# Notes on Implementing MC's Method

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**Contents**



## I. EXPLICIT RUNGE-KUTTA SCHEME

For the fundamental therorem of calculus,

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(y(\tau),\tau)d\tau = y(t_n) + h\int_0^1 f(y(t_n+h\tau), t_n+h\tau)d\tau \tag{1}$$

replace the integral with a quadrature approximation

$$y_{n+1} = y_n + h\Sigma_{i=1}^{s} b_i f(y(t_n+c_ih), t_n+c_ih) \tag{2}$$

then we have to construct an approximation, denoted by $Y_i \simeq y(t_n+c_ih)$. With explicit Runge-Kutta methods you construct $Y_i$ using

$$Y_1 = y_n, \quad f(Y_1,t_n), \quad f(Y_2, t_n+hc_2), \quad ..., \quad f(Y_{i-1}, t_n+hc_{i-1}). \tag{3}$$

Then

$$\boxed{\begin{aligned} Y_s &= y_n + h\Sigma_{i=1}^{s-1} a_{si} f(Y_i, t_n+hc_i). \qquad (4)\\ y_{n+1} &= y_n + h\Sigma_{i=1}^{s} b_i f(Y_i, t_n+hc_i) \qquad (5)\end{aligned}}$$

More

$$y' = f(y,t) \tag{6}$$

$$y'' = \frac{d}{dt}y' = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right) f = \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} \tag{7}$$

$$\begin{aligned} y''' &= \frac{d}{dt}y'' = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial^2 f}{\partial t^2} + f\frac{\partial^2 f}{\partial t\partial y} + \frac{\partial f}{\partial t}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial y\partial t} + f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y} + f^2\frac{\partial^2 f}{\partial y^2} \\ &= \frac{\partial^2 f}{\partial t^2} + f^2\frac{\partial^2 f}{\partial y^2} + 2f\frac{\partial^2 f}{\partial t\partial y} + \frac{\partial f}{\partial t}\frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2 \end{aligned} \tag{8}$$

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Consider $t_n = 0, y(t) = t \Rightarrow y(0) = 0, y' = 1$. For first order accuracy, need to obtain the exact solution of $y(t) = t$.

$$Y_i = y_n + h\Sigma_{j=1}^{i-1} a_{ij} \simeq y(t_n + c_i h) = y_n + hc_i, \tag{9}$$

$$y_{n+1} = y_n + h\Sigma_{i=1}^{s} b_i = y_n + h. \tag{10}$$

$$\Rightarrow \begin{cases} \Sigma_{j=1}^{i-1} a_{ij} = c_i, \\ \Sigma_{i=1}^{s} b_i = 1. \end{cases} \Rightarrow \{ \; a_{21} = c_2 \tag{11}$$

where is necessary for first-order accuracy.

For up to the 3-order conditions, it sufices to study the case of **autonomous** differential equations, $y' = f(y)$. Then

$$y' = f, \tag{12}$$

$$y'' = (f\frac{\partial}{\partial y})f = ff_y, \tag{13}$$

$$y''' = (f\frac{\partial}{\partial y})(ff_y) = ff_y f_y + f^2 f_{yy} \tag{14}$$

### A. Taylor expansion of $k_i$ for RK4 up to $\mathcal{O}(h^3)$

Following the convention of Mongwane, we defined

$$k_i = hf(Y_i, t_n + hc_i) \tag{15}$$

then

$$k_1 = hy', \tag{16}$$

$$k_2 = hy' + \frac{h^2}{2}y'' + \frac{h^3}{8}(y''' - f_y y''), \tag{17}$$

$$k_3 = hy' + \frac{h^2}{2}y'' + \frac{h^3}{8}(y''' + f_y y''). \tag{18}$$

where $f_y y'' \equiv (y'')^2/y' = \frac{4(k_3^{(c)} - k_2^{(c)})}{h^3}$, and $y', y'', y'''$ can be represented with $k_i^{(c)}$ of coarse grid using the dense output formula.

### B. Taylor expansion of $Y_i$ for RK4 up to $\mathcal{O}(h^3)$

Similarly, we can also expand $Y_i$ instead of $k_i$, and we have

$$Y_1 = y_n, \tag{19}$$

$$Y_2 = y_n + \frac{h}{2}y', \tag{20}$$

$$Y_3 = y_n + \frac{h}{2}y' + \frac{h^2}{4}y'' + \frac{h^3}{16}(y''' - f_y y'') \tag{21}$$

$$Y_4 = y_n + hy' + \frac{h^2}{2}y'' + \frac{h^3}{8}(y''' + f_y y'') \tag{22}$$

where $f_y y'' \equiv (y'')^2/y' = \frac{4(k_3^{(c)} - k_2^{(c)})}{h^3}$, and $y', y'', y'''$ can be represented with $k_i^{(c)}$ of coarse grid using the dense output formula,

$$y(t_n + \theta h) = y_n + \Sigma_{i=1}^{4} b_i(\theta) k_i^{(c)} + \mathcal{O}(h^4), \tag{23}$$

$$\frac{d^{(m)}}{dt^{(m)}} y(t_n + \theta h) = \frac{1}{h^m}\Sigma_{i=1}^{s} k_i^{(c)} \frac{d^{(m)}}{d\theta^{(m)}} b_i(\theta) + \mathcal{O}(h^{4-m}). \tag{24}$$

where $b_1(\theta) = \theta - \frac{3}{2}\theta^2 + \frac{2}{3}\theta^3$, $b_2(\theta) = b_3(\theta) = \theta^2 - \frac{2}{3}\theta^3$, $b_4(\theta) = -\frac{1}{2}\theta^2 + \frac{2}{3}\theta^3$.

*1. Pseudo code*

1. integrate coarse grid from $t_n$ to $t_n + h^{(c)}$ and store $k_i^{(c)}$ somewhere,
2. interpolate in time for $y, y', y'', y'''$ using (23)-(24)
   (a) at $\theta = 0.0$ for the first fine step,
   (b) at $\theta = 0.5$ for the second fine step,
3. calculate $Y_i$ for the fine grid using (19)-(22).
4. interpolate in space to fill $Y_i$ in the fine ghost points.