Chapter 4

Logistic Regression

Contents

4.1 Generative Classifiers

If we are distinguishing cat from dog images using a Generative Classifier, we build a model of what is in a cat image.

- Knows about whiskers, ears, eyes.
- Assigns a probability to any image to determine how cat-like is that image?

Similarly, build a model of what is in a dog image. Now given a new image, run both models and see which one fits better.

4.2 Discriminative Classifiers

If we are distinguishing cat from dog images using a Discriminative Classifier.

- Just try to distinguish dogs from cats.
  - Oh look, dogs have collars.
  - Ignore everything else.

4.3 Generative vs Discriminative Classifiers

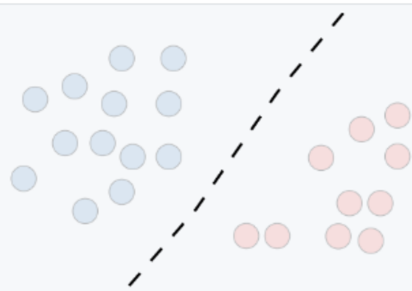
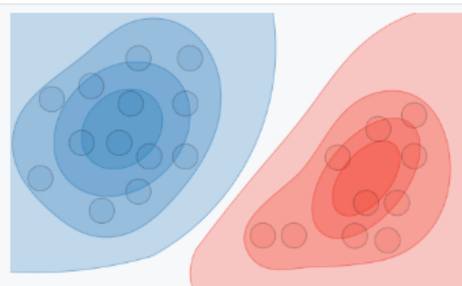
	Discriminative model	**Generative model**
Goal	Directly estimate $P(y\|x)$	Estimate $P(x\|y)$ to then deduce $P(y\|x)$
What's learned	Decision boundary	Probability distributions of the data
Illustration		
Examples	Regressions, SVMs	GDA, Naive Bayes

Figure 4.1: Differences between Generative and Discriminative classifiers.

Generative Classifiers (Naïve Bayes) –

- Assume some functional form for conditional independence.
- Estimate parameters of $P(D|h)$, $P(h)$ directly from training data.
- Use Bayes' rule to calculate $P(h|D)$.

Why not learn $P(h|D)$ or the decision boundary directly? Discriminative Classifiers (Logistic Regression) –

- Assume some functional form for $P(h|D)$ or for the decision boundary.
- Estimate parameters of $P(h|D)$ directly from training data.

4.4 Learning a Logistic Regression Classifier

Given n input-output pairs –

1. A feature representation of the input. For each input observation x_i, a vector of features $[x_1, x_2, \ldots, x_d]$.
2. A classification function that computes y, the estimated class, via $P(y|x)$, using the sigmoid of softmax functions.
3. An objective function for learning, like cross-entropy loss.
4. An algorithm for optimizing the objective function, like stochastic gradient ascent/descent.

4.5 Logistic Regression

Logistic Regression assumes the following function form for $P(y|x)$:

$$P(y=1|x) = \frac{1}{1 + e^{-(\sum_i w_i x_i + b)}}$$

$$\begin{aligned}
P(y=1|x) &= \frac{1}{1 + e^{-(\sum_i w_i x_i + b)}} \\
&= \frac{e^{(\sum_i w_i x_i + b)}}{e^{(\sum_i w_i x_i + b)} + 1} \\
P(y=0|x) &= 1 - \frac{1}{1 + e^{(\sum_i w_i x_i + b)}} \\
&= \frac{1}{e^{(\sum_i w_i x_i + b)} + 1} \\
\frac{P(y=1|x)}{P(y=0|x)} &= e^{(\sum_i w_i x_i + b)} > 1 \\
&\Rightarrow \sum_i w_i x_i + b > 0
\end{aligned}$$

Logistic Regression is a linear classifier. Turning a probability into a classifier using the logistic function:

$$y_{LR} \begin{cases} 1 & \text{if } P(y=1|x) \geq 0.5 \quad \leftarrow w_i x_i + b \geq 0 \\ 0 & \text{otherwise} \quad \leftarrow w_i x_i + b < 0 \end{cases}$$

4.6 LR Example

Suppose we are doing binary sentiment classification on movie review text, and we would like to know whether to assign the sentiment class position $= 1$ or negative $= 0$ to the following review:

It's hokey. There are virtually no surprises, and the writing is second-rate. So why was is so enjoyable? For one thing, the case is great. Another nice touch is the music. I was overcome with the urge to get off the couch and start dancing. It sucked me in, and it'll do the same to you.

