

SIMULATING THE HESTON MODEL VIA  
THE QE METHOD WITH A SPECIFIED ERROR TOLERANCE

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## LIST OF SYMBOLS

Symbol	Definition
$X_t$	Asset price process
$V_t$	Stochastic volatility process
$W_1, W_2$	Two standard Brownian motions
$\rho$	Correlation constant between $W_1$ and $W_2$
$\mu$	Drift
$\kappa$	Speed of mean reversion
$\theta$	Value of the long-term variance
$\nu$	Volatility of variance
$T$	Time to maturity
$\Delta$	Time step size
$S_0$	Initial price
$K$	Strike price
$V_0$	Initial variance
$r$	Risk-free interest rate
$c$	Mean shift



## ABSTRACT

The Quadratic Exponential (QE) model is a market standard simulation method for the Heston stochastic volatility model. We identify certain numerical problems with the standard discretization and modify the original method to correct these problems. We implement our modified QE scheme for the Heston model in the Guaranteed Automatic Integration Library (GAIL)—a suite of algorithms that includes Monte Carlo and quasi-Monte Carlo methods for multidimensional integration and computation of means. GAIL computes answers to satisfy user-defined error tolerances. We also implement variance reduction techniques for our modified QE scheme in GAIL. The numerical results show that our modified scheme is fast and accurate, and satisfies the user-defined error tolerances.

## CHAPTER 1

### INTRODUCTION

#### 1.1 Background

In the financial market, observed volatility of the asset price is not a constant. For more accurate option pricing, we need an algorithm that simulates the volatility process. There are several well-known stochastic volatility models: the Hull-White model [9], the Scott-Chesney model [3], the Heston model [7] and the SABR model [6]. We focus on the Heston model since it is one of the most widely used stochastic volatility models. Specifically, the Quadratic Exponential (QE) scheme is used to simulate the volatility process and the Broadie-Kaya scheme gives the discretization of the asset price process.

The QE scheme was developed by Andersen [1]. It is a market standard simulation method for the Heston model. Its attractiveness lies in its efficiency. It relies on simple probability density functions and needs a moderate amount of storage. Depending on the value of the volatility of the variance, the QE scheme approximates its distribution using either a Gaussian or an exponential distribution. After obtaining the value of the variance at each time step, the Broadie-Kaya scheme is used to simulate the dynamics of the asset price process. The Broadie-Kaya scheme does not satisfy an equivalent discrete-time martingale condition. The martingale property can be attained by adjusting the Broadie-Kaya scheme.

#### 1.2 Our contributions

For the numerical scheme part, we notice that the standard implementation gives inaccurate results when the volatility of the variance of the asset price process is very small or zero. This inaccuracy is accentuated when the initial variance does not start from the value of the long-term variance. We identify this problem and fix

it by finding a more accurate approximation of the time integral of the volatility. We also make a change of variables in the QE scheme to avoid round-off errors in the discretization scheme of the asset price process.

We implement the modified scheme in the Guaranteed Automatic Integration Library (GAIL) , which theoretically guarantees the result and stops algorithms automatically according to a user-defined error tolerance [8]. The GAIL includes a suite of algorithms that applies the Monte Carlo methods for multidimensional integration and computation of means. The financial application modules of GAIL are under construction, and our work is aimed to add efficient and accurate algorithms of calculating stochastic volatility model in the asset path class. We also apply importance sampling to reduce the variance of the approximation. We compare the results with different options at different strike prices. Our numerical experiments show that the new algorithm is accurate and fast.

### **1.3 Outline of the thesis**

The organization of the thesis is as follows. In Chapter 2 we introduce the Heston model, the QE scheme and the Broadie-Kaya scheme. In Chapter 3, we derive our modified QE algorithm and provide its justification. In Chapter 4, we explain variance reduction techniques and present numerical tests. Finally, we summarize our work and discuss possible future work.

