

# MATH 252 - Introduction to Differential Equations

## Notes

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# Chapter 1

## Introduction to Differential Equations

### 1.1 Terminology and Notation

**Differential equation (D.E.)** – An equation in which at least one derivative of an unknown function.

**Order of the D.E.** – The highest order of derivative in the D.E.

#### 1.1.1 – Example

$$4y'' + e^x y' - 3yy' = \sin(x)$$

An example of a partial differential equation is:

$$\frac{\partial T}{\partial x} + x^2 \frac{\partial T}{\partial y} = x + y$$

however, we won't study these in this course.

#### 1.1.2 – Linear vs Non-Linear DE's

**Linear D.E.** – The dependent variable and all of its derivatives in the D.E. are in separate terms to the 1<sup>st</sup> power.  $y^{(n)}$  or  $\frac{d^n y}{dx^n}$  where  $n \neq 1$  are non-first power.

$$4y'' + e^x y' - 3yy' = \sin(x)$$

is a non-linear D.E. while

$$4y'' + e^x y' - 3y = \sin(x)$$

is linear.

The general formula of a linear D.E. would look like

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = g(x)$$

**Solution** – a function  $\phi(x)$  and an interval  $I$  for which the D.E. is satisfied when  $y = \phi(x)$  for all  $x$  in  $I$ .

It may be the case that the natural domain of  $\phi(x)$  is larger than  $I$ .

### 1.1.3 – Example

$y' = -\frac{1}{x^2}$  has a solution  $\phi(x) = \frac{1}{x}$  on  $I = (0, \infty)$  but the domain of  $\phi(x) = (-\infty, 0) \cup (0, \infty)$ .

Practice:

$$\frac{d^2x}{dt^2} + 16x = 0$$

Show (*Verify* not derive)  $x(t) = c_1 \sin(4t)$  is a solution on  $(-\infty, \infty)$  where  $c$  is any real parameter.

$$\begin{aligned} x &= c_1 \sin(4t) \\ \frac{dx}{dt} &= 4c_1 \cos(4t) \\ \frac{d^2x}{dt^2} &= -16c_1 \sin(4t) \\ \text{LHS} &= \frac{d^2x}{dt^2} + 16x \\ &= -16c_1 \sin(4t) + 16(c_1 \sin(4t)) \\ &= 0 = \text{RHS} \end{aligned}$$

But the equation  $x = c_2 \cos(4t)$  would also be a solution. If you have 2 equations that are both solutions, you could add them together and you would still have a solution.  $x = c_1 \sin(4t) + c_2 \cos(4t)$  is a solution for all parameters  $c_1$  and  $c_2$ . In fact, this is the general solution to the D.E.

The D.E.

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

Show  $y = (\frac{1}{4}x^2 + C)^2$  is a one parameter family of solutions

$$\begin{aligned} \text{LHS} &= \frac{dy}{dx} = 2 \left( \frac{1}{4}x^2 + C \right) \times \frac{1}{2}x \\ &= x \left( \frac{1}{4}x^2 + C \right) \\ \text{RHS} &= xy^{\frac{1}{2}} = x \left( \left( \frac{1}{4}x^2 + C \right)^2 \right)^{\frac{1}{2}} \\ &= x \left( \frac{1}{4}x^2 + C \right) \\ \text{LHS} &= \text{RHS} \end{aligned}$$

But there is another solution: namely  $y(x) = 0$  for all  $x$ . This is called the “trivial solution”.

## 1.2 Initial Value Problems (IVP)

1st order IVP is a 1st order D.E. together with one extra condition:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

2nd order IVP

$$y'' = f(x, y, y')$$

Initial conditions:

- $y(x_0) = y_0$
- $y'(x_0) = y_1$

### 1.2.1 – Example

$$y' = y \text{ and } y(0) = 3$$

$y = ce^x$  is a one-parameter family of solutions

$$\frac{d}{dx}(ce^x) = ce^x = y$$

$$\begin{aligned} ce^1 &= -2 \\ c &= -\frac{2}{e} \\ y &= \left(-\frac{2}{e}\right)e^x \\ y &= -2e^{x-1} \end{aligned}$$

### 1.2.2 – Example

$$\text{D.E.: } y' + 2xy^2 = 0 \text{ and } y(0) = 1$$

Given that you have the solution:  $y = \frac{1}{x^2+C}$ , Solve:

$$\begin{aligned} -1 &= \frac{1}{(0)^2 + c} \\ -1 &= \frac{1}{c} \\ -1 \times c &= 1 \\ c &= -1 \\ y &= \frac{1}{x^2 - 1}, I = (-1, 1) \end{aligned}$$

### 1.2.3 – Example

D.E.:  $y' + 2xy^2 = 0$  and  $y(0) = 1$

#### Example

$$x'' + 16x = 0 \text{ and } x\left(\frac{\pi}{2}\right) = 5 \text{ and } x'\left(\frac{\pi}{2}\right) = -4$$

$$\begin{aligned} x &= c_1 \cos(4t) + c_2 \sin(4t) \\ 5 &= c_1 \cos(4t) + c_2 \sin(4t) \\ &= c_1 \cos(2\pi) + c_2 \sin(2\pi) \\ &= c_1(1) + c_2(0) \\ &= c_1 \\ x' &= -4c_1 \sin(4t) + 4c_2 \cos(4t) \\ -4 &= -4c_1 \sin\left(4\left(\frac{\pi}{2}\right)\right) + 4c_2 \cos\left(4\left(\frac{\pi}{2}\right)\right) \\ &= -4c_1 \sin(2\pi) + 4c_2 \cos(2\pi) \\ &= -4c_1(0) + 4c_2(1) \\ &= 4c_2 \\ -1 &= c_2 \end{aligned}$$

Reasonable Question: Given a 1st order IVP, can we say whether a solution *exists* or not and, if a solution exists, is it *unique*.

**Theorem:** Given  $y' = f(x, y)$  and  $y(x_0) = y_0$ , if  $f(x, y)$  and  $\frac{\partial f}{\partial y}$  are both continuous on a rectangle  $R$  containing  $(x_0, y_0)$  in its interior, then there exists an interval  $I = (x_0 - h, x_0 + h)$  where  $h > 0$  such that there exists a unique solution to IVP on  $I$ .

### 1.2.4 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(1) = 2$$

- $f(x, y) = xy^{\frac{1}{2}}$  is continuous everywhere its defined  $y \geq 0$
- $\frac{\partial f}{\partial y} = x^{\frac{1}{2}}y^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$  is continuous everywhere its defined  $y > 0$

### 1.2.5 – Example

$$y' = xy^{\frac{1}{2}} \text{ and } y(0) = 0$$

- $f(x, y) = xy^{\frac{1}{2}}$  is continuous for all  $x$  and  $y \geq 0$
- $\frac{\partial f}{\partial y} = \frac{x}{2y}$  is continuous for all  $x$  and  $y > 0$ .
- ***Theorem does not give any conclusion.***

# Chapter 2

## First-Order Differential Equations

### 2.1 Solution Curves Without a Solution

Given a 1st order D.E.  $y' = f(x, y)$ ,  $y'$  is the slope of the tangent line at any point  $(x_0, y_0)$  on a solution curve

#### 2.1.1 – Example

$$y' = f(x, y) = x + y$$

- $f(0, 0)=0$
- $f(1, 0)=1$

### 2.1.2 – Slope/Direction Fields

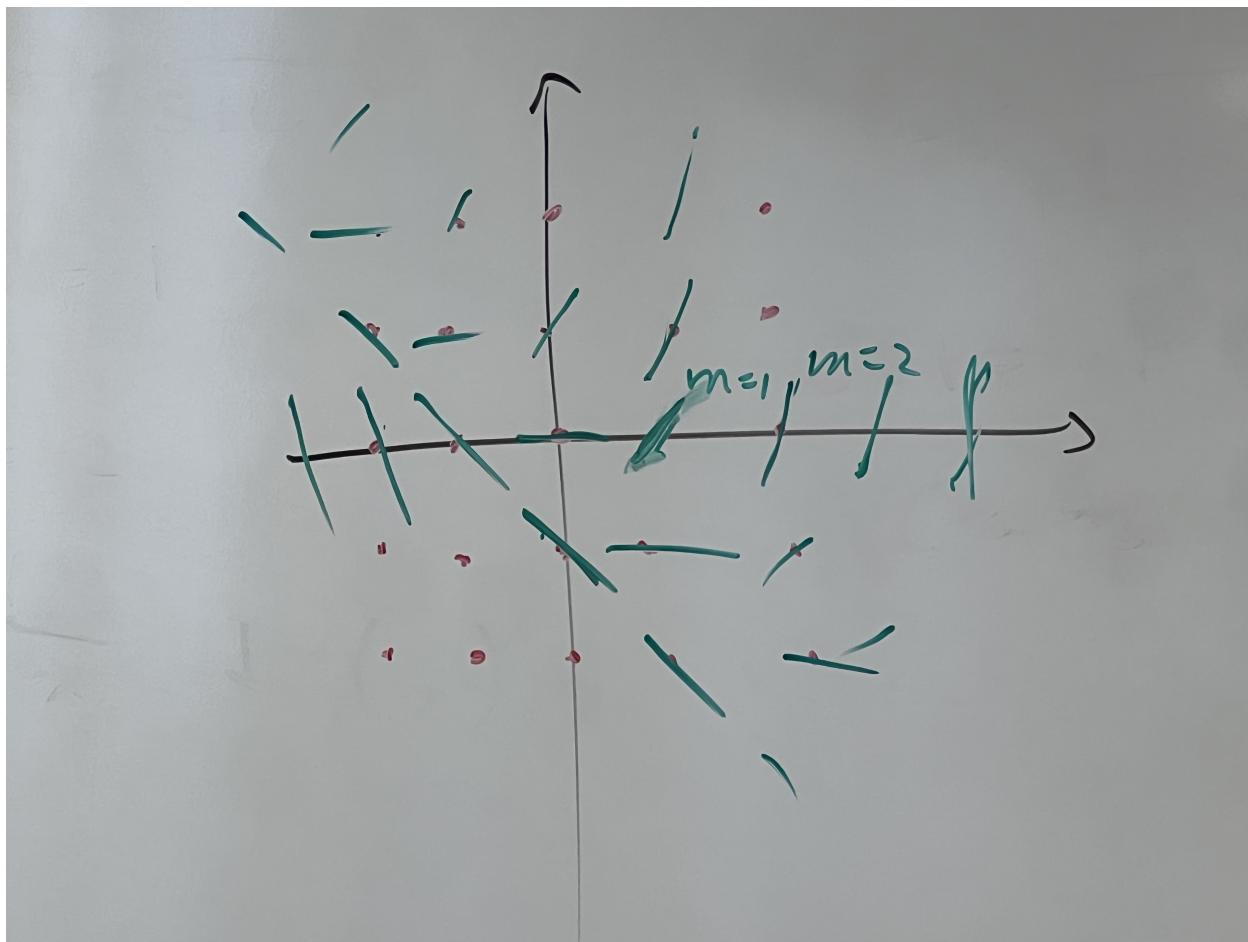


Figure 2.1: The direction field for the previous example

If the function  $f(x, y)$  in the D.E.  $y' = f(x, y)$  is reasonably simple so that we can solve  $f(x, y) = 0$ , we can make a “phase portrait diagram”. We will also assume  $f(x, y)$  only involves the  $y$ -variable.

### 2.1.3 – Example

$$\begin{aligned}y' &= (y + 2)(y - 3)(y - 5) \\f(x, y) &= (y + 2)(y - 3)(y - 5)\end{aligned}$$

An “equilibrium solution” is a solution where  $y$  is a constant. In this example:  $y = 3$ ,  $y = 5$ ,  $y = -2$  are each constant functions.

