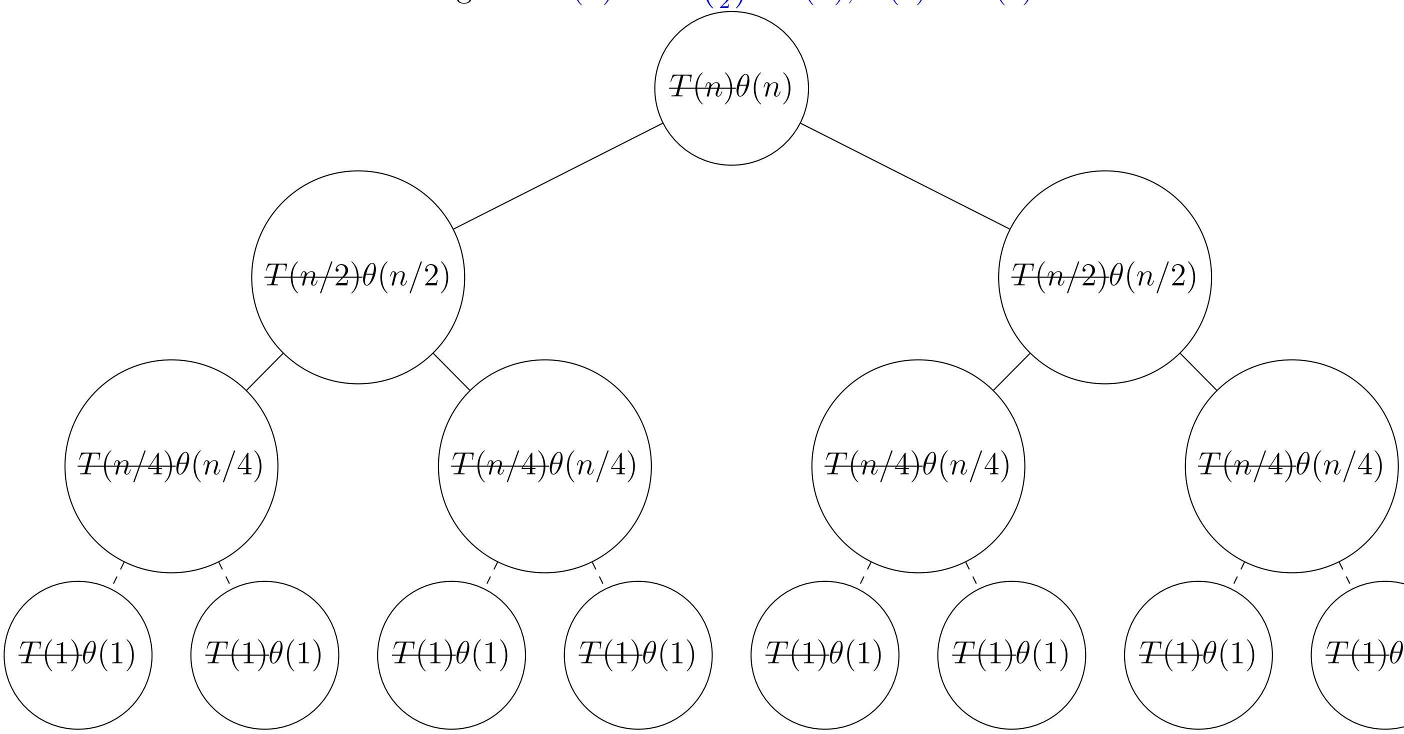
1. A recurrence relation describes runtime function recursively for a recursive algorithm. Write a recurrence relation for the Merge sort algorithm. HINT: try to count the number of executions of each statement and the cost of each Recurrence relations for merge sort lines: $2 = c_1, 3 = c_2, 4 = 5 = T\left(\frac{n}{2}\right)$ and $O(n)$ as the runtime for merge

$$T(n) = c_1 + c_2 + T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + O(n)$$
$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

Solving Recurrence Relations – Recurrence Tree Method

We solve a recurrence relation to get a function in its closed (non-recursive) form. The recurrence tree method is a visual method of repeatedly substituting in the recurrence relation for $T(n)$ on smaller and smaller n until you reach the base case, and then summing up all the nodes in the tree.

2. Draw the recurrence tree for Mergesort $T(n) = 2T\left(\frac{n}{2}\right) + \theta(n)$, $T(1) = O(1)$



Divide and Conquer Algorithms

- Divide – divide the problem into sub-problems that can be solved independently
- Conquer – recursively solve each sub-problem
- Combine – possibly necessary, combine solutions into sub-problems

Not all problems can be solved with the divide and conquer approach. Maybe sub-problems are not independent, or solutions to sub-problems cannot be combined to find solution to main problem.

3. Write a recursive algorithm for Binary Search. Write and solve its recurrence relation.Divide & Conquer & Combine Runtime analysis:

Algorithm 4.1 Binary Search Algorithm

```
function BS(A,key,i,j)                    ▷ Initial call: BS((A (sorted), key, 1, n))
    if i ≤ j then                         ▷ Handle Base case not found
        k ← ⌊(i+j)/2⌋
        if A[k] == key then
            return key
        end if
    end if
end function
```

$$T(n) = O(1) + T\left(\frac{n}{2}\right)$$
$$T(1) = O(1)$$

4. Write a recursive algorithm for Selection Sort (or insertion sort or bubble sort). Write and solve its recurrence relation.

Algorithm 4.2 Selection Sort Algorithm

```
function Select(A, i, j)        ▷ Not a stable sorting algorithm. Initial call: Select(A, 1, n)
    if i < j then                                            ▷ Base case, 1 item is sorted.
        minSoFar = A[i]
        kk=i
        for k=i+1; k≤j; k++ do
            if A[k] < minSoFar then
                minSoFar ← A[k]
                kk ← k
            end if
        end for
        Swap A[i] with A[k]
        Select(A, i+1, j)
    end if
end function
```

Runtime analysis: $T(n) = T(n-1) + O(n)$, $T(1) = O(1)$

5. Describe an efficient divide and conquer algorithm to count the number of times a character appears in a string of length n.

Algorithm 4.3 Count of char in string

```
function FIND_COUNT(S, from, to, char)
    if from < to then
        mid ← (from+to)/2
        x ← Find_Count(A, from, mid, char)
        y ← Find_Count(A, mid+1, to, char)
        return x + y
    else                                        ▷ 1 character left
        if S[from]==char then return 1
        elsereturn 0
        end if
    end if
end function
```

Inductive Proofs

(needed in next lecture to prove a solution to a recurrence relation)

6) What are the three steps in an inductive proof?

- Prove for base case
- Assume true for n, prove for larger n

7) Use an inductive proof to show the sum of the first n integers is $\frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1(2)}{2}$$

$$1 = 1 \sum_{k=1}^{m} k \qquad = \frac{m(m+1)}{2}$$

$$\sum_{k=1}^{m+1} k = \frac{(m+1)(m+2)}{2}$$

$$\sum_{k=1}^{m} k + \sum_{m+1}^{m+1} = \frac{(m)(m+1)}{2} + (m+1)$$

$$= \frac{(m)(m+1)}{2} + (m+1)$$