

# MATH 252 - Introduction to Differential Equations

## Notes

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# Chapter 1

## Introduction to Differential Equations

### 1.1 Terminology and Notation

**Differential equation (D.E.)** – An equation in which at least one derivative of an unknown function.

**Order of the D.E.** – The highest order of derivative in the D.E.

#### 1.1.1 – Example

$$4y'' + e^x y' - 3yy' = \sin(x)$$

An example of a partial differential equation is:

$$\frac{\partial T}{\partial x} + x^2 \frac{\partial T}{\partial y} = x + y$$

however, we won't study these in this course.

#### 1.1.2 – Linear vs Non-Linear DE's

**Linear D.E.** – The dependent variable and all of its derivatives in the D.E. are in separate terms to the 1<sup>st</sup> power.  $y^{(n)}$  or  $\frac{d^n y}{dx^n}$  where  $n \neq 1$  are non-first power.

$$4y'' + e^x y' - 3yy' = \sin(x)$$

is a non-linear D.E. while

$$4y'' + e^x y' - 3y = \sin(x)$$

is linear.

The general formula of a linear D.E. would look like

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = g(x)$$

**Solution** – a function  $\phi(x)$  and an interval  $I$  for which the D.E. is satisfied when  $y = \phi(x)$  for all  $x$  in  $I$ .

It may be the case that the natural domain of  $\phi(x)$  is larger than  $I$ .

### 1.1.3 – Example

$y' = -\frac{1}{x^2}$  has a solution  $\phi(x) = \frac{1}{x}$  on  $I = (0, \infty)$  but the domain of  $\phi(x) = (-\infty, 0) \cup (0, \infty)$ .

Practice:

$$\frac{d^2x}{dt^2} + 16x = 0$$

Show (*Verify* not derive)  $x(t) = c_1 \sin(4t)$  is a solution on  $(-\infty, \infty)$  where  $c$  is any real parameter.

$$\begin{aligned} x &= c_1 \sin(4t) \\ \frac{dx}{dt} &= 4c_1 \cos(4t) \\ \frac{d^2x}{dt^2} &= -16c_1 \sin(4t) \\ \text{LHS} &= \frac{d^2x}{dt^2} + 16x \\ &= -16c_1 \sin(4t) + 16(c_1 \sin(4t)) \\ &= 0 = \text{RHS} \end{aligned}$$

But the equation  $x = c_2 \cos(4t)$  would also be a solution. If you have 2 equations that are both solutions, you could add them together and you would still have a solution.  $x = c_1 \sin(4t) + c_2 \cos(4t)$  is a solution for all parameters  $c_1$  and  $c_2$ . In fact, this is the general solution to the D.E.

The D.E.

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

Show  $y = (\frac{1}{4}x^2 + C)^2$  is a one parameter family of solutions

$$\begin{aligned} \text{LHS} &= \frac{dy}{dx} = 2 \left( \frac{1}{4}x^2 + C \right) \times \frac{1}{2}x \\ &= x \left( \frac{1}{4}x^2 + C \right) \\ \text{RHS} &= xy^{\frac{1}{2}} = x \left( \left( \frac{1}{4}x^2 + C \right)^2 \right)^{\frac{1}{2}} \\ &= x \left( \frac{1}{4}x^2 + C \right) \\ \text{LHS} &= \text{RHS} \end{aligned}$$

But there is another solution: namely  $y(x) = 0$  for all  $x$ . This is called the “trivial solution”.

## 1.2 Initial Value Problems (IVP)

1st order IVP is a 1st order D.E. together with one extra condition:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

2nd order IVP

$$y'' = f(x, y, y')$$

Initial conditions:

- $y(x_0) = y_0$
- $y'(x_0) = y_1$

### 1.2.1 – Example

$$y' = y \text{ and } y(0) = 3$$

$y = ce^x$  is a one-parameter family of solutions

$$\frac{d}{dx}(ce^x) = ce^x = y$$

$$\begin{aligned} ce^1 &= -2 \\ c &= -\frac{2}{e} \\ y &= \left(-\frac{2}{e}\right)e^x \\ y &= -2e^{x-1} \end{aligned}$$

### 1.2.2 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Given that you have the solution:  $y = \frac{1}{x^2+C}$ , Solve:

$$\begin{aligned} -1 &= \frac{1}{(0)^2 + c} \\ -1 &= \frac{1}{c} \\ -1 \times c &= 1 \\ c &= -1 \\ y &= \frac{1}{x^2 - 1}, I = (-1, 1) \end{aligned}$$

### 1.2.3 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

#### Example

$$x'' + 16x = 0 \text{ and } x\left(\frac{\pi}{2}\right) = 5 \text{ and } x'\left(\frac{\pi}{2}\right) = -4$$

$$x = c_1 \cos(4t) + c_2 \sin(4t)$$

$$5 = c_1 \cos(4t) + c_2 \sin(4t)$$

$$= c_1 \cos(2\pi) + c_2 \sin(2\pi)$$

$$= c_1(1) + c_2(0)$$

$$= c_1$$

$$x' = -4c_1 \sin(4t) + 4c_2 \cos(4t)$$

$$-4 = -4c_1 \sin\left(4\left(\frac{\pi}{2}\right)\right) + 4c_2 \cos\left(4\left(\frac{\pi}{2}\right)\right)$$

$$= -4c_1 \sin(2\pi) + 4c_2 \cos(2\pi)$$

$$= -4c_1(0) + 4c_2(1)$$

$$= 4c_2$$

$$-1 = c_2$$

Reasonable Question: Given a 1st order IVP, can we say whether a solution *exists* or not and, if a solution exists, is it *unique*.

**Theorem:** Given  $y' = f(x, y)$  and  $y(x_0) = y_0$ , if  $f(x, y)$  and  $\frac{\partial f}{\partial y}$  are both continuous on a rectangle  $R$  containing  $(x_0, y_0)$  in its interior, then there exists an interval  $I = (x_0 - h, x_0 + h)$  where  $h > 0$  such that there exists a unique solution to IVP on  $I$ .

### 1.2.4 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(1) = 2$$

- $f(x, y) = xy^{\frac{1}{2}}$  is continuous everywhere its defined  $y \geq 0$
- $\frac{\partial f}{\partial y} = x^{\frac{1}{2}}y^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$  is continuous everywhere its defined  $y > 0$

### 1.2.5 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(0) = 0$$

- $f(x, y) = xy^{\frac{1}{2}}$  is continuous for all  $x$  and  $y \geq 0$
- $\frac{\partial f}{\partial y} = \frac{x}{2y}$  is continuous for all  $x$  and  $y > 0$ .
- **Theorem does not give any conclusion.**

# Chapter 2

## First-Order Differential Equations

### 2.1 Solution Curves Without a Solution

Given a 1st order D.E.  $y' = f(x, y)$ ,  $y'$  is the slope of the tangent line at any point  $(x_0, y_0)$  on a solution curve

#### 2.1.1 – Example

$$y' = f(x, y) = x + y$$

- $f(0, 0)=0$
- $f(1, 0)=1$

### 2.1.2 – Slope/Direction Fields

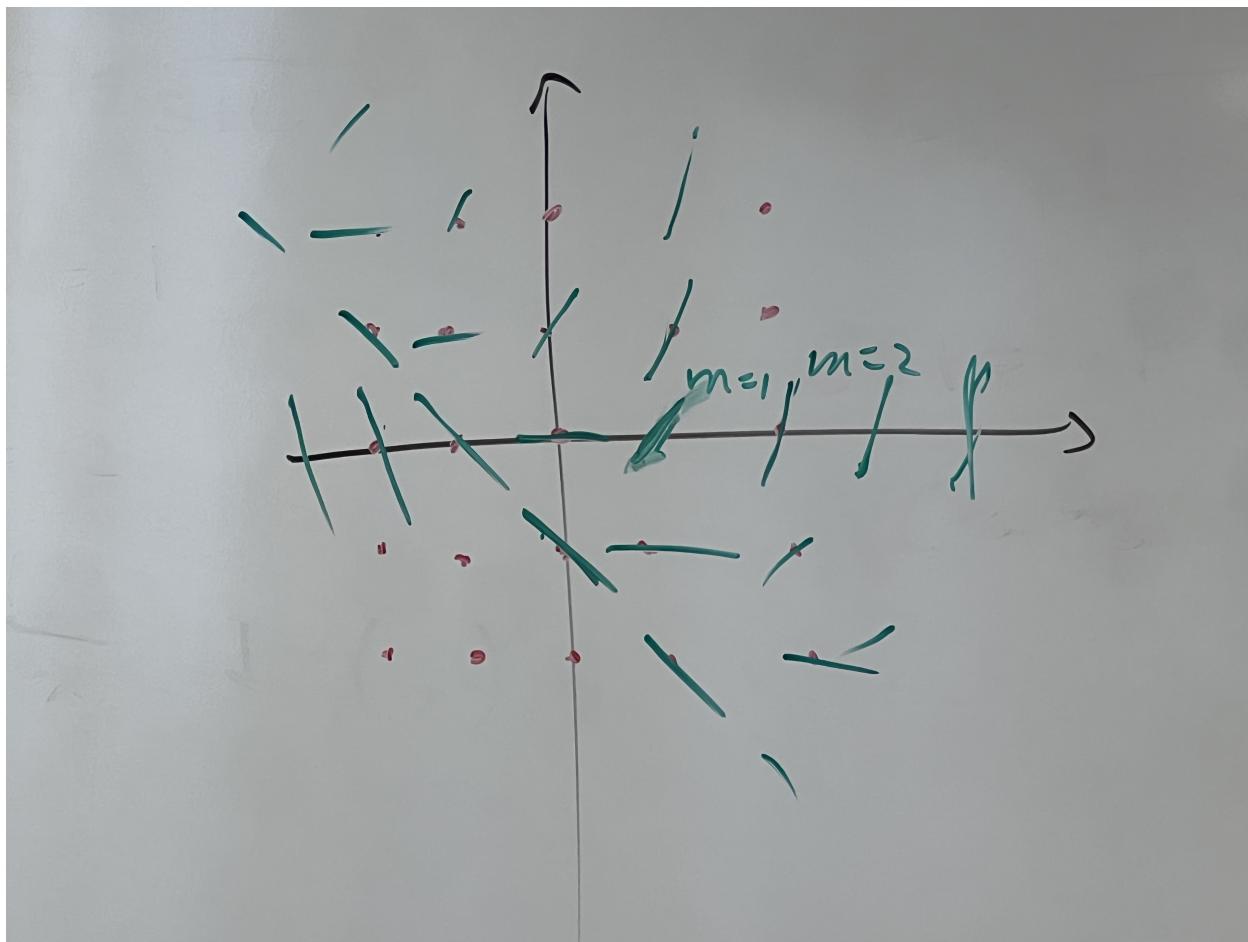


Figure 2.1: The direction field for the previous example

If the function  $f(x, y)$  in the D.E.  $y' = f(x, y)$  is reasonably simple so that we can solve  $f(x, y) = 0$ , we can make a “phase portrait diagram”. We will also assume  $f(x, y)$  only involves the  $y$ -variable.

### 2.1.3 – Example

$$\begin{aligned} y' &= (y + 2)(y - 3)(y - 5) \\ f(x, y) &= (y + 2)(y - 3)(y - 5) \end{aligned}$$

An “equilibrium solution” is a solution where  $y$  is a constant. In this example:  $y = 3$ ,  $y = 5$ ,  $y = -2$  are each constant functions.

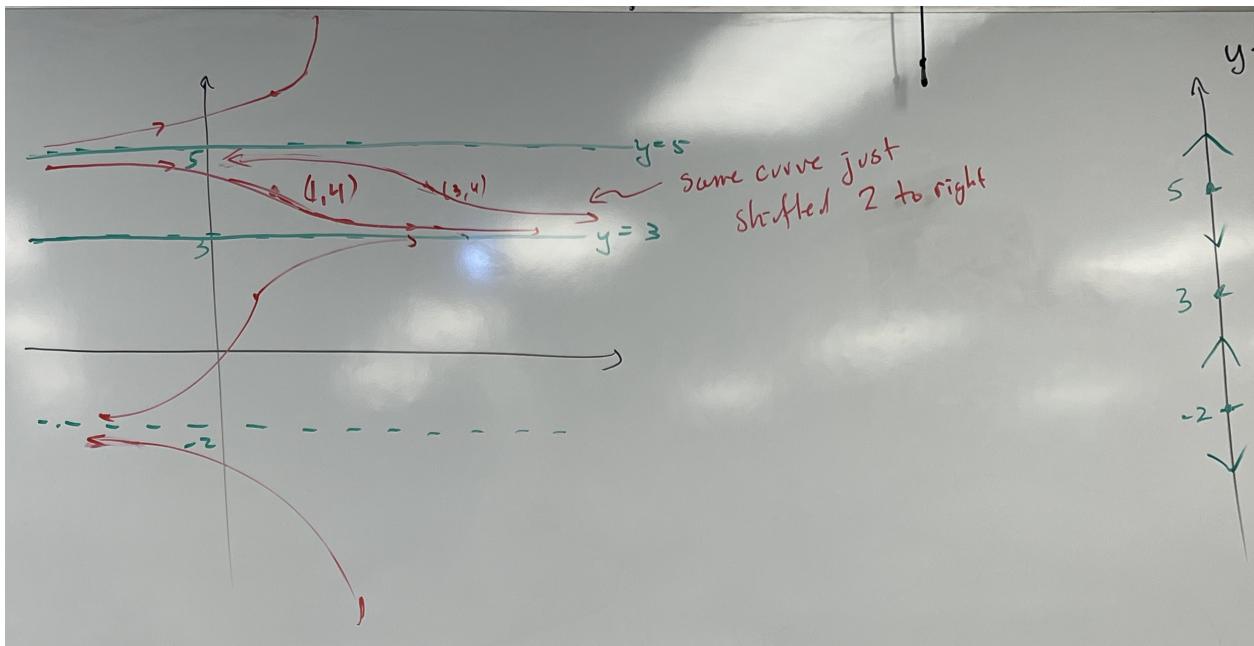


Figure 2.2: The equilibrium solution for the previous example.

The area around  $y = 5$  is an unstable equilibrium since the solutions diverge and go in separate directions away from  $y = 5$ . The area around  $y = 3$  is a stable equilibrium because the slopes above and below it converge to  $y = 3$ . The area around  $y = -2$  is semi-stable, since all the slopes around it will converge in one direction, but the point isn't always  $y = -2$ .

## 2.2 Separable Differential Equations

Separable D.E.s are DE's  $\frac{dy}{dx} = f(x, y)$  where  $f(x, y)$  can be factored as  $f(x, y) = g(x)h(y)$ .

$$\frac{dy}{dx} = (1 + y^2)x^3 \text{ is separable}$$

$$\frac{dy}{dx} = \sin(xy) \text{ is not separable}$$

$$\frac{dy}{dx} = x^3y \text{ is separable}$$

$$\begin{aligned} \frac{5}{xy} \frac{dy}{dx} &= (x^2 + y)e^y \\ \frac{dy}{dx} &= \frac{xy(x^2 + y)e^y}{5} \\ &= \frac{x(x^2 + y)}{5} \times ye^y \end{aligned}$$

### 2.2.1 – Method of Solution

“Separate the variable” to get  $\frac{1}{h(y)}dy = g(x)d$  or  $p(y)dy = g(x)dx$  where  $p(y) = \frac{1}{h(y)}$ .  
**Integrate both sides**

$$\int p(y)dy = \int g(x)dx \text{ and if possible, solve for } y$$

### 2.2.2 – Example

$$\begin{aligned} \frac{dy}{dx} &= (1 + y^2)x^3 \\ \int \frac{1}{1 + y^2}dy &= \int x^3dx \\ \tan^{-1}(y) + C_1 &= \frac{x^4}{4} + C_2 \\ \tan^{-1}(y) &= \frac{x^4}{4} + C_2 - C_1 \\ \tan^{-1}(y) &= \frac{x^4}{4} + C \\ y &= \tan\left(\frac{x^4}{4} + C\right) \end{aligned}$$

### 2.2.3 – Example

Problem 12 from the textbook.

$$\begin{aligned} \sin(3x)dx + 2y \cos^3(3x)dy &= 0 \\ \int -2ydy &= \int \frac{\sin(3x)}{\cos^3(x)}dx \\ &= \int \tan(3x) \sec^2(3x)dx \\ &= \int u \frac{1}{3}du \text{ where } u = \tan(3x), du = 3 \sec^2(3x)dx \\ -2 \int ydy &= \frac{1}{3} \int u du + C \\ -y^2 &= \frac{u^2}{6} + C \\ &= \frac{\tan^2(3x)}{6} + C \\ \frac{\tan^2(3x)}{6} + y^2 &= -C \\ \frac{\tan^2(3x)}{6} + y^2 &= C \end{aligned}$$

Problem 25 from the textbook.

$$x^2 \frac{dy}{dx} = y - xy, y(-1) = -1$$

$$x^2 \frac{dy}{dx} = y - xy$$

$$x^2 \frac{dy}{dx} = y(1 - x)$$

$$\frac{dy}{y} = \frac{(1-x)}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{(1-x)}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{x^2} dx - \int \frac{x}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{x^2} dx - \int \frac{1}{x} dx$$

$$\ln |y| + C_1 = -\frac{1}{x} + C_2 - \ln |x| + C_3$$

$$\ln |y| = -\frac{1}{x} - \ln |x| + C$$

$$y = e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C$$

$$y = e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C$$

$$y = e^{-\frac{1}{x}} \times \frac{1}{|x|} \times e^C$$

$$y = \frac{1}{|x|} e^{C-\frac{1}{x}}$$

$$-1 = \frac{1}{|-1|} e^{C-\frac{1}{-1}}$$

$$-1 = \frac{1}{1} e^{C-(-1)}$$

$$-1 = e^{C+1}$$

## 2.3 First Order Linear Differential Equations

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\left. \frac{dy}{dx} + P(x)y = f(x) \right\} \text{ Standard form of a 1st-order linear DE}$$

We will try to find a function  $\mu(x)$  such that by multiplying the D.E. by an integrating factor (I.F.)  $\mu(x)$ :

$$\mu(x) \frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

such that the LHS is an exact derivative, Observe:

$$\frac{d}{dx}(\mu(x)y) = \mu(x) \frac{dy}{dx} + \frac{dy}{dx}y$$

from which we see

$$\begin{aligned}\mu(x)P(x) &= \frac{d\mu}{dx} \\ P(x)dx &= \frac{d\mu}{\mu(x)} \\ \int P(x)dx &= \int \frac{d\mu}{\mu} \\ \int P(x)dx &= \ln \mu \\ \ln \mu &= \int P(x)dx \\ \mu &= e^{\int P(x)dx}\end{aligned}$$

### 2.3.1 – Example

$$\begin{aligned}x \frac{dy}{dx} - 4y &= x^6 e^x \\ \text{Standard form: } \frac{dy}{dx} - \frac{4}{x}y &= x^5 e^x \\ P(x) &= -\frac{4}{x} \\ \mu &= e^{\int -\frac{4}{x} dx} \\ &= e^{-4 \ln x} \\ &= e^{\ln x^{-4}} \\ &= x^{-4} \\ \text{I.F.} &= \mu = x^{-4}\end{aligned}$$

Now multiply the standard form of the given D.E. by  $x^{-4}$ .

$$\begin{aligned}x^{-4} \frac{dy}{dx} - x^{-4} \frac{4}{x}y &= x^{-4} x^5 e^x \\ x^{-4} \frac{dy}{dx} - x^{-4} \frac{4}{x}y &= x e^x \\ \int \frac{d}{dx}(x^{-4}y) &= \int x e^x \\ x^{-4}y &= \int x e^x\end{aligned}$$

### 2.3.2 – Example

$$\begin{aligned}
 & (x^2 - 9) \frac{dy}{dx} + xy = 0 \\
 & (x^2 - 9) \frac{dy}{dx} + xy = 0 \\
 & \frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0 \\
 & P(x) = \frac{x}{x^2 - 9} \\
 & \int P(x)dx = \int \frac{x}{x^2 - 9}dx \\
 & \int P(x)dx = \int \frac{1}{u - 9} \frac{du}{2} \\
 & \int P(x)dx = \frac{1}{2} \int \frac{1}{u - 9} du \\
 & \int P(x)dx = \frac{1}{2} \ln|u - 9| \\
 & \int P(x)dx = \frac{1}{2} \ln|x^2 - 9| \\
 & \mu = e^{\frac{1}{2} \ln|x^2 - 9|} \\
 & \mu = e^{\ln|(x^2 - 9)^{\frac{1}{2}}|} \\
 & \mu = (x^2 - 9)^{\frac{1}{2}} \\
 & \mu = \sqrt{x^2 - 9} \\
 & \sqrt{x^2 - 9} \left( \frac{dy}{dx} + \frac{x}{x^2 - 9}y \right) = \sqrt{x^2 - 9}(0) \\
 & \sqrt{x^2 - 9} \frac{dy}{dx} + \frac{x}{\sqrt{x^2 - 9}}y = 0 \\
 & \int \frac{d}{dx} \left( y\sqrt{x^2 - 9} \right) = \int 0 \\
 & y\sqrt{x^2 - 9} = C \\
 & y = \frac{C}{\sqrt{x^2 - 9}}
 \end{aligned}$$

## 2.4 Exact Equations

1st Order D.E. in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

Given a function

$$z = f(x, y)$$

, the total differential,  $dz$ , is defined as

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

### 2.4.1 – Method

See if we can find a function  $f(x, y)$  such that

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

If we can do this, then the D.E. is equivalent to

$$df = 0 \Rightarrow f(x, y) = c$$

is an implicit solution of D.E.

