

Chapter 8

Systems of Linear First-Order Differential Equations

8.2 Solving Homogenous Systems

8.2.3 – Complex EigenValues

8.2.4 – Example

$$\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} 6 - \lambda & -1 \\ 5 & 4 - \lambda \end{vmatrix} \\ &= (6 - \lambda)(4 - \lambda) - (-1)(5) \\ &= 24 - 6\lambda - 4\lambda + \lambda^2 + 5 \\ &= \lambda^2 - 10\lambda + 29 \\ &= (\lambda - 5)^2 + 4 \\ (\lambda - 5)^2 &= -4 \\ \lambda - 5 &= \pm 2i \\ \lambda &= 5 \pm 2i \end{aligned}$$

For $\lambda_1 = 5 + 2i$

$$\begin{aligned}
 \left[\begin{array}{cc|c} 6 - \lambda_1 & -1 & 0 \\ 5 & 4 - \lambda_1 & 0 \end{array} \right] &= \left[\begin{array}{cc|c} 6 - 5 - 2i & -1 & 0 \\ 5 & 4 - 5 - 2i & 0 \end{array} \right] \\
 &= \left[\begin{array}{cc|c} 1 - 2i & -1 & 0 \\ 5 & -1 - 2i & 0 \end{array} \right] \\
 &= \left(r_1 \leftarrow \frac{r_1}{1 - 2i} \right) \left[\begin{array}{cc|c} 1 & -\frac{1}{1 - 2i} & 0 \\ 5 & -1 - 2i & 0 \end{array} \right] \\
 &= \left(r_1 \leftarrow r_1 \times \frac{1 + 2i}{1 + 2i} \right) \left[\begin{array}{cc|c} 1 & -\frac{1}{5} - \frac{2}{5}i^{\textcolor{red}{1}} & 0 \\ 5 & -1 - 2i & 0 \end{array} \right] \\
 &= (r_2 \leftarrow r_2 - 5r_1) \left[\begin{array}{cc|c} 1 & -\frac{1}{5} - \frac{2}{5}i & 0 \\ 0 & 0 & 0 \end{array} \right]
 \end{aligned}$$

$$k_1 + \left(-\frac{1}{5} - \frac{2}{5}i \right) k_2 = 0$$

$$\text{Let } k_2 = 1 - 2i$$

$$k_1 + \left(-\frac{1}{5} - \frac{2}{5}i \right) (1 - 2i) = 0$$

$$k_1 + (-1)^{\textcolor{red}{2}} = 0$$

$$k_1 = 1$$

$$\mathbf{k}_1 = \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix}$$

This, along with an eigenvector \mathbf{k}_2 for $\lambda_2 = 5 - 2i$, can be expressed in terms of Real Matrices as

$$\begin{aligned}
 \mathbf{X}_1 &= [\beta_1 \cos(\beta t) - \beta_2 \sin(\beta t)] e^{\alpha t} \\
 \mathbf{X}_2 &= [\beta_2 \cos(\beta t) - \beta_1 \sin(\beta t)] e^{\alpha t}
 \end{aligned} \tag{8.1}$$

where α is the real part of the eigen value, β is the coefficient of the imaginary part of the eigen value,

$$\beta_1 = \text{Real Part } (k_1) = \text{Re} \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\beta_2 = \text{Imaginary Part } (k_1) = \text{Im} \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

8.3 General Ideas

(Don't worry about detailed calculation)

¹

$$-\frac{1}{1 - 2i} \times \frac{1 + 2i}{1 + 2i} = -\frac{1 + 2i}{(1 - 2i)(1 + 2i)} = -\frac{1 + 2i}{1 - 4i^2} = -\frac{1 + 2i}{1 - (-4)} = -\frac{1 + 2i}{1 + 4} = -\frac{1 + 2i}{5}$$

²Essesntially undid the complex conjgate multiplication.

Non-homogenous system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

where \mathbf{F} is a non-zero column vector.

8.3.1 – 1st Solve the Complimentary DE

$$\mathbf{X}' = \mathbf{A}\mathbf{X},$$

General solution:

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$$

8.3.2 – Next, find one particular solution

The general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$ is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$