MATH 252.01　　　　　**Differential Equations**　　　　　**Test #2**

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1. One solution to the differential equation

$$y'' - \tan(x)y' - \sec^2(x)y = 0$$

is $y_1(x) = \sec(x)$ on the interval $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Find a solution $y_2(x)$ of the given DE which is linearly independent from $y_1(x)$. You may find some of the following facts helpful.

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C \qquad\qquad \int \tan(x)dx = \ln|\sec(x)| + C$$

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx \\
&= \sec(x) \int \frac{e^{-\int -\tan(x)dx}}{\sec^2(x)}dx \\
&= \sec(x) \int \frac{e^{\int \tan(x)dx}}{\sec^2(x)}dx \\
&= \sec(x) \int \frac{e^{(\ln|\sec(x)|)}}{\sec^2(x)}dx \\
&= \sec(x) \int \frac{e^{(\ln(\sec(x)))}}{\sec^2(x)}dx \\
&= \sec(x) \int \frac{\sec(x)}{\sec^2(x)}dx \\
&= \sec(x) \int \frac{1}{\sec(x)}dx \\
&= \sec(x) \int \cos(x)dx \\
&= \sec(x)(\sin(x)) \\
&= \frac{1}{\cos x}(\sin(x)) \\
&= \tan x
\end{aligned}
$$

2. Determine the general solution of the following differential equation.

$$y'' + 3y' - 10y = 20x^2 - 2x$$

$$
\begin{aligned}
y'' + 3y' - 10y &= 0 \\
m^2 e^{mx} + 3m e^{mx} - 10 e^{mx} &= 0 \\
m^2 + 3m - 10 &= 0 \\
(m + 5)(m - 2) &= 0
\end{aligned}
$$

$$m_1 + 5 = 0 \qquad m_2 - 2 = 0$$
$$m_1 = -5 \qquad m_2 = 2$$
$$y_1 = e^{-5x} \qquad y_2 = e^{2x}$$
$$y_c = c_1 e^{-5x} + c_2 e^{2x}$$
$$y_p = Ax^4 + Bx^3$$
$$y_p' = 4Ax^3 + 3Bx^2$$
$$y_p'' = 12Ax^2 + 6Bx$$
$$12Ax^2 + 6Bx = 20x^2 - 2x$$
$$6Ax^2 + 3Bx = 10x^2 - x$$
$$6A = 10 \qquad 3B = -1$$
$$A = \frac{10}{6} \qquad B = \frac{-1}{3}$$
$$A = \frac{5}{3} \qquad B = -\frac{1}{3}$$
$$y_p = \frac{5}{3}x^4 - \frac{1}{3}x^3$$
$$y = y_c + y_p$$
$$= c_1 e^{-5x} + c_2 e^{2x} + \frac{5}{3}x^4 - \frac{1}{3}x^3$$

3. Suppose that you are given a 6th order linear differential equation with constant coefficients

$$a_6 y^{(6)} + a_5 y^{(5)} + \cdots + a_1 y' + a_0 y = f(x)$$

and that the auxiliary equation of the complementary differential equation can be written in factored forms as

$$m^3(m+3)(m-(2+5i))(m-(2-5i)) = 0$$

.

(a) Write the general solution y_c of the complementary differential equation.

$$m_1^3 = 0 \qquad m_2 + 3 = 0 \qquad m_3 - (2+5i) = 0 \qquad m_4 - (2-5i) = 0$$
$$m_1 = 0 \qquad m_2 = -3 \qquad m_3 = 2 + 5i \qquad m_4 = 2 - 5i$$
$$y_1 = e^{0x} \qquad y_2 = e^{-3x} \qquad y_3 = e^{2x}\cos(5x) \qquad y_4 = e^{2x}\sin(5x)$$

$$y_5 = y_1 \int \frac{e^{-\int y_1 dx}}{(y_1)^2}$$
$$= 1 \int \frac{e^{-\int 1 dx}}{(1)^2}$$
$$= \int \frac{e^{-x}}{1}$$
$$= \int e^{-x}$$

$$y_c = c_1 e^{0x} + c_2 e^{-3x} + c_3 e^{2x} \cos(5x) + c_4 e^{2x} \sin(5x)$$
$$= c_1(1) + c_2 e^{-3x} + c_3 e^{2x} \cos(5x) + c_4 e^{2x} \sin(5x)$$
$$= c_1 + c_2 e^{-3x} + c_3 e^{2x} \cos(5x) + c_4 e^{2x} \sin(5x)$$

