

MATH 252 - Introduction to Differential Equations

Notes

Len Washington III

October 4, 2023

Contents

1	Introduction to Diff-Eq	4
1.1	Terminology and Notation	4
1.1.1	Example	4
1.1.2	Linear vs Non-Linear DE's	4
1.1.3	Example	5
1.2	Initial Value Problems (IVP)	6
1.2.1	Example	6
1.2.2	Example	6
1.2.3	Example	7
1.2.4	Example	7
1.2.5	Example	7
2	First-Order Differential Equations	8
2.1	Solution Curves Without a Solution	8
2.1.1	Example	8
2.1.2	Slope/Direction Fields	9
2.1.3	Example	9
2.2	Separable D.E.s	10
2.2.1	Method of Solution	11
2.2.2	Example	11
2.2.3	Example	11
2.3	First Order Linear D.E.'s	12
2.3.1	Example	13
2.3.2	Example	14
2.4	Exact Equations	14
2.4.1	Method	15
2.4.2	Example	15
2.4.3	Example	16
2.4.4	What can you do if $M_y \neq N_x$	17
2.4.5	Example	17
2.5	Substitution Methods	18
2.5.1	Example	18
2.5.2	Example	19
2.5.3	Substitution Rule	19
2.5.4	Example	19

2.5.5	Bernoulli Equation	20
2.5.6	Example	21
2.5.7	Example	22
3	Modeling using DEs	24
3.1	Linear DE Modeling	24
3.1.1	Standard Problems	24
3.1.2	Population Model	24
3.1.3	Example	25
3.1.4	Radioactive Decay	26
3.1.5	Mixture Problems	26
4	Higher Order DEs	28
4.1	Linear Equations	28
4.1.1	Example	28
4.2	Reduction of Order	29
4.2.1	Example	29
4.2.2	Example	31
4.2.3	Example	33
4.3	Higher Order DEs with Constant Coefficients	33
4.3.1	Example	33
4.3.2	Example	34
4.3.3	Example	34
4.3.4	Example	34
4.3.5	Euler's Formula	35
4.3.6	Example	36
4.3.7	Example	36
4.3.8	Example	37
4.4	Nonhomogeneous, Linear DE with Constant Coefficients	37
4.4.1	Method of Undetermined Coefficients	37
4.4.2	Example	37
4.4.3	Example	39
4.4.4	Method of Undetermined Coefficients 2	40
4.4.5	Steps	40
4.4.6	Example	41
4.4.7	Example	43
4.4.8	What would you guess for the form of y_p ?	44
4.6	Variation of Parameters Method	44
4.6.1	1st Step: General solution of complementary DE	44
4.6.2	Example	44
4.6.3	3×3 Determinants	46

5 Modeling with Higher-Order DEs	47
5.1 Spring-Mass Problems	47
5.1.1 Example	49
5.1.2 Example	50
5.1.3 Undamped Motion	50
5.1.4 Free, Damped Motion	51
5.1.5 Driven Motion	52
5.1.6 Example	53
6 Series Solutions of Linear Equations	55
6.1 Solution by Infinite Series	55
6.1.1 Review of Infinite Series Facts	55
6.1.2 Ratio Test	56
6.1.3 Example	56

Chapter 1

Introduction to Differential Equations

1.1 Terminology and Notation

Differential equation (D.E.) – An equation in which at least one derivative of an unknown function.

Order of the D.E. – The highest order of derivative in the D.E.

1.1.1 – Example

$$4y'' + e^x y' - 3yy' = \sin(x)$$

An example of a partial differential equation is:

$$\frac{\partial T}{\partial x} + x^2 \frac{\partial T}{\partial y} = x + y$$

however, we won't study these in this course.

1.1.2 – Linear vs Non-Linear DE's

Linear D.E. – The dependent variable and all of its derivatives in the D.E. are in separate terms to the 1st power. $y^{(n)}$ or $\frac{d^n y}{dx^n}$ where $n \neq 1$ are non-first power.

$$4y'' + e^x y' - 3yy' = \sin(x)$$

is a non-linear D.E. while

$$4y'' + e^x y' - 3y = \sin(x)$$

is linear.

The general formula of a linear D.E. would look like

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = g(x)$$

Solution – a function $\phi(x)$ and an interval I for which the D.E. is satisfied when $y = \phi(x)$ for all x in I .

It may be the case that the natural domain of $\phi(x)$ is larger than I .

1.1.3 – Example

$y' = -\frac{1}{x^2}$ has a solution $\phi(x) = \frac{1}{x}$ on $I = (0, \infty)$ but the domain of $\phi(x) = (-\infty, 0) \cup (0, \infty)$.

Practice:

$$\frac{d^2x}{dt^2} + 16x = 0$$

Show (*Verify* not derive) $x(t) = c_1 \sin(4t)$ is a solution on $(-\infty, \infty)$ where c is any real parameter.

$$\begin{aligned} x &= c_1 \sin(4t) \\ \frac{dx}{dt} &= 4c_1 \cos(4t) \\ \frac{d^2x}{dt^2} &= -16c_1 \sin(4t) \\ \text{LHS} &= \frac{d^2x}{dt^2} + 16x \\ &= -16c_1 \sin(4t) + 16(c_1 \sin(4t)) \\ &= 0 = \text{RHS} \end{aligned}$$

But the equation $x = c_2 \cos(4t)$ would also be a solution. If you have 2 equations that are both solutions, you could add them together and you would still have a solution. $x = c_1 \sin(4t) + c_2 \cos(4t)$ is a solution for all parameters c_1 and c_2 . In fact, this is the general solution to the D.E.

The D.E.

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

Show $y = (\frac{1}{4}x^2 + C)^2$ is a one parameter family of solutions

$$\begin{aligned} \text{LHS} &= \frac{dy}{dx} = 2 \left(\frac{1}{4}x^2 + C \right) \times \frac{1}{2}x \\ &= x \left(\frac{1}{4}x^2 + C \right) \\ \text{RHS} &= xy^{\frac{1}{2}} = x \left(\left(\frac{1}{4}x^2 + C \right)^2 \right)^{\frac{1}{2}} \\ &= x \left(\frac{1}{4}x^2 + C \right) \\ \text{LHS} &= \text{RHS} \end{aligned}$$

But there is another solution: namely $y(x) = 0$ for all x . This is called the “trivial solution”.

1.2 Initial Value Problems (IVP)

1st order IVP is a 1st order D.E. together with one extra condition:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

2nd order IVP

$$y'' = f(x, y, y')$$

Initial conditions:

- $y(x_0) = y_0$
- $y'(x_0) = y_1$

1.2.1 – Example

$$y' = y \text{ and } y(0) = 3$$

$y = ce^x$ is a one-parameter family of solutions

$$\frac{d}{dx}(ce^x) = ce^x = y$$

$$\begin{aligned} ce^1 &= -2 \\ c &= -\frac{2}{e} \\ y &= \left(-\frac{2}{e}\right)e^x \\ y &= -2e^{x-1} \end{aligned}$$

1.2.2 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Given that you have the solution: $y = \frac{1}{x^2+C}$, Solve:

$$\begin{aligned} -1 &= \frac{1}{(0)^2 + c} \\ -1 &= \frac{1}{c} \\ -1 \times c &= 1 \\ c &= -1 \\ y &= \frac{1}{x^2 - 1}, I = (-1, 1) \end{aligned}$$

1.2.3 – Example

D.E.: $y' + 2xy^2 = 0$ and $y(0) = 1$

Example

$$x'' + 16x = 0 \text{ and } x\left(\frac{\pi}{2}\right) = 5 \text{ and } x'\left(\frac{\pi}{2}\right) = -4$$

$$\begin{aligned} x &= c_1 \cos(4t) + c_2 \sin(4t) \\ 5 &= c_1 \cos(4t) + c_2 \sin(4t) \\ &= c_1 \cos(2\pi) + c_2 \sin(2\pi) \\ &= c_1(1) + c_2(0) \\ &= c_1 \\ x' &= -4c_1 \sin(4t) + 4c_2 \cos(4t) \\ -4 &= -4c_1 \sin\left(4\left(\frac{\pi}{2}\right)\right) + 4c_2 \cos\left(4\left(\frac{\pi}{2}\right)\right) \\ &= -4c_1 \sin(2\pi) + 4c_2 \cos(2\pi) \\ &= -4c_1(0) + 4c_2(1) \\ &= 4c_2 \\ -1 &= c_2 \end{aligned}$$

Reasonable Question: Given a 1st order IVP, can we say whether a solution *exists* or not and, if a solution exists, is it *unique*.

Theorem: Given $y' = f(x, y)$ and $y(x_0) = y_0$, if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous on a rectangle R containing (x_0, y_0) in its interior, then there exists an interval $I = (x_0 - h, x_0 + h)$ where $h > 0$ such that there exists a unique solution to IVP on I .

1.2.4 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(1) = 2$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous everywhere its defined $y \geq 0$
- $\frac{\partial f}{\partial y} = x^{\frac{1}{2}}y^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$ is continuous everywhere its defined $y > 0$

1.2.5 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(0) = 0$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous for all x and $y \geq 0$
- $\frac{\partial f}{\partial y} = \frac{x}{2y}$ is continuous for all x and $y > 0$.
- ***Theorem does not give any conclusion.***

Chapter 2

First-Order Differential Equations

2.1 Solution Curves Without a Solution

Given a 1st order D.E. $y' = f(x, y)$, y' is the slope of the tangent line at any point (x_0, y_0) on a solution curve

