Error Terms for Logarithmic Hamiltonian

似鳥啓吾

2018 年 10 月 31 日

A time transformed logarithmic Hamiltonian is

$$\Gamma = \ln(p_0 + T) - \ln U, \tag{1}$$

where T is kinetic energy, U a negative potential, p_0 a time momentum of which value is negative of total energy $p_0 = -(T - U)$. $\mathcal{H} = \mathcal{T} + \mathcal{V}$ Partial derivatives are

$$\frac{\partial}{\partial p_0}\Gamma = \frac{1}{p_0 + T}, \tag{2a}$$
$$\frac{\partial}{\partial p_i}\Gamma = \frac{1}{p_0 + T} v_i, \tag{2b}$$
$$\frac{\partial}{\partial q_0}\Gamma = 0, \tag{2c}$$
$$\frac{\partial}{\partial p_i}\Gamma = -\frac{1}{U} f_i, \tag{2d}$$

with v_i is velocity $v_i = p_i/m_i = \partial T/\partial p_i$ and f_i force $f_i = \partial U/\partial q_i$.

A Poisson bracket is

$$\{A,\ B\} = \sum_{i=0}^{N} \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i} \right). \tag{3}$$

Now let us investigate

$$\begin{aligned} \{\ln(p_0 + T),\ -\ln U\} &= \sum_{i=1}^{N} \frac{\partial}{\partial p_i} \left(\ln(p_0 + T)\right) \frac{\partial}{\partial q_i} \left(\ln U\right) \\ &= \sum_{i=1}^{N} \frac{1}{p_0 + T} \frac{1}{U} (v_i \cdot f_i). \end{aligned} \tag{4}$$

For $\partial/\partial q_0 = 0$, thus $i = 0$ is not included in the summation range. Now,

$$\frac{\partial}{\partial p_i} \{\ln(p_0 + T),\ -\ln U\} = \frac{1}{p_0 + T}\frac{1}{U} \left[\frac{f_i}{m_i} - \frac{\sum_j \left(v_j \cdot f_j\right)}{p_0 + T} v_i \right], \tag{5}$$

$$\frac{\partial}{\partial q_i} \{\ln(p_0 + T),\ -\ln U\} = \frac{1}{p_0 + T}\frac{1}{U} \left[\left(\sum_j \frac{\partial f_j}{\partial q_i} \cdot v_j \right) - \frac{\sum_j \left(v_j \cdot f_j\right)}{U} f_j \right], \tag{6}$$

and

$$\left\{\{\ln(p_0+T),\ -\ln U\},\ -\ln U\right\} = \sum_i \frac{\partial}{\partial p_i}\{\ln(p_0+T),\ -\ln U\}\cdot\frac{\partial}{\partial q_i}\ln U$$
$$= \frac{1}{p_0+T}\frac{1}{U^2}\left[\sum_i \frac{{f_i}^2}{m_i} - \frac{\left[\sum_j\left(v_j\cdot f_j\right)\right]^2}{p_0+T}\right], \tag{7}$$

$$\left\{\{-\ln U,\ \ln(p_0+T)\},\ -\ln(p_0+T)\right\} = -\sum_i \frac{\partial}{\partial q_i}\{\ln(p_0+T),\ -\ln U\}\cdot\frac{\partial}{\partial p_i}\ln(p_0+T)$$
$$= -\frac{1}{(p_0+T)^2}\frac{1}{U}\left[\left(\sum_{i,j}\frac{\partial f_j}{\partial q_i}\cdot v_j\cdot v_i\right) - \frac{\left[\sum_j\left(v_j\cdot f_j\right)\right]^2}{U}\right]. \tag{8}$$

Consider a Kepler problem

$$U = p_0 + T = \frac{\mu}{r},\quad m_i = 1,\quad f_i = -\frac{\mu r_i}{r^3},\quad \frac{\partial f_i}{\partial q_j} = -\mu\left(\frac{\delta_{ij}}{r^3} - \frac{3r_ir_j}{r^5}\right).$$

Then,

$$\{\{\mathcal{T},\ \mathcal{V}\},\ \mathcal{V}\} = \frac{1}{\mu^2}\left[\frac{\mu}{r} - \left(\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r}\right)^2\right], \tag{9}$$
$$\{\{\mathcal{V},\ \mathcal{T}\},\ \mathcal{T}\} = \frac{1}{\mu^2}\left[v^2 - 2\left(\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r}\right)^2\right]. \tag{10}$$

Thus,

$$\{\{\mathcal{T},\ \mathcal{V}\},\ \mathcal{V}\} - \frac{1}{2}\{\{\mathcal{V},\ \mathcal{T}\},\ \mathcal{T}\} = -\frac{1}{\mu^2}\left[\frac{1}{2}v^2 - \frac{\mu}{r}\right] = \frac{p_0}{\mu^2}, \tag{11}$$

is proportional to the leading error term of the DKD mode.