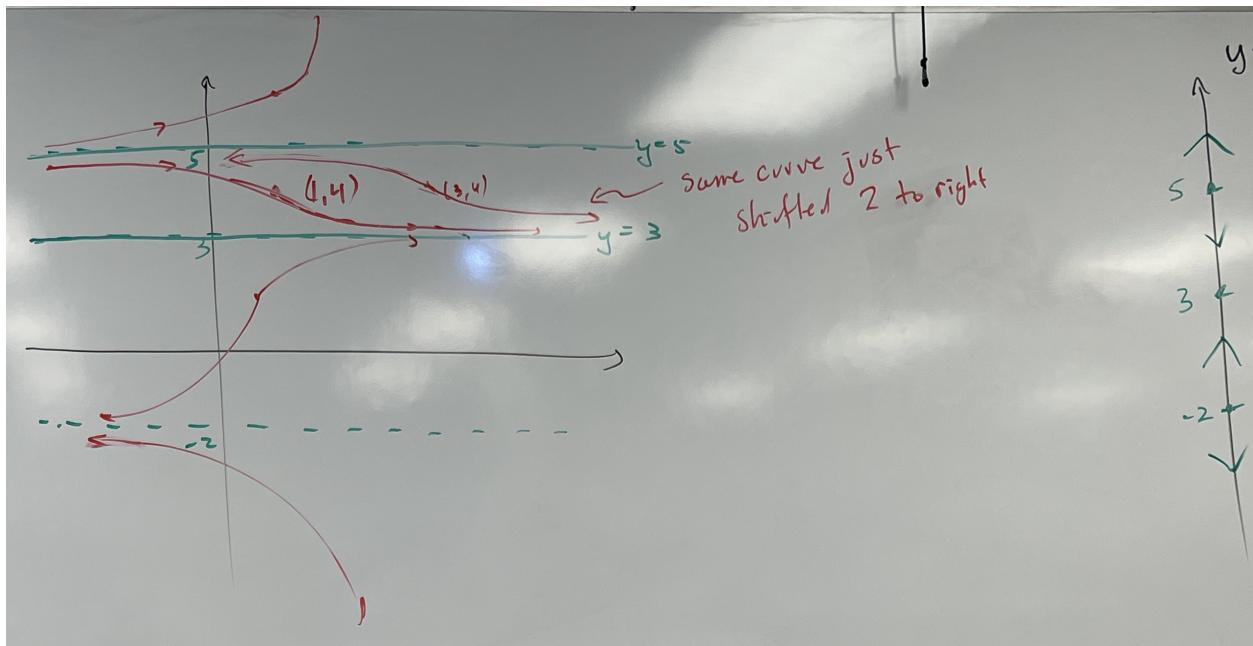


Figure 2.2: The equilibrium solution for the previous example.

The area around  $y = 5$  is an unstable equilibrium since the solutions diverge and go in separate directions away from  $y = 5$ . The area around  $y = 3$  is a stable equilibrium because the slopes above and below it converge to  $y = 3$ . The area around  $y = -2$  is semi-stable, since all the slopes around it will converge in one direction, but the point isn't always  $y = -2$ .

## 2.2 Separable Differential Equations

Separable D.E.s are DE's  $\frac{dy}{dx} = f(x, y)$  where  $f(x, y)$  can be factored as  $f(x, y) = g(x)h(y)$ .

$$\frac{dy}{dx} = (1 + y^2)x^3 \text{ is separable}$$

$$\frac{dy}{dx} = \sin(xy) \text{ is not separable}$$

$$\frac{dy}{dx} = x^3y \text{ is not separable}$$

$$\frac{5}{xy} \frac{dy}{dx} = (x^2 + y) e^y$$

$$\frac{dy}{dx} = \frac{xy(x^2 + y)e^y}{5}$$

$$= \frac{x(x^2 + y)}{5} \times ye^y$$

### 2.2.1 – Method of Solution

“Separate the variable” to get  $\frac{1}{h(y)}dy = g(x)d$  or  $p(y)dy = g(x)dx$  where  $p(y) = \frac{1}{h(y)}$ . Integrate both sides

$$\int p(y)dy = \int g(x)dx \text{ and if possible, solve for } y$$

### 2.2.2 – Example

$$\begin{aligned} \frac{dy}{dx} &= (1 + y^2)x^3 \\ \int \frac{1}{1 + y^2}dy &= \int x^3dx \\ \tan^{-1}(y) + C_1 &= \frac{x^4}{4} + C_2 \\ \tan^{-1}(y) &= \frac{x^4}{4} + C_2 - C_1 \\ \tan^{-1}(y) &= \frac{x^4}{4} + C \\ y &= \tan\left(\frac{x^4}{4} + C\right) \end{aligned}$$

### 2.2.3 – Example

Problem 12 from the textbook.

$$\begin{aligned} \sin(3x)dx + 2y \cos^3(3x)dy &= 0 \\ \int -2ydy &= \int \frac{\sin(3x)}{\cos^3(x)}dx \\ &= \int \tan(3x) \sec^2(3x)dx \\ &= \int u \frac{1}{3}du \text{ where } u = \tan(3x), \ du = 3 \sec^2(3x)dx \\ -2 \int ydy &= \frac{1}{3} \int u \ du + C \\ -y^2 &= \frac{u^2}{6} + C \\ &= \frac{\tan^2(3x)}{6} + C \\ \frac{\tan^2(3x)}{6} + y^2 &= -C \\ \frac{\tan^2(3x)}{6} + y^2 &= C \end{aligned}$$

Problem 25 from the textbook.

$$x^2 \frac{dy}{dx} = y - xy, y(-1) = -1$$

$$x^2 \frac{dy}{dx} = y - xy$$

$$x^2 \frac{dy}{dx} = y(1 - x)$$

$$\frac{dy}{y} = \frac{(1-x)}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{(1-x)}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{x^2} dx - \int \frac{x}{x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{x^2} dx - \int \frac{1}{x} dx$$

$$\ln |y| + C_1 = -\frac{1}{x} + C_2 - \ln |x| + C_3$$

$$\ln |y| = -\frac{1}{x} - \ln |x| + C$$

$$y = e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C$$

$$y = e^{-\frac{1}{x}} \times e^{-\ln|x|} \times e^C$$

$$y = e^{-\frac{1}{x}} \times \frac{1}{|x|} \times e^C$$

$$y = \frac{1}{|x|} e^{C-\frac{1}{x}}$$

$$-1 = \frac{1}{|-1|} e^{C-\frac{1}{-1}}$$

$$-1 = \frac{1}{1} e^{C-(-1)}$$

$$-1 = e^{C+1}$$

## 2.3 First Order Linear Differential Equations

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\left. \frac{dy}{dx} + P(x)y = f(x) \right\} \text{ Standard form of a 1st-order linear DE}$$

We will try to find a function  $\mu(x)$  such that by multiplying the D.E. by an integrating factor (I.F.)  $\mu(x)$ :

$$\mu(x) \frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

such that the LHS is an exact derivative, Observe:

$$\frac{d}{dx}(\mu(x)y) = \mu(x) \frac{dy}{dx} + \frac{dy}{dx}y$$

from which we see

$$\begin{aligned}\mu(x)P(x) &= \frac{d\mu}{dx} \\ P(x)dx &= \frac{d\mu}{\mu(x)} \\ \int P(x)dx &= \int \frac{d\mu}{\mu} \\ \int P(x)dx &= \ln \mu \\ \ln \mu &= \int P(x)dx \\ \mu &= e^{\int P(x)dx}\end{aligned}$$

### 2.3.1 – Example

$$\begin{aligned}x \frac{dy}{dx} - 4y &= x^6 e^x \\ \text{Standard form: } \frac{dy}{dx} - \frac{4}{x}y &= x^5 e^x \\ P(x) &= -\frac{4}{x} \\ \mu &= e^{\int -\frac{4}{x} dx} \\ &= e^{-4 \ln x} \\ &= e^{\ln x^{-4}} \\ &= x^{-4} \\ \text{I.F.} &= \mu = x^{-4}\end{aligned}$$

Now multiply the standard form of the given D.E. by  $x^{-4}$ .

$$\begin{aligned}x^{-4} \frac{dy}{dx} - x^{-4} \frac{4}{x}y &= x^{-4} x^5 e^x \\ x^{-4} \frac{dy}{dx} - x^{-4} \frac{4}{x}y &= x e^x \\ \int \frac{d}{dx}(x^{-4}y) &= \int x e^x \\ x^{-4}y &= \int x e^x\end{aligned}$$

### 2.3.2 – Example

$$\begin{aligned}
 & (x^2 - 9) \frac{dy}{dx} + xy = 0 \\
 & (x^2 - 9) \frac{dy}{dx} + xy = 0 \\
 & \frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0 \\
 & P(x) = \frac{x}{x^2 - 9} \\
 & \int P(x)dx = \int \frac{x}{x^2 - 9}dx \\
 & \int P(x)dx = \int \frac{1}{u - 9} \frac{du}{2} \\
 & \int P(x)dx = \frac{1}{2} \int \frac{1}{u - 9} du \\
 & \int P(x)dx = \frac{1}{2} \ln|u - 9| \\
 & \int P(x)dx = \frac{1}{2} \ln|x^2 - 9| \\
 & \mu = e^{\frac{1}{2} \ln|x^2 - 9|} \\
 & \mu = e^{\ln|(x^2 - 9)^{\frac{1}{2}}|} \\
 & \mu = (x^2 - 9)^{\frac{1}{2}} \\
 & \mu = \sqrt{x^2 - 9} \\
 & \sqrt{x^2 - 9} \left( \frac{dy}{dx} + \frac{x}{x^2 - 9}y \right) = \sqrt{x^2 - 9}(0) \\
 & \sqrt{x^2 - 9} \frac{dy}{dx} + \frac{x}{\sqrt{x^2 - 9}}y = 0 \\
 & \int \frac{d}{dx} \left( y\sqrt{x^2 - 9} \right) = \int 0 \\
 & y\sqrt{x^2 - 9} = C \\
 & y = \frac{C}{\sqrt{x^2 - 9}}
 \end{aligned}$$

## 2.4 Exact Equations

1st Order D.E. in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

Given a function

$$z = f(x, y)$$

, the total differential,  $dz$ , is defined as

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

### 2.4.1 – Method

See if we can find a function  $f(x, y)$  such that

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

If we can do this, then the D.E. is equivalent to

$$df = 0 \Rightarrow f(x, y) = c$$

is an implicit solution of D.E.

Assume that  $M$  and  $N$  have continuous 1st order partials (assuming  $f$  exists)

$$\left. \begin{array}{l} My = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} dy = f_{xy} \\ Nx = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} dy = f_{yx} \end{array} \right\} \text{Theorem tells us these are equal}$$

This provides a quick test to check if the D.E. is exact or not.

### 2.4.2 – Example

$$2xydx + (x^2 - 1)dy = 0$$

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

To check if the D.E. is exact

$$M_y = 2x = N_x$$

We now know there exists a function  $f(x, y)$  with

$$\begin{aligned} \frac{\partial f}{\partial x} &= M = 2xy \\ \frac{\partial f}{\partial y} &= N = x^2 - 1 \end{aligned}$$

$$\begin{aligned}
 f_M(x, y) &= \int \frac{\partial f}{\partial x} dx \\
 &= \int 2xy dx \\
 &= x^2y + \phi(y) \\
 \frac{\partial f}{\partial y} (x^2y + \phi(y)) &= x^2 - 1 \text{ required to equal } N \\
 x^2 + \phi'(y) &= x^2 - 1 \\
 \phi'(y) &= -1 \\
 \phi(y) &= \int -1 dy \\
 &= -y \\
 f(x, y) &= x^2y - y \\
 d(f(x, y)) &= 0 \\
 f(x, y) &= c \\
 x^2y - y = c &\text{ is an implicit solution of the D.E.}
 \end{aligned}$$

Note: the  $f_M$  format is just there to show which partial equation was integrated. It was made by me and, as far as I know, is not standardly known.

### 2.4.3 – Example

$$\begin{aligned}
 (e^{2y} - y \cos(xy)) dx + (2xe^{2y} - x \cos(xy) + 2y) dy &= 0 \\
 M_y &= N_x \\
 \frac{\partial}{\partial y} (e^{2y} - y \cos(xy)) &= \frac{\partial}{\partial x} (2xe^{2y} - x \cos(xy) + 2y) \\
 2e^{2y} - [\cos(xy) - y \sin(xy) \times x] &= 2e^{2y} - (\cos(xy) - x \sin(xy) \times y) + 0 \\
 2e^{2y} - \cos(xy) + xy \sin(xy) &= 2e^{2y} - \cos(xy) + xy \sin(xy) \\
 \frac{\partial f}{\partial x} &= M = e^{2y} - y \cos(xy) \\
 \frac{\partial f}{\partial y} &= N = 2xe^{2y} - x \cos(xy) + 2y \\
 f_N(x, y) &= \int \frac{\partial f}{\partial y} dy \\
 &= \int (2xe^{2y} - x \cos(xy) + 2y) dy \\
 &= \frac{2xe^{2y}}{2} - \frac{x \sin(xy)}{x} + 2 \times \frac{y^2}{2} + \phi(x) \\
 &= xe^{2y} - \sin(xy) + y^2 + \phi(x)
 \end{aligned}$$

Take the  $\partial x$  of this and equate with  $M$ :

$$\begin{aligned} M &= \frac{\partial}{\partial x} (xe^{2y} - \sin(xy) + y^2 + \phi(x)) \\ e^{2y} - y \cos(xy) &= e^{2y} - y \cos(xy) + 0 + \phi'(x) \\ 0 &= \phi'(x) \\ \phi(x) &= c \end{aligned}$$

So  $f(x, y) = c_2$  is the solution

$$xe^{2y} - \sin(xy) + y^2 = c$$

$$dx = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

#### 2.4.4 – What can you do if $M_y \neq N_x$

Sometimes you can multiply the DE by an integrating factor  $\mu(x, y)$  to get an exact DE.

