MATH 252 - Introduction to Differential Equations Notes

Len Washington III

December 7, 2023

Contents

Chapter 1

Introduction to Differential Equations

1.1 Terminology and Notation

Differential equation (D.E.) – An equation in which at least one derivative of an unknown function.
Order of the D.E. – The highest order of derivative in the D.E.

1.1.1 – Example

$$4y'' + e^x y' - 3yy' = \sin(x)$$

An example of a partial differential equation is:

$$\frac{\partial T}{\partial x} + x^2 \frac{\partial T}{\partial y} = x + y$$

however, we won't study these in this course.

1.1.2 – Linear vs Non-Linear DE's

Linear D.E. – The dependent variable and all of its derivatives in the D.E. are in separate terms to the 1^{st} power. $y^{(n)}$ or $\frac{d^n y}{dx^n}$ where $n \neq 1$ are non-first power.

$$4y'' + e^x y' - 3yy' = \sin(x)$$

is a non-linear D.E. while

$$4y'' + e^x y' - 3y = \sin(x)$$

is linear.
The general formula of a linear D.E. would look like

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = g(x)$$

Solution – a function $\phi(x)$ and an interval I for which the D.E. is satisfied when $y = \phi(x)$ for all x in I.
It may be the case that the natural domain of $\phi(x)$ is larger than I.

1.1.3 – Example

$y' = -\frac{1}{x^2}$ has a solution $\phi(x) = \frac{1}{x}$ on $I = (0, \infty)$ but the domain of $\phi(x) = (-\infty, 0) \cup (0, \infty)$.

Practice:

$$\frac{d^2x}{dt^2} + 16x = 0$$

Show (*Verify* not derive) $x(t) = c_1 \sin(4t)$ is a solution on $(-\infty, \infty)$ where c is any real parameter.

$$\begin{aligned} x &= c_1 \sin(4t) \\ \frac{dx}{dt} &= 4c_1 \cos(4t) \\ \frac{d^2x}{dt^2} &= -16c_1 \sin(4t) \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \frac{d^2x}{dt^2} + 16x \\ &= -16c_1 \sin(4t) + 16(c_1 \sin(4t)) \\ &= 0 = \text{RHS} \end{aligned}$$

But the equation $x = c_2 \cos(4t)$ would also be a solution. If you have 2 equations that are both solutions, you could add them together and you would still have a solution. $x = c_1 \sin(4t) + c_2 \cos(4t)$ is a solution for all parameters c_1 and c_2. *In fact, this is the general solution to the D.E.*

The D.E.

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

Show $y = \left(\frac{1}{4}x^2 + C\right)^2$ is a one parameter family of solutions

$$\begin{aligned} \text{LHS} = \frac{dy}{dx} &= 2\left(\frac{1}{4}x^2 + C\right) \times \frac{1}{2}x \\ &= x\left(\frac{1}{4}x^2 + C\right) \\ \text{RHS} = xy^{\frac{1}{2}} &= x\left(\left(\frac{1}{4}x^2 + C\right)^2\right)^{\frac{1}{2}} \\ &= x\left(\frac{1}{4}x^2 + C\right) \\ \text{LHS} &= \text{RHS} \end{aligned}$$

But there is another solution: namely $y(x) = 0$ for all x. This is called the "trivial solution".

1.2 Initial Value Problems (IVP)

1st order IVP is a 1st order D.E. together with one extra condition:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

2nd order IVP

$$y'' = f(x, y, y')$$

Initial conditions:

- $y(x_0) = y_0$
- $y'(x_0) = y_1$

1.2.1 – Example

$$y' = y \text{ and } y(0) = 3$$

$y = ce^x$ is a one-parameter family of solutions

$$\frac{d}{dx}(ce^x) = ce^x = y$$

$$\begin{aligned} ce^1 &= -2 \\ c &= -\frac{2}{e} \\ y &= \left(-\frac{2}{e}\right) e^x \\ y &= -2e^{x-1} \end{aligned}$$

1.2.2 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Given that you have the solution: $y = \frac{1}{x^2+C}$, Solve:

$$\begin{aligned} -1 &= \frac{1}{(0)^2 + c} \\ -1 &= \frac{1}{c} \\ -1 \times c &= 1 \\ c &= -1 \\ y &= \frac{1}{x^2 - 1}, I = (-1, 1) \end{aligned}$$

1.2.3 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Example

$$x'' + 16x = 0 \text{ and } x(\frac{\pi}{2}) = 5 \text{ and } x'(\frac{\pi}{2}) = -4$$

$$\begin{aligned}
x &= c_1 \cos(4t) + c_2 \sin(4t) \\
5 &= c_1 \cos(4t) + c_2 \sin(4t) \\
&= c_1 \cos(2\pi) + c_2 \sin(2\pi) \\
&= c_1(1) + c_2(0) \\
&= c_1 \\
x' &= -4c_1 \sin(4t) + 4c_2 \cos(4t) \\
-4 &= -4c_1 \sin\left(4\left(\frac{\pi}{2}\right)\right) + 4c_2 \cos\left(4\left(\frac{\pi}{2}\right)\right) \\
&= -4c_1 \sin(2\pi) + 4c_2 \cos(2\pi) \\
&= -4c_1(0) + 4c_2(1) \\
&= 4c_2 \\
-1 &= c_2
\end{aligned}$$

Reasonable Question: Given a 1st order IVP, can we say whether a solution *exists* or not and, if a solution exists, is it *unique*.

Theorem: Given $y' = f(x, y)$ and $y(x_0) = y_0$, if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous on a rectangle R containing (x_0, y_0) in its interior, then there exists an interval $I = (x_0 - h, x_0 + h)$ where $h > 0$ such that there exists a unique solution to IVP on I.

1.2.4 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(1) = 2$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous everywhere its defined $y \geq 0$
- $\frac{\partial f}{\partial y} = x\frac{1}{2}y^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$ is continuous everywhere its defined $y > 0$

1.2.5 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(0) = 0$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous for all x and $y \geq 0$
- $\frac{\partial f}{\partial} = \frac{x}{2y}$ is continuous for all x and $y > 0$.
- ***Theorem does not give any conclusion.***

Chapter 2

First-Order Differential Equations

2.1 Solution Curves Without a Solution

Given a 1st order D.E. $y' = f(x, y)$, y' is the slope of the tangent line at any point (x_0, y_0) on a solution curve

2.1.1 – Example

$$y' = f(x, y) = x + y$$

- $f(0, 0)$=0
- $f(1, 0)$=1

2.1.2 – Slope/Direction Fields

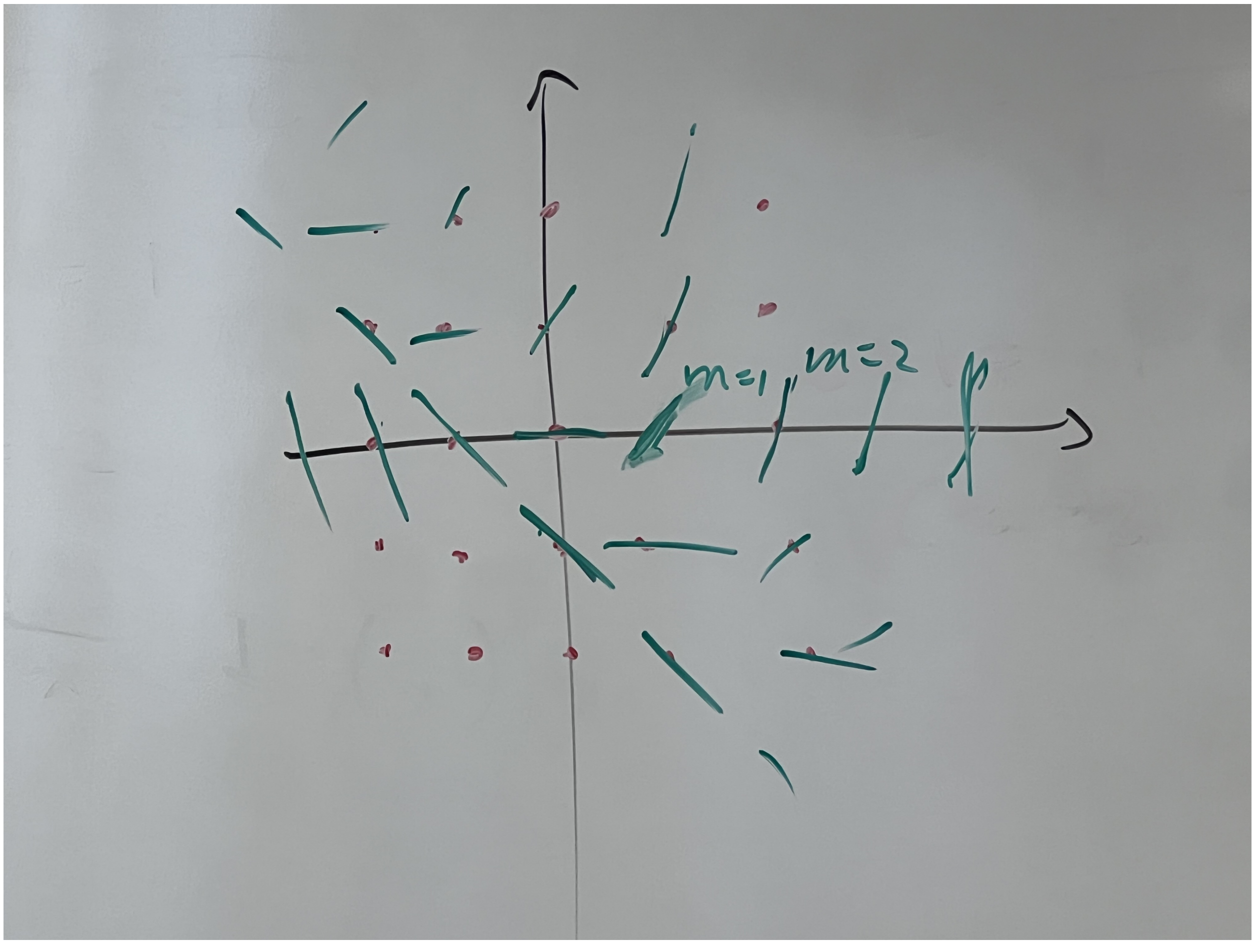

Figure 2.1: The direction field for the previous example

If the function $f(x,y)$ in the D.E. $y' = f(x,y)$ is reasonably simple so that we can solve $f(x,y) = 0$, we can make a "phase portrait diagram". We will also assume $f(x,y)$ only involves the y-variable.

2.1.3 – Example

$$y' = (y+2)(y-3)(y-5)$$
$$f(x,y) = (y+2)(y-3)(y-5)$$

An "equilibrium solution" is a solution where y is a constant. In this example: $y = 3$, $y = 5$, $y = -2$ are each constant functions.

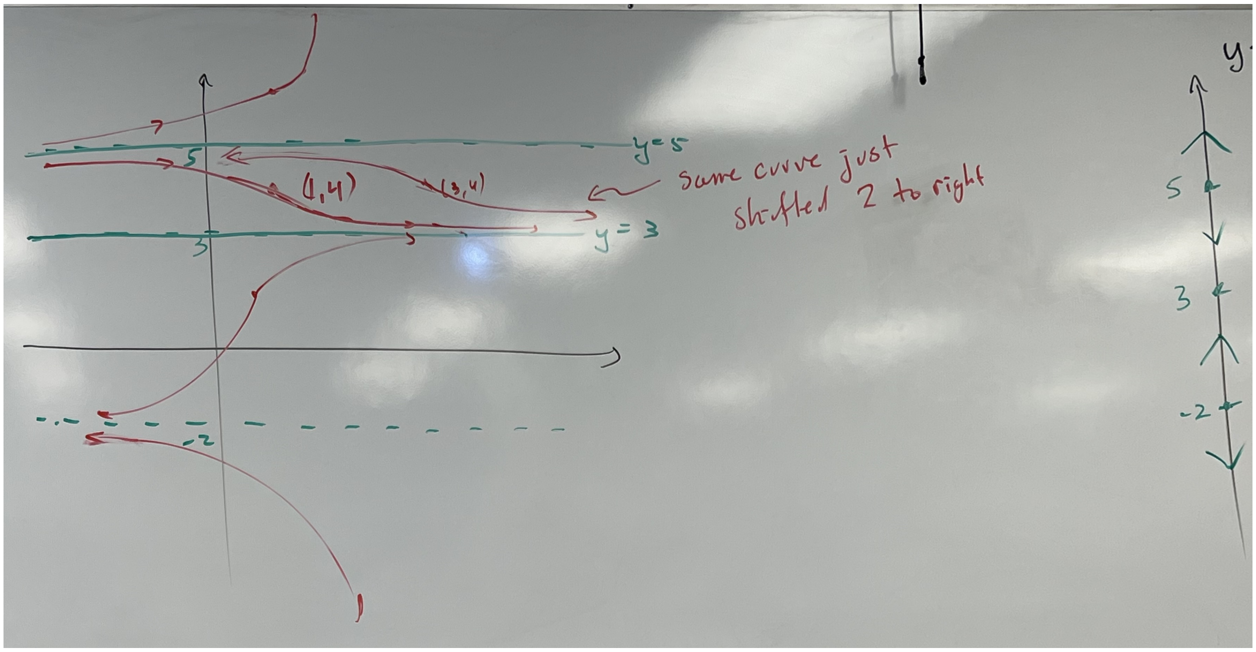

Figure 2.2: The equilibrium solution for the previous example.

The area around $y = 5$ is an unstable equilibrium since the solutions diverge and go in separate directions away from $y = 5$. The area around $y = 3$ is a stable equilibrium because the slopes above and below it converge to $y = 3$. The area around $y = -2$ is semi-stable, since all the slopes around it will converge in one direction, but the point isn't always $y = -2$.

2.2 Separable Differential Equations

Separable D.E.s are DE's $\frac{dy}{dx} = f(x, y)$ where $f(x, y)$ can be factored as $f(x, y) = g(x)h(y)$.

$$\frac{dy}{dx} = (1 + y^2)x^3 \text{ is separable}$$
$$\frac{dy}{dx} = \sin(xy) \text{ is } \textit{not} \text{ separable}$$
$$\frac{dy}{dx} = x^3y \text{ is separable}$$

$$\begin{aligned}
\frac{5}{xy}\frac{dy}{dx} &= \left(x^2 + y\right)e^y \\
\frac{dy}{dx} &= \frac{xy\left(x^2 + y\right)e^y}{5} \\
&= \frac{x\left(x^2 + y\right)}{5} \times ye^y
\end{aligned}$$

2.2.1 – Method of Solution

"Separate the variable" to get $\frac{1}{h(y)}dy = g(x)d$ or $p(y)dy = g(x)dx$ where $p(y) = \frac{1}{h(y)}$.
Integrate both sides

$$\int p(y)dy = \int g(x)dx \text{ and if possible, solve for } y$$

2.2.2 – Example

$$\begin{aligned}
\frac{dy}{dx} &= \left(1+y^2\right)x^3 \\
\int \frac{1}{1+y^2}dy &= \int x^3 dx \\
\tan^{-1}(y) + C_1 &= \frac{x^4}{4} + C_2 \\
\tan^{-1}(y) &= \frac{x^4}{4} + C_2 - C_1 \\
\tan^{-1}(y) &= \frac{x^4}{4} + C \\
y &= \tan\left(\frac{x^4}{4} + C\right)
\end{aligned}$$

2.2.3 – Example

Problem 12 from the textbook.

$$\sin(3x)dx + 2y\cos^3(3x)dy = 0$$

$$\begin{aligned}
\int -2ydy &= \int \frac{\sin(3x)}{\cos^3(x)}dx \\
&= \int \tan(3x)\sec^2(3x)dx \\
&= \int u\ \frac{1}{3}du \text{ where } u = \tan(3x),\ du = 3\sec^2(3x)dx \\
-2\int ydy &= \frac{1}{3}\int u\ du + C \\
-y^2 &= \frac{u^2}{6} + C \\
&= \frac{\tan^2(3x)}{6} + C
\end{aligned}$$

$$\frac{\tan^2(3x)}{6} + y^2 = -C$$

$$\frac{\tan^2(3x)}{6} + y^2 = C$$

Problem 25 from the textbook.

$$x^2\frac{dy}{dx} = y - xy, y(-1) = -1$$

$$\begin{aligned}
x^2\frac{dy}{dx} &= y - xy \\
x^2\frac{dy}{dx} &= y(1-x) \\
\frac{dy}{y} &= \frac{(1-x)}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{(1-x)}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{1}{x^2}dx - \int \frac{x}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{1}{x^2}dx - \int \frac{1}{x}dx \\
\ln|y| + C_1 &= -\frac{1}{x} + C_2 - \ln|x| + C_3 \\
\ln|y| &= -\frac{1}{x} - \ln|x| + C \\
y &= e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C \\
y &= e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C \\
y &= e^{-\frac{1}{x}} \times \frac{1}{|x|} \times e^C \\
y &= \frac{1}{|x|}e^{C-\frac{1}{x}} \\
-1 &= \frac{1}{|-1|}e^{C-\frac{1}{-1}} \\
-1 &= \frac{1}{1}e^{C-(-1)} \\
-1 &= e^{C+1}
\end{aligned}$$

2.3 First Order Linear Differential Equations

$$\begin{aligned}
a_1(x)\frac{dy}{dx} + a_0(x)y &= g(x) \\
\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y &= \frac{g(x)}{a_1(x)}
\end{aligned}$$

$$\left.\frac{dy}{dx} + P(x)y = f(x)\right\} \text{ Standard form of a 1st-order linear DE}$$

We will try to find a function $\mu(x)$ such that by multiplying the D.E. by an integrating factor (I.F.) $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

such that the LHS is an exact derivative, Observe:

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)\frac{dy}{dx} + \frac{dy}{dx}y$$

from which we see

$$\begin{aligned}
\mu(x)P(x) &= \frac{d\mu}{dx} \\
P(x)dx &= \frac{d\mu}{\mu(x)} \\
\int P(x)dx &= \int \frac{d\mu}{\mu} \\
\int P(x)dx &= \ln\mu \\
\ln\mu &= \int P(x)dx \\
\mu &= e^{\int P(x)dx}
\end{aligned}$$

2.3.1 – Example

$$\begin{aligned}
x\frac{dy}{dx} - 4y &= x^6e^x \\
\text{Standard form: } \frac{dy}{dx} - \frac{4}{x}y &= x^5e^x \\
P(x) &= -\frac{4}{x} \\
\mu &= e^{\int \frac{-4}{x}dx} \\
&= e^{-4\ln x} \\
&= e^{\ln x^{-4}} \\
&= x^{-4} \\
\text{I.F.} = \mu &= x^{-4}
\end{aligned}$$

Now multiply the standard form of the given D.E. by x^{-4}.

$$\begin{aligned}
x^{-4}\frac{dy}{dx} - x^{-4}\frac{4}{x}y &= x^{-4}x^5e^x \\
x^{-4}\frac{dy}{dx} - x^{-4}\frac{4}{x}y &= xe^x \\
\int \frac{d}{dx}\left(x^{-4}y\right) &= \int xe^x \\
x^{-4}y &= \int xe^x
\end{aligned}$$

2.3.2 – Example

$$\left(x^2-9\right)\frac{dy}{dx}+xy=0$$

$$\begin{aligned}
\left(x^2-9\right)\frac{dy}{dx}+xy&=0\\
\frac{dy}{dx}+\frac{x}{x^2-9}y&=0\\
P(x)&=\frac{x}{x^2-9}\\
\int P(x)dx&=\int\frac{x}{x^2-9}dx\\
\int P(x)dx&=\int\frac{1}{u-9}\frac{du}{2}\\
\int P(x)dx&=\frac{1}{2}\int\frac{1}{u-9}du\\
\int P(x)dx&=\frac{1}{2}\ln|u-9|\\
\int P(x)dx&=\frac{1}{2}\ln|x^2-9|\\
\mu&=e^{\frac{1}{2}\ln|x^2-9|}\\
\mu&=e^{\ln|\left(x^2-9\right)^{\frac{1}{2}}|}\\
\mu&=\left(x^2-9\right)^{\frac{1}{2}}\\
\mu&=\sqrt{x^2-9}\\
\sqrt{x^2-9}\left(\frac{dy}{dx}+\frac{x}{x^2-9}y\right)&=\sqrt{x^2-9}(0)\\
\sqrt{x^2-9}\frac{dy}{dx}+\frac{x}{\sqrt{x^2-9}}y&=0\\
\int\frac{d}{dx}\left(y\sqrt{x^2-9}\right)&=\int 0\\
y\sqrt{x^2-9}&=C\\
y&=\frac{C}{\sqrt{x^2-9}}
\end{aligned}$$

2.4 Exact Equations

1st Order D.E. in differential form

$$M(x,y)dx+N(x,y)dy=0$$

Given a function

$$z = f(x, y)$$

, the total differential, dz, is defined as

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

2.4.1 – Method

See if we can find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

If we can do this, then the D.E. is equivalent to

$$df = 0 \Rightarrow f(x, y) = c$$

is an implicit solution of D.E.

