

MATH 252 - Introduction to Differential Equations Notes

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Chapter 1

Introduction to Differential Equations

1.1 Terminology and Notation

Differential equation (D.E.) – An equation in which at least one derivative of an unknown function.

Order of the D.E. – The highest order of derivative in the D.E.

1.1.1 – Example

$$4y'' + e^x y' - 3yy' = \sin(x)$$

An example of a partial differential equation is:

$$\frac{\partial T}{\partial x} + x^2 \frac{\partial T}{\partial y} = x + y$$

however, we won't study these in this course.

1.1.2 – Linear vs Non-Linear DE's

Linear D.E. – The dependent variable and all of its derivatives in the D.E. are in separate terms to the 1st power. $y^{(n)}$ or $\frac{d^n y}{dx^n}$ where $n \neq 1$ are non-first power.

$$4y'' + e^x y' - 3yy' = \sin(x)$$

is a non-linear D.E. while

$$4y'' + e^x y' - 3y = \sin(x)$$

is linear.

The general formula of a linear D.E. would look like

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = g(x)$$

Solution – a function $\phi(x)$ and an interval I for which the D.E. is satisfied when $y = \phi(x)$ for all x in I .

It may be the case that the natural domain of $\phi(x)$ is larger than I .

1.1.3 – Example

$y' = -\frac{1}{x^2}$ has a solution $\phi(x) = \frac{1}{x}$ on $I = (0, \infty)$ but the domain of $\phi(x) = (-\infty, 0) \cup (0, \infty)$.

Practice:

$$\frac{d^2x}{dt^2} + 16x = 0$$

Show (Verify not derive) $x(t) = c_1 \sin(4t)$ is a solution on $(-\infty, \infty)$ where c is any real parameter.

$$\begin{aligned} x &= c_1 \sin(4t) \\ \frac{dx}{dt} &= 4c_1 \cos(4t) \\ \frac{d^2x}{dt^2} &= -16c_1 \sin(4t) \\ \text{LHS} &= \frac{d^2x}{dt^2} + 16x \\ &= -16c_1 \sin(4t) + 16(c_1 \sin(4t)) \\ &= 0 = \text{RHS} \end{aligned}$$

But the equation $x = c_2 \cos(4t)$ would also be a solution. **If you have 2 equations that are both solutions, you could add them together and you would still have a solution.** $x = c_1 \sin(4t) + c_2 \cos(4t)$ is a solution for all parameters c_1 and c_2 . *In fact, this is the general solution to the D.E.*

The D.E.

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

Show $y = \left(\frac{1}{4}x^2 + C\right)^2$ is a one parameter family of solutions

$$\begin{aligned} \text{LHS} &= \frac{dy}{dx} = 2 \left(\frac{1}{4}x^2 + C \right) \times \frac{1}{2}x \\ &= x \left(\frac{1}{4}x^2 + C \right) \\ \text{RHS} &= xy^{\frac{1}{2}} = x \left(\left(\frac{1}{4}x^2 + C \right)^2 \right)^{\frac{1}{2}} \\ &= x \left(\frac{1}{4}x^2 + C \right) \\ \text{LHS} &= \text{RHS} \end{aligned}$$

But there is another solution: namely $y(x) = 0$ for all x . This is called the “trivial solution”.

1.2 Initial Value Problems (IVP)

1st order IVP is a 1st order D.E. together with one extra condition:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

2nd order IVP

$$y'' = f(x, y, y')$$

Initial conditions:

- $y(x_0) = y_0$
- $y'(x_0) = y_1$

1.2.1 – Example

$$y' = y \text{ and } y(0) = 3$$

$y = ce^x$ is a one-parameter family of solutions

$$\frac{d}{dx}(ce^x) = ce^x = y$$

$$ce^1 = -2$$

$$c = -\frac{2}{e}$$

$$y = \left(-\frac{2}{e}\right) e^x$$

$$y = -2e^{x-1}$$

1.2.2 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Given that you have the solution: $y = \frac{1}{x^2+C}$, Solve:

$$-1 = \frac{1}{(0)^2 + c}$$

$$-1 = \frac{1}{c}$$

$$-1 \times c = 1$$

$$c = -1$$

$$y = \frac{1}{x^2 - 1}, I = (-1, 1)$$

1.2.3 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Example

$$x'' + 16x = 0 \text{ and } x\left(\frac{\pi}{2}\right) = 5 \text{ and } x'\left(\frac{\pi}{2}\right) = -4$$

$$x = c_1 \cos(4t) + c_2 \sin(4t)$$

$$5 = c_1 \cos(4t) + c_2 \sin(4t)$$

$$= c_1 \cos(2\pi) + c_2 \sin(2\pi)$$

$$= c_1(1) + c_2(0)$$

$$= c_1$$

$$x' = -4c_1 \sin(4t) + 4c_2 \cos(4t)$$

$$-4 = -4c_1 \sin\left(4\left(\frac{\pi}{2}\right)\right) + 4c_2 \cos\left(4\left(\frac{\pi}{2}\right)\right)$$

$$= -4c_1 \sin(2\pi) + 4c_2 \cos(2\pi)$$

$$= -4c_1(0) + 4c_2(1)$$

$$= 4c_2$$

$$-1 = c_2$$

Reasonable Question: Given a 1st order IVP, can we say whether a solution *exists* or not and, if a solution exists, is it *unique*.

Theorem: Given $y' = f(x, y)$ and $y(x_0) = y_0$, if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous on a rectangle R containing (x_0, y_0) in its interior, then there exists an interval $I = (x_0 - h, x_0 + h)$ where $h > 0$ such that there exists a unique solution to IVP on I .

