CS 430 - Lecture Activity Notes

Len Washington III

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List of Algorithms

Opening Questions

1. Algorithm A takes 5 seconds to sort 1000 records, and Algorithm B takes 10 seconds to sort 1000 records. You have the code for both algorithms. When deciding which algorithm to use to sort up to 1,000,000 records, why might Algorithm B be the better choice? Algorithm B might grow slower than algorithm A when more data is added.

2. Why is it helpful to sometimes define a problem in its most basic mathematical terms? Understanding the problem space might help with thinking of algorithmic solutions since there might be a possible shortcut rather than the brute force algorithm. It also helps to know if the problem is easy, which might mean that the brute force method would work effectively.

3. In your own words, explain what a loop invariant is. A statement that's true for every iteration of a loop. It never changes. It's a way to prove an iterative algorithm correctly.

4. What are the three kinds of growth in run time analysis we may do on an algorithm. Best case, worst case, average case.

5. For recursive algorithms, what do we need to define and solve to do the runtime analysis? Recurrence relation. Base recursive?

Sorting as a Case Study for Algorithms Analysis

1. Write pseudocode for one of these iterative sorts: InsertionSort, BubbleSort, SelectionSort. Then draw pictures for a sample run on 5 random numbers showing comparisons/swaps.

2. Write the loop invariant for your sort and prove your sort is correct by proving Initialization, Maintenance, and Termination.
   An important property of sorting algorithms is whether or not it is stable: numbers with the same value appear in the output array in the same order as they do in the input array.
   Before the i^{th} iteration, all of the items that were originally before position i are now sorted. After the i^{th} iteration, the original items $1 \rightarrow i$ are sorted (that doesn't mean they're in their final position)

   Initialization: Before the $i = 1$ iteration, $A[0]$ is sorted.
   Maintenance: Line 2 correctly sets the current starting position, lines 3 and 4 verifies the element is in the current position and walks down the item.
   Termination: Before the $i = n$ iteration, which doesn't exist, all items before position n are now sorted.

3. Is your sort stable? (Is the relative order of duplicate values maintained when the algorithm is done?) This is stable because in line 3, the values are only swapped if one value is less than the other. But if the values are the same, the swap won't occur.
   Resource (memory or runtime) Use Analysis – Resource use analysis usually depends on the size of the input to the algorithm. You can write a function T(n) that matches the behavior of the resource use of the algorithm.NOTE: For recursive algorithms we develop and solve a recurrence relation to find the T(n), the resource use function.

4. For the iterative sort you wrote above, construct a run time analysis function T(n) by assigning a different constant (c1, c2, c3, etc) to each type of statement (the run time for statements of that type), and counting how many times each statement executes for an input size n. Then sum up the constants times the execution counts. You may need to define variables other than n if there are event controlled iterations with an unknown number of loops.

$$T(n) = nc_1 + (n-1)c_2 + c_3 \sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n-1} (t_i - 1)\,[c_4 + c_2]$$

   where t_i depends on initial order of the array. Worst case: opposite order where $t_i = i$. Best case: if the array is already sorted where $t_i = 1$.

Table 1.1: Constants

Line #	Constant	Number of Executions
1	c_1	n
2	c_2	$n-1$
3	c_3	t_i
4	c_4	
5	c_5	

t_i is how far the i^{th} item needs to walk ($0 \rightarrow (i-1)$)

For a more descriptive analysis we need to consider the various generic execution flows and input structures that result in those generic execution flows. There are 3 more descriptive resource use analyses: 1) worst case (usually used) 2) average case (sometimes used) 3) best case (hardly ever used)

5. For your T(n) function from above determine the best case, worst case and average case T(n)s.
   Best Case ($t_i = 1$):

$$\begin{aligned} T(n) &= nc_1 + (n-1)c_2 + c_3\sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n-1}(t_i - 1)\left[c_4 + c_2\right] \\ &= nc_1 + (n-1)c_2 + c_3\sum_{i=1}^{n-1}(1) + \sum_{i=1}^{n-1}(1-1)\left[c_4 + c_2\right] \\ &= nc_1 + (n-1)c_2 + c_3(n-1) + \sum_{i=1}^{n-1}(0)\left[c_4 + c_2\right] \\ &= nc_1 + (n-1)c_2 + c_3(n-1) \\ &= nc_1 + (n-1)(c_2 + c_3) \end{aligned}$$

   Worst Case ($t_i = i$):

$$\begin{aligned} T(n) &= nc_1 + (n-1)c_2 + c_3\sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n-1}(t_i - 1)\left[c_4 + c_2\right] \\ &= nc_1 + (n-1)c_2 + c_3\sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1}(i-1)\left[c_4 + c_2\right] \end{aligned}$$

Asymptotic Analysis

- To simplify comparing the resource usage of different algorithms for the same problem.
- Ignore machine dependent constants; look at the growth of $T(n)$ as $n \rightarrow \infty$.
- As you double n, what does $T(n)$ do?? Double?? Square??

Theta (Θ) Notation (more details in future lectures)

- Drop lower order terms;
- Ignore leading constants
- Concentrates on the growth
- Tight bound on growth

Ω is lowerbound, O (read as Big-O) is upperbound

1. For your best case, average cast, worst case $T(n)$ functions from above, give the asymptotic function (Θ notation)

$$T(n) = c_1 n + c_2(n-1) + c_3 \sum_{i=1}^{n-1} t_i + c_4 \sum_{1}^{n-1} (t_i - 1)$$

Worst case: $t_i = i \rightarrow O(n^2)$

Best case: $t_i = i \rightarrow O(n)$

2. Given the problem sizes and worst case runtime for one of the problem sizes, and what you know about each algorithm, predict the missing runtimes.

	$n = 100$	$n = 200$	$n = 400$	$n = 800$
Linear search $O(n)$	10 seconds	20	40	80
Binary search $O(\lg n)$	7	8 seconds	9	10
Insertion Sort $O(n^2)$	20	80	320 seconds	1,280

For Binary search, the steps increase logarithmically, meaning that for each iteration, the solution is cut in half. This means that (starting with $n = 200$), after one iteration the size has been cut in half, which would give you the $n = 100$ problem. So the time would only increase with 1 iteration for doubling the size. The 1-second increase was an estimation for how long each iteration was.

Opening Questions - Average Case Runtime

3. How did we approach case runtime analysis of iterative algorithms previously? How can we improve on this? Insertion sort – more or less, the inner loop needs to walk half way down on every other iteration.

Expectation of a Random Variable

A random variable is a variable that maps an outcome of a random process to a number. Examples:

- Flipping a coin. If heads: $X = 1$, if tails: $X = 0$
- Y =sum of 7 rolls of a fair die (there is only 1-way to get a sum of 7, get 1's on every roll. But there are multiple ways you could get a sum of 12, so this would be a random variable.)
- Z =in insertion sort, the number of swaps needed to move the ith item to its correct position in items 1 through $(i-1)$. Range$(Z) = [0, i-1]$ (assuming all equally likely)

The expected value of a random variable X is the sum over all outcomes of the value of the outcome times the probability of the outcome.

$$E(X) = \sum_{s \in S} X(s)\, p(s)$$

- s is the outcome
- X is the random variable (assigning a number to a certain outcome.)
- p is the probability

4. What is the expected outcome when you roll a fair die once? What about a loaded die where the probability of a side coming up is the value of the side divided by 21?

$$1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{21}{6} = 3.5$$

$$1\left(\frac{1}{21}\right)+2\left(\frac{2}{21}\right)+3\left(\frac{3}{21}\right)+4\left(\frac{4}{21}\right)+5\left(\frac{5}{21}\right)+6\left(\frac{6}{21}\right) = \frac{1+4+9+16+25+36}{21} = \frac{91}{21} = \frac{13}{3} = 4.\bar{3}$$

5. Calculate the expected outcome when you roll a fair die twice and sum the results. Do this two different ways. There are 36 possible combinations of rolls. $2 = (\{1,1\}) = \frac{1}{36}, 3 = (\{1,2\};\{2,1\}) = \frac{2}{36}, \ldots$

Now let's use expectation of a random variable to improve our average case runtime for insertion sort (similar for bubble sort or selection sort).

- Sort n distinct elements using insertion sort
- X_i is the random variable equal to the number of comparisons used to insert a_i into the proper position after the first $i-1$ elements have already been sorted. $1 \leq X_i \leq i-1$

$E(X_i)$ is the expected number of comparisons to insert a_i into the proper position after the first $i-1$ elements have been sorted.
$E(X) = E(X_2) + E(X_3) + \cdots + E(X_n)$ is the expected number of comparisons to complete the sort (our new average case runtime function).

6. Write equations for the following and simplify.

$$\sum_{j=1}^{n} j = \frac{(n)(n+1)}{2}$$

- $E(X_i)$Outcomes: $1, 2, 3, \ldots, (i-1)$, Probability for each: $\frac{1}{i-1}$

$$\begin{aligned} E(X_i) &= \left(\frac{1}{i-1}\right)1 + \left(\frac{1}{i-1}\right)2 + \left(\frac{1}{i-1}\right)3 + \cdots + \left(\frac{1}{i-1}\right)(i-1) \\ &= \left(\frac{1}{i-1}\right)\sum_{j=1}^{i-1} j \\ &= \left(\frac{1}{i-1}\right)\left[\frac{(i)(i+1)}{2} - i\right] \\ &= \frac{(i)(i+1)}{2(i-1)} - \frac{i}{i-1} \end{aligned}$$

- $E(X)$

$$\sum_{i=2}^{n}\left(\frac{i}{2}\right) = O(n^2)$$

7. What if the data is not random?

Asymptotic Analysis (more details)

BIG-O Notation – Upper bound on growth of a runtime function

- $f(n) \in O(g(n))$ reads as "f(n) is big-O of g(n)"

If there exists C, n_0 such that $0 \leq f(n) \leq CG(n)$ when $n \geq n_0$ and $c > 0$.