2.1.1 – Example

$$y' = f(x, y) = x + y$$

- $f(0, 0)=0$
- $f(1, 0)=1$

2.1.2 – Slope/Direction Fields

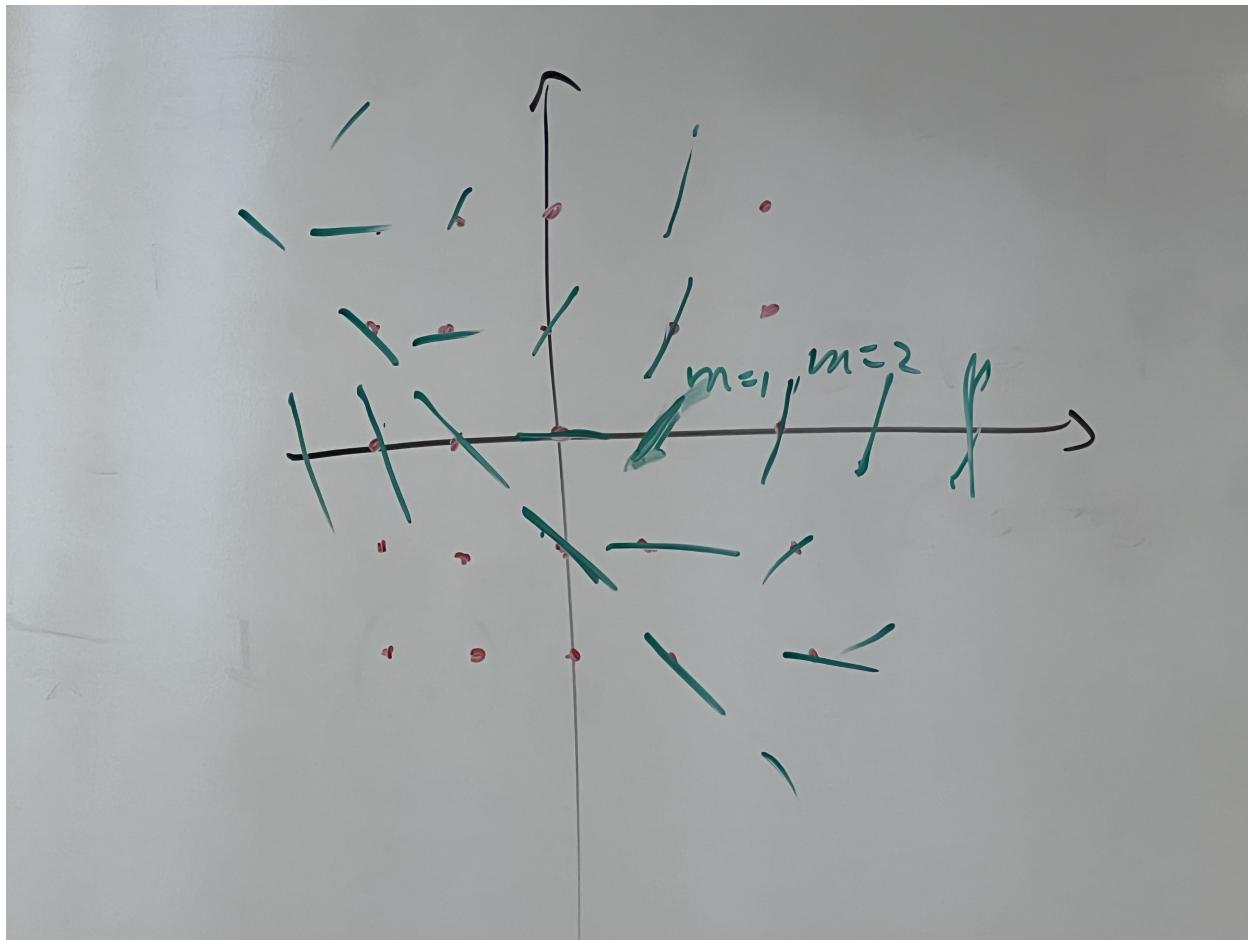


Figure 2.1: The direction field for the previous example

If the function $f(x, y)$ in the D.E. $y' = f(x, y)$ is reasonably simple so that we can solve $f(x, y) = 0$, we can make a “phase portrait diagram”. We will also assume $f(x, y)$ only involves the y -variable.

2.1.3 – Example

$$\begin{aligned} y' &= (y + 2)(y - 3)(y - 5) \\ f(x, y) &= (y + 2)(y - 3)(y - 5) \end{aligned}$$

An “equilibrium solution” is a solution where y is a constant. In this example: $y = 3$, $y = 5$, $y = -2$ are each constant functions.

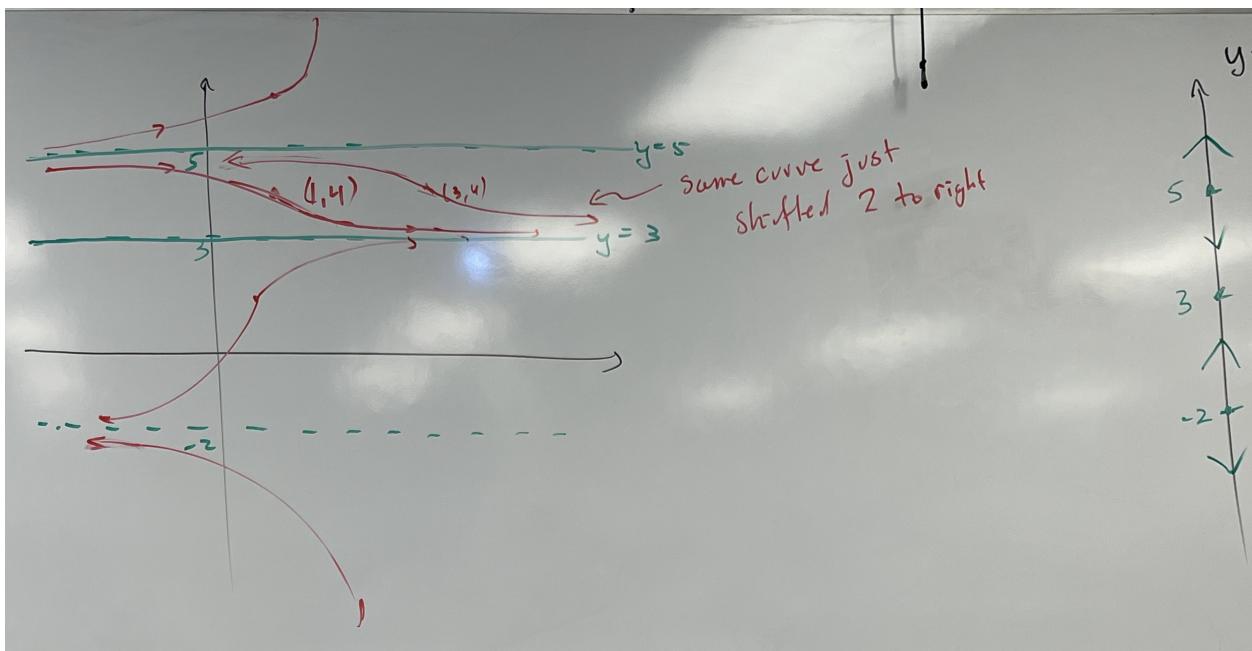


Figure 2.2: The equilibrium solution for the previous example.

The area around $y = 5$ is an unstable equilibrium since the solutions diverge and go in separate directions away from $y = 5$. The area around $y = 3$ is a stable equilibrium because the slopes above and below it converge to $y = 3$. The area around $y = -2$ is semi-stable, since all the slopes around it will converge in one direction, but the point isn't always $y = -2$.

2.2 Separable Differential Equations

Separable D.E.s are DE's $\frac{dy}{dx} = f(x, y)$ where $f(x, y)$ can be factored as $f(x, y) = g(x)h(y)$.

$$\frac{dy}{dx} = (1 + y^2)x^3 \text{ is separable}$$

$$\frac{dy}{dx} = \sin(xy) \text{ is not separable}$$

$$\frac{dy}{dx} = x^3y \text{ is not separable}$$

$$\frac{5}{xy} \frac{dy}{dx} = (x^2 + y) e^y$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy(x^2 + y)e^y}{5} \\ &= \frac{x(x^2 + y)}{5} \times ye^y \end{aligned}$$

2.2.1 – Method of Solution

“Separate the variable” to get $\frac{1}{h(y)}dy = g(x)d$ or $p(y)dy = g(x)dx$ where $p(y) = \frac{1}{h(y)}$. **Integrate both sides**

$$\int p(y)dy = \int g(x)dx \text{ and if possible, solve for } y$$

2.2.2 – Example

$$\begin{aligned} \frac{dy}{dx} &= (1 + y^2)x^3 \\ \int \frac{1}{1 + y^2}dy &= \int x^3dx \\ \tan^{-1}(y) + C_1 &= \frac{x^4}{4} + C_2 \\ \tan^{-1}(y) &= \frac{x^4}{4} + C_2 - C_1 \\ \tan^{-1}(y) &= \frac{x^4}{4} + C \\ y &= \tan\left(\frac{x^4}{4} + C\right) \end{aligned}$$

2.2.3 – Example

Problem 12 from the textbook.

$$\begin{aligned} \sin(3x)dx + 2y \cos^3(3x)dy &= 0 \\ \int -2ydy &= \int \frac{\sin(3x)}{\cos^3(x)}dx \\ &= \int \tan(3x) \sec^2(3x)dx \\ &= \int u \frac{1}{3}du \text{ where } u = \tan(3x), du = 3 \sec^2(3x)dx \\ -2 \int ydy &= \frac{1}{3} \int u du + C \\ -y^2 &= \frac{u^2}{6} + C \\ &= \frac{\tan^2(3x)}{6} + C \\ \frac{\tan^2(3x)}{6} + y^2 &= -C \\ \frac{\tan^2(3x)}{6} + y^2 &= C \end{aligned}$$

Problem 25 from the textbook.

$$x^2 \frac{dy}{dx} = y - xy, y(-1) = -1$$

$$x^2 \frac{dy}{dx} = y - xy$$

$$x^2 \frac{dy}{dx} = y(1 - x)$$

$$\frac{dy}{y} = \frac{(1-x)}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{(1-x)}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{x^2} dx - \int \frac{x}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{x^2} dx - \int \frac{1}{x} dx$$

$$\ln |y| + C_1 = -\frac{1}{x} + C_2 - \ln |x| + C_3$$

$$\ln |y| = -\frac{1}{x} - \ln |x| + C$$

$$y = e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C$$

$$y = e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C$$

$$y = e^{-\frac{1}{x}} \times \frac{1}{|x|} \times e^C$$

$$y = \frac{1}{|x|} e^{C-\frac{1}{x}}$$

$$-1 = \frac{1}{|-1|} e^{C-\frac{1}{-1}}$$

$$-1 = \frac{1}{1} e^{C-(-1)}$$

$$-1 = e^{C+1}$$

2.3 First Order Linear Differential Equations

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\left. \frac{dy}{dx} + P(x)y = f(x) \right\} \text{ Standard form of a 1st-order linear DE}$$

We will try to find a function $\mu(x)$ such that by multiplying the D.E. by an integrating factor (I.F.) $\mu(x)$:

$$\mu(x) \frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

such that the LHS is an exact derivative, Observe:

$$\frac{d}{dx}(\mu(x)y) = \mu(x) \frac{dy}{dx} + \frac{dy}{dx}y$$

from which we see

$$\begin{aligned}\mu(x)P(x) &= \frac{d\mu}{dx} \\ P(x)dx &= \frac{d\mu}{\mu(x)} \\ \int P(x)dx &= \int \frac{d\mu}{\mu} \\ \int P(x)dx &= \ln \mu \\ \ln \mu &= \int P(x)dx \\ \mu &= e^{\int P(x)dx}\end{aligned}$$

2.3.1 – Example

$$\begin{aligned}x \frac{dy}{dx} - 4y &= x^6 e^x \\ \text{Standard form: } \frac{dy}{dx} - \frac{4}{x}y &= x^5 e^x \\ P(x) &= -\frac{4}{x} \\ \mu &= e^{\int -\frac{4}{x} dx} \\ &= e^{-4 \ln x} \\ &= e^{\ln x^{-4}} \\ &= x^{-4} \\ \text{I.F.} &= \mu = x^{-4}\end{aligned}$$

Now multiply the standard form of the given D.E. by x^{-4} .

$$\begin{aligned}x^{-4} \frac{dy}{dx} - x^{-4} \frac{4}{x}y &= x^{-4} x^5 e^x \\ x^{-4} \frac{dy}{dx} - x^{-4} \frac{4}{x}y &= x e^x \\ \int \frac{d}{dx}(x^{-4}y) &= \int x e^x \\ x^{-4}y &= \int x e^x\end{aligned}$$

2.3.2 – Example

$$\begin{aligned}
 & (x^2 - 9) \frac{dy}{dx} + xy = 0 \\
 & (x^2 - 9) \frac{dy}{dx} + xy = 0 \\
 & \frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0 \\
 & P(x) = \frac{x}{x^2 - 9} \\
 & \int P(x)dx = \int \frac{x}{x^2 - 9}dx \\
 & \int P(x)dx = \int \frac{1}{u-9} \frac{du}{2} \\
 & \int P(x)dx = \frac{1}{2} \int \frac{1}{u-9}du \\
 & \int P(x)dx = \frac{1}{2} \ln|u-9| \\
 & \int P(x)dx = \frac{1}{2} \ln|x^2-9| \\
 & \mu = e^{\frac{1}{2} \ln|x^2-9|} \\
 & \mu = e^{\ln|(x^2-9)^{\frac{1}{2}}|} \\
 & \mu = (x^2 - 9)^{\frac{1}{2}} \\
 & \mu = \sqrt{x^2 - 9} \\
 & \sqrt{x^2 - 9} \left(\frac{dy}{dx} + \frac{x}{x^2 - 9}y \right) = \sqrt{x^2 - 9}(0) \\
 & \sqrt{x^2 - 9} \frac{dy}{dx} + \frac{x}{\sqrt{x^2 - 9}}y = 0 \\
 & \int \frac{d}{dx} \left(y\sqrt{x^2 - 9} \right) = \int 0 \\
 & y\sqrt{x^2 - 9} = C \\
 & y = \frac{C}{\sqrt{x^2 - 9}}
 \end{aligned}$$

2.4 Exact Equations

1st Order D.E. in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

Given a function

$$z = f(x, y)$$

, the total differential, dz , is defined as

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

2.4.1 – Method

See if we can find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

If we can do this, then the D.E. is equivalent to

$$df = 0 \Rightarrow f(x, y) = c$$

is an implicit solution of D.E.

