After Lecture 03 & 04

– Answer any questions on HW1
Practice Problems (all taken from previous exams)

1. Which of the following is time complexity of fun()?

```
int fun(int n){
        int count = 0;
        for(int i = 0; i < n; i++){
                for(int j = i; j > 0; j--){
                        count = count + 1;
                }
        }
        return count;
}
```

   a) $O(n)$
   b) $O(n^2)$
   c) $O(n \log n)$
   d) $O(n \log(n) \log(n))$

2. Consider the following function. What is the returned value of the above function?

```
int unknown(int n){
        int i, j, k=0;
        for(i = n/2; i <= n; i++){
                for(j=2; j <= n; j=j*2){
                        k = k + n/2;
                }
        }
        return k;
}
```

   a) $\Theta(n^2)$
   b) $\Theta(n^2 \log n)$
   c) $\Theta(n^3)$
   d) $\Theta(n^3 \log n)$

3. What is the worst-case auxiliary space complexity (including stack space for recursion) of merge sort?

   a) $O(1)$
   b) $O(\log n)$
   c) $O(n)$
   d) $O(n \log n)$

4. Choose the incorrect statement about merge sort from the following:
    a) It is a comparison-based sort.
    b) It's runtime is dependent on input order.
    c) It is not an in-place (all the operations are on the original array) algorithm.
    d) It is a stable algorithm.

5. Use the definition of big-O to prove or disprove.
    5a) is $2^{n+1}? = ?O\left(2^n\right)$True:

$$2^{n+1} = C2^n$$
$$2 = C \text{ if c } \geq 2$$

    5b) is $2^{2n}? = ?O\left(2^n\right)$False:

$$2^{2n} = C2^n$$
$$2^n = C$$

However 2^n has an infinite range so it cannot be upperbounded

6. Although merge sort runs in $\Theta(n \lg n)$ worst-case time and insertion sort runs in $\Theta(n^2)$ worst-case time, the constant factors in insertion sort make it faster for small n. Thus, it makes sense to use insertion sort within merge sort when sub-problems become sufficiently small. Consider a modification to merge sort in which n/k sub-lists of length k are sorted using insertion sort and then merged using the standard merging mechanism, where k is a value to be determined.
    6a) Show the n/k sub-lists; each of length k, can be sorted by insertion sort in $\Theta(nk)$ worst-case time.

$$k \rightarrow \Theta\left(k^2\right)$$
$$\frac{n}{k} \rightarrow \Theta\left(\frac{n}{k} \times k^2\right) = \Theta(nk)$$

    6b) Show that the sub-lists can be merged in $\Theta(n \lg \left(\frac{n}{k}\right))$ worst-case time.

$$\frac{n}{k} \rightarrow \Theta\left(\frac{n}{2k}\right) \text{ for merging}$$
$$\Theta(2k) \text{ m?? for } 2k \text{ elements}$$
$$\Theta\left(\frac{n}{2k} \times 2k\right) = \Theta\left(n\right) \Theta\left(\frac{n}{4k} \times 4k\right) = \Theta\left(n\right)$$

    6c) Given that the modified algorithm runs in $\Theta(nk + n \lg \left(\frac{n}{k}\right))$ worst-case time, what is the largest asymptotic (Θ-notation) value of k as a function of n for which the modified algorithm has the same asymptotic running time as standard merge sort?
    6d) How should k be chosen in practice?

7. The Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, … is defined recursively as

Algorithm 1 The Fibonacci sequence

```
function FIB($n$)    ▷ This mathematical definition leads naturally to a recursive algorithm
    if n $\le$ 1 then return n
    end if
    return FIB(n-1) + FIB(n-2)
end function
```

7a) Write the recurrence relation, $T(n)$, for the asymptotic runtime for procedure FIB(n) shown above, and solve the recurrence relation to show that $T(n) = O\left(2^{n-2}\right)$.

7b) Another recursive procedure which computes the nth Fibonacci number is below.

Algorithm 2

```
function F1(n)
    if n < 2 then return n
    else return F2(2, n, 1, 1)
    end if
end function

function F2(i,n,x,y)
    if i $\le$ n then
        F2(i+1, n, y, x+y)
    end if
    return x
end function
```

Trace out the algorithm as it computes $F1(1)$, $F1(2)$, $F1(3)$, $F1(4)$, explain how the algorithm works, and then compare its asymptotic runtime to the time for procedure FIB(n).

8. Use mathematical induction to show that when n is an exact power 2, the solution of the recurrence

$$T(n) = \begin{cases} 2 & \text{if } n = 2 \\ 2T(n/2) + n & \text{if } n = 2^k, \text{ for } k > 1 \end{cases}$$

is $T(n) = n \lg n$