GENZ AND KEISTER TEST FUNCTIONS

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There are seven test functions below, each test function should accept a vector input $\boldsymbol{x} = [x_1 \cdots x_d]$ and return an output of the function values. The integration hyperbox is $[a_1 \cdots a_d; b_1 \cdots b_d]$, where d is the dimension of the input vector.

1. Genz "Oscillatory"

The integrand f is

$$f(\boldsymbol{x}) = \cos\left(2\pi r + \sum_{i=1}^{d} \alpha_i x_i\right) \tag{1}$$

Thus the true solution is:

if $d = 1$

$$I = \int f(\boldsymbol{x}) d\boldsymbol{x} = \int_{a_1}^{b_1} \cos\left(2\pi r + \alpha_1 x_1\right) dx_1 = \sin\left(2\pi r + \alpha_1 x_1\right)/\alpha_1 \Big|_{a_1}^{b_1} \tag{2}$$

if $d = 2$

$$I = \int f(\boldsymbol{x}) d\boldsymbol{x} = \int_{a_2}^{b_2} \int_{a_1}^{b_1} \cos\left(2\pi r + \alpha_1 x_1 + \alpha_2 x_2\right) dx_1 dx_2 \tag{3}$$

$$= \int_{a_2}^{b_2} (\sin\left(2\pi r + \alpha_1 b_1 + \alpha_2 x_2\right) - \sin\left(2\pi r + \alpha_1 a_1 + \alpha_2 x_2\right))/\alpha_1 dx_2 \tag{4}$$

$$= (-\cos\left(2\pi r + \alpha_1 b_1 + \alpha_2 x_2\right) + \cos\left(2\pi r + \alpha_1 a_1 + \alpha_2 x_2\right))/(\alpha_1 \alpha_2)\Big|_{a_2}^{b_2} \tag{5}$$

For generalized form of the integral, we use the iterative method to calculate the true solution. the algorithm is

```
% iterative method to calculate the integral of Genz test function "Oscillatory"
        s = zeros(2^dim,1);
        sign = zeros(2^dim,1);
        s(1) = 2*pi*r+hyperbox(2,:)*alpha(1:dim)';
        sign(1) = 1;
        for i = 1:dim
            s(2^(i-1)+1:2^i) = s(1:2^(i-1))-alpha(dim-i+1)*(hyperbox(2,dim-i+1)-hyperbox(1,dim-i+1));
            sign(2^(i-1)+1:2^i) = -sign(1:2^(i-1));
        end
```

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```
        switch mod(dim,4)
            case 1
                f_true = sum(sign.*sin(s))/prod(alpha(1:dim));
            case 2
                f_true = sum(sign.*(-cos(s)))/prod(alpha(1:dim));
            case 3
                f_true = sum(sign.*(-sin(s)))/prod(alpha(1:dim));
            case 0
                f_true = sum(sign.*cos(s))/prod(alpha(1:dim));
        end
```

2. Genz "Product Peak"

The integrand f is

$$f(\boldsymbol{x}) = \frac{1}{\prod_{i=1}^{d}(\alpha_i^2 + (x_i - \beta_i)^2)} \tag{6}$$

The integral is

$$I = \int f(\boldsymbol{x})d\boldsymbol{x} = \int_{a_d}^{b_d} \cdots \int_{a_1}^{b_1} \frac{1}{\prod_{i=1}^{d}(\alpha_i^2 + (x_i - \beta_i)^2)} dx_1 \cdots dx_d = \prod_{i=1}^{d} \frac{\arctan(\frac{x_i-\beta_i}{\alpha_i})}{\alpha_i}\Bigg|_{a_i}^{b_i} \tag{7}$$

3. Genz "Corner Peak"

The integrand f is

$$f(\boldsymbol{x}) = \frac{1}{(1 + \sum_{i=1}^{d}(\alpha_i x_i))^{d+r}} \tag{8}$$

if $d = 1$

$$I = \int f(\boldsymbol{x})d\boldsymbol{x} = \int_{a_1}^{b_1} \frac{1}{(1+\alpha_1 x_1)^{1+r}} dx_1 = -\frac{1}{\alpha_1 r(1+\alpha_1 x_1)^r}\Bigg|_{a_1}^{b_1} \tag{9}$$

if $d = 2$

$$I = \int f(\boldsymbol{x})d\boldsymbol{x} = \frac{1}{r(r+1)\alpha_1\alpha_2}\left((1+\alpha_1 b_1 + \alpha_2 x_2)^{-r} - (1 + \alpha_1 a_1 + \alpha_2 x_2)^{-r}\right)\Bigg|_{a_2}^{b_2} \tag{10}$$

For generalized form of the integral, we use the iterative method to calculate the true solution. the algorithm is

```
% iterative method to calculate the integral of Genz test function "Corner Peak"
        s = zeros(2^dim,1);
        sign = zeros(2^dim,1);
```

