

Notes on Implementing MC's Method

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I. EXPLICIT RUNGE-KUTTA SCHEME

To integrate the ODE

$$\frac{dy}{dt} = f(y, t), \quad (1)$$

we start from the fundamental theorem of calculus

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(y(\tau), \tau) d\tau = y(t_n) + h \int_0^1 f(y(t_n + h\tau), t_n + h\tau) d\tau. \quad (2)$$

We can replace the integral with a quadrature approximation

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i f(y(t_n + c_i h), t_n + c_i h), \quad (3)$$

where we have to construct an approximation, denoted by $Y_i \simeq y(t_n + c_i h)$. With explicit Runge-Kutta method we construct Y_i using

$$y_n, \quad f(Y_1, t_n + hc_1), \quad f(Y_2, t_n + hc_2), \quad \dots, \quad f(Y_{i-1}, t_n + hc_{i-1}). \quad (4)$$

Then the explicit Runge-Kutta method can be sum up as the following,

$$Y_s = y_n + h \sum_{i=1}^{s-1} a_{si} f(Y_i, t_n + hc_i), \quad (5)$$

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i f(Y_i, t_n + hc_i). \quad (6)$$

We can also introduce the intermediate slopes as $k_i := f(Y_i, t_n + hc_i)$, then

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i k_i. \quad (7)$$

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Further more for the chain rule,

$$y' = f(y, t) \quad (8)$$

$$y'' = \frac{d}{dt}y' = \left(\frac{\partial}{\partial t} + f \frac{\partial}{\partial y} \right) f = \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \quad (9)$$

$$\begin{aligned} y''' &= \frac{d}{dt}y'' = \left(\frac{\partial}{\partial t} + f \frac{\partial}{\partial y} \right) \left(\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \right) \\ &= \frac{\partial^2 f}{\partial t^2} + f \frac{\partial^2 f}{\partial t \partial y} + \frac{\partial f}{\partial t} \frac{\partial f}{\partial y} + f \frac{\partial^2 f}{\partial y \partial t} + f \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} + f^2 \frac{\partial^2 f}{\partial y^2} \\ &= \frac{\partial^2 f}{\partial t^2} + 2f \frac{\partial^2 f}{\partial t \partial y} + \frac{\partial f}{\partial t} \frac{\partial f}{\partial y} + f^2 \frac{\partial^2 f}{\partial y^2} + f \left(\frac{\partial f}{\partial y} \right)^2 \end{aligned} \quad (10)$$

Consider the case $t_n = 0$, $y(t) = t \Rightarrow y(0) = 0, y' = 1$, for the first order accuracy, the above scheme need to obtain the exact solution of $y(t) = t$.

$$Y_i = y_n + h \Sigma_{j=1}^{i-1} a_{ij} \simeq y(t_n + c_i h) = y_n + h c_i, \quad (11)$$

$$y_{n+1} = y_n + h \Sigma_{i=1}^s b_i = y_n + h. \quad (12)$$

$$\Rightarrow \begin{cases} \Sigma_{j=1}^{i-1} a_{ij} = c_i, \\ \Sigma_{i=1}^s b_i = 1, \end{cases} \quad (13)$$

$a_{21} = c_2$ for example.

For up to the 3-order conditions, it sufices to study the case of **autonomous** differential equations, $y' = f(y)$, where the chain rule become

$$y' = f, \quad (14)$$

$$y'' = \left(f \frac{\partial}{\partial y} \right) f = f f_y, \quad (15)$$

$$y''' = \left(f \frac{\partial}{\partial y} \right) (f f_y) = f f_y^2 + f^2 f_{yy} \quad (16)$$

