MATH 252 - Introduction to Differential Equations Notes

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Contents

Chapter 1

Introduction to Differential Equations

1.1 Terminology and Notation

Differential equation (D.E.) – An equation in which at least one derivative of an unknown function.
Order of the D.E. – The highest order of derivative in the D.E.

1.1.1 – Example

$$4y'' + e^x y' - 3yy' = \sin(x)$$

An example of a partial differential equation is:

$$\frac{\partial T}{\partial x} + x^2 \frac{\partial T}{\partial y} = x + y$$

however, we won't study these in this course.

1.1.2 – Linear vs Non-Linear DE's

Linear D.E. – The dependent variable and all of its derivatives in the D.E. are in separate terms to the 1^{st} power. $y^{(n)}$ or $\frac{d^n y}{dx^n}$ where $n \neq 1$ are non-first power.

$$4y'' + e^x y' - 3yy' = \sin(x)$$

is a non-linear D.E. while

$$4y'' + e^x y' - 3y = \sin(x)$$

is linear.
The general formula of a linear D.E. would look like

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = g(x)$$

Solution – a function $\phi(x)$ and an interval I for which the D.E. is satisfied when $y = \phi(x)$ for all x in I.
It may be the case that the natural domain of $\phi(x)$ is larger than I.

1.1.3 – Example

$y' = -\frac{1}{x^2}$ has a solution $\phi(x) = \frac{1}{x}$ on $I = (0, \infty)$ but the domain of $\phi(x) = (-\infty, 0) \cup (0, \infty)$.

Practice:

$$\frac{d^2x}{dt^2} + 16x = 0$$

Show (*Verify* not derive) $x(t) = c_1 \sin(4t)$ is a solution on $(-\infty, \infty)$ where c is any real parameter.

$$\begin{aligned}
x &= c_1 \sin(4t) \\
\frac{dx}{dt} &= 4c_1 \cos(4t) \\
\frac{d^2x}{dt^2} &= -16c_1 \sin(4t) \\
\text{LHS} = \frac{d^2x}{dt^2} &+ 16x \\
&= -16c_1 \sin(4t) + 16(c_1 \sin(4t)) \\
&= 0 = \text{RHS}
\end{aligned}$$

But the equation $x = c_2 \cos(4t)$ would also be a solution. If you have 2 equations that are both solutions, you could add them together and you would still have a solution. $x = c_1 \sin(4t) + c_2 \cos(4t)$ is a solution for all parameters c_1 and c_2. *In fact, this is the general solution to the D.E.*

The D.E.

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

Show $y = \left(\frac{1}{4}x^2 + C\right)^2$ is a one parameter family of solutions

$$\begin{aligned}
\text{LHS} = \frac{dy}{dx} &= 2\left(\frac{1}{4}x^2 + C\right) \times \frac{1}{2}x \\
&= x\left(\frac{1}{4}x^2 + C\right) \\
\text{RHS} = xy^{\frac{1}{2}} &= x\left(\left(\frac{1}{4}x^2 + C\right)^2\right)^{\frac{1}{2}} \\
&= x\left(\frac{1}{4}x^2 + C\right) \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

But there is another solution: namely $y(x) = 0$ for all x. This is called the "trivial solution".

1.2 Initial Value Problems (IVP)

1st order IVP is a 1st order D.E. together with one extra condition:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

2nd order IVP

$$y'' = f(x, y, y')$$

Initial conditions:

- $y(x_0) = y_0$
- $y'(x_0) = y_1$

1.2.1 – Example

$$y' = y \text{ and } y(0) = 3$$

$y = ce^x$ is a one-parameter family of solutions

$$\frac{d}{dx}(ce^x) = ce^x = y$$

$$\begin{aligned} ce^1 &= -2 \\ c &= -\frac{2}{e} \\ y &= \left(-\frac{2}{e}\right) e^x \\ y &= -2e^{x-1} \end{aligned}$$

1.2.2 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Given that you have the solution: $y = \frac{1}{x^2+C}$, Solve:

$$\begin{aligned} -1 &= \frac{1}{(0)^2 + c} \\ -1 &= \frac{1}{c} \\ -1 \times c &= 1 \\ c &= -1 \\ y &= \frac{1}{x^2 - 1}, I = (-1, 1) \end{aligned}$$

1.2.3 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Example

$$x'' + 16x = 0 \text{ and } x(\frac{\pi}{2}) = 5 \text{ and } x'(\frac{\pi}{2}) = -4$$

$$\begin{aligned}
x &= c_1\cos(4t) + c_2\sin(4t) \\
5 &= c_1\cos(4t) + c_2\sin(4t) \\
&= c_1\cos(2\pi) + c_2\sin(2\pi) \\
&= c_1(1) + c_2(0) \\
&= c_1 \\
x' &= -4c_1\sin(4t) + 4c_2\cos(4t) \\
-4 &= -4c_1\sin\left(4\left(\frac{\pi}{2}\right)\right) + 4c_2\cos\left(4\left(\frac{\pi}{2}\right)\right) \\
&= -4c_1\sin(2\pi) + 4c_2\cos(2\pi) \\
&= -4c_1(0) + 4c_2(1) \\
&= 4c_2 \\
-1 &= c_2
\end{aligned}$$

Reasonable Question: Given a 1st order IVP, can we say whether a solution *exists* or not and, if a solution exists, is it *unique*.
Theorem: Given $y' = f(x, y)$ and $y(x_0) = y_0$, if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous on a rectangle R containing (x_0, y_0) in its interior, then there exists an interval $I = (x_0 - h, x_0 + h)$ where $h > 0$ such that there exists a unique solution to IVP on I.

1.2.4 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(1) = 2$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous everywhere its defined $y \geq 0$
- $\frac{\partial f}{\partial y} = x\frac{1}{2}y^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$ is continuous everywhere its defined $y > 0$

1.2.5 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(0) = 0$$

- $f(x, y) = xy^{\frac{1}{2}}$ is continuous for all x and $y \geq 0$
- $\frac{\partial f}{\partial} = \frac{x}{2y}$ is continuous for all x and $y > 0$.
- ***Theorem does not give any conclusion.***

Chapter 2

First-Order Differential Equations

2.1 Solution Curves Without a Solution

Given a 1st order D.E. $y' = f(x, y)$, y' is the slope of the tangent line at any point (x_0, y_0) on a solution curve

