

Chapter 2

First-Order Differential Equations

2.4 Exact Equations

1st Order D.E. in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

Given a function

$$z = f(x, y)$$

, the total differential, dz , is defined as

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

2.4.1 – Method

See if we can find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

If we can do this, then the D.E. is equivalent to

$$df = 0 \Rightarrow f(x, y) = c$$

is an implicit solution of D.E.

Assume that M and N have continuous 1st order partials (assuming f exists)

$$\left. \begin{array}{l} My = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} dy = f_{xy} \\ Nx = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} dy = f_{yx} \end{array} \right\} \text{Theorem tells us these are equal}$$

This provides a quick test to check if the D.E. is exact or not.

2.4.2 – Example

$$2xydx + (x^2 - 1) dy = 0$$

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

To check if the D.E. is exact

$$M_y = 2x = N_x$$

We now know there exists a function $f(x, y)$ with

$$\frac{\partial f}{\partial x} = M = 2xy$$

$$\frac{\partial f}{\partial y} = N = x^2 - 1$$

$$f_M(x, y) = \int \frac{\partial f}{\partial x} dx$$

$$= \int 2xy dx$$

$$= x^2 y + \phi(y)$$

$$\frac{\partial f}{\partial y} (x^2 y + \phi(y)) = x^2 - 1 \text{ required to equal } N$$

$$x^2 + \phi'(y) = x^2 - 1$$

$$\phi'(y) = -1$$

$$\phi(y) = \int -1 dy$$

$$= -y$$

$$f(x, y) = x^2 y - y$$

$$d(f(x, y)) = 0$$

$$f(x, y) = c$$

$$x^2 y - y = c \text{ is an implicit solution of the D.E.}$$

Note: the f_M format is just there to show which partial equation was integrated. It was made by me and, as far as I know, is not standardly known.

2.4.3 – Example

$$(e^{2y} - y \cos(xy)) dx + (2xe^{2y} - x \cos(xy) + 2y) dy = 0$$

$$M_y = N_x$$

$$\frac{\partial}{\partial y} (e^{2y} - y \cos(xy)) = \frac{\partial}{\partial x} (2xe^{2y} - x \cos(xy) + 2y)$$

$$2e^{2y} - [\cos(xy) - y \sin(xy) \times x] = 2e^{2y} - (\cos(xy) - x \sin(xy) \times y) + 0$$

$$2e^{2y} - \cos(xy) + xy \sin(xy) = 2e^{2y} - \cos(xy) + xy \sin(xy)$$

$$\begin{aligned}
 \frac{\partial f}{\partial x} &= M = e^{2y} - y \cos(xy) \\
 \frac{\partial f}{\partial y} &= N = 2xe^{2y} - x \cos(xy) + 2y \\
 f_N(x, y) &= \int \frac{\partial f}{\partial y} dy \\
 &= \int (2xe^{2y} - x \cos(xy) + 2y) dy \\
 &= \frac{2xe^{2y}}{2} - \frac{x \sin(xy)}{x} + 2 \times \frac{y^2}{2} + \phi(x) \\
 &= xe^{2y} - \sin(xy) + y^2 + \phi(x)
 \end{aligned}$$

Take the ∂x of this and equate with M :

$$\begin{aligned}
 M &= \frac{\partial}{\partial x} (xe^{2y} - \sin(xy) + y^2 + \phi(x)) \\
 e^{2y} - y \cos(xy) &= e^{2y} - y \cos(xy) + 0 + \phi'(x) \\
 0 &= \phi'(x) \\
 \phi(x) &= c
 \end{aligned}$$

So $f(x, y) = c_2$ is the solution

$$xe^{2y} - \sin(xy) + y^2 = c$$

$$dx = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

2.4.4 – What can you do if $M_y \neq N_x$

Sometimes you can multiply the DE by an integrating factor $\mu(x, y)$ to get an exact DE.

If

$$\frac{M_y - N_x}{N}$$

is a function of only x , then

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

will be an I.F.

If

$$\frac{N_x - M_y}{M}$$

is a function of only y , then

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

will be an I.F.

2.4.5 – Example

$$xydx + (2x^2 + 3y^2 - 20) dy = 0$$

$$M_y = x$$

$$N_x = 4x$$

$$M_y \neq N_x$$

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy}$$

$$= \frac{3x}{xy}$$

$$= \frac{3}{y} \text{ is a function of just } y$$

So:

$$\mu = e^{\int \frac{3}{y} dy}$$

$$= e^{3 \ln y}$$

$$= y^3$$

$$xy^4 dx + y^3 (2x^2 + 3y^2 - 20) dy = 0(y^3)$$

$$xy^4 dx + (2x^2 y^3 + 3y^5 - 20y^3) dy =$$

$$M_y = N_x$$

$$4xy^3 = 4xy^3$$

$$\frac{\partial f}{\partial x}$$