Assume that  $M$  and  $N$  have continuous 1st order partials (assuming  $f$  exists)

$$\left. \begin{array}{l} My = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} dy = f_{xy} \\ Nx = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} dy = f_{yx} \end{array} \right\} \text{Theorem tells us these are equal}$$

This provides a quick test to check if the D.E. is exact or not.

### 2.4.2 – Example

$$2xydx + (x^2 - 1)dy = 0$$

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

To check if the D.E. is exact

$$M_y = 2x = N_x$$

We now know there exists a function  $f(x, y)$  with

$$\begin{aligned} \frac{\partial f}{\partial x} &= M = 2xy \\ \frac{\partial f}{\partial y} &= N = x^2 - 1 \end{aligned}$$

$$\begin{aligned}
 f_M(x, y) &= \int \frac{\partial f}{\partial x} dx \\
 &= \int 2xy dx \\
 &= x^2y + \phi(y) \\
 \frac{\partial f}{\partial y} (x^2y + \phi(y)) &= x^2 - 1 \text{ required to equal } N \\
 x^2 + \phi'(y) &= x^2 - 1 \\
 \phi'(y) &= -1 \\
 \phi(y) &= \int -1 dy \\
 &= -y \\
 f(x, y) &= x^2y - y \\
 d(f(x, y)) &= 0 \\
 f(x, y) &= c \\
 x^2y - y = c &\text{ is an implicit solution of the D.E.}
 \end{aligned}$$

Note: the  $f_M$  format is just there to show which partial equation was integrated. It was made by me and, as far as I know, is not standardly known.

### 2.4.3 – Example

$$\begin{aligned}
 (e^{2y} - y \cos(xy)) dx + (2xe^{2y} - x \cos(xy) + 2y) dy &= 0 \\
 M_y &= N_x \\
 \frac{\partial}{\partial y} (e^{2y} - y \cos(xy)) &= \frac{\partial}{\partial x} (2xe^{2y} - x \cos(xy) + 2y) \\
 2e^{2y} - [\cos(xy) - y \sin(xy) \times x] &= 2e^{2y} - (\cos(xy) - x \sin(xy) \times y) + 0 \\
 2e^{2y} - \cos(xy) + xy \sin(xy) &= 2e^{2y} - \cos(xy) + xy \sin(xy) \\
 \frac{\partial f}{\partial x} &= M = e^{2y} - y \cos(xy) \\
 \frac{\partial f}{\partial y} &= N = 2xe^{2y} - x \cos(xy) + 2y \\
 f_N(x, y) &= \int \frac{\partial f}{\partial y} dy \\
 &= \int (2xe^{2y} - x \cos(xy) + 2y) dy \\
 &= \frac{2xe^{2y}}{2} - \frac{x \sin(xy)}{x} + 2 \times \frac{y^2}{2} + \phi(x) \\
 &= xe^{2y} - \sin(xy) + y^2 + \phi(x)
 \end{aligned}$$

Take the  $\partial x$  of this and equate with  $M$ :

$$\begin{aligned} M &= \frac{\partial}{\partial x} (xe^{2y} - \sin(xy) + y^2 + \phi(x)) \\ e^{2y} - y \cos(xy) &= e^{2y} - y \cos(xy) + 0 + \phi'(x) \\ 0 &= \phi'(x) \\ \phi(x) &= c \end{aligned}$$

So  $f(x, y) = c_2$  is the solution

$$xe^{2y} - \sin(xy) + y^2 = c$$

$$dx = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

#### 2.4.4 – What can you do if $M_y \neq N_x$

Sometimes you can multiply the DE by an integrating factor  $\mu(x, y)$  to get an exact DE.

If

$$\frac{M_y - N_x}{N}$$

is a function of only  $x$ , then

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

will be an I.F.

If

$$\frac{N_x - M_y}{M}$$

is a function of only  $y$ , then

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

will be an I.F.

#### 2.4.5 – Example

$$xydx + (2x^2 + 3y^2 - 20) dy = 0$$

$$\begin{aligned}
M_y &= x \\
N_x &= 4x \\
M_y &\neq N_x \\
\frac{N_x - M_y}{M} &= \frac{4x - x}{xy} \\
&= \frac{3x}{xy} \\
&= \frac{3}{y} \text{ is a function of just } y
\end{aligned}$$

So:

$$\begin{aligned}
\mu &= e^{\int \frac{3}{y} dy} \\
&= e^{3 \ln y} \\
&= y^3 \\
xy^4 dx + y^3(2x^2 + 3y^2 - 20) dy &= 0(y^3) \\
xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3) dy &= \\
M_y &= N_x \\
4xy^3 &= 4xy^3 \\
\frac{\partial f}{\partial x} &
\end{aligned}$$

## 2.5 Substitution Methods

Taking a D.E. that's not:

- Separable
- 1st Order Linear
- Exact

and making a substitution to turn the new D.E. into one of these.

**Theorem:** Given a D.E.

$$M(x, y)dx + N(x, y)dy = 0$$

A function  $f(x, y)$  is said to be homogenous of order  $\alpha$  if  $f(tx, ty) = t^\alpha f(x, y)$ .

### 2.5.1 – Example

Given:

$$f(x, y) = x^3 + 5xy^2 - y^3$$

Then:

$$\begin{aligned}
 f(tx, ty) &= (tx)^3 + 5(tx)(ty)^2 - (ty)^3 \\
 &= t^3x^3 + 5t^3xy^2 - t^3y^3 \\
 &= t^3(x^3 + 5xy^2 - y^3) \\
 &= t^3f(x, y)
 \end{aligned}$$

### 2.5.2 – Example

$$\begin{aligned}
 f(x, y) &= \frac{x+y}{x^2+y^2} \\
 f(tx, ty) &= \frac{tx+ty}{(tx)^2+(ty)^2} \\
 f(tx, ty) &= \frac{tx+ty}{x^2t^2+y^2t^2} \\
 f(tx, ty) &= \frac{t}{t^2} \times \frac{x+y}{x^2+y^2} \\
 f(tx, ty) &= \frac{t}{t^2}f(x, y) \\
 f(tx, ty) &= \frac{1}{t}f(x, y)
 \end{aligned}$$

$f(x, y) = \frac{x+y}{x^2+y^2}$  is homogenous of order  $\alpha = -1$

### 2.5.3 – Substitution Rule

If  $M(x, y)$  and  $N(x, y)$  are homogenous, each of the same order, then  $u = \frac{y}{x}$  i.e.,  $y = ux$  or  $v = \frac{x}{y}$  (i.e.  $x = vy$ ) will produce a separable D.E.

### 2.5.4 – Example

Solve the separable D.E. and then back-substitute

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

$$\begin{aligned}
M(x, y) &= x^2 + y^2 & N &= x^2 - xy \\
M_y &= 2y & N_x &= 2x - y \\
M_y &\neq N_x \\
M(tx, ty) &= (tx)^2 + (ty)^2 \\
&= t^2x^2 + t^2y^2 \\
&= t^2(x^2 + y^2) \\
&= t^2M(x, y) \quad M \text{ is homogeneous of order 2 and so is } N \\
u &= \frac{y}{x} \\
y &= ux \\
dy &= udx + xdu \\
(x^2 + (ux)^2)dx + (x^2 - x(ux))(udx + xdu) &= 0 \\
(x^2 + u^2x^2)dx + (x^2 - ux^2)(udx + xdu) &= 0 \\
(1 + u^2)x^2dx + x^2(1 - u)(udx + xdu) &= 0 \\
(1 + u^2)x^2dx + x^2(udx + xdu - u^2dx - uxdx) &= 0 \\
x^2(1dx + u^2dx + udx + xdu - u^2dx - uxdx) &= 0 \\
x^2(1dx + u^2dx - u^2dx + udx + xdu - uxdx) &= 0 \\
x^2(1dx + udx + xdu - uxdx) &= 0 \\
x^2(1 + u)dx + x^3(1 - u)du &= 0 \\
\int \frac{1}{x}dx &= \int -\frac{1-u}{1+u}du \\
&= \int \frac{u-1}{u+1}du \\
&= \int \frac{u+(1-2)}{u+1}du \\
&= \int \left( \frac{u+1}{u+1} - \frac{2}{u+1} \right) du \\
&= \int \left( 1 - \frac{2}{u+1} \right) du \\
\ln|x| &= \int \left( 1 - \frac{2}{u+1} \right) du \\
&= u - 2 \ln|u+1| + C \\
\ln|x| &= \frac{y}{x} - 2 \ln \left| \frac{y}{x} + 1 \right| + C
\end{aligned}$$

### 2.5.5 – Bernoulli Equation

**Theorem:** An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where  $n \neq 0, 1$  is called a Bernoulli Equation. The substitution

$$u = y^{1-n}$$

will transform the D.E. into a 1st order linear.

### 2.5.6 – Example

$$\begin{aligned} x \frac{dy}{dx} + y &= x^2 y^2 \\ \frac{dy}{dx} + \frac{y}{x} &= x y^2 \end{aligned}$$

is a **Bernoulli equation** with  $n = 2$ .

$$\begin{aligned} u &= y^{1-2} \\ &= y^{-1} \\ &= \frac{1}{y} \\ \frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\ &= -1 y^{-2} \frac{dy}{dx} \\ &= -\frac{1}{y^2} \frac{dy}{dx} \\ -y^{-2} \frac{dy}{dx} + -y^{-2} \times \frac{y}{x} &= -y^{-2} \times x y^2 \\ -y^{-2} \frac{dy}{dx} + -\frac{1}{x} y^{-1} &= -x \\ \frac{du}{dx} - \frac{1}{x} u &= -x \end{aligned}$$

$$\begin{aligned}
\text{I.F.} &= \mu = e^{P(x)dx} \\
&= e^{-\int \frac{1}{x} dx} \\
&= e^{-\ln|x|} \\
&= e^{\ln|x^{-1}|} \\
&= x^{-1} \\
\frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u &= -1 \\
\frac{d}{dx} \left( \frac{1}{x} u \right) &= -1 \\
\int \frac{d}{dx} \left( \frac{1}{x} u \right) dx &= \int -1 dx \\
\frac{1}{x} u &= \int -1 dx \\
\frac{1}{x} u &= -x + C \\
\frac{1}{x} \times 1y &= -x + C \\
\frac{1}{x(-x+C)} &= y \\
y &= \frac{1}{Cx-x^2}
\end{aligned}$$

**Theorem:** If the D.E. can be expressed as

$$\frac{dy}{dx} = f(Ax + by + C)$$

for particular numbers  $A, B, C$ , then let

$$u = Ax + By + C$$

to get a separable D.E.

### 2.5.7 – Example

$$\frac{dy}{dx} = (-2x + y)^2 - 7, y(0) = 0$$

$$\begin{aligned} u &= -2x + y \\ \frac{du}{dx} &= \frac{dy}{dx} \times \frac{du}{dy} \\ &= -2 + \frac{dy}{dx} \\ \frac{du}{dx} + 2 &= \frac{dy}{dx} \\ \frac{du}{dx} + 2 &= u^2 - 7 \\ \frac{du}{dx} &= u^2 - 9 \\ \frac{du}{u^2 - 9} &= dx \\ \int \frac{du}{u^2 - 9} &= \int dx \\ \int \frac{du}{(u+3)(u-3)} &= x + C \\ \int \frac{du}{(u+3)(u-3)} &= x + C \end{aligned}$$

# Chapter 3

## Modeling using DEs

### 3.1 Linear DE Modeling

#### 3.1.1 – Standard Problems

- 1) Population Growth (or decline)
- 2) Radioactive Decay
- 3) Newton's Law of Cooling
- 4) Mixture Problems

#### 3.1.2 – Population Model

Assume the rate of population change is proportional to the size of the population

$$P(t) = \text{population at time } t$$

$$\frac{dP}{dt} = kP$$

$$\frac{\frac{dP}{dt}}{P} = k \text{ is the relative growth rate of the population}$$

$$\begin{aligned}
 \frac{dP}{dt} &= kP \\
 \frac{dP}{P} &= kdt \\
 \int \frac{dP}{P} &= \int kdt \\
 \ln|P| &= kt + C \\
 |P| &= e^{kt+C} \\
 |P| &= e^{kt}e^C \\
 |P| &= Ae^{kt} \text{ where } A > 0 \\
 P &= \pm Ae^{kt} \\
 P &= Be^{kt} \text{ where } B \neq 0 \\
 P &= De^{kt} \text{ where } D \text{ can be any real number}
 \end{aligned}$$

The constant can become any number because 0 would be a valid rate of population change, it means that the population size isn't changing.

### 3.1.3 – Example

If, initially at 2 p.m., there are 1,000 bacteria on a petri dish and at 4 p.m., there are 2,000 bacteria. Assuming constant relative growth rate, how many bacteria are there at 5 p.m.?  $P(t)$  = population  $t$  hours after 2 p.m.

$$\begin{aligned}
 P(t) &= Ae^{kt} \\
 1000 &= Ae^{(0)k} \\
 1000 &= Ae^0 \\
 1000 &= A(1) \\
 A &= 1000
 \end{aligned}$$

$$\begin{aligned}
 P(2) &= 2000 \\
 P(2) &= 1000e^{2k} \\
 2000 &= 1000e^{2k} \\
 2 &= e^{2k} \\
 \ln(2) &= 2k \\
 k &= \frac{\ln(2)}{2}
 \end{aligned}$$

$$\begin{aligned}
P(t) &= 1000e^{\frac{\ln(2)}{2}t} \\
P(3) &= 1000e^{\frac{\ln(2)}{2}(3)} \\
&= 1000e^{1.5\ln(2)} \\
&= 1000e^{\ln(2^{1.5})} \\
&= 1000(2^{1.5}) \\
&= 2000(\sqrt{2}) \\
P(3) &\approx 2828.427(\sqrt{2}) \\
P(t) &= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\ln(2^{\frac{t}{2}})} \\
&= 1000 \times 2^{\frac{t}{2}}
\end{aligned}$$

### 3.1.4 – Radioactive Decay

$$m(t) = m_0 e^{kt} \text{ where } k < 0$$

The Half-Life is the amount of time it takes for half of the original amount to remain:

$$\frac{1}{2}A_0 = A_0 e^{kt} \Rightarrow \frac{1}{2} = e^{kt}$$

### 3.1.5 – Mixture Problems

#### Setup

Initially, the container has 200 gallons of brine solution (salt-water) of concentration  $\frac{10 \text{ lbs}}{200 \text{ gallons}} = 0.05 \frac{\text{lbs}}{\text{gallon}}$ . A solution of  $\frac{5 \text{ lbs}}{200 \text{ gallons}} = 0.025 \frac{\text{lbs}}{\text{gallon}}$  is poured into the initial container at a rate of  $\frac{4 \text{ gallons}}{\text{min}}$ . How many pounds of salt are there in the container after 2 hours.

Let  $A(t) = \# \text{ lbs of salt } t \text{ minutes after the process starts}$

$\frac{dA}{dt}$  = The rate of change of # lbs of salt

$$\begin{aligned}
 \frac{dA}{dt} &= 0.025 \frac{\text{lbs}}{\text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \Big\} \text{rate in} \\
 &\quad - \frac{A(t) \text{lbs}}{200 \text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \Big\} \text{rate out} \\
 &= (0.025)4 \frac{\text{lbs}}{\text{min}} - \frac{4A(t)}{200} \frac{\text{lbs}}{\text{min}} \\
 &= 0.1 \frac{\text{lbs}}{\text{min}} - \frac{A(t)}{50} \frac{\text{lbs}}{\text{min}} \\
 &= 0.1 - \frac{A(t)}{50} \\
 \frac{dA}{dt} + \frac{1}{50}A &= 0.1 \\
 \mu &= e^{\int P(t)dt} \\
 &= e^{\int \frac{1}{50}dt} \\
 &= e^{\frac{t}{50}} \\
 e^{\frac{t}{50}} \left( \frac{dA}{dt} \right) + e^{\frac{t}{50}} \left( \frac{1}{50}A \right) &= e^{\frac{t}{50}}(0.1) \\
 \frac{d}{dt} \left( e^{\frac{t}{50}} A \right) &= e^{\frac{t}{50}}(0.1) \\
 \int \frac{d}{dt} \left( e^{\frac{t}{50}} A \right) &= \int \frac{1}{10} e^{\frac{t}{50}} \\
 e^{\frac{t}{50}} A &= \frac{1}{10} \times \frac{e^{\frac{t}{50}}}{\frac{1}{50}} + C \\
 e^{\frac{t}{50}} A &= 5e^{\frac{t}{50}} + C \\
 A(t) &= 5 + C e^{-\frac{t}{50}} \\
 &= 5 + C e^{-0.02t} \\
 A(120) &= 5 + C e^{-0.02(120)} \\
 &= 5 + C e^{-2.4}
 \end{aligned}$$

**Note:** This section was created after all sections through **Section 7.5** were taught.

### 3.3 Applications: Modelling with a System of Linear DEs

$$\begin{aligned}
 x_1'' + 3x_1' + x_2'' - 5x_2 &= e^t \\
 x_1'' - 4x_2'' + 6x_2' &= \sin t
 \end{aligned}$$

Two unknown functions  $x_1(t)$  and  $x_2(t)$ . We learn methods to solve such systems in 4.4 and 7.6.

