Example 1 Pendulum

$\phi(t)$ = angular displacement.
$\phi(0) = \theta$ = initial angular displacement.
Let t_p = period of the pendulum (time for 1 complete cycle)
What factors determine $\mathbf{t_p}$ ($[t_p] = T$)?

1. g
   - $[g] = LT^{-2}$
2. m – mass
   - $[m] = M$
3. r – length
   - $[r] = L$
4. θ = initial displacement
   - $[\theta] = 1$
5. air resistance (drag)
6. friction in hinge

Assume $t_p = f(m, g, r, \theta)$ Dimension reduction:

$$\begin{aligned}
[t_p] &= [m^a g^b r^c \theta^d] \\
[t_p] &= [m]^a [g]^b [r]^c [\theta]^d \\
M^0 L^0 T^1 &= M^a \left(LT^2\right)^b L^c \left(M^0 L^0 T^0\right)^d \\
&= M^a L^b T^{2b} L^c M^{0d} L^{0d} T^{0d} \\
&= M^a L^b T^{2b} L^c (1) \\
&= M^a L^{b+c} T^{2b}
\end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right] = (r_2 \leftarrow 2r_2 + r_3) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right]
$$

$$
= (r_2 \leftrightarrow r_3) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \end{array}\right]
$$

$$
= \left(r_2 \leftarrow -\frac{r_2}{2}\right) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 2 & 0 & 1 \end{array}\right]
$$

$$
= \left(r_3 \leftarrow \frac{r_3}{2}\right) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \end{array}\right]
$$

$$
a = 0 \quad b = -\frac{1}{2} \quad c = \frac{1}{2} \quad d = \text{arbitrary constant}
$$

$$
\begin{aligned}
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} &= \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ d \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \quad d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= g^{-\frac{1}{2}} r^{\frac{1}{2}} + \qquad\qquad \theta^1 \\
&= \sqrt{g^{-1} r} + \theta \\
&= \sqrt{\frac{r}{g}} + \theta
\end{aligned}
$$

Note the dimensions:

$$
\begin{aligned}
\left[\sqrt{\frac{r}{g}}\right] &= \sqrt{\frac{[r]}{[g]}} \\
&= \sqrt{\frac{L}{LT^{-2}}} \\
&= \sqrt{\frac{1}{T^{-2}}} \\
&= \sqrt{T^2} \\
&= T \\
[\theta] &= 1
\end{aligned}
$$

$$
\text{nullspace} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \text{non-dimensional quantity!}
$$

$$t_p = \alpha\sqrt{\frac{r}{g}}\theta^d$$

where α is a non-dimensional constant ($[\alpha] = 1$)

$$t_p = \sqrt{\frac{r}{g}}\left[\alpha, d_1\theta^d + d_2\theta^d + d_3\theta^d + \dots\right]$$

where all the terms within the brackets can be defined as $h(\theta)$ (and the α's are hidden in $h(\theta)$ as well). So:

$$t_p = \sqrt{\frac{r}{g}}h(\theta)$$

where $h(\theta)$ can be found using data. The completely dimensionless form would be:

$$\frac{t_p}{\sqrt{\frac{r}{g}}} = h(\theta)$$

Example 2 Pendulum ODE

Derivation of ODE describing motion.

Newton's 2nd Law - rotational form

τ = torque (force)

- $[\tau] = \frac{ML}{T^2}$

I = moment of inertia (like mass; measures how difficult it is to change the rotational force)

α = angular acceleration (a)

- $\frac{1}{T^2}$

$$\alpha = \frac{\tau}{I} \text{ or } \tau = I\alpha$$

What is the dimension for I?

$$\begin{aligned}[\tau] &= [I\alpha]\\ &= [I][\alpha]\\ \frac{ML^2}{T^2} &= [I]\frac{1}{T^2}\\ ML^2 &= [I]\end{aligned}$$

In physics, I = mass $\times$ square of the distance from axis = mr^2

$$\begin{aligned}
\tau &= I\frac{d^2\psi}{dt^2} \\
&= \vec{r} \times \vec{f} \\
&= -mgr\sin(\psi) \\
I\frac{d^2\psi}{dt^2} &= -mgr\sin(\psi) \\
mr^2\frac{d^2\psi}{dt^2} &= -mgr\sin(\psi) \\
r\frac{d^2\psi}{dt^2} &= -g\sin(\psi) \\
r\frac{d^2\psi}{dt^2} + g\sin(\psi) &= 0 \\
\frac{d^2\psi}{dt^2} + \frac{g}{r}\sin(\psi) &= 0
\end{aligned}$$

Assuming that θ is small, $\sin(\psi) \approx \psi$.

$$\frac{d^2\psi}{dt^2} + \frac{g}{r}\psi = 0, \quad \psi(0) = \theta, \quad \psi'(0) = 0$$