Assume that M and N have continuous 1st order partials (assuming f exists)

$$\left.\begin{array}{lll} My & = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}dy = & f_{xy} \\ Nx & = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}dy = & f_{yx} \end{array}\right\} \text{Theorem tells us these are equal}$$

This provides a quick test to check if the D.E. is exact or not.

2.4.2 – Example

$$2xydx + \left(x^2 - 1\right)dy = 0$$
$$M(x, y) = 2xy \qquad N(x, y) = x^2 - 1$$

To check if the D.E. is exact

$$M_y = 2x = N_x$$

We now know there exists a function $f(x, y)$ with

$$\begin{aligned} \frac{\partial f}{\partial x} = M = & \quad 2xy \\ \frac{\partial f}{\partial y} = N = & \quad x^2 - 1 \end{aligned}$$

$$\begin{aligned}
f_M(x,y) &= \int \frac{\partial f}{\partial x} dx \\
&= \int 2xy dx \\
&= x^2y + \phi(y) \\
\frac{\partial f}{\partial y}\left(x^2y+\phi(y)\right) &= x^2 - 1 \text{ required to equal } N \\
x^2 + \phi'(y) &= x^2 - 1 \\
\phi'(y) &= -1 \\
\phi(y) &= \int -1 dy \\
&= -y \\
f(x,y) &= x^2y - y \\
d\left(f(x,y)\right) &= 0 \\
f(x,y) &= c \\
x^2y - y &= c \text{ is an implicit solution of the D.E.}
\end{aligned}$$

Note: the f_M format is just there to show which partial equation was integrated. It was made by me and, as far as I know, is not standardly known.

2.4.3 – Example

$$\begin{aligned}
\left(e^{2y} - y\cos(xy)\right)dx + (2xe^{2y} - x\cos(xy) + 2y)dy = 0 \\
M_y = N_x \\
\frac{\partial}{\partial y}\left(e^{2y} - y\cos(xy)\right) = \frac{\partial}{\partial x}\left(2xe^{2y} - x\cos(xy) + 2y\right) \\
2e^{2y} - [\cos(xy) - y\sin(xy) \times x] = 2e^{2y} - (\cos(xy) - x\sin(xy) \times y) + 0 \\
2e^{2y} - \cos(xy) + xy\sin(xy) = 2e^{2y} - \cos(xy) + xy\sin(xy)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= M = e^{2y} - y\cos(xy) \\
\frac{\partial f}{\partial y} &= N = 2xe^{2y} - x\cos(xy) + 2y \\
f_N(x,y) &= \int \frac{\partial f}{\partial y} dy \\
&= \int \left(2xe^{2y} - x\cos(xy) + 2y\right) dy \\
&= \frac{2xe^{2y}}{2} - \frac{x\sin(xy)}{x} + 2 \times \frac{y^2}{2} + \phi(x) \\
&= xe^{2y} - \sin(xy) + y^2 + \phi(x)
\end{aligned}$$

Take the ∂x of this and equate with M:

$$\begin{aligned}
M &= \frac{\partial}{\partial x}\left(xe^{2y} - \sin(xy) + y^2 + \phi(x)\right) \\
e^{2y} - y\cos(xy) &= e^{2y} - y\cos(xy) + 0 + \phi'(x) \\
0 &= \phi'(x) \\
\phi(x) &= c
\end{aligned}$$

So $f(x, y) = c_2$ is the solution

$$xe^{2y} - \sin(xy) + y^2 = c$$

$$dx = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

2.4.4 – What can you do if $M_y \neq N_x$

Sometimes you can multiply the DE by an integrating factor $\mu(x, y)$ to get an exact DE.

If

$$\frac{M_y - N_x}{N}$$

is a function of only x, then

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

will be an I.F.

If

$$\frac{N_x - M_y}{M}$$

is a function of only y, then

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

will be an I.F.

2.4.5 – Example

$$xydx + \left(2x^2 + 3y^2 - 20\right) dy = 0$$

$$\begin{aligned}
M_y &= x \\
N_x &= 4x \\
M_y &\neq N_x \\
\frac{N_x - M_y}{M} &= \frac{4x - x}{xy} \\
&= \frac{3x}{xy} \\
&= \frac{3}{y} \text{ is a function of just } y
\end{aligned}$$

So:

$$\begin{aligned}
\mu &= e^{\int \frac{3}{y} dy} \\
&= e^{3 \ln y} \\
&= y^3 \\
xy^4 dx + y^3 \left(2x^2 + 3y^2 - 20\right) dy &= 0(y^3) \\
xy^4 dx + \left(2x^2 y^3 + 3y^5 - 20y^3\right) dy &= \\
M_y &= N_x \\
4xy^3 &= 4xy^3 \\
\frac{\partial f}{\partial x}
\end{aligned}$$

2.5 Substitution Methods

Taking a D.E. that's not:

- Separable
- 1st Order Linear
- Exact

and making a substitution to turn the new D.E. into one of these.

Theorem: Given a D.E.

$$M(x,y)dx + N(x,y)dy = 0$$

A function $f(x,y)$ is said to be homogenous of order α if $f(tx,ty) = t^\alpha f(x,y)$.

2.5.1 – Example

Given:

$$f(x,y) = x^3 + 5xy^2 - y^3$$

Then:

$$\begin{aligned} f(tx,ty) &= (tx)^3 + 5(tx)(ty)^2 - (ty)^3 \\ &= t^3x^3 + 5t^3xy^2 - t^3y^3 \\ &= t^3\left(x^3 + 5xy^2 - y^3\right) \\ &= t^3 f(x,y) \end{aligned}$$

2.5.2 – Example

$$\begin{aligned} f(x,y) &= \frac{x+y}{x^2+y^2} \\ f(tx,ty) &= \frac{tx+ty}{(tx)^2+(ty)^2} \\ f(tx,ty) &= \frac{tx+ty}{x^2t^2+y^2t^2} \\ f(tx,ty) &= \frac{t}{t^2} \times \frac{x+y}{x^2+y^2} \\ f(tx,ty) &= \frac{t}{t^2} f(x,y) \\ f(tx,ty) &= \frac{1}{t} f(x,y) \end{aligned}$$

$f(x,y) = \frac{x+y}{x^2+y^2}$ is homogenous of order $\alpha = -1$

2.5.3 – Substitution Rule

If $M(x,y)$ and $N(x,y)$ are homogenous, each of the same order, then $u = \frac{y}{x}$ i.e., $y = ux$ or $v = \frac{x}{y}$ (i.e. $x = vy$) will produce a separable D.E.

2.5.4 – Example

Solve the separable D.E. and then back-substitute

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

$$
\begin{aligned}
M(x,y) = x^2 + y^2 &\qquad N = x^2 - xy \\
M_y = 2y &\qquad N_x = 2x - y \\
M_y &\neq N_x \\
M(tx,ty) &= (tx)^2 + (ty)^2 \\
&= t^2x^2 + t^2y^2 \\
&= t^2(x^2+y^2) \\
&= t^2M(x,y) \ M \text{ is homogeneous of order 2 and so is } N \\
u &= \frac{y}{x} \\
y &= ux \\
dy &= udx + xdu \\
(x^2 + (ux)^2)dx + (x^2 - x(ux))(udx + xdu) &= 0 \\
(x^2 + u^2x^2)dx + (x^2 - ux^2)(udx + xdu) &= 0 \\
(1+u^2)x^2dx + x^2(1-u)(udx + xdu) &= 0 \\
(1+u^2)x^2dx + x^2\left(udx + xdu - u^2dx - uxdu\right) &= 0 \\
x^2\left(1dx + u^2dx + udx + xdu - u^2dx - uxdu\right) &= 0 \\
x^2\left(1dx + u^2dx - u^2dx + udx + xdu - uxdu\right) &= 0 \\
x^2\left(1dx + udx + xdu - uxdu\right) &= 0 \\
x^2(1+u)dx + x^3(1-u)du &= 0 \\
\int \frac{1}{x}dx &= \int -\frac{1-u}{1+u}du \\
&= \int \frac{u-1}{u+1}du \\
&= \int \frac{u+(1-2)}{u+1}du \\
&= \int \left(\frac{u+1}{u+1} - \frac{2}{u+1}\right)du \\
&= \int \left(1 - \frac{2}{u+1}\right)du \\
\ln|x| &= \int \left(1 - \frac{2}{u+1}\right)du \\
&= u - 2\ln|u+1| + C \\
\ln|x| &= \frac{y}{x} - 2\ln\left|\frac{y}{x} + 1\right| + C
\end{aligned}
$$

2.5.5 – Bernoulli Equation

Theorem: An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where $n \neq 0, 1$ is called a Bernoulli Equation. The substitution

$$u = y^{1-n}$$

will transform the D.E. into a 1st order linear.

2.5.6 – Example

$$\begin{aligned} x\frac{dy}{dx} + y &= x^2y^2 \\ \frac{dy}{dx} + \frac{y}{x} &= xy^2 \end{aligned}$$

is a Bernoulli equation with $n = 2$.

$$\begin{aligned} u &= y^{1-2} \\ &= y^{-1} \\ &= \frac{1}{y} \\ \frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\ &= -1y^{-2}\frac{dy}{dx} \\ &= -\frac{1}{y^2}\frac{dy}{dx} \\ -y^{-2}\frac{dy}{dx} + -y^{-2} \times \frac{y}{x} &= -y^{-2} \times xy^2 \\ -y^{-2}\frac{dy}{dx} + -\frac{1}{x}y^{-1} &= -x \\ \frac{du}{dx} - \frac{1}{x}u &= -x \end{aligned}$$

$$\begin{aligned}
\text{I.F.} = \mu &= e^{P(x)dx} \\
&= e^{-\int \frac{1}{x}dx} \\
&= e^{-\ln|x|} \\
&= e^{\ln|x^{-1}|} \\
&= x^{-1} \\
\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u &= -1 \\
\frac{d}{dx}\left(\frac{1}{x}u\right) &= -1 \\
\int \frac{d}{dx}\left(\frac{1}{x}u\right) &= \int -1dx \\
\frac{1}{x}u &= \int -1dx \\
\frac{1}{x}u &= -x + C \\
\frac{1}{x} \times 1y &= -x + C \\
\frac{1}{x(-x+C)} &= y \\
y &= \frac{1}{Cx - x^2}
\end{aligned}$$

Theorem: If the D.E. can be expressed as

$$\frac{dy}{dx} = f(Ax + by + C)$$

for particular numbers A, B, C, then let

$$u = Ax + By + C$$

to get a separable D.E.

2.5.7 – Example

$$\frac{dy}{dx} = (-2x + y)^2 - 7, y(0) = 0$$

$$\begin{aligned}
u &= -2x + y \\
\frac{du}{dx} &= \frac{dy}{dx} \times \frac{du}{dy} \\
&= -2 + \frac{dy}{dx} \\
\frac{du}{dx} + 2 &= \frac{dy}{dx} \\
\frac{du}{dx} + 2 &= u^2 - 7 \\
\frac{du}{dx} &= u^2 - 9 \\
\frac{du}{u^2 - 9} &= dx \\
\int \frac{du}{u^2 - 9} &= \int dx \\
\int \frac{du}{(u+3)(u+9)} &= x + C \\
\int \frac{du}{(u+3)(u+9)} &= x + C
\end{aligned}$$

Chapter 3

Modeling using DEs

3.1 Linear DE Modeling

3.1.1 – Standard Problems

1) Population Growth (or decline)

2) Radioactive Decay

3) Newton's Law of Cooling

4) Mixture Problems

3.1.2 – Population Model

Assume the rate of population change is proportional to the size of the population

$$P(t) = \text{ population at time } t$$

$$\frac{dP}{dt} = kP$$

$$\frac{\frac{dP}{dt}}{P} = k \text{ is the relative growth rate of the population}$$

$$\frac{dP}{dt} = kP$$
$$\frac{dP}{P} = kdt$$
$$\int \frac{dP}{P} = \int kdt$$
$$ln|P| = kt + C$$
$$|P| = e^{kt+C}$$
$$|P| = e^{kt}e^{C}$$
$$|P| = Ae^{kt} \text{ where } A > 0$$
$$P = \pm Ae^{kt}$$
$$P = Be^{kt} \text{ where } B \neq 0$$
$$P = De^{kt} \text{ where } D \text{ can be any real number}$$

The constant can become any number because 0 would be a valid rate of population change, it means that the population size isn't changing.

3.1.3 – Example

If, initially at 2 p.m., there are 1,000 bacteria on a petri dish and at 4 p.m., there are 2,000 bacteria. Assuming constant relative growth rate, how many bacteria are there at 5 p.m.?
$P(t)$ = population t hours after 2 p.m.

$$P(t) = Ae^{kt}$$
$$1000 = Ae^{(0)k}$$
$$1000 = Ae^{0}$$
$$1000 = A(1)$$
$$A = 1000$$

$$P(2) = 2000$$
$$P(2) = 1000e^{2k}$$
$$2000 = 1000e^{2k}$$
$$2 = e^{2k}$$
$$\ln(2) = 2k$$
$$k = \frac{\ln(2)}{2}$$

$$\begin{aligned}
P(t) &= 1000e^{\frac{\ln(2)}{2}t} \\
P(3) &= 1000e^{\frac{\ln(2)}{2}(3)} \\
&= 1000e^{1.5\ln(2)} \\
&= 1000e^{\ln(2^{1.5})} \\
&= 1000(2^{1.5}) \\
&= 2000(\sqrt{2}) \\
P(3) &\approx 2828.427(\sqrt{2}) \\
P(t) &= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\ln(2^{\frac{t}{2}})} \\
&= 1000 \times 2^{\frac{t}{2}}
\end{aligned}$$

3.1.4 – Radioactive Decay

$$m(t) = m_0e^{kt} \text{ where } k < 0$$

The Half-Life is the amount of time it takes for half of the original amount to remain:

$$\frac{1}{2}A_0 = A_0e^{kt} \Rightarrow \frac{1}{2} = e^{kt}$$

3.1.5 – Mixture Problems

Setup

Initially, the container has 200 gallons of brine solution (salt-water) of concentration $\frac{10 \text{ lbs}}{200 \text{ gallons}} = 0.05\frac{\text{lbs}}{\text{gallon}}$. A solution of $\frac{5 \text{ lbs}}{200 \text{ gallons}} = 0.025\frac{\text{lbs}}{\text{gallon}}$ is poured into the initial container at a rate of $\frac{4 \text{ gallons}}{\text{min}}$. How many pounds of salt are there in the container after 2 hours.

Let $A(t) = \#$ lbs of salt t minutes after the precess starts

$\frac{dA}{dt}$ = The rate of change of # lbs of salt

$$\begin{aligned}
\frac{dA}{dt} &= 0.025\frac{\text{lbs}}{\text{gal}} \times 4\frac{\text{gal}}{\text{min}}\Big\}\text{ rate in} \\
&\quad - \frac{A(t)\text{lbs}}{200\text{gal}} \times 4\frac{\text{gal}}{\text{min}}\Big\}\text{ rate out} \\
&= (0.025)4\frac{\text{lbs}}{\text{min}} - \frac{4A(t)}{200}\frac{\text{lbs}}{\text{min}} \\
&= 0.1\frac{\text{lbs}}{\text{min}} - \frac{A(t)}{50}\frac{\text{lbs}}{\text{min}} \\
&= 0.1 - \frac{A(t)}{50} \\
\frac{dA}{dt} + \frac{1}{50}A &= 0.1 \\
\mu &= e^{\int P(t)dt} \\
&= e^{\int \frac{1}{50}dt} \\
&= e^{\frac{t}{50}} \\
e^{\frac{t}{50}}\left(\frac{dA}{dt}\right) + e^{\frac{t}{50}}\left(\frac{1}{50}A\right) &= e^{\frac{t}{50}}(0.1) \\
\frac{d}{dt}\left(e^{\frac{t}{50}}A\right) &= e^{\frac{t}{50}}(0.1) \\
\int \frac{d}{dt}\left(e^{\frac{t}{50}}A\right) &= \int \frac{1}{10}e^{\frac{t}{50}} \\
e^{\frac{t}{50}}A &= \frac{1}{10} \times \frac{e^{\frac{t}{50}}}{\frac{1}{50}} + C \\
e^{\frac{t}{50}}A &= 5e^{\frac{t}{50}} + C \\
A(t) &= 5 + Ce^{-\frac{t}{50}} \\
&= 5 + Ce^{-0.02t} \\
A(120) &= 5 + Ce^{-0.02(120)} \\
&= 5 + Ce^{-2.4}
\end{aligned}$$

Note: This section was created after all sections through Section 7.5 were taught.