1a. Use the definition of big-O to show $2n^2$ is big-O n^3 (find a C and n_0 that works in the above)

$$\begin{aligned} f(n) &< cg(n) \\ 2n^2 &\leq cn^3 \text{ when } n > n_0(n > 1) \text{ is } 2n^2 < cn^3 \\ 2(2^2) &\leq c(2^3) \end{aligned}$$

1b. Use the definition of big-O to show $T(n) = 3n^3 - 4n^2 + 3\lg n - n = O(n^3)$

$$0 < 3n^3 - 4n^2 + 3\lg n - n < cn^3 (\text{ find } c > 0 \text{ and } n_0 > 0 \text{ such that } n > n_0)$$

$$\begin{aligned} 3n^3 - 4n^2 + 3\lg n - n? &< 3n^3 \\ -4n^2 + 3\lg n - n? &< 0 \\ \forall n > 0, -4n^2 &\text{ is negative} \end{aligned}$$

Note: The ? < means is this true

Omega Ω Notation – Lower Bound $g(n) = \Omega(g(n))$
$0 \leq cg(n) \leq f(n)$
$c?$ $n_0 < n$

2. Use the definition of omega to show $n^{\frac{1}{2}} = \Omega(\log n)$

$$n > n_0, c > 0$$
$$c \lg(n) \leq n^{\frac{1}{2}}$$
$$n = 4, c \log_2(4)? \leq 4^{\frac{1}{2}}$$
$$2c \leq 2$$
$$n = 16, c \log_2(16)? \leq 16^{\frac{1}{2}}$$
$$4c \leq 4$$
$$2^{clog_2 n} \leq 2^{\sqrt{n}}$$
$$2^c 2^{log_2 n} \leq 2^{\sqrt{n}}$$
$$n2^c \leq 2^{\sqrt{n}}$$

Theta θ Notation – Strict Bound $f(n) = \theta(g(n))$
$c_1 g(n) \leq f(n) \leq c_2 g(n)$
$c_1? < c_2?$

3a. Use the definition of theta to show $3n^3 - 4n^2 + 37n = \theta(n^3)$

$$c_1 n^3 \leq 3n^3 - 4n^2 + 37n \quad \leq c_2 n^3$$
$$2n^3? \leq 3n^3 - 4n^2 + 37n$$
$$0? \leq n^3 - 4n^2 + 37n$$
$$\text{True for } n > 0$$
$$3n^3 - 4n^2 + 37n \leq c_2 n^3$$
$$3n^3 - 4n^2 + 37n \leq 3n^3$$
$$-4n^2 + 37n \leq 0$$
$$n(-4n + 37) \leq 0 \ \forall n \geq 10$$
$$c_1 = 2, n_1 = 0, c_2 = 3, n_2 = 10$$

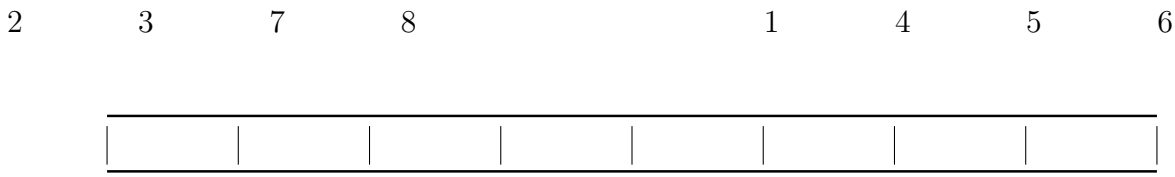
3b. Use the definition of theta to show $n^2 + 3n^3 = \theta(n^3)$

$$c_1 n^3 \leq n^2 + 3n^3 \quad \leq c_2 n^3$$
$$c_1 n \leq 1 + 3n \leq \quad c_2 n$$
$$c_1 = 2 \quad c_2 = 4 \ \forall n > 1$$

Recursive Sorting – Mergesort

- divide and conquer (and combine) approach, recursive algorithm
- key idea: you can merge sorted lists of total length n in $\theta(n)$ linear time
- base case: a list of length one element is sorted
- For all sorting algorithms, the base case is $n = 1$.

1. Demonstrate how you can merge two sorted sub-lists total n items with n compares/copies. How much memory do we need to do this? Write pseudocode to do this.



Algorithm 3.1 Merge algorithm

```
1:  function MERGE(A, i, j, k) ▷ i is the index of the lower sorted sequence, j is the index of the upper sorted, k is the end
2:      B ← array of size k
3:      l ← 0
4:      i_end ← i-1
5:      while i < i_end && j ≤ k do
6:          if A[i] < A[j] then
7:              B[l] ← A[i]
8:              i++
9:              l++
10:         else
11:             B[l] ← A[j]
12:             j++
13:             l++
14:         end if
15:     end while
16: end function
```

Algorithm 3.2 Merge sort algorithm

```
1: function MERGESORT(A, p, r)                ▷ Initial call Mergesort(A, 1, n)
2:     if p < r then
3:         q ← (p+r)/2                        ▷ Integer division
4:         MergeSort(A, p, q)                 ▷ Recursively sort 1st half
5:         MergeSort(A, q + 1, r)             ▷ Recursively sort 2nd half
6:         Merge(A, p, q, r)                  ▷ Merge 2 sorted sub-lists
7:     end if
8: end function
```

2. Demonstrate Mergesort on this data:

	3	41	52	26	38	57	09	49
1.	3(p=1, q=4)	41	52	26	38	57	09	49(
2.	3(p=1, q=2)	41	52	26(r=4)				
3.	3(p=1, q=1)	41 (r=2)						

I don't even know how I can write this part in LaTeX

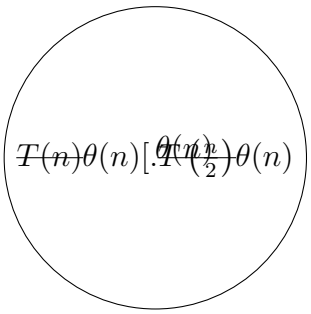
1. A recurrence relation describes runtime function recursively for a recursive algorithm. Write a recurrence relation for the Merge sort algorithm. HINT: try to count the number of executions of each statement and the cost of each Recurrence relations for merge sort lines: $2 = c_1, 3 = c_2, 4 = 5 = T\left(\frac{n}{2}\right)$ and $O(n)$ as the runtime for merge

$$T(n) = c_1 + c_2 + T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + O(n)$$
$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

Solving Recurrence Relations – Recurrence Tree Method

We solve a recurrence relation to get a function in its closed (non-recursive) form. The recurrence tree method is a visual method of repeatedly substituting in the recurrence relation for $T(n)$ on smaller and smaller n until you reach the base case, and then summing up all the nodes in the tree.

2. Draw the recurrence tree for Mergesort $T(n) = 2T\left(\frac{n}{2}\right) + \theta(n)$, $T(1) = O(1)$



Divide and Conquer Algorithms

- Divide – divide the problem into sub-problems that can be solved independently
- Conquer – recursively solve each sub-problem
- Combine – possibly necessary, combine solutions into sub-problems

Not all problems can be solved with the divide and conquer approach. Maybe sub-problems are not independent, or solutions to sub-problems cannot be combined to find solution to main problem.

3. Write a recursive algorithm for Binary Search. Write and solve its recurrence relation.Divide & Conquer & Combine Runtime analysis:

Algorithm 4.1 Binary Search Algorithm

```
1: function BS(A,key,i, j)                ▷ Initial call: BS(A (sorted), key, 1, n)
2:     if i ≤ j then                      ▷ Handle Base case not found
3:         k ← ⌊(i+j)/2⌋
4:         if A[k] == key then
5:             return key
6:         end if
7:     end if
8: end function
```

$$T(n) = O(1) + T\left(\frac{n}{2}\right)$$
$$T(1) = O(1)$$

4. Write a recursive algorithm for Selection Sort (or insertion sort or bubble sort). Write and solve its recurrence relation.

Algorithm 4.2 Selection Sort Algorithm

```
1: function SELECT(A, i, j)      ▷ Not a stable sorting algorithm. Initial call: Select(A, 1, n)
2:     if i < j then                                  ▷ Base case, 1 item is sorted.
3:         minSoFar = A[i]
4:         kk=i
5:         for k=i+1; k≤j; k++ do
6:             if A[k] < minSoFar then
7:                 minSoFar ← A[k]
8:                 kk ← k
9:             end if
10:        end for
11:        Swap A[i] with A[k]
12:        Select(A, i+1, j)
13:    end if
14: end function
```

Runtime analysis: $T(n) = T(n-1) + O(n)$, $T(1) = O(1)$

5. Describe an efficient divide and conquer algorithm to count the number of times a character appears in a string of length n.

Algorithm 4.3 Count of char in string

```
1:  function FIND_COUNT(S, from, to, char)
2:      if from < to then
3:          mid ← (from+to)/2
4:          x ← Find_Count(A, from, mid, char)
5:          y ← Find_Count(A, mid+1, to, char)
6:          return x + y
7:      else                                          ▷ 1 character left
8:          if S[from]==char thenreturn 1
9:          elsereturn 0
10:         end if
11:     end if
12: end function
```

Inductive Proofs

(needed in next lecture to prove a solution to a recurrence relation)

6) What are the three steps in an inductive proof?

- Prove for base case
- Assume true for n, prove for larger n

7) Use an inductive proof to show the sum of the first n integers is $\frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1(2)}{2}$$

$$1 = 1 \sum_{k=1}^{m} k \qquad = \frac{m(m+1)}{2}$$

$$\sum_{k=1}^{m+1} k = \frac{(m+1)(m+2)}{2}$$

$$\sum_{k=1}^{m} k + \sum_{m+1}^{m+1} = \frac{(m)(m+1)}{2} + (m+1)$$

$$= \frac{(m)(m+1)}{2} + (m+1)$$

Opening Questions

1. Define Big-O, Omega and Theta notation

2. In your own words explain what a recurrence relation is, what do we use recurrence relations for, why do we solve recurrence relations?

Recurrence Relation Solution Approach—Guess and prove by induction

Guess (or are given Hint) at form of solution, probe it is the solution

- Using definition of BIG-O or θ
- Using induction
  - Probe Base Case (if boundary condition given)
  - Assume true for some n
  - Prove true for a larger n

Example:

$$T(n) = 4T(n/2) + 2 \text{ guess } T(n) = O(n^3) \text{ ??}$$

Assume $T(k) \leq ck^3$ for some $k < n$, use assumption with $k = n/2$, then prove it for $k = n$
$T(n/2) \leq c(n/2)^3$ merge with recurrence

$$T(n) \leq 4c(n/2)^3 + n$$
$$T(n) \leq \frac{c}{2n^3} + n$$
$$T(n) \leq cn^3 - \left(\frac{c}{2n^3} - n\right)$$
$$\left(\frac{c}{2n^3} - n\right) > 0 \text{ if } c \geq 2 \text{ and } n > 1$$
$$T(n) \leq cn^3 - \text{(something positive)}$$
$$T(n) \leq cn^3$$
$$T(n) = O(n^3)$$

1. $$T(n) = 4T(n/2) + n^3 \text{ guess } T(n) = \Theta(n^3)??$$

2. $$T(n) = 4T(n/2) + n \text{ guess } T(n) = O(n^2)??$$

Recurrence Relation Solution Approach - Iteration Method (repeated substitution)

Convert the recurrence relation to summation using repeated substitution (Iterations)

- Keys to Iteration Method
  - # of times iterated to get $T(1)$
  - Find the pattern in terms and simplify to summation

EXAMPLE

$$
\begin{aligned}
T(n) &= 4T\left(\frac{n}{2}\right) + n \\
T(n) &= n + 4T\left(\frac{n}{2}\right) \\
&= n + 4\left(4T\left(\frac{n}{4}\right) + \frac{n}{2}\right) \\
&= n + 4\left(\frac{n}{2} + 4\left(\frac{n}{4} + 4T\left(\frac{n}{8}\right)\right)\right) \\
&= (n + 2n + 4n + 64)T\left(\frac{n}{8}\right) \\
T(n) &= n + 2n + 4n + \cdots + 4^{\lg_2(n)}T(1) \\
&= n \sum_{k=0}^{\lg(n-1)} 2^k + \theta(n^2) \\
&= n\left(\frac{2^{\lg(n)} - 1}{2 - 1}\right) \\
T(n) &= n^2 - n \\
&= O(n^2)
\end{aligned}
$$

3. $T(n) = 3T(n/3) + \lg n$ proof by iteration/repeated substitution

4. $$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

5. $$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n^2$$

6. $$T(n) = T(n-1) + n$$

Recurrence Relation Solution Approach – Master Method

For solving recurrences of the form $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Compare growth of $f(n)$ to $n^{\log_b(a)}$

Case 1)

$$f(n) \leq cn^{\log_b(a)} \; T(n) = \Theta(n^{\log_b(a)})$$

Case 2)

$$c_1 n^{\log_b(a)} \leq f(n) \leq c_2 n^{\log_b(a)} \; T(n) = \Theta(n^{\log_b(a)} \lg(n))$$

Case 3)

$$f(n) \geq cn^{\log_b(a)} \; T(n) = \Theta(f(n))$$

7. $$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

8. $$T(n) = 7T\left(\frac{n}{2}\right) + O(n^2)$$

9. $$T(n) = 4T\left(\frac{n}{2}\right) + n^3$$

Opening Questions

1. Mergesort is $\Theta(O(n \lg n))$ runtime in best case, worst case and average case. How much memory is needed for Mergesort on input size n? $O(n)$

2. Mergesort does all the work of sorting items in the Merge function, after recursively splitting the collection down to the base case. Briefly explain the difference with Quicksort. All the work for Mergesort is being done after the recursive calls, where as Quicksort does the work before the recursive calls.

