Chapter 8

Systems of Linear First-Order Differential Equations

8.2 Solving Homogenous Systems

8.2.3 – Complex EigenValues

8.2.4 – Example

$$\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

$$\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} \\
&= (6-\lambda)(4-\lambda) - (-1)(5) \\
&= 24 - 6\lambda - 4\lambda + \lambda^2 + 5 \\
&= \lambda^2 - 10\lambda + 29 \\
&= (\lambda - 5)^2 + 4
\end{aligned}$$

$$\begin{aligned}
(\lambda - 5)^2 &= -4 \\
\lambda - 5 &= \pm 2i \\
\lambda &= 5 \pm 2i
\end{aligned}$$

For $\lambda_1 = 5 + 2i$

$$\left[\begin{array}{cc|c} 6-\lambda_1 & -1 & 0 \\ 5 & 4-\lambda_1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 6-5-2i & -1 & 0 \\ 5 & 4-5-2i & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right]$$
$$= \left(r_1 \leftarrow \frac{r_1}{1-2i}\right) \left[\begin{array}{cc|c} 1 & -\frac{1}{1-2i} & 0 \\ 5 & -1-2i & 0 \end{array}\right]$$
$$= \left(r_1 \leftarrow r_1 \times \frac{1+2i}{1+2i}\right) \left[\begin{array}{cc|c} 1 & -\frac{1}{5}-\frac{2}{5}i^{1} & 0 \\ 5 & -1-2i & 0 \end{array}\right]$$
$$= (r_2 \leftarrow r_2 - 5r_1) \left[\begin{array}{cc|c} 1 & -\frac{1}{5}-\frac{2}{5}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$k_1 + \left(-\frac{1}{5} - \frac{2}{5}i\right) k_2 = 0$$
$$\text{Let } k_2 = 1 - 2i$$
$$k_1 + \left(-\frac{1}{5} - \frac{2}{5}i\right)(1-2i) = 0$$
$$k_1 + (-1)^{2} = 0$$
$$k_1 = 1$$
$$\mathbf{k}_1 = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

This, along with an eigenvector $\mathbf{k_2}$ for $\lambda_2 = 5-2i$, can be expressed in terms of Real Matrices as

$$\begin{aligned} \mathbf{X_1} &= [\beta_1 \cos(\beta t) - \beta_2 \sin(\beta t)]\, e^{\alpha t} \\ \mathbf{X_2} &= [\beta_2 \cos(\beta t) - \beta_1 \sin(\beta t)]\, e^{\alpha t} \end{aligned} \tag{8.1}$$

where α is the real part of the eigen value, β is the coefficient of the imaginary part of the eigen value,

$\beta_1 = \text{Real Part } (k_1) = Re \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$\beta_2 = \text{Imaginary Part } (k_1) = Im \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

8.3 General Ideas

(Don't worry about detailed calculation)

[1]
$$-\frac{1}{1-2i} \times \frac{1+2i}{1+2i} = -\frac{1+2i}{(1-2i)(1+2i)} = -\frac{1+2i}{1-4i^2} = -\frac{1+2i}{1-(-4)} = -\frac{1+2i}{1+4} = -\frac{1+2i}{5}$$

[2]Essesntially undid the complex conjgate multiplication.

Non-homogenous system:

$$\mathbf{X'} = \mathbf{AX} + \mathbf{F}$$

where $\mathbf{F}$ is a non-0 column vector.

8.3.1 – 1st Solve the Complimentary DE

$$\mathbf{X'} = \mathbf{AX},$$

General solution:

$$\mathbf{X} = c_1\mathbf{X_1} + c_2\mathbf{X_2} + \cdots + c_n\mathbf{X_n}$$

8.3.2 – Next, find one particular solution

The general solution of $\mathbf{X'} = \mathbf{AX} + \mathbf{F}$ is

$$\mathbf{X} = \mathbf{X_c} + \mathbf{X_p}$$