3.3 Applications: Modelling with a System of Linear DEs

$$\begin{aligned}
x_1'' + 3x_1' + x_2'' - 5x_2 &= e^t \\
x_1'' - 4x_2'' + 6x_2' &= \sin t
\end{aligned}$$

Two unknown functions $x_1(t)$ and $x_2(t)$. We learn methods to solve such systems in 4.4 and 7.6.
There is usually a sequence of isotopes the initial isotope transforms through. Suppose we have Decay Series of the form

$$X \to Y \to Z \text{ (where } X \text{ is a stable isotope)}$$

The decay rates $X \to Y$ and $Y \to Z$ can be significantly different.

$$\frac{dX}{dt} = k_1 X$$
$$\frac{dY}{dt} = k_2 Y$$

where $X(t)$ = mass of isotope X at time t and $Y(t)$ = mass of isotope Y at time t.
Assume the relative decay rate of a radioactive substance is a constant

$$\frac{\frac{dX}{dt}}{X} = k_1, \quad k_1 < 0$$
$$\frac{\frac{dY}{dt}}{Y} = k_2, \quad k_2 < 0$$

We'll define

$$\lambda_1 = -k_1 \quad so\ \lambda_1 > 0$$
$$\lambda_2 = -k_2 \quad so\ \lambda_2 > 0$$

Notation:

$$X \to^{k_1} \to Y \to^{k_2} \to Z$$

is the same as

$$X \to^{-\lambda_1} \to Y \to^{-\lambda_2} \to Z$$

3.3.1 – Example

$$\begin{aligned}
Dx + (D+2)y &= 0 & (D-3)x - 2y &= 0 \\
D(D-3)x + (D-3)(D+2)y &= 0 & D(D-3)x - D(2y) &= 0 \\
D(x'-3x) + (D-3)(y'+2y) &= 0 & D(x'-3x) - 2y' &= 0 \\
x'' - 3x' + y'' + 2y' - 3y' - 6y &= 0 & x'' - 3x' - 2y' &= 0 \\
x'' - 3x' + y'' - y' - 6y &= 0 & x'' - 3x' - 2y' &= 0
\end{aligned}$$

$$\begin{aligned}
x'' - 3x' + y'' - y' - 6y &= x'' - 3x' - 2y' \\
y'' - y' - 6y &= -2y' \\
y'' + y' - 6y &= 0
\end{aligned}$$

3.3.2 – Example

Suppose now, you start with 500 grams (.5 kg) of X and 0 grams of Y and Z. If $k_1 = -.01$ (per year) and $k_2 = -.003$ (per year). Determine how much Z there will be after $t = 1,000$ years.

$$\frac{dX}{dt} = -\lambda_2 Y \qquad \frac{dY}{dt} = \lambda_1 X - \lambda_2 Y \qquad \frac{dZ}{dt} = \lambda_2 Y$$

$$\begin{aligned} DX + \lambda_1 X = 0 \Rightarrow (D + \lambda_1)X & = 0 \\ DY - \lambda_1 X + \lambda_2 Y = 0 \Rightarrow (D + \lambda_2)Y - \lambda_1 X & = 0 \\ DZ + \lambda_2 Y = 0 \Rightarrow DZ + \lambda_2 Y & = 0 \end{aligned}$$

$\frac{dX}{dt} = -\lambda_1 X$ is separable and 1st order linear, which means $X = c_1 e^{-\lambda_1 t}$, and we know the initial value is 500g, so $X(t) = 500e^{-.01t}$.

$$\begin{aligned} \frac{dY}{dt} &= \lambda_1 X - \lambda_2 Y \\ &= \lambda_1 \times c_1 e^{-\lambda_1 t} - \lambda_2 Y \\ &= .01 \times 500e^{-.01t} - \lambda_2 Y \\ \frac{dY}{dt} + \lambda_2 Y &= 5e^{-.01t} \\ Y' + .003Y &= 5e^{-.01t} \end{aligned}$$

$$\begin{aligned} Y_c(t) &= c_2 e^{-\lambda_2 t} \\ y_p(t) &= Ae^{-0.1t} \\ y_p'(t) &= -0.1Ae^{-0.1t} \\ -0.1Ae^{-0.1t} + 0.03 \times Ae^{-0.1t} &= 5e^{-.01t} \\ -0.1A + 0.03A &= 5 \\ -0.07A &= 5 \\ A &= \frac{5}{-0.07} \end{aligned}$$

$$\begin{aligned} Y(t) &= Y_c(t) + Y_p(t) \\ &= c_2 e^{-\lambda_2 t} + Ae^{-0.1t} \\ &= 0e^{-0.03t} - \frac{5}{0.07}e^{-0.1t} \\ &= -\frac{500}{7}e^{-0.1t} \end{aligned}$$

and a similar method can be done for $Z(t)$.

3.3.3 – Other Application: Mixture Problems

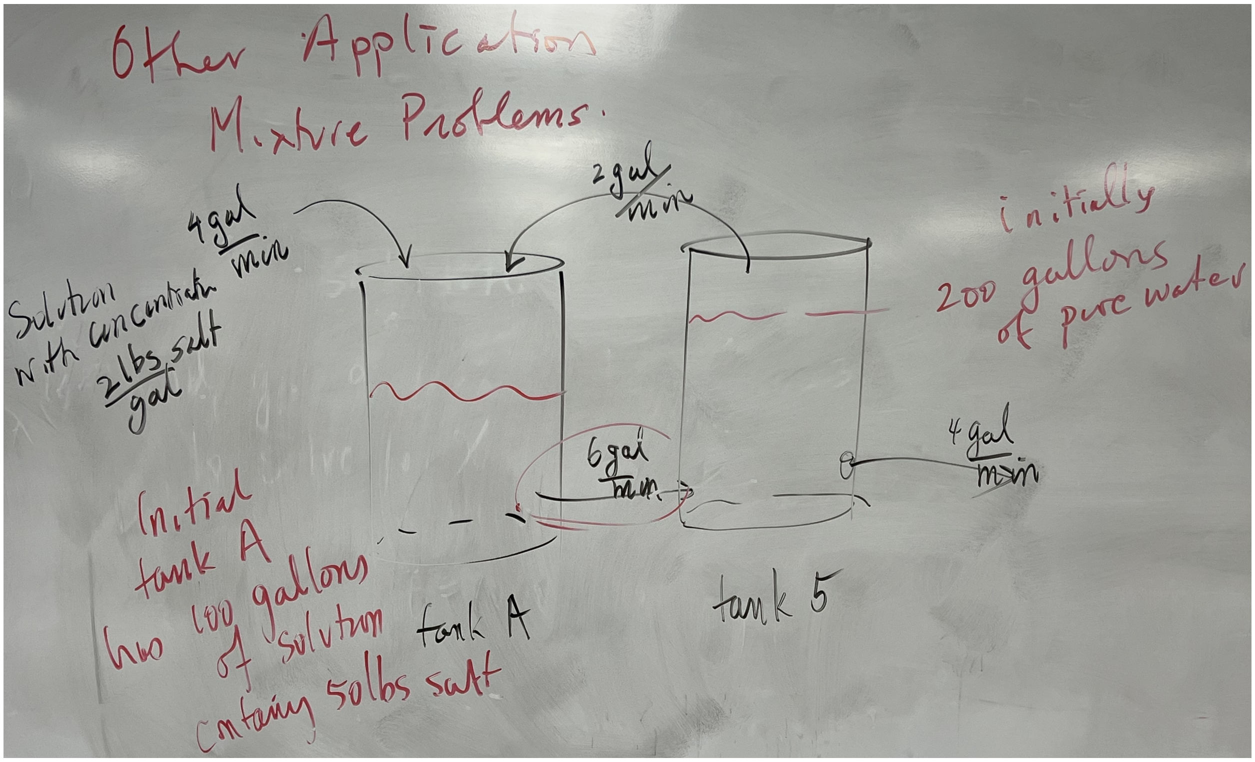

Figure 3.1

Solution with concentration 2 $\frac{\text{lbs of salt}}{\text{gal}}$. Initial tank A has 100 gallons of solution containing 50 lbs of salt.

Let $x_1(t) = \#$ of lbs of salt in tank A at time t.
Let $x_2(t) = \#$ of lbs of salt in tank B at time t.
Tank A will always have 100 gallons of solution (6 gallons in = 6 gallons out) some for tank B with $V = 200$ gal

$$\begin{aligned}\frac{dx_1}{dt} &= \text{rate salt in to tank } A - \text{rate salt } \textbf{out} \text{ of tank } B \\ &= 4\frac{gal}{min} \times 2\frac{lbs}{gal} + 2\frac{gal}{min} \times \frac{\#\text{ lbs salt in } B}{200\ gal} - \frac{6\ gal}{min} \times \frac{x_1}{100\ gal} \\ &= 8\frac{lbs}{min} + \frac{\#\text{ lbs salt in } B}{100\ min} - \frac{3x_1}{50\ min} \\ &= 8 + \frac{x_2}{100} - \frac{3x_1}{50}\end{aligned}$$

Chapter 4

Higher Order Differential Equations

4.1 Linear Equations

An nth order DE is linear if it has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0 y = g(x)$$

Theorem: If all the coefficient functions are continuous and $a_n(x)$ is not 0 on an interval I and $g(x)$ is continuous, then any initial value problem

$$DE + y(x_0) = y_0$$

has a unique solution on the interval I if $g(x) = 0$. i.e.

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

then the DE is said to be homogeneous.

4.1.1 – Example

$$y'' - 3y' - 4y = 0$$

Show $y_1 = e^{4x}$ is a solution and $y_2 = e^{-x}$ is a solution.

$$y_1 = e^{4x}$$

$$y_1' = 4e^{4x}$$

$$y_1'' = 16e^{4x}$$

$$\begin{aligned}
16e^{4x} - 3(4e^{4x}) - 4e^{4x} &= 0 \\
16e^{4x} - 12e^{4x} - 4e^{4x} &= 0 \\
e^{4x}(16 - 12 - 4) &= 0 \\
e^{4x}(0) &= 0 \\
0 &= 0
\end{aligned}$$

$$y_3 = 6y_1 = 6e^{4x}$$

$$y_3' = 6y_1' = 24e^{4x}$$

$$y_3'' = 6y_1'' = 96e^{4x}$$

$$\begin{aligned} 96e^{4x} - 3(24e^{4x}) - 4(6e^{4x}) &= 0 \\ 96e^{4x} - 72e^{4x} - 24e^{4x} &= 0 \\ e^{4x}(96 - 72 - 24) &= 0 \\ e^{4x}(0) &= 0 \\ 0 &= 0 \end{aligned}$$

Theorem: Superposition Principle: if $y_1, y_2, \ldots, y_m$ are each solutions of an nth order Linear, homongenous DE, then $c_1y_1 + c_2y_2 + \cdots + c_my_m$ will also be a solution for any constants $c_1, c_2, \ldots, c_m$.

Our goal is to express the general solution in as concise a way as possible.

Linear combination – a collection of solutions $y_1, y_2, \ldots, y_m$ is linearly independent is if the only way $c_1y_1+c_2y_2+\cdots+c_my_m = 0$ is iff (if and only if) all of the constants $c_1, c_2, \ldots, c_m = 0$. Otherwise we say $y_1, y_2, \ldots, y_m$ are linearly dependent.

Theorem: If the DE is an nth order Linear Homogeneous equation then there will exist a collection of n linearly independent solutions $y_1, y_2, \ldots, y_n$ and the general solution will be $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$

One way to check for linear independence is to compile the Wronskian

$$W(y_1, y_2, \ldots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

4.2 Reduction of Order

If you have one solution to a 2^{nd} order linear homogenous DE, then it is possible to use that function to construct a 2^{nd} Linear Independent solution to the DE.

4.2.1 – Example

For example, the DE

$$y'' - y = 0$$

One solution is $y = e^x$ on $(-\infty, \infty)$.

Idea: We look for y_2 of the form

$$y_2(x) = u(x)y_1(x) \text{ where } u(x) \text{ is not a constant}$$

The general solution is of the form:

$$y = c_1y_1 + c_2y_2$$

where y_1 and y_2 are linearly independent solutions.

To find $u(x)$, we substitute this into the DE

$$\begin{aligned}
y_2 &= u(x)y_1(x) \\
y_2' &= u(x)y_1'(x) + u'(x)y_2(x) \\
y_2'' &= u(x)y_1''(x) + u'(x)y_1(x) + u'(x)y_2'(x) + u''(x)y_1(x) \\
&= uy_1'' + 2u'y_1' + u''y_1
\end{aligned}$$

So $y'' - y = 0$ becomes

$$\begin{aligned}
uy_1'' + 2u'y_1' + u''y_1 - uy_1 &= 0 \text{ when we sub } y = y_2 \quad = u_{y1} \\
u(e^x)'' + 2u'(e^x)' + u''(e^x) - u(e^x) &= 0 \\
ue^x + 2u'e^x + u''e^x - ue^x &= 0 \\
2u'e^x + u''e^x &= 0 \\
e^x(2u' + u'') &= 0 \\
2u' + u'' &= 0
\end{aligned}$$

Let $w = u'$

$$\begin{aligned}
2w + w' &= 0 \\
2w + \frac{dw}{dx} &= 0 \\
\frac{dw}{dx} &= -2w \\
\frac{dw}{w} &= -2dx \\
\int \frac{dw}{w} &= \int -2dx \\
\ln|w| &= -2x \\
w &= e^{-2x} \\
u' &= e^{-2x} \\
\int u' &= \int e^{-2x} \\
u &= -\frac{1}{2}e^{-2x} \\
y_2 &= uy_1 \\
&= -\frac{1}{2}e^{-2x} \times e^{x} \\
&= -\frac{1}{2}e^{-x}
\end{aligned}$$

Double check that y_2 is a solution of the DE

$$\begin{aligned}
y_2 &= -\frac{1}{2}e^{-x} \\
y_2' &= \frac{1}{2}e^{-x} \\
y_2'' &= -\frac{1}{2}e^{-x} \\
y_2'' - y &= -\frac{1}{2}e^{-x} - \left(-\frac{1}{2}e^{-x}\right) \\
&= -\frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} \\
&= 0
\end{aligned}$$

In general,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

put into standard form by dividing by $a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$, the same method as in our example leads to the formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \tag{4.1}$$

4.2.2 – Example

Part 1

$$x^2y'' - 3xy' + 4y = 0$$

Verify that $y_1 = x^2$ is a solution $y_1' = 2x, y_1'' = 2$.

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ x^2(2) - 3x(2x) + 4(x^2) &= 0 \\ 2x^2 - 6x^2 + 4x^2 &= 0 \\ 6x^2 - 6x^2 &= 0 \\ 0 &= 0 \end{aligned}$$

Part 2

Find a linearly independent solution $y_2(x)$.

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ y'' - \frac{3}{x}y' + \frac{4}{x^2}y &= 0 \\ P(x) &= -\frac{3}{x} \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int \frac{3}{x}dx}}{(y_1(x))^2} dx \\ y_2 &= y_1 \int \frac{e^{3\ln|x|}}{(y_1(x))^2} dx \\ y_2 &= y_1 \int \frac{e^{\ln|x^3|}}{(y_1(x))^2} dx \\ y_2 &= x^2 \int \frac{x^3}{(x^2)^2} dx \\ y_2 &= x^2 \int \frac{x^3}{x^4} dx \\ y_2 &= x^2 \int \frac{1}{x} dx \\ y_2 &= x^2 \ln|x| + C \end{aligned}$$

Part 3: Double check that y_2 is a solution of the DE

$$\begin{aligned}
y_2 &= x^2 \ln|x| \\
y_2' &= x^2 \times \frac{1}{x} + 2x \ln|x| \\
y_2'' &= 1 + 2x\frac{1}{x} + 2\ln|x| \\
&= 1 + 2 + 2\ln|x| \\
&= 3 + 2\ln|x|
\end{aligned}$$

So the LHS DE becomes

$$\begin{aligned}
x^2\left(3 + 2\ln|x|\right) - 3x\left(x + 2x\ln|x|\right) + 4x^2\ln|x| &= 3x^2 + 2x^2\ln|x| - 3x^2 - 6x^2\ln|x| + 4x^2\ln|x| \\
&= 3x^2 - 3x^2 + 2x^2\ln|x| - 6x^2\ln|x| + 4x^2\ln|x| \\
&= 3x^2 - 3x^2 + 2x^2\ln|x| - 6x^2\ln|x| + 4x^2\ln|x| \\
&= 0 + x^2\ln|x|(2 - 6 + 4) \\
&= x^2\ln|x|(0) \\
&= 0
\end{aligned}$$

Write the general solution of the DE including the interval of the solution

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1x^2 + c_2x^2\left(\ln|x| + C\right) \\
&= c_1x^2 + c_2x^2\ln|x| + Cc_2x^2 \\
\text{just } y &= c_1x^2 + c_2x^2\ln|x| \text{ on } I = (0,\infty), y(2) = 3, y'(2) = 5
\end{aligned}$$

4.2.3 – Example

$$\begin{aligned}
3y'' + y' - 4y &= 0 \\
y &= e^{mx} \\
y' &= me^{mx} \\
y'' &= m^2e^{mx}
\end{aligned}$$

$$\begin{aligned}
3y'' + y' - 4y &= 3m^2e^{mx} + me^{mx} - 4e^{mx} \\
&= e^{mx}(3m^2 + m - 4) \\
&= e^{mx}(3m^2 + 4)(m - 1)
\end{aligned}$$

$$m = 1 \quad m = -\frac{4}{3}$$

$$y_1 = e^x, y_2 = e^{-\frac{4}{3}x}$$

4.3 Higher Order, Linear, Homogeneous DE with Constant Coefficients

4.3.1 – Example

$$3y^{(4)} - 2y''' + 7y' + 8y = 0$$

Theorems in 4.1 tell us that the general solution is of the form $y = c_1 y_1$.
Conjecture: A solution of the form $y = e^{mx} \Rightarrow y' = me^{mx}$.

4.3.2 – Example

$$\begin{aligned} 5y' - 4y &= 0 \\ y' - \frac{4}{5}y &= 0 \\ me^{mx} - \frac{4}{5}e^{mx} &= 0 \\ e^{mx}\left(m - \frac{4}{5}\right) &= 0 \\ m - \frac{4}{5} &= 0 \\ m &= \frac{4}{5} \end{aligned}$$

$y = c_1 e^{\frac{4}{5}x}$ is the general solution of the DE

4.3.3 – Example

$$\begin{aligned} y'' + 5y' - 6y &= 0 \\ y(m^2 e^{mx}) + 5(me^{mx}) - 6e^{mx} &= 0 \\ e^{mx}\left(m^2 y + 5m - 6\right) &= 0 \\ m^2 y + 5m - 6 &= 0 \\ (m+6)(m-1) &= 0 \\ m + 6 = 0 & \qquad m - 1 = 0 \\ m = -6 & \qquad m = 1 \\ y_1 = e^{-6x} & \qquad y_2 = e^{x} \end{aligned}$$

These are Linearly Independent (L.I), Therefore:

$$y = c_1 e^{-6x} + c_2 e^{x}$$

4.3.4 – Example

$$
\begin{aligned}
y'' - 6y' + 9y &= 0 \\
m^2e^{mx} - 6\left(me^{mx}\right) + 9e^{mx} &= 0 \\
m^2 - 6m + 9 &= 0 \\
(m-3)^2 &= 0 \; m = 3 \text{ is a repeated root} \\
m - 3 &= 0 \\
m &= 3 \\
y_1 = e^{3x} \qquad y_2 &= e^{3x} \text{ are linearly dependent}
\end{aligned}
$$

Use the Reduction of order function:

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \\
&= e^{3x} \int \frac{e^{-\int -6dx}}{(e^{3x})^2} dx \\
&= e^{3x} \int \frac{e^{\int 6dx}}{e^{6x}} dx \\
&= e^{3x} \int \frac{e^{6x}}{e^{6x}} dx \\
&= e^{3x} \int 1dx \\
&= e^{3x}x \\
&= xe^{3x}
\end{aligned}
$$

Always works out for this solution if e^{m_1x} is a solution and m_1 is a root of multiplicity k than $y_1 = e^{m_1x}, y_2 = xe^{m_1x}, \dots, y_k = x^{k-1}e^{mx}$ are linear solutions.

$$
\begin{aligned}
y'' + 9y &= 0 \\
m^2 + 9 &= 0 \\
m^2 &= -9 \\
m &= \sqrt{-9} \text{ No real solutions} \\
m &= \pm\sqrt{-9} \\
m &= \pm 3i \\
y &= c_1e^{3ix} + c_2e^{-3ix} \text{ where } c_1 \;\&\; c_2 \text{ arbitrary complex numbers}
\end{aligned}
$$

We'd rather only deal with real-valued solutions.