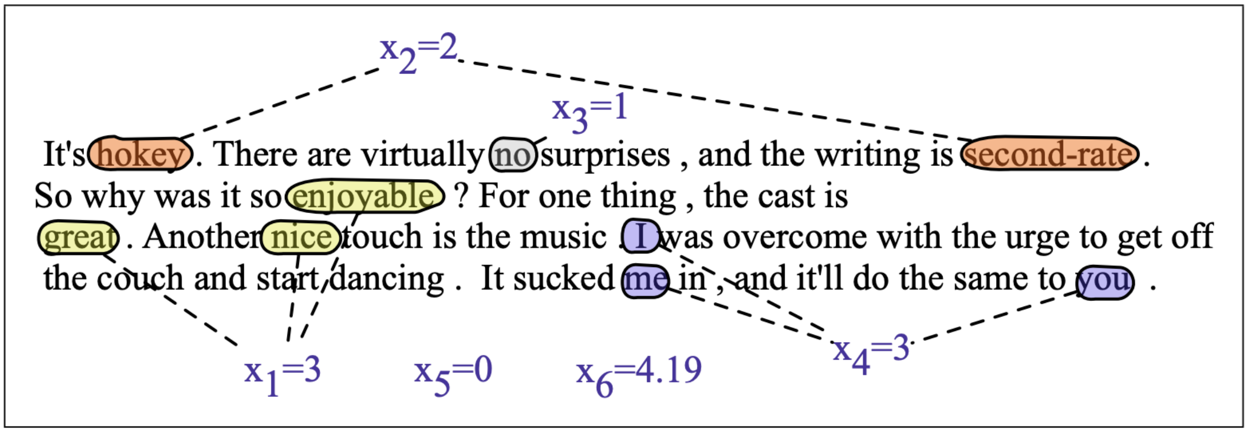


Figure 4.2: LR Example.

x_1	count(positive lexicon words $\in$ doc)	3
x_2	count(negative lexicon words $\in$ doc)	2
x_3	$\begin{cases} 1 & \text{if "no"} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$	1
x_4	count(1st and 2nd pronouns $\in$ doc)	3
x_5	$\begin{cases} 1 & \text{if "!"} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$	0
x_6	$\ln(\text{word count of doc})$	$\ln(66) = 4.19$

Figure 4.3: Feature vector for the LR Example.

4.7 Sentiment Classification

Let's assume for the moment that we've already learned a real-valued weight for each of these features, and that the 6 weights corresponding to the 6 features are $[2.5, -5.0, -1.2, 0.5, 2.0, 0.7]$, while $b = 0.1$.

$$
\begin{aligned}
P(+ve|x) &= P(y=1|x) \\
&= \frac{1}{1+e^{(\sum_i w_i x_i + b)}} \\
&= \frac{1}{1+e^{-(2.5(3)+(-5)(2)+(-1.2)(1)+0.5(3)+2.0(0)+0.7(4.19)+0.1)}} \\
&= 0.30 \\
P(-ve|x) &= P(y=0|x) \\
&= 1 - P(y=1|x) \\
&= 1 - 0.70 \\
&= 0.30
\end{aligned}
$$

Since $P(+ve|x) > P(-ve|x)$, the output sentiment class is positive.

4.8 Training Logistic Regression

We'll focus on binary classification. We parameterize (w_i, b) as θ:

$$
\begin{aligned}
P(y_i = 0|x_i, \theta) &= \frac{1}{e^{\sum_i w_i x_i + b} + 1} \\
P(y_i = 1|x_i, \theta) &= \frac{e^{\sum_i w_i x_i + b}}{e^{\sum_i w_i x_i + b} + 1} \\
P(y_i|x_i, \theta) &= \frac{e^{y_i \sum_i w_i x_i + b}}{e^{\sum_i w_i x_i + b} + 1}
\end{aligned}
$$

How do we learn parameters θ?

