Notes on Z4c

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Contents

I. DERIVATION

- $D_i\beta^i$:

$$D_i\beta^i = \frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}\beta^i\right) = \chi^{3/2}\partial_i\left(\chi^{-3/2}\beta^i\right) = \partial_i\beta^i - \frac{3}{2}\chi^{-1}\beta^i\partial_i\chi \tag{1}$$

where $\chi = \gamma^{-1/3}$.

- $D_iD_j\alpha$:

$$\begin{aligned}
\Gamma^k{}_{ij} &= \frac{1}{2}\gamma^{kl}(\partial_i\gamma_{jl} + \partial_j\gamma_{li} - \partial_l\gamma_{ij}) &(2)\\
&= \frac{1}{2}\tilde{\gamma}^{kl}\left[(\partial_i\tilde{\gamma}_{jl} - \partial_i\ln\chi\tilde{\gamma}_{jl}) + (\partial_j\tilde{\gamma}_{li} - \partial_j\ln\chi\tilde{\gamma}_{li}) - (\partial_l\tilde{\gamma}_{ij} - \partial_l\ln\chi\tilde{\gamma}_{ij})\right] &(3)\\
&= \tilde{\Gamma}^k{}_{ij} - \frac{1}{2}(\partial_i\ln\chi\delta^k{}_j + \partial_j\ln\chi\delta^k{}_i - \tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\partial_l\ln\chi) &(4)
\end{aligned}$$

where $\partial_l\gamma_{ij} = \partial_l(\chi^{-1}\tilde{\gamma}_{ij}) = \chi^{-1}(\partial_l\tilde{\gamma}_{ij} - \chi^{-1}\partial_l\chi\tilde{\gamma}_{ij}) = \chi^{-1}(\partial_l\tilde{\gamma}_{ij} - \partial_l\ln\chi\tilde{\gamma}_{ij})$. Then,

$$\begin{aligned}
D_iD_j\alpha &= \partial_i\partial_j\alpha - \Gamma^k{}_{ij}\partial_k\alpha &(5)\\
&= \partial_i\partial_j\alpha - \left[\tilde{\Gamma}^k{}_{ij} - \frac{1}{2}(\partial_i\ln\chi\delta^k{}_j + \partial_j\ln\chi\delta^k{}_i - \tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\partial_l\ln\chi)\right]\partial_k\alpha &(6)\\
&= \partial_i\partial_j\alpha - \tilde{\Gamma}^k{}_{ij}\partial_k\alpha + \frac{1}{2}(\partial_i\ln\chi\partial_j\alpha + \partial_j\ln\chi\partial_i\alpha - \tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\partial_l\ln\chi\partial_k\alpha) &(7)\\
&= \partial_i\partial_j\alpha - \tilde{\Gamma}^k{}_{ij}\partial_k\alpha + \partial_{(i}\ln\chi\partial_{j)}\alpha - \frac{1}{2}\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\partial_l\ln\chi\partial_k\alpha &(8)
\end{aligned}$$

- $\partial_k\tilde{A}^{ij}$:

$$\begin{aligned}
\partial_k\tilde{A}^{ij} &= \partial_k\left(\tilde{\gamma}^{il}\tilde{\gamma}^{jm}\tilde{A}_{lm}\right) = \partial_k\tilde{\gamma}^{il}\tilde{\gamma}^{jm}\tilde{A}_{lm} + \tilde{\gamma}^{il}\partial_k\tilde{\gamma}^{jm}\tilde{A}_{lm} + \tilde{\gamma}^{il}\tilde{\gamma}^{jm}\partial_k\tilde{A}_{lm} &(9)\\
&= -\tilde{\gamma}^{ip}\tilde{\gamma}^{lq}\partial_k\tilde{\gamma}_{pq}\tilde{\gamma}^{jm}\tilde{A}_{lm} - \tilde{\gamma}^{il}\tilde{\gamma}^{jp}\tilde{\gamma}^{mq}\partial_k\tilde{\gamma}_{pq}\tilde{A}_{lm} + \tilde{\gamma}^{il}\tilde{\gamma}^{jm}\partial_k\tilde{A}_{lm} &(10)\\
&= -\tilde{\gamma}^{ip}\tilde{A}^{jq}\partial_k\tilde{\gamma}_{pq} - \tilde{\gamma}^{jp}\tilde{A}^{iq}\partial_k\tilde{\gamma}_{pq} + \tilde{\gamma}^{il}\tilde{\gamma}^{jm}\partial_k\tilde{A}_{lm} &(11)\\
&= -2\tilde{\gamma}^{l(i}\tilde{A}^{j)m}\partial_k\tilde{\gamma}_{lm} + \tilde{\gamma}^{il}\tilde{\gamma}^{jm}\partial_k\tilde{A}_{lm} &(12)
\end{aligned}$$

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A. Erik's implementation

1. Note

- Evolve $\bar{g}_{ij} = \tilde{g}_{ij} - \delta ij$,
- Evolve $\bar{\chi} = \chi - 1$,
- Evolve $\bar{\alpha} = \alpha - 1$,

