Notes on Runge-Kutta method

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I. EXPLICIT RUNGE-KUTTA SCHEME

$$y' = f(y,t) \tag{1}$$

$$y'' = \frac{d}{dt}y' = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right) f = \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} \tag{2}$$

A. Two-stage, second-order RK

$$Y_1 = y_n \tag{3}$$

$$Y_2 = y_n + ha_{21}f(Y_1, t_n) \tag{4}$$

$$Y_3 = y_n + hb_1 f(Y_1, t_n) + hb_2 f(Y_2, t_n + c_2 h) \tag{5}$$

Taylar expand about (y_n, t_n):

$$y_{n+1} = y_n + hb_1 f(y_n, t_n) + hb_2\left[f + h\left(a_{21}\frac{\partial f}{\partial y}f + c_2\frac{\partial f}{\partial t}\right)\right] + \mathcal{O}(h^3) \tag{6}$$

$$= y_n + h(b_1 + b_2)f + h^2 b_2\left(a_{21}\frac{\partial f}{\partial y}f + c_2\frac{\partial f}{\partial t}\right) + \mathcal{O}(h^3) \tag{7}$$

since

$$f(Y_2, t_n + c_2 h) = f(y_n + ha_{21}f(Y_1, t_n), t_n + c_2 h) \tag{8}$$

$$= f(y_n, t_n) + h\left[\frac{\partial f}{\partial y}(y_n, t_n)a_{21}f(y_n, t_n) + \frac{\partial f}{\partial t}(y_n, t_n)c_2\right] + \mathcal{O}(h^2) \tag{9}$$

Compare to the Tayler expansion

$$y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \mathcal{O}(h^3) \tag{10}$$

$$= y_n + hf + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right) + \mathcal{O}(h^3) \tag{11}$$

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we have the order condition

$$b_1 + b_2 = 1 \tag{12}$$

$$b_2 a_{21} = \frac{1}{2} \tag{13}$$

$$b_2 c_2 = \frac{1}{2} \tag{14}$$

B. Three-stage, third-order RK