1.2.4 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(1) = 2$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous everywhere its defined $y \geq 0$
- $\frac{\partial f}{\partial y} = x^{\frac{1}{2}}y^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$ is continuous everywhere its defined $y > 0$

1.2.5 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(0) = 0$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous for all x and $y \geq 0$
- $\frac{\partial f}{\partial y} = \frac{x}{2y}$ is continuous for all x and $y > 0$.
- **Theorem does not give any conclusion.**

Chapter 2

First-Order Differential Equations

2.1 Solution Curves Without a Solution

Given a 1st order D.E. $y' = f(x, y)$, y' is the slope of the tangent line at any point (x_0, y_0) on a solution curve

2.1.1 – Example

$$y' = f(x, y) = x + y$$

- $f(0, 0) = 0$
- $f(1, 0) = 1$

2.1.2 – Slope/Direction Fields

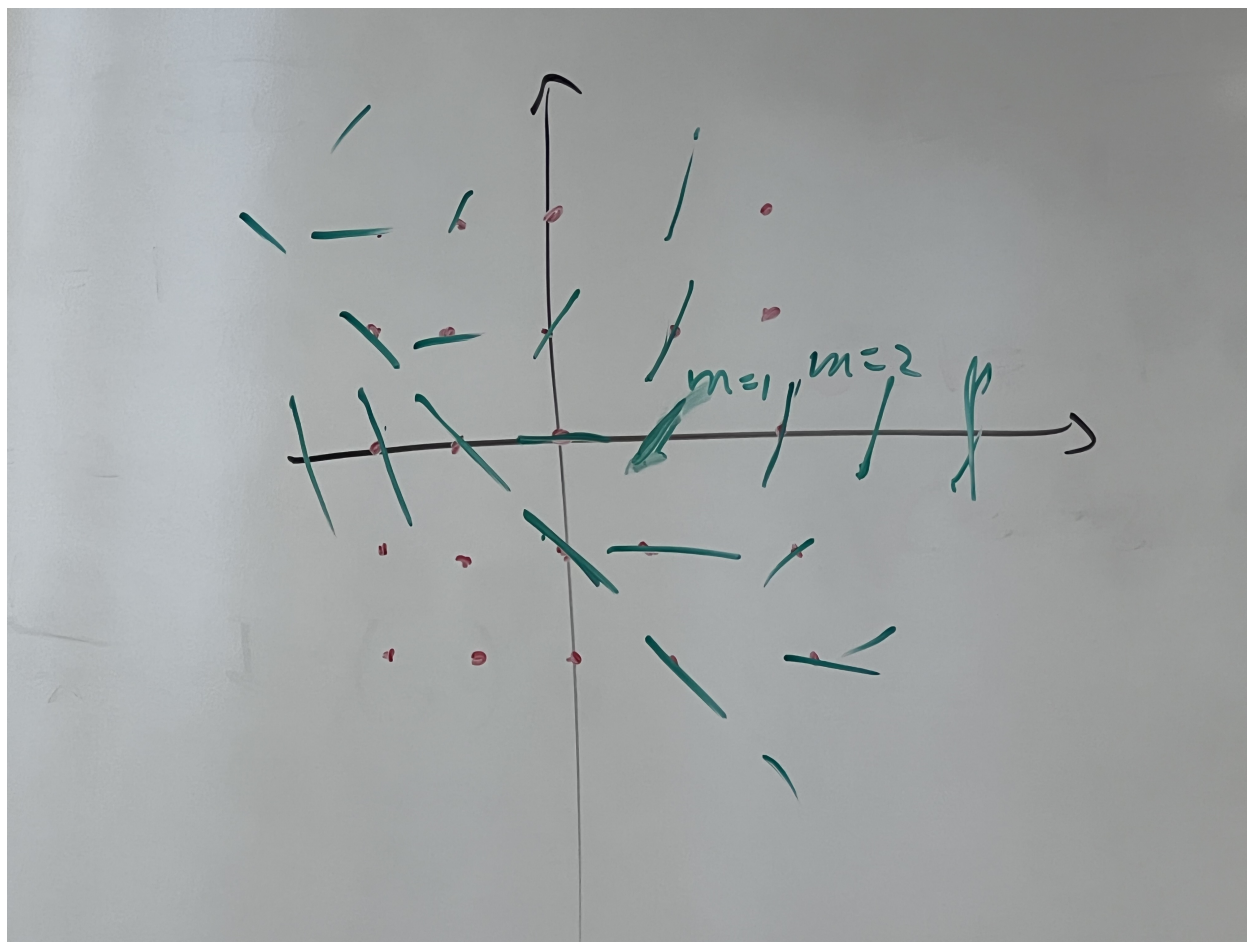


Figure 2.1: The direction field for the previous example

If the function $f(x, y)$ in the D.E. $y' = f(x, y)$ is reasonably simple so that we can solve $f(x, y) = 0$, we can make a “phase portrait diagram”. We will also assume $f(x, y)$ only involves the y -variable.

2.1.3 – Example

$$y' = (y + 2)(y - 3)(y - 5)$$

$$f(x, y) = (y + 2)(y - 3)(y - 5)$$

An “equilibrium solution” is a solution where y is a constant. In this example: $y = 3$, $y = 5$, $y = -2$ are each constant functions.