Assume that M and N have continuous 1st order partials (assuming f exists)

$$\left. \begin{array}{l} My = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} dy = f_{xy} \\ Nx = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} dy = f_{yx} \end{array} \right\} \text{Theorem tells us these are equal}$$

This provides a quick test to check if the D.E. is exact or not.

2.4.2 – Example

$$2xydx + (x^2 - 1)dy = 0$$

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

To check if the D.E. is exact

$$M_y = 2x = N_x$$

We now know there exists a function $f(x, y)$ with

$$\begin{aligned} \frac{\partial f}{\partial x} &= M = 2xy \\ \frac{\partial f}{\partial y} &= N = x^2 - 1 \end{aligned}$$

$$\begin{aligned}
 f_M(x, y) &= \int \frac{\partial f}{\partial x} dx \\
 &= \int 2xy dx \\
 &= x^2y + \phi(y) \\
 \frac{\partial f}{\partial y} (x^2y + \phi(y)) &= x^2 - 1 \text{ required to equal } N \\
 x^2 + \phi'(y) &= x^2 - 1 \\
 \phi'(y) &= -1 \\
 \phi(y) &= \int -1 dy \\
 &= -y \\
 f(x, y) &= x^2y - y \\
 d(f(x, y)) &= 0 \\
 f(x, y) &= c \\
 x^2y - y = c &\text{ is an implicit solution of the D.E.}
 \end{aligned}$$

Note: the f_M format is just there to show which partial equation was integrated. It was made by me and, as far as I know, is not standardly known.

2.4.3 – Example

$$\begin{aligned}
 (e^{2y} - y \cos(xy)) dx + (2xe^{2y} - x \cos(xy) + 2y) dy &= 0 \\
 M_y &= N_x \\
 \frac{\partial}{\partial y} (e^{2y} - y \cos(xy)) &= \frac{\partial}{\partial x} (2xe^{2y} - x \cos(xy) + 2y) \\
 2e^{2y} - [\cos(xy) - y \sin(xy) \times x] &= 2e^{2y} - (\cos(xy) - x \sin(xy) \times y) + 0 \\
 2e^{2y} - \cos(xy) + xy \sin(xy) &= 2e^{2y} - \cos(xy) + xy \sin(xy) \\
 \frac{\partial f}{\partial x} &= M = e^{2y} - y \cos(xy) \\
 \frac{\partial f}{\partial y} &= N = 2xe^{2y} - x \cos(xy) + 2y \\
 f_N(x, y) &= \int \frac{\partial f}{\partial y} dy \\
 &= \int (2xe^{2y} - x \cos(xy) + 2y) dy \\
 &= \frac{2xe^{2y}}{2} - \frac{x \sin(xy)}{x} + 2 \times \frac{y^2}{2} + \phi(x) \\
 &= xe^{2y} - \sin(xy) + y^2 + \phi(x)
 \end{aligned}$$

Take the ∂x of this and equate with M :

$$\begin{aligned} M &= \frac{\partial}{\partial x} (xe^{2y} - \sin(xy) + y^2 + \phi(x)) \\ e^{2y} - y \cos(xy) &= e^{2y} - y \cos(xy) + 0 + \phi'(x) \\ 0 &= \phi'(x) \\ \phi(x) &= c \end{aligned}$$

So $f(x, y) = c_2$ is the solution

$$xe^{2y} - \sin(xy) + y^2 = c$$

$$dx = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

2.4.4 – What can you do if $M_y \neq N_x$

Sometimes you can multiply the DE by an integrating factor $\mu(x, y)$ to get an exact DE.

If

$$\frac{M_y - N_x}{N}$$

is a function of only x , then

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

will be an I.F.

If

$$\frac{N_x - M_y}{M}$$

is a function of only y , then

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

will be an I.F.

2.4.5 – Example

$$xydx + (2x^2 + 3y^2 - 20) dy = 0$$

$$M_y = x$$

$$N_x = 4x$$

$$M_y \neq N_x$$

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy}$$

$$= \frac{3x}{xy}$$

$$= \frac{3}{y} \text{ is a function of just } y$$

So:

$$\begin{aligned}\mu &= e^{\int \frac{3}{y} dy} \\ &= e^{3 \ln y} \\ &= y^3 \\ xy^4 dx + y^3 (2x^2 + 3y^2 - 20) dy &= 0(y^3) \\ xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3) dy &= \\ M_y &= N_x \\ 4xy^3 &= 4xy^3 \\ \frac{\partial f}{\partial x} &\end{aligned}$$

2.5 Substitution Methods

Taking a D.E. that's not:

- Separable
- 1st Order Linear
- Exact

and making a substitution to turn the new D.E. into one of these.

Theorem: Given a D.E.

$$M(x, y)dx + N(x, y)dy = 0$$

A function $f(x, y)$ is said to be homogenous of order α if $f(tx, ty) = t^\alpha f(x, y)$.

2.5.1 – Example

Given:

$$f(x, y) = x^3 + 5xy^2 - y^3$$

Then:

$$\begin{aligned}f(tx, ty) &= (tx)^3 + 5(tx)(ty)^2 - (ty)^3 \\ &= t^3x^3 + 5t^3xy^2 - t^3y^3 \\ &= t^3(x^3 + 5xy^2 - y^3) \\ &= t^3f(x, y)\end{aligned}$$

2.5.2 – Example

$$\begin{aligned}
 f(x, y) &= \frac{x + y}{x^2 + y^2} \\
 f(tx, ty) &= \frac{tx + ty}{(tx)^2 + (ty)^2} \\
 f(tx, ty) &= \frac{tx + ty}{x^2 t^2 + y^2 t^2} \\
 f(tx, ty) &= \frac{t}{t^2} \times \frac{x + y}{x^2 + y^2} \\
 f(tx, ty) &= \frac{t}{t^2} f(x, y) \\
 f(tx, ty) &= \frac{1}{t} f(x, y)
 \end{aligned}$$

$f(x, y) = \frac{x+y}{x^2+y^2}$ is homogenous of order $\alpha = -1$

2.5.3 – Substitution Rule

If $M(x, y)$ and $N(x, y)$ are homogenous, each of the same order, then $u = \frac{y}{x}$ i.e., $y = ux$ or $v = \frac{x}{y}$ (i.e. $x = vy$) will produce a separable D.E.

2.5.4 – Example

Solve the separable D.E. and then back-substitute

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

$$\begin{aligned}
 M(x, y) &= x^2 + y^2 & N &= x^2 - xy \\
 M_y &= 2y & N_x &= 2x - y \\
 M_y &\neq N_x \\
 M(tx, ty) &= (tx)^2 + (ty)^2 & & \\
 &= t^2x^2 + t^2y^2 & & \\
 &= t^2(x^2 + y^2) & & \\
 &= t^2M(x, y) & M \text{ is homogeneous of order 2 and so is } N \\
 u &= \frac{y}{x} & & \\
 y &= ux & & \\
 dy &= udx + xdu & & \\
 (x^2 + (ux)^2)dx + (x^2 - x(ux))(udx + xdu) &= 0 & & \\
 (x^2 + u^2x^2)dx + (x^2 - ux^2)(udx + xdu) &= 0 & & \\
 (1 + u^2)x^2dx + x^2(1 - u)(udx + xdu) &= 0 & & \\
 (1 + u^2)x^2dx + x^2(udx + xdu - u^2dx - uxdu) &= 0 & & \\
 x^2(1dx + u^2dx + udx + xdu - u^2dx - uxdu) &= 0 & & \\
 x^2(1dx + u^2dx - u^2dx + udx + xdu - uxdu) &= 0 & & \\
 x^2(1dx + udx + xdu - uxdu) &= 0 & & \\
 x^2(1 + u)dx + x^3(1 - u)du &= 0 & & \\
 \int \frac{1}{x}dx &= \int -\frac{1-u}{1+u}du & & \\
 &= \int \frac{u-1}{u+1}du & & \\
 &= \int \frac{u+(1-2)}{u+1}du & & \\
 &= \int \left(\frac{u+1}{u+1} - \frac{2}{u+1} \right) du & & \\
 &= \int \left(1 - \frac{2}{u+1} \right) du & & \\
 \ln|x| &= \int \left(1 - \frac{2}{u+1} \right) du & & \\
 &= u - 2 \ln|u+1| + C & & \\
 \ln|x| &= \frac{y}{x} - 2 \ln \left| \frac{y}{x} + 1 \right| + C & &
 \end{aligned}$$

2.5.5 – Bernoulli Equation

Theorem: An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where $n \neq 0, 1$ is called a Bernoulli Equation. The substitution

$$u = y^{1-n}$$

will transform the D.E. into a 1st order linear.

2.5.6 – Example

$$\begin{aligned} x \frac{dy}{dx} + y &= x^2 y^2 \\ \frac{dy}{dx} + \frac{y}{x} &= x y^2 \end{aligned}$$

is a **Bernoulli equation** with $n = 2$.

$$\begin{aligned} u &= y^{1-2} \\ &= y^{-1} \\ &= \frac{1}{y} \\ \frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\ &= -1y^{-2} \frac{dy}{dx} \\ &= -\frac{1}{y^2} \frac{dy}{dx} \\ -y^{-2} \frac{dy}{dx} + -y^{-2} \times \frac{y}{x} &= -y^{-2} \times xy^2 \\ -y^{-2} \frac{dy}{dx} + -\frac{1}{x} y^{-1} &= -x \\ \frac{du}{dx} - \frac{1}{x} u &= -x \end{aligned}$$

$$\begin{aligned}\text{I.F.} &= \mu = e^{P(x)dx} \\ &= e^{-\int \frac{1}{x} dx} \\ &= e^{-\ln|x|} \\ &= e^{\ln|x^{-1}|} \\ &= x^{-1}\end{aligned}$$

$$\begin{aligned}\frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u &= -1 \\ \frac{d}{dx} \left(\frac{1}{x} u \right) &= -1 \\ \int \frac{d}{dx} \left(\frac{1}{x} u \right) dx &= \int -1 dx \\ \frac{1}{x} u &= \int -1 dx \\ \frac{1}{x} u &= -x + C \\ \frac{1}{x} \times 1y &= -x + C \\ \frac{1}{x(-x+C)} &= y \\ y &= \frac{1}{Cx-x^2}\end{aligned}$$

Theorem: If the D.E. can be expressed as

$$\frac{dy}{dx} = f(Ax + by + C)$$

for particular numbers A, B, C , then let

$$u = Ax + By + C$$

to get a separable D.E.