4.9 Cross-Entropy Loss

- We want to know how far is the classifier output $\hat{y}$ from the true output y. Let's call this difference $L(\hat{y}, y)$.
- Since there are only 2 discrete outcomes (0 or 1), we can express the probability $P(y|x)$ from our classifiers as:

$$P(y|x) = \hat{y}^y \cdot (1-\hat{y})^{1-y}$$

- Goal: maximize the probability of the correct label $P(y|x)$.
- Maximize:

$$
\begin{aligned}
P(y|x) &= \hat{y}^y \cdot (1-\hat{y})^{1-y} \\
\log(P(y|x)) &= \log\left(\hat{y}^y \cdot (1-\hat{y})^{1-y}\right) \\
&= y\log(\hat{y}) + (1-y)\log(1-\hat{y})
\end{aligned}
$$

- We want to minimize the cross-entropy loss:

$$
\begin{aligned}
\textbf{Minimize}: L_{CE}(\hat{y}, y) &= -\log P(y|x) \\
&= -[y\log(\hat{y}) + (1-y)\log(1-\hat{y})] \\
\min_{\theta} L_{CE}(\hat{y}, y) &= -[y\log(\hat{y}) + (1-y)\log(1-\hat{y})] \\
&= -\left[y\log\left(\frac{e^{\sum_i w_i x_i + b}}{1+e^{\sum_i w_i x_i + b}}\right) + (1-y)\log\left(1 - \frac{e^{\sum_i w_i x_i + b}}{1+e^{\sum_i w_i x_i + b}}\right)\right] \\
&= -\left[y\left(\sum_i w_i x_i + b - \log\left(1 + e^{\sum_i w_i x_i + b}\right)\right) + (1-y)\left(-\log\left(1 + e^{\sum_i w_i x_i + b}\right)\right)\right] \\
&= -\left[y\left(\sum_i w_i x_i + b\right) - \log\left(1 + e^{\sum_i w_i x_i + b}\right)\right] \\
&= \log\left(1 + e^{\sum_i w_i x_i + b}\right) + y\left(\sum_i w_i x_i + b\right)
\end{aligned}
$$

4.10 Minimizing Cross-Entropy Loss

$$\min_{\theta} L_{CE}(\hat{y}, y)$$

- Minimizing loss function $L_{CE}(\hat{y}, y)$ is a convex optimization problem.
- Convex function have a global minimum.
- Concave function have a global maxima.

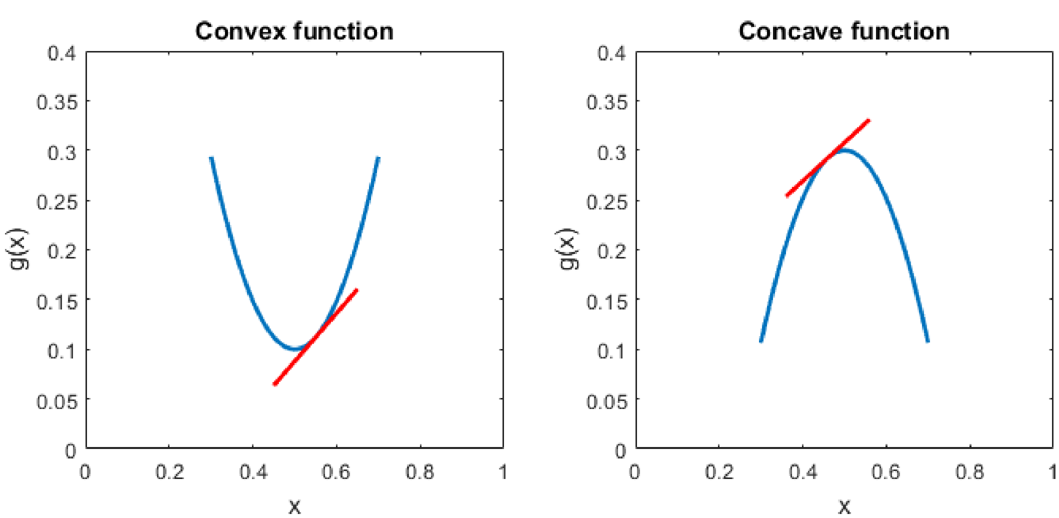


Figure 4.4: An example of a convec and concave function.

4.11 Optimizing a Convex/Concave Function

- Maximum of a concave function is equivalent to the minimum of a convex function.
- Gradient Ascent is used for finding the maximum of a concave function.
- Gradient Descent is used for finding the minimum of a convex function.

