Chapter 2

First-Order Differential Equations

First Order Linear Differential Equations

$$
\begin{aligned}
a_1(x)\frac{dy}{dx} + a_0(x)y &= g(x) \\
\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y &= \frac{g(x)}{a_1(x)}
\end{aligned}
$$

$$\left.\frac{dy}{dx} + P(x)y = f(x)\right\} \text{ Standard form of a 1st-order linear DE}$$

We will try to find a function $\mu(x)$ such that by multiplying the D.E. by an integrating factor (I.F.) $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

such that the LHS is an exact derivative, Observe:

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)\frac{dy}{dx} + \frac{dy}{dx}y$$

from which we see

$$
\begin{aligned}
\mu(x)P(x) &= \frac{d\mu}{dx} \\
P(x)dx &= \frac{d\mu}{\mu(x)} \\
\int P(x)dx &= \int \frac{d\mu}{\mu} \\
\int P(x)dx &= \ln\mu \\
\ln\mu &= \int P(x)dx \\
\mu &= e^{\int P(x)dx}
\end{aligned}
$$

2.3.1 – Example

$$
\begin{aligned}
x\frac{dy}{dx} - 4y &= x^6 e^x \\
\text{Standard form: } \frac{dy}{dx} - \frac{4}{x}y &= x^5 e^x \\
P(x) &= -\frac{4}{x} \\
\mu &= e^{\int \frac{-4}{x} dx} \\
&= e^{-4\ln x} \\
&= e^{\ln x^{-4}} \\
&= x^{-4} \\
\textcolor{red}{\text{I.F.}} = \mu &= x^{-4}
\end{aligned}
$$

Now multiply the standard form of the given D.E. by x^{-4}.

$$
\begin{aligned}
x^{-4}\frac{dy}{dx} - x^{-4}\frac{4}{x}y &= x^{-4}x^5 e^x \\
x^{-4}\frac{dy}{dx} - x^{-4}\frac{4}{x}y &= xe^x \\
\int \frac{d}{dx}\left(x^{-4}y\right) &= \int xe^x \\
x^{-4}y &= \int xe^x
\end{aligned}
$$

2.3.2 – Example

$$
\left(x^2 - 9\right)\frac{dy}{dx} + xy = 0
$$

$$
\begin{aligned}
\left(x^2 - 9\right)\frac{dy}{dx} + xy &= 0 \\
\frac{dy}{dx} + \frac{x}{x^2 - 9}y &= 0 \\
P(x) &= \frac{x}{x^2 - 9} \\
\int P(x)dx &= \int \frac{x}{x^2 - 9}dx \\
\int P(x)dx &= \int \frac{1}{u - 9}\frac{du}{2} \\
\int P(x)dx &= \frac{1}{2}\int \frac{1}{u - 9}du \\
\int P(x)dx &= \frac{1}{2}\ln|u - 9| \\
\int P(x)dx &= \frac{1}{2}\ln|x^2 - 9|
\end{aligned}
$$

$$\mu = e^{\frac{1}{2}\ln|x^2-9|}$$

$$\mu = e^{\ln\left|\left(x^2-9\right)^{\frac{1}{2}}\right|}$$

$$\mu = \left(x^2-9\right)^{\frac{1}{2}}$$

$$\mu = \sqrt{x^2-9}$$

$$\sqrt{x^2-9}\left(\frac{dy}{dx}+\frac{x}{x^2-9}y\right) = \sqrt{x^2-9}(0)$$

$$\sqrt{x^2-9}\frac{dy}{dx}+\frac{x}{\sqrt{x^2-9}}y = 0$$

$$\int \frac{d}{dx}\left(y\sqrt{x^2-9}\right) = \int 0$$

$$y\sqrt{x^2-9} = C$$

$$y = \frac{C}{\sqrt{x^2-9}}$$