

# Chapter 4

## Higher Order Differential Equations

### 4.2 Reduction of Order

If you have one solution to a 2<sup>nd</sup> order linear homogenous DE, then it is possible to use that function to construct a 2<sup>nd</sup> Linear Independent solution to the DE.

#### 4.2.1 – Example

For example, the DE

$$y'' - y = 0$$

One solution is  $y = e^x$  on  $(-\infty, \infty)$ .

Idea: We look for  $y_2$  of the form

$$y_2(x) = u(x)y_1(x) \text{ where } u(x) \text{ is not a constant}$$

The general solution is of the form:

$$y = c_1y_1 + c_2y_2$$

where  $y_1$  and  $y_2$  are linearly independent solutions.

To find  $u(x)$ , we substitute this into the DE

$$\begin{aligned} y_2 &= u(x)y_1(x) \\ y_2' &= u(x)y_1'(x) + u'(x)y_1(x) \\ y_2'' &= u(x)y_1''(x) + u'(x)y_1'(x) + u'(x)y_1'(x) + u''(x)y_1(x) \\ &= uy_1'' + 2u'y_1' + u''y_1 \end{aligned}$$

So  $y'' - y = 0$  becomes

$$\begin{aligned}
 uy_1'' + 2u'y_1' + u''y_1 - uy_1 &= 0 \text{ when we sub } y = y_2 = uy_1 \\
 u(e^x)'' + 2u'(e^x)' + u''(e^x) - u(e^x) &= 0 \\
 ue^x + 2u'e^x + u''e^x - ue^x &= 0 \\
 2u'e^x + u''e^x &= 0 \\
 e^x(2u' + u'') &= 0 \\
 2u' + u'' &= 0
 \end{aligned}$$

Let  $w = u'$

$$\begin{aligned}
 2w + w' &= 0 \\
 2w + \frac{dw}{dx} &= 0 \\
 \frac{dw}{dx} &= -2w \\
 \frac{dw}{w} &= -2dx \\
 \int \frac{dw}{w} &= \int -2dx \\
 \ln |w| &= -2x \\
 w &= e^{-2x} \\
 u' &= e^{-2x} \\
 \int u' &= \int e^{-2x} \\
 u &= -\frac{1}{2}e^{-2x} \\
 y_2 &= uy_1 \\
 &= -\frac{1}{2}e^{-2x} \times e^x \\
 &= -\frac{1}{2}e^{-x}
 \end{aligned}$$

Double check that  $y_2$  is a solution of the DE

$$\begin{aligned}
 y_2 &= -\frac{1}{2}e^{-x} \\
 y_2' &= \frac{1}{2}e^{-x} \\
 y_2'' &= -\frac{1}{2}e^{-x} \\
 y_2'' - y &= -\frac{1}{2}e^{-x} - \left(-\frac{1}{2}e^{-x}\right) \\
 &= -\frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} \\
 &= 0
 \end{aligned}$$

In general,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

put into standard form by dividing by  $a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

where  $P(x) = \frac{a_1(x)}{a_2(x)}$  and  $Q(x) = \frac{a_0(x)}{a_2(x)}$ , the same method as in our **example** leads to the formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \quad (4.1)$$

### 4.2.2 – Example

#### Part 1

$$x^2y'' - 3xy' + 4y = 0$$

Verify that  $y_1 = x^2$  is a solution  $y'_1 = 2x, y''_1 = 2$ .

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ x^2(2) - 3x(2x) + 4(x^2) &= 0 \\ 2x^2 - 6x^2 + 4x^2 &= 0 \\ 6x^2 - 6x^2 &= 0 \\ 0 &= 0 \end{aligned}$$

#### Part 2

Find a linearly independent solution  $y_2(x)$ .

$$x^2 y'' - 3xy' + 4y = 0$$

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$P(x) = -\frac{3}{x}$$

$$y_2 = y_1 \int \frac{e^{\int \frac{3}{x} dx}}{(y_1(x))^2} dx$$

$$y_2 = y_1 \int \frac{e^{3 \ln |x|}}{(y_1(x))^2} dx$$

$$y_2 = y_1 \int \frac{e^{\ln |x^3|}}{(y_1(x))^2} dx$$

$$y_2 = x^2 \int \frac{x^3}{(x^2)^2} dx$$

$$y_2 = x^2 \int \frac{x^3}{x^4} dx$$

$$y_2 = x^2 \int \frac{1}{x} dx$$

$$y_2 = x^2 \ln |x| + C$$

**Part 3: Double check that  $y_2$  is a solution of the DE**

$$y_2 = x^2 \ln |x|$$

$$y_2' = x^2 \times \frac{1}{x} + 2x \ln |x|$$

$$y_2'' = 1 + 2x \frac{1}{x} + 2 \ln |x|$$

$$= 1 + 2 + 2 \ln |x|$$

$$= 3 + 2 \ln |x|$$

So the LHS DE becomes

$$\begin{aligned} x^2 (3 + 2 \ln |x|) - 3x (x + 2x \ln |x|) + 4x^2 \ln |x| &= 3x^2 + 2x^2 \ln |x| - 3x^2 - 6x^2 \ln |x| + 4x^2 \ln |x| \\ &= 3x^2 - 3x^2 + 2x^2 \ln |x| - 6x^2 \ln |x| + 4x^2 \ln |x| \\ &= 3x^2 - 3x^2 + 2x^2 \ln |x| - 6x^2 \ln |x| + 4x^2 \ln |x| \\ &= 0 + x^2 \ln |x| (2 - 6 + 4) \\ &= x^2 \ln |x| (0) \\ &= 0 \end{aligned}$$

Write the general solution of the DE including the interval of the solution

$$\begin{aligned}
 y &= c_1 y_1 + c_2 y_2 \\
 &= c_1 x^2 + c_2 x^2 (\ln |x| + C) \\
 &= c_1 x^2 + c_2 x^2 \ln |x| + C c_2 x^2 \\
 \text{just } y &= c_1 x^2 + c_2 x^2 \ln |x| \text{ on } I = (0, \infty), y(2) = 3, y'(2) = 5
 \end{aligned}$$

### 4.2.3 – Example

$$\begin{aligned}
 3y'' + y' - 4y &= 0 \\
 y &= e^{mx} \\
 y' &= m e^{mx} \\
 y'' &= m^2 e^{mx} \\
 3y'' + y' - 4y &= 3m^2 e^{mx} + m e^{mx} - 4e^{mx} \\
 &= e^{mx} (3m^2 + m - 4) \\
 &= e^{mx} (3m^2 + 4)(m - 1) \\
 m = 1 \quad m &= -\frac{4}{3} \\
 y_1 &= e^x, y_2 = e^{-\frac{4}{3}x}
 \end{aligned}$$