4.12 Gradients

- The gradient of a function is a vector pointing in the direction of the greatest increase in a function.

Gradient Ascent: Find the gradient of the function at the current point and move in the same direction.

Gradient Descent: Find the gradient of the function at the current point and move in the opposite direction.

4.13 Gradient Descent for Logistic Regression

- Let us represent $\hat{y} = f(x, \theta)$
- Gradient:

$$\nabla_\theta L(f(x,\theta),y) = \left[\frac{\partial L(f(x,\theta),y)}{\partial b}, \frac{\partial L(f(x,\theta),y)}{\partial w_1}, \frac{\partial L(f(x,\theta),y)}{\partial w_2}, \dots, \frac{\partial L(f(x,\theta),y)}{\partial w_d}\right] \tag{4.1}$$

- Update Rule:

$$\begin{aligned} \Delta\theta &= \eta \cdot \nabla_\theta L(f(x,\theta),y) \\ \theta_{t+1} &= \theta_t - \eta \cdot \frac{\partial}{\partial(w,b)} L(f(x,\theta),y) \end{aligned} \tag{4.2}$$

Gradient descent algorithm will iterate until $\Delta\theta < \epsilon$.

$$\begin{aligned} L_{CE}(f(x,\theta),y) &= \log\left(1 + e^{\sum_i w_i x_i + b}\right) - y\left(\sum_i w_i x_i + b\right) \\ \theta_{t+1} &= \theta_t - \eta \cdot \frac{\partial}{\partial(w,b)} L(f(x,\theta),y) \\ &= \theta_t - \eta \cdot x_i \left[\frac{e^{\sum_i w_i x_i + b}}{1 + e^{\sum_i w_i x_i + b}} - y\right] \\ &= \theta_t - \eta \cdot x_i \left[\hat{P}(y=1|x,\theta_t) - y\right] \end{aligned}$$

4.14 Learning Rate

- η is a hyperparameter.
- Large $\eta \Rightarrow$ Fast convergence but larger residual error. Also, possible oscillations.
- Small $\eta \Rightarrow$ Slow convergence but small residual error.

4.15 Batch Training

- Stochastic gradient descent is called stochastic because it chooses a single random example at a time, moving the weights to improve performance on that single example.
- This results in very choppy movements, so it's common to compute the gradient over batches of training instances rather than a single instance.
- Training data: $\{x_i, y_i\}_{i=i...n}$ where $x_i = (x_{i1}, x_{i2}, \ldots, x_{id})$, n is the total instances in a batch and d is the dimension of an instance.

$$\theta_{t+1} = \theta_t - \frac{\eta}{n} \times \sum_{i=1}^{n} xij \left[\frac{1}{1 + e^{-\theta^T \mathbf{x}}} - y_i \right] \tag{4.3}$$

4.16 Understanding the Sigmoid

- Large weights lead to overfitting.
- Penalizing larger weights can reduce overfitting.

4.17 Regularization

- Regularization is used to avoid overfitting.
- The weights for features will attempt to perfectly fit details of the training set, modeling even noisy data that just accidentally correlate with the class. The problem is called overfitting.
- A good model should generalize well from the training data to the unseen test set, but a model that overfits will have poor generalization.
- To avoid overfitting, a new regularization term $R(\theta)$ is added to the loss function.

$$\min_{\theta} L_{reg}(\hat{y}, y) = -\frac{1}{n} \sum_{i=1}^{n} [y \log(\hat{y}) + (1 - y) \log(1 - \hat{y})] + \lambda R(\theta) \tag{4.4}$$

4.18 L1 Regularization

- L1 Regularization is also called Lasso Regularization.
- Uses the L1 norm (Manhattan distance) of the weights.

$$R(\theta) = ||\theta||_1 = \sum_{j=0}^{d} |\theta_j|$$

$$\min_{\theta} L_{reg}(\hat{y}, y) = -\frac{1}{n}\sum_{i=1}^{n}[y\log(\hat{y}) + (1-y)\log(1-\hat{y})] + \lambda\sum_{j=0}^{d}|\theta_j| \tag{4.5}$$

4.19 L2 Regularization

- L2 Regularization is also called Ridge Regularization.
- Uses the square of the L2 (Euclidean) norm of the weights.

$$R(\theta) = ||\theta||_2^2 = \sum_{j=0}^{d} \theta_j^2$$

$$\min_{\theta} L_{reg}(\hat{y}, y) = -\frac{1}{n}\sum_{i=1}^{n}[y\log(\hat{y}) + (1-y)\log(1-\hat{y})] + \lambda\sum_{j=0}^{d}\theta_j^2 \tag{4.6}$$

4.20 Example – Spam Recognition

Let us apply logistic regression on the spam email recognition problem, assuming $\eta = 3.0$ and starting with $\theta_{w,b} = [0, 0, 0, 0, 0, 0]$.