## CHAPTER 2

### OPTIONS PRICED BY THE HESTON STOCHASTIC VOLATILITY MODEL

#### 2.1 The Heston model

Steven L. Heston introduced a stochastic volatility model which describes the evolution of the volatility of an asset price process [7]. Compared to fixed volatility models, stochastic volatility models more accurately explain data from the real world. The Heston model assumes the following dynamics of stock price and volatility processes:

$$dX_t = \mu X_t dt + \sqrt{V_t} X_t dW_t^1, \quad (2.1)$$

$$dV_t = \kappa(\theta - V_t) dt + \nu \sqrt{V_t} dW_t^2. \quad (2.2)$$

The first equation describes the asset price process  $X_t$  and the second equation gives the evolution of the variance process  $V_t$ . Two standard Brownian motions,  $W^1$  and  $W^2$ , are set with a correlation constant  $\rho$ . The drift, the mean reversion speed of the variance, the long-term variance, and the volatility of the variance are denoted by  $\mu$ ,  $\kappa$ ,  $\theta$  and  $\nu$  respectively.

Applying the Ito's formula to (2.1), we have an equivalent form of the stock price process:

$$d \ln(X_t) = \left(\mu - \frac{V_t}{2}\right) dt + \sqrt{V_t} dW_t^1. \quad (2.3)$$

The asset price process  $X$  is governed by a geometric Brownian motion with stochastic volatility.

#### 2.2 The Quadratic exponential scheme for stochastic volatility

We apply the Quadratic Exponential scheme illustrated in Andersen [1] to simulate the volatility process. Let  $\hat{V}$  denote the discrete-time approximation of  $V$ . Detailed steps are listed below.

1. Given  $\hat{V}_t$ , compute  $m$  and  $s^2$  from the following equations:

$$m = \theta + (\hat{V}_t - \theta)e^{-\kappa\Delta},$$

$$s^2 = \frac{\hat{V}_t \nu^2 e^{-\kappa\Delta}}{\kappa} \left(1 - e^{-\kappa\Delta}\right) + \frac{\theta \nu^2}{2\kappa} \left(1 - e^{-\kappa\Delta}\right)^2.$$

2. Compute  $\psi = s^2/m^2$ .
3. Draw a uniform random number  $U_V$ .
4. If  $\psi \leq 1.5^1$ :

- (a) Compute  $a$  and  $b > 0$  from the following equations:

$$b^2 = 2\psi^{-1} - 1 + \sqrt{2\psi^{-1} - 1} \sqrt{2\psi^{-1} - 1} \geq 0, \quad a = \frac{m}{1 + b^2}.$$

- (b) Compute  $Z_V = \Phi^{-1}(U_V)$ , where  $\Phi$  is the standard normal distribution function.

- (c) Set  $\hat{V}_{t+\Delta} = a(b + Z_V)^2$ .

5. Otherwise, if  $\psi > 1.5$ :

- (a) Compute  $\beta$  and  $p$  according to the equations

$$p = \frac{\psi - 1}{\psi + 1} \in [0, 1), \quad \beta = \frac{1 - p}{m} = \frac{2}{m(\psi + 1)} > 0.$$

- (b) Set  $\hat{V}_{t+\Delta} = \Psi^{-1}(U_V; p, \beta)$ , where  $\Psi$  is the exponential distribution with Dirac measure.

$$\Psi(x) = p + (1 - p)(1 - e^{-\beta x}), \quad x \geq 0,$$

$$\Psi^{-1}(u) = \Psi^{-1}(u; p, \beta) = \begin{cases} 0, & 0 \leq u \leq p, \\ \beta^{-1} \ln\left(\frac{1-p}{1-u}\right), & p < u \leq 1. \end{cases}$$

---

<sup>1</sup>The constant 1.5 is a typical value allowed in the QE scheme. The exact choice for this constant doesn't affect the quality of the overall simulation significantly [1].

