# Chapter 8

# Systems of Linear First-Order Differential Equations

## 8.1 Preliminary Theory – Linear Systems

In this chapter, we assume the system can be put in the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

Further assume $g_1, g_2, \ldots, g_n$ are linear with respect to $x_1$, $x_2$, $\ldots$, $x_n$.

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

In matrix notation:

$$X' = AX + F$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$F = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} = [a_{ij}(t)] \quad \begin{matrix} i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \end{matrix}$$

In general, if all the $a_{ij}(t)$'s and $f_i(t)$'s are continuous on an interval $I$, then the IVP $X' = AX + F$ has a unique solution:

$$\begin{aligned} x_1(t_0) &= w_1 \\ x_2(t_0) &= w_2 \\ &\dots \\ x_n(t_0) &= w_n \end{aligned}$$

where $w_1, w_2, \dots, w_n$ are just numbers.

If the initial conditions aren't given, then we want the general solution

$$X = c_1 X_1 + c_2 X_2 + \dots + c_n X_n$$

where each $X_i$ is a solution of $X' = Ax$ and $\{X_1(t), X_2(t), \dots, X_n(t)\}$[1] is a linearly independent collection of solutions. This will be true iff the Wronskian$(X_1, X_2, \dots, X_n)$ is non-zero[2]. The set $\{X_1, X_2, \dots, X_n\}$ is called a fundamental set.

[1] For a system of $n$ equations

[2] Wronskian$(X_1, X_2, \dots, X_n) = \det(X_1, X_2, \dots, X_n)$