Chapter 7

Method of Laplace Transforms for Solving DE's

7.2 Solving I.V.T by using Laplace Transform

Take $\mathscr{L}$ of both sides of the DE

7.2.1 – Example

$$y'' - 3y' + 2y = e^{-4t},\ y(0) = 1,\ y'(0) = 5$$
$$\mathscr{L}\{y''\} - \mathscr{L}\{3y'\} + \mathscr{L}\{2y\} = \mathscr{L}\{e^{-4t}\}$$

We need more formulas first.

$$\begin{aligned} u &= e^{-st} & dv &= f'(x)dt \\ du &= -se^{-st}dt & v &= f(x) \end{aligned}$$

$$\begin{aligned} \mathscr{L}\{f'(t)\} &= \int_0^\infty e^{st} f'(t)dt \\ &= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty -sf(t)e^{st}dt \\ &= -f(t)e^{-st} + s\int_0^\infty f(t)e^{st}dt \\ &= -f(t)e^{-st} + s\mathscr{L}\{f(t)\} \\ &= -f(t)e^{-st} + sF(s) \\ &= -f(0)e^{-s(0)} + sF(s) \\ &= -f(0)(1) + sF(s) \\ &= -f(0) + sF(s) \end{aligned}$$

$$\begin{aligned}
\mathscr{L}\{f''(t)\} &= \mathscr{L}\{(f'(t))'\} \\
&= s\mathscr{L}\{f'(t)\} - f'(0) \\
&= s\left(-f(0) + sF(s)\right) - f'(0) \\
&= -sf(0) + s^2F(s) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0) \\
&= s^2\mathscr{L}\{f\} - sf(0) - f'(0)
\end{aligned}$$

In general

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)\cdots - s^{n-n}f^{n-1}(0) \tag{7.1}$$

or

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)\cdots - f^{n-1}(0) \tag{7.2}$$

So the DE transforms to

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) - 3\left(sY(s) - y(0)\right) + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - s(1) - 5 - 3\left(sY(s) - 1\right) + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - s - 5 - 3sY(s) + 3 + 2Y(s) &= \frac{1}{s+4} \\
s^2Y(s) - 3sY(s) + 2Y(s) - s - 5 + 3 &= \frac{1}{s+4} \\
Y(s)\left(s^2 - 3s + 2\right) - s - 2 &= \frac{1}{s+4} \\
Y(s)\left(s^2 - 3s + 2\right) &= \frac{1}{s+4} + s + 2 \\
Y(s) &= \frac{\frac{1}{s+4} + s + 2}{(s^2 - 3s + 2)} \\
&= \frac{1 + (s+2)(s+4)}{(s+4)\left(s^2 - 3s + 2\right)} \\
&= \frac{1 + s^2 + 6s + 8}{(s+4)(s-1)(s-2)} \\
&= \frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)}
\end{aligned}$$

$$\begin{aligned}
\frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)} &= \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2} \\
&= \frac{A(s-1)(s-2)}{(s+4)(s-1)(s-2)} + \frac{B(s+4)(s-2)}{(s+4)(s-1)(s-2)} + \frac{C(s+4)(s-1)}{(s+4)(s-1)(s-2)} \\
s^2 + 6s + 9 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1) \\
(s+3)^2 &= A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1)
\end{aligned}$$

$$\begin{aligned}
s^2+6s+9 &= A(s-1)(s-2)+B(s+4)(s-2)+C(s+4)(s-1)\\
(-4)^2+6(-4)+9 &= A(-4-1)(-4-2)+B(-4+4)(-4-2)+C(-4+4)(-4-1)\\
16-24+9 &= A(-5)(-6)+B(0)(-6)+C(0)(-5)\\
1 &= 30A\\
A &= \frac{1}{30}\\
(s+3)^2 &= A(s-1)(s-2)+B(s+4)(s-2)+C(s+4)(s-1)\\
(1+3)^2 &= A(1-1)(1-2)+B(1+4)(1-2)+C(1+4)(1-1)\\
4^2 &= A(0)(-1)+B(5)(-1)+C(5)(0)\\
16 &= -5B\\
B &= \frac{-16}{5}\\
(s+3)^2 &= A(s-1)(s-2)+B(s+4)(s-2)+C(s+4)(s-1)\\
(2+3)^2 &= A(2-1)(2-2)+B(2+4)(2-2)+C(2+4)(2-1)\\
5^2 &= A(1)(0)+B(6)(0)+C(6)(1)\\
25 &= 6C\\
C &= \frac{6}{25}
\end{aligned}$$

Note:

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \to \mathscr{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$$

$$\begin{aligned}
Y(s) &= \frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\\
&= \frac{\frac{1}{30}}{s+4}+\frac{\frac{-16}{5}}{s-1}+\frac{\frac{6}{25}}{s-2}\\
y(t) &= \mathscr{L}^{-1}\{Y(s)\}\\
&= \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}-\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{6}{25}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}\\
&= \frac{1}{30}e^{-4t}-\frac{16}{5}e^{t}+\frac{6}{25}e^{2t}
\end{aligned}$$

7.2.2 – Finding Inverse-Laplace Transform

7.2.3 – Example

$$\begin{aligned}
\mathscr{L}\left\{\frac{1}{s^4}\right\} &= \frac{1}{3!}\mathscr{L}\left\{\frac{3!}{s^4}\right\} \\
&= \frac{1}{3!}\mathscr{L}\left\{\frac{3!}{s^{3+1}}\right\} \\
&= \frac{1}{3!}t^3 \\
&= \frac{1}{6}t^3
\end{aligned}$$

7.2.4 – Example

$$\begin{aligned}
\mathscr{L}\left\{\frac{5}{s^2+49}\right\} &= \frac{5}{7}\mathscr{L}\left\{\frac{7}{s^2+49}\right\} \\
&= \frac{5}{7}\sin(7t)
\end{aligned}$$

7.2.5 – Example

$$\begin{aligned}
\mathscr{L}\left\{\frac{(s+1)^3}{s^4}\right\} &= \mathscr{L}\left\{\frac{s^3+3s^2+3s+1}{s^4}\right\} \\
&= \mathscr{L}\left\{\frac{s^3}{s^4}\right\} + \mathscr{L}\left\{\frac{3s^2}{s^4}\right\} + \mathscr{L}\left\{\frac{3s}{s^4}\right\} + \mathscr{L}\left\{\frac{1}{s^4}\right\} \\
&= \mathscr{L}\left\{\frac{1}{s}\right\} + \mathscr{L}\left\{\frac{3}{s^2}\right\} + \mathscr{L}\left\{\frac{3}{s^3}\right\} + \mathscr{L}\left\{\frac{1}{s^4}\right\} \\
&= \mathscr{L}\left\{\frac{1}{s}\right\} + 3\mathscr{L}\left\{\frac{1}{s^2}\right\} + 3\mathscr{L}\left\{\frac{1}{s^3}\right\} + \mathscr{L}\left\{\frac{1}{s^4}\right\} \\
&= \mathscr{L}\left\{\frac{1}{s}\right\} + 3\mathscr{L}\left\{\frac{1}{s^2}\right\} + \frac{3}{2!}\mathscr{L}\left\{\frac{2!}{s^3}\right\} + \frac{1}{3!}\mathscr{L}\left\{\frac{3!}{s^4}\right\} \\
&= 1 + 3t + \frac{3}{2!}t^2 + \frac{1}{3!}t^3 \\
&= 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3
\end{aligned}$$