```
        s(1) = 1+hyperbox(2,:)*alpha(1:dim)';
        sign(1) = 1;
        for i = 1:dim
            s(2^(i-1)+1:2^i) = s(1:2^(i-1))-alpha(dim-i+1)*(hyperbox(2,dim-i+1)-hyperbox(1,dim-i+1));
            sign(2^(i-1)+1:2^i) = -sign(1:2^(i-1));
        end
        f_true = (-1)^dim*sum(sign.*s.^(-r))/prod(alpha(1:dim))/prod(r:r+dim-1);
```

4. Genz "Gaussian"

The integrand f is

$$f(\boldsymbol{x}) = \exp\left(-\sum_{i=1}^{d}\left(\alpha_i^2(x_i-\beta_i)^2\right)\right) \tag{11}$$

The integral is

$$I = \int f(\boldsymbol{x})d\boldsymbol{x} = \int_{a_d}^{b_d}\cdots\int_{a_1}^{b_1}\exp\left(-\sum_{i=1}^{d}(\alpha_i^2(x_i-\beta_i)^2)\right)dx_1\cdots dx_d \tag{12}$$

$$= \prod_{i=1}^{d}\int_{a_i}^{b_i}\exp(-(\alpha_i^2(x_i-\beta_i)^2))dx_i = \prod_{i=1}^{d}\frac{\sqrt{\pi}}{2\alpha_i}\operatorname{erf}(\alpha_i(x_i-\beta_i))\Big|_{a_i}^{b_i} \tag{13}$$

5. Genz "Continuous"

The integrand f is

$$f(\boldsymbol{x}) = \exp\left(-\sum_{i=1}^{d}(\alpha_i\,|x_i-\beta_i|)\right) \tag{14}$$

The integral is

$$I = \int f(\boldsymbol{x})d\boldsymbol{x} = \int_{a_d}^{b_d}\cdots\int_{a_1}^{b_1}\exp\left(-\sum_{i=1}^{d}(\alpha_i\,|x_i-\beta_i|)\right)dx_1\cdots dx_d \tag{15}$$

$$= \prod_{i=1}^{d}\int_{a_i}^{b_i}\exp(-(\alpha_i\,|x_i-\beta_i|))dx_i \tag{16}$$

$$= \prod_{i=1}^{d}\begin{cases}\int_{a_i}^{b_i}\exp(-\alpha_i(x_i-\beta_i))dx_i \text{ if } a_i\ge\beta_i\\ \int_{a_i}^{\beta_i}\exp(-\alpha_i(\beta_i-x_i))dx_i+\int_{\beta_i}^{b_i}\exp(-\alpha_i(x_i-\beta_i))dx_i \text{ if } b_i\ge\beta_i\ge a_i\\ \int_{a_i}^{b_i}\exp(-\alpha_i(\beta_i-x_i))dx_i \text{ if } b_i\le\beta_i\end{cases} \tag{17}$$

$$= \prod_{i=1}^{d}\begin{cases}-\frac{1}{\alpha_i}\exp(-\alpha_i(x_i-\beta_i))|_{a_i}^{b_i} \text{ if } a_i\ge\beta_i\\ \frac{1}{\alpha_i}\exp(-\alpha_i(\beta_i-x_i))|_{a_i}^{\beta_i}+(-\frac{1}{\alpha_i})\exp(-\alpha_i(x_i-\beta_i))|_{\beta_i}^{b_i} \text{ if } b_i\ge\beta_i\ge a_i\\ \frac{1}{\alpha_i}\exp(-\alpha_i(\beta_i-x_i))|_{a_i}^{b_i} \text{ if } b_i\le\beta_i\end{cases} \tag{18}$$

6. Genz "Discontinuous"

$$f(\boldsymbol{x}) = \begin{cases} \exp\left(-\sum_{i=1}^{d}(\alpha_i x_i)\right) & \text{if } \beta_i < x_i \text{ for } i = 1\cdots d \\ 0 \text{ otherwise} \end{cases} \tag{19}$$

The integral is

$$I = \int f(\boldsymbol{x})d\boldsymbol{x} = \int_{a_d}^{b_d}\cdots\int_{a_1}^{b_1}\exp\left(-\sum_{i=1}^{d}(\alpha_i x_i)\right)dx_1\cdots dx_d \tag{20}$$

$$= \prod_{i=1}^{d}\int_{a_i}^{b_i}\exp(-\alpha_i x_i)dx_i = \begin{cases} \prod_{i=1}^{d} -\frac{\exp(-\alpha_i x_i)}{\alpha_i}\Big|_{a_i}^{b_i} \text{ if } \beta_i < x_i \text{ for } i = 1\cdots d \\ 0 \text{ otherwise} \end{cases} \tag{21}$$

7. Keister Test function

Keister considered the following multidimensional integral that has applications in physics:

$$\int_{R^s}\cos(\|x\|)e^{-\|x\|^2}dx = \pi^{s/2}\int_{[0,1)^s}\cos\left(\sqrt{\sum_{j=1}^{s}\frac{[\Phi^{-1}(y_i)]^2}{2}}\right)dy \tag{22}$$

where Φ demotes the standard Gaussian distribution function and the norm is 2-norm. The integrand is:

$$f(\boldsymbol{x}) = \pi^{d/2}\cos\left(\sqrt{\frac{1}{2}\sum_{i=1}^{d}(\Phi^{-1}(x_i))^2}\right) \tag{23}$$

where x_i are in unit interval.

The true solution is calculated by Mathematica for dimension 1-8

$$I_1 = 1.3803884470431429$$

$$I_2 = 1.808186634594926$$

$$I_3 = \frac{\pi^{3/2}}{2\exp(1/4)}$$

$$I_4 = 2.165929302574506$$

$$I_5 = \frac{\pi^{5/2}}{12\exp(1/4)}$$

$$I_6 = -2.3273037292979391292$$

$$I_7 = \frac{-31\pi^{7/2}}{120\exp(1/4)}$$

$$I_8 = -30.609075003558562675$$