2.1.1 – Example

$$y' = f(x, y) = x + y$$

- $f(0, 0)$=0
- $f(1, 0)$=1

2.1.2 – Slope/Direction Fields

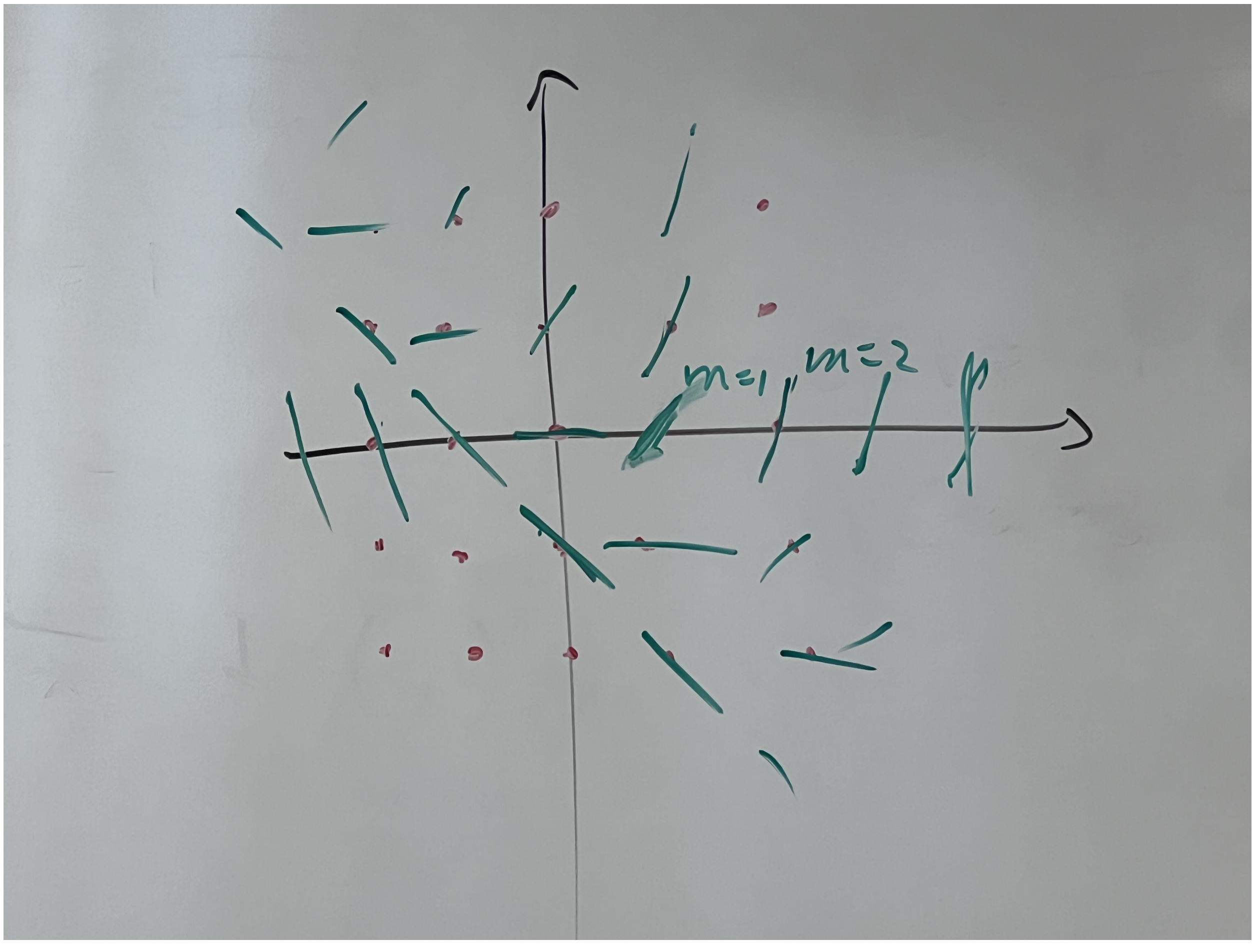

Figure 2.1: The direction field for the previous example

If the function $f(x, y)$ in the D.E. $y' = f(x, y)$ is reasonably simple so that we can solve $f(x, y) = 0$, we can make a "phase portrait diagram". We will also assume $f(x, y)$ only involves the y-variable.

2.1.3 – Example

$$y' = (y + 2)(y - 3)(y - 5)$$
$$f(x, y) = (y + 2)(y - 3)(y - 5)$$

An "equilibrium solution" is a solution where y is a constant. In this example: $y = 3$, $y = 5$, $y = -2$ are each constant functions.

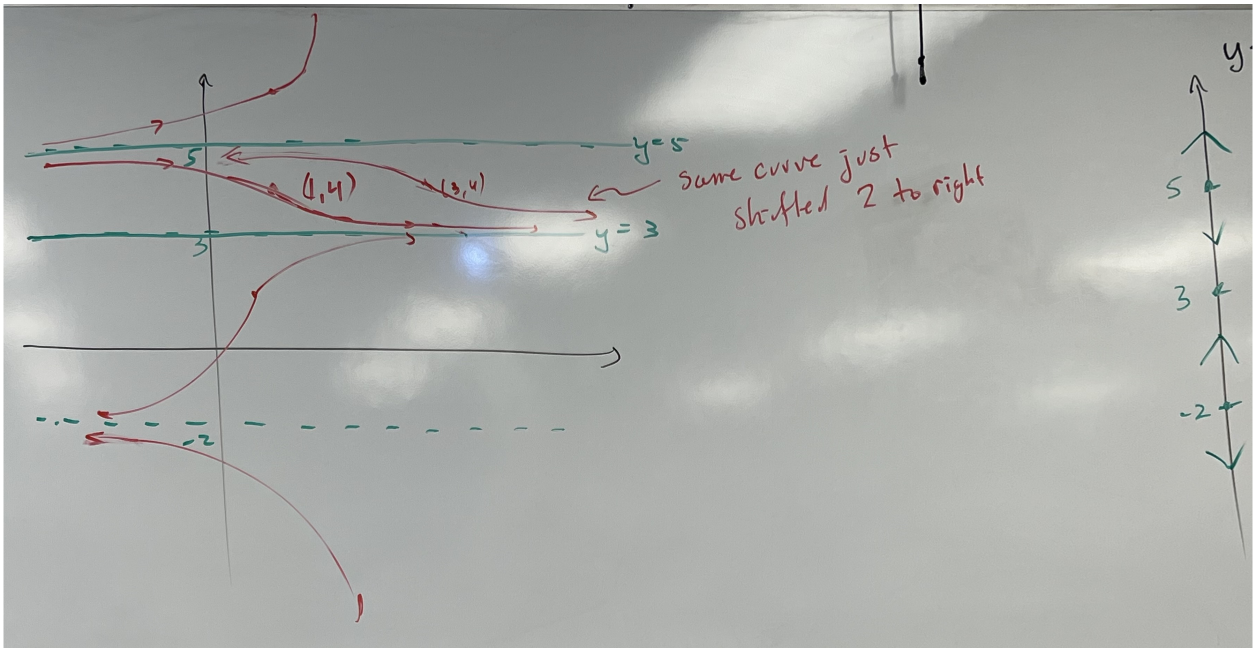

Figure 2.2: The equilibrium solution for the previous example.

The area around $y = 5$ is an unstable equilibrium since the solutions diverge and go in separate directions away from $y = 5$. The area around $y = 3$ is a stable equilibrium because the slopes above and below it converge to $y = 3$. The area around $y = -2$ is semi-stable, since all the slopes around it will converge in one direction, but the point isn't always $y = -2$.

2.2 Separable Differential Equations

Separable D.E.s are DE's $\frac{dy}{dx} = f(x, y)$ where $f(x, y)$ can be factored as $f(x, y) = g(x)h(y)$.

$$\frac{dy}{dx} = (1 + y^2)x^3 \text{ is separable}$$

$$\frac{dy}{dx} = \sin(xy) \text{ is } \textit{not} \text{ separable}$$

$$\frac{dy}{dx} = x^3 y \text{ is } \textit{not} \text{ separable}$$

$$\begin{aligned}\frac{5}{xy}\frac{dy}{dx} &= \left(x^2 + y\right)e^y \\ \frac{dy}{dx} &= \frac{xy\left(x^2+y\right)e^y}{5} \\ &= \frac{x\left(x^2+y\right)}{5} \times ye^y\end{aligned}$$

2.2.1 – Method of Solution

"Separate the variable" to get $\frac{1}{h(y)}dy = g(x)d$ or $p(y)dy = g(x)dx$ where $p(y) = \frac{1}{h(y)}$.**Integrate both sides**

$$\int p(y)dy = \int g(x)dx \text{ and if possible, solve for } y$$

2.2.2 – Example

$$\begin{aligned}
\frac{dy}{dx} &= \left(1+y^2\right)x^3 \\
\int \frac{1}{1+y^2}dy &= \int x^3 dx \\
\tan^{-1}(y) + C_1 &= \frac{x^4}{4} + C_2 \\
\tan^{-1}(y) &= \frac{x^4}{4} + C_2 - C_1 \\
\tan^{-1}(y) &= \frac{x^4}{4} + C \\
y &= \tan\left(\frac{x^4}{4} + C\right)
\end{aligned}$$

2.2.3 – Example

Problem 12 from the textbook.

$$\sin(3x)dx + 2y\cos^3(3x)dy = 0$$

$$\begin{aligned}
\int -2y dy &= \int \frac{\sin(3x)}{\cos^3(x)}dx \\
&= \int \tan(3x)\sec^2(3x)dx \\
&= \int u\ \frac{1}{3}du \text{ where } u = \tan(3x),\ du = 3\sec^2(3x)dx \\
-2\int y dy &= \frac{1}{3}\int u\ du + C \\
-y^2 &= \frac{u^2}{6} + C \\
&= \frac{\tan^2(3x)}{6} + C
\end{aligned}$$

$$\frac{\tan^2(3x)}{6} + y^2 = -C$$

$$\frac{\tan^2(3x)}{6} + y^2 = C$$

Problem 25 from the textbook.

$$x^2\frac{dy}{dx} = y - xy, y(-1) = -1$$

$$\begin{aligned}
x^2\frac{dy}{dx} &= y - xy \\
x^2\frac{dy}{dx} &= y(1-x) \\
\frac{dy}{y} &= \frac{(1-x)}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{(1-x)}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{1}{x^2}dx - \int \frac{x}{x^2}dx \\
\int \frac{dy}{y} &= \int \frac{1}{x^2}dx - \int \frac{1}{x}dx \\
\ln|y| + C_1 &= -\frac{1}{x} + C_2 - \ln|x| + C_3 \\
\ln|y| &= -\frac{1}{x} - \ln|x| + C \\
y &= e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C \\
y &= e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C \\
y &= e^{-\frac{1}{x}} \times \frac{1}{|x|} \times e^C \\
y &= \frac{1}{|x|}e^{C-\frac{1}{x}} \\
-1 &= \frac{1}{|-1|}e^{C-\frac{1}{-1}} \\
-1 &= \frac{1}{1}e^{C-(-1)} \\
-1 &= e^{C+1}
\end{aligned}$$