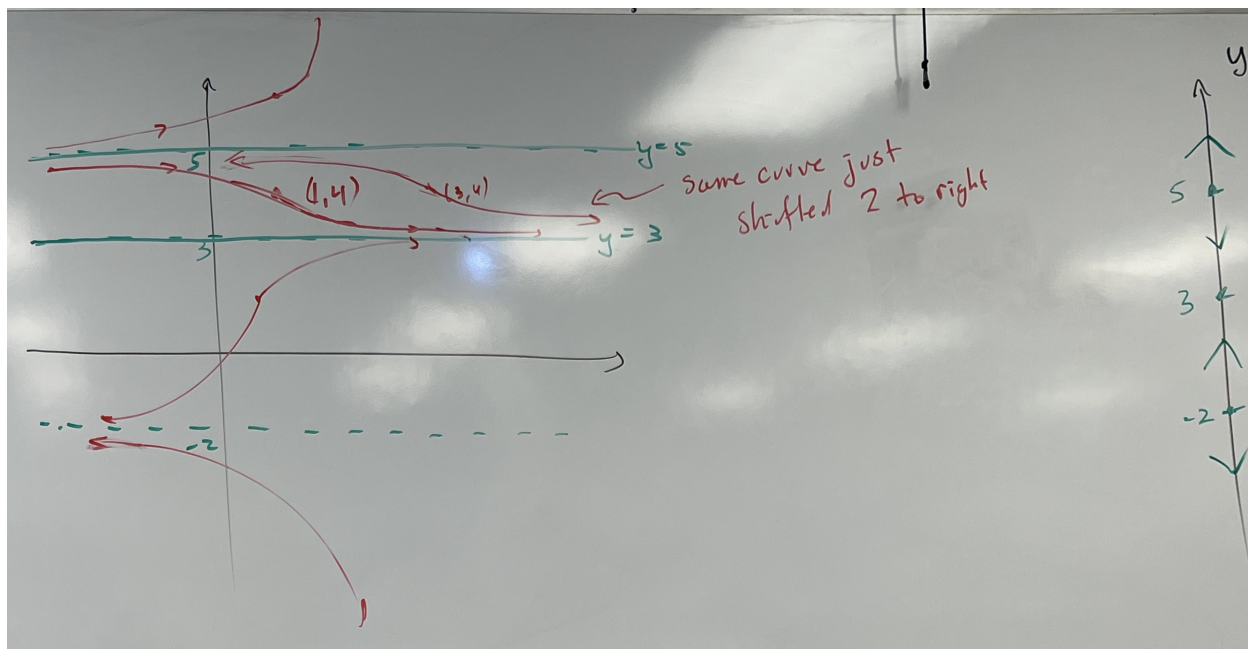


Figure 2.2: The equilibrium solution for the previous example.

The area around $y = 5$ is an unstable equilibrium since the solutions diverge and go in separate directions away from $y = 5$. The area around $y = 3$ is a stable equilibrium because the slopes above and below it converge to $y = 3$. The area around $y = -2$ is semi-stable, since all the slopes around it will converge in one direction, but the point isn't always $y = -2$.

2.2 Separable Differential Equations

Separable D.E.s are DE's $\frac{dy}{dx} = f(x, y)$ where $f(x, y)$ can be factored as $f(x, y) = g(x)h(y)$.

$$\frac{dy}{dx} = (1 + y^2)x^3 \text{ is separable}$$

$$\frac{dy}{dx} = \sin(xy) \text{ is not separable}$$

$$\frac{dy}{dx} = x^3y \text{ is not separable}$$

$$\frac{5}{xy} \frac{dy}{dx} = (x^2 + y) e^y$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy(x^2 + y)e^y}{5} \\ &= \frac{x(x^2 + y)}{5} \times ye^y \end{aligned}$$

2.2.1 – Method of Solution

“Separate the variable” to get $\frac{1}{h(y)} dy = g(x) dx$ or $p(y) dy = g(x) dx$ where $p(y) = \frac{1}{h(y)}$. **Integrate both sides**

$$\int p(y) dy = \int g(x) dx \text{ and if possible, solve for } y$$

2.2.2 – Example

$$\begin{aligned} \frac{dy}{dx} &= (1 + y^2) x^3 \\ \int \frac{1}{1 + y^2} dy &= \int x^3 dx \\ \tan^{-1}(y) + C_1 &= \frac{x^4}{4} + C_2 \\ \tan^{-1}(y) &= \frac{x^4}{4} + C_2 - C_1 \\ \tan^{-1}(y) &= \frac{x^4}{4} + C \\ y &= \tan\left(\frac{x^4}{4} + C\right) \end{aligned}$$

2.2.3 – Example

Problem 12 from the textbook.

$$\begin{aligned} \sin(3x) dx + 2y \cos^3(3x) dy &= 0 \\ \int -2y dy &= \int \frac{\sin(3x)}{\cos^3(x)} dx \\ &= \int \tan(3x) \sec^2(3x) dx \\ &= \int u \frac{1}{3} du \text{ where } u = \tan(3x), \quad du = 3 \sec^2(3x) dx \\ -2 \int y dy &= \frac{1}{3} \int u du + C \\ -y^2 &= \frac{u^2}{6} + C \\ &= \frac{\tan^2(3x)}{6} + C \\ \frac{\tan^2(3x)}{6} + y^2 &= -C \\ \frac{\tan^2(3x)}{6} + y^2 &= C \end{aligned}$$

Problem 25 from the textbook.

$$x^2 \frac{dy}{dx} = y - xy, y(-1) = -1$$

$$x^2 \frac{dy}{dx} = y - xy$$

$$x^2 \frac{dy}{dx} = y(1 - x)$$

$$\frac{dy}{y} = \frac{(1 - x)}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{(1 - x)}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{x^2} dx - \int \frac{x}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{x^2} dx - \int \frac{1}{x} dx$$

$$\ln |y| + C_1 = -\frac{1}{x} + C_2 - \ln |x| + C_3$$

$$\ln |y| = -\frac{1}{x} - \ln |x| + C$$

$$y = e^{-\frac{1}{x}} \times e^{-\ln |x|} \times e^C$$

$$y = e^{-\frac{1}{x}} \times e^{-\ln |x|} \times e^C$$

$$y = e^{-\frac{1}{x}} \times \frac{1}{|x|} \times e^C$$

$$y = \frac{1}{|x|} e^{C - \frac{1}{x}}$$

$$-1 = \frac{1}{|-1|} e^{C - \frac{1}{-1}}$$

$$-1 = \frac{1}{1} e^{C - (-1)}$$

$$-1 = e^{C+1}$$

2.3 First Order Linear Differential Equations

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\left. \frac{dy}{dx} + P(x)y = f(x) \right\} \text{ Standard form of a 1st-order linear DE}$$