4.3.5 – Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\begin{aligned} e^{i3x} &= \cos(3x) + i\sin(3x) \\ e^{-i3x} &= \cos(-3x) + i\sin(-3x) \\ e^{-i3x} &= \cos(3x) - i\sin(3x) \\ e^{i3x} + e^{-i3x} &= \cos(3x) + i\sin(3x) + \cos(3x) - i\sin(3x) \\ e^{i3x} + e^{-i3x} &= 2\cos(3x) \\ Y_1 = \frac{1}{2}e^{i3x} + \frac{1}{2}e^{-i3x} &= \cos(3x) \\ Y_2 &= \sin(3x) \\ \frac{1}{2i}y_1 - \frac{1}{2i}y_2 &= \sin(3x) \end{aligned}$$

General solution:

$$\begin{aligned} y &= C_1Y_1 + C_2Y_2 \\ &= C_1\cos(3x) + C_2\sin(3x) \end{aligned}$$

where C_1 and C_2 are complex numbers that generate all complex-valued solutions of the DE

4.3.6 – Example

$$\begin{aligned} y'' + 25y &= 0 \\ m^2e^{mx} + 25e^{mx} &= 0 \\ m^2 + 25 &= 0 \\ m^2 &= -25 \\ m &= \pm 5i \end{aligned}$$

General solution

$$\begin{aligned} y_1 &= c_1y_1 + c_2y_2 \\ &= c_1\cos(5x) + c_2\sin(5x) \end{aligned}$$

4.3.7 – Example

$$\begin{aligned}
y'' + 2y' + 6y &= 0 \\
m^2 + 2m + 6 &= 0 \\
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-2 \pm \sqrt{2^2 - 4(1)(6)}}{2(1)} \\
&= \frac{-2 \pm \sqrt{4 - 24}}{2} \\
&= \frac{-2 \pm \sqrt{-20}}{2} \\
&= \frac{-2 \pm \sqrt{4} \times \sqrt{-5}}{2} \\
&= \frac{-2 \pm 2\sqrt{-5}}{2} \\
&= -1 \pm \sqrt{-5} \\
&= -1 \pm \sqrt{5}i \\
y_1 &= e^{(-1+\sqrt{5}i)x} \\
&= e^{-x} e^{i\sqrt{5}x} \\
&= e^{-x} \cos(\sqrt{5}x) \\
y_2 &= e^{(-1-\sqrt{5}i)x} \\
&= e^{-x} e^{-i\sqrt{5}x} \\
&= e^{-x} \sin(\sqrt{5}x)
\end{aligned}$$

So the general solution is

$$y = c_1 e^{-x} \cos(\sqrt{5}x) + c_2 e^{-x} \sin(\sqrt{5}x)$$

In general, if $m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$ are roots of the auxiliary equation, then $\begin{aligned} y_1 &= e^{\alpha x}\cos(\beta x) \\ y_2 &= e^{\alpha x}\sin(\beta x) \end{aligned}$ are solutions.

4.3.8 – Example

$$\begin{aligned}
y^{(4)} - 16y &= 0 \\
m^4 - 16 &= 0 \\
(m^2 - 4)(m^2 + 4) &= 0 \\
(m - 2)(m + 2)(m^2 + 4) &= 0 \\
&\quad m = 2 : y_1 = e^{2x} \\
&\quad m = -2 : y_1 = e^{-2x} \\
&\quad m = 2i : \cos(2x), \sin(2x)
\end{aligned}$$

4.4 Nonhomogeneous, Linear DE with Constant Coefficients

4.4.1 – Method of Undetermined Coefficients

Section 4.5 gives another approach but it is a bit more abstract

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0y = g(x) \text{ where } g(x) \neq 0$$

Theorem: If we can find any one particular solution y_p of this DE $(y_p + y_c)$, where y_c is the solution of the complementary DE (the same LHS$= 0$ instead of $g(x)$), is also a solution of the non-homogeneous DE, then the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n + y_p \end{aligned}$$

where you use Section 4.3 methods for the c_iy_i's.

4.4.2 – Example

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

Step 1: Find the General Solution y_c of the complimentary DE

$y'' + 4y' - 2y = 0$
Aux equation:

$$\begin{aligned} m^2 + 4m - 2 &= 0 \\ m^2 + 4m + 4 &= 6 \\ (m+2)^2 &= 6 \\ m + 2 &= \pm\sqrt{6} \\ m &= -2 \pm \sqrt{6} \\ y_1 &= e^{(-2+\sqrt{6})x} \\ y_2 &= e^{(-2-\sqrt{6})x} \end{aligned}$$

Step 2: Find a particular solution y_p of given DE

Educated Guess:

$$y_p = Ax^2 + Bx + C$$

for some coefficients A, B, C. For the moment, they're undetermined coefficients.

Plugging in the y_p, we get

$$\begin{aligned} y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}$$

So,

$$\begin{aligned} 2A + 4(2Ax + B) - 2\left(Ax^2 + Bx + C\right) &= 2x^2 - 3x + 6 \\ 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C &= 2x^2 - 3x + 6 \\ -2Ax^2 + 8Ax - 2Bx + 2A + 4B - 2C &= 2x^2 - 3x + 6 \\ -2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) &= 2x^2 - 3x + 6 \\ -2A &= 2 \\ 8A - 2B &= -3 \\ 2A + 4B - 2C &= 6 \\ -2A &= 2 \\ A &= -1 \\ 8(-1) - 2B &= -3 \\ -8 - 2B &= -3 \\ 8 + 2B &= 3 \\ 2B &= -5 \\ B &= -\frac{5}{2} \\ 2(-1) + 4(-\frac{5}{2}) - 2C &= 6 \\ -2 + -10 - 2C &= 6 \\ -2C &= 18 \\ C &= -9 \end{aligned}$$

Step 3: Check

$$\begin{aligned} y_p' &= 2(-1)x + \left(-\frac{5}{2}\right) \\ &= -2x - \frac{5}{2} \\ y_p'' &= 2(-1) \\ &= -2 \\ y'' + 4y' - 2y &= -2 + 4\left(-2x - \frac{5}{2}\right) - 2\left(-x^2 - \frac{5}{2}x - 9\right) \\ &= -2 - 8x - 10 + 2x^2 + 5x + 18 \\ &= 2x^2 - 8x + 5x - 10 + 18 - 2 \\ &= 2x^2 - 3x + 6 \end{aligned}$$

4.4.3 – Example

$$y'' - y' + y = 2\sin(3x)$$

Step 1: Find the General Solution y_c of the complimentary DE

Aux equation:

$$\begin{aligned}
m^2 - m + 1 &= 0 \\
m &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \\
&= \frac{1 \pm \sqrt{1-4}}{2} \\
&= \frac{1 \pm \sqrt{-3}}{2} \\
&= \frac{1 \pm \sqrt{3}i}{2} \\
m_1 &= \frac{1 + \sqrt{3}i}{2} \\
m_2 &= \frac{1 - \sqrt{3}i}{2} \\
y_1 &= e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) \\
y_2 &= e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)
\end{aligned}$$

Step 2: Guess $y_p = A\sin(3x) + B\cos(3x)$

Plug into the DE

$$\overbrace{-9A\sin(3x) - 9B\cos(3x)}^{y''} - \left(\overbrace{3A\cos(3x) - 3B\sin(3x)}^{y'}\right) + \overbrace{A\sin(3x) + B\cos(3x)}^{y} = 2\sin(3x)$$

4.4.4 – Method of Undetermined Coefficients 2

For Solving Linear, Non-homogeneous DE with constant coefficients

$$a_2y'' + a_1y' + a_0y = f(x)$$

Standard Form:

$$y'' + a_1y' + a_0y = g(x)$$

4.4.5 – Steps

Step 1) Solve $y'' + a_1 y' + a_0 y = 0$ called the general solution y_c.

Step 2) Find one particular solution y_p of the given DE and the general solution is

$$y = y_c + y_p$$

This method can only be used when $g(x)$ is a polynomial (An exponetial (i.e. e^{kx}), sines or cosines or sums of products of these types of functions)

4.4.6 – Example

$$y'' - 3y' - 4y = 4\cos(3x)$$

1st solve:

$$\begin{aligned}
y'' - 3y' - 4y &= 0 \\
m^2 e^{mx} - 3me^{mx} - 4e^{mx} &= 0 \\
m^2 - 3m - 4 &= 0 \\
(m-4)(m+1) &= 0 \\
m - 4 = 0 \qquad & m + 1 = 0 \\
m = 4 \qquad & m = -1 \\
y_c = c_1 e^{4x} + c_2 e^{-x} &
\end{aligned}$$

$$\begin{aligned}
y &= A\cos(3x) + B\sin(3x) \\
y' &= -3A\sin(3x) + 3B\cos(3x) \\
y'' &= -9A\cos(3x) - 9B\sin(3x)
\end{aligned}$$

$$\begin{aligned}
y'' - 3y' - 4y &= 4\cos(3x) \\
(-9A\cos(3x) - 9B\sin(3x)) - 3(-3A\sin(3x) + 3B\cos(3x)) - 4(A\cos(3x) + B\sin(3x)) &= 4\cos(3x) \\
-9A\cos(3x) - 9B\sin(3x) + 9A\sin(3x) - 9B\cos(3x) - 4A\cos(3x) - 4B\sin(3x) &= 4\cos(3x) \\
-9A\cos(3x) - 9B\cos(3x) - 4A\cos(3x) - 9B\sin(3x) + 9A\sin(3x) - 4B\sin(3x) &= 4\cos(3x) \\
\cos(3x)(-9A - 9B - 4A) + \sin(3x)(-9B + 9A - 4B) &= 4\cos(3x) \\
\cos(3x)(-13A - 9B) + \sin(3x)(9A - 13B) &= 4\cos(3x) \\
\begin{cases} -13A & -9B & = 4 \\ 9A & -13B & = 0 \end{cases} & \text{Solve simultaneously}
\end{aligned}$$

One way to solve Linear Systems of Equations is called Cramer's Rule.

$$\det \begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}$$

$$\begin{aligned}
A &= \frac{\begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}} \\
&= \frac{4(-13) - 0(-9)}{-13(-13) - 9(-9)} \\
&= \frac{-52 - 0}{169 + 81} \\
&= -\frac{52}{250} \\
&= -\frac{26}{125} \\
B &= \frac{\begin{bmatrix} -13 & 4 \\ 9 & 0 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}} \\
&= \frac{-13(0) - 4(9)}{250} \\
&= \frac{0 - 36}{250} \\
&= -\frac{36}{250} \\
&= -\frac{18}{125}
\end{aligned}$$

Check:

$$\begin{aligned}
(-13)\left(-\frac{26}{125}\right) + (-9)\left(-\frac{18}{125}\right) ?&= 4 \\
\frac{338}{125} + \frac{162}{125} ?&= 4 \\
\frac{500}{125} &= 4 \\
9\left(-\frac{26}{125}\right) + (-13)\left(-\frac{18}{125}\right) ?&= 0 \\
-\frac{234}{125} + \frac{234}{125} ?&= 0 \\
0 &= 0
\end{aligned}$$

So

$$y = -\frac{26}{125}\cos(3x) - \frac{18}{125}\sin(3x) + c_1 e^{4x} + c_2 e^{-x}$$

is the general solution to the given DE.

4.4.7 – Example

$$y'' - 5y' + 4y = 8e^x$$

If we try:

$$\begin{aligned} y_p &= Ae^x \\ Ae^x - 5Ae^x + 4Ae^x &= 8e^x \\ e^x(A - 5A + 4A) &= 8e^x \\ A - 5A + 4A &= 8 \\ 0A &= 8 \\ &\text{has no solution.} \end{aligned}$$

Solve

$$y'' - 5y' + 4y = 0$$

1st

$$\begin{aligned} m^2 - 5m + 4 &= 0 \\ (m-1)(m-4) &= 0 \\ m - 1 = 0 &\qquad m - 4 = 0 \\ m = 1 &\qquad m = 4 \\ y_1 = e^{1mx} &\qquad y_2 = e^{4mx} \\ y_1 = e^{mx} &\qquad y_2 = e^{4mx} \end{aligned}$$

$$y_c = c_2e^{mx} + c_2e^{4mx} \text{ hole at } Ae^x \text{ is } c_1 = A \quad c_2 = 0$$

Suppose we have a 5th order DE with

$$a_5y^{(5)} + a_4y^{(4)} + \cdots + a_1y' + a_0y = g(x)$$

and the auxiliary equation factors as

$$m^2(m-3)(m-(2+i))(m-(2-i)) = 0$$

$$m = 0 \text{ (multiplicity 2)} \qquad m = 3 \qquad m = 2 + i \qquad m = 2 - i$$

Step 1

Write the general solution to the complimentary DE

$$\begin{aligned} y_1 = e^{0x} &= & 1 \\ y_2 = xe^{0x} &= & x \\ y_3 = e^{3x} &= & e^{3x} \\ y_4 = e^{(2+i)x} &= & e^{2x}\cos(x) \\ y_5 = e^{(2-i)x} &= & e^{2x}\sin(x) \end{aligned}$$

$$y_c = c_1 + c_2x + c_3e^{3x} + e^{2x}\cos(x) + e^{2x}\sin(x)$$

4.4.8 – What would you guess for the form of y_p?

If

(ii) $g(x) = e^{5x} \Rightarrow y_p = Ae^{5x}$

(iii) $g(x) = e^{3x} \Rightarrow y_p = Axe^{3x}$ (because e^{3x} is in y_c)

(iv) $g(x) = 5e^{2x}\sin(x) \Rightarrow y_p = \left(Ae^{2x}\cos(x) + Be^{2x}\sin(x)\right)x$

(v) $g(x) = 6x^2e^{4x} \Rightarrow y_p = \left(Ax^2 + Bx + C\right)e^{4x}$

(vi) $g(x) = x^2e^{3x} \Rightarrow y_p = \left(Ax^2 + Bx + C\right)e^{3x}x$
$\left(Ax^2 + Bx + C\right)e^{3x}$

Table 4.1: Particular Solutions for Undetermined Coefficients

g(x)		**Form of $\mathbf{y_p}$**
1.	1 (any constant)	A
2.	$5x+7$	$Ax+B$
3.	$3x^2-2$	Ax^2+Bx+C
4.	x^3-x+1	Ax^3+Bx^2+Cx+E
5.	$\sin 4x$	$A\cos 4x + B\sin 4x$
6.	$\cos 4x$	$A\cos 4x + B\sin 4x$
7.	e^{5x}	Ae^{5x}
8.	$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9.	x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10.	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11.	$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x + (Ex^2+Fx+G)\sin 4x$
12.	$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$

4.6 Variation of Parameters Method

$$y'' + P(x)y' + Q(x)y = f(x)$$

will only work on problems where $P(x)$ and $Q(x)$ are constants.

4.6.1 – 1st Step: General solution of complementary DE

$$y = c_1y_1 + c_2y_2$$

Guess a solution to the non-homogeneous of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions of x.

This theory produces

$$u_1' = \frac{W_1}{W} \text{ and } u_2' = \frac{W_2}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

4.6.2 – Example

$$4y'' + 36y = \csc(3x)$$

$$\begin{aligned}
4y'' + 36y &= \csc(3x) \\
y'' + 9y &= \frac{\csc(3x)}{4} \\
m^2e^{mx} + 9e^{mx} &= 0 \\
m^2 + 9 &= 0 \\
m^2 &= -9 \\
m &= \pm\sqrt{-9} \\
&= \pm 3i
\end{aligned}$$

$$\begin{aligned}
y_1 &= e^{0x}\cos(3x) & y_2 &= e^{0x}\sin(3x) \\
y_1 &= 1\cos(3x) & y_2 &= 1\sin(3x) \\
y_1 &= \cos(3x) & y_2 &= \sin(3x)
\end{aligned}$$

$$y_c = c_1\cos(3x) + c_2\sin(3x)$$

Guess

$$y_p = u_1y_1 + u_2y_2$$

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

where

$$\begin{aligned}
W &= \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix} \\
&= (3\cos(3x))(\cos(3x)) - (\sin(3x))(-3\sin(3x)) \\
&= 3\cos^2(3x) + 3\sin^2(3x) \\
&= 3\left(\cos^2(3x) + \sin^2(3x)\right) \\
&= 3\,(1) \\
&= 3
\end{aligned}$$

$$
\begin{aligned}
W_1 &= \begin{vmatrix} 0 & \sin(3x) \\ \frac{1}{4}\csc(3x) & \cos(3x) \end{vmatrix} \\
&= 0\cos(3x) - \sin(3x)\left(\frac{\csc(3x)}{4}\right) \\
&= -\frac{\sin(3x)\csc(3x)}{4} \\
&= -\frac{1}{4} \\
W_2 &= \begin{vmatrix} \cos(3x) & 0 \\ \sin(3x) & \frac{1}{4}\csc(3x) \end{vmatrix} \\
&= \frac{1}{4}\csc(3x)\cos(3x) - 0\sin(3x) \\
&= \frac{\cos(3x)}{4\sin(3x)} \\
&= \frac{1}{4}\cot(3x)
\end{aligned}
$$

$$
\begin{aligned}
u_1' &= \frac{W_1}{W} & u_2' &= \frac{W_2}{W} \\
u_1' &= \frac{-\frac{1}{4}}{3} & u_2' &= \frac{\frac{1}{4}\cot(3x)}{3} \\
u_1' &= -\frac{1}{12} & u_2 &= \frac{1\cot(3x)}{12} \\
u_1' &= -\frac{1}{12} & u_2 &= \frac{1\cos(3x)}{12\sin(3x)} \\
u_1 &= \int -\frac{1}{12}dx & u_2 &= \int \frac{1\cos(3x)}{12\sin(3x)}dx \\
u_1 &= -\frac{x}{12} & u_2 &= \frac{1}{12}\int \frac{1}{3}\frac{dv}{v} \\
& & u_2 &= \frac{1}{36}\ln|v| \\
& & u_2 &= \frac{1}{36}\ln|\sin(3x)|
\end{aligned}
$$

$$
\begin{aligned}
y_p &= u_1y_1 + u_2y_2 \\
&= -\frac{x}{12}\cos(3x) + \frac{1}{36}\ln|\sin(3x)|\sin(3x) \\
&= -\frac{x\cos(3x)}{12} + \frac{\sin(3x)}{36}\ln|\sin(3x)|
\end{aligned}
$$

$$
\begin{aligned}
y &= y_c + y_p \\
&= c_1\cos(3x) + c_2\sin(3x) - \frac{x\cos(3x)}{12} + \frac{\sin(3x)}{36}\ln|\sin(3x)|
\end{aligned}
$$

4.6.3 – 3×3 Determinants

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ f & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Matrix of Signs

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

4.9 Systems of Higher Order Linear DEs

$$\text{Solve}\begin{cases} x' & +y' & +2y & = 0 \\ x' & -3x & -2y & = 0 \end{cases}$$

Need to find $x(t)$ and $y(t)$ that simultaneously solve these equations.

