1. Consider the autonomous differential equation below.

   (a) Determine the equilibrium solutions of the differential equation. $y = 3$, $y = -2$, $y = 5$

   (b) Draw a phase portrait diagram for the differential equation.

   (c) For each equilibrium solutions, state whether it is stable, unstable, or semistable.

$$\frac{dy}{dx} = (3 - y)(y + 2)^2(y - 5)$$

2. Consider the DE

$$y'' + 2y' = 8y$$

   (i) Verify that all members of the two parameter family of function $y_1 = c_1e^{2x} + c_2e^{-4x}$ are solutions of the DE.

$$\begin{aligned}
y_1 &= c_1e^{2x} + c_2e^{-4x} \\
y_1' &= 2c_1e^{2x} - 4c_2e^{-4x} \\
y_1'' &= 4c_1e^{2x} + 16c_2e^{-4x} \\
y'' + 2y' &= 8y \\
4c_1e^{2x} + 16c_2e^{-4x} + 2\left(2c_1e^{2x} - 4c_2e^{-4x}\right) &= 8\left(c_1e^{2x} + c_2e^{-4x}\right) \\
4c_1e^{2x} + 16c_2e^{-4x} + 4c_1e^{2x} - 8c_2e^{-4x} &= 8c_1e^{2x} + 8c_2e^{-4x} \\
8c_1e^{2x} + 8c_2e^{-4x} &= 8c_1e^{2x} + 8c_2e^{-4x}
\end{aligned}$$

   (ii) Find a member of the family $y_1 = c_1e^{2x} + c_2e^{-4x}$ that satisfies the initial conditions: $y(0) = 4$ and $y'(0) = 2$.

$$\begin{aligned}
4 &= c_1e^{2(0)} + c_2e^{-4(0)} \\
4 &= c_1e^{0} + c_2e^{0} \\
4 &= c_1(1) + c_2(1) \\
4 &= c_1 + c_2 \\
c_1 &= 4 - c_2 \\
2 &= 2c_1e^{2(0)} - 4c_2e^{-4(0)} \\
2 &= 2c_1e^{0} - 4c_2e^{0} \\
2 &= 2c_1(1) - 4c_2(1) \\
1 &= c_1 - 2c_2 \\
c_1 &= 1 + 2c_2
\end{aligned}$$

$$
\begin{aligned}
4 - c_2 &= 1 + 2c_2 \\
4 &= 1 + 3c_2 \\
3 &= 3c_2 \\
c_2 &= 1 \\
c_1 &= 1 + 2(1) \\
c_1 &= 1 + 2 \\
c_1 &= 3 \\
y_1 &= 3e^{2x} + 1e^{-4x} \\
&= 3e^{2x} + e^{-4x}
\end{aligned}
$$

3. For each of the differential equations given below, indicate whether or not the equation is separable, exact, or first order linear by writing "yes" or "no" in the chart below. You may use the space below the chart for scratch work.

Differential Equation	Separable	Exact	First Order Linear
$\frac{dy}{dx} = y - xy$	yes	no/yes*	yes
$ydx = (y - xy^2)dy$	no	yes	no
$\left(x^2 + \frac{2y}{x}\right) dx = (3 - \ln(x^2))dy$			

4. Solve the DE:

$$y\frac{dy}{dx} = x + xy^2$$

$$
\begin{aligned}
y\frac{dy}{dx} &= x + xy^2 \\
ydy &= \left(x + xy^2\right) dx \\
\left(x + xy^2\right) dx - ydy &= 0 \\
M = x + xy^2 &\quad N = -y \\
M_y = 0 + 2xy &\quad N_x = 0
\end{aligned}
$$

5. Solve the DE:

$$x^2y' + 4xy = \cos(x^3)$$

$$
\begin{aligned}
x^2\frac{dy}{dx} + 4xy &= \cos(x^3) \\
\frac{dy}{dx} + \frac{4}{x}y &= \frac{\cos(x^3)}{x^2} \\
P(x) &= \frac{4}{x} \\
\mu &= e^{\int P(x)dx} \\
&= e^{\int \frac{4}{x}dx} \\
&= e^{4ln|x|} \\
&= e^{ln|x^4|} \\
&= x^4 \\
x^4\frac{dy}{dx} + 4x^3y &= \cos(x^3)x^2 \\
\frac{d}{dx}\left(x^4y\right) &= \cos(x^3)x^2 \\
\int \frac{d}{dx}\left(x^4y\right) &= \int \cos(x^3)x^2dx \\
x^4y &= \int \cos(x^3)x^2dx \\
x^4y &= \int \frac{1}{3}\cos(u)du \\
x^4y &= \frac{1}{3}\sin(u) + \frac{1}{3}C \\
x^4y &= \frac{\sin(x^3)}{3} + C \\
y &= \frac{\sin(x^3)}{3x^4} + \frac{C}{x^4}
\end{aligned}
$$

6. Solve the DE:

$$(x^3 + y^3)dx + (3xy^2 + \sin(y))dy = 0$$

$$
\begin{aligned}
(x^3 + y^3)dx + (3xy^2 + \sin(y))dy = 0 \\
M(x,y) = x^3 + y^3 \quad N(x,y) = 3xy^2 + \sin(y) \\
M_y(x,y) = 3y^2 \quad N_x(x,y) = 3y^2 + 0 \\
M_y(x,y) &= N_x(x,y) \\
f(x,y) &= \int (x^3 + y^3)dx \\
f(x,y) &= \frac{x^4}{4} + xy^3 + \phi(y) \\
\frac{\partial f}{\partial y}\left(\frac{x^4}{4} + xy^3 + \phi(y)\right) &= 3xy^2 + \sin(y)
\end{aligned}
$$

$$\begin{aligned}
3xy^2 + \phi'(y) &= 3xy^2 + \sin(y) \\
\phi'(y) &= \sin(y) \\
\int \phi'(y)dy &= \int \sin(y)dy \\
\phi(y) &= -\cos(y) \\
f(x,y) &= \frac{x^4}{4} + xy^3 - \cos(y) \\
f'(x,y) &= 0 \\
f(x,y) &= c \\
\frac{x^4}{4} + xy^3 - \cos(y) &= c
\end{aligned}$$

7. In each part below, a DE is given which is not separable, exact, nor first order linear. Make an appropriate substitution to transform the given DE into one which is either separable, exact, or first order linear. Simplify the transformed DE to show that it is either separable, exact, or first order linear; but do not solve the transformed DE.

(i)

$$y' = \frac{2x+y}{y+2x}$$

$$\begin{aligned}
y' &= \frac{2x+y}{y+2x} \\
y' &= 1 \\
\frac{dy}{dx} &= 1 \\
dy &= dx
\end{aligned}$$

Separable

(ii)

$$y' = \frac{2x+y}{2y+x}$$

(iii)

$$\frac{dy}{dx} = y(xy^3 - 1)$$

8. Suppose that a large container initially contains a solution of brine which has 50 pounds of salt and volume 400 gallons. A second brine solution of concentration 2 pounds of salt per gallon is added at a rate of 4 gallons per minute. And the well mixed solution in the container flows out also at a rate of 4 gallons per minute. Construct an initial value problem whose solution $A(t)$ would represent the pounds of salt in the container t minutes after the described process began. Do not solve initial value problem.