2.3 First Order Linear Differential Equations

$$\begin{aligned}
a_1(x)\frac{dy}{dx} + a_0(x)y &= g(x) \\
\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y &= \frac{g(x)}{a_1(x)}
\end{aligned}$$

$$\left.\frac{dy}{dx} + P(x)y = f(x)\right\} \text{ Standard form of a 1st-order linear DE}$$

We will try to find a function $\mu(x)$ such that by multiplying the D.E. by an integrating factor (I.F.) $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

such that the LHS is an exact derivative, Observe:

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)\frac{dy}{dx} + \frac{dy}{dx}y$$

from which we see

$$\begin{aligned}
\mu(x)P(x) &= \frac{d\mu}{dx} \\
P(x)dx &= \frac{d\mu}{\mu(x)} \\
\int P(x)dx &= \int \frac{d\mu}{\mu} \\
\int P(x)dx &= \ln\mu \\
\ln\mu &= \int P(x)dx \\
\mu &= e^{\int P(x)dx}
\end{aligned}$$

2.3.1 – Example

$$\begin{aligned}
x\frac{dy}{dx} - 4y &= x^6e^x \\
\text{Standard form: } \frac{dy}{dx} - \frac{4}{x}y &= x^5e^x \\
P(x) &= -\frac{4}{x} \\
\mu &= e^{\int \frac{-4}{x}dx} \\
&= e^{-4\ln x} \\
&= e^{\ln x^{-4}} \\
&= x^{-4} \\
\text{I.F.} = \mu &= x^{-4}
\end{aligned}$$

Now multiply the standard form of the given D.E. by x^{-4}.

$$\begin{aligned}
x^{-4}\frac{dy}{dx} - x^{-4}\frac{4}{x}y &= x^{-4}x^5e^x \\
x^{-4}\frac{dy}{dx} - x^{-4}\frac{4}{x}y &= xe^x \\
\int \frac{d}{dx}\left(x^{-4}y\right) &= \int xe^x \\
x^{-4}y &= \int xe^x
\end{aligned}$$

2.3.2 – Example

$$\left(x^2-9\right)\frac{dy}{dx}+xy=0$$

$$\begin{aligned}
\left(x^2-9\right)\frac{dy}{dx}+xy&=0\\
\frac{dy}{dx}+\frac{x}{x^2-9}y&=0\\
P(x)&=\frac{x}{x^2-9}\\
\int P(x)dx&=\int\frac{x}{x^2-9}dx\\
\int P(x)dx&=\int\frac{1}{u-9}\frac{du}{2}\\
\int P(x)dx&=\frac{1}{2}\int\frac{1}{u-9}du\\
\int P(x)dx&=\frac{1}{2}\ln|u-9|\\
\int P(x)dx&=\frac{1}{2}\ln|x^2-9|\\
\mu&=e^{\frac{1}{2}\ln|x^2-9|}\\
\mu&=e^{\ln|\left(x^2-9\right)^{\frac{1}{2}}|}\\
\mu&=\left(x^2-9\right)^{\frac{1}{2}}\\
\mu&=\sqrt{x^2-9}\\
\sqrt{x^2-9}\left(\frac{dy}{dx}+\frac{x}{x^2-9}y\right)&=\sqrt{x^2-9}(0)\\
\sqrt{x^2-9}\frac{dy}{dx}+\frac{x}{\sqrt{x^2-9}}y&=0\\
\int\frac{d}{dx}\left(y\sqrt{x^2-9}\right)&=\int 0\\
y\sqrt{x^2-9}&=C\\
y&=\frac{C}{\sqrt{x^2-9}}
\end{aligned}$$

2.4 Exact Equations

1st Order D.E. in differential form

$$M(x,y)dx+N(x,y)dy=0$$

Given a function

$$z=f(x,y)$$

, the total differential, dz, is defined as

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

2.4.1 – Method

See if we can find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

If we can do this, then the D.E. is equivalent to

$$df = 0 \Rightarrow f(x, y) = c$$

is an implicit solution of D.E.
Assume that M and N have continuous 1st order partials (assuming f exists)

$$\left.\begin{array}{rl} My & = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} dy = \quad f_{xy} \\ Nx & = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} dy = \quad f_{yx} \end{array}\right\} \text{Theorem tells us these are equal}$$

This provides a quick test to check if the D.E. is exact or not.

2.4.2 – Example

$$2xydx + \left(x^2 - 1\right) dy = 0$$
$$M(x, y) = 2xy \qquad N(x, y) = x^2 - 1$$

To check if the D.E. is exact

$$M_y = 2x = N_x$$

We now know there exists a function $f(x, y)$ with

$$\begin{aligned} \frac{\partial f}{\partial x} = M = & \quad 2xy \\ \frac{\partial f}{\partial y} = N = & \quad x^2 - 1 \end{aligned}$$

$$\begin{aligned}
f_M(x,y) &= \int \frac{\partial f}{\partial x} dx \\
&= \int 2xy dx \\
&= x^2y + \phi(y) \\
\frac{\partial f}{\partial y}\left(x^2y + \phi(y)\right) &= x^2 - 1 \text{ required to equal } N \\
x^2 + \phi'(y) &= x^2 - 1 \\
\phi'(y) &= -1 \\
\phi(y) &= \int -1 dy \\
&= -y \\
f(x,y) &= x^2y - y \\
d\left(f(x,y)\right) &= 0 \\
f(x,y) &= c \\
x^2y - y &= c \text{ is an implicit solution of the D.E.}
\end{aligned}$$

Note: the f_M format is just there to show which partial equation was integrated. It was made by me and, as far as I know, is not standardly known.

2.4.3 – Example

$$\left(e^{2y} - y\cos(xy)\right)dx + (2xe^{2y} - x\cos(xy) + 2y)dy = 0$$

$$M_y = N_x$$

$$\begin{aligned}
\frac{\partial}{\partial y}\left(e^{2y} - y\cos(xy)\right) &= \frac{\partial}{\partial x}\left(2xe^{2y} - x\cos(xy) + 2y\right) \\
2e^{2y} - [\cos(xy) - y\sin(xy) \times x] &= 2e^{2y} - (\cos(xy) - x\sin(xy) \times y) + 0 \\
2e^{2y} - \cos(xy) + xy\sin(xy) &= 2e^{2y} - \cos(xy) + xy\sin(xy)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= M = e^{2y} - y\cos(xy) \\
\frac{\partial f}{\partial y} &= N = 2xe^{2y} - x\cos(xy) + 2y \\
f_N(x,y) &= \int \frac{\partial f}{\partial y} dy \\
&= \int \left(2xe^{2y} - x\cos(xy) + 2y\right) dy \\
&= \frac{2xe^{2y}}{2} - \frac{x\sin(xy)}{x} + 2 \times \frac{y^2}{2} + \phi(x) \\
&= xe^{2y} - \sin(xy) + y^2 + \phi(x)
\end{aligned}$$

Take the ∂x of this and equate with M:

$$\begin{aligned}
M &= \frac{\partial}{\partial x}\left(xe^{2y} - \sin(xy) + y^2 + \phi(x)\right) \\
e^{2y} - y\cos(xy) &= e^{2y} - y\cos(xy) + 0 + \phi'(x) \\
0 &= \phi'(x) \\
\phi(x) &= c
\end{aligned}$$

So $f(x, y) = c_2$ is the solution

$$xe^{2y} - \sin(xy) + y^2 = c$$

$$dx = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

2.4.4 – What can you do if $M_y \neq N_x$

Sometimes you can multiply the DE by an integrating factor $\mu(x, y)$ to get an exact DE.

If

$$\frac{M_y - N_x}{N}$$

is a function of only x, then

$$\mu = e^{\int \frac{M_y - N_x}{N}dx}$$

will be an I.F.
If

$$\frac{N_x - M_y}{M}$$

is a function of only y, then

$$\mu = e^{\int \frac{N_x - M_y}{M}dy}$$

will be an I.F.

