Chapter 2

First-Order Differential Equations

2.4 Exact Equations

1st Order D.E. in differential form

$$M(x,y)dx + N(x,y)dy = 0$$

Given a function

$$z = f(x,y)$$

, the total differential, dz, is defined as

$$dx = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

2.4.1 – Method

See if whe can find a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

If we can do this, then the D.E. is equivalent to

$$df = 0 \Rightarrow f(x,y) = c$$

is an implicit solution of D.E.

Assume that M and N have continuous 1st order partials (assuming f exists)

$$\left.\begin{array}{rll} My & = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}dy = & f_{xy} \\ Nx & = \frac{\partial}{\partial x}\frac{\partial f}{\partial Y}dy = & f_{yx} \end{array}\right\} \text{Theorem tells use these are equal}$$

This provides a quick test to check if the D.E. is exact or not.

2.4.2 – Example

$$2xydx + \left(x^2 - 1\right) dy = 0$$
$$M(x,y) = 2xyN(x,y) = x^2 - 1$$

To check if the D.E. is exact

$$M_y = 2x = N_x$$

We now know there exists a function $f(x,y)$ with

$$\frac{\partial f}{\partial x} = M = \quad 2xy$$
$$\frac{\partial f}{\partial y} = N = \quad x^2 - 1$$

$$\begin{aligned}
f_M(x,y) &= \int \frac{\partial f}{\partial x} dx \\
&= \int 2xydx \\
&= x^2y + \phi(y) \\
\frac{\partial f}{\partial y}\left(x^2y + \phi(y)\right) &= x^2 - 1 \text{ required to equal } N \\
x^2 + \phi'(y) &= x^2 - 1 \\
\phi'(y) &= -1 \\
\phi(y) &= \int -1dy \\
&= -y \\
f(x,y) &= x^2y - y \\
d\left(f(x,y)\right) &= 0 \\
f(x,y) &= c \\
x^2y - y &= c \text{ is an implicit solution of the D.E.}
\end{aligned}$$

Note: the f_M format is just there to show which partial equation was integrated. It was made by me and, as far as I know, not standardly known.

2.4.3 – Example

$$\begin{aligned}
\left(e^{2y} - y\cos(xy)\right) dx + (2xe^{2y} - x\cos(xy) + 2y)dy &= 0 \\
M_y &= N_x \\
\frac{\partial}{\partial y}\left(e^{2y} - y\cos(xy)\right) &= \frac{\partial}{\partial x}\left(2xe^{2y} - x\cos(xy) + 2y\right) \\
2e^{2y} - [\cos(xy) - y\sin(xy) \times x] &= 2e^{2y} - (\cos(xy) - x\sin(xy) \times y) + 0 \\
2e^{2y} - \cos(xy) + xy\sin(xy) &= 2e^{2y} - \cos(xy) + xy\sin(xy)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= M = e^{2y} - y\cos(xy) \\
\frac{\partial f}{\partial y} &= N = 2xe^{2y} - x\cos(xy) + 2y \\
f_N(x,y) &= \int \frac{\partial f}{\partial y} dy \\
&= \int \left(2xe^{2y} - x\cos(xy) + 2y\right) dy \\
&= \frac{2xe^{2y}}{2} - \frac{x\sin(xy)}{x} + 2 \times \frac{y^2}{2} + \phi(x) \\
&= xe^{2y} - \sin(xy) + y^2 + \phi(x)
\end{aligned}$$

Take the ∂x of this and equate with M:

$$\begin{aligned}
M &= \frac{\partial}{\partial x}\left(xe^{2y} - \sin(xy) + y^2 + \phi(x)\right) \\
e^{2y} - y\cos(xy) &= e^{2y} - y\cos(xy) + 0 + \phi'(x) \\
0 &= \phi'(x) \\
\phi(x) &= c
\end{aligned}$$

So $f(x,y) = c_2$ is the solution

$$xe^{2y} - \sin(xy) + y^2 = c$$

$$dx = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

2.4.4 – What can you do if $M_y \neq N_x$

Sometimes you can multiply the DE by an integrating factor $\mu(x,y)$ to get an exact DE. If

$$\frac{M_y - N_x}{N}$$

is a function of only x, then

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

will be an I.F.

If

$$\frac{N_x - M_y}{M}$$

is a function of only y, then

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

will be an I.F.

2.4.5 – Example

$$xydx + \left(2x^2 + 3y^2 - 20\right) dy = 0$$

$$\begin{aligned}
M_y &= x \\
N_x &= 4x \\
M_y &\neq N_x \frac{N_x - M_y}{M} &= \frac{4x - x}{xy} \\
&= \frac{3x}{xy} \\
&= \frac{3}{y} \text{ is a function of just } y
\end{aligned}$$

So:

$$\begin{aligned}
\mu &= e^{\int \frac{3}{y} dy} \\
&= e^{3 \ln y} \\
&= y^3 \\
xy^4 dx + y^3 \left(2x^2 + 3y^2 - 20\right) dy &= 0(y^3) \\
xy^4 dx + \left(2x^2y^3 + 3y^5 - 20y^3\right) dy &= \\
M_y &= N_x \\
4xy^3 &= 4xy^3 \frac{\partial f}{\partial x}
\end{aligned}$$