Contents

Dynamics

$$\frac{dS}{dt} =$$

$$\frac{dI}{dt} = \lambda(N-I)I - \mu I \tag{1}$$

Dimensional reduction (see lecture 19) $x = F(\tau, R_0)$

$$\begin{aligned}
I &= Nx \\
\frac{dI}{dt} &= N\frac{dx}{dt} \\
\frac{dI}{d\tau}\frac{d\tau}{dt} &= N\frac{dx}{d\tau}\frac{d\tau}{dt} \\
\frac{dI}{d\tau} &= N\frac{dx}{d\tau}
\end{aligned}$$

$$\frac{dx}{dt} = R_0(1-x)x - x \tag{2}$$

R_e = effective reproductive number. i.e. number of new infections by a single individual at a specific time. R_e is a function of time!

Locate the rest points of (2)

$$\begin{aligned}
0 &= g(x) \\
&= -x + R_0 x(1-x) \\
&= x\left[-1 + R_0(1-x)\right]
\end{aligned}$$

$$x = 0 \qquad \begin{aligned}
-1 + R_0(1-x) &= 0 \\
R_0(1-x) &= 1 \\
1 - x &= \frac{1}{R_0} \\
x &= \frac{1}{R_0} + 1
\end{aligned}$$

Rest or equilibrium:

$$x = 1 - \frac{1}{R_0} \Rightarrow \begin{cases} > 0 & \text{if } R_0 > 1 \\ < 0 & \text{if } R_0 < 1 \end{cases}$$

Test stability by looking at $g'(x)$.

$$\begin{aligned} g'(x) &= -1 + R_0(1-2x) \\ g'(0) &= -1 + R_0(1-2(0)) \\ &= -1 + R_0(1-2(0)) \\ &= -1 + R_0(1-0) \\ &= -1 + R_0(1) \\ &= -1 + R_0 \\ &\Rightarrow \begin{cases} > 0 & R_0 > 1 \Rightarrow \text{unstable} \\ < 0 & R_0 < 1 \Rightarrow \text{stable} \end{cases} \end{aligned}$$

$$\begin{aligned} g'\left(1 - \frac{1}{R_0}\right) &= -1 + R_0\left(1 - 2\left(1 - \frac{1}{R_0}\right)\right) \\ &= -1 + R_0\left(1 - 2 + \frac{2}{R_0}\right) \\ &= -1 + R_0\left(-1 + \frac{2}{R_0}\right) \\ &= -1 - R_0 + 2 \\ &= 1 - R_0 \\ &\Rightarrow \begin{cases} < 0 & R_0 > 1 \Rightarrow \text{stable} \\ > 0 & R_0 < 1 \Rightarrow \text{unstable} \end{cases} \end{aligned}$$

SIS model - alternate model

$$\frac{dS}{dt} = -\beta \frac{I}{N} S + \mu I \tag{3}$$

where β is the average # of contacts per unit time, $\frac{I}{N}$ is the probability of selecting an infective, μ is the recovery rate, and $\frac{1}{\mu}$ is the $E[infected\ time]$. $\beta\frac{I}{N}$ if the force of infection ...

$$\begin{aligned} R_0 &\propto \left(\frac{\text{infections}}{\text{contact}}\right)\left(\frac{\text{contacts}}{\text{time}}\right)\left(\frac{\text{time}}{\text{infection}}\right) \\ &= 1 \times \beta \times \frac{1}{\mu} \\ &= \frac{\beta}{\mu} \end{aligned}$$

Early COVID-19 Model

SIR

S Susceptible

I Infected

R Recovered (permanent immunity) or removed (death)

$$S \xrightarrow{\beta} I \xrightarrow{\mu} R$$

$\frac{I}{N}$ = probability of selecting an infected. $\frac{BI}{N}$ = force of infection. $\beta\frac{I}{N}S$Rate of new infectives.

$$\frac{dS}{dt} = -\beta\frac{I}{N} \tag{4}$$