# Chapter 6

# Series Solutions of Linear Equations

## 6.2 Second Order, Linear Homogenous DE

$$
\begin{aligned}
a_2(x)y'' + a_1(x)y' + a_0(x)y &= 0 \\
y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y &= 0 \\
y'' + P(x)y' + Q(x)y &= 0
\end{aligned}
$$

$x$'s for which $a_2(x) \neq 0$ will be called ordinary points. $x$'s for which $a_2(x) = 0$ will be called singular points.

Existence of Power Series **Theorem:** If $x_0$ is an ordinary point of the DE, then there exists two, linearly independent solution $y_1$, $y_2$ which are both in the form of power series

$$
\sum_{n=0}^{\infty} c_n (x - x_0)^n
$$

and these series will have radius of convergence of at least the distance from $x_0$ to the singular point of the DE.

### 6.2.1 – Example

Consider

$$
(x^2 + 2x + y)y'' + xy' - 6y = 0
$$

(i) What are the singular points of the DE?

$$\begin{aligned} x^2 + 2x + 5 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm \sqrt{16} \times \sqrt{-1}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

So $-1 + 2i$ and $-1 - 2i$ are the only singular points.

(ii) Is there a power series solution centered at $x_0 = 0$? Yes, since $x_0 = 0$ is an ordinary point, you can find

$$y_1 = \sum_{n=0}^{\infty} c_n x^n$$

and

$$y_2 = \sum_{n=0}^{\infty} d_n x^n,$$

two linearly independent solutions.

(iii) What is the minimum the radius could be for these series? As stated in the theorem, the radius is at minimum the distance from $x_0$ to the singular point. If you have complex singular points, calculate the distance using the complex plane graph. $\sqrt{(-1-0)^2 + (2-0)^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$.

- How about if we want series

$$\sum_{n=0}^{\infty} c_n (x-3)^2$$

$$\sqrt{(-1-3)^2 + (-2-0)^2} = \sqrt{(-4)^2 + (-2)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

### 6.2.2 – Example

Use Power Series centered at 0 (Maclaurin Series) to solve the DE:

$$y'' - xy = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$
$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} c_n x^n &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= \sum_{k+2=2}^{\infty} c_{k+2}(k+2)(k+2-1)x^k - \sum_{k-1=0}^{\infty} c_{k-1}x^k \\
&= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\
&= \sum_{k=0}^{1} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\
&= c_{0+2}(0+2)(0+1)x^0 + \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1)x^k - c_{k-1}x^k\right] \\
&= c_2(2)(1)(1) + \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] \\
&= 2c_2 + \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] \\
&= 2(0) + \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] \\
&= 0 + \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] \\
&= \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] = 0
\end{aligned}$$

$$\begin{aligned}
c_{k+2}(k+2)(k+1) - c_{k-1} &= 0 \\
c_{k+2}(k+2)(k+1) &= c_{k-1} \\
c_{k+2} &= \frac{c_{k-1}}{(k+2)(k+1)}
\end{aligned}$$

$$
\begin{aligned}
c_0 &= \text{arbitrary}\\
c_1 &= \text{arbitrary}\\
c_2 &= 0\\
c_3 &= \frac{c_0}{(1+2)(1+1)} = \frac{c_0}{3\times 2}\\
c_4 &= \frac{c_1}{(2+2)(2+1)} = \frac{c_1}{4\times 3}\\
c_5 &= \frac{c_2}{(3+2)(3+1)} = \frac{0}{5\times 4} = 0\\
c_6 &= \frac{c_3}{(4+2)(4+1)} = \frac{c_0}{3\times 2}\times\frac{1}{6\times 5} = \frac{c_0}{6\times 5\times 3\times 2}\\
c_7 &= \frac{c_4}{(5+2)(5+1)} = \frac{c_1}{4\times 3}\times\frac{1}{7\times 6} = \frac{c_1}{7\times 6\times 4\times 3}\\
c_8 &= \frac{c_5}{(6+2)(6+1)} = \frac{0}{8\times 7} = 0
\end{aligned}
$$

$$
\begin{aligned}
y &= c_0y_1 + c_1y_2\\
&= c_0\left(1+\frac{1}{3\times 2}x^3+\frac{1}{6\times 5\times 3\times 2}x^6+\dots\right)+c_1\left(x+\frac{1}{4\times 3}x^4+\frac{1}{7\times 6\times 4\times 3}+\dots\right)\\
&= c_0\left(1+\frac{1}{3\times 2}x^3+\frac{4}{6\times 5\times 4\times 3\times 2}x^6+\dots\right)+c_1\left(x+\frac{2}{4\times 3\times 2}x^4+\frac{2(5)}{7\times 6\times 5\times 4\times 3\times 2}+\dots\right)
\end{aligned}
$$

## 6.2.3 – Example

$$(x^2+1)y''+xy'-y=0$$

$$
\begin{aligned}
(x^2+1)y''+xy'-y&=0\\
y''+\frac{x}{x^2+1}y'-\frac{1}{x^2+1}y&=0
\end{aligned}
$$

Ordinary points:

$$
\begin{aligned}
x^2+1&=0\\
x^2&=-1\\
x&=\pm\sqrt{-1}\\
x&=\pm i
\end{aligned}
$$

$$
\begin{aligned}
y&=\sum_{n=0}^{\infty}c_nx^n\\
y'&=\sum_{n=1}^{\infty}c_nnx^{n-1}\\
y''&=\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}
\end{aligned}
$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \frac{x}{x^2+1}\sum_{n=1}^{\infty} c_n n x^{n-1} - \frac{1}{x^2+1}\sum_{n=0}^{\infty} c_n x^n = 0$$