If

$$\frac{M_y - N_x}{N}$$

is a function of only  $x$ , then

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

will be an I.F.

If

$$\frac{N_x - M_y}{M}$$

is a function of only  $y$ , then

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

will be an I.F.

#### 2.4.5 – Example

$$xydx + (2x^2 + 3y^2 - 20) dy = 0$$

$$M_y = x$$

$$N_x = 4x$$

$$M_y \neq N_x$$

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy}$$

$$= \frac{3x}{xy}$$

$$= \frac{3}{y} \text{ is a function of just } y$$

So:

$$\begin{aligned}
 \mu &= e^{\int \frac{3}{y} dy} \\
 &= e^{3 \ln y} \\
 &= y^3 \\
 xy^4 dx + y^3 (2x^2 + 3y^2 - 20) dy &= 0(y^3) \\
 xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3) dy &= \\
 M_y &= N_x \\
 4xy^3 &= 4xy^3 \\
 \frac{\partial f}{\partial x} &
 \end{aligned}$$

## 2.5 Substitution Methods

Taking a D.E. that's not:

- Separable
- 1st Order Linear
- Exact

and making a substitution to turn the new D.E. into one of these.

**Theorem:** Given a D.E.

$$M(x, y)dx + N(x, y)dy = 0$$

A function  $f(x, y)$  is said to be homogenous of order  $\alpha$  if  $f(tx, ty) = t^\alpha f(x, y)$ .

### 2.5.1 – Example

Given:

$$f(x, y) = x^3 + 5xy^2 - y^3$$

Then:

$$\begin{aligned}
 f(tx, ty) &= (tx)^3 + 5(tx)(ty)^2 - (ty)^3 \\
 &= t^3x^3 + 5t^3xy^2 - t^3y^3 \\
 &= t^3(x^3 + 5xy^2 - y^3) \\
 &= t^3f(x, y)
 \end{aligned}$$

### 2.5.2 – Example

$$\begin{aligned}
 f(x, y) &= \frac{x + y}{x^2 + y^2} \\
 f(tx, ty) &= \frac{tx + ty}{(tx)^2 + (ty)^2} \\
 f(tx, ty) &= \frac{tx + ty}{x^2t^2 + y^2t^2} \\
 f(tx, ty) &= \frac{t}{t^2} \times \frac{x + y}{x^2 + y^2} \\
 f(tx, ty) &= \frac{t}{t^2} f(x, y) \\
 f(tx, ty) &= \frac{1}{t} f(x, y)
 \end{aligned}$$

$f(x, y) = \frac{x+y}{x^2+y^2}$  is homogenous of order  $\alpha = -1$

### 2.5.3 – Substitution Rule

If  $M(x, y)$  and  $N(x, y)$  are homogenous, each of the same order, then  $u = \frac{y}{x}$  i.e.,  $y = ux$  or  $v = \frac{x}{y}$  (i.e.  $x = vy$ ) will produce a separable D.E.

### 2.5.4 – Example

Solve the separable D.E. and then back-substitute

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

$$\begin{aligned}
 M(x, y) &= x^2 + y^2 & N &= x^2 - xy \\
 M_y &= 2y & N_x &= 2x - y \\
 M_y &\neq N_x \\
 M(tx, ty) &= (tx)^2 + (ty)^2 \\
 &= t^2x^2 + t^2y^2 \\
 &= t^2(x^2 + y^2) \\
 &= t^2M(x, y) \quad M \text{ is homogeneous of order 2 and so is } N \\
 u &= \frac{y}{x} \\
 y &= ux \\
 dy &= udx + xdu \\
 (x^2 + (ux)^2)dx + (x^2 - x(ux))(udx + xdu) &= 0 \\
 (x^2 + u^2x^2)dx + (x^2 - ux^2)(udx + xdu) &= 0 \\
 (1 + u^2)x^2dx + x^2(1 - u)(udx + xdu) &= 0 \\
 (1 + u^2)x^2dx + x^2(udx + xdu - u^2dx - uxdx) &= 0 \\
 x^2(1dx + u^2dx + udx + xdu - u^2dx - uxdx) &= 0 \\
 x^2(1dx + u^2dx - u^2dx + udx + xdu - uxdx) &= 0 \\
 x^2(1dx + udx + xdu - uxdx) &= 0 \\
 x^2(1 + u)dx + x^3(1 - u)du &= 0 \\
 \int \frac{1}{x}dx &= \int -\frac{1-u}{1+u}du \\
 &= \int \frac{u-1}{u+1}du \\
 &= \int \frac{u+(1-2)}{u+1}du \\
 &= \int \left( \frac{u+1}{u+1} - \frac{2}{u+1} \right) du \\
 &= \int \left( 1 - \frac{2}{u+1} \right) du \\
 \ln|x| &= \int \left( 1 - \frac{2}{u+1} \right) du \\
 &= u - 2 \ln|u+1| + C \\
 \ln|x| &= \frac{y}{x} - 2 \ln \left| \frac{y}{x} + 1 \right| + C
 \end{aligned}$$

### 2.5.5 – Bernoulli Equation

**Theorem:** An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where  $n \neq 0, 1$  is called a Bernoulli Equation. The substitution

$$u = y^{1-n}$$

will transform the D.E. into a 1st order linear.

### 2.5.6 – Example

$$\begin{aligned} x \frac{dy}{dx} + y &= x^2 y^2 \\ \frac{dy}{dx} + \frac{y}{x} &= x y^2 \end{aligned}$$

is a **Bernoulli equation** with  $n = 2$ .

$$\begin{aligned} u &= y^{1-2} \\ &= y^{-1} \\ &= \frac{1}{y} \\ \frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\ &= -1 y^{-2} \frac{dy}{dx} \\ &= -\frac{1}{y^2} \frac{dy}{dx} \\ -y^{-2} \frac{dy}{dx} + -y^{-2} \times \frac{y}{x} &= -y^{-2} \times x y^2 \\ -y^{-2} \frac{dy}{dx} + -\frac{1}{x} y^{-1} &= -x \\ \frac{du}{dx} - \frac{1}{x} u &= -x \end{aligned}$$

$$\begin{aligned}
\text{I.F.} &= \mu = e^{P(x)dx} \\
&= e^{-\int \frac{1}{x} dx} \\
&= e^{-\ln|x|} \\
&= e^{\ln|x^{-1}|} \\
&= x^{-1} \\
\frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u &= -1 \\
\frac{d}{dx} \left( \frac{1}{x} u \right) &= -1 \\
\int \frac{d}{dx} \left( \frac{1}{x} u \right) dx &= \int -1 dx \\
\frac{1}{x} u &= \int -1 dx \\
\frac{1}{x} u &= -x + C \\
\frac{1}{x} \times 1y &= -x + C \\
\frac{1}{x(-x+C)} &= y \\
y &= \frac{1}{Cx-x^2}
\end{aligned}$$

**Theorem:** If the D.E. can be expressed as

$$\frac{dy}{dx} = f(Ax + by + C)$$

for particular numbers  $A, B, C$ , then let

$$u = Ax + By + C$$

to get a separable D.E.

### 2.5.7 – Example

$$\frac{dy}{dx} = (-2x + y)^2 - 7, y(0) = 0$$

$$\begin{aligned} u &= -2x + y \\ \frac{du}{dx} &= \frac{dy}{dx} \times \frac{du}{dy} \\ &= -2 + \frac{dy}{dx} \\ \frac{du}{dx} + 2 &= \frac{dy}{dx} \\ \frac{du}{dx} + 2 &= u^2 - 7 \\ \frac{du}{dx} &= u^2 - 9 \\ \frac{du}{u^2 - 9} &= dx \\ \int \frac{du}{u^2 - 9} &= \int dx \\ \int \frac{du}{(u+3)(u-3)} &= x + C \\ \int \frac{du}{(u+3)(u-3)} &= x + C \end{aligned}$$

# Chapter 3

## Modeling using DEs

### 3.1 Linear DE Modeling

#### 3.1.1 – Standard Problems

- 1) Population Growth (or decline)
- 2) Radioactive Decay
- 3) Newton's Law of Cooling
- 4) Mixture Problems

#### 3.1.2 – Population Model

Assume the rate of population change is proportional to the size of the population

$$P(t) = \text{population at time } t$$

$$\frac{dP}{dt} = kP$$

$\frac{dP}{dt} / P = k$  is the relative growth rate of the population

$$\begin{aligned}
 \frac{dP}{dt} &= kP \\
 \frac{dP}{P} &= kdt \\
 \int \frac{dP}{P} &= \int kdt \\
 \ln|P| &= kt + C \\
 |P| &= e^{kt+C} \\
 |P| &= e^{kt}e^C \\
 |P| &= Ae^{kt} \text{ where } A > 0 \\
 P &= \pm Ae^{kt} \\
 P &= Be^{kt} \text{ where } B \neq 0 \\
 P &= De^{kt} \text{ where } D \text{ can be any real number}
 \end{aligned}$$

The constant can become any number because 0 would be a valid rate of population change, it means that the population size isn't changing.

### 3.1.3 – Example

If, initially at 2 p.m., there are 1,000 bacteria on a petri dish and at 4 p.m., there are 2,000 bacteria. Assuming constant relative growth rate, how many bacteria are there at 5 p.m.?  $P(t)$  = population  $t$  hours after 2 p.m.

$$\begin{aligned}
 P(t) &= Ae^{kt} \\
 1000 &= Ae^{(0)k} \\
 1000 &= Ae^0 \\
 1000 &= A(1) \\
 A &= 1000
 \end{aligned}$$

$$\begin{aligned}
 P(2) &= 2000 \\
 P(2) &= 1000e^{2k} \\
 2000 &= 1000e^{2k} \\
 2 &= e^{2k} \\
 \ln(2) &= 2k \\
 k &= \frac{\ln(2)}{2}
 \end{aligned}$$

$$\begin{aligned}
P(t) &= 1000e^{\frac{\ln(2)}{2}t} \\
P(3) &= 1000e^{\frac{\ln(2)}{2}(3)} \\
&= 1000e^{1.5\ln(2)} \\
&= 1000e^{\ln(2^{1.5})} \\
&= 1000(2^{1.5}) \\
&= 2000(\sqrt{2}) \\
P(3) &\approx 2828.427(\sqrt{2}) \\
P(t) &= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\frac{t}{2}\ln(2)} \\
&= 1000e^{\ln(2^{\frac{t}{2}})} \\
&= 1000 \times 2^{\frac{t}{2}}
\end{aligned}$$

### 3.1.4 – Radioactive Decay

$$m(t) = m_0 e^{kt} \text{ where } k < 0$$

The Half-Life is the amount of time it takes for half of the original amount to remain:

$$\frac{1}{2}A_0 = A_0 e^{kt} \Rightarrow \frac{1}{2} = e^{kt}$$

### 3.1.5 – Mixture Problems

#### Setup

Initially, the container has 200 gallons of brine solution (salt-water) of concentration  $\frac{10 \text{ lbs}}{200 \text{ gallons}} = 0.05 \frac{\text{lbs}}{\text{gallon}}$ . A solution of  $\frac{5 \text{ lbs}}{200 \text{ gallons}} = 0.025 \frac{\text{lbs}}{\text{gallon}}$  is poured into the initial container at a rate of  $\frac{4 \text{ gallons}}{\text{min}}$ . How many pounds of salt are there in the container after 2 hours.

