Testing Samples

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1 Serre SS of a fibration on 3-sphere

Consider the following sequence

$$K(\mathbf{Z},2) \to F \to S^3 \to K(\mathbf{Z},3)$$

and the fibration on the left. The second page of the Serre SS of this fibration is of the form

$$E_2^{0,2n} = E_2^{3,2n} = \mathbf{Z},\ n \geq 0.$$

In other words, $E_2 = \operatorname{Ext}[a] \otimes \mathbf{Z}[t]$ with $a \in E_2^{3,0}$ and $t \in E_2^{0,2}$. The differentials are generated under Leibniz rule by

$$d_3(t) = a,$$

so the map from $E_3^{0,2n} = \langle t^n \rangle$ to $E_2^{3,2(n-1)} = \langle t^{n-1}a \rangle$ is just multiplication by n. There are no higher differentials.

2 Serre SS of the Path fibration of K(Z, 3)

Fix the base field $\mathbf{F} = \mathbf{Z}/3$. We have

$$H^*K(\mathbf{Z},2) = \mathbf{F}[t],$$

where $|t| = 2$, and

$$H^*K(\mathbf{Z},3) = \operatorname{Ext}[a_0, a_1, \cdots] \otimes \mathbf{F}[b_0, b_1, \cdots],$$

where $|a_i| = 1 + 2 \cdot 3^i$, $|b_i| = 2 + 2 \cdot 3^{i+1}$. Then the second page of the Serre SS of the fibration

$$K(\mathbf{Z},2) \to * \to K(\mathbf{Z},3),$$

is the bigraded algebra generated by the first algebra as the Y-axis and the second as the X-axis.

In other words, we have

$$t \in E_2^{0,2}, \; a_i \in E_2^{1+2\cdot 3^i,0}, \; b_i \in E_2^{2+2\cdot 3^{i+1},0}.$$

The grading of differentials on page r is $(r, 1-r)$, and there are two classes of differentials

$$\begin{aligned} d_{1+2\cdot 3^i}(t^{3^i}) &= a_i, \\ d_{1+4\cdot 3^i}(t^{2\cdot 3^i} a_i) &= b_i. \end{aligned}$$

For example, the first few differentials are

$$\begin{aligned} d_3(t) &= a_0, \\ d_7(t^3) &= a_1, \\ d_{19}(t^9) &= a_2, \\ d_{55}(t^{27}) &= a_3, \end{aligned}$$

and

$$\begin{aligned} d_5 t^2 a_0 &= b_0, \\ d_{13}(t^6 a_1) &= b_1, \\ d_{37}(t^{18} a_2) &= b_2, \\ d_{109}(t^{54} a_3) &= b_3. \end{aligned}$$

All other differentials are generated under the Leibniz rule by the above two classes and trivial differentials. Explicitly, on page $r = 1 + 2 \cdot 3^i$, if x factors as $\tilde{x}(t^{3^i})^j$ (and cannot factors further), then $d_{1+2\cdot 3^i}(\tilde{x}) = 0$ and

$$d_{1+2\cdot 3^i}(x) = \tilde{x} j (t^{3^i})^{j-1} a_i.$$

Similarly for the case $r = 1 + 4 \cdot 3^i$.

This spectral sequence will converge to $H^*(*) = \mathbf{Z}$. That is, every nonzero element of nontrivial bidegree is either killed or disregarded. To convince yourself of this, let's proceed as follows.

Claim 1. *On page 3, t^i is disregarded unless $i = 0 \mod 3$; $t^i a_0$ is killed unless $i = 2 \mod 3$.*

Proof. Recall we are over $\mathbf{F} = \mathbf{Z}/3$. If $i \neq 0 \mod 3$, then $i = \pm 1$ and by Leibniz rule

$$d_3(t^i) = i t^{i-1} d_3(t) = i t^{i-1} a_0 = \pm t^{i-1} a_0;$$

if $i = 0 \mod 3$, then

$$d_3(t^i) = i t^{i-1} d_3(t) = 0.$$

□

Claim 2. *On page 5, $d_5(t^{3k+2}a_0) = t^{3k}b_0$, i.e. the remaining classes of the form $t^i a_0$ is disregarded, while $t^{3k}b_0$ is killed.*

Proof. Note t^i does not exist on page 5 unless $i = 3k$. So we split $t^{3k+2}a_0$ as $t^{3k} \cdot t^2 a_0$. As $d_5(t^{3k}) = 0$, we apply the Leibniz rule. □

Claim 3. *All other $t^i b_0$ is disregarded on page 3.*

Proof. Note $d_3(b_0) = 0$. As $i \neq 3k$, $t^i b_0$ supports a nontrivial differential

$$d_3(t^i b_0) = \pm t^{i-1}a_0 b_0.$$

□

Similarly, we have the following 3 claims.

Claim 4. *On page 7, t^{3i} is disregarded unless $i = 0 \mod 3$; $t^{3i}a_0$ is killed unless $i = 2 \mod 3$.*

Claim 5. *On page 13, $d_{13}(t^{3(3k+2)}a_1) = t^{3k}b_1$, i.e. the remaining classes of the form $t^i a_0$ is disregarded, while $t^{3k}b_1$ is killed.*

Claim 6. *All other $t^i b_1$ is disregarded on page 7.*

Inductively, we see that every element of the following forms is either killed or disregarded

$$t^i, t^i a_j, t^i b_j \text{ and } \text{ for } i \neq 0 \mod 3,\ t^{i-1}a_j b_j.$$

Similarly arguments can show that other elements are also either killed or disregarded.