Chapter 6

Series Solutions of Linear Equations

6.1 Solving Linear DE's without constant coefficients

6.1.1 – Idea of Method

We will try to find a solution of the DE in the form of a power series

$$y = \sum_{n=0}^{\infty} c_n x^n \text{ (centered at 0)}$$

or

$$y = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ (centered at } a\text{)}$$

When you substitute this into the DE you get recurrence relationships for the coefficients c_0, c_1, ... Once you've found the coefficients in terms of either c_0, or c_0, c_1 where $c_0 \neq c_1$. Then you should determine where the series converges.

An example of a recurrence relation is the Fibonacci Sequence

$$F_{n+2} = F_n + F_{n+1}$$

6.1.2 – Example

Use this method to solve the DE

$$y' + y = 0$$

Assume

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

then

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$\sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

Shift the index of the first summation such that both have terms x^n. To do this, we'll make the substitution $k = n - 1 \Rightarrow n = k + 1$

$$\begin{aligned}
\sum_{k+1=1}^{\infty} c_{k+1}(k+1)x^{(k+1)-1} + \sum_{k=0}^{\infty} c_k x^k &= 0 \\
\sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + \sum_{k=0}^{\infty} c_k x^k &= 0 \\
\sum_{k=0}^{\infty} \left[c_{k+1}(k+1)x^k + c_k x^k\right] &= 0 \\
\sum_{k=0}^{\infty} \left[(k+1)c_{k+1} + c_k\right] x^k &= 0
\end{aligned}$$

This implies $(k+1)c_{k+1} + c_k = 0$ for all $k = 0, 1, 2, \ldots$ [1] $\Rightarrow c_{k+1} = \frac{-c_k}{k+1}$ for all $k = 0, 1, 2, \ldots$.

$$\begin{aligned}
c_0 &= c_0 \\
c_1 &= -\frac{c_0}{1} \\
&= -c_0 \\
c_2 &= -\frac{c_1}{2} \\
&= -\frac{-c_0}{2} \\
&= \frac{-1^2 c_0}{2} \\
&= \frac{c_0}{2} \\
c_3 &= -\frac{c_2}{3} \\
&= -\frac{\frac{c_0}{2}}{3} \\
&= \frac{-c_0}{6}
\end{aligned}$$

Conjecture: It is apparent that

$$c_n + \frac{(-1)^n c_0}{n!}$$

[1] since the only power series that equals 0 is $\sum_{k=0}^{\infty} 0x^k$

Plugging into the DE

$$
\begin{aligned}
y = \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{n!} x^n \\
&= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n
\end{aligned}
$$

By the Ratio Test

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{\left| \frac{(-1)^{n+1}}{(n+1)!} x^{n+1} \right|}{\left| \frac{(-1)^n}{n!} x^n \right|} \\
&= \lim_{n\to\infty} \left| (-1) \frac{x\ n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \frac{|x|\ n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{|x|}{n+1} \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} \\
&= |x| \times 0 \\
&= 0 \text{ The series converges everywhere}
\end{aligned}
$$

6.1.3 – Power Series of Basic Functions

$$
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots
$$

Our answer in the DE is

$$
y = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}
$$