Let  $A(t) = \# \text{ lbs of salt } t \text{ minutes after the process starts}$

$\frac{dA}{dt}$  = The rate of change of # lbs of salt

$$\begin{aligned} \frac{dA}{dt} &= 0.025 \frac{\text{lbs}}{\text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \Bigg\} \text{rate in} \\ &\quad - \frac{A(t) \text{lbs}}{200 \text{gal}} \times 4 \frac{\text{gal}}{\text{min}} \Bigg\} \text{rate out} \\ &= (0.025)4 \frac{\text{lbs}}{\text{min}} - \frac{4A(t)}{200} \frac{\text{lbs}}{\text{min}} \\ &= 0.1 \frac{\text{lbs}}{\text{min}} - \frac{A(t)}{50} \frac{\text{lbs}}{\text{min}} \\ &= 0.1 - \frac{A(t)}{50} \end{aligned}$$

$$\begin{aligned} \frac{dA}{dt} + \frac{1}{50}A &= 0.1 \\ \mu &= e^{\int P(t)dt} \\ &= e^{\int \frac{1}{50}dt} \\ &= e^{\frac{t}{50}} \end{aligned}$$

$$e^{\frac{t}{50}} \left( \frac{dA}{dt} \right) + e^{\frac{t}{50}} \left( \frac{1}{50}A \right) = e^{\frac{t}{50}}(0.1)$$

$$\begin{aligned} \frac{d}{dt} \left( e^{\frac{t}{50}} A \right) &= e^{\frac{t}{50}}(0.1) \\ \int \frac{d}{dt} \left( e^{\frac{t}{50}} A \right) dt &= \int \frac{1}{10} e^{\frac{t}{50}} dt \end{aligned}$$

$$e^{\frac{t}{50}} A = \frac{1}{10} \times \frac{e^{\frac{t}{50}}}{\frac{1}{50}} + C$$

$$e^{\frac{t}{50}} A = 5e^{\frac{t}{50}} + C$$

$$\begin{aligned} A(t) &= 5 + C e^{-\frac{t}{50}} \\ &= 5 + C e^{-0.02t} \end{aligned}$$

$$\begin{aligned} A(120) &= 5 + C e^{-0.02(120)} \\ &= 5 + C e^{-2.4} \end{aligned}$$

# Chapter 4

## Higher Order Differential Equations

### 4.1 Linear Equations

An  $n$ th order DE is linear if it has the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1(x) \frac{dy}{dx} + a_0 y = g(x)$$

**Theorem:** If all the coefficient functions are continuous and  $a_n(x)$  is not 0 on an interval  $I$  and  $g(x)$  is continuous, then any initial value problem

$$DE + y(x_0) = y_0$$

has a unique solution on the interval  $I$  if  $g(x) = 0$ . i.e.

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

then the DE is said to be homogeneous.

#### 4.1.1 – Example

$$y'' - 3y' - 4y = 0$$

Show  $y_1 = e^{4x}$  is a solution and  $y_2 = e^{-x}$  is a solution.

$$y_1 = e^{4x}$$

$$y'_1 = 4e^{4x}$$

$$y''_1 = 16e^{4x}$$

$$16e^{4x} - 3(4e^{4x}) - 4e^{4x} = 0$$

$$16e^{4x} - 12e^{4x} - 4e^{4x} = 0$$

$$e^{4x}(16 - 12 - 4) = 0$$

$$e^{4x}(0) = 0$$

$$0 = 0$$

$$y_3 = 6y_1 = 6e^{4x}$$

$$y'_3 = 6y'_1 = 24e^{4x}$$

$$y''_3 = 6y''_1 = 96e^{4x}$$

$$96e^{4x} - 3(24e^{4x}) - 4(6e^{4x}) = 0$$

$$96e^{4x} - 72e^{4x} - 24e^{4x} = 0$$

$$e^{4x}(96 - 72 - 24) = 0$$

$$e^{4x}(0) = 0$$

$$0 = 0$$

**Theorem:** Superposition Principle: if  $y_1, y_2, \dots, y_m$  are each solutions of an  $n$ th order Linear, homogenous DE, then  $c_1y_1 + c_2y_2 + \dots + c_my_m$  will also be a solution for any constants  $c_1, c_2, \dots, c_m$ .

Our goal is to express the general solution in as concise a way as possible.

**Linear combination** – a collection of solutions  $y_1, y_2, \dots, y_m$  is linearly independent if the only way  $c_1y_1 + c_2y_2 + \dots + c_my_m = 0$  is iff (if and only if) all of the constants  $c_1, c_2, \dots, c_m = 0$ . Otherwise we say  $y_1, y_2, \dots, y_m$  are linearly dependent.

**Theorem:** If the DE is an  $n$ th order Linear Homogeneous equation then there will exist a collection of  $n$  linearly independent solutions  $y_1, y_2, \dots, y_n$  and the general solution will be  $y_c = c_1y_1 + c_2y_2 + \dots + c_ny_n$

One way to check for linear independence is to compile the Wronskian

$$W(y_1, y_2, \dots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

## 4.2 Reduction of Order

If you have one solution to a 2<sup>nd</sup> order linear homogenous DE, then it is possible to use that function to construct a 2<sup>nd</sup> Linear Independent solution to the DE.

### 4.2.1 – Example

For example, the DE

$$y'' - y = 0$$

One solution is  $y = e^x$  on  $(-\infty, \infty)$ .

Idea: We look for  $y_2$  of the form

$$y_2(x) = u(x)y_1(x) \text{ where } u(x) \text{ is not a constant}$$

The general solution is of the form:

$$y = c_1y_1 + c_2y_2$$

where  $y_1$  and  $y_2$  are linearly independent solutions.

To find  $u(x)$ , we substitute this into the DE

$$\begin{aligned} y_2 &= u(x)y_1(x) \\ y'_2 &= u(x)y'_1(x) + u'(x)y_2(x) \\ y''_2 &= u(x)y''_1(x) + u'(x)y_1(x) + u'(x)y'_2(x) + u''(x)y_1(x) \\ &= uy''_1 + 2u'y'_1 + u''y_1 \end{aligned}$$

So  $y'' - y = 0$  becomes

$$\begin{aligned} uy''_1 + 2u'y'_1 + u''y_1 - uy_1 &= 0 \text{ when we sub } y = y_2 = u_{y_1} \\ u(e^x)'' + 2u'(e^x)' + u''(e^x) - u(e^x) &= 0 \\ ue^x + 2u'e^x + u''e^x - ue^x &= 0 \\ 2u'e^x + u''e^x &= 0 \\ e^x(2u' + u'') &= 0 \\ 2u' + u'' &= 0 \end{aligned}$$

Let  $w = u'$

$$\begin{aligned} 2w + w' &= 0 \\ 2w + \frac{dw}{dx} &= 0 \\ \frac{dw}{dx} &= -2w \\ \frac{dw}{w} &= -2dx \\ \int \frac{dw}{w} &= \int -2dx \\ \ln|w| &= -2x \\ w &= e^{-2x} \\ u' &= e^{-2x} \\ \int u' &= \int e^{-2x} \\ u &= -\frac{1}{2}e^{-2x} \\ y_2 &= uy_1 \\ &= -\frac{1}{2}e^{-2x} \times e^x \\ &= -\frac{1}{2}e^{-x} \end{aligned}$$

**Double check that  $y_2$  is a solution of the DE**

$$\begin{aligned}y_2 &= -\frac{1}{2}e^{-x} \\y'_2 &= \frac{1}{2}e^{-x} \\y''_2 &= -\frac{1}{2}e^{-x} \\y''_2 - y &= -\frac{1}{2}e^{-x} - \left(-\frac{1}{2}e^{-x}\right) \\&= -\frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} \\&= 0\end{aligned}$$

In general,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

put into standard form by dividing by  $a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

where  $P(x) = \frac{a_1(x)}{a_2(x)}$  and  $Q(x) = \frac{a_0(x)}{a_2(x)}$ , the same method as in our **example** leads to the formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \quad (4.1)$$

## 4.2.2 – Example

### Part 1

$$x^2y'' - 3xy' + 4y = 0$$

Verify that  $y_1 = x^2$  is a solution  $y'_1 = 2x, y''_1 = 2$ .

$$\begin{aligned}x^2y'' - 3xy' + 4y &= 0 \\x^2(2) - 3x(2x) + 4(x^2) &= 0 \\2x^2 - 6x^2 + 4x^2 &= 0 \\6x^2 - 6x^2 &= 0 \\0 &= 0\end{aligned}$$

**Part 2**

Find a linearly independent solution  $y_2(x)$ .

$$\begin{aligned}x^2y'' - 3xy' + 4y &= 0 \\y'' - \frac{3}{x}y' + \frac{4}{x^2}y &= 0 \\P(x) &= -\frac{3}{x} \\y_2 &= y_1 \int \frac{e^{\int \frac{3}{x} dx}}{(y_1(x))^2} dx \\y_2 &= y_1 \int \frac{e^{3 \ln|x|}}{(y_1(x))^2} dx \\y_2 &= y_1 \int \frac{e^{\ln|x^3|}}{(y_1(x))^2} dx \\y_2 &= x^2 \int \frac{x^3}{(x^2)^2} dx \\y_2 &= x^2 \int \frac{x^3}{x^4} dx \\y_2 &= x^2 \int \frac{1}{x} dx \\y_2 &= x^2 \ln|x| + C\end{aligned}$$

**Part 3: Double check that  $y_2$  is a solution of the DE**

$$\begin{aligned}y_2 &= x^2 \ln|x| \\y'_2 &= x^2 \times \frac{1}{x} + 2x \ln|x| \\y''_2 &= 1 + 2x \frac{1}{x} + 2 \ln|x| \\&= 1 + 2 + 2 \ln|x| \\&= 3 + 2 \ln|x|\end{aligned}$$

So the LHS DE becomes

$$\begin{aligned}x^2(3 + 2 \ln|x|) - 3x(x + 2x \ln|x|) + 4x^2 \ln|x| &= 3x^2 + 2x^2 \ln|x| - 3x^2 - 6x^2 \ln|x| + 4x^2 \ln|x| \\&= 3x^2 - 3x^2 + 2x^2 \ln|x| - 6x^2 \ln|x| + 4x^2 \ln|x| \\&= 3x^2 - 3x^2 + 2x^2 \ln|x| - 6x^2 \ln|x| + 4x^2 \ln|x| \\&= 0 + x^2 \ln|x|(2 - 6 + 4) \\&= x^2 \ln|x|(0) \\&= 0\end{aligned}$$

Write the general solution of the DE including the interval of the solution

$$\begin{aligned}
 y &= c_1 y_1 + c_2 y_2 \\
 &= c_1 x^2 + c_2 x^2 (\ln |x| + C) \\
 &= c_1 x^2 + c_2 x^2 \ln |x| + C c_2 x^2 \\
 \text{just } y &= c_1 x^2 + c_2 x^2 \ln |x| \text{ on } I = (0, \infty), y(2) = 3, y'(2) = 5
 \end{aligned}$$

### 4.2.3 – Example

$$\begin{aligned}
 3y'' + y' - 4y &= 0 \\
 y &= e^{mx} \\
 y' &= me^{mx} \\
 y'' &= m^2 e^{mx} \\
 3y'' + y' - 4y &= 3m^2 e^{mx} + me^{mx} - 4e^{mx} \\
 &= e^{mx}(3m^2 + m - 4) \\
 &= e^{mx}(3m^2 + 4)(m - 1) \\
 m = 1 \quad m &= -\frac{4}{3} \\
 y_1 &= e^x, y_2 = e^{-\frac{4}{3}x}
 \end{aligned}$$

## 4.3 Higher Order, Linear, Homogeneous DE with Constant Coefficients

### 4.3.1 – Example

$$3y^{(4)} - 2y''' + 7y' + 8y = 0$$

Theorems in 4.1 tell us that the general solution is of the form  $y = c_1 y_1$ . **Conjecture:** A solution of the form  $y = e^{mx} \Rightarrow y' = me^{mx}$ .