Quicksort

A recursive divide and conquer algorithm.

- base case: a list of length one element is sorted.
- Divide "Partition" array into 2 sub-arrays with small #'s in the beginning, large #'s in second and known index dividing them What defines small and large? The dividing index may not be the middle.
- Conquer - recursively sort each sub-array
- Combine - Nothing to do

1. Write pseudocode for Quicksort (similar to Mergesort).

Algorithm 6.1 Quicksort

```
1: function QSort(A, p, r)                    ▷ The initial call is QSort(A, 1, n)
2:     if p < r then
3:         q ← Partition(A, p, r)    ▷ Moves small to left and large to right, q is the seperator
4:         QSort(A, p, q-1)                   ▷ The value at A[q] is the pivot, q is the idnex
5:         QSort(A, q+1, r)
6:     end if
7: end function
```

Partition idea (you should be able to do this in place): pick the last element in the current array as the "pivot", the number used to decide large or small. Then make a single pass of the array to move the "small" numbers before the "large" numbers and keep the "large" numbers after the "small" numbers. Then put the "pivot" between the two subarrays and return the location of the pivot.

2. Write iterative pseudocode for Partition. How much memory is needed?

Algorithm 6.2 Partition: Worst case runtime $O(n)$

```
1:  function PARTITION(A,p,r)
2:      pivot ← A[r]
3:      i ← p − 1
4:      for j=p dor-1
5:          if A[j] ≤ pivot then
6:              Swap A[i+1] with A[j]
7:              increment i
8:          end if
9:      end for
10:     Swap A[i+1] with A[r]
11:     return i+1
12: end function
```

Quicksort Runtime:

$$\begin{aligned} T(n) &= O(n) + T(\text{small items}) + T(\text{large items}) \\ T(1) &= O(1) + (O \leftrightarrow n-1) + (n-1 \leftrightarrow O) \end{aligned}$$

If All Items $<$ Pivot

$$\begin{aligned} T(n) &= O(n) + T(n-1) + T(0) \\ &= O(1) + T(1-1) + O(1) \\ &= O(1) + T(0) + O(1) \\ &= O(1) \end{aligned}$$

$$\begin{aligned} T(n) &= O(n) + T(n-1) \\ &= O(n) + O(n-1) + T(n-2) \\ &= O(n) + O(n-1) + O(n-2) + T(n-3) \\ &= O(n) + O(n-1) + O(n-2) + O(n-3) + \cdots + T(1) \\ &= O(n) + O(n-1) + O(n-2) + O(n-3) + \cdots + O(1) \\ &= O\left(n^2\right) \end{aligned}$$

If Pivot is Median

$$
\begin{aligned}
T(n) &= O(n) + 2T\left(\frac{n}{2}\right) \\
&\quad \text{(Master method: } a = 2, b = 2, f(n) = O(n)\text{)} \\
&= n^{\log_b(a)} \\
&= n^{\log_2(2)} \\
&= n^1 \\
&= O(n \lg n)
\end{aligned}
$$

What if pivot divides items $\frac{1}{10}$ on oneside and $\frac{9}{10}$ on other

$$
\begin{aligned}
T(n) &= T\left(\frac{n}{10}\right) + T\left(\frac{n}{10}\right) + O(n) \\
&\dots \\
&= \log_{\frac{10}{9}}(n) \\
&= O(n \lg n)
\end{aligned}
$$

As long as the split is some sort of ratio, you get $O(n \lg n)$ as the runtime for quicksort. Only in the rare case of all the data being on one side of the partition that gives $O(n^2)$ as the partition. To fix this, pick 3 items randomly, chose the middle of those 3, swap that middle with the last index and you'll be guaranteed to have $O(n \lg n)$.

Algorithm 6.3 Partition Improved

```
1:  function PartitionI(A,p,r)
2:      b, c, d are random indexes from A.
3:      Swap A[r] with Median(A[b], A[c], A[d])
4:      pivot ← A[r]
5:      i ← p − 1
6:      for j=p dor-1
7:          if A[j] ≤ pivot then
8:              Swap A[i+1] with A[j]
9:              increment i
10:         end if
11:     end for
12:     Swap A[i+1] with A[r]
13:     return i+1
14: end function
```

3. Demonstrate Partition on this array.
4. What do you think the best possible outcome would be for a call to Partition, and why? What about worst possible outcome?
5. Write (and solve) recurrence relations for Quicksort in the best case partition and worst case partition.
6. What is there is a pretty bad, but not awful, partition at every call. Try always a 9 to 1 split from partition. Write and solve recurrence relation.

With all the other sorts we could describe a particular input order that would yield worst case run time.

7. How can we avoid a particular input order yielding worst case run time for quicksort?

Visual Sorting Software By A. Alegoz, previous CS430 student (1.8Mb zipped, Win only)http://www.cs.iit.edu/~cs430/IITSort.zip

Opening Questions

1. Explain the difference between the Binary Search Tree Parent-Child value relationship and the Heap Property Parent-Child value relationship. BST: the right node is larger than the parent node and the left node is smaller than the parent node. In Heaps, the parent is either larger than both children (MaxHeap) or smaller than both children (MinHeap).

2. Binary search trees are a dynamic data structure that uses left-child and right-child pointers to represent the tree. How is this different from a heap? In a heap, you can take the index of a node, apply $\lfloor \frac{\text{node}}{2} \rfloor$ to get the parent node. So you can wall up or down a heap whereas a binary tree can only walk down.

Heaps

Since a heap is a nearly complete binary tree and will always grow and shrink the rightmost bottom leaf, you can implement a heap with an array instead of needing pointers (as is needed for a binary search tree which can grow/shrink at any node. Example of a MaxHeap showing array implementation:

1. At what index position is the largest element in a MaxHeap? We have to know how to easily move around a Heap. Can you devise a formula to relate the index of a parent to the indexes of its children? How about a formula for the index of a child to the index of its parent? The largest element in a Max Heap is at index 1 (given 1-based indexing). The formula is index_parent $= \left\lfloor \frac{\text{child\_index}}{2} \right\rfloor$.

2. If a heap was one larger, where does the tree have to gain a node from when done? If a heap was one smaller, where does the three have to lose a node when done?

3. Considering your answer to #2, try to devise a way to ExtractMax from this maxheap. What is the runtime in terms of heapsize?

Algorithm 7.1 ExtractMax

```
1: function ExtractMax(heap)                    ▷ 1 based indexing, not 0
2:     Save the value at the root
3:     Move heap[heapsize] to heap[1]
4:     MaxHeapify(heap)
5: end function
```

Save the current value at the root. Then move 5 (which is at the heapsize index) to the root, and then heapify.

4. Considering your answer to #2, try to devise a way to Insert(20) into this maxheap. What is the runtime in terms of heapsize?

Both the above ExtractMax and Insert assumed we already had a heap. To efficiently build a heap, we put all the items to insert in an array. Call MaxHeapify (walk value down) from index $\frac{\text{heapsize}}{2}$ up to the root (index 1).

5. Write pseudocode for this BuildHeap algorithm and demonstrate on this data. What is the runtime in terms of n, the size of B? $B = [15\ 8\ 4\ 9\ 3\ 16]$ B is not a max heap.

Algorithm 7.2 BuildHeap

```
1: function BuildHeap(heap)
2:     MaxHeapify(heap, 1, heapsize/2) ▷ Precondition: Both children of that index are valid max heaps
3:                                     ▷ Any index position larger than heapsize/2 is a leaf→ valid max heap
4: end function
```

Runtime: must call $MaxHeapify$: $\frac{heapsize}{2}$ times. $MaxHeapify$ in the worst case runs at $O(\lg heapsize) = O(\lg n)$. However, working at $heapsize$, at most $MaxHeapify$ works once (incase the only child is larger than the node).

$$\begin{aligned}\theta(n) &= \lg(n) + 2\lg(n-1) + 4\lg(n-2) + \cdots + \left(\frac{n}{4}\right)(1) + \frac{n}{2}(0) \\ &= \text{linear time.}\end{aligned}$$

6. Write the loop invariant for BuildHeap and prove that it works.After the jth iteration (index of the item we call MaxHeapify on: (range $\frac{n}{2} \to 1$). Before call on MaxHeapify on jth iindex in the heap $\frac{n}{2} \to 1$, $\frac{n}{2} + 1 \to n$ are max heaps, which means child of j index are max heaps.
INIT: We know all index positions $> n/2$ are leafs or empty, therefore, they are already maxheaps.
MAINT: If $p(j) \to p(next\_j)$

7. How can we use a maxHeap and extractMax to sort? Build the heap in $O(n)$ time, then extractMax which saves the root, then sorts the heap. The initially saved root gets moved to another array into the index $heapsize - 1$. Then extract max is run again, and the saved value is moved into $heapsize - 2$. The runtime is $O(n \lg n)$.

Opening Questions

- If we have 17 distinct items to sort, what is the height of the decision tree that represents all possible orderings of the 17 items? The height of the decision tree is $\lceil \log_2(17!) \rceil = \lceil 48.337603311133 \rceil = 49$ and the possible orderings of the 17 items is 17!. However, $\log_2(n!) = \theta(n \lg n)$.