There is usually a sequence of isotopes the initial isotope transforms through. Suppose we have Decay Series of the form

$$X \rightarrow Y \rightarrow Z \text{ ( where } X \text{ is a stable isotope)}$$

The decay rates  $X \rightarrow Y$  and  $Y \rightarrow Z$  can be significantly different.

$$\begin{aligned}\frac{dX}{dt} &= k_1 X \\ \frac{dY}{dt} &= k_2 Y\end{aligned}$$

where  $X(t) = \text{mass of isotope } X \text{ at time } t$  and  $Y(t) = \text{mass of isotope } Y \text{ at time } t$ . Assume the relative decay rate of a radioactive substance is a constant

$$\begin{aligned}\frac{\frac{dX}{dt}}{X} &= k_1, \quad k_1 < 0 \\ \frac{\frac{dY}{dt}}{Y} &= k_2, \quad k_2 < 0\end{aligned}$$

We'll define

$$\lambda_1 = -k_1 \text{ so } \lambda_1 > 0$$

$$\lambda_2 = -k_2 \text{ so } \lambda_2 > 0$$

Notation:

$$X \xrightarrow{k_1} Y \xrightarrow{k_2} Z$$

is the same as

$$X \xrightarrow{-\lambda_1} Y \xrightarrow{-\lambda_2} Z$$

### 3.3.1 – Example

$$\begin{array}{ll} Dx + (D + 2)y = 0 & (D - 3)x - 2y = 0 \\ D(D - 3)x + (D - 3)(D + 2)y = 0 & D(D - 3)x - D(2y) = 0 \\ D(x' - 3x) + (D - 3)(y' + 2y) = 0 & D(x' - 3x) - 2y' = 0 \\ x'' - 3x' + y'' + 2y' - 3y' - 6y = 0 & x'' - 3x' - 2y' = 0 \\ x'' - 3x' + y'' - y' - 6y = 0 & x'' - 3x' - 2y' = 0 \end{array}$$

$$\begin{aligned}x'' - 3x' + y'' - y' - 6y &= x'' - 3x' - 2y' \\ y'' - y' - 6y &= -2y' \\ y'' + y' - 6y &= 0\end{aligned}$$

### 3.3.2 – Example

Suppose now, you start with 500 grams (.5 kg) of  $X$  and 0 grams of  $Y$  and  $Z$ . If  $k_1 = -.01$  (per year) and  $k_2 = -.003$  (per year). Determine how much  $Z$  there will be after  $t = 1,000$  years.

$$\frac{dX}{dt} = -\lambda_2 Y \quad \frac{dY}{dt} = \lambda_1 X - \lambda_2 Y \quad \frac{dZ}{dt} = \lambda_2 Y$$

$$\begin{aligned} DX + \lambda_1 X &= 0 \Rightarrow (D + \lambda_1)X &= 0 \\ DY - \lambda_1 X + \lambda_2 Y &= 0 \Rightarrow (D + \lambda_2)Y - \lambda_1 X &= 0 \\ DZ + \lambda_2 Y &= 0 \Rightarrow DZ + \lambda_2 Y &= 0 \end{aligned}$$

$\frac{dX}{dt} = -\lambda_1 X$  is separable and 1st order linear, which means  $X = c_1 e^{-\lambda_1 t}$ , and we know the initial value is 500g, so  $X(t) = 500e^{-0.01t}$ .

$$\begin{aligned} \frac{dY}{dt} &= \lambda_1 X - \lambda_2 Y \\ &= \lambda_1 \times c_1 e^{-\lambda_1 t} - \lambda_2 Y \\ &= .01 \times 500 e^{-0.01t} - \lambda_2 Y \\ \frac{dY}{dt} + \lambda_2 Y &= 5e^{-0.01t} \\ Y' + .003Y &= 5e^{-0.01t} \\ Y_c(t) &= c_2 e^{-\lambda_2 t} \\ y_p(t) &= Ae^{-0.1t} \\ y'_p(t) &= -0.1Ae^{-0.1t} \\ -0.1Ae^{-0.1t} + 0.03 \times Ae^{-0.1t} &= 5e^{-0.01t} \\ -0.1A + 0.03A &= 5 \\ -0.07A &= 5 \\ A &= \frac{5}{-0.07} \\ Y(t) &= Y_c(t) + Y_p(t) \\ &= c_2 e^{-\lambda_2 t} + Ae^{-0.1t} \\ &= 0e^{-0.03t} - \frac{5}{0.07}e^{-0.1t} \\ &= -\frac{500}{7}e^{-0.1t} \end{aligned}$$

and a similar method can be done for  $Z(t)$ .

### 3.3.3 – Other Application: Mixture Problems

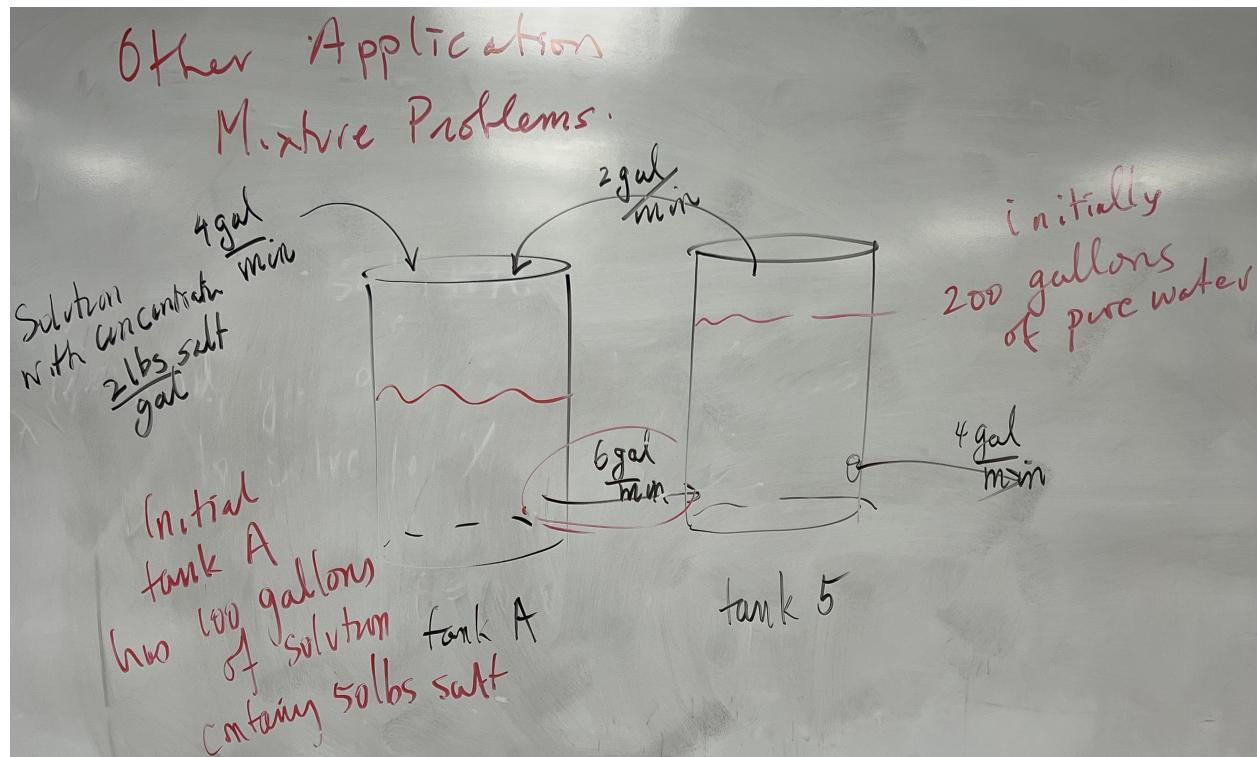


Figure 3.1

Solution with concentration  $2 \frac{\text{lbs of salt}}{\text{gal}}$ . Initial tank  $A$  has 100 gallons of solution containing 50 lbs of salt.

Let  $x_1(t) = \#$  of lbs of salt in tank  $A$  at time  $t$ .

Let  $x_2(t) = \#$  of lbs of salt in tank  $B$  at time  $t$ .

Tank  $A$  will always have 100 gallons of solution (6 gallons in = 6 gallons out) some for tank  $B$  with  $V = 200$  gal

$$\begin{aligned}
 \frac{dx_1}{dt} &= \text{rate salt in to tank } A - \text{rate salt out of tank } B \\
 &= 4 \frac{\text{gal}}{\text{min}} \times 2 \frac{\text{lbs}}{\text{gal}} + 2 \frac{\text{gal}}{\text{min}} \times \frac{\# \text{ lbs salt in } B}{200 \text{ gal}} - \frac{6 \text{ gal}}{\text{min}} \times \frac{x_1}{100 \text{ gal}} \\
 &= 8 \frac{\text{lbs}}{\text{min}} + \frac{\# \text{ lbs salt in } B}{100 \text{ min}} - \frac{3x_1}{50 \text{ min}} \\
 &= 8 + \frac{x_2}{100} - \frac{3x_1}{50}
 \end{aligned}$$

# Chapter 4

## Higher Order Differential Equations

### 4.1 Linear Equations

An  $n$ th order DE is linear if it has the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1(x) \frac{dy}{dx} + a_0 y = g(x)$$

**Theorem:** If all the coefficient functions are continuous and  $a_n(x)$  is not 0 on an interval  $I$  and  $g(x)$  is continuous, then any initial value problem

$$DE + y(x_0) = y_0$$

has a unique solution on the interval  $I$  if  $g(x) = 0$ . i.e.

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

then the DE is said to be homogeneous.

#### 4.1.1 – Example

$$y'' - 3y' - 4y = 0$$

Show  $y_1 = e^{4x}$  is a solution and  $y_2 = e^{-x}$  is a solution.

$$y_1 = e^{4x}$$

$$y'_1 = 4e^{4x}$$

$$y''_1 = 16e^{4x}$$

$$16e^{4x} - 3(4e^{4x}) - 4e^{4x} = 0$$

$$16e^{4x} - 12e^{4x} - 4e^{4x} = 0$$

$$e^{4x}(16 - 12 - 4) = 0$$

$$e^{4x}(0) = 0$$

$$0 = 0$$

$$y_3 = 6y_1 = 6e^{4x}$$

$$y'_3 = 6y'_1 = 24e^{4x}$$

$$y''_3 = 6y''_1 = 96e^{4x}$$

$$96e^{4x} - 3(24e^{4x}) - 4(6e^{4x}) = 0$$

$$96e^{4x} - 72e^{4x} - 24e^{4x} = 0$$

$$e^{4x}(96 - 72 - 24) = 0$$

$$e^{4x}(0) = 0$$

$$0 = 0$$

**Theorem:** Superposition Principle: if  $y_1, y_2, \dots, y_m$  are each solutions of an  $n$ th order Linear, homogenous DE, then  $c_1y_1 + c_2y_2 + \dots + c_my_m$  will also be a solution for any constants  $c_1, c_2, \dots, c_m$ .

Our goal is to express the general solution in as concise a way as possible.

**Linear combination** – a collection of solutions  $y_1, y_2, \dots, y_m$  is linearly independent if the only way  $c_1y_1 + c_2y_2 + \dots + c_my_m = 0$  is iff (if and only if) all of the constants  $c_1, c_2, \dots, c_m = 0$ . Otherwise we say  $y_1, y_2, \dots, y_m$  are linearly dependent.

**Theorem:** If the DE is an  $n$ th order Linear Homogeneous equation then there will exist a collection of  $n$  linearly independent solutions  $y_1, y_2, \dots, y_n$  and the general solution will be  $y_c = c_1y_1 + c_2y_2 + \dots + c_ny_n$

One way to check for linear independence is to compute the Wronskian

$$W(y_1, y_2, \dots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

## 4.2 Reduction of Order

If you have one solution to a 2<sup>nd</sup> order linear homogenous DE, then it is possible to use that function to construct a 2<sup>nd</sup> Linear Independent solution to the DE.

### 4.2.1 – Example

For example, the DE

$$y'' - y = 0$$

One solution is  $y = e^x$  on  $(-\infty, \infty)$ .

Idea: We look for  $y_2$  of the form

$$y_2(x) = u(x)y_1(x) \text{ where } u(x) \text{ is not a constant}$$

The general solution is of the form:

$$y = c_1 y_1 + c_2 y_2$$

where  $y_1$  and  $y_2$  are linearly independent solutions.

To find  $u(x)$ , we substitute this into the DE

$$\begin{aligned} y_2 &= u(x)y_1(x) \\ y'_2 &= u(x)y'_1(x) + u'(x)y_2(x) \\ y''_2 &= u(x)y''_1(x) + u'(x)y_1(x) + u'(x)y'_2(x) + u''(x)y_1(x) \\ &= uy''_1 + 2u'y'_1 + u''y_1 \end{aligned}$$

So  $y'' - y = 0$  becomes

$$\begin{aligned} uy''_1 + 2u'y'_1 + u''y_1 - uy_1 &= 0 \text{ when we sub } y = y_2 = u y_1 \\ u(e^x)'' + 2u'(e^x)' + u''(e^x) - u(e^x) &= 0 \\ ue^x + 2u'e^x + u''e^x - ue^x &= 0 \\ 2u'e^x + u''e^x &= 0 \\ e^x(2u' + u'') &= 0 \\ 2u' + u'' &= 0 \end{aligned}$$

Let  $w = u'$

$$\begin{aligned}
 2w + w' &= 0 \\
 2w + \frac{dw}{dx} &= 0 \\
 \frac{dw}{dx} &= -2w \\
 \frac{dw}{w} &= -2dx \\
 \int \frac{dw}{w} &= \int -2dx \\
 \ln|w| &= -2x \\
 w &= e^{-2x} \\
 u' &= e^{-2x} \\
 \int u' &= \int e^{-2x} \\
 u &= -\frac{1}{2}e^{-2x} \\
 y_2 &= uy_1 \\
 &= -\frac{1}{2}e^{-2x} \times e^x \\
 &= -\frac{1}{2}e^{-x}
 \end{aligned}$$

**Double check that  $y_2$  is a solution of the DE**

$$\begin{aligned}
 y_2 &= -\frac{1}{2}e^{-x} \\
 y'_2 &= \frac{1}{2}e^{-x} \\
 y''_2 &= -\frac{1}{2}e^{-x} \\
 y''_2 - y &= -\frac{1}{2}e^{-x} - \left(-\frac{1}{2}e^{-x}\right) \\
 &= -\frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} \\
 &= 0
 \end{aligned}$$

In general,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

put into standard form by dividing by  $a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

where  $P(x) = \frac{a_1(x)}{a_2(x)}$  and  $Q(x) = \frac{a_0(x)}{a_2(x)}$ , the same method as in our **example** leads to the

formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \quad (4.1)$$

### 4.2.2 – Example

#### Part 1

$$x^2y'' - 3xy' + 4y = 0$$

Verify that  $y_1 = x^2$  is a solution  $y'_1 = 2x, y''_1 = 2$ .

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ x^2(2) - 3x(2x) + 4(x^2) &= 0 \\ 2x^2 - 6x^2 + 4x^2 &= 0 \\ 6x^2 - 6x^2 &= 0 \\ 0 &= 0 \end{aligned}$$

#### Part 2

Find a linearly independent solution  $y_2(x)$ .

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ y'' - \frac{3}{x}y' + \frac{4}{x^2}y &= 0 \\ P(x) &= -\frac{3}{x} \\ y_2 &= y_1 \int \frac{e^{\int \frac{3}{x}dx}}{(y_1(x))^2} dx \\ y_2 &= y_1 \int \frac{e^{3\ln|x|}}{(y_1(x))^2} dx \\ y_2 &= y_1 \int \frac{e^{\ln|x^3|}}{(y_1(x))^2} dx \\ y_2 &= x^2 \int \frac{x^3}{(x^2)^2} dx \\ y_2 &= x^2 \int \frac{x^3}{x^4} dx \\ y_2 &= x^2 \int \frac{1}{x} dx \\ y_2 &= x^2 \ln|x| + C \end{aligned}$$

**Part 3: Double check that  $y_2$  is a solution of the DE**

$$\begin{aligned}y_2 &= x^2 \ln |x| \\y'_2 &= x^2 \times \frac{1}{x} + 2x \ln |x| \\y''_2 &= 1 + 2x \frac{1}{x} + 2 \ln |x| \\&= 1 + 2 + 2 \ln |x| \\&= 3 + 2 \ln |x|\end{aligned}$$

So the LHS DE becomes

$$\begin{aligned}x^2(3 + 2 \ln |x|) - 3x(x + 2x \ln |x|) + 4x^2 \ln |x| &= 3x^2 + 2x^2 \ln |x| - 3x^2 - 6x^2 \ln |x| + 4x^2 \ln |x| \\&= 3x^2 - 3x^2 + 2x^2 \ln |x| - 6x^2 \ln |x| + 4x^2 \ln |x| \\&= 3x^2 - 3x^2 + 2x^2 \ln |x| - 6x^2 \ln |x| + 4x^2 \ln |x| \\&= 0 + x^2 \ln |x|(2 - 6 + 4) \\&= x^2 \ln |x|(0) \\&= 0\end{aligned}$$

**Write the general solution of the DE including the interval of the solution**

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 x^2 + c_2 x^2 (\ln |x| + C) \\&= c_1 x^2 + c_2 x^2 \ln |x| + C c_2 x^2 \\&\text{just } y = c_1 x^2 + c_2 x^2 \ln |x| \text{ on } I = (0, \infty), y(2) = 3, y'(2) = 5\end{aligned}$$

#### 4.2.3 – Example

$$\begin{aligned}3y'' + y' - 4y &= 0 \\y &= e^{mx} \\y' &= me^{mx} \\y'' &= m^2 e^{mx} \\3y'' + y' - 4y &= 3m^2 e^{mx} + me^{mx} - 4e^{mx} \\&= e^{mx}(3m^2 + m - 4) \\&= e^{mx}(3m^2 + 4)(m - 1) \\m = 1 \quad m = -\frac{4}{3} \\y_1 &= e^x, y_2 = e^{-\frac{4}{3}x}\end{aligned}$$

## 4.3 Higher Order, Linear, Homogeneous DE with Constant Coefficients

### 4.3.1 – Example

$$3y^{(4)} - 2y''' + 7y' + 8y = 0$$

Theorems in 4.1 tell us that the general solution is of the form  $y = c_1 y_1$ .