### 2.3 The Broadie-Kaya discretization scheme for the asset price process

We will use the Broadie-Kaya discretization scheme to simulate the asset price process. To begin with, we give a brief derivation of such scheme. Details are presented in Andersen [1].

First we integrate the SDE of  $V_t$  for a bias-free scheme:

$$V_{t+\Delta} = V_t + \int_t^{t+\Delta} \kappa(\theta - V_u) du + \nu \int_t^{t+\Delta} \sqrt{V_u} dW_u^2,$$

which can be written as

$$\int_t^{t+\Delta} \sqrt{V_u} dW_u^2 = \nu^{-1} \left( V_{t+\Delta} - V_t - \int_t^{t+\Delta} \kappa(\theta - V_u) du \right). \quad (2.4)$$

Recall (2.3), by the Cholesky decomposition, we have

$$d \ln X_t = (\mu - \frac{1}{2} V_t) dt + \sqrt{V_t} (\rho dW_t^2 + \sqrt{1 - \rho^2} dW_t),$$

where  $W$  is a Brownian motion independent of  $W^2$ . An integral form of the above equation is

$$\begin{aligned} \ln X_{t+\Delta} = \ln X_t + \mu \Delta - \frac{1}{2} \int_t^{t+\Delta} V_u du + \rho \int_t^{t+\Delta} \sqrt{V_u} dW_u^2 \\ + \sqrt{1 - \rho^2} \int_t^{t+\Delta} \sqrt{V_u} dW_u. \end{aligned}$$

Substituting (2.4) into the above equation, we obtain

$$\begin{aligned} \ln X_{t+\Delta} = \ln X_t + \mu \Delta + \frac{\rho}{\nu} (V_{t+\Delta} - V_t - \kappa \theta \Delta) + \left( \frac{\kappa \rho}{\nu} - \frac{1}{2} \right) \int_t^{t+\Delta} V_u du \\ + \sqrt{1 - \rho^2} \int_t^{t+\Delta} \sqrt{V_u} dW_u. \end{aligned} \quad (2.5)$$

Next, we need to find appropriate approximations for the integrals in (2.5). Inspired by the Trapezoidal rule, we write

$$\int_t^{t+\Delta} V_u du \approx \Delta [\gamma V_t + (1 - \gamma) V_{t+\Delta}], \quad (2.6)$$

where the parameter  $\gamma$  remains to be specified. In the Broadie-Kaya discretization scheme, the coefficient  $\gamma$  is usually taken to be  $1/2$ , but we propose an alternative in Section 3.1. The Itô integral  $\int_t^{t+\Delta} \sqrt{V_u} dW_u$  is Gaussian with mean zero and variance  $\int_t^{t+\Delta} V_u du$ . Hence, we have

$$\int_t^{t+\Delta} \sqrt{V_u} dW_u = \sqrt{\int_t^{t+\Delta} V_u du} \cdot Z \approx \sqrt{\Delta[\gamma V_t + (1-\gamma)V_{t+\Delta}]} \cdot Z, \quad (2.7)$$

where  $Z$  is a standard Gaussian random variable which is independent of  $V$ . Approximation (2.7), combined with (2.5) and (2.6) and Section 2.2, provides us the following discretization scheme

$$\ln \hat{X}_{t+\Delta} = \ln \hat{X}_t + K_0 + K_1 \hat{V}_t + K_2 \hat{V}_{t+\Delta} + \sqrt{K_3 \hat{V}_t + K_4 \hat{V}_{t+\Delta}} \cdot Z. \quad (2.8)$$

Parameters  $K_0, K_1, K_2, K_3$  and  $K_4$  are given by

$$\begin{aligned} K_0 &= -\frac{\rho\kappa\theta}{\nu}\Delta, & K_1 &= \gamma\Delta\left(\frac{\kappa\rho}{\nu} - \frac{1}{2}\right) - \frac{\rho}{\nu}, & K_2 &= (1-\gamma)\Delta\left(\frac{\kappa\rho}{\nu} - \frac{1}{2}\right) + \frac{\rho}{\nu}, \\ K_3 &= \gamma\Delta(1-\rho^2) & \text{and} & & K_4 &= (1-\gamma)\Delta(1-\rho^2). \end{aligned}$$