Table 4.1

	and	vaccine	the	of	nigeria	y
Email **a**	1	1	0	1	1	1
Email **b**	0	0	1	1	0	0
Email **c**	0	1	1	0	0	1
Email **d**	1	0	0	1	0	0
Email **e**	1	0	1	0	1	1
Email **f**	1	0	1	1	0	0

1 entails that a word (i.e., "and") is present in an email (i.e. "Email **a**") and 0 entails that a word is absent in an email.

Table 4.2

	$x_0 = 1$	x_1 = and	x_2 = vaccine	x_3 = the	x_4 = of	x_5 = nigeria	y
Email **a**	1	1	1	0	1	1	1
Email **b**	1	0	0	1	1	0	0
Email **c**	1	0	1	1	0	0	1
Email **d**	1	1	0	0	1	0	0
Email **e**	1	1	0	1	0	1	1
Email **f**	1	1	0	1	1	0	0

The column x_0 was added to account for this bias b.
$x = [x_0, x_1, x_2, x_3, x_4, x_5]$, $\theta = [b, w_1, w_2, w_3, w_4, w_5]$

4.21 Training Phase

$$\theta_{t+1} = \theta_t - \frac{\eta}{n} \times \sum_{i=1}^{n} x_{ij} \left[\frac{1}{1 + e^{-\theta^T \mathbf{x}}} - y_i \right]$$

1) Calculate the factor $-\theta^T \mathbf{x}$ for every example in the dataset.

2) Calculate the factor $\sum_{i=1}^{n} x_{ij} \left[\frac{1}{1+e^{-\theta^T \mathbf{x}}} - y_i \right]$ for every example in the dataset, for every θ

3) Compute every θ

Table 4.3

x	y	$\theta^T \mathbf{x}$	$\left(\frac{1}{1+e^{-\theta^T \mathbf{x}}} - y\right) x_0$
[1, 1, 1, 0, 1, 1]	**1**	$[0,0,0,0,0,0] \times [1,1,1,0,1,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 1 = -0.5$
[1, 0, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,0,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 1 = 0.5$
[1, 0, 1, 1, 0, 0]	**1**	$[0,0,0,0,0,0] \times [1,0,1,1,0,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 1 = -0.5$
[1, 1, 0, 0, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,1,0,0,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 1 = 0.5$
[1, 1, 0, 1, 0, 1]	**1**	$[0,0,0,0,0,0] \times [1,1,0,1,0,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 1 = -0.5$
[1, 1, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,1,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 1 = 0.5$

$$\begin{aligned} \sum_{i=1}^{6} x_{i0} \left[\frac{1}{1 + e^{\theta^T \mathbf{x}}} - y_i \right] &= -0.5 + 0.5 - 0.5 + 0.5 - 0.5 + 0.5 \\ &= 0.0 \end{aligned}$$

Table 4.4

x	y	$\theta^T \mathbf{x}$	$\left(\frac{1}{1+e^{-\theta^T \mathbf{x}}} - y\right) x_1$
[1, 1, 1, 0, 1, 1]	**1**	$[0,0,0,0,0,0] \times [1,1,1,0,1,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 1 = -0.5$
[1, 0, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,0,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 0 = 0$
[1, 0, 1, 1, 0, 0]	**1**	$[0,0,0,0,0,0] \times [1,0,1,1,0,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 0 = 0$
[1, 1, 0, 0, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,1,0,0,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 1 = 0.5$
[1, 1, 0, 1, 0, 1]	**1**	$[0,0,0,0,0,0] \times [1,1,0,1,0,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 1 = -0.5$
[1, 1, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,1,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 1 = 0.5$

$$\sum_{i=1}^{6} x_{i1} \left[\frac{1}{1 + e^{\theta^{\mathbf{T}} \mathbf{x}}} - y_i \right] = -0.5 + 0 + 0 + 0.5 - 0.5 + 0.5$$
$$= 0.0$$