2.5.7 – Example

$$\frac{dy}{dx} = (-2x + y)^2 - 7, y(0) = 0$$

$$\begin{aligned} u &= -2x + y \\ \frac{du}{dx} &= \frac{dy}{dx} \times \frac{du}{dy} \\ &= -2 + \frac{dy}{dx} \\ \frac{du}{dx} + 2 &= \frac{dy}{dx} \\ \frac{du}{dx} + 2 &= u^2 - 7 \\ \frac{du}{u^2 - 9} &= dx \\ \int \frac{du}{u^2 - 9} &= \int dx \\ \int \frac{du}{(u+3)(u-9)} &= x + C \\ \int \frac{du}{(u+3)(u-9)} &= x + C \end{aligned}$$

Chapter 3

Modeling using DEs

3.1 Linear DE Modeling

3.1.1 – Standard Problems

- 1) Population Growth (or decline)
- 2) Radioactive Decay
- 3) Newton's Law of Cooling
- 4) Mixture Problems

3.1.2 – Population Model

Assume the rate of population change is proportional to the size of the population

$$P(t) = \text{population at time } t$$

$$\frac{dP}{dt} = kP$$

$\frac{\frac{dP}{dt}}{P} = k$ is the relative growth rate of the population

$$\begin{aligned}
 \frac{dP}{dt} &= kP \\
 \frac{dP}{P} &= kdt \\
 \int \frac{dP}{P} &= \int kdt \\
 \ln|P| &= kt + C \\
 |P| &= e^{kt+C} \\
 |P| &= e^{kt}e^C \\
 |P| &= Ae^{kt} \text{ where } A > 0 \\
 P &= \pm Ae^{kt} \\
 P &= Be^{kt} \text{ where } B \neq 0 \\
 P &= De^{kt} \text{ where } D \text{ can be any real number}
 \end{aligned}$$

The constant can become any number because 0 would be a valid rate of population change, it means that the population size isn't changing.

3.1.3 – Example

If, initially at 2 p.m., there are 1,000 bacteria on a petri dish and at 4 p.m., there are 2,000 bacteria. Assuming constant relative growth rate, how many bacteria are there at 5 p.m.? $P(t)$ = population t hours after 2 p.m.

$$\begin{aligned}
 P(t) &= Ae^{kt} \\
 1000 &= Ae^{(0)k} \\
 1000 &= Ae^0 \\
 1000 &= A(1) \\
 A &= 1000
 \end{aligned}$$

$$\begin{aligned}
 P(2) &= 2000 \\
 P(2) &= 1000e^{2k} \\
 2000 &= 1000e^{2k} \\
 2 &= e^{2k} \\
 \ln(2) &= 2k \\
 k &= \frac{\ln(2)}{2}
 \end{aligned}$$

$$\begin{aligned}
P(t) &= 1000e^{\frac{\ln(2)}{2}t} \\
P(3) &= 1000e^{\frac{\ln(2)}{2}(3)} \\
&= 1000e^{1.5\ln(2)} \\
&= 1000e^{\ln(2^{1.5})} \\
&= 1000(2^{1.5}) \\
&= 2000(\sqrt{2}) \\
P(3) &\approx 2828.427(\sqrt{2}) \\
P(t) &= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\ln(2^{\frac{t}{2}})} \\
&= 1000 \times 2^{\frac{t}{2}}
\end{aligned}$$

3.1.4 – Radioactive Decay

$$m(t) = m_0 e^{kt} \text{ where } k < 0$$

The Half-Life is the amount of time it takes for half of the original amount to remain:

$$\frac{1}{2}A_0 = A_0 e^{kt} \Rightarrow \frac{1}{2} = e^{kt}$$

3.1.5 – Mixture Problems

Setup

Initially, the container has 200 gallons of brine solution (salt-water) of concentration $\frac{10 \text{ lbs}}{200 \text{ gallons}} = 0.05 \frac{\text{lbs}}{\text{gallon}}$. A solution of $\frac{5 \text{ lbs}}{200 \text{ gallons}} = 0.025 \frac{\text{lbs}}{\text{gallon}}$ is poured into the initial container at a rate of $\frac{4 \text{ gallons}}{\text{min}}$. How many pounds of salt are there in the container after 2 hours.

Let $A(t) = \# \text{ lbs of salt } t \text{ minutes after the process starts}$

$\frac{dA}{dt}$ = The rate of change of # lbs of salt

$$\begin{aligned}\frac{dA}{dt} &= 0.025 \frac{\text{lbs}}{\text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \Bigg\} \text{rate in} \\ &\quad - \frac{A(t) \text{lbs}}{200 \text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \Bigg\} \text{rate out} \\ &= (0.025)4 \frac{\text{lbs}}{\text{min}} - \frac{4A(t)}{200} \frac{\text{lbs}}{\text{min}} \\ &= 0.1 \frac{\text{lbs}}{\text{min}} - \frac{A(t)}{50} \frac{\text{lbs}}{\text{min}} \\ &= 0.1 - \frac{A(t)}{50}\end{aligned}$$

$$\begin{aligned}\frac{dA}{dt} + \frac{1}{50}A &= 0.1 \\ \mu &= e^{\int P(t)dt} \\ &= e^{\int \frac{1}{50}dt} \\ &= e^{\frac{t}{50}}\end{aligned}$$

$$\begin{aligned}e^{\frac{t}{50}} \left(\frac{dA}{dt} \right) + e^{\frac{t}{50}} \left(\frac{1}{50}A \right) &= e^{\frac{t}{50}}(0.1) \\ \frac{d}{dt} \left(e^{\frac{t}{50}} A \right) &= e^{\frac{t}{50}}(0.1) \\ \int \frac{d}{dt} \left(e^{\frac{t}{50}} A \right) dt &= \int \frac{1}{10} e^{\frac{t}{50}} dt \\ e^{\frac{t}{50}} A &= \frac{1}{10} \times \frac{e^{\frac{t}{50}}}{\frac{1}{50}} + C \\ e^{\frac{t}{50}} A &= 5e^{\frac{t}{50}} + C \\ A(t) &= 5 + Ce^{-\frac{t}{50}} \\ &= 5 + Ce^{-0.02t} \\ A(120) &= 5 + Ce^{-0.02(120)} \\ &= 5 + Ce^{-2.4}\end{aligned}$$

Chapter 4

Higher Order Differential Equations

4.1 Linear Equations

An n th order DE is linear if it has the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1(x) \frac{dy}{dx} + a_0 y = g(x)$$

Theorem: If all the coefficient functions are continuous and $a_n(x)$ is not 0 on an interval I and $g(x)$ is continuous, then any initial value problem

$$DE + y(x_0) = y_0$$

has a unique solution on the interval I if $g(x) = 0$. i.e.

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

then the DE is said to be homogeneous.

4.1.1 – Example

$$y'' - 3y' - 4y = 0$$

Show $y_1 = e^{4x}$ is a solution and $y_2 = e^{-x}$ is a solution.

$$y_1 = e^{4x}$$

$$y'_1 = 4e^{4x}$$

$$y''_1 = 16e^{4x}$$

$$16e^{4x} - 3(4e^{4x}) - 4e^{4x} = 0$$

$$16e^{4x} - 12e^{4x} - 4e^{4x} = 0$$

$$e^{4x}(16 - 12 - 4) = 0$$

$$e^{4x}(0) = 0$$

$$0 = 0$$

$$y_3 = 6y_1 = 6e^{4x}$$

$$y'_3 = 6y'_1 = 24e^{4x}$$

$$y''_3 = 6y''_1 = 96e^{4x}$$

$$96e^{4x} - 3(24e^{4x}) - 4(6e^{4x}) = 0$$

$$96e^{4x} - 72e^{4x} - 24e^{4x} = 0$$

$$e^{4x}(96 - 72 - 24) = 0$$

$$e^{4x}(0) = 0$$

$$0 = 0$$

Theorem: Superposition Principle: if y_1, y_2, \dots, y_m are each solutions of an n th order Linear, homogenous DE, then $c_1y_1 + c_2y_2 + \dots + c_my_m$ will also be a solution for any constants c_1, c_2, \dots, c_m .

Our goal is to express the general solution in as concise a way as possible.

Linear combination – a collection of solutions y_1, y_2, \dots, y_m is linearly independent if the only way $c_1y_1 + c_2y_2 + \dots + c_my_m = 0$ is iff (if and only if) all of the constants $c_1, c_2, \dots, c_m = 0$. Otherwise we say y_1, y_2, \dots, y_m are linearly dependent.

Theorem: If the DE is an n th order Linear Homogeneous equation then there will exist a collection of n linearly independent solutions y_1, y_2, \dots, y_n and the general solution will be $y_c = c_1y_1 + c_2y_2 + \dots + c_ny_n$

One way to check for linear independence is to compile the Wronskian

$$W(y_1, y_2, \dots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

4.2 Reduction of Order

If you have one solution to a 2nd order linear homogenous DE, then it is possible to use that function to construct a 2nd Linear Independent solution to the DE.

4.2.1 – Example

For example, the DE

$$y'' - y = 0$$

One solution is $y = e^x$ on $(-\infty, \infty)$.

Idea: We look for y_2 of the form

$$y_2(x) = u(x)y_1(x) \text{ where } u(x) \text{ is not a constant}$$

The general solution is of the form:

$$y = c_1y_1 + c_2y_2$$

where y_1 and y_2 are linearly independent solutions.

To find $u(x)$, we substitute this into the DE

$$\begin{aligned} y_2 &= u(x)y_1(x) \\ y'_2 &= u(x)y'_1(x) + u'(x)y_2(x) \\ y''_2 &= u(x)y''_1(x) + u'(x)y_1(x) + u'(x)y'_2(x) + u''(x)y_1(x) \\ &= uy''_1 + 2u'y'_1 + u''y_1 \end{aligned}$$

So $y'' - y = 0$ becomes

$$\begin{aligned} uy''_1 + 2u'y'_1 + u''y_1 - uy_1 &= 0 \text{ when we sub } y = y_2 = uy_1 \\ u(e^x)'' + 2u'(e^x)' + u''(e^x) - u(e^x) &= 0 \\ ue^x + 2u'e^x + u''e^x - ue^x &= 0 \\ 2u'e^x + u''e^x &= 0 \\ e^x(2u' + u'') &= 0 \\ 2u' + u'' &= 0 \end{aligned}$$

Let $w = u'$

$$\begin{aligned} 2w + w' &= 0 \\ 2w + \frac{dw}{dx} &= 0 \\ \frac{dw}{dx} &= -2w \\ \frac{dw}{w} &= -2dx \\ \int \frac{dw}{w} &= \int -2dx \\ \ln|w| &= -2x \\ w &= e^{-2x} \\ u' &= e^{-2x} \\ \int u' &= \int e^{-2x} \\ u &= -\frac{1}{2}e^{-2x} \\ y_2 &= uy_1 \\ &= -\frac{1}{2}e^{-2x} \times e^x \\ &= -\frac{1}{2}e^{-x} \end{aligned}$$

Double check that y_2 is a solution of the DE

$$\begin{aligned}y_2 &= -\frac{1}{2}e^{-x} \\y'_2 &= \frac{1}{2}e^{-x} \\y''_2 &= -\frac{1}{2}e^{-x} \\y''_2 - y &= -\frac{1}{2}e^{-x} - \left(-\frac{1}{2}e^{-x}\right) \\&= -\frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} \\&= 0\end{aligned}$$

