CS 581

Advanced Artificial Intelligence

February 28, 2024

Announcements / Reminders

- **Please follow the Week 08 To Do List instructions (if you haven't already)**

- **Programming Assignment #01: due on Sunday 03/03 at 11:59 PM CST**

Plan for Today

- ~~Reinforcement Learning~~ [Delayed]
- Bayes Networks
- Decision Networks
- Inference in Probabilistic Networks

Bayes Networks

Independence

Two events A and B are independent if:

$$P(A \cap B) = P(A) * P(B)$$

So (from conditional probability formula):

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) * P(B)}{P(B)} = P(A)$$

Disjointment vs. Independence

Concept	Meaning	Formulas
Disjoint	Events A and B cannot occur at the same time	$A \cap B = \varnothing$ $P(A \cup B) = P(A) + P(B)$
Independent	Event A does not give any information about event B	$P(A \mid B) = P(A)$ $P(B \mid A) = P(B)$ $P(A \cap B) = P(A) * P(B)$

Conditional Independence

Random variable X is conditionally independent of random variable Y given Z if for all $x \in Dx$, for all $y \in Dy$, and for all $z \in Dz$, such that

$$P(Y = y \wedge Z = z) > 0 \textbf{ and } P(Y = y' \wedge Z = z) > 0$$

$$P(X = x \mid Y = y \wedge Z = z) = P(X = x \mid Y = y' \wedge Z = z)$$

In other words, given a value of Z, knowing Y's value DOES NOT affect your belief in value of X.

Conditional Independence

The following four statements are equivalent as long as conditional probabilities:

1. X **is conditionally independent of** Y **given** Z
2. Y **is conditionally independent of** X **given** Z
3. $P(X \mid Y, Z) = P(X \mid Z)$
4. $P(X, Y \mid Z) = P(X \mid Z) * P(Y \mid Z)$

Conditional Independence

Consider three random variables: P(owerful), H(appy), R(ich)

with domains:

$$D_P = \{\text{powerful, powerless}\},\ D_H = \{\text{happy, unhappy}\},\ D_R = \{\text{rich, poor}\}$$

Now, when:

$P(H = \text{happy}, R = \text{rich}) > 0$ **and** $P(H = \text{unhappy}, R = \text{rich}) > 0$

and:

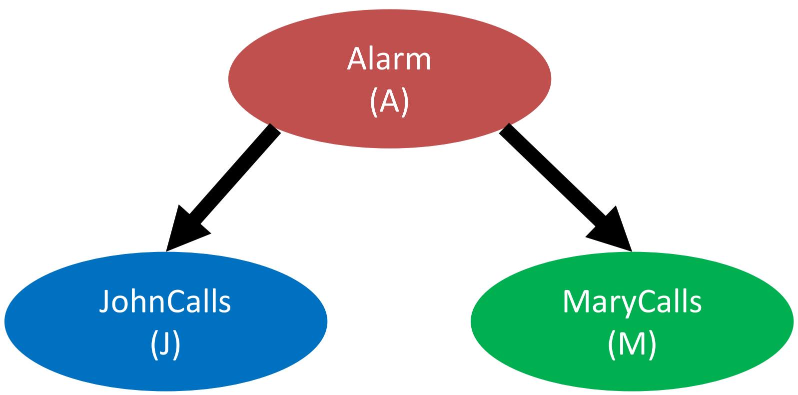
$$P(P = \text{powerful} \mid H = \text{happy}, R = \text{rich}) = P(P = \text{powerful} \mid H = \text{unhappy}, R = \text{rich})$$

In other words, given a value of R, knowing Y's value DOES NOT affect your belief in the value of X.

"Being un/happy does not make you less powerful, if you are rich."

More On Conditional Independence

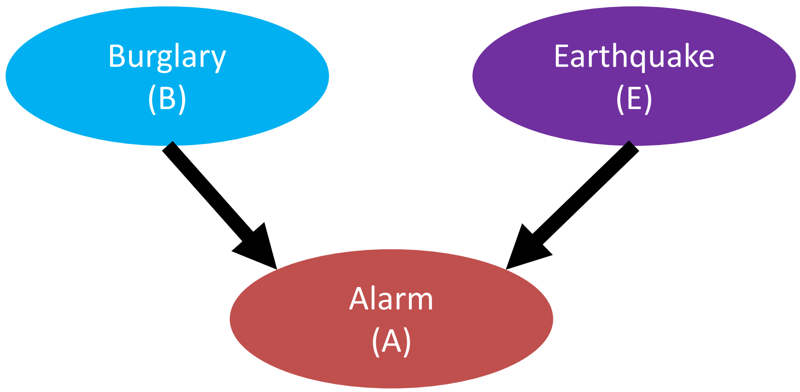
Common Cause:



JohnCalls and MaryCalls are NOT independent

JohnCalls and MaryCalls are CONDITIONALLY independent given Alarm

Common Effect:



Burglary and Earthquake are independent

Burglary and Earthquake are NOT CONDITIONALLY independent given Alarm

Marginal Probability

Marginal probability: the probability of an event occurring $\mathrm{P(A)}$.

It may be thought of as an unconditional probability.

It is not conditioned on another event.

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)	Conditional probabilities
true	true	$P(H \mid e) * P(e) \approx 0.074$	$P(H \mid e) = \dfrac{P(e \mid H) * P(H)}{P(e)} = \dfrac{6/18 * 18/81}{13/81} \approx 0.462$
true	false	$P(H \mid \neg e) * P(\neg e) \approx 0.148$	$P(H \mid \neg e) = \dfrac{P(\neg e \mid H) * P(H)}{P(\neg e)} = \dfrac{12/18 * 18/81}{68/81} \approx 0.176$
false	true	$P(\neg H \mid e) * P(e) \approx 0.086$	$P(\neg H \mid e) = \dfrac{P(e \mid \neg H) * P(\neg H)}{P(e)} = \dfrac{7/63 * 63/81}{13/81} \approx 0.538$
false	false	$P(\neg H \mid \neg e) * P(\neg e) \approx 0.691$	$P(\neg H \mid \neg e) = \dfrac{P(\neg e \mid \neg H) * P(\neg H)}{P(\neg e)} = \dfrac{56/63 * 63/81}{68/81} \approx 0.824$
		SUM = 1	

Joint probabilities calculated using the Product Rule:

$$P(A \wedge B) = P(A \mid B) * P(B)$$

Conditional probabilities calculated using Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}$$

Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(grad = true \wedge female = true) = P(H, e) = P(H \wedge e) = P(H \mid e) * P(e) \approx 0.074$
true	false	$P(grad = true \wedge female = false) = P(H, \neg e) = P(H \mid \neg e) * P(\neg e) \approx 0.148$
false	true	$P(grad = false \wedge female = true) = P(\neg H, e) = P(\neg H \mid e) * P(e) \approx 0.086$
false	false	$P(grad = false \wedge female = false) = P(\neg H, \neg e) = P(\neg H \mid \neg e) * P(\neg e) \approx 0.691$
		SUM = 1

Joint probabilities calculated using the Product Rule:

$$P(A \wedge B) = P(A \mid B) * P(B)$$

Conditional probabilities calculated using Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}$$

Joint Probability Distribution

H: grad	e: female	$P(H, e) = P(H \wedge e)$: $P(grad \wedge female)$
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

If we know the joint probability distribution, we can infer:

- **marginal probabilities** $P(H)$, $P(\neg H)$, $P(e)$, **and** $P(\neg e)$
- **conditional probabilities** $P(H \mid e)$, $P(H \mid \neg e)$, $P(\neg H \mid e)$, **and** $P(\neg H \mid \neg e)$

Joint Probability: Marginalization

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

Probability P(H):

$$P(H) = P(grad = true) = 0.074 + 0.148 \approx 18 / 81$$

Probability P(H): **"sum of all probabilities where** H **true"**

Joint Probability: Marginalization

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

Probability P(e):

$$P(e) = P(female = true) = 0.074 + 0.086 \approx 13 / 81$$

Probability P(e): **"sum of all probabilities where** e **true"**

Joint Probability: Conditionals

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

From product rule:

$$P(H \wedge e) = P(H \mid e) * P(e)$$

we can derive:

$$P(H \mid e) = \frac{P(H \wedge e)}{P(e)}$$

Joint Probability: Conditionals

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	0.074
true	false	0.148
false	true	0.086
false	false	0.691
		SUM = 1

From product rule:

$$P(H \wedge e) = P(H \mid e) * P(e)$$

we can derive:

$$P(H \mid e) = \frac{P(H \wedge e)}{P(e)} = \frac{0.074}{0.074 + 0.086} \approx 0.462$$

Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

Joint probabilities calculated using the Product Rule:

$$P(A \wedge B) = P(A \mid B) * P(B)$$

Conditional probabilities calculated using Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}$$

Complex Joint Distributions

Consider a complex joint probability distribution involving N random variables P_1, P_2, P_3, ..., P_{N-1}, Pp_N .

	N Random Variables						
	P_1	P_2	P_3	…	P_{N-1}	P_N	Joint Probability
2^N Possible Worlds (Models)	true	true	true	…	true	true	0.001
	true	true	true	…	true	false	0.201
	true	true	false	…	false	true	0.022
	…	…	…	…	…	…	…
	false	false	true	…	true	false	0.004
	false	false	true	…	false	true	0.301
	false	false	false	…	false	false	0.025

2^N values

Non-binary / Non-Boolean RVs

Some Random Variables are going to have more than two possible, discrete, values:

- height **-> short, average, tall**
- size **-> S, M, L, XL**
- streetLight **-> green, orange, red**
- vision **-> 20/20, 15/15, etc.**
- continent **-> Africa, Antarctica, Asia, Australia, Europe, North America, South America**

Non-binary RVs increase the complexity.

This May Be Impossible to Manage!

	N Random Variables						
	P_1	P_2	P_3	…	P_{N-1}	P_N	Joint Probability
2^N Possible Worlds (Models)	true	true	true	…	true	true	false
	true	true	true	…	true	false	true
	true	true	false	…	false	true	false
	…	…	…	…	…	…	…
	false	false	true	…	true	false	true
	false	false	true	…	false	true	true
	false	false	false	…	false	false	false

2^N values

Independent Variable

		Toothache		¬Toothache	
¬Cloudy		Catch	¬Catch	Catch	¬Catch
	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

		Toothache		¬Toothache	
Cloudy		Catch	¬Catch	Catch	¬Catch
	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

Let's introduce another random variable Cloudy **representing some weather conditions. It is difficult to imagine the other random variables here (**Toothache**,** Cavity**,** Catch**) being dependent on** Cloudy **and vice versa.**

Independent Variable

		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
¬Cloudy	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576
		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
Cloudy	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

Let's try to calculate the following probability:

$$P(Toothache, Catch, Cavity, Cloudy)$$

using the Product Rule:

$$P(Toothache, Catch, Cavity, Cloudy) =$$
$$= P(Cloudy \mid Toothache, Catch, Cavity) * P(Toothache, Catch, Cavity)$$

Independent Variable

		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
¬Cloudy	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576
		Toothache		**¬Toothache**	
		Catch	**¬Catch**	**Catch**	**¬Catch**
Cloudy	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

It's hard to imagine Cloudy influencing other variables, so:

$$P(Cloudy \mid Toothache, Catch, Cavity) = P(Cloudy)$$

and then:

$$P(Toothache, Catch, Cavity, Cloudy) =$$
$$= P(Cloudy) * P(Toothache, Catch, Cavity)$$

Independent Variable / Factoring

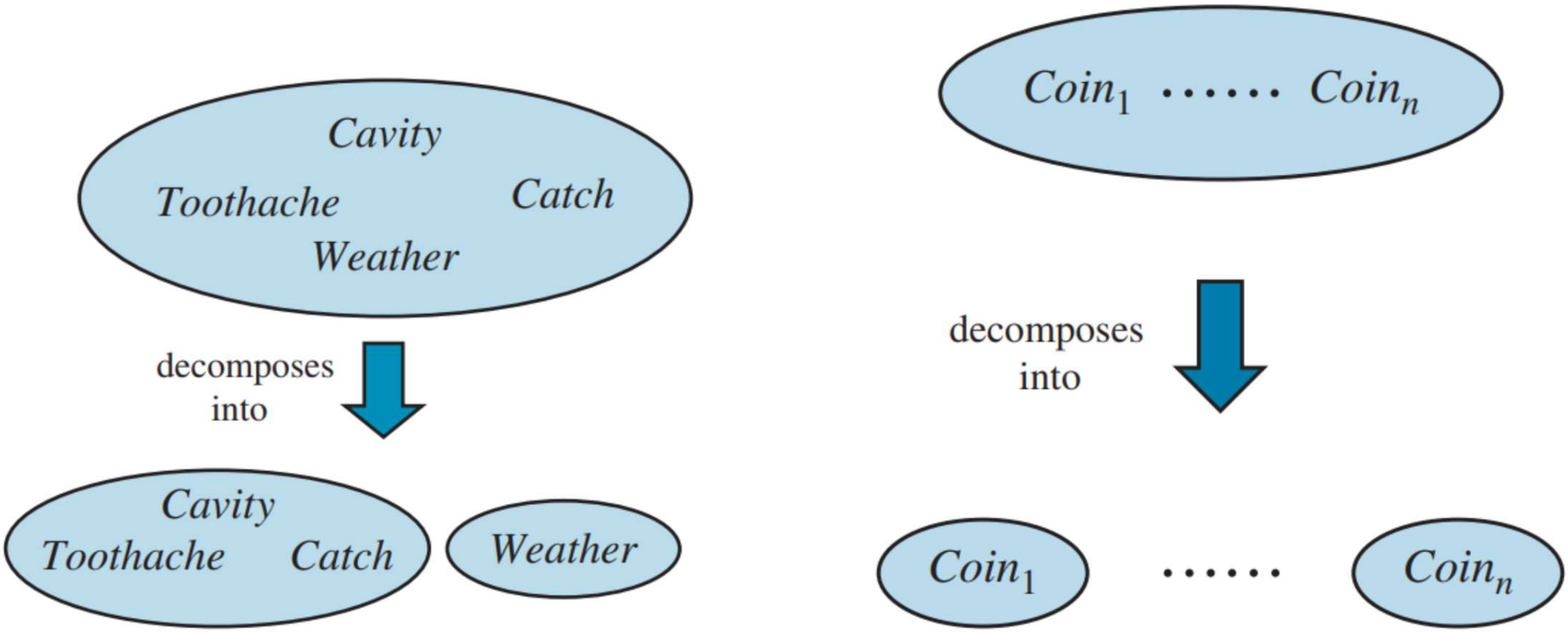
		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
¬Cloudy	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576
		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
Cloudy	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

It's hard to imagine Cloudy influencing other variables, so:

$$P(Toothache, Catch, Cavity, Cloudy) =$$
$$= P(Cloudy) * P(Toothache, Catch, Cavity)$$

This shows that Cloudy is INDEPENDENT of other variables and factoring can be applied.

Factoring / Decomposition



Bayes' Rule

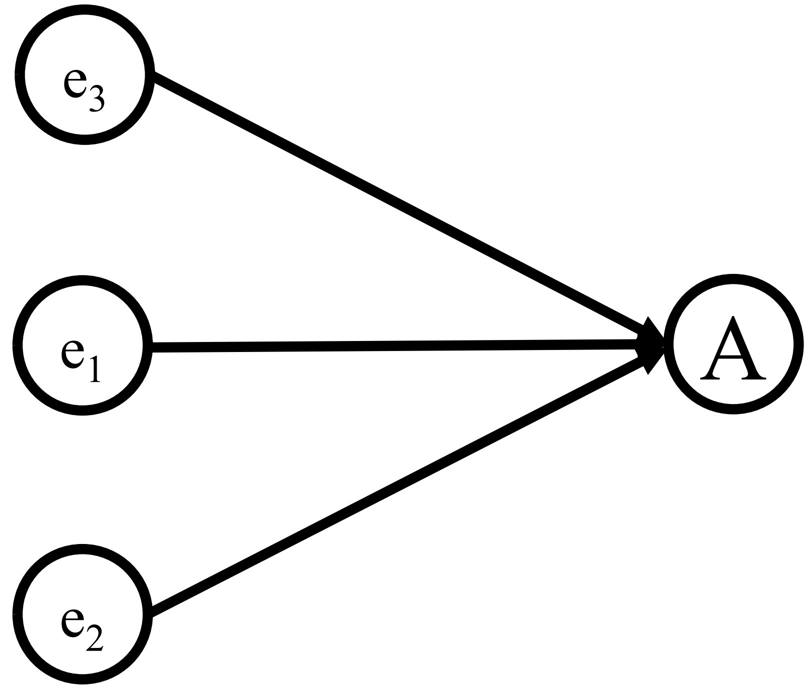
$$\boldsymbol{P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}}$$

Prior vs. Posterior Probabilities

Prior Probability

$$P(A)$$

Posterior Probability



$$P(A \mid \text{parents}(A))$$

Use Chain Rule To Decompose

N Random Variables						Joint Probability
P_1	P_2	P_3	…	P_{N-1}	P_N	
true	true	true	…	true	true	false
true	true	true	…	true	false	true
true	true	false	…	false	true	false
…	…	…	…	…	…	…
false	false	true	…	true	false	true
false	false	true	…	false	true	true
false	false	false	…	false	false	false

Chain Rule

Conditional probabilities can be used to decompose conjunctions using the chain rule. For any random variables $f_1, f_2, \ldots, f_n$ **:**

$$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n) \;= \prod_{i=1}^{n} P(f_i \mid f_1 \wedge \ldots \wedge \; f_{i-1})$$

$$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n) =$$

$$P(f_1) *$$

$$P(f_2 \mid f_1) *$$

$$P(f_3 \mid f_1 \wedge f_2)$$

$$* \ldots *$$

$$P(f_n \mid f_1 \wedge \; \ldots \; \wedge f_{n-1})$$

Chain Rule

Conditional probabilities can be used to decompose joint probabilities using the chain rule. For any random variables $f_1, f_2, \ldots, f_n$ and values $x_1, x_2, \ldots, x_n$:

$$P(f_1 = x_1, f_2 = x_2, \ldots, f_n = x_n) =$$

$$P(f_1 = x_1) *$$

$$P(f_2 \mid f_1 = x_1) *$$

$$P(f_3 \mid f_1 = x_1, f_2 = x_2) *$$

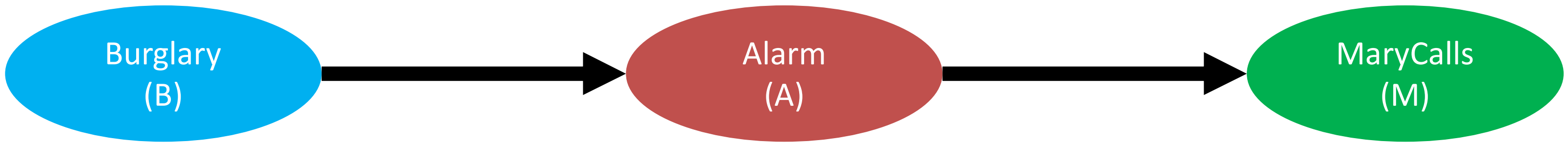
$$\ldots$$

$$P(f_n = x_n \mid f_1 = x_1, \ldots, f_{n-1} = x_{n-1}) =$$

$$= \prod_{i=1}^{n} P(f_i = x_i \mid f_1 = x_1, \ldots, f_{i-1} = x_{i-1})$$

Conditional Independence

Causal Chain:



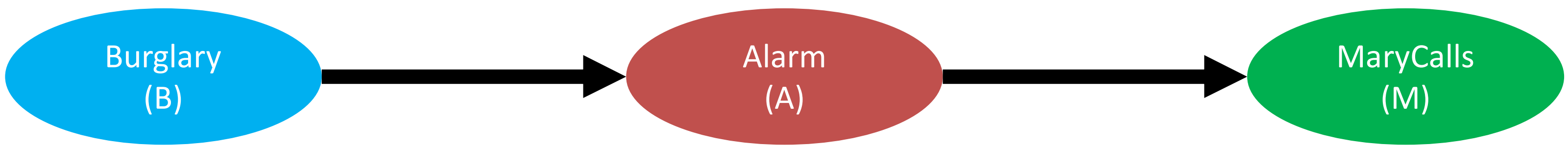
$$P(M \mid A, B) = \frac{P(B, A, M)}{P(B, A)} = \frac{P(B) * P(A \mid B) * P(M \mid A)}{P(B) * P(A \mid B)} = P(M \mid A)$$

Burglary and MaryCalls are CONDITIONALLY independent given Alarm.

If Alarm is given, what "happened before" Alarm does not directly influence MaryCalls.

Conditional Independence

Causal Chain:



by Conditional Probability formula

Joint Probabilities

by Chain Rule:

$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n) = \prod_{i=1}^{n} P(f_i \mid parents(f_i))$

$$P(M \mid A, B) = \frac{P(B, A, M)}{P(B, A)} = \frac{P(B) * P(A \mid B) * P(M \mid A)}{P(B) * P(A \mid B)} = P(M \mid A)$$

Burglary and MaryCalls are CONDITIONALLY independent given Alarm.

If Alarm is given, what "happened before" Alarm does not directly influence MaryCalls.

Chain Rule

Conditional probabilities can be used to decompose conjunctions using the chain rule. For any propositions (random variables) $f_1, f_2, \dots, f_n$ **:**

$$P(f_1 \wedge f_2 \wedge \dots \wedge f_n) = \prod_{i=1}^{n} P(f_i \mid f_1 \wedge f_2 \wedge \dots \wedge f_{i-1})$$

However, it can be rewritten as:

$$P(f_1 \wedge f_2 \wedge \dots \wedge f_n) = \prod_{i=1}^{n} P(f_i \mid parents(f_i))$$

because with conditional independence(s) considered:

$$\prod_{i=1}^{n} P(f_i \mid f_1 \wedge f_2 \wedge \dots \wedge f_{i-1}) = \prod_{i=1}^{n} P(f_i \mid parents(f_i))$$

Chain Rule

Conditional probabilities can be used to decompose conjunctions using the chain rule. For any propositions (random variables) $f_1, f_2, \ldots, f_n$ **:**

$$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n) =$$

$$P(f_1) \;*$$

$$P(f_2 \mid f_1) \;*$$

$$P(f_3 \mid f_1 \wedge f_2) \;*$$

$$\ldots$$

$$P(f_n \mid f_1 \wedge f_2 \wedge \ldots \wedge f_{n-1}) =$$

$$= \prod_{i=1}^{n} P(f_i \mid Parents(f_i))$$ $\leftarrow$ Enabled by conditional independence

Parents of Random Variable f_i

Parents of random variable f_i ($parents(f_i)$) is a minimal set of predecessors of f_i in the total ordering such that the other predecessors of f_i are conditionally independent of f_i given $parents(f_i)$.

A set of all predecessors of f_i : $A = \{f_1, f_2, \ldots, f_{i-1}\}$

A set of all parents of f_i : B

A set of all non-parents (predecessors NOT in B) of f_i : C

$$A = \{f_1, f_2, \ldots, f_{i-1}\} = B \cup C \; where \; B \cap C = \varnothing$$

when $parents(f_i)$ are given (all their values are known).

Parents of Random Variable f_i

Parents of random variable f_i ($parents(f_i)$) is a minimal set of predecessors of f_i in the total ordering such that the other predecessors of f_i are conditionally independent of f_i given $parents(f_i)$.

