Chapter 6

Continuous Random Variables

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LO 6.5: **Calculate and interpret probabilities or a random variable that follows the normal distribution.**

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LO 6.7: **Calculate and interpret probabilities or a random variable that follows the lognormal distribution.**

6.1 Continuous Random Variables and the Uniform Probability Distribution

LO 6.1 **Describe a continuous random variable.**

- Remember that random variables may be classified as

  Discrete The random variable assumes a countable number of distinct values.

  Continuous The random variable is characterized by (infinitely) uncountable values within any interval.

- When computing probabilities for a continuous random variable, keep in mind that $P(X = x) = 0$.
  - We cannot assign a nonzero probability to each infinitely uncountable value and still have the probabilities sum to one.

- Thus, since $P(X = a)$ and $P(X = b)$ both equal zero, the following holds true for continuous random variables:

$$P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$$

6.1.1 Probability Density Function $f(x)$ of a continuous random variable X

- Describes the relative likelihood that X assumes a value within a general interval (e.g., $P(a \leq X \leq b)$), where
  - $f(x) > 0$ for all possible values of X.
  - The area under $f(x)$ over all values of x equals 1.

6.1.2 Cumulative Density Function $F(x)$ of a continuous random variable X

- For any value x of the random variable X, the cumulative distribution function $F(x)$ is computes as:

$$F(X) = P(X \leq x)$$

- As a result:

$$P(a \leq X \leq b) = F(b) - F(a)$$

6.1.3 The Continuous Uniform Distribution

LO 6.2 **Describe a continuous uniform distribution and calculate associated probabilities.**

- Describe a random variable that has an equally likely chance of assuming a value within a specified range.
- Probability density function:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \text{ and} \\ 0 & \text{for } x < a \text{ or } x > b \end{cases} \tag{6.1}$$

  where a and b are the lower and upper limits, respectively.

- The expected value and standard deviation of X are:

$$E(X) = \mu = \frac{a+b}{2} \tag{6.2}$$

$$\begin{aligned} \mathrm{SD}(X) &= \sigma \\ &= \sqrt{\frac{(b-a)^2}{12}} \end{aligned} \tag{6.3}$$

6.1.4 Graph of the continuous uniform distribution

- The values of a and b on the horizontal axis represent the lower and upper limits, respectively.
- The height of the distribution does not directly represent a probability.
- It is the area under $f(x)$ that corresponds to probability.

Cumulative function:

$$\begin{aligned} P(X > x_1) &= \text{base} \times \text{height} \\ &= (b - x_1) \times \frac{1}{b-a} \end{aligned}$$

6.2 The Normal Distribution

- For a random variable X with mean μ and variance σ^2:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{6.4}$$

6.2.1 The Normal Distribution

LO 6.3 **Explain the characteristics of the normal distribution.**

- Symmetric
- Bell-shaped
- Closely approximates the probability distribution of a wide range of random variables, such as the
  - Heights and weights of newborn babies
  - Scores on SAT
  - Cumulative debt of college graduates
- Serves as the cornerstone of statistical inference.

6.2.2 Characteristics of the Normal Distribution

- Symmetric about its mean
  - Mean = Median = Mode
- Asymptotic–that is, the tail gets closer and closer to the horizontal axis but never touches it.
- The normal distribution is completely described by two parameters: μ and σ^2.

  μ is the population mean which describes the central location of the distribution.
  σ^2 is the population variance which describes the dispersion of the distribution.

6.2.3 Probability Density Function of the Normal Distribution

- For a random variable X with mean μ and variance σ^2:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{6.5}$$

6.2.4 The Standard Normal (Z) Distribution

LO 6.4 **Use the standard normal table of the z-table.**

- A special case of the normal distribution:
  - Mean μ is qual to zero ($E(X) = 0$).
  - Standard deviation σ is equal to 1 (SD$(Z) = 1$).