In general,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

put into standard form by dividing by $a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$, the same method as in our **example** leads to the formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \quad (4.1)$$

4.2.2 – Example

Part 1

$$x^2y'' - 3xy' + 4y = 0$$

Verify that $y_1 = x^2$ is a solution $y'_1 = 2x$, $y''_1 = 2$.

$$\begin{aligned}x^2y'' - 3xy' + 4y &= 0 \\x^2(2) - 3x(2x) + 4(x^2) &= 0 \\2x^2 - 6x^2 + 4x^2 &= 0 \\6x^2 - 6x^2 &= 0 \\0 &= 0\end{aligned}$$

Part 2

Find a linearly independent solution $y_2(x)$.

$$\begin{aligned}
 & x^2 y'' - 3xy' + 4y = 0 \\
 & y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0 \\
 & P(x) = -\frac{3}{x} \\
 & y_2 = y_1 \int \frac{e^{\int \frac{3}{x} dx}}{(y_1(x))^2} dx \\
 & y_2 = y_1 \int \frac{e^{3 \ln|x|}}{(y_1(x))^2} dx \\
 & y_2 = y_1 \int \frac{e^{\ln|x^3|}}{(y_1(x))^2} dx \\
 & y_2 = x^2 \int \frac{x^3}{(x^2)^2} dx \\
 & y_2 = x^2 \int \frac{x^3}{x^4} dx \\
 & y_2 = x^2 \int \frac{1}{x} dx \\
 & y_2 = x^2 \ln|x| + C
 \end{aligned}$$

Part 3: Double check that y_2 is a solution of the DE

$$\begin{aligned}
 y_2 &= x^2 \ln|x| \\
 y'_2 &= x^2 \times \frac{1}{x} + 2x \ln|x| \\
 y''_2 &= 1 + 2x \frac{1}{x} + 2 \ln|x| \\
 &= 1 + 2 + 2 \ln|x| \\
 &= 3 + 2 \ln|x|
 \end{aligned}$$

So the LHS DE becomes

$$\begin{aligned}
 x^2(3 + 2 \ln|x|) - 3x(x + 2x \ln|x|) + 4x^2 \ln|x| &= 3x^2 + 2x^2 \ln|x| - 3x^2 - 6x^2 \ln|x| + 4x^2 \ln|x| \\
 &= 3x^2 - 3x^2 + 2x^2 \ln|x| - 6x^2 \ln|x| + 4x^2 \ln|x| \\
 &= 3x^2 - 3x^2 + 2x^2 \ln|x| - 6x^2 \ln|x| + 4x^2 \ln|x| \\
 &= 0 + x^2 \ln|x|(2 - 6 + 4) \\
 &= x^2 \ln|x|(0) \\
 &= 0
 \end{aligned}$$

Write the general solution of the DE including the interval of the solution

$$\begin{aligned}
 y &= c_1 y_1 + c_2 y_2 \\
 &= c_1 x^2 + c_2 x^2 (\ln |x| + C) \\
 &= c_1 x^2 + c_2 x^2 \ln |x| + C c_2 x^2 \\
 \text{just } y &= c_1 x^2 + c_2 x^2 \ln |x| \text{ on } I = (0, \infty), y(2) = 3, y'(2) = 5
 \end{aligned}$$

4.2.3 – Example

$$\begin{aligned}
 3y'' + y' - 4y &= 0 \\
 y &= e^{mx} \\
 y' &= me^{mx} \\
 y'' &= m^2 e^{mx} \\
 3y'' + y' - 4y &= 3m^2 e^{mx} + me^{mx} - 4e^{mx} \\
 &= e^{mx}(3m^2 + m - 4) \\
 &= e^{mx}(3m^2 + 4)(m - 1) \\
 m = 1 \quad m &= -\frac{4}{3} \\
 y_1 &= e^x, y_2 = e^{-\frac{4}{3}x}
 \end{aligned}$$

4.3 Higher Order, Linear, Homogeneous DE with Constant Coefficients

4.3.1 – Example

$$3y^{(4)} - 2y''' + 7y' + 8y = 0$$

Theorems in 4.1 tell us that the general solution is of the form $y = c_1 y_1$. **Conjecture:** A solution of the form $y = e^{mx} \Rightarrow y' = me^{mx}$.

4.3.2 – Example

$$\begin{aligned}
 5y' - 4y &= 0 \\
 y' - \frac{4}{5}y &= 0 \\
 me^{mx} - \frac{4}{5}e^{mx} &= 0 \\
 e^{mx} \left(m - \frac{4}{5} \right) &= 0 \\
 m - \frac{4}{5} &= 0 \\
 m &= \frac{4}{5}
 \end{aligned}$$

$y = c_1 e^{\frac{4}{5}x}$ is the general solution of the DE

4.3.3 – Example

$$\begin{aligned}
 y'' + 5y' - 6y &= 0 \\
 y(m^2 e^{mx}) + 5(me^{mx}) - 6e^{mx} &= 0 \\
 e^{mx} (m^2 y + 5m - 6) &= 0 \\
 m^2 y + 5m - 6 &= 0 \\
 (m + 6)(m - 1) &= 0 \\
 m + 6 = 0 &\quad m - 1 = 0 \\
 m = -6 &\quad m = 1 \\
 y_1 = e^{-6x} &\quad y_2 = e^x \text{ These are Linearly Independent (L.I)}
 \end{aligned}$$

Therefore:

$$y = c_1 e^{-6x} + c_2 e^x$$

4.3.4 – Example

$$\begin{aligned}
 y'' - 6y' + 9y &= 0 \\
 m^2 e^{mx} - 6(me^{mx}) + 9e^{mx} &= 0 \\
 m^2 - 6m + 9 &= 0 \\
 (m - 3)^2 &= 0 \quad m = 3 \text{ is a repeated root} \\
 m - 3 &= 0 \\
 m &= 3 \\
 y_1 = e^{3x} &\quad y_2 = e^{3x} \text{ are linearly dependent}
 \end{aligned}$$

Use the Reduction of order function:

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \\
&= e^{3x} \int \frac{e^{-\int -6dx}}{(e^{3x})^2} dx \\
&= e^{3x} \int \frac{e^{\int 6dx}}{e^{6x}} dx \\
&= e^{3x} \int \frac{e^{6x}}{e^{6x}} dx \\
&= e^{3x} \int 1 dx \\
&= e^{3x} x \\
&= xe^{3x}
\end{aligned}$$

Always works out for this solution if $e^{m_1 x}$ is a solution and m_1 is a root of multiplicity k than $y_1 = e^{m_1 x}, y_2 = xe^{m_1 x}, \dots, y_k = x^{k-1}e^{m_1 x}$ are linear solutions.

$$y'' + 9y = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \sqrt{-9} \text{ No real solutions}$$

$$m = \pm\sqrt{-9}$$

$$m = \pm 3i$$

$$y = c_1 e^{3ix} + c_2 e^{-3ix} \text{ where } c_1 \& c_2 \text{ arbitrary complex numbers}$$

We'd rather only deal with real-valued solutions.

4.3.5 – Euler's Formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$e^{i3x} = \cos(3x) + i \sin(3x)$$

$$e^{-i3x} = \cos(-3x) + i \sin(-3x)$$

$$e^{-i3x} = \cos(3x) - i \sin(3x)$$

$$e^{i3x} + e^{-i3x} = \cos(3x) + i \sin(3x) + \cos(3x) - i \sin(3x)$$

$$e^{i3x} + e^{-i3x} = 2 \cos(3x)$$

$$Y_1 = \frac{1}{2}e^{i3x} + \frac{1}{2}e^{-i3x} = \cos(3x)$$

$$Y_2 = \sin(3x)$$

$$\frac{1}{2i}y_1 - \frac{1}{2i}y_2 = \sin(3x)$$

General solution:

$$\begin{aligned}
y &= C_1 Y_1 + C_2 Y_2 \\
&= C_1 \cos(3x) + C_2 \sin(3x)
\end{aligned}$$

where C_1 and C_2 are complex numbers that generate all complex-valued solutions of the DE

4.3.6 – Example

$$\begin{aligned}
 y'' + 25y &= 0 \\
 m^2 e^{mx} + 25e^{mx} &= 0 \\
 m^2 + 25 &= 0 \\
 m^2 &= -25 \\
 m &= \pm 5i \\
 \text{General solution} \\
 y_1 &= c_1 y_1 + c_2 y_2 \\
 &= c_1 \cos(5x) + c_2 \sin(5x)
 \end{aligned}$$

4.3.7 – Example

$$\begin{aligned}
 y'' + 2y' + 6y &= 0 \\
 m^2 + 2m + 6 &= 0 \\
 \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-2 \pm \sqrt{2^2 - 4(1)(6)}}{2(1)} \\
 &= \frac{-2 \pm \sqrt{4 - 24}}{2} \\
 &= \frac{-2 \pm \sqrt{-20}}{2} \\
 &= \frac{-2 \pm \sqrt{4 \times -5}}{2} \\
 &= \frac{-2 \pm 2\sqrt{-5}}{2} \\
 &= -1 \pm \sqrt{-5} \\
 &= -1 \pm \sqrt{5}i \\
 y_1 &= e^{(-1+\sqrt{5}i)x} \\
 &= e^{-x} e^{i\sqrt{5}x} \\
 &= e^{-x} \cos(\sqrt{5}x) \\
 y_2 &= e^{(-1-\sqrt{5}i)x} \\
 &= e^{-x} e^{-i\sqrt{5}x} \\
 &= e^{-x} \sin(\sqrt{5}x)
 \end{aligned}$$

So the general solution is

$$y = c_1 e^{-x} \cos(\sqrt{5}x) + c_2 e^{-x} \sin(\sqrt{5}x)$$

In general, if $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$ are roots of the auxiliary equation, then $y_1 = e^{\alpha x} \cos(\beta x)$
 $y_2 = e^{\alpha x} \sin(\beta x)$ are solutions.

4.3.8 – Example

$$\begin{aligned} y^{(4)} - 16y &= 0 \\ m^4 - 16 &= 0 \\ (m^2 - 4)(m^2 + 4) &= 0 \\ (m - 2)(m + 2)(m^2 + 4) &= 0 \\ m = 2 : y_1 &= e^{2x} \\ m = -2 : y_1 &= e^{-2x} \\ m = 2i : \cos(2x), \sin(2x) & \end{aligned}$$

4.4 Nonhomogeneous, Linear DE with Constant Coefficients

4.4.1 – Method of Undetermined Coefficients

Section 4.5 gives another approach but it is a bit more abstract

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1(x) \frac{dy}{dx} + a_0 y = g(x) \text{ where } g(x) \neq 0$$

Theorem: If we can find any one particular solution y_p of this DE ($y_p + y_c$), where y_c is the solution of the complementary DE (the same LHS= 0 instead of $g(x)$), is also a solution of the non-homogeneous DE, then the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n + y_p \end{aligned}$$

where you use [Section 4.3](#) methods for the $c_i y_i$'s.

4.4.2 – Example

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

Step 1: Find the General Solution y_c of the complimentary DE

$y'' + 4y' - 2y = 0$ Aux equation:

$$\begin{aligned}m^2 + 4m - 2 &= 0 \\m^2 + 4m + 4 &= 6 \\(m + 2)^2 &= 6 \\m + 2 &= \pm\sqrt{6} \\m &= -2 \pm \sqrt{6} \\y_1 &= e^{(-2+\sqrt{6})x} \\y_2 &= e^{(-2-\sqrt{6})x}\end{aligned}$$

Step 2: Find a particular solution y_p of given DE

Educated Guess:

$$y_p = Ax^2 + Bx + C$$

for some coefficients A, B, C . For the moment, they're undetermined coefficients.

