Chapter 7

Method of Laplace Transforms for Solving DE's

7.3 Operational Rules Part 2

Three More Rules

7.3.1 – Rule 1

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}F(s) \Rightarrow \mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s) \tag{7.1}$$

7.3.2 – Rule 2

Is there a way to break up $\mathscr{L}$ over a product of functions?

$$\begin{aligned} \mathscr{L}\{f(t)g(t)\}? &= \mathscr{L}\{f(t)\} \times \mathscr{L}\{g(t)\} \\ \mathscr{L}\left\{t^2 \times t^3\right\}? &= \mathscr{L}\left\{t^2\right\} \times \mathscr{L}\left\{t^3\right\} \\ \mathscr{L}\left\{t^5\right\}? &= \frac{2!}{s^{2+1}} \times \frac{3!}{s^{3+1}} \\ \frac{5!}{s^{5+1}}? &= \frac{2}{s^3} \times \frac{6}{s^4} \\ \frac{120}{s^6} &\neq \frac{12}{s^7} \end{aligned}$$

If we define the convolution production of $f(t)$ and $g(t)$ as

$$(f \times g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

then

$$\mathscr{L}\{(f \times g)(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} \tag{7.2}$$

7.3.3 – Rule 3

If $f(t)$ is periodic with period T, then

$$\mathscr{L}\left\{f(t)\right\} = \frac{\int_0^t e^{-st} f(t) dt}{1 - e^{-sT}} \tag{7.3}$$

7.3.4 – Example

$$\begin{aligned}
&\mathscr{L}\left\{t \times \sin(kt)\right\} \\
\mathscr{L}\left\{t \times \sin(kt)\right\} &= -\frac{d}{ds}\mathscr{L}\left\{\sin(kt)\right\} \\
&= -\frac{d}{ds}\left(\frac{k}{s^2 + k^2}\right) \\
&= -\frac{\frac{d}{ds}k \times \left(s^2 + k^2\right) - k\frac{d}{ds}\left(s^2 + k^2\right)}{\left(s^2 + k^2\right)^2} \\
&= -\frac{0\left(s^2 + k^2\right) - k(2s)}{\left(s^2 + k^2\right)^2} \\
&= -\frac{-2sk}{\left(s^2 + k^2\right)^2} \\
&= \frac{2sk}{\left(s^2 + k^2\right)^2}
\end{aligned}$$

7.3.5 – Example

$$x'' + 16x = \cos(4t), \quad x(0) = 0, \quad x'(0) = 1$$

$$\begin{aligned}
x'' + 16x &= \cos(4t) \\
\mathscr{L}\left\{x''\right\} + 16\mathscr{L}\left\{x\right\} &= \mathscr{L}\left\{\cos(4t)\right\} \\
s^2 X(s) - sx(0) - x'(0) + 16X(s) &= \frac{s}{s^2 + 4^2} \\
X(s)\left(s^2 + 16\right) - s(0) - 1 &= \frac{s}{s^2 + 16} \\
X(s)\left(s^2 + 16\right) - 1 &= \frac{s}{s^2 + 16} \\
X(s)\left(s^2 + 16\right) &= \frac{s}{s^2 + 16} + 1
\end{aligned}$$

$$\begin{aligned}
X(s) &= \frac{s}{\left(s^2 + 16\right)^2} + \frac{1}{s^2 + 16} \\
\mathscr{L}^{-1}\left\{X(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{s}{\left(s^2 + 16\right)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2 + 16}\right\} \\
x(t) &= \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{8s}{\left(s^2 + 16\right)^2}\right\} + \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{4}{s^2 + 4^2}\right\} \\
&= \frac{1}{8}t\sin(4t) + \frac{1}{4}\sin(4t)
\end{aligned}$$

7.3.6 – Example

Find $e^t \sin(t)$

$$e^t \sin(t) = \int_0^t e^\tau \sin(t-\tau) d\tau$$

$$u = e^\tau \qquad dv = \sin(t-\tau)d\tau$$

$$du = e^\tau d\tau \qquad \begin{aligned} v &= \int \sin(t-\tau)d\tau \\ &= \frac{\tau\cos(t-\tau)}{\tau} \\ &= \cos(t-\tau) \end{aligned}$$

$$\begin{aligned} e^t \sin(t) &= \int_0^t e^\tau \sin(t-\tau)d\tau \\ &= e^\tau \cos(t-\tau)\Big|_0^t - \int_0^t e^\tau \cos(t-\tau)d\tau \end{aligned}$$