Since this is a 2nd order, homogenous function, we can assume $\psi = e^{\lambda t}$

$$\begin{aligned}
0 &= \lambda^2 e^{\lambda t} + \frac{g}{r}e^{\lambda t} \\
&= e^{\lambda t}\left(\lambda^2 + \frac{g}{r}\right) \\
&= \lambda^2 + \frac{g}{r} \\
-\lambda^2 &= \frac{g}{r} \\
\lambda^2 &= -\frac{g}{r} \\
\lambda &= \pm\sqrt{-\frac{g}{r}} \\
&= \pm\sqrt{\frac{g}{r}} \times \sqrt{-1} \\
&= \pm\sqrt{\frac{g}{r}}i
\end{aligned}$$

$$\text{The fundamental set is } \left\{e^{i\sqrt{\frac{g}{r}}t}, e^{-i\sqrt{\frac{g}{r}}t}\right\}$$

Their real form (from ODE course) is

$$\psi = c_1\cos\left(\sqrt{\frac{g}{r}}t\right) + c_2\sin\left(\sqrt{\frac{g}{r}}t\right) \tag{1}$$

c_1 and c_2 can be found using the initial conditions $\psi(0) = \theta$ and $\psi'(0) = 0$.

$$\begin{aligned}
\psi(0) &= c_1 \cos\left(\sqrt{\frac{g}{r}}(0)\right) + c_2 \sin\left(\sqrt{\frac{g}{r}}(0)\right) \\
\theta &= c_1 \cos(0) + c_2 \sin(0) \\
\theta &= c_1(1) + c_2(0) \\
\theta &= c_1 \\
c_1 &= \theta
\end{aligned}$$

$$\begin{aligned}
\psi'(t) &= -\sqrt{\frac{g}{r}} c_1 \sin\left(\sqrt{\frac{g}{r}}t\right) + \sqrt{\frac{g}{r}} c_2 \cos\left(\sqrt{\frac{g}{r}}t\right) \\
\psi'(0) &= -\sqrt{\frac{g}{r}} c_1 \sin\left(\sqrt{\frac{g}{r}}(0)\right) + \sqrt{\frac{g}{r}} c_2 \cos\left(\sqrt{\frac{g}{r}}(0)\right) \\
0 &= -\sqrt{\frac{g}{r}} c_1 \sin(0) + \sqrt{\frac{g}{r}} c_2 \cos(0) \\
0 &= -\sqrt{\frac{g}{r}} c_1(0) + \sqrt{\frac{g}{r}} c_2(1) \\
0 &= \sqrt{\frac{g}{r}} c_2 \\
0 &= c_2 \\
c_2 &= 0
\end{aligned}$$

$$\begin{aligned}
\psi &= c_1 \cos\left(\sqrt{\frac{g}{r}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{r}}t\right) \\
\psi &= \theta \cos\left(\sqrt{\frac{g}{r}}t\right) + 0 \sin\left(\sqrt{\frac{g}{r}}t\right) \\
\psi &= \theta \cos\left(\sqrt{\frac{g}{r}}t\right)
\end{aligned}$$

Buckingham's Pi Theorem

Let $q =$ physical quantity (measurable). q depends on n parameters and variables $p_1, \ldots, p_n$

$$q = f(p_1, p_2, \ldots, p_n)$$

Assume fundamental dimensions h, T, M.

$$\begin{aligned}
[q] &= L^{l_0} T^{t_0} M^{m_0} \\
[p_i] &= L^{l_i} T^{t_i} M^{m_i} \\
[q] &= [p_1]^{a_1} [p_2]^{a_2} \ldots [p_n]^{a_n}
\end{aligned}$$

Objective: find a_j where $j = 1, 2, \ldots, n$

$$L^{l_0}T^{t_0}M^{m_0} = \left(L^{l_1}T^{t_1}M^{m_1}\right)^{a_1} \times \left(L^{l_2}T^{t_2}M^{m_2}\right)^{a_2} \times \cdots \times \left(L^{l_n}T^{t_n}M^{m_n}\right)^{a_n}$$

Equate exponents!

$$L : l_0 = l_1a_1 + l_2a_2 + \cdots + l_na_n$$
$$T : t_0 = t_1a_1 + t_2a_2 + \cdots + t_na_n$$
$$M : m_0 = m_1a_1 + m_2a_2 + \cdots + m_na_n$$

$$A = \begin{bmatrix} l_1 & l_2 & \dots & l_n \\ t_1 & t_2 & \dots & t_n \\ m_1 & m_2 & \dots & m_n \end{bmatrix}_{3\times n} \qquad \vec{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1} \qquad \vec{b} = \begin{bmatrix} l_0 \\ t_0 \\ m_0 \end{bmatrix}_{3\times 1}$$

In general, $n > 3$ – undetermined $\Rightarrow \infty$ set of solutions.

$$A\vec{x} = \vec{b} \text{ has a solution of form}$$
$$\vec{x} = \vec{x_p} + \gamma_1\vec{x_1} + \gamma_2\vec{x_2} + \cdots + \gamma_k\vec{x_k}$$

where γ is arbitrary, $\vec{x_p}$ is the particular solution, and $\{\vec{x_j}\}$ is the null space of A.

