# SPLIT COVERS FOR CERTAIN REPRESENTATIONS OF CLASSICAL GROUPS

by

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PH.D. THESIS

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# ABSTRACT

Let $R(G)$ denote the category of smooth representations of a $p$-adic group. Bernstein has constructed an indexing set $\mathcal{B}(G)$ such that $R(G)$ decomposes into a direct sum over $\mathfrak{s} \in \mathcal{B}(G)$ of full subcategories $R^{\mathfrak{s}}(G)$ known as Bernstein subcategories. Bushnell and Kutzko have developed a method to study the representations contained in a given subcategory. One attempts to associate to that subcategory a smooth irreducible representation $(\tau, W)$ of a compact open subgroup $J < G$. If the functor $V \mapsto \mathrm{Hom}_J(W, V)$ is an equivalence of categories from $R^{\mathfrak{s}}(G) \to \mathcal{H}(G, \tau) - \mathrm{mod}$ we call $(J, \tau)$ a type.

Given a Levi subgroup $L < G$ and a type $(J_L, \tau_L)$ for a subcategory of representations on $L$, Bushnell and Kutzko further show that one can construct a type on $G$ that "lies over" $(J_L, \tau_L)$ by constructing an object known as a cover. In particular, a cover implements induction of $\mathcal{H}(L, \tau_L)$-modules in a manner compatible with parabolic induction of $L$-representations.

In this thesis I construct a cover for certain representations of the Siegel Levi subgroup of $Sp(2k)$ over an archimedean local field of characteristic zero. In particular, the representations I consider are twisted by highly ramified characters. This compliments work of Bushnell, Goldberg, and Stevens on covers in the self-dual case. My construction is quite concrete, and I also show that the cover I construct has a useful property known as splitness. In fact, I prove a fairly general theorem characterizing when covers are split.

# PUBLIC ABSTRACT

Number theory studies the integers (i.e., $1, 2, 3, \ldots$) and their properties. The integers have been studied for over three thousand years, so most of the interesting questions that remain are quite difficult. Thus, to find answers, mathematicians have been forced to invent sophisticated new mathematical tools. One fruitful strategy has been to notice that the integers live inside bigger sets of numbers, like the rational numbers (i.e. fractions), and the real numbers (i.e. the whole number line), and use the extra structure of these bigger sets to better understand the integers.

The real numbers can be built from the rationals via a process that can be thought of as "filling in the holes between the numbers." In the late $19^{\text{th}}$ century mathematicians began to investigate other ways to fill in these holes. In fact, there is precisely one way for each prime number $p$. For each $p$, this "filling in" produces a set of numbers called the $p$-adic numbers. The $p$-adic numbers have proved a useful tool in number theory. A key motivation in modern research is the idea that if one can get information about the real numbers and the $p$-adic numbers for all primes $p$, one should be able to translate this into information about the integers. This goal has inspired much of the recent progress in modern number theory.

In my thesis I study the structure of certain sets of matrices with entries in the $p$-adic numbers. I construct tools that can be used to analyze their structure, and I prove a general result about the nature of these tools. It is my goal that these constructions be used to calculate further number theoretic data.

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# CHAPTER 1
# SMOOTH REPRESENTATIONS OF $P$-ADIC GROUPS

This thesis deals with the theory of types and covers. This theory was introduced by Colin Bushnell and Phil Kutzko to understand the representation theory of algebraic groups over non-archimedean local fields. Questions about representations of such groups arise naturally in the context of the Langlands program, and types have proven a useful tool in this context. In this section we review the basic theory of smooth representations of locally profinite groups.

## 1.1 Smooth Representations

Let $G$ be a topological group. We say $G$ is *locally profinite* if it is locally compact and totally disconnected. A collection of open sets $\{U_\alpha\}$ containing some common point $x \in G$ is called a *neighborhood basis* of $x$ if, given any open set $O$ containing $x$, there is some $U_\alpha$ contained in $O$. Given a neighborhood basis of a single point we can form a basis for the topology on $G$ by taking cosets. It is a classical result that $G$ is locally profinite if and only if there is a neighborhood basis of the identity in $G$ consisting of compact, open, normal subgroups of $G$ [5]. For the remainder of this section we assume $G$ is locally profinite.

By a representation of $G$ we will mean a pair $(\pi, V)$ where $V$ is a complex vector space and $\pi : G \to \mathrm{End}(V)$ is a group homomorphism. We often suppress $V$ in our notation when it is not necessary for the surrounding discussion. Let $(\pi, V)$ and $(\sigma, W)$ be representations of $G$. A morphism in the category of representations is given by a linear map $\varphi : V \to W$ with $\varphi(\pi(x)v) = \sigma(x)\varphi(v)$ for all $v \in V$ and $x \in G$. Such morphisms are called *intertwining maps.* We call a subspace of $V$ a *subrepresentation* if it is invariant under the action of $G$ via $\pi$. We will often call a subrepresentation a $G$-*space* or $G$-*subspace.*

We say a vector $v \in V$ is *smooth* if it is fixed by an open compact subgroup of $G$. Let $V^\infty$ denote the subset of smooth vectors in $V$. Then $V^\infty$ is a $G$-subspace of $V$, and we say that the representation $(\pi, V)$ is *smooth* if $V = V^\infty$. We denote the category of smooth representations of $G$ by $R(G)$. The maps in this category are simply $G$-intertwining maps, and subrepresentations are simply $G$ invariant subspaces.

Let $\varphi : G \to H$ be a homomorphism of groups. One can use $\varphi$ to transfer representations of $H$ to representations of $G$ via a process known as *inflation.* Given a representation $\pi$ of $H$, one defines the inflated representation $\sigma$ of $G$ by $\sigma(g) = \pi(\varphi(g))$.

**Proposition 1.1.** Let $G$ be a locally profinite group with an open compact subgroup $J$. Then smooth irreducible representations of $J$ are precisely inflations of irreducible representations of finite quotients of $J$. In particular, they are finite dimensional.

*Proof.* Let $(\tau, W)$ be an irreducible representation of $J$ and pick a non-zero vector $w \in W$. Since $\tau$ is smooth, $w$ is fixed by a non-trivial open subgroup $J_0 < J$. Cosets of $J_0$ form an open cover of $J$, so by compactness $J/J_0$ must be finite. Since $\tau$ is irreducible, $W$ is spanned by $Jw$, which is equal to $Sw$ for a set of coset representatives of $J/J_0$. Thus $W$ is finite dimensional.

Let $w_1, \ldots, w_n$ be a basis for $W$, and for each $i$ let $J_i$ be a non-trivial open subgroup fixing $w_i$. Set $K$ be the subgroup of $J$ that fixes all of $W$. Then $K$ is normal since it is the kernel of a group action, and $K$ contains $\bigcap_i J_i$, so it is open. Thus $\tau$ is inflated from a representation of $J/K$, which is finite since $J$ is compact and $K$ is open. □

**Lemma 1.1.** Let $J < G$ be a compact open subgroup and $(\pi, V)$ a smooth representation of $J$. Then $\pi$ is semisimple. In particular, $V$ is a sum of irreducible invariant subspaces.

*Proof.* Let $w \in V$ and let $W$ be the $G$-space generated by $w$. As in the proof of proposition 1.1 there is a normal, open, compact subgroup $J' < J$ that acts trivially on $W$. Thus the action of $J$ on $W$ is given by inflation of a representation of the finite group $J/J'$. Complex representations of finite groups are semisimple, so $W$ is a sum of irreducible representations of $J$. But $w$ was arbitrary, so this completes the proof. □

Write $\hat{J}$ for the set of isomorphism classes of smooth irreducible representations of a compact open group $J < G$. Let $\tau \in \hat{J}$ and define $V^\tau$ to be the sum of all $J$-subrepresentations of $V$ that are isomorphic to $\tau$. We call $V^\tau$ the $\tau$-isotypic subspace of $V$. By semisimplicity $V$ is the sum of its $\tau$-isotypic subspaces, where $\tau$ ranges over $\hat{J}$, and $V^\tau \cap V^\rho = \{0\}$ for $\tau \neq \rho \in \hat{J}$, since otherwise we would have $\tau \cong \rho$. Thus

$$V = \bigoplus_{\tau \in \hat{J}} V^\tau.$$

**Definition 1.1.** We call a representation $(\pi, V) \in R(G)$ *admissible* if $V^\tau$ is finite dimensional for every open compact subgroup $J < G$ and every $\tau \in \hat{J}$.

Let $(\pi, V)$ be a smooth representation of $G$. Define $V^*$ to be the dual space to $V$ and write $\langle \cdot, \cdot \rangle$ for the canonical pairing between $V^*$ and $V$. We define a representation $\pi^*$ on $V^*$ by $\langle \pi^*(g)\phi, v\rangle = \langle \phi, \pi(g^{-1})v\rangle$ for $\phi \in V^*$ and $v \in V$. Set $\breve{V} = (V^*)^\infty$, the set of smooth vectors in $V^*$ under the representation $\pi^*$, and write $\breve{\pi}$ for the representation given by $\pi^*$ restricted to $\breve{V}$. Then $\breve{\pi}$ is a smooth representation of $G$ called the *contragrediant representation* to $\pi$. One can show that $\pi$ is admissible if and only if $\pi$ is isomorphic to its own double contragrediant [2].

## 1.2 Induction and Restriction

Let $H$ be a closed subgroup of $G$ and let $(\sigma, W)$ be a representation of $H$. We can use $\sigma$ to construct a representation $(\rho, X)$ of $G$. Define $X$ to be the set of all functions $f : G \to W$ satisfying

1. $f(xy) = \sigma(x)f(y)$ for $x \in H$ and $y \in G$
2. there exists an open subgroup $J$ of $G$, possibly depending on $f$, such that $f(xj) = f(x)$ for all $j \in J$.

We let $\rho$ act on $X$ by right translation: $(\rho(x)f)(y) = f(yx)$ for all $x,\ y \in G$. This makes $(\rho, X)$ a smooth representation of $G$. We call the functor $(\sigma, W) \mapsto (\rho, X)$ *smooth induction* and denote it by $\mathrm{Ind}_H^G$.

We say a set $A \subset G$ is compact modulo $H$ if the image of $A$ under the canonical projection $\pi_H : G \to G/H$ is compact in the quotient topology on $G/H$. We say a function on $G$ is compactly supported modulo $H$ if the support of $f$ is compact modulo $H$ as a set.

The representation $\mathrm{Ind}_H^G\sigma$ has an important subspace consisting of only those functions whose support is compact modulo $H$. This subspace is $G$-stable because $\mathrm{supp}\,(\rho(g)f) = g^{-1}\mathrm{supp}\,(f)$. We call the representation of $G$ on this subspace $c$–$\mathrm{Ind}_H^G\sigma$ and refer to the functor $c$–$\mathrm{Ind}_H^G$ as *compact induction*. Given a vector $w \in W$, we write $f_w$ to denote the unique function in $c$–$\mathrm{Ind}_H^G\sigma$ that is supported on $H$ and satisfies $f_w(1) = w$. The map $w \mapsto f_w$ gives an $H$-isomorphism from $W$ to functions in $c$–$\mathrm{Ind}_H^G\sigma$ supported on $H$. Note that if $G/H$ is compact then compact induction and smooth induction are equal.

Technically, to fully define the induction and compact induction functors, we must specify their action on intertwining maps. Let $\varphi \in \mathrm{Hom}_H(\tau, \sigma)$. We define a map in $\mathrm{Hom}_G(\mathrm{Ind}\tau, \mathrm{Ind}\sigma)$ by postcomposing with $\varphi$. That is, given $\phi \in \mathrm{Ind}\tau$, we define an element of $\mathrm{Ind}\sigma$ by $\varphi \circ \phi$. The same definition works for compact induction.

We will now state some basic results from category theory.

**Definition 1.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and pick functors $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ and $\mathfrak{G} : \mathcal{D} \to \mathcal{C}$. Then we say the pair $(\mathfrak{F}, \mathfrak{G})$ is an adjoint pair if for each pair of objects $M \in Ob(\mathcal{C})$ and $N \in Ob(\mathcal{D})$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(M, \mathfrak{G}(N)) \cong \mathrm{Hom}_{\mathcal{D}}(\mathfrak{F}(M), N).$$

In this case we say that $\mathfrak{F}$ is a left adjoint to $\mathfrak{G}$ and $\mathfrak{G}$ is a right adjoint to $F$.

**Proposition 1.2.** Let $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ be a functor.

- If $\mathfrak{F}$ has a left adjoint then it is right exact, and if it has a right adjoint then it is left exact [12, p. 257].

- If $\mathfrak{G}$ and $\mathfrak{G}'$ are both right (or left) adjoints to $\mathfrak{F}$ they are naturally equivalent. In particular, $\mathfrak{G}(N) \cong \mathfrak{G}'(N)$ for all $N \in Ob(\mathcal{D})$ [12, p. 257]. This fact is referred to as *adjoint uniqueness*.

Returning to the context of smooth representations, we have a functor from $R(G)$ to $R(H)$ given by restriction. We denote this functor by $\mathrm{res}_H^G$; however, given a representation $\pi$ of $G$ we will simply write $\pi$ for its restriction to $H$ when this is unlikely to cause confusion. The following two theorems tell us that smooth induction and compact induction, respectively provide right and left adjoints to restriction.