Plugging in the y_p , we get

$$\begin{aligned}y'_p &= 2Ax + B \\y''_p &= 2A\end{aligned}$$

So,

$$\begin{aligned}
 2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) &= 2x^2 - 3x + 6 \\
 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C &= 2x^2 - 3x + 6 \\
 -2Ax^2 + 8Ax - 2Bx + 2A + 4B - 2C &= 2x^2 - 3x + 6 \\
 -2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) &= 2x^2 - 3x + 6 \\
 -2A &= 2 \\
 8A - 2B &= -3 \\
 2A + 4B - 2C &= 6 \\
 -2A &= 2 \\
 A &= -1 \\
 8(-1) - 2B &= -3 \\
 -8 - 2B &= -3 \\
 8 + 2B &= 3 \\
 2B &= -5 \\
 B &= -\frac{5}{2} \\
 2(-1) + 4\left(-\frac{5}{2}\right) - 2C &= 6 \\
 -2 + -10 - 2C &= 6 \\
 -2C &= 18 \\
 C &= -9
 \end{aligned}$$

Step 3: Check

$$\begin{aligned}
 y'_p &= 2(-1)x + \left(-\frac{5}{2}\right) \\
 &= -2x - \frac{5}{2} \\
 y''_p &= 2(-1) \\
 &= -2 \\
 y'' + 4y' - 2y &= -2 + 4\left(-2x - \frac{5}{2}\right) - 2\left(-x^2 - \frac{5}{2}x - 9\right) \\
 &= -2 - 8x - 10 + 2x^2 + 5x + 18 \\
 &= 2x^2 - 8x + 5x - 10 + 18 - 2 \\
 &= 2x^2 - 3x + 6
 \end{aligned}$$

4.4.3 – Example

$$y'' - y' + y = 2 \sin(3x)$$

Step 1: Find the General Solution y_c of the complimentary DE

Aux equation:

$$\begin{aligned}
 m^2 - m + 1 &= 0 \\
 m &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \\
 &= \frac{1 \pm \sqrt{1 - 4}}{2} \\
 &= \frac{1 \pm \sqrt{-3}}{2} \\
 &= \frac{1 \pm \sqrt{3}i}{2} \\
 m_1 &= \frac{1 + \sqrt{3}i}{2} \\
 m_2 &= \frac{1 - \sqrt{3}i}{2} \\
 y_1 &= e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) \\
 y_2 &= e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)
 \end{aligned}$$

Step 2: Guess $y_p = A \sin(3x) + B \cos(3x)$

Plug into the DE

$$\begin{aligned}
 y'' &\quad - y' &+ y &= 2 \sin(3x) \\
 \overbrace{-9A \sin(3x) - 9B \cos(3x)} &- \left(\overbrace{3A \cos(3x) - 3B \sin(3x)} \right) &+ \overbrace{A \sin(3x) + B \cos(3x)} &= 2 \sin(3x)
 \end{aligned}$$

4.3.4 – Method of Undetermined Coefficients 2

For Solving Linear, Non-homogeneous DE with constant coefficients

$$a_2y'' + a_1y' + a_0y = f(x)$$

Standard Form:

$$y'' + a_1y' + a_0y = g(x)$$

4.3.5 – Steps

Step 1) Solve $y'' + a_1y' + a_0y = 0$ called the general solution y_c .

Step 2) Find one particular solution y_p of the given DE and the general solution is

$$y = y_c + y_p$$

This method can only be used when $g(x)$ is a polynomial (An exponential (i.e. e^{kx}), sines or cosines or sums of products of these types of functions)

4.3.6 – Example

$$y'' - 3y' - 4y = 4 \cos(3x)$$

1st solve:

$$\begin{aligned} y'' - 3y' - 4y &= 0 \\ m^2 e^{mx} - 3me^{mx} - 4e^{mx} &= 0 \\ m^2 - 3m - 4 &= 0 \\ (m - 4)(m + 1) &= 0 \\ m - 4 &= 0 \quad m + 1 = 0 \\ m &= 4 \quad m = -1 \\ y_c &= c_1 e^{4x} + c_2 e^{-x} \end{aligned}$$

$$y = A \cos(3x) + B \sin(3x)$$

$$y' = -3A \sin(3x) + 3B \cos(3x)$$

$$y'' = -9A \cos(3x) - 9B \sin(3x)$$

$$y'' - 3y' - 4y = 4 \cos(3x)$$

$$(-9A \cos(3x) - 9B \sin(3x)) - 3(-3A \sin(3x) + 3B \cos(3x)) - 4(A \cos(3x) + B \sin(3x)) = 4 \cos(3x)$$

$$-9A \cos(3x) - 9B \sin(3x) + 9A \sin(3x) - 9B \cos(3x) - 4A \cos(3x) - 4B \sin(3x) = 4 \cos(3x)$$

$$-9A \cos(3x) - 9B \cos(3x) - 4A \cos(3x) - 9B \sin(3x) + 9A \sin(3x) - 4B \sin(3x) = 4 \cos(3x)$$

$$\cos(3x)(-9A - 9B - 4A) + \sin(3x)(-9B + 9A - 4B) = 4 \cos(3x)$$

$$\cos(3x)(-13A - 9B) + \sin(3x)(9A - 13B) = 4 \cos(3x)$$

$$\begin{cases} -13A & -9B = 4 \\ 9A & -13B = 0 \end{cases} \text{ Solve simultaneously}$$

One way to solve Linear Systems of Equations is called Cramer's Rule.

$$\begin{aligned} \det \begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix} \\ A &= \frac{\begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}} \\ &= \frac{4(-13) - 0(-9)}{-13(-13) - 9(-9)} \\ &= \frac{-52 - 0}{169 + 81} \\ &= -\frac{52}{250} \\ &= -\frac{26}{125} \\ B &= \frac{\begin{bmatrix} -13 & 4 \\ 9 & 0 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}} \\ &= \frac{-13(0) - 4(9)}{250} \\ &= \frac{0 - 36}{250} \\ &= -\frac{36}{250} \\ &= -\frac{18}{125} \end{aligned}$$

Check:

$$\begin{aligned} (-13) \left(-\frac{26}{125} \right) + (-9) \left(-\frac{18}{125} \right) ? &= 4 \\ \frac{338}{125} + \frac{162}{125} ? &= 4 \\ \frac{500}{125} &= 4 \\ 9 \left(-\frac{26}{125} \right) + (-13) \left(-\frac{18}{125} \right) ? &= 0 \\ -\frac{234}{125} + \frac{234}{125} ? &= 0 \\ 0 &= 0 \end{aligned}$$

So

$$y = -\frac{26}{125} \cos(3x) - \frac{18}{125} \sin(3x) + c_1 e^{4x} + c_2 e^{-x}$$

is the general solution to the given DE.

4.3.7 – Example

$$y'' - 5y' + 4y = 8e^x$$

If we try:

$$\begin{aligned} y_p &= Ae^x \\ Ae^x - 5Ae^x + 4Ae^x &= 8e^x \\ e^x(A - 5A + 4) &= 8e^x \\ 0 &= 8e^x \text{ has no solution.} \end{aligned}$$

Solve

$$y'' - 5y' + 4y = 0$$

1st

$$\begin{aligned} m^2 - 5m + 4 &= 0 \\ (m - 1)(m - 4) &= 0 \\ m - 1 = 0 &\quad m - 4 = 0 \\ m = 1 &\quad m = 4 \\ y_1 = e^{1mx} &\quad y_2 = e^{4mx} \\ y_1 = e^{mx} &\quad y_2 = e^{4mx} \\ y_c = c_1 e^{mx} + c_2 e^{4mx} &\quad \text{hole at } Ae^x \text{ is } c_1 = A \quad c_2 = 0 \end{aligned}$$

Suppose we have a 5th order DE with

$$a_5 y^{(5)} + a_4 y^{(4)} + \cdots + a_1 y' + a_0 y = g(x)$$

and the auxiliary equation factors as

$$m^2(m - 3)(m - (2 + i))(m - (2 - i)) = 0$$

$$m = 0 \text{ (multiplicity 2)} \quad m = 3 \quad m = 2 + i \quad m = 2 - i$$

Step 1

Write the general solution to the complimentary DE

$$\begin{aligned} y_1 &= e^{0x} = 1 \\ y_2 &= x e^{0x} = x \\ y_3 &= e^{3x} = e^{3x} \\ y_4 &= e^{(2+i)x} = e^{2x} \cos(x) \\ y_5 &= e^{(2-i)x} = e^{2x} \sin(x) \end{aligned}$$

$$y_c = c_1 + c_2 x + c_3 e^{3x} + e^{2x} \cos(x) + e^{2x} \sin(x)$$

4.3.8 – What would you guess for the form of y_p ?

If

$$(ii) \ g(x) = e^{5x} \Rightarrow y_p = Ae^{5x}$$

$$(iii) \ g(x) = e^{3x} \Rightarrow y_p = Axe^{3x} \ (\text{because } e^{3x} \text{ is in } y_c)$$

$$(iv) \ g(x) = 5e^{2x} \sin(x) \Rightarrow y_p = (Ae^{2x} \cos(x) + Be^{2x} \sin(x)) x$$

$$(v) \ g(x) = 6x^2 e^{4x} \Rightarrow y_p = (Ax^2 + Bx + C) e^{4x}$$

$$(vi) \ g(x) = x^2 e^{3x} \Rightarrow y_p = (Ax^2 + Bx + C) e^{3x} x$$

4.6 Variation of Parameters Method

$$y'' + P(x)y' + Q(x)y = f(x)$$

will only work on problems where $P(x)$ and $Q(x)$ are constants.