So: when $parents(f_i)$ are given, f_i probabilistically depends on each of its parents ($parents(f_i)$), but is independent of its other predecessors. That is

$$parents(f_i) \subseteq \{f_1, f_2, \ldots, f_{i-1}\}$$

such that:

$$P(f_i \mid f_1 \wedge f_2 \wedge \ldots \wedge f_{i-1}) = P(f_i \mid parents(f_i))$$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(grad = true \wedge female = true) = P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) \approx 0.074$
true	false	$P(grad = true \wedge female = false) = P(H, \neg e) = P(H) * P(\neg e \mid H) \approx 0.148$
false	true	$P(grad = false \wedge female = true) = P(\neg H, e) = P(\neg H) * P(e \mid \neg H) \approx 0.086$
false	false	$P(grad = false \wedge female = false) = P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) \approx 0.691$
		SUM = 1

Joint probabilities calculated using the Chain Rule:

$$P(f_1 \wedge f_2) = \prod_{i=1}^{2} P(f_i \mid f_1 \wedge \ldots \wedge f_{i-1})$$

$$P(f_1 \wedge f_2) = P(f_1) * P(f_2 \mid f_1)$$

so: $P(grad \wedge female) = P(H \wedge e) = P(H) * P(e \mid H)$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) = 18 / 81 * 6 / 18 \approx 0.074$
true	false	$P(H, \neg e) = P(H) * P(\neg e \mid H) = 18 / 81 * 12 / 18 \approx 0.148$
false	true	$P(\neg H, e) = P(\neg H) * P(e \mid \neg H) = 63 / 81 * 7 / 63 \approx 0.086$
false	false	$P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63 / 81 * 56 / 63 \approx 0.691$
		SUM = 1

Joint probabilities calculated using the Chain Rule:

$$P(f_1 \wedge f_2) = \prod_{i=1}^{2} P(f_i \mid f_1 \wedge \ldots \wedge f_{i-1})$$

$$P(f_1 \wedge f_2) = P(f_1) * P(f_2 \mid f_1)$$

so: $P(grad \wedge female) = P(H \wedge e) = P(H) * P(e \mid H)$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
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false	false	$P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63/81 * 56/63 \approx 0.691$
		SUM = 1

Joint probabilities calculated using the Chain Rule:

$$P(f_1 \wedge f_2) = \prod_{i=1}^{2} P(f_i \mid parents(f_i))$$

$$P(f_1 \wedge f_2) = P(f_1) * P(f_2 \mid parents(f_i))$$

so: $P(H \wedge e) = P(H) * P(e \mid parents(e)) = P(H) * P(e \mid H)$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(H,e) = P(H \wedge e) = P(H) * P(e \mid H) = 18/81 * 6/18 \approx 0.074$
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false	true	$P(\neg H,e) = P(\neg H) * P(e \mid \neg H) = 63/81 * 7/63 \approx 0.086$
false	false	$P(\neg H,\neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63/81 * 56/63 \approx 0.691$
		SUM = 1

$$P(H \wedge e) = P(H) * P(e \mid parents(e)) = \boxed{P(H)} * \boxed{P(e \mid H)}$$

H: grad	¬H: ¬grad
18 / 81 ≈ 0.22	63 / 81 ≈ 0.78

H: grad	e: female	P(e | H)
true	true	6 / 18 ≈ 0.333
true	false	12 / 18 ≈ 0.667
false	true	7 / 63 ≈ 0.111
false	false	56 / 63 ≈ 0.889

Conditional Probability Table (CPT)

Bayes Network: Factorization

Chain rule AND definition of $parents(f_i)$ gives us:

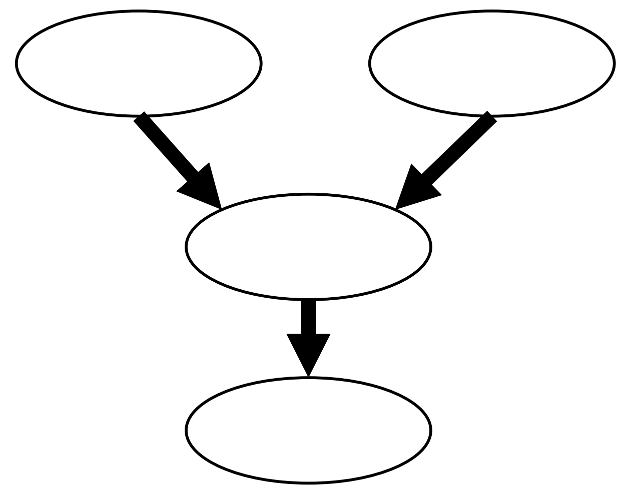
$$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n) = \prod_{i=1}^{n} P(f_i \mid parents(f_i))$$

Joint probability distribution **=** **Product of conditional probabilities after factorization of joint probability distribution**

N Random Variables						Joint Probability
P_1	P_2	P_3	...	P_{N-1}	P_N	
true	true	true	...	true	true	0.0011
true	true	true	...	true	false	0.0451
true	true	false	...	false	true	0.1011
...	...	...	...	...	...	...
false	false	true	...	true	false	0.0909
false	false	true	...	false	true	0.0651
false	false	false	...	false	false	0.2021

Joint probability distribution

Factorization



Bayes Network: graph representation of joint probability distribution factorization

Bayesian (Belief) Network

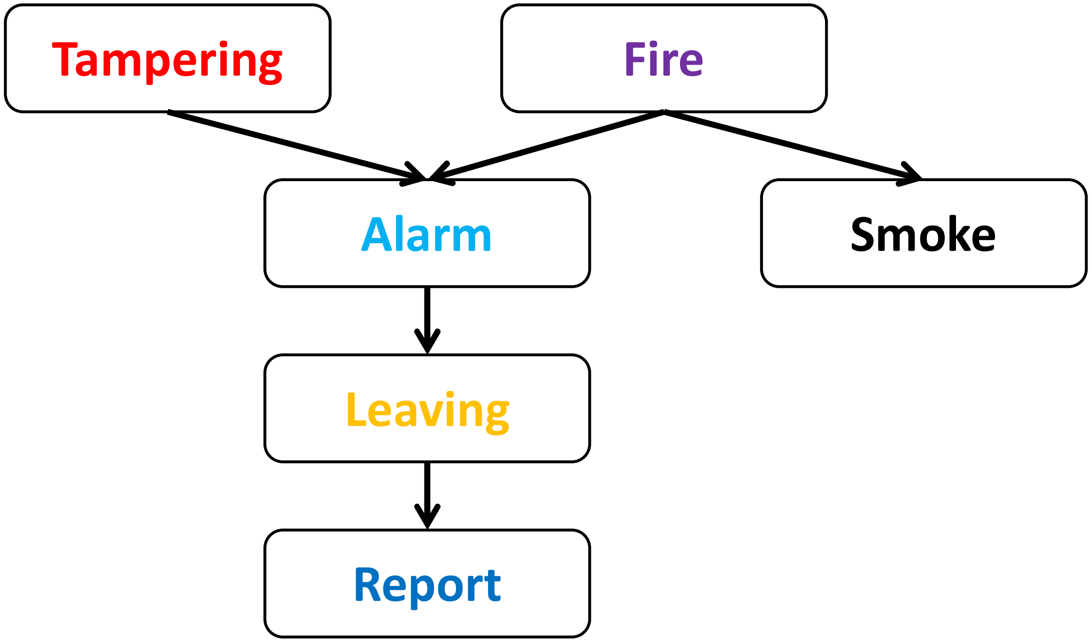
A Bayesian belief network describes the joint probability distribution for a set of variables.

A Bayesian network is an acyclic, directed graph (DAG), where the nodes are random variables (propositions). There is an edge (arc) from each elements of $\text{parents}(X_i)$ into X_i. Associated with the Bayesian network is a set of conditional probability distributions - the conditional probability of each variable given its parents (which includes the prior probabilities of those variables with no parents).

Consists of:

- **a graph (DAG) with nodes corresponding to random variables**
- **a domain for each random variable**
- **a set of conditional probability distributions $P(X_i \mid \text{parents}(X_i))$**

Bayesian (Belief) Network: Example



Random Variables (Propositions):

- Tampering: true if the alarm is tampered with
- Fire: true if there is a fire
- Alarm: true if the alarm sounds
- Smoke: true if there is smoke
- Leaving: true if people leaving the building at once
- Report: true if someone who left the building reports fire

Domain for all variables: $\{\text{true}, \text{false}\}$

NOTE: RVs don't have to be Boolean

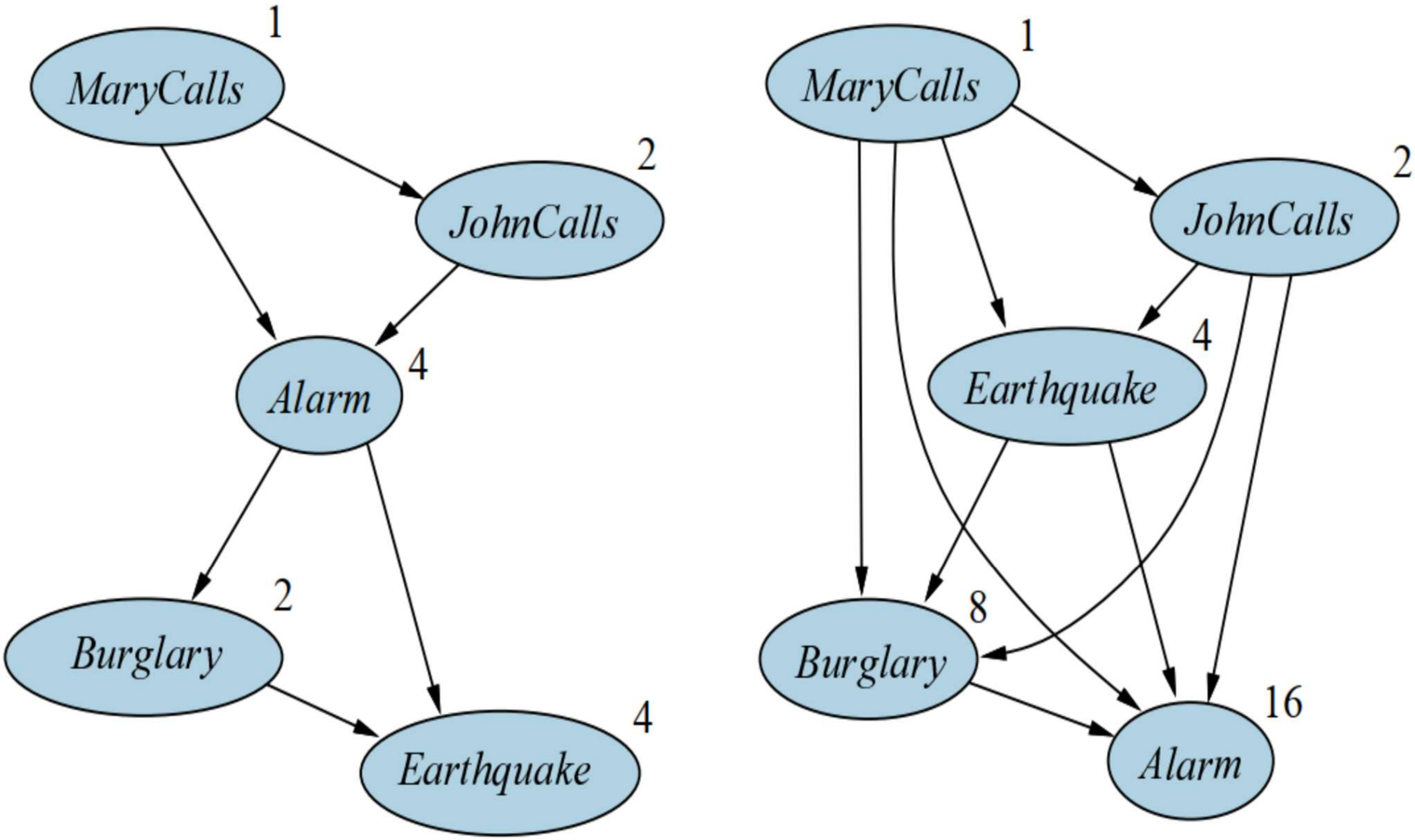
Building Bayesian (Belief) Network

1. Order Random Variables (ordering matters!)
2. Create network nodes for each Random Variable
3. Add edges between parent nodes and children nodes
   - For every node node X_i:
     - choose a minimal set S of parents for X_i
     - for each parent node Y in S add an edge from Y to X_i

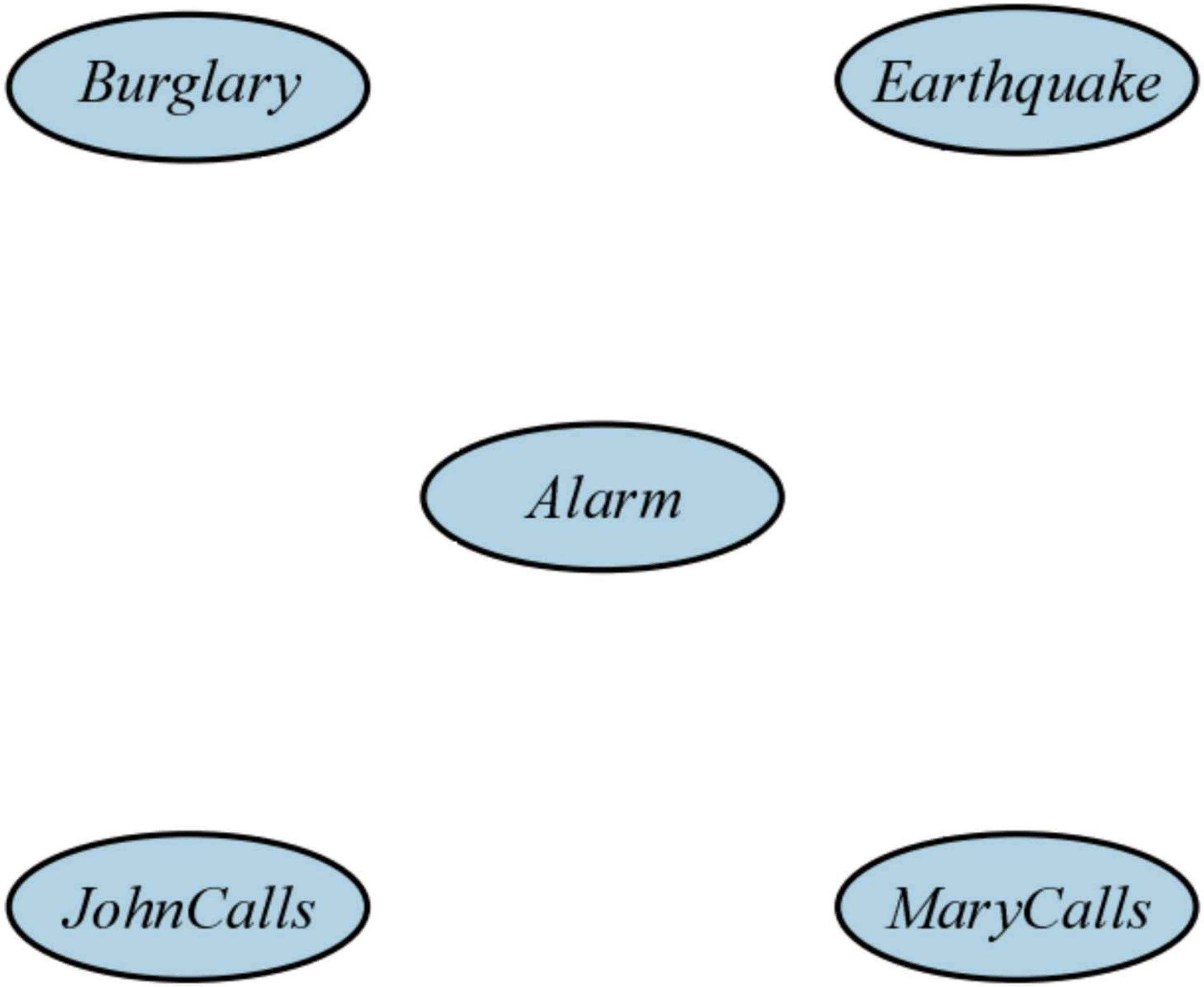
4. Add Conditional Probability Tables

Make it compact / sparse: choose your Random Variable ordering wisely.

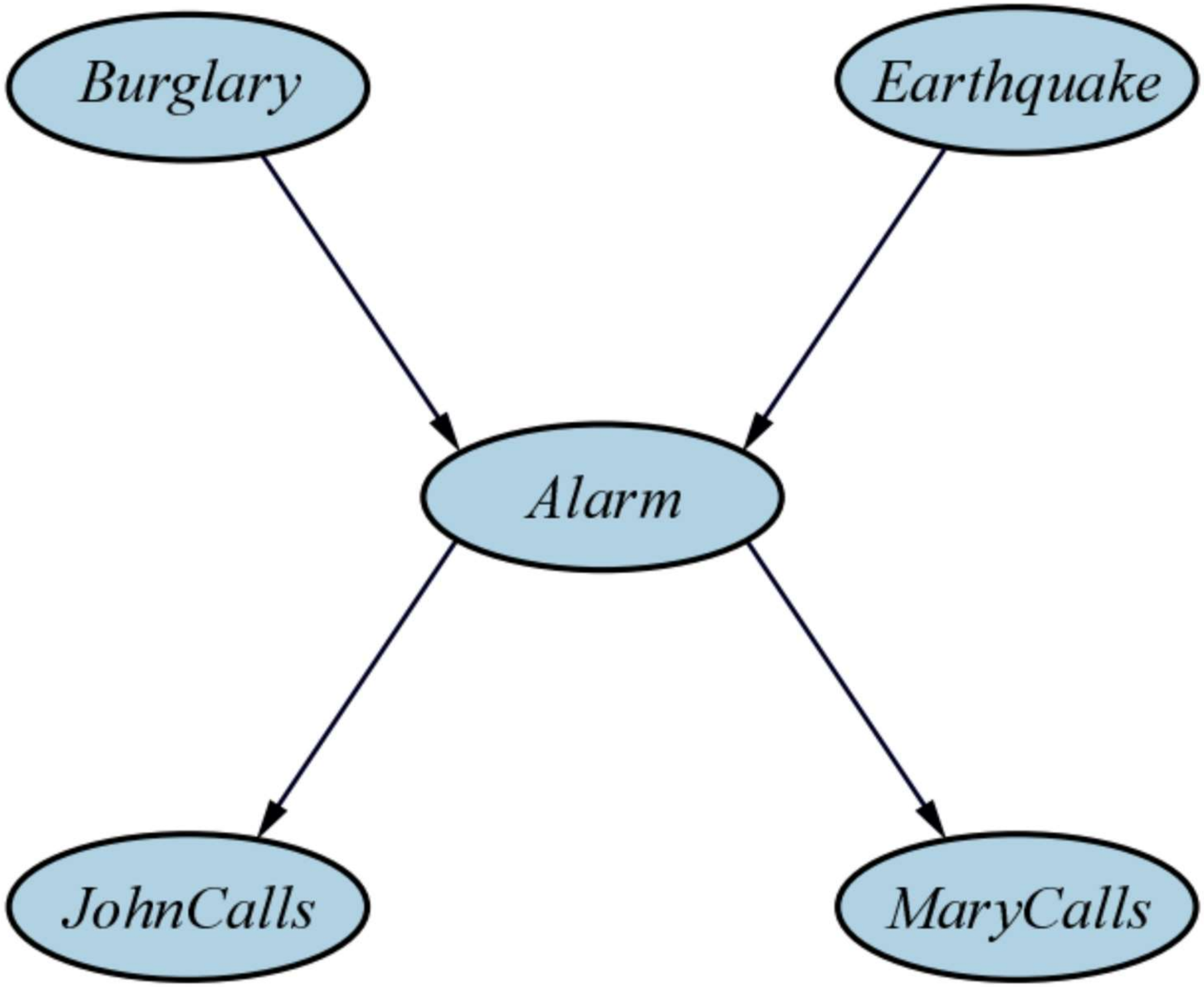
Ordering Matters!



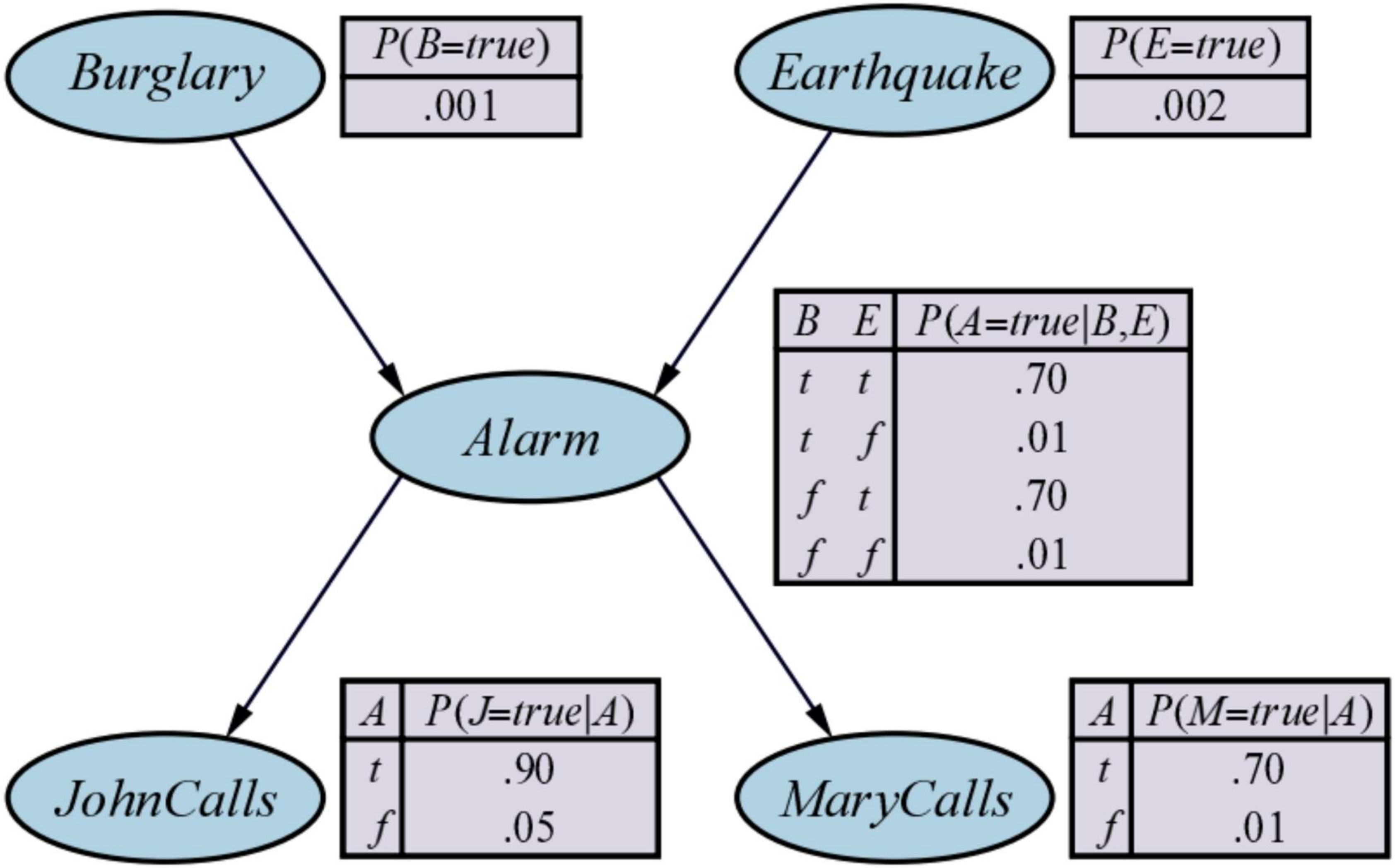
Create Vertices / Node / Random Vars



Add Edges



Add Conditional Probability Tables



P(B=true)
.001

P(E=true)
.002

B	E	P(A=true|B,E)
t	t	.70
t	f	.01
f	t	.70
f	f	.01

A	P(J=true|A)
t	.90
f	.05

A	P(M=true|A)
t	.70
f	.01

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) = 18 / 81 * 6 / 18 \approx 0.074$
true	false	$P(H, \neg e) = P(H) * P(\neg e \mid H) = 18 / 81 * 12 / 18 \approx 0.148$
false	true	$P(\neg H, e) = P(\neg H) * P(e \mid \neg H) = 63 / 81 * 7 / 63 \approx 0.086$
false	false	$P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63 / 81 * 56 / 63 \approx 0.691$
		SUM = 1

$$P(H \wedge e) = P(H) * P(e \mid parents(e)) = \boxed{P(H)} * \boxed{P(e \mid H)}$$

H: grad	¬H: ¬grad
18 / 81 ≈ 0.22	63 / 81 ≈ 0.78

H: grad	e: female	P(e | H)
true	true	6 / 18 ≈ 0.333
true	false	12 / 18 ≈ 0.667
false	true	7 / 63 ≈ 0.111
false	false	56 / 63 ≈ 0.889

Conditional Probability Table (CPT)

