CS 430 Lecture 27 Activities

Shortest Path Problem

How to find the shortest route between two points on a map.

Input

- Directed graph $G = (V, E)$
- Weight function $w : E \to \mathbf{R}$

Weight of path

$$\begin{aligned} p &= \langle v_0, v_1, \ldots, v_k \rangle \\ &= \sum_{i=1}^{k} w(v_{i-1}, v_i) \\ &= \text{sum of edge weights on path } p. \end{aligned}$$

Shortest-path weight u to v

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \overset{p}{\leadsto} v & \text{if there exists a path } u \leadsto v, \} \\ \infty & \text{otherwise.} \end{cases}$$

Shortest path u to v is any path p such that $w(p) = \delta(u, v)$.
Variants

- Single-source: Find shortest paths from a given source vertex $s \in V$ to every vertex $v \in V$.
- Single-destination: Find shortest paths to a given destination vertex.
- Single-pair: Find shortest path from u to v. No way known that's better in worst case than solving single-source.
- All-pairs: Find shortest path from u to v for all $u, v \in V$. We'll see algorithms for all-pairs in the next chapter.

Negative-weight edges – OK, as long as no negative-weight cycles are reachable from the source.

- If we have a negative-weight cycle, just keep going around it, and get $w(s, v) = -\infty$ for all v on the cycle.
- But OK if the negative-weight cycle is not reachable form the source.
- Some algorithms work only if there are no negative-weight edges in the graph.

1. What would the brute force approach be to solve the shortest path problem, and what is its run time?

$$1 \text{ Path of 1 edge}$$
$$|V|-2 \text{ Paths of 2 edges}$$
$$(|V|-2)(|V|-3) \text{ Paths of 3 edges}$$
$$\vdots$$
$$|V|! \text{ Paths of V edges}$$

2. Prove optimal substructure for the shortest path problem. Since shortest paths contain shortest subpaths (optimal solution to subproblem must in in optimal answer to problem.) $G\{V, E\}$, weight function on edge given source v, find shortest path to u

Output of single-source shortest-path algorithm For each vertex $v \in V$:

- $d[v] = \delta(s, v)$, Initially, $d[v] = \infty$; reduces as algorithms progress. But always maintain $d[v] \leftarrow \delta(s, v)$. Call $d[v]$ a shortest-path estimate.
- $\pi[v]$ = predecessor of v on a shortest path from s. If no predecessor, $\pi[v] = NIL$, π induces a tree–shortest-path tree.

Initialization – All the shortest-paths algorithms start with INIT-SINGLE-SOURCE.

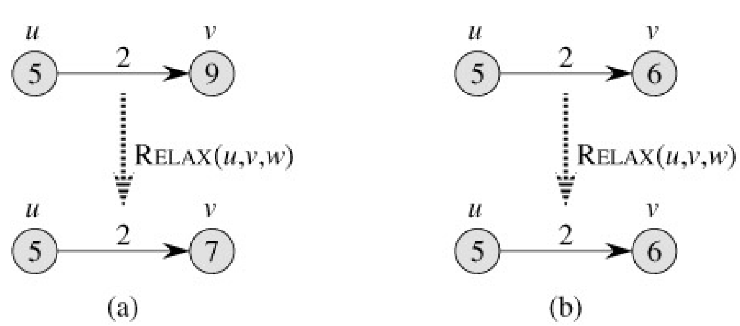
Algorithm 27.1 Single Source Initialization

function INIT-SINGLE-SOURCE(V, s)
    for all $v \in V$ **do**
        $d[v] \leftarrow \infty$
        $\pi[v] \leftarrow$ NIL
    end for
    $d[s] \leftarrow 0$
end function

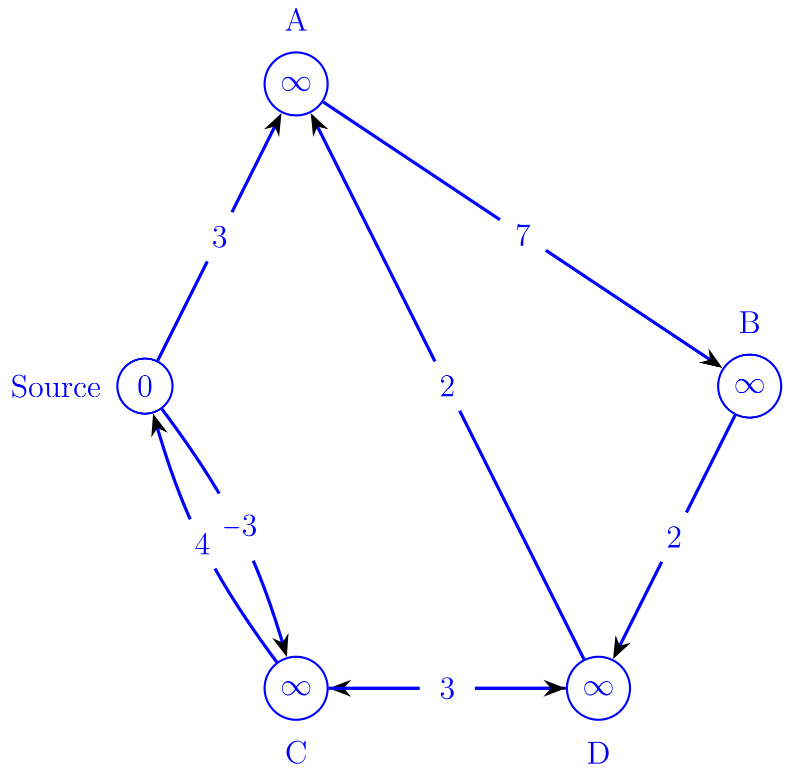
Relaxing an edge (u, v) - Can we improve the shortest-path estimate (best seen so far) from the source s to v be going through u and taking edge (u, v)?

