Chapter 7

Method of Laplace Transforms for Solving DE's

Chapter Goals

- Given a DE, Perform a Calculus-based rule for finding the laplace transformation of DE.
- Solve this new equation algebraically.
- Find the inverse-Laplace transformation to get our solution to the IVP.

7.1 Definition of Laplace Transform

Given a function $f(t)$, the Laplace Transform of $f(t)$ is

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

7.1.1 – Laplace Transformations of basic Functions

(1)

$$\begin{aligned}
\mathscr{L}\{1\} &= \int_0^\infty e^{-st}(1)dt \\
&= \int_0^\infty e^{-st}dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}dt \\
&= \lim_{b\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^b \\
&= \lim_{b\to\infty} \frac{e^{-s(b)}}{-s} - \frac{e^{-s(0)}}{-s} \\
&= -\frac{e^{-s(0)}}{-s} + \lim_{b\to\infty} \frac{e^{-s(b)}}{-s} \\
&= -\frac{e^0}{-s} + \frac{1}{-s}\lim_{b\to\infty} e^{-s}e^b \\
&= \frac{1}{s} - \frac{e^{-s}}{s}\lim_{b\to\infty} e^b \\
&= \frac{1}{s} \text{ for } s > 0
\end{aligned}$$

(2)

$$\begin{aligned}
\mathscr{L}\{k\} &= \int_0^\infty e^{-st}kdt \\
&= k\int_0^\infty e^{-st}dt \\
&= k\frac{1}{s} \\
&= \frac{k}{s}
\end{aligned}$$

The Laplace Transform is a *linear operator*, in other words,

(3)

$$\mathscr{L}\{f(t)+g(t)\} = \mathscr{L}\{f(t)\} + \mathscr{L}\{g(t)\}$$

(4)

$$\mathscr{L}\{kf(t)\} = k\mathscr{L}\{f(t)\}$$

(5)

$$\begin{aligned}
\mathscr{L}\{e^{2t}\} &= \int_0^\infty e^{-st} \times e^{2t} dt \\
&= \int_0^\infty e^{(2-s)t} dt \\
&= \int_0^\infty e^{-(s-2)t} dt \\
u &= -(s-2)t \\
du &= -(s-2)dt \\
&= \int_0^\infty e^u \frac{du}{-(s-2)} \\
&= \frac{1}{-(s-2)} \int_0^\infty e^u du \\
&= \frac{1}{-(s-2)} e^u \Big|_0^\infty \\
&= \frac{1}{-(s-2)} e^{-(s-2)t} \Big|_0^\infty \\
&= \lim_{b\to\infty} \frac{1}{-(s-2)} e^{-(s-2)b} - \frac{1}{-(s-2)} e^{-(s-2)0} \\
&= -\frac{1}{-(s-2)} e^0 + \lim_{b\to\infty} \frac{1}{-(s-2)} e^{-(s-2)b} \\
&= -\frac{1}{-(s-2)}(1) \\
&= \frac{1}{s-2}
\end{aligned}$$

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a} \text{ for } s > a$$

(6)

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

(7)

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2}$$

(8)

$$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}$$

Table 7.1: Transforms of Some Basic Functions

$$\mathscr{L}\left\{1\right\} = \frac{1}{s} \tag{7.1}$$

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \tag{7.2}$$

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a} \tag{7.3}$$

$$\mathscr{L}\left\{\sin kt\right\} = \frac{k}{s^2+k^2} \tag{7.4}$$

$$\mathscr{L}\left\{\cos kt\right\} = \frac{s}{s^2+k^2} \tag{7.5}$$

$$\mathscr{L}\left\{\sinh kt\right\} = \frac{k}{s^2-k^2} \tag{7.6}$$

$$\mathscr{L}\left\{\cosh kt\right\} = \frac{s}{s^2-k^2} \tag{7.7}$$