Create Vertices / Node / Random Vars

Create Vertices / Node / Random Vars

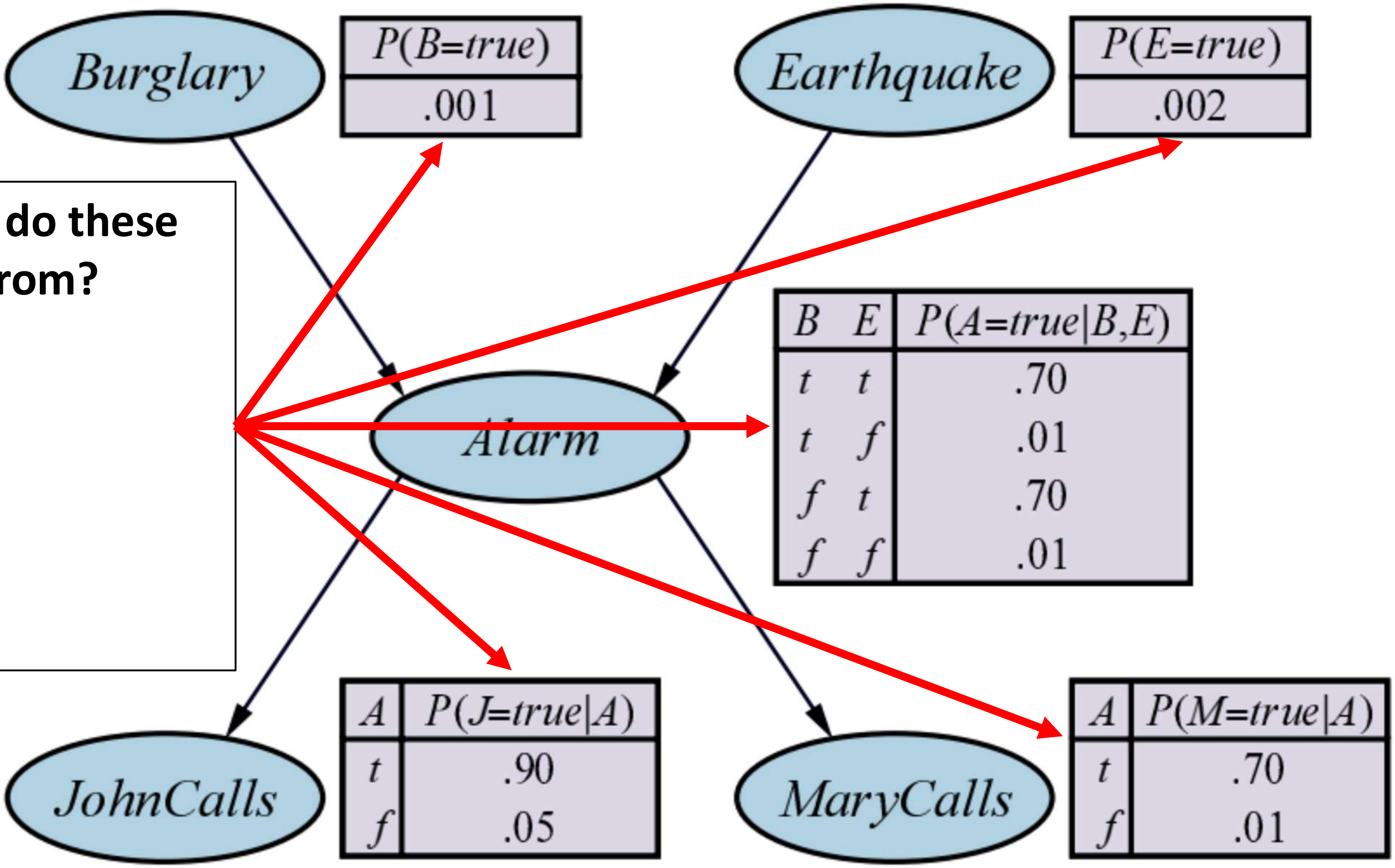
Add Edges

Add Conditional Probability Tables

H: grad	¬H: ¬grad
$18 / 81 \approx 0.22$	$63 / 81 \approx 0.78$

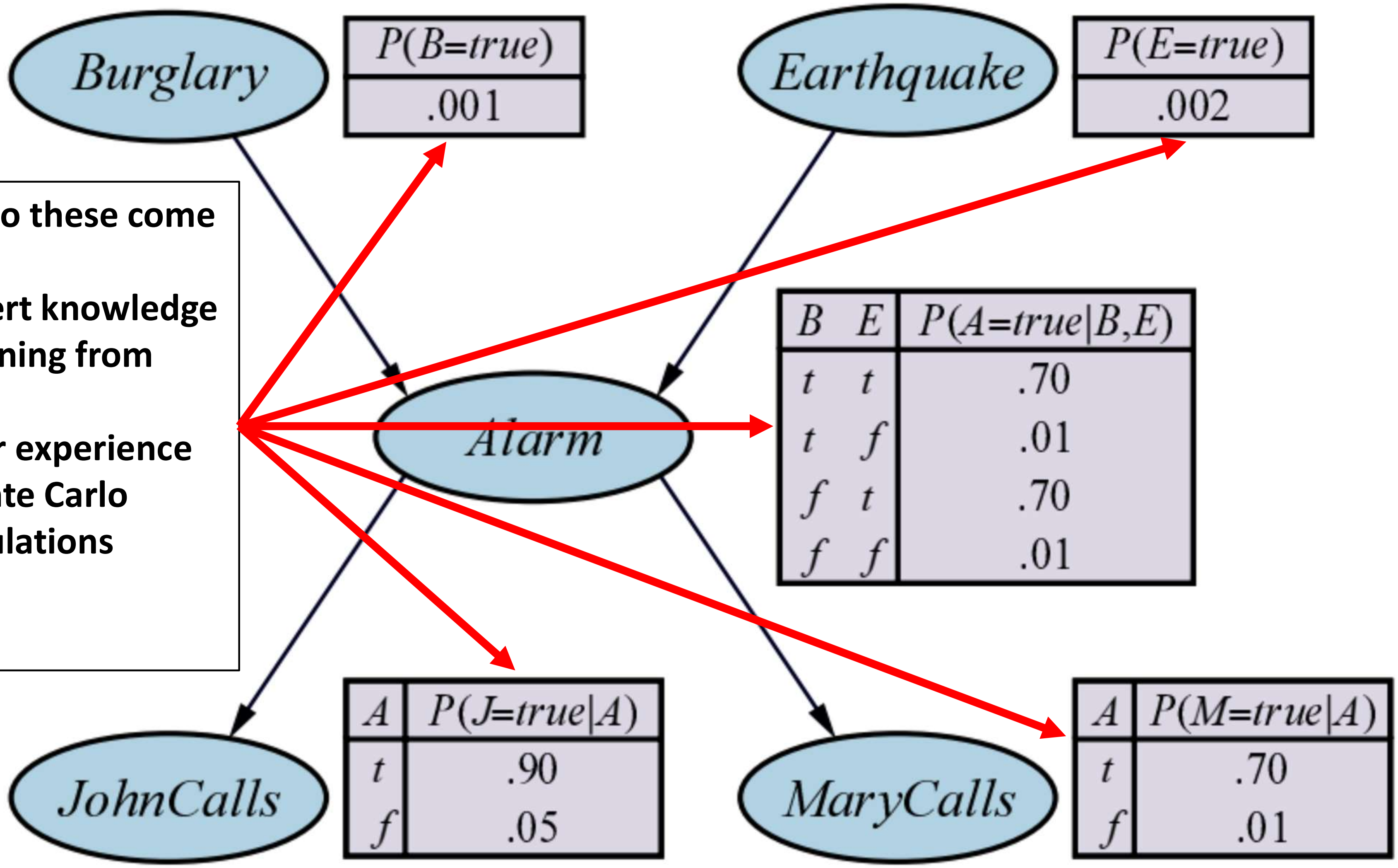
H: grad	e: female	P(e | H)
true	true	$6 / 18 \approx 0.333$
true	false	$12 / 18 \approx 0.667$
false	true	$7 / 63 \approx 0.111$
false	false	$56 / 63 \approx 0.889$

Add Conditional Probability Tables



Where do these come from?

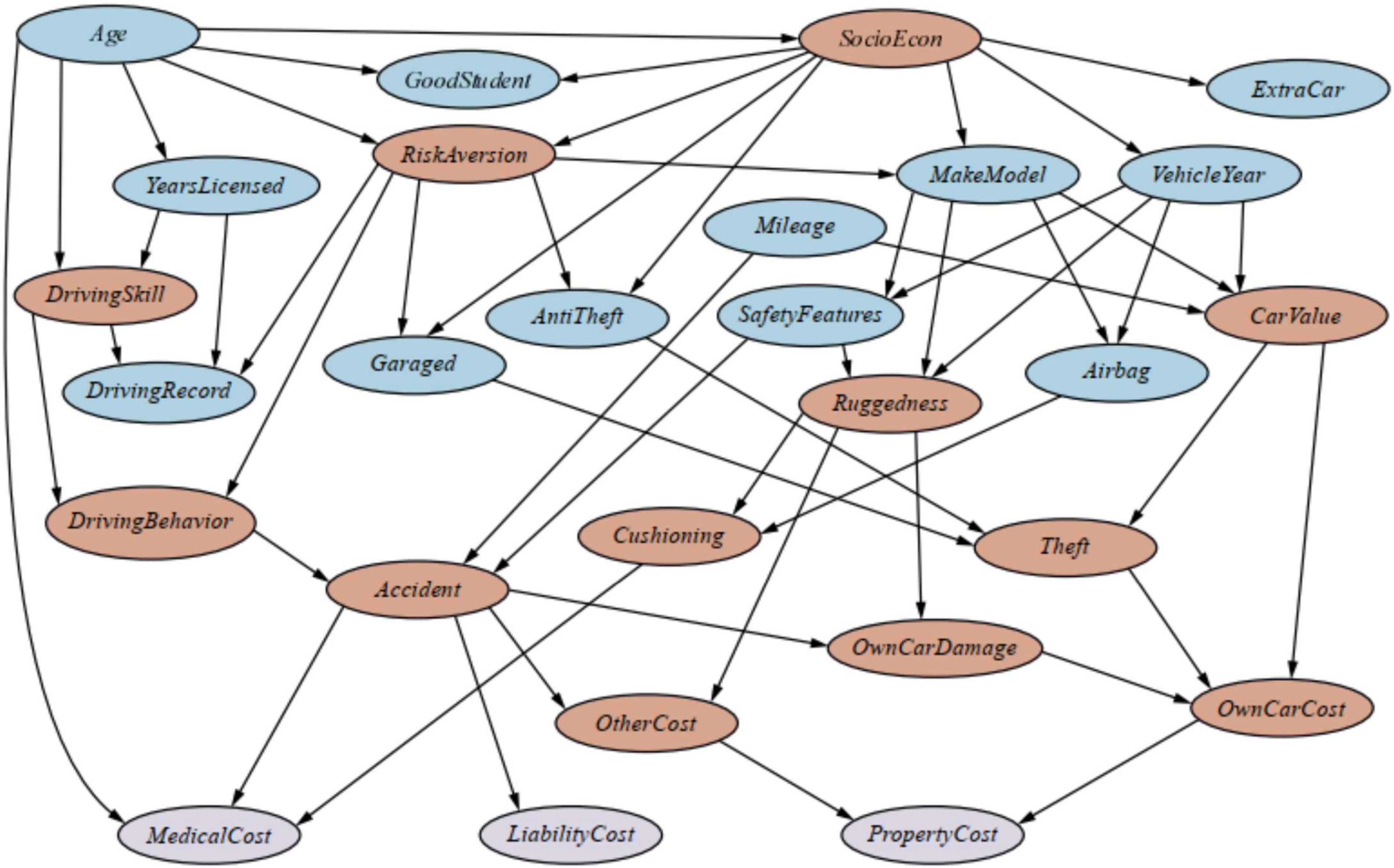
Add Conditional Probability Tables



Where do these come from?

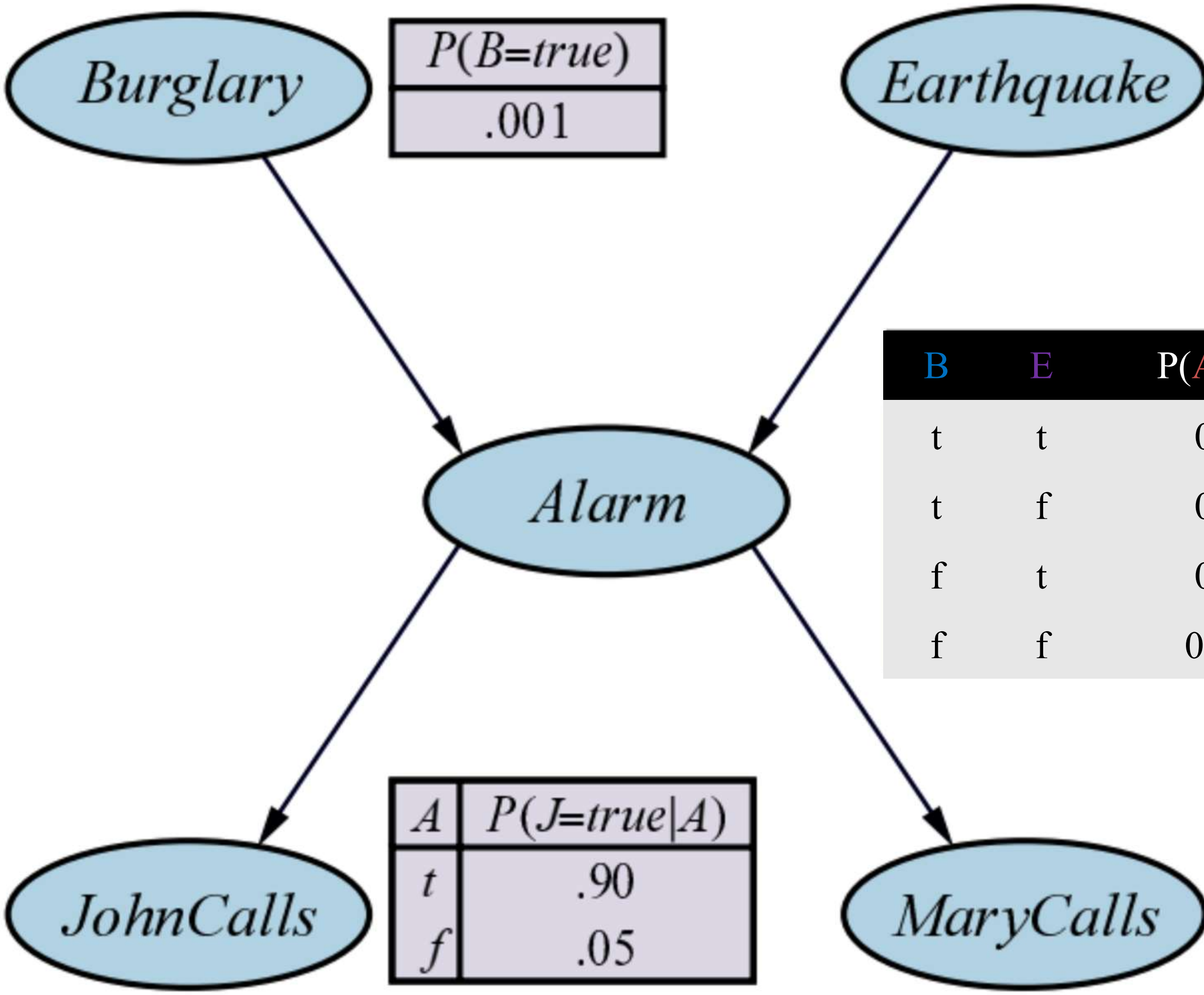
- Expert knowledge
- Learning from data
- Prior experience
- Monte Carlo simulations
- etc.

Bayesian Network: Car Insurance



Inference in Bayes Networks

Inference In Bayes Networks



P(B=true)
.001

P(E=true)
.002

B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

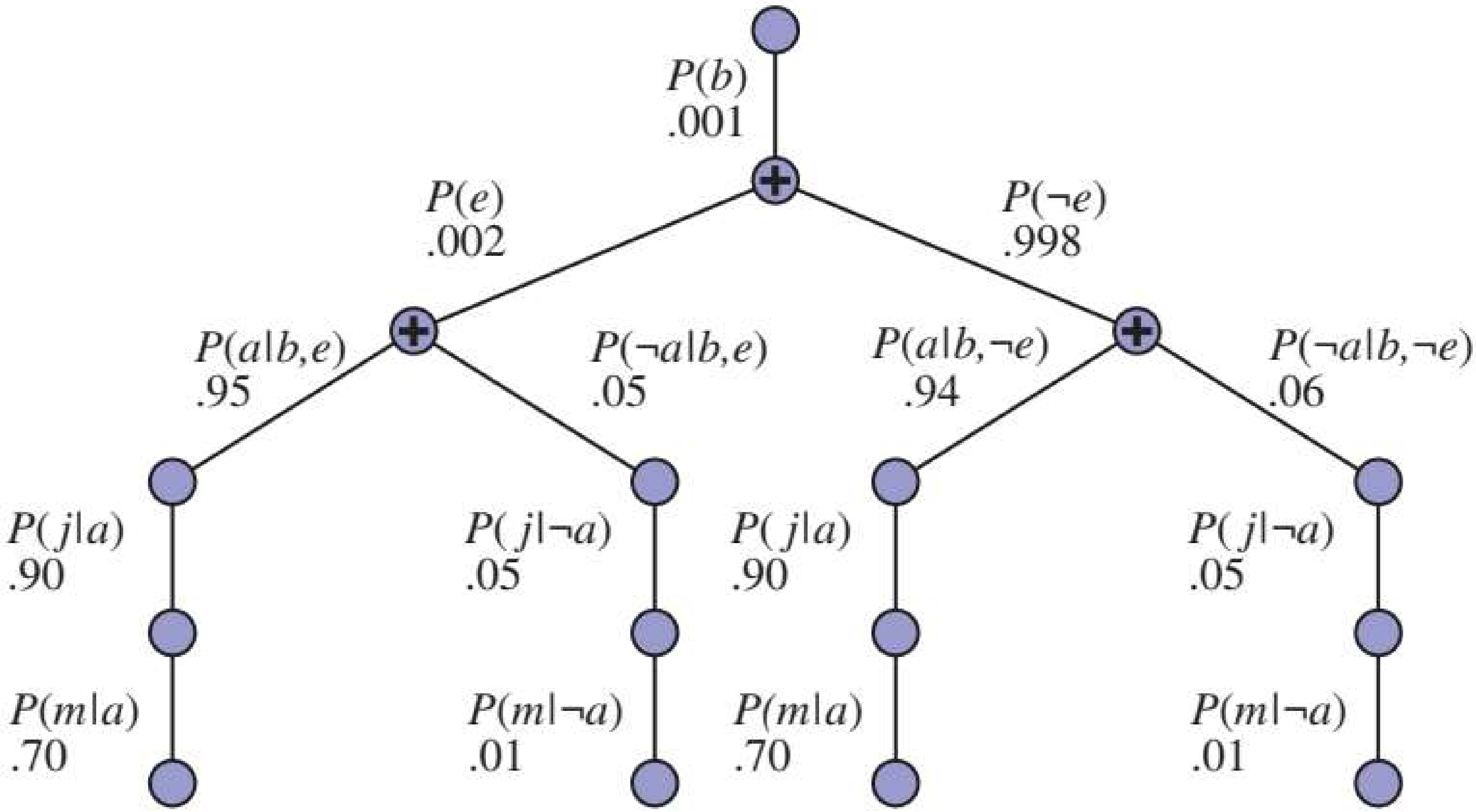
A	P(J=true|A)
t	.90
f	.05

A	P(M=true|A)
t	.70
f	.01

Inference by Enumeration: Example

Query (what is the probability distribution for the following conditional P()?):

$$\boldsymbol{P}(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$



Joint Probability: Marginalization

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

Probability P(H)**:**

$$P(H) = P(grad = true) = 0.074 + 0.148 \approx 18 / 81$$

Probability P(H)**: "sum of all probabilities where** H **true"**

Joint Probability: Marginalization

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

Probability P(e)**:**

$$P(e) = P(female = true) = 0.074 + 0.086 \approx 13 / 81$$

Probability P(e)**: "sum of all probabilities where** e **true"**

Joint Probability: Conditionals

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	0.074
true	false	0.148
false	true	0.086
false	false	0.691
		SUM = 1

From product rule:

$$P(H \wedge e) = P(H \mid e) * P(e)$$

we can derive:

$$P(H \mid e) = \frac{P(H \wedge e)}{P(e)} = \frac{0.074}{0.074 + 0.086} \approx 0.462$$

General Inference Procedure

Given:

- **a query involving a single variable X (in our example: Cavity),**
- **a <u>list</u> of evidence variables E (in our example: just Toothache),**
- **a <u>list</u> of observed values e for E,**
- **a list of remaining unobserved variables Y (in our example: just Catch),**

where X, E, and Y together are a COMPLETE set of variables for the domain, the probability $P(X \mid E)$ can be evaluated as:

$$\boldsymbol{P}(X \mid \boldsymbol{e}) = \alpha * \boldsymbol{P}(X, \boldsymbol{e}) = \alpha * \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{e}, \boldsymbol{y})$$

where ys are all possible values for Ys, α - normalization constant.

$P(X, e, y)$ is a subset of probabilities from the joint distribution

Inference

Query:

$$P(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$

Given:

- **a query involving a single variable** X
- **a <u>list</u> of evidence variables** K,
- **a <u>list</u> of observed values** k **for** K,
- **a list of remaining unobserved variables** Y

the probability P(X | K) **can be evaluated as:**

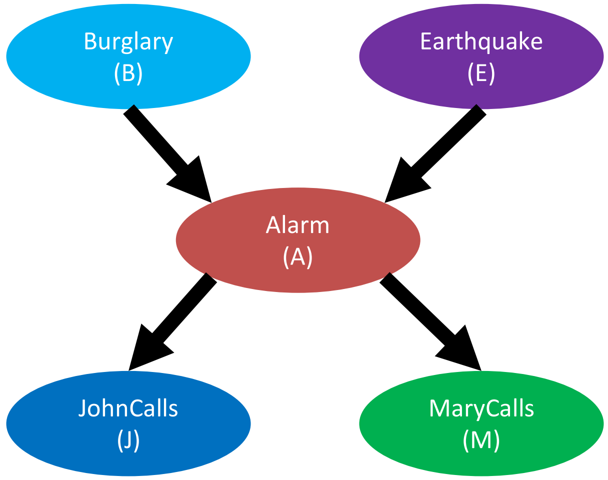
$$P(X \mid \boldsymbol{k}) = \alpha * P(X, \boldsymbol{k})$$

$$= \alpha * \sum_{y} P(X, \boldsymbol{k}, \boldsymbol{y})$$

where ys **are all possible values for** Ys, α **- normalization constant.**

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query:

$$\boldsymbol{P}(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$

Given:

- **a query involving a single variable X:** *Burglary*
- **a <u>list</u> of evidence variables K:** *JohnCalls*, *MaryCalls*
- **a <u>list</u> of observed values k for K:** *johnCalls*, *maryCalls*
- **a list of remaining unobserved variables Y:** *Earthquake*, *Alarm*

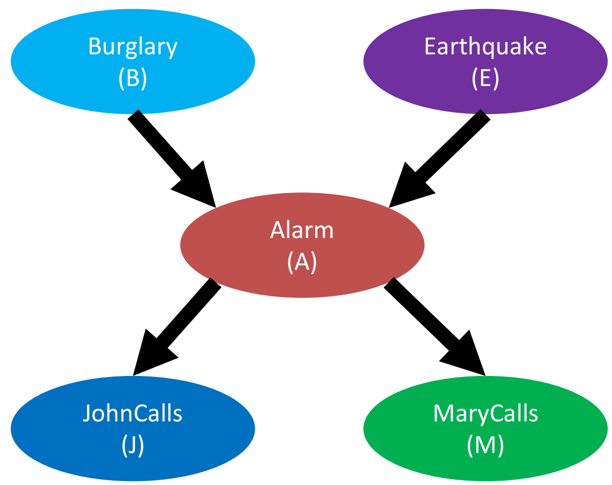
the probability P(X | K) can be evaluated as:

$$\boldsymbol{P}(X \mid \boldsymbol{k}) = \alpha * \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{k}, \boldsymbol{y})$$

where ys are all possible values for Ys, α - normalization constant.

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query:

$$P(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$

Given:

- **a query involving a single variable** X**:** B
- **a <u>list</u> of evidence variables** $\mathbf{K}$**:** J, M
- **a <u>list</u> of observed values** $\mathbf{k}$ **for** $\mathbf{K}$**:** j, m
- **a list of remaining unobserved variables** $\mathbf{Y}$**:** E, A

the probability $\mathrm{P}(X \mid \mathbf{K})$ **can be evaluated as:**

$$\boldsymbol{P}(X \mid \boldsymbol{k}) = \alpha * \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{k}, \boldsymbol{y})$$

where $\mathbf{y}$**s are all possible values for** $\mathbf{Y}$**s,** α **- normalization constant.**

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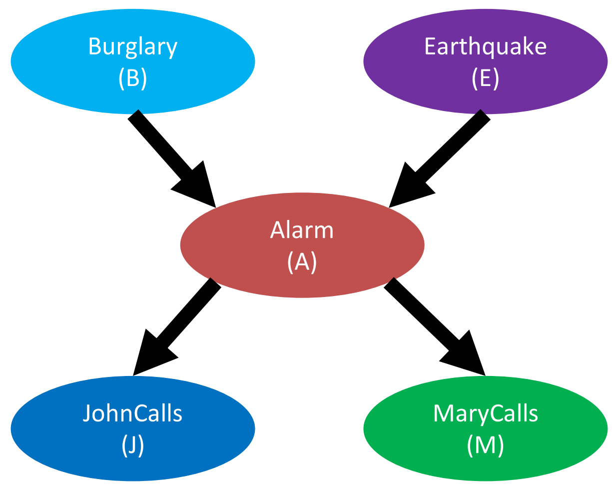

B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query:

$$P(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$

Given:

- **a query involving a single variable** B
- **a <u>list</u> of evidence variables** $\mathbf{K}$**:** J, M
- **a <u>list</u> of observed values** $\mathbf{k}$ **for** $\mathbf{K}$**:** j, m
- **a list of remaining unobserved variables** $\mathbf{Y}$**:** E, A

the probability $\mathrm{P}(\mathrm{B} \mid \mathrm{J}, \mathrm{M})$ **can be evaluated as:**

$$\boldsymbol{P}(B \mid j, m) = \alpha * \sum_{e} \sum_{a} \boldsymbol{P}(B, j, m, e, a)$$

where $\mathbf{y}$**s are all possible values for** $\mathbf{Y}$**s,** α **- normalization constant.**

P(B)	P(¬B)
0.001	0.999

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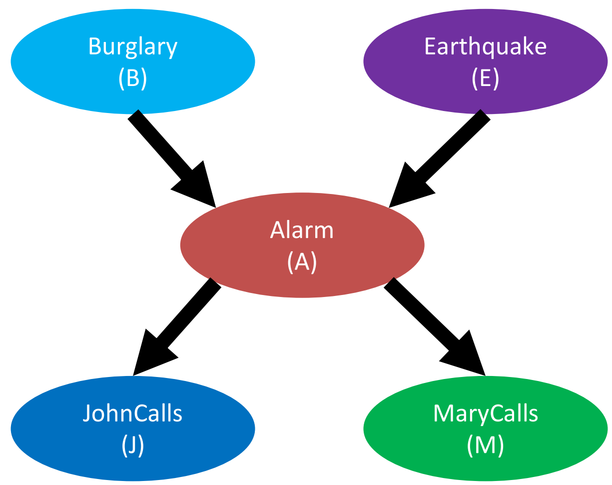

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f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$

Given:

- **a query involving a single variable** B
- **a <u>list</u> of evidence variables** $\mathbf{K}$**:** J, M
- **a <u>list</u> of observed values** $\mathbf{k}$ **for** $\mathbf{K}$**:** j, m
- **a list of remaining unobserved variables** $\mathbf{Y}$**:** E, A

the query can be evaluated as:

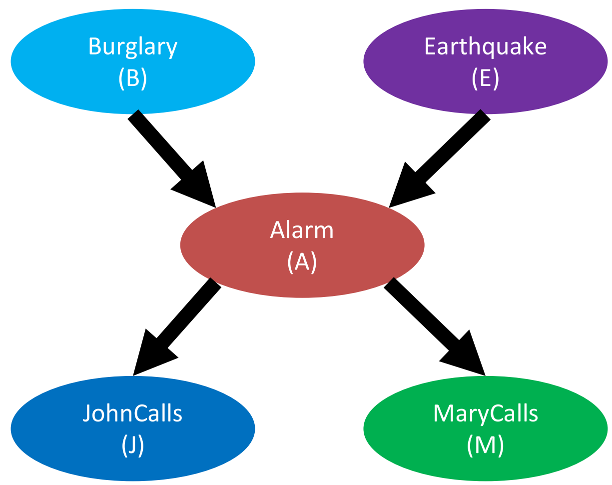
$$P(b \mid j, m) = \alpha * \sum_{e} \sum_{a} P(b, j, m, e, a)$$

By Chain rule:

$$P(b, j, m, e \ a)$$
$$= P(b) * P(e) * P(a|b, e) * P(j|a) * P(m|a)$$

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$

Given:

- **a query involving a single variable** B
- **a list of evidence variables** $\mathbf{K}$: J, M
- **a list of observed values** $\mathbf{k}$ **for** $\mathbf{K}$: j, m
- **a list of remaining unobserved variables** $\mathbf{Y}$: E, A

