Chapter 4

Higher Order Differential Equations

4.4 Nonhomogeneous, Linear DE with Constant Coefficients

4.4.1 – Method of Undetermined Coefficients 2

For Solving Linear, Non-homogeneous DE with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Standard Form:

$$y'' + a_1 y' + a_0 y = g(x)$$

4.4.2 – Steps

Step 1) Solve $y'' + a_1 y' + a_0 y = 0$ called the general solution y_c.

Step 2) Find one particular solution y_p of the given DE and the general solution is

$$y = y_c + y_p$$

This method can only be used when $g(x)$ is a polynomial (An exponetial (i.e. e^{kx}), sines or cosines or sums of products of these types of functions)

4.4.3 – Example

$$y'' - 3y' - 4y = 4\cos(3x)$$

1st solve:

$$\begin{aligned}
y'' - 3y' - 4y &= 0 \\
m^2 e^{mx} - 3me^{mx} - 4e^{mx} &= 0 \\
m^2 - 3m - 4 &= 0 \\
(m-4)(m+1) &= 0 \\
m - 4 = 0 &\qquad m + 1 = 0 \\
m = 4 &\qquad m = -1 \\
y_c = c_1 e^{4x} + c_2 e^{-x}
\end{aligned}$$

$$\begin{aligned}
y &= A\cos(3x) + B\sin(3x) \\
y' &= -3A\sin(3x) + 3B\cos(3x) \\
y'' &= -9A\cos(3x) - 9B\sin(3x)
\end{aligned}$$

$$\begin{aligned}
y'' - 3y' - 4y &= 4\cos(3x) \\
(-9A\cos(3x) - 9B\sin(3x)) - 3\left(-3A\sin(3x) + 3B\cos(3x)\right) - 4\left(A\cos(3x) + B\sin(3x)\right) &= 4\cos(3x) \\
-9A\cos(3x) - 9B\sin(3x) + 9A\sin(3x) - 9B\cos(3x) - 4A\cos(3x) - 4B\sin(3x) &= 4\cos(3x) \\
-9A\cos(3x) - 9B\cos(3x) - 4A\cos(3x) - 9B\sin(3x) + 9A\sin(3x) - 4B\sin(3x) &= 4\cos(3x) \\
\cos(3x)(-9A - 9B - 4A) + \sin(3x)(-9B + 9A - 4B) &= 4\cos(3x) \\
\cos(3x)(-13A - 9B) + \sin(3x)(9A - 13B) &= 4\cos(3x) \\
\begin{cases} -13A & -9B & = 4 \\ 9A & -13B & = 0 \end{cases} &\text{Solve simultaneously}
\end{aligned}$$

One way to solve Linear Systems of Equations is called Cramer's Rule.

$$\det \begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}$$

$$\begin{aligned}
A &= \frac{\begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}} \\
&= \frac{4(-13) - 0(-9)}{-13(-13) - 9(-9)} \\
&= \frac{-52 - 0}{169 + 81} \\
&= -\frac{52}{250} \\
&= -\frac{26}{125} \\
B &= \frac{\begin{bmatrix} -13 & 4 \\ 9 & 0 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}} \\
&= \frac{-13(0) - 4(9)}{250} \\
&= \frac{0 - 36}{250} \\
&= -\frac{36}{250} \\
&= -\frac{18}{125}
\end{aligned}$$

Check:

$$\begin{aligned}
(-13)\left(-\frac{26}{125}\right) + (-9)\left(-\frac{18}{125}\right) ? &= 4 \\
\frac{338}{125} + \frac{162}{125} ? &= 4 \\
\frac{500}{125} &= 4 \\
9\left(-\frac{26}{125}\right) + (-13)\left(-\frac{18}{125}\right) ? &= 0 \\
-\frac{234}{125} + \frac{234}{125} ? &= 0 \\
0 &= 0
\end{aligned}$$

So

$$y = -\frac{26}{125}\cos(3x) - \frac{18}{125}\sin(3x) + c_1 e^{4x} + c_2 e^{-x}$$

is the general solution to the given DE.

