Chapter 4

Higher Order Differential Equations

4.6 Variation of Parameters Method

$$y'' + P(x)y' + Q(x)y = f(x)$$

will only work on problems where $P(x)$ and $Q(x)$ are constants.

4.6.1 – 1st Step: General solution of complementary DE

$$y = c_1 y_1 + c_2 y_2$$

Guess a solution to the non-homogeneous of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions of x.

This theory produces

$$u_1' = \frac{W_1}{W} \text{ and } u_2' = \frac{W_2}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

4.6.2 – Example

$$4y'' + 36y = \csc(3x)$$

$$\begin{aligned}
4y'' + 36y &= \csc(3x) \\
y'' + 9y &= \frac{\csc(3x)}{4} \\
m^2 e^{mx} + 9e^{mx} &= 0 \\
m^2 + 9 &= 0 \\
m^2 &= -9 \\
m &= \pm\sqrt{-9} \\
&= \pm 3i
\end{aligned}$$

$$\begin{aligned}
y_1 &= e^{0x}\cos(3x) & y_2 &= e^{0x}\sin(3x) \\
y_1 &= 1\cos(3x) & y_2 &= 1\sin(3x) \\
y_1 &= \cos(3x) & y_2 &= \sin(3x)
\end{aligned}$$

$$y_c = c_1\cos(3x) + c_2\sin(3x)$$

Guess

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

where

$$\begin{aligned}
W &= \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix} \\
&= (3\cos(3x))(\cos(3x)) - (\sin(3x))(-3\sin(3x)) \\
&= 3\cos^2(3x) + 3\sin^2(3x) \\
&= 3\left(\cos^2(3x) + \sin^2(3x)\right) \\
&= 3\,(1) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
W_1 &= \begin{vmatrix} 0 & \sin(3x) \\ \frac{1}{4}\csc(3x) & \cos(3x) \end{vmatrix} \\
&= 0\cos(3x) - \sin(3x)\left(\frac{\csc(3x)}{4}\right) \\
&= -\frac{\sin(3x)\csc(3x)}{4} \\
&= -\frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
W_2 &= \begin{vmatrix} \cos(3x) & 0 \\ \sin(3x) & \frac{1}{4}\csc(3x) \end{vmatrix} \\
&= \frac{1}{4}\csc(3x)\cos(3x) - 0\sin(3x) \\
&= \frac{\cos(3x)}{4\sin(3x)} \\
&= \frac{1}{4}\cot(3x)
\end{aligned}$$

$$
\begin{aligned}
u_1' &= \frac{W_1}{W} & u_2' &= \frac{W_2}{W} \\
u_1' &= \frac{-\frac{1}{4}}{3} & u_2' &= \frac{\frac{1}{4}\cot(3x)}{3} \\
u_1' &= -\frac{1}{12} & u_2 &= \frac{1\cot(3x)}{12} \\
u_1' &= -\frac{1}{12} & u_2 &= \frac{1\cos(3x)}{12\sin(3x)} \\
u_1 &= \int -\frac{1}{12}dx & u_2 &= \int \frac{1\cos(3x)}{12\sin(3x)}dx \\
u_1 &= -\frac{x}{12} & u_2 &= \frac{1}{12}\int \frac{1}{3}\frac{dv}{v} \\
& & u_2 &= \frac{1}{36}\ln|v| \\
& & u_2 &= \frac{1}{36}\ln|\sin(3x)|
\end{aligned}
$$

$$
\begin{aligned}
y_p &= u_1y_1 + u_2y_2 \\
&= -\frac{x}{12}\cos(3x) + \frac{1}{36}\ln|\sin(3x)|\sin(3x) \\
&= -\frac{x\cos(3x)}{12} + \frac{\sin(3x)}{36}\ln|\sin(3x)|
\end{aligned}
$$

$$
\begin{aligned}
y &= y_c + y_p \\
&= c_1\cos(3x) + c_2\sin(3x) - \frac{x\cos(3x)}{12} + \frac{\sin(3x)}{36}\ln|\sin(3x)|
\end{aligned}
$$

4.6.3 – 3×3 Determinants

$$
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ f & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}
$$

Matrix of Signs

$$
\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}
$$