Chapter 3

Modeling using DE

3.1 Linear DE Modeling

3.1.1 – Standard Problems

1) Population Growth (or decline)

2) Radioactive Decay

3) Newton's Law of Cooling

4) Mixture Problems

3.1.2 – Population Model

Assume the rate of population change is proportional to the size of the population

$$P(t) = \text{ population at time } t$$

$$\frac{dP}{dt} = kP$$

$$\frac{\frac{dP}{dt}}{P} = k \text{ is the relative growth rate of the population}$$

$$
\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{dP}{P} &= kdt \\
\int \frac{dP}{P} &= \int kdt \\
ln|P| &= kt + C \\
|P| &= e^{kt+C} \\
|P| &= e^{kt}e^{C} \\
|P| &= Ae^{kt} \text{ where } A > 0 \\
P &= \pm Ae^{kt} \\
P &= Be^{kt} \text{ where } B \neq 0 \\
P &= De^{kt} \text{ where } D \text{ can be any real number}
\end{aligned}
$$

The constant can become any number because 0 would be a valid rate of population change, it means that the population size isn't changing.

3.1.3 – Example

If, initially at 2 p.m., there are 1,000 bacteria on a petri dish and at 4 p.m., there are 2,000 bacteria. Assuming constant relative growth rate, how many bacteria are there at 5 p.m.?

$P(t)$ = population t hours after 2 p.m.

$$\begin{aligned}
P(t) &= Ae^{kt} \\
1000 &= Ae^{(0)k} \\
1000 &= Ae^{0} \\
1000 &= A(1) \\
A &= 1000 \\
P(2) &= 2000 \\
P(2) &= 1000e^{2k} \\
2000 &= 1000e^{2k} \\
2 &= e^{2k} \\
\ln(2) &= 2k \\
k &= \frac{\ln(2)}{2} \\
P(t) &= 1000e^{\frac{\ln(2)}{2}t} \\
P(3) &= 1000e^{\frac{\ln(2)}{2}(3)} \\
&= 1000e^{1.5\ln(2)} \\
&= 1000e^{\ln(2^{1.5})} \\
&= 1000(2^{1.5}) \\
&= 2000(\sqrt{2}) \\
P(3) &\approx 2828.427(\sqrt{2}) \\
P(t) &= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\ln(2^{\frac{t}{2}})} \\
&= 1000 \times 2^{\frac{t}{2}}
\end{aligned}$$

3.1.4 – Radioactive Decay

$$m(t) = m_0e^{kt} \text{ where } k < 0$$

The Half-Life is the amount of time it takes for half of the original amount to remain:

$$\frac{1}{2}A_0 = A_0e^{kt} \Rightarrow \frac{1}{2} = e^{kt}$$

3.1.5 – Mixture Problems

Setup

Initially, the container has 200 gallons of brine solution (salt-water) of concentration $\frac{10 \text{ lbs}}{200 \text{ gallons}} = 0.05\frac{\text{lbs}}{\text{gallon}}$. A solution of $\frac{5 \text{ lbs}}{200 \text{ gallons}} 0.025\frac{\text{lbs}}{\text{gallon}}$ is poured into the initial container at a rate of $\frac{4 \text{ gallons}}{\text{min}}$. How many pounds of salt are there in the container after 2 hours.

Let $y(t) = \#$ lbs of salt t minutes after the precess starts $\frac{dy}{dt}$ = The rate of change of # lbs of salt

$$
\begin{aligned}
\frac{dy}{dt} &= \left.0.025\frac{\text{lbs}}{\text{gal}} \times 4\frac{\text{gal}}{\text{min}}\right\}\text{rate in} \\
&\quad - \left.\frac{y(t)\text{lbs}}{200\text{gal}} \times 4\frac{\text{gal}}{\text{min}}\right\}\text{rate out} \\
&= (0.025)4\frac{\text{lbs}}{\text{min}} - \frac{4y(t)}{200}\frac{\text{lbs}}{\text{min}} \\
&= 0.1\frac{\text{lbs}}{\text{min}} - \frac{y(t)}{50}\frac{\text{lbs}}{\text{min}} \\
&= 0.1 - \frac{y(t)}{50} \\
\frac{dy}{dt} + \frac{1}{50}y &= 0.1 \\
\mu &= e^{\int P(t)dt} \\
&= e^{\int \frac{1}{50}dt} \\
&= e^{\frac{t}{50}} \\
e^{\frac{t}{50}}\left(\frac{dy}{dt}\right) + e^{\frac{t}{50}}\left(\frac{1}{50}y\right) &= e^{\frac{t}{50}}(0.1) \\
\frac{d}{dt}\left(e^{\frac{t}{50}}y\right) &= e^{\frac{t}{50}}(0.1) \\
\int \frac{d}{dt}\left(e^{\frac{t}{50}}y\right) &= \int \frac{1}{10}e^{\frac{t}{50}} \\
e^{\frac{t}{50}}y &= \frac{1}{10} \times \frac{e^{\frac{t}{50}}}{\frac{1}{50}} + C \\
e^{\frac{t}{50}}y &= 5e^{\frac{t}{50}} + C \\
y &= 5 + Ce^{-\frac{t}{50}} \\
&= 5 + Ce^{-0.02t} \\
y(120) &= 5 + Ce^{-0.02(120)} \\
&= 5 + Ce^{-2.4}
\end{aligned}
$$