Chapter 3

Modeling using DE

3.3 Applications: Modelling with a System of Linear DEs

$$x_1'' + 3x_1' + x_2'' - 5x_2 = e^t$$
$$x_1'' - 4x_2'' + 6x_2' = \sin t$$

Two unknown functions $x_1(t)$ and $x_2(t)$. We learn methods to solve such systems in 4.4 and 7.6.
There is usually a sequence of isotopes the initial isotope transforms through. Suppose we have Decay Series of the form

$$X \to Y \to Z \text{ (where } X \text{ is a stable isotope)}$$

The decay rates $X \to Y$ and $Y \to Z$ can be significantly different.

$$\frac{dX}{dt} = k_1 X$$
$$\frac{dY}{dt} = k_2 Y$$

where $X(t) =$ mass of isotope X at time t and $Y(t) =$ mass of isotope Y at time t.
Assume the relative decay rate of a radioactive substance is a constant

$$\frac{\frac{dX}{dt}}{X} = k_1, \quad k_1 < 0$$
$$\frac{\frac{dY}{dt}}{Y} = k_2, \quad k_2 < 0$$

We'll define

$$\lambda_1 = -k_1 \quad so\ \lambda_1 > 0$$
$$\lambda_2 = -k_2 \quad so\ \lambda_2 > 0$$

Notation:

$$X \to^{k_1} \to Y \to^{k_2} \to Z$$

is the same as

$$X \to^{-\lambda_1} \to Y \to^{-\lambda_2} \to Z$$

3.3.1 – Example

$$\begin{aligned} Dx+(D+2)y&=0 & (D-3)x-2y&=0\\ D(D-3)x+(D-3)(D+2)y&=0 & D(D-3)x-D(2y)&=0\\ D(x'-3x)+(D-3)(y'+2y)&=0 & D(x'-3x)-2y'&=0\\ x''-3x'+y''+2y'-3y'-6y&=0 & x''-3x'-2y'&=0\\ x''-3x'+y''-y'-6y&=0 & x''-3x'-2y'&=0 \end{aligned}$$

$$\begin{aligned} x''-3x'+y''-y'-6y&=x''-3x'-2y'\\ y''-y'-6y&=-2y'\\ y''+y'-6y&=0 \end{aligned}$$

3.3.2 – Example

Suppose now, you start with 500 grams (.5 kg) of X and 0 grams of Y and Z. If $k_1=-.01$ (per year) and $k_2=-.003$ (per year). Determine how much Z there will be after $t=1,000$ years.

$$\frac{dX}{dt}=-\lambda_2 Y \qquad \frac{dY}{dt}=\lambda_1 X-\lambda_2 Y \qquad \frac{dZ}{dt}=\lambda_2 Y$$

$$\begin{aligned} DX+\lambda_1 X=0 \Rightarrow (D+\lambda_1)X &=0\\ DY-\lambda_1 X+\lambda_2 Y=0 \Rightarrow (D+\lambda_2)Y-\lambda_1 X &=0\\ DZ+\lambda_2 Y=0 \Rightarrow DZ+\lambda_2 Y &=0 \end{aligned}$$

$\frac{dX}{dt}=-\lambda_1 X$ is separable and 1st order linear, which means $X=c_1e^{-\lambda_1 t}$, and we know the initial value is 500g, so $X(t)=500e^{-.01t}$.

$$\begin{aligned} \frac{dY}{dt}&=\lambda_1 X-\lambda_2 Y\\ &=\lambda_1\times c_1e^{-\lambda_1 t}-\lambda_2 Y\\ &=.01\times 500e^{-.01t}-\lambda_2 Y\\ \frac{dY}{dt}+\lambda_2 Y&=5e^{-.01t}\\ Y'+.003Y&=5e^{-.01t}\\ Y_c(t)&=c_2e^{-\lambda_2 t}\\ y_p(t)&=Ae^{-0.1t}\\ y_p'(t)&=-0.1Ae^{-0.1t}\\ -0.1Ae^{-0.1t}+0.03\times Ae^{-0.1t}&=5e^{-.01t}\\ -0.1A+0.03A&=5\\ -0.07A&=5\\ A&=\frac{5}{-0.07} \end{aligned}$$

$$
\begin{aligned}
Y(t) &= Y_c(t) + Y_p(t) \\
&= c_2 e^{-\lambda_2 t} + Ae^{-0.1t} \\
&= 0e^{-0.03t} - \frac{5}{0.07}e^{-0.1t} \\
&= -\frac{500}{7}e^{-0.1t}
\end{aligned}
$$

and a similar method can be done for $Z(t)$.

