Chapter 8

Systems of Linear First-Order Differential Equations

8.2 Solving Homogenous Systems

$$\mathbf{X}' = \mathbf{AX}$$

where $\mathbf{A}$ is a constant matrix.

Look for solutions of the form

$$\mathbf{X} = \mathbf{k}e^{\lambda t}$$

where λ is an eigenvalue and the k vector is the corresponding eigenvector.

Plug this into the Matrix Equation to get

$$\begin{aligned}
\mathbf{k}\lambda e^{\lambda t} &= A\mathbf{k}e^{\lambda t} \\
0 &= A\mathbf{k}e^{\lambda t} - \mathbf{k}\lambda e^{\lambda t} \\
&= (A\mathbf{k} - \mathbf{k}\lambda)e^{\lambda t} \\
&= A\mathbf{k} - \mathbf{k}\lambda \\
&= (A - \lambda I)\mathbf{k}
\end{aligned}$$

which would have either $\mathbf{k} = \mathbf{0}$ (trivial solution) or $\mathbf{k}$ is an eigenvector of $\mathbf{A}$ and λ is a corresponding eigenvalue, there are linearly-independent.

8.2.1 – Example

$$\frac{dx}{dt} = 2x + 3y \qquad \frac{dy}{dt} = 2x + y$$

Write as

$$\mathbf{X}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \mathbf{X}.$$

Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

Real distinct eigenvalues

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix} \\ &= (2-\lambda)(1-\lambda) - (3)(2) \\ &= \lambda^2 - 3\lambda + 2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1) \end{aligned}$$

$$\begin{aligned} \lambda_1 - 4 &= 0 & \lambda_2 + 1 &= 0 \\ \lambda_1 &= 4 & \lambda_2 &= -1 \end{aligned}$$

For $\lambda_1 = 4$ $(\mathbf{A} - 4I)\mathbf{k} = 0$

$$\begin{aligned} \left[\begin{array}{cc|c} 2-\lambda_1 & 3 & 0 \\ 2 & 1-\lambda_1 & 0 \end{array}\right] &= \left[\begin{array}{cc|c} 2-4 & 3 & 0 \\ 2 & 1-4 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$$\begin{aligned} -2k_1 + 3k_2 &= 0 \\ 3k_2 &= 2k_1 \\ 3(2) &= 2k_1 \\ 6 &= 2k_1 \\ 3 &= k_1 \end{aligned}$$

$$\mathbf{k_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

For $\lambda_2 = -1$ $(\mathbf{A} + I)\mathbf{k} = 0$

$$\begin{aligned} \left[\begin{array}{cc|c} 2-\lambda_2 & 3 & 0 \\ 2 & 1-\lambda_2 & 0 \end{array}\right] &= \left[\begin{array}{cc|c} 2-(-1) & 3 & 0 \\ 2 & 1-(-1) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$$\begin{aligned} 3k_1 + 3k_2 &= 0 \\ 3k_1 &= -3k_2 \\ k_1 &= -k_2 \\ k_1 &= -(-1) \\ k_1 &= 1 \end{aligned}$$

$$\mathbf{k_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{X_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} \quad \mathbf{X_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

are both solutions and by a Theorem, $\mathbf{X_1}$ and $\mathbf{X_2}$ are linearly independent.
The general solution is

$$X = c_1\mathbf{X_1} + c_2\mathbf{X_2} \tag{8.1}$$

for particular constants c_1 and c_2.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3c_1e^{4t} + c_2e^{-t} \\ 2c_1e^{4t} - c_2e^{-t} \end{bmatrix}$$

These values come from the matrix of eigenvectors, in this case

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1e^{\lambda_1 t} \\ c_2e^{\lambda_2 t} \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1e^{4t} \\ c_2e^{-t} \end{bmatrix}$$

8.2.2 – Example

$$\mathbf{X}' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \mathbf{X}$$

Eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{bmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{bmatrix} \\ &= (3-\lambda)(-9-\lambda) - (2)(-18) \\ &= \lambda^2 + 6\lambda - 27 + 36 \\ &= \lambda^2 + 6\lambda + 9 \\ &= (\lambda+3)^2 \end{aligned}$$

The multiplicity of $(\lambda + 3)^2 = 0$ is 2, so

$$\lambda_1 = -3 \quad \lambda_2 = -3$$

Eigenvectors:

$$
\begin{aligned}
[A - \lambda_1 I|0] &= \begin{bmatrix} 3-\lambda_1 & -18 \\ 2 & -9-\lambda_1 \end{bmatrix} \\
&= \begin{bmatrix} 3-(-3) & -18 \\ 2 & -9-(-3) \end{bmatrix} \\
&= \begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -6 \\ 2 & -6 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -6 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \\
1k_1 - 3k_2 &= 0 \\
k_1 &= 3k_2 \\
k_1 &= 3(1) \\
k_1 &= 3 \\
\mathbf{k_1} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\end{aligned}
$$

The 2nd, Linear Independent solution is

$$\mathbf{k_2} = \mathbf{k_1} t e^{\lambda_1 t} + P e^{\lambda_1}$$

$(A - \lambda I)P = K$