4.9.1 – Method: Systematic Elimination

Change from using prime notation to indicate derivatives to the D operator notation.

$$x' \Rightarrow Dx \quad , \quad y' + 2y \Rightarrow (D+2)y$$

$$\begin{aligned}
\left\{\begin{array}{ll} Dx + (D+2)y & = 0 \\ (D-3)x - 2y & = 0 \end{array}\right\} &= \left\{\begin{array}{ll} (D-3)\,[Dx + (D+2)y] & = 0 \\ D\,[(D-3)x - 2y] & = 0 \end{array}\right\} \\
&= \left\{\begin{array}{ll} (D^2 - 3D)x + (D^2 - D - 6)y & = 0 \\ (D^2 - 3D)x - 2Dy & = 0 \end{array}\right\} \\
&= \left\{\begin{array}{ll} x'' - 3x' + y'' - y' - 6y & = 0 \\ x'' - 3x' - 2y' & = 0 \end{array}\right\}
\end{aligned}$$

$$\begin{aligned}
x'' - 3x' + y'' - y' - 6y &= x'' - 3x' - 2y' \\
y'' - y' - 6y &= -2y' \\
y'' + y' - 6y &= 0 \\
m^2e^{mx} + me^{mx} - 6e^{mx} &= 0 \\
m^2 + m - 6 &= 0 \\
(m+3)(m-2) &= 0
\end{aligned}$$

$$\begin{array}{rl} m_1 + 3 = 0 & \quad m_2 - 2 = 0 \\ m_1 = -3 & \quad m_2 = 2 \\ y_1 = e^{-3t} & \quad y_2 = e^{2t} \end{array}$$

$$y(t) = c_1e^{-3t} + c_2e^{2t}$$

Using the same ideas, we can find $x(t)$, namely:

$$\left\{\begin{array}{ll} Dx + (D+2)y & = 0 \\ (D-3)x - 2y & = 0 \end{array}\right\} = \left\{\begin{array}{ll} (-2)\left[Dx + (D+2)y\right] & = 0 \\ (D+2)\left[(D-3)x - 2y\right] & = 0 \end{array}\right\}$$
$$= \left\{\begin{array}{ll} -2Dx - (2D+4)y & = 0 \\ (D^2 - D - 6)x - (2D+4)y & = 0 \end{array}\right\}$$
$$= \left\{\begin{array}{lll} -2x' & -2y' - 4y & = 0 \\ x'' - x' - 6x & -2y' - 4y & = 0 \end{array}\right\}$$

$$\begin{aligned} x'' - x' - 6x - 2y' - 4y &= -2x' - 2y' - 4y \\ x'' - x' - 6x &= -2x' \\ x'' + x' - 6x &= 0 \\ n^2 e^{nt} + ne^{nt} - 6e^{nt} &= 0 \\ n^2 + n - 6 &= 0 \\ (n+3)(n-2) &= 0 \end{aligned}$$

$$\begin{array}{ll} n_1 + 3 = 0 & n_2 - 2 = 0 \\ n_1 = -3 & n_2 = 2 \\ x_1 = e^{-3t} & x_2 = e^{2t} \end{array}$$

$$x(t) = c_3 e^{-3t} + c_4 e^{2t}$$

However, the 4 constants c_1, c_2, c_3, c_4 are not completely independent of each other. To see their dependency, plug $x(t)$ & $y(t)$ into each of the original equations.

$$\begin{aligned} c_3 e^{-3t} - 3c_1 e^{-3t} + c_1 e^{-3t} &= 0 \\ c_4 e^{2t} - 3c_1 e^{-3t} + c_1 e^{-3t} &= 0 \\ c_3 e^{-3t} + 2c_1 e^{2t} + c_1 e^{2t} &= 0 \\ c_4 e^{2t} + 2c_1 e^{2t} + c_1 e^{2t} &= 0 \\ &\cdots \\ -(3c_3 + c_1)e^{-3t} + (2c_4 + 4c_2)e^{2t} &= 0 \end{aligned}$$

for this to be true for all t, you need

$$\left\{\begin{array}{ll} 3c_3 + c_1 = 0 & \Rightarrow c_3 = -\frac{1}{3}c_1 \\ 2c_4 + 4c_2 = 0 & \Rightarrow c_4 = -2c_2 \end{array}\right.$$

4.9.2 – Example

$$\left\{\begin{array}{ll} x' - 4x + y'' & = t^2 \\ x' + x + y' & = 0 \end{array}\right.$$

$$\left\{\begin{array}{ll} x' - 4x + y'' & = t^2 \\ x' + x + y' & = 0 \end{array}\right\} = \left\{\begin{array}{ll} (D-4)x + D^2y & = t^2 \\ (D+1)x + Dy & = 0 \end{array}\right\}$$
$$= \left\{\begin{array}{ll} (D-4)x + D^2y & = t^2 \\ D(D+1)x + D^2y & = 0 \end{array}\right\}$$
$$= \left\{\begin{array}{ll} (D-4)x + D^2y & = t^2 \\ (D^2+D)x + D^2y & = 0 \end{array}\right\}$$

$$\begin{aligned} (D-4)x + D^2y - (D^2+D)x - D^2y &= t^2 - 0 \\ x' - 4x - x'' - x' &= t^2 - 0 \\ -4x - x'' &= t^2 \\ x'' + 4x &= -t^2 \end{aligned}$$

First Solve $x'' + 4x = 0$

Guess $x = e^{mt}$

$$\begin{aligned} m^2 + 4 &= 0 \\ m^2 &= -4 \\ m &= \pm\sqrt{-4} \\ m &= \pm 2i \\ x(t) &= c_1\cos(2t) + c_2\sin(2t) \end{aligned}$$

Next find one particular solution:

Chapter 5

Modeling with Higher-Order Differential Equations

5.1 Spring-Mass Problems

Suppose that there are no forces affecting the motion other than the gravitational force and the spring force.

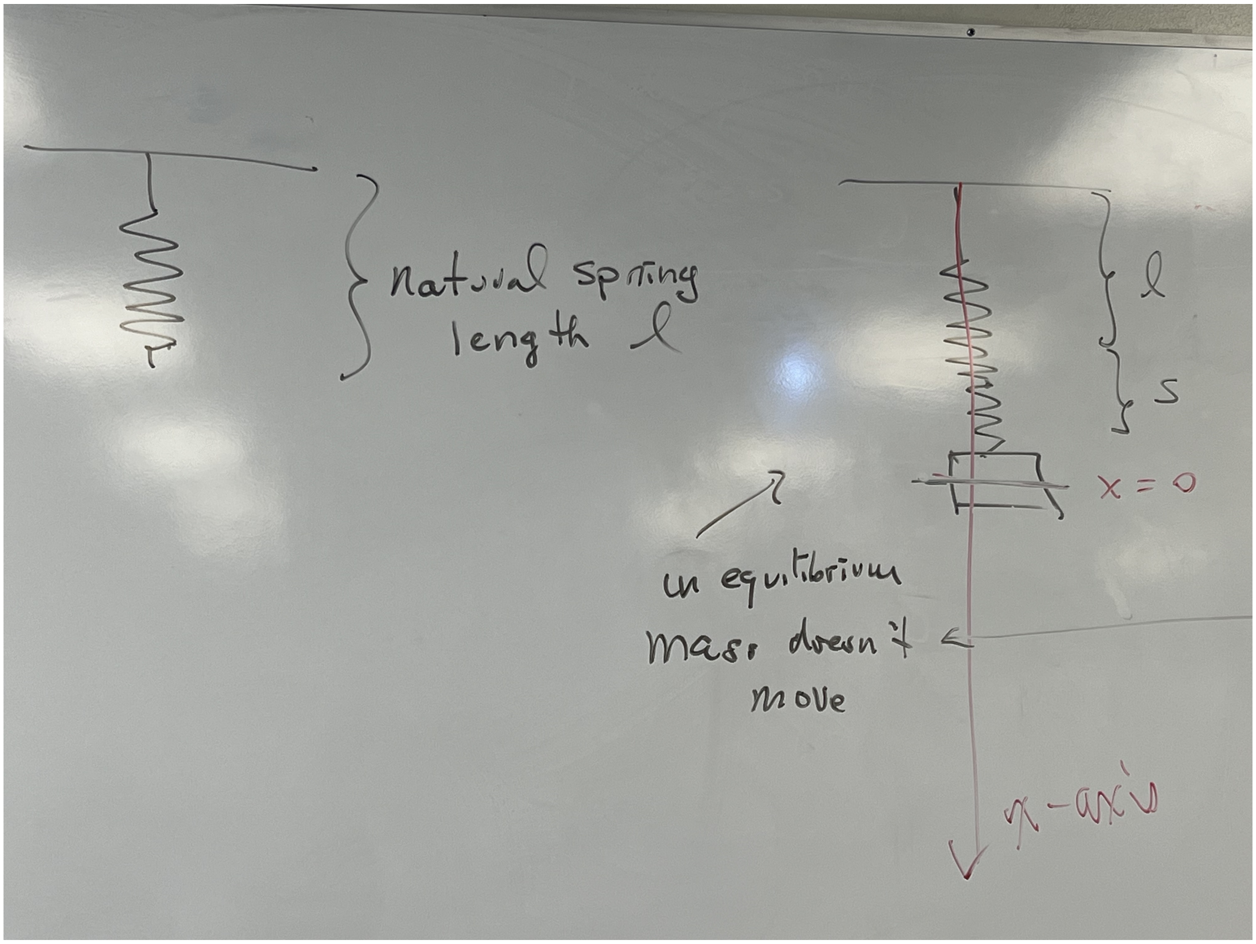

Figure 5.1: Diagram of a spring in equilibrium.

In Equilibrium, mass doesn't move.

$$F_g = -F_{spring} \rightarrow mg = -F_{spring} \rightarrow -mg = -kx\text{(Hooke's Law)} \tag{5.1}$$

where $g \approx 9.8m/s^2$ if m is in kg, else $g \approx 32.1ft/sec^2 \rightarrow 32ft/sec^2$.

In equilibrium:

$$F_{net} = mg - ks = 0$$

, so

$$k = \frac{mg}{s}$$

In general:

$$F_{net} = m \times acceleration \rightarrow m\frac{d^2x}{dt^2}$$

$$\begin{aligned}
mg + (-k)(x+s) &= m\frac{d^2x}{dt^2} \\
mg - kx - ks &= m\frac{d^2x}{dt^2} \\
(mg - ks) - kx &= m\frac{d^2x}{dt^2} \\
0 - kx &= m\frac{d^2x}{dt^2}
\end{aligned}$$

So the differential equation is:

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{5.2}$$

, a 2nd order, homogeneous DE with a constant coefficient.

5.1.1 – Example

$$\begin{aligned}
m\frac{d^2x}{dt^2} + kx &= 0 \\
\frac{d^2x}{dt^2} + \frac{k}{m}x &= 0
\end{aligned}$$

Guess:

$$x = e^{lt}$$

$$\begin{aligned}
m\frac{d^2x}{dt^2} + kx &= 0 \\
l^2e^{lt} + \frac{k}{m}e^{lt} &= 0 \\
l^2 + \frac{k}{m} &= 0 \\
l^2 &= -\frac{k}{m} \\
l &= \pm\sqrt{-\frac{k}{m}} \\
l &= 0 \pm \sqrt{\frac{k}{m}i}
\end{aligned}$$

$$\begin{aligned}
x_1 &= e^{0t}\cos\left(\frac{k}{m}t\right) & x_2 &= e^{0t}\sin\left(\frac{k}{m}t\right) \\
x_1 &= \cos\left(\frac{k}{m}t\right) & x_2 &= \sin\left(\frac{k}{m}t\right) \\
\text{Let } \omega & & &= \sqrt{\frac{k}{m}} \\
x_1 &= \cos\left(\omega^2t\right) & x_2 &= \sin\left(\omega^2t\right)
\end{aligned}$$

5.1.2 – Example

Mass weighs 2lbs, stretch spring 6 inches

$$
\begin{aligned}
F &= ma \\
2lbs &= m \times 32\frac{\text{ft}}{\text{sec}^2} \\
m &= \frac{2lbs}{32\frac{\text{ft}}{\text{sec}^2}} \\
m &= \frac{1}{16}\text{ slug} \\
F_{spring} &= kx \\
&= k(6in) \\
2lbs &= \frac{k}{2} \\
k &= 4\frac{\text{lbs}}{\text{ft}} \\
\omega &= \sqrt{\frac{4}{\frac{1}{16}}} \\
\omega &= \sqrt{64} \\
\omega &= 8
\end{aligned}
$$

5.1.3 – Undamped Motion

$$
\begin{aligned}
m\frac{d^2x}{dt^2} + kx &= 0 \\
\frac{d^2x}{dt^2} + \frac{k}{m}x &= 0 \\
\frac{d^2x}{dt^2} + \omega^2 x &= 0 \text{ where } \omega = \sqrt{\frac{k}{m}} \\
n^2e^{nt} + \omega^2 e^{nt} &= 0 \text{ where } x = e^{nt} \\
n^2 + \omega^2 &= 0 \\
n^2 &= -\omega^2 \\
n &= 0 \pm \sqrt{-\omega^2} \\
&= 0 \pm \omega i
\end{aligned}
$$

$$x_1 = \cos(\omega t) \qquad x_2 = \sin(\omega t)$$

General solution:

$$x = c_1\cos(\omega t) + c_2\sin(\omega t)$$

5.1.4 – Free, Damped Motion

Assume in addition to F_{spring} and $F_{gravity}$ that there is a force damping the motion which is directly proportional, in the opposite direction, to the mass's velocity.

$$\begin{aligned} m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0 \\ \frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \\ \frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \omega^2 x = 0 \end{aligned} \tag{5.3}$$

where β is the drag coefficient. If we substitute $2\lambda = \frac{\beta}{m} \Rightarrow \lambda = \frac{\beta}{2m}$

$$\begin{aligned} \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x &= 0 \\ n^2e^{nt} + 2\lambda ne^{nt} + \omega^2 e^{nt} &= 0 \\ n^2 + 2\lambda n + \omega^2 &= 0 \\ n^2 + 2\lambda n &= -\omega^2 \\ n^2 + 2\lambda n + \lambda^2 &= \lambda^2 - \omega^2 \\ (n+\lambda)^2 &= -\omega^2 + \lambda^2 \\ n + \lambda &= \pm\sqrt{\lambda^2 - \omega^2} \\ n &= -\lambda \pm \sqrt{\lambda^2 - \omega^2} \end{aligned}$$

Case 1: $\lambda^2 > \omega^2$

then there are two distinct real solutions where $n_1 < 0$ and $n_2 < 0$.

$$\begin{aligned} n_1 &= -\lambda + \sqrt{\lambda^2 - \omega^2} \\ n_2 &= -\lambda - \sqrt{\lambda^2 - \omega^2} \end{aligned}$$

$$\begin{aligned} x_1 &= e^{n_1 t} & x_2 &= e^{n_2 t} \\ x_1 &= e^{t\left(-\lambda + \sqrt{\lambda^2 - \omega^2}\right)} & x_2 &= e^{t\left(-\lambda - \sqrt{\lambda^2 - \omega^2}\right)} \end{aligned}$$

General solution:

$$\begin{aligned} x &= c_1 e^{t\left(-\lambda + \sqrt{\lambda^2 - \omega^2}\right)} + c_2 e^{t\left(-\lambda - \sqrt{\lambda^2 - \omega^2}\right)} \\ &= c_1 e^{-t\lambda} e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\lambda} e^{-t\sqrt{\lambda^2 - \omega^2}} \\ &= e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\sqrt{\lambda^2 - \omega^2}}\right) \end{aligned} \tag{5.4}$$

Case 2: $\lambda = \omega$

$$\begin{aligned} n &= -\lambda \pm \sqrt{\lambda^2 - \omega^2} \\ n &= -\lambda \pm \sqrt{0} \\ n &= -\lambda \end{aligned}$$

where λ has multiplicity 2.

$$x_1 = e^{-\lambda t} \qquad x_2 = te^{-\lambda t}$$

General solution:

$$\begin{aligned} x &= c_1 e^{-\lambda t} + c_2 t e^{-\lambda t} \\ &= e^{-\lambda t}\left(1 + c_2 t\right) \end{aligned} \tag{5.5}$$

Case 3: $\lambda^2 < \omega^2$

then

$$x_1 = e^{-\lambda t}\cos\left(\sqrt{\omega^2-\lambda^2}t\right) \qquad x_2 = e^{-\lambda t}\sin\left(\sqrt{\omega^2-\lambda^2}t\right)$$

$$\begin{aligned} x &= c_1 e^{-\lambda t}\cos\left(\sqrt{\omega^2-\lambda^2}t\right) + c_2 e^{-\lambda t}\sin\left(\sqrt{\omega^2-\lambda^2}t\right) \\ &= e^{-\lambda t}\left(c_1\cos\left(\sqrt{\omega^2-\lambda^2}t\right) + c_2\sin\left(\sqrt{\omega^2-\lambda^2}t\right)\right) \\ &= e^{-\lambda t}\left(A\sin\left(\sqrt{\omega^2-\lambda^2}t\right) + \phi\right) \end{aligned} \tag{5.6}$$

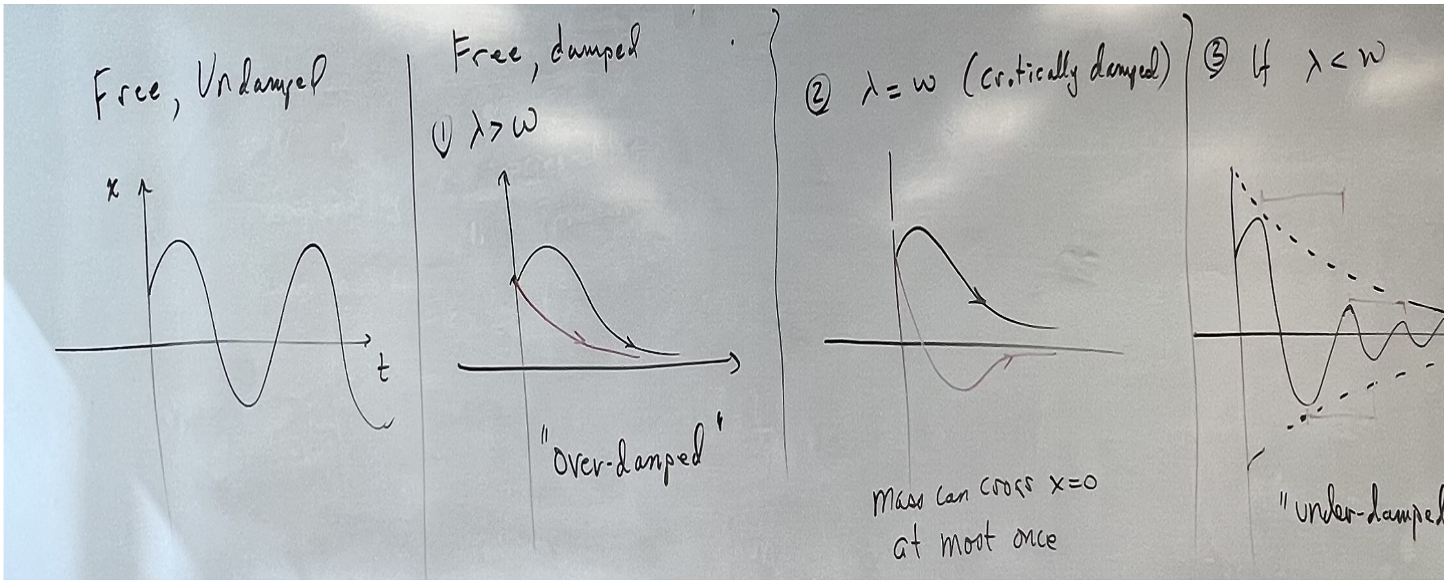

Figure 5.2: Graphs depicting each type of motion of a spring.

