Chapter 2

First-Order Differential Equations

2.5 Substitution Methods

Taking a D.E. that's not:

- Separable
- 1st Order Linear
- Exact

and making a substitution to turn the new D.E. into one of these

Theorem: Given a D.E.

$$M(x,y)dx + N(x,y)dy = 0$$

A function $f(x,y)$ is said to be homogenous of order α if $f(tx,ty) = t^{\alpha} f(x,y)$.

2.5.1 – Example

Given:

$$f(x,y) = x^3 + 5xy^2 - y^3$$

Then:

$$\begin{aligned} f(tx,ty) &= (tx)^3 + 5(tx)(ty)^2 - (ty)^3 \\ &= t^3x^3 + 5t^3xy^2 - t^3y^3 \\ &= t^3\left(x^3 + 5xy^2 - y^3\right) \\ &= t^3 f(x,y) \end{aligned}$$

2.5.2 – Example

$$
\begin{aligned}
f(x,y) &= \frac{x+y}{x^2+y^2} \\
f(tx,ty) &= \frac{tx+ty}{(tx)^2+(ty)^2} \\
f(tx,ty) &= \frac{tx+ty}{x^2t^2+y^2t^2} \\
f(tx,ty) &= \frac{t}{t^2} \times \frac{x+y}{x^2+y^2} \\
f(tx,ty) &= \frac{t}{t^2} f(x,y) \\
f(tx,ty) &= \frac{1}{t} f(x,y)
\end{aligned}
$$

$f(x,y) = \frac{x+y}{x^2+y^2}$ is homogenous of order $\alpha = -1$

2.5.3 – Substitution Rule

If $M(x,y)$ and $N(x,y)$ are homogenous, each of the same order, then $u = \frac{y}{x}$ i.e., $y = ux$ or $v = \frac{x}{y}$ (i.e. $x = vy$) will produce a separable D.E.

2.5.4 – Example

Solve the separable D.E. and then back-substitute

$$(x^2+y^2)dx + (x^2-xy)dy = 0$$

$$
\begin{aligned}
M(x,y) &= x^2+y^2 \quad N = x^2 - xy \\
M_y &= 2y \quad N_x = 2x - y \\
M_y &\neq N_x \\
M(tx,ty) &= (tx)^2 + (ty)^2 \\
&= t^2x^2 + t^2y^2 \\
&= t^2(x^2+y^2) \\
&= t^2M(x,y) \text{ } M \text{ is homogeneous of order 2 and so is } N \\
u &= \frac{y}{x} \\
y &= ux \\
dy &= udx + xdu \\
(x^2+(ux)^2)dx + (x^2 - x(ux))(udx+xdu) &= 0 \\
(x^2+u^2x^2)dx + (x^2-ux^2)(udx+xdu) &= 0 \\
(1+u^2)x^2dx + x^2(1-u)(udx+xdu) &= 0 \\
(1+u^2)x^2dx + x^2\left(udx + xdu - u^2dx - uxdu\right) &= 0 \\
x^2\left(1dx + u^2dx + udx + xdu - u^2dx - uxdu\right) &= 0 \\
x^2\left(1dx + u^2dx - u^2dx + udx + xdu - uxdu\right) &= 0 \\
x^2\left(1dx + udx + xdu - uxdu\right) &= 0 \\
x^2(1+u)dx + x^3(1-u)du &= 0 \\
\int \frac{1}{x}dx &= \int -\frac{1-u}{1+u}du \\
&= \int \frac{u-1}{u+1}du \\
&= \int \frac{u+(1-2)}{u+1}du \\
&= \int \left(\frac{u+1}{u+1} - \frac{2}{u+1}\right)du \\
&= \int \left(1 - \frac{2}{u+1}\right)du \\
\ln|x| &= \int \left(1 - \frac{2}{u+1}\right)du \\
&= u - 2\ln|u+1| + C \\
\ln|x| &= \frac{y}{x} - 2\ln\left|\frac{y}{x}+1\right| + C
\end{aligned}
$$

Theorem: An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where $n \neq 0, 1$ is called a Bernoulli Equation. The substitution

$$u = y^{1-n}$$

will transform the D.E. into a 1st order linear.

2.5.5 – Example

$$
\begin{aligned}
x\frac{dy}{dx} + y &= x^2y^2 \\
\frac{dy}{dx} + \frac{y}{x} &= xy^2
\end{aligned}
$$

is a Bernoulli equation with $n = 2$.

$$\begin{aligned}
u &= y^{1-2} \\
&= y^{-1} \\
&= \frac{1}{y} \\
\frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\
&= -1y^{-2}\frac{dy}{dx} \\
&= -\frac{1}{y^2}\frac{dy}{dx} \\
-y^{-2}\frac{dy}{dx} + -y^{-2} \times \frac{y}{x} &= -y^{-2} \times xy^2 \\
-y^{-2}\frac{dy}{dx} + -\frac{1}{x}y^{-1} &= -x \\
\frac{du}{dx} - \frac{1}{x}u = -x & \\
\text{I.F.} = \mu &= e^{P(x)dx} \\
&= e^{-\int \frac{1}{x}dx} \\
&= e^{-\ln|x|} \\
&= e^{\ln|x^{-1}|} \\
&= x^{-1} \\
\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u &= -1 \\
\frac{d}{dx}\left(\frac{1}{x}u\right) &= -1 \\
\int \frac{d}{dx}\left(\frac{1}{x}u\right) &= \int -1dx \\
\frac{1}{x}u &= \int -1dx \\
\frac{1}{x}u &= -x + C \\
\frac{1}{x} \times 1y &= -x + C \\
\frac{1}{x(-x+C)} &= y \\
y &= \frac{1}{Cx - x^2}
\end{aligned}$$

Theorem: If the D.E. can be expressed as

$$\frac{dy}{dx} = f(Ax + by + C)$$

for particular numbers A, B, C, then let

$$u = Ax + By + C$$

to get a separable D.E.

2.5.6 – Example

$$\begin{aligned}
\frac{dy}{dx} = (-2x+y)^2 - 7, y(0) = 0 \\
u &= -2x + y \\
\frac{du}{dx} &= \frac{dy}{dx} \times \frac{du}{dy} \\
&= -2 + \frac{dy}{dx} \\
\frac{du}{dx} + 2 &= \frac{dy}{dx} \\
\frac{du}{dx} + 2 &= u^2 - 7 \\
\frac{du}{dx} &= u^2 - 9 \\
\frac{du}{u^2 - 9} &= dx \\
\int \frac{du}{u^2 - 9} &= \int dx \\
\int \frac{du}{(u+3)(u+9)} &= x + C \\
\int \frac{du}{(u+3)(u+9)} &= x + C
\end{aligned}$$