Moreover,  $\hat{X}$  denotes the discrete-time approximation of  $X$ . Given the simulation scheme for  $V$ , the discretization scheme for  $\ln X$  can be generated in the following fashion:

1. Given  $\hat{V}_t$ , generate  $\hat{V}_{t+\Delta}$  using the QE scheme.
2. Given  $\ln \hat{X}_t$ ,  $\hat{V}_t$  and the value for  $\hat{V}_{t+\Delta}$  computed in Step 2, compute  $\ln \hat{X}_{t+\Delta}$  from (2.8) using  $Z = \Phi^{-1}(U)$ , where  $U$  is independent of  $U_V$  used for  $\hat{V}_{t+\Delta}$ .

## 2.4 Martingale correction

In mathematical finance, the existence of a risk-neutral measure (or equivalent martingale measure) corresponds to an arbitrage-free market. Under the equivalent martingale measure, each price process equals to the expectation of its discounted

price process. From the numerical point of view, the influence whether the asset price process  $X$  is a martingale or not is minor. Given its significant importance in mathematical finance, we present the martingale correction due to Andersen [1] as follows:

$$\ln \hat{X}_{t+\Delta} = \ln \hat{X}_t + K_0^* + K_1 \hat{V}_t + K_2 \hat{V}_{t+\Delta} + \sqrt{K_3 \hat{V}_t + K_4 \hat{V}_{t+\Delta}} \cdot Z,$$

where

$$K_0^* = \begin{cases} -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1 + \frac{1}{2}K_3)\hat{V}_t, & \psi \leq 1.5, \\ -\ln\left(\frac{\beta(1-p)}{\beta-A}\right) - (K_1 + \frac{1}{2}K_3)\hat{V}_t, & \psi > 1.5 \end{cases}$$

and  $A = K_2 + \frac{1}{2}K_4$ .



### CHAPTER 3

#### OVERCOMING NUMERICAL ERRORS FOR SMALL VOLATILITY OF VARIANCE

Before we introduce our modified QE scheme<sup>2</sup>, we first show the problem of the original scheme with a plot. When  $\nu$  is equal or close to zero, the discretization scheme introduced previously gives inaccurate answers, especially when the initial variance,  $V_0$ , does not equal to the long-term variance  $\theta$ .

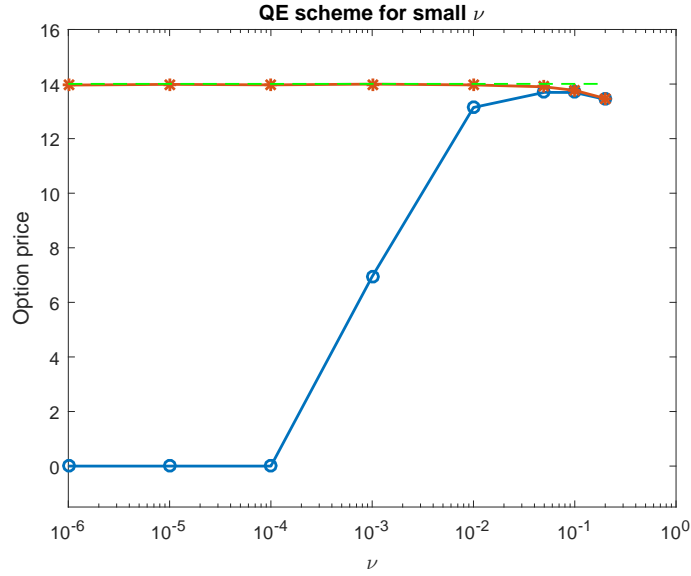


Figure 3.1. European call option price calculate by the QE scheme and our modified QE scheme for an initial price of \$80, a strike price of \$100, zero interest rate,  $T = 2$ ,  $\kappa = 1$ ,  $\rho = -0.3$ ,  $\theta = 0.09$ ,  $V_0 = 0.36$  and relative tolerance of 1%. Stars denote the prices calculated by our modified QE scheme when the relative tolerance of 1% is met. Circles denote the prices calculated by the original QE scheme. The dashed line denotes the option price calculated by the Black-Scholes formula.