### 4.3.2 – Example

$$\begin{aligned}
 5y' - 4y &= 0 \\
 y' - \frac{4}{5}y &= 0 \\
 me^{mx} - \frac{4}{5}e^{mx} &= 0 \\
 e^{mx} \left( m - \frac{4}{5} \right) &= 0 \\
 m - \frac{4}{5} &= 0 \\
 m &= \frac{4}{5}
 \end{aligned}$$

$y = c_1 e^{\frac{4}{5}x}$  is the general solution of the DE

### 4.3.3 – Example

$$\begin{aligned}
 y'' + 5y' - 6y &= 0 \\
 y(m^2 e^{mx}) + 5(me^{mx}) - 6e^{mx} &= 0 \\
 e^{mx} (m^2 y + 5m - 6) &= 0 \\
 m^2 y + 5m - 6 &= 0 \\
 (m + 6)(m - 1) &= 0 \\
 m + 6 = 0 &\quad m - 1 = 0 \\
 m = -6 &\quad m = 1 \\
 y_1 = e^{-6x} &\quad y_2 = e^x \text{ These are Linearly Independent (L.I)}
 \end{aligned}$$

Therefore:

$$y = c_1 e^{-6x} + c_2 e^x$$

### 4.3.4 – Example

$$\begin{aligned}
 y'' - 6y' + 9y &= 0 \\
 m^2 e^{mx} - 6(me^{mx}) + 9e^{mx} &= 0 \\
 m^2 - 6m + 9 &= 0 \\
 (m - 3)^2 &= 0 \quad m = 3 \text{ is a repeated root} \\
 m - 3 &= 0 \\
 m &= 3 \\
 y_1 = e^{3x} &\quad y_2 = e^{3x} \text{ are linearly dependent}
 \end{aligned}$$

Use the Reduction of order function:

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \\
&= e^{3x} \int \frac{e^{-\int -6dx}}{(e^{3x})^2} dx \\
&= e^{3x} \int \frac{e^{\int 6dx}}{e^{6x}} dx \\
&= e^{3x} \int \frac{e^{6x}}{e^{6x}} dx \\
&= e^{3x} \int 1 dx \\
&= e^{3x} x \\
&= xe^{3x}
\end{aligned}$$

Always works out for this solution if  $e^{m_1 x}$  is a solution and  $m_1$  is a root of multiplicity  $k$  than  $y_1 = e^{m_1 x}, y_2 = xe^{m_1 x}, \dots, y_k = x^{k-1}e^{m_1 x}$  are linear solutions.

$$y'' + 9y = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \sqrt{-9} \text{ No real solutions}$$

$$m = \pm\sqrt{-9}$$

$$m = \pm 3i$$

$$y = c_1 e^{3ix} + c_2 e^{-3ix} \text{ where } c_1 \& c_2 \text{ arbitrary complex numbers}$$

We'd rather only deal with real-valued solutions.

#### 4.3.5 – Euler's Formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$e^{i3x} = \cos(3x) + i \sin(3x)$$

$$e^{-i3x} = \cos(-3x) + i \sin(-3x)$$

$$e^{-i3x} = \cos(3x) - i \sin(3x)$$

$$e^{i3x} + e^{-i3x} = \cos(3x) + i \sin(3x) + \cos(3x) - i \sin(3x)$$

$$e^{i3x} + e^{-i3x} = 2 \cos(3x)$$

$$Y_1 = \frac{1}{2}e^{i3x} + \frac{1}{2}e^{-i3x} = \cos(3x)$$

$$Y_2 = \sin(3x)$$

$$\frac{1}{2i}y_1 - \frac{1}{2i}y_2 = \sin(3x)$$

General solution:

$$\begin{aligned}
y &= C_1 Y_1 + C_2 Y_2 \\
&= C_1 \cos(3x) + C_2 \sin(3x)
\end{aligned}$$

where  $C_1$  and  $C_2$  are complex numbers that generate all complex-valued solutions of the DE

### 4.3.6 – Example

$$\begin{aligned}
 y'' + 25y &= 0 \\
 m^2 e^{mx} + 25e^{mx} &= 0 \\
 m^2 + 25 &= 0 \\
 m^2 &= -25 \\
 m &= \pm 5i \\
 \text{General solution} \\
 y_1 &= c_1 y_1 + c_2 y_2 \\
 &= c_1 \cos(5x) + c_2 \sin(5x)
 \end{aligned}$$

### 4.3.7 – Example

$$\begin{aligned}
 y'' + 2y' + 6y &= 0 \\
 m^2 + 2m + 6 &= 0 \\
 \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-2 \pm \sqrt{2^2 - 4(1)(6)}}{2(1)} \\
 &= \frac{-2 \pm \sqrt{4 - 24}}{2} \\
 &= \frac{-2 \pm \sqrt{-20}}{2} \\
 &= \frac{-2 \pm \sqrt{4 \times -5}}{2} \\
 &= \frac{-2 \pm 2\sqrt{-5}}{2} \\
 &= -1 \pm \sqrt{-5} \\
 &= -1 \pm \sqrt{5}i \\
 y_1 &= e^{(-1+\sqrt{5}i)x} \\
 &= e^{-x} e^{i\sqrt{5}x} \\
 &= e^{-x} \cos(\sqrt{5}x) \\
 y_2 &= e^{(-1-\sqrt{5}i)x} \\
 &= e^{-x} e^{-i\sqrt{5}x} \\
 &= e^{-x} \sin(\sqrt{5}x)
 \end{aligned}$$

So the general solution is

$$y = c_1 e^{-x} \cos(\sqrt{5}x) + c_2 e^{-x} \sin(\sqrt{5}x)$$

In general, if  $m_1 = \alpha + i\beta$ ,  $m_2 = \alpha - i\beta$  are roots of the auxiliary equation, then  $y_1 = e^{\alpha x} \cos(\beta x)$   
 $y_2 = e^{\alpha x} \sin(\beta x)$  are solutions.

### 4.3.8 – Example

$$\begin{aligned} y^{(4)} - 16y &= 0 \\ m^4 - 16 &= 0 \\ (m^2 - 4)(m^2 + 4) &= 0 \\ (m - 2)(m + 2)(m^2 + 4) &= 0 \\ m = 2 : y_1 &= e^{2x} \\ m = -2 : y_1 &= e^{-2x} \\ m = 2i : \cos(2x), \sin(2x) & \end{aligned}$$

## 4.4 Nonhomogeneous, Linear DE with Constant Coefficients

### 4.4.1 – Method of Undetermined Coefficients

Section 4.5 gives another approach but it is a bit more abstract

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1(x) \frac{dy}{dx} + a_0 y = g(x) \text{ where } g(x) \neq 0$$

**Theorem:** If we can find any one particular solution  $y_p$  of this DE ( $y_p + y_c$ ), where  $y_c$  is the solution of the complementary DE (the same LHS= 0 instead of  $g(x)$ ), is also a solution of the non-homogeneous DE, then the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n + y_p \end{aligned}$$

where you use [Section 4.3](#) methods for the  $c_i y_i$ 's.

### 4.4.2 – Example

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

**Step 1: Find the General Solution  $y_c$  of the complimentary DE**

$y'' + 4y' - 2y = 0$  Aux equation:

$$\begin{aligned} m^2 + 4m - 2 &= 0 \\ m^2 + 4m + 4 &= 6 \\ (m + 2)^2 &= 6 \\ m + 2 &= \pm\sqrt{6} \\ m &= -2 \pm \sqrt{6} \\ y_1 &= e^{(-2+\sqrt{6})x} \\ y_2 &= e^{(-2-\sqrt{6})x} \end{aligned}$$

**Step 2: Find a particular solution  $y_p$  of given DE**

Educated Guess:

$$y_p = Ax^2 + Bx + C$$

for some coefficients  $A, B, C$ . For the moment, they're undetermined coefficients.

Plugging in the  $y_p$ , we get

$$\begin{aligned} y'_p &= 2Ax + B \\ y''_p &= 2A \end{aligned}$$

So,

$$\begin{aligned}
 2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) &= 2x^2 - 3x + 6 \\
 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C &= 2x^2 - 3x + 6 \\
 -2Ax^2 + 8Ax - 2Bx + 2A + 4B - 2C &= 2x^2 - 3x + 6 \\
 -2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) &= 2x^2 - 3x + 6 \\
 -2A &= 2 \\
 8A - 2B &= -3 \\
 2A + 4B - 2C &= 6 \\
 -2A &= 2 \\
 A &= -1 \\
 8(-1) - 2B &= -3 \\
 -8 - 2B &= -3 \\
 8 + 2B &= 3 \\
 2B &= -5 \\
 B &= -\frac{5}{2} \\
 2(-1) + 4\left(-\frac{5}{2}\right) - 2C &= 6 \\
 -2 + -10 - 2C &= 6 \\
 -2C &= 18 \\
 C &= -9
 \end{aligned}$$

### Step 3: Check

$$\begin{aligned}
 y'_p &= 2(-1)x + \left(-\frac{5}{2}\right) \\
 &= -2x - \frac{5}{2} \\
 y''_p &= 2(-1) \\
 &= -2 \\
 y'' + 4y' - 2y &= -2 + 4\left(-2x - \frac{5}{2}\right) - 2\left(-x^2 - \frac{5}{2}x - 9\right) \\
 &= -2 - 8x - 10 + 2x^2 + 5x + 18 \\
 &= 2x^2 - 8x + 5x - 10 + 18 - 2 \\
 &= 2x^2 - 3x + 6
 \end{aligned}$$

### 4.4.3 – Example

$$y'' - y' + y = 2 \sin(3x)$$

**Step 1: Find the General Solution  $y_c$  of the complimentary DE**

Aux equation:

$$\begin{aligned}
 m^2 - m + 1 &= 0 \\
 m &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \\
 &= \frac{1 \pm \sqrt{1-4}}{2} \\
 &= \frac{1 \pm \sqrt{-3}}{2} \\
 &= \frac{1 \pm \sqrt{3}i}{2} \\
 m_1 &= \frac{1 + \sqrt{3}i}{2} \\
 m_2 &= \frac{1 - \sqrt{3}i}{2} \\
 y_1 &= e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) \\
 y_2 &= e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)
 \end{aligned}$$

**Step 2: Guess  $y_p = A \sin(3x) + B \cos(3x)$**

Plug into the DE

$$\underbrace{y''}_{-\overbrace{9A \sin(3x) - 9B \cos(3x)}^{= 0}} - \underbrace{y'}_{-\overbrace{(3A \cos(3x) - 3B \sin(3x))}^{= 0}} + \underbrace{y}_{\overbrace{A \sin(3x) + B \cos(3x)}^{= 0}} = 2 \sin(3x)$$

#### 4.3.4 – Method of Undetermined Coefficients 2

For Solving Linear, Non-homogeneous DE with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Standard Form:

$$y'' + a_1 y' + a_0 y = g(x)$$

#### 4.3.5 – Steps

Step 1) Solve  $y'' + a_1 y' + a_0 y = 0$  called the general solution  $y_c$ .

Step 2) Find one particular solution  $y_p$  of the given DE and the general solution is

$$y = y_c + y_p$$

This method can only be used when  $g(x)$  is a polynomial (An exponential (i.e.  $e^{kx}$ ), sines or cosines or sums of products of these types of functions)

### 4.3.6 – Example

$$y'' - 3y' - 4y = 4 \cos(3x)$$

1st solve:

$$\begin{aligned} y'' - 3y' - 4y &= 0 \\ m^2 e^{mx} - 3m e^{mx} - 4e^{mx} &= 0 \\ m^2 - 3m - 4 &= 0 \\ (m - 4)(m + 1) &= 0 \\ m - 4 &= 0 \quad m + 1 = 0 \\ m &= 4 \quad m = -1 \\ y_c &= c_1 e^{4x} + c_2 e^{-x} \end{aligned}$$

$$\begin{aligned} y &= A \cos(3x) + B \sin(3x) \\ y' &= -3A \sin(3x) + 3B \cos(3x) \\ y'' &= -9A \cos(3x) - 9B \sin(3x) \end{aligned}$$

$$\begin{aligned} y'' - 3y' - 4y &= 4 \cos(3x) \\ (-9A \cos(3x) - 9B \sin(3x)) - 3(-3A \sin(3x) + 3B \cos(3x)) - 4(A \cos(3x) + B \sin(3x)) &= 4 \cos(3x) \\ -9A \cos(3x) - 9B \sin(3x) + 9A \sin(3x) - 9B \cos(3x) - 4A \cos(3x) - 4B \sin(3x) &= 4 \cos(3x) \\ -9A \cos(3x) - 9B \cos(3x) - 4A \cos(3x) - 9B \sin(3x) + 9A \sin(3x) - 4B \sin(3x) &= 4 \cos(3x) \\ \cos(3x)(-9A - 9B - 4A) + \sin(3x)(-9B + 9A - 4B) &= 4 \cos(3x) \\ \cos(3x)(-13A - 9B) + \sin(3x)(9A - 13B) &= 4 \cos(3x) \\ \left\{ \begin{array}{l} -13A - 9B = 4 \\ 9A - 13B = 0 \end{array} \right. \text{ Solve simultaneously} \end{aligned}$$

One way to solve Linear Systems of Equations is called Cramer's Rule.

$$\det \begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}$$

$$A = \frac{\begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}}$$

$$= \frac{4(-13) - 0(-9)}{-13(-13) - 9(-9)}$$

$$= \frac{-52 - 0}{169 + 81}$$

$$= -\frac{52}{250}$$

$$= -\frac{26}{125}$$

$$B = \frac{\begin{bmatrix} -13 & 4 \\ 9 & 0 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}}$$

$$= \frac{-13(0) - 4(9)}{250}$$

$$= \frac{0 - 36}{250}$$

$$= -\frac{36}{250}$$

$$= -\frac{18}{125}$$

Check:

$$(-13) \left( -\frac{26}{125} \right) + (-9) \left( -\frac{18}{125} \right) ? = 4$$

$$\frac{338}{125} + \frac{162}{125} ? = 4$$

$$\frac{500}{125} = 4$$

$$9 \left( -\frac{26}{125} \right) + (-13) \left( -\frac{18}{125} \right) ? = 0$$

$$-\frac{234}{125} + \frac{234}{125} ? = 0$$

$$0 = 0$$

So

$$y = -\frac{26}{125} \cos(3x) - \frac{18}{125} \sin(3x) + c_1 e^{4x} + c_2 e^{-x}$$

is the general solution to the given DE.