2.4.5 – Example

$$\begin{aligned}
xydx + \left(2x^2 + 3y^2 - 20\right)dy &= 0 \\
M_y &= x \\
N_x &= 4x \\
M_y &\neq N_x \\
\frac{N_x - M_y}{M} &= \frac{4x - x}{xy} \\
&= \frac{3x}{xy} \\
&= \frac{3}{y} \text{ is a function of just } y
\end{aligned}$$

So:

$$
\begin{aligned}
\mu &= e^{\int \frac{3}{y} dy} \\
&= e^{3 \ln y} \\
&= y^3 \\
xy^4 dx + y^3 \left(2x^2 + 3y^2 - 20\right) dy &= 0(y^3) \\
xy^4 dx + \left(2x^2 y^3 + 3y^5 - 20y^3\right) dy &= \\
M_y &= N_x \\
4xy^3 &= 4xy^3 \\
\frac{\partial f}{\partial x}
\end{aligned}
$$

2.5 Substitution Methods

Taking a D.E. that's not:

- Separable
- 1st Order Linear
- Exact

and making a substitution to turn the new D.E. into one of these.

Theorem: Given a D.E.

$$M(x,y)dx + N(x,y)dy = 0$$

A function $f(x,y)$ is said to be homogenous of order α if $f(tx,ty) = t^\alpha f(x,y)$.

2.5.1 – Example

Given:

$$f(x,y) = x^3 + 5xy^2 - y^3$$

Then:

$$
\begin{aligned}
f(tx,ty) &= (tx)^3 + 5(tx)(ty)^2 - (ty)^3 \\
&= t^3x^3 + 5t^3xy^2 - t^3y^3 \\
&= t^3\left(x^3 + 5xy^2 - y^3\right) \\
&= t^3 f(x,y)
\end{aligned}
$$

2.5.2 – Example

$$\begin{aligned}
f(x,y) &= \frac{x+y}{x^2+y^2} \\
f(tx,ty) &= \frac{tx+ty}{(tx)^2+(ty)^2} \\
f(tx,ty) &= \frac{tx+ty}{x^2t^2+y^2t^2} \\
f(tx,ty) &= \frac{t}{t^2} \times \frac{x+y}{x^2+y^2} \\
f(tx,ty) &= \frac{t}{t^2} f(x,y) \\
f(tx,ty) &= \frac{1}{t} f(x,y)
\end{aligned}$$

$f(x,y) = \frac{x+y}{x^2+y^2}$ is homogenous of order $\alpha = -1$

2.5.3 – Substitution Rule

If $M(x,y)$ and $N(x,y)$ are homogenous, each of the same order, then $u = \frac{y}{x}$ i.e., $y = ux$ or $v = \frac{x}{y}$ (i.e. $x = vy$) will produce a separable D.E.

2.5.4 – Example

Solve the separable D.E. and then back-substitute

$$(x^2+y^2)dx + (x^2-xy)dy = 0$$

$$\begin{aligned}
M(x,y) = x^2 + y^2 \qquad & N = x^2 - xy \\
M_y = 2y \qquad & N_x = 2x - y \\
M_y &\neq N_x \\
M(tx,ty) &= (tx)^2 + (ty)^2 \\
&= t^2x^2 + t^2y^2 \\
&= t^2(x^2+y^2) \\
&= t^2M(x,y) \text{ } M \text{ is homogeneous of order 2 and so is } N \\
u &= \frac{y}{x} \\
y &= ux \\
dy &= udx + xdu \\
(x^2 + (ux)^2)dx + (x^2 - x(ux))(udx + xdu) &= 0 \\
(x^2 + u^2x^2)dx + (x^2 - ux^2)(udx + xdu) &= 0 \\
(1+u^2)x^2dx + x^2(1-u)(udx + xdu) &= 0 \\
(1+u^2)x^2dx + x^2\left(udx + xdu - u^2dx - uxdu\right) &= 0 \\
x^2\left(1dx + u^2dx + udx + xdu - u^2dx - uxdu\right) &= 0 \\
x^2\left(1dx + u^2dx - u^2dx + udx + xdu - uxdu\right) &= 0 \\
x^2\left(1dx + udx + xdu - uxdu\right) &= 0 \\
x^2(1+u)dx + x^3(1-u)du &= 0 \\
\int \frac{1}{x}dx &= \int -\frac{1-u}{1+u}du \\
&= \int \frac{u-1}{u+1}du \\
&= \int \frac{u+(1-2)}{u+1}du \\
&= \int \left(\frac{u+1}{u+1} - \frac{2}{u+1}\right)du \\
&= \int \left(1 - \frac{2}{u+1}\right)du \\
\ln|x| &= \int \left(1 - \frac{2}{u+1}\right)du \\
&= u - 2\ln|u+1| + C \\
\ln|x| &= \frac{y}{x} - 2\ln\left|\frac{y}{x}+1\right| + C
\end{aligned}$$

2.5.5 – Bernoulli Equation

Theorem: An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where $n \neq 0, 1$ is called a Bernoulli Equation. The substitution

$$u = y^{1-n}$$

will transform the D.E. into a 1st order linear.

2.5.6 – Example

$$\begin{aligned} x\frac{dy}{dx} + y &= x^2y^2 \\ \frac{dy}{dx} + \frac{y}{x} &= xy^2 \end{aligned}$$

is a Bernoulli equation with $n = 2$.

$$\begin{aligned} u &= y^{1-2} \\ &= y^{-1} \\ &= \frac{1}{y} \\ \frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\ &= -1y^{-2}\frac{dy}{dx} \\ &= -\frac{1}{y^2}\frac{dy}{dx} \\ -y^{-2}\frac{dy}{dx} + -y^{-2} \times \frac{y}{x} &= -y^{-2} \times xy^2 \\ -y^{-2}\frac{dy}{dx} + -\frac{1}{x}y^{-1} &= -x \\ \frac{du}{dx} - \frac{1}{x}u &= -x \end{aligned}$$

$$
\begin{aligned}
\text{I.F.} = \mu &= e^{P(x)dx} \\
&= e^{-\int \frac{1}{x}dx} \\
&= e^{-\ln|x|} \\
&= e^{\ln|x^{-1}|} \\
&= x^{-1} \\
\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u &= -1 \\
\frac{d}{dx}\left(\frac{1}{x}u\right) &= -1 \\
\int \frac{d}{dx}\left(\frac{1}{x}u\right) &= \int -1dx \\
\frac{1}{x}u &= \int -1dx \\
\frac{1}{x}u &= -x + C \\
\frac{1}{x} \times 1y &= -x + C \\
\frac{1}{x(-x+C)} &= y \\
y &= \frac{1}{Cx - x^2}
\end{aligned}
$$

Theorem: If the D.E. can be expressed as

$$\frac{dy}{dx} = f(Ax + by + C)$$

for particular numbers A, B, C, then let

$$u = Ax + By + C$$

to get a separable D.E.

2.5.7 – Example

$$\frac{dy}{dx} = (-2x + y)^2 - 7, y(0) = 0$$

$$
\begin{aligned}
u &= -2x + y \\
\frac{du}{dx} &= \frac{dy}{dx} \times \frac{du}{dy} \\
&= -2 + \frac{dy}{dx} \\
\frac{du}{dx} + 2 &= \frac{dy}{dx} \\
\frac{du}{dx} + 2 &= u^2 - 7 \\
\frac{du}{dx} &= u^2 - 9 \\
\frac{du}{u^2 - 9} &= dx \\
\int \frac{du}{u^2 - 9} &= \int dx \\
\int \frac{du}{(u+3)(u+9)} &= x + C \\
\int \frac{du}{(u+3)(u+9)} &= x + C
\end{aligned}
$$

Chapter 3

Modeling using DEs

3.1 Linear DE Modeling

3.1.1 – Standard Problems

1) Population Growth (or decline)
2) Radioactive Decay
3) Newton's Law of Cooling
4) Mixture Problems

3.1.2 – Population Model

Assume the rate of population change is proportional to the size of the population

$$P(t) = \text{ population at time } t$$

$$\frac{dP}{dt} = kP$$

$$\frac{\frac{dP}{dt}}{P} = k \text{ is the relative growth rate of the population}$$

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{dP}{P} &= kdt \\
\int \frac{dP}{P} &= \int kdt \\
ln|P| &= kt + C \\
|P| &= e^{kt+C} \\
|P| &= e^{kt}e^{C} \\
|P| &= Ae^{kt} \text{ where } A > 0 \\
P &= \pm Ae^{kt} \\
P &= Be^{kt} \text{ where } B \neq 0 \\
P &= De^{kt} \text{ where } D \text{ can be any real number}
\end{aligned}$$

The constant can become any number because 0 would be a valid rate of population change, it means that the population size isn't changing.