**Theorem 1.1** (Frobenious reciprocity)**.** Let $H$ be a closed subgroup of $G$. Let $\pi$ be a representation of $H$ and $\sigma$ a representation of $G$. Then one has an isomorphism of complex vector spaces

$$\mathrm{Hom}_H(\pi, \sigma) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma).$$

*Proof.* Let $\varphi : \pi \to \sigma$ be an $H$-intertwining map. Define a $G$-intertwining map $\varphi_* \pi \to \mathrm{Ind}_H^G \sigma$ by $\varphi_*(v)(g) = \varphi(\pi(g)v)$. To see that the map $\varphi \mapsto \varphi_*$ is an isomporphism we will construct an inverse. Let $\psi : \pi \to \mathrm{Ind}_H^G \sigma$ be a $G$-intertwining map. Then the map $v \mapsto \psi(v)(1)$ is an $H$-intertwining map from $\pi$ to $\sigma$. Associating this map to $\psi$ gives the desired inverse. $\square$

**Theorem 1.2** (Frobenious reciprocity for compact induction)**.** Now suppose that $H$ is an open subgroup of $G$. (This implies that $H$ is also closed). As in the previous theorem, let $\sigma$ be a representation of $H$ and $\pi$ a representation of $G$. Then one has an isomporphism of complex vector spaces

$$\mathrm{Hom}_H(\sigma, \pi) \cong \mathrm{Hom}_G(c\text{–}\mathrm{Ind}_H^G \sigma, \pi).$$

*Proof.* Let $\varphi : \sigma \to \pi$ be an $H$-intertwining map. For $w$ in the vector space for $\sigma$ define $\varphi_*(f_w) = \varphi(w)$. The entire representation c-$\mathrm{Ind}_H^G\sigma$ is generated as a vector space by $G$ translates of such $f_w$, so $\varphi_*$ extends to a unique $G$-intertwining map $\varphi_* : c\text{–}\mathrm{Ind}_H^G\sigma \to \pi$. Again we will construct an inverse to $\varphi \mapsto \varphi_*$. Let $\phi : c\text{–}\mathrm{Ind}_H^G \to \pi$ be a $G$-intertwining map. The inverse we seek is provided by that map that sends $\phi$ to $[w \mapsto \phi(f_w)]$. □

NOTE: There is another way to write the map $\varphi \mapsto \varphi_*$ in the above proof. Let $\varphi \in \mathrm{hom}_H(\sigma, \pi)$ and $f \in c\text{–}\mathrm{Ind}_H^G\sigma$. Then we define

$$\varphi_*(f) = \int_{G/H} \pi(x)\varphi\left(f(x^{-1})\right) dx. \tag{1.1}$$

The integrand is $H$-invariant, so $\varphi_*$ is well defined. One checks that the inverse map described in the proof above is also an inverse to equation 1.1. This will be useful in our discussion of types.

**Proposition 1.3.** Given an open subgroup $H < G$, the functors $\mathrm{Ind}_H^G$ and $c\text{–}\mathrm{Ind}_H^G$ preserve sums and are additive and exact.

*Proof.* The first two properties are clear. The functor Ind is left exact because it has a right adjoint (theorem 1.1), and similarly $c$–Ind is right exact because it has a right adjoint (theorem 1.2).

Suppose that $f : V \to W$ is a surjective map of smooth $H$-representations and let $\phi \in \mathrm{Ind}_H^G W$. Let $S$ be a set of coset representatives of $G/H$. For each $x \in S$ we can find $v_x \in V$ such that $f(v_x) = \phi(x)$. Define $\psi \in \mathrm{Ind}_H^G V$ to be the unique function with $\psi(x) = v_x$ for each $x \in S$. Then $\phi = f \circ \psi$, so Ind takes surjective maps to surjective maps. Thus Ind is exact.

Now let $f : V \to W$ be an injective map of smooth $H$-representations and let $\phi,\ \psi \in c\text{–}\mathrm{Ind}_H^G V$. Suppose $\phi \neq \psi$. Then there exists some $x \in G$ with $\phi(x) \neq \psi$ of $x$. Since $f$ is injective, $f \circ \phi(x) \neq f \circ \psi(x)$, so $c$–Ind sends $f$ to an injective map. Thus $c$–Ind is exact, so we are done. □

## 1.3 The Hecke Algebra

We now discuss an important algebra associated with $G$. We say a function $f : G \to \mathbb{C}$ is smooth if the inverse image of each point in $\mathbb{C}$ is open. Define $\mathcal{H}(G)$ to be the set of all smooth, compactly supported, complex valued functions on $G$. We see that $\mathcal{H}(G)$ is a complex vector space. Put a multiplication operation on $\mathcal{H}(G)$ by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy \qquad f,\ g \in \mathcal{H}(G),$$

where $dx$ denotes Haar measure on $G$. Since $f$ and $g$ are smooth and compactly supported, this integral reduces to a finite sum for each $x$, so there are no analytic issues involved in its convergence.

**Lemma 1.2.** For each $f \in \mathcal{H}(G)$ there exists an open compact subgroup $K < G$ such that $f$ is constant on left cosets of $K$.

*Proof.* Pick $x \in G$. Because $f$ is smooth, there exists an open compact subgroup $K_x < G$ such that $f$ is constant on $xK_x$. The support of $f$ is compact, so we may pick a finite set of points $\{x_1, \dots, x_n\}$ such that supp $f = \cup_i x_i K_{x_i}$. Then $K = \cap_i K_{x_i}$ is a finite intersection of open groups. Thus it is open and is the desired group. $\square$

**Proposition 1.4.** Let $f,\ g,\ h \in \mathcal{H}(G)$. Then the following holds

1. $f * g$ is smooth and compactly supported
2. $(f * g) * h = f * (g * h)$.

Thus $\mathcal{H}(G)$ is an associative complex algebra.

*Proof.* Pick $f,\ g \in \mathcal{H}(G)$. A simple computation shows that supp $(f*g) \subset$ (supp $f$)(supp $g$). The set supp $f \times$ supp $g$ is compact, and the product map $G \times G \to G$ is continuous, so (supp $f$)(supp $g$) is the continuous image of a compact set and hence compact. For $g$, let $K$ be the group guaranteed by lemma 1.2. From the definition of $*$ it is easy to see that $f * g$ is also constant on left cosets of $K$. Thus $f * g$ is smooth, so the first claim is proved.

To prove the second claim we calculate

$$\begin{aligned}(f * g) * h(x) &= \int_G f(y) \int_G g(z) h(z^{-1} y^{-1} x) dz dy \quad \text{(substitute } z \mapsto y^{-1} z\text{)} \\ &= \int_G \int_G f(y) g(y^{-1} z) dy h(z^{-1} x) dz = f * (g * h)(x).\end{aligned}$$

$\square$

**Proposition 1.5.** Let $(\tau, W)$ be an irreducible representation of a compact open subgroup $J < G$. Define a function $e_\tau \in \mathcal{H}(G)$ by

$$e_\tau(x) = \begin{cases} \frac{\mathrm{Tr}(\tau(x))}{\mathrm{Vol}(J)} & x \in J \\ \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Then $e_\tau$ is an idempotent in $\mathcal{H}(G)$ and given a smooth representation $(\pi, V) \in R(G)$ we have $\pi(e_\tau)V = V^\tau$. (Recall that an element $e$ of a ring is called an *idempotent* if $e^2 = e$.)

It is often useful to choose this $\tau$ to be the trivial representation. We write $e_J$ for the corresponding idempotent. The existence of so many idempotents in $\mathcal{H}(G)$ plays a fundamental role in its study. We turn briefly to some more general ring-theoretic considerations.

**Definition 1.3.** A ring $R$ (possibly without identity) is called *idempotented* if for each finite collection of elements $r_1,\ r_2, \ldots,\ r_n \in R$ there exists an idempotent $e \in R$ such that $er_ie = r_i$ for $1 \leq i \leq n$.

**Lemma 1.3.** The ring $\mathcal{H}(G)$ is idempotented.

*Proof.* Let $f_1, \ldots,\ f_n \in \mathcal{H}(G)$. By lemma 1.2, for each $i$ there exists an open compact subgroup $K_i < G$ such that $f_i$ is constant on cosets of $K_i$. Set $K = \cap_i K_i$ and let $e_K$ be the corresponding idempotent defined in equation 1.2. Then it is easy to check that $e_K * f_i * e_K = f_i$ for each $i$, so $\mathcal{H}(G)$ is idempotented. □

We call an $S$-module $M$ *nondegenerate* if $SM = M$. Idempotented rings should be viewed as being quite close to rings with identity. For example, given any subring $A$ in an idempotented ring $R$ one has $RA \supset A$. The following proposition is a less trivial illustration of this similarity.

**Proposition 1.6.** Let $R \subset S$ be idempotented rings. Let $N$ be a nondegenerate

$R$-module and $M$ a nondegenerate $S$-module. Then there is a natural isomorphism

$$\mathrm{Hom}_S(S \otimes_R N, M) \cong \mathrm{Hom}_R(N, M).$$

That is, the functor $S \otimes_R -$ is a left adjoint to restriction.

*Proof.* We define the isomorphism by sending $\varphi \in \mathrm{Hom}_S(S \otimes_R N, M)$ to a map $\varphi^* \in \mathrm{Hom}_R(N, M)$. For $n \in N$ we define $\varphi^*(n) = \varphi(e \otimes n)$ where $e \in R$ is an idempotent that fixes $n$. To check that this is well defined, suppose $e' \in R$ is an idempotent with $e'n = n$. Then since $R$ is idempotented there exists an idempotent $h \in R$ with $he = e$ and $he' = e'$. Thus

$$(e - e') \otimes n = h(e - e') \otimes n = h \otimes (e - e')n = 0.$$

Define a map in the opposite direction by sending an $R$-map $\psi$ to an $S$-map $\psi_*$ where $\psi_*(s \otimes n) = s\psi(n)$ for $s \in S$, $n \in N$. Viewed as a map on $S \times N$, the homomorphism $\psi_*$ is $R$-balanced because $\psi$ is an $R$-map, so it is well defined on $S \otimes_R N$. We must now check that $\psi \mapsto \psi_*$ and $\varphi \mapsto \varphi^*$ are inverse maps. Given $s \in S$ and $n \in N$, we calculate

$$(\varphi^*)_*(s \otimes n) = s\varphi^*(n) = s\varphi(e \otimes n)\varphi(se \otimes n) = \varphi(s \otimes en) = \varphi(s \otimes n)$$

$$(\psi_*)^*(n) = \psi_*(e \otimes n) = e\psi(n) = \psi(en) = \psi(n),$$

and this finishes the proof. $\square$

We now relate the module theory of $\mathcal{H}(G)$ to the representation theory of $G$. Let $(\pi, V)$ be a smooth representation of $G$. We define an action of $\mathcal{H}(G)$ on $V$ by

$$\pi(f)v = \int_G f(g)\pi(g)v dg \quad f \in \mathcal{H}(G),\ v \in V. \tag{1.3}$$

Once again, by the smoothness of $f$ and $\pi$, and the compact support of $f$, this integral reduces to a finite sum, so convergence is not an issue. This action makes each smooth $G$ representation an $\mathcal{H}(G)$ module.

**Theorem 1.3.** There is an equivalence of categories between smooth representations of $G$ and nondegenerate modules over $\mathcal{H}(G)$.

*Proof.* Let $(\pi, V)$ be a $G$ representation, which we make into an $\mathcal{H}(G)$ module via equation 1.3. We must show that this module is nondegenerate. Pick $v \in V$. Since $\pi$ is smooth, there exists an open subgroup $J < G$ that fixes $v$. Then one sees that $e_J v = v$, so $v \in \mathcal{H}(G)V$, so $V$ is a nondegenerate $\mathcal{H}(G)$ module.

Now let $M$ be a nondegenerate $\mathcal{H}(G)$ module. Let $m \in M$ and $g \in G$. Since $M$ is nondegenerate there exists some $f \in \mathcal{H}(G)$ and $m' \in M$ so that $f.m' = m$. Since $f$ is smooth and compactly supported, and since open compact subgroups of $G$ form a neighborhood basis of the identity, we can find an open compact subgroup $J < G$ that $f$ is constant on cosets of $J$. Then $e_J f = f$. Thus $e_J f m' = f m' = m$, so $e_J m = m$. Let $e_{gJ}$ be the characteristic function of $gJ$ normalized by the volume of $J$ (note that in general this function is not an idempotent). We define a representation of $G$ on $M$ by $\pi(g)m = e_{gJ}m$. One checks that this representation is well defined, and that it provides an inverse to the functor defined in the previous paragraph. $\square$

The final proposition of this section is essentially an application of proposition 1.6 to the context of smooth representations.

**Proposition 1.7.** Let $(\sigma, V)$ be a smooth representation of an open subgroup $H < G$.

Then we have a canonical isomorphism of $G$-representations

$$c\text{–}\mathrm{Ind}_H^G \sigma \cong \mathcal{H}(G) \otimes_{\mathcal{H}(H)} \sigma.$$

*Proof.* Applying proposition 1.6 with $R = \mathcal{H}(H)$ and $S = \mathcal{H}(G)$ we see that $\mathcal{H}(G) \otimes_{\mathcal{H}(H)} -$ is a left adjoint to restriction. Combining theorem 1.3 and theorem 1.2 we see that compact induction is also a left adjoint to restriction. Thus the proposition follows from adjoint uniqueness. □

### 1.4 Parabolic Induction

Reductive algebraic groups (see definition 1.6) contain certain preferred subgroups known as Levi subgroups, which are also reductive. For example, Levi subgroups of $GL(n)$ are isomorphic to products of copies of $GL(m)$ for $m \leq n$, Levi subgroups of $SL(n)$ are isomorphic to products of copies of $SL(m)$ for $m \leq n$, and Levi subgroups of classical groups are isomorphic to products of general linear groups and smaller classical groups. This suggests that we could construct representations of reductive groups from our knowledge of the representation theory of smaller reductive groups by induction from Levi subgroups.

There is an obstacle: in general if $G$ is a reductive group and $L$ is a Levi subgroup then the space $G/L$ is not compact. In particular, admissible representations (recall definition 1.1) need not be sent to admissible representations. The appropriate solution was developed by Harish Chandra: each Levi subgroup $L$ is contained in a so-called parabolic subgroup $P$. In fact, $P = LN$ where $N$ is a certain normal subgroup of $P$. Thus one can inflate a representation from $L$ to $P$ by making it trivial on $N$. In general $G/P$ is compact, so one can then safely induce from $P$ to $G$. This process is called parabolic induction.

The program is then to construct smooth irreducible representations of $G$ by realizing them as subrepresentations of representations induced from Levi subgroups. In this context, the building blocks are irreducible representations that do not arise in this manner for any Levi subgroup of $G$. Such representations are called supercuspidal. Understanding the irreducible representations of $G$ then falls into two parts: constructing the irreducible supercuspidals and understanding the subquotients of representations parabolically induced from Levi subgroups. Giving precise definitions of these objects and stating some of their basic properties will be the task of this section.

### 1.4.1 Algebraic Groups

We must recall some results from the theory of algebraic groups. Most of the content in this section will be stated without proof. For a thorough account of this material, the reader can consult the book of Springer [13].