Table 4.5

x	y	$\theta^T \mathbf{x}$	$\left(\frac{1}{1+e^{-\theta^T \mathbf{x}}} - y\right) x_2$
[1, 1, 1, 0, 1, 1]	**1**	$[0,0,0,0,0,0] \times [1,1,1,0,1,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 1 = -0.5$
[1, 0, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,0,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 0 = 0$
[1, 0, 1, 1, 0, 0]	**1**	$[0,0,0,0,0,0] \times [1,0,1,1,0,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 1 = -0.5$
[1, 1, 0, 0, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,1,0,0,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 0 = 0$
[1, 1, 0, 1, 0, 1]	**1**	$[0,0,0,0,0,0] \times [1,1,0,1,0,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right) \times 0 = 0$
[1, 1, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0] \times [1,1,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right) \times 0 = 0$

$$\sum_{i=1}^{6} x_{i2} \left[\frac{1}{1 + e^{\theta^{\mathbf{T}} \mathbf{x}}} - y_i \right] = -0.5 + 0 - 0.5 + 0 + 0 + 0$$
$$= -1.0$$

Table 4.6

x	y	$\theta^T\mathbf{x}$	$\left(\frac{1}{1+e^{-\theta^T\mathbf{x}}} - y\right)x_3$
[1, 1, 1, 0, 1, 1]	**1**	$[0,0,0,0,0,0]\times[1,1,1,0,1,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 0 = 0$
[1, 0, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,0,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 1 = 0.5$
[1, 0, 1, 1, 0, 0]	**1**	$[0,0,0,0,0,0]\times[1,0,1,1,0,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 1 = -0.5$
[1, 1, 0, 0, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,1,0,0,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 0 = 0$
[1, 1, 0, 1, 0, 1]	**1**	$[0,0,0,0,0,0]\times[1,1,0,1,0,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 1 = -0.5$
[1, 1, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,1,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 1 = 0.5$

$$\begin{aligned}\sum_{i=1}^{6} x_{i3}\left[\frac{1}{1+e^{\theta^\mathbf{T}\mathbf{x}}} - y_i\right] &= 0 + 0.5 - 0.5 + 0 - 0.5 + 0.5 \\ &= 0.0\end{aligned}$$

Table 4.7

x	y	$\theta^T\mathbf{x}$	$\left(\frac{1}{1+e^{-\theta^T\mathbf{x}}} - y\right)x_4$
[1, 1, 1, 0, 1, 1]	**1**	$[0,0,0,0,0,0]\times[1,1,1,0,1,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 1 = -0.5$
[1, 0, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,0,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 1 = 0.5$
[1, 0, 1, 1, 0, 0]	**1**	$[0,0,0,0,0,0]\times[1,0,1,1,0,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 0 = 0$
[1, 1, 0, 0, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,1,0,0,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 1 = 0.5$
[1, 1, 0, 1, 0, 1]	**1**	$[0,0,0,0,0,0]\times[1,1,0,1,0,1] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 0 = 0$
[1, 1, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,1,0,1,1,0] = 0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 1 = 0.5$

$$\begin{aligned}\sum_{i=1}^{6} x_{i4}\left[\frac{1}{1+e^{\theta^\mathbf{T}\mathbf{x}}} - y_i\right] &= -0.5 + 0.5 + 0 + 0.5 + 0 + 0.5 \\ &= 1.0\end{aligned}$$

Table 4.8

x	y	$\theta^T\mathbf{x}$	$\left(\frac{1}{1+e^{-\theta^T\mathbf{x}}} - y\right)x_5$
[1, 1, 1, 0, 1, 1]	**1**	$[0,0,0,0,0,0]\times[1,1,1,0,1,1]=0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 1 = -0.5$
[1, 0, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,0,0,1,1,0]=0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 0 = 0$
[1, 0, 1, 1, 0, 0]	**1**	$[0,0,0,0,0,0]\times[1,0,1,1,0,0]=0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 0 = 0$
[1, 1, 0, 0, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,1,0,0,1,0]=0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 0 = 0$
[1, 1, 0, 1, 0, 1]	**1**	$[0,0,0,0,0,0]\times[1,1,0,1,0,1]=0$	$\left(\frac{1}{1+e^0} - \mathbf{1}\right)\times 1 = -0.5$
[1, 1, 0, 1, 1, 0]	**0**	$[0,0,0,0,0,0]\times[1,1,0,1,1,0]=0$	$\left(\frac{1}{1+e^0} - \mathbf{0}\right)\times 0 = 0$