**Conjecture:** A solution of the form  $y = e^{mx} \Rightarrow y' = me^{mx}$ .

### 4.3.2 – Example

$$\begin{aligned} 5y' - 4y &= 0 \\ y' - \frac{4}{5}y &= 0 \\ me^{mx} - \frac{4}{5}e^{mx} &= 0 \\ e^{mx} \left(m - \frac{4}{5}\right) &= 0 \\ m - \frac{4}{5} &= 0 \\ m &= \frac{4}{5} \end{aligned}$$

$y = c_1 e^{\frac{4}{5}x}$  is the general solution of the DE

### 4.3.3 – Example

$$\begin{aligned} y'' + 5y' - 6y &= 0 \\ y(m^2 e^{mx}) + 5(me^{mx}) - 6e^{mx} &= 0 \\ e^{mx} (m^2 y + 5m - 6) &= 0 \\ m^2 y + 5m - 6 &= 0 \\ (m + 6)(m - 1) &= 0 \\ m + 6 = 0 &\quad m - 1 = 0 \\ m = -6 &\quad m = 1 \\ y_1 = e^{-6x} &\quad y_2 = e^x \end{aligned}$$

These are **Linearly Independent (L.I.)**, Therefore:

$$y = c_1 e^{-6x} + c_2 e^x$$

### 4.3.4 – Example

$$\begin{aligned}
 & y'' - 6y' + 9y = 0 \\
 & m^2 e^{mx} - 6(m e^{mx}) + 9e^{mx} = 0 \\
 & m^2 - 6m + 9 = 0 \\
 & (m - 3)^2 = 0 \quad m = 3 \text{ is a repeated root} \\
 & m - 3 = 0 \\
 & m = 3 \\
 & y_1 = e^{3x} \quad y_2 = e^{3x} \text{ are linearly dependent}
 \end{aligned}$$

Use the Reduction of order function:

$$\begin{aligned}
 y_2 &= y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \\
 &= e^{3x} \int \frac{e^{-\int -6dx}}{(e^{3x})^2} dx \\
 &= e^{3x} \int \frac{e^{\int 6dx}}{e^{6x}} dx \\
 &= e^{3x} \int \frac{e^{6x}}{e^{6x}} dx \\
 &= e^{3x} \int 1 dx \\
 &= e^{3x} x \\
 &= x e^{3x}
 \end{aligned}$$

Always works out for this solution if  $e^{m_1 x}$  is a solution and  $m_1$  is a root of multiplicity  $k$  than  $y_1 = e^{m_1 x}, y_2 = x e^{m_1 x}, \dots, y_k = x^{k-1} e^{m_1 x}$  are linear solutions.

$$\begin{aligned}
 & y'' + 9y = 0 \\
 & m^2 + 9 = 0 \\
 & m^2 = -9 \\
 & m = \sqrt{-9} \text{ No real solutions} \\
 & m = \pm\sqrt{-9} \\
 & m = \pm 3i \\
 & y = c_1 e^{3ix} + c_2 e^{-3ix} \text{ where } c_1 \text{ & } c_2 \text{ arbitrary complex numbers}
 \end{aligned}$$

We'd rather only deal with real-valued solutions.

### 4.3.5 – Euler's Formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$\begin{aligned}
e^{i3x} &= \cos(3x) + i \sin(3x) \\
e^{-i3x} &= \cos(-3x) + i \sin(-3x) \\
e^{-i3x} &= \cos(3x) - i \sin(3x) \\
e^{i3x} + e^{-i3x} &= \cos(3x) + i \sin(3x) + \cos(3x) - i \sin(3x) \\
e^{i3x} + e^{-i3x} &= 2 \cos(3x) \\
Y_1 = \frac{1}{2}e^{i3x} + \frac{1}{2}e^{-i3x} &= \cos(3x) \\
Y_2 &= \sin(3x) \\
\frac{1}{2i}y_1 - \frac{1}{2i}y_2 &= \sin(3x)
\end{aligned}$$

General solution:

$$\begin{aligned}
y &= C_1 Y_1 + C_2 Y_2 \\
&= C_1 \cos(3x) + C_2 \sin(3x)
\end{aligned}$$

where  $C_1$  and  $C_2$  are complex numbers that generate all complex-valued solutions of the DE

#### 4.3.6 – Example

$$\begin{aligned}
y'' + 25y &= 0 \\
m^2 e^{mx} + 25e^{mx} &= 0 \\
m^2 + 25 &= 0 \\
m^2 &= -25 \\
m &= \pm 5i
\end{aligned}$$

General solution

$$\begin{aligned}
y_1 &= c_1 y_1 + c_2 y_2 \\
&= c_1 \cos(5x) + c_2 \sin(5x)
\end{aligned}$$

### 4.3.7 – Example

$$\begin{aligned}
 y'' + 2y' + 6y &= 0 \\
 m^2 + 2m + 6 &= 0 \\
 \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-2 \pm \sqrt{2^2 - 4(1)(6)}}{2(1)} \\
 &= \frac{-2 \pm \sqrt{4 - 24}}{2} \\
 &= \frac{-2 \pm \sqrt{-20}}{2} \\
 &= \frac{-2 \pm \sqrt{4} \times \sqrt{-5}}{2} \\
 &= \frac{-2 \pm 2\sqrt{-5}}{2} \\
 &= -1 \pm \sqrt{-5} \\
 &= -1 \pm \sqrt{5}i \\
 y_1 &= e^{(-1+\sqrt{5}i)x} \\
 &= e^{-x}e^{i\sqrt{5}x} \\
 &= e^{-x} \cos(\sqrt{5}x) \\
 y_2 &= e^{(-1-\sqrt{5}i)x} \\
 &= e^{-x}e^{-i\sqrt{5}x} \\
 &= e^{-x} \sin(\sqrt{5}x)
 \end{aligned}$$

So the general solution is

$$y = c_1 e^{-x} \cos(\sqrt{5}x) + c_2 e^{-x} \sin(\sqrt{5}x)$$

In general, if  $m_1 = \alpha+i\beta$ ,  $m_2 = \alpha-i\beta$  are roots of the auxiliary equation, then  $y_1 = e^{\alpha x} \cos(\beta x)$   
 $y_2 = e^{\alpha x} \sin(\beta x)$  are solutions.

### 4.3.8 – Example

$$\begin{aligned}
 y^{(4)} - 16y &= 0 \\
 m^4 - 16 &= 0 \\
 (m^2 - 4)(m^2 + 4) &= 0 \\
 (m - 2)(m + 2)(m^2 + 4) &= 0 \\
 m = 2 : y_1 &= e^{2x} \\
 m = -2 : y_1 &= e^{-2x} \\
 m = 2i : \cos(2x), \sin(2x) &
 \end{aligned}$$

## 4.4 Nonhomogeneous, Linear DE with Constant Coefficients

### 4.4.1 – Method of Undetermined Coefficients

Section 4.5 gives another approach but it is a bit more abstract

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1(x) \frac{dy}{dx} + a_0 y = g(x) \text{ where } g(x) \neq 0$$

**Theorem:** If we can find any one particular solution  $y_p$  of this DE ( $y_p + y_c$ ), where  $y_c$  is the solution of the complementary DE (the same LHS= 0 instead of  $g(x)$ ), is also a solution of the non-homogeneous DE, then the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n + y_p \end{aligned}$$

where you use [Section 4.3](#) methods for the  $c_i y_i$ 's.

### 4.4.2 – Example

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

**Step 1: Find the General Solution  $y_c$  of the complimentary DE**

$$y'' + 4y' - 2y = 0$$

Aux equation:

$$\begin{aligned} m^2 + 4m - 2 &= 0 \\ m^2 + 4m + 4 &= 6 \\ (m + 2)^2 &= 6 \\ m + 2 &= \pm\sqrt{6} \\ m &= -2 \pm \sqrt{6} \\ y_1 &= e^{(-2+\sqrt{6})x} \\ y_2 &= e^{(-2-\sqrt{6})x} \end{aligned}$$

**Step 2: Find a particular solution  $y_p$  of given DE**

Educated Guess:

$$y_p = Ax^2 + Bx + C$$

for some coefficients  $A, B, C$ . For the moment, they're undetermined coefficients.

Plugging in the  $y_p$ , we get

$$\begin{aligned}y'_p &= 2Ax + B \\y''_p &= 2A\end{aligned}$$

So,

$$\begin{aligned}2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) &= 2x^2 - 3x + 6 \\2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C &= 2x^2 - 3x + 6 \\-2Ax^2 + 8Ax - 2Bx + 2A + 4B - 2C &= 2x^2 - 3x + 6 \\-2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) &= 2x^2 - 3x + 6 \\-2A &= 2 \\8A - 2B &= -3 \\2A + 4B - 2C &= 6 \\-2A &= 2 \\A &= -1 \\8(-1) - 2B &= -3 \\-8 - 2B &= -3 \\8 + 2B &= 3 \\2B &= -5 \\B &= -\frac{5}{2} \\2(-1) + 4\left(-\frac{5}{2}\right) - 2C &= 6 \\-2 + -10 - 2C &= 6 \\-2C &= 18 \\C &= -9\end{aligned}$$

### Step 3: Check

$$\begin{aligned}y'_p &= 2(-1)x + \left(-\frac{5}{2}\right) \\&= -2x - \frac{5}{2} \\y''_p &= 2(-1) \\&= -2 \\y'' + 4y' - 2y &= -2 + 4\left(-2x - \frac{5}{2}\right) - 2\left(-x^2 - \frac{5}{2}x - 9\right) \\&= -2 - 8x - 10 + 2x^2 + 5x + 18 \\&= 2x^2 - 8x + 5x - 10 + 18 - 2 \\&= 2x^2 - 3x + 6\end{aligned}$$

### 4.4.3 – Example

$$y'' - y' + y = 2 \sin(3x)$$

**Step 1: Find the General Solution  $y_c$  of the complimentary DE**

Aux equation:

$$\begin{aligned} m^2 - m + 1 &= 0 \\ m &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1 - 4}}{2} \\ &= \frac{1 \pm \sqrt{-3}}{2} \\ &= \frac{1 \pm \sqrt{3}i}{2} \\ m_1 &= \frac{1 + \sqrt{3}i}{2} \\ m_2 &= \frac{1 - \sqrt{3}i}{2} \\ y_1 &= e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) \\ y_2 &= e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) \end{aligned}$$

**Step 2: Guess  $y_p = A \sin(3x) + B \cos(3x)$**

Plug into the DE

$$\underbrace{y''}_{-9A \sin(3x) - 9B \cos(3x)} - \underbrace{y'}_{(3A \cos(3x) - 3B \sin(3x))} + \underbrace{y}_{A \sin(3x) + B \cos(3x)} = 2 \sin(3x)$$

### 4.4.4 – Method of Undetermined Coefficients 2

For Solving Linear, Non-homogeneous DE with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Standard Form:

$$y'' + a_1 y' + a_0 y = g(x)$$

### 4.4.5 – Steps

Step 1) Solve  $y'' + a_1y' + a_0y = 0$  called the general solution  $y_c$ .

Step 2) Find one particular solution  $y_p$  of the given DE and the general solution is

$$y = y_c + y_p$$

This method can only be used when  $g(x)$  is a polynomial (An exponential (i.e.  $e^{kx}$ ), sines or cosines or sums of products of these types of functions)

### 4.4.6 – Example

$$y'' - 3y' - 4y = 4 \cos(3x)$$

1st solve:

$$\begin{aligned} y'' - 3y' - 4y &= 0 \\ m^2 e^{mx} - 3me^{mx} - 4e^{mx} &= 0 \\ m^2 - 3m - 4 &= 0 \\ (m - 4)(m + 1) &= 0 \\ m - 4 &= 0 \quad m + 1 = 0 \\ m &= 4 \quad m = -1 \\ y_c &= c_1 e^{4x} + c_2 e^{-x} \end{aligned}$$

$$y = A \cos(3x) + B \sin(3x)$$

$$y' = -3A \sin(3x) + 3B \cos(3x)$$

$$y'' = -9A \cos(3x) - 9B \sin(3x)$$

$$y'' - 3y' - 4y = 4 \cos(3x)$$

$$(-9A \cos(3x) - 9B \sin(3x)) - 3(-3A \sin(3x) + 3B \cos(3x)) - 4(A \cos(3x) + B \sin(3x)) = 4 \cos(3x)$$

$$-9A \cos(3x) - 9B \sin(3x) + 9A \sin(3x) - 9B \cos(3x) - 4A \cos(3x) - 4B \sin(3x) = 4 \cos(3x)$$

$$-9A \cos(3x) - 9B \cos(3x) - 4A \cos(3x) - 9B \sin(3x) + 9A \sin(3x) - 4B \sin(3x) = 4 \cos(3x)$$

$$\cos(3x)(-9A - 9B - 4A) + \sin(3x)(-9B + 9A - 4B) = 4 \cos(3x)$$

$$\cos(3x)(-13A - 9B) + \sin(3x)(9A - 13B) = 4 \cos(3x)$$

$$\begin{cases} -13A & -9B = 4 \\ 9A & -13B = 0 \end{cases} \text{ Solve simultaneously}$$

One way to solve Linear Systems of Equations is called Cramer's Rule.

$$\det \begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}$$

$$A = \frac{\begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}}$$

$$= \frac{4(-13) - 0(-9)}{-13(-13) - 9(-9)}$$

$$= \frac{-52 - 0}{169 + 81}$$

$$= -\frac{52}{250}$$

$$= -\frac{26}{125}$$

$$B = \frac{\begin{bmatrix} -13 & 4 \\ 9 & 0 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}}$$

$$= \frac{-13(0) - 4(9)}{250}$$

$$= \frac{0 - 36}{250}$$

$$= -\frac{36}{250}$$

$$= -\frac{18}{125}$$

Check:

$$(-13) \left( -\frac{26}{125} \right) + (-9) \left( -\frac{18}{125} \right) ? = 4$$

$$\frac{338}{125} + \frac{162}{125} ? = 4$$

$$\frac{500}{125} = 4$$

$$9 \left( -\frac{26}{125} \right) + (-13) \left( -\frac{18}{125} \right) ? = 0$$

$$-\frac{234}{125} + \frac{234}{125} ? = 0$$

$$0 = 0$$

So

$$y = -\frac{26}{125} \cos(3x) - \frac{18}{125} \sin(3x) + c_1 e^{4x} + c_2 e^{-x}$$

is the general solution to the given DE.

### 4.4.7 – Example

$$y'' - 5y' + 4y = 8e^x$$

If we try:

$$\begin{aligned} y_p &= Ae^x \\ Ae^x - 5Ae^x + 4Ae^x &= 8e^x \\ e^x(A - 5A + 4) &= 8e^x \\ A - 5A + 4 &= 8 \\ A - 5A - 4 &= 0 \end{aligned}$$

has no solution.

Solve

$$y'' - 5y' + 4y = 0$$

1st

$$\begin{aligned} m^2 - 5m + 4 &= 0 \\ (m - 1)(m - 4) &= 0 \\ m - 1 = 0 &\quad m - 4 = 0 \\ m = 1 &\quad m = 4 \\ y_1 = e^{1mx} &\quad y_2 = e^{4mx} \\ y_1 = e^{mx} &\quad y_2 = e^{4mx} \\ y_c = c_2 e^{mx} + c_2 e^{4mx} &\quad \text{hole at } Ae^x \text{ is } c_1 = A \quad c_2 = 0 \end{aligned}$$

Suppose we have a 5th order DE with

$$a_5 y^{(5)} + a_4 y^{(4)} + \cdots + a_1 y' + a_0 y = g(x)$$

and the auxiliary equation factors as

$$m^2(m - 3)(m - (2 + i))(m - (2 - i)) = 0$$

$$m = 0 \quad (\text{multiplicity 2}) \quad m = 3 \quad m = 2 + i \quad m = 2 - i$$

#### Step 1

Write the general solution to the complimentary DE

$$\begin{aligned} y_1 &= e^{0x} = 1 \\ y_2 &= x e^{0x} = x \\ y_3 &= e^{3x} = e^{3x} \\ y_4 &= e^{(2+i)x} = e^{2x} \cos(x) \\ y_5 &= e^{(2-i)x} = e^{2x} \sin(x) \end{aligned}$$

$$y_c = c_1 + c_2 x + c_3 e^{3x} + e^{2x} \cos(x) + e^{2x} \sin(x)$$

#### 4.4.8 – What would you guess for the form of $y_p$ ?

If

$$(ii) \ g(x) = e^{5x} \Rightarrow y_p = Ae^{5x}$$

$$(iii) \ g(x) = e^{3x} \Rightarrow y_p = Axe^{3x} \ (\text{because } e^{3x} \text{ is in } y_c)$$

$$(iv) \ g(x) = 5e^{2x} \sin(x) \Rightarrow y_p = (Ae^{2x} \cos(x) + Be^{2x} \sin(x)) x$$

$$(v) \ g(x) = 6x^2 e^{4x} \Rightarrow y_p = (Ax^2 + Bx + C) e^{4x}$$

$$(vi) \ g(x) = x^2 e^{3x} \Rightarrow y_p = (Ax^2 + Bx + C) e^{3x} x$$

Table 4.1: Particular Solutions for Undetermined Coefficients

<b>g(x)</b>	<b>Form of <math>y_p</math></b>
1. 1 (any constant)	$A$
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. $e^{5x}$	$Ae^{5x}$
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. $x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

## 4.6 Variation of Parameters Method

$$y'' + P(x)y' + Q(x)y = f(x)$$

will only work on problems where  $P(x)$  and  $Q(x)$  are constants.

#### 4.6.1 – 1st Step: General solution of complementary DE

$$y = c_1 y_1 + c_2 y_2$$

Guess a solution to the non-homogeneous of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where  $u_1$  and  $u_2$  are functions of  $x$ .

This theory produces

$$u'_1 = \frac{W_1}{W} \text{ and } u'_2 = \frac{W_2}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix}$$

#### 4.6.2 – Example

$$4y'' + 36y = \csc(3x)$$

$$4y'' + 36y = \csc(3x)$$

$$y'' + 9y = \frac{\csc(3x)}{4}$$

$$m^2 e^{mx} + 9e^{mx} = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \pm\sqrt{-9}$$

$$= \pm 3i$$

$$y_1 = e^{0x} \cos(3x) \quad y_2 = e^{0x} \sin(3x)$$

$$y_1 = 1 \cos(3x) \quad y_2 = 1 \sin(3x)$$

$$y_1 = \cos(3x) \quad y_2 = \sin(3x)$$

$$y_c = c_1 \cos(3x) + c_2 \sin(3x)$$

Guess

$$y_p = u_1 y_1 + u_2 y_2$$

$$u'_1 = \frac{W_1}{W} \quad u'_2 = \frac{W_2}{W}$$

where

$$\begin{aligned} W &= \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3 \sin(3x) & 3 \cos(3x) \end{vmatrix} \\ &= (3 \cos(3x))(\cos(3x)) - (\sin(3x))(-3 \sin(3x)) \\ &= 3 \cos^2(3x) + 3 \sin^2(3x) \\ &= 3 (\cos^2(3x) + \sin^2(3x)) \\ &= 3(1) \\ &= 3 \end{aligned}$$

$$\begin{aligned}
W_1 &= \begin{vmatrix} 0 & \sin(3x) \\ \frac{1}{4} \csc(3x) & \cos(3x) \end{vmatrix} \\
&= 0 \cos(3x) - \sin(3x) \left( \frac{\csc(3x)}{4} \right) \\
&= -\frac{\sin(3x) \csc(3x)}{4} \\
&= -\frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
W_2 &= \begin{vmatrix} \cos(3x) & 0 \\ \sin(3x) & \frac{1}{4} \csc(3x) \end{vmatrix} \\
&= \frac{1}{4} \csc(3x) \cos(3x) - 0 \sin(3x) \\
&= \frac{\cos(3x)}{4 \sin(3x)} \\
&= \frac{1}{4} \cot(3x)
\end{aligned}$$

$$\begin{aligned}
u'_1 &= \frac{W_1}{W} & u'_2 &= \frac{W_2}{W} \\
u'_1 &= \frac{-\frac{1}{4}}{3} & u'_2 &= \frac{\frac{1}{4} \cot(3x)}{3} \\
u'_1 &= -\frac{1}{12} & u_2 &= \frac{1 \cot(3x)}{12} \\
u'_1 &= -\frac{1}{12} & u_2 &= \frac{1 \cos(3x)}{12 \sin(3x)} \\
u_1 &= \int -\frac{1}{12} dx & u_2 &= \int \frac{1 \cos(3x)}{12 \sin(3x)} dx \\
u_1 &= -\frac{x}{12} & u_2 &= \frac{1}{12} \int \frac{1}{3} \frac{dv}{v} \\
&& u_2 &= \frac{1}{36} \ln |v| \\
&& u_2 &= \frac{1}{36} \ln |\sin(3x)|
\end{aligned}$$

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= -\frac{x}{12} \cos(3x) + \frac{1}{36} \ln |\sin(3x)| \sin(3x) \\
&= -\frac{x \cos(3x)}{12} + \frac{\sin(3x)}{36} \ln |\sin(3x)|
\end{aligned}$$

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 \cos(3x) + c_2 \sin(3x) - \frac{x \cos(3x)}{12} + \frac{\sin(3x)}{36} \ln |\sin(3x)|
\end{aligned}$$

### 4.6.3 – $3 \times 3$ Determinants

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ f & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

#### Matrix of Signs

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

## 4.9 Systems of Higher Order Linear DEs

$$\text{Solve} \left\{ \begin{array}{l} x' + y' + 2y = 0 \\ x' - 3x - 2y = 0 \end{array} \right.$$

Need to find  $x(t)$  and  $y(t)$  that simultaneously solve these equations.