Let $k$ be an algebraically closed field and let $k^n$ be $n$-dimensional affine space over $k$. Suppose a variety $G \subset k^n$ has the structure of a group. We say $G$ is an *algebraic group* if the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are rational maps.

Set $A = k[X_1, \ldots, X_n]$ and let $I(G)$ denote the ideal associated to $G$ consisting of all polynomials that are trivial on $G$. Call $A[G] = A/I(G)$ the *coordinate ring* of

$G$. An element of $A[G]$ is determined by the values it takes on elements of $G$, so we may define an action of $G$ on $A[G]$ by $xf(y) = f(xy)$ for $f \in A[G]$ and $x,\ y \in G$. This identifies elements of $G$ with elements of the endomorphism ring of $A[G]$. We say an element $r$ of a ring is unipotent if $r - 1$ is nillpotent.

**Definition 1.4.** An element $x \in G$ is called *unipotent* if it corresponds to a unipotent element of the endomorphism ring of $A[G]$.

**Definition 1.5.** The *radical* of an algebraic group is the connected component (in the Zariski topology) containing the identity of the maximal normal solvable subgroup. The *unipotent radical* is the group of all unipotent elements in the radical.

**Definition 1.6.** An algebraic group is called *reductive* if its unipotent radical is trivial.

Many important groups are reductive, including general and special linear groups as well as orthogonal, symplectic, and unitary groups. An example of a non-reductive group is the group of invertible upper triangular $n \times n$ matrices for $n \geq 2$.

**Definition 1.7.** A *Borel* subgroup $B$ of an algebraic group $G$ is a maximal connected solvable group of $G$. If a subgroup $P < G$ contains $B$ we call $P$ a *parabolic subgroup.*

**Proposition 1.8** (Levi decomposition)**.** Let $P$ be a parabolic subgroup of some reductive algebraic group. Write $N$ for the unipotent radical of $P$. Then there exists a reductive subgroup $L < G$ called a *Levi component* of $P$ such that $P = LN$. In fact, one has $P \cong L \ltimes N$. Further, all Levi subgroups of $P$ are isomorphic to one another.

### 1.4.2 Parabolic Subgroups of $Sp(2k, F)$

In this document we will be particularly concerned with the symplectic group $Sp(2k, F)$ for $k > 0$ an integer. This section will define this group and give a concrete description of its parabolic subgroups. Let $F$ be a non-archimedean local field of characteristic zero and $V$ a $2k$-dimensional vector space over $F$. Fix a nondegenerate, antisymmetric, bilinear form $\langle \cdot, \cdot \rangle$ on $V$. The *symplectic group* associated to this form is the set of invertible matrices in $x \in \mathrm{End}_F(V) \cong GL(2k, F)$ with the property that

$$\langle xv, xw \rangle = \langle v, w \rangle$$

for all $v, w \in V$.

We say a subspace $W \subset V$ is *isotropic* if $\langle v, w \rangle = 0$ for all $v,\ w \in W$. Let $W$ be an isotropic subspace of $V$ of maximal dimension. A *flag* of $W$ means a sequence of subspaces $\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = W$. If we define $P$ to be the set of matrices in $Sp(2k, F)$ that stabilize each $V_i$, then $P$ will be a parabolic subgroup of $Sp(2k, F)$. Fix a basis $\{e_1, \ldots, e_k\}$ for $W$ and set $V_i$ to be the subspace of $W$ generated by $e_1, \ldots, e_i$. Then associated to each sequence $1 \leq r_1 \lneq \cdots \lneq r_m \lneq k$ we have the subgroup of $G$ fixing the flag $V_{r_1}, \ldots, V_{r_m}$. This subgroup is always parabolic, containing the Borel subgroup that fixes the maximal flag $V_1, \ldots, V_k$. We call such parabolic subgroups the *standard parabolic subgroups* $Sp(2k, F)$ relative to the basis $e_1, \ldots, e_k$. The unipotent radical of a standard parabolic is the subgroup of matrices that act trivially on each quotient $V_i/V_{i-1}$ of the associated flag. Any parabolic subgroup of $Sp(2k, F)$ is isomorphic, and in fact conjugate, to a unique standard parabolic [7, p. 120].

Let $Id$ denote the $k \times k$-identity matrix. Let $\{e_1, \ldots, e_{2k}\}$ be a basis for $V$. We define an antisymmetric form on $V$ with respect to this basis by setting

$$J = \begin{pmatrix} & Id \\ -Id & \end{pmatrix}$$

$$\langle v, w \rangle = {}^t v J w$$

where we view $v,\ w \in V$ as column vectors. With respect to this form,

$$Sp(2k, F) = \left\{ g \in GL(2k, F) : gJ^t g = J \right\}.$$

Further, $\{e_1, \ldots, e_k\}$ spans a maximal isotropic subspace of $V$. Call this subspace $W$.

If we pick the flag consisting only of $W$, we get a parabolic subgroup $P$ known as the *Siegel parabolic.* The Siegel parabolic consists of all matrices contained in $Sp(2k)$ of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

where each entry represents a $k \times k$ block. Further, the unipotent radical $N$ will consist of matrices where the two blocks on the diagonal are simply the $k \times k$ identity matrix, while there is a standard choice of Levi component $L$ consisting of all matrices where the block above the diagonal is zero. By checking the condition that such matrices be contained in $Sp(2k, F)$, one can calculate

$$N = \left\{ \begin{pmatrix} 1 & Y \\ & 1 \end{pmatrix} : Y \in \mathrm{Mat}_k(F) \text{ is symmetric} \right\}$$
$$L = \left\{ \begin{pmatrix} X & \\ & {}^t X^{-1} \end{pmatrix} : Y \in GL(k, F) \right\}.$$

Thus, in particular, for the Siegel parabolic $L \cong GL(k, F)$.

### 1.4.3 Parabolic Induction

Let $P$ be a parabolic subgroup of $G$ with Levi component $L$ and unipotent radical $N$. Let $\sigma$ be a smooth irreducible representation of $L$. If we induce $\sigma$ to $G$ we can construct representations of $G$ from representations of smaller reductive groups. However, as discussed above, it is problematic to simply induce from $L$ to $G$. To remedy this issue, we first inflate $\sigma$ from $L \cong P/N$ to $P$ and then smoothly induce this inflated representation from $P$ to $G$. We refer to this process as *parabolic induction* and denote the resulting representation by $i_P^G(\sigma)$.

**Definition 1.8.** Let $\pi$ be an irreducible representation of $G$. We say $\pi$ is *supercuspidal* if it is not a subrepresentation of $i_P^G(\sigma)$ for any parabolic subgroup $P$ and representation $\sigma$ of the Levi component of $P$.

Supercuspidal representations should be thought of as the building blocks in the sense that any smooth irreducible representation of $G$ is a subrepresentation of a representation parabolically induced from some supercuspidal.

Parabolic induction has a left adjoint known as *Jaquet restriction.* Just as parabolic induction is a composition of two processes, inflation and induction, so also Jaquet restriction is a composition of two processes, restriction and "taking coinvariants." Let $(\pi, V)$ be a smooth $G$-representation. Define $V(N)$ to be the $N$-subspace of $V$ spanned by vectors of the form $\pi(n)v - v$ for $v \in V$, $n \in N$. Then the *Jaquet restriction functor*, $r_N$, is defined by

$$r_N(V) = V/V(N).$$

Because $N$ is normalized by $L$, $L$ stabilizes $V(N)$, so there is a natural action of $L$ on $r_N(V)$. Further, given a map of of $G$-spaces $f : V \to W$ we have $f(V(N)) \subset W(N)$. Thus $f$ passes to a map $r_N(f) : r_N(V) \to r_N(W)$, so $r_N$ is a functor.

**Proposition 1.9.** Given any pair of smooth representations $(\sigma, W)$ of $H$ and $(\pi, V)$ of $G$, there is a natural isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{Hom}_L(r_N(\pi), \sigma) \cong \mathrm{Hom}_G(\pi, i_P^G\sigma).$$

*Proof.* Let $\sigma$ also denote the inflation of $\sigma$ from $L$ to $P$. Then by Frobenious reciprocity

$$\mathrm{Hom}_G(\pi, i_P^G\sigma) \cong \mathrm{Hom}_P(\pi, \sigma).$$

Let $\varphi \in \mathrm{Hom}_P(\pi, \sigma)$, $n \in N$, and $v \in V$. Then $\varphi(\pi(n)v - v) = \sigma(n)\varphi(v) - \varphi(v) = 0$. Thus $\varphi$ factors through the quotient $V/V(N)$ so

$$\mathrm{Hom}_P(\pi, \sigma) \cong \mathrm{Hom}_L(r_N(\pi), \sigma).$$

$\square$

**Lemma 1.4.** The functors $r_N$ and $i_P^G$ are exact.

*Proof.* The functor $\mathrm{Ind}_P^G$ has already been shown to be exact, so we must show that the inflation functor from $R(L)$ to $R(P)$ is exact. This functor acts trivially on the vector spaces and maps in an exact sequence of $L$-representations, so exactness is clear.

Jaquet restriction has a right adjoint, so it is left exact. Suppose $f : V \to W$ is a surjective map of $G$-spaces. Then given an element $w + W(N) \in r_N(W)$ there is an element $v \in V$ with $f(v) = w$. Thus $r_N(f)(v + V(N)) = w + W(N)$, so $r_N$ is exact. $\square$

# CHAPTER 2
# TYPES AND COVERS

Bushnell and Kutzko have developed a method known as the theory of types used to understand the structure of the category $R(G)$. This theory builds on a result of Bernstein showing that $R(G)$ decomposes into a certain product of subcategories. A type associated to one of these subcategories is a smooth irreducible representation $\tau$ of an open compact subgroup $J < G$ that "sees" the representations contained in that subcategory in a sense which will be made precise. If a type exists then the subcategory is equivalent to the category of left modules over an associative, unital, complex algebra associated with $\tau$.

Types were first constructed by Kutzko for supercuspidal representations of $GL(2)$ as a part of his proof of the local Langlands correspondence for $GL(2)$ [2]. They were later constructed for all representations of $GL(n)$ and $SL(n)$ by Bushnell and Kutzko, and for certain self-dual representations of classical groups by Kutzko, Stevens and Goldberg, as well as for other representations of classical groups by Stevens and Miyauchi [3, 4, 8, 10].

In this chapter I will describe the Bernstein decomposition. I will also discuss the theory of types with a particular emphasis on the notion of a cover and on the role played by idempotents in the Hecke algebra. This material largely follows the description of types in [1].

## 2.1 The Bernstein Center

Once again let $F$ be a non-archimedean local field of characteristic zero, and let $G$ denote the $F$-points of a reductive algebraic group. We call a smooth homomorphism $\chi : G \to \mathbb{C}^\times$ a *character* of $G$ if $|\chi| \equiv 1$ and a *quasicharacter* otherwise. If a quasicharacter $\chi$ is trivial on all compact subgroups of $G$ we say it is *unramified*. Write $\mathfrak{X}(G)$ for the set of unramified quasicharacters of $G$.

Let $H$ be a subgroup of $G$ and $\sigma$ a representation of $H$, and fix an element $x \in G$. Then we define the *conjugate representation* $\sigma^x$ as follows. Set $H^x = x^{-1}Hx$ and define $\sigma^x(h) = \sigma(xhx^{-1})$ for $h \in H^x$.

**Definition 2.1.** For $i = 1,\ 2$ let $L_i$ be a Levi subgroup of $G$ and let $\sigma_i$ be a supercuspidal representation of $L_i$. If there exists an element $x \in G$ and a character $\chi \in \mathfrak{X}(L_1)$ such that $L_1 \cong L_2^x$ and $\pi_1 \otimes \chi \cong \pi_2^x$ we say that $\sigma_1$ and $\sigma_2$ are *inertially equivalent* and write $\sigma_1 \sim \sigma_2$.

Write $\mathcal{B}(G)$ for the set of inertial equivalence classes of supercuspidal representations of Levi subgroups of $G$. Note that we include $G$ itself in the list of Levi subgroups.

Given an inertial equivalence class $\mathfrak{s} \in \mathcal{B}(\mathrm{G})$ we define $R^{\mathfrak{s}}(G)$ to be the full subcategory of $R(G)$ whose objects are representations having the property that each of their irreducible subquotients is a subquotient of some representation parabolically induced from an element of $\mathfrak{s}$. We call $R^{\mathfrak{s}}(G)$ the *Bernstein component* of $R(G)$ corresponding to $\mathfrak{s}$. When we wish to specify $\mathfrak{s}$ by a representative $(L, \sigma)$ we write $\mathfrak{s} = [L, \sigma]_G$. The presence of $G$ in this notation is important since this choice determines which elements we are allowed to conjugate by. Each subcategory $R^{\mathfrak{s}}(G)$ is closed under forming arbitrary sums and products as well as taking quotients and submodules.

It is a result of Bernstein that $R(G)$ decomposes into a product of subcategories

$$R(G) = \prod_{\mathfrak{s} \in \mathcal{B}} R^{\mathfrak{s}}(G).$$

This means the following: let $(\pi, V)$ be a smooth representation of $G$ and for each $\mathfrak{s} \in \mathcal{B}(G)$ define $V^{\mathfrak{s}}$ to be the maximal $G$-stable subspace of $V$ contained in $R^{\mathfrak{s}}(G)$ (note that $V^{\mathfrak{s}}$ exists and is unique because each Bernstein component is closed under sums). Then $V = \prod_{\mathfrak{s} \in \mathcal{B}} V^{\mathfrak{s}}$ and this decomposition is unique with respect to the property that each summand is contained in a distinct Bernstein component. In fact, each $V^{\mathfrak{s}}$ is the largest $G$-subspace of $V$ contained in $R^{\mathfrak{s}}(G)$. Further, if $\mathfrak{s}$ and $\mathfrak{s}'$ are distinct inertial equivalence classes with $\pi \in R^{\mathfrak{s}}(G)$ and $\pi' \in R^{\mathfrak{s}'}(G)$ then $\mathrm{Hom}_G(\pi, \pi') = 0$ [6].

We recall a definition from category theory. Let $\mathcal{A}$ be an additive category. We define the center of $\mathcal{A}$ to be $\mathrm{End}_{\mathcal{A}}(id_{\mathcal{A}})$, the collection of natural transformations from the identity functor to itself. The center forms a ring with addition being given by addition of morphisms, and multiplication by composition of natural transformations. We write $Z(\mathcal{A})$ for the center of $\mathcal{A}$.