3.1.3 – Example

If, initially at 2 p.m., there are 1,000 bacteria on a petri dish and at 4 p.m., there are 2,000 bacteria. Assuming constant relative growth rate, how many bacteria are there at 5 p.m.? $P(t)$ = population t hours after 2 p.m.

$$\begin{aligned}
P(t) &= Ae^{kt} \\
1000 &= Ae^{(0)k} \\
1000 &= Ae^{0} \\
1000 &= A(1) \\
A &= 1000
\end{aligned}$$

$$\begin{aligned}
P(2) &= 2000 \\
P(2) &= 1000e^{2k} \\
2000 &= 1000e^{2k} \\
2 &= e^{2k} \\
\ln(2) &= 2k \\
k &= \frac{\ln(2)}{2}
\end{aligned}$$

$$
\begin{aligned}
P(t) &= 1000e^{\frac{\ln(2)}{2}t} \\
P(3) &= 1000e^{\frac{\ln(2)}{2}(3)} \\
&= 1000e^{1.5\ln(2)} \\
&= 1000e^{\ln(2^{1.5})} \\
&= 1000(2^{1.5}) \\
&= 2000(\sqrt{2}) \\
P(3) &\approx 2828.427(\sqrt{2}) \\
P(t) &= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\ln(2^{\frac{t}{2}})} \\
&= 1000 \times 2^{\frac{t}{2}}
\end{aligned}
$$

3.1.4 – Radioactive Decay

$$m(t) = m_0e^{kt} \text{ where } k < 0$$

The Half-Life is the amount of time it takes for half of the original amount to remain:

$$\frac{1}{2}A_0 = A_0e^{kt} \Rightarrow \frac{1}{2} = e^{kt}$$

3.1.5 – Mixture Problems

Setup

Initially, the container has 200 gallons of brine solution (salt-water) of concentration $\frac{10 \text{ lbs}}{200 \text{ gallons}} = 0.05\frac{\text{lbs}}{\text{gallon}}$. A solution of $\frac{5 \text{ lbs}}{200 \text{ gallons}}0.025\frac{\text{lbs}}{\text{gallon}}$ is poured into the initial container at a rate of $\frac{4 \text{ gallons}}{\text{min}}$. How many pounds of salt are there in the container after 2 hours.

Let $A(t) = \#$ lbs of salt t minutes after the precess starts

$\frac{dA}{dt}$ = The rate of change of # lbs of salt

$$
\begin{aligned}
\frac{dA}{dt} &= \left. 0.025\frac{\text{lbs}}{\text{gal}} \times 4\frac{\text{gal}}{\text{min}} \right\} \text{rate in} \\
&\quad \left. - \frac{A(t)\text{lbs}}{200\text{gal}} \times 4\frac{\text{gal}}{\text{min}} \right\} \text{rate out} \\
&= (0.025)4\frac{\text{lbs}}{\text{min}} - \frac{4A(t)}{200}\frac{\text{lbs}}{\text{min}} \\
&= 0.1\frac{\text{lbs}}{\text{min}} - \frac{A(t)}{50}\frac{\text{lbs}}{\text{min}} \\
&= 0.1 - \frac{A(t)}{50} \\
\frac{dA}{dt} + \frac{1}{50}A &= 0.1 \\
\mu &= e^{\int P(t)dt} \\
&= e^{\int \frac{1}{50}dt} \\
&= e^{\frac{t}{50}} \\
e^{\frac{t}{50}}\left(\frac{dA}{dt}\right) + e^{\frac{t}{50}}\left(\frac{1}{50}A\right) &= e^{\frac{t}{50}}(0.1) \\
\frac{d}{dt}\left(e^{\frac{t}{50}}A\right) &= e^{\frac{t}{50}}(0.1) \\
\int \frac{d}{dt}\left(e^{\frac{t}{50}}A\right) &= \int \frac{1}{10}e^{\frac{t}{50}} \\
e^{\frac{t}{50}}A &= \frac{1}{10} \times \frac{e^{\frac{t}{50}}}{\frac{1}{50}} + C \\
e^{\frac{t}{50}}A &= 5e^{\frac{t}{50}} + C \\
A(t) &= 5 + Ce^{-\frac{t}{50}} \\
&= 5 + Ce^{-0.02t} \\
A(120) &= 5 + Ce^{-0.02(120)} \\
&= 5 + Ce^{-2.4}
\end{aligned}
$$

Chapter 4

Higher Order Differential Equations

4.1 Linear Equations

An nth order DE is linear if it has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0 y = g(x)$$

Theorem: If all the coefficient functions are continuous and $a_n(x)$ is not 0 on an interval I and $g(x)$ is continuous, then any initial value problem

$$DE + y(x_0) = y_0$$

has a unique solution on the interval I if $g(x) = 0$. i.e.

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

then the DE is said to be homogeneous.

4.1.1 – Example

$$y'' - 3y' - 4y = 0$$

Show $y_1 = e^{4x}$ is a solution and $y_2 = e^{-x}$ is a solution.

$$y_1 = e^{4x}$$

$$y_1' = 4e^{4x}$$

$$y_1'' = 16e^{4x}$$

$$\begin{aligned} 16e^{4x} - 3(4e^{4x}) - 4e^{4x} &= 0 \\ 16e^{4x} - 12e^{4x} - 4e^{4x} &= 0 \\ e^{4x}(16 - 12 - 4) &= 0 \\ e^{4x}(0) &= 0 \\ 0 &= 0 \end{aligned}$$

$$y_3 = 6y_1 = 6e^{4x}$$

$$y_3' = 6y_1' = 24e^{4x}$$

$$y_3'' = 6y_1'' = 96e^{4x}$$

$$\begin{aligned} 96e^{4x} - 3(24e^{4x}) - 4(6e^{4x}) &= 0 \\ 96e^{4x} - 72e^{4x} - 24e^{4x} &= 0 \\ e^{4x}(96 - 72 - 24) &= 0 \\ e^{4x}(0) &= 0 \\ 0 &= 0 \end{aligned}$$

Theorem: Superposition Principle: if $y_1, y_2, \ldots, y_m$ are each solutions of an nth order Linear, homongenous DE, then $c_1y_1 + c_2y_2 + \cdots + c_my_m$ will also be a solution for any constants $c_1, c_2, \ldots, c_m$.
Our goal is to express the general solution in as concise a way as possible.

Linear combination – a collection of solutions $y_1, y_2, \ldots, y_m$ is linearly independent is if the only way $c_1y_1+c_2y_2+\cdots+c_my_m = 0$ is iff (if and only if) all of the constants $c_1, c_2, \ldots, c_m = 0$. Otherwise we say $y_1, y_2, \ldots, y_m$ are linearly dependent.

Theorem: If the DE is an nth order Linear Homogeneous equation then there will exist a collection of n linearly independent solutions $y_1, y_2, \ldots, y_n$ and the general solution will be $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$

One way to check for linear independence is to compile the Wronskian

$$W(y_1, y_2, \ldots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

4.2 Reduction of Order

If you have one solution to a 2^{nd} order linear homogenous DE, then it is possible to use that function to construct a 2^{nd} Linear Independent solution to the DE.

4.2.1 – Example

For example, the DE

$$y'' - y = 0$$

One solution is $y = e^x$ on $(-\infty, \infty)$.
Idea: We look for y_2 of the form

$$y_2(x) = u(x)y_1(x) \text{ where } u(x) \text{ is not a constant}$$

The general solution is of the form:

$$y = c_1y_1 + c_2y_2$$

where y_1 and y_2 are linearly independent solutions.
To find $u(x)$, we substitute this into the DE

$$\begin{aligned} y_2 &= u(x)y_1(x) \\ y_2' &= u(x)y_1'(x) + u'(x)y_2(x) \\ y_2'' &= u(x)y_1''(x) + u'(x)y_1(x) + u'(x)y_2'(x) + u''(x)y_1(x) \\ &= uy_1'' + 2u'y_1' + u''y_1 \end{aligned}$$

So $y'' - y = 0$ becomes

$$\begin{aligned} uy_1'' + 2u'y_1' + u''y_1 - uy_1 &= 0 \text{ when we sub } y = y_2 \quad = u_{y1} \\ u(e^x)'' + 2u'(e^x)' + u''(e^x) - u(e^x) &= 0 \\ ue^x + 2u'e^x + u''e^x - ue^x &= 0 \\ 2u'e^x + u''e^x &= 0 \\ e^x(2u' + u'') &= 0 \\ 2u' + u'' &= 0 \end{aligned}$$