### 4.9.1 – Method: Systematic Elimination

Change from using prime notation to indicate derivatives to the  $D$  operator notation.

$$x' \Rightarrow Dx, \quad y' + 2y \Rightarrow (D + 2)y$$

$$\begin{aligned} \left\{ \begin{array}{l} Dx + (D + 2)y = 0 \\ (D - 3)x - 2y = 0 \end{array} \right\} &= \left\{ \begin{array}{l} (D - 3)[Dx + (D + 2)y] = 0 \\ D[(D - 3)x - 2y] = 0 \end{array} \right\} \\ &= \left\{ \begin{array}{l} (D^2 - 3D)x + (D^2 - D - 6)y = 0 \\ (D^2 - 3D)x - 2Dy = 0 \end{array} \right\} \\ &= \left\{ \begin{array}{l} x'' - 3x' + y'' - y' - 6y = 0 \\ x'' - 3x' - 2y' = 0 \end{array} \right\} \end{aligned}$$

$$x'' - 3x' + y'' - y' - 6y = x'' - 3x' - 2y'$$

$$y'' - y' - 6y = -2y'$$

$$y'' + y' - 6y = 0$$

$$m^2 e^{mx} + m e^{mx} - 6e^{mx} = 0$$

$$m^2 + m - 6 = 0$$

$$(m + 3)(m - 2) = 0$$

$$m_1 + 3 = 0 \quad m_2 - 2 = 0$$

$$m_1 = -3 \quad m_2 = 2$$

$$y_1 = e^{-3t} \quad y_2 = e^{2t}$$

$$y(t) = c_1 e^{-3t} + c_2 e^{2t}$$

Using the same ideas, we can find  $x(t)$ , namely:

$$\begin{aligned} \left\{ \begin{array}{l} Dx + (D+2)y = 0 \\ (D-3)x - 2y = 0 \end{array} \right\} &= \left\{ \begin{array}{l} (-2)[Dx + (D+2)y] = 0 \\ (D+2)[(D-3)x - 2y] = 0 \end{array} \right\} \\ &= \left\{ \begin{array}{l} -2Dx - (2D+4)y = 0 \\ (D^2 - D - 6)x - (2D+4)y = 0 \end{array} \right\} \\ &= \left\{ \begin{array}{l} -2x' - 2y' - 4y = 0 \\ x'' - x' - 6x - 2y' - 4y = 0 \end{array} \right\} \end{aligned}$$

$$x'' - x' - 6x - 2y' - 4y = -2x' - 2y' - 4y$$

$$x'' - x' - 6x = -2x'$$

$$x'' + x' - 6x = 0$$

$$n^2 e^{nt} + n e^{nt} - 6e^{nt} = 0$$

$$n^2 + n - 6 = 0$$

$$(n+3)(n-2) = 0$$

$$n_1 + 3 = 0 \quad n_2 - 2 = 0$$

$$n_1 = -3 \quad n_2 = 2$$

$$x_1 = e^{-3t} \quad x_2 = e^{2t}$$

$$x(t) = c_3 e^{-3t} + c_4 e^{2t}$$

However, the 4 constants  $c_1, c_2, c_3, c_4$  are not completely independent of each other. To see their dependency, plug  $x(t)$  &  $y(t)$  into each of the original equations.

$$\begin{aligned} c_3 e^{-3t} - 3c_1 e^{-3t} + c_1 e^{-3t} &= 0 \\ c_4 e^{2t} - 3c_1 e^{-3t} + c_1 e^{-3t} &= 0 \\ c_3 e^{-3t} + 2c_1 e^{2t} + c_1 e^{2t} &= 0 \\ c_4 e^{2t} + 2c_1 e^{2t} + c_1 e^{2t} &= 0 \\ &\dots \\ -(3c_3 + c_1)e^{-3t} + (2c_4 + 4c_2)e^{2t} &= 0 \end{aligned}$$

for this to be true for all  $t$ , you need

$$\begin{cases} 3c_3 + c_1 = 0 \Rightarrow c_3 = -\frac{1}{3}c_1 \\ 2c_4 + 4c_2 = 0 \Rightarrow c_4 = -2c_2 \end{cases}$$

#### 4.9.2 – Example

$$\begin{cases} x' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$$

$$\begin{cases} x' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases} = \begin{cases} (D - 4)x + D^2y = t^2 \\ (D + 1)x + Dy = 0 \end{cases}$$

$$= \begin{cases} (D - 4)x + D^2y = t^2 \\ D(D + 1)x + D^2y = 0 \end{cases}$$

$$= \begin{cases} (D - 4)x + D^2y = t^2 \\ (D^2 + D)x + D^2y = 0 \end{cases}$$

$$(D - 4)x + D^2y - (D^2 + D)x - D^2y = t^2 - 0$$

$$x' - 4x - x'' - x' = t^2 - 0$$

$$-4x - x'' = t^2$$

$$x'' + 4x = -t^2$$

**First Solve**  $x'' + 4x = 0$

Guess  $x = e^{mt}$

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm\sqrt{-4}$$

$$m = \pm 2i$$

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

**Next find one particular solution:**

# Chapter 5

## Modeling with Higher-Order Differential Equations

### 5.1 Spring-Mass Problems

Suppose that there are no forces affecting the motion other than the gravitational force and the spring force.

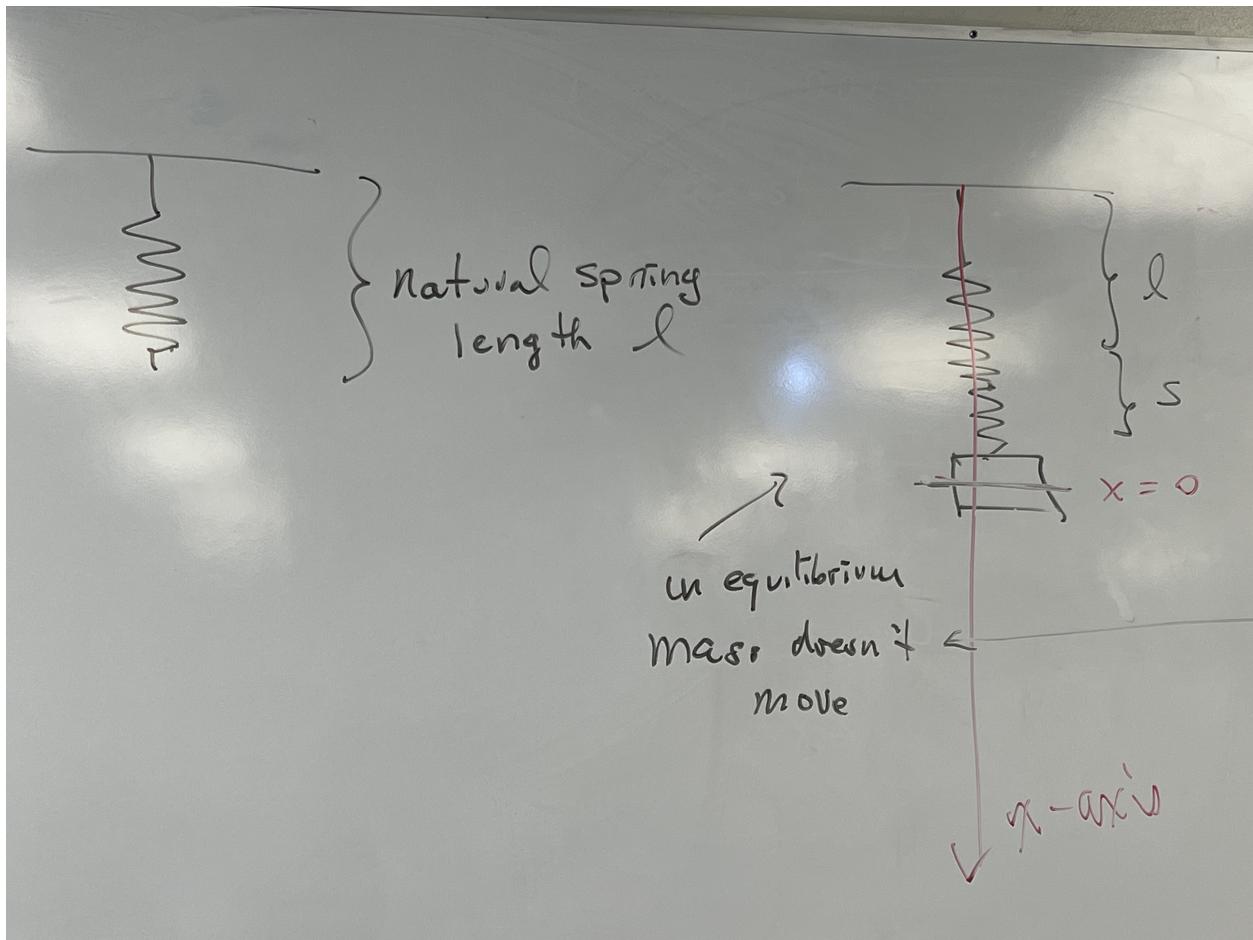


Figure 5.1: Diagram of a spring in equilibrium.

In Equilibrium, mass doesn't move.

$$F_g = -F_{spring} \rightarrow mg = -F_{spring} \rightarrow -mg = -kx \quad (\text{Hooke's Law}) \quad (5.1)$$

where  $g \approx 9.8m/s^2$  if m is in kg, else  $g \approx 32.1ft/sec^2 \rightarrow 32ft/sec^2$ .

In equilibrium:

$$F_{net} = mg - ks = 0$$

, so

$$k = \frac{mg}{s}$$

In general:

$$F_{net} = m \times \text{acceleration} \rightarrow m \frac{d^2x}{dt^2}$$

$$\begin{aligned}
 mg + (-k)(x + s) &= m \frac{d^2x}{dt^2} \\
 mg - kx - ks &= m \frac{d^2x}{dt^2} \\
 (mg - ks) - kx &= m \frac{d^2x}{dt^2} \\
 0 - kx &= m \frac{d^2x}{dt^2}
 \end{aligned}$$

So the differential equation is:

$$m \frac{d^2x}{dt^2} + kx = 0 \quad (5.2)$$

, a 2nd order, homogeneous DE with a constant coefficient.

### 5.1.1 – Example

$$\begin{aligned}
 m \frac{d^2x}{dt^2} + kx &= 0 \\
 \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0
 \end{aligned}$$

Guess:

$$x = e^{lt}$$

$$\begin{aligned}
 m \frac{d^2x}{dt^2} + kx &= 0 \\
 l^2 e^{lt} + \frac{k}{m} e^{lt} &= 0 \\
 l^2 + \frac{k}{m} &= 0 \\
 l^2 &= -\frac{k}{m} \\
 l &= \pm \sqrt{-\frac{k}{m}} \\
 l &= 0 \pm \sqrt{\frac{k}{m}} i
 \end{aligned}$$

$$x_1 = e^{0t} \cos \left( \frac{k}{m} t \right) \qquad x_2 = e^{0t} \sin \left( \frac{k}{m} t \right)$$

$$x_1 = \cos \left( \frac{k}{m} t \right) \qquad x_2 = \sin \left( \frac{k}{m} t \right)$$

$$\text{Let } \omega = \sqrt{\frac{k}{m}}$$

$$x_1 = \cos (\omega^2 t) \qquad x_2 = \sin (\omega^2 t)$$

### 5.1.2 – Example

Mass weighs 2lbs, stretch spring 6 inches

$$F = ma$$

$$2\text{lbs} = m \times 32 \frac{\text{ft}}{\text{sec}^2}$$

$$m = \frac{2\text{lbs}}{32 \frac{\text{ft}}{\text{sec}^2}}$$

$$m = \frac{1}{16} \text{ slug}$$

$$\begin{aligned} F_{spring} &= kx \\ &= k(6\text{in}) \end{aligned}$$

$$2\text{lbs} = \frac{k}{2}$$

$$k = 4 \frac{\text{lbs}}{\text{ft}}$$

$$\omega = \sqrt{\frac{4}{\frac{1}{16}}}$$

$$\omega = \sqrt{64}$$

$$\omega = 8$$

### 5.1.3 – Undamped Motion

$$m \frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \text{ where } \omega = \sqrt{\frac{k}{m}}$$

$$n^2 e^{nt} + \omega^2 e^{nt} = 0 \text{ where } x = e^{nt}$$

$$n^2 + \omega^2 = 0$$

$$n^2 = -\omega^2$$

$$n = 0 \pm \sqrt{-\omega^2}$$

$$= 0 \pm \omega i$$

$$x_1 = \cos(\omega t) \quad x_2 = \sin(\omega t)$$

General solution:

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

### 5.1.4 – Free, Damped Motion

Assume in addition to  $F_{spring}$  and  $F_{gravity}$  that there is a force damping the motion which is directly proportional, in the opposite direction, to the mass's velocity.

$$\begin{aligned} m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx &= 0 \\ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m} x &= 0 \\ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \omega^2 x &= 0 \end{aligned} \tag{5.3}$$

where  $\beta$  is the drag coefficient. If we substitute  $2\lambda = \frac{\beta}{m} \Rightarrow \lambda = \frac{\beta}{2m}$

$$\begin{aligned} \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x &= 0 \\ n^2 e^{nt} + 2\lambda n e^{nt} + \omega^2 e^{nt} &= 0 \\ n^2 + 2\lambda n + \omega^2 &= 0 \\ n^2 + 2\lambda n = -\omega^2 & \\ n^2 + 2\lambda n + \lambda^2 = \lambda^2 - \omega^2 & \\ (n + \lambda)^2 = -\omega^2 + \lambda^2 & \\ n + \lambda = \pm \sqrt{\lambda^2 - \omega^2} & \\ n = -\lambda \pm \sqrt{\lambda^2 - \omega^2} & \end{aligned}$$

**Case 1:**  $\lambda^2 > \omega^2$

then there are two distinct real solutions where  $n_1 < 0$  and  $n_2 < 0$ .

$$\begin{aligned} n_1 &= -\lambda + \sqrt{\lambda^2 - \omega^2} \\ n_2 &= -\lambda - \sqrt{\lambda^2 - \omega^2} \\ x_1 &= e^{n_1 t} \quad x_2 = e^{n_2 t} \\ x_1 &= e^{t(-\lambda + \sqrt{\lambda^2 - \omega^2})} \quad x_2 = e^{t(-\lambda - \sqrt{\lambda^2 - \omega^2})} \end{aligned}$$

General solution:

$$\begin{aligned} x &= c_1 e^{t(-\lambda + \sqrt{\lambda^2 - \omega^2})} + c_2 e^{t(-\lambda - \sqrt{\lambda^2 - \omega^2})} \\ &= c_1 e^{-t\lambda} e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\lambda} e^{-t\sqrt{\lambda^2 - \omega^2}} \\ &= e^{-\lambda t} \left( c_1 e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\sqrt{\lambda^2 - \omega^2}} \right) \end{aligned} \tag{5.4}$$

**Case 2:**  $\lambda = \omega$

$$\begin{aligned} n &= -\lambda \pm \sqrt{\lambda^2 - \omega^2} \\ n &= -\lambda \pm \sqrt{0} \\ n &= -\lambda \end{aligned}$$

where  $\lambda$  has multiplicity 2.

$$x_1 = e^{-\lambda t} \quad x_2 = te^{-\lambda t}$$

General solution:

$$\begin{aligned} x &= c_1 e^{-\lambda t} + c_2 t e^{-\lambda t} \\ &= e^{-\lambda t} (1 + c_2 t) \end{aligned} \tag{5.5}$$

**Case 3:**  $\lambda^2 < \omega^2$

then

$$\begin{aligned} x_1 &= e^{-\lambda t} \cos(\sqrt{\omega^2 - \lambda^2} t) & x_2 &= e^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t) \\ x &= c_1 e^{-\lambda t} \cos(\sqrt{\omega^2 - \lambda^2} t) + c_2 e^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t) \\ &= e^{-\lambda t} \left( c_1 \cos(\sqrt{\omega^2 - \lambda^2} t) + c_2 \sin(\sqrt{\omega^2 - \lambda^2} t) \right) \\ &= e^{-\lambda t} \left( A \sin(\sqrt{\omega^2 - \lambda^2} t) + \phi \right) \end{aligned} \tag{5.6}$$

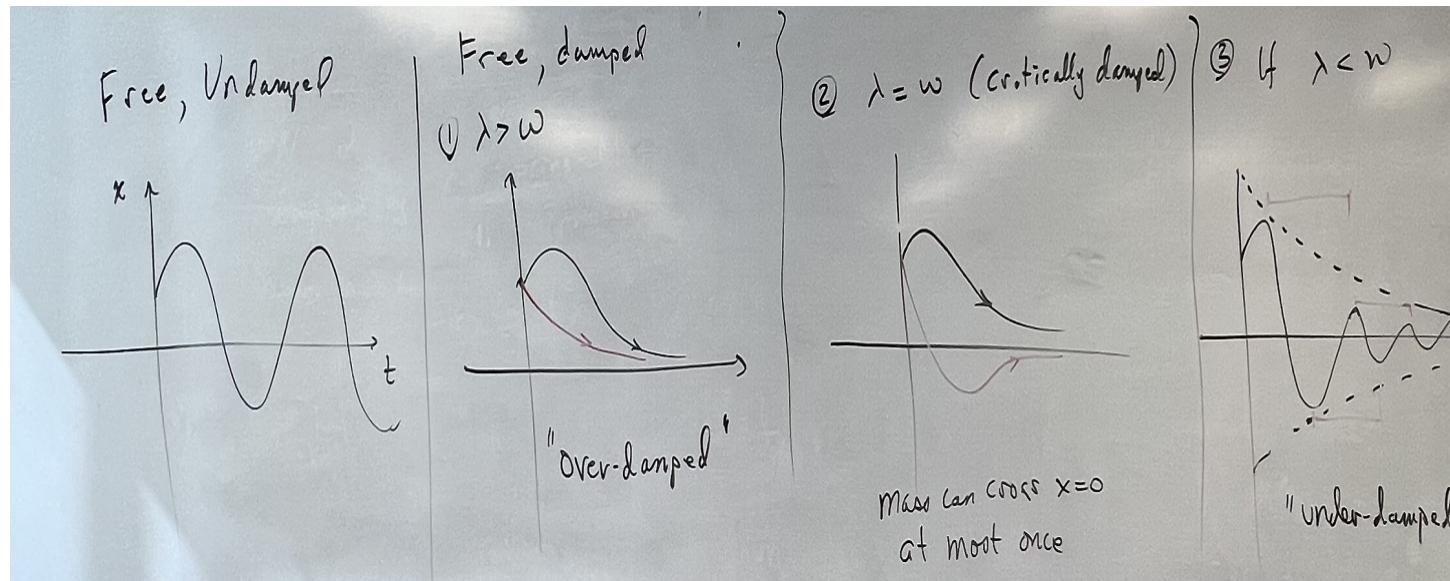


Figure 5.2: Graphs depicting each type of motion of a spring.