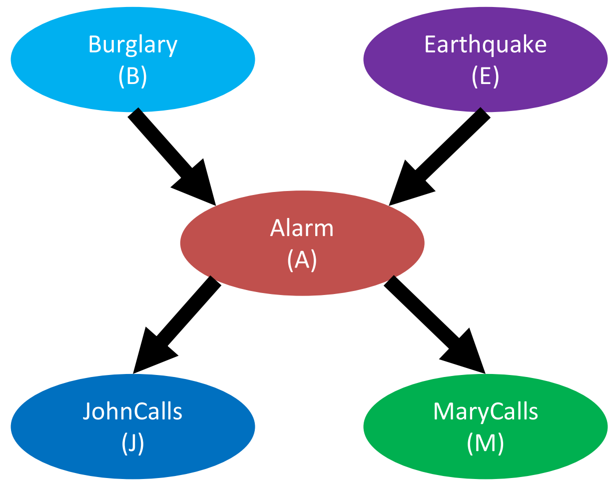
the query can be evaluated as:

$P(b \mid j, m)$

$$= \alpha * \sum_e \sum_a P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$$

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



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t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
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Inference

Query (let's change it a bit for simplicity):

$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$

Given:

- **a query involving a single variable** B
- **a <u>list</u> of evidence variables K:** J, M
- **a <u>list</u> of observed values k for K:** j, m
- **a list of remaining unobserved variables Y:** E, A

the query can be evaluated as:

$$P(b \mid j, m)$$

$$= \alpha * \sum_{e} \sum_{a} P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$$

$$= \alpha * P(b) * \sum_{e} P(e) * \sum_{a} P(a|b,e) * P(j|a) * P(m|a)$$

P(B)	P(¬B)
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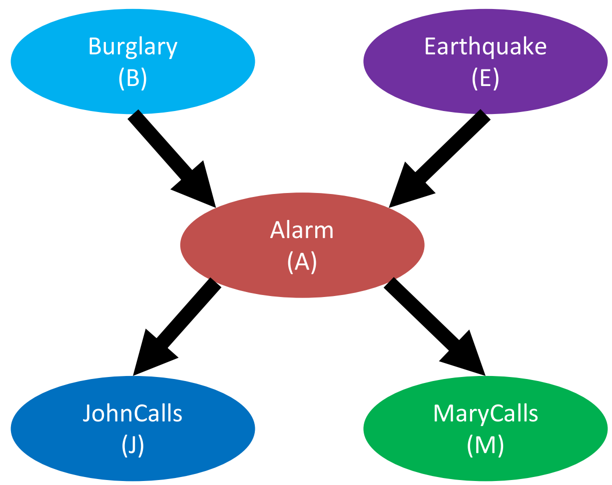

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A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$

Query rewritten:

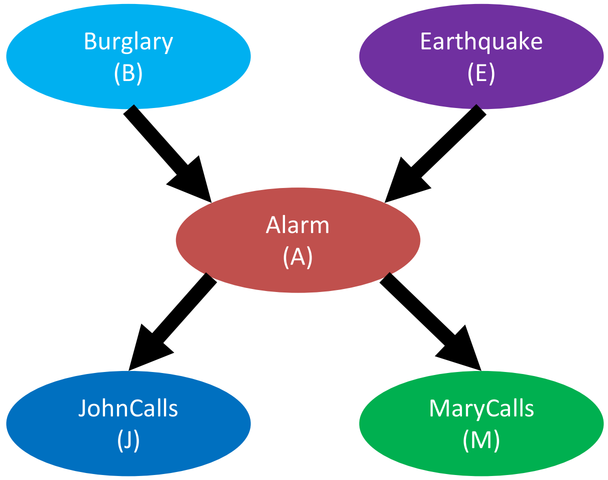
$P(b \mid j, m)$

$= \alpha * \sum_{e} \sum_{a} P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$

$= \alpha * P(b) * \sum_{e} P(e) * \sum_{a} P(a|b,e) * P(j|a) * P(m|a)$

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



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t	f	0.94
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Inference

Query (let's change it a bit for simplicity):

$$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$$

Query rewritten:

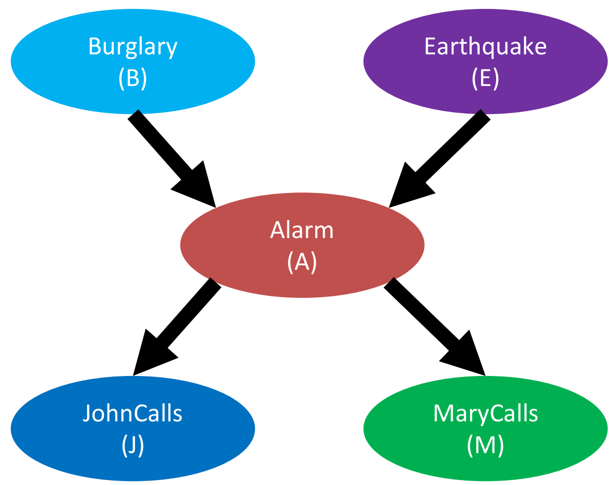
$$P(b \mid j, m)$$

$$= \alpha * \sum_{e} \sum_{a} P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$$

$$= \alpha * \boxed{P(b)} * \sum_{e} P(e) * \sum_{a} P(a|b,e) * P(j|a) * P(m|a)$$

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



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Inference

Query (let's change it a bit for simplicity):

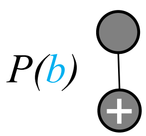
$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$

Query rewritten:

$P(b \mid j, m)$

$= \alpha * \sum_{e} \sum_{a} P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$

$= \alpha * P(b) * \boxed{\sum_{e}} P(e) * \sum_{a} P(a|b,e) * P(j|a) * P(m|a)$



P(B)	P(¬B)
0.001	0.999

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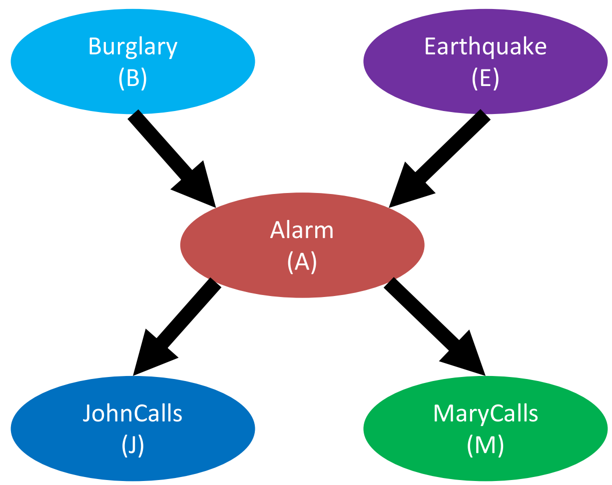
0.002	0.998



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A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

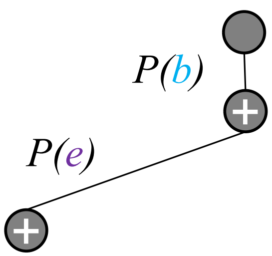
$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$

Query rewritten:

$$P(b \mid j, m)$$

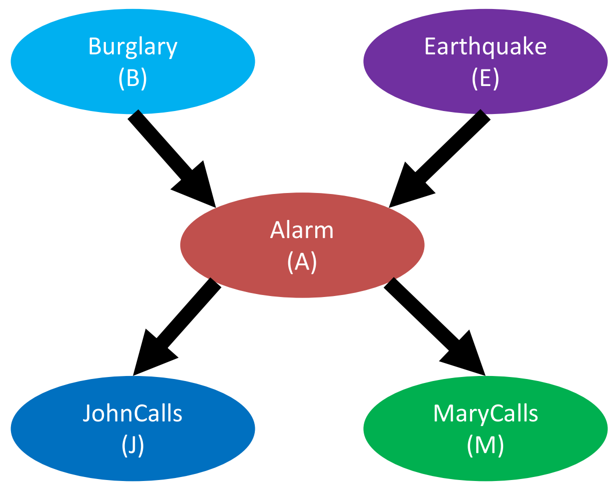
$$= \alpha * \sum_{e} \sum_{a} P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$$

$$= \alpha * P(b) * \boxed{\sum_{e} P(e) *} \sum_{a} P(a|b,e) * P(j|a) * P(m|a)$$



P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

$P(Burglary = true \mid JohnCalls = true \land MaryCalls = true)$

Query rewritten:

$$P(b \mid j, m)$$

$$= \alpha * \sum_e \sum_a P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$$

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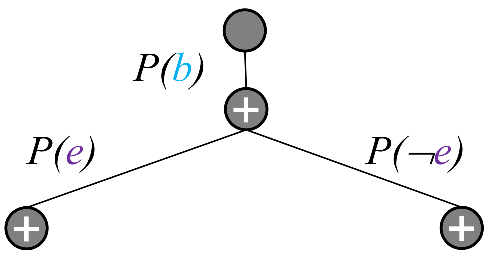

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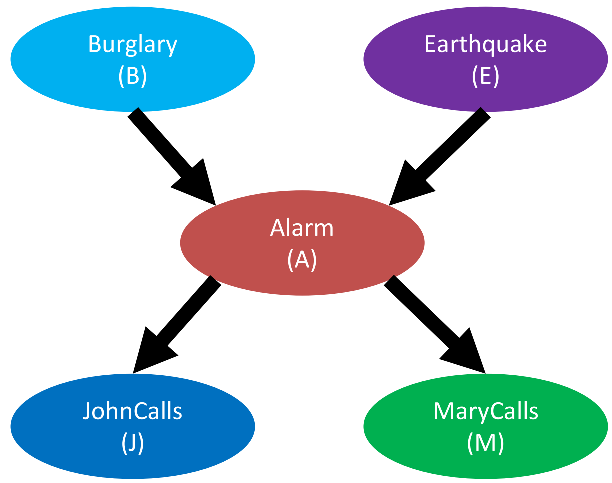

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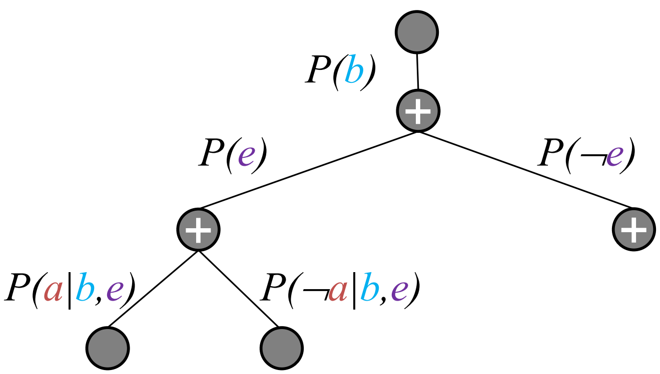

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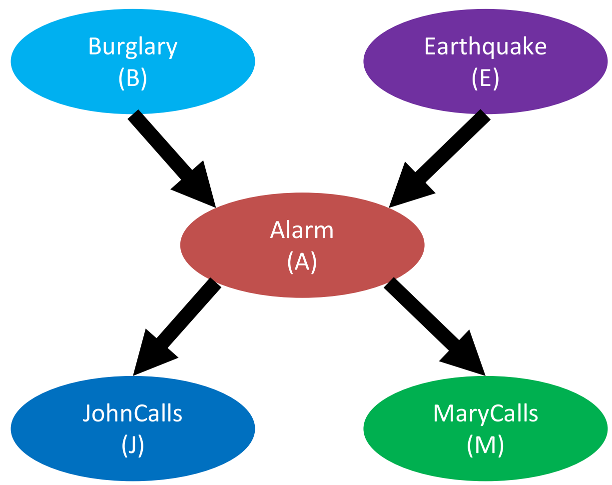

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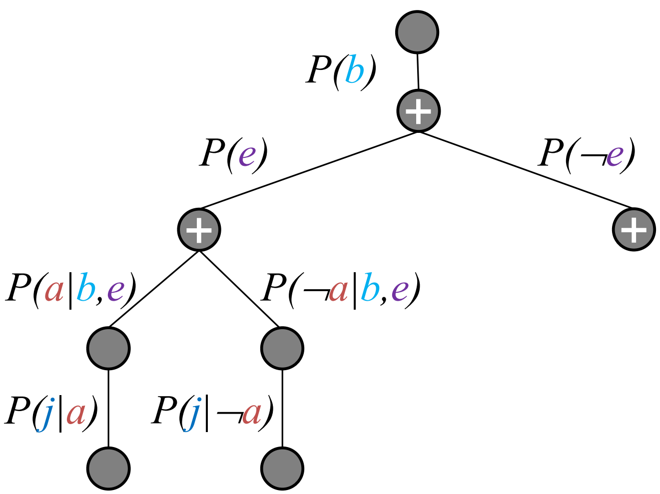

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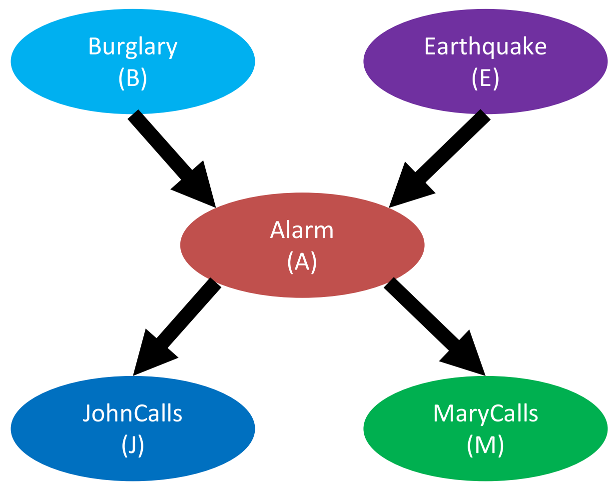

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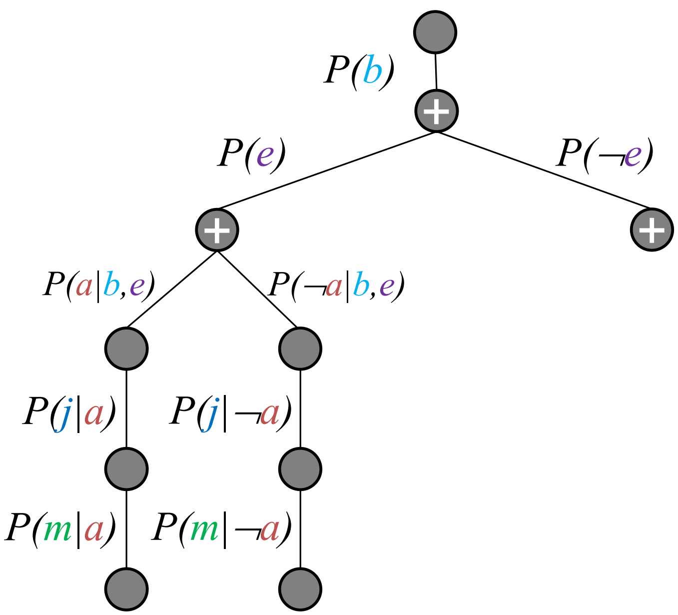

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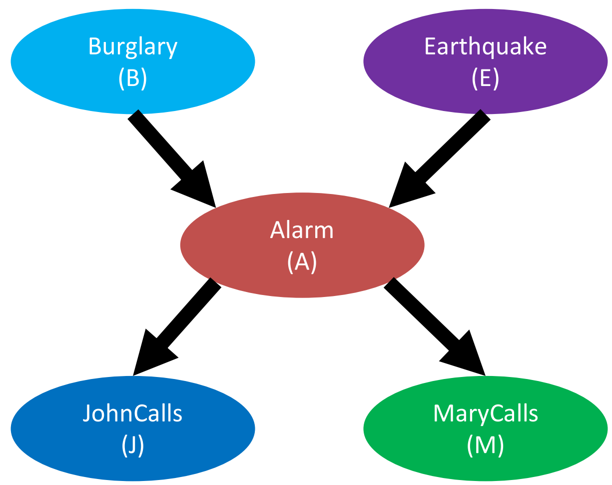

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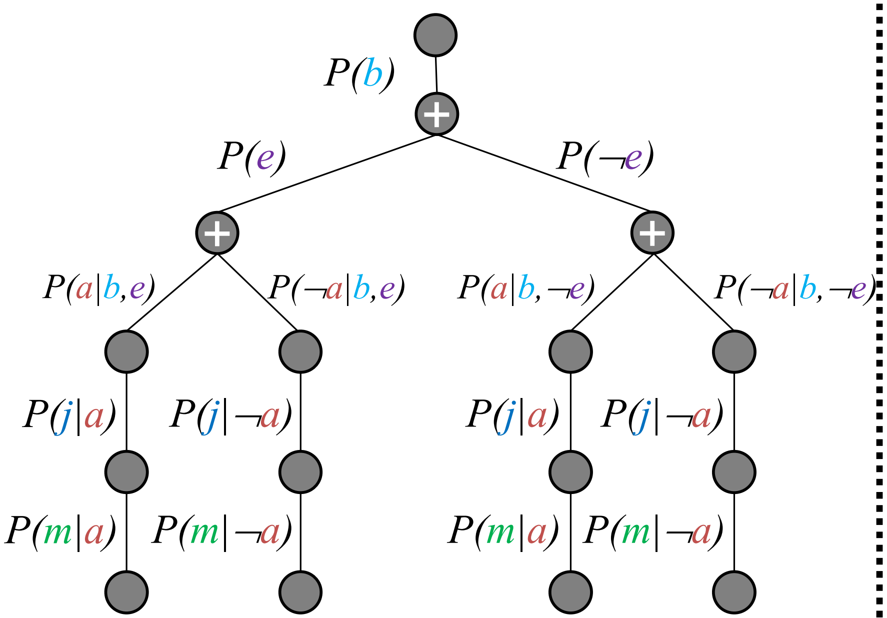

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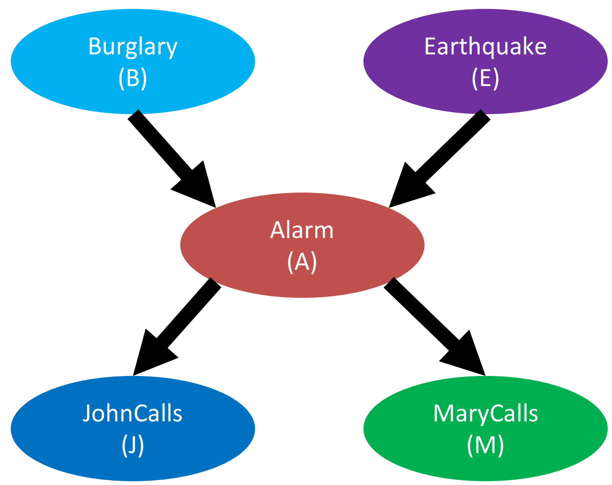

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Inference

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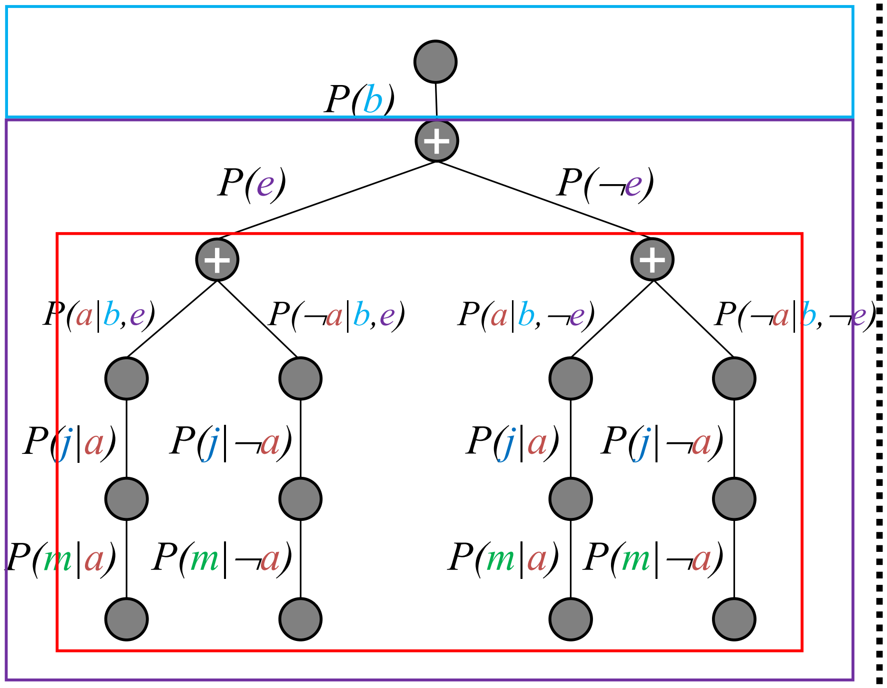
$P(b \mid j, m)$

$= \alpha * \sum_e \sum_a P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$

$= \alpha * \boxed{P(b)} * \sum_e P(e) * \boxed{\sum_a P(a|b,e) * P(j|a) * P(m|a)}$



P(B)	P(¬B)
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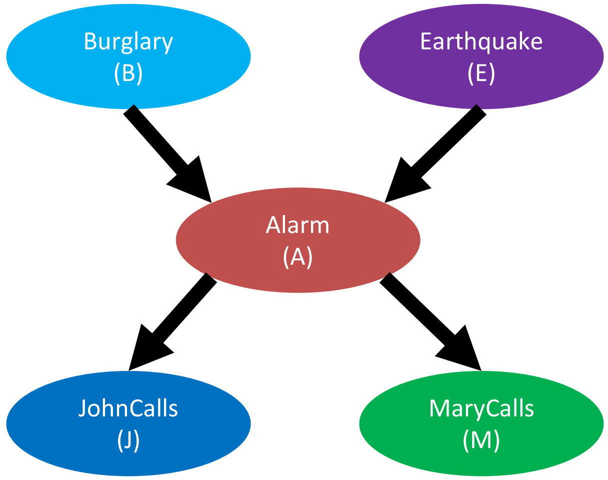
0.002	0.998



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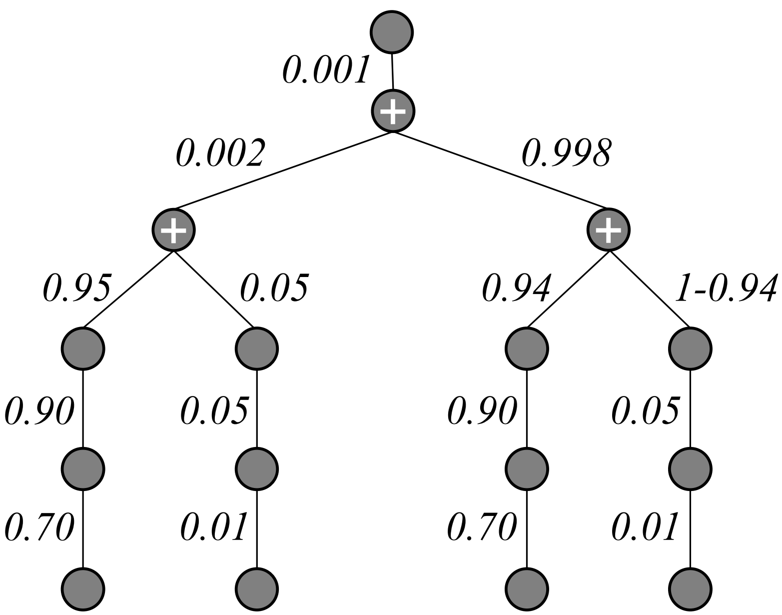

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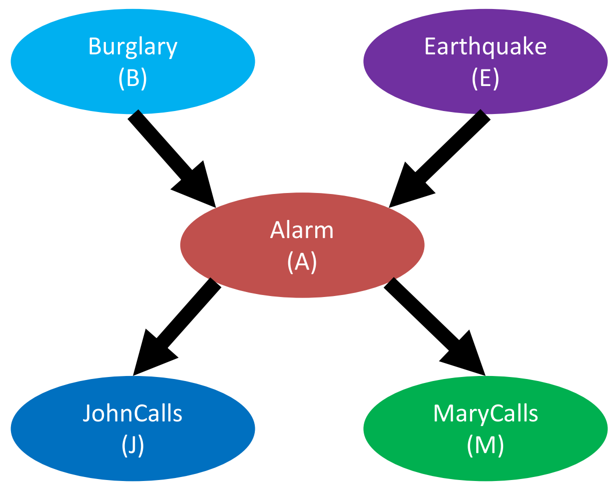

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f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

$$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$$

We can now calculate:

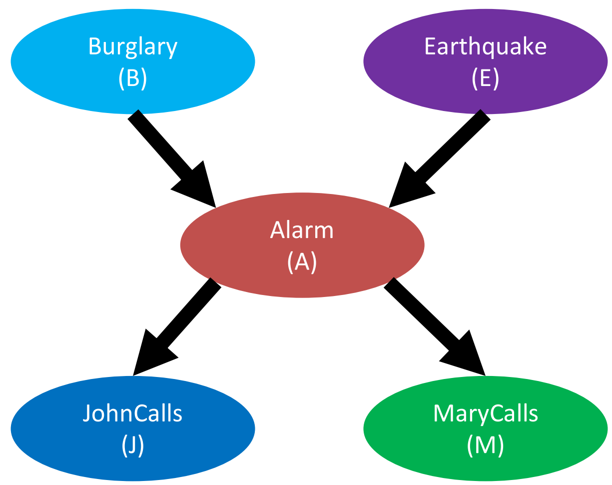
$$P(b \mid j, m) = \alpha * 0.00059224$$

And through similar process (not shown):

$$P(\neg b \mid j, m) = \alpha * 0.0014919$$

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (now we can get joint distribution):

$$\boldsymbol{P}(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$

We can now calculate:

