Chapter 6

Series Solutions of Linear Equations

6.1 Solution by Infinite Series

2nd order linear DE with (possibly) variable coefficients

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

$$y'' + P(x)y' + Q(x)y = F(x)$$

6.1.1 – Review of Infinite Series Facts

Maclaurin Series

$$\sum_{n=0}^{\infty} a_n x^n$$

Power series centered at 0

Taylor Series

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

Centered at a = 0
It's a theorem that power series either

(1) Converge all real numbers x on the interval $I = (-\infty, \infty)$ and the radius of convergence is $R = \infty$

(2) Converge only when $x = a$ on the interval $I = [a, a]$ and the radius of convergence is $R = 0 = \{a\}$

(3) The series converges on an interval centered at a finite, non-zero radius $R = (a - R, a + R)$

6.1.2 – Ratio Test

Use the Ratio Test to determine which of these 3 cases occurs in a specific problem.
The 3 cases of the ratio test are:

$L < 1$, the series converges

$L > 1$, the series diverges

$L = 1$,

6.1.3 – Example

Determine the radius and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n(n+1)}$$

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{3^{n+1}(n+2)}\right|}{\left|\frac{x^n}{3^n(n+1)}\right|} \\
&= \lim_{n\to\infty} \frac{|x|}{3}\frac{n+1}{n+2} \\
&= \frac{|x|}{3}\lim_{n\to\infty} \frac{n+1}{n+2} \\
&= \frac{|x|}{3}(1) \\
&= \frac{|x|}{3} \\
\frac{|x|}{3} &< 1 \\
|x| &< 3 \\
I &= (-3,3)
\end{aligned}$$