### 5.1.5 – Driven Motion (not Free Motion)

Can be both damped or **undamped**.

Imagine the support is oscillating up and down due to an external force.

If the motion is:

## Undamped

Assume that  $\gamma \neq \omega$

$$\begin{aligned}
& \frac{d^2x}{dt^2} + \omega^2 x = F(t) \\
&= F_0 \sin(\gamma t) \\
& \frac{d^2x}{dt^2} + \omega^2 x = 0 \\
& \vdots \\
& x = c_1 \cos(\omega t) + c_2 \sin(\omega t) \\
& \frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin(\gamma t) \\
& \textcolor{red}{x_p(t)} = A \cos(\gamma t) + B \sin(\gamma t) \\
& -A\gamma^2 \cos(\gamma t) - B\gamma^2 \sin(\gamma t) + \omega^2(A \cos(\gamma t) + B \sin(\gamma t)) = F_0 \sin(\gamma t) \\
& -A\gamma^2 \cos(\gamma t) - B\gamma^2 \sin(\gamma t) + A\omega^2 \cos(\gamma t) + B\omega^2 \sin(\gamma t) = F_0 \sin(\gamma t) \\
& \cos(\gamma t)(-A\gamma^2 + A\omega^2) = 0 \quad \cos(\gamma t) \quad B(\omega^2 - \gamma^2) = F_0 \\
& -A\gamma^2 + A\omega^2 = 0 \quad B = \frac{F_0}{\omega^2 - \gamma^2} \\
& A(\omega^2 - \gamma^2) = 0 \quad B = \frac{F_0}{\omega^2 - \gamma^2}
\end{aligned}$$

We assumed that  $\gamma \neq \omega$ , which forces  $A = 0$  for the left equation to work.

$$A = 0 \quad B = \frac{F_0}{\omega^2 - \gamma^2}$$

General solution:

$$\begin{aligned}
x(t) &= x_c(t) + x_p(t) \\
&= c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2} \sin(\gamma t)
\end{aligned} \tag{5.7}$$

If  $\gamma$  is almost equal to  $\omega$ , then  $\frac{F_0}{\omega^2 - \gamma^2}$  is a large constant. This situation is called Resonance.

### 5.1.6 – Example

Driven, Damped, Spring-Mass

$$\begin{aligned}
& \frac{1}{5} \frac{d^2x}{dt^2} + 1.2 \frac{dx}{dt} + 2x = 5 \cos(4t), \quad x(0) = \frac{1}{2}, \quad x'(0) = 0 \\
& \frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 10x = 25 \cos(4t) \\
& \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)
\end{aligned}$$

where

$$\lambda = 3, \omega = \sqrt{10}, F(t) = 25 \cos(4t)$$

For instance, if the mass = 2kgs, then the spring constant is  $k = 20$  and  $\beta = 12$ .

$$\sqrt{10} = \sqrt{\frac{k}{m}} \Rightarrow 10 = \frac{k}{m} \Rightarrow 10m = k \Rightarrow k = 10(2) = 20$$

$$\beta = 2\lambda m = 2(3)(2) = 6(2) = 12$$

**To solve the complimentary DE**

, guess  $x = e^{nt}$

$$n^2 e^{nt} + 6n e^{nt} + 10 e^{nt} = 0$$

$$n^2 + 6n + 10 = 0$$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 40}}{2}$$

$$= \frac{-6 \pm \sqrt{-4}}{2}$$

$$= \frac{-6 \pm 2i}{2}$$

$$= -3 \pm i$$

$$x_1 = e^{-3t} \cos(t) \quad x_2 = e^{-3t} \sin(t)$$

**To find a particular solution**

Guess

$$x_p = A \cos(4t) + B \sin(4t)$$

$$x'_p = -4A \sin(4t) + 4B \cos(4t)$$

$$x''_p = -16A \cos(4t) - 16B \sin(4t)$$

$$-16A \cos(4t) - 16B \sin(4t) + 6(-4A \sin(4t) + 4B \cos(4t)) + 10(A \cos(4t) + B \sin(4t)) = 25 \cos(4t)$$

$$-16A \cos(4t) - 16B \sin(4t) - 24A \sin(4t) + 24B \cos(4t) + 10A \cos(4t) + 10B \sin(4t) = 25 \cos(4t)$$

$$-16A \cos(4t) + 10A \cos(4t) + 24B \cos(4t) - 16B \sin(4t) - 24A \sin(4t) + 10B \sin(4t) = 25 \cos(4t)$$

$$-6A \cos(4t) + 24B \cos(4t) - 6B \sin(4t) - 24A \sin(4t) = 25 \cos(4t)$$

$$\cos(4t)(-6A + 24B) + \sin(4t)(-6B - 24A) = 25 \cos(4t)$$

# Chapter 6

## Series Solutions of Linear Equations

### 6.1 Solution by Infinite Series

2nd order linear DE with (possibly) variable coefficients

$$\begin{aligned} a_2(x)y'' + a_1(x)y' + a_0(x)y &= f(x) \\ y'' + P(x)y' + Q(x)y &= F(x) \end{aligned}$$

#### 6.1.1 – Review of Infinite Series Facts

##### Maclaurin Series

$$\sum_{n=0}^{\infty} a_n x^n$$

Power series centered at 0

##### Taylor Series

$$\sum_{n=0}^{\infty} a_n (x - a)^n$$

Centered at  $a = 0$

It's a theorem that power series either

- (1) Converge all real numbers  $x$  on the interval  $I = (-\infty, \infty)$  and the radius of convergence is  $R = \infty$
- (2) Converge only when  $x = a$  on the interval  $I = [a, a]$  and the radius of convergence is  $R = 0 = \{a\}$
- (3) The series converges on an interval centered at a finite, non-zero radius  $R = (a - R, a + R)$

### 6.1.2 – Ratio Test

Use the Ratio Test to determine which of these 3 cases occurs in a specific problem.

The 3 cases of the ratio test are:

$L < 1$  , the series converges

$L > 1$  , the series diverges

$L = 1$  , the series could converge or diverge (you have to check)

### 6.1.3 – Example

Determine the radius and interval of convergence for

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{x^n}{3^n(n+1)} \\ L &= \lim_{n \rightarrow \infty} \frac{\left| \frac{x^{n+1}}{3^{n+1}(n+2)} \right|}{\left| \frac{x^n}{3^n(n+1)} \right|} \\ &= \lim_{n \rightarrow \infty} \frac{|x| n + 1}{3 n + 2} \\ &= \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{n + 1}{n + 2} \\ &= \frac{|x|}{3} (1) \\ &= \frac{|x|}{3} \\ \frac{|x|}{3} &< 1 \\ |x| &< 3 \\ I &= (-3, 3) \end{aligned}$$

### 6.1.4 – Idea of Method

We will try to find a solution of the DE in the form of a power series

$$y = \sum_{n=0}^{\infty} c_n x^n \text{ (centered at 0)}$$

or

$$y = \sum_{n=0}^{\infty} c_n (x - a)^n \text{ (centered at } a\text{)}$$

When you substitute this into the DE you get recurrence relationships for the coefficients  $c_0, c_1, \dots$ . Once you've found the coefficients in terms of either  $c_0$ , or  $c_0, c_1$  where  $c_0 \neq c_1$ . Then you should determine where the series converges.

An example of a recurrence relation is the Fibonacci Sequence

$$F_{n+2} = F_n + F_{n+1}$$

### 6.1.5 – Example

Use this method to solve the DE

$$y' + y = 0$$

Assume

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots = \sum_{n=1}^{\infty} c_n n x^{n-1} \\ &\quad \sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0 \end{aligned}$$

Shift the index of the first summation such that both have terms  $x^n$ . To do this, we'll make the substitution  $k = n - 1 \Rightarrow n = k + 1$

$$\begin{aligned} \sum_{k+1=1}^{\infty} c_{k+1}(k+1)x^{(k+1)-1} + \sum_{k=0}^{\infty} c_k x^k &= 0 \\ \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + \sum_{k=0}^{\infty} c_k x^k &= 0 \\ \sum_{k=0}^{\infty} [c_{k+1}(k+1)x^k + c_k x^k] &= 0 \\ \sum_{k=0}^{\infty} [(k+1)c_{k+1} + c_k] x^k &= 0 \end{aligned}$$

This implies  $(k+1)c_{k+1} + c_k = 0$  for all  $k = 0, 1, 2, \dots$ <sup>1</sup>  $\Rightarrow c_{k+1} = \frac{-c_k}{k+1}$  for all  $k = 0, 1, 2, \dots$

---

<sup>1</sup>since the only power series that equals 0 is  $\sum_{k=0}^{\infty} 0x^k$

$$\begin{aligned}
c_0 &= c_0 \\
c_1 &= -\frac{c_0}{1} \\
&= -c_0 \\
c_2 &= -\frac{c_1}{2} \\
&= -\frac{-c_0}{2} \\
&= \frac{-1^2 c_0}{2} \\
&= \frac{c_0}{2} \\
c_3 &= -\frac{c_2}{3} \\
&= -\frac{\frac{c_0}{2}}{3} \\
&= \frac{-c_0}{6}
\end{aligned}$$

**Conjecture:** It is apparent that

$$c_n + \frac{(-1)^n c_0}{n!}$$

**Plugging into the DE**

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{n!} x^n \\
&= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n
\end{aligned}$$

By the Ratio Test

$$\begin{aligned}
 L &= \lim_{n \rightarrow \infty} \frac{\left| \frac{(-1)^{n+1}}{(n+1)!} x^{n+1} \right|}{\left| \frac{(-1)^n}{n!} x^n \right|} \\
 &= \lim_{n \rightarrow \infty} \left| (-1) \frac{x n!}{(n+1)!} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{|x| n!}{(n+1)!} \\
 &= \lim_{n \rightarrow \infty} \frac{|x|}{n+1} \\
 &= |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} \\
 &= |x| \times 0 \\
 &= 0 \text{ The series converges everywhere}
 \end{aligned}$$

### 6.1.6 – Power Series of Basic Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Our answer in the DE is

$$y = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}$$

## 6.2 Second Order, Linear Homogenous DE

$$\begin{aligned}
 a_2(x)y'' + a_1(x)y' + a_0(x)y &= 0 \\
 y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y &= 0 \\
 y'' + P(x)y' + Q(x)y &= 0
 \end{aligned}$$

$x$ 's for which  $a_2(x) \neq 0$  will be called ordinary points.  $x$ 's for which  $a_2(x) = 0$  will be called singular points.

**Existence of Power Series Theorem:** If  $x_0$  is an ordinary point of the DE, then there exists two, linearly independent solution  $y_1, y_2$  which are both in the form of power series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

and these series will have radius of convergence of at least the distance from  $x_0$  to the singular point of the DE.

### 6.2.1 – Example

Consider

$$(x^2 + 2x + 5)y'' + xy' - 6y = 0$$

- (i) What are the singular points of the DE?

$$\begin{aligned} x^2 + 2x + 5 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm \sqrt{16} \times \sqrt{-1}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

So  $-1 + 2i$  and  $-1 - 2i$  are the only singular points.

- (ii) Is there a power series solution centered at  $x_0 = 0$ ? Yes, since  $x_0 = 0$  is an ordinary point, you can find

$$y_1 = \sum_{n=0}^{\infty} c_n x^n$$

and

$$y_2 = \sum_{n=0}^{\infty} d_n x^n,$$

two linearly independent solutions.

- (iii) What is the minimum the radius could be for these series? As stated in the theorem, the radius is at minimum the distance from  $x_0$  to the singular point. If you have complex singular points, calculate the distance using the complex plane graph.  $\sqrt{(-1 - 0)^2 + (2 - 0)^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$ .

- How about if we want series

$$\sum_{n=0}^{\infty} c_n (x - 3)^2$$

$$\sqrt{(-1 - 3)^2 + (-2 - 0)^2} = \sqrt{(-4)^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$$

### 6.2.2 – Example

Use Power Series centered at 0 ([Maclaurin Series](#)) to solve the DE:

$$y'' - xy = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\begin{aligned} \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - x \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= \sum_{k+2=2}^{\infty} c_{k+2}(k+2)(k+2-1)x^k - \sum_{k-1=0}^{\infty} c_{k-1}x^k \\ &= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\ &= \sum_{k=0}^1 c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\ &= c_{0+2}(0+2)(0+1)x^0 + \sum_{k=1}^{\infty} [c_{k+2}(k+2)(k+1)x^k - c_{k-1}x^k] \\ &= c_2(2)(1)(1) + \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] \\ &= 2c_2 + \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] \\ &= 2(0) + \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] \\ &= 0 + \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] \\ &= \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] = 0 \end{aligned}$$

$$c_{k+2}(k+2)(k+1) - c_{k-1} = 0$$

$$c_{k+2}(k+2)(k+1) = c_{k-1}$$

$$c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)}$$

$$c_0 = \text{arbitrary}$$

$$c_1 = \text{arbitrary}$$

$$c_2 = 0$$

$$c_3 = \frac{c_0}{(1+2)(1+1)} = \frac{c_0}{3 \times 2}$$

$$c_4 = \frac{c_1}{(2+2)(2+1)} = \frac{c_1}{4 \times 3}$$

$$c_5 = \frac{c_2}{(3+2)(3+1)} = \frac{0}{5 \times 4} = 0$$

$$c_6 = \frac{c_3}{(4+2)(4+1)} = \frac{c_0}{3 \times 2} \times \frac{1}{6 \times 5} = \frac{c_0}{6 \times 5 \times 3 \times 2}$$

$$c_7 = \frac{c_4}{(5+2)(5+1)} = \frac{c_1}{4 \times 3} \times \frac{1}{7 \times 6} = \frac{c_1}{7 \times 6 \times 4 \times 3}$$

$$c_8 = \frac{c_5}{(6+2)(6+1)} = \frac{0}{8 \times 7} = 0$$

$$y = c_0 y_1 + c_1 y_2$$

$$= c_0 \left( 1 + \frac{1}{3 \times 2} x^3 + \frac{1}{6 \times 5 \times 3 \times 2} x^6 + \dots \right) + c_1 \left( x + \frac{1}{4 \times 3} x^4 + \frac{1}{7 \times 6 \times 4 \times 3} + \dots \right)$$

$$= c_0 \left( 1 + \frac{1}{3 \times 2} x^3 + \frac{4}{6 \times 5 \times 4 \times 3 \times 2} x^6 + \dots \right) + c_1 \left( x + \frac{2}{4 \times 3 \times 2} x^4 + \frac{2(5)}{7 \times 6 \times 5 \times 4 \times 3 \times 2} + \dots \right)$$

### 6.2.3 – Example

$$(x^2 + 1)y'' + xy' - y = 0$$

$$(x^2 + 1)y'' + xy' - y = 0$$

$$y'' + \frac{x}{x^2 + 1} y' - \frac{1}{x^2 + 1} y = 0$$

Ordinary points:

$$x^2 + 1 = 0$$

$$x^2 = -1$$

$$x = \pm\sqrt{-1}$$

$$x = \pm i$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \frac{x}{x^2+1} \sum_{n=1}^{\infty} c_n n x^{n-1} - \frac{1}{x^2+1} \sum_{n=0}^{\infty} c_n x^n = 0$$

# Chapter 7

## Method of Laplace Transforms for Solving DE's

### Chapter Goals

- Given a DE, Perform a Calculus-based rule for finding the laplace transformation of DE.
- Solve this new equation algebraically.
- Find the inverse-Laplace transformation to get our solution to the IVP.

### 7.1 Definition of Laplace Transform

Given a function  $f(t)$ , the Laplace Transform of  $f(t)$  is

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

### 7.1.1 – Laplace Transformations of basic Functions

(1)

$$\begin{aligned}
 \mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)dt \\
 &= \int_0^\infty e^{-st}dt \\
 &= \lim_{b \rightarrow \infty} \int_0^b e^{-st}dt \\
 &= \lim_{b \rightarrow \infty} \left. \frac{e^{-st}}{-s} \right|_0^b \\
 &= \lim_{b \rightarrow \infty} \frac{e^{-s(b)}}{-s} - \frac{e^{-s(0)}}{-s} \\
 &= -\frac{e^{-s(0)}}{-s} + \lim_{b \rightarrow \infty} \frac{e^{-s(b)}}{-s} \\
 &= -\frac{e^0}{-s} + \frac{1}{-s} \lim_{b \rightarrow \infty} e^{-s} e^b \\
 &= \frac{1}{s} - \frac{e^{-s}}{s} \lim_{b \rightarrow \infty} e^b \\
 &= \frac{1}{s} \text{ for } s > 0
 \end{aligned}$$

(2)

$$\begin{aligned}
 \mathcal{L}\{k\} &= \int_0^\infty e^{-st}kdt \\
 &= k \int_0^\infty e^{-st}dt \\
 &= k \frac{1}{s} \\
 &= \frac{k}{s}
 \end{aligned}$$

The Laplace Transform is a *linear operator*, in other words,

(3)

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

(4)

$$\mathcal{L}\{kf(t)\} = k\mathcal{L}\{f(t)\}$$

(5)

$$\begin{aligned}
\mathcal{L}\{e^{2t}\} &= \int_0^\infty e^{-st} \times e^{2t} dt \\
&= \int_0^\infty e^{(2-s)t} dt \\
&= \int_0^\infty e^{-(s-2)t} dt \\
u &= -(s-2)t \\
du &= -(s-2)dt \\
&= \int_0^\infty e^u \frac{du}{-(s-2)} \\
&= \frac{1}{-(s-2)} \int_0^\infty e^u du \\
&= \frac{1}{-(s-2)} e^u \Big|_0^\infty \\
&= \frac{1}{-(s-2)} e^{-(s-2)t} \Big|_0^\infty \\
&= \lim_{b \rightarrow \infty} \frac{1}{-(s-2)} e^{-(s-2)b} - \frac{1}{-(s-2)} e^{-(s-2)0} \\
&= -\frac{1}{-(s-2)} e^0 + \lim_{b \rightarrow \infty} \frac{1}{-(s-2)} e^{-(s-2)b} \\
&= -\frac{1}{-(s-2)} (1) \\
&= \frac{1}{s-2} \\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \text{ for } s > a
\end{aligned}$$

(6)

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

(7)

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2}$$

(8)

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}$$

Table 7.1: Transforms of Some Basic Functions

$\mathcal{L}\{1\} = \frac{1}{s}$	(7.1)
$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$	(7.2)
$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$	(7.3)
$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$	(7.4)
$\mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$	(7.5)
$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2 - k^2}$	(7.6)
$\mathcal{L}\{\cosh kt\} = \frac{s}{s^2 - k^2}$	(7.7)