### 4.3.7 – Example

$$y'' - 5y' + 4y = 8e^x$$

If we try:

$$\begin{aligned} y_p &= Ae^x \\ Ae^x - 5Ae^x + 4Ae^x &= 8e^x \\ e^x(A - 5A + 4) &= 8e^x \\ 0 &= 8e^x \text{ has no solution.} \end{aligned}$$

Solve

$$y'' - 5y' + 4y = 0$$

1st

$$\begin{aligned} m^2 - 5m + 4 &= 0 \\ (m - 1)(m - 4) &= 0 \\ m - 1 = 0 &\quad m - 4 = 0 \\ m = 1 &\quad m = 4 \\ y_1 = e^{1mx} &\quad y_2 = e^{4mx} \\ y_1 = e^{mx} &\quad y_2 = e^{4mx} \\ y_c = c_2 e^{mx} + c_2 e^{4mx} &\text{ hole at } Ae^x \text{ is } c_1 = A \quad c_2 = 0 \end{aligned}$$

Suppose we have a 5th order DE with

$$a_5 y^{(5)} + a_4 y^{(4)} + \cdots + a_1 y' + a_0 y = g(x)$$

and the auxiliary equation factors as

$$m^2(m - 3)(m - (2 + i))(m - (2 - i)) = 0$$

$$m = 0 \text{ (multiplicity 2)} \quad m = 3 \quad m = 2 + i \quad m = 2 - i$$

#### Step 1

Write the general solution to the complimentary DE

$$\begin{aligned} y_1 &= e^{0x} = 1 \\ y_2 &= x e^{0x} = x \\ y_3 &= e^{3x} = e^{3x} \\ y_4 &= e^{(2+i)x} = e^{2x} \cos(x) \\ y_5 &= e^{(2-i)x} = e^{2x} \sin(x) \end{aligned}$$

$$y_c = c_1 + c_2 x + c_3 e^{3x} + e^{2x} \cos(x) + e^{2x} \sin(x)$$

### 4.3.8 – What would you guess for the form of $y_p$ ?

If

$$(ii) \ g(x) = e^{5x} \Rightarrow y_p = Ae^{5x}$$

$$(iii) \ g(x) = e^{3x} \Rightarrow y_p = Axe^{3x} \ (\text{because } e^{3x} \text{ is in } y_c)$$

$$(iv) \ g(x) = 5e^{2x} \sin(x) \Rightarrow y_p = (Ae^{2x} \cos(x) + Be^{2x} \sin(x)) x$$

$$(v) \ g(x) = 6x^2 e^{4x} \Rightarrow y_p = (Ax^2 + Bx + C) e^{4x}$$

$$(vi) \ g(x) = x^2 e^{3x} \Rightarrow y_p = (Ax^2 + Bx + C) e^{3x} x$$

## 4.6 Variation of Parameters Method

$$y'' + P(x)y' + Q(x)y = f(x)$$

will only work on problems where  $P(x)$  and  $Q(x)$  are constants.

### 4.6.1 – 1st Step: General solution of complementary DE

$$y = c_1 y_1 + c_2 y_2$$

Guess a solution to the non-homogeneous of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where  $u_1$  and  $u_2$  are functions of  $x$ . This theory produces

$$u'_1 = \frac{W_1}{W} \text{ and } u'_2 = \frac{W_2}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix}$$

### 4.6.2 – Example

$$4y'' + 36y = \csc(3x)$$

$$4y'' + 36y = \csc(3x)$$

$$y'' + 9y = \frac{\csc(3x)}{4}$$

$$m^2 e^{mx} + 9e^{mx} = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$\begin{aligned} m &= \pm\sqrt{-9} \\ &= \pm 3i \end{aligned}$$

$$\begin{aligned}
y_1 &= e^{0x} \cos(3x) & y_2 &= e^{0x} \sin(3x) \\
y_1 &= 1 \cos(3x) & y_2 &= 1 \sin(3x) \\
y_1 &= \cos(3x) & y_2 &= \sin(3x) \\
y_c &= c_1 \cos(3x) + c_2 \sin(3x)
\end{aligned}$$

Guess

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
u'_1 &= \frac{W_1}{W} & u'_2 &= \frac{W_2}{W}
\end{aligned}$$

where

$$\begin{aligned}
W &= \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3 \sin(3x) & 3 \cos(3x) \end{vmatrix} \\
&= (3 \cos(3x))(\cos(3x)) - (\sin(3x))(-3 \sin(3x)) \\
&= 3 \cos^2(3x) + 3 \sin^2(3x) \\
&= 3(\cos^2(3x) + \sin^2(3x)) \\
&= 3(1) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
W_1 &= \begin{vmatrix} 0 & \sin(3x) \\ \frac{1}{4} \csc(3x) & \cos(3x) \end{vmatrix} \\
&= 0 \cos(3x) - \sin(3x) \left( \frac{\csc(3x)}{4} \right) \\
&= -\frac{\sin(3x) \csc(3x)}{4} \\
&= -\frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
W_2 &= \begin{vmatrix} \cos(3x) & 0 \\ \sin(3x) & \frac{1}{4} \csc(3x) \end{vmatrix} \\
&= \frac{1}{4} \csc(3x) \cos(3x) - 0 \sin(3x) \\
&= \frac{\cos(3x)}{4 \sin(3x)} \\
&= \frac{1}{4} \cot(3x)
\end{aligned}$$

$$\begin{aligned}
u'_1 &= \frac{W_1}{W} & u'_2 &= \frac{W_2}{W} \\
u'_1 &= \frac{-\frac{1}{4}\cot(3x)}{3} & u'_2 &= \frac{\frac{1}{4}\cot(3x)}{3} \\
u'_1 &= -\frac{1}{12} & u_2 &= \frac{1\cot(3x)}{12} \\
u'_1 &= -\frac{1}{12} & u_2 &= \frac{1\cos(3x)}{12\sin(3x)} \\
u_1 &= \int -\frac{1}{12}dx & u_2 &= \int \frac{1\cos(3x)}{12\sin(3x)}dx \\
u_1 &= -\frac{x}{12} & u_2 &= \frac{1}{12} \int \frac{1}{3} \frac{dv}{v} \\
&& u_2 &= \frac{1}{36} \ln|v| \\
&& u_2 &= \frac{1}{36} \ln|\sin(3x)|
\end{aligned}$$

$$\begin{aligned}
y_p &= u_1y_1 + u_2y_2 \\
&= -\frac{x}{12} \cos(3x) + \frac{1}{36} \ln|\sin(3x)| \sin(3x) \\
&= -\frac{x \cos(3x)}{12} + \frac{\sin(3x)}{36} \ln|\sin(3x)|
\end{aligned}$$

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 \cos(3x) + c_2 \sin(3x) - \frac{x \cos(3x)}{12} + \frac{\sin(3x)}{36} \ln|\sin(3x)|
\end{aligned}$$

### 4.6.3 – 3×3 Determinants

$$\left| \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \right| = a \left| \begin{array}{cc} e & f \\ h & i \end{array} \right| - b \left| \begin{array}{cc} d & f \\ f & i \end{array} \right| + c \left| \begin{array}{cc} d & e \\ g & h \end{array} \right|$$

#### Matrix of Signs

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

# Chapter 5

## Modeling with Higher-Order Differential Equations

### 5.1 Spring-Mass Problems

Suppose that there are no forces affecting the motion other than the gravitational force and the spring force.

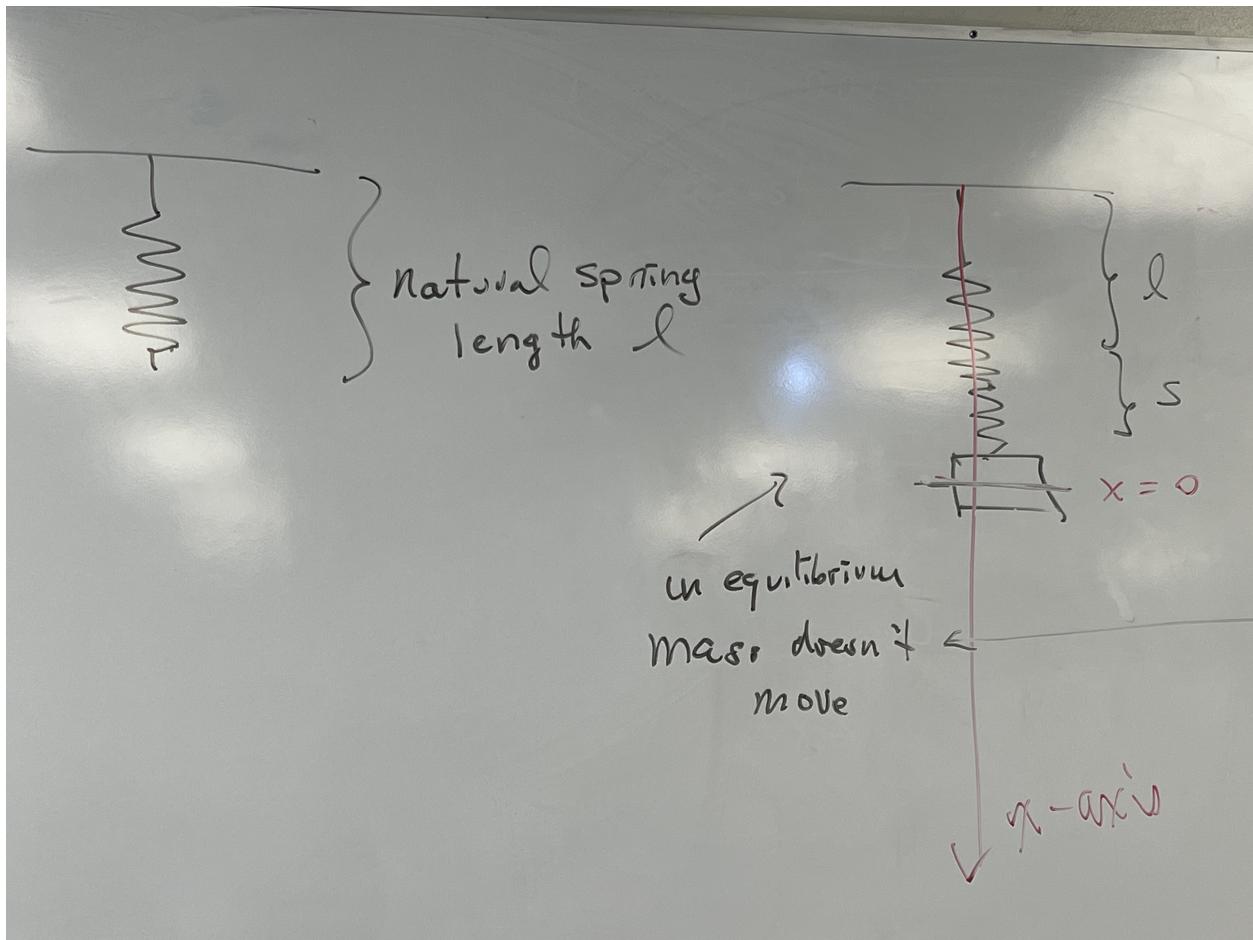


Figure 5.1: Diagram of a spring in equilibrium.

In Equilibrium, mass doesn't move.

$$F_g = -F_{spring} \rightarrow mg = -F_{spring} \rightarrow -mg = -kx \text{ (Hooke's Law)}$$

where  $g \approx 9.8m/s^2$  if m is in kg, else  $g \approx 32.1ft/sec^2$ .

