Chapter 8

Systems of Linear First-Order Differential Equations

8.1 Preliminary Theory – Linear Systems

In this chapter, we assume the system can be put in the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

Further assume $g_1, g_2, \ldots, g_n$ are linear with respect to x_1, x_2, $\ldots$, x_n.

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

In matrix notation:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

where

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} = [a_{ij}(t)] \quad \begin{matrix} i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \end{matrix}$$

In general, if all the $a_{ij}(t)$'s and $f_i(t)$'s are continuous on an interval I, then the IVP $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$ has a unique solution:

$$\begin{aligned} x_1(t_0) &= w_1 \\ x_2(t_0) &= w_2 \\ &\cdots \\ x_n(t_0) &= w_n \end{aligned}$$

where $w_1, w_2, \dots, w_n$ are just numbers.

If the initial conditions aren't given, then we want the general solution

$$\mathbf{X} = c_1\mathbf{X_1} + c_2\mathbf{X_2} + \cdots + c_n\mathbf{X_n}$$

where each X_i is a solution of $X' = Ax$ and $\{X_1(t), X_2(t), \dots, X_n(t)\}$[1] is a linearly independent collection of solutions. This will be true iff the Wronskian$(X_1, X_2, \dots, X_n)$ is non-zero[2]. The set $\{X_1, X_2, \dots, X_n\}$ is called a fundamental set.

[1] For a system of n equations

[2] Wronskian$(X_1, X_2, \dots, X_n) = \det(X_1, X_2, \dots, X_n)$