5.1.5 – Driven Motion (not Free Motion)

Can be both damped or undamped.

Imagine the support is oscillating up and down due to an external force.

If the motion is:

Undamped

Assume that $\gamma \neq \omega$

$$\begin{aligned}\frac{d^2x}{dt^2} + \omega^2 x &= F(t)\\ &= F_0\sin(\gamma t)\end{aligned}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\vdots$$

$$x = c_1\cos(\omega t) + c_2\sin(\omega t)$$

$$\begin{aligned}\frac{d^2x}{dt^2} + \omega^2 x &= F_0\sin(\gamma t)\\ x_p(t) &= A\cos(\gamma t) + B\sin(\gamma t)\\ -A\gamma^2\cos(\gamma t) - B\gamma^2\sin(\gamma t) + \omega^2(A\cos(\gamma t) + B\sin(\gamma t)) &= F_0\sin(\gamma t)\\ -A\gamma^2\cos(\gamma t) - B\gamma^2\sin(\gamma t) + A\omega^2\cos(\gamma t) + B\omega^2\sin(\gamma t) &= F_0\sin(\gamma t)\end{aligned}$$

$$\begin{aligned}\cos(\gamma t)\left(-A\gamma^2 + A\omega^2\right) &= 0\cos(\gamma t) & B(\omega^2-\gamma^2) &= F_0\\ -A\gamma^2 + A\omega^2 &= 0 & B &= \frac{F_0}{\omega^2-\gamma^2}\\ A(\omega^2-\gamma^2) &= 0 & B &= \frac{F_0}{\omega^2-\gamma^2}\end{aligned}$$

We assumed that $\gamma \neq \omega$, which forces $A = 0$ for the left equation to work.

$$A = 0 \qquad B = \frac{F_0}{\omega^2-\gamma^2}$$

General solution:

$$\begin{aligned}x(t) &= x_c(t) + x_p(t)\\ &= c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{F_0}{\omega^2-\gamma^2}\sin(\gamma t)\end{aligned} \tag{5.7}$$

If γ is almost equal to ω, then $\frac{F_0}{\omega^2-\gamma^2}$ is a large constant. This situation is called Resonance.

5.1.6 – Example

Driven, Damped, Spring-Mass

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos(4t), \quad x(0) = \frac{1}{2}, \quad x'(0) = 0$$

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 25\cos(4t)$$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

where

$$\lambda = 3, \omega = \sqrt{10}, F(t) = 25\cos(4t)$$

For instance, if the mass = 2kgs, then the spring constant is $k = 20$ and $\beta = 12$.

$$\sqrt{10} = \sqrt{\frac{k}{m}} \Rightarrow 10 = \frac{k}{m} \Rightarrow 10m = k \Rightarrow k = 10(2) = 20$$
$$\beta = 2\lambda m = 2(3)(2) = 6(2) = 12$$

To solve the complimentary DE

, guess $x = e^{nt}$

$$\begin{aligned} n^2e^{nt} + 6ne^{nt} + 10e^{nt} &= 0 \\ n^2 + 6n + 10 &= 0 \end{aligned}$$

$$\begin{aligned} n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36 - 40}}{2} \\ &= \frac{-6 \pm \sqrt{-4}}{2} \\ &= \frac{-6 \pm 2i}{2} \\ &= -3 \pm i \end{aligned}$$

$$x_1 = e^{-3t}\cos(t) \qquad x_2 = e^{-3t}\sin(t)$$

To find a particular solution

Guess

$$\begin{aligned} x_p &= A\cos(4t) + B\sin(4t) \\ x_p' &= -4A\sin(4t) + 4B\cos(4t) \\ x_p'' &= -16A\cos(4t) - 16B\sin(4t) \end{aligned}$$

$$\begin{aligned} -16A\cos(4t) - 16B\sin(4t) + 6\left(-4A\sin(4t) + 4B\cos(4t)\right) + 10\left(A\cos(4t) + B\sin(4t)\right) &= 25\cos(4t) \\ -16A\cos(4t) - 16B\sin(4t) - 24A\sin(4t) + 24B\cos(4t) + 10A\cos(4t) + 10B\sin(4t) &= 25\cos(4t) \\ -16A\cos(4t) + 10A\cos(4t) + 24B\cos(4t) - 16B\sin(4t) - 24A\sin(4t) + 10B\sin(4t) &= 25\cos(4t) \\ -6A\cos(4t) + 24B\cos(4t) - 6B\sin(4t) - 24A\sin(4t) &= 25\cos(4t) \\ \cos(4t)(-6A + 24B) + \sin(4t)(-6B - 24A) &= 25\cos(4t) \end{aligned}$$

Chapter 6

Series Solutions of Linear Equations

6.1 Solution by Infinite Series

2nd order linear DE with (possibly) variable coefficients

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$
$$y'' + P(x)y' + Q(x)y = F(x)$$

6.1.1 – Review of Infinite Series Facts

Maclaurin Series

$$\sum_{n=0}^{\infty} a_n x^n$$

Power series centered at 0

Taylor Series

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

Centered at a = 0

It's a theorem that power series either

(1) Converge all real numbers x on the interval $I = (-\infty, \infty)$ and the radius of convergence is $R = \infty$

(2) Converge only when $x = a$ on the interval $I = [a, a]$ and the radius of convergence is $R = 0 = \{a\}$

(3) The series converges on an interval centered at a finite, non-zero radius $R = (a - R, a + R)$

6.1.2 – Ratio Test

Use the Ratio Test to determine which of these 3 cases occurs in a specific problem.
The 3 cases of the ratio test are:

$L < 1$, the series converges

$L > 1$, the series diverges

$L = 1$, the series could converge or diverge (you have to check)

6.1.3 – Example

Determine the radius and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n(n+1)}$$

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{3^{n+1}(n+2)}\right|}{\left|\frac{x^n}{3^n(n+1)}\right|} \\
&= \lim_{n\to\infty} \frac{|x|}{3}\frac{n+1}{n+2} \\
&= \frac{|x|}{3} \lim_{n\to\infty} \frac{n+1}{n+2} \\
&= \frac{|x|}{3}(1) \\
&= \frac{|x|}{3} \\
\frac{|x|}{3} &< 1 \\
|x| &< 3 \\
I &= (-3, 3)
\end{aligned}$$

6.1.4 – Idea of Method

We will try to find a solution of the DE in the form of a power series

$$y = \sum_{n=0}^{\infty} c_n x^n \text{ (centered at 0)}$$

or

$$y = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ (centered at } a\text{)}$$

When you substitute this into the DE you get recurrence relationships for the coefficients c_0, c_1, ... Once you've found the coefficients in terms of either c_0, or c_0, c_1 where $c_0 \neq c_1$. Then you should determine where the series converges.

An example of a recurrence relation is the Fibonacci Sequence

$$F_{n+2} = F_n + F_{n+1}$$

6.1.5 – Example

Use this method to solve the DE

$$y' + y = 0$$

Assume

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

then

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$\sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

Shift the index of the first summation such that both have terms x^n. To do this, we'll make the substitution $k = n - 1 \Rightarrow n = k + 1$

$$\sum_{k+1=1}^{\infty} c_{k+1}(k+1)x^{(k+1)-1} + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)x^{k} + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=0}^{\infty} \left[c_{k+1}(k+1)x^{k} + c_k x^k\right] = 0$$

$$\sum_{k=0}^{\infty} \left[(k+1)c_{k+1} + c_k\right] x^k = 0$$

This implies $(k+1)c_{k+1} + c_k = 0$ for all $k = 0, 1, 2, \dots$ [1] $\Rightarrow c_{k+1} = \frac{-c_k}{k+1}$ for all $k = 0, 1, 2, \dots$.

[1] since the only power series that equals 0 is $\sum_{k=0}^{\infty} 0x^k$

$$\begin{aligned}
c_0 &= c_0 \\
c_1 &= -\frac{c_0}{1} \\
&= -c_0 \\
c_2 &= -\frac{c_1}{2} \\
&= -\frac{-c_0}{2} \\
&= \frac{-1^2 c_0}{2} \\
&= \frac{c_0}{2} \\
c_3 &= -\frac{c_2}{3} \\
&= -\frac{\frac{c_0}{2}}{3} \\
&= \frac{-c_0}{6}
\end{aligned}$$

Conjecture: It is apparent that

$$c_n + \frac{(-1)^n c_0}{n!}$$

Plugging into the DE

$$\begin{aligned}
y = \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{n!} x^n \\
&= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n
\end{aligned}$$

By the Ratio Test

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{(n+1)!}x^{n+1}\right|}{\left|\frac{(-1)^n}{n!}x^n\right|} \\
&= \lim_{n\to\infty} \left|(-1)\frac{x\ n!}{(n+1)!}\right| \\
&= \lim_{n\to\infty} \frac{|x|\ n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{|x|}{n+1} \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} \\
&= |x| \times 0 \\
&= 0 \text{ The series converges everywhere}
\end{aligned}
$$

6.1.6 – Power Series of Basic Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Our answer in the DE is

$$y = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}$$

6.2 Second Order, Linear Homogenous DE

$$
\begin{aligned}
a_2(x)y'' + a_1(x)y' + a_0(x)y &= 0 \\
y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y &= 0 \\
y'' + P(x)y' + Q(x)y &= 0
\end{aligned}
$$

x's for which $a_2(x) \neq 0$ will be called ordinary points. x's for which $a_2(x) = 0$ will be called singular points.

Existence of Power Series **Theorem:** If x_0 is an ordinary point of the DE, then there exists two, linearly independent solution y_1, y_2 which are both in the form of power series

$$\sum_{n=0}^{\infty} c_n(x - x_0)^n$$

and these series will have radius of convergence of at least the distance from x_0 to the singular point of the DE.

6.2.1 – Example

Consider

$$(x^2 + 2x + 5)y'' + xy' - 6y = 0$$

(i) What are the singular points of the DE?

$$\begin{aligned} x^2 + 2x + 5 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm \sqrt{16} \times \sqrt{-1}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

So $-1 + 2i$ and $-1 - 2i$ are the only singular points.

(ii) Is there a power series solution centered at $x_0 = 0$? Yes, since $x_0 = 0$ is an ordinary point, you can find

$$y_1 = \sum_{n=0}^{\infty} c_n x^n$$

and

$$y_2 = \sum_{n=0}^{\infty} d_n x^n,$$

two linearly independent solutions.

(iii) What is the minimum the radius could be for these series? As stated in the theorem, the radius is at minimum the distance from x_0 to the singular point. If you have complex singular points, calculate the distance using the complex plane graph. $\sqrt{(-1-0)^2 + (2-0)^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$.

- How about if we want series

$$\sum_{n=0}^{\infty} c_n (x-3)^2$$

$$\sqrt{(-1-3)^2 + (-2-0)^2} = \sqrt{(-4)^2 + (-2)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

6.2.2 – Example

Use Power Series centered at 0 (Maclaurin Series) to solve the DE:

$$y'' - xy = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} c_n x^n &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= \sum_{k+2=2}^{\infty} c_{k+2}(k+2)(k+2-1)x^k - \sum_{k-1=0}^{\infty} c_{k-1}x^k \\
&= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\
&= \sum_{k=0}^{1} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\
&= c_{0+2}(0+2)(0+1)x^0 + \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1)x^k - c_{k-1}x^k\right] \\
&= c_2(2)(1)(1) + \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] \\
&= 2c_2 + \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] \\
&= 2(0) + \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] \\
&= 0 + \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] \\
&= \sum_{k=1}^{\infty} x^k \left[c_{k+2}(k+2)(k+1) - c_{k-1}\right] = 0
\end{aligned}$$

$$\begin{aligned}
c_{k+2}(k+2)(k+1) - c_{k-1} &= 0 \\
c_{k+2}(k+2)(k+1) &= c_{k-1} \\
c_{k+2} &= \frac{c_{k-1}}{(k+2)(k+1)}
\end{aligned}$$

$$\begin{aligned}
c_0 &= \text{arbitrary} \\
c_1 &= \text{arbitrary} \\
c_2 &= 0 \\
c_3 &= \frac{c_0}{(1+2)(1+1)} = \frac{c_0}{3 \times 2} \\
c_4 &= \frac{c_1}{(2+2)(2+1)} = \frac{c_1}{4 \times 3} \\
c_5 &= \frac{c_2}{(3+2)(3+1)} = \frac{0}{5 \times 4} = 0 \\
c_6 &= \frac{c_3}{(4+2)(4+1)} = \frac{c_0}{3 \times 2} \times \frac{1}{6 \times 5} = \frac{c_0}{6 \times 5 \times 3 \times 2} \\
c_7 &= \frac{c_4}{(5+2)(5+1)} = \frac{c_1}{4 \times 3} \times \frac{1}{7 \times 6} = \frac{c_1}{7 \times 6 \times 4 \times 3} \\
c_8 &= \frac{c_5}{(6+2)(6+1)} = \frac{0}{8 \times 7} = 0
\end{aligned}$$

$$\begin{aligned}
y &= c_0 y_1 + c_1 y_2 \\
&= c_0 \left(1 + \frac{1}{3 \times 2} x^3 + \frac{1}{6 \times 5 \times 3 \times 2} x^6 + \dots\right) + c_1 \left(x + \frac{1}{4 \times 3} x^4 + \frac{1}{7 \times 6 \times 4 \times 3} + \dots\right) \\
&= c_0 \left(1 + \frac{1}{3 \times 2} x^3 + \frac{4}{6 \times 5 \times 4 \times 3 \times 2} x^6 + \dots\right) + c_1 \left(x + \frac{2}{4 \times 3 \times 2} x^4 + \frac{2(5)}{7 \times 6 \times 5 \times 4 \times 3 \times 2} + \dots\right)
\end{aligned}$$

6.2.3 – Example

$$(x^2+1)y'' + xy' - y = 0$$

$$\begin{aligned}
(x^2+1)y'' + xy' - y &= 0 \\
y'' + \frac{x}{x^2+1} y' - \frac{1}{x^2+1} y &= 0
\end{aligned}$$

Ordinary points:

$$\begin{aligned}
x^2 + 1 &= 0 \\
x^2 &= -1 \\
x &= \pm\sqrt{-1} \\
x &= \pm i
\end{aligned}$$

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
y' &= \sum_{n=1}^{\infty} c_n n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}
\end{aligned}$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \frac{x}{x^2+1}\sum_{n=1}^{\infty} c_n n x^{n-1} - \frac{1}{x^2+1}\sum_{n=0}^{\infty} c_n x^n = 0$$

Chapter 7

Method of Laplace Transforms for Solving DE's

Chapter Goals

- Given a DE, Perform a Calculus-based rule for finding the laplace transformation of DE.
- Solve this new equation algebraically.
- Find the inverse-Laplace transformation to get our solution to the IVP.

7.1 Definition of Laplace Transform

Given a function $f(t)$, the Laplace Transform of $f(t)$ is

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

7.1.1 – Laplace Transformations of basic Functions

(1)

$$\begin{aligned}
\mathscr{L}\{1\} &= \int_0^\infty e^{-st}(1)dt \\
&= \int_0^\infty e^{-st}dt \\
&= \lim_{b\to\infty}\int_0^b e^{-st}dt \\
&= \lim_{b\to\infty}\left.\frac{e^{-st}}{-s}\right|_0^b \\
&= \lim_{b\to\infty}\frac{e^{-s(b)}}{-s} - \frac{e^{-s(0)}}{-s} \\
&= -\frac{e^{-s(0)}}{-s} + \lim_{b\to\infty}\frac{e^{-s(b)}}{-s} \\
&= -\frac{e^0}{-s} + \frac{1}{-s}\lim_{b\to\infty} e^{-s}e^b \\
&= \frac{1}{s} - \frac{e^{-s}}{s}\lim_{b\to\infty} e^b \\
&= \frac{1}{s} \text{ for } s > 0
\end{aligned}$$

(2)

$$\begin{aligned}
\mathscr{L}\{k\} &= \int_0^\infty e^{-st}kdt \\
&= k\int_0^\infty e^{-st}dt \\
&= k\frac{1}{s} \\
&= \frac{k}{s}
\end{aligned}$$

The Laplace Transform is a *linear operator*, in other words,

(3)

$$\mathscr{L}\{f(t)+g(t)\} = \mathscr{L}\{f(t)\} + \mathscr{L}\{g(t)\}$$

(4)

$$\mathscr{L}\{kf(t)\} = k\mathscr{L}\{f(t)\}$$

(5)

$$
\begin{aligned}
\mathscr{L}\{e^{2t}\} &= \int_0^\infty e^{-st} \times e^{2t} dt \\
&= \int_0^\infty e^{(2-s)t} dt \\
&= \int_0^\infty e^{-(s-2)t} dt \\
u &= -(s-2)t \\
du &= -(s-2)dt \\
&= \int_0^\infty e^u \frac{du}{-(s-2)} \\
&= \frac{1}{-(s-2)} \int_0^\infty e^u du \\
&= \frac{1}{-(s-2)} e^u \Big|_0^\infty \\
&= \frac{1}{-(s-2)} e^{-(s-2)t} \Big|_0^\infty \\
&= \lim_{b\to\infty} \frac{1}{-(s-2)} e^{-(s-2)b} - \frac{1}{-(s-2)} e^{-(s-2)0} \\
&= -\frac{1}{-(s-2)} e^0 + \lim_{b\to\infty} \frac{1}{-(s-2)} e^{-(s-2)b} \\
&= -\frac{1}{-(s-2)} (1) \\
&= \frac{1}{s-2}
\end{aligned}
$$

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a} \text{ for } s > a$$

(6)

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

(7)

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2}$$

(8)

$$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}$$

Table 7.1: Transforms of Some Basic Functions

$$\mathscr{L}\{1\} = \frac{1}{s} \tag{7.1}$$
$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \tag{7.2}$$
$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \tag{7.3}$$
$$\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2} \tag{7.4}$$
$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2} \tag{7.5}$$
$$\mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2} \tag{7.6}$$
$$\mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2} \tag{7.7}$$