4.4.4 – Example

$$y'' - 5y' + 4y = 8e^x$$

If we try:

$$\begin{aligned} y_p &= Ae^x \\ Ae^x - 5Ae^x + 4Ae^x &= 8e^x \\ e^x(A - 5A + 4) &= 8e^x \\ A - 5A + 4 &= 8 \\ A - 5A - 4 &= 0 \\ &\text{has no solution.} \end{aligned}$$

Solve

$$y'' - 5y' + 4y = 0$$

1st

$$\begin{aligned} m^2 - 5m + 4 &= 0 \\ (m-1)(m-4) &= 0 \\ m - 1 &= 0 \qquad m - 4 = 0 \\ m &= 1 \qquad m = 4 \\ y_1 &= e^{1mx} \qquad y_2 = e^{4mx} \\ y_1 &= e^{mx} \qquad y_2 = e^{4mx} \end{aligned}$$

$$y_c = c_2e^{mx} + c_2e^{4mx} \text{ hole at } Ae^x \text{ is } c_1 = A \quad c_2 = 0$$

Suppose we have a 5th order DE with

$$a_5y^{(5)} + a_4y^{(4)} + \cdots + a_1y' + a_0y = g(x)$$

and the auxiliary equation factors as

$$m^2(m-3)(m-(2+i))(m-(2-i)) = 0$$

$$m = 0 \text{ (multiplicity 2)} \qquad m = 3 \qquad m = 2 + i \qquad m = 2 - i$$

Step 1

Write the general solution to the complimentary DE

$$\begin{aligned} y_1 &= e^{0x} = & 1 \\ y_2 &= xe^{0x} = & x \\ y_3 &= e^{3x} = & e^{3x} \\ y_4 &= e^{(2+i)x} = & e^{2x}\cos(x) \\ y_5 &= e^{(2-i)x} = & e^{2x}\sin(x) \end{aligned}$$

$$y_c = c_1 + c_2x + c_3e^{3x} + e^{2x}\cos(x) + e^{2x}\sin(x)$$

4.4.5 – What would you guess for the form of y_p?

If

(ii) $g(x) = e^{5x} \Rightarrow y_p = Ae^{5x}$

(iii) $g(x) = e^{3x} \Rightarrow y_p = Axe^{3x}$ (because e^{3x} is in y_c)

(iv) $g(x) = 5e^{2x}\sin(x) \Rightarrow y_p = \left(Ae^{2x}\cos(x) + Be^{2x}\sin(x)\right)x$

(v) $g(x) = 6x^2e^{4x} \Rightarrow y_p = \left(Ax^2 + Bx + C\right)e^{4x}$

(vi) $g(x) = x^2e^{3x} \Rightarrow y_p = \left(Ax^2 + Bx + C\right)e^{3x}x$
$\left(Ax^2 + Bx + C\right)e^{3x}$

Table 4.1: Particular Solutions for Undetermined Coefficients

g(x)		**Form of $\mathbf{y_p}$**
1.	1 (any constant)	A
2.	$5x + 7$	$Ax + B$
3.	$3x^2 - 2$	$Ax^2 + Bx + C$
4.	$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5.	$\sin 4x$	$A\cos 4x + B\sin 4x$
6.	$\cos 4x$	$A\cos 4x + B\sin 4x$
7.	e^{5x}	Ae^{5x}
8.	$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9.	x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10.	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11.	$5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12.	$xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

4.6 Methods of Variation of Parameters

This method can be used for linear, non-homogenous, DE with constant coefficients and any $g(x)$ function.

4.6.1 – Example

Suppose you have a 2nd order DE

$$y'' + P(x)y' + Q(x)y = g(x)$$

1st solve complementary DE

$$y_c = c_1y_1 + c_2y_2$$

Guess

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

for some functions $u_1(x)$ and $u_2(x)$. Plug into DE, make an additional assumption on $u_1(x)$, $u_2(x)$. Get

$$u_1(x) = \frac{W_1}{W}$$

and

$$u_2' = \frac{W_2}{W}$$