In each part below a different function $f(x)$ is given for the above differential equation. Give the form of a particular solution y_p of the DE that one should look for according to the method of Undetermined Coefficients. (Do not attempt to solve for the coefficients)

(a)
$$f(x) = e^{-3x}$$

(b)
$$f(x) = 4x + 7$$

(c)
$$f(x) = \cos(6x)$$

(d)
$$f(x) = e^{2x} \sin(5x)$$

(e)
$$f(x) = xe^{-3x}$$

4. Find power series representations of two linearly independent solutions of the following differential equation. The solution to this problem does allow for one to indicate the pattern of the coefficients in the power series.

$$y'' - xy' - y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$
$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$0 = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k+2=2}^{\infty} (k+2)(k+2-1)c_{k+2} x^{k+2-2} - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{1}(k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{1} c_k x^k - \sum_{k=1}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{1}(k+2)(k+1)c_{k+2} x^k - \sum_{k=0}^{1} c_k x^k + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2} x^k - kc_k x^k - c_k x^k$$

$$= (0+2)(0+1)c_{0+2} x^0 - c_0 x^0 + \sum_{k=1}^{\infty} x^k \left[(k+2)(k+1)c_{k+2} - kc_k - c_k\right]$$

$$= (2)(1)c_2(1) - c_0(1) + \sum_{k=1}^{\infty} x^k \left[(k+2)(k+1)c_{k+2} - (k+1)c_k\right]$$

$$= 2c_2 - c_0 + \sum_{k=1}^{\infty} x^k \left[(k+2)(k+1)c_{k+2} - (k+1)c_k\right]$$

$$2c_2 - c_0 = 0$$
$$2c_2 = c_0$$
$$c_2 = \frac{c_0}{2}$$
$$(k+2)(k+1)c_{k+2} - (k+1)c_k = 0$$
$$(k+2)(k+1)c_{k+2} = (k+1)c_k$$
$$c_{k+2} = \frac{(k+1)c_k}{(k+2)(k+1)}$$
$$= \frac{c_k}{k+2}$$

5. Use the method of Laplace Transforms to solve the following initial value problem.

$$y'' + 2y' - 3y = 4, \quad y(0) = 1, \quad y'(0) = -2$$

6. Find the indicated Laplace Transforms or Inverse Laplace Transforms. If you need to make a partial fraction expansion, you may leave your final answer in terms of the "unknown" constants A, B, C, etc. (i.e., you do not need to solve for the constants in the expansion).

(a)

$$\mathcal{L}^{-1}\left\{t^5 - 4\cos(5t) + 2e^{-6t} + 7\right\}$$

(b)

$$\mathcal{L}^{-1}\left\{\frac{3s+2}{s^3(s^2+9)}\right\}$$

7. Use the definition of the Laplace transform to find the Laplace transform of the function $f(t)$ defined by

$$f(t) = \begin{cases} 2 & \text{if } 0 \le t < 1 \\ 3 & \text{if } t \ge 1 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}(2)\,dt + \int_1^\infty e^{-st}(3)\,dt$$

$$= 2\int_0^1 e^{-st}\,dt + 3\int_1^\infty e^{-st}\,dt$$

$$= 2\int_0^1 e^{-st} + 3\int_1^\infty e^{-st}\,dt$$

$$= 2 \times \left.\frac{e^{-st}}{-s}\right|_0^1 + 3 \times \left.\frac{e^{-st}}{-s}\right|_1^\infty$$

$$= 2\left(\frac{e^{-s(1)}}{-s} - \frac{e^{-s(0)}}{-s}\right) + 3\left(\lim_{b\to\infty}\frac{e^{-s(b)}}{-s} - \frac{e^{-s(1)}}{-s}\right)$$

$$= 2\left(\frac{e^{-s} - e^0}{-s}\right) + 3\left(-\frac{e^{-s}}{-s} + \lim_{b\to\infty}\frac{1}{-se^{sb}}\right)$$

$$= 2\left(\frac{e^0 - e^{-s}}{s}\right) + 3\left(-\frac{e^{-s}}{-s} + 0\right)$$

$$= 2\left(\frac{e^0 - e^{-s}}{s}\right) + \frac{3e^{-s}}{s}$$

$$= \frac{2(1) - 2e^{-s}}{s} + \frac{3e^{-s}}{s}$$

$$= \frac{2(1) - 2e^{-s} + 3e^{-s}}{s}$$

$$= \frac{2 + e^{-s}}{s}$$