Algorithm 27.2 Relaxing an Edge

function RELAX(u, v, w)
    if $d[v] > d[u] + w(u, v)$ **then**
        $d[v] \leftarrow d[u] + w(u, v)$
        $\pi[v] \leftarrow u$
    end if
end function



The algorithms differ in the order and how many times they relax each edge.



Shortest Path Algorithm - Bellman-Ford

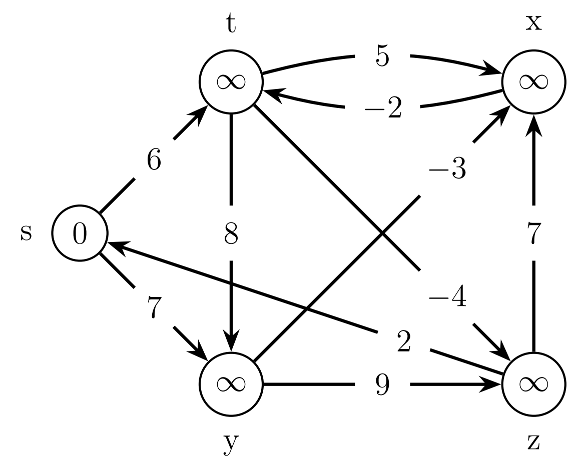
The most straightforward of the "relax an edge" algorithms.[1] Relaxes the edges in a fixed order (any fixed order) $|v| - 1$ times.[2] Not a greedy algorithm.

- Allows negative-weight edges.
- Computers $d[v]$ and $\pi[v]$ for all $v \in V$.
- Returns TRUE if no negative-weight cycles are reachable from s, FALSE otherwise.

[1] Works on any graph with no negative cycles, and the algorithm finds the negative cycles.
[2] Most # of edges on short path.

Algorithm 27.3 Bellman-Form Shortest Path Algorithm $O(VE) = O(V^3)$

```
function BELLMAN-FORD(V, E, w, s)
    INIT-SINGLE-SOURCE(V, s)
    for i ← 1 to |V| − 1 do
        for all edge (u, v) ∈ E do
            RELAX(u, v, w)
        end for
    end for
    for all edge (u, v) ∈ E do
        ▷ All edges, in any order, same order each time
        if d[v] > d[u] + w(u, v) then        ▷ Must be a negative cycle
            return FALSE
        end if
        return TRUE
    end for
end function
```



3. Execute Bellman-Ford on the above graph from source s for this edge order $(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$. Update the $d[v]$ and $\pi[v]$ values for each iteration.

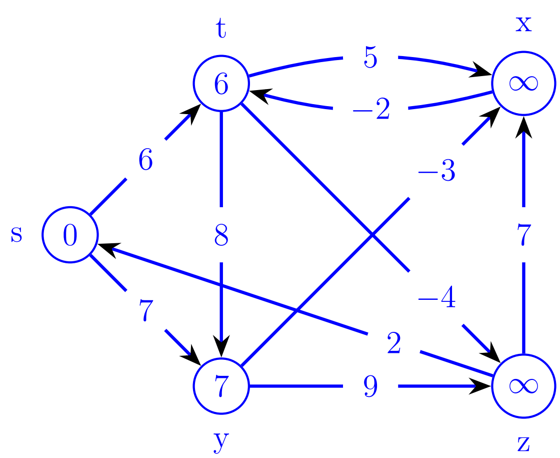


Figure 27.1: Example of the Execution of Bellman-Ford Step 1.

4. What is the runtime of Bellman-Ford? $O(VE) = O(V^3)$

5. Prove Bellman-Ford is correct.
   Values you get on each pass how quickly it converges depends on order of relaxation. But guaranteed to converged after $|V| - 1$ passes, assume no negative-weight cycles.