}
\tag{1.307}
$$

$$
\begin{aligned}
C_l &= \sum_{k=1}^2 p_k B_{k,l}^{-1} = (q, p) \\
\det(L) = C_i p_l \delta_{i,l} - C_{i,l} p_l &= \begin{pmatrix} pq & -pq \\ -pq & pq \end{pmatrix} \to (\det(L))_i = pq
\end{aligned}
\tag{1.308}
$$

$$
N_\Gamma((p, q)) = \frac{1}{pq} \frac{p^{q-1} q^{p-1}}{p! q!} \cdot pq = \frac{p^{q-1} q^{p-1}}{p! q!}
\tag{1.309}
$$

Which is same as the results of previous calculation.