**Lemma 2.1.** Let $R$ be a unital ring and let $\mathcal{A}$ denote the category of left $R$-modules. Then the center of $\mathcal{A}$ is isomorphic to the center of $R$.

NOTE: When $R$ is unital, we will always require that the the identity element of $R$

act trivially on any $R$-module.

*Proof.* An element $\rho \in Z(\mathcal{A})$ is a collection of morphisms $\rho_M \in \mathrm{End}_R(M)$, one for each object $M \in Ob(\mathcal{A})$, that commute with morphisms. Define a map $Z(\mathcal{A}) \to R$ by $\rho \mapsto \rho_R(1)$. We will show that this map is the desired isomorphism.

We first show the map is into $Z(R)$. For each $r \in R$ we define a map $\varphi_r \in \mathrm{End}_R(R)$ by $\varphi_r(s) = sr$ for $s \in R$. Though $\varphi_r$ may not be a ring map, it is always a morphism of left $R$-modules, so we have the following commutative diagram.

$$\begin{tikzcd} R \arrow[r, "\rho_R"] \arrow[d, "\varphi_r"] & R \arrow[d, "\varphi_r"] \\ R \arrow[r, "\rho_R"] & R \end{tikzcd}$$

Thus for each $r \in R$ we have $\rho_R(1)r = \rho_R(r) = r\rho_R(1)$, so $\rho_R(1) \in Z(R)$.

We next show onto. Given $r \in Z(R)$, define a natural equivalence $\rho^r \in Z(\mathcal{A})$ by $\rho^r_M(m) = rm$ for each $R$-module $M$ and each element $m \in M$. Then $\rho^r_R(1) = r$.

It remains to show that each natural equivalence $\rho \in Z(\mathcal{A})$ is determined by the value $\rho_R(1)$. Let $M$ be an arbitrary $R$-module and pick an arbitrary element $m \in M$. We define an $R$-module map $\psi_m : R \to M$ by $\psi_m(r) = rm$. Then the following diagram must commute.

$$\begin{tikzcd} R \arrow[r, "\rho_R"] \arrow[d, "\psi_m"] & R \arrow[d, "\psi_m"] \\ M \arrow[r, "\rho_M"] & M \end{tikzcd}$$

Plugging in 1 we get $\rho_M(m) = \rho_R(1)m$. Since $M$, $m$, and $\rho$ were arbitrary, the proof is done. $\square$

Bernstein has given a description of the center of the category $R^{\mathfrak{s}}(G)$. Let $L$ be a Levi subgroup of $G$, and let $\sigma \in R(L)$ be an irreducible supercuspidal representation.

Set $\mathfrak{s}_L = [L,\sigma]_L$ and $\mathfrak{s} = [L,\sigma]_G$. Let $N_G(L)$ denote the normalizer in $G$ of $L$ and define

$$N_G(\mathfrak{s}_L) = \{x \in N_G(L) : \sigma^x \cong \sigma \otimes \nu \text{ for some } \nu \in \mathfrak{X}(L)\}$$

$$W(G,\mathfrak{s}_L) = N_G(\mathfrak{s}_L)/L$$

$$D_\sigma = \{\sigma \otimes \nu : \nu \in \mathfrak{X}(L)\}.$$

The group $W(G,\mathfrak{s}_L)$ is a subgroup of the Weyl group $W(G,L) = N_G(L)/L$, so it is finite. The normalizer $N_G(L)$ acts on smooth representations of $L$ by conjugation. Since $L$ acts trivially, this action passes to an action of the Weyl group. By definition, $W(G,\mathfrak{s}_L)$ is precisely the subgroup of $W(G,L)$ that preserves $D_\sigma$ under this action.

**Theorem 2.1** (Bernstein)**.** The center of the category $R^{\mathfrak{s}}(G)$ is isomorphic to the ring of regular functions on the complex variety $D_\sigma/W(G,\mathfrak{s}_L)$ [6, thm. 2.13].

## 2.2 The Algebra $\mathcal{H}(G,\tau)$

The previous section tells us that to study all smooth representations of $G$ it suffices to study each Bernstein component separately. Bushnell and Kutzko have proposed a method to accomplish this, known as the theory of types, that gives a procedure by which we can associate a given Bernstein component to the category of modules over a certain algebra that we will call $\mathcal{H}(G,\rho)$. This algebra is relatively amenable to analysis, so, when it can be carried out, this procedure is quite useful. In this section we define the algebra in question and prove some of its fundamental properties.

Let $J$ and $K$ be compact open subgroups of $G$, and let $(\tau,W)$ and $(\rho,U)$ be smooth representations of $J$ and $K$, respectively. We define the space of functions

$$\mathcal{H}(G;\tau,\rho)=\{f:G\to \mathrm{Hom}_{\mathbb{C}}(W,U): f(jxk)=\tau(j)f(x)\rho(k) \text{ for } j\in J,$$
$$k\in K,\ x\in G, \text{ and } f \text{ is compactly supported}\}.$$

At the core of Mackey theory in the $p$-adic setting is the following result, which is due to Kutzko in this setting [9].

**Theorem 2.2.** There is a canonical isomorphism

$$\mathcal{H}(G;\tau,\rho)\cong \mathrm{Hom}_G(c\text{–}\mathrm{Ind}_J^G\tau, c\text{–}\mathrm{Ind}_K^G\rho).$$

*Proof.* We send a function $f\in\mathcal{H}(G;\tau,\rho)$ to an intertwining operator $A_f$ defined by

$$A_f\phi(x)=\int_G f(y)\phi(y^{-1}x)dy,\ x\in G,\ \phi\in c\text{–}\mathrm{Ind}\tau.$$

Recall the function $f_w$ defined in section 1.2 for a vector $w\in W$. We use it to associate a function $f_A$ to an intertwining operator $A$ defined by $f_A(x)(w)=Af_w(x)$ for $x\in G$ and $w\in W$. One checks that the maps $f\mapsto A_f$ and $A\mapsto f_A$ are well defined and inverses of one another. □

We consider tuples $(J,\tau)$ where $J$ is a compact open subgroup of $G$ and $\tau$ is an irreducible representation of $J$. Given $x\in G$ we can again form the conjugate representation $\tau^x$ of $J^x$. We say $x$ *intertwines* $\tau$ if $\mathrm{Hom}_{J\cap J^x}(\tau,\tau^x)\neq\emptyset$. Because $J$ is compact, $\tau|(J\cap J^x)$ is semisimple. Thus, if $x$ intertwines $\tau$, $\tau|(J\cap J^x)$ must have irreducible subrepresentations $\tau_1,\ \tau_2$ satisfying $\tau_1\cong\tau_2^x$. This implies $\tau_1^{x^{-1}}\cong\tau_2$ as representations of $J\cap J^{x^{-1}}$, so we see that $x^{-1}$ also intertwines $\tau$. It is useful to know that the set of elements intertwining $\tau$ is closed under taking inverses.

The intertwining of $\tau$ is governed by an algebra $\mathcal{H}(G, \tau)$ known as the *Hecke algebra* associated with $\tau$ or the algebra of $\tau$-spherical operators. Recall that $\check{\tau}$ denotes the contragrediant representation to $\tau$. The elements of $\mathcal{H}(G, \tau)$ are compactly supported functions $f : G \mapsto \mathrm{End}_J(\check{\tau})$ satisfying $f(j_1 x j_2) = \check{\tau}(j_1) f(x) \check{\tau}(j_2)$ for $j_1,\ j_2 \in J$ and $x \in G$. Multiplication is convolution, given by the formula

$$f * g(x) = \int_G f(y) g(y^{-1}x) dy.$$

Note that, somewhat counterintuitively, $\mathcal{H}(G, \tau) = \mathcal{H}(G; \check{\tau}, \check{\tau})$. We can equally well define the perhaps more familiar algebra of functions $\mathcal{H}(G, \check{\tau})$, which takes values in $\mathrm{End}_J(\tau)$. Write $W$ for the vector space acted on by $\tau$ and let $\langle \cdot, \cdot \rangle$ represent the canonical pairing between $W$ and $\check{W}$. Given an operator $a \in \mathrm{End}(\check{W})$ we define an operator $\check{a} \in \mathrm{End}(W)$ by

$$\langle \check{a} w, \phi \rangle = \langle w, a\phi \rangle \quad \forall w \in W,\ \phi \in \check{W}.$$

Since $\tau$ is smooth and irreducible and $J$ is compact, $W$ is finite dimensional, so there are no analytic issues involved in this definition. For $x \in J$ we calculate

$$\langle \check{\tau}(x)\check{}\, w, \phi \rangle = \langle w, \check{\tau}(x)\phi \rangle = \langle \tau(x^{-1}) w, \phi \rangle$$

so given $f \in \mathcal{H}(G, \tau)$ we may define $\check{f} \in \mathcal{H}(G, \check{\tau})$ by $\check{f}(x) = f(x^{-1})\check{}$. The map $\mathcal{H}(G, \tau) \to \mathcal{H}(G, \check{\tau})$ given by $f \mapsto \check{f}$ is a vector space isomorphism and satisfies $f_1 * f_2 \mapsto \check{f}_2 * \check{f}_1$, so it is an anti-isomorphism of algebras. Note that this map inverts the support of functions.

**Proposition 2.1.** A function in $\mathcal{H}(G, \tau)$ has support equal to a finite union of $J$-double cosets. For a given $x \in G$ there exists a function $f \in \mathcal{H}(G, \tau)$ with $f(x) \neq 0$ if and only if $x$ intertwines $\tau$. In fact, there is a canonical isomorphism of vector spaces between the space of functions in $\mathcal{H}(G,\tau)$ that are supported only on $JxJ$ and the space $\mathrm{Hom}_{J\cap J^x}(\tau,\tau^x)$.

*Proof.* The first assertion follows from the definition of the spherical Hecke algebra. The facts that the anti-isomorphism $f \mapsto \breve{f}$ inverts the support of functions and that the set of $x$ intertwining $\tau$ is closed under inverses show that it suffices to prove the second assertion for $\mathcal{H}(G,\breve{\tau})$.

Given a function $f \in \mathcal{H}(G,\breve{\tau})$ supported at $x$ one checks that $f(x) \in \mathrm{Hom}_{J\cap J^x}(\tau,\tau^x)$, so $x$ intertwines $\tau$. Conversely, given a non-trivial intertwining map in $\varphi \in \mathrm{Hom}_{J\cap J^x}(\tau,\tau^x)$, we can define a function $f \in \mathcal{H}(G,\breve{\tau})$ supported on $JxJ$ by $f(j_1xj_2) = \sigma^x(j_1)\varphi(x)\sigma(j_2)$ for $j_1,\ j_2 \in J$. These two processes are inverse to one another, so they give the desired isomorphism. $\square$

The following description of $\tau$-spherical functions supported at $x \in G$ will be convenient in the sequel.

**Lemma 2.2.** Let $x \in G$ and suppose that $f \in \mathcal{H}(G,\tau)$ is supported only on the double coset $JxJ$. Set $T = f(x)$. Then $f$ is completely determined by $T$. Conversely, for a given choice of $T \in \mathrm{End}_J(\breve{\tau})$ we define $f$ by the formula $f(k_1xk_2) = \breve{\tau}(k_1)T\breve{\tau}(k_2)$ for $k_1, k_2 \in J$. This is well defined if and only if $\breve{\tau}^x(k)\circ T = T\circ\breve{\tau}(k)$ for all $k \in J^x \cap J$.

*Proof.* That $f$ is determined by $T$ is clear, so we turn to the second assertion. To prove "if," let $kxk' = jxj'$ for $k, k', j, j' \in J$. Then $j^{-1}k = xj'k'^{-1}x^{-1}$, so

$$\begin{aligned} f(kxk') &= \breve{\tau}(k) \circ T \circ \breve{\tau}(k') = \breve{\tau}(j) \circ \breve{\tau}(j^{-1}k) \circ T \circ \breve{\tau}(k') \\ &= \breve{\tau}(j) \circ \breve{\tau}^x(j'k'^{-1}) \circ T \circ \breve{\tau}(k') = \breve{\tau}(j) \circ T \circ \breve{\tau}(j'k'^{-1}) \circ \breve{\tau}(k') = f(jxj'). \end{aligned}$$

Thus $f$ is well defined.

To check "only if," simply take $k \in J \cap J^x$, write $f(xk) = f(xkx^{-1}x)$, and apply the equivariance of $f$. □

## 2.3 Types

We are now in position to define the notion of a type. Let $W$ denote the vector space on which $\tau$ acts. Define the space of $\tau$-coinvariants of $\pi$ to be $V_\tau = \mathrm{Hom}_J(W, V)$. Then there is a natural action of $\mathcal{H}(G, \tau)$ on $V_\tau$ given by

$$(f \cdot \phi)(w) = \int_G \pi(g)\phi(\breve{f}(g^{-1})w)dg, \quad f \in \mathcal{H}(G, \tau),\ \phi \in V_\tau,\ w \in W. \tag{2.1}$$

This action arises as follows. Theorem 2.2 identifies $\mathcal{H}(G; \tau, \tau)$ with $\mathrm{End}_G(c\text{–Ind}\tau)$, giving a left action of $\mathcal{H}(G; \tau, \tau)$ on $c$–Ind$\tau$. Thus the anti-isomorphism $\mathcal{H}(G, \tau) \to \mathcal{H}(G; \tau, \tau)$ described in section 2.2 makes $c$–Ind$\tau$ a right module for $\mathcal{H}(G, \tau)$. Applying Frobenious reciprocity to $V_\tau$ we get

$$V_\tau \cong \mathrm{Hom}_G(c\text{–Ind}\tau, \pi)$$

so we have a left action of $\mathcal{H}(G, \tau)$ on $V_\tau$ given by acting on the right on the argument of an intertwining map in $\mathrm{Hom}_G(c\text{–Ind}\tau, \pi)$. Tracing through these definitions and using equation 1.1 one arrives at equation 2.1.

This action allows us to define a functor

$$M_\tau : R(G) \to \mathcal{H}(G,\tau)\text{-mod}$$

$$\pi \mapsto V_\tau.$$

We wish to understand the degree to which $M_\tau$ can be made to capture the representation theory of $G$.