7.2 Solving I.V.T by using Laplace Transform

Take $\mathscr{L}$ of both sides of the DE

7.2.1 – Example

$$y'' - 3y' + 2y = e^{-4t},\ y(0) = 1,\ y'(0) = 5$$
$$\mathscr{L}\{y''\} - \mathscr{L}\{3y'\} + \mathscr{L}\{2y\} = \mathscr{L}\{e^{-4t}\}$$

We need more formulas first.

$$\begin{aligned} u &= e^{-st} & dv &= f'(x)dt \\ du &= -se^{-st}dt & v &= f(x) \end{aligned}$$

$$\begin{aligned} \mathscr{L}\{f'(t)\} &= \int_0^\infty e^{st} f'(t)dt \\ &= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty -sf(t)e^{st}dt \\ &= -f(t)e^{-st} + s\int_0^\infty f(t)e^{st}dt \\ &= -f(t)e^{-st} + s\mathscr{L}\{f(t)\} \\ &= -f(t)e^{-st} + sF(s) \\ &= -f(0)e^{-s(0)} + sF(s) \\ &= -f(0)(1) + sF(s) \\ &= -f(0) + sF(s) \end{aligned}$$

$$\begin{aligned}
\mathscr{L}\{f''(t)\} &= \mathscr{L}\{(f'(t))'\} \\
&= s\mathscr{L}\{f'(t)\} - f'(0) \\
&= s\left(-f(0) + sF(s)\right) - f'(0) \\
&= -sf(0) + s^2F(s) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0) \\
&= s^2\mathscr{L}\{f\} - sf(0) - f'(0)
\end{aligned}$$

In general

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)\cdots - s^{n-n}f^{n-1}(0) \tag{7.8}$$

or

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)\cdots - f^{n-1}(0) \tag{7.9}$$

So the DE transforms to

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) - 3\left(sY(s) - y(0)\right) + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - s(1) - 5 - 3\left(sY(s) - 1\right) + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - s - 5 - 3sY(s) + 3 + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - 3sY(s) + 2Y(s) - s - 5 + 3 &= \frac{1}{s+4} \\
Y(s)\left(s^2 - 3s + 2\right) - s - 2 &= \frac{1}{s+4} \\
Y(s)\left(s^2 - 3s + 2\right) &= \frac{1}{s+4} + s + 2 \\
Y(s) &= \frac{\frac{1}{s+4} + s + 2}{(s^2 - 3s + 2)} \\
&= \frac{1 + (s+2)(s+4)}{(s+4)\left(s^2 - 3s + 2\right)} \\
&= \frac{1 + s^2 + 6s + 8}{(s+4)(s-1)(s-2)} \\
&= \frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)}
\end{aligned}$$

$$\begin{aligned}
\frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)} &= \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2} \\
&= \frac{A(s-1)(s-2)}{(s+4)(s-1)(s-2)} + \frac{B(s+4)(s-2)}{(s+4)(s-1)(s-2)} + \frac{C(s+4)(s-1)}{(s+4)(s-1)(s-2)} \\
s^2 + 6s + 9 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1) \\
(s+3)^2 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1)
\end{aligned}$$

$$\begin{aligned}
s^2+6s+9 &= A(s-1)(s-2)+B(s+4)(s-2)+C(s+4)(s-1)\\
(-4)^2+6(-4)+9 &= A(-4-1)(-4-2)+B(-4+4)(-4-2)+C(-4+4)(-4-1)\\
16-24+9 &= A(-5)(-6)+B(0)(-6)+C(0)(-5)\\
1 &= 30A\\
A &= \frac{1}{30}\\
(s+3)^2 &= A(s-1)(s-2)+B(s+4)(s-2)+C(s+4)(s-1)\\
(1+3)^2 &= A(1-1)(1-2)+B(1+4)(1-2)+C(1+4)(1-1)\\
4^2 &= A(0)(-1)+B(5)(-1)+C(5)(0)\\
16 &= -5B\\
B &= \frac{-16}{5}\\
(s+3)^2 &= A(s-1)(s-2)+B(s+4)(s-2)+C(s+4)(s-1)\\
(2+3)^2 &= A(2-1)(2-2)+B(2+4)(2-2)+C(2+4)(2-1)\\
5^2 &= A(1)(0)+B(6)(0)+C(6)(1)\\
25 &= 6C\\
C &= \frac{6}{25}
\end{aligned}$$

Note:

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \rightarrow \mathscr{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$$

$$\begin{aligned}
Y(s) &= \frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\\
&= \frac{\frac{1}{30}}{s+4}+\frac{\frac{-16}{5}}{s-1}+\frac{\frac{6}{25}}{s-2}\\
y(t) &= \mathscr{L}^{-1}\{Y(s)\}\\
&= \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}-\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{6}{25}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}\\
&= \frac{1}{30}e^{-4t}-\frac{16}{5}e^{t}+\frac{6}{25}e^{2t}
\end{aligned}$$

7.2.2 – Finding Inverse-Laplace Transform

7.2.3 – Example

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{s^4}\right\} &= \frac{1}{3!}\mathscr{L}\left\{\frac{3!}{s^4}\right\}\\ &= \frac{1}{3!}\mathscr{L}\left\{\frac{3!}{s^{3+1}}\right\}\\ &= \frac{1}{3!}t^3\\ &= \frac{1}{6}t^3\end{aligned}$$

7.2.4 – Example

$$\begin{aligned}\mathscr{L}\left\{\frac{5}{s^2+49}\right\} &= \frac{5}{7}\mathscr{L}\left\{\frac{7}{s^2+49}\right\}\\ &= \frac{5}{7}\sin(7t)\end{aligned}$$

7.2.5 – Example

$$\begin{aligned}\mathscr{L}\left\{\frac{(s+1)^3}{s^4}\right\} &= \mathscr{L}\left\{\frac{s^3+3s^2+3s+1}{s^4}\right\}\\ &= \mathscr{L}\left\{\frac{s^3}{s^4}\right\}+\mathscr{L}\left\{\frac{3s^2}{s^4}\right\}+\mathscr{L}\left\{\frac{3s}{s^4}\right\}+\mathscr{L}\left\{\frac{1}{s^4}\right\}\\ &= \mathscr{L}\left\{\frac{1}{s}\right\}+\mathscr{L}\left\{\frac{3}{s^2}\right\}+\mathscr{L}\left\{\frac{3}{s^3}\right\}+\mathscr{L}\left\{\frac{1}{s^4}\right\}\\ &= \mathscr{L}\left\{\frac{1}{s}\right\}+3\mathscr{L}\left\{\frac{1}{s^2}\right\}+3\mathscr{L}\left\{\frac{1}{s^3}\right\}+\mathscr{L}\left\{\frac{1}{s^4}\right\}\\ &= \mathscr{L}\left\{\frac{1}{s}\right\}+3\mathscr{L}\left\{\frac{1}{s^2}\right\}+\frac{3}{2!}\mathscr{L}\left\{\frac{2!}{s^3}\right\}+\frac{1}{3!}\mathscr{L}\left\{\frac{3!}{s^4}\right\}\\ &= 1+3t+\frac{3}{2!}t^2+\frac{1}{3!}t^3\\ &= 1+3t+\frac{3}{2}t^2+\frac{1}{6}t^3\end{aligned}$$

7.3 Operational Rules

7.3.1 – Operational Rules Part 1

Even with the reuses we know, a problem like

$$\mathscr{L}\left\{e^{4t}t^3\right\}$$

would require us to go to the definition until we learn some rules.

$$\begin{aligned}\mathscr{L}\left\{e^{4t}t^3\right\} &= \int_0^\infty e^{-st}e^{4t}t^3 dt \\ &= \int_0^\infty e^{-(s-4)t}t^3 dt\end{aligned}$$

Compare with

$$\mathscr{L}\left\{t^3\right\} = \int_0^\infty e^{-st}t^3 dt = F(s)$$

where $f(t) = t^3$

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a) \tag{7.10}$$

where $F(s) = \mathscr{L}\left\{f(t)\right\}$

$$\mathscr{L}\left\{e^{4t}t^3\right\} = \frac{3!}{(s-4)^4}$$

7.3.2 – Example

$$\begin{aligned}\mathscr{L}\left\{e^{-3t}\sin(5t)\right\} &= \mathscr{L}\left\{\sin(5t)\right\}\Big|_{s\to s+3} \\ &= \frac{5}{s^2+25}\Big|_{s\to s+3} \\ &= \frac{5}{(s+3)^2+25}\Big|_{s\to s+3} \\ &= \frac{5}{s^2+6s+9+25} \\ &= \frac{5}{s^2+6s+34}\end{aligned}$$

7.3.3 – Example

$$\begin{aligned}\mathscr{L}\left\{e^{-2t}\cos(6t)\right\} &= F(s+2) \\ &= \frac{s+2}{(s+2)^2+36} \\ &= \frac{s+2}{s^2+4s+4+36} \\ &= \frac{s+2}{s^2+4s+40}\end{aligned}$$

7.3.4 – Example

Find

$$\mathscr{L}^{-1}\left\{\frac{s+3}{s^2-8s+97}\right\}$$

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{s+3}{s^2-8s+97}\right\} &= \mathscr{L}^{-1}\left\{\frac{s+3}{s^2-8s+16+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s+3}{(s-4)^2+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s+3+4-4}{(s-4)^2+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s-4}{(s-4)^2+81}\right\} + \mathscr{L}^{-1}\left\{\frac{3+4}{(s-4)^2+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s-4}{(s-4)^2+81}\right\} + \mathscr{L}^{-1}\left\{\frac{7}{(s-4)^2+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s-4}{(s-4)^2+9^2}\right\} + \mathscr{L}^{-1}\left\{\frac{7}{(s-4)^2+9^2}\right\} \\
&= e^{4t}\cos(9t) + \frac{7}{9}\mathscr{L}^{-1}\left\{\frac{9}{(s-4)^2+9^2}\right\} \\
&= e^{4t}\cos(9t) + \frac{7}{9}e^{4t}\sin(9t)
\end{aligned}$$

7.3.5 – Operational Rules Part 2

Involves taking the Laplace transform of a function shifted on the t-axis.
This can be written in terms of the Heavyside Function (or Unit Step function)

$$\mathscr{U}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

$$f(t-2) \times \mathscr{U}(t-2)$$

is "off when $t < 2$ and on when $t \geq 2$.

7.3.6 – Example

$$
\begin{aligned}
\mathscr{L}\left\{f(t-2)U(t-2)\right\} &= \int_0^\infty e^{-st} f(t-2)U(t-2)dt \\
&= \int_0^2 e^{-st} f(t-2)U(t-2)dt + \int_2^\infty e^{-st} f(t-2)U(t-2)dt \\
&= \int_0^2 e^{-st} f(t-2)(0)dt + \int_2^\infty e^{-st} f(t-2)(1)dt \\
&= \int_0^2 0dt + \int_2^\infty e^{-st} f(t-2)dt \\
&= \int_2^\infty e^{-st} f(t-2)dt \\
(\text{ Let } v = t-2,\ dv = dt) \quad &= \int_0^\infty e^{-s(v+2)} f(v)dv \\
&= \int_0^\infty e^{-sv} e^{-2s} f(v)dv \\
&= e^{-2s} \int_0^\infty e^{-sv} f(v)dv
\end{aligned}
$$

Shifting Theorem Shifting on t-axis

$$\mathscr{L}\left\{f(t-a)\mathscr{U}(t-a)\right\} = e^{-as}\mathscr{L}\left\{f(t)\right\}$$

Proof:

$$
\begin{aligned}
\mathscr{L}\left\{f(t-a)\mathscr{U}(t-a)\right\} &= \int_0^\infty e^{-st} f(t-a)\mathscr{U}(t-a) \\
&= \int_0^a 0dt + \int_a^\infty e^{-st} f(t-a)dt \\
\text{Let } \tau = t-a,\ d\tau = dt \quad &= 0 + \int_a^\infty e^{-s(\tau+a)} f(\tau)d\tau \\
&= e^{-sa} \int_a^\infty e^{-s\tau} f(\tau)d\tau \\
&= e^{-sa}\mathscr{L}\left\{f(\tau)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L}\left\{f(\tau)\mathscr{U}(\tau)\right\} &= e^{-as}\mathscr{L}\left\{f(\tau)\right\} \\
\mathscr{L}\left\{f(t-a)\mathscr{U}(t-a)\right\} &= e^{-as}\mathscr{L}\left\{f(t-a)\right\} \qquad (7.11)
\end{aligned}
$$

$$
\begin{aligned}
f(t-a)\mathscr{U}(t-a) &= \mathscr{L}^{-1}\left\{e^{-as}F(s)\right\} \\
f(t-a)\mathscr{U}(t-a) &= \mathscr{L}^{-1}\left\{e^{-as}\mathscr{L}\left\{f(t)\right\}\right\} \qquad (7.12)
\end{aligned}
$$

7.4 Operational Rules Part 2

Three More Rules

7.4.1 – Rule 1

$$\mathscr{L}\left\{tf(t)\right\} = -\frac{d}{ds}F(s) \Rightarrow \mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}F(s) \tag{7.13}$$

7.4.2 – Rule 2

Is there a way to break up $\mathscr{L}$ over a product of functions?

$$\begin{aligned} \mathscr{L}\left\{f(t)g(t)\right\}? &= \mathscr{L}\left\{f(t)\right\} \times \mathscr{L}\left\{g(t)\right\} \\ \mathscr{L}\left\{t^2 \times t^3\right\}? &= \mathscr{L}\left\{t^2\right\} \times \mathscr{L}\left\{t^3\right\} \\ \mathscr{L}\left\{t^5\right\}? &= \frac{2!}{s^{2+1}} \times \frac{3!}{s^{3+1}} \\ \frac{5!}{s^{5+1}}? &= \frac{2}{s^3} \times \frac{6}{s^4} \\ \frac{120}{s^6} &\neq \frac{12}{s^7} \end{aligned}$$

If we define the convolution production of $f(t)$ and $g(t)$ as

$$(f \times g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

then

$$\mathscr{L}\left\{(f \times g)(t)\right\} = \mathscr{L}\left\{f(t)\right\}\mathscr{L}\left\{g(t)\right\} \tag{7.14}$$

7.4.3 – Rule 3

If $f(t)$ is periodic with period T, then

$$\mathscr{L}\left\{f(t)\right\} = \frac{\int_0^t e^{-st}f(t)dt}{1-e^{-sT}} \tag{7.15}$$

7.4.4 – Example

$$\mathscr{L}\left\{t \times \sin(kt)\right\}$$

$$\begin{aligned}
\mathscr{L}\left\{t \times \sin(kt)\right\} &= -\frac{d}{ds}\mathscr{L}\left\{\sin(kt)\right\} \\
&= -\frac{d}{ds}\left(\frac{k}{s^2+k^2}\right) \\
&= -\frac{\frac{d}{ds}k \times \left(s^2+k^2\right) - k\frac{d}{ds}\left(s^2+k^2\right)}{\left(s^2+k^2\right)^2} \\
&= -\frac{0\left(s^2+k^2\right) - k(2s)}{\left(s^2+k^2\right)^2} \\
&= -\frac{-2sk}{\left(s^2+k^2\right)^2} \\
&= \frac{2sk}{\left(s^2+k^2\right)^2}
\end{aligned}$$

7.4.5 – Example

$$x'' + 16x = \cos(4t), \quad x(0) = 0, \quad x'(0) = 1$$

$$\begin{aligned}
x'' + 16x &= \cos(4t) \\
\mathscr{L}\left\{x''\right\} + 16\mathscr{L}\left\{x\right\} &= \mathscr{L}\left\{\cos(4t)\right\} \\
s^2X(s) - sx(0) - x'(0) + 16X(s) &= \frac{s}{s^2+4^2} \\
X(s)\left(s^2+16\right) - s(0) - 1 &= \frac{s}{s^2+16} \\
X(s)\left(s^2+16\right) - 1 &= \frac{s}{s^2+16} \\
X(s)\left(s^2+16\right) &= \frac{s}{s^2+16} + 1
\end{aligned}$$

$$\begin{aligned}
X(s) &= \frac{s}{\left(s^2+16\right)^2} + \frac{1}{s^2+16} \\
\mathscr{L}^{-1}\left\{X(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{s}{\left(s^2+16\right)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+16}\right\} \\
x(t) &= \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{8s}{\left(s^2+16\right)^2}\right\} + \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} \\
&= \frac{1}{8}t\sin(4t) + \frac{1}{4}\sin(4t)
\end{aligned}$$

7.4.6 – Example

Find $e^t \sin(t)$

$$e^t \sin(t) = \int_0^t e^{\tau}\sin(t-\tau)d\tau$$

$$u = e^{\tau} \qquad dv = \sin(t-\tau)d\tau$$

$$du = e^{\tau}d\tau \qquad \begin{aligned} v &= \int \sin(t-\tau)d\tau \\ &= \frac{\tau\cos(t-\tau)}{\tau} \\ &= \cos(t-\tau) \end{aligned}$$

$$\begin{aligned} e^t\sin(t) &= \int_0^t e^{\tau}\sin(t-\tau)d\tau \\ &= e^{\tau}\cos(t-\tau)\Big|_0^t - \int_0^t e^{\tau}\cos(t-\tau)d\tau \end{aligned}$$

$$h = e^{\tau} \qquad dj = \sin(t-\tau)d\tau$$

$$dh = e^{\tau}d\tau \qquad \begin{aligned} j &= \int \cos(t-\tau)d\tau \\ &= \frac{\sin(t-\tau)}{-1} \\ &= -\sin(t-\tau) \end{aligned}$$

$$\begin{aligned} \int_0^t e^{\tau}\sin(t-\tau) &= e^{\tau}\cos(t-\tau)\Big|_{\tau=0}^t - \left(-e^{\tau}\sin(t-\tau)\Big|_{\tau=0}^t - \int_0^t e^{\tau}(-\sin(t-\tau))d\tau\right) \\ &= e^t\cos(1) - e^0\cos(t) + e^t(0) - e^0\sin(t) - \int_0^t e^{\tau}\sin(t-\tau)d\tau \end{aligned}$$

$$2\int_0^t e^{\tau}\sin(t-\tau)d\tau = e^t - \cos(t) - sin(t)$$

So

$$e^t\sin(t) = \frac{e^t - \cos(t) - \sin(t)}{2}$$

$$
\begin{aligned}
\mathscr{L}\left\{e^t \sin(t)\right\} &= \mathscr{L}\left\{\frac{e^t - \cos(t) - \sin(t)}{2}\right\} \\
&= \frac{1}{2}\mathscr{L}\left\{e^t - \cos(t) - \sin(t)\right\} \\
&= \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1^2} - \frac{1}{s^2+1^2}\right) \\
&= \frac{1}{2}\left(\frac{s^2+1}{(s-1)(s^2+1)} - \frac{s(s-1)}{(s-1)(s^2+1)} - \frac{s-1}{(s-1)(s^2+1)}\right) \\
&= \frac{1}{2}\left(\frac{s^2+1-s(s-1)-s-1}{(s-1)(s^2+1)}\right) \\
&= \frac{s^2+1-s^2+s-s+1}{2(s-1)(s^2+1)} \\
&= \frac{s^2-s^2+s-s+1+1}{2(s-1)(s^2+1)} \\
&= \frac{1+1}{2(s-1)(s^2+1)} \\
&= \frac{2}{2(s-1)(s^2+1)} \\
&= \frac{1}{(s-1)(s^2+1)}
\end{aligned}
$$

