

Chapter 7

Method of Laplace Transforms for Solving DE's

7.5 The Dirac Delta Function

$$\delta_a(t - t_0) = \begin{cases} 0 & \text{if } t < t_0 - a \\ \frac{1}{2a} & \text{if } t_0 - a \leq t \leq t_0 + a \\ 0 & \text{if } t > t_0 + a \end{cases}$$
$$\int_0^{\infty} f(t)\delta(t - t_0)dt = f(t_0) \quad (7.1)$$

So

$$\begin{aligned} \mathcal{L}\{\delta(t - t_0)\} &= \int_0^{\infty} e^{-st}\delta(t - t_0)dt \\ &= e^{-st_0} \end{aligned} \quad (7.2)$$

7.5.1 – Example

$$\begin{aligned} y'' + y &= 4\delta(t - 2\pi) \\ y(0) &= 1 \quad y'(0) = 0 \\ y'' + y &= 4\delta(t - 2\pi) \\ \mathcal{L}\{y''\} + \mathcal{L}\{y\} &= 4\mathcal{L}\{\delta(t - 2\pi)\} \\ s^2Y(s) - sy(0) - y'(0) + Y(s) &= 4e^{-s(2\pi)} \\ Y(s)(s^2 + 1) - s(1) - 0 &= 4e^{-2\pi s} \\ Y(s)(s^2 + 1) - s &= 4e^{-2\pi s} \\ Y(s)(s^2 + 1) &= 4e^{-2\pi s} + s \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{4e^{-2\pi s} + s}{s^2 + 1} \\ &= \frac{4e^{-2\pi s}}{s^2 + 1} + \frac{s}{s^2 + 1} \\ &= 4e^{-2\pi s} \frac{e^s}{s^2 + 1^2} + \frac{s}{s^2 + 1} \\ y(t) &= \mathcal{L}^{-1} \left\{ 4e^{-2\pi s} \frac{1}{s^2 + 1^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} \\ &= 4 \sin(t - 2\pi) \mathcal{U}(t - 2\pi) + \cos(t) \end{aligned}$$