Chapter 2

First-Order Differential Equations

2.1 Test

2.2 Separable Differential Equations

Separable D.E.s are DE's $\frac{dy}{dx} = f(x, y)$ where $f(x, y)$ can be factored as $f(x, y) = g(x)h(y)$.

$$\frac{dy}{dx} = (1 + y^2)x^3 \text{ is separable}$$
$$\frac{dy}{dx} = \sin(xy) \text{ is } \textit{not} \text{ separable}$$
$$\frac{dy}{dx} = x^3 y \text{ is separable}$$

$$\begin{aligned} \frac{5}{xy}\frac{dy}{dx} &= \left(x^2 + y\right) e^y \\ \frac{dy}{dx} &= \frac{xy\left(x^2 + y\right) e^y}{5} \\ &= \frac{x\left(x^2 + y\right)}{5} \times ye^y \end{aligned}$$

2.2.1 – Method of Solution

"Separate the variable" to get $\frac{1}{h(y)}dy = g(x)d$ or $p(y)dy = g(x)dx$ where $p(y) = \frac{1}{h(y)}$.
Integrate both sides

$$\int p(y)dy = \int g(x)dx \text{ and if possible, solve for } y$$

2.2.2 – Example

$$\begin{aligned}
\frac{dy}{dx} &= \left(1+y^2\right)x^3 \\
\int \frac{1}{1+y^2}dy &= \int x^3 dx \\
\tan^{-1}(y) + C_1 &= \frac{x^4}{4} + C_2 \\
\tan^{-1}(y) &= \frac{x^4}{4} + C_2 - C_1 \\
\tan^{-1}(y) &= \frac{x^4}{4} + C \\
y &= \tan\left(\frac{x^4}{4} + C\right)
\end{aligned}$$

2.2.3 – Example

Problem 12 from the textbook.

$$\begin{aligned}
&\sin(3x)dx + 2y\cos^3(3x)dy = 0 \\
\int -2y dy &= \int \frac{\sin(3x)}{\cos^3(x)}dx \\
&= \int \tan(3x)\sec^2(3x)dx \\
&= \int u\ \frac{1}{3}du \text{ where } u = \tan(3x),\ du = 3\sec^2(3x)dx \\
-2\int y dy &= \frac{1}{3}\int u\ du + C \\
-y^2 &= \frac{u^2}{6} + C \\
&= \frac{\tan^2(3x)}{6} + C \\
\frac{\tan^2(3x)}{6} + y^2 &= -C \\
\frac{\tan^2(3x)}{6} + y^2 &= C
\end{aligned}$$

Problem 25 from the textbook.

$$x^2\frac{dy}{dx} = y - xy, y(-1) = -1$$

$$
\begin{aligned}
x^2\frac{dy}{dx} &= y - xy \\
x^2\frac{dy}{dx} &= y(1-x) \\
\frac{dy}{y} &= \frac{(1-x)}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{(1-x)}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{1}{x^2}dx - \int \frac{x}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{1}{x^2}dx - \int \frac{1}{x}dx \\
\ln|y| + C_1 &= -\frac{1}{x} + C_2 - \ln|x| + C_3 \\
\ln|y| &= -\frac{1}{x} - \ln|x| + C \\
y &= e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C \\
y &= e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C \\
y &= e^{-\frac{1}{x}} \times \frac{1}{|x|} \times e^C \\
y &= \frac{1}{|x|}e^{C-\frac{1}{x}} \\
-1 &= \frac{1}{|-1|}e^{C-\frac{1}{-1}} \\
-1 &= \frac{1}{1}e^{C-(-1)} \\
-1 &= e^{C+1}
\end{aligned}
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