We calculate the price of European call options with different values of  $\nu$ . Neither the QE scheme nor the exact sampling works for small  $\nu$ . When  $\nu$  equals zero and the initial variance  $V_0$  does not equal to the long-term variance  $\theta$ ,  $V_t$  is

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<sup>2</sup>From this chapter, when we mention the QE scheme (modified QE scheme), we refer to the QE scheme (modified QE scheme) and the discretization scheme (modified discretization scheme) for  $X$ .

deterministic and changes over time. We set the variance in the Black-Scholes formula to be the same as the long-term variance  $\theta$  and the correct option price should be close to this price when  $\nu$  is small. However, the original QE scheme gives zero option price when  $\nu$  equals zero and prices close to zero when  $\nu$  is close to zero. It is clear from Figure 3.1 that the original QE scheme deviates from the prices calculated by the Black-Scholes model and our modified QE scheme when  $\nu$  is smaller than approximately 0.05.

### 3.1 Finding a $\gamma$ that is accurate for the approximation of the time integral of $V$

From (2.5) and (2.7) note that the Broadie-Kaya scheme in integral form is written as

$$\begin{aligned} \ln X_{t+\Delta} = & \ln X_t + \frac{\rho}{\nu}(V_{t+\Delta} - V_t - \kappa\theta\Delta) + \left(\frac{\kappa\rho}{\nu} - \frac{1}{2}\right) \int_t^{t+\Delta} V_u \, du \\ & + \sqrt{(1 - \rho^2) \int_t^{t+\Delta} V_u \, du} \cdot Z. \end{aligned} \quad (3.1)$$

We need to approximate the time-integral of  $V$ . Rather than simply setting  $\gamma = 1/2$ , we want to find the value of  $\gamma$  that makes (2.6) exact for  $\nu = 0$  and show that the error for the approximation of the integral of  $V$  is of order  $o(\nu\Delta)$ . The factor of  $\nu$  in the error will then cancel the factor of  $1/\nu$  in (3.1). So, the simulation will be accurate when  $\nu$  is close to 0.

When  $\nu = 0$ , the stochastic differential equation (2.2) becomes deterministic and has the solution

$$V_t = \theta + (V_0 - \theta)e^{-\kappa t}. \quad (3.2)$$

Then, we have

$$\begin{aligned}
\int_t^{t+\Delta} V_u \, du &= \int_t^{t+\Delta} (V_u - \theta) \, du + \theta \Delta \\
&= \int_t^{t+\Delta} (V_0 - \theta) e^{-\kappa u} \, du + \theta \Delta \\
&= \frac{-(V_0 - \theta) e^{-\kappa(t+\Delta)} + (V_0 - \theta) e^{-\kappa t}}{\kappa} + \theta \Delta.
\end{aligned} \tag{3.3}$$