4.6.1 – 1st Step: General solution of complementary DE

$$y = c_1 y_1 + c_2 y_2$$

Guess a solution to the non-homogeneous of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions of x . This theory produces

$$u'_1 = \frac{W_1}{W} \text{ and } u'_2 = \frac{W_2}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix}$$

4.6.2 – Example

$$4y'' + 36y = \csc(3x)$$

$$4y'' + 36y = \csc(3x)$$

$$y'' + 9y = \frac{\csc(3x)}{4}$$

$$m^2 e^{mx} + 9e^{mx} = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \pm\sqrt{-9}$$

$$= \pm 3i$$

$$\begin{aligned}
y_1 &= e^{0x} \cos(3x) & y_2 &= e^{0x} \sin(3x) \\
y_1 &= 1 \cos(3x) & y_2 &= 1 \sin(3x) \\
y_1 &= \cos(3x) & y_2 &= \sin(3x) \\
y_c &= c_1 \cos(3x) + c_2 \sin(3x)
\end{aligned}$$

Guess

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
u'_1 &= \frac{W_1}{W} & u'_2 &= \frac{W_2}{W}
\end{aligned}$$

where

$$\begin{aligned}
W &= \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3 \sin(3x) & 3 \cos(3x) \end{vmatrix} \\
&= (3 \cos(3x))(\cos(3x)) - (\sin(3x))(-3 \sin(3x)) \\
&= 3 \cos^2(3x) + 3 \sin^2(3x) \\
&= 3 (\cos^2(3x) + \sin^2(3x)) \\
&= 3(1) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
W_1 &= \begin{vmatrix} 0 & \sin(3x) \\ \frac{1}{4} \csc(3x) & \cos(3x) \end{vmatrix} \\
&= 0 \cos(3x) - \sin(3x) \left(\frac{\csc(3x)}{4} \right) \\
&= -\frac{\sin(3x) \csc(3x)}{4} \\
&= -\frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
W_2 &= \begin{vmatrix} \cos(3x) & 0 \\ \sin(3x) & \frac{1}{4} \csc(3x) \end{vmatrix} \\
&= \frac{1}{4} \csc(3x) \cos(3x) - 0 \sin(3x) \\
&= \frac{\cos(3x)}{4 \sin(3x)} \\
&= \frac{1}{4} \cot(3x)
\end{aligned}$$

$$\begin{aligned}
u'_1 &= \frac{W_1}{W} & u'_2 &= \frac{W_2}{W} \\
u'_1 &= \frac{-\frac{1}{4}\cot(3x)}{3} & u'_2 &= \frac{\frac{1}{4}\cot(3x)}{3} \\
u'_1 &= -\frac{1}{12} & u_2 &= \frac{1\cot(3x)}{12} \\
u'_1 &= -\frac{1}{12} & u_2 &= \frac{1\cos(3x)}{12\sin(3x)} \\
u_1 &= \int -\frac{1}{12}dx & u_2 &= \int \frac{1\cos(3x)}{12\sin(3x)}dx \\
u_1 &= -\frac{x}{12} & u_2 &= \frac{1}{12} \int \frac{1}{3} \frac{dv}{v} \\
&& u_2 &= \frac{1}{36} \ln|v| \\
&& u_2 &= \frac{1}{36} \ln|\sin(3x)|
\end{aligned}$$

$$\begin{aligned}
y_p &= u_1y_1 + u_2y_2 \\
&= -\frac{x}{12} \cos(3x) + \frac{1}{36} \ln|\sin(3x)| \sin(3x) \\
&= -\frac{x \cos(3x)}{12} + \frac{\sin(3x)}{36} \ln|\sin(3x)|
\end{aligned}$$

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 \cos(3x) + c_2 \sin(3x) - \frac{x \cos(3x)}{12} + \frac{\sin(3x)}{36} \ln|\sin(3x)|
\end{aligned}$$

4.6.3 – 3×3 Determinants

$$\left| \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \right| = a \left| \begin{array}{cc} e & f \\ h & i \end{array} \right| - b \left| \begin{array}{cc} d & f \\ f & i \end{array} \right| + c \left| \begin{array}{cc} d & e \\ g & h \end{array} \right|$$

Matrix of Signs

$$\left[\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array} \right]$$

Chapter 5

Modeling with Higher-Order Differential Equations

5.1 Spring-Mass Problems

Suppose that there are no forces affecting the motion other than the gravitational force and the spring force.

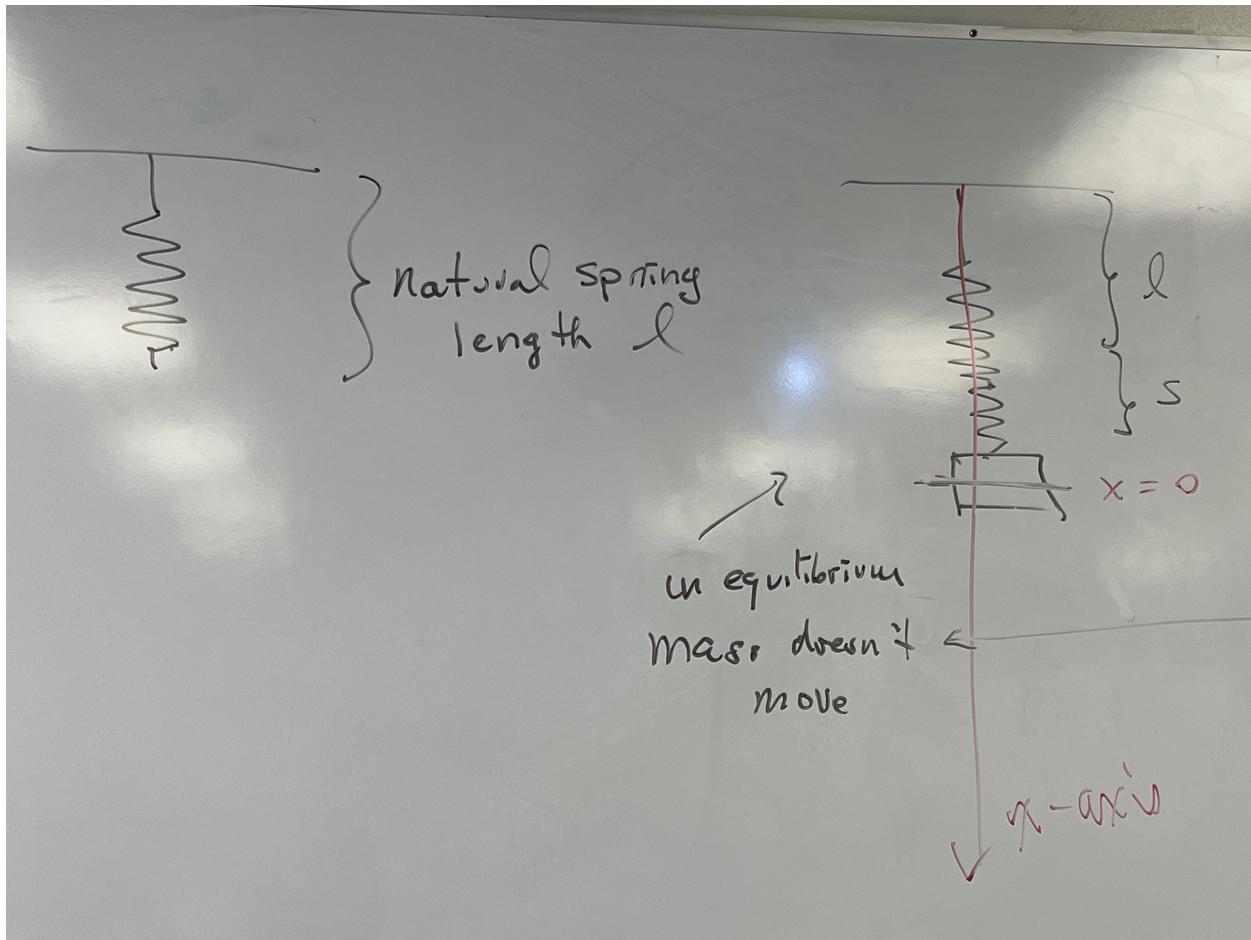


Figure 5.1: Diagram of a spring in equilibrium.

In Equilibrium, mass doesn't move.

$$F_g = -F_{spring} \rightarrow mg = -F_{spring} \rightarrow -mg = -kx \text{ (Hooke's Law)}$$

where $g \approx 9.8m/s^2$ if m is in kg, else $g \approx 32.1 \text{ ft/sec}^2$.

In equilibrium:

$$F_{net} = mg - ks = 0$$

, so

$$k = \frac{mg}{s}$$

In general:

$$F_{net} = m \times \text{acceleration} \rightarrow m \frac{d^2x}{dt^2}$$

$$\begin{aligned}
 mg + (-k)(x + s) &= m \frac{d^2x}{dt^2} \\
 mg - kx - ks &= m \frac{d^2x}{dt^2} \\
 (mg - ks) - kx &= m \frac{d^2x}{dt^2} \\
 0 - kx &= m \frac{d^2x}{dt^2}
 \end{aligned}$$

So the differential equation is:

$$m \frac{d^2x}{dt^2} + kx = 0 \quad (5.1)$$

, a 2nd order, homogeneous DE with a constant coefficient.

5.1.1 – Example

$$\begin{aligned}
 m \frac{d^2x}{dt^2} + kx &= 0 \\
 \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0
 \end{aligned}$$

Guess:

$$\begin{aligned}
 x &= e^{lt} \\
 m \frac{d^2x}{dt^2} + kx &= 0 \\
 l^2 e^{lt} + \frac{k}{m} e^{lt} &= 0 \\
 l^2 + \frac{k}{m} &= 0 \\
 l^2 &= -\frac{k}{m} \\
 l &= \pm \sqrt{-\frac{k}{m}} \\
 l &= 0 \pm \sqrt{\frac{k}{m}} i
 \end{aligned}$$

$$\begin{aligned}
 x_1 &= e^{0t} \cos \left(\frac{k}{m} t \right) & x_2 &= e^{0t} \sin \left(\frac{k}{m} t \right) \\
 x_1 &= \cos \left(\frac{k}{m} t \right) & x_2 &= \sin \left(\frac{k}{m} t \right)
 \end{aligned}$$

$$\text{Let } \omega = \sqrt{\frac{k}{m}}$$

$$x_1 = \cos(\omega^2 t) \quad x_2 = \sin(\omega^2 t)$$

5.1.2 – Example

Mass weighs 2lbs, stretch spring 6 inches

$$F = ma$$

$$2\text{lbs} = m \times 32 \frac{\text{ft}}{\text{sec}^2}$$

$$m = \frac{2\text{lbs}}{32 \frac{\text{ft}}{\text{sec}^2}}$$

$$m = \frac{1}{16} \text{ slug}$$

$$\begin{aligned} F_{spring} &= kx \\ &= k(6\text{in}) \end{aligned}$$

$$2\text{lbs} = \frac{k}{2}$$

$$k = 4 \frac{\text{lbs}}{\text{ft}}$$

$$\omega = \sqrt{\frac{4}{\frac{1}{16}}}$$

$$\omega = \sqrt{64}$$

$$\omega = 8$$

5.1.3 – Undamped Motion

$$m \frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2x = 0 \text{ where } \omega = \sqrt{\frac{k}{m}}$$

$$n^2 e^{nt} + \omega^2 e^{nt} = 0 \text{ where } x = e^{nt}$$

$$n^2 + \omega^2 = 0$$

$$n^2 = -\omega^2$$

$$n = 0 \pm \sqrt{-\omega^2}$$

$$= 0 \pm \omega i$$

$$x_1 = \cos(\omega t) \quad x_2 = \sin(\omega t)$$

General solution:

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

5.1.4 – Free, Damped Motion

Assume in addition to F_{spring} and $F_{gravity}$ that there is a force damping the motion which is directly proportional, in the opposite direction, to the mass's velocity.

$$\begin{aligned} m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx &= 0 \\ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m}x &= 0 \\ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \omega^2 x &= 0 \end{aligned} \tag{5.2}$$

where β is the drag coefficient. If we substitute $2\lambda = \frac{\beta}{m} \Rightarrow \lambda = \frac{\beta}{2m}$

$$\begin{aligned} \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x &= 0 \\ n^2 e^{nt} + 2\lambda n e^{nt} + \omega^2 e^{nt} &= 0 \\ n^2 + 2\lambda n + \omega^2 &= 0 \\ n^2 + 2\lambda n = -\omega^2 & \\ n^2 + 2\lambda n + \lambda^2 = \lambda^2 - \omega^2 & \\ (n + \lambda)^2 = -\omega^2 + \lambda^2 & \\ n + \lambda = \pm \sqrt{\lambda^2 - \omega^2} & \\ n = -\lambda \pm \sqrt{\lambda^2 - \omega^2} & \end{aligned}$$

Case 1: $\lambda^2 - \omega^2 > 0$

then there are two distinct real solutions where $n_1 < 0$ and $n_2 < 0$.

