Chapter 4

Higher Order Differential Equations

4.2 Reduction of Order

If you have one solution to a 2^{nd} order linear homogenous DE, then it is possible to use that function to construct a 2^{nd} Linear Independent solution to the DE.

4.2.1 – Example

For example, the DE

$$y'' - y = 0$$

One solution is $y = e^x$ on $(-\infty, \infty)$.

Idea: We look for y_2 of the form

$$y_2(x) = u(x)y_1(x) \text{ where } u(x) \text{ is not a constant}$$

The general solution is of the form:

$$y = c_1 y_1 + c_2 y_2$$

where y_1 and y_2 are linearly independent solutions.

To find $u(x)$, we substitute this into the DE

$$\begin{aligned}
y_2 &= u(x)y_1(x) \\
y_2' &= u(x)y_1'(x) + u'(x)y_2(x) \\
y_2'' &= u(x)y_1''(x) + u'(x)y_1(x) + u'(x)y_2'(x) + u''(x)y_1(x) \\
&= uy_1'' + 2u'y_1' + u''y_1
\end{aligned}$$

So $y'' - y = 0$ becomes

$$\begin{aligned}
uy_1'' + 2u'y_1' + u''y_1 - uy_1 &= 0 \text{ when we sub } y = y_2 \quad = u_{y1} \\
u(e^x)'' + 2u'(e^x)' + u''(e^x) - u(e^x) &= 0 \\
ue^x + 2u'e^x + u''e^x - ue^x &= 0 \\
2u'e^x + u''e^x &= 0 \\
e^x(2u' + u'') &= 0 \\
2u' + u'' &= 0
\end{aligned}$$

Let $w = u'$

$$\begin{aligned}
2w + w' &= 0 \\
2w + \frac{dw}{dx} &= 0 \\
\frac{dw}{dx} &= -2w \\
\frac{dw}{w} &= -2dx \\
\int \frac{dw}{w} &= \int -2dx \\
\ln|w| &= -2x \\
w &= e^{-2x} \\
u' &= e^{-2x} \\
\int u' &= \int e^{-2x} \\
u &= -\frac{1}{2}e^{-2x} \\
y_2 &= uy_1 \\
&= -\frac{1}{2}e^{-2x} \times e^{x} \\
&= -\frac{1}{2}e^{-x}
\end{aligned}$$

Double check that y_2 is a solution of the DE

$$\begin{aligned}
y_2 &= -\frac{1}{2}e^{-x} \\
y_2' &= \frac{1}{2}e^{-x} \\
y_2'' &= -\frac{1}{2}e^{-x} \\
y_2'' - y &= -\frac{1}{2}e^{-x} - \left(-\frac{1}{2}e^{-x}\right) \\
&= -\frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} \\
&= 0
\end{aligned}$$

In general,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

put into standard form by dividing by $a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$, the same method as in our example leads to the formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \tag{4.1}$$

4.2.2 – Example

Part 1

$$x^2y'' - 3xy' + 4y = 0$$

Verify that $y_1 = x^2$ is a solution $y_1' = 2x, y_1'' = 2$.

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ x^2(2) - 3x(2x) + 4(x^2) &= 0 \\ 2x^2 - 6x^2 + 4x^2 &= 0 \\ 6x^2 - 6x^2 &= 0 \\ 0 &= 0 \end{aligned}$$

Part 2

Find a linearly independent solution $y_2(x)$.

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ y'' - \frac{3}{x}y' + \frac{4}{x^2}y &= 0 \\ P(x) &= -\frac{3}{x} \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int \frac{3}{x}dx}}{(y_1(x))^2} dx \\ y_2 &= y_1 \int \frac{e^{3\ln|x|}}{(y_1(x))^2} dx \\ y_2 &= y_1 \int \frac{e^{\ln|x^3|}}{(y_1(x))^2} dx \\ y_2 &= x^2 \int \frac{x^3}{(x^2)^2} dx \\ y_2 &= x^2 \int \frac{x^3}{x^4} dx \\ y_2 &= x^2 \int \frac{1}{x} dx \\ y_2 &= x^2 \ln|x| + C \end{aligned}$$

4.2.3 – Part 3: Double check that y_2 is a solution of the DE

$$\begin{aligned} y_2 &= x^2 \ln|x| \\ y_2' &= x^2 \times \frac{1}{x} + 2x \ln|x| \\ y_2'' &= 1 + 2x\frac{1}{x} + 2\ln|x| \\ &= 1 + 2 + 2\ln|x| \\ &= 3 + 2\ln|x| \end{aligned}$$

So the LHS DE becomes

$$\begin{aligned} x^2\left(3 + 2\ln|x|\right) - 3x\left(x + 2x\ln|x|\right) + 4x^2\ln|x| &= 3x^2 + 2x^2\ln|x| - 3x^2 - 6x^2\ln|x| + 4x^2\ln|x| \\ &= 3x^2 - 3x^2 + 2x^2\ln|x| - 6x^2\ln|x| + 4x^2\ln|x| \\ &= 3x^2 - 3x^2 + 2x^2\ln|x| - 6x^2\ln|x| + 4x^2\ln|x| \\ &= 0 \end{aligned}$$

Write the general solution of the DE including the interval of the solution

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1x^2 + c_2x^2\left(\ln|x| + C\right) \\ &= c_1x^2 + c_2x^2\ln|x| + Cc_2x^2 \\ \text{just } y &= c_1x^2 + c_2x^2\ln|x| \text{ on } I = (0,\infty)\,, y(2) = 3, y'(2) = 5 \end{aligned}$$

4.2.4 – Example

$$3y'' + y' - 4y = 0$$

$$\begin{aligned} y &= e^{mx} \\ y' &= me^{mx} \\ y'' &= m^2e^{mx} \end{aligned}$$

$$\begin{aligned} 3y'' + y' - 4y &= 3m^2e^{mx} + me^{mx} - 4e^{mx} \\ &= e^{mx}(3m^2 + m - 4) \\ &= e^{mx}(3m^2 + 4)(m - 1) \end{aligned}$$

$$m = 1 \quad m = -\frac{4}{3}$$

$$y_1 = e^x, y_2 = e^{-\frac{4}{3}x}$$