We want to find an expression of  $\theta$  written in terms of  $V_t$  and  $V_{t+\Delta}$ . Setting time to be  $t + \Delta$  in (3.2) and multiply  $e^{\kappa\Delta}$  on both sides, we obtain

$$e^{\kappa\Delta}(V_{t+\Delta} - \theta) = (V_0 - \theta)e^{-\kappa t}.$$

This provides us an expression for  $\theta$  as

$$\theta = \frac{V_t - e^{\kappa\Delta} V_{t+\Delta}}{1 - e^{\kappa\Delta}}.$$

By substituting the above expression of  $\theta$  into (3.3), we obtain

$$\begin{aligned}
\int_t^{t+\Delta} V_u \, du &= \frac{V_t - V_{t+\Delta}}{\kappa} + \Delta \frac{V_t - e^{\kappa\Delta} V_{t+\Delta}}{1 - e^{\kappa\Delta}} \\
&= \Delta[\gamma V_t + (1 - \gamma) V_{t+\Delta}],
\end{aligned}$$

where

$$\gamma = \frac{1 - e^{\kappa\Delta} + \kappa\Delta}{\kappa\Delta(1 - e^{\kappa\Delta})}, \quad 0 \leq \gamma \leq 1. \tag{3.4}$$

As  $\kappa\Delta \rightarrow 0$ ,  $\gamma \rightarrow \frac{1}{2}$ . When  $\kappa\Delta$  goes to zero, our scheme becomes the Trapezoidal rule. We want to apply the value of  $\gamma$  in (3.4) for all values of  $\nu$  to approximate the integral  $\int_t^{t+\Delta} V_u \, du$ . In Appendix A, it is shown that

$$\int_t^{t+\Delta} V_u \, du = \Delta[\gamma V_t + (1 - \gamma) V_{t+\Delta}] + O(\nu \Delta^{1.5}).$$

### 3.2 Change of variables to avoid overflow

Another problem with the formula (3.1) is that  $\nu$  appears in the denominator. This may lead to numerical error when  $\nu$  is small. To avoid this error we make a

change of variables of  $s^2, \psi, b^{-2}, a, b, K_0, K_1, K_2, K_3$ , and  $K_4$ . For parameters in the QE scheme, we make a change of variables:

$$\begin{aligned}\tilde{s}^2 &= \nu^{-2}s^2 = \frac{\hat{V}_t e^{-\kappa\Delta}}{\kappa}(1 - e^{-\kappa\Delta}) + \frac{\theta}{2\kappa}(1 - e^{-\kappa\Delta})^2, \\ \tilde{\psi} &= \nu^{-2}\psi = \frac{\frac{\hat{V}_t e^{-\kappa\Delta}}{\kappa}(1 - e^{-\kappa\Delta}) + \frac{\theta}{2\kappa}(1 - e^{-\kappa\Delta})^2}{(\theta + (\hat{V}_t - \theta)e^{-\kappa\Delta})^2}, \\ \tilde{b}^{-2} &= \nu^{-2}b^{-2} = \frac{\tilde{\psi}}{2\sqrt{1 - \frac{\tilde{\psi}\nu^2}{2}}\left(1 + \sqrt{1 - \frac{\tilde{\psi}\nu^2}{2}}\right)}, \\ \tilde{a} &= \nu^{-2}a = \frac{m\tilde{b}^{-2}}{1 + \nu^2\tilde{b}^{-2}}.\end{aligned}$$

As we see, new variables  $\tilde{s}^2$  and  $\tilde{\psi}$  are free of  $\nu$ . Moreover, as  $\nu \rightarrow 0$ ,  $\tilde{a}$  and  $\tilde{b}$  have finite limits. After making the change of variables, when  $\nu$  goes to 0, our new variables have finite limits. Now we use new variables that we defined to calculate  $\hat{X}$ . Recall (2.8):

$$\ln \hat{X}_{t+\Delta} = \ln \hat{X}_t + K_0 + K_1 \hat{V}_t + K_2 \hat{V}_{t+\Delta} + \sqrt{K_3 \hat{V}_t + K_4 \hat{V}_{t+\Delta}} \cdot Z.$$

Observe the expression of parameters we discussed in Section 2.3, the terms including  $\nu^{-1}$  will cause overflow when  $\nu$  is close or equal to zero. We want to define a new variable of stochastic volatility process  $V$  in a way that there is no  $\nu^{-1}$  in our modified discretization scheme.