We will try to find a function $\mu(x)$ such that by multiplying the D.E. by an integrating factor (I.F.) $\mu(x)$:

$$\mu(x) \frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

such that the LHS is an exact derivative, Observe:

$$\frac{d}{dx} (\mu(x)y) = \mu(x) \frac{dy}{dx} + \frac{d\mu}{dx} y$$

from which we see

$$\mu(x)P(x) = \frac{d\mu}{dx}$$

$$P(x)dx = \frac{d\mu}{\mu(x)}$$

$$\int P(x)dx = \int \frac{d\mu}{\mu}$$

$$\int P(x)dx = \ln \mu$$

$$\ln \mu = \int P(x)dx$$

$$\mu = e^{\int P(x)dx}$$

2.3.1 – Example

$$x \frac{dy}{dx} - 4y = x^6 e^x$$

Standard form: $\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x$

$$P(x) = -\frac{4}{x}$$

$$\mu = e^{\int \frac{-4}{x} dx}$$

$$= e^{-4 \ln x}$$

$$= e^{\ln x^{-4}}$$

$$= x^{-4}$$

$$\text{I.F.} = \mu = x^{-4}$$

Now multiply the standard form of the given D.E. by x^{-4} .

$$x^{-4} \frac{dy}{dx} - x^{-4} \frac{4}{x} y = x^{-4} x^5 e^x$$

$$x^{-4} \frac{dy}{dx} - x^{-4} \frac{4}{x} y = x e^x$$

$$\int \frac{d}{dx} (x^{-4} y) = \int x e^x$$

$$x^{-4} y = \int x e^x$$

2.3.2 – Example

$$(x^2 - 9) \frac{dy}{dx} + xy = 0$$

$$(x^2 - 9) \frac{dy}{dx} + xy = 0$$

$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0$$

$$P(x) = \frac{x}{x^2 - 9}$$

$$\int P(x)dx = \int \frac{x}{x^2 - 9}dx$$

$$\int P(x)dx = \int \frac{1}{u - 9} \frac{du}{2}$$

$$\int P(x)dx = \frac{1}{2} \int \frac{1}{u - 9} du$$

$$\int P(x)dx = \frac{1}{2} \ln |u - 9|$$

$$\int P(x)dx = \frac{1}{2} \ln |x^2 - 9|$$

$$\mu = e^{\frac{1}{2} \ln |x^2 - 9|}$$

$$\mu = e^{\ln |(x^2 - 9)^{\frac{1}{2}}|}$$

$$\mu = (x^2 - 9)^{\frac{1}{2}}$$

$$\mu = \sqrt{x^2 - 9}$$

$$\sqrt{x^2 - 9} \left(\frac{dy}{dx} + \frac{x}{x^2 - 9}y \right) = \sqrt{x^2 - 9}(0)$$

$$\sqrt{x^2 - 9} \frac{dy}{dx} + \frac{x}{\sqrt{x^2 - 9}}y = 0$$

$$\int \frac{d}{dx} (y\sqrt{x^2 - 9}) = \int 0$$

$$y\sqrt{x^2 - 9} = C$$

$$y = \frac{C}{\sqrt{x^2 - 9}}$$

2.4 Exact Equations

1st Order D.E. in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

Given a function

$$z = f(x, y)$$

, the total differential, dz , is defined as

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

2.4.1 – Method

See if we can find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

If we can do this, then the D.E. is equivalent to

$$df = 0 \Rightarrow f(x, y) = c$$

is an implicit solution of D.E.

Assume that M and N have continuous 1st order partials (assuming f exists)

$$\left. \begin{array}{l} My = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} dy = f_{xy} \\ Nx = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} dy = f_{yx} \end{array} \right\} \text{Theorem tells us these are equal}$$

This provides a quick test to check if the D.E. is exact or not.

2.4.2 – Example

$$2xydx + (x^2 - 1) dy = 0$$

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

To check if the D.E. is exact

$$M_y = 2x = N_x$$

We now know there exists a function $f(x, y)$ with

$$\begin{aligned} \frac{\partial f}{\partial x} &= M = 2xy \\ \frac{\partial f}{\partial y} &= N = x^2 - 1 \end{aligned}$$

$$\begin{aligned}
 f_M(x, y) &= \int \frac{\partial f}{\partial x} dx \\
 &= \int 2xy dx \\
 &= x^2 y + \phi(y) \\
 \frac{\partial f}{\partial y} (x^2 y + \phi(y)) &= x^2 - 1 \text{ required to equal } N \\
 x^2 + \phi'(y) &= x^2 - 1 \\
 \phi'(y) &= -1 \\
 \phi(y) &= \int -1 dy \\
 &= -y \\
 f(x, y) &= x^2 y - y \\
 d(f(x, y)) &= 0 \\
 f(x, y) &= c \\
 x^2 y - y &= c \text{ is an implicit solution of the D.E.}
 \end{aligned}$$

Note: the f_M format is just there to show which partial equation was integrated. It was made by me and, as far as I know, is not standardly known.

