

# Chapter 4

## Higher Order Differential Equations

### 4.4 Nonhomogeneous, Linear DE with Constant Coefficients

#### 4.3.1 – Method of Undetermined Coefficients 2

For Solving Linear, Non-homogeneous DE with constant coefficients

$$a_2y'' + a_1y' + a_0y = f(x)$$

Standard Form:

$$y'' + a_1y' + a_0y = g(x)$$

#### 4.3.2 – Steps

Step 1) Solve  $y'' + a_1y' + a_0y = 0$  called the general solution  $y_c$ .

Step 2) Find one particular solution  $y_p$  of the given DE and the general solution is

$$y = y_c + y_p$$

This method can only be used when  $g(x)$  is a polynomial (An exponential (i.e.  $e^{kx}$ ), sines or cosines or sums of products of these types of functions)

#### 4.3.3 – Example

$$y'' - 3y' - 4y = 4\cos(3x)$$

1st solve:

$$\begin{aligned}
 y'' - 3y' - 4y &= 0 \\
 m^2 e^{mx} - 3m e^{mx} - 4e^{mx} &= 0 \\
 m^2 - 3m - 4 &= 0 \\
 (m - 4)(m + 1) &= 0 \\
 m - 4 = 0 \quad m + 1 = 0 \\
 m = 4 \quad m = -1 \\
 y_c &= c_1 e^{4x} + c_2 e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 y &= A \cos(3x) + B \sin(3x) \\
 y' &= -3A \sin(3x) + 3B \cos(3x) \\
 y'' &= -9A \cos(3x) - 9B \sin(3x)
 \end{aligned}$$

$$\begin{aligned}
 y'' - 3y' - 4y &= 4 \cos(3x) \\
 (-9A \cos(3x) - 9B \sin(3x)) - 3(-3A \sin(3x) + 3B \cos(3x)) - 4(A \cos(3x) + B \sin(3x)) &= 4 \cos(3x) \\
 -9A \cos(3x) - 9B \sin(3x) + 9A \sin(3x) - 9B \cos(3x) - 4A \cos(3x) - 4B \sin(3x) &= 4 \cos(3x) \\
 -9A \cos(3x) - 9B \cos(3x) - 4A \cos(3x) - 9B \sin(3x) + 9A \sin(3x) - 4B \sin(3x) &= 4 \cos(3x) \\
 \cos(3x)(-9A - 9B - 4A) + \sin(3x)(-9B + 9A - 4B) &= 4 \cos(3x) \\
 \cos(3x)(-13A - 9B) + \sin(3x)(9A - 13B) &= 4 \cos(3x) \\
 \begin{cases} -13A & -9B & = 4 \\ 9A & -13B & = 0 \end{cases} &\text{Solve simultaneously}
 \end{aligned}$$

One way to solve Linear Systems of Equations is called Cramer's Rule.

$$\det \begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}$$

$$A = \frac{\begin{bmatrix} 4 & -9 \\ 0 & -13 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}}$$

$$= \frac{4(-13) - 0(-9)}{-13(-13) - 9(-9)}$$

$$= \frac{-52 - 0}{169 + 81}$$

$$= -\frac{52}{250}$$

$$= -\frac{26}{125}$$

$$B = \frac{\begin{bmatrix} -13 & 4 \\ 9 & 0 \end{bmatrix}}{\begin{bmatrix} -13 & -9 \\ 9 & -13 \end{bmatrix}}$$

$$= \frac{-13(0) - 4(9)}{250}$$

$$= \frac{0 - 36}{250}$$

$$= -\frac{36}{250}$$

$$= -\frac{18}{125}$$

Check:

$$(-13) \left( -\frac{26}{125} \right) + (-9) \left( -\frac{18}{125} \right) ? = 4$$

$$\frac{338}{125} + \frac{162}{125} ? = 4$$

$$\frac{500}{125} = 4$$

$$9 \left( -\frac{26}{125} \right) + (-13) \left( -\frac{18}{125} \right) ? = 0$$

$$-\frac{234}{125} + \frac{234}{125} ? = 0$$

$$0 = 0$$

So

$$y = -\frac{26}{125} \cos(3x) - \frac{18}{125} \sin(3x) + c_1 e^{4x} + c_2 e^{-x}$$

is the general solution to the given DE.