Define  $\tilde{V}_t = \hat{V}_t - \theta$  and write  $\hat{V}_{t+\Delta}$  in terms of new variables:

$$\hat{V}_{t+\Delta} = \frac{\theta + \tilde{V}_t e^{-\kappa\Delta}}{1 + \nu^2 \tilde{b}^{-2}}(1 + \nu \tilde{b}^{-1} Z_V)^2.$$

Then, we have the expression of  $\tilde{V}_{t+\Delta}$  as

$$\tilde{V}_{t+\Delta} = \frac{\theta\nu[2\tilde{b}^{-1}Z_V + \nu\tilde{b}^{-2}(Z_V^2 - 1)] + \tilde{V}_t e^{-\kappa\Delta}(1 + \nu\tilde{b}^{-1}Z_V)^2}{1 + \nu^2\tilde{b}^{-2}}.$$

Define  $\mathring{V}_{t+\Delta} = (\tilde{V}_{t+\Delta} - \tilde{V}_t e^{-\kappa\Delta})/\nu$ ,

$$\begin{aligned}\mathring{V}_{t+\Delta} &= \frac{1}{\nu}[\tilde{V}_{t+\Delta} - (\theta + \tilde{V}_t e^{-\kappa\Delta})] \\ &= (\theta + \tilde{V}_t e^{-\kappa\Delta}) \left[ \frac{2\tilde{b}^{-1}Z_V}{1 + \nu^2\tilde{b}^{-2}} + \nu \frac{\tilde{b}^{-2}(Z_V^2 - 1)}{1 + \nu^2\tilde{b}^{-2}} \right].\end{aligned}$$

In order to calculate  $\ln \hat{X}$ , we note that

$$\gamma \tilde{V}_t + (1 - \gamma) \tilde{V}_{t+\Delta} = \frac{1 - e^{-\kappa \Delta}}{\kappa \Delta} \tilde{V}_t + (1 - \gamma) \nu \dot{V}_{t+\Delta}.$$

Substituting the above equation into the following equation:

$$\begin{aligned} K_0 + K_1 \hat{V}_t + K_2 \hat{V}_{t+\Delta} &= (K_0 + \theta K_1 + \theta K_2) + K_1 \tilde{V}_t + K_2 \tilde{V}_{t+\Delta} \\ &= -\theta \frac{\Delta}{2} - \frac{\Delta}{2} [\gamma \tilde{V}_t + (1 - \gamma) \tilde{V}_{t+\Delta}] + \frac{\rho \kappa \Delta e^{\kappa \Delta}}{e^{\kappa \Delta} - 1} \dot{V}_{t+\Delta} \\ &= -\theta \frac{\Delta}{2} - \frac{1 - e^{-\kappa \Delta}}{2\kappa} \tilde{V}_t - \left[ \frac{\nu(1 - e^{\kappa \Delta} + \kappa \Delta e^{\kappa \Delta})}{2\kappa(1 - e^{\kappa \Delta})} + \frac{\rho \kappa \Delta e^{\kappa \Delta}}{1 - e^{\kappa \Delta}} \right] \dot{V}_{t+\Delta}. \end{aligned} \quad (3.5)$$

Moreover,

$$\begin{aligned} K_3 \hat{V}_t + K_4 \hat{V}_{t+\Delta} &= \Delta(1 - \rho^2) [\theta + \gamma \tilde{V}_t + (1 - \gamma) \tilde{V}_{t+\Delta}] \\ &= \Delta(1 - \rho^2) \left\{ \theta + \left[ \frac{1 - e^{-\kappa \Delta}}{\kappa \Delta} \tilde{V}_t + (1 - \gamma) \nu \dot{V}_{t+\Delta} \right] \right\}. \end{aligned} \quad (3.6)$$