2.4.3 – Example

$$\begin{aligned}
 (e^{2y} - y \cos(xy)) dx + (2xe^{2y} - x \cos(xy) + 2y) dy &= 0 \\
 M_y &= N_x \\
 \frac{\partial}{\partial y} (e^{2y} - y \cos(xy)) &= \frac{\partial}{\partial x} (2xe^{2y} - x \cos(xy) + 2y) \\
 2e^{2y} - [\cos(xy) - y \sin(xy) \times x] &= 2e^{2y} - (\cos(xy) - x \sin(xy) \times y) + 0 \\
 2e^{2y} - \cos(xy) + xy \sin(xy) &= 2e^{2y} - \cos(xy) + xy \sin(xy) \\
 \frac{\partial f}{\partial x} = M &= e^{2y} - y \cos(xy) \\
 \frac{\partial f}{\partial y} = N &= 2xe^{2y} - x \cos(xy) + 2y \\
 f_N(x, y) &= \int \frac{\partial f}{\partial y} dy \\
 &= \int (2xe^{2y} - x \cos(xy) + 2y) dy \\
 &= \frac{2xe^{2y}}{2} - \frac{x \sin(xy)}{x} + 2 \times \frac{y^2}{2} + \phi(x) \\
 &= xe^{2y} - \sin(xy) + y^2 + \phi(x)
 \end{aligned}$$

Take the ∂x of this and equate with M :

$$\begin{aligned} M &= \frac{\partial}{\partial x} (xe^{2y} - \sin(xy) + y^2 + \phi(x)) \\ e^{2y} - y \cos(xy) &= e^{2y} - y \cos(xy) + 0 + \phi'(x) \\ 0 &= \phi'(x) \\ \phi(x) &= c \end{aligned}$$

So $f(x, y) = c_2$ is the solution

$$xe^{2y} - \sin(xy) + y^2 = c$$

$$dx = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

2.4.4 – What can you do if $M_y \neq N_x$

Sometimes you can multiply the DE by an integrating factor $\mu(x, y)$ to get an exact DE.

If

$$\frac{M_y - N_x}{N}$$

is a function of only x , then

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

will be an I.F.

If

$$\frac{N_x - M_y}{M}$$

is a function of only y , then

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

will be an I.F.

2.4.5 – Example

$$xydx + (2x^2 + 3y^2 - 20) dy = 0$$

$$M_y = x$$

$$N_x = 4x$$

$$M_y \neq N_x$$

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy}$$

$$= \frac{3x}{xy}$$

$$= \frac{3}{y} \text{ is a function of just } y$$

So:

$$\begin{aligned}
 \mu &= e^{\int \frac{3}{y} dy} \\
 &= e^{3 \ln y} \\
 &= y^3 \\
 xy^4 dx + y^3 (2x^2 + 3y^2 - 20) dy &= 0(y^3) \\
 xy^4 dx + (2x^2 y^3 + 3y^5 - 20y^3) dy &= \\
 M_y &= N_x \\
 4xy^3 &= 4xy^3 \\
 \frac{\partial f}{\partial x}
 \end{aligned}$$

2.5 Substitution Methods

Taking a D.E. that's not:

- Separable
- 1st Order Linear
- Exact

and making a substitution to turn the new D.E. into one of these.

Theorem: Given a D.E.

$$M(x, y)dx + N(x, y)dy = 0$$

A function $f(x, y)$ is said to be homogenous of order α if $f(tx, ty) = t^\alpha f(x, y)$.

2.5.1 – Example

Given:

$$f(x, y) = x^3 + 5xy^2 - y^3$$

Then:

$$\begin{aligned}
 f(tx, ty) &= (tx)^3 + 5(tx)(ty)^2 - (ty)^3 \\
 &= t^3 x^3 + 5t^3 xy^2 - t^3 y^3 \\
 &= t^3 (x^3 + 5xy^2 - y^3) \\
 &= t^3 f(x, y)
 \end{aligned}$$

2.5.2 – Example

$$\begin{aligned}
 f(x, y) &= \frac{x + y}{x^2 + y^2} \\
 f(tx, ty) &= \frac{tx + ty}{(tx)^2 + (ty)^2} \\
 f(tx, ty) &= \frac{tx + ty}{x^2t^2 + y^2t^2} \\
 f(tx, ty) &= \frac{t}{t^2} \times \frac{x + y}{x^2 + y^2} \\
 f(tx, ty) &= \frac{t}{t^2} f(x, y) \\
 f(tx, ty) &= \frac{1}{t} f(x, y)
 \end{aligned}$$

$f(x, y) = \frac{x+y}{x^2+y^2}$ is homogenous of order $\alpha = -1$

2.5.3 – Substitution Rule

If $M(x, y)$ and $N(x, y)$ are homogenous, each of the same order, then $u = \frac{y}{x}$ i.e., $y = ux$ or $v = \frac{x}{y}$ (i.e. $x = vy$) will produce a separable D.E.

2.5.4 – Example

Solve the separable D.E. and then back-substitute

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

$$M(x, y) = x^2 + y^2 \quad N = x^2 - xy$$

$$M_y = 2y \quad N_x = 2x - y$$

$$M_y \neq N_x$$

$$M(tx, ty) = (tx)^2 + (ty)^2$$

$$= t^2x^2 + t^2y^2$$

$$= t^2(x^2 + y^2)$$

$$= t^2M(x, y) \quad M \text{ is homogeneous of order 2 and so is } N$$

$$u = \frac{y}{x}$$

$$y = ux$$

$$dy = udx + xdu$$

$$(x^2 + (ux)^2)dx + (x^2 - x(ux))(udx + xdu) = 0$$

$$(x^2 + u^2x^2)dx + (x^2 - ux^2)(udx + xdu) = 0$$

$$(1 + u^2)x^2dx + x^2(1 - u)(udx + xdu) = 0$$

$$(1 + u^2)x^2dx + x^2(udx + xdu - u^2dx - uxdu) = 0$$

$$x^2(1dx + u^2dx + udx + xdu - u^2dx - uxdu) = 0$$

$$x^2(1dx + u^2dx - u^2dx + udx + xdu - uxdu) = 0$$

$$x^2(1dx + udx + xdu - uxdu) = 0$$

$$x^2(1 + u)dx + x^3(1 - u)du = 0$$

$$\int \frac{1}{x} dx = \int -\frac{1 - u}{1 + u} du$$

$$= \int \frac{u - 1}{u + 1} du$$

$$= \int \frac{u + (1 - 2)}{u + 1} du$$

$$= \int \left(\frac{u + 1}{u + 1} - \frac{2}{u + 1} \right) du$$

$$= \int \left(1 - \frac{2}{u + 1} \right) du$$

$$\ln|x| = \int \left(1 - \frac{2}{u + 1} \right) du$$

$$= u - 2 \ln|u + 1| + C$$

$$\ln|x| = \frac{y}{x} - 2 \ln \left| \frac{y}{x} + 1 \right| + C$$

2.5.5 – Bernoulli Equation

Theorem: An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where $n \neq 0, 1$ is called a Bernoulli Equation. The substitution

$$u = y^{1-n}$$

will transform the D.E. into a 1st order linear.