In equilibrium:

$$F_{net} = mg - ks = 0$$

, so

$$k = \frac{mg}{s}$$

In general:

$$F_{net} = m \times \text{acceleration} \rightarrow m \frac{d^2x}{dt^2}$$

$$\begin{aligned}
 mg + (-k)(x + s) &= m \frac{d^2x}{dt^2} \\
 mg - kx - ks &= m \frac{d^2x}{dt^2} \\
 (mg - ks) - kx &= m \frac{d^2x}{dt^2} \\
 0 - kx &= m \frac{d^2x}{dt^2}
 \end{aligned}$$

So the differential equation is:

$$m \frac{d^2x}{dt^2} + kx = 0 \quad (5.1)$$

, a 2nd order, homogeneous DE with a constant coefficient.

### 5.1.1 – Example

$$\begin{aligned}
 m \frac{d^2x}{dt^2} + kx &= 0 \\
 \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0
 \end{aligned}$$

Guess:

$$\begin{aligned}
 x &= e^{lt} \\
 m \frac{d^2x}{dt^2} + kx &= 0 \\
 l^2 e^{lt} + \frac{k}{m} e^{lt} &= 0 \\
 l^2 + \frac{k}{m} &= 0 \\
 l^2 &= -\frac{k}{m} \\
 l &= \pm \sqrt{-\frac{k}{m}} \\
 l &= 0 \pm \sqrt{\frac{k}{m}}i
 \end{aligned}$$

$$\begin{aligned}
 x_1 &= e^{0t} \cos \left( \frac{k}{m} t \right) & x_2 &= e^{0t} \sin \left( \frac{k}{m} t \right) \\
 x_1 &= \cos \left( \frac{k}{m} t \right) & x_2 &= \sin \left( \frac{k}{m} t \right)
 \end{aligned}$$

Let  $\omega = \sqrt{\frac{k}{m}}$

$$x_1 = \cos(\omega^2 t) \quad x_2 = \sin(\omega^2 t)$$

### 5.1.2 – Example

Mass weighs 2lbs, stretch spring 6 inches

$$F = ma$$

$$\begin{aligned} 2\text{lbs} &= m \times 32 \frac{\text{ft}}{\text{sec}^2} \\ m &= \frac{2\text{lbs}}{32 \frac{\text{ft}}{\text{sec}^2}} \\ m &= \frac{1}{16} \text{ slug} \end{aligned}$$

$$\begin{aligned} F_{spring} &= kx \\ &= k(6\text{in}) \end{aligned}$$

$$\begin{aligned} 2\text{lbs} &= \frac{k}{2} \\ k &= 4 \frac{\text{lbs}}{\text{ft}} \\ \omega &= \sqrt{\frac{4}{\frac{1}{16}}} \\ \omega &= \sqrt{64} \\ \omega &= 8 \end{aligned}$$

### 5.1.3 – Undamped Motion

$$\begin{aligned} m \frac{d^2x}{dt^2} + kx &= 0 \\ \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0 \\ \frac{d^2x}{dt^2} + \omega^2x &= 0 \text{ where } \omega = \sqrt{\frac{k}{m}} \\ n^2 e^{nt} + \omega^2 e^{nt} &= 0 \text{ where } x = e^{nt} \\ n^2 + \omega^2 &= 0 \\ n^2 &= -\omega^2 \\ n &= 0 \pm \sqrt{-\omega^2} \\ &= 0 \pm \omega i \end{aligned}$$

$$x_1 = \cos(\omega t) \quad x_2 = \sin(\omega t)$$

General solution:

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

### 5.1.4 – Free, Damped Motion

Assume in addition to  $F_{spring}$  and  $F_{gravity}$  that there is a force damping the motion which is directly proportional, in the opposite direction, to the mass's velocity.

$$\begin{aligned} m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx &= 0 \\ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m}x &= 0 \\ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \omega^2 x &= 0 \end{aligned} \tag{5.2}$$

where  $\beta$  is the drag coefficient. If we substitute  $2\lambda = \frac{\beta}{m} \Rightarrow \lambda = \frac{\beta}{2m}$

$$\begin{aligned} \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x &= 0 \\ n^2 e^{nt} + 2\lambda n e^{nt} + \omega^2 e^{nt} &= 0 \\ n^2 + 2\lambda n + \omega^2 &= 0 \\ n^2 + 2\lambda n &= -\omega^2 \\ n^2 + 2\lambda n + \lambda^2 &= \lambda^2 - \omega^2 \\ (n + \lambda)^2 &= -\omega^2 + \lambda^2 \\ n + \lambda &= \pm \sqrt{\lambda^2 - \omega^2} \\ n &= -\lambda \pm \sqrt{\lambda^2 - \omega^2} \end{aligned}$$

**Case 1:**  $\lambda^2 - \omega^2 > 0$

then there are two distinct real solutions where  $n_1 < 0$  and  $n_2 < 0$ .

$$\begin{aligned} n_1 &= -\lambda + \sqrt{\lambda^2 - \omega^2} \\ n_2 &= -\lambda - \sqrt{\lambda^2 - \omega^2} \\ x_1 &= e^{n_1 t} \quad x_2 = e^{n_2 t} \\ x_1 &= e^{t(-\lambda + \sqrt{\lambda^2 - \omega^2})} \quad x_2 = e^{t(-\lambda - \sqrt{\lambda^2 - \omega^2})} \end{aligned}$$

General solution:

$$\begin{aligned} x &= c_1 e^{t(-\lambda + \sqrt{\lambda^2 - \omega^2})} + c_2 e^{t(-\lambda - \sqrt{\lambda^2 - \omega^2})} \\ &= c_1 e^{-t\lambda} e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\lambda} e^{-t\sqrt{\lambda^2 - \omega^2}} \\ &= e^{-t\lambda} \left( c_1 e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\sqrt{\lambda^2 - \omega^2}} \right) \end{aligned} \tag{5.3}$$

**Case 2:**  $\lambda = \omega$

$$\begin{aligned} n &= -\lambda \pm \sqrt{\lambda^2 - \omega^2} \\ n &= -\lambda \pm \sqrt{0} \\ n &= -\lambda \end{aligned}$$

where  $\lambda$  has multiplicity 2.

$$x_1 = e^{-\lambda t} \quad x_2 = te^{-\lambda t}$$

General solution:

$$\begin{aligned} x &= c_1 e^{-\lambda t} + c_2 t e^{-\lambda t} \\ &= e^{-\lambda t} (1 + c_2 t) \end{aligned} \tag{5.4}$$

**Case 3:**  $0 < \lambda < \omega$

then

$$\begin{aligned} x_1 &= e^{-\lambda t} \cos(\sqrt{\omega^2 - \lambda^2} t) & x_2 &= e^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t) \\ x &= c_1 e^{-\lambda t} \cos(\sqrt{\omega^2 - \lambda^2} t) + c_2 e^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t) \\ &= e^{-\lambda t} \left( c_1 \cos(\sqrt{\omega^2 - \lambda^2} t) + c_2 \sin(\sqrt{\omega^2 - \lambda^2} t) \right) \\ &= e^{-\lambda t} \left( A \sin(\sqrt{\omega^2 - \lambda^2} t) + \phi \right) \end{aligned} \tag{5.5}$$

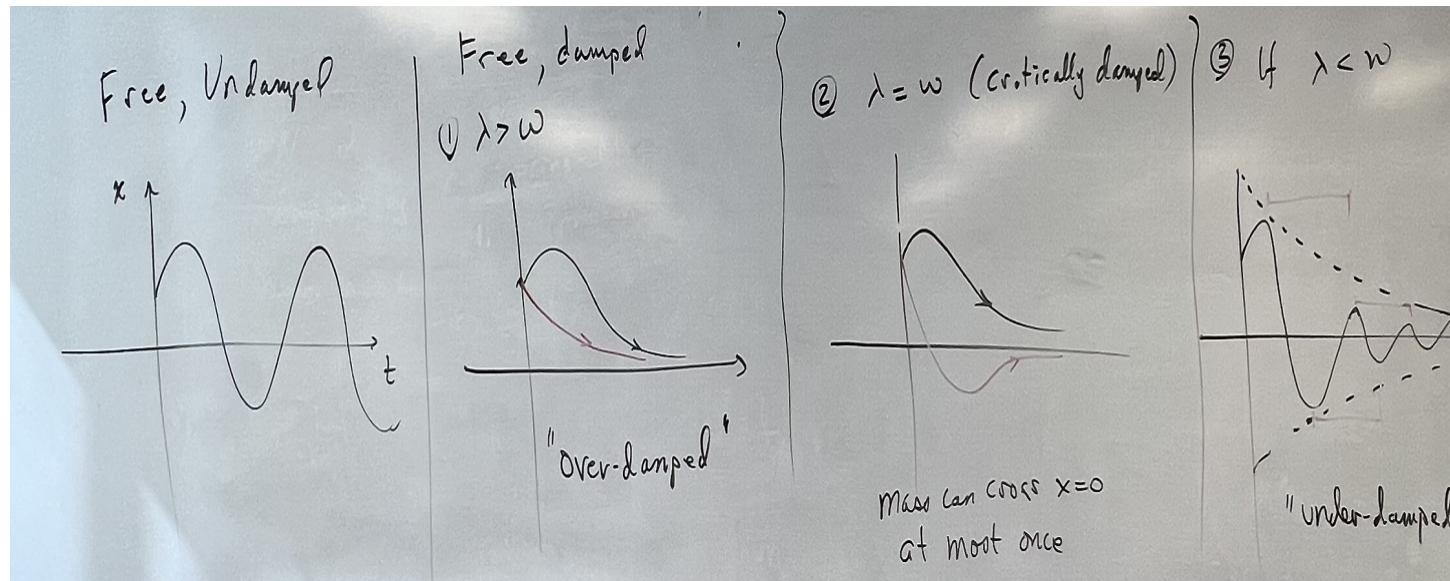


Figure 5.2:

### 5.1.5 – Driven Motion (not Free Motion)

Can be both damped or **undamped**.

Imagine the support is oscillating up and down due to an external force. If the motion is:

## Undamped

Assume that  $\gamma \neq \omega$

$$\begin{aligned}
 \frac{d^2x}{dt^2} + \omega^2 x &= F(t) \\
 &= F_0 \sin(\gamma t) \\
 \frac{d^2x}{dt^2} + \omega^2 x &= 0 \\
 &\vdots \\
 x &= c_1 \cos(\omega t) + c_2 \sin(\omega t) \\
 \frac{d^2x}{dt^2} + \omega^2 x &= F_0 \sin(\gamma t) \\
 \textcolor{red}{x_p(t)} &= A \cos(\gamma t) + B \sin(\gamma t) \\
 -A\gamma^2 \cos(\gamma t) - B\gamma^2 \sin(\gamma t) + \omega^2(A \cos(\gamma t) + B \sin(\gamma t)) &= F_0 \sin(\gamma t) \\
 -A\gamma^2 \cos(\gamma t) - B\gamma^2 \sin(\gamma t) + A\omega^2 \cos(\gamma t) + B\omega^2 \sin(\gamma t) &= F_0 \sin(\gamma t) \\
 \cos(\gamma t) (-A\gamma^2 + A\omega^2) = 0 \cos(\gamma t) &\quad B(\omega^2 - \gamma^2) = F_0 \\
 -A\gamma^2 + A\omega^2 = 0 &\quad B = \frac{F_0}{\omega^2 - \gamma^2} \\
 A(\omega^2 - \gamma^2) = 0 &\quad B = \frac{F_0}{\omega^2 - \gamma^2}
 \end{aligned}$$

We assumed that  $\gamma \neq \omega$ , which forces  $A = 0$  for the left equation to work.

$$A = 0 \quad B = \frac{F_0}{\omega^2 - \gamma^2}$$

General solution:

$$\begin{aligned}
 x(t) &= x_c(t) + x_p(t) \\
 &= c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2} \sin(\gamma t)
 \end{aligned} \tag{5.6}$$

If  $\gamma$  is almost equal to  $\omega$ , then  $\frac{F_0}{\omega^2 - \gamma^2}$  is a large constant. This situation is called Resonance.