Given $(\pi, V) \in R(G)$, we define $V[\tau]$ to be the $G$-subspace of $V$ generated by the $\tau$-isotypic subspace $V^\tau$. Define the full subcategory $R^\tau(G) \subset R(G)$ by $(\pi, V) \in R^\tau(G)$ if $V[\tau] = V$. Given a collection of objects $R \subset R(G)$, we define $\mathrm{Irr}R$ to be the set of irreducible representations contained in $R$.

**Definition 2.2.** Let $\mathfrak{s} \in \mathcal{B}(G)$. A pair $(J, \tau)$ is called an $\mathfrak{s}$-type if $\mathrm{Irr}R^{\mathfrak{s}}(G) = \mathrm{Irr}R^\tau(G)$.

**Proposition 2.2** (Bushnell, Kutzko)**.** Fix a pair $(J, \tau)$ and $\mathfrak{s} \in \mathcal{B}(G)$. Then

1. $\tau$ is an $\mathfrak{s}$-type if and only if $R^{\mathfrak{s}}(G) = R^\tau(G)$ [1, lem. 3.4].

2. if $\tau$ is an $\mathfrak{s}$-type then $M_\tau$ is an equivalence of categories from $R^{\mathfrak{s}}(G)$ to $\mathcal{H}(G,\tau)$-mod [1, sec. 4.2].

### 2.3.1 Covers

The basic program for the classification smooth irreducible representations of $G$ is to understand the supercuspidal representations of Levi subgroups of $G$ and then study the resulting composition factors when they are parabolically induced to representations of $G$. It is natural to ask if this process has an analogue in the setting of types. In particular, given a type for a supercuspidal Bernstein component of a Levi subgroup $L < G$, we would like to be able to construct a type for the corresponding category of induced representations. This is accomplished via the notion of a cover.

Let $P$ be a parabolic subgroup of $G$ with unipotent radical $N$ and Levi component $L$. Each parabolic subgroup has an *opposite parabolic* $\bar{P}$ with radical $\bar{N}$ satisfying $P \cap \bar{P} = L$ [7, p. 70]. For example, in the case of the Siegel parabolic for $Sp(2k, F)$ as defined in section 1.4.2, $\bar{P}$ simply consists of matrices whose transposes are in $P$, and similarly for $\bar{N}$.

Let $J < G$ be a compact open subgroup and let $\tau$ be an irreducible representation of $J$. We say the pair $(J, \tau)$ is *decomposed* with respect to $P$ if

1. $J = (J \cap \bar{N})(J \cap L)(J \cap N)$ (this is called an *Iwahori decomposition* for $J$)
2. the groups $J \cap N,\ J \cap \bar{N}$ are contained in the kernel of $\tau$

We define an element $\zeta \in G$ to be *strongly positive* for $P$ if

1. $\zeta N \zeta^{-1} \subset N$
2. $\zeta \bar{N} \zeta^{-1} \supset \bar{N}$
3. given any pair of compact open subgroups $H,\ K \subset N$ there exists an integer $n$ such that $\zeta^n H \zeta^{-n} \subset K$

**Definition 2.3.** Now let $\sigma$ be an irreducible supercuspidal representation of $L$ and set $\mathfrak{s}_L = [L, \sigma]_L$ and $\mathfrak{s} = [L, \sigma]_G$. Suppose $(J_L, \tau_L)$ is a type for $\mathfrak{s}_L$. We say that $(J, \tau)$ *covers* $(J_L, \tau_L)$ if

1. $(J, \tau)$ is decomposed with respect to $P$

2. $J \cap L \cong J_L$ and $\tau|(J \cap L) \cong \tau_L$

3. there exists an element $\zeta \in Z(L)$ that is strongly positive for $P$ and invertible functions $f,\ g \in \mathcal{H}(G,\tau)$ that are supported at $\zeta$ and $\zeta^{-1}$, respectively.

**Theorem 2.3** (Bushnell, Kutzko)**.** If $(J,\tau)$ covers $(J_L,\tau_L)$ then $(J,\tau)$ is a type for $\mathfrak{s}$ [1, thm 8.3].

When $(J,\tau)$ covers some $(J_L,\tau_L)$ one wishes to study $\mathcal{H}(G,\tau)$ in terms of $\mathcal{H}(L,\tau_L)$. Bushnell and Kutzko have shown that there is an injective homomorphism $t_p : \mathcal{H}(L,\tau_L) \to \mathcal{H}(G,\tau)$ giving the following commutative square of functors [1, cor. 8.4].

$$\begin{tikzcd} R^{\mathfrak{s}}(G) \arrow[r, "M_\tau"] & \mathcal{H}(G,\tau)-\mathrm{mod} \\ R^{\mathfrak{s}_L}(L) \arrow[u, "i_P^G"] \arrow[r, "M_{\tau_L}"] & \mathcal{H}(L,\tau_L)-\mathrm{mod} \arrow[u, "(t_p)_*"'] \end{tikzcd}$$

Here $(t_p)_*$ is induction of modules defined by $(t_P)_*(M) = \mathrm{Hom}_{\mathcal{H}(L,\tau_L)}(\mathcal{H}(G,\tau), M)$ for $M$ an $\mathcal{H}(L,\tau_L)$-module. If $t_p$ is surjective as well as injective we say the cover $(J,\tau)$ is *split*. From the above diagram we see that this implies that, in particular, parabolic induction sends irreducible representations to irreducible representations.

There is a similar diagram for Jaquet restriction [1, cor. 8.4]. Let $N$ be the unipotent radical of $P$ and write $t_P^*$ for the functor from $\mathcal{H}(G,\tau)$-mod to $\mathcal{H}(L,\tau_L)$-mod given by restriction of scalars via $t_P$. Then the following diagram commutes.

$$\begin{tikzcd} R^{\mathfrak{s}}(G) \arrow[r, "M_\tau"] \arrow[d, "r_N"'] & \mathcal{H}(G,\tau)-\mathrm{mod} \arrow[d, "t_p^*"] \\ R^{\mathfrak{s}_L}(L) \arrow[r, "M_{\tau_L}"] & \mathcal{H}(L,\tau_L)-\mathrm{mod} \end{tikzcd}$$

In particular, this diagram tells us that given $(\pi, V) \in R^{\mathfrak{s}}(G)$ we have

$$V_\tau \cong (r_N(V))_{\tau_L}, \tag{2.2}$$

where we are viewing $V_\tau$ as an $\mathcal{H}(L, \tau_L)$-module via $t_p^*$. Finally, we comment that either of these diagrams could be deduced from the other using uniqueness of adjoint functors combined with the fact that $\left(r_N, i_P^G\right)$ and $(t_P^*, (t_P)_*)$ are adjoint pairs.

## 2.4 Idempotents in the Hecke Algebra

In this section we describe an alternate view of types that will be necessary for a finiteness result in section 4. Here we will often write $\mathcal{H}$ for $\mathcal{H}(G)$ when this will cause no confusion. Most of the material in this section is due to Bushnell and Kutzko and can be found in [1, sec. 3].

There are two natural representations of $G$ on $\mathcal{H}(G)$:

$$(L(x)f)(g) = f(x^{-1}g)$$

$$(R(x)f)(g) = f(gx).$$

We call $L$ and $R$ the left and right regular representations, respectively. It is easy to check that under proposition 1.3 the left regular representation corresponds to viewing $\mathcal{H}$ as a left $\mathcal{H}$-module under convolution. View $\mathcal{H}$ as a $G$-representation under the left regular action. Then by section 2.1 $\mathcal{H}$ decomposes into a product of subrepresentations $\mathcal{H}^{\mathfrak{s}} \in R^{\mathfrak{s}}(G)$ for $\mathfrak{s} \in \mathcal{B}(G)$.

**Proposition 2.3.** The spaces $\mathcal{H}^{\mathfrak{s}}$ are two-sided ideals of $\mathcal{H}$. Further, if $(\pi, V)$ is a smooth $G$-representation then $V^{\mathfrak{s}} = \mathcal{H}^{\mathfrak{s}}V$ for any $\mathfrak{s} \in \mathcal{B}(G)$.

*Proof.* Since $\mathcal{H}^{\mathfrak{s}}$ is a $G$-subspace of $\mathcal{H}$ under $L$, we know that it is a left $\mathcal{H}$-submodule. Now for all $h, g \in G$ the $\mathcal{H}$ automorphisms $L(h)$ and $R(g)$ commute with each other. Thus for each $g \in G$ the space $R(g)\mathcal{H}^{\mathfrak{s}}$ is isomorphic to $\mathcal{H}^{\mathfrak{s}}$. By section 2.1 $\mathrm{Hom}_{\mathcal{H}}(\mathcal{H}^{\mathfrak{s}}, \mathcal{H}^{\mathfrak{t}}) = \{0\}$ if $\mathfrak{s} \neq \mathfrak{t}$, so $R(g)\mathcal{H}^{\mathfrak{s}} = \mathcal{H}^{\mathfrak{s}}$. Thus $\mathcal{H}^{\mathfrak{s}}$ is a right ideal as well.

For $\mathfrak{s} \neq \mathfrak{t} \in \mathcal{B}(G)$ we thus have $\mathcal{H}^{\mathfrak{s}} \star \mathcal{H}^{\mathfrak{t}} \subset \mathcal{H}^{\mathfrak{s}} \cap \mathcal{H}^{\mathfrak{t}} = \{0\}$, so $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}^{\mathfrak{s}} \star \mathcal{H}$. Any smooth $G$-representation is a nondegenerate $\mathcal{H}(G)$-module and hence a quotient of a direct sum of copies of $\mathcal{H}$. This proves the second assertion. □

Let $\mathfrak{s}$ be an inertial equivalence class for $G$, and let $(J, \tau)$ be a type for $\mathfrak{s}$. Recall the idempotent $e_\tau$ associated to $\tau$ in section 1.3. Then proposition 1.5 makes $V^\rho$ a module of over the subalgebra $e_\rho \star \mathcal{H} \star e\rho$ of $\mathcal{H}$. Continue viewing $\mathcal{H}$ as a $G$-space under the right-regular action. By proposition 2.2, $\mathcal{H}^{\mathfrak{s}}$ is precisely the $G$-subspace of $\mathcal{H}$, or $\mathcal{H}(G)$ submodule, generated by $\mathcal{H}^\tau$. But $\mathcal{H}^\tau = e_\tau \star \mathcal{H}$, so we conclude

$$\mathcal{H}^{\mathfrak{s}} = \mathcal{H} \star e_\tau \star \mathcal{H}. \tag{2.3}$$

In the remainder of this section we will state a series of results with the goal of relating $e_\rho \star \mathcal{H} \star e_\rho$ and $\mathcal{H}(G, \rho)$. Let $(\rho, W)$ be an irreducible representation of a compact open subgroup $K$ of $G$.

**Lemma 2.3.** Let $\mathcal{H}(K)$ be the subalgebra of $\mathcal{H}(G)$ consisting of functions with support in $K$. Then the map $e_\rho \star \mathcal{H}(K) \star e_\rho \to \mathrm{End}_{\mathbb{C}}(W)$ is an isomorphism.

*Proof.* We recall a result of Bernside on finite dimensional algebras. Let $V$ be a complex vector space. Then a collection of linear operators on $V$ generates $\mathrm{End}(V)$ as an algebra if and only if the operators have no common, non-trivial, invariant subspace. Thus, because $W$ is irreducible as an $\mathcal{H}(K)$-module, and hence as an $e_\rho \star \mathcal{H}(K) \star e_\rho$-module, $e_\rho \star \mathcal{H}(K) \star e_\rho \to \mathrm{End}_{\mathbb{C}}(W)$ must be surjective.

Now pick a non-trivial element $f \in \mathcal{H}(K)$ and suppose $f$ acts trivially on $W$. Then $e_\rho \star \mathcal{H}(K) = \mathcal{H}(K)^\rho \cong \oplus W$, so $f \star e_\rho \star \mathcal{H}(K) = 0$. Thus, in particular, $f \star e_\rho = 0$, so $e_\rho \star \mathcal{H}(K) \star e_\rho$ acts faithfully on $W$. □

**Proposition 2.4.** Let $W^*$ denote the dual space of $W$. Then there is an isomorphism of $G$-representations

$$c\text{–}\mathrm{Ind}_K^G(\rho) \otimes_{\mathbb{C}} W^* \cong \mathcal{H}(G) \star e_\rho.$$

*Proof.* We will need to use the standard isomorphism $W \otimes W^* \cong \mathrm{End}_{\mathbb{C}}(W)$. Also, recall from proposition 1.7 the isomorphism $c\text{–}\mathrm{Ind}_K^G \rho \cong \mathcal{H}(G) \otimes_{\mathcal{H}(K)} W$. This gives us

$$\begin{aligned} c\text{–}\mathrm{Ind}_K^G \rho \otimes_{\mathbb{C}} W^* &\cong \mathcal{H}(G) \otimes_{\mathcal{H}(K)} W \otimes_{\mathbb{C}} W^* \\ &\cong \mathcal{H}(G) \otimes_{\mathcal{H}(K)} \mathrm{End}_{\mathbb{C}}(W) \\ &\cong \mathcal{H}(G) \otimes_{\mathcal{H}(K)} e_\rho \star \mathcal{H}(K) \star e_\rho. \end{aligned}$$

To finish the proof, one checks that $\mathcal{H}(G) \star e_\rho \cong \mathcal{H}(G) \star e_\rho \star \mathcal{H}(K) \star e_\rho$. □

**Proposition 2.5.** There is a canonical isomorphism of unital complex algebras

$$\mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(W) \cong e_\rho \star \mathcal{H}(G) \star e_\rho.$$

*Proof.* Recall that $\mathcal{H}(G, \rho) \cong \mathrm{End}_G(c\text{–}\mathrm{Ind}_K^G \rho)$, where the action of $\mathcal{H}(G, \rho)$ is given by convolution. Since the action of $G$ on $c\text{–}\mathrm{Ind}_K^G \rho \otimes_{\mathbb{C}} W^*$ is on the first factor, we have an isomorphism of $G$-modules

$$\mathrm{End}_G\left(c\text{–}\mathrm{Ind}_K^G\rho \otimes_{\mathbb{C}} W^*\right) \cong \mathcal{H}(G,\rho) \otimes \mathrm{End}_{\mathbb{C}}(W).$$

However, by proposition (2.4), this endomorphism ring is also isomorphic to

$$\mathrm{End}_G(\mathcal{H}(G) \star e\rho) = \mathrm{End}_{\mathcal{H}(G)}(\mathcal{H}(G) \star e\rho).$$

An $\mathcal{H}(G)$ map $\varphi \in \mathrm{End}_{\mathcal{H}(G)}(\mathcal{H}(G) \star e_\rho)$ is completely determined by its action on $e_\rho$. Further, $\varphi(e_\rho) = \varphi(e_\rho \star e_\rho \star e_\rho) = e_\rho \star \varphi(e_\rho) \star e_\rho \in e_\rho \star \mathcal{H}(G) \star e_\rho$. Thus the restriction map

$$\mathrm{End}_{\mathcal{H}(G)}(\mathcal{H}(G) \star e_\rho) \to \mathrm{End}_{e_\rho \star \mathcal{H}(G) \star e_\rho}(e_\rho \star \mathcal{H}(G) \star e_\rho) \cong e_\rho \star \mathcal{H}(G) \star e_\rho$$

is an isomorphism. This completes the proof. $\square$

# CHAPTER 3
# CONSTRUCTION OF THE COVER

We now specialize to $G = Sp(2k, F)$ with respect to the antisymmetric form $J$ described in section 1.4.2. Let $P$ denote the standard Siegel parabolic with respect to $J$, $N$ its unipotent radical, and $L$ the standard Levi subgroup. The goal of this section is to construct a cover for certain representations of $L$.