Comparison Sorts

The sorted order they determine is based only on comparisons between the input elements. That is, given two elements a_i and a_j, we perform one of the tests $a_i < a_j$, $a_i = a_j$, or $a_i > a_j$ to determine their relative order.
Claim: We must make at least $\Omega(n \lg n)$ comparisons in the general case.
The decision-tree model

- Abstraction of any comparison sort.
- Represents comparisons made by a specific sorting algorithm on inputs of a given size.
- Abstracts away everything else: control and data movement
- View the tree as if the algorithm splits in two at each node, based on the information it has determined up to that point; The tree models all possible execution traces.

Each internal node is labeled by indices of array elements **from their original positions**. Each leaf is labeled by the permutation of orders that the algorithm determines.

1. How many possible permutations of n items are there (at least one of them must be sorted)? In the decision tree model, how many leaves are there for n items? For a binary tree to have that many leaves, how many levels does it need? Why do we care about how many levels are in the decision tree? There are n! possible permutations. A binary tree would have 2^{height} levels.

Linear Time Sorting – Counting Sort

If we do not use comparison of keys to sort data, how can we sort data? It seems comparing items is necessary for sorting.

- Counting sort assumes that each of the n input elements is an integer in the range 0 to k, for some integer k. When $k = O(n)$, the sort runs in $T(n)$ time (best if the size of the range is $k \lll n$)
- For each input element x, count how many elements are equal to x then use this to count how many elements are $\leq x$. This information can be used to place element x directly into its position in the output array.
- Does not sort in place, needs 2^{nd} array size n and 3^{rd} array size k

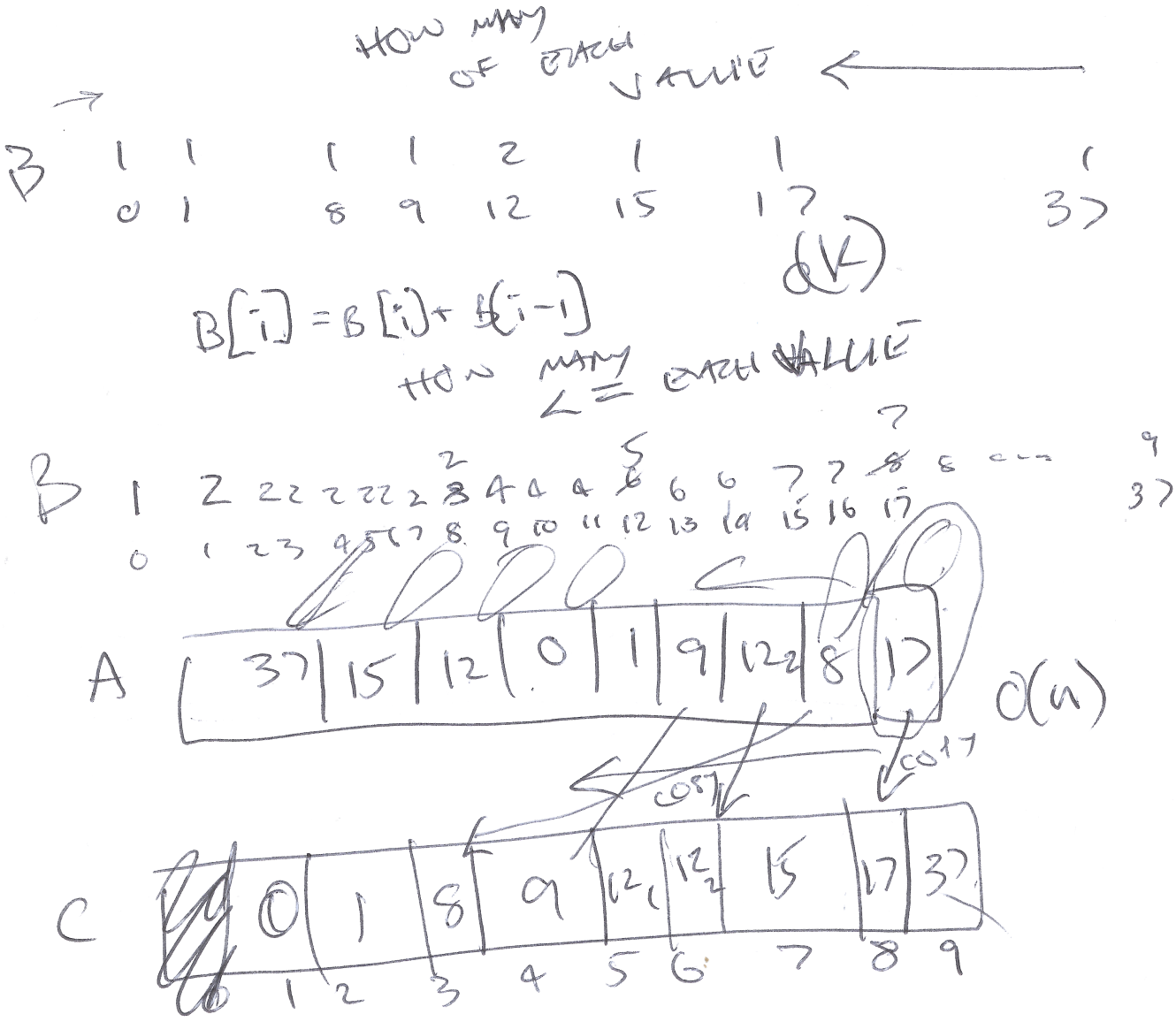


Figure 8.1: Counting sort demonstration.

2. Write pseudocode for counting sort.

Algorithm 8.1 Counting Sort

```
1: function CountingSort(A)  ▷ A=original array, B=size-k counters, C=size-n sorted data
2:     for i = 1...n do
3:         B[A[i]]++                                              ▷ counters
4:     end for
5:     for j = 1...k do
6:         B[j] = B[j] + B[j − 1]                                 ▷ ≤ counters
7:     end for
8:     for i = n...1 do
9:         C[B[A[i]]] = A[i]
10:        B[A[i]]--
11:    end for
12: end function
```

3. Demo counting sort on this data:

2	5	3	0	2	3	0	3

Counting sort is best on large amounts of data where the range of the data is small. Like many single digit numbers. However, we can use counting sort multiple times to sort multiple digit numbers, starting with the right most digit, etc. This is called Radix Sort.

Table 8.1: Radix Sort

329		72**0**		7**2**0		3**29**	
457		35**5**		3**2**9		3**55**	
657		43**6**		4**3**6		4**36**	
839	→	45**7**	→	8**3**9	→	4**57**	
436		65**7**		3**5**5		6**57**	
720		32**9**		4**5**7		7**20**	
355		83**9**		6**5**7		8**39**	

Each $O(k)$ where k is $0 \to 9$. This makes the overall $O = O(n+\text{digits}*k) = O(n+10k)$, but k is insignificant to n, so it's essentially $O(n)$.

An important property of counting sort is that it is **stable**: numbers with the same value appear in the output array in the same order as they do in the input array. That is, it breaks ties between two numbers by the rule that whichever number appears first in the input array appears first in the output array. Normally, the property of stability is important only when satellite data are carried around with the element being sorted. Counting sort's ability is important for another reason: counting sort is often used as a subroutine in radix sort. As we shall see in the next section, in order for radix sort to work correctly, counting sort must be stable.

4. Which of these sorts can be stable:

- Insertion Sort
- Selection Sort ← you could potentially jump an item passed its duplicate
- Bubble Sort
- Merge Sort ← depends on the merge
- Quick Sort ← the paritioning might not be stable and an item can be jumped passed its duplicate.
- Heap Sort

Opening Questions

1. Order Statistics: Select the ith smallest of n elements (the element with rank i)
   - $i = 1$: minimum; 1^{st} order statistic, $O(n)$
   - $i = n$: maximum; n^{st} order statistic, $O(n)$
   - $i = \lfloor \frac{n+1}{2} \rfloor$ or $\lceil \frac{n+1}{2} \rceil$: median Sort: $O(n \lg n)$, then take $A\left[\frac{n}{2}\right]$ for the median, $O(1)$

How fast can we solve the problem for various i values?

Randomized Algorithm for finding the ith Element

1. Think about partition (with a random choice of the pivot"median of 3") from quicksort. Can you think of a way to use that and comparing the final location of the pivot to i, and then divide and conquer?

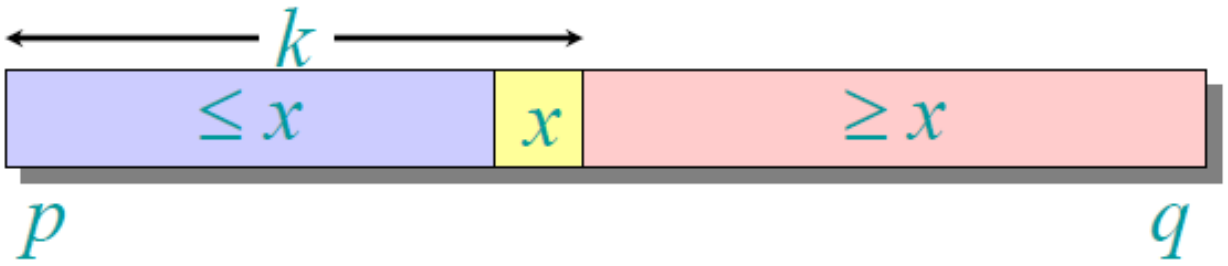


Figure 9.1:

Most cases $O(n)$, $A[k] = x$, the pivot. Three possibilities:

$i = k$ Done, $A[k]$ is the i smallest value

$i < k$ Redo Partition from indexes p to $k-1$, because we know that the ith smallest must be smaller than the pivot at index k, still looking for the ith smallest.

$i > k$ Redo Partition from indexes $k+1$ to q, not looking for the $i-k$ smallest, since we threw out k elements that we know are smaller than the ith element.

