Appendix B

B

2.1 Eigenvalues & Eigenvectors of a Square Matrix

Given a square matrix A, we look for a non-zero column vector k and a number λ such that $A_{n \times n} k_{n \times 1} = \lambda k_{n \times 1}$.

If such λ and k exist, λ is called an eigenvalue for the matrix A and k is the corresponding eigenvector.

2.1.1 – Example

Verify that

$$k = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

is an eigenvector for the matrix

$$A = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

and determine the corresponding eigenvalue.
Calculate

$$Ak = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0(1) + -1(-1) + -3(1) \\ 2(1) + 3(-1) + 3(1) \\ -2(1) + 1(-1) + 1(1) \end{bmatrix} \\
&= \begin{bmatrix} 0 + 1 - 3 \\ 2 - 3 + 3 \\ -2 - 1 + 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \\
&= -2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\
&= 2k
\end{aligned}$$

Therefore k is an eigenvector corresponding to eigenvalue $\lambda = -2$.

Notice that any non-zero multiple of k would also be an eigenvector corresponding to $\lambda = -2$.
Proof:

$$\begin{aligned}
A(5k) &= 5Ak \\
&= 5(-2)k \\
&= (-2)(5k)
\end{aligned}$$

2.1.2 – Example

Find the eigenvalues, and for each, a corresponding eigenvector for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Theory: If $Ak = \lambda k$ for a non-zero k, then $Ak - \lambda k = 0 \Rightarrow (A - \lambda)k = 0$ for a non-zero k. However, we cannot subtract a number from a matrix. Instead, the equation would be $(A - \lambda I)k = 0$.
This would mean that the matrix $A - \lambda I$ is singular (not invertible). This can be checked by ensuring that $\det(A - \lambda I) = 0$

$$
\begin{aligned}
A - \lambda I &= \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{bmatrix} \\
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} \\
&= 1 \begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} - 0 \begin{vmatrix} 1-\lambda & 2 \\ -1 & -2 \end{vmatrix} + (-1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix} \\
&= 1\left[(6)(-2) - (-1)(-1-\lambda)\right] - 0 + (-1-\lambda)\left[(1-\lambda)(-1-\lambda) - (2)(6)\right] \\
&= 1\left[-12 + -1 - \lambda\right] + (-1-\lambda)\left[-1 - \lambda + \lambda + \lambda^2 - 12\right] \\
&= 1\left[-13 - \lambda\right] + (-1-\lambda)\left[\lambda^2 - 1 - 12\right] \\
&= -13 - \lambda + (-1-\lambda)\left[\lambda^2 - 13\right] \\
&= -13 - \lambda - \lambda^2 + 13 - \lambda^3 + 13\lambda \\
&= -\lambda - \lambda^2 - \lambda^3 + 13\lambda \\
&= -\lambda^3 - \lambda^2 + 13\lambda - \lambda \\
&= -\lambda^3 - \lambda^2 + 12\lambda \\
&= -\lambda\left(\lambda^2 + \lambda - 12\right) \\
&= -\lambda\left(\lambda + 4\right)\left(\lambda - 3\right)
\end{aligned}
$$

When this is 0, λ is an eigenvalue.

$$-\lambda\left(\lambda + 4\right)\left(\lambda - 3\right) = 0$$

$$
\begin{array}{ccc}
-\lambda_1 = 0 & \lambda_2 + 4 = 0 & \lambda_3 - 3 = 0 \\
\lambda_1 = 0 & \lambda_2 = -4 & \lambda_3 = 3
\end{array}
$$

Next, for each eigenvalue λ_1, λ_2, λ_3, find a corresponding eigenvalue k_1, k_2, k_3.

For $\lambda_1 = 0$

$$\begin{aligned}
0 &= (A - \lambda_1 I)k_1 \\
&= \begin{bmatrix} 1-\lambda_1 & 2 & 1 \\ 6 & -1-\lambda_1 & 0 \\ -1 & -2 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\
&= \begin{bmatrix} 1-0 & 2 & 1 \\ 6 & -1-0 & 0 \\ -1 & -2 & -1-0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] &= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] (r_3 \leftarrow r_3 + r_1) \\
&= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -13 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] (r_2 \leftarrow r_2 - 6r_1) \\
&= \dots \\
&= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -13 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
k_1 + 2k_2 + 1k_3 &= 0 \\
-13k_2 - 6k_3 &= 0
\end{aligned}$$

Let $k_3 = 1$, then $k_2 = \frac{-6}{13}k_3 = -\frac{6}{13}$ and $k_1 = -2k_2 - k_3 = \frac{12}{13} - 1 = -\frac{1}{13}$

$$k_1 = \begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}$$