## 7.2 Solving I.V.T by using Laplace Transform

Take  $\mathcal{L}$  of both sides of the DE

### 7.2.1 – Example

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{3y'\} + \mathcal{L}\{2y\} = \mathcal{L}\{e^{-4t}\}$$

We need more formulas first.

$$\begin{aligned} u &= e^{-st} & dv &= f'(x)dt \\ du &= -se^{-st}dt & v &= f(x) \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \int_0^\infty e^{st} f'(t) dt \\ &= f(t)e^{-st} \Big|_0^\infty - \int_0^\infty -sf(t)e^{st} dt \\ &= -f(t)e^{-st} + s \int_0^\infty f(t)e^{st} dt \\ &= -f(t)e^{-st} + s\mathcal{L}\{f(t)\} \\ &= -f(t)e^{-st} + sF(s) \\ &= -f(0)e^{-s(0)} + sF(s) \\ &= -f(0)(1) + sF(s) \\ &= -f(0) + sF(s) \end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= \mathcal{L}\{(f'(t))'\} \\
&= s\mathcal{L}\{f'(t)\} - f'(0) \\
&= s(-f(0) + sF(s)) - f'(0) \\
&= -sf(0) + s^2F(s) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0) \\
&= s^2\mathcal{L}\{f\} - sf(0) - f'(0)
\end{aligned}$$

In general

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s^{n-n} f^{n-1}(0) \quad (7.8)$$

or

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{n-1}(0) \quad (7.9)$$

So the DE transforms to

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - s(1) - 5 - 3(sY(s) - 1) + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - s - 5 - 3sY(s) + 3 + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - 3sY(s) + 2Y(s) - s - 5 + 3 &= \frac{1}{s+4} \\
Y(s)(s^2 - 3s + 2) - s - 2 &= \frac{1}{s+4} \\
Y(s)(s^2 - 3s + 2) &= \frac{1}{s+4} + s + 2 \\
Y(s) &= \frac{\frac{1}{s+4} + s + 2}{(s^2 - 3s + 2)} \\
&= \frac{1 + (s+2)(s+4)}{(s+4)(s^2 - 3s + 2)} \\
&= \frac{1 + s^2 + 6s + 8}{(s+4)(s-1)(s-2)} \\
&= \frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)}
\end{aligned}$$

$$\begin{aligned}
\frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)} &= \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2} \\
&= \frac{A(s-1)(s-2)}{(s+4)(s-1)(s-2)} + \frac{B(s+4)(s-2)}{(s+4)(s-1)(s-2)} + \frac{C(s+4)(s-1)}{(s+4)(s-1)(s-2)} \\
s^2 + 6s + 9 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1) \\
(s+3)^2 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1)
\end{aligned}$$

$$\begin{aligned}
s^2 + 6s + 9 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1) \\
(-4)^2 + 6(-4) + 9 &= A(-4-1)(-4-2) + B(-4+4)(-4-2) + C(-4+4)(-4-1) \\
16 - 24 + 9 &= A(-5)(-6) + B(0)(-6) + C(0)(-5) \\
1 &= 30A \\
A &= \frac{1}{30} \\
(s+3)^2 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1) \\
(1+3)^2 &= A(1-1)(1-2) + B(1+4)(1-2) + C(1+4)(1-1) \\
4^2 &= A(0)(-1) + B(5)(-1) + C(5)(0) \\
16 &= -5B \\
B &= \frac{-16}{5} \\
(s+3)^2 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1) \\
(2+3)^2 &= A(2-1)(2-2) + B(2+4)(2-2) + C(2+4)(2-1) \\
5^2 &= A(1)(0) + B(6)(0) + C(6)(1) \\
25 &= 6C \\
C &= \frac{6}{25}
\end{aligned}$$

Note:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\begin{aligned}
Y(s) &= \frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)} \\
&= \frac{\frac{1}{30}}{s+4} + \frac{\frac{-16}{5}}{s-1} + \frac{\frac{6}{25}}{s-2} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - \frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{6}{25}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \\
&= \frac{1}{30}e^{-4t} - \frac{16}{5}e^t + \frac{6}{25}e^{2t}
\end{aligned}$$

### 7.2.2 – Finding Inverse-Laplace Transform

#### 7.2.3 – Example

$$\begin{aligned}\mathcal{L} \left\{ \frac{1}{s^4} \right\} &= \frac{1}{3!} \mathcal{L} \left\{ \frac{3!}{s^4} \right\} \\ &= \frac{1}{3!} \mathcal{L} \left\{ \frac{3!}{s^{3+1}} \right\} \\ &= \frac{1}{3!} t^3 \\ &= \frac{1}{6} t^3\end{aligned}$$

#### 7.2.4 – Example

$$\begin{aligned}\mathcal{L} \left\{ \frac{5}{s^2 + 49} \right\} &= \frac{5}{7} \mathcal{L} \left\{ \frac{7}{s^2 + 49} \right\} \\ &= \frac{5}{7} \sin(7t)\end{aligned}$$

#### 7.2.5 – Example

$$\begin{aligned}\mathcal{L} \left\{ \frac{(s+1)^3}{s^4} \right\} &= \mathcal{L} \left\{ \frac{s^3 + 3s^2 + 3s + 1}{s^4} \right\} \\ &= \mathcal{L} \left\{ \frac{s^3}{s^4} \right\} + \mathcal{L} \left\{ \frac{3s^2}{s^4} \right\} + \mathcal{L} \left\{ \frac{3s}{s^4} \right\} + \mathcal{L} \left\{ \frac{1}{s^4} \right\} \\ &= \mathcal{L} \left\{ \frac{1}{s} \right\} + \mathcal{L} \left\{ \frac{3}{s^2} \right\} + \mathcal{L} \left\{ \frac{3}{s^3} \right\} + \mathcal{L} \left\{ \frac{1}{s^4} \right\} \\ &= \mathcal{L} \left\{ \frac{1}{s} \right\} + 3\mathcal{L} \left\{ \frac{1}{s^2} \right\} + 3\mathcal{L} \left\{ \frac{1}{s^3} \right\} + \mathcal{L} \left\{ \frac{1}{s^4} \right\} \\ &= \mathcal{L} \left\{ \frac{1}{s} \right\} + 3\mathcal{L} \left\{ \frac{1}{s^2} \right\} + \frac{3}{2!} \mathcal{L} \left\{ \frac{2!}{s^3} \right\} + \frac{1}{3!} \mathcal{L} \left\{ \frac{3!}{s^4} \right\} \\ &= 1 + 3t + \frac{3}{2!} t^2 + \frac{1}{3!} t^3 \\ &= 1 + 3t + \frac{3}{2} t^2 + \frac{1}{6} t^3\end{aligned}$$

## 7.3 Operational Rules

### 7.3.1 – Operational Rules Part 1

Even with the reuses we know, a problem like

$$\mathcal{L} \{ e^{4t} t^3 \}$$

would require us to go to the definition until we learn some rules.

$$\begin{aligned}\mathcal{L} \{e^{4t}t^3\} &= \int_0^\infty e^{-st}e^{4t}t^3 dt \\ &= \int_0^\infty e^{-(s-4)t}t^3 dt\end{aligned}$$

Compare with

$$\mathcal{L} \{t^3\} = \int_0^\infty e^{-st}t^3 dt = F(s)$$

where  $f(t) = t^3$

$$\mathcal{L} \{e^{at}f(t)\} = F(s-a) \quad (7.10)$$

where  $F(s) = \mathcal{L} \{f(t)\}$

$$\mathcal{L} \{e^{4t}t^3\} = \frac{3!}{(s-4)^4}$$

### 7.3.2 – Example

$$\begin{aligned}\mathcal{L} \{e^{-3t} \sin(5t)\} &= \mathcal{L} \{\sin(5t)\} \Big|_{s \rightarrow s+3} \\ &= \frac{5}{s^2 + 25} \Big|_{s \rightarrow s+3} \\ &= \frac{5}{(s+3)^2 + 25} \Big|_{s \rightarrow s+3} \\ &= \frac{5}{s^2 + 6s + 9 + 25} \\ &= \frac{5}{s^2 + 6s + 34}\end{aligned}$$

### 7.3.3 – Example

$$\begin{aligned}\mathcal{L} \{e^{-2t} \cos(6t)\} &= F(s+2) \\ &= \frac{s+2}{(s+2)^2 + 36} \\ &= \frac{s+2}{s^2 + 4s + 4 + 36} \\ &= \frac{s+2}{s^2 + 4s + 40}\end{aligned}$$

### 7.3.4 – Example

Find

$$\begin{aligned}
 & \mathcal{L}^{-1} \left\{ \frac{s+3}{s^2 - 8s + 97} \right\} \\
 \mathcal{L}^{-1} \left\{ \frac{s+3}{s^2 - 8s + 97} \right\} &= \mathcal{L}^{-1} \left\{ \frac{s+3}{s^2 - 8s + 16 + 81} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{s+3}{(s-4)^2 + 81} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{s+3+4-4}{(s-4)^2 + 81} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{s-4}{(s-4)^2 + 81} \right\} + \mathcal{L}^{-1} \left\{ \frac{3+4}{(s-4)^2 + 81} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{s-4}{(s-4)^2 + 81} \right\} + \mathcal{L}^{-1} \left\{ \frac{7}{(s-4)^2 + 81} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{s-4}{(s-4)^2 + 9^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{7}{(s-4)^2 + 9^2} \right\} \\
 &= e^{4t} \cos(9t) + \frac{7}{9} \mathcal{L}^{-1} \left\{ \frac{9}{(s-4)^2 + 9^2} \right\} \\
 &= e^{4t} \cos(9t) + \frac{7}{9} e^{4t} \sin(9t)
 \end{aligned}$$

### 7.3.5 – Operational Rules Part 2

Involves taking the Laplace transform of a function shifted on the  $t$ -axis.

This can be written in terms of the Heavyside Function (or Unit Step function)

$$U(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

$$f(t-2) \times U(t-2)$$

is “off when  $t < 2$  and on when  $t \geq 2$ .

### 7.3.6 – Example

$$\begin{aligned}
 \mathcal{L}\{f(t-2)U(t-2)\} &= \int_0^\infty e^{-st} f(t-2)U(t-2)dt \\
 &= \int_0^2 e^{-st} f(t-2)U(t-2)dt + \int_2^\infty e^{-st} f(t-2)U(t-2)dt \\
 &= \int_0^2 e^{-st} f(t-2)(0)dt + \int_2^\infty e^{-st} f(t-2)(1)dt \\
 &= \int_0^2 0dt + \int_2^\infty e^{-st} f(t-2)dt \\
 &= \int_2^\infty e^{-st} f(t-2)dt \\
 (\text{ Let } v = t-2, dv = dt) \quad &= \int_0^\infty e^{-s(v+2)} f(v)dv \\
 &= \int_0^\infty e^{-sv} e^{-2s} f(v)dv \\
 &= e^{-2s} \int_0^\infty e^{-sv} f(v)dv
 \end{aligned}$$

Shifting Theorem Shifting on  $t$ -axis

$$\mathcal{L}\{f(t-a)U(t-a)\} = e^{-as} \mathcal{L}\{f(t)\}$$

Proof:

$$\begin{aligned}
 \mathcal{L}\{f(t-a)U(t-a)\} &= \int_0^\infty e^{-st} f(t-a)U(t-a)dt \\
 &= \int_0^a 0dt + \int_a^\infty e^{-st} f(t-a)dt \\
 \text{Let } \tau = t-a, d\tau = dt \quad &= 0 + \int_a^\infty e^{-s(\tau+a)} f(\tau)d\tau \\
 &= e^{-sa} \int_a^\infty e^{-s\tau} f(\tau)d\tau \\
 &= e^{-sa} \mathcal{L}\{f(\tau)\}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{L}\{f(\tau)U(\tau)\} &= e^{-as} \mathcal{L}\{f(\tau)\} \\
 \mathcal{L}\{f(t-a)U(t-a)\} &= e^{-as} \mathcal{L}\{f(t-a)\} \tag{7.11}
 \end{aligned}$$

$$\begin{aligned}
 f(t-a)U(t-a) &= \mathcal{L}^{-1}\{e^{-as}F(s)\} \\
 f(t-a)U(t-a) &= \mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f(t)\}\} \tag{7.12}
 \end{aligned}$$

## 7.4 Operational Rules Part 2

Three More Rules

### 7.4.1 – Rule 1

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s) \Rightarrow \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s) \quad (7.13)$$

### 7.4.2 – Rule 2

Is there a way to break up  $\mathcal{L}$  over a product of functions?

$$\begin{aligned}\mathcal{L}\{f(t)g(t)\} &= \mathcal{L}\{f(t)\} \times \mathcal{L}\{g(t)\} \\ \mathcal{L}\{t^2 \times t^3\} &= \mathcal{L}\{t^2\} \times \mathcal{L}\{t^3\} \\ \mathcal{L}\{t^5\} &= \frac{2!}{s^{2+1}} \times \frac{3!}{s^{3+1}} \\ \frac{5!}{s^{5+1}} &= \frac{2}{s^3} \times \frac{6}{s^4} \\ \frac{120}{s^6} &\neq \frac{12}{s^7}\end{aligned}$$

If we define the convolution production of  $f(t)$  and  $g(t)$  as

$$(f \times g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

then

$$\mathcal{L}\{(f \times g)(t)\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} \quad (7.14)$$

### 7.4.3 – Rule 3

If  $f(t)$  is periodic with period  $T$ , then

$$\mathcal{L}\{f(t)\} = \frac{\int_0^t e^{-st} f(t)dt}{1 - e^{-sT}} \quad (7.15)$$

### 7.4.4 – Example

$$\mathcal{L}\{t \times \sin(kt)\}$$

$$\begin{aligned}
\mathcal{L}\{t \times \sin(kt)\} &= -\frac{d}{ds} \mathcal{L}\{\sin(kt)\} \\
&= -\frac{d}{ds} \left( \frac{k}{s^2 + k^2} \right) \\
&= -\frac{\frac{d}{ds} k \times (s^2 + k^2) - k \frac{d}{ds} (s^2 + k^2)}{(s^2 + k^2)^2} \\
&= -\frac{0(s^2 + k^2) - k(2s)}{(s^2 + k^2)^2} \\
&= -\frac{-2sk}{(s^2 + k^2)^2} \\
&= \frac{2sk}{(s^2 + k^2)^2}
\end{aligned}$$

#### 7.4.5 – Example

$$x'' + 16x = \cos(4t), \quad x(0) = 0, \quad x'(0) = 1$$

$$x'' + 16x = \cos(4t)$$

$$\mathcal{L}\{x''\} + 16\mathcal{L}\{x\} = \mathcal{L}\{\cos(4t)\}$$

$$s^2 X(s) - sx(0) - x'(0) + 16X(s) = \frac{s}{s^2 + 4^2}$$

$$X(s)(s^2 + 16) - s(0) - 1 = \frac{s}{s^2 + 16}$$

$$X(s)(s^2 + 16) - 1 = \frac{s}{s^2 + 16}$$

$$X(s)(s^2 + 16) = \frac{s}{s^2 + 16} + 1$$

$$X(s) = \frac{s}{(s^2 + 16)^2} + \frac{1}{s^2 + 16}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 16)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 16}\right\}$$

$$x(t) = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2 + 4^2}\right\}$$

$$= \frac{1}{8}t \sin(4t) + \frac{1}{4} \sin(4t)$$

#### 7.4.6 – Example

Find  $e^t \sin(t)$

$$e^t \sin(t) = \int_0^t e^\tau \sin(t - \tau) d\tau$$

$$u = e^\tau \quad dv = \sin(t - \tau)d\tau$$

$$\begin{aligned} du &= e^\tau d\tau & v &= \int \sin(t - \tau)d\tau \\ && &= \frac{\tau \cos(t - \tau)}{\tau} \\ && &= \cos(t - \tau) \end{aligned}$$

$$\begin{aligned} e^t \sin(t) &= \int_0^t e^\tau \sin(t - \tau)d\tau \\ &= e^\tau \cos(t - \tau) \Big|_0^t - \int_0^t e^\tau \cos(t - \tau)d\tau \end{aligned}$$

$$h = e^\tau \quad dj = \sin(t - \tau)d\tau$$

$$\begin{aligned} dh &= e^\tau d\tau & j &= \int \cos(t - \tau)d\tau \\ && &= \frac{\sin(t - \tau)}{-1} \\ && &= -\sin(t - \tau) \end{aligned}$$

$$\begin{aligned} \int_0^t e^\tau \sin(t - \tau) &= e^\tau \cos(t - \tau) \Big|_{\tau=0}^t - \left( -e^\tau \sin(t - \tau) \Big|_{\tau=0}^t - \int_0^t e^\tau (-\sin(t - \tau))d\tau \right) \\ &= e^t \cos(1) - e^0 \cos(t) + e^t(0) - e^0 \sin(t) - \int_0^t e^\tau \sin(t - \tau)d\tau \\ 2 \int_0^t e^\tau \sin(t - \tau)d\tau &= e^t - \cos(t) - \sin(t) \end{aligned}$$

So

$$e^t \sin(t) = \frac{e^t - \cos(t) - \sin(t)}{2}$$

$$\begin{aligned}
\mathcal{L}\{e^t \sin(t)\} &= \mathcal{L}\left\{\frac{e^t - \cos(t) - \sin(t)}{2}\right\} \\
&= \frac{1}{2}\mathcal{L}\{e^t - \cos(t) - \sin(t)\} \\
&= \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1^2} - \frac{1}{s^2+1^2}\right) \\
&= \frac{1}{2}\left(\frac{s^2+1}{(s-1)(s^2+1)} - \frac{s(s-1)}{(s-1)(s^2+1)} - \frac{s-1}{(s-1)(s^2+1)}\right) \\
&= \frac{1}{2}\left(\frac{s^2+1-s(s-1)-s-1}{(s-1)(s^2+1)}\right) \\
&= \frac{s^2+1-s^2+s-s+1}{2(s-1)(s^2+1)} \\
&= \frac{s^2-s^2+s-s+1+1}{2(s-1)(s^2+1)} \\
&= \frac{1+1}{2(s-1)(s^2+1)} \\
&= \frac{2}{2(s-1)(s^2+1)} \\
&= \frac{1}{(s-1)(s^2+1)}
\end{aligned}$$

### 7.4.7 – Example

$$\begin{aligned}
\mathcal{L}\{(f \times g)(t)\} &= \mathcal{L}\{f(t)\} \times \mathcal{L}\{g(t)\} \\
(f \times g)(t) &= \mathcal{L}^{-1}\{F(s) \times G(s)\}
\end{aligned}$$

Determine

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$$

using the Convolution Theorem.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} \\
\sin(A)\sin(B) &= \frac{1}{2}(\cos(A-B) + \cos(A+B)) \\
&= \frac{1}{k^2}\mathcal{L}\left\{\frac{1 \times k}{s^2+k^2}\right\}
\end{aligned}$$