2.5.6 – Example

$$\begin{aligned}x \frac{dy}{dx} + y &= x^2 y^2 \\ \frac{dy}{dx} + \frac{y}{x} &= xy^2\end{aligned}$$

is a Bernoulli equation with $n = 2$.

$$\begin{aligned}u &= y^{1-2} \\ &= y^{-1} \\ &= \frac{1}{y} \\ \frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\ &= -1y^{-2} \frac{dy}{dx} \\ &= -\frac{1}{y^2} \frac{dy}{dx} \\ -y^{-2} \frac{dy}{dx} + -y^{-2} \times \frac{y}{x} &= -y^{-2} \times xy^2 \\ -y^{-2} \frac{dy}{dx} + -\frac{1}{x} y^{-1} &= -x \\ \frac{du}{dx} - \frac{1}{x} u &= -x\end{aligned}$$

$$\begin{aligned}
 \text{I.F.} = \mu &= e^{P(x)dx} \\
 &= e^{-\int \frac{1}{x} dx} \\
 &= e^{-\ln|x|} \\
 &= e^{\ln|x^{-1}|} \\
 &= x^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u &= -1 \\
 \frac{d}{dx} \left(\frac{1}{x} u \right) &= -1 \\
 \int \frac{d}{dx} \left(\frac{1}{x} u \right) &= \int -1 dx \\
 \frac{1}{x} u &= \int -1 dx \\
 \frac{1}{x} u &= -x + C \\
 \frac{1}{x} \times 1y &= -x + C \\
 \frac{1}{x(-x + C)} &= y \\
 y &= \frac{1}{Cx - x^2}
 \end{aligned}$$

Theorem: If the D.E. can be expressed as

$$\frac{dy}{dx} = f(Ax + by + C)$$

for particular numbers A , B , C , then let

$$u = Ax + By + C$$

to get a separable D.E.

2.5.7 – Example

$$\frac{dy}{dx} = (-2x + y)^2 - 7, y(0) = 0$$

$$u = -2x + y$$

$$\frac{du}{dx} = \frac{dy}{dx} \times \frac{du}{dy}$$

$$= -2 + \frac{dy}{dx}$$

$$\frac{du}{dx} + 2 = \frac{dy}{dx}$$

$$\frac{du}{dx} + 2 = u^2 - 7$$

$$\frac{du}{dx} = u^2 - 9$$

$$\frac{du}{u^2 - 9} = dx$$

$$\int \frac{du}{u^2 - 9} = \int dx$$

$$\int \frac{du}{(u+3)(u+9)} = x + C$$

$$\int \frac{du}{(u+3)(u+9)} = x + C$$

Chapter 3

Modeling using DE

3.1 Linear DE Modeling

3.1.1 – Standard Problems

- 1) Population Growth (or decline)
- 2) Radioactive Decay
- 3) Newton's Law of Cooling
- 4) Mixture Problems

3.1.2 – Population Model

Assume the rate of population change is proportional to the size of the population

$P(t)$ = population at time t

$$\frac{dP}{dt} = kP$$

$\frac{\frac{dP}{dt}}{P} = k$ is the relative growth rate of the population

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{dP}{P} &= kdt \\
\int \frac{dP}{P} &= \int kdt \\
\ln|P| &= kt + C \\
|P| &= e^{kt+C} \\
|P| &= e^{kt}e^C \\
|P| &= Ae^{kt} \text{ where } A > 0 \\
P &= \pm Ae^{kt} \\
P &= Be^{kt} \text{ where } B \neq 0 \\
P &= De^{kt} \text{ where } D \text{ can be any real number}
\end{aligned}$$

The constant can become any number because 0 would be a valid rate of population change, it means that the population size isn't changing.

3.1.3 – Example

If, initially at 2 p.m., there are 1,000 bacteria on a petri dish and at 4 p.m., there are 2,000 bacteria. Assuming constant relative growth rate, how many bacteria are there at 5 p.m.? $P(t)$ = population t hours after 2 p.m.

$$\begin{aligned}
P(t) &= Ae^{kt} \\
1000 &= Ae^{(0)k} \\
1000 &= Ae^0 \\
1000 &= A(1) \\
A &= 1000 \\
P(2) &= 2000 \\
P(2) &= 1000e^{2k} \\
2000 &= 1000e^{2k} \\
2 &= e^{2k} \\
\ln(2) &= 2k \\
k &= \frac{\ln(2)}{2}
\end{aligned}$$

$$\begin{aligned}
P(t) &= 1000e^{\frac{\ln(2)}{2}t} \\
P(3) &= 1000e^{\frac{\ln(2)}{2}(3)} \\
&= 1000e^{1.5\ln(2)} \\
&= 1000e^{\ln(2^{1.5})} \\
&= 1000(2^{1.5}) \\
&= 2000(\sqrt{2}) \\
P(3) &\approx 2828.427(\sqrt{2}) \\
P(t) &= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\ln(2^{\frac{t}{2}})} \\
&= 1000 \times 2^{\frac{t}{2}}
\end{aligned}$$

3.1.4 – Radioactive Decay

$$m(t) = m_0e^{kt} \text{ where } k < 0$$

The Half-Life is the amount of time it takes for half of the original amount to remain:

$$\frac{1}{2}A_0 = A_0e^{kt} \Rightarrow \frac{1}{2} = e^{kt}$$

3.1.5 – Mixture Problems

Setup

Initially, the container has 200 gallons of brine solution (salt-water) of concentration $\frac{10 \text{ lbs}}{200 \text{ gallons}} = 0.05 \frac{\text{lbs}}{\text{gallon}}$. A solution of $\frac{5 \text{ lbs}}{200 \text{ gallons}} 0.025 \frac{\text{lbs}}{\text{gallon}}$ is poured into the initial container at a rate of $\frac{4 \text{ gallons}}{\text{min}}$. How many pounds of salt are there in the container after 2 hours.

