

Chapter 2

First-Order Differential Equations

2.5 Substitution Methods

Taking a D.E. that's not:

- Separable
- 1st Order Linear
- Exact

and making a substitution to turn the new D.E. into one of these.

Theorem: Given a D.E.

$$M(x, y)dx + N(x, y)dy = 0$$

A function $f(x, y)$ is said to be homogenous of order α if $f(tx, ty) = t^\alpha f(x, y)$.

2.5.1 – Example

Given:

$$f(x, y) = x^3 + 5xy^2 - y^3$$

Then:

$$\begin{aligned} f(tx, ty) &= (tx)^3 + 5(tx)(ty)^2 - (ty)^3 \\ &= t^3x^3 + 5t^3xy^2 - t^3y^3 \\ &= t^3(x^3 + 5xy^2 - y^3) \\ &= t^3f(x, y) \end{aligned}$$

2.5.2 – Example

$$\begin{aligned}f(x, y) &= \frac{x + y}{x^2 + y^2} \\f(tx, ty) &= \frac{tx + ty}{(tx)^2 + (ty)^2} \\f(tx, ty) &= \frac{tx + ty}{x^2t^2 + y^2t^2} \\f(tx, ty) &= \frac{t}{t^2} \times \frac{x + y}{x^2 + y^2} \\f(tx, ty) &= \frac{t}{t^2} f(x, y) \\f(tx, ty) &= \frac{1}{t} f(x, y)\end{aligned}$$

$f(x, y) = \frac{x+y}{x^2+y^2}$ is homogenous of order $\alpha = -1$

2.5.3 – Substitution Rule

If $M(x, y)$ and $N(x, y)$ are homogenous, each of the same order, then $u = \frac{y}{x}$ i.e., $y = ux$ or $v = \frac{x}{y}$ (i.e. $x = vy$) will produce a separable D.E.

2.5.4 – Example

Solve the separable D.E. and then back-substitute

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

$$M(x, y) = x^2 + y^2 \quad N = x^2 - xy$$

$$M_y = 2y \quad N_x = 2x - y$$

$$M_y \neq N_x$$

$$M(tx, ty) = (tx)^2 + (ty)^2$$

$$= t^2x^2 + t^2y^2$$

$$= t^2(x^2 + y^2)$$

$$= t^2M(x, y) \quad M \text{ is homogeneous of order 2 and so is } N$$

$$u = \frac{y}{x}$$

$$y = ux$$

$$dy = udx + xdu$$

$$(x^2 + (ux)^2)dx + (x^2 - x(ux))(udx + xdu) = 0$$

$$(x^2 + u^2x^2)dx + (x^2 - ux^2)(udx + xdu) = 0$$

$$(1 + u^2)x^2dx + x^2(1 - u)(udx + xdu) = 0$$

$$(1 + u^2)x^2dx + x^2(udx + xdu - u^2dx - uxdu) = 0$$

$$x^2(1dx + u^2dx + udx + xdu - u^2dx - uxdu) = 0$$

$$x^2(1dx + u^2dx - u^2dx + udx + xdu - uxdu) = 0$$

$$x^2(1dx + udx + xdu - uxdu) = 0$$

$$x^2(1 + u)dx + x^3(1 - u)du = 0$$

$$\int \frac{1}{x}dx = \int -\frac{1 - u}{1 + u}du$$

$$= \int \frac{u - 1}{u + 1}du$$

$$= \int \frac{u + (1 - 2)}{u + 1}du$$

$$= \int \left(\frac{u + 1}{u + 1} - \frac{2}{u + 1} \right) du$$

$$= \int \left(1 - \frac{2}{u + 1} \right) du$$

$$\ln|x| = \int \left(1 - \frac{2}{u + 1} \right) du$$

$$= u - 2 \ln|u + 1| + C$$

$$\ln|x| = \frac{y}{x} - 2 \ln \left| \frac{y}{x} + 1 \right| + C$$

2.5.5 – Bernoulli Equation

Theorem: An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where $n \neq 0, 1$ is called a Bernoulli Equation. The substitution

$$u = y^{1-n}$$

will transform the D.E. into a 1st order linear.

2.5.6 – Example

$$x \frac{dy}{dx} + y = x^2 y^2$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2$$

is a Bernoulli equation with $n = 2$.

$$\begin{aligned} u &= y^{1-2} \\ &= y^{-1} \\ &= \frac{1}{y} \\ \frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\ &= -1y^{-2} \frac{dy}{dx} \\ &= -\frac{1}{y^2} \frac{dy}{dx} \\ -y^{-2} \frac{dy}{dx} + -y^{-2} \times \frac{y}{x} &= -y^{-2} \times xy^2 \\ -y^{-2} \frac{dy}{dx} + -\frac{1}{x} y^{-1} &= -x \\ \frac{du}{dx} - \frac{1}{x} u &= -x \end{aligned}$$

$$\begin{aligned}
 \text{I.F.} = \mu &= e^{P(x)dx} \\
 &= e^{-\int \frac{1}{x} dx} \\
 &= e^{-\ln|x|} \\
 &= e^{\ln|x^{-1}|} \\
 &= x^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u &= -1 \\
 \frac{d}{dx} \left(\frac{1}{x} u \right) &= -1 \\
 \int \frac{d}{dx} \left(\frac{1}{x} u \right) &= \int -1 dx \\
 \frac{1}{x} u &= \int -1 dx \\
 \frac{1}{x} u &= -x + C \\
 \frac{1}{x} \times 1y &= -x + C \\
 \frac{1}{x(-x + C)} &= y \\
 y &= \frac{1}{Cx - x^2}
 \end{aligned}$$

Theorem: If the D.E. can be expressed as

$$\frac{dy}{dx} = f(Ax + by + C)$$

for particular numbers A, B, C , then let

$$u = Ax + By + C$$

to get a separable D.E.

2.5.7 – Example

$$\frac{dy}{dx} = (-2x + y)^2 - 7, y(0) = 0$$

$$u = -2x + y$$

$$\begin{aligned}\frac{du}{dx} &= \frac{dy}{dx} \times \frac{du}{dy} \\ &= -2 + \frac{dy}{dx}\end{aligned}$$

$$\frac{du}{dx} + 2 = \frac{dy}{dx}$$

$$\frac{du}{dx} + 2 = u^2 - 7$$

$$\frac{du}{dx} = u^2 - 9$$

$$\frac{du}{u^2 - 9} = dx$$

$$\int \frac{du}{u^2 - 9} = \int dx$$

$$\int \frac{du}{(u+3)(u+9)} = x + C$$

$$\int \frac{du}{(u+3)(u+9)} = x + C$$