

Math 486/522 - Homework 3

Fall 2024

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1. In class (Lecture 5), we considered a simplified drug path diffusion model. Let $u(x, t)$ (M/L in 1 space dim.) be the concentration of a drug at depth x from the surface of the skin at time t . We found that u satisfies the diffusion equation problem

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0$$

$$\text{IC: } u(x, 0) = 0, \quad \text{BC: } u(0, t) = u_0, \quad u(\infty, t) = 0.$$

Using the similarity variable $\eta = \frac{x}{\sqrt{Dt}}$, we converted the PDE to an ODE and derived the solution

$$u(x, t) = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}\right), \quad \operatorname{erfc}(y) = 1 - \frac{2}{\sqrt{\pi}} \int_0^y e^{-r^2} dr. \quad (1)$$

If $u_0 = 10$ and $D = 1/4$ then

- (a) Make a graph of $u(2, t)$, i.e. at depth $x = 2$, from $t = 0$ to $t = 100$. Does this make sense? [Problem 1a answer here.](#)
 - (b) Find the first time t when the concentration at depth $x = 2$ is equal to 4. [Problem 1b answer here.](#)
 - (c) On the same set of axes, plot $u(x, t)$ from $x = 0$ to $x = 10$ for the sequence of times $t = 1, 5, 10, 50$. Please label each graph with its appropriate time t . [Problem 1c answer here.](#)
2. Consider a drug delivery system in which the drug is delivered with a constant flow or flux $J = -D \frac{\partial u}{\partial x}$ at the surface of the skin, $x = 0$. You can think of it as a drug pump at $x = 0$. We assume $u(x, t)$ (M/L in 1 space dim.) is the concentration satisfying a chemical diffusion problem with a constant flow or flux at the boundary $x = 0$:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0$$

$$\text{IC: } u(x, 0) = 0, \quad \text{BC: } D \frac{\partial u}{\partial x}(0, t) = -A, \quad u(\infty, t) = 0.$$

- (a) What are the dimensions of D and A ? [Problem 2a answer here.](#)
- (b) Use dimensional reduction to derive a form of $u(x, t)$. Is there a similarity variable, if so call it η . Be sure your similarity variable vanishes at $x = 0$ so $\eta = 0$. [Problem 2b answer here.](#)
- (c) Use the class notes to transform the diffusion problem into an ODE-BVP. Verify that the boundary conditions are consistent with the IC and BC of the diffusion problem. [Problem 2c answer here.](#)

3. A more realistic model of the drug patch has the drug diffusing in the skin with diffusion coefficient D but also degrading according to a first order chemical reaction with rate constant k , lookup a first chemical reaction online. The chemical concentration of the drug is $u(x, t)$ (M/L) in the skin and the patch at $x = 0$ provides a concentration $u(0, t) = Ae^{-kt}$. $k > 0$, also decaying in time. The PDE problem for the concentration u of the drug is:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - ku, \quad x > 0, \quad t > 0 \quad (2)$$

$$\text{IC: } u(x, 0) = 0 \quad (3)$$

$$\text{BC: } u(0, t) = Ae^{-kt}, \quad u(\infty, t) = 0. \quad (4)$$

- (a) Describe the meaning of each term on the right side of equation (3). [Problem 3a answer here.](#)
- (b) If the concentration is independent of x , i.e. a simple first order chemical reaction, state and solve the ODE for kinetics of the reaction assuming as initial concentration of A . You can look for a solution of (3) that only depends on t . [Problem 3b answer here.](#)
- (c) Suppose the boundary condition at $x = 0$ in (5) is changed to $u(0, t) = A$. Use (3) to find an equation for the long-time or steady-state concentration, i.e. $U(x) = \lim_{t \rightarrow \infty} u(x, t)$, i.e. independent of t . Find $U(x)$. [Problem 3c answer here.](#)
- (d) In the original problem (3)-(5), find the dimensions of A , D and k . [Problem 3d answer here.](#)
- (e) Use dimensional analysis to identify three dimensionless quantities that involve the variables and parameters in the problem. [Problem 3e answer here.](#)
4. ODE review. Please review Dawkin's ODE notes, if needed. Classify and solve:
- (a) Find the general solution $y(t) : \frac{dy}{dt} = 2t - y$. [Problem 4a answer here.](#)
- (b) Solve the initial value problem $y(t) : \frac{dy}{dt} - 2ty = 2, y(0) = 0$ and write your solution in terms of the error function: $\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-z^2} dz$. [Problem 4b answer here.](#)
- (c) Find the general solution $y(t)$ to:
- (i) $y'' - 4y = 0$ [Problem 4c\(i\) answer here.](#)
- (ii) $y'' + 5y' + 6y = 0$ [Problem 4c\(ii\) answer here.](#)