Let $A(t) = \# \text{ lbs of salt } t \text{ minutes after the process starts}$

$\frac{dA}{dt}$ = The rate of change of # lbs of salt

$$\begin{aligned}\frac{dA}{dt} &= 0.025 \frac{\text{lbs}}{\text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \left\} \text{rate in} \right. \\ &\quad - \frac{A(t)\text{lbs}}{200\text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \left\} \text{rate out} \right. \\ &= (0.025)4 \frac{\text{lbs}}{\text{min}} - \frac{4A(t)}{200} \frac{\text{lbs}}{\text{min}} \\ &= 0.1 \frac{\text{lbs}}{\text{min}} - \frac{A(t)}{50} \frac{\text{lbs}}{\text{min}} \\ &= 0.1 - \frac{A(t)}{50}\end{aligned}$$

$$\frac{dA}{dt} + \frac{1}{50}A = 0.1$$

$$\mu = e^{\int P(t)dt}$$

$$= e^{\int \frac{1}{50}dt}$$

$$= e^{\frac{t}{50}}$$

$$e^{\frac{t}{50}} \left(\frac{dA}{dt} \right) + e^{\frac{t}{50}} \left(\frac{1}{50}A \right) = e^{\frac{t}{50}}(0.1)$$

$$\frac{d}{dt} \left(e^{\frac{t}{50}}A \right) = e^{\frac{t}{50}}(0.1)$$

$$\int \frac{d}{dt} \left(e^{\frac{t}{50}}A \right) = \int \frac{1}{10} e^{\frac{t}{50}}$$

$$e^{\frac{t}{50}}A = \frac{1}{10} \times \frac{e^{\frac{t}{50}}}{\frac{1}{50}} + C$$

$$e^{\frac{t}{50}}A = 5e^{\frac{t}{50}} + C$$

$$\begin{aligned}A(t) &= 5 + Ce^{-\frac{t}{50}} \\ &= 5 + Ce^{-0.02t}\end{aligned}$$

$$\begin{aligned}A(120) &= 5 + Ce^{-0.02(120)} \\ &= 5 + Ce^{-2.4}\end{aligned}$$

Chapter 4

Higher Order Differential Equations

4.1 Linear Equations

An n th order DE is linear if it has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0 y = g(x)$$

Theorem: If all the coefficient functions are continuous and $a_n(x)$ is not 0 on an interval I and $g(x)$ is continuous, then any initial value problem

$$DE + y(x_0) = y_0$$

has a unique solution on the interval I if $g(x) = 0$. i.e.

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

then the DE is said to be homogeneous.

4.1.1 – Example

$$y'' - 3y' - 4y = 0$$

Show $y_1 = e^{4x}$ is a solution and $y_2 = e^{-x}$ is a solution.

$$y_1 = e^{4x}$$

$$y_1' = 4e^{4x}$$

$$y_1'' = 16e^{4x}$$

$$16e^{4x} - 3(4e^{4x}) - 4e^{4x} = 0$$

$$16e^{4x} - 12e^{4x} - 4e^{4x} = 0$$

$$e^{4x}(16 - 12 - 4) = 0$$

$$e^{4x}(0) = 0$$

$$0 = 0$$

$$y_3 = 6y_1 = 6e^{4x}$$

$$y'_3 = 6y'_1 = 24e^{4x}$$

$$y''_3 = 6y''_1 = 96e^{4x}$$

$$96e^{4x} - 3(24e^{4x}) - 4(6e^{4x}) = 0$$

$$96e^{4x} - 72e^{4x} - 24e^{4x} = 0$$

$$e^{4x}(96 - 72 - 24) = 0$$

$$e^{4x}(0) = 0$$

$$0 = 0$$

Theorem: Superposition Principle: if y_1, y_2, \dots, y_m are each solutions of an n th order Linear, homogenous DE, then $c_1y_1 + c_2y_2 + \dots + c_my_m$ will also be a solution for any constants c_1, c_2, \dots, c_m .

Our goal is to express the general solution in as concise a way as possible.

Linear combination – a collection of solutions y_1, y_2, \dots, y_m is linearly independent if the only way $c_1y_1 + c_2y_2 + \dots + c_my_m = 0$ is iff (if and only if) all of the constants $c_1, c_2, \dots, c_m = 0$. Otherwise we say y_1, y_2, \dots, y_m are linearly dependent.

Theorem: If the DE is an n th order Linear Homogeneous equation then there will exist a collection of n linearly independent solutions y_1, y_2, \dots, y_n and the general solution will be $y_c = c_1y_1 + c_2y_2 + \dots + c_ny_n$

One way to check for linear independence is to compile the Wronskian

$$W(y_1, y_2, \dots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

4.2 Reduction of Order

If you have one solution to a 2nd order linear homogenous DE, then it is possible to use that function to construct a 2nd Linear Independent solution to the DE.

4.2.1 – Example

For example, the DE

$$y'' - y = 0$$

One solution is $y = e^x$ on $(-\infty, \infty)$.

Idea: We look for y_2 of the form

$$y_2(x) = u(x)y_1(x) \text{ where } u(x) \text{ is not a constant}$$

The general solution is of the form:

$$y = c_1y_1 + c_2y_2$$

where y_1 and y_2 are linearly independent solutions.