$$\begin{aligned} n_1 &= -\lambda + \sqrt{\lambda^2 - \omega^2} \\ n_2 &= -\lambda - \sqrt{\lambda^2 - \omega^2} \\ x_1 &= e^{n_1 t} \quad x_2 = e^{n_2 t} \\ x_1 &= e^{t(-\lambda + \sqrt{\lambda^2 - \omega^2})} \quad x_2 = e^{t(-\lambda - \sqrt{\lambda^2 - \omega^2})} \end{aligned}$$

General solution:

$$\begin{aligned} x &= c_1 e^{t(-\lambda + \sqrt{\lambda^2 - \omega^2})} + c_2 e^{t(-\lambda - \sqrt{\lambda^2 - \omega^2})} \\ &= c_1 e^{-t\lambda} e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\lambda} e^{-t\sqrt{\lambda^2 - \omega^2}} \\ &= e^{-\lambda t} \left(c_1 e^{-t\lambda} e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\lambda} e^{-t\sqrt{\lambda^2 - \omega^2}} \right) \end{aligned} \tag{5.3}$$

Case 2: $\lambda = \omega$

$$\begin{aligned} n &= -\lambda \pm \sqrt{\lambda^2 - \omega^2} \\ n &= -\lambda \pm \sqrt{0} \\ n &= -\lambda \end{aligned}$$

where λ has multiplicity 2.

$$x_1 = e^{-\lambda t} \quad x_2 = te^{-\lambda t}$$

General solution:

$$\begin{aligned} x &= c_1 e^{-\lambda t} + c_2 t e^{-\lambda t} \\ &= e^{-\lambda t} (1 + c_2 t) \end{aligned} \tag{5.4}$$

Case 3: $0 < \lambda < \omega$

then

$$\begin{aligned} x_1 &= e^{-\lambda t} \cos(\sqrt{\omega^2 - \lambda^2} t) & x_2 &= e^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t) \\ x &= c_1 e^{-\lambda t} \cos(\sqrt{\omega^2 - \lambda^2} t) + c_2 e^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t) \\ &= e^{-\lambda t} \left(c_1 \cos(\sqrt{\omega^2 - \lambda^2} t) + c_2 \sin(\sqrt{\omega^2 - \lambda^2} t) \right) \\ &= e^{-\lambda t} (A \sin(\sqrt{\omega^2 - \lambda^2} t) + \phi) \end{aligned} \tag{5.5}$$

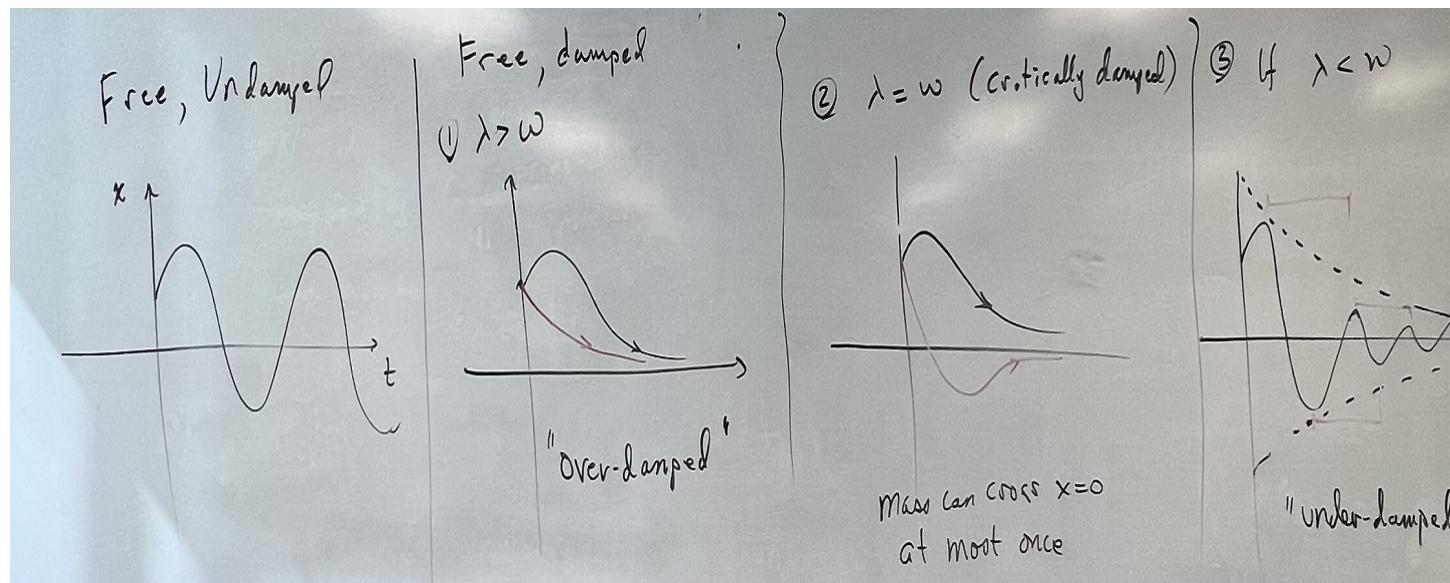


Figure 5.2:

5.1.5 – Driven Motion (not Free Motion)

Can be both damped or **undamped**.

Imagine the support is oscillating up and down due to an external force. If the motion is:

Undamped

Assume that $\gamma \neq \omega$

$$\begin{aligned}
 \frac{d^2x}{dt^2} + \omega^2 x &= F(t) \\
 &= F_0 \sin(\gamma t) \\
 \frac{d^2x}{dt^2} + \omega^2 x &= 0 \\
 &\vdots \\
 x &= c_1 \cos(\omega t) + c_2 \sin(\omega t) \\
 \frac{d^2x}{dt^2} + \omega^2 x &= F_0 \sin(\gamma t) \\
 \textcolor{red}{x_p(t)} &= A \cos(\gamma t) + B \sin(\gamma t) \\
 -A\gamma^2 \cos(\gamma t) - B\gamma^2 \sin(\gamma t) + \omega^2(A \cos(\gamma t) + B \sin(\gamma t)) &= F_0 \sin(\gamma t) \\
 -A\gamma^2 \cos(\gamma t) - B\gamma^2 \sin(\gamma t) + A\omega^2 \cos(\gamma t) + B\omega^2 \sin(\gamma t) &= F_0 \sin(\gamma t) \\
 \cos(\gamma t)(-A\gamma^2 + A\omega^2) = 0 \cos(\gamma t) &\quad B(\omega^2 - \gamma^2) = F_0 \\
 -A\gamma^2 + A\omega^2 = 0 &\quad B = \frac{F_0}{\omega^2 - \gamma^2} \\
 A(\omega^2 - \gamma^2) = 0 &\quad B = \frac{F_0}{\omega^2 - \gamma^2}
 \end{aligned}$$

We assumed that $\gamma \neq \omega$, which forces $A = 0$ for the left equation to work.

$$A = 0 \quad B = \frac{F_0}{\omega^2 - \gamma^2}$$

General solution:

$$\begin{aligned}
 x(t) &= x_c(t) + x_p(t) \\
 &= c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2} \sin(\gamma t)
 \end{aligned} \tag{5.6}$$

If γ is almost equal to ω , then $\frac{F_0}{\omega^2 - \gamma^2}$ is a large constant. This situation is called Resonance.

5.1.6 – Example

Driven, Damped, Spring-Mass

$$\begin{aligned}
 \frac{1}{5} \frac{d^2x}{dt^2} + 1.2 \frac{dx}{dt} + 2x &= 5 \cos(4t), x(0) = \frac{1}{2}, x'(0) = 0 \\
 \frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 10x &= 25 \cos(4t) \\
 \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x &= F(t)
 \end{aligned}$$

where

$$\lambda = 3, \omega = \sqrt{10}, F(t) = 25 \cos(4t)$$

For instance, if the mass = 2kgs, then the spring constant is $k = 20$ and $\beta = 12$.

$$\sqrt{10} = \sqrt{\frac{k}{m}} \Rightarrow 10 = \frac{k}{m} \Rightarrow 10m = k \Rightarrow k = 10(2) = 20$$

$$\beta = 2\lambda m = 2(3)(2) = 6(2) = 12$$

To solve the complimentary DE

, guess $x = e^{nt}$

$$n^2 e^{nt} + 6n e^{nt} + 10 e^{nt} = 0$$

$$n^2 + 6n + 10 = 0$$

$$\begin{aligned} n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36 - 40}}{2} \\ &= \frac{-6 \pm \sqrt{-4}}{2} \\ &= \frac{-6 \pm 2i}{2} \\ &= -3 \pm i \end{aligned}$$

$$x_1 = e^{-3t} \cos(t) \quad x_2 = e^{-3t} \sin(t)$$

To find a particular solution

Guess

$$x_p = A \cos(4t) + B \sin(4t)$$

$$x'_p = -4A \sin(4t) + 4B \cos(4t)$$

$$x''_p = -16A \cos(4t) - 16B \sin(4t)$$

$$-16A \cos(4t) - 16B \sin(4t) + 6(-4A \sin(4t) + 4B \cos(4t)) + 10(A \cos(4t) + B \sin(4t)) = 25 \cos(4t)$$

$$-16A \cos(4t) - 16B \sin(4t) - 24A \sin(4t) + 24B \cos(4t) + 10A \cos(4t) + 10B \sin(4t) = 25 \cos(4t)$$

$$-16A \cos(4t) + 10A \cos(4t) + 24B \cos(4t) - 16B \sin(4t) - 24A \sin(4t) + 10B \sin(4t) = 25 \cos(4t)$$

$$-6A \cos(4t) + 24B \cos(4t) - 6B \sin(4t) - 24A \sin(4t) = 25 \cos(4t)$$

$$\cos(4t)(-6A + 24B) + \sin(4t)(-6B - 24A) = 25 \cos(4t)$$

Chapter 6

Series Solutions of Linear Equations

6.1 Solution by Infinite Series

2nd order linear DE with (possibly) variable coefficients

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

$$y'' + P(x)y' + Q(x)y = F(x)$$

6.1.1 – Review of Infinite Series Facts

Maclaurin Series

$$\sum_{n=0}^{\infty} a_n x^n$$

Power series centered at 0

Taylor Series

$$\sum_{n=0}^{\infty} a_n (x - a)^n$$

Centered at $a = 0$ It's a theorem that power series either

- (1) Converge all real numbers x on the interval $I = (-\infty, \infty)$ and the radius of convergence is $R = \infty$
- (2) Converge only when $x = a$ on the interval $I = [a, a]$ and the radius of convergence is $R = 0 = \{a\}$
- (3) The series converges on an interval centered at a finite, non-zero radius $R = (a - R, a + R)$

6.1.2 – Ratio Test

Use the Ratio Test to determine which of these 3 cases occurs in a specific problem. The 3 cases of the ratio test are:

$L < 1$, the series converges

$L > 1$, the series diverges

$L = 1$,

6.1.3 – Example

Determine the radius and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n(n+1)}$$

$$L = \lim_{n \rightarrow \infty} \frac{\left| \frac{x^{n+1}}{3^{n+1}(n+2)} \right|}{\left| \frac{x^n}{3^n(n+1)} \right|}$$

$$= \lim_{n \rightarrow \infty} \frac{|x|}{3} \frac{n+1}{n+2}$$

$$= \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{n+1}{n+2}$$

$$= \frac{|x|}{3} (1)$$

$$= \frac{|x|}{3}$$

$$\frac{|x|}{3} < 1$$

$$|x| < 3$$

$$I = (-3, 3)$$