3.3.3 – Other Application: Mixture Problems

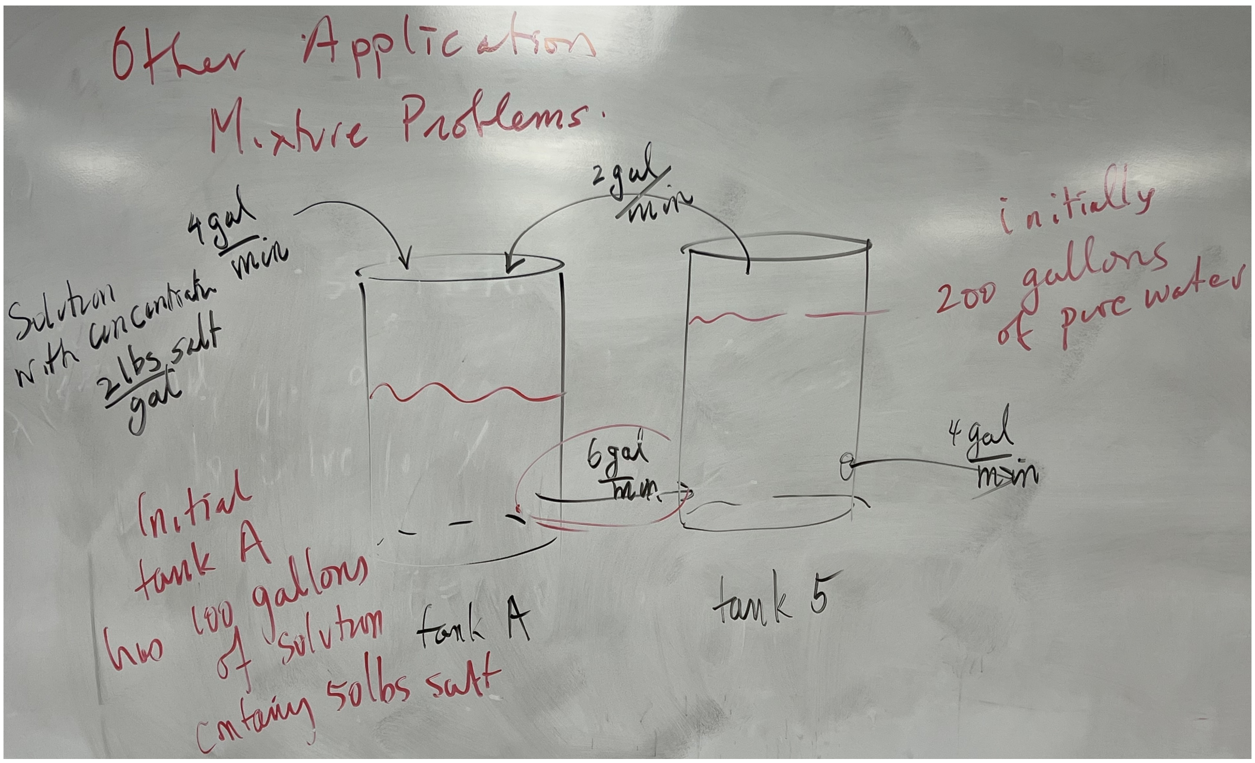

Figure 3.1

Solution with concentration $2\ \frac{\text{lbs of salt}}{\text{gal}}$. Initial tank A has 100 gallons of solution containing 50 lbs of salt.

Let $x_1(t) = \#$ of lbs of salt in tank A at time t.
Let $x_2(t) = \#$ of lbs of salt in tank B at time t.
Tank A will always have 100 gallons of solution (6 gallons in = 6 gallons out) some for tank B with $V = 200$ gal

$$\begin{aligned}
\frac{dx_1}{dt} &= \text{rate salt in to tank } A - \text{rate salt } \textbf{out} \text{ of tank } B \\
&= 4\frac{gal}{min} \times 2\frac{lbs}{gal} + 2\frac{gal}{min} \times \frac{\#\text{ lbs salt in } B}{200\ gal} - \frac{6\ gal}{min} \times \frac{x_1}{100\ gal} \\
&= 8\frac{lbs}{min} + \frac{\#\text{ lbs salt in } B}{100\ min} - \frac{3x_1}{50\ min} \\
&= 8 + \frac{x_2}{100} - \frac{3x_1}{50}
\end{aligned}$$