### 5.1.6 – Example

Driven, Damped, Spring-Mass

$$\begin{aligned}
 \frac{1}{5} \frac{d^2x}{dt^2} + 1.2 \frac{dx}{dt} + 2x &= 5 \cos(4t), x(0) = \frac{1}{2}, x'(0) = 0 \\
 \frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 10x &= 25 \cos(4t) \\
 \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x &= F(t)
 \end{aligned}$$

where

$$\lambda = 3, \omega = \sqrt{10}, F(t) = 25 \cos(4t)$$

For instance, if the mass = 2kgs, then the spring constant is  $k = 20$  and  $\beta = 12$ .

$$\sqrt{10} = \sqrt{\frac{k}{m}} \Rightarrow 10 = \frac{k}{m} \Rightarrow 10m = k \Rightarrow k = 10(2) = 20$$

$$\beta = 2\lambda m = 2(3)(2) = 6(2) = 12$$

### To solve the complimentary DE

, guess  $x = e^{nt}$

$$n^2 e^{nt} + 6ne^{nt} + 10e^{nt} = 0$$

$$n^2 + 6n + 10 = 0$$

$$\begin{aligned} n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36 - 40}}{2} \\ &= \frac{-6 \pm \sqrt{-4}}{2} \\ &= \frac{-6 \pm 2i}{2} \\ &= -3 \pm i \end{aligned}$$

$$x_1 = e^{-3t} \cos(t) \quad x_2 = e^{-3t} \sin(t)$$

### To find a particular solution

Guess

$$x_p = A \cos(4t) + B \sin(4t)$$

$$x'_p = -4A \sin(4t) + 4B \cos(4t)$$

$$x''_p = -16A \cos(4t) - 16B \sin(4t)$$

$$-16A \cos(4t) - 16B \sin(4t) + 6(-4A \sin(4t) + 4B \cos(4t)) + 10(A \cos(4t) + B \sin(4t)) = 25 \cos(4t)$$

$$-16A \cos(4t) - 16B \sin(4t) - 24A \sin(4t) + 24B \cos(4t) + 10A \cos(4t) + 10B \sin(4t) = 25 \cos(4t)$$

$$-16A \cos(4t) + 10A \cos(4t) + 24B \cos(4t) - 16B \sin(4t) - 24A \sin(4t) + 10B \sin(4t) = 25 \cos(4t)$$

$$-6A \cos(4t) + 24B \cos(4t) - 6B \sin(4t) - 24A \sin(4t) = 25 \cos(4t)$$

$$\cos(4t)(-6A + 24B) + \sin(4t)(-6B - 24A) = 25 \cos(4t)$$

# Chapter 6

## Series Solutions of Linear Equations

### 6.1 Solution by Infinite Series

2nd order linear DE with (possibly) variable coefficients

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

$$y'' + P(x)y' + Q(x)y = F(x)$$

#### 6.1.1 – Review of Infinite Series Facts

##### Maclaurin Series

$$\sum_{n=0}^{\infty} a_n x^n$$

Power series centered at 0

##### Taylor Series

$$\sum_{n=0}^{\infty} a_n (x - a)^n$$

Centered at  $a = 0$  It's a theorem that power series either

- (1) Converge all real numbers  $x$  on the interval  $I = (-\infty, \infty)$  and the radius of convergence is  $R = \infty$
- (2) Converge only when  $x = a$  on the interval  $I = [a, a]$  and the radius of convergence is  $R = 0 = \{a\}$
- (3) The series converges on an interval centered at a finite, non-zero radius  $R = (a - R, a + R)$

### 6.1.2 – Ratio Test

Use the Ratio Test to determine which of these 3 cases occurs in a specific problem. The 3 cases of the ratio test are:

$L < 1$  , the series converges

$L > 1$  , the series diverges

$L = 1$  ,

### 6.1.3 – Example

Determine the radius and interval of convergence for

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{x^n}{3^n(n+1)} \\ L &= \lim_{n \rightarrow \infty} \frac{\left| \frac{x^{n+1}}{3^{n+1}(n+2)} \right|}{\left| \frac{x^n}{3^n(n+1)} \right|} \\ &= \lim_{n \rightarrow \infty} \frac{|x|}{3} \frac{n+1}{n+2} \\ &= \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{n+1}{n+2} \\ &= \frac{|x|}{3}(1) \\ &= \frac{|x|}{3} \\ \frac{|x|}{3} &< 1 \\ |x| &< 3 \\ I &= (-3, 3) \end{aligned}$$

### 6.1.4 – Idea of Method

We will try to find a solution of the DE in the form of a power series

$$y = \sum_{n=0}^{\infty} c_n x^n \text{ (centered at 0)}$$

or

$$y = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ (centered at } a)$$

When you substitute this into the DE you get recurrence relationships for the coefficients  $c_0, c_1, \dots$ . Once you've found the coefficients in terms of either  $c_0$ , or  $c_0, c_1$  where  $c_0 \neq c_1$ . Then you should determine where the series converges.

An example of a recurrence relation is the Fibonacci Sequence

$$F_{n+2} = F_n + F_{n+1}$$

### 6.1.5 – Example

Use this method to solve the DE

$$y' + y = 0$$

Assume

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots = \sum_{n=1}^{\infty} c_n n x^{n-1} \\ &\quad \sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0 \end{aligned}$$

Shift the index of the first summation such that both have terms  $x^n$ . To do this, we'll make the substitution  $k = n - 1 \Rightarrow n = k + 1$

$$\begin{aligned} \sum_{k+1=1}^{\infty} c_{k+1}(k+1)x^{(k+1)-1} + \sum_{k=0}^{\infty} c_k x^k &= 0 \\ \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + \sum_{k=0}^{\infty} c_k x^k &= 0 \\ \sum_{k=0}^{\infty} [c_{k+1}(k+1)x^k + c_k x^k] &= 0 \\ \sum_{k=0}^{\infty} [(k+1)c_{k+1} + c_k] x^k &= 0 \end{aligned}$$

This implies  $(k+1)c_{k+1} + c_k = 0$  for all  $k = 0, 1, 2, \dots$ . <sup>1</sup>  $\Rightarrow c_{k+1} = \frac{-c_k}{k+1}$  for all  $k = 0, 1, 2, \dots$ .

$$\begin{aligned} c_0 &= c_0 \\ c_1 &= -\frac{c_0}{1} \\ &= -c_0 \\ c_2 &= -\frac{c_1}{2} \\ &= -\frac{-c_0}{2} \\ &= \frac{-1^2 c_0}{2} \\ &= \frac{c_0}{2} \\ c_3 &= -\frac{c_2}{3} \\ &= -\frac{\frac{c_0}{2}}{3} \\ &= \frac{-c_0}{6} \end{aligned}$$

**Conjecture:** It is apparent that

$$c_n + \frac{(-1)^n c_0}{n!}$$

**Plugging into the DE**

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{n!} x^n \\ &= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \end{aligned}$$

---

<sup>1</sup>since the only power series that equals 0 is  $\sum_{k=0}^{\infty} 0x^k$

By the Ratio Test

$$\begin{aligned}
 L &= \lim_{n \rightarrow \infty} \frac{\left| \frac{(-1)^{n+1}}{(n+1)!} x^{n+1} \right|}{\left| \frac{(-1)^n}{n!} x^n \right|} \\
 &= \lim_{n \rightarrow \infty} \left| (-1) \frac{x n!}{(n+1)!} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{|x| n!}{(n+1)!} \\
 &= \lim_{n \rightarrow \infty} \frac{|x|}{n+1} \\
 &= |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} \\
 &= |x| \times 0 \\
 &= 0 \text{ The series converges everywhere}
 \end{aligned}$$

### 6.1.6 – Power Series of Basic Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Our answer in the DE is

$$y = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}$$

## 6.2 Second Order, Linear Homogenous DE

$$\begin{aligned}
 a_2(x)y'' + a_1(x)y' + a_0(x)y &= 0 \\
 y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y &= 0 \\
 y'' + P(x)y' + Q(x)y &= 0
 \end{aligned}$$

$x$ 's for which  $a_2(x) \neq 0$  will be called ordinary points.  $x$ 's for which  $a_2(x) = 0$  will be called singular points.

**Existence of Power SeriesTheorem:** If  $x_0$  is an ordinary point of the DE, then there exists two, linearly independent solution  $y_1, y_2$  which are both in the form of power series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

and these series will have radius of convergence of at least the distance from  $x_0$  to the singular point of the DE.

### 6.2.1 – Example

Consider

$$(x^2 + 2x + y)y'' + xy' - 6y = 0$$

- (i) What are the singular points of the DE?

$$\begin{aligned} x^2 + 2x + 5 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm \sqrt{16} \times \sqrt{-1}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

So  $-1 + 2i$  and  $-1 - 2i$  are the only singular points.

- (ii) Is there a power series solution centered at  $x_0 = 0$ ? Yes, since  $x_0 = 0$  is an ordinary point, you can find

$$y_1 = \sum_{n=0}^{\infty} c_n x^n$$

and

$$y_2 = \sum_{n=0}^{\infty} d_n x^n,$$

two linearly independent solutions.

- (iii) What is the minimum the radius could be for these series? As stated in the theorem, the radius is at minimum the distance from  $x_0$  to the singular point. If you have complex singular points, calculate the distance using the complex plane graph.  $\sqrt{(-1 - 0)^2 + (2 - 0)^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$ .

- How about if we want series

$$\sum_{n=0}^{\infty} c_n (x - 3)^2$$

$$\sqrt{(-1 - 3)^2 + (-2 - 0)^2} = \sqrt{(-4)^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$$

### 6.2.2 – Example

Use Power Series centered at 0 ([Maclaurin Series](#)) to solve the DE:

$$y'' - xy = 0$$

$$\begin{aligned}
 y &= \sum_{n=0}^{\infty} c_n x^n \\
 y' &= \sum_{n=1}^{\infty} c_n n x^{n-1} \\
 y'' &= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} \\
 \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - x \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} \\
 &= \sum_{k+2=2}^{\infty} c_{k+2}(k+2)(k+2-1)x^k - \sum_{k-1=0}^{\infty} c_{k-1}x^k \\
 &= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\
 &= \sum_{k=0}^1 c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\
 &= c_{0+2}(0+2)(0+1)x^0 + \sum_{k=1}^{\infty} [c_{k+2}(k+2)(k+1)x^k - c_{k-1}x^k] \\
 &= c_2(2)(1)(1) + \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] \\
 &= 2c_2 + \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] \\
 &= 2(0) + \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] \\
 &= 0 + \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] \\
 &= \sum_{k=1}^{\infty} x^k [c_{k+2}(k+2)(k+1) - c_{k-1}] = 0
 \end{aligned}$$

$$\begin{aligned}
 c_{k+2}(k+2)(k+1) - c_{k-1} &= 0 \\
 c_{k+2}(k+2)(k+1) &= c_{k-1} \\
 c_{k+2} &= \frac{c_{k-1}}{(k+2)(k+1)}
 \end{aligned}$$

$c_0 = \text{arbitrary}$

$c_1 = \text{arbitrary}$

$c_2 = 0$

$$c_3 = \frac{c_0}{(1+2)(1+1)} = \frac{c_0}{3 \times 2}$$

$$c_4 = \frac{c_1}{(2+2)(2+1)} = \frac{c_1}{4 \times 3}$$

$$c_5 = \frac{c_2}{(3+2)(3+1)} = \frac{0}{5 \times 4} = 0$$

$$c_6 = \frac{c_3}{(4+2)(4+1)} = \frac{c_0}{3 \times 2} \times \frac{1}{6 \times 5} = \frac{c_0}{6 \times 5 \times 3 \times 2}$$

$$c_7 = \frac{c_4}{(5+2)(5+1)} = \frac{c_1}{4 \times 3} \times \frac{1}{7 \times 6} = \frac{c_1}{7 \times 6 \times 4 \times 3}$$

$$c_8 = \frac{c_5}{(6+2)(6+1)} = \frac{0}{8 \times 7} = 0$$

$$y = c_0 y_1 + c_1 y_2$$

$$= c_0 \left( 1 + \frac{1}{3 \times 2} x^3 + \frac{1}{6 \times 5 \times 3 \times 2} x^6 + \dots \right) + c_1 \left( x + \frac{1}{4 \times 3} x^4 + \frac{1}{7 \times 6 \times 4 \times 3} + \dots \right)$$

$$= c_0 \left( 1 + \frac{1}{3 \times 2} x^3 + \frac{4}{6 \times 5 \times 4 \times 3 \times 2} x^6 + \dots \right) + c_1 \left( x + \frac{2}{4 \times 3 \times 2} x^4 + \frac{2(5)}{7 \times 6 \times 5 \times 4 \times 3 \times 2} + \dots \right)$$

### 6.2.3 – Example

$$(x^2 + 1)y'' + xy' - y = 0$$

$$(x^2 + 1)y'' + xy' - y = 0$$

$$y'' + \frac{x}{x^2 + 1} y' - \frac{1}{x^2 + 1} y = 0$$

Ordinary points:

$$x^2 + 1 = 0$$

$$x^2 = -1$$

$$x = \pm\sqrt{-1}$$

$$x = \pm i$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \frac{x}{x^2+1} \sum_{n=1}^{\infty} c_n n x^{n-1} - \frac{1}{x^2+1} \sum_{n=0}^{\infty} c_n x^n = 0$$