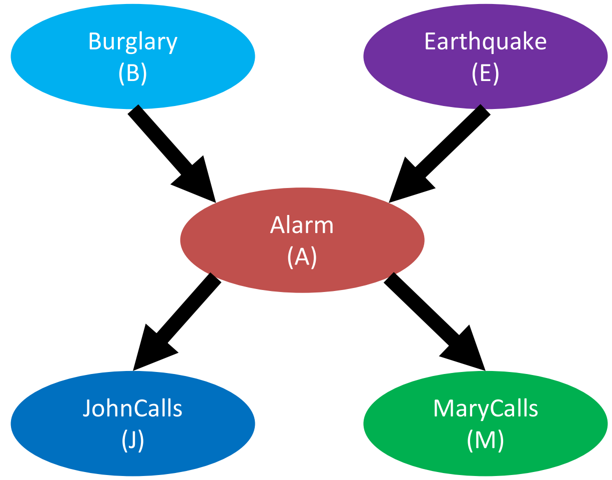
$$\boldsymbol{P}(B \mid j, m) = \alpha * < 0.00059224, 0.0014919 >$$

And after normalization:

$$\boldsymbol{P}(B \mid j, m) \approx < 0.284, 0.716 >$$

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998

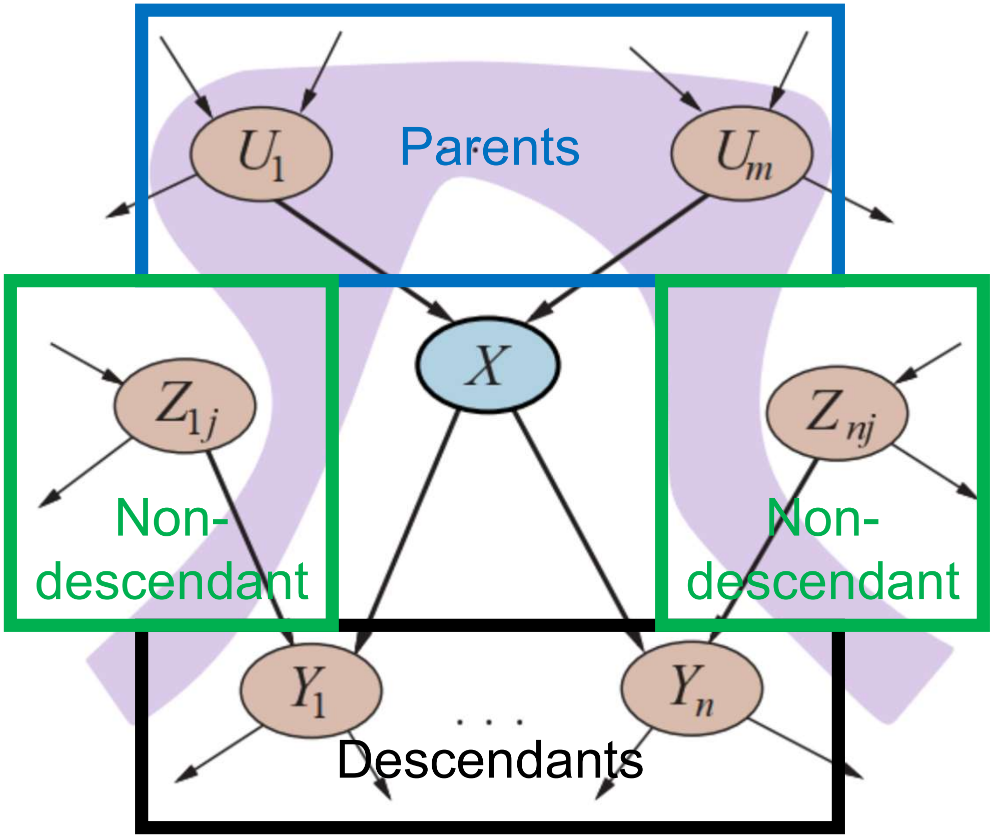


B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

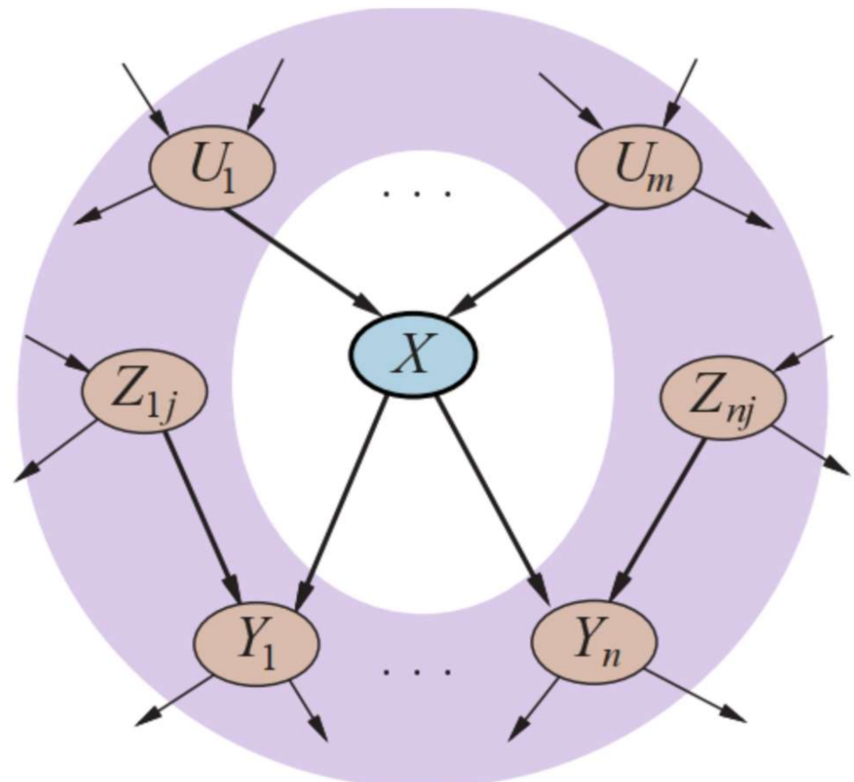
A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

More On Conditional Independence



Node X is conditionally independent of its non-descendants given its parents.



Node X is conditionally independent of ALL other nodes in the network its given its Markov blanket.

Why do we care?

An unconstrained joint probability distribution with N **binary** variables involves 2^{N} probabilities.

Bayesian network with at most k parents per each node (N) involves $\mathrm{N} * 2^{\mathrm{k}}$ probabilities ($\mathrm{k} < \mathrm{N}$).

Decision Networks

Decision Theory

- **Decisions**: every plan (**actions**) leads to an outcome (state)
- **Agents have preferences (preferred outcomes)**
- **Preferences $\rightarrow$ outcome utilities**
- **Agents have degrees of belief (probabilities) for actions**

Decision theory = probability theory + utility theory

Decision Theory

- Decisions: every plan (actions) leads to an outcome (state)
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- Preferences $\rightarrow$ outcome utilities
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Decision theory = $\underbrace{\text{probability theory}}_{\text{BELIEFS}}$ + $\underbrace{\text{utility theory}}_{\text{DESIRES}}$

Maximum Expected (Average) Utility

$$MEU(M) = P(S10) * U(S10) + P(S15) * U(S15) + P(S55) * U(S55)$$

Environment could be in one of those states:

A = 4, B = 7, C = 0 → Plan X
A = 4, B = 7, C = 1 → Plan Y
A = 4, B = 7, C = 2 → Plan X
A = 4, B = 7, C = 3 → Plan Z

Action M
- Outcome S10: P(S10) = 0.1 U(S10) = 2
- Outcome S15: P(S15) = 0.4 U(S15) = 3
- Outcome S55: P(S55) = 0.5 U(S55) = 1

Pick highest MEU action

Action N
- Outcome S20: P(S20) = 0.7 U(S20) = 8
- Outcome S15: P(S15) = 0.2 U(S15) = 4
- Outcome S12: P(S12) = 0.1 U(S12) = 5

$$MEU(N) = P(S20) * U(S20) + P(S15) * U(S15) + P(S12) * U(S12)$$

Agents Decisions

Recall that agent ACTIONS change the state:

- **if we are in state** s
- **action** a **is expected to**
- **lead to another state** s' **(outcome)**

Given uncertainty about the current state s **and action outcome** s' **we need to define the following:**

- **probability (belief) of being in state** s**:** $P(s)$
- **probability (belief) of action** a **leading to outcome** s'**:** $P(s' \mid s, a)$

Now:

$$P(s' \mid s, a) = P(\textsc{Result}(a) = s') = \sum_s P(s) * P(s' \mid s, a)$$

Expected Action Utility

The expected utility of an action a given the evidence is the average utility value of all possible outcomes s' of action a, weighted by their probability (belief) of occurence:

$$\mathrm{EU}(\mathrm{a}) = \sum_{s'} \sum_{s} \mathrm{P}(\mathrm{s}) * \mathrm{P}(\mathrm{s}' \mid \mathrm{s}, \mathrm{a}) * \mathrm{U}(\mathrm{s}') = \sum_{s'} \mathrm{P}(Result(a) = s') * \mathrm{U}(\mathrm{s}')$$

Rational agent should choose an action that maximizes the expected utility:

$$\text{chosen action} = \underset{\mathrm{a}}{\text{argmax}}\ \mathrm{EU}(\mathrm{a})$$

State Utility Function

Agent's preferences (desires) are captured by the Utility function $U(s)$.

Utility function assigns a value to each state s to express how desirable this state is to the agent.

Decision Network (Influence Diagram)

Decision networks (also called influence diagrams) are structures / mechanisms for making rational decisions.

Decision networks are based on Bayesian networks, but include additional nodes that represent actions and utilities.

Decision Networks

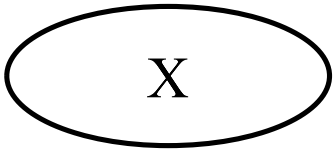
The most basic decision network needs to include:

- **information about current state** s
- **possible actions**
- **resulting state** s' **(after applying chosen action** a**)**
- **utility of the resulting state** $U(s')$

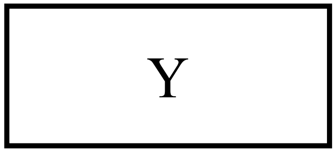
Decision Network Nodes

Decision networks are built using the following nodes:

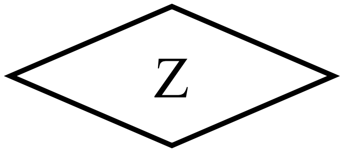
- **chance nodes:**



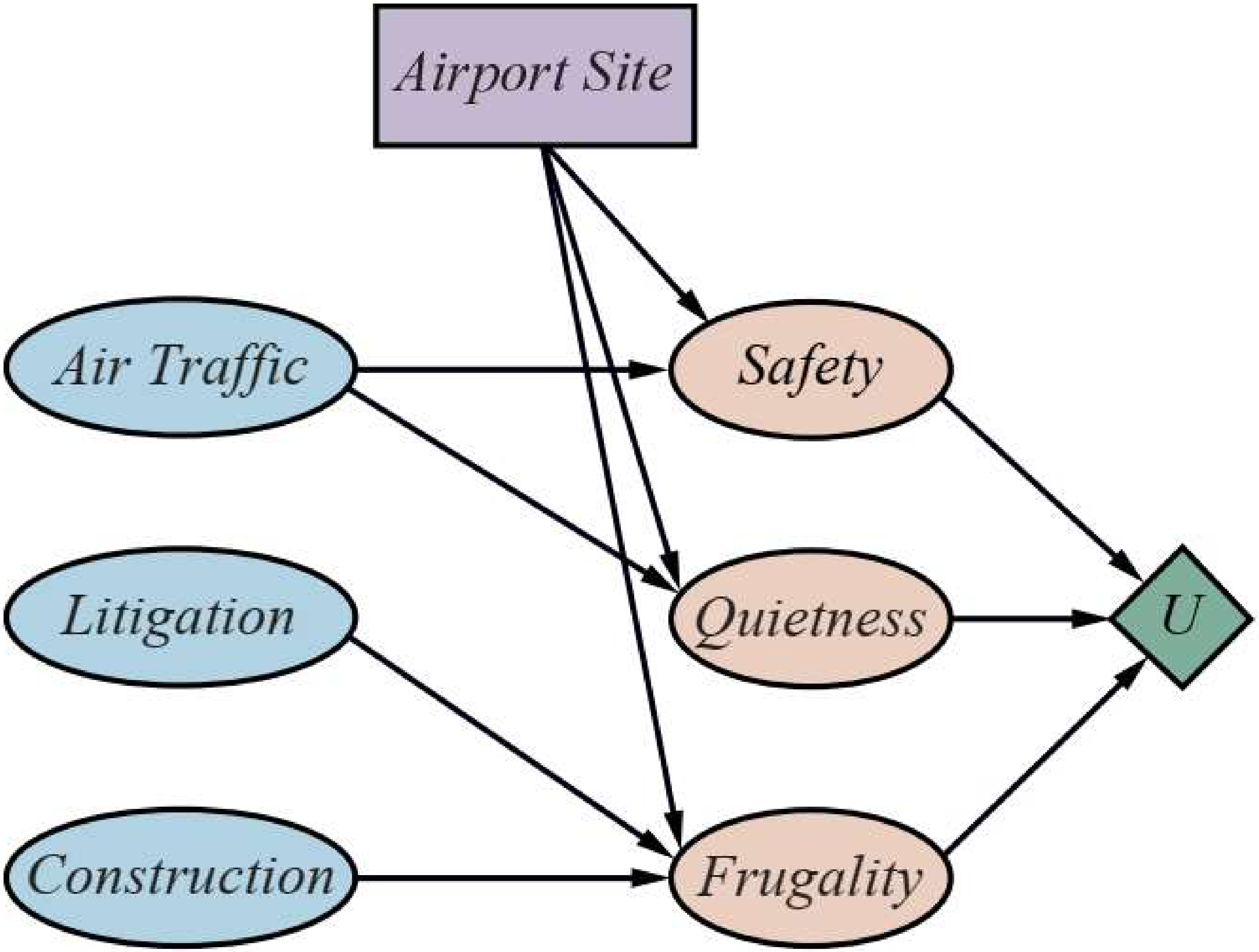
- **decision nodes:**



- **utility (or value) nodes**



Decision Network: Example



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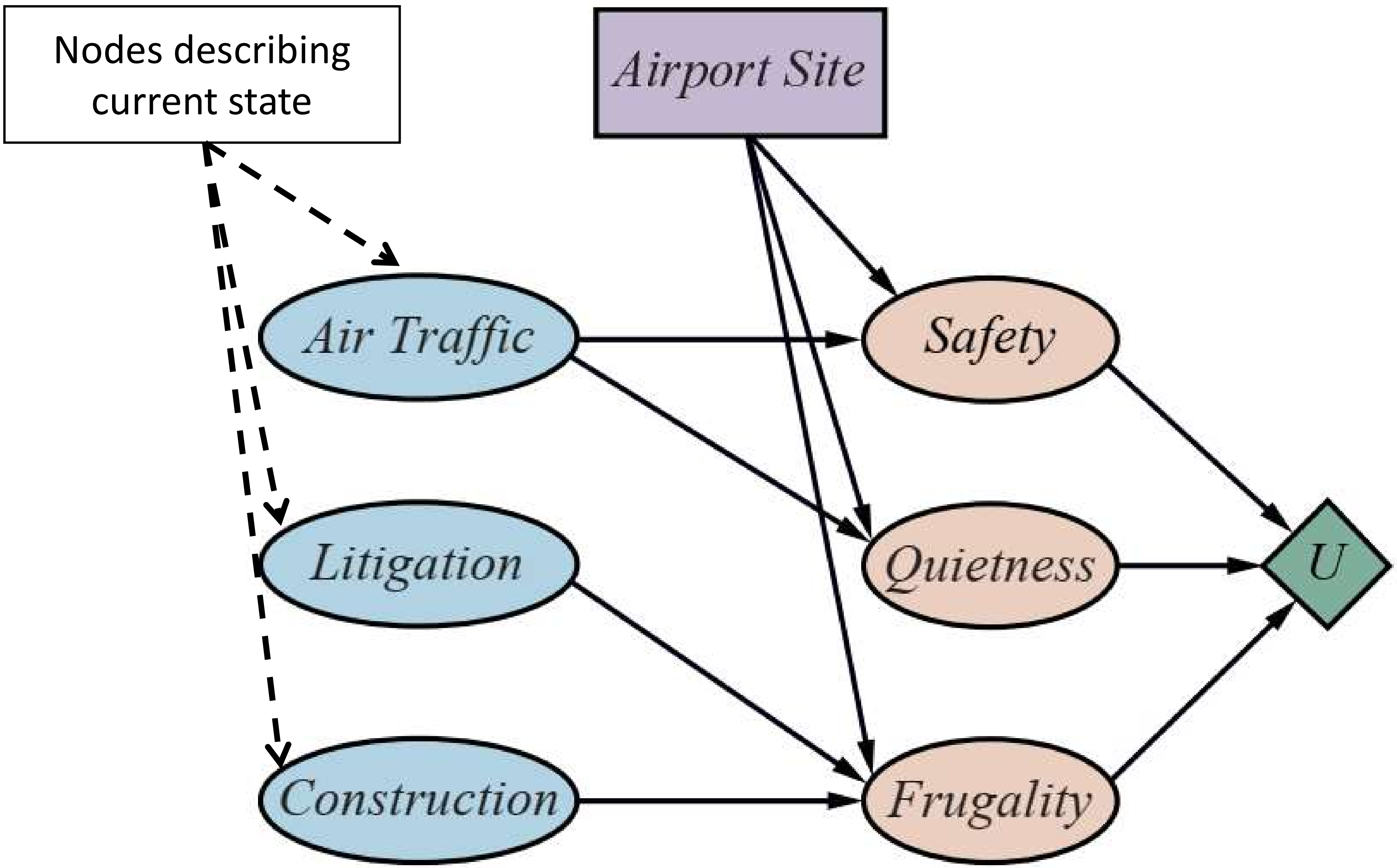

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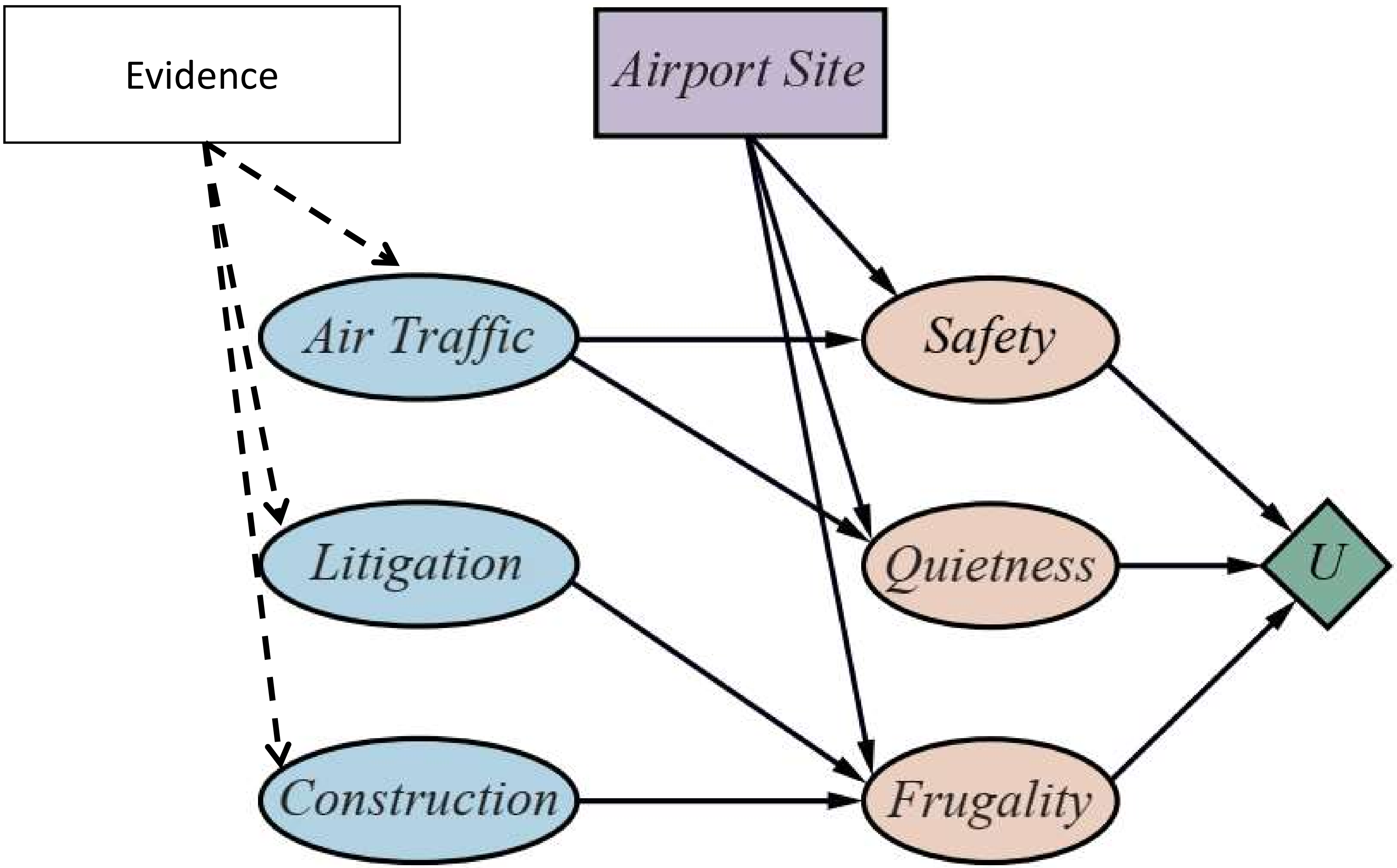

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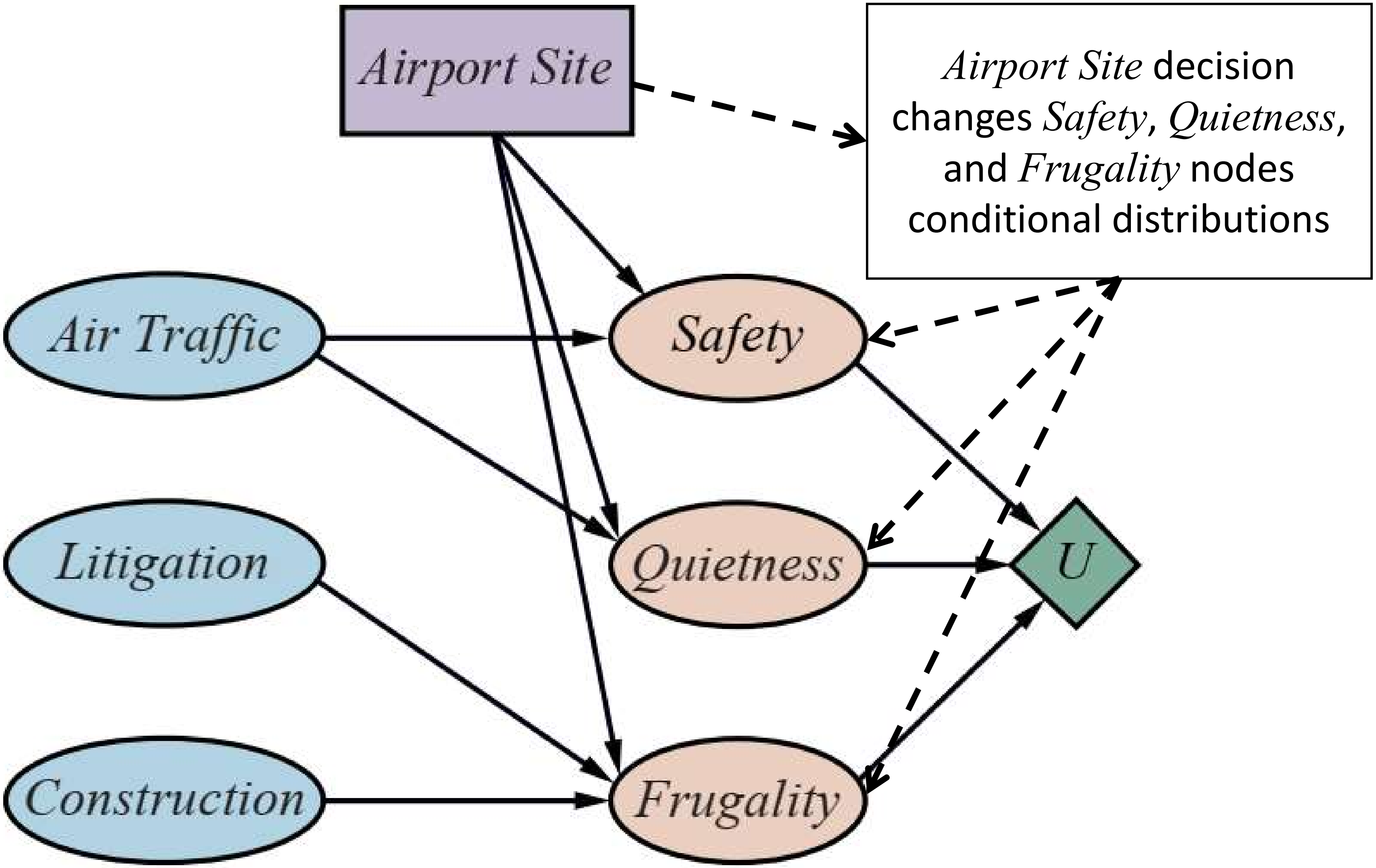