To find $u(x)$, we substitute this into the DE

$$\begin{aligned} y_2 &= u(x)y_1(x) \\ y_2' &= u(x)y_1'(x) + u'(x)y_1(x) \\ y_2'' &= u(x)y_1''(x) + u'(x)y_1'(x) + u''(x)y_1(x) + u'(x)y_1'(x) \\ &= uy_1'' + 2u'y_1' + u''y_1 \end{aligned}$$

So $y'' - y = 0$ becomes

$$\begin{aligned} uy_1'' + 2u'y_1' + u''y_1 - uy_1 &= 0 \text{ when we sub } y = y_2 = uy_1 \\ u(e^x)'' + 2u'(e^x)' + u''(e^x) - u(e^x) &= 0 \\ ue^x + 2u'e^x + u''e^x - ue^x &= 0 \\ 2u'e^x + u''e^x &= 0 \\ e^x(2u' + u'') &= 0 \\ 2u' + u'' &= 0 \end{aligned}$$

Let $w = u'$

$$\begin{aligned} 2w + w' &= 0 \\ 2w + \frac{dw}{dx} &= 0 \\ \frac{dw}{dx} &= -2w \\ \frac{dw}{w} &= -2dx \\ \int \frac{dw}{w} &= \int -2dx \\ \ln |w| &= -2x \\ w &= e^{-2x} \\ u' &= e^{-2x} \\ \int u' &= \int e^{-2x} \\ u &= -\frac{1}{2}e^{-2x} \\ y_2 &= uy_1 \\ &= -\frac{1}{2}e^{-2x} \times e^x \\ &= -\frac{1}{2}e^{-x} \end{aligned}$$

Double check that y_2 is a solution of the DE

$$\begin{aligned}
 y_2 &= -\frac{1}{2}e^{-x} \\
 y_2' &= \frac{1}{2}e^{-x} \\
 y_2'' &= -\frac{1}{2}e^{-x} \\
 y_2'' - y &= -\frac{1}{2}e^{-x} - \left(-\frac{1}{2}e^{-x}\right) \\
 &= -\frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} \\
 &= 0
 \end{aligned}$$

In general,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

put into standard form by dividing by $a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$, the same method as in our **example** leads to the formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \quad (4.1)$$

4.2.2 – Example

Part 1

$$x^2y'' - 3xy' + 4y = 0$$

Verify that $y_1 = x^2$ is a solution $y_1' = 2x, y_1'' = 2$.

$$\begin{aligned}
 x^2y'' - 3xy' + 4y &= 0 \\
 x^2(2) - 3x(2x) + 4(x^2) &= 0 \\
 2x^2 - 6x^2 + 4x^2 &= 0 \\
 6x^2 - 6x^2 &= 0 \\
 0 &= 0
 \end{aligned}$$

Part 2

Find a linearly independent solution $y_2(x)$.

$$x^2 y'' - 3xy' + 4y = 0$$

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$P(x) = -\frac{3}{x}$$

$$y_2 = y_1 \int \frac{e^{\int \frac{3}{x} dx}}{(y_1(x))^2} dx$$

$$y_2 = y_1 \int \frac{e^{3 \ln |x|}}{(y_1(x))^2} dx$$

$$y_2 = y_1 \int \frac{e^{\ln |x^3|}}{(y_1(x))^2} dx$$

$$y_2 = x^2 \int \frac{x^3}{(x^2)^2} dx$$

$$y_2 = x^2 \int \frac{x^3}{x^4} dx$$

$$y_2 = x^2 \int \frac{1}{x} dx$$

$$y_2 = x^2 \ln |x| + C$$

Part 3: Double check that y_2 is a solution of the DE

$$y_2 = x^2 \ln |x|$$

$$y_2' = x^2 \times \frac{1}{x} + 2x \ln |x|$$

$$y_2'' = 1 + 2x \frac{1}{x} + 2 \ln |x|$$

$$= 1 + 2 + 2 \ln |x|$$

$$= 3 + 2 \ln |x|$$

So the LHS DE becomes

$$\begin{aligned} x^2 (3 + 2 \ln |x|) - 3x (x + 2x \ln |x|) + 4x^2 \ln |x| &= 3x^2 + 2x^2 \ln |x| - 3x^2 - 6x^2 \ln |x| + 4x^2 \ln |x| \\ &= 3x^2 - 3x^2 + 2x^2 \ln |x| - 6x^2 \ln |x| + 4x^2 \ln |x| \\ &= 3x^2 - 3x^2 + 2x^2 \ln |x| - 6x^2 \ln |x| + 4x^2 \ln |x| \\ &= 0 + x^2 \ln |x| (2 - 6 + 4) \\ &= x^2 \ln |x| (0) \\ &= 0 \end{aligned}$$

Write the general solution of the DE including the interval of the solution

$$\begin{aligned}
 y &= c_1 y_1 + c_2 y_2 \\
 &= c_1 x^2 + c_2 x^2 (\ln |x| + C) \\
 &= c_1 x^2 + c_2 x^2 \ln |x| + C c_2 x^2 \\
 \text{just } y &= c_1 x^2 + c_2 x^2 \ln |x| \text{ on } I = (0, \infty), y(2) = 3, y'(2) = 5
 \end{aligned}$$

4.2.3 – Example

$$\begin{aligned}
 3y'' + y' - 4y &= 0 \\
 y &= e^{mx} \\
 y' &= m e^{mx} \\
 y'' &= m^2 e^{mx} \\
 3y'' + y' - 4y &= 3m^2 e^{mx} + m e^{mx} - 4e^{mx} \\
 &= e^{mx} (3m^2 + m - 4) \\
 &= e^{mx} (3m^2 + 4)(m - 1) \\
 m = 1 \quad m &= -\frac{4}{3} \\
 y_1 &= e^x, y_2 = e^{-\frac{4}{3}x}
 \end{aligned}$$