### 7.4.8 – Example

Solve the Integral Equation

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau \quad \text{for } f(t)$$

Don't forget that the  $\int_0^t f(\tau)e^{t-\tau}d\tau$  is  $f(t) \times e^t$ . Take  $\mathcal{L}$  of both sides

$$\begin{aligned}
 F(s) &= 3 \frac{2!}{s^{2+1}} - \frac{1}{s - (-1)} - F(s) \frac{1}{s - 1} \\
 &= \frac{6}{s^3} - \frac{1}{s + 1} - F(s) \frac{1}{s - 1} \\
 F(s) + F(s) \frac{1}{s - 1} &= \frac{6}{s^3} - \frac{1}{s + 1} \\
 F(s) \left(1 + \frac{1}{s - 1}\right) &= \frac{6}{s^3} - \frac{1}{s + 1} \\
 F(s) \left(\frac{s - 1}{s - 1} + \frac{1}{s - 1}\right) &= \frac{6}{s^3} - \frac{1}{s + 1} \\
 F(s) \left(\frac{s - 1 + 1}{s - 1}\right) &= \frac{6}{s^3} - \frac{1}{s + 1} \\
 F(s) \left(\frac{s}{s - 1}\right) &= \frac{6}{s^3} - \frac{1}{s + 1} \\
 F(s) &= \frac{6(s - 1)}{s^3 s} - \frac{s - 1}{(s + 1)s} \\
 &= \frac{6s - 6}{s^4} - \frac{s - 1}{s^2 + s} \\
 &= \dots \\
 &= \frac{6s + 6 - s^3}{s^3(s + 1)}
 \end{aligned}$$

## 7.5 The Dirac Delta Function

$$\begin{aligned}
 \delta_a(t - t_0) &= \begin{cases} 0 & \text{if } t < t_0 - a \\ \frac{1}{2a} & \text{if } t_0 - a \leq t \leq t_0 + a \\ 0 & \text{if } t > t_0 + a \end{cases} \\
 \int_0^\infty f(t)\delta(t - t_0)dt &= f(t_0)
 \end{aligned} \tag{7.16}$$

So

$$\begin{aligned}
 \mathcal{L}\{\delta(t - t_0)\} &= \int_0^\infty e^{-st}\delta(t - t_0)dt \\
 &= e^{-st_0}
 \end{aligned} \tag{7.17}$$

### 7.5.1 – Example

$$y'' + y = 4\delta(t - 2\pi)$$

$$\begin{aligned}
y(0) &= 1 \quad y'(0) = 0 \\
y'' + y &= 4\delta(t - 2\pi) \\
\mathcal{L}\{y''\} + \mathcal{L}\{y\} &= 4\mathcal{L}\{\delta(t - 2\pi)\} \\
s^2 Y(s) - sy(0) - y'(0) + Y(s) &= 4e^{-s(2\pi)} \\
Y(s)(s^2 + 1) - s(1) - 0 &= 4e^{-2\pi s} \\
Y(s)(s^2 + 1) - s &= 4e^{-2\pi s} \\
Y(s)(s^2 + 1) &= 4e^{-2\pi s} + s \\
Y(s) &= \frac{4e^{-2\pi s} + s}{s^2 + 1} \\
&= \frac{4e^{-2\pi s}}{s^2 + 1} + \frac{s}{s^2 + 1} \\
&= 4e^{-2\pi} \frac{e^s}{s^2 + 1^2} + \frac{s}{s^2 + 1} \\
y(t) &= \mathcal{L}^{-1} \left\{ 4e^{-2\pi s} \frac{1}{s^2 + 1^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} \\
&= 4 \sin(t - 2\pi) \mathcal{U}(t - 2\pi) + \cos(t)
\end{aligned}$$

# Chapter 8

## Systems of Linear First-Order Differential Equations

### 8.1 Preliminary Theory – Linear Systems

In this chapter, we assume the system can be put in the form

$$\begin{aligned}\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \dots, x_n) \\ \frac{dx_2}{dt} &= g_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= g_n(t, x_1, x_2, \dots, x_n)\end{aligned}$$

Further assume  $g_1, g_2, \dots, g_n$  are linear with respect to  $x_1, x_2, \dots, x_n$ .

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + f_1(t) \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + f_n(t)\end{aligned}$$

In matrix notation:

$$X' = AX + F$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$F = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} = [a_{ij}(t)] \quad i = 1, 2, \dots, n$$

$$j = 1, 2, \dots, n$$

In general, if all the  $a_{ij}(t)$ 's and  $f_i(t)$ 's are continuous on an interval  $I$ , then the IVP  $X' = AX + F$  has a unique solution:

$$x_1(t_0) = w_1$$

$$x_2(t_0) = w_2$$

...

$$x_n(t_0) = w_n$$

where  $w_1, w_2, \dots, w_n$  are just numbers.

If the initial conditions aren't given, then we want the general solution

$$X = c_1 X_1 + c_2 X_2 + \dots + c_n X_n$$

where each  $X_i$  is a solution of  $X' = Ax$  and  $\{X_1(t), X_2(t), \dots, X_n(t)\}$ <sup>1</sup> is a linearly independent collection of solutions. This will be true iff the Wronskian( $X_1, X_2, \dots, X_n$ ) is non-zero<sup>2</sup>. The set  $\{X_1, X_2, \dots, X_n\}$  is called a fundamental set.

## 8.2 Solving Homogenous Systems

$$\mathbf{X}' = A\mathbf{X}$$

where  $A$  is a constant matrix.

Look for solutions of the form

$$\mathbf{X} = \mathbf{k}e^{\lambda t}$$

where  $\lambda$  is an eigenvalue and the  $k$  vector is the corresponding eigenvector.

Plug this into the Matrix Equation to get

$$\begin{aligned} \mathbf{k}\lambda e^{\lambda t} &= A\mathbf{k}e^{\lambda t} \\ 0 &= A\mathbf{k}e^{\lambda t} - \mathbf{k}\lambda e^{\lambda t} \\ &= (A\mathbf{k} - \mathbf{k}\lambda)e^{\lambda t} \\ &= A\mathbf{k} - \mathbf{k}\lambda \\ &= (A - \lambda I)\mathbf{k} \end{aligned}$$

---

<sup>1</sup>For a system of  $n$  equations

<sup>2</sup>Wronskian( $X_1, X_2, \dots, X_n$ ) =  $\det(X_1, X_2, \dots, X_n)$

which would have either  $\mathbf{k} = \mathbf{0}$  (trivial solution) or  $\mathbf{k}$  is an eigenvector of  $\mathbf{A}$  and  $\lambda$  is a corresponding eigenvalue, there are linearly-independent.

### 8.2.1 – Example

$$\frac{dx}{dt} = 2x + 3y \quad \frac{dy}{dt} = 2x + y$$

Write as

$$\mathbf{X}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \mathbf{X}.$$

Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

Real distinct eigenvalues

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{bmatrix} 2 - \lambda & 3 \\ 2 & 1 - \lambda \end{bmatrix} \\ &= (2 - \lambda)(1 - \lambda) - (3)(2) \\ &= \lambda^2 - 3\lambda + 2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1) \end{aligned}$$

$$\begin{aligned} \lambda_1 - 4 &= 0 & \lambda_2 + 1 &= 0 \\ \lambda_1 &= 4 & \lambda_2 &= -1 \end{aligned}$$

For  $\lambda_1 = 4$   $(\mathbf{A} - 4I)\mathbf{k} = 0$

$$\begin{aligned} \left[ \begin{array}{cc|c} 2 - \lambda_1 & 3 & 0 \\ 2 & 1 - \lambda_1 & 0 \end{array} \right] &= \left[ \begin{array}{cc|c} 2 - 4 & 3 & 0 \\ 2 & 1 - 4 & 0 \end{array} \right] \\ &= \left[ \begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array} \right] \\ &= \left[ \begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array} \right] \\ &= \left[ \begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right] \end{aligned}$$

$$\begin{aligned} -2k_1 + 3k_2 &= 0 \\ 3k_2 &= 2k_1 \\ 3(2) &= 2k_1 \\ 6 &= 2k_1 \\ 3 &= k_1 \end{aligned}$$

$$\mathbf{k}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

For  $\lambda_2 = -1$   $(\mathbf{A} + I)\mathbf{k} = 0$

$$\begin{array}{c} \left[ \begin{array}{cc|c} 2 - \lambda_2 & 3 & 0 \\ 2 & 1 - \lambda_2 & 0 \end{array} \right] = \left[ \begin{array}{cc|c} 2 - (-1) & 3 & 0 \\ 2 & 1 - (-1) & 0 \end{array} \right] \\ = \left[ \begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array} \right] \\ = \left[ \begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array} \right] \\ = \left[ \begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right] \end{array}$$

$$\begin{aligned} 3k_1 + 3k_2 &= 0 \\ 3k_1 &= -3k_2 \\ k_1 &= -k_2 \\ k_1 &= -(-1) \\ k_1 &= 1 \end{aligned}$$

$$\mathbf{k}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{X}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} \quad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

are both solutions and by a Theorem,  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are linearly independent.  
The general solution is

$$X = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 \quad (8.1)$$

for particular constants  $c_1$  and  $c_2$ .

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3c_1 e^{4t} + c_2 e^{-t} \\ 2c_1 e^{4t} - c_2 e^{-t} \end{bmatrix}$$

These values come from the matrix of eigenvectors, in this case

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{4t} \\ c_2 e^{-t} \end{bmatrix}$$

### 8.2.2 – Example

$$\mathbf{X}' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \mathbf{X}$$

Eigenvalues:

$$\begin{aligned}
 0 &= \det(A - \lambda I) \\
 &= \begin{bmatrix} 3 - \lambda & -18 \\ 2 & -9 - \lambda \end{bmatrix} \\
 &= (3 - \lambda)(-9 - \lambda) - (2)(-18) \\
 &= \lambda^2 + 6\lambda - 27 + 36 \\
 &= \lambda^2 + 6\lambda + 9 \\
 &= (\lambda + 3)^2
 \end{aligned}$$

The multiplicity of  $(\lambda + 3)^2 = 0$  is 2, so

$$\lambda_1 = -3 \quad \lambda_2 = -3$$

Eigenvectors:

$$\begin{aligned}
 [A - \lambda_1 I | 0] &= \begin{bmatrix} 3 - \lambda_1 & -18 \\ 2 & -9 - \lambda_1 \end{bmatrix} \\
 &= \begin{bmatrix} 3 - (-3) & -18 \\ 2 & -9 - (-3) \end{bmatrix} \\
 &= \begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \\
 &= \begin{bmatrix} 2 & -6 \\ 2 & -6 \end{bmatrix} \\
 &= \begin{bmatrix} 2 & -6 \\ 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \\
 1k_1 - 3k_2 &= 0 \\
 k_1 &= 3k_2 \\
 k_1 &= 3(1) \\
 k_1 &= 3 \\
 \mathbf{k}_1 &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}
 \end{aligned}$$

The 2nd, Linear Independent solution is

$$\mathbf{k}_2 = \mathbf{k}_1 t e^{\lambda_1 t} + P e^{\lambda_1 t}$$

$$(A - \lambda I)P = K$$

## 8.2 Solving Homogenous Systems

### 8.2.3 – Complex EigenValues

#### 8.2.4 – Example

$$\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} 6 - \lambda & -1 \\ 5 & 4 - \lambda \end{vmatrix} \\ &= (6 - \lambda)(4 - \lambda) - (-1)(5) \\ &= 24 - 6\lambda - 4\lambda + \lambda^2 + 5 \\ &= \lambda^2 - 10\lambda + 29 \\ &= (\lambda - 5)^2 + 4 \\ (\lambda - 5)^2 &= -4 \\ \lambda - 5 &= \pm 2i \\ \lambda &= 5 \pm 2i \end{aligned}$$

For  $\lambda_1 = 5 + 2i$

$$\begin{aligned} \left[ \begin{array}{cc|c} 6 - \lambda_1 & -1 & 0 \\ 5 & 4 - \lambda_1 & 0 \end{array} \right] &= \left[ \begin{array}{cc|c} 6 - 5 - 2i & -1 & 0 \\ 5 & 4 - 5 - 2i & 0 \end{array} \right] \\ &= \left[ \begin{array}{cc|c} 1 - 2i & -1 & 0 \\ 5 & -1 - 2i & 0 \end{array} \right] \\ &= \left( r_1 \leftarrow \frac{r_1}{1 - 2i} \right) \left[ \begin{array}{cc|c} 1 & -\frac{1}{1-2i} & 0 \\ 5 & -1 - 2i & 0 \end{array} \right] \\ &= \left( r_1 \leftarrow r_1 \times \frac{1+2i}{1+2i} \right) \left[ \begin{array}{cc|c} 1 & -\frac{1}{5} - \frac{2}{5}i^3 & 0 \\ 5 & -1 - 2i & 0 \end{array} \right] \\ &= (r_2 \leftarrow r_2 - 5r_1) \left[ \begin{array}{cc|c} 1 & -\frac{1}{5} - \frac{2}{5}i & 0 \\ 0 & 0 & 0 \end{array} \right] \end{aligned}$$

$$k_1 + \left( -\frac{1}{5} - \frac{2}{5}i \right) k_2 = 0$$

Let  $k_2 = 1 - 2i$

$$k_1 + \left( -\frac{1}{5} - \frac{2}{5}i \right) (1 - 2i) = 0$$

$$k_1 + (-1)^4 = 0$$

$$k_1 = 1$$

$$\mathbf{k}_1 = \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix}$$

This, along with an eigenvector  $\mathbf{k}_2$  for  $\lambda_2 = 5 - 2i$ , can be expressed in terms of Real Matrices as

$$\begin{aligned} \mathbf{X}_1 &= [\beta_1 \cos(\beta t) - \beta_2 \sin(\beta t)] e^{\alpha t} \\ \mathbf{X}_2 &= [\beta_2 \cos(\beta t) - \beta_1 \sin(\beta t)] e^{\alpha t} \end{aligned} \quad (8.2)$$

where  $\alpha$  is the real part of the eigen value,  $\beta$  is the coefficient of the imaginary part of the eigen value,

$$\begin{aligned} \beta_1 &= \text{Real Part } (k_1) = \operatorname{Re} \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \beta_2 &= \text{Imaginary Part } (k_1) = \operatorname{Im} \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -2 \end{bmatrix} \end{aligned}$$

## 8.3 General Ideas

(Don't worry about detailed calculation)

Non-homogenous system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

where  $\mathbf{F}$  is a non-0 column vector.

### 8.3.1 – 1st Solve the Complimentary DE

$$\mathbf{X}' = \mathbf{A}\mathbf{X},$$

General solution:

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + \cdots + c_n \mathbf{X}_n$$

<sup>3</sup>

$$-\frac{1}{1-2i} \times \frac{1+2i}{1+2i} = -\frac{1+2i}{(1-2i)(1+2i)} = -\frac{1+2i}{1-4i^2} = -\frac{1+2i}{1-(-4)} = -\frac{1+2i}{1+4} = -\frac{1+2i}{5}$$

<sup>4</sup>Essentially undid the complex conjugate multiplication.

**8.3.2 – Next, find one particular solution**

The general solution of  $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$  is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

# Appendix B

## 2.1 Eigenvalues & Eigenvectors of a Square Matrix

Given a square matrix  $A$ , we look for a non-zero column vector  $k$  and a number  $\lambda$  such that  $A_{n \times n} k_{n \times 1} = \lambda k_{n \times 1}$ .

If such  $\lambda$  and  $k$  exist,  $\lambda$  is called an eigenvalue for the matrix  $A$  and  $k$  is the corresponding eigenvector.

### 2.1.1 – Example

Verify that

$$k = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

is an eigenvector for the matrix

$$A = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

and determine the corresponding eigenvalue.

Calculate

$$Ak = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0(1) + -1(-1) + -3(1) \\ 2(1) + 3(-1) + 3(1) \\ -2(1) + 1(-1) + 1(1) \end{bmatrix} \\
&= \begin{bmatrix} 0 + 1 - 3 \\ 2 - 3 + 3 \\ -2 - 1 + 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \\
&= -2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\
&= 2k
\end{aligned}$$

Therefore  $k$  is an eigenvector corresponding to eigenvalue  $\lambda = -2$ .

Notice that any non-zero multiple of  $k$  would also be an eigenvector corresponding to  $\lambda = -2$ .  
Proof:

$$\begin{aligned}
A(5k) &= 5Ak \\
&= 5(-2)k \\
&= (-2)(5k)
\end{aligned}$$

### 2.1.2 – Example

Find the eigenvalues, and for each, a corresponding eigenvector for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Theory: If  $Ak = \lambda k$  for a non-zero  $k$ , then  $Ak - \lambda k = 0 \Rightarrow (A - \lambda I)k = 0$  for a non-zero  $k$ . However, we cannot subtract a number from a matrix. Instead, the equation would be  $(A - \lambda I)k = 0$ .

This would mean that the matrix  $A - \lambda I$  is singular (not invertible). This can be checked by ensuring that  $\det(A - \lambda I) = 0$

$$\begin{aligned}
A - \lambda I &= \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{bmatrix} \\
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} \\
&= 1 \begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} - 0 \begin{vmatrix} 1-\lambda & 2 \\ -1 & -2 \end{vmatrix} + (-1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix} \\
&= 1[(6)(-2) - (-1)(-1-\lambda)] - 0 + (-1-\lambda)[(1-\lambda)(-1-\lambda) - (2)(6)] \\
&= 1[-12 + -1 - \lambda] + (-1 - \lambda)[-1 - \lambda + \lambda + \lambda^2 - 12] \\
&= 1[-13 - \lambda] + (-1 - \lambda)[\lambda^2 - 1 - 12] \\
&= -13 - \lambda + (-1 - \lambda)[\lambda^2 - 13] \\
&= -13 - \lambda - \lambda^2 + 13 - \lambda^3 + 13\lambda \\
&= -\lambda - \lambda^2 - \lambda^3 + 13\lambda \\
&= -\lambda^3 - \lambda^2 + 13\lambda - \lambda \\
&= -\lambda^3 - \lambda^2 + 12\lambda \\
&= -\lambda(\lambda^2 + \lambda - 12) \\
&= -\lambda(\lambda + 4)(\lambda - 3)
\end{aligned}$$

When this is 0,  $\lambda$  is an eigenvalue.

$$-\lambda(\lambda + 4)(\lambda - 3) = 0$$

$$\begin{aligned}
-\lambda_1 &= 0 & \lambda_2 + 4 &= 0 & \lambda_3 - 3 &= 0 \\
\lambda_1 &= 0 & \lambda_2 &= -4 & \lambda_3 &= 3
\end{aligned}$$

Next, for each eigenvalue  $\lambda_1, \lambda_2, \lambda_3$ , find a corresponding eigenvalue  $k_1, k_2, k_3$ .

For  $\lambda_1 = 0$

$$\begin{aligned} 0 &= (A - \lambda_1 I)k_1 \\ &= \begin{bmatrix} 1 - \lambda_1 & 2 & 1 \\ 6 & -1 - \lambda_1 & 0 \\ -1 & -2 & -1 - \lambda_1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 - 0 & 2 & 1 \\ 6 & -1 - 0 & 0 \\ -1 & -2 & -1 - 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 6 & -1 & 0 & | & 0 \\ -1 & -2 & -1 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 6 & -1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad (r_3 \leftarrow r_3 + r_1)$$

$$= \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & -13 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad (r_2 \leftarrow r_2 - 6r_1)$$

$$= \dots$$

$$= \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -13 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} k_1 + 2k_2 + 1k_3 &= 0 \\ -13k_2 - 6k_3 &= 0 \end{aligned}$$

Let  $k_3 = 1$ , then  $k_2 = \frac{-6}{13}k_3 = -\frac{6}{13}$  and  $k_1 = -2k_2 - k_3 = \frac{12}{13} - 1 = -\frac{1}{13}$

$$k_1 = \begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}$$