Chapter 7

Method of Laplace Transforms for Solving DE's

7.3 Operational Rules Part 1

Even with the reuses we know, a problem like

$$\mathscr{L}\left\{e^{4t}t^3\right\}$$

would require us to go to the definition until we learn some rules.

$$\begin{aligned}\mathscr{L}\left\{e^{4t}t^3\right\} &= \int_0^\infty e^{-st}e^{4t}t^3 dt \\ &= \int_0^\infty e^{-(s-4)t}t^3 dt\end{aligned}$$

Compare with

$$\mathscr{L}\left\{t^3\right\} = \int_0^\infty e^{-st}t^3 dt = F(s)$$

where $f(t) = t^3$

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a) \tag{7.1}$$

where $F(s) = \mathscr{L}\{f(t)\}$

$$\mathscr{L}\left\{e^{4t}t^3\right\} = \frac{3!}{(s-4)^4}$$

7.3.1 – Example

$$
\begin{aligned}
\mathscr{L}\left\{e^{-3t}\sin(5t)\right\} &= \mathscr{L}\left\{\sin(5t)\right\}\Big|_{s\to s+3} \\
&= \frac{5}{s^2+25}\Big|_{s\to s+3} \\
&= \frac{5}{(s+3)^2+25}\Big|_{s\to s+3} \\
&= \frac{5}{s^2+6s+9+25} \\
&= \frac{5}{s^2+6s+34}
\end{aligned}
$$

7.3.2 – Example

$$
\begin{aligned}
\mathscr{L}\left\{e^{-2t}\cos(6t)\right\} &= F(s+2) \\
&= \frac{s+2}{(s+2)^2+36} \\
&= \frac{s+2}{s^2+4s+4+36} \\
&= \frac{s+2}{s^2+4s+40}
\end{aligned}
$$

7.3.3 – Example

Find

$$
\mathscr{L}^{-1}\left\{\frac{s+3}{s^2-8s+97}\right\}
$$

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{s+3}{s^2-8s+97}\right\} &= \mathscr{L}^{-1}\left\{\frac{s+3}{s^2-8s+16+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s+3}{(s-4)^2+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s+3+4-4}{(s-4)^2+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s-4}{(s-4)^2+81}\right\} + \mathscr{L}^{-1}\left\{\frac{3+4}{(s-4)^2+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s-4}{(s-4)^2+81}\right\} + \mathscr{L}^{-1}\left\{\frac{7}{(s-4)^2+81}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s-4}{(s-4)^2+9^2}\right\} + \mathscr{L}^{-1}\left\{\frac{7}{(s-4)^2+9^2}\right\} \\
&= e^{4t}\cos(9t) + \frac{7}{9}\mathscr{L}^{-1}\left\{\frac{9}{(s-4)^2+9^2}\right\} \\
&= e^{4t}\cos(9t) + \frac{7}{9}e^{4t}\sin(9t)
\end{aligned}$$

Involves taking the Laplace transform of a function shifted on the t-axis.
This can be written in terms of the Heavyside Function (or Unit Step function)

$$U(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

$$f(t-2) \times U(t-2)$$

is “off when $t < 2$ and on when $t \geq 2$.

7.3.4 – Example

$$\begin{aligned}
\mathscr{L}\left\{f(t-2)U(t-2)\right\} &= \int_0^\infty e^{-st} f(t-2)U(t-2)dt \\
&= \int_0^2 e^{-st} f(t-2)U(t-2)dt + \int_2^\infty e^{-st} f(t-2)U(t-2)dt \\
&= \int_0^2 e^{-st} f(t-2)(0)dt + \int_2^\infty e^{-st} f(t-2)(1)dt \\
&= \int_0^2 0dt + \int_2^\infty e^{-st} f(t-2)dt \\
&= \int_2^\infty e^{-st} f(t-2)dt \\
(\text{ Let } v = t-2,\ dv = dt) \quad &= \int_0^\infty e^{-s(v+2)} f(v)dv \\
&= \int_0^\infty e^{-sv} e^{-2s} f(v)dv \\
&= e^{-2s} \int_0^\infty e^{-sv} f(v)dv
\end{aligned}$$