2. Equations

$$
\begin{aligned}
\bar{\gamma}^{ij} &= (\bar{\gamma}_{ij} + \delta_{ij})^{-1} - \delta^{ij} && (13)\\
\partial_k \bar{\gamma}^{ij} \equiv \partial_k \tilde{\gamma}^{ij} &= -\tilde{\gamma}^{il}\tilde{\gamma}^{jm}\partial_k \tilde{\gamma}_{lm} = -(\bar{\gamma}^{il} + \delta^{il})(\bar{\gamma}^{jm} + \delta^{jm})\partial_k \bar{\gamma}_{lm} && (14)\\
\tilde{\Gamma}_{kij} &= \frac{1}{2}\left(\partial_i \bar{\gamma}_{jk} + \partial_j \bar{\gamma}_{ki} - \partial_k \bar{\gamma}_{ij}\right) && (15)\\
\tilde{\Gamma}_{ki}{}^{j} &= \tilde{\Gamma}_{kil}(\bar{\gamma}^{lj} + \delta^{lj}) && (16)\\
\tilde{\Gamma}^{k}{}_{ij} &= (\bar{\gamma}^{kl} + \delta^{kl})\tilde{\Gamma}_{lij} && (17)\\
(\tilde{\Gamma}_d)^k &= (\bar{\gamma}^{ij} + \delta^{ij})\tilde{\Gamma}^{k}{}_{ij} && (18)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{D}_i \tilde{D}_j \chi &= \partial_i \partial_j \chi - \tilde{\Gamma}^{k}{}_{ij}\partial_k \chi && (19)\\
\gamma^{ij} &= (1 + \bar{\chi})(\bar{\gamma}^{ij} + \delta^{ij}) && (20)\\
\partial_k \gamma_{ij} &= -\chi^{-2}\partial_k \chi \tilde{\gamma}_{ij} + \chi^{-1}\partial_k \tilde{\gamma}_{ij} = -(1+\bar{\chi})^{-2}\partial_k \bar{\chi}(\bar{\gamma}_{ij} + \delta_{ij}) + (1+\bar{\chi})^{-1}\partial_k \bar{\gamma}_{ij} && (21)\\
\Gamma_{kij} &= \frac{1}{2}\left(\partial_i \gamma_{jk} + \partial_j \gamma_{ki} - \partial_k \gamma_{ij}\right) && (22)\\
\Gamma^{k}{}_{ij} &= \gamma^{kl}\Gamma_{lij} && (23)\\
D_i D_j \alpha &= \partial_i \partial_j \bar{\alpha} - \Gamma^{k}{}_{ij}\partial_k \bar{\alpha} && (24)
\end{aligned}
$$

$$
\tilde{R}^{\chi}_{ij} = \frac{1}{2(1+\bar{\chi})}\tilde{D}_i \tilde{D}_j \chi + \frac{1}{2(1+\bar{\chi})}(\bar{\gamma}_{ij} + \delta_{ij})\tilde{D}^l \tilde{D}_l \chi - \frac{1}{4(1+\bar{\chi})^2}\partial_i \bar{\chi}\partial_j \bar{\chi} - \frac{3}{4(1+\bar{\chi})^2}(\bar{\gamma}_{ij} + \delta_{ij})(\bar{\gamma}^{kl} + \delta^{kl})\partial_k \bar{\chi}\partial_l \bar{\chi} \quad (25)
$$

$$
\begin{aligned}
\tilde{R}_{ij} &= -\frac{1}{2}(\bar{\gamma}^{lm} + \delta^{lm})\partial_l \partial_m \bar{\gamma}_{ij} + (\bar{\gamma}_{k(i} + \delta_{k(i})\partial_{j)}\tilde{\Gamma}^k + (\tilde{\Gamma}_d)^k \tilde{\Gamma}_{(ij)k} + (2\tilde{\Gamma}^{k}{}_{l(i}\tilde{\Gamma}_{j)k}{}^{l} + \tilde{\Gamma}^{k}{}_{im}\tilde{\Gamma}_{kj}{}^{m}) && (26)\\
R_{ij} &= \tilde{R}^{\chi}_{ij} + \tilde{R}_{ij} && (27)\\
R &= \gamma^{ij}R_{ij} && (28)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{A}^{ij} &= (\bar{\gamma}^{il} + \delta^{il})(\bar{\gamma}^{jm} + \delta^{jm})\tilde{A}_{lm} && (29)\\
\partial_k \tilde{A}^{ij} &= \partial_k\left(\tilde{\gamma}^{il}\tilde{\gamma}^{jm}\tilde{A}_{lm}\right) = \partial_k \bar{\gamma}^{il}(\bar{\gamma}^{jm} + \delta^{jm})\tilde{A}_{lm} + (\bar{\gamma}^{il} + \delta^{il})\partial_k \bar{\gamma}^{jm}\tilde{A}_{lm} + (\bar{\gamma}^{il} + \delta^{il})(\bar{\gamma}^{jm} + \delta^{jm})\partial_k \tilde{A}_{lm} && (30)
\end{aligned}
$$

II. MORE

[1]

[1] Roger Alexander. Solving ordinary differential equations i: Nonstiff problems (e. hairer, sp norsett, and g. wanner). Siam Review, 32(3):485, 1990.