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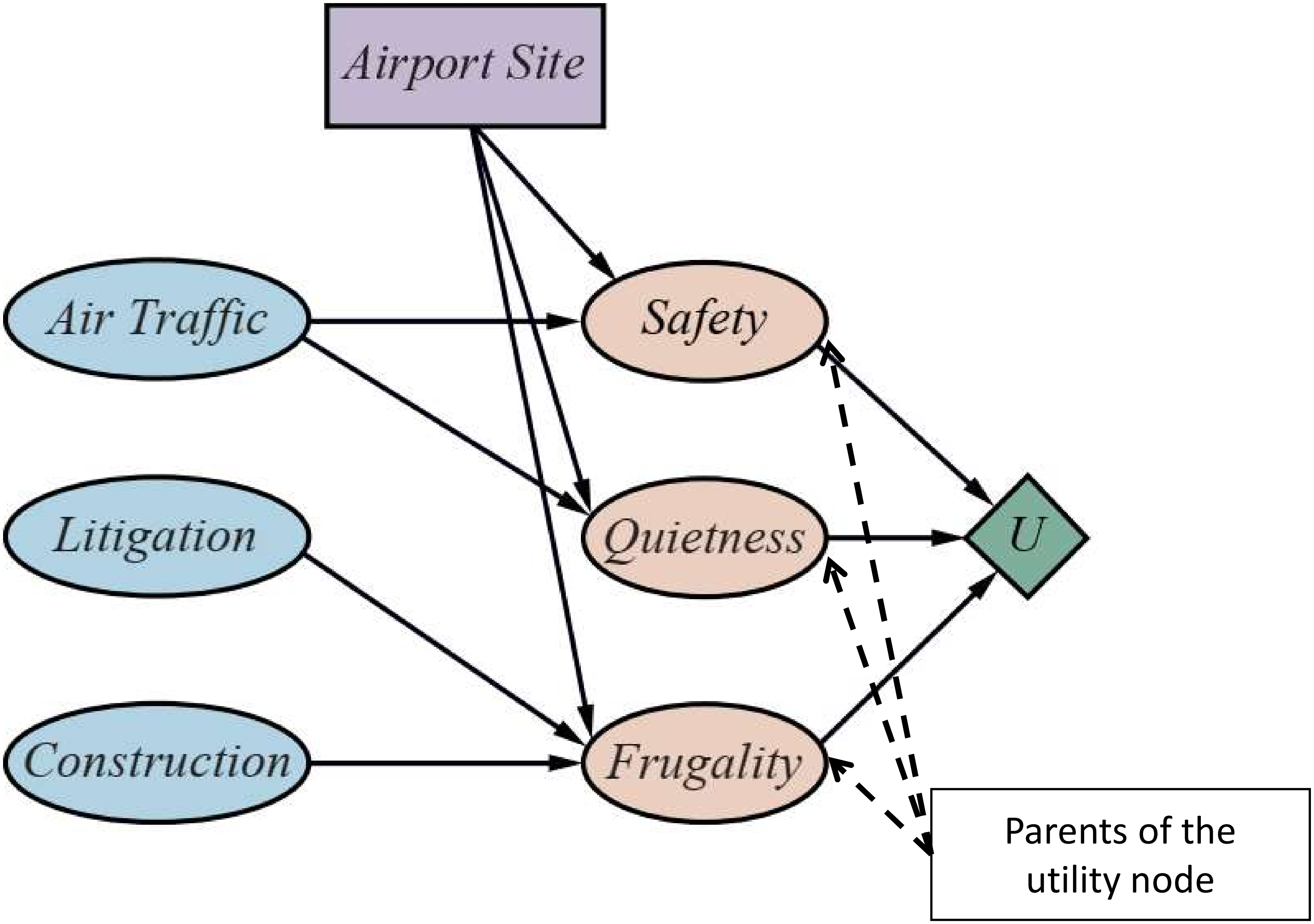

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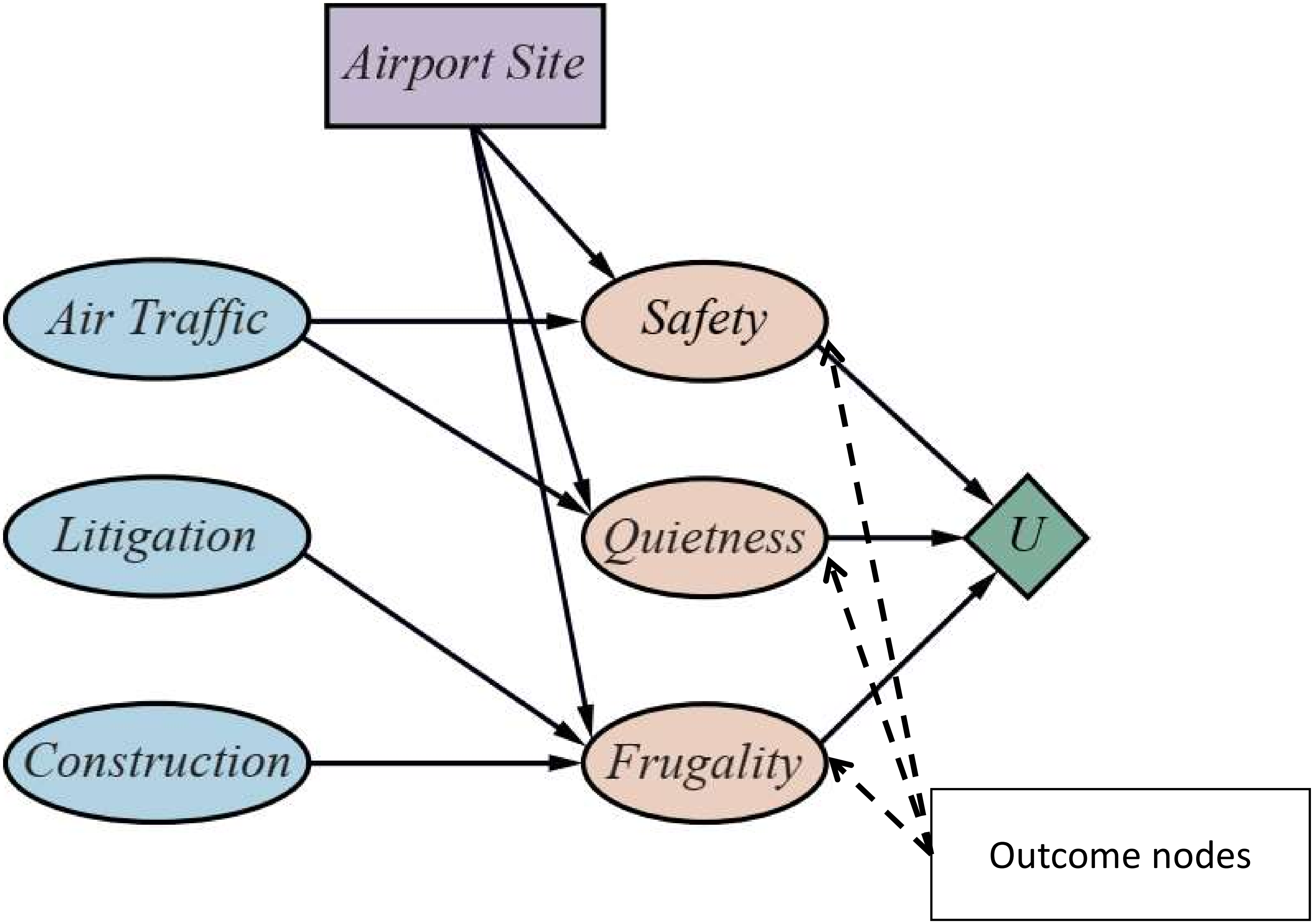

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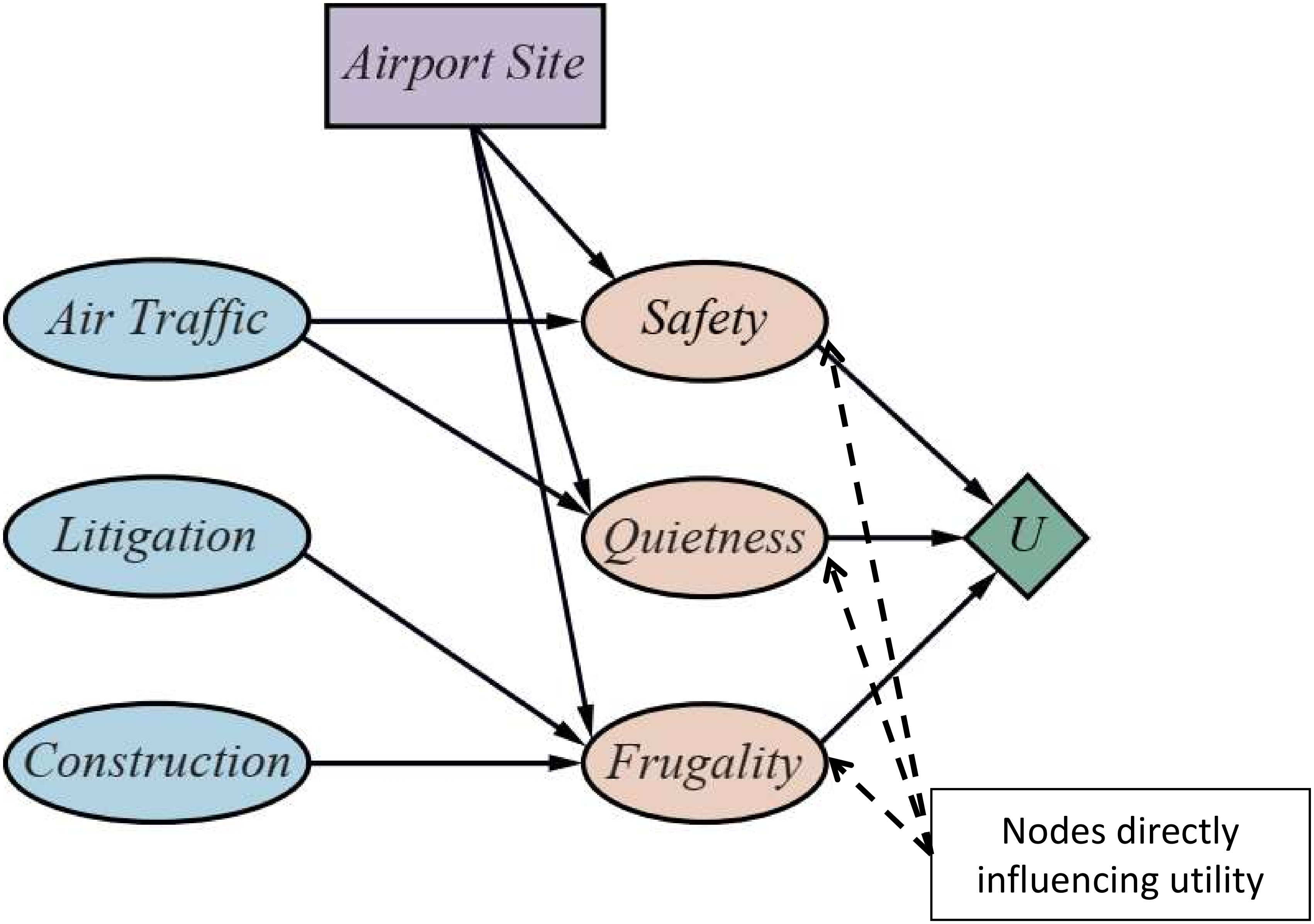
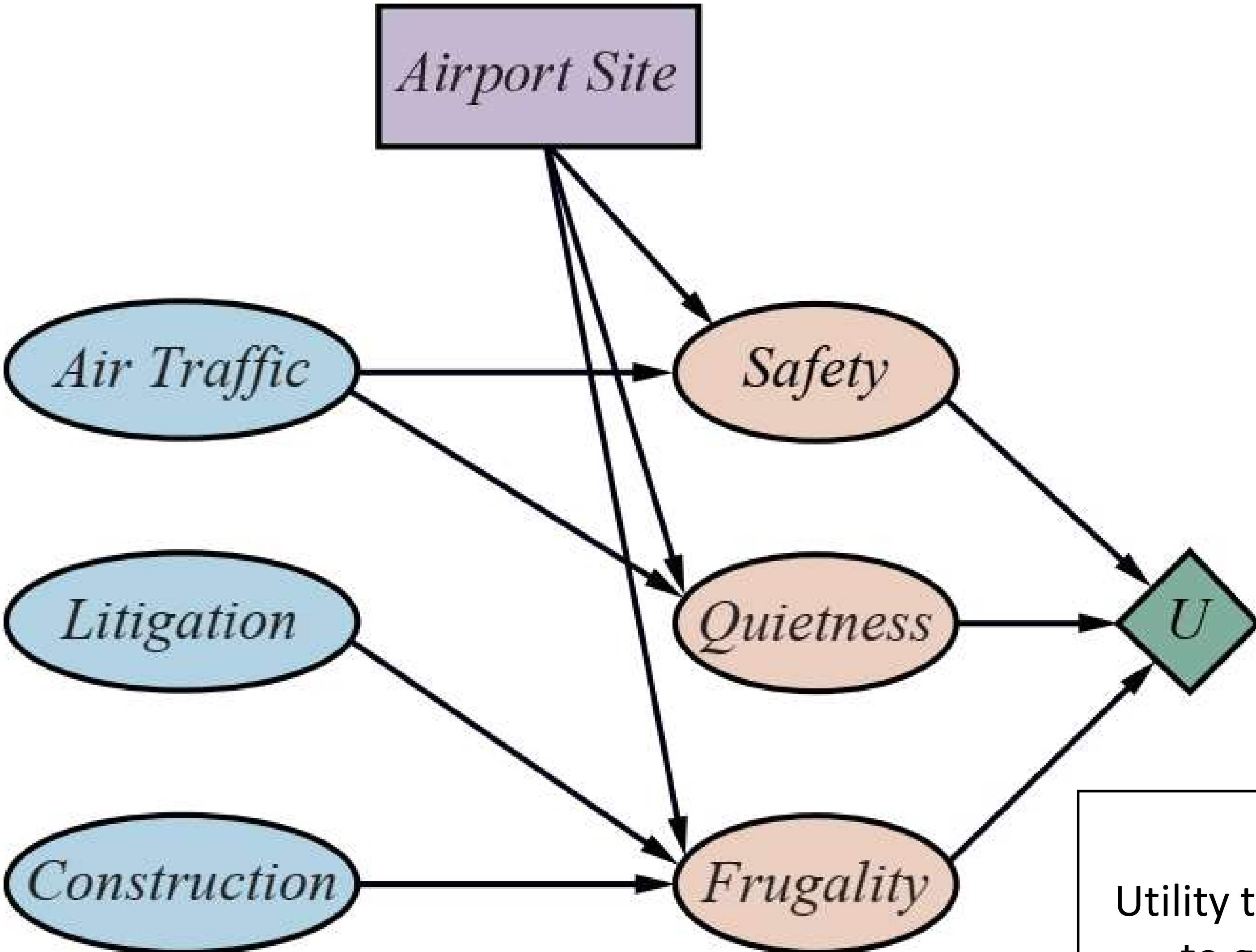


Decision Network: Evaluation

The algorithm for decision network evaluation is as follows:

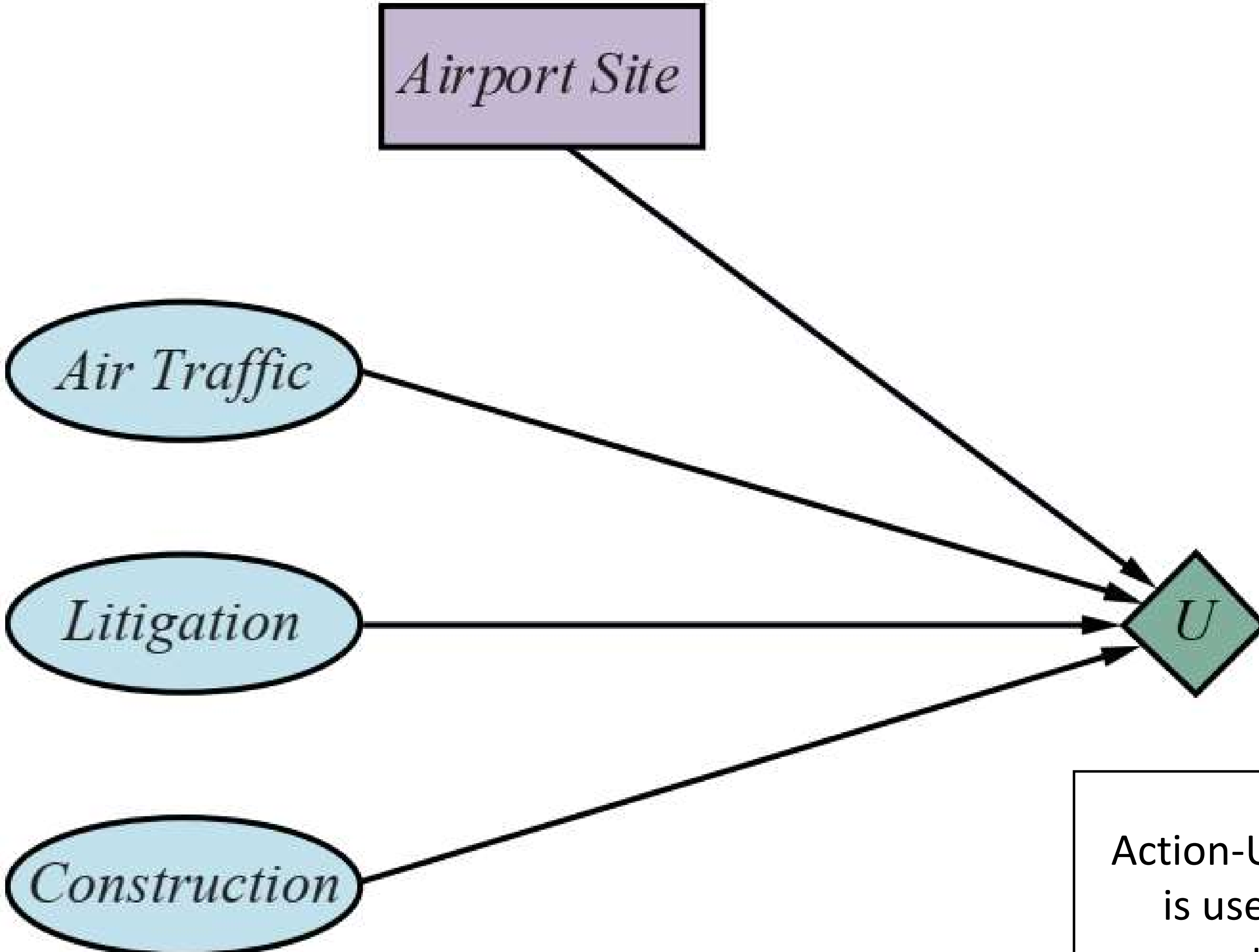
1. Set the evidence variables for the current state
2. For each possible value a of decision node:
   a. Set the decision node to that value
   b. Calculate the posterior probabilities for the parent nodes of the utility node
   c. Calculate the utility for the action / value a
3. Return the action with highest utility

Decision Network: Example



Utility table is used to get utility

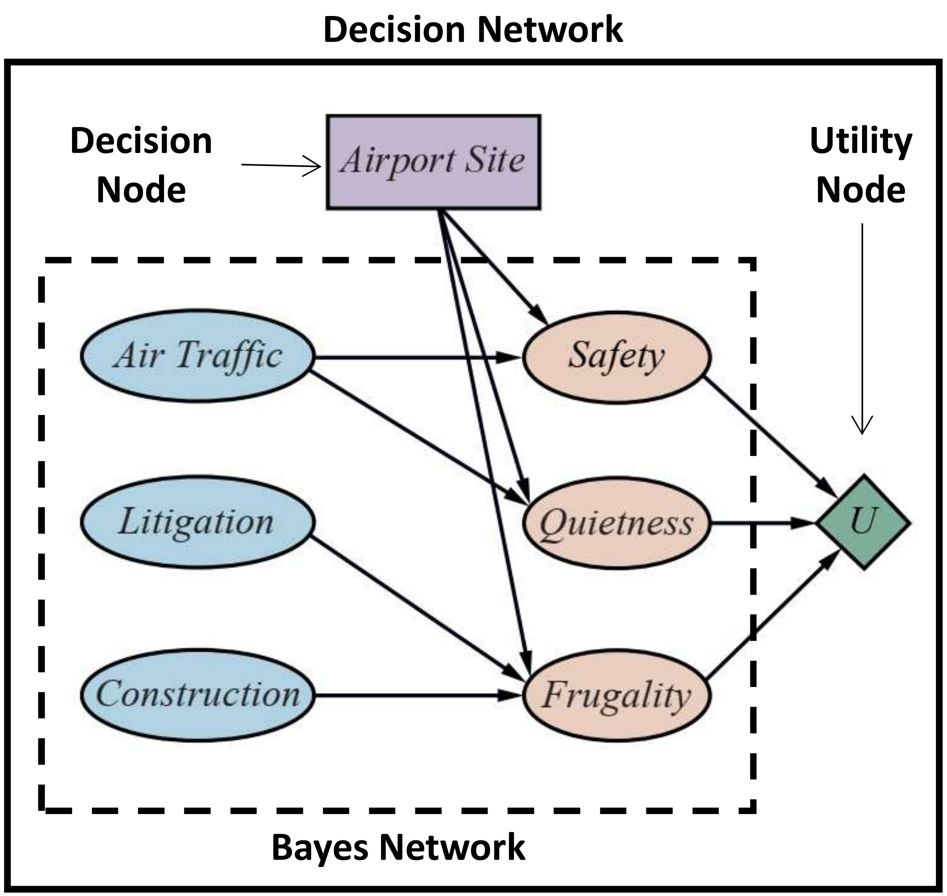
Decision Network: Simplified Form



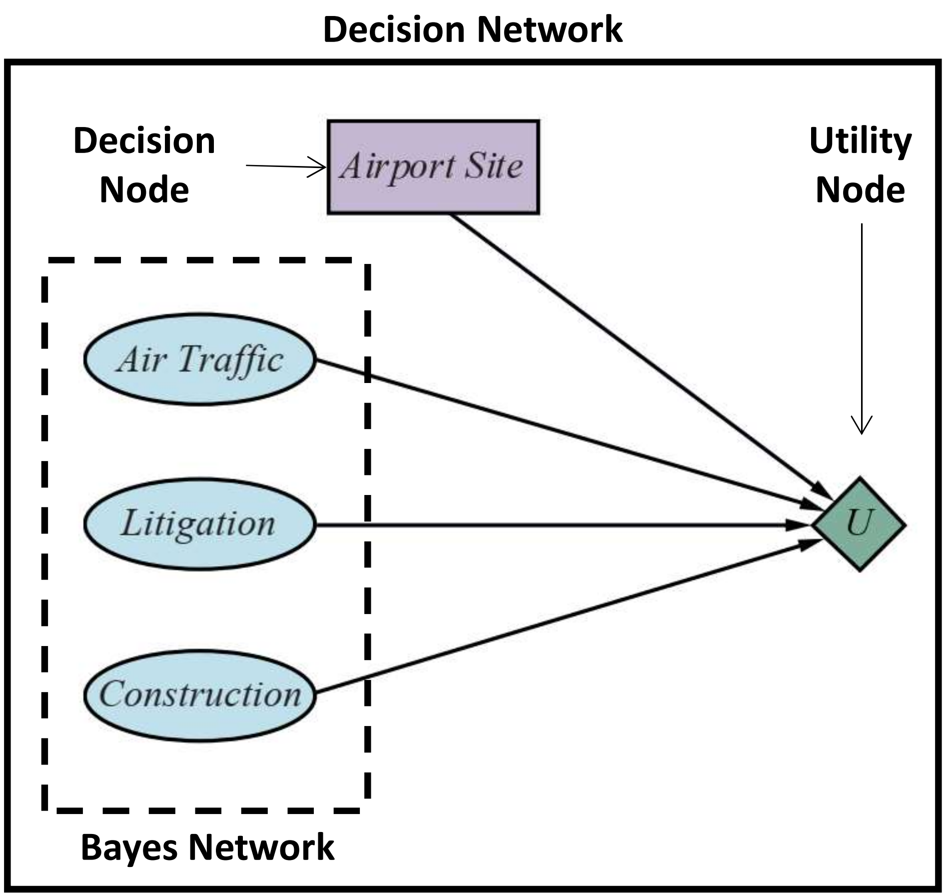
Action-Utility table is used to get expected utility

(Single-Stage) Decision Networks

General Structure

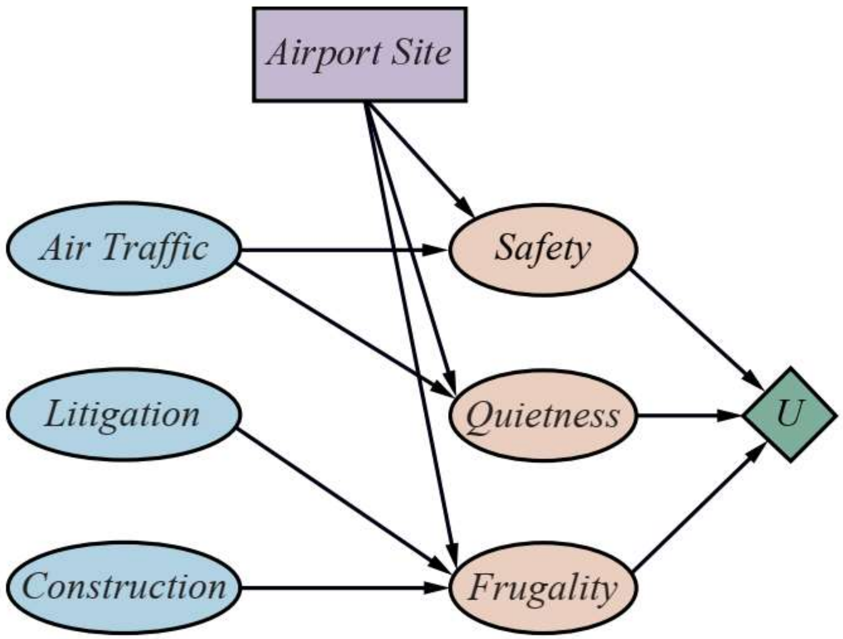


Simplified Structure



(Single-Stage) Decision Networks

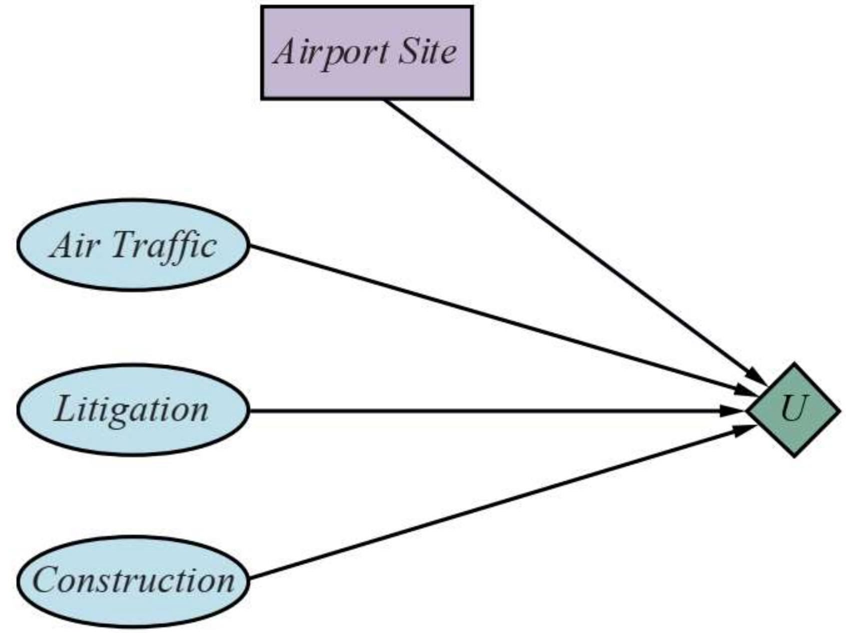
General Structure



Utility Table

S	low	low	low	low	high	high	high	high
Q	low	low	high	high	low	low	high	high
F	low	high	low	high	low	high	low	high
U	10	20	5	50	70	150	100	200