We will not construct a cover for every supercuspidal type on $L$, but will instead focus on the case where the representation of $L$ is twisted by a highly ramified character (see section 3.2 for definitions). This setting is of interest as it is the context for stability of gamma factors (see [**?**] for more details on gamma factors and the Langlands-Shahidi method). It is hoped that splitness of a cover can play an algebraic role similar to stability, and that the existence of a split cover can be used in an explicit local calculation of the stable gamma factor. This thesis represents the first step towards such a calculation.

## 3.1 A Type on $L$

Let $\sigma \in R(L)$ be supercuspidal, and set $\mathfrak{s}_L = [L, \sigma]_L$. Our first task is to choose a type for $\mathfrak{s}_L$. Recall from section 1.4.2 that $L \cong GL(k, F)$. Bushnell and Kutzko have constructed a type for $\mathfrak{s}_L$ by showing that $\sigma$ is induced from a smooth irreducible representation $\tilde{\lambda}$ of an open subgroup $\tilde{J} < L$ that is compact modulo the center $Z(L)$. Let $L^\circ$ be the subgroup of elements of $L$ with unit determinant and set $J = \tilde{J} \cap L^\circ$. Set $\lambda$ to be $\tilde{\lambda}$ restricted to $J$. Then Bushnell and Kutzko further show that $\lambda$ is irreducible and is intertwined by an element $x \in G$ if and only if $x \in \tilde{J}$. Finally, they show that $J$ is normal in $\tilde{J}$ and that the quotient $\tilde{J}/J$ is abelian [3].

**Lemma 3.1.** Let $\tilde{J} < L$ be a subgroup that is compact mod $Z(L)$ and contains $Z(L)$. Then subgroup $J = \tilde{J} \cap L^\circ$ is the unique maximal compact subgroup of $\tilde{J}$ and $J \cap Z(L) = Z(L) \cap L^\circ$.

*Proof.* Any compact subgroup of $L$ is contained in $L^\circ$ since otherwise its image under the map $x \mapsto |\det(x)|$ would not be compact. On the other hand, the image of $J$ in $L/Z$ is a subgroup of $\tilde{J}/Z$, so it is compact. Thus there exists a compact subset $C \subset L$ so that $J = C \cdot (Z \cap J)$. By the definition of $J$, $Z \cap J = Z \cap \tilde{J} \cap L^\circ$, but $Z \subset \tilde{J}$, so $Z \cap \tilde{J} = Z$. A scalar matrix has unit determinant if and only if the element on the diagonal is a unit. Thus $L^\circ \cap Z$ consists of scalar matrices with unit diagonal and so is isomorphic to $\mathcal{O}^\times$. Putting all this together, we have $Z \cap J = Z \cap L^\circ \cong \mathcal{O}^\times$, so $J \cong C \cdot \mathcal{O}^\times$. This is a product of compact groups, so $J$ is compact. The second assertion is clear. $\square$

**Lemma 3.2.** The representation $\lambda$ is a type for $\mathfrak{s}_L$.

*Proof.* We must show that an irreducible representation $\sigma' \in R(L)$ contains $\lambda$ if and only if $\sigma' \in R^{\mathfrak{s}_L}(L)$. Since $\mathfrak{s}_L = [L, \sigma]_L$, irreducible representations in $R^{\mathfrak{s}_L}(L)$ are all of the form $\sigma \otimes \chi$ for some unramified quasicharacter $\chi \in \mathfrak{X}(L)$. The representation $\tilde{\lambda}$ clearly contains $\lambda$, and $\sigma$ is induced from $\tilde{\lambda}$, so $\sigma$ contains $\lambda$. Thus, because $\chi$ is trivial on $J$, each $\sigma \otimes \chi$ also contains $\lambda$. $\square$

Rather than this type, it will be convenient to choose a related type with a specified group.

**Proposition 3.1.** We can choose a type $(K_L, \tau_L)$ for $\mathfrak{s}_L$ with $K_L = GL(k, \mathcal{O})$.

*Proof.* By lemma, 3.1 $J$ is the unique maximal compact subgroup of $\tilde{J}$. If necessary, we conjugate $\tilde{\lambda}$ so that $J < K_L$. Set $\tau_L = c\text{–Ind}_J^{K_L}\lambda$. Now $x \in K_L$ intertwines $\lambda$ if and only if $x \in \tilde{J}$, but $\tilde{J} \cap K_L = J$ since $J$ is the maximal compact subgroup of $\tilde{J}$, so by Mackey's theorem $\tau_L$ is irreducible. Let $\sigma \in R(L)$ be irreducible. Then by Frobenious reciprocity

$$\mathrm{Hom}_J(\lambda, \sigma) = \mathrm{Hom}_{K_L}(\tau_L, \sigma),$$

so $\sigma$ contains $\tau_L$ if and only if it contains $\lambda$. But $\sigma$ contains $\lambda$ if and only if $\sigma$ is an unramified twist of $\pi$, so $\tau_L$ is a type for $\mathfrak{s}_L$. □

Finally, we wish to determine the structure of the Hecke algebra $\mathcal{H}(L, \tau_L)$.

**Lemma 3.3.** The algebra $\mathcal{H}(L, \tau_L)$ is isomorphic to $\mathcal{H}(L, \lambda)$.

*Proof.* By theorem 2.2

$$\mathcal{H}(L, \tau_L) \cong \mathrm{End}_L(c\text{–Ind}_{K_L}^L \tau_L),$$

$$\mathcal{H}(L, \lambda) \cong \mathrm{End}_L(c\text{–Ind}_J^L \lambda).$$

By transitivity of induction $c\text{–Ind}\tau_L \cong c\text{–Ind}\lambda$, so these two endomorphism rings are isomorphic and we conclude the lemma. □

**Proposition 3.2.** The algebra $\mathcal{H}(L, \tau_L)$ is a PID.

*Proof.* By lemma 3.3 we must show that $\mathcal{H}(L,\lambda)$ is a PID. Unramified quasicharacters of $L$ are all of the form $x \mapsto |\det(x)|^s$ for $s \in \mathbb{C}$. Thus $\mathfrak{X}(L)$ is isomorphic to $\mathbb{C}$ as a complex variety. Recall the set $D_\sigma$ from section 2.1. Because $W(L,\tau_L)$ is trivial we have $D_\sigma/W(L,\tau_L) = D_\sigma \cong (X)(L)$. Thus by theorem 2.1 the center of the category $R^{\tau_L}(L)$ is isomorphic to the ring of regular functions $\mathbb{C}[X,X^{-1}]$. This ring is a localization of the PID $\mathbb{C}[X]$, so it is a PID. Applying lemma 2.1, we see that the center of $\mathcal{H}(L,\lambda)$ is isomorphic to $\mathbb{C}[X,X^{-1}]$, so it remains to show that $\mathcal{H}(L,\lambda)$ is commutative.

Recall that $x \in G$ intertwines $\lambda$ if and only if $x \in \tilde{J}$. Thus $\mathcal{H}(L,\lambda) = \mathcal{H}(\tilde{J},\lambda)$. Fix $x \in \tilde{J}$. Since $J < \tilde{J}$ is a normal subgroup, $J^x \cap J = J$. Thus, because $\lambda$ and $\lambda^x$ are irreducible, the space $\mathrm{Hom}_{J\cap J^x}(\lambda^x,\lambda) = \mathrm{Hom}_J(\lambda^x,\lambda)$ is at most one-dimensional. Since $x$ intertwines $\lambda$ it is precisely one-dimensional. This means that $\mathcal{H}(L,\lambda)$ is isomorphic to the group algebra $\mathbb{C}[\tilde{J}/J]$, but this group is abelian, so we are done. $\square$

### 3.2 The Cover

We do not construct a cover for all $\mathfrak{s}_L$. Let $(K_L,\tau_L)$ be the $\mathfrak{s}_L$-type constructed in lemma (3.1). Given a multiplicative character $\chi : F^\times \to \mathbb{C}^\times$ we define the twisted representation $\sigma \otimes \chi$ by $\sigma \otimes \chi(g) = \chi(\det(g))\sigma(g)$. Set $\mathfrak{s}_L \otimes \chi = [L, \sigma\otimes\chi]_L$. Then irreducible elements of $R^{\mathfrak{s}}_L(L)$ are of the form $\sigma \otimes \eta$ while irreducible elements of $R^{\mathfrak{s}_L\otimes\chi}(L)$ are of the form $\sigma\otimes\chi\otimes\eta$ where $\eta$ ranges over unramified quasicharacters of $L$. Thus $(K_L,\tau_L\otimes\chi)$ is a type for $\mathfrak{s}_L\otimes\chi$ since $\tau_L\otimes\chi$ is contained in an irreducible representation if and only if that representation is an element of $R^{\mathfrak{s}_L\otimes\chi}(L)$. Our goal in this section is to construct a cover for this type in the case that $\chi$ is sufficiently highly ramified.

Let us fix an appropriate character. Let $\chi$ be a multiplicative character of $F$ with conductor $n$. That is, $n$ is the smallest integer such that $1+\mathfrak{p}^n \in \ker(\chi)$. Our "highly ramified" condition is that $n \geq 2$ and that $n$ is large enough that

$$Id + Mat_k(\mathfrak{p}^{n-1}) < \ker(\tau_L). \tag{3.1}$$

Note that open compact subgroups of the form $Id + Mat_k(\mathfrak{p}^n)$ form a neighborhood basis of the identity in $K_L$, so such an $n$ must exist because $\tau_L$ is a smooth finite-dimensional representation.

**Lemma 3.4.** With $\chi$ as above there exists $\alpha \in 1+\mathfrak{p}^{n-1}$ such that $\chi^2(\alpha) \neq 1$.

*Proof.* Since the conductor of $\chi$ is $n$ we may pick $x \in \mathfrak{p}^{n-1}$ so that $\chi(1+x) \neq 1$. Let $\alpha = 1 + \frac{x}{2}$. Then $\alpha \in 1+\mathfrak{p}^{n-1}$ since the residue characteristic of $F$ is not equal to 2. Further,

$$\chi^2(\alpha) = \chi\left(1+x+\frac{x^2}{4}\right) = \chi(1+x)\chi\left(1+(1+x)^{-1}\frac{x^2}{4}\right),$$

but $\frac{x^2}{4} \in \mathfrak{p}^n$ because $n \geq 2$. Thus

$$1+(1+x)^{-1}\frac{x^2}{4} \in \ker(\chi),$$

so $\chi^2(\alpha) = \chi(1+x) \neq 1$. $\square$

We now construct the group for our cover. Let $\varphi$ denote the homomorphism given by reduction modulo $\mathfrak{p}^n$, and $I = Sp(2k, \mathcal{O}) \cap P$. Define $J = \varphi^{-1}(\varphi(I))$. Then $J < Sp(2k, \mathcal{O})$, so $J$ is compact and $\ker\varphi$ is open. Thus $J$ is open. In fact, we can see that $J$ is simply the set of $g \in G$ with integral entries such that $g_{ij} \in \mathfrak{p}^{\mathfrak{n}}$ if $i \geq k+1$ and $j \leq k$. Our initial definition makes it clear that $J$ is a group. Note that this definition does not require any knowledge of the image $\varphi(Sp(2k, \mathcal{O}))$.

**Lemma 3.5.** The group $J$ has an Iwahori decomposition. That is

$$J = (J \cap \bar{N})(J \cap L)(J \cap N).$$

*Proof.* Let $g \in J$. Then we must show that $g = \bar{n}p$ for $\bar{n} \in J \cap \bar{N}$ and $p \in J \cap P$. If we can prove this, then the decomposition $P = LN$ of our parabolic subgroup will give us the complete Iwahori decomposition. Let

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \bar{n} = \begin{pmatrix} Id & \\ Y & Id \end{pmatrix},$$

where we are writing our matrices in $k \times k$ blocks. Note that the image of $g$ is block upper triangular modulo $\mathfrak{p}^n$. Let $A'$ denote the upper left $k \times k$ block of $g^{-1}$. Then since $gg^{-1} = Id$ we have $AA' \equiv Id \bmod \mathfrak{p}^n$. Put another way, $AA' \in Id + Mat_k(\mathfrak{p}^n)$, and hence $AA'$ is invertible. Thus $A$ is invertible with inverse $A'(AA')^{-1} \in GL(k, \mathcal{O})$.