7.4.7 – Example

$$
\begin{aligned}
\mathscr{L}\left\{(f \times g)(t)\right\} &= \mathscr{L}\left\{f(t)\right\} \times \mathscr{L}\left\{g(t)\right\} \\
(f \times g)(t) &= \mathscr{L}^{-1}\left\{F(s) \times G(s)\right\}
\end{aligned}
$$

Determine

$$\mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+k^2\right)^2}\right\}$$

using the Convolution Theorem.

$$\mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+k^2\right)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\}$$

$$
\begin{aligned}
\sin(A)\sin(B) &= \frac{1}{2}\left(\cos(A-B)+\cos(A+B)\right) \\
&= \frac{1}{k^2}\mathscr{L}\left\{\frac{1 \times k}{s^2+k^2}\right\}
\end{aligned}
$$

7.4.8 – Example

Solve the Integral Equation

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}\,d\tau \quad \text{for } f(t)$$

Don't forget that the $\int_0^t f(\tau)e^{t-\tau}d\tau$ is $f(t)\times e^t$. Take $\mathscr{L}$ of both sides

$$\begin{aligned}
F(s) &= 3\frac{2!}{s^{2+1}} - \frac{1}{s-(-1)} - F(s)\frac{1}{s-1} \\
&= \frac{6}{s^3} - \frac{1}{s+1} - F(s)\frac{1}{s-1} \\
F(s) + F(s)\frac{1}{s-1} &= \frac{6}{s^3} - \frac{1}{s+1} \\
F(s)\left(1 + \frac{1}{s-1}\right) &= \frac{6}{s^3} - \frac{1}{s+1} \\
F(s)\left(\frac{s-1}{s-1} + \frac{1}{s-1}\right) &= \frac{6}{s^3} - \frac{1}{s+1} \\
F(s)\left(\frac{s-1+1}{s-1}\right) &= \frac{6}{s^3} - \frac{1}{s+1} \\
F(s)\left(\frac{s}{s-1}\right) &= \frac{6}{s^3} - \frac{1}{s+1}
\end{aligned}$$

$$\begin{aligned}
F(s) &= \frac{6(s-1)}{s^3 s} - \frac{s-1}{(s+1)s} \\
&= \frac{6s-6}{s^4} - \frac{s-1}{s^2+s} \\
&= \dots \\
&= \frac{6s+6-s^3}{s^3(s+1)}
\end{aligned}$$

7.5 The Dirac Delta Function

$$\delta_a(t-t_0) = \begin{cases} 0 & \text{if } t < t_0 - a \\ \frac{1}{2a} & \text{if } t_0 - a \le t \le t_0 + a \\ 0 & \text{if } t > t_0 + a \end{cases}$$

$$\int_0^\infty f(t)\delta(t-t_0)dt = f(t_0) \tag{7.16}$$

So

$$\begin{aligned}
\mathscr{L}\left\{\delta(t-t_0)\right\} &= \int_0^\infty e^{-st}\delta(t-t_0)dt \\
&= e^{-st_0}
\end{aligned} \tag{7.17}$$

7.5.1 – Example

$$y'' + y = 4\delta(t-2\pi)$$

$$\begin{aligned}
y(0) = 1 \quad y'(0) &= 0 \\
y'' + y &= 4\delta(t - 2\pi) \\
\mathscr{L}\left\{y''\right\} + \mathscr{L}\left\{y\right\} &= 4\mathscr{L}\left\{\delta(t - 2\pi)\right\} \\
s^2Y(s) - sy(0) - y'(0) + Y(s) &= 4e^{-s(2\pi)} \\
Y(s)(s^2 + 1) - s(1) - 0 &= 4e^{-2\pi s} \\
Y(s)(s^2 + 1) - s &= 4e^{-2\pi s} \\
Y(s)(s^2 + 1) &= 4e^{-2\pi s} + s
\end{aligned}$$

$$\begin{aligned}
Y(s) &= \frac{4e^{-2\pi s} + s}{s^2 + 1} \\
&= \frac{4e^{-2\pi s}}{s^2 + 1} + \frac{s}{s^2 + 1} \\
&= 4e^{-2\pi} \frac{e^s}{s^2 + 1^2} + \frac{s}{s^2 + 1} \\
y(t) &= \mathscr{L}^{-1}\left\{4e^{-2\pi s} \frac{1}{s^2 + 1^2}\right\} + \mathscr{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} \\
&= 4\sin(t - 2\pi)\mathscr{U}(t - 2\pi) + \cos(t)
\end{aligned}$$

Chapter 8

Systems of Linear First-Order Differential Equations

8.1 Preliminary Theory – Linear Systems

In this chapter, we assume the system can be put in the form

$$\begin{aligned}\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n)\\ \frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n)\\ &\vdots\\ \frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)\end{aligned}$$

Further assume $g_1, g_2, \ldots, g_n$ are linear with respect to x_1, x_2, $\ldots$, x_n.

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\\ &\vdots\\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)\end{aligned}$$

In matrix notation:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

where

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} = [a_{ij}(t)] \quad \begin{matrix} i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \end{matrix}$$

In general, if all the $a_{ij}(t)$'s and $f_i(t)$'s are continuous on an interval I, then the IVP $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$ has a unique solution:

$$\begin{aligned} x_1(t_0) &= w_1 \\ x_2(t_0) &= w_2 \\ &\dots \\ x_n(t_0) &= w_n \end{aligned}$$

where $w_1, w_2, \dots, w_n$ are just numbers.

If the initial conditions aren't given, then we want the general solution

$$\mathbf{X} = c_1\mathbf{X_1} + c_2\mathbf{X_2} + \dots + c_n\mathbf{X_n}$$

where each X_i is a solution of $X' = Ax$ and $\{X_1(t), X_2(t), \dots, X_n(t)\}$[1] is a linearly independent collection of solutions. This will be true iff the Wronskian$(X_1, X_2, \dots, X_n)$ is non-zero[2]. The set $\{X_1, X_2, \dots, X_n\}$ is called a fundamental set.

8.2 Solving Homogenous Systems

$$\mathbf{X}' = \mathbf{AX}$$

where $\mathbf{A}$ is a constant matrix.

Look for solutions of the form

$$\mathbf{X} = \mathbf{k}e^{\lambda t}$$

where λ is an eigenvalue and the k vector is the corresponding eigenvector.

Plug this into the Matrix Equation to get

$$\begin{aligned} \mathbf{k}\lambda e^{\lambda t} &= A\mathbf{k}e^{\lambda t} \\ 0 &= A\mathbf{k}e^{\lambda t} - \mathbf{k}\lambda e^{\lambda t} \\ &= (A\mathbf{k} - \mathbf{k}\lambda)e^{\lambda t} \\ &= A\mathbf{k} - \mathbf{k}\lambda \\ &= (A - \lambda I)\mathbf{k} \end{aligned}$$

[1] For a system of n equations

[2] Wronskian$(X_1, X_2, \dots, X_n) = \det(X_1, X_2, \dots, X_n)$

which would have either $\mathbf{k} = \mathbf{0}$ (trivial solution) or $\mathbf{k}$ is an eigenvector of $\mathbf{A}$ and λ is a corresponding eigenvalue, there are linearly-independent.

8.2.1 – Example

$$\frac{dx}{dt} = 2x + 3y \qquad \frac{dy}{dt} = 2x + y$$

Write as

$$\mathbf{X}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \mathbf{X}.$$

Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

Real distinct eigenvalues

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix} \\ &= (2-\lambda)(1-\lambda) - (3)(2) \\ &= \lambda^2 - 3\lambda + 2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1) \end{aligned}$$

$$\begin{aligned} \lambda_1 - 4 &= 0 & \lambda_2 + 1 &= 0 \\ \lambda_1 &= 4 & \lambda_2 &= -1 \end{aligned}$$

For $\lambda_1 = 4$ $(\mathbf{A} - 4I)\mathbf{k} = 0$

$$\begin{aligned} \left[\begin{array}{cc|c} 2-\lambda_1 & 3 & 0 \\ 2 & 1-\lambda_1 & 0 \end{array}\right] &= \left[\begin{array}{cc|c} 2-4 & 3 & 0 \\ 2 & 1-4 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$$\begin{aligned} -2k_1 + 3k_2 &= 0 \\ 3k_2 &= 2k_1 \\ 3(2) &= 2k_1 \\ 6 &= 2k_1 \\ 3 &= k_1 \end{aligned}$$

$$\mathbf{k_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

For $\lambda_2 = -1$ $(\mathbf{A} + I)\mathbf{k} = 0$

$$\left[\begin{array}{cc|c} 2-\lambda_2 & 3 & 0 \\ 2 & 1-\lambda_2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2-(-1) & 3 & 0 \\ 2 & 1-(-1) & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} 3k_1 + 3k_2 &= 0 \\ 3k_1 &= -3k_2 \\ k_1 &= -k_2 \\ k_1 &= -(-1) \\ k_1 &= 1 \end{aligned}$$

$$\mathbf{k_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{X_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} \quad \mathbf{X_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

are both solutions and by a Theorem, $\mathbf{X_1}$ and $\mathbf{X_2}$ are linearly independent.
The general solution is

$$X = c_1\mathbf{X_1} + c_2\mathbf{X_2} \tag{8.1}$$

for particular constants c_1 and c_2.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3c_1e^{4t} + c_2e^{-t} \\ 2c_1e^{4t} - c_2e^{-t} \end{bmatrix}$$

These values come from the matrix of eigenvectors, in this case

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1e^{\lambda_1 t} \\ c_2e^{\lambda_2 t} \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1e^{4t} \\ c_2e^{-t} \end{bmatrix}$$

8.2.2 – Example

$$\mathbf{X}' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \mathbf{X}$$

Eigenvalues:

$$\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \begin{bmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{bmatrix} \\
&= (3-\lambda)(-9-\lambda) - (2)(-18) \\
&= \lambda^2 + 6\lambda - 27 + 36 \\
&= \lambda^2 + 6\lambda + 9 \\
&= (\lambda+3)^2
\end{aligned}$$

The multiplicity of $(\lambda+3)^2 = 0$ is 2, so

$$\lambda_1 = -3 \qquad \lambda_2 = -3$$

Eigenvectors:

$$\begin{aligned}
[A - \lambda_1 I|0] &= \begin{bmatrix} 3-\lambda_1 & -18 \\ 2 & -9-\lambda_1 \end{bmatrix} \\
&= \begin{bmatrix} 3-(-3) & -18 \\ 2 & -9-(-3) \end{bmatrix} \\
&= \begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -6 \\ 2 & -6 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -6 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \\
1k_1 - 3k_2 &= 0 \\
k_1 &= 3k_2 \\
k_1 &= 3(1) \\
k_1 &= 3 \\
\mathbf{k_1} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\end{aligned}$$

The 2nd, Linear Independent solution is

$$\mathbf{k_2} = \mathbf{k_1} t e^{\lambda_1 t} + P e^{\lambda_1}$$

$(A - \lambda I)P = K$

8.2 Solving Homogenous Systems

8.2.3 – Complex EigenValues

8.2.4 – Example

$$\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} \\ &= (6-\lambda)(4-\lambda) - (-1)(5) \\ &= 24 - 6\lambda - 4\lambda + \lambda^2 + 5 \\ &= \lambda^2 - 10\lambda + 29 \\ &= (\lambda - 5)^2 + 4 \end{aligned}$$

$$\begin{aligned} (\lambda - 5)^2 &= -4 \\ \lambda - 5 &= \pm 2i \\ \lambda &= 5 \pm 2i \end{aligned}$$

For $\lambda_1 = 5 + 2i$

$$\begin{aligned} \left[\begin{array}{cc|c} 6-\lambda_1 & -1 & 0 \\ 5 & 4-\lambda_1 & 0 \end{array}\right] &= \left[\begin{array}{cc|c} 6-5-2i & -1 & 0 \\ 5 & 4-5-2i & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right] \\ &= \left(r_1 \leftarrow \frac{r_1}{1-2i}\right) \left[\begin{array}{cc|c} 1 & -\frac{1}{1-2i} & 0 \\ 5 & -1-2i & 0 \end{array}\right] \\ &= \left(r_1 \leftarrow r_1 \times \frac{1+2i}{1+2i}\right) \left[\begin{array}{cc|c} 1 & -\frac{1}{5} - \frac{2}{5}i^{3} & 0 \\ 5 & -1-2i & 0 \end{array}\right] \\ &= (r_2 \leftarrow r_2 - 5r_1) \left[\begin{array}{cc|c} 1 & -\frac{1}{5} - \frac{2}{5}i & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$$k_1 + \left(-\frac{1}{5} - \frac{2}{5}i\right) k_2 = 0$$

$$\text{Let } k_2 = 1 - 2i$$

$$k_1 + \left(-\frac{1}{5} - \frac{2}{5}i\right)(1 - 2i) = 0$$

$$k_1 + (-1)^{4} = 0$$

$$k_1 = 1$$

$$\mathbf{k}_1 = \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix}$$

This, along with an eigenvector $\mathbf{k_2}$ for $\lambda_2 = 5 - 2i$, can be expressed in terms of Real Matrices as

$$\begin{aligned} \mathbf{X_1} &= [\beta_1 \cos(\beta t) - \beta_2 \sin(\beta t)]\, e^{\alpha t} \\ \mathbf{X_2} &= [\beta_2 \cos(\beta t) - \beta_1 \sin(\beta t)]\, e^{\alpha t} \end{aligned} \tag{8.2}$$

where α is the real part of the eigen value, β is the coefficient of the imaginary part of the eigen value,

$\beta_1 = \text{Real Part } (k_1) = Re \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$\beta_2 = \text{Imaginary Part } (k_1) = Im \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

8.3 General Ideas

(Don't worry about detailed calculation)
Non-homogenous system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

where $\mathbf{F}$ is a non-zero column vector.

8.3.1 – 1st Solve the Complimentary DE

$$\mathbf{X}' = \mathbf{A}\mathbf{X},$$

General solution:

$$\mathbf{X} = c_1\mathbf{X_1} + c_2\mathbf{X_2} + \cdots + c_n\mathbf{X_n}$$

---

[3]

$$-\frac{1}{1-2i} \times \frac{1+2i}{1+2i} = -\frac{1+2i}{(1-2i)(1+2i)} = -\frac{1+2i}{1-4i^2} = -\frac{1+2i}{1-(-4)} = -\frac{1+2i}{1+4} = -\frac{1+2i}{5}$$

[4] Essesntially undid the complex conjgate multiplication.

8.3.2 – Next, find one particular solution

The general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$ is

$$\mathbf{X} = \mathbf{X_c} + \mathbf{X_p}$$

Appendix B

2.1 Eigenvalues & Eigenvectors of a Square Matrix

Given a square matrix A, we look for a non-zero column vector k and a number λ such that $A_{n\times n}k_{n\times 1} = \lambda k_{n\times 1}$.

If such λ and k exist, λ is called an eigenvalue for the matrix A and k is the corresponding eigenvector.

2.1.1 – Example

Verify that

$$k = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

is an eigenvector for the matrix

$$A = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

and determine the corresponding eigenvalue.
Calculate

$$Ak = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0(1) + -1(-1) + -3(1) \\ 2(1) + 3(-1) + 3(1) \\ -2(1) + 1(-1) + 1(1) \end{bmatrix} \\
&= \begin{bmatrix} 0 + 1 - 3 \\ 2 - 3 + 3 \\ -2 - 1 + 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \\
&= -2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\
&= 2k
\end{aligned}$$

Therefore k is an eigenvector corresponding to eigenvalue $\lambda = -2$.

Notice that any non-zero multiple of k would also be an eigenvector corresponding to $\lambda = -2$.
Proof:

$$\begin{aligned}
A(5k) &= 5Ak \\
&= 5(-2)k \\
&= (-2)(5k)
\end{aligned}$$

2.1.2 – Example

Find the eigenvalues, and for each, a corresponding eigenvector for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Theory: If $Ak = \lambda k$ for a non-zero k, then $Ak - \lambda k = 0 \Rightarrow (A - \lambda)k = 0$ for a non-zero k. However, we cannot subtract a number from a matrix. Instead, the equation would be $(A - \lambda I)k = 0$.
This would mean that the matrix $A - \lambda I$ is singular (not invertible). This can be checked by ensuring that $\det(A - \lambda I) = 0$

$$
\begin{aligned}
A - \lambda I &= \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} \\
&= 1\begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} - 0\begin{vmatrix} 1-\lambda & 2 \\ -1 & -2 \end{vmatrix} + (-1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix} \\
&= 1\left[(6)(-2) - (-1)(-1-\lambda)\right] - 0 + (-1-\lambda)\left[(1-\lambda)(-1-\lambda) - (2)(6)\right] \\
&= 1\left[-12 + -1 - \lambda\right] + (-1-\lambda)\left[-1 - \lambda + \lambda + \lambda^2 - 12\right] \\
&= 1\left[-13 - \lambda\right] + (-1-\lambda)\left[\lambda^2 - 1 - 12\right] \\
&= -13 - \lambda + (-1-\lambda)\left[\lambda^2 - 13\right] \\
&= -13 - \lambda - \lambda^2 + 13 - \lambda^3 + 13\lambda \\
&= -\lambda - \lambda^2 - \lambda^3 + 13\lambda \\
&= -\lambda^3 - \lambda^2 + 13\lambda - \lambda \\
&= -\lambda^3 - \lambda^2 + 12\lambda \\
&= -\lambda\left(\lambda^2 + \lambda - 12\right) \\
&= -\lambda\left(\lambda + 4\right)\left(\lambda - 3\right)
\end{aligned}
$$

When this is 0, λ is an eigenvalue.

$$-\lambda\left(\lambda + 4\right)\left(\lambda - 3\right) = 0$$

$$
\begin{array}{ccc}
-\lambda_1 = 0 & \lambda_2 + 4 = 0 & \lambda_3 - 3 = 0 \\
\lambda_1 = 0 & \lambda_2 = -4 & \lambda_3 = 3
\end{array}
$$

Next, for each eigenvalue λ_1, λ_2, λ_3, find a corresponding eigenvalue k_1, k_2, k_3.

For $\lambda_1 = 0$

$$\begin{aligned}
0 &= (A - \lambda_1 I)k_1 \\
&= \begin{bmatrix} 1-\lambda_1 & 2 & 1 \\ 6 & -1-\lambda_1 & 0 \\ -1 & -2 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\
&= \begin{bmatrix} 1-0 & 2 & 1 \\ 6 & -1-0 & 0 \\ -1 & -2 & -1-0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] &= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] (r_3 \leftarrow r_3 + r_1) \\
&= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -13 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] (r_2 \leftarrow r_2 - 6r_1) \\
&= \dots \\
&= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -13 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
k_1 + 2k_2 + 1k_3 &= 0 \\
-13k_2 - 6k_3 &= 0
\end{aligned}$$

Let $k_3 = 1$, then $k_2 = \frac{-6}{13}k_3 = -\frac{6}{13}$ and $k_1 = -2k_2 - k_3 = \frac{12}{13} - 1 = -\frac{1}{13}$

$$k_1 = \begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}$$