Let $w = u'$

$$\begin{aligned} 2w + w' &= 0 \\ 2w + \frac{dw}{dx} &= 0 \\ \frac{dw}{dx} &= -2w \\ \frac{dw}{w} &= -2dx \\ \int \frac{dw}{w} &= \int -2dx \\ \ln|w| &= -2x \\ w &= e^{-2x} \\ u' &= e^{-2x} \\ \int u' &= \int e^{-2x} \\ u &= -\frac{1}{2}e^{-2x} \\ y_2 &= uy_1 \\ &= -\frac{1}{2}e^{-2x} \times e^x \\ &= -\frac{1}{2}e^{-x} \end{aligned}$$

Double check that y_2 is a solution of the DE

$$\begin{aligned} y_2 &= -\frac{1}{2}e^{-x} \\ y_2' &= \frac{1}{2}e^{-x} \\ y_2'' &= -\frac{1}{2}e^{-x} \\ y_2'' - y &= -\frac{1}{2}e^{-x} - \left(-\frac{1}{2}e^{-x}\right) \\ &= -\frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} \\ &= 0 \end{aligned}$$

In general,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

put into standard form by dividing by $a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$, the same method as in our example leads to the formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \tag{4.1}$$

4.2.2 – Example

Part 1

$$x^2y'' - 3xy' + 4y = 0$$

Verify that $y_1 = x^2$ is a solution $y_1' = 2x, y_1'' = 2$.

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ x^2(2) - 3x(2x) + 4(x^2) &= 0 \\ 2x^2 - 6x^2 + 4x^2 &= 0 \\ 6x^2 - 6x^2 &= 0 \\ 0 &= 0 \end{aligned}$$

Part 2

Find a linearly independent solution $y_2(x)$.

$$x^2y'' - 3xy' + 4y = 0$$
$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$
$$P(x) = -\frac{3}{x}$$
$$y_2 = y_1 \int \frac{e^{\int \frac{3}{x}dx}}{(y_1(x))^2}dx$$
$$y_2 = y_1 \int \frac{e^{3\ln|x|}}{(y_1(x))^2}dx$$
$$y_2 = y_1 \int \frac{e^{\ln|x^3|}}{(y_1(x))^2}dx$$
$$y_2 = x^2 \int \frac{x^3}{(x^2)^2}dx$$
$$y_2 = x^2 \int \frac{x^3}{x^4}dx$$
$$y_2 = x^2 \int \frac{1}{x}dx$$
$$y_2 = x^2 \ln|x| + C$$

Part 3: Double check that y_2 is a solution of the DE

$$y_2 = x^2 \ln|x|$$
$$y_2' = x^2 \times \frac{1}{x} + 2x\ln|x|$$
$$y_2'' = 1 + 2x\frac{1}{x} + 2\ln|x|$$
$$= 1 + 2 + 2\ln|x|$$
$$= 3 + 2\ln|x|$$

So the LHS DE becomes

$$\begin{aligned} x^2(3 + 2\ln|x|) - 3x(x + 2x\ln|x|) + 4x^2\ln|x| &= 3x^2 + 2x^2\ln|x| - 3x^2 - 6x^2\ln|x| + 4x^2\ln|x| \\ &= 3x^2 - 3x^2 + 2x^2\ln|x| - 6x^2\ln|x| + 4x^2\ln|x| \\ &= 3x^2 - 3x^2 + 2x^2\ln|x| - 6x^2\ln|x| + 4x^2\ln|x| \\ &= 0 + x^2\ln|x|(2 - 6 + 4) \\ &= x^2\ln|x|(0) \\ &= 0 \end{aligned}$$

Write the general solution of the DE including the interval of the solution

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1x^2 + c_2x^2\left(\ln|x| + C\right) \\ &= c_1x^2 + c_2x^2\ln|x| + Cc_2x^2 \\ \text{just } y &= c_1x^2 + c_2x^2\ln|x| \text{ on } I = (0,\infty)\,, y(2) = 3, y'(2) = 5 \end{aligned}$$

4.2.3 – Example

$$3y'' + y' - 4y = 0$$

$$\begin{aligned} y &= e^{mx} \\ y' &= me^{mx} \\ y'' &= m^2e^{mx} \end{aligned}$$

$$\begin{aligned} 3y'' + y' - 4y &= 3m^2e^{mx} + me^{mx} - 4e^{mx} \\ &= e^{mx}(3m^2 + m - 4) \\ &= e^{mx}(3m^2 + 4)(m - 1) \\ m = 1 \quad m &= -\frac{4}{3} \end{aligned}$$

$$y_1 = e^x, y_2 = e^{-\frac{4}{3}x}$$

4.3 Higher Order, Linear, Homogeneous DE with Constant Coefficients

4.3.1 – Example

$$3y^{(4)} - 2y''' + 7y' + 8y = 0$$

Theorems in 4.1 tell us that the general solution is of the form $y = c_1y_1$.**Conjecture:** A solution of the form $y = e^{mx} \Rightarrow y' = me^{mx}$.

4.3.2 – Example

$$\begin{aligned} 5y' - 4y &= 0 \\ y' - \frac{4}{5}y &= 0 \\ me^{mx} - \frac{4}{5}e^{mx} &= 0 \\ e^{mx}\left(m - \frac{4}{5}\right) &= 0 \\ m - \frac{4}{5} &= 0 \\ m &= \frac{4}{5} \end{aligned}$$

$y = c_1 e^{\frac{4}{5}x}$ is the general solution of the DE

4.3.3 – Example

$$\begin{aligned} y'' + 5y' - 6y &= 0 \\ y(m^2e^{mx}) + 5(me^{mx}) - 6e^{mx} &= 0 \\ e^{mx}\left(m^2y + 5m - 6\right) &= 0 \\ m^2y + 5m - 6 &= 0 \\ (m+6)(m-1) &= 0 \\ m + 6 = 0 &\qquad m - 1 = 0 \\ m = -6 &\qquad m = 1 \\ y_1 = e^{-6x} &\qquad y_2e^{x} \text{ These are Linearly Independent (L.I)} \end{aligned}$$

Therefore:

$$y = c_1e^{-6x} + c_2e^{x}$$

4.3.4 – Example

$$\begin{aligned} y'' - 6y' + 9y &= 0 \\ m^2e^{mx} - 6\left(me^{mx}\right) + 9e^{mx} &= 0 \\ m^2 - 6m + 9 &= 0 \\ (m-3)^2 &= 0 \; m = 3 \text{ is a repeated root} \\ m - 3 &= 0 \\ m &= 3 \\ y_1 = e^{3x} &\qquad y_2 = e^{3x} \text{ are linearly dependent} \end{aligned}$$

Use the Reduction of order function:

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \\
&= e^{3x} \int \frac{e^{-\int -6dx}}{(e^{3x})^2} dx \\
&= e^{3x} \int \frac{e^{\int 6dx}}{e^{6x}} dx \\
&= e^{3x} \int \frac{e^{6x}}{e^{6x}} dx \\
&= e^{3x} \int 1dx \\
&= e^{3x} x \\
&= xe^{3x}
\end{aligned}$$

Always works out for this solution if $e^{m_1 x}$ is a solution and m_1 is a root of multiplicity k than $y_1 = e^{m_1 x}, y_2 = xe^{m_1 x}, \ldots, y_k = x^{k-1}e^{mx}$ are linear solutions.

$$\begin{aligned}
y'' + 9y &= 0 \\
m^2 + 9 &= 0 \\
m^2 &= -9 \\
m &= \sqrt{-9} \text{ No real solutions} \\
m &= \pm\sqrt{-9} \\
m &= \pm 3i \\
y &= c_1 e^{3ix} + c_2 e^{-3ix} \text{ where } c_1 \& c_2 \text{ arbitrary complex numbers}
\end{aligned}$$

We'd rather only deal with real-valued solutions.