We wish to pick $Y$ so that $\bar{n}^{-1}g \in P$. Observe that $\bar{n}^{-1}$ is simply $\bar{n}$ with $Y$ replaced by $-Y$. A simple calculation shows that we must set $Y = CA^{-1}$. Setting $p = \bar{n}^{-1}g$, we have $g = \bar{n}p$ as desired. We must show that $\bar{n}, p \in J$. Since $A \in GL(k, \mathcal{O})$ and $C \in Mat_k(\mathfrak{p}^n)$, we have $Y \in Mat_k(\mathfrak{p}^n)$, so in particular it remains to show $Y$ is symmetric. Let $q$ denote the antisymmetric $2k \times 2k$ matrix

$$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}.$$

Then because $g \in G$ we have

$$gJ^t g = J = \bar{n}pq^t p^t \bar{n} \tag{3.2}$$

$$\Rightarrow pq^t p = \bar{n}^{-1} q^t \bar{n}^{-1}. \tag{3.3}$$

When we calculate the lower right $k \times k$ block of equation (3.3) we get $Y - {}^tY = 0$. Thus $Y$ is symmetric, so $\bar{n} \in \bar{N} \cap J$. Furthermore, $p = \bar{n}^{-1}g$ and $g \in J$, so $p \in P \cap J$. $\square$

We define a representation $\tau$ of $J$ acting on the vector space of $\tau_L$ by writing

$$\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \tau_L(A). \tag{3.4}$$

This is a well defined representation since

$$\begin{aligned} \tau\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\right) &= \tau_L(AA' + BC') \\ &= \tau_L(AA')\tau_L\left(Id + (AA')^{-1}BC'\right) \end{aligned}$$

and $Id + (AA')^{-1}BC'$ is in the kernel of $\tau_L$ by our condition on $n$. We identify $J \cap L$ with $K_L$. Then from the definition of $\tau$ we see that $\tau|(J \cap N)$ and $\tau|(J \cap \bar{N})$ are trivial and that $\tau|K_L \equiv \tau_L$. Finally, we define the twisted representation $\tau \otimes \chi$ by replacing $\tau_L$ with $\tau_L \otimes \chi$ in equation (3.4). We have now defined the representation that will be our cover, and we must show that it is a cover. This is the main theorem of this section.

**Theorem 3.1.** The pair $(J, \tau \otimes \chi)$ is a cover for $(K_L, \tau_L \otimes \chi)$.

The remainder of this section is devoted to proving this theorem. We have already proved conditions (i) and (ii) of definition 2.3, so it remains to check condition (iii). That is, we must construct invertible elements of $\mathcal{H}(G, \tau \otimes \chi)$ supported at positive and negative powers of a strongly positive element for $P$. Set

$$\zeta = \begin{pmatrix} \varpi Id & \\ & \varpi^{-1} Id \end{pmatrix}.$$

Then $\zeta \in G$ and $\zeta$ is strongly positive with respect to $P$.

Note that conjugation by $\zeta$ or $\zeta^{-1}$ does not affect the upper left $k \times k$ block of a $2k \times 2k$ matrix, so given $x \in J \cap J^{\zeta^{\pm 1}}$ we have $(\tau \otimes \chi)(x) = (\tau \otimes \chi)^{\zeta^{\pm 1}}(x)$. Thus $\zeta$ and $\zeta^{-1}$ intertwine $\tau$, so there are $\tau$-spherical functions $f_\zeta$ and $f_{\zeta^{-1}}$ supported on $J\zeta J$ and $J\zeta^{-1}J$, respectively. Still using the fact that conjugation by $\zeta^{\pm 1}$ does not affect $\tau \otimes \chi$, we may apply lemma 2.2 to pick $f_\zeta(\zeta) = f_{\zeta^{-1}}(\zeta^{-1}) = 1$.

We will show that $f_\zeta$ and $f_{\zeta^{-1}}$ are inverse to one another under convolution. The identity element in $\mathcal{H}(G, \tau)$ is the function given by $1/\mathrm{Vol}(J)$ on $J$ and 0 elsewhere. We must show that $f_\zeta * f_{\zeta^{-1}}$ is the identity. By definition,

$$(f_\zeta * f_{\zeta^{-1}})(Id) = \int_G f_\zeta(x) f_{\zeta^{-1}}(x^{-1}) dx.$$

This integral is supported on $J\zeta J$, but on $J\zeta J$ we see that $f_\zeta(x) = f_{\zeta^{-1}}(x)^{-1}$, so the integral simplifies to $\mathrm{Vol}(J\zeta J)$. Thus, up to normalization, $(f_\zeta * f_{\zeta^{-1}})(Id) = 1/\mathrm{Vol}(J)$. The support of the convolution of two functions is contained in the product of their supports, so supp $(f_\zeta * f_{\zeta^{-1}}) \subset (J\zeta J)(J\zeta^{-1}J) \subset J\bar{N}J$, where the second containment follows from the positivity of $\zeta$ combined with the Iwahori decomposition of $J$. To show that $f_{\zeta^{-1}} = f_\zeta^{-1}$, it remains to show that this support is simply $J$. We will need the following technical lemma.

**Lemma 3.6.** Given a symmetric matrix $Y \in Mat_k(F) \setminus Mat_k(\mathfrak{p}^n)$ there exists a matrix $A \in GL(k, \mathcal{O})$ such that

$$AY^tA = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}, \quad a \notin \mathfrak{p}^n. \tag{3.5}$$

*Proof.* Let $M$ be a free $\mathcal{O}$-module of rank $k$ with basis $v_1, \dots, v_k$, and define a symmetric bilinear form on $M$ by $B(v_i, v_j) = Y_{i,j}$. Let $w_1, \dots, w_k$ be an alternate basis for $M$, and let $A \in GL(k, \mathcal{O})$ be the change of basis matrix. That is, $Av_i = w_i$. Then it is easily checked that the matrix for $B$ with respect to $\{w_i\}$ is given by $AY^tA$.

Let $(B(M, M)) = (a)$. Since $Y \notin Mat_k(\mathfrak{p}^n)$ we have $v(a) < n$. Pick $x, y \in M$ so that $B(x, y) = a$. Then

$$B(x + y, x + y) = B(x, x) + 2B(x, y) + B(y, y),$$

so either $B(x, x)$, $B(y, y)$, or $B(x + y, x + y)$ generates $(a)$. Thus, adjusting $a$ by a unit if necessary, we may pick a single element $x \in M$ with $B(x, x) = a$.

Given a submodule $N \subset M$ we define the submodule

$$N^\perp = \{m \in M : B(m, n) = 0 \text{ for all } n \in N\}.$$

Suppose $M = N \oplus N'$. We say the sum is perpendicular and write $M = N \perp N'$ if $N' \subset N^\perp$ (equivalently, if $N \subset N'^\perp$). Now let $y \in M$ and set $b = B(x, y)$. Then $c = b/a$ is an integer and $B(x, cx) = ac = b$. Thus $y - cx \in \langle x \rangle^\perp$, so $y = cx + (y - cx) \in \langle x \rangle + \langle x \rangle^\perp$. Thus $M = \langle x \rangle \perp \langle x \rangle^\perp$. Now set $w_1 = x$, and let $w_2, \dots, w_k$ be a basis for $\langle x \rangle^\perp$. Then written with respect to $\{w_i\}$ $B$ is of the same form as $AY^tA$ in equation 3.5, so we are done. □

It is worth noting that by iterating lemma 3.6 we have in fact shown that $Y$ can be put into a diagonal form; however, we will not need this result.

Define the $2k \times 2k$ matrix

$$\bar{n}(Y) = \begin{pmatrix} 1 & \\ Y & 1 \end{pmatrix} \tag{3.6}$$

for $Y$ a $k \times k$ matrix. By proposition 3.3, a $\tau$-spherical function can only be supported at $x \in G$ if $x$ intertwines $\tau$. Thus the following proposition completes the proof that $f_\zeta * f_{\zeta^{-1}}$ is supported only on $J$, and hence that $(J, \tau \otimes \chi)$ is a cover.

**Proposition 3.3.** Let $Y$ be a $k \times k$ symmetric matrix not contained in $Mat_k(\mathfrak{p}^n)$. Then $\bar{n}(Y)$ does not intertwine $\tau \otimes \chi$.

*Proof.* We first show that it suffices to consider $Y$ of the form

$$Y = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

where $a \notin \mathfrak{p}^n$. By lemma 3.3 an element $g \in G$ intertwines $\tau \otimes \chi$ iff each other element of the double coset $JgJ$ intertwines $\tau \otimes \chi$. In particular, invoking our choice of type allowed by Lemma 3.1, if we take $A \in GL(k, \mathcal{O})$ then we have

$$\begin{pmatrix} {}^tA^{-1} & \\ & A \end{pmatrix} \in J,$$

$$\begin{pmatrix} {}^tA^{-1} & \\ & A \end{pmatrix} \bar{n}(Y) \begin{pmatrix} {}^tA & \\ & A^{-1} \end{pmatrix} = \bar{n}(AY^tA).$$

Thus we may replace $Y$ with $AY^tA$ for any $A \in GL(k, \mathcal{O})$, so we apply lemma 3.6. We now have two cases.

*Case 1, $a \notin \mathfrak{p}$*: Let $\alpha \in 1 + \mathfrak{p}^{n-1}$ be the element guaranteed by lemma 3.4. Define $k \times k$ matrices $A$ and $X$, and a $2k \times 2k$ matrix $j$ by

$$A = \begin{pmatrix} \alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} (\alpha - \alpha^{-1})a^{-1} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$j = \begin{pmatrix} A & X \\ & A^{-1} \end{pmatrix}.$$

Then $j \in J$. The following identities will be useful

$$A - XY = A^{-1}$$

$$YX + A^{-1} = A$$

$$YA - YXY - A^{-1}Y = 0.$$

To check them, it is helpful to note that all three matrices involved are block diagonal with a $1 \times 1$ block followed by a $k-1 \times k-1$ block. In particular, to see the third identity note that $YA - A^{-1}Y$ is already zero on the lower block, and $-YXY$ is also zero on the lower block. On the upper block these matrices give $a\alpha - a(\alpha - \alpha^{-1})a^{-1}a - \alpha^{-1}a$, which is zero, so the identity holds. Note that $\bar{n}(Y)^{-1} = \bar{n}(-Y)$. These three identities are precisely what is needed to show that

$$\bar{n}(Y) \cdot j \cdot \bar{n}(-Y) = \begin{pmatrix} A^{-1} & X \\ & A \end{pmatrix}.$$

Thus, by our choice of $\alpha$, we have $(\tau \otimes \chi)(j) = \tau_L(A)\chi(\det(A)) = \chi(\alpha) \neq \chi(\alpha^{-1}) =$

$\tau\left(\bar{n}(Y) \cdot j \cdot \bar{n}(Y)^{-1}\right)$. Note that we are using the condition on $n$ in equation 3.1 to see that $\tau_L(A)$ and $\tau_L(A^{-1})$ are trivial.

*Case 2,* $a \in \mathfrak{p}$: Using the same $\alpha$ as above, set

$$b = \frac{1-\alpha}{a}$$

$$X = \begin{pmatrix} b & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$Y' = \begin{pmatrix} \frac{a^2 b}{1-ab} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$j = \begin{pmatrix} 1 & X \\ Y' & 1 + XY' \end{pmatrix}.$$

Then $j \in J$. We will need the following identities

$$1 - XY = A$$

$$YX + XY' + 1 = A^{-1}$$

$$Y' - YXY - XY'Y = 0.$$

As in case 1, checking that these identities hold for the lower block is trivial. The upper block of each identity reduces to the corresponding identity below:

$$1 - ba = \alpha$$

$$ab + b\frac{a^2 b}{1-ab} + 1 = \alpha^{-1}$$

$$\frac{a^2b}{1-ab} - aba - ab\frac{a^2b}{1-ab} = 0,$$

and these can be checked using the definition of $b$. Once again, the first set of identities is precisely what is needed to show

$$\bar{n}(Y) \cdot j \cdot \bar{n}(-Y) = \begin{pmatrix} A & X \\ & A^{-1} \end{pmatrix}.$$

Thus $(\tau \otimes \chi)(j) = 1 \neq \chi(\alpha) = (\tau \otimes \chi)(\bar{n}(Y) \cdot j \cdot \bar{n}(Y)^{-1})$. $\square$

# CHAPTER 4
# A SPLITNESS THEOREM

The goal of this chapter is to prove that the cover constructed in section (3.2) is split, i.e., the injection $t_P : \mathcal{H}(L, \tau_L) \to \mathcal{H}(G, \tau)$ is an isomorphism. We will, in fact, prove a more general theorem regarding the splitness of covers.

We first recall a result of Alan Roche on parabolic induction. This result answers the question: when is parabolic induction restricted to a given Bernstein component an equivalence of categories? Let $G$ be a reductive $p$-adic group and let $L < M < G$ be Levi subgroups of $G$. Choose parabolic subgroups $P < Q < G$ with Levi components $L$ and $M$, respectively. Let $\sigma$ be a supercuspidal representation of $L$ and set $\mathfrak{s}_L = [L, \sigma]_L$, $\mathfrak{s} = [L, \sigma]_M$, and $\mathfrak{t} = [L, \sigma]_G$. Define the following groups

$$\begin{aligned} N_G(\mathfrak{s}_L) &= \{g \in N_G(L) : \sigma^g \cong \sigma \otimes \nu \text{ for some } \nu \in \mathfrak{X}(L)\} \\ N_M(\mathfrak{s}_L) &= N_G(\mathfrak{s}_L) \cap M \\ W_G(\mathfrak{s}_L) &= N_G(\mathfrak{s}_L)/L \\ W_M(\mathfrak{s}_L) &= N_M(\mathfrak{s}_L)/L. \end{aligned}$$

The main result of [11] is:

**Theorem 4.1** (Roche)**.** The parabolic induction functor $i_Q^G : R^{\mathfrak{s}}(M) \to R^{\mathfrak{t}}(G)$ is an equivalence of categories if and only if $W_M(\mathfrak{s}_L) = W_G(\mathfrak{s}_L)$ [11, thm. 3.1].

**Corollary 4.1.** The functor $i_L^G : R^{\tau_L}(L) \to R^{\mathfrak{t}}(G)$ is an equivalence of categories if and only if $N_G(\mathfrak{s}_L) = L$.

