# Chapter 4

# Higher Order Differential Equations

## 4.9 Systems of Higher Order Linear DEs

$$\text{Solve}\left\{\begin{array}{llll} x' & +y' & +2y & =0 \\ x' & -3x & -2y & =0 \end{array}\right.$$

Need to find $x(t)$ and $y(t)$ that simultaneously solve these equations.

### 4.9.1 – Method: Systematic Elimination

Change from using prime notation to indicate derivatives to the $D$ operator notation.

$$x' \Rightarrow Dx \quad , \quad y' + 2y \Rightarrow (D+2)y$$

$$\left\{\begin{array}{ll} Dx + (D+2)y & = 0 \\ (D-3)x - 2y & = 0 \end{array}\right\} = \left\{\begin{array}{ll} (D-3)\,[Dx + (D+2)y] & = 0 \\ D\,[(D-3)x - 2y] & = 0 \end{array}\right\}$$

$$= \left\{\begin{array}{ll} (D^2-3D)x + (D^2-D-6)y & = 0 \\ (D^2-3D)x - 2Dy & = 0 \end{array}\right\}$$

$$= \left\{\begin{array}{ll} x'' - 3x' + y'' - y' - 6y & = 0 \\ x'' - 3x' - 2y' & = 0 \end{array}\right\}$$

$$\begin{aligned} x'' - 3x' + y'' - y' - 6y &= x'' - 3x' - 2y' \\ y'' - y' - 6y &= -2y' \\ y'' + y' - 6y &= 0 \\ m^2e^{mx} + me^{mx} - 6e^{mx} &= 0 \\ m^2 + m - 6 &= 0 \\ (m+3)(m-2) &= 0 \end{aligned}$$

$$\begin{array}{rr} m_1 + 3 = 0 & \quad m_2 - 2 = 0 \\ m_1 = -3 & \quad m_2 = 2 \\ y_1 = e^{-3t} & \quad y_2 = e^{2t} \end{array}$$

$$y(t) = c_1e^{-3t} + c_2e^{2t}$$

Using the same ideas, we can find $x(t)$, namely:

$$\left\{\begin{array}{ll} Dx+(D+2)y & =0 \\ (D-3)x-2y & =0 \end{array}\right\} = \left\{\begin{array}{ll} (-2)\left[Dx+(D+2)y\right] & =0 \\ (D+2)\left[(D-3)x-2y\right] & =0 \end{array}\right\}$$
$$= \left\{\begin{array}{ll} -2Dx-(2D+4)y & =0 \\ (D^2-D-6)x-(2D+4)y & =0 \end{array}\right\}$$
$$= \left\{\begin{array}{lll} -2x' & -2y'-4y & =0 \\ x''-x'-6x & -2y'-4y & =0 \end{array}\right\}$$

$$\begin{aligned} x''-x'-6x-2y'-4y &= -2x'-2y'-4y \\ x''-x'-6x &= -2x' \\ x''+x'-6x &= 0 \\ n^2e^{nt}+ne^{nt}-6e^{nt} &= 0 \\ n^2+n-6 &= 0 \\ (n+3)(n-2) &= 0 \end{aligned}$$

$$\begin{array}{rl} n_1+3=0 & n_2-2=0 \\ n_1=-3 & n_2=2 \\ x_1=e^{-3t} & x_2=e^{2t} \end{array}$$

$$x(t)=c_3e^{-3t}+c_4e^{2t}$$

However, the 4 constants $c_1$, $c_2$, $c_3$, $c_4$ are not completely independent of each other. To see their dependency, plug $x(t)$ & $y(t)$ into each of the original equations.

$$\begin{aligned} c_3e^{-3t}-3c_1e^{-3t}+c_1e^{-3t} &= 0 \\ c_4e^{2t}-3c_1e^{-3t}+c_1e^{-3t} &= 0 \\ c_3e^{-3t}+2c_1e^{2t}+c_1e^{2t} &= 0 \\ c_4e^{2t}+2c_1e^{2t}+c_1e^{2t} &= 0 \\ &\dots \\ -(3c_3+c_1)e^{-3t}+(2c_4+4c_2)e^{2t} &= 0 \end{aligned}$$

for this to be true for all $t$, you need

$$\left\{\begin{array}{ll} 3c_3+c_1=0 & \Rightarrow c_3=-\frac{1}{3}c_1 \\ 2c_4+4c_2=0 & \Rightarrow c_4=-2c_2 \end{array}\right.$$

## 4.9.2 – Example

$$\left\{\begin{array}{ll} x'-4x+y'' & =t^2 \\ x'+x+y' & =0 \end{array}\right.$$

$$\left\{\begin{array}{ll} x' - 4x + y'' & = t^2 \\ x' + x + y' & = 0 \end{array}\right\} = \left\{\begin{array}{ll} (D-4)x + D^2 y & = t^2 \\ (D+1)x + Dy & = 0 \end{array}\right\}$$
$$= \left\{\begin{array}{ll} (D-4)x + D^2 y & = t^2 \\ D(D+1)x + D^2 y & = 0 \end{array}\right\}$$
$$= \left\{\begin{array}{ll} (D-4)x + D^2 y & = t^2 \\ (D^2+D)x + D^2 y & = 0 \end{array}\right\}$$
$$\begin{aligned} (D-4)x + D^2 y - (D^2+D)x - D^2 y &= t^2 - 0 \\ x' - 4x - x'' - x' &= t^2 - 0 \\ -4x - x'' &= t^2 \\ x'' + 4x &= -t^2 \end{aligned}$$

**First Solve** $x'' + 4x = 0$

Guess $x = e^{mt}$

$$\begin{aligned} m^2 + 4 &= 0 \\ m^2 &= -4 \\ m &= \pm\sqrt{-4} \\ m &= \pm 2i \\ x(t) &= c_1 \cos(2t) + c_2 \sin(2t) \end{aligned}$$

**Next find one particular solution:**