Chapter 1

Introduction to Differential Equations

1.2 Initial Value Problems (IVP)

1st order IVP is a 1st order D.E. together with one extra condition:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

2nd order IVP

$$y'' = f(x, y, y')$$

Initial conditions:

- $y(x_0) = y_0$
- $y'(x_0) = y_1$

1.2.1 – Example

$$y' = y \text{ and } y(0) = 3$$

$y = ce^x$ is a one-parameter family of solutions

$$\frac{d}{dx}(ce^x) = ce^x = y$$

$$\begin{aligned} ce^1 &= -2 \\ c &= -\frac{2}{e} \\ y &= \left(-\frac{2}{e}\right) e^x \\ y &= -2e^{x-1} \end{aligned}$$

1.2.2 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Given that you have the solution: $y = \frac{1}{x^2+C}$, Solve:

$$\begin{aligned}
-1 &= \frac{1}{(0)^2 + c} \\
-1 &= \frac{1}{c} \\
-1 \times c &= 1 \\
c &= -1 \\
y &= \frac{1}{x^2 - 1}, I = (-1, 1)
\end{aligned}$$

1.2.3 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Example

$$x'' + 16x = 0 \text{ and } x(\frac{\pi}{2}) = 5 \text{ and } x'(\frac{\pi}{2}) = -4$$

$$\begin{aligned}
x &= c_1 \cos(4t) + c_2 \sin(4t) \\
5 &= c_1 \cos(4t) + c_2 \sin(4t) \\
&= c_1 \cos(2\pi) + c_2 \sin(2\pi) \\
&= c_1(1) + c_2(0) \\
&= c_1 \\
x' &= -4c_1 \sin(4t) + 4c_2 \cos(4t) \\
-4 &= -4c_1 \sin\left(4\left(\frac{\pi}{2}\right)\right) + 4c_2 \cos\left(4\left(\frac{\pi}{2}\right)\right) \\
&= -4c_1 \sin(2\pi) + 4c_2 \cos(2\pi) \\
&= -4c_1(0) + 4c_2(1) \\
&= 4c_2 \\
-1 &= c_2
\end{aligned}$$

Reasonable Question: Given a 1st order IVP, can we say whether a solution *exists* or not and, if a solution exists, is it *unique*.

Theorem: Given $y' = f(x, y)$ and $y(x_0) = y_0$, if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous on a rectangle R containing (x_0, y_0) in its interior, then there exists an interval $I = (x_0 - h, x_0 + h)$ where $h > 0$ such that there exists a unique solution to IVP on I.

1.2.4 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(1) = 2$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous everywhere its defined $y \geq 0$
- $\frac{\partial f}{\partial y} = x\frac{1}{2}y^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$ is continuous everywhere its defined $y > 0$

1.2.5 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(0) = 0$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous for all x and $y \geq 0$
- $\frac{\partial f}{\partial} = \frac{x}{2y}$ is continuous for all x and $y > 0$.
- ***Theorem does not give any conclusion.***