*Proof.* This is merely the simplification of theorem 4.1 to the situation $P = Q$. □

## 4.1 A Finiteness Result

We now prove a few technical results to be used in the main theorem of this section. We continue with $G$ a reductive $p$-adic group and fix a parabolic subgroup $P$ with Levi component $L$ and unipotent radical $N$. Write $\bar{N}$ for the unipotent radical opposite to $N$. Let $\sigma \in R(L)$ be supercuspidal, and let $(J_L, \tau_L)$ be a type for $[L, \sigma]_L$. Let $(J, \tau)$ be a cover for $(J_L, \tau_L)$. Fix a maximal compact subgroup $K < G$ containing $J$ and such that $K_L = K \cap L$ is a maximal compact subgroup of $L$ and $K = (K \cap \bar{N}) \cdot K_L \cdot (K \cap N)$. Let $r_N : R(G) \to R(L)$ denote Jaquet restriction.

**Proposition 4.1.** In the setting of the previous paragraph, let $(\rho, U)$ be an irreducible representation of $K$, and define $\rho_L$ to be the representation on $U^{K \cap N}$ given by restriction of $\rho$ to $K_L$. Then $r_N(c\text{–}\mathrm{Ind}_K^G \rho) \cong c\text{–}\mathrm{Ind}_{K_L}^L \rho_L$.

*Proof.* Let $\delta_N$ denote the modular character of $P$. We define a map $\Phi : r_N(c\text{–}\mathrm{Ind}_K^G \rho) \to c\text{–}\mathrm{Ind}_{K_L}^L \rho_L$ by

$$\Phi f(x) = \delta_N(x) \int_N f(xn) dn \quad \text{for } x \in L.$$

This map is clearly linear, so to check that it is well defined on cosets it suffices to show that an element of $c\text{–}\mathrm{Ind}_K^G \rho(N)$ is sent to zero. Let $f \in c\text{–}\mathrm{Ind}_K^G \rho$ and $n' \in N$. Then

$$\Phi(\pi(n')f - f)(x) = \delta_N(x) \left( \int_N f(xnn')dn - \int_N f(xn)dn \right) = 0$$

because $dn$ is a Haarr measure for $N$.

We next show that the integrand of $\Phi$ is compactly supported. Because $\Phi$ is linear, it suffices to work with a spanning set consisting of functions of the form

$$f_{g,w}(x) = \begin{cases} \rho(k)w & \text{for } x = kg \in Kg \\ 0 & \text{otherwise} \end{cases}$$

where $g \in L$ and $w \in U$. The support of $\Phi f_{g,w}$ is contained in $KgN \cap L$. Since $g \in G = KLN$ we may take $g \in G$. By the Iwahori decomposition of $K$ we have $x \in \big((K \cap \bar{N}) \cdot K_L g N\big) \cap L = K_L g$, which is open and compact. Thus, by the smoothness of $f$, the integral defining $\Phi f(x)$ reduces to a finite sum, so it converges.

To see that $\Phi f$ is contained in $c$–$\mathrm{Ind}_{K_L}^{L} \rho_L$, we must check that it is smooth and right-equivariant with respect to $(\rho_L, K_L)$. Under right translation $\Phi f$ is stabilized by $\ker \rho \cap L$, so it is smooth. One easily checks that $\Phi f(kx) = \rho_L(k)\Phi f(x)$ for $k \in K_L$.

The representation $c$–$\mathrm{Ind}\rho_L$ is spanned by functions of the form

$$F_{g,w}(x) = \begin{cases} \rho_L(k)w & \text{for } x = kg \in K_L g \\ 0 & \text{otherwise} \end{cases}$$

where $g \in L$ and $w \in U^{K \cap N}$. Let $p$ denote the projection $U \to U^{K \cap N}$ with respect to the decomposition $U = U^{K \cap N} \oplus U(K \cap N)$. Then for $g \in M$ one calculates $\Phi f_{g,w} = F_{g,p(w)}$, so for $w \in U^{K \cap N}$ we have $\Phi f_{g,w} = F_{g,w}$. Thus $\Phi$ is surjective.

Finally, we must show that $\Phi$ is injective. By the definition of $r_N$, we must show that if $\Phi f \equiv 0$ then $f \in (c\text{–}\mathrm{Ind}\rho)(N)$. As mentioned above, the support of any $f_{g,w}$ is contained in the double coset $KgN$, and our function $f$ is a linear combination of $f_{g,w}$. Thus we may assume $\mathrm{supp}\, f \subset KgN$ for some $g \in G$. We now require a lemma.

We now claim that the map $x \mapsto (KxN) \cap L$ induces a bijection from $K \backslash G/N$ to $K_L \backslash L$. If this map is well defined, it is clearly surjective and injective. Thus all we must show is that $(KxN) \cap L$ is a right $K_L$ coset for all $x \in G$. Write $x = x_k x_l x_n$ for $x_k \in K$, $x_l \in L$, and $x_n \in N$. Then, using the Iwahori decomposition of $K$ and the fact that $x_l$ normalizes $N$, we have

$$(KxN) \cap L = ((K \cap \bar{N} K_L x_l N) \cap L = K_L x_l.$$

Thus we may as well take $g \in L$. Let $\Sigma$ denote the right translation action on $c$–$\mathrm{Ind}_K^G \rho$. The function $f$ is compactly supported, hence a finite sum of $\Sigma(n_i) f_{g,w_i}$ for $n_i \in N$, $w_i \in U$. Thus $r_N(f) = f_{g,w}$ where $w = \sum_i w_i$. Similarly, $\Phi f = \Phi f_{g,w} = F_{g,p(w)}$, so we must have $p(w) = 0$. Thus $w \in U(K \cap N)$, so there exists a finite set of $n_j \in N$ such that $w = \sum_j \rho(n_j) w_j - w_j$. We deduce that

$$f_{g,w} = \sum_j (\Sigma(g^{-1} n_j g) f_{g,w_j} - f_{g,w_j}).$$

This certainly lies in $(c\text{–Ind}\rho)(N)$, so the proof is complete. □

Write $\mathscr{A} = \mathcal{H}(G, \tau)$ and $\mathscr{B} = \mathcal{H}(L, \tau_L)$. Using $t_P$ we view $\mathscr{B}$ as a subalgebra of $\mathscr{A}$. The following proposition is the main result of this section.

**Proposition 4.2.** $\mathscr{A}$ is finitely generated as a $\mathscr{B}$-module.

*Proof.* Set $\rho = c$–$\mathrm{Ind}_J^K \tau$ and define $\rho_L$ as in proposition (4.1). Define $Z = \mathcal{H}(L, \tau_L, \rho_L)$. This proof will proceed in two parts: first we show that $\mathscr{A} \cong Z$, then we show that $Z$ is finitely generated as a $\mathscr{B}$-module. By theorem 2.2, $Z \cong \mathrm{Hom}_L(c\text{–}\mathrm{Ind}_{J_L}^L \tau_L, c\text{–}\mathrm{Ind}_{K_L}^L \rho_L)$.

Thus, by Frobenious reciprocity for compact induction,

$$Z \cong \mathrm{Hom}_{J_L}(\tau_L, c\text{–Ind}\rho_L) = (c\text{–Ind}\rho_L)_{\tau_L}. \tag{4.1}$$

Similarly, by theorem 2.2 combined with Frobenious reciprocity, we have

$$\mathscr{A} \cong \mathrm{End}_G(c\text{–Ind}_J^G\tau) \cong (c\text{–Ind}\tau)_\tau. \tag{4.2}$$

By equation 2.2, $(c\text{–Ind}\tau)_\tau \cong ((c\text{–Ind}\tau)_N)_{\tau_L}$. Thus, by transitivity of induction, we have $\mathscr{A} \cong ((c\text{–Ind}_K^G\rho)_N)_{\tau_L}$, and by proposition (4.1) this is isomorphic to $(c\text{–Ind}_{K_L}^L\rho_L)_{\tau_L}$. Combining this result with equations (4.1) and (4.2), we get $Z \cong \mathscr{A}$.

We now show that $Z$ is finitely generated as a $\mathscr{B}$-module. Write $\mathcal{H} = \mathcal{H}(L)$. Let $U$ be the vector space of $\rho_L$. By proposition 2.5 we have an isomorphism of $\mathcal{H}$-modules

$$\mathcal{H} \star e_{\rho_L} \cong c\text{–Ind}_{K_L}^L\rho_L \otimes_{\mathbb{C}} U^* \cong \bigoplus_{\dim_{\mathbb{C}} U} c\text{–Ind}\rho_L.$$

Thus, by equation (4.1), it suffices to show that $(\mathcal{H} \star e_{\rho_L})_{\tau_L}$ is finitely generated as a $\mathscr{B}$-module. It is clear that the map

$$(\mathcal{H} \star e_{\rho_L})_{\tau_L} \otimes_{\mathbb{C}} W \to (\mathcal{H} \star e_{\rho_L})^{\tau_L}$$

$$\phi \otimes w \mapsto \phi(w)$$

is an isomorphism. Further, $(\mathcal{H} \star e_{\rho_L})^{\tau_L} \cong e_{\tau_L} \star \mathcal{H} \star e_{\rho_L}$.

Because $\tau_L$ is a type for some inertial equivalence class $\mathfrak{s}_L$, equation 2.3 tells us that $\mathcal{H}^{\mathfrak{s}_L} = \mathcal{H} \star e_{\tau_L} \star \mathcal{H}$. Proposition 2.3 further tells us that there is a two-sided ideal $\mathfrak{a} \subset \mathcal{H}$ such that $\mathcal{H} = \mathcal{H} \star e_{\tau_L} \star \mathcal{H} \oplus \mathfrak{a}$. In particular, we can write

$$e_{\rho_L} = \sum_i f_i \star e_{\tau_L} \star h_i + a$$

for $f_i,\ h_i \in \mathcal{H}$ and $a \in \mathfrak{a}$. Since $\mathfrak{a}^{\mathfrak{s}_L} = \{0\}$ we have $e_{\tau_L} \star a = 0$, so

$$\begin{aligned} e_{\tau_L} \star \mathcal{H} \star e_{\rho_L} &= e_{\tau_L} \star \mathcal{H} \star \left( \sum_i f_i \star e_{\tau_L} \star h_i \right) \\ &\subset \bigoplus_i e_{\tau_L} \star \mathcal{H} \star e_{\tau_L} \star h_i. \end{aligned}$$

Each summand of this space is isomorphic to $e_{\tau_L} \star \mathcal{H} \star e_{\tau_L}/\mathrm{ann}(h_i)$, where $\mathrm{ann}(h) = \{f \in \mathcal{H} : f \star h = 0\}$ for $h \in \mathcal{H}$. Thus $e_{\tau_L} \star \mathcal{H} \star e_{\rho_L}$ is a subspace of a finite sum of quotients of the noetherian ring $e_{\tau_L} \star \mathcal{H} \star e_{\tau_L}$, and hence is finitely generated over that ring. This completes the proof. □

## 4.2 The Splitness Theorem

Theorem 4.2 is the main result of this chapter. Combined with the theorem of Roche on parabolic induction (theorem 4.1) it shows that the cover constructed in chapter 3 is split.

**Theorem 4.2** (Splitness theorem)**.** Let $G$ be a reductive $p$-adic group, and let $L < G$ be a Levi subgroup contained in a parabolic subgroup $P$. Fix an irreducible supercuspidal representation $\sigma \in R(L)$ and set $\mathfrak{s}_L = [L, \sigma]_L$ and $\mathfrak{s} = [L, \sigma]_G$. Let $(J_L, \tau_L)$ be a type for $\mathfrak{s}_L$ covered by $(J, \tau)$ and such that the algebra $\mathcal{H}(L, \tau_L)$ is a PID. Then $(J, \tau)$ is a split cover for $(J_L, \tau_L)$ (i.e., $t_P : \mathcal{H}(L, \tau_L) \to \mathcal{H}(G, \tau)$ is an isomorphism) if and only if $i_P^G : R^{\mathfrak{s}_L} \to R^{\mathfrak{s}}$ is an equivalence of categories.

NOTE: By the work of Bushnell and Kutzko, if $L$ is the Levi subgroup of the Siegel parabolic then it is known to have a type with $\mathcal{H}(L, \tau_L)$ a PID, so this is not an unreasonable condition. It is the hope of the author that this restriction can ultimately be lifted.

*Proof.* Clearly if $t_P$ is an isomorphism then $(t_P)_*$ is an equivalence of categories, so, by section 2.3.1, $i_P^G$ is also an equivalence of categories.

To prove the converse we continue in the notation of this section. Assume that $i_P^G$, and hence $(t_P)_*$ is an equivalence of categories. Using $t_P$ we view $\mathscr{B}$ as a subalgebra of $\mathscr{A}$. Our task is then to prove that $\mathscr{A} = \mathscr{B}$. The functor $t_P^*$ is a right adjoint for $(t_P)_*$, so it must the case that $(t_P)_* \circ t_P^*$ is naturally equivalent to the identity functor. Thus, in particular, $(t_P)_* \circ t_P^*(\mathscr{A}) = \mathrm{Hom}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}) \cong \mathscr{A}$. By proposition 4.2 $\mathscr{A}$ is a finitely generated $\mathscr{B}$-module, and by assumption $\mathscr{B}$ is a PID, so the fundamental theorem of finitely generated modules over a PID tells us that $\mathscr{A} = \mathscr{B}$. ∎

**Corollary 4.2.** The cover $(J, \tau \otimes \chi)$ for the type $(J_L, \tau_L \otimes \chi)$ constructed in section 3.2 is split.

*Proof.* By proposition 3.2, $(J, \tau \otimes \chi)$ satisfies the conditions of theorem 4.2, so we must show that $i_P^G$ is an equivalence of categories. By corollary 4.1, this is equivalent to showing that $N_G(\mathfrak{s}_L) = L$. Because $P$ is maximal the Weyl group $N_G(L)/L$ has order two, so we need only check that a permutation matrix $w_0$ contained in the nontrivial coset does not normalize $\sigma \otimes \chi$. As usual, we view the character $\chi$ of $L \cong GL(k)$ as a multiplicative character, which we also write $\chi \in F^\times$, composed with the determinant. Let $\omega_\sigma$ denote the central character of $\sigma$. The central character of $\sigma \otimes \chi$ is then $\omega_\sigma \chi^k$, while the central character of the conjugate representation is

$(\omega_\sigma \otimes \chi)^{w_0} = \omega_\sigma^{-1}\chi^{-k}$. The highly ramified condition given by equation 3.1, combined with lemma 3.4, guarantees that these characters are not equal. $\square$

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