Contents

Given 5 data points ()

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{b}$$

Defines a place that $\vec{b}$ is not in. The closest value would be the projection such that $\vec{b} - A\vec{y} \perp A\vec{x}$ where $\vec{y}$ is the least square solution.

$$\textsc{Residual} = r(\vec{x}) = b - A\vec{x} \tag{1}$$

$r = 0 \Rightarrow \vec{x}$ is solution $A\vec{x} = \vec{b}$
Measure $\vec{r}$: $||\vec{r}(x)|| = ||\vec{b} - A\vec{x}||$.
Goal min $||r(x)||$
Details: $A_{m\times n} \Rightarrow A^T_{n\times m}$

$$(A+B)^T = A^T + B^T \tag{2}$$
$$(AB)^T = B^T A^T \tag{3}$$

$\perp \Rightarrow$ orthogonal.
Vectors $\vec{u}$ abd $\vec{v}$ are orthogonal if $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = 0$.

$$(\vec{b} - A\vec{y}) \perp \{A\vec{x} | \vec{x} \in \mathbb{R}^2\}$$

$$\begin{aligned}
(A\vec{x})\cdot(\vec{b} - A\vec{y}) &= 0\\
(A\vec{x})^T(\vec{b} - A\vec{y}) &= 0\\
\vec{x}^T A^T(\vec{b} - A\vec{y}) &= 0\\
A^T(\vec{b} - A\vec{y}) &= 0\\
A^T\vec{b} - A^T A\vec{y} &= 0\\
A^T_{n\times m}\vec{b} &= A^T_{n\times m}A_{m\times n}\vec{y}
\end{aligned}$$

Example 1

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$A\vec{v} = \vec{b}$ is inconsistent.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Solve normal equation: $A^T A y = A^T b$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\begin{aligned}
\left[\begin{array}{cc|c} 3 & 1 & 6 \\ 1 & 3 & 4 \end{array}\right] &= (r_2 \leftarrow r_2 - \frac{r_1}{3}) \left[\begin{array}{cc|c} 3 & 1 & 6 \\ 0 & \frac{8}{3} & 2 \end{array}\right] \\
&= \left(r_2 \leftarrow \frac{3}{8} r_2\right) \left[\begin{array}{cc|c} 3 & 1 & 6 \\ 0 & 1 & \frac{3}{4} \end{array}\right] \\
&= (r_1 \leftarrow r_1 - r_2) \left[\begin{array}{cc|c} 3 & 0 & \frac{25}{4} \\ 0 & 1 & \frac{3}{4} \end{array}\right] \\
&= \left(r_1 \leftarrow \frac{r_1}{3}\right) \left[\begin{array}{cc|c} 1 & 0 & \frac{25}{12} \\ 0 & 1 & \frac{3}{4} \end{array}\right]
\end{aligned}$$

Data: $(t_1, y_1), \ldots, (t_m, y_m)$ is the training set. Formula: $y = a + bt$

Example 2

$(t, y) = (1)$

$$\begin{aligned}
a &+ b(1) = 2 \\
a &+ b(-1) = 1 \\
a &+ b(1) = 3
\end{aligned}$$

$$SSE \Rightarrow r(x) = e_1^2 + e_2^2 + \cdots + e_m^2 \tag{4}$$

$$RMSE = \sqrt{\frac{SSE}{m}} \tag{5}$$

Data points: $\{(-1, 1), (0, 0), (1, 0), (2, -2)\}$

Example 3 Fit to a straight line

$y = a + bt$

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$