# Chapter 7

# Method of Laplace Transforms for Solving DE's

## 7.3 The Dirac Delta Function

$$\delta_a(t-t_0) = \begin{cases} 0 & \text{if } t < t_0 - a \\ \frac{1}{2a} & \text{if } t_0 - a \le t \le t_0 + a \\ 0 & \text{if } t > t_0 + a \end{cases}$$

$$\int_0^\infty f(t)\delta(t-t_0)dt = f(t_0) \tag{7.1}$$

So

$$\begin{aligned} \mathscr{L}\left\{\delta(t-t_0)\right\} &= \int_0^\infty e^{-st}\delta(t-t_0)dt \\ &= e^{-st_0} \end{aligned} \tag{7.2}$$

### 7.3.1 – Example

$$y'' + y = 4\delta(t-2\pi)$$

$$\begin{aligned} y(0) = 1 \;\; y'(0) &= 0 \\ y'' + y &= 4\delta(t-2\pi) \\ \mathscr{L}\left\{y''\right\} + \mathscr{L}\left\{y\right\} &= 4\mathscr{L}\left\{\delta(t-2\pi)\right\} \\ s^2Y(s) - sy(0) - y'(0) + Y(s) &= 4e^{-s(2\pi)} \\ Y(s)(s^2+1) - s(1) - 0 &= 4e^{-2\pi s} \\ Y(s)(s^2+1) - s &= 4e^{-2\pi s} \\ Y(s)(s^2+1) &= 4e^{-2\pi s} + s \end{aligned}$$

$$\begin{aligned}
Y(s) &= \frac{4e^{-2\pi s} + s}{s^2 + 1} \\
&= \frac{4e^{-2\pi s}}{s^2 + 1} + \frac{s}{s^2 + 1} \\
&= 4e^{-2\pi} \frac{e^{s}}{s^2 + 1^2} + \frac{s}{s^2 + 1} \\
y(t) &= \mathscr{L}^{-1}\left\{4e^{-2\pi s} \frac{1}{s^2 + 1^2}\right\} + \mathscr{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} \\
&= 4\sin(t - 2\pi)\mathscr{U}(t - 2\pi) + \cos(t)
\end{aligned}$$