2. Demonstrate on this array to find i =th smallest element

6	10	13	5	8	3	2	11

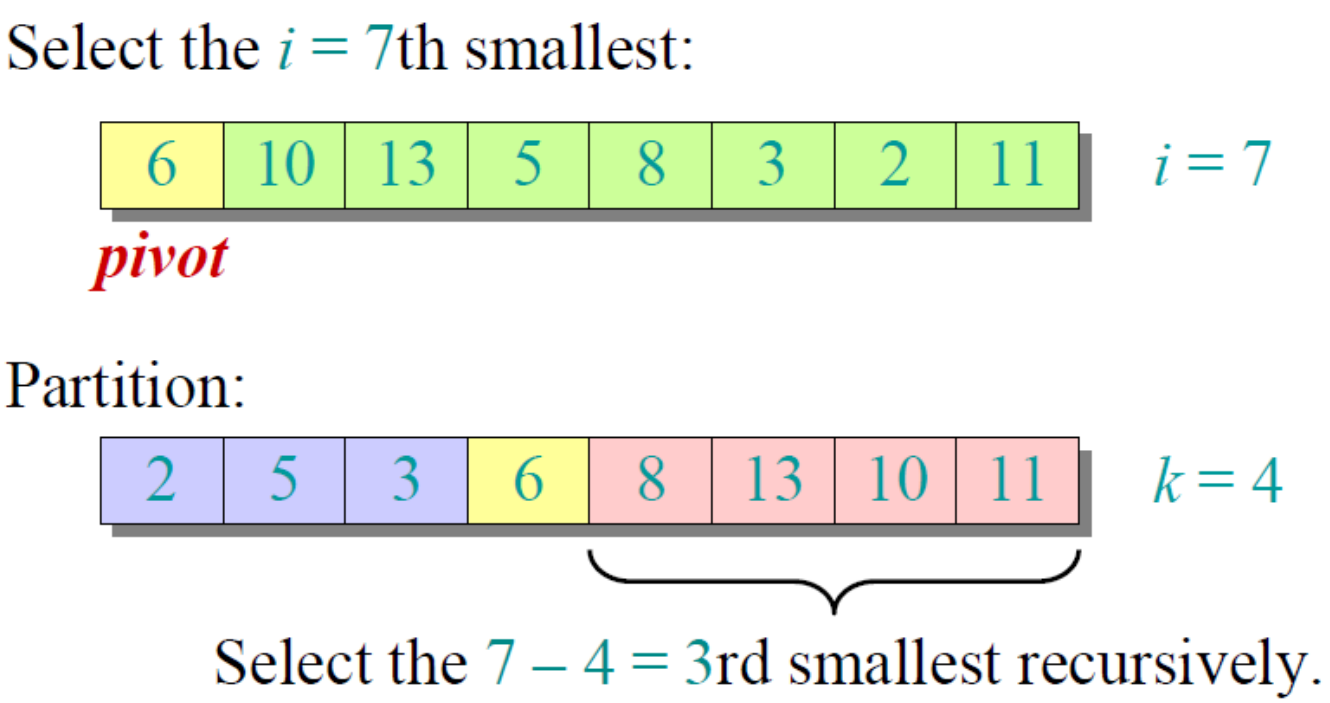


Figure 9.2:

Algorithm 9.1 Statistical Order to find the Smallest Element in Linear Time

```
1: function OrderStat(A, p, q, i)                ▷ Initial call: OrderStat(A, 1, n, i)
2:     k ← Partition(A, p, q)                    ▷ O(size A from p → q)
3:     if i = k then
4:         return A[k]
5:     else if i < k then
6:         return OrderStat(A, p, k − 1, i)
7:     else                                      ▷ i > k
8:         return OrderStat(A, k + 1, q, i − k)
9:     end if
10: end function
```

Runtime Analysis:

$$\begin{aligned} T(\text{size of subproblem}) &= T(n) \\ &= O(n) + T\left(\text{size of subproblem: } \frac{n}{2} \to n-1\right) \\ T(1) &= O(1) \\ \text{Master method}: T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\ a &= 1 \\ b &= 2 \\ f(n) &= n^{\log_2(1)} = 0 \\ &= 1 \\ \Theta(f(n)) &= \Theta(n) \end{aligned}$$

Worst case:

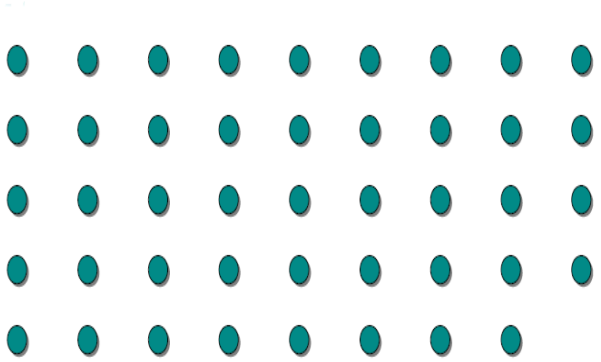
$$\begin{aligned} T(n) &= T(n-1) + O(1) \\ &= O\left(\frac{n(n-1)}{2}\right) \\ &= O\left(n^2\right) \end{aligned}$$

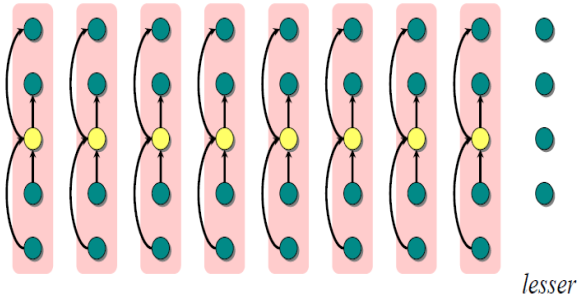
3. What is the worst case running time if you find the ith smallest element?
   Is there an algorithm to find the ith smallest element that runs in linear time in the worst case?

Algorithm 9.2 Select Smallest Element in Linear Time

```
1: function SELECT(i, n)
2:     Divide the n elements into groups of 5. Find the median of each 5-element group by hand.
3:     Recursively SELECT the median x of the ⌊n/5⌋ group medians to be the pivot.
4:     Partition around the pivot x. Let k =RANK(k).
5:     if i = k then
6:         return x
7:     else if i < k then
8:         Recursively SELECT the ith smallest element in the lower part
9:     else
10:        Recursively SELECT the (i − k)th smallest element in the upper part
11:    end if
12: end function
```

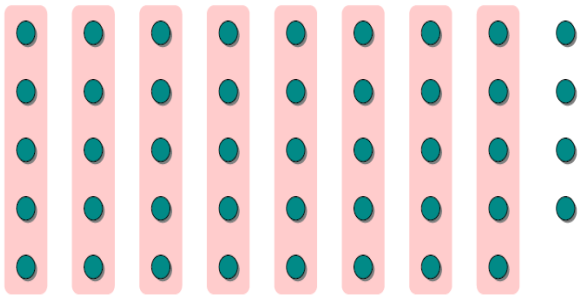
Choosing the pivot



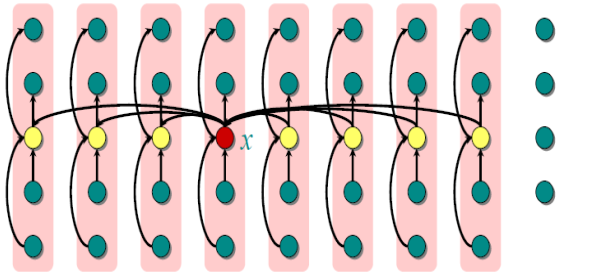


1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.

Choosing the pivot

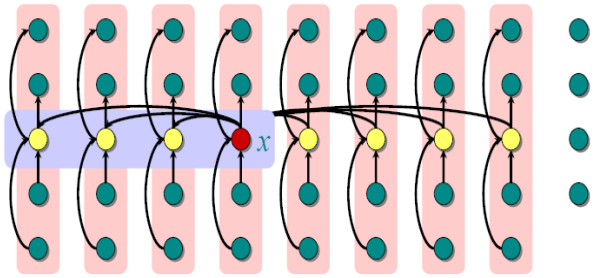


1. Divide the n elements into groups of 5.



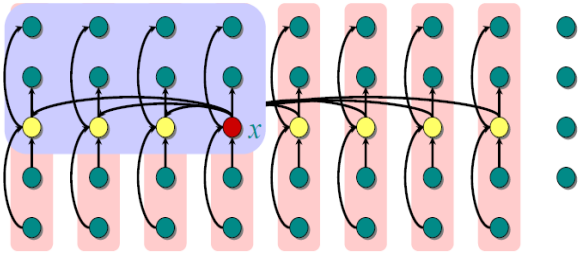
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

Analysis



At least half the group medians are $\le x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

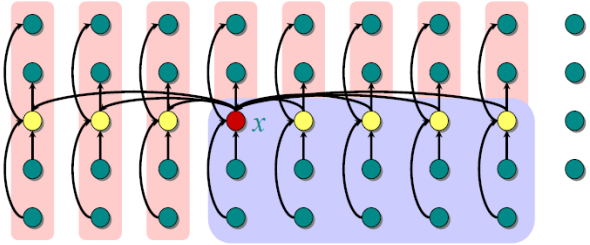
Analysis



At least half the group medians are $\le x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\le x$.

Analysis



At least half the group medians are $\le x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\le x$.
- Similarly, at least $3\lfloor n/10 \rfloor$ elements are $\ge x$.

Developing the recurrence

$T(n)$ SELECT(i, n)

$\Theta(n)$ 1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.

$T(n/5)$ 2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

$\Theta(n)$ 3. Partition around the pivot x. Let $k = \text{rank}(x)$.

$T(7n/10)$ 4. **if** $i = k$ **then return** x
elseif $i < k$
then recursively SELECT the ith smallest element in the lower part
else recursively SELECT the $(i–k)$th smallest element in the upper part

Solving the recurrence **Substitution:** $T(n) \leq cn$

$$\begin{aligned} T(n) &= T\left(\frac{1}{5}n\right) + T\left(\frac{7}{10}\right) + \Theta(n) \\ T(n) &\leq \frac{1}{5}cn + \frac{7}{10}cn + \Theta(n) \\ &= \frac{9}{10}cn + \Theta(n) \\ &= cn - \left(\frac{1}{10}cn - \Theta(n)\right) \\ &\leq cn \end{aligned}$$

if c is chosen large enough to handle the $\Theta(n)$.

In practice, this algorithm runs slowly, because the constant in front of n is large.

Would we use this approach to find the median to partition around in Quicksort, and achieve in worst-case $\Theta(n \log n)$ time? **No**, too many calls to $O(cn)$ Select with a big c. We can do this if we need to find the median once or a couple of times, but not inside of Partition.

Opening Questions

1. In your own words explain how you insert a new key in a binary search tree. When you have a value, you compare the value you're inserting to the value in the node. If the inserting value is less than the value, you recursively call the Insert function with the left node. If the value was greater, you recursively call Insert on the right node.

2. Give an example of a series of 5 keys inserted one at a time into a binary search tree that will yield a tree of height 5.

   5[.7[.9[.8[.8.5 The runtime of Insert is $O(h)$ where h is the height of the tree.

3. If we do a left rotate on node X below, explain which left and/or right child links need to be changed.

   X[.Y b c Y[.X a This tree maintains the relationships of the first tree. c was already larger than Y, which is maintained. b was smaller than Y but larger than X, which has also been maintained. But now the height of C has been decreased by 1.

Tree depth vs Tree Height

The length of the path from the root r to a node x is the depth of x in T. The height of a node in a tree is the number of edges on the longest simple downward path from the node to a leaf, and the height of a tree is the height of its root. The height of a tree is also equal to the largest depth of any node in the tree.

