Chapter 1

Introduction to Differential Equations

1.1 Terminology and Notation

Differential equation (D.E.) – An equation in which at least one derivative of an unknown function.
Order of the D.E. – The highest order of derivative in the D.E.

1.1.1 – Example

$$4y'' + e^x y' - 3yy' = \sin(x)$$

An example of a partial differential equation is:

$$\frac{\partial T}{\partial x} + x^2 \frac{\partial T}{\partial y} = x + y$$

however, we won't study these in this course.

1.1.2 – Linear vs Non-Linear DE's

Linear D.E. – The dependent variable and all of its derivatives in the D.E. are in separate terms to the 1^{st} power. $y^{(n)}$ or $\frac{d^n y}{dx^n}$ where $n \neq 1$ are non-first power.

$$4y'' + e^x y' - 3yy' = \sin(x)$$

is a non-linear D.E. while

$$4y'' + e^x y' - 3y = \sin(x)$$

is linear.
The general formula of a linear D.E. would look like

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = g(x)$$

Solution – a function $\phi(x)$ and an interval I for which the D.E. is satisfied when $y = \phi(x)$ for all x in I.
It may be the case that the natural domain of $\phi(x)$ is larger than I.

1.1.3 – Example

$y' = -\frac{1}{x^2}$ has a solution $\phi(x) = \frac{1}{x}$ on $I = (0, \infty)$ but the domain of $\phi(x) = (-\infty, 0) \cup (0, \infty)$.

Practice:

$$\frac{d^2x}{dt^2} + 16x = 0$$

Show (*Verify* not derive) $x(t) = c_1 \sin(4t)$ is a solution on $(-\infty, \infty)$ where c is any real parameter.

$$\begin{aligned}
x &= c_1 \sin(4t) \\
\frac{dx}{dt} &= 4c_1 \cos(4t) \\
\frac{d^2x}{dt^2} &= -16c_1 \sin(4t) \\
\text{LHS} = \frac{d^2x}{dt^2} &+ 16x \\
&= -16c_1 \sin(4t) + 16(c_1 \sin(4t)) \\
&= 0 = \text{RHS}
\end{aligned}$$

But the equation $x = c_2 \cos(4t)$ would also be a solution. If you have 2 equations that are both solutions, you could add them together and you would still have a solution. $x = c_1 \sin(4t) + c_2 \cos(4t)$ is a solution for all parameters c_1 and c_2. *In fact, this is the general solution to the D.E.*

The D.E.

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

Show $y = \left(\frac{1}{4}x^2 + C\right)^2$ is a one parameter family of solutions

$$\begin{aligned}
\text{LHS} = \frac{dy}{dx} &= 2\left(\frac{1}{4}x^2 + C\right) \times \frac{1}{2}x \\
&= x\left(\frac{1}{4}x^2 + C\right) \\
\text{RHS} = xy^{\frac{1}{2}} &= x\left(\left(\frac{1}{4}x^2 + C\right)^2\right)^{\frac{1}{2}} \\
&= x\left(\frac{1}{4}x^2 + C\right) \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

But there is another solution: namely $y(x) = 0$ for all x. This is called the "trivial solution".