$$h = e^\tau \qquad dj = \sin(t-\tau)d\tau$$

$$dh = e^\tau d\tau \qquad \begin{aligned} j &= \int \cos(t-\tau)d\tau \\ &= \frac{\sin(t-\tau)}{-1} \\ &= -\sin(t-\tau) \end{aligned}$$

$$\begin{aligned} \int_0^t e^\tau \sin(t-\tau) &= e^\tau \cos(t-\tau)\Big|_{\tau=0}^t - \left(-e^\tau \sin(t-\tau)\Big|_{\tau=0}^t - \int_0^t e^\tau(-\sin(t-\tau))d\tau\right) \\ &= e^t\cos(1) - e^0\cos(t) + e^t(0) - e^0\sin(t) - \int_0^t e^\tau \sin(t-\tau)d\tau \end{aligned}$$

$$2\int_0^t e^\tau \sin(t-\tau)d\tau = e^t - \cos(t) - sin(t)$$

So

$$e^t \sin(t) = \frac{e^t - \cos(t) - \sin(t)}{2}$$

$$\begin{aligned}
\mathscr{L}\left\{e^t \sin(t)\right\} &= \mathscr{L}\left\{\frac{e^t - \cos(t) - \sin(t)}{2}\right\} \\
&= \frac{1}{2}\mathscr{L}\left\{e^t - \cos(t) - \sin(t)\right\} \\
&= \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1^2} - \frac{1}{s^2+1^2}\right) \\
&= \frac{1}{2}\left(\frac{s^2+1}{(s-1)(s^2+1)} - \frac{s(s-1)}{(s-1)(s^2+1)} - \frac{s-1}{(s-1)(s^2+1)}\right) \\
&= \frac{1}{2}\left(\frac{s^2+1-s(s-1)-s-1}{(s-1)(s^2+1)}\right) \\
&= \frac{s^2+1-s^2+s-s+1}{2(s-1)(s^2+1)} \\
&= \frac{s^2-s^2+s-s+1+1}{2(s-1)(s^2+1)} \\
&= \frac{1+1}{2(s-1)(s^2+1)} \\
&= \frac{2}{2(s-1)(s^2+1)} \\
&= \frac{1}{(s-1)(s^2+1)}
\end{aligned}$$

7.3.7 – Example

$$\begin{aligned}
\mathscr{L}\left\{(f \times g)(t)\right\} &= \mathscr{L}\left\{f(t)\right\} \times \mathscr{L}\left\{g(t)\right\} \\
(f \times g)(t) &= \mathscr{L}^{-1}\left\{F(s) \times G(s)\right\}
\end{aligned}$$

Determine

$$\mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+k^2\right)^2}\right\}$$

using the Convolution Theorem.

$$\mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+k^2\right)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\}$$

$$\begin{aligned}
\sin(A)\sin(B) &= \frac{1}{2}\left(\cos(A-B)+\cos(A+B)\right) \\
&= \frac{1}{k^2}\mathscr{L}\left\{\frac{1 \times k}{s^2+k^2}\right\}
\end{aligned}$$

7.3.8 – Example

Solve the Integral Equation

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau \quad \text{for } f(t)$$

Don't forget that the $\int_0^t f(\tau)e^{t-\tau}d\tau$ is $f(t) \times e^t$. Take $\mathscr{L}$ of both sides

$$\begin{aligned}
F(s) &= 3\frac{2!}{s^{2+1}} - \frac{1}{s-(-1)} - F(s)\frac{1}{s-1} \\
&= \frac{6}{s^3} - \frac{1}{s+1} - F(s)\frac{1}{s-1} \\
F(s) + F(s)\frac{1}{s-1} &= \frac{6}{s^3} - \frac{1}{s+1} \\
F(s)\left(1 + \frac{1}{s-1}\right) &= \frac{6}{s^3} - \frac{1}{s+1} \\
F(s)\left(\frac{s-1}{s-1} + \frac{1}{s-1}\right) &= \frac{6}{s^3} - \frac{1}{s+1} \\
F(s)\left(\frac{s-1+1}{s-1}\right) &= \frac{6}{s^3} - \frac{1}{s+1} \\
F(s)\left(\frac{s}{s-1}\right) &= \frac{6}{s^3} - \frac{1}{s+1}
\end{aligned}$$

$$\begin{aligned}
F(s) &= \frac{6(s-1)}{s^3 s} - \frac{s-1}{(s+1)s} \\
&= \frac{6s-6}{s^4} - \frac{s-1}{s^2+s} \\
&= \dots \\
&= \frac{6s+6-s^3}{s^3(s+1)}
\end{aligned}$$