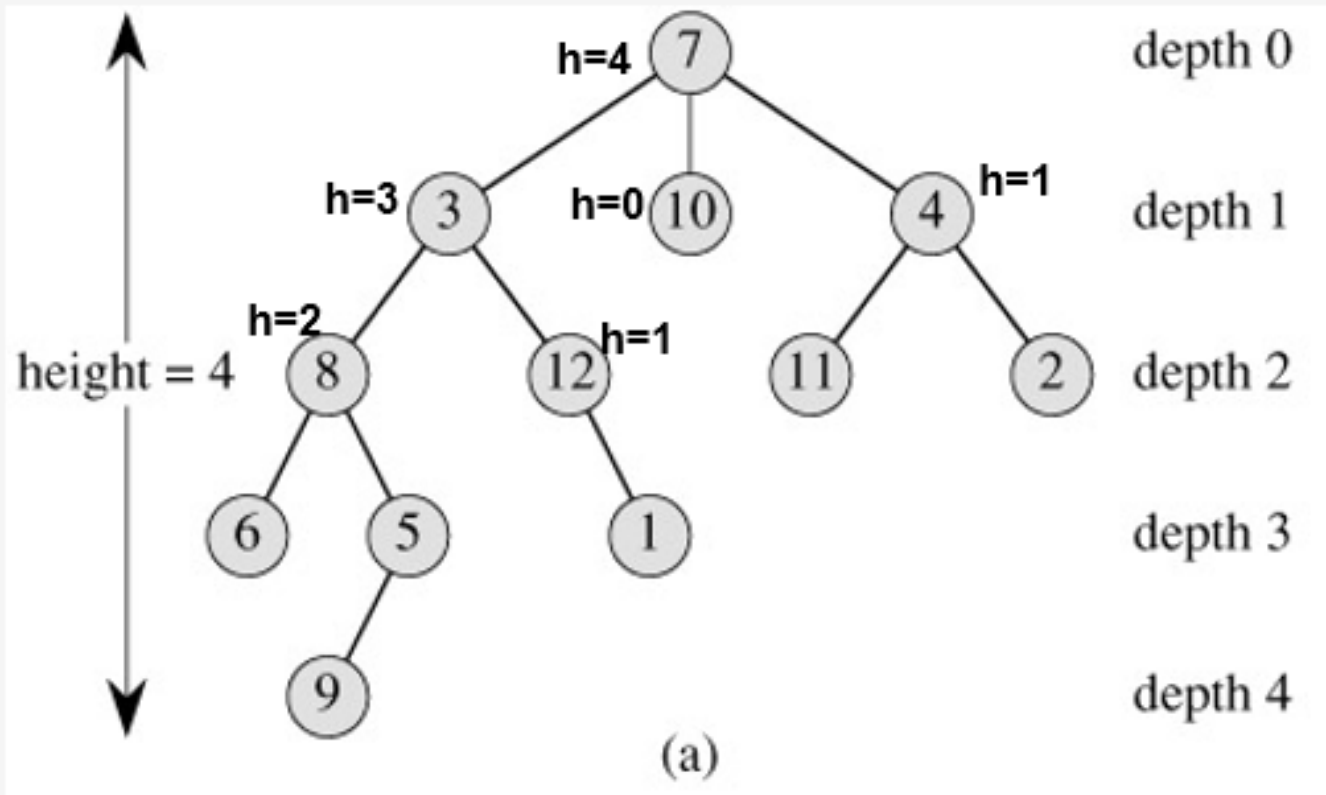
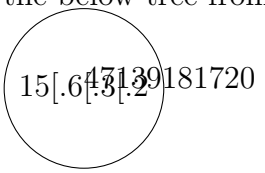


Figure 10.1: Height vs Depth of a Tree

BST Operations

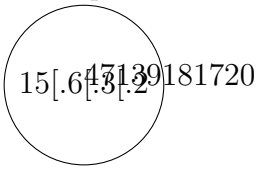
1) Write pseudocode for BST Successor (next larger value) (or Predecessor). Demonstrate on the below tree from node 15 and then node 13.



Algorithm 10.1 Binary Search Tree Successor (Best: $O(1)$, Worst: $O(h)$)

```
1: function SUCCESSOR(p: Node)  ▷ The Successor is usually the min of the graph in the right node, essentially, right than left-left-...-left.
2:     if p.right exists then
3:         return MIN(p.right)
4:     else
5:         go up until you go up on left pointer
6:         if never go up left then
7:             return null                          ▷ You started in the largest value.
8:         end if
9:     end if
10: end function
```

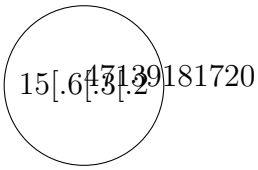
2) Write pseudocode for BST Insert. Demonstrate on the below tree to insert 5 and then 19.



Algorithm 10.2 Binary Search Tree Insert ($O(h)$)

```
1:  function INSERT(key)
2:      if root == null then
3:          root ← NODE(key)
4:          return
5:      end if
6:      p ← root
7:      q ← null                                  ▷ Following the parent node
8:      while p ≠ null do
9:          if p.val > key then
10:             q ← p
11:             p ← p.left                        ▷ Go Left
12:         else
13:             q ← p
14:             p ← p.right                       ▷ Go Right
15:         end if
16:     end while
17:     if q.val > key then
18:         q.left ← NODE(key)
19:     else
20:         q.right ← NODE(key)
21:     end if
22: end function
```

3) What are the three possible cases when deleting a node from a BST?



- No children
  - Set the $parent.left$ or $parent.right$ (depending on where you came from) to null; $O(1)$.
- One child
  - Make the parent of the child point to the child of the node you're deleting (essentially splicing out the node you're deleting); $O(1)$.

- Two children
  - Swap the predecessor or successor value into the node, and then deleting the value where the predecessor—successor was.

Algorithm 10.3 Delete A Node with 2 Children from a BST

```
1: function DELETE2CHILD(p: Node)
2:     successor ← SUCCESSOR(p)                ▷ Or Predeccesor, O(h)
3:     p.value ← successor.value               ▷ O(1)
4:     DELETE(successor)                       ▷ O(h)
5: end function
```

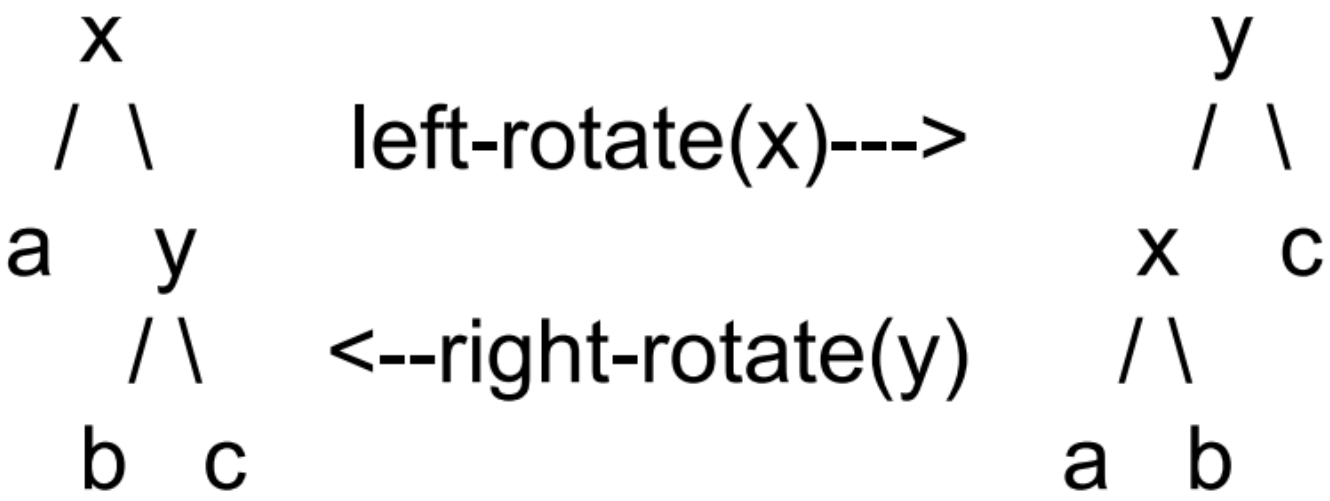
4) Write pseudocode for BST Delete (assume you already have a pointer to the node)

Algorithm 10.4 Binary Search Tree Delete

```
1: function DELETE(p: Node)                    ▷
2:
3: end function
```

BST Rotations

Local operation in a search tree that maintains the BST property and possibly alters the height of the BST.
x and y are nodes; a, b, c are sub trees



5. Write pseudocode for LeftRotate (or RightRotate). What is the worst case runtime?

Algorithm 10.5 Binary Search Tree Left Rotate

```
1: function LeftRotate(p:Node, z:Node)   ▷ To make this right rotate, switch all lefts with rights and rights with left.
2:     savey ← p.right                   ▷ z is the parent of p
3:     if z.left = p then
4:         z.left ← savey
5:     else
6:         z.right ← savey
7:     end if
8:     p.right ← z.left
9:     y.left ← p
10:    savey.left ← p
11: end function
```

Opening Questions

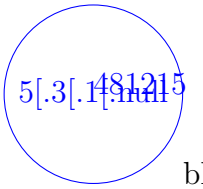
1. What is the main problem with Binary Search Trees that Red-Black Trees correct? Explain briefly (2–3 sentences) how Red-Black Trees correct this problem with Binary Search Trees. The biggest problem is that the tree may be a straight line that would cause the height to be $O(n)$. Red-Black trees maintain this balance by keeping track of the colors and allows the tree to be better balanced using a specific set of rules.

2. For the balanced binary search trees, why is it important that we can show that a rotation at a node is $O(1)$ (i.e. not dependent on the size of the BST) We want to make fixes that are not dependent on the size of a binary tree.

Red-Black Trees

Red-Black Properties

1. Every node is colored either red or black
2. The root is black
3. Every null pointer descending from a leaf is considered to be a null black leaf node
4. If a node is red, then both of its children are black
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes (black height)



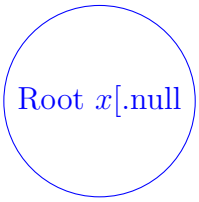
black-height of a node $bh(x)$ – The number of black nodes on any path from, but not including, a node x down to a null black leaf node. (Black height counts black nodes including null leaves, height doesn't count null nodes.)

1. If a node x has $bh(x) = 3$, what is its largest and smallest possible height (distance to the farthest leaf) in the BST? 2–5

2. Prove using induction and red-black tree properties. A red-black tree with n internal nodes (nodes with values), none null nodes (n key values) has height at most $2\lg(n+1)$

Part A) First show the sub-tree rooted at node x has at least $2^{bh(x)} - 1$ internal nodes. Use induction. Relating the black height of a node to how many values at least must be in that subtree.

Base Case: $bh(x) = 0$



This tree has $bh = 1$, with the only internal node being the root.

$$\begin{aligned} 2^{\text{bh(x)}} - 1 &= 2^0 - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Induction Step: Assuming at least $2^{bh(x)} - 1$ keys in some subtree rooted at x.
If a node is red, $bh((parent)) = bh((child))$ else a node is black, then the $bh((parent)) = bh((child)) + 1$.

$$\left(2^{bh(x)} - 1\right)\left(2^{bh(x)} - 1\right) + 1 = 2 \times \left(2^{bh(x)} - 1\right) + 1$$

Part B) Let h be height of a Red-Black Tree, by property four, at least half of the nodes on path from root to leaf are black

$$bh(root) \geq \frac{h}{2}$$

Use that and Part A to show

$$h \leq 2\log(n+1)$$

$bh(root) = k$ at least $2^k - 1$.

$$\begin{aligned} 2^{bh(root)} - 1 &\leq n \\ 2^{h/2} - 1 &\leq n \\ 2^{h/2} &\leq n + 1 \\ \frac{h}{2} &\leq \lg(n+1) \\ h &\leq 2\lg(n+1) \\ h &= O(\lg(n+1)) \end{aligned}$$

3. Which BST operations change for a red-black tree and which do not change? What do the operations that change need to be aware of and why?