**4.3.4 – Example**

$$y'' - 5y' + 4y = 8e^x$$

If we try:

$$\begin{aligned} y_p &= Ae^x \\ Ae^x - 5Ae^x + 4Ae^x &= 8e^x \\ e^x(A - 5A + 4) &= 8e^x \\ 0 &= 8e^x \text{ has no solution.} \end{aligned}$$

Solve

$$y'' - 5y' + 4y = 0$$

1st

$$\begin{aligned} m^2 - 5m + 4 &= 0 \\ (m - 1)(m - 4) &= 0 \\ m - 1 = 0 \quad m - 4 &= 0 \\ m = 1 \quad m &= 4 \\ y_1 = e^{1mx} \quad y_2 &= e^{4mx} \\ y_1 = e^{mx} \quad y_2 &= e^{4mx} \\ y_c = c_1 e^{mx} + c_2 e^{4mx} \end{aligned}$$

Suppose we have a 5th order DE with

$$a_5 y^{(5)} + a_4 y^{(4)} + \cdots + a_1 y' + a_0 y = g(x)$$

and the auxiliary equation factors as

$$\begin{aligned} m^2(m - 3)(m - (2 + i))(m - (2 - i)) &= 0 \\ m = 0 \text{ (multiplicity 2)} \quad m = 3 \quad m = 2 + i \quad m = 2 - i \end{aligned}$$

**Step 1**

Write the general solution to the complimentary DE

$$\begin{aligned} y_1 = e^{0x} &= 1 \\ y_2 = x e^{0x} &= x \\ y_3 = e^{3x} &= e^{3x} \\ y_4 = e^{(2+i)x} &= e^{2x} \cos(x) \\ y_5 = e^{(2-i)x} &= e^{2x} \sin(x) \\ y_c &= c_1 + c_2 x + c_3 e^{3x} + e^{2x} \cos(x) + e^{2x} \sin(x) \end{aligned}$$

**4.3.5 – What would you guess for the form of  $y_p$ ?**

If

$$(ii) \quad g(x) = e^{5x} \Rightarrow y_p = Ae^{5x}$$

$$(iii) \quad g(x) = e^{3x} \Rightarrow y_p = Axe^{3x} \text{ (because } e^{3x} \text{ is in } y_c)$$

$$(iv) \quad g(x) = 5e^{2x} \sin(x) \Rightarrow y_p = (Ae^{2x} \cos(x) + Be^{2x} \sin(x)) x$$

$$(v) \quad g(x) = 6x^2 e^{4x} \Rightarrow y_p = (Ax^2 + Bx + C) e^{4x}$$

$$(vi) \quad g(x) = x^2 e^{3x} \Rightarrow y_p = (Ax^2 + Bx + C) e^{3x} x$$

$$(Ax^2 + Bx + C) e^{3x}$$

**4.6 Methods of Variation of Parameters**

This method can be used for linear, non-homogenous, DE with constant coefficients and any  $g(x)$  function.

**4.6.1 – Example**

Suppose you have a 2nd order DE

$$y'' + P(x)y' + Q(x)y = g(x)$$

1st solve complementary DE

$$y_c = c_1 y_1 + c_2 y_2$$

Guess

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

for some functions  $u_1(x)$  and  $u_2(x)$ . Plug into DE, make an additional assumption on  $u_1(x)$ ,  $u_2(x)$ . Get

$$u_1(x) = \frac{W_1}{W}$$

and

$$u_2' = \frac{W_2}{W}$$