Simplified Structure



Action-Utility Table (not all columns shown)

AT	low	low	low	---	---	high	high	high
L	low	low	high	---	---	low	high	high
C	low	high	low	---	---	high	low	high
AS	A	A	A	---	---	B	B	B
U	10	20	5	---	---	150	100	200

Decision Network: Evaluation

The algorithm for decision network evaluation is as follows:

1. Set the evidence variables for the current state
2. For each possible value a of decision node:
   a. Set the decision node to that value
   b. Calculate the posterior probabilities for the parent nodes of the utility node
   c. Calculate the utility for the action / value a
3. Return the action with highest utility

Agent's Decisions

Recall that agent ACTIONS change the state:

- **if we are in state** s
- **action** a **is expected to**
- **lead to another state** s' **(outcome)**

Given uncertainty about the current state s **and action outcome** s' **we need to define the following:**

- **probability (belief) of being in state** s**:** $P(s)$
- **probability (belief) of action** a **leading to outcome** s'**:** $P(s' \mid s, a)$

Now:

$$P(s' \mid s, a) = P(\textsc{Result}(a) = s') = \sum_s P(s) * P(s' \mid s, a)$$

Expected Action Utility

The expected utility of an action a **given the evidence is the average utility value of all possible outcomes** s' **of action** a, **weighted by their probability (belief) of occurence:**

$$\mathrm{EU}(a) = \sum_{s'} \sum_{s} \mathrm{P}(s) * \mathrm{P}(s' \mid s, a) * \mathrm{U}(s') = \sum_{s'} \mathrm{P}(Result(a) = s') * \mathrm{U}(s')$$

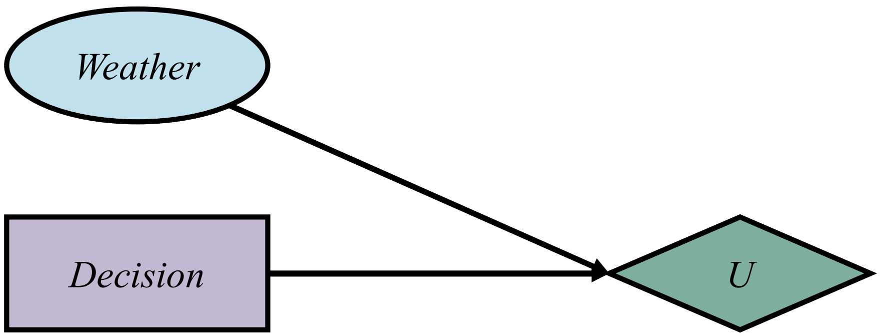
Rational agent should choose an action that maximizes the expected utility:

$$\text{chosen action} = \underset{a}{\text{argmax}}\ \mathrm{EU}(a)$$

Decision Networks: Example

Decision: take umbrella

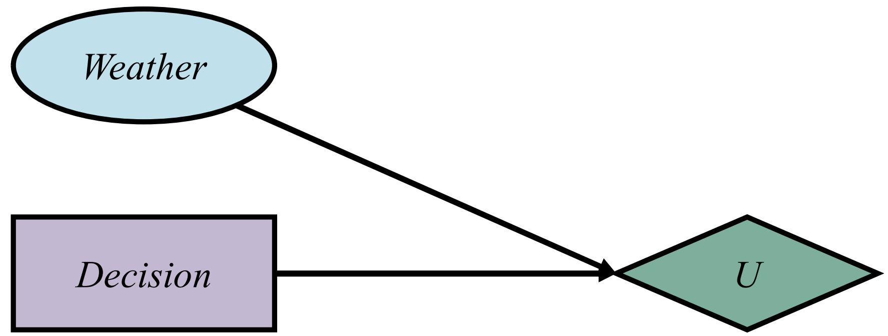
P(W=rain)	P(W=sun)
0.30	0.70



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Decision: leave umbrella

P(W=rain)	P(W=sun)
0.30	0.70



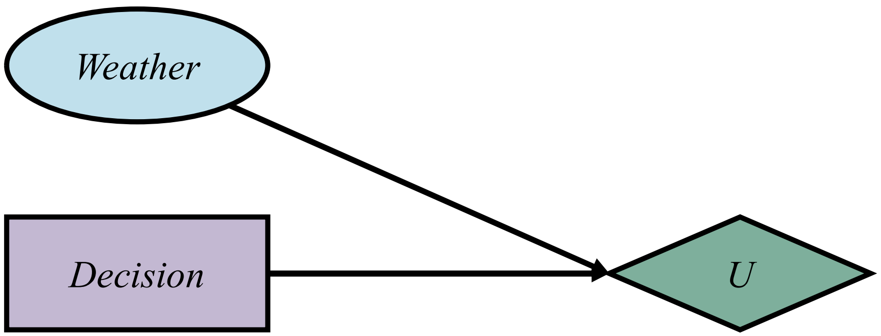
D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Decision Networks: Example

Decision: take umbrella

$$EU(a) = \sum_{s'} P(Result(a) = s') * U(s')$$

P(W=rain)	P(W=sun)
0.30	0.70



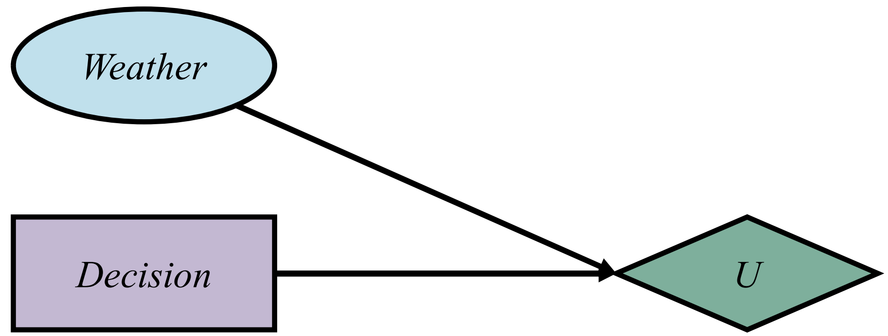
D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

$$EU(take) = ???$$

Decision: leave umbrella

$$EU(a) = \sum_{s'} P(Result(a) = s') * U(s')$$

P(W=rain)	P(W=sun)
0.30	0.70



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

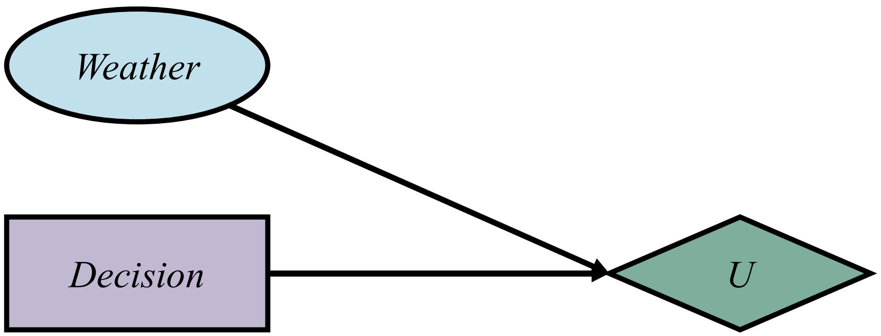
$$EU(leave) = ???$$

Decision Networks: Example

Decision: take umbrella

$$EU(a) = \sum_{s'} P(Result(a) = s') * U(s')$$

P(W=rain)	P(W=sun)
0.30	0.70



S_1': D = take, W = sun
S_2': D = take, W = rain
EU(take) =
P(Result(take) = S_1')*U(S_1') +
P(Result(take) = S_2')*U(S_2') =
0.70 * 20 + 0.30 * 70 = 35

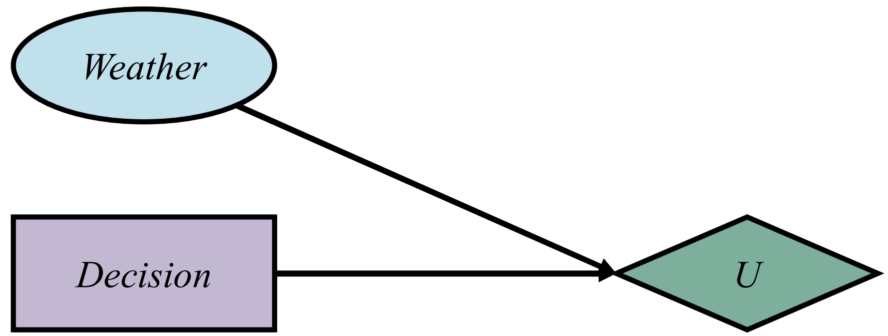
D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

$$EU(take) = 35$$

Decision: leave umbrella

$$EU(a) = \sum_{s'} P(Result(a) = s') * U(s')$$

P(W=rain)	P(W=sun)
0.30	0.70



S_3': D = leave, W = sun
S_4': D = leave, W = rain
EU(leave) =
P(Result(leave) = S_3')*U(S_3') +
P(Result(leave) = S_4')*U(S_4') =
0.70 * 100 + 0.30 * 0 = 70

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

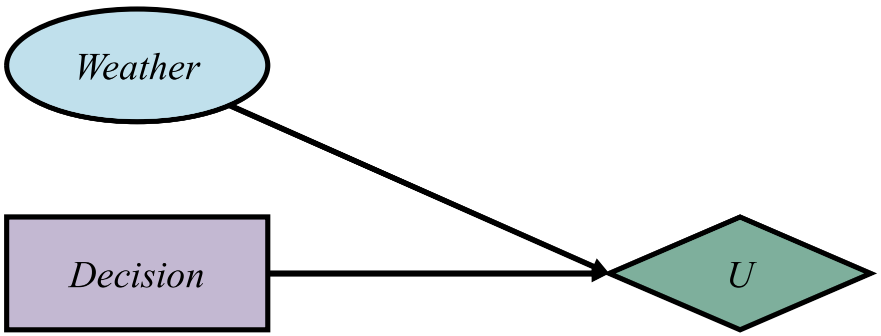
$$EU(leave) = 70$$

Decision Networks: Example

Which action to choose: take or leave Umbrella?

$EU(a) = \sum_{s'} P(Result(a) = s') * U(s')$

P(W=rain)	P(W=sun)
0.30	0.70

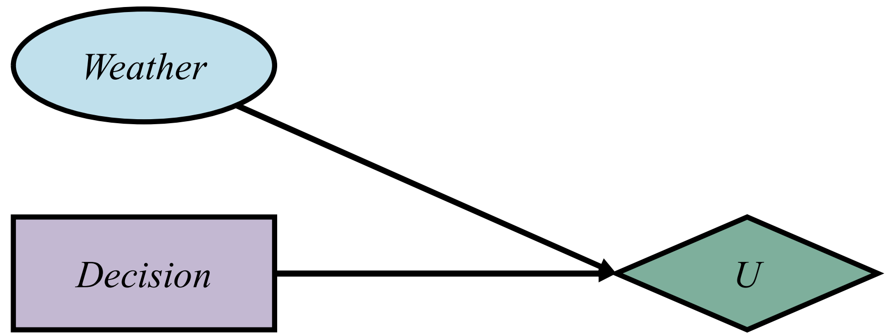


S_1': D = take, W = sun
S_2': D = take, W = rain
EU(take) =
P(Result(take) = S_1')*U(S_1') +
P(Result(take) = S_2')*U(S_2') =
0.70 * 20 + 0.30 * 70 = 35

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

$EU(a) = \sum_{s'} P(Result(a) = s') * U(s')$

P(W=rain)	P(W=sun)
0.30	0.70



S_3': D = leave, W = sun
S_4': D = leave, W = rain
EU(leave) =
P(Result(leave) = S_3')*U(S_3') +
P(Result(leave) = S_4') U(S_4') =
0.70 * 100 + 0.30 * 0 = 70

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

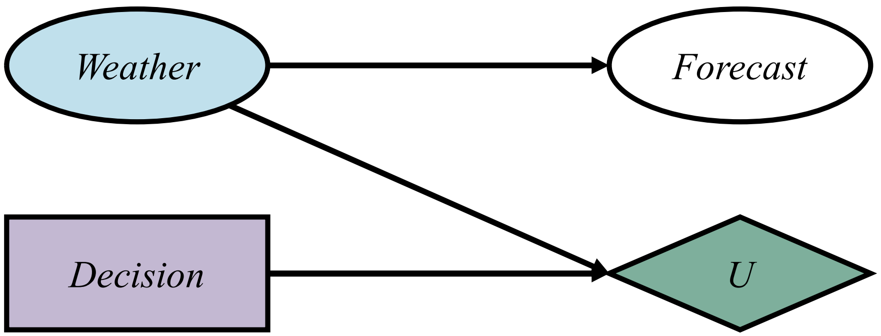
$$\text{action} = \underset{a}{\text{argmax}}\ EU(a) \mid \max(EU(take), \underline{EU(leave)}) = \max(35, \mathbf{70}) \rightarrow leave$$

Decision Networks: Example

Decision: take umbrella

P(W=rain)	P(W=sun)
0.30	0.70

P(F=sun)	P(F=rain)
0.59	0.41

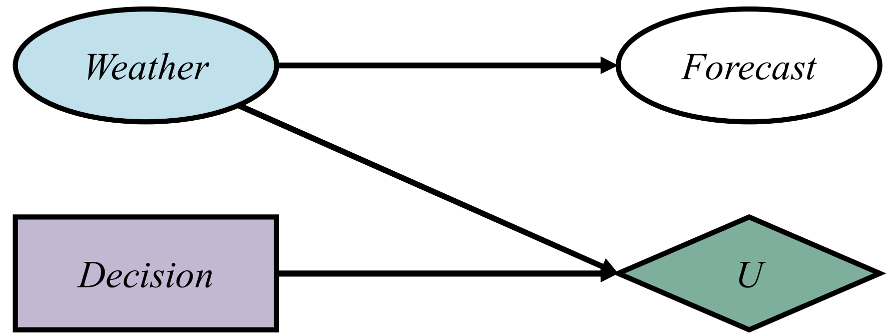


D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Decision: leave umbrella

P(W=rain)	P(W=sun)
0.30	0.70

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

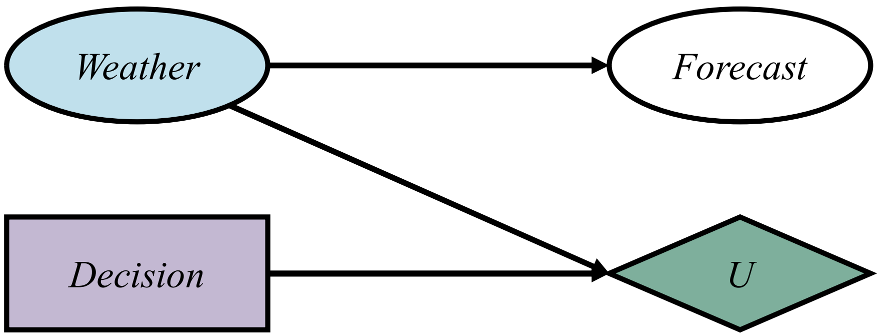
Decision Networks: Example

Decision: take umbrella given e

$\mathrm{EU}(a \mid \mathrm{e}) = \sum_{s'} \mathrm{P}(Result(a) = s' \mid \mathrm{e}) * \mathrm{U}(s')$

P(rain | F)	P(sun | F)
???	???

P(F=sun)	P(F=rain)
0.59	0.41



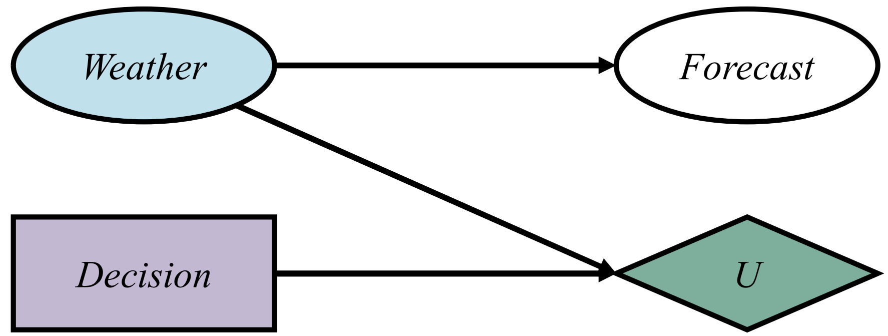
D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Decision: leave umbrella given e

$\mathrm{EU}(a \mid \mathrm{e}) = \sum_{s'} \mathrm{P}(Result(a) = s' \mid \mathrm{e}) * \mathrm{U}(s')$

P(rain | F)	P(sun | F)
???	???

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

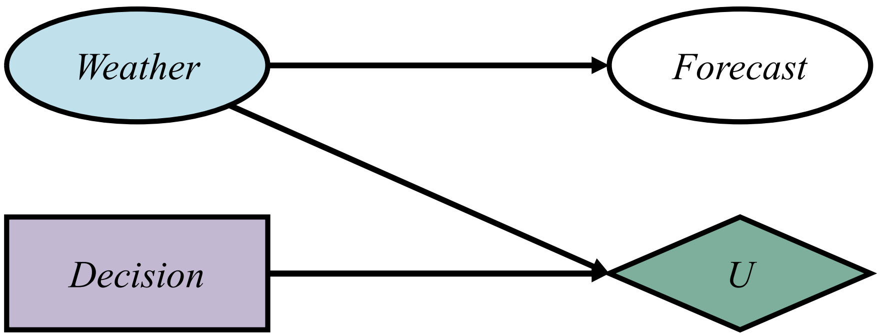
Decision Networks: Example

Decision: take umbrella given e

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(W=rain)	P(W=sun)
0.30	0.70

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Conditional probabilities

Assume that we are given:

F	W	P(F|W)
sun	sun	0.80
rain	sun	0.20
sun	rain	0.10
rain	rain	0.90

By Bayes' Theorem:

$$P(W = sun \mid F = sun) = \frac{P(F = sun \mid W = sun) * P(W = sun)}{P(F = sun)} = \frac{0.80 * 0.70}{0.59} = 0.95$$

$$P(W = sun \mid F = rain) = \frac{P(F = rain \mid W = sun) * P(W = sun)}{P(F = rain)} = \frac{0.20 * 0.70}{0.41} = 0.34$$

$$P(W = rain \mid F = sun) = \frac{P(F = sun \mid W = rain) * P(W = rain)}{P(F = sun)} = \frac{0.10 * 0.30}{0.59} = 0.05$$

$$P(W = rain \mid F = rain) = \frac{P(F = rain \mid W = rain) * P(W = rain)}{P(F = rain)} = \frac{0.90 * 0.30}{0.41} = 0.66$$

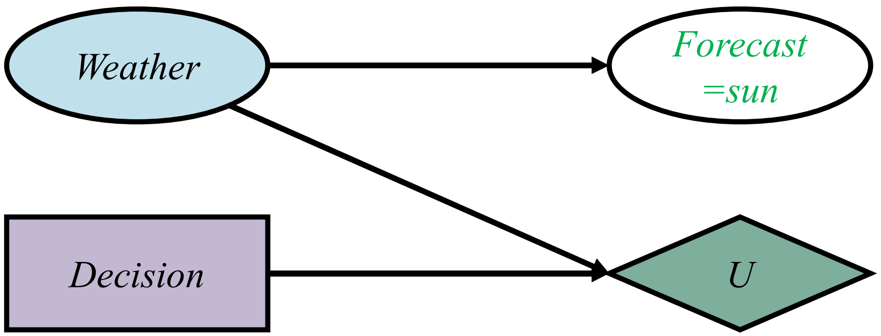
Decision Networks: Example

Decision: take umbrella given sun

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.05	0.95

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

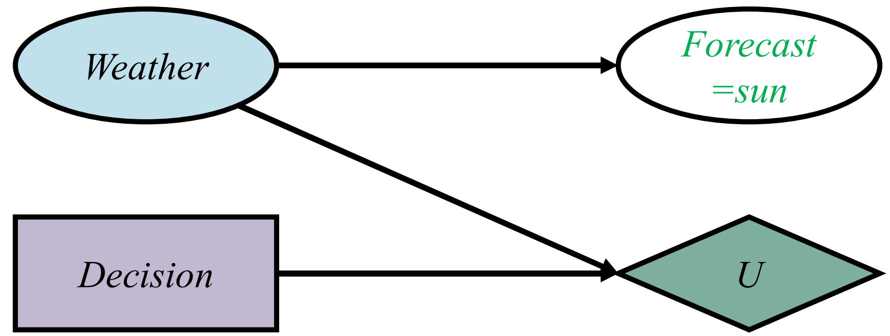
EU(take given sun forecast) = ???

Decision: leave umbrella given sun

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.05	0.95

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

EU(leave given sun forecast) = ???

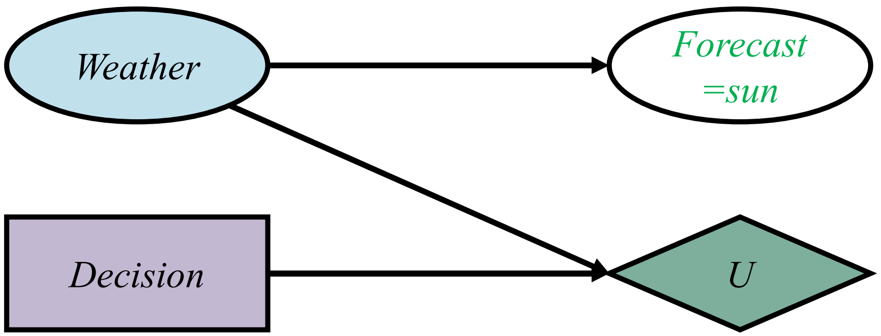
Decision Networks: Example

Decision: take umbrella given sun

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.05	0.95

P(F=sun)	P(F=rain)
0.59	0.41



S_1': D = take, W = sun
S_2': D = take, W = rain
EU(take) =
P(Result(take)=S_1'|e)*U(S_1') +
P(Result(take)=S_2'|e)*U(S_2') =
0.95 * 20 + 0.05 * 70 = 22.5

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

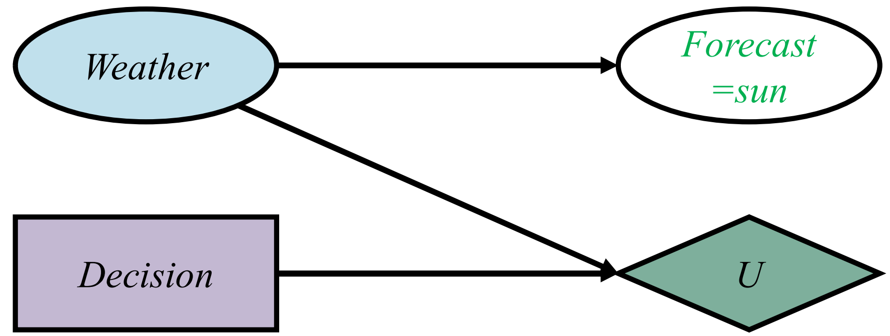
EU(take given sun forecast) = 22.5

Decision: leave umbrella given sun

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.05	0.95

P(F=sun)	P(F=rain)
0.59	0.41



S_3': D = leave, W = sun
S_4': D = leave, W = rain
EU(leave) =
P(Result(leave)=S_3'|e)*U(S_3') +
P(Result(leave)=S_4'|e)*U(S_4') =
0.95 * 100 + 0.05 * 0 = 95

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

EU(leave given sun forecast) = 95

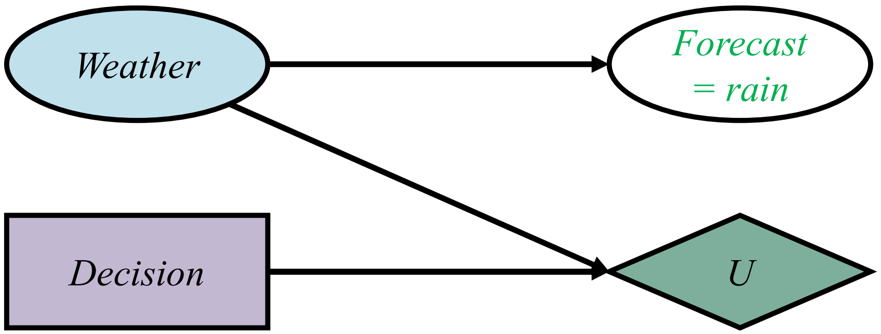
Decision Networks: Example

Decision: take umbrella given rain

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.66	0.34

P(F=sun)	P(F=rain)
0.59	**0.41**



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

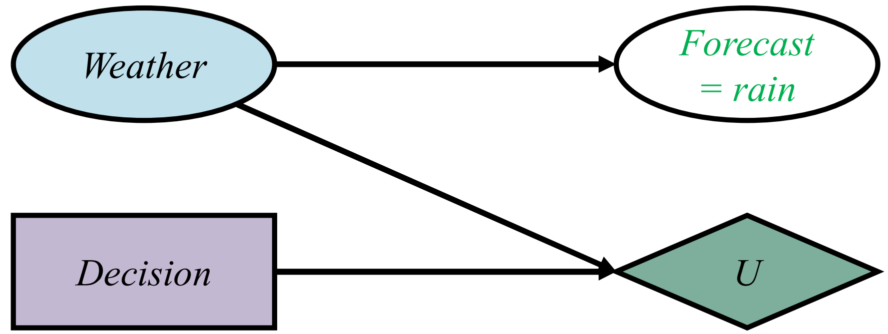
EU(take given rain forecast) = ???

