Example 1 Pendulum

$\phi(t)$ = angular displacement.
$\phi(0) = \theta$ = initial angular displacement.
Let t_ϕ = period of the pendulum (time for 1 complete cycle)

What factors determine $\mathbf{t}_\phi$ ($[t_\phi = T]$)?

1. g
   - $[g] = LT^{-2}$
2. m – mass
   - $[m] = M$
3. r – length
   - $[r] = L$
4. θ = initial displacement
   - $[\theta] = 1$
5. air resistance (drag)
6. friction in hinge

Assume $t_\phi = f(m, g, r, \theta)$ Dimension reduction:

$$\begin{aligned}
[t_\phi] &= [m^a g^b r^c \theta^d] \\
[t_\phi] &= [m]^a [g]^b [r]^c [\theta]^d \\
M^0 L^0 T^1 &= M^a \left(LT^2\right)^b L^c \left(M^0 L^0 T^0\right)^d \\
&= M^a L^b T^{2b} L^c M^{0d} L^{0d} T^{0d} \\
&= M^a L^b T^{2b} L^c (1) \\
&= M^a L^{b+c} T^{2b}
\end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right] = (r_2 \leftarrow 2r_2 + r_3) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$= (r_2 \leftrightarrow r_3) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right]$$

$$= \left(r_2 \leftarrow -\frac{r_2}{2}\right) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 2 & 0 & 0 \end{array}\right]$$

$$= \left(r_3 \leftarrow \frac{r_3}{2}\right) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$a = 0 \quad b = -\frac{1}{2} \quad c = 0 \quad d = k$$

$$t_\phi = m^0 g^{-\frac{1}{2}} r^c \theta^d$$

Buckingham's Pi Theorem

Let $q =$ physical quantity (measurable). q depends on n parameters $p_1, \ldots, p_n$

$$q = f(p_1, p_2, \ldots, p_n)$$

Assume fundamental dimensions h, T, M.

$$[q] = L^{l_0} T^{t_0} M^{m_0}$$
$$[p_i] = L^{l_i} T^{t_i} M^{m_i}$$
$$[q] = [p_1]^{a_1} [p_2]^{a_2} \ldots [p_n]^{a_n}$$

Objective: find a_j where $j = 1, 2, \ldots, n$

$$L^{l_0} T^{t_0} M^{m_0} = \left(L^{l_1} T^{t_1} M^{m_1}\right)^{a_1} \times \left(L^{l_2} T^{t_2} M^{m_2}\right)^{a_2} \times \cdots \times \left(L^{l_n} T^{t_n} M^{m_n}\right)^{a_n}$$

$$L : l_0 =$$

$$A\vec{x} = \vec{b} \text{ has a solution of form}$$
$$\vec{x} = \vec{x_\phi} + \gamma_1 \vec{x_1} + \gamma_2 \vec{x_2} + \cdots + \gamma_n \vec{x_n}$$

where γ is arbitrary, $\vec{x_\phi}$ is the particular solution, and $\{\vec{x_j}\}$ is the null space of A.

Example 2

Drag on a sphere within a fluid. The sphere has radius R moving at velocity v with a resistant drag force D_F. What factors play a role?

$R =$ radius $[L]$

$v =$ velocity $[LT^{-1}]$

$\rho =$ density $[ML^{-3}]$

$\mu =$ dynamic viscosity $[ML^{-1}T^{-1}]$

$$\begin{aligned}
D_F &= f(R, v, \rho, \mu) \\
[D_F] &= [R]^a[v]^b[\rho]^c[\mu]^d \\
MLT^{-2} &= L^a \left(LT^{-1}\right)^b \left(ML^{-3}\right)^c \left(ML^{-1}T^{-1}\right)^d \\
MLT^{-2} &= L^a \times L^bT^{-b} \times M^cL^{-3c} \times M^dL^{-d}T^{-d} \\
MLT^{-2} &= M^{c+d}L^{a+b-3c-d}T^{-b-d}
\end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\left[\begin{array}{cccc|c} 1 & 1 & -3 & -1 & 1 \\ 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] &= (r_2 \leftarrow -r_2) \left[\begin{array}{cccc|c} 1 & 1 & -3 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \\
&= (r_1 \leftarrow r_1 - r_2) \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \\
&= (r_1 \leftarrow r_1 + 3r_3) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]
\end{aligned}$$

$$\begin{array}{ccc} a + d = 0 & b + d = 2 & c + d = 1 \\ a = -d & b = -d + 2 & c = -d + 1 \end{array}$$

Example 3 T-Rex Top Speed

How fast can a Tyrannosaurus Rex (T-Rex) walk or run?
Develop a model!
Find $v =$ velocity $[v] = \left[\frac{L}{T}\right]$
What variables are important?

m mass, $[M]$

g gravity, $[LT^{-2}]$

h hip length, $[L]$

s stride length – distance between 2 steps, $[s] = L$

$$\begin{aligned}
v &= f(s, h, g, m) \\
[v] &= [s^a h^b g^c m^d] \\
LT^{-1} &= L^a L^b \left(LT^{-2}\right)^c M^d \\
LT^{-1} &= L^{a+b} L^c T^{-2c} M^d \\
LT^{-1} &= L^{a+b+c} T^{-2c} M^d
\end{aligned}$$

Equate exponents:

$$\begin{aligned}
L: &\quad 1 = a + b + c \\
T: &\quad -1 = -2c \\
M: &\quad 0 = d
\end{aligned}$$