- $\vec{x_j}$ correspond to exponents that yield dimensionless quantities

$$\text{defines: } \Pi_j, j = 1, \dots, k$$

- $\vec{x_p}$ exponents that yield a quantity with $[q]$, call it Q

So

$$q = QF(\Pi_1, \Pi_1, \dots, \Pi_k)$$
$$\pi_0 = \frac{q}{Q} = QF(\Pi_1, \Pi_1, \dots, \Pi_k)$$

where the second equation is all in terms of dimensionless quantities

Example 3 Drag on a sphere

Drag on a sphere within a fluid. The sphere has radius R moving at velocity v with a resistant drag force D_F. What factors play a role?

$R =$ radius $[L]$

$v =$ velocity $[LT^{-1}]$

$\rho =$ density $[ML^{-3}]$

$\mu =$ dynamic viscosity $[ML^{-1}T^{-1}]$

$$\begin{aligned}
D_F &= f(R, v, \rho, \mu) \\
[D_F] &= [R]^a [v]^b [\rho]^c [\mu]^d \\
MLT^{-2} &= L^a \left(LT^{-1}\right)^b \left(ML^{-3}\right)^c \left(ML^{-1}T^{-1}\right)^d \\
MLT^{-2} &= L^a \times L^b T^{-b} \times M^c L^{-3c} \times M^d L^{-d} T^{-d} \\
MLT^{-2} &= M^{c+d} L^{a+b-3c-d} T^{-b-d}
\end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\left[\begin{array}{cccc|c} 1 & 1 & -3 & -1 & 1 \\ 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] &= (r_2 \leftarrow -r_2) \left[\begin{array}{cccc|c} 1 & 1 & -3 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \\
&= (r_1 \leftarrow r_1 - r_2) \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & -1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \\
&= (r_1 \leftarrow r_1 + 3r_3) \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]
\end{aligned}$$

$$a + d = 2 \qquad b + d = 2 \qquad c + d = 1$$

$$a = 2 - d \qquad b = -d + 2 \qquad c = -d + 1$$

$$\begin{aligned}
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} &= \begin{bmatrix} 2-d \\ 2-d \\ 1-d \\ d \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\
&= R^2 v^2 \rho^1 + R^{-1} v^{-1} \rho^{-1} \mu^1 \\
&= R^2 v^2 \rho + \frac{\mu}{Rv\rho} \\
&= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + {\color{red}-d} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \\
&= R^2 v^2 \rho^1 + R^1 v^1 \rho^1 \mu^{-1} \\
&= R^2 v^2 \rho + \frac{Rv\rho}{\mu}
\end{aligned}$$

$$D_F = R^2 v^2 \rho F \left(\frac{Rv\rho}{\mu}\right)$$

where $\frac{Rv\rho}{\mu}$ represents the Reynold's number Re

Reynold's number

$$Re = \frac{Rv\rho}{\mu} = \frac{\text{inertial}}{\text{viscous}} \tag{2}$$

When Re is small, there will be laminar (smooth) flow, when Re is large, the flow will be turbulent (having vortices).

Example 4 T-Rex Top Speed

How fast can a Tyrannosaurus Rex (T-Rex) walk or run?
Develop a model!
Find $v =$ velocity $[v] = \left[\frac{L}{T}\right]$
What variables are important?

m mass, $[M]$

g gravity, $[LT^{-2}]$

h hip length, $[L]$

s stride length – distance between 2 steps, $[s] = L$

$$\begin{aligned} v &= f(s, h, g, m) \\ [v] &= [s^a h^b g^c m^d] \\ LT^{-1} &= L^a L^b \left(LT^{-2}\right)^c M^d \\ LT^{-1} &= L^{a+b} L^c T^{-2c} M^d \\ LT^{-1} M^0 &= L^{a+b+c} T^{-2c} M^d \end{aligned}$$

Equate exponents:

$$\begin{aligned} L: &\quad 1 = a + b + c \\ T: &\quad -1 = -2c \\ M: &\quad 0 = d \text{ (independent of mass)} \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \end{array}\right] = \left(r_2 \leftarrow -\frac{r_2}{2}\right) \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$$\begin{aligned} a + b + c = 1 &\qquad b = \text{ free } c = \frac{1}{2} \\ a + b + \frac{1}{2} = 1 &\qquad b = \text{ free } c = \frac{1}{2} \\ a = -b + \frac{1}{2} &\qquad b = \text{ free } c = \frac{1}{2} \end{aligned}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - b \\ b \\ \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
$$= s^{\frac{1}{2}} g^{\frac{1}{2}} + s^{-1} h^{1}$$
$$= \sqrt{sg} + \frac{h}{s}$$

$$b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow (-1) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow s^1 h^{-1} \Rightarrow \frac{s}{h}$$

$$v = \sqrt{gh} f \left(\frac{s}{h} \right)$$

Froude number

As found in the previous example:

$$\frac{v}{\sqrt{gh}} = \frac{\text{inertial force}}{\text{gravitational force}}$$