$$\begin{aligned}\sum_{i=1}^{6} x_{i5}\left[\frac{1}{1+e^{\theta^\mathbf{T}\mathbf{x}}} - y_i\right] &= -0.5 + 0 + 0 + 0 - 0.5 + 0 \\ &= -1.0\end{aligned}$$

$$\begin{aligned}\theta_1 &= \theta_0 - \frac{\eta}{n} \times \sum_{i=1}^{n} x_{ij}\left[\frac{1}{1+e^{-\theta^T x}} - y_i\right] \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\end{aligned}$$

4.22 Testing Phase

Let us test logistic regression on the spam email recognition problem using the $\theta = [0, 0, 0.5, 0, -0.5, 0.5]$.

Table 4.9

x	y	$\theta^T \mathbf{x}$	$\mathbf{P} = \left(\frac{1}{1+e^{-\theta^T \mathbf{x}}} - y\right)$	**Predicted Class**
[1, 1, 1, 0, 1, 1]	**1**	$[0, 0, 0.5, 0, -0.5, 0.5] \times [1, 1, 1, 0, 1, 1] = 0.5$	0.622459331	**1**
[1, 0, 0, 1, 1, 0]	**0**	$[0, 0, 0.5, 0, -0.5, 0.5] \times [1, 0, 0, 1, 1, 0] = -0.5$	0.377540669	**0**
[1, 0, 1, 1, 0, 0]	**1**	$[0, 0, 0.5, 0, -0.5, 0.5] \times [1, 0, 1, 1, 0, 0] = 0.5$	0.622459331	**1**
[1, 1, 0, 0, 1, 0]	**0**	$[0, 0, 0.5, 0, -0.5, 0.5] \times [1, 1, 0, 0, 1, 0] = -0.5$	0.377540669	**0**
[1, 1, 0, 1, 0, 1]	**1**	$[0, 0, 0.5, 0, -0.5, 0.5] \times [1, 1, 0, 1, 0, 1] = 0.5$	0.622459331	**1**
[1, 1, 0, 1, 1, 0]	**0**	$[0, 0, 0.5, 0, -0.5, 0.5] \times [1, 1, 0, 1, 1, 0] = -0.5$	0.377540669	**0**

No Misclassification

4.23 Multinomial Logistic Regression

- The loss function for multinomial logistic regression generalizes the loss function for binary logistic regression from 2 to K classes.
- The true label y is a vector with K elements, each corresponding to a class, with $y_c = 1$ if the correct class is c, with all other elements of y being 0.
- The classifier will produce an estimate vector with K elements $\hat{y}$, each element $\hat{y}_k$ of which represents the estimated probability $P(y_k = 1|x)$.

$$\textsc{Softmax}(z_i) = \frac{\exp(z_i)}{\sum_{j=1}^{K} \exp z_j} \quad 1 \le i \le K \tag{4.7}$$

$$L_{CE}(\hat{y}, y) = -\sum_{k=1}^{K} y_k \log(\hat{y}_k) \tag{4.8}$$

$$\begin{aligned} L_{CE}(\hat{y}, y) &= -\log(\hat{y}_c) \\ &= -\log\left(\frac{e^{\sum_{i=1}^{d} w_c x_i + b_c}}{\sum_{j=1}^{K} e^{\sum_{i=1}^{d} w_j x_i + b_j}}\right) \end{aligned} \tag{4.9}$$

$$\frac{\partial L_{CE}}{\partial(w_k, b_k)} = x_i \left[\frac{e^{\sum_{i=1}^{d} w_k x_i + b_k}}{\sum_{j=1}^{K} e^{\sum_{i=1}^{d} w_j x_i + b_j}} - y_k\right] \tag{4.10}$$

4.24 Conclusion

Logistic Regression –

- is a discriminative classifier,

- is a linear classifier,
- optimizes by minimizing the cross-entropy loss via gradient descent,
- trains parameters:
  - begins with initial weight vector,
  - modifies it iteratively to minimize the loss function.