A. Dense Output

Consider formulas of the form for a 4-stage Runge-Kutta method [1]

$$u(\theta) = y_n + h \Sigma_{i=1}^4 b_i(\theta) k_i. \quad (17)$$

where k_i is defined below eq. (6), and $b_i(\theta)$ are polynomials to be determined such that $u(\theta) - y(x_0 + \theta h) = \mathcal{O}(h^{p^*+1})$. We write the polynomials $b_j(\theta)$ as

$$b_j(\theta) = \Sigma_{q=1}^{p^*} b_{jq} \theta^q. \quad (18)$$

For the 4-stage RK4 with $p^* = 3$ the order condition produce a unique solution

$$b_1(\theta) = \theta - \frac{3}{2}\theta^2 + \frac{2}{3}\theta^3, \quad (19)$$

$$b_2(\theta) = b_3(\theta) = \theta^2 - \frac{2}{3}\theta^3, \quad (20)$$

$$b_4(\theta) = -\frac{1}{2}\theta^2 + \frac{2}{3}\theta^3. \quad (21)$$

The dense output formula can be summarized as [2]

$$y(t_n + \theta h) = y_n + h \Sigma_{i=1}^4 b_i(\theta) k_i + \mathcal{O}(h^4), \quad (22)$$

$$\frac{d^{(m)}}{dt^{(m)}} y(t_n + \theta h) = \frac{1}{h^{(m-1)}} \Sigma_{i=1}^s k_i \frac{d^{(m)}}{d\theta^{(m)}} b_i(\theta) + \mathcal{O}(h^{4-m}). \quad (23)$$

B. Taylor expansion of k_i for RK4 up to $\mathcal{O}(h^2)$

The Taylor expansion of k_i around t_n are [2]

$$k_1 = y', \quad (24)$$

$$k_2 = y' + \frac{h}{2}y'' + \frac{h^2}{8}(y''' - f_y y''), \quad (25)$$

$$k_3 = y' + \frac{h}{2}y'' + \frac{h^2}{8}(y''' + f_y y''). \quad (26)$$

where $f_y y'' \equiv (y'')^2 / y' = \frac{4(k_3^{(c)} - k_2^{(c)})}{h_{(c)}^2}$, and y', y'', y''' can be represented with $k_i^{(c)}$ of coarse grid using the dense output formula (22)-(23).

C. Taylor expansion of Y_i for RK4 up to $\mathcal{O}(h^3)$

Similarly, we can also expand Y_i instead of k_i [3]

$$Y_1 = y_n, \quad (27)$$

$$Y_2 = y_n + \frac{h}{2}y', \quad (28)$$

$$Y_3 = y_n + \frac{h}{2}y' + \frac{h^2}{4}y'' + \frac{h^3}{16}(y''' - f_y y'') \quad (29)$$

$$Y_4 = y_n + hy' + \frac{h^2}{2}y'' + \frac{h^3}{8}(y''' + f_y y'') \quad (30)$$

where $f_y y'' \equiv (y'')^2 / y' = \frac{4(k_3^{(c)} - k_2^{(c)})}{h_{(c)}^2}$, and y', y'', y''' can be represented with $k_i^{(c)}$ of coarse grid using the dense output formula (22)-(23).

1. Pseudocode of MC's method

1. Integrate coarse grid from t_n to $t_n + h^{(c)}$ and store $k_i^{(c)}$ somewhere.
2. Interpolate in time for y, y', y'', y''' using (22)-(23) with stored $k_i^{(c)}$
 - (a) at $\theta = 0.0$ for the first fine step,
 - (b) at $\theta = 0.5$ for the second fine step.
3. Calculate Y_i for the first and second fine steps using (27)-(30).
4. Interpolate in space to fill Y_i in the fine ghost points.
5. Repeat.

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- [1] Roger Alexander. Solving ordinary differential equations i: Nonstiff problems (e. hairer, sp norsett, and g. wanner). Siam Review, 32(3):485, 1990.
 - [2] Bishop Mongwane. Toward a consistent framework for high order mesh refinement schemes in numerical relativity. General Relativity and Gravitation, 47:1–21, 2015.
 - [3] Peter McCorquodale and Phillip Colella. A high-order finite-volume method for conservation laws on locally refined grids. Communications in Applied Mathematics and Computational Science, 6(1):1–25, 2011.