

# Chapter 3

## Modeling using DE

### 3.1 Linear DE Modeling

#### 3.1.1 – Standard Problems

- 1) Population Growth (or decline)
- 2) Radioactive Decay
- 3) Newton's Law of Cooling
- 4) Mixture Problems

#### 3.1.2 – Population Model

Assume the rate of population change is proportional to the size of the population

$P(t)$  = population at time  $t$

$$\frac{dP}{dt} = kP$$

$\frac{\frac{dP}{dt}}{P} = k$  is the relative growth rate of the population

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{dP}{P} &= kdt \\
\int \frac{dP}{P} &= \int kdt \\
\ln|P| &= kt + C \\
|P| &= e^{kt+C} \\
|P| &= e^{kt}e^C \\
|P| &= Ae^{kt} \text{ where } A > 0 \\
P &= \pm Ae^{kt} \\
P &= Be^{kt} \text{ where } B \neq 0 \\
P &= De^{kt} \text{ where } D \text{ can be any real number}
\end{aligned}$$

The constant can become any number because 0 would be a valid rate of population change, it means that the population size isn't changing.

### 3.1.3 – Example

If, initially at 2 p.m., there are 1,000 bacteria on a petri dish and at 4 p.m., there are 2,000 bacteria. Assuming constant relative growth rate, how many bacteria are there at 5 p.m.?  
 $P(t)$  = population  $t$  hours after 2 p.m.

$$\begin{aligned}
P(t) &= Ae^{kt} \\
1000 &= Ae^{(0)k} \\
1000 &= Ae^0 \\
1000 &= A(1) \\
A &= 1000 \\
P(2) &= 2000 \\
P(2) &= 1000e^{2k} \\
2000 &= 1000e^{2k} \\
2 &= e^{2k} \\
\ln(2) &= 2k \\
k &= \frac{\ln(2)}{2}
\end{aligned}$$

$$\begin{aligned}
P(t) &= 1000e^{\frac{\ln(2)}{2}t} \\
P(3) &= 1000e^{\frac{\ln(2)}{2}(3)} \\
&= 1000e^{1.5\ln(2)} \\
&= 1000e^{\ln(2^{1.5})} \\
&= 1000(2^{1.5}) \\
&= 2000(\sqrt{2}) \\
P(3) &\approx 2828.427(\sqrt{2}) \\
P(t) &= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\ln(2^{\frac{t}{2}})} \\
&= 1000 \times 2^{\frac{t}{2}}
\end{aligned}$$

### 3.1.4 – Radioactive Decay

$$m(t) = m_0e^{kt} \text{ where } k < 0$$

The Half-Life is the amount of time it takes for half of the original amount to remain:

$$\frac{1}{2}A_0 = A_0e^{kt} \Rightarrow \frac{1}{2} = e^{kt}$$

### 3.1.5 – Mixture Problems

#### Setup

Initially, the container has 200 gallons of brine solution (salt-water) of concentration  $\frac{10 \text{ lbs}}{200 \text{ gallons}} = 0.05 \frac{\text{lbs}}{\text{gallon}}$ . A solution of  $\frac{5 \text{ lbs}}{200 \text{ gallons}} = 0.025 \frac{\text{lbs}}{\text{gallon}}$  is poured into the initial container at a rate of  $\frac{4 \text{ gallons}}{\text{min}}$ . How many pounds of salt are there in the container after 2 hours.

Let  $A(t) = \# \text{ lbs of salt } t \text{ minutes after the process starts}$

$\frac{dA}{dt}$  = The rate of change of # lbs of salt

$$\begin{aligned}\frac{dA}{dt} &= 0.025 \frac{\text{lbs}}{\text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \left\} \text{rate in} \right. \\ &\quad - \frac{A(t)\text{lbs}}{200\text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \left\} \text{rate out} \right. \\ &= (0.025)4 \frac{\text{lbs}}{\text{min}} - \frac{4A(t)}{200} \frac{\text{lbs}}{\text{min}} \\ &= 0.1 \frac{\text{lbs}}{\text{min}} - \frac{A(t)}{50} \frac{\text{lbs}}{\text{min}} \\ &= 0.1 - \frac{A(t)}{50}\end{aligned}$$

$$\frac{dA}{dt} + \frac{1}{50}A = 0.1$$

$$\mu = e^{\int P(t)dt}$$

$$= e^{\int \frac{1}{50}dt}$$

$$= e^{\frac{t}{50}}$$

$$e^{\frac{t}{50}} \left( \frac{dA}{dt} \right) + e^{\frac{t}{50}} \left( \frac{1}{50}A \right) = e^{\frac{t}{50}}(0.1)$$

$$\frac{d}{dt} \left( e^{\frac{t}{50}}A \right) = e^{\frac{t}{50}}(0.1)$$

$$\int \frac{d}{dt} \left( e^{\frac{t}{50}}A \right) = \int \frac{1}{10} e^{\frac{t}{50}}$$

$$e^{\frac{t}{50}}A = \frac{1}{10} \times \frac{e^{\frac{t}{50}}}{\frac{1}{50}} + C$$

$$e^{\frac{t}{50}}A = 5e^{\frac{t}{50}} + C$$

$$\begin{aligned}A(t) &= 5 + Ce^{-\frac{t}{50}} \\ &= 5 + Ce^{-0.02t}\end{aligned}$$

$$\begin{aligned}A(120) &= 5 + Ce^{-0.02(120)} \\ &= 5 + Ce^{-2.4}\end{aligned}$$