Chapter 4

Higher Order Differential Equations

4.1 Linear Equations

An nth order DE is linear if it has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0y = g(x)$$

Theorem: If all the coefficient functions are continuous and $a_n(x)$ is not 0 on an interval I and $g(x)$ is continuous, then any initial value problem

$$DE + y(x_0) = y_0$$

has a unique solution on the interval I if $g(x) = 0$. i.e.

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

then the DE is said to be homogeneous.

4.1.1 – Example

$$y'' - 3y' - 4y = 0$$

Show $y_1 = e^{4x}$ is a solution and $y_2 = e^{-x}$ is a solution.

$$y_1 = e^{4x}$$

$$y_1' = 4e^{4x}$$

$$y_1'' = 16e^{4x}$$

$$\begin{aligned}
16e^{4x} - 3(4e^{4x}) - 4e^{4x} &= 0 \\
16e^{4x} - 12e^{4x} - 4e^{4x} &= 0 \\
e^{4x}(16 - 12 - 4) &= 0 \\
e^{4x}(0) &= 0 \\
0 &= 0
\end{aligned}$$

$$y_3 = 6y_1 = 6e^{4x}$$

$$y_3' = 6y_1' = 24e^{4x}$$

$$y_3'' = 6y_1'' = 96e^{4x}$$

$$\begin{aligned} 96e^{4x} - 3(24e^{4x}) - 4(6e^{4x}) &= 0 \\ 96e^{4x} - 72e^{4x} - 24e^{4x} &= 0 \\ e^{4x}(96 - 72 - 24) &= 0 \\ e^{4x}(0) &= 0 \\ 0 &= 0 \end{aligned}$$

Theorem: Superposition Principle: if $y_1, y_2, \ldots, y_m$ are each solutions of an nth order Linear, homongenous DE, then $c_1y_1 + c_2y_2 + \cdots + c_my_m$ will also be a solution for any constants $c_1, c_2, \ldots, c_m$.

Our goal is to express the general solution in as concise a way as possible.

Linear combination – a collection of solutions $y_1, y_2, \ldots, y_m$ is linearly independent is if the only way $c_1y_1+c_2y_2+\cdots+c_my_m = 0$ is iff (if and only if) all of the constants $c_1, c_2, \ldots, c_m = 0$. Otherwise we say $y_1, y_2, \ldots, y_m$ are linearly dependent.

Theorem: If the DE is an nth order Linear Homogeneous equation then there will exist a collection of n linearly independent solutions $y_1, y_2, \ldots, y_n$ and the general solution will be $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$

One way to check for linear independence is to compile the Wronskian

$$W(y_1, y_2, \ldots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$