Decision: leave umbrella given rain

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.66	0.34

P(F=sun)	P(F=rain)
0.59	**0.41**



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

EU(leave given rain forecast) = ???

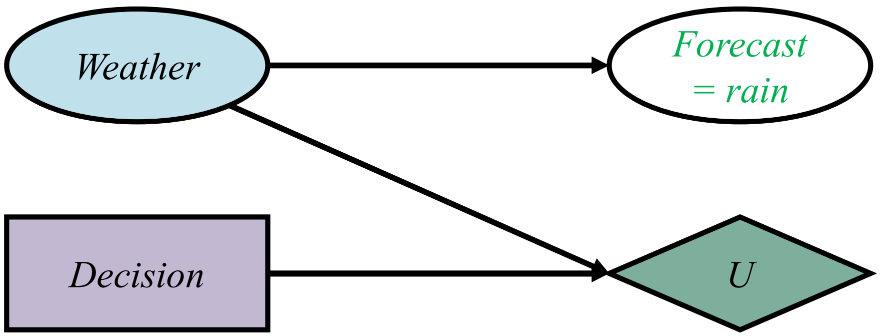
Decision Networks: Example

Decision: take umbrella given rain

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.66	0.34

P(F=sun)	P(F=rain)
0.59	**0.41**



S_1': D = take, W = sun
S_2': D = take, W = rain
EU(take) =
P(Result(take)=S_1'|e)*U(S_1') +
P(Result(take)=S_2'|e)*U(S_2') =
0.34 * 20 + 0.66 * 70 = 53

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

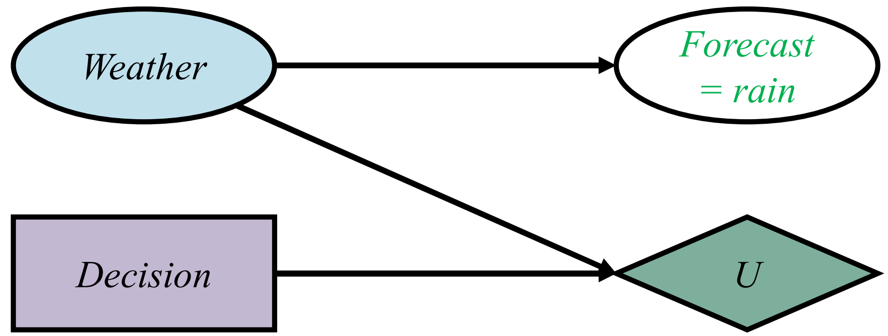
EU(take given rain forecast) = 53

Decision: leave umbrella given rain

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.66	0.34

P(F=sun)	P(F=rain)
0.59	**0.41**



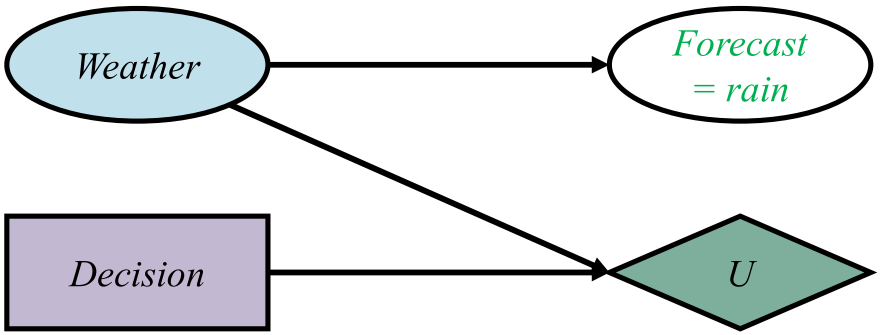
S_3': D = leave, W = sun
S_4': D = leave, W = rain
EU(leave) =
P(Result(leave)=S_3'|e)*U(S_3') +
P(Result(leave)=S_4'|e)*U(S_4') =
0.34 * 100 + 0.66 * 0 = 34

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

EU(leave given rain forecast) = 34

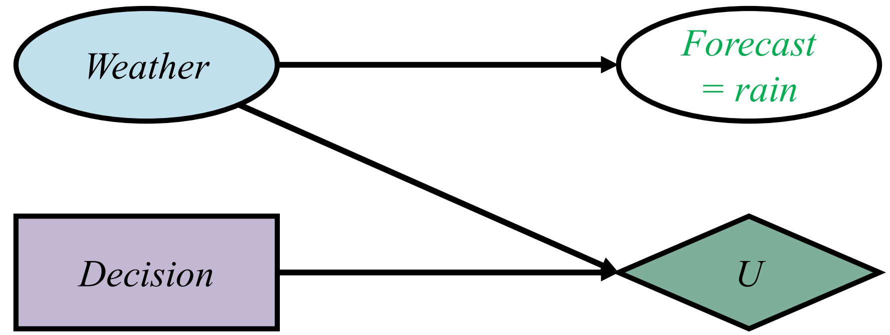
Decision Networks: Example

Decision: take umbrella given rain



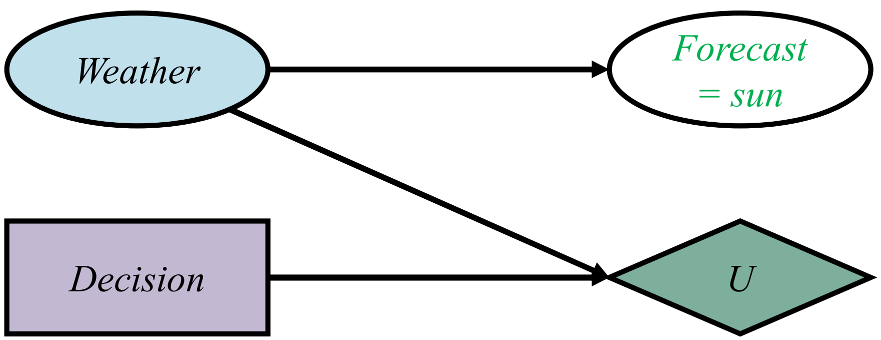
EU(take given rain forecast) = 53

Decision: leave umbrella given rain



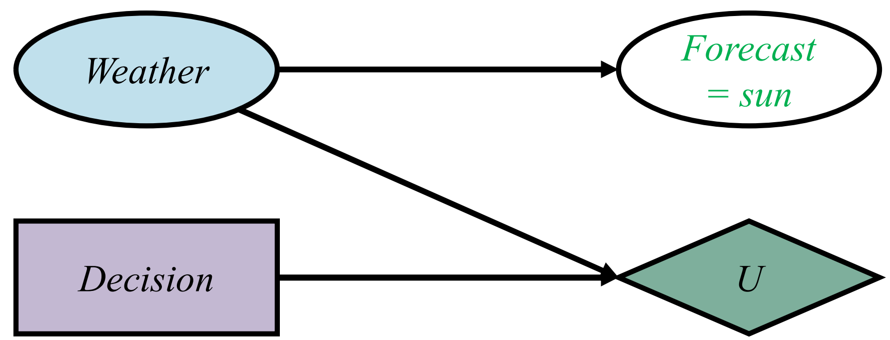
EU(leave given rain forecast) = 34

Decision: take umbrella given sun



EU(take given sun forecast) = 22.5

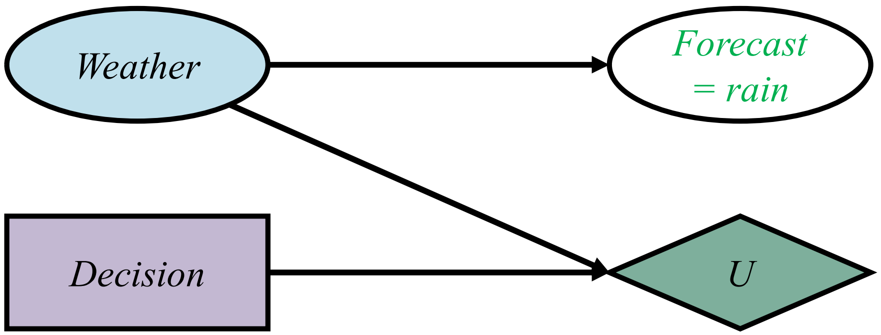
Decision: leave umbrella given sun



EU(leave given sun forecast) = 95

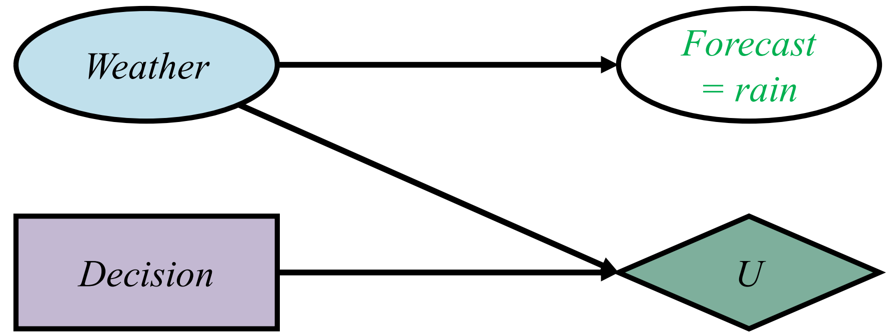
Decision Networks: Example

Decision:take umbrella given rain



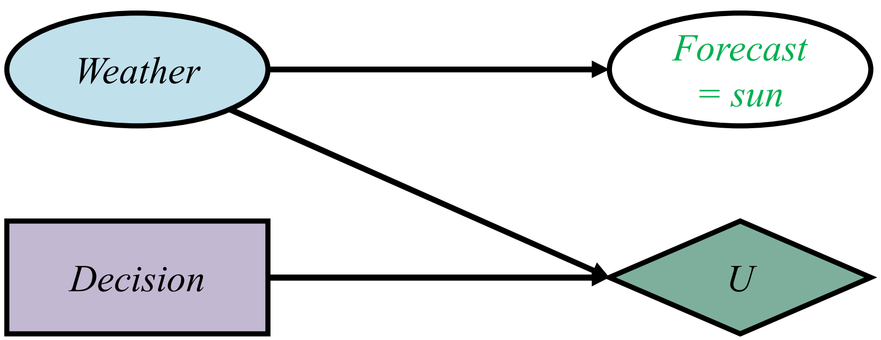
EU(take given rain forecast) = 53

Decision:leave umbrella given rain



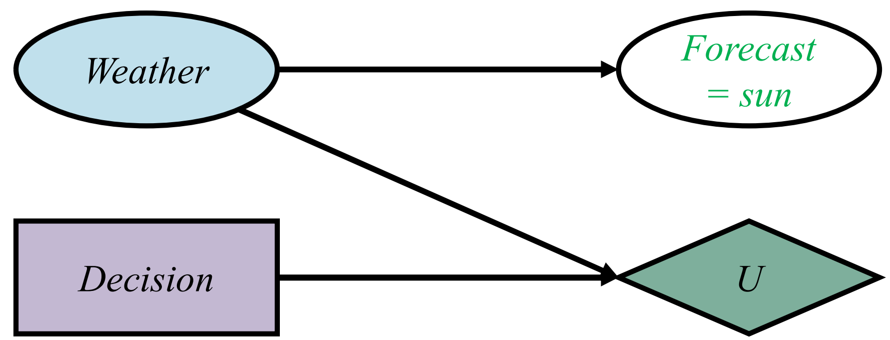
EU(leave given rain forecast) = 34

Decision:take umbrella given sun



EU(take given sun forecast) = 22.5

Decision:leave umbrella given sun



EU(leave given sun forecast) = 95

Value of Perfect Information

The value/utility of best action α without additional evidence (information) is :

$$\mathrm{MEU}(\alpha) = \max_{a} \sum_{s'} \mathrm{P}(Result(a) = s') * \mathrm{U}(s')$$

If we include new evidence/information ($\mathrm{E_j} = \mathrm{e_j}$) given by some variable Ej, value/utility of best action α becomes:

$$\mathrm{MEU}(a_{e_j} \mid e_j) = \max_{a} \sum_{s'} \mathrm{P}(Result(a) = s' \mid e_j) * \mathrm{U}(s')$$

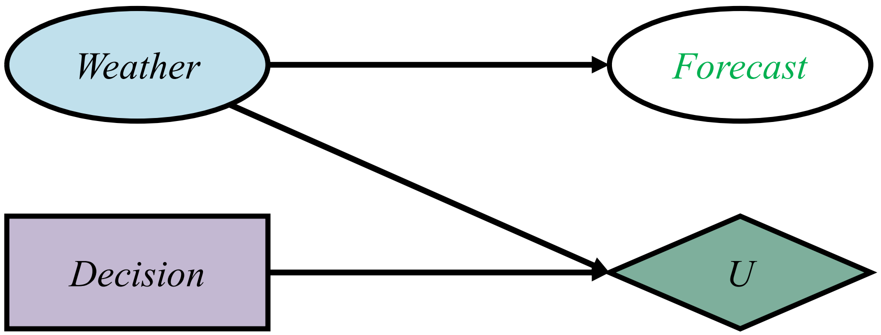
The value of additional evidence/information from Ej is:

$$\mathrm{VPI}(E_j) = \left(\sum_{e_j} \mathrm{P}(E_j = e_j) * \mathrm{MEU}(a_{e_j} \mid E_j = e_j)\right) - MEU(a)$$

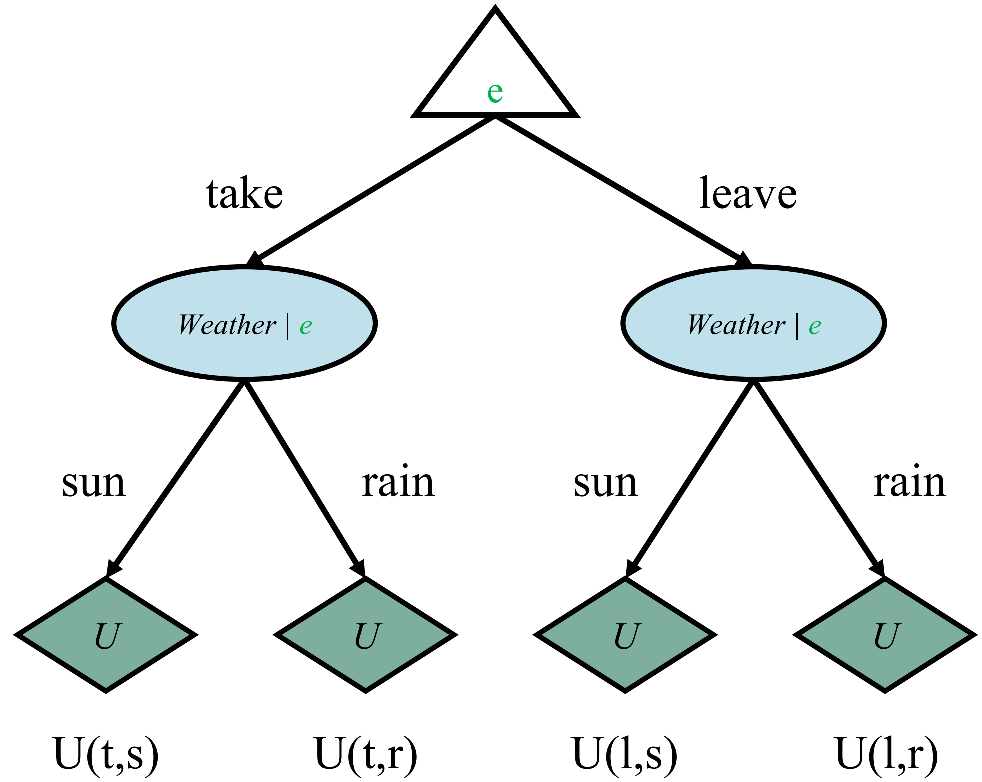
using our current beliefs about the world.

Decision Network: Example

Decision network



Outcome tree



The value of best action α without additional evidence

$$\text{MEU}(\alpha) = \text{MEU}(\text{leave}) = 70$$

With evidence information ($\text{E}_\text{j} = \text{e}_\text{j}$) given by Forecast:

$$\text{MEU}(a_{e_1} \mid e_1) = \text{MEU}(\text{take} \mid F = rain) = 53$$

$$\text{MEU}(a_{e_2} \mid e_2) = \text{MEU}(\text{leave} \mid F = sun) = 95$$

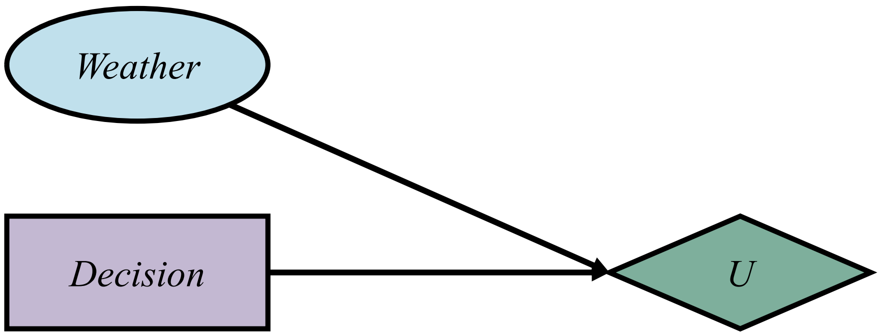
The value of additional evidence / information from F is:

$$\text{VPI}(E_j) = \left(\sum_{e_j} \text{P}(E_j = e_j) * \text{MEU}(a_{e_j} \mid E_j = e_j)\right) - MEU(a)$$

$$\text{VPI}(F) = (\text{P}(F = rain) * \text{MEU}(take \mid F = rain) + \text{P}(F = sun) * \text{MEU}(\text{leave} \mid F = sun)) - MEU(leave) = (0.41 * 53 + 0.59 * 95) - 70 = 7.78$$

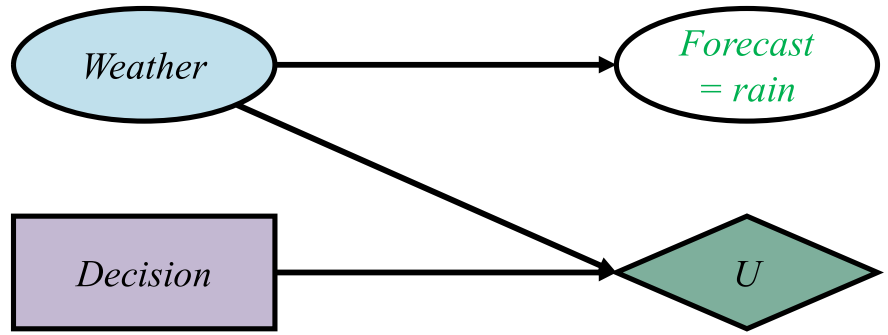
Decision Networks: Example

Decision: leave umbrella



$$\text{EU}(\text{leave}) = 70$$

Decision: take umbrella given rain



$$\text{EU}(\text{take given rain forecast}) = 53$$

The value of best action α without additional evidence

$$\text{MEU}(\alpha) = \text{MEU}(\text{leave}) = 70$$

With evidence information ($\mathrm{E_j} = \mathrm{e_j}$) given by Forecast:

$$\text{MEU}(a_{e_1} \mid e_1) = \text{MEU}(\text{take} \mid F = rain) = 53$$

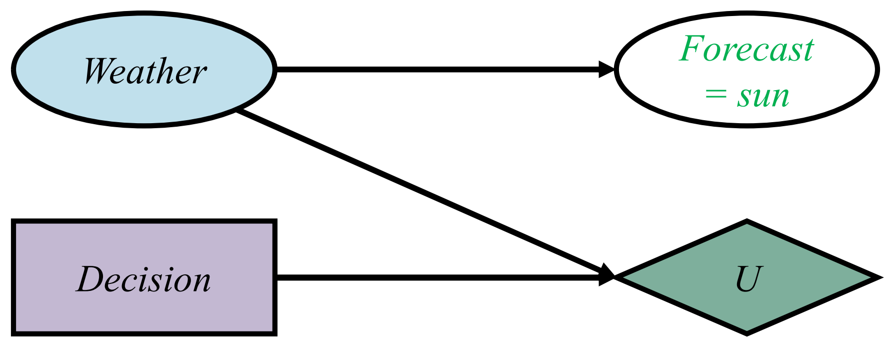
$$\text{MEU}(a_{e_2} \mid e_2) = \text{MEU}(\text{leave} \mid F = sun) = 95$$

The value of additional evidence / information from F is:

$$\text{VPI}(E_j) = \left(\textstyle\sum_{e_j} \text{P}(E_j = e_j) * \text{MEU}(a_{e_j} \mid E_j = e_j)\right) - MEU(a)$$

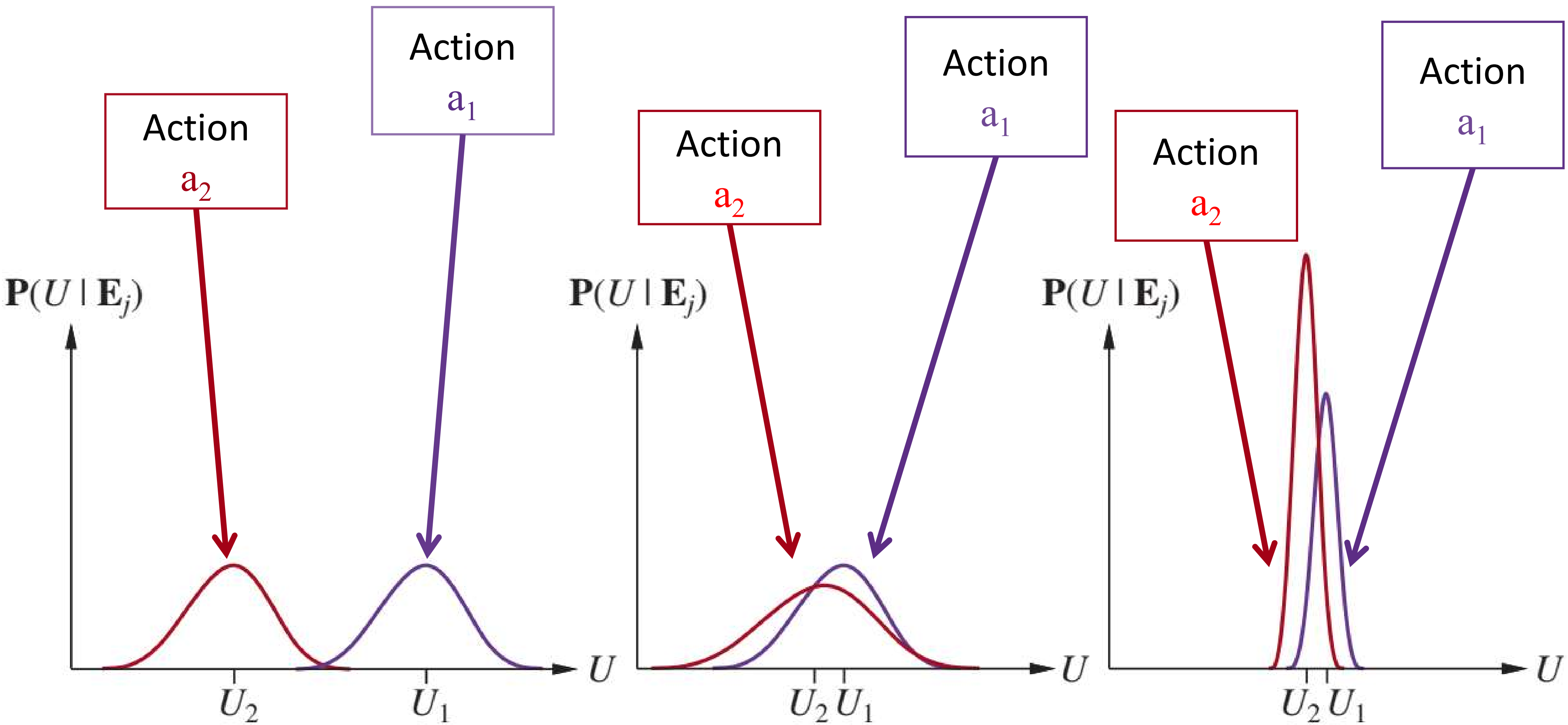
$$\text{VPI}(F) = (\text{P}(F = rain) * \text{MEU}(take \mid F = rain) + \text{P}(F = sun) * \text{MEU}(\text{leave} \mid F = sun)) - MEU(leave) = (0.41 * 53 + 0.59 * 95) - 70 = 7.78$$

Decision: leave umbrella given sun



$$\text{EU}(\text{leave given sun forecast}) = 95$$

Utility & Value of Perfect Information



New information will not help here.

New information may help a lot here.

New information may help a bit here.

VPI Properties

Given a decision network with possible observations $\mathrm{E_j}$ (sources of new information / evidence):

- **The expected value of information is nonnegative:**

$$\forall_{\mathrm{j}}\ \mathrm{VPI}(E_j) \geq 0$$

- **VPI is not additive:**

$$\mathrm{VPI}(E_j, E_k) \neq \mathrm{VPI}(E_j) + \mathrm{VPI}(E_k)$$

- **VPI is order-independent:**

$$\mathrm{VPI}(E_j,\ E_k) = VPI(E_j) + VPI(\ E_k \mid E_j) = VPI(E_k) + VPI(\ E_j \mid E_k) = VPI(E_k\ ,\ E_j\ )$$

Information Gathering Agent

function INFORMATION-GATHERING-AGENT($percept$) **returns** an $action$
  persistent: D, a decision network

  integrate $percept$ into D
  $j \leftarrow$ the value that maximizes $VPI(E_j)\ /\ C(E_j)$
  if $VPI(E_j)\ >\ C(E_j)$
    then return $Request(E_j)$
  else return the best action from D