4.3.5 – Euler's Formula

$$\begin{aligned}
e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\
e^{i3x} &= \cos(3x) + i\sin(3x) \\
e^{-i3x} &= \cos(-3x) + i\sin(-3x) \\
e^{-i3x} &= \cos(3x) - i\sin(3x) \\
e^{i3x} + e^{-i3x} &= \cos(3x) + i\sin(3x) + \cos(3x) - i\sin(3x) \\
e^{i3x} + e^{-i3x} &= 2\cos(3x) \\
Y_1 = \frac{1}{2}e^{i3x} + \frac{1}{2}e^{-i3x} &= \cos(3x) \\
Y_2 &= \sin(3x) \\
\frac{1}{2i}y_1 - \frac{1}{2i}y_2 &= \sin(3x)
\end{aligned}$$

General solution:

$$\begin{aligned}
y &= C_1 Y_1 + C_2 Y_2 \\
&= C_1 \cos(3x) + C_2 \sin(3x)
\end{aligned}$$

where C_1 and C_2 are complex numbers that generate all complex-valued solutions of the DE

4.3.6 – Example

$$\begin{aligned}
y'' + 25y &= 0 \\
m^2 e^{mx} + 25e^{mx} &= 0 \\
m^2 + 25 &= 0 \\
m^2 &= -25 \\
m &= \pm 5i \\
&\text{General solution} \\
y_1 &= c_1 y_1 + c_2 y_2 \\
&= c_1 \cos(5x) + c_2 \sin(5x)
\end{aligned}$$

4.3.7 – Example

$$\begin{aligned}
y'' + 2y' + 6y &= 0 \\
m^2 + 2m + 6 &= 0 \\
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-2 \pm \sqrt{2^2 - 4(1)(6)}}{2(1)} \\
&= \frac{-2 \pm \sqrt{4 - 24}}{2} \\
&= \frac{-2 \pm \sqrt{-20}}{2} \\
&= \frac{-2 \pm \sqrt{4} \times \sqrt{-5}}{2} \\
&= \frac{-2 \pm 2\sqrt{-5}}{2} \\
&= -1 \pm \sqrt{-5} \\
&= -1 \pm \sqrt{5}i \\
y_1 &= e^{(-1+\sqrt{5}i)x} \\
&= e^{-x} e^{i\sqrt{5}x} \\
&= e^{-x} \cos(\sqrt{5}x) \\
y_2 &= e^{(-1-\sqrt{5}i)x} \\
&= e^{-x} e^{-i\sqrt{5}x} \\
&= e^{-x} \sin(\sqrt{5}x)
\end{aligned}$$

So the general solution is

$$y = c_1 e^{-x} \cos(\sqrt{5}x) + c_2 e^{-x} \sin(\sqrt{5}x)$$

In general, if $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$ are roots of the auxiliary equation, then $\begin{aligned} y_1 &= e^{\alpha x}\cos(\beta x) \\ y_2 &= e^{\alpha x}\sin(\beta x) \end{aligned}$ are solutions.

4.3.8 – Example

$$\begin{aligned}
y^{(4)} - 16y &= 0 \\
m^4 - 16 &= 0 \\
(m^2-4)(m^2+4) &= 0 \\
(m-2)(m+2)(m^2+4) &= 0 \\
m &= 2 : y_1 = e^{2x} \\
m &= -2 : y_1 = e^{-2x} \\
m &= 2i : \cos(2x), \sin(2x)
\end{aligned}$$

4.4 Nonhomogeneous, Linear DE with Constant Coefficients

4.4.1 – Method of Undetermined Coefficients

Section 4.5 gives another approach but it is a bit more abstract

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0y = g(x) \text{ where } g(x) \neq 0$$

Theorem: If we can find any one particular solution y_p of this DE $(y_p + y_c)$, where y_c is the solution of the complementary DE (the same LHS$=0$ instead of $g(x)$), is also a solution of the non-homogeneous DE, then the general solution is

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n + y_p
\end{aligned}$$

where you use Section 4.3 methods for the c_iy_i's.

4.4.2 – Example

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

Step 1: Find the General Solution y_c of the complimentary DE

$y'' + 4y' - 2y = 0$Aux equation:

$$\begin{aligned} m^2 + 4m - 2 &= 0 \\ m^2 + 4m + 4 &= 6 \\ (m+2)^2 &= 6 \\ m + 2 &= \pm\sqrt{6} \\ m &= -2 \pm \sqrt{6} \\ y_1 &= e^{(-2+\sqrt{6})x} \\ y_2 &= e^{(-2-\sqrt{6})x} \end{aligned}$$

Step 2: Find a particular solution y_p of given DE

Educated Guess:

$$y_p = Ax^2 + Bx + C$$

for some coefficients A, B, C. For the moment, they're undetermined coefficients. Plugging in the y_p, we get

$$\begin{aligned} y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}$$

So,

$$\begin{aligned}
2A + 4(2Ax + B) - 2\left(Ax^2 + Bx + C\right) &= 2x^2 - 3x + 6 \\
2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C &= 2x^2 - 3x + 6 \\
-2Ax^2 + 8Ax - 2Bx + 2A + 4B - 2C &= 2x^2 - 3x + 6 \\
-2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) &= 2x^2 - 3x + 6 \\
-2A &= 2 \\
8A - 2B &= -3 \\
2A + 4B - 2C &= 6 \\
-2A &= 2 \\
A &= -1 \\
8(-1) - 2B &= -3 \\
-8 - 2B &= -3 \\
8 + 2B &= 3 \\
2B &= -5 \\
B &= -\frac{5}{2} \\
2(-1) + 4(-\frac{5}{2}) - 2C &= 6 \\
-2 + -10 - 2C &= 6 \\
-2C &= 18 \\
C &= -9
\end{aligned}$$

Step 3: Check

$$\begin{aligned}
y_p' &= 2(-1)x + \left(-\frac{5}{2}\right) \\
&= -2x - \frac{5}{2} \\
y_p'' &= 2(-1) \\
&= -2 \\
y'' + 4y' - 2y &= -2 + 4\left(-2x - \frac{5}{2}\right) - 2\left(-x^2 - \frac{5}{2}x - 9\right) \\
&= -2 - 8x - 10 + 2x^2 + 5x + 18 \\
&= 2x^2 - 8x + 5x - 10 + 18 - 2 \\
&= 2x^2 - 3x + 6
\end{aligned}$$

4.4.3 – Example

$$y'' - y' + y = 2\sin(3x)$$

Step 1: Find the General Solution y_c of the complimentary DE

Aux equation:

$$\begin{aligned} m^2 - m + 1 &= 0 \\ m &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1-4}}{2} \\ &= \frac{1 \pm \sqrt{-3}}{2} \\ &= \frac{1 \pm \sqrt{3}i}{2} \\ m_1 &= \frac{1 + \sqrt{3}i}{2} \\ m_2 &= \frac{1 - \sqrt{3}i}{2} \\ y_1 &= e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) \\ y_2 &= e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) \end{aligned}$$

Step 2: Guess $y_p = A\sin(3x) + B\cos(3x)$

Plug into the DE

$$\overbrace{-9A\sin(3x) - 9B\cos(3x)}^{y''} - \left(\overbrace{3A\cos(3x) - 3B\sin(3x)}^{y'}\right) + \overbrace{A\sin(3x) + B\cos(3x)}^{y} = 2\sin(3x)$$

4.3.4 – Method of Undetermined Coefficients 2

For Solving Linear, Non-homogeneous DE with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Standard Form:

$$y'' + a_1 y' + a_0 y = g(x)$$

4.3.5 – Steps

Step 1) Solve $y'' + a_1 y' + a_0 y = 0$ called the general solution y_c.

Step 2) Find one particular solution y_p of the given DE and the general solution is

$$y = y_c + y_p$$

This method can only be used when $g(x)$ is a polynomial (An exponetial (i.e. e^{kx}), sines or cosines or sums of products of these types of functions)

4.3.6 – Example

$$y'' - 3y' - 4y = 4\cos(3x)$$

1st solve:

$$\begin{aligned} y'' - 3y' - 4y &= 0 \\ m^2e^{mx} - 3me^{mx} - 4e^{mx} &= 0 \\ m^2 - 3m - 4 &= 0 \\ (m-4)(m+1) &= 0 \\ m - 4 = 0 &\qquad m + 1 = 0 \\ m = 4 &\qquad m = -1 \\ y_c = c_1e^{4x} &+ c_2e^{-x} \end{aligned}$$

$$\begin{aligned} y &= A\cos(3x) + B\sin(3x) \\ y' &= -3A\sin(3x) + 3B\cos(3x) \\ y'' &= -9A\cos(3x) - 9B\sin(3x) \end{aligned}$$

$$\begin{aligned} y'' - 3y' - 4y &= 4\cos(3x) \\ (-9A\cos(3x) - 9B\sin(3x)) - 3(-3A\sin(3x) + 3B\cos(3x)) - 4(A\cos(3x) + B\sin(3x)) &= 4\cos(3x) \\ -9A\cos(3x) - 9B\sin(3x) + 9A\sin(3x) - 9B\cos(3x) - 4A\cos(3x) - 4B\sin(3x) &= 4\cos(3x) \\ -9A\cos(3x) - 9B\cos(3x) - 4A\cos(3x) - 9B\sin(3x) + 9A\sin(3x) - 4B\sin(3x) &= 4\cos(3x) \\ \cos(3x)(-9A - 9B - 4A) + \sin(3x)(-9B + 9A - 4B) &= 4\cos(3x) \\ \cos(3x)(-13A - 9B) + \sin(3x)(9A - 13B) &= 4\cos(3x) \\ \begin{cases} -13A & -9B & = 4 \\ 9A & -13B & = 0 \end{cases} &\text{Solve simultaneously} \end{aligned}$$