6.2.5 Standard Normal Table (Z-Table)

- Gives the cumulative probabilities $P(Z \leq z)$ for positive and negative values of z.
- Since the random variable Z is symmetric about its mean of 0,

$$P(Z < 0) = P(Z > 0) = 0.5$$

- To obtain the $P(Z < z)$, read down the z-column first, then across the top.

6.2.6 Finding the Probability for a Given z-Value

- Transform normally distributed random variables into standard normal random variables and use the z-table to compute the relevant probabilities.
- The z-table provides cumulative probabilities $P(Z \leq z)$ for a given z.

6.3 Solving Problems with the Normal Distribution

LO 6.5 **Calculate and interpret probabilities or a random variable that follows the normal distribution.**

6.3.1 The Normal Transformation

- Any normally distributed random variable X with mean μ and standard deviation σ can be transformed into the standard normal random variable Z as:

$$Z = \frac{X-\mu}{\sigma} \text{ with corresponding values } z = \frac{x-\mu}{\sigma} \tag{6.6}$$

- As constructed: $E(Z) = 0$ and $\text{SD}(Z) = 1$.
- A z-value specifies by how many standard deviations the corresponding x value falls above ($z > 0$) or below ($z < 0$) the mean.
  - A positive z indicates by how many standard deviations the corresponding x lies above μ.
  - A zero z indicates that the corresponding x equals μ.
  - A negative z indicates by how many standard deviations the corresponding x lies below μ.

6.3.2 Use the Inverse Transformation to Compute Probabilities for Given x values

- A standard normal variable Z can be transformed to the normally distributed random variable X with mean μ and standard deviation σ as

$$X = \mu + Z\sigma \text{ with corresponding values } x = \mu + z\sigma \tag{6.7}$$

6.4 Other Continuous Probability Distributions

LO 6.6 **Calculate and interpret probabilities or a random variable that follows the exponential distribution.**

6.4.1 Exponential Distribution

- A random variable X follows the exponential distribution if its probability density function is:

$$f(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0 \tag{6.8}$$

  where λ is the rate parameter and $E(X) = \text{SD}(X) = \frac{1}{\lambda}$.
- The cumulative distribution function is:

$$P(X \leq x) = 1 - e^{-\lambda x} \tag{6.9}$$

6.4.2 The Lognormal Distribution

LO 6.7 **Calculate and interpret probabilities or a random variable that follows the lognormal distribution.**

- Defined for a positive random variable, the lognormal distribution is positively skewed.
- Useful for describing variables such as

- Income
- Real estate values
- Asset prices

- Failure rate may increase or decrease over time.
- Let X be a normally distributed random variable with mean μ and standard deviation σ. The random variable $Y = e^X$ follows the lognormal distribution with a probability density function as

$$f(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(y)-\mu)^2}{2\sigma^2}\right) \text{ for } y > 0 \tag{6.10}$$

- The lognormal distribution is clearly positively skewed for $\sigma > 1$. For $\sigma < 1$, the lognormal distribution somewhat resembles to normal distribution.

6.4.3 Expected values and standard deviations of the lognormal and normal distributions

- Let X be a normal random variable with mean μ and standard deviation σ and let $Y = e^X$ by the corresponding lognormal variable. The mean μ_y and standard deviation σ_Y or Y are derived as:

$$\mu_Y = \exp\left(\frac{2\mu + \sigma^2}{2}\right) \tag{6.11}$$

$$\sigma_Y = \sqrt{(\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)} \tag{6.12}$$

- Equivalently, the mean and standard deviation of the normal variable $X = \ln(Y)$ are derived as

$$\mu = \ln\left(\frac{\mu_Y^2}{\sqrt{\mu_Y^2 + \sigma_Y^2}}\right) \tag{6.13}$$

$$\sigma = \sqrt{\ln\left(1 + \frac{\sigma_Y^2}{\mu_Y^2}\right)} \tag{6.14}$$