- Search – Won't change, you'd ignore the colors
- Insert – Update
- Delete – Update
- Predecessor – Won't change
- Successor – Won't change
- Minimum – Won't change
- Maximum – Won't change
- Rotations – Update

Red-Black Tree Insert

Similar to BST Insert, assume we start with a valid red-black tree.

1. Locate leaf position to insert new node
2. Color new node red and create 2 new black null leafs below newly inserted red node
3. If parent of new insert was _____ (fill in the blank, black or red), then done. ELSE procedure to recolor nodes and perform rotations to maintain red-black-properties.

There are three cases if Red-Black Property #4 when insert a red node Z (or changed color of a node to red) and its parent is also red.

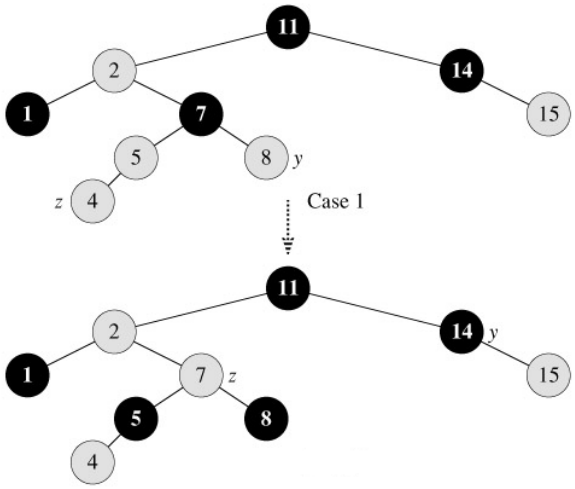


Figure 11.1: Broken Red-Black Property #4: Case #1

Node Z (red) is a left or right child and its parent is red and its uncle is red (the children of nodes value 4 5 8 must all be black, or null black). Swap the colors of a parent node and both its children, preserving the black height property at all nodes.

- Change Z's parent and uncle to black
- Change Z's grandparent to red
- No effect on black height on any node
- Z's grandparent is now Z and check again for property #4 (two reds in a row) still broken at new node Z (possible non-terminal case, need loop or recursion)

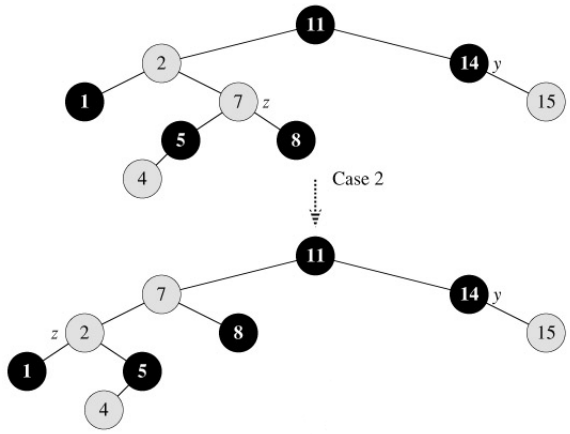


Figure 11.2: Broken Red-Black Property #4: Case #2

Node Z is a right child and its parent is red and its uncle is NOT red.
Do a single rotation, preserving the black height property at all nodes.

- Rotate left on parent of Z.
- Re-label old parent of Z as Z and continue to case #3.

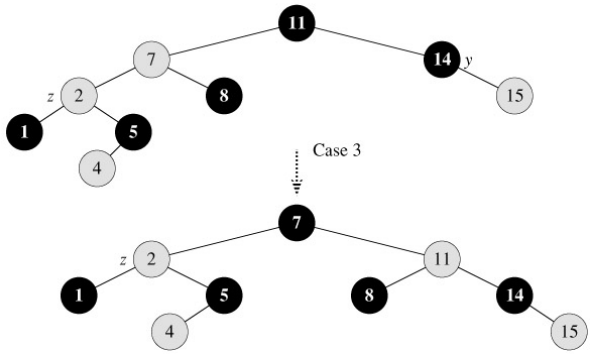


Figure 11.3: Broken Red-Black Property #4: Case #3

Node Z is a left child and its parent is red and its uncle is NOT red.
Do a single rotation and swap the colors of a parent node and both its children, preserving the black height property at all nodes.

- Rotate right on grandparent of Z
- Color old parent of Z black
- Color old grandparent of Z red

Opening Questions

1. What do you think the issue we need to handle when deleting a node from a red-black tree? How does red-black delete differ from a BST delete? If the actaul node deleted is red, then you're done. If the node you're deleting is black, then you potentially could have two red nodes in a row. If you color the deleted nodes child black (if it was red), then you're done. There would only be 0 or 1 children, not 2. If the child was already black, you add an extra "blackness" to temporarily fix the black-height problem, meaning the node counts for 2 black nodes.

Red-Black Tree Delete

- Think of V as having an "extra" unit of blackness. This extra blackness must be absorbed into the tree (by a red node), or propagated up to the root and out of the tree. There are four cases – our examples and "rules" assume that V is a left child. There are symmetric cases for V as a right child

Terminology in Examples

- The node just deleted was U
- The node that replaces it is V, which has an extra unit of blackness
- The parent of V is P
- The sibling of V is S

Figure 12.1: Red Black Tree Graphics Definitions

- V's sibling, S is Red
  - Rotate S around P and recolor S & P
- NOT a terminal case – One of the other cases with now apply
- All other cases apply when S is Black

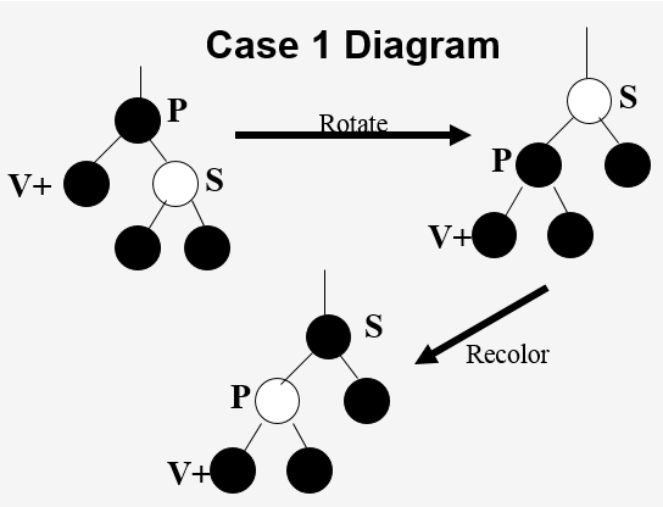


Figure 12.2: Red Black Tree Delete Case #1

- V's sibling, S is black and has two black children.
  - Recolor S to be Red
  - P absorbs V's extra blackness
    - If P was Red, make it black, we're done
    - If P was Black, it now has extra blackness and problem has been propagated up the tree

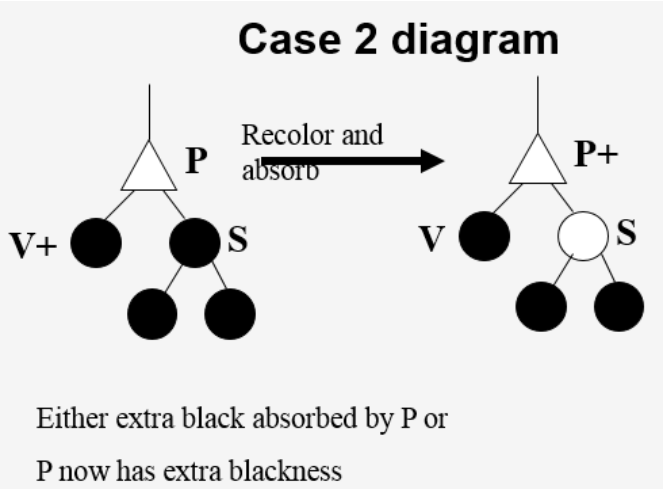


Figure 12.3: Red Black Tree Delete Case #2

- S is black
- S's RIGHT child is RED (Left child either color)
  - Rotate S around P
  - Swap colors of S and P, and color S's Right child Black
- This is the terminal case – we're done

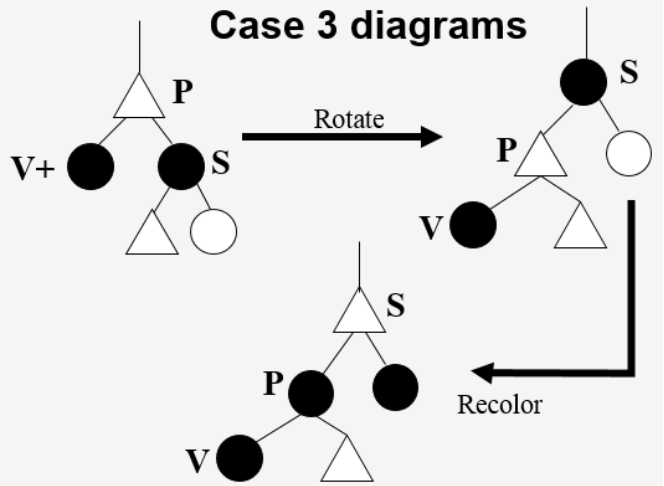


Figure 12.4: Red Black Tree Delete Case #3

- S is Black, S's right child is Black and S's left child is Red
  - Rotate S's left child around S
  - Swap color of S and S's left child
  - Now in case 3

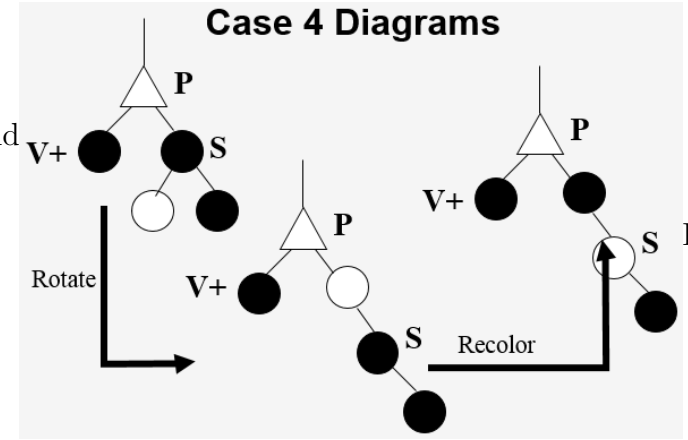


Figure 12.5: Red Black Tree Delete Case #4

Black Visualization:

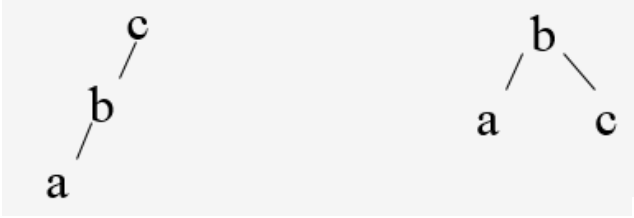
- http://gauss.ececs.uc.edu/RedBlack/redblack.html
- https://www.cs.usfca.edu/~galles/visualization/RedBlack.html

AVL Trees

An AVL tree is a special type of binary tree that is always "partially" balanced. The criteria that is used to determine the "level" of "balanced-ness" is the difference between the heights of sub-trees of every node in the tree. The "height" of the tree is the "number of levels" in the tree.AN AVL tree is a special binary tree in which the difference between the height of the right and left sub-trees (of any node) is never more than one.