One way to solve Linear Systems of Equations is called Cramer's Rule.

$$\det \begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}$$

$$\begin{aligned}
A &= \frac{\begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}} \\
&= \frac{4(-13) - 0(-9)}{-13(-13) - 9(-9)} \\
&= \frac{-52 - 0}{169 + 81} \\
&= -\frac{52}{250} \\
&= -\frac{26}{125} \\
B &= \frac{\begin{bmatrix} -13 & 4 \\ 9 & 0 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}} \\
&= \frac{-13(0) - 4(9)}{250} \\
&= \frac{0 - 36}{250} \\
&= -\frac{36}{250} \\
&= -\frac{18}{125}
\end{aligned}$$

Check:

$$\begin{aligned}
(-13)\left(-\frac{26}{125}\right) + (-9)\left(-\frac{18}{125}\right) &? = 4 \\
\frac{338}{125} + \frac{162}{125} &? = 4 \\
\frac{500}{125} &= 4 \\
9\left(-\frac{26}{125}\right) + (-13)\left(-\frac{18}{125}\right) &? = 0 \\
-\frac{234}{125} + \frac{234}{125} &? = 0 \\
0 &= 0
\end{aligned}$$

So

$$y = -\frac{26}{125}\cos(3x) - \frac{18}{125}\sin(3x) + c_1 e^{4x} + c_2 e^{-x}$$

is the general solution to the given DE.

4.3.7 – Example

$$y'' - 5y' + 4y = 8e^x$$

If we try:

$$\begin{aligned} y_p &= Ae^x \\ Ae^x - 5Ae^x + 4Ae^x &= 8e^x \\ e^x(A - 5A + 4) &= 8e^x \\ 0 &= 8e^x \text{ has no solution.} \end{aligned}$$

Solve

$$y'' - 5y' + 4y = 0$$

1st

$$\begin{aligned} m^2 - 5m + 4 &= 0 \\ (m-1)(m-4) &= 0 \\ m - 1 = 0 &\qquad m - 4 = 0 \\ m = 1 &\qquad m = 4 \\ y_1 = e^{1mx} &\qquad y_2 = e^{4mx} \\ y_1 = e^{mx} &\qquad y_2 = e^{4mx} \\ y_c = c_2e^{mx} &+ c_2e^{4mx} \text{ hole at } Ae^x \text{ is } c_1 = A \quad c_2 = 0 \end{aligned}$$

Suppose we have a 5th order DE with

$$a_5y^{(5)} + a_4y^{(4)} + \cdots + a_1y' + a_0y = g(x)$$

and the auxiliary equation factors as

$$m^2(m-3)(m-(2+i))(m-(2-i)) = 0$$

$$m = 0 \text{ (multiplicity 2)} \qquad m = 3 \qquad m = 2 + i \qquad m = 2 - i$$

Step 1

Write the general solution to the complimentary DE

$$\begin{aligned} y_1 = e^{0x} = &\quad 1 \\ y_2 = xe^{0x} = &\quad x \\ y_3 = e^{3x} = &\quad e^{3x} \\ y_4 = e^{(2+i)x} = &\quad e^{2x}\cos(x) \\ y_5 = e^{(2-i)x} = &\quad e^{2x}\sin(x) \end{aligned}$$

$$y_c = c_1 + c_2x + c_3e^{3x} + e^{2x}\cos(x) + e^{2x}\sin(x)$$

4.3.8 – What would you guess for the form of y_p?

If

(ii) $g(x) = e^{5x} \Rightarrow y_p = Ae^{5x}$

(iii) $g(x) = e^{3x} \Rightarrow y_p = Axe^{3x}$ (because e^{3x} is in y_c)

(iv) $g(x) = 5e^{2x}\sin(x) \Rightarrow y_p = \left(Ae^{2x}\cos(x) + Be^{2x}\sin(x)\right)x$

(v) $g(x) = 6x^2e^{4x} \Rightarrow y_p = \left(Ax^2 + Bx + C\right)e^{4x}$

(vi) $g(x) = x^2e^{3x} \Rightarrow y_p = \left(Ax^2 + Bx + C\right)e^{3x}x$
$\left(Ax^2 + Bx + C\right)e^{3x}$

4.6 Variation of Parameters Method

$$y'' + P(x)y' + Q(x)y = f(x)$$

will only work on problems where $P(x)$ and $Q(x)$ are constants.

4.6.1 – 1st Step: General solution of complementary DE

$$y = c_1y_1 + c_2y_2$$

Guess a solution to the non-homogeneous of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions of x.This theory produces

$$u_1' = \frac{W_1}{W} \text{ and } u_2' = \frac{W_2}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

4.6.2 – Example

$$\begin{aligned}
4y'' + 36y &= \csc(3x) \\
4y'' + 36y &= \csc(3x) \\
y'' + 9y &= \frac{\csc(3x)}{4} \\
m^2e^{mx} + 9e^{mx} &= 0 \\
m^2 + 9 &= 0 \\
m^2 &= -9 \\
m &= \pm\sqrt{-9} \\
&= \pm 3i
\end{aligned}$$

$$y_1 = e^{0x}\cos(3x) \qquad y_2 = e^{0x}\sin(3x)$$
$$y_1 = 1\cos(3x) \qquad y_2 = 1\sin(3x)$$
$$y_1 = \cos(3x) \qquad y_2 = \sin(3x)$$
$$y_c = c_1\cos(3x) + c_2\sin(3x)$$

Guess

$$y_p = u_1 y_1 + u_2 y_2$$
$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

where

$$\begin{aligned}
W &= \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix} \\
&= (3\cos(3x))(\cos(3x)) - (\sin(3x))(-3\sin(3x)) \\
&= 3\cos^2(3x) + 3\sin^2(3x) \\
&= 3\left(\cos^2(3x) + \sin^2(3x)\right) \\
&= 3\,(1) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
W_1 &= \begin{vmatrix} 0 & \sin(3x) \\ \frac{1}{4}\csc(3x) & \cos(3x) \end{vmatrix} \\
&= 0\cos(3x) - \sin(3x)\left(\frac{\csc(3x)}{4}\right) \\
&= -\frac{\sin(3x)\csc(3x)}{4} \\
&= -\frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
W_2 &= \begin{vmatrix} \cos(3x) & 0 \\ \sin(3x) & \frac{1}{4}\csc(3x) \end{vmatrix} \\
&= \frac{1}{4}\csc(3x)\cos(3x) - 0\sin(3x) \\
&= \frac{\cos(3x)}{4\sin(3x)} \\
&= \frac{1}{4}\cot(3x)
\end{aligned}$$

$$\begin{aligned} u_1' &= \frac{W_1}{W} & u_2' &= \frac{W_2}{W} \\ u_1' &= \frac{-frac14}{3} & u_2' &= \frac{\frac{1}{4}\cot(3x)}{3} \\ u_1' &= -\frac{1}{12} & u_2 &= \frac{1\cot(3x)}{12} \\ u_1' &= -\frac{1}{12} & u_2 &= \frac{1\cos(3x)}{12\sin(3x)} \\ u_1 &= \int -\frac{1}{12}dx & u_2 &= \int \frac{1\cos(3x)}{12\sin(3x)}dx \\ u_1 &= -\frac{x}{12} & u_2 &= \frac{1}{12}\int \frac{1}{3}\frac{dv}{v} \\ & & u_2 &= \frac{1}{36}\ln|v| \\ & & u_2 &= \frac{1}{36}\ln|\sin(3x)| \end{aligned}$$

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= -\frac{x}{12}\cos(3x) + \frac{1}{36}\ln|\sin(3x)|\sin(3x) \\ &= -\frac{x\cos(3x)}{12} + \frac{\sin(3x)}{36}\ln|\sin(3x)| \end{aligned}$$

$$\begin{aligned} y &= y_c + y_p \\ &= c_1\cos(3x) + c_2\sin(3x) - \frac{x\cos(3x)}{12} + \frac{\sin(3x)}{36}\ln|\sin(3x)| \end{aligned}$$

4.6.3 – 3×3 Determinants

$$\left| \begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix} \right|$$