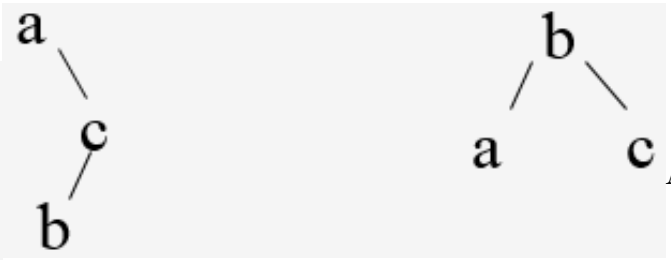
1. How do you think we could keep track of the height of the right and left sub-trees of every node?
2. If we find an imbalance, how can we correct it without adding any significant cost to the insert of delete?

Single Rotations
The imbalance is left-left (or right-right)



Perform single right rotation at c (R-rotation)
Similar idea for single left rotation (L-rotation)

Double Rotations
The imbalance is left-right (or right-left)



Perform right rotation at c then left rotation at a (RL-rotation)
Similar idea for left rotation then right rotation (LR-Rotation)

AVL

Visualization: https://visualgo.net/bn/bst

Opening Questions

Previously we discussed the order-statistic selection problem: given a collection of data, find the kth largest element. We saw that this is possible in $O(n)$ time for each kth element you are trying to find. The naïve method would just be to sort the data in $O(n \lg n)$ and then access each kth element in $O(1)$ time.

1. Which of these two methods would you use if you knew you would be asked to find multiple kth largest elements from a set of static data? If you're doing $\geq \lg n$ queries, it would be better to sort the data and then constantly choose the value based off of the index. If you're doing $< \lg n$, it would be better to run the $O(n)$ each time.
2. What if our collection of data is changing (dynamic), would either of these approaches work efficiently for a collection of data that has inserts and deletes happening? No, you would have to re-run either algorithm for either.

Augmenting Data Structures

For particular applications, it is useful to modify a standard data structure to support additional functionality. Sometimes the modification is as simple as by storing additional information in it, and by modifying the ordinary operations of the data structure to maintain the additional information.

Dynamic Order-Statistic Trees (Augmenting Balanced Trees)

1. What can we do with a binary search tree (and more efficiently with a balanced binary search tree)? We can insert and delete items, and if it's balanced, we can do that in $O(\lg n)$.

2. Consider the naïve method of finding the kth largest item is to sort the array and then access the kth item in $O(1)$ time. Can we do this with a (balanced) BST? What is we augment it (HINT: recall how counting sort worked)? A BST can do this by keeping track of the size of the subtree.

3. What is the recursive formula to find the size of a subtree at node x?

Algorithm 13.1 Size of a subtree at a node

```
1: function SubtreeSize(node)                    ▷ The size of null leaves is 0.
2:     size ← 1
3:     if node.left ≠ null then
4:         size ← size+ SubtreeSize(node.left)
5:     end if
6:     if node.right ≠ null then
7:         size ← size+ SubtreeSize(node.right)
8:     end if
9:     return size
10: end function
```

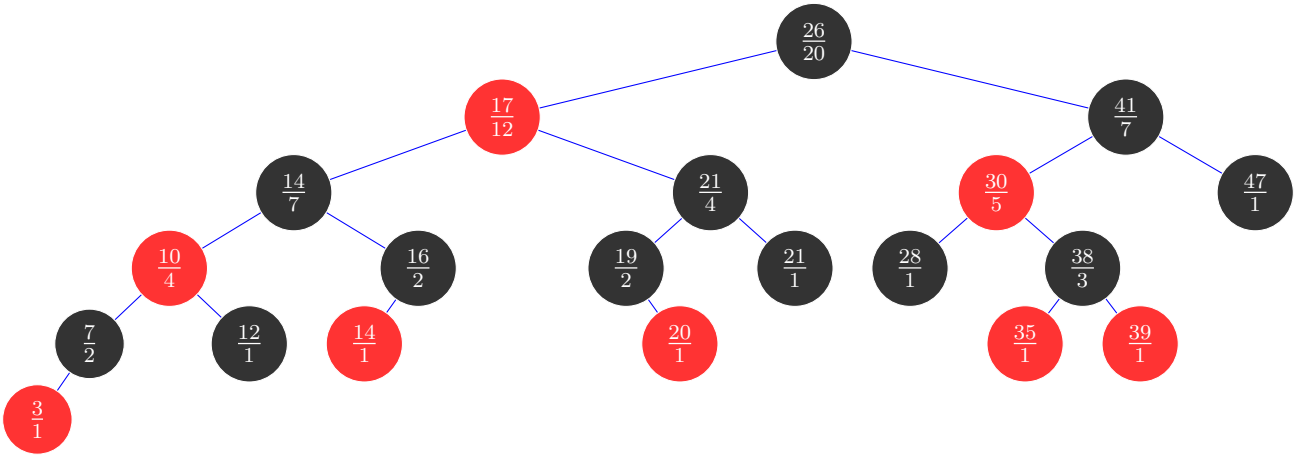


Figure 13.1: An order-statistic tree, which is an augmented red-black tree. In addition to its usual attributes, each node x has an attribute $x.size$, which is the number of nodes, other than the sentinel, in the subtree rooted at x.

4. Discuss in detail how would you keep the size at a node correct when you insert a new node, and possibly rotate to keep the tree balanced?

Algorithm 13.2 Left Rotation of a BST

```
1: function LEFT-ROTATE(T, x)
2:     y ← x.right                                  ▷ Set y
3:     x.right ← y.left ▷ Turn y's left subtree into x's right subtree
4:     if y.left ≠ T.nil then
5:         y.left.p ← x
6:     end if
7:     y.p ← x.p                                    ▷ link x's parent to y
8:     if x.p == T.nil then
9:         T.root ← y
10:    else if x == x.p.left then
11:        x.p.left ← y
12:    else
13:        x.p.right ← y
14:    end if
15:    y.left ← x                                   ▷ Put x on y's left
16:    x.p ← y
17: end function
18:
19:
```

Algorithm 13.3 Insert a node into a BST

```
1: function TREE-INSERT(T, z)
2:     y ← null
3:     x ← T.root
4:     while x ≠ null do
5:         y ← x
6:         if z.key < x.key then
7:             x = x.left
8:         else
9:             x = x.right
10:        end if
11:    end while
12:    z.p ← y
13:    if y == null then
14:        T.root ← z                               ▷ Tree T wa
15:    else if z.key < y.key then
16:        y.left ← z
17:    else
18:        y.right ← z
19:    end if
20: end function
```

After

you would add or delete the node from the BST, when you go back up the recursion, you can re-count the sizes after any rotations that may be necessary. (In any rotations, the only sizes that need to get updated are the nodes that are rotating, the size of the parent would stay the same.)

5. Discuss in general how would you keep the size at a node correct when you delete a node, and possibly rotate to keep the tree balanced? You can maintain these sizes on deletions and rotations. As the Red-Black deletion rules says, if the node you're deleting has 0 or 1 children, then the node was in the search path. If the node has two children, you would have to find the successor and update the search path.

6. How can we use the augmented data at each node (size) in a balanced binary search tree to solve the kth largest item problem? We can use the size of a node to check how many values are less than or greater than it (by subtracting the size of the subtree from the size of the root), which can help us determine if the required rank is within that subtree. If not, we can cut out that subtree to reduce the runtime.

7. How can we use the augmented data at each node (size) in a balanced binary search tree so when given a pointer to a node in the tree, we can determine its rank (the index position of the node of the tree data in sorted order)?

Algorithm 13.4 Order Statistic with a BST for the ith smallest

```
1: function SelectSmallest(x:Node, i:int)      ▷ Initial call: SelectSmallest(tree.root, i)
2:     y ← SubtreeSize(x.left) +1
3:     if i == y then
4:         return x.value
5:     else if i < y then
6:         SelectSmallest(x.left, i)
7:     else
8:         SelectSmallest(x.right, i − y)
9:     end if
10: end function
```

Algorithm 13.5 Order Statistic with a BST for the kth largest

```
1: function SelectLargest(x:Node, k:int)      ▷ Equivalent to SelectSmallest(x, n − k)
2:     y ← SubtreeSize(x.right) +1
3:     if k == y then
4:         return x.value
5:     else if k < y then
6:         SelectLargest(x.right, k)
7:     else
8:         SelectLargest(x.left, k − y)
9:     end if
10: end function
```

8. Why not just use this approach always (not just with dynamically changing data) instead of the $O(n)$ one? The runtime to build and insert into the initial BST would be $O(n \lg n)$.

9. Can we use this approach on a regular BST? No, we need a balanced BST.

Opening Questions

1. Briefly explain what two properties a problem must have so that a dynamic programming algorithm will work.

2. Previously we have learned that divide-and-conquer algorithms partition a problem into independent sub-problems, solve each sub-problem recursively, and then combine their solutions to solve the original problem. Briefly, how are dynamic programming algorithms similar and how are they different from divide-and-conquer algorithms?

3. Why does it matter how we parenthesize a chain of matrix multiplications? We get the right answer any way we associate the matrices for multiplication. i.e. If A, B and C are matrices of correct dimensions for multiplication, then $(A \times B)C = A(B \times C)$.

Dynamic Programming

Dynamic Programming Steps

1. Define structure of optimal solution, including what are the largest sub-problems.
2. Recursively define optimal solution
3. Compute solution using table bottom up
4. Construct Optimal Solution

Optimal Matrix Chain Multiplication (optimal parenthesization)

1. How many ways are there to parenthesize (two at a time multiplication) 4 matrices $A \times B \times C \times D$?
2. Step 1: Generically define the structure of the optimal solution to the Matrix Chain Multiplication problem. The optimal way to multiply n matrices A_1 through A_n is:
3. Step 2: Recursively define the optimal solution. Assume $P(1, n)$ is the optimal cost answer. Make sure you include the base case.
4. Use proof by contradiction to show that Matrix Chain Multiplication problem has optimal substructure, i.e. the optimal answer to the problem must contain optimal answers to sub-problems.
5. Step 3: Compute solution using a table bottom up for the Matrix Chain Multiplication problem. Use you answer to question 3 above. Note the overlapping sub-problems as you go.
6. Step 4: Construct Optimal solution

A	B	C	D
2×4	4×6	6×3	3×7