Now we can rewrite the discretization scheme. For the discretization scheme of  $X$  when  $\nu^2 \tilde{\psi} \leq 1.5$ :

$$\begin{aligned} \ln \hat{X}_{t+\Delta} &= \ln \hat{X}_t - \theta \frac{\Delta}{2} - \frac{1 - e^{-\kappa \Delta}}{2\kappa} \tilde{V}_t - \left[ \frac{\nu(1 - e^{\kappa \Delta} + \kappa \Delta e^{\kappa \Delta})}{2\kappa(1 - e^{\kappa \Delta})} + \frac{\rho \kappa \Delta e^{\kappa \Delta}}{1 - e^{\kappa \Delta}} \right] \dot{V}_{t+\Delta} \\ &\quad + \sqrt{\Delta(1 - \rho^2) \left\{ \theta + \left[ \frac{1 - e^{-\kappa \Delta}}{\kappa \Delta} \tilde{V}_t + (1 - \gamma) \nu \dot{V}_{t+\Delta} \right] \right\}} \cdot Z. \end{aligned} \quad (3.7)$$

Since the discretization scheme is inaccurate only when  $\nu$  is small, we will only apply our modifications to  $\nu^2 \tilde{\psi} \leq 1.5$ . For the  $\nu^2 \tilde{\psi} > 1.5$  case, the scheme remains unchanged:

$$\ln \hat{X}_{t+\Delta} = \ln \hat{X}_t + K_0 + \frac{1}{\nu} K_1 \hat{V}_t + \frac{1}{\nu} K_2 \hat{V}_{t+\Delta} + \sqrt{K_3 \hat{V}_t + K_4 \hat{V}_{t+\Delta}} \cdot Z. \quad (3.8)$$

### 3.3 The modified QE and Broadie-Kaya discretization schemes

Now we can propose our modified scheme for the Heston model:

1. Given  $\tilde{V}_t$ , compute  $m$  and  $\tilde{s}^2$  from following equations

$$m = \Theta + \tilde{V}_t e^{-\kappa\Delta},$$

$$\tilde{s}^2 = \frac{(\tilde{V}_t + \theta)e^{-\kappa\Delta}}{\kappa} \left(1 - e^{-\kappa\Delta}\right) + \frac{\theta}{2\kappa} \left(1 - e^{-\kappa\Delta}\right)^2.$$

2. Compute  $\tilde{\psi} = \tilde{s}^2/m^2$ .
3. Generate two independent normal random variables  $Z_V$  and  $Z$  from GAIL.
4. If  $\nu^2\tilde{\psi} \leq 1.5$ :

- (a) Compute  $\tilde{a}$  and  $\tilde{b}$  from following equations

$$\tilde{b}^{-2} = \frac{\tilde{\psi}}{2\sqrt{1 - \frac{1}{2}\tilde{\psi}\nu^2} \left(1 + \sqrt{1 - \frac{1}{2}\tilde{\psi}\nu^2}\right)},$$

$$\tilde{a} = \frac{m\tilde{b}^{-2}}{1 + \nu^2\tilde{b}^{-2}}.$$

- (b) Set  $\tilde{V}_{t+\Delta} = -\theta + \tilde{a}(\tilde{b} + \nu Z_V)^2$ .
- (c) Compute  $\mathring{V}_{t+\Delta} = (\theta + \tilde{V}_t e^{-\kappa\Delta}) \left[ \frac{2\tilde{b}^{-1}Z_V}{1+\nu^2\tilde{b}^{-2}} + \frac{\nu\tilde{b}^{-2}(Z_V^2-1)}{1+\nu^2\tilde{b}^{-2}} \right]$ .
- (d) Given  $\tilde{V}_t$ ,  $\ln \hat{X}_t$  and the value for  $\mathring{V}_{t+\Delta}$ , compute  $\ln \hat{X}_{t+\Delta}$  from (3.7).

5. Otherwise, if  $\nu^2\tilde{\psi} > 1.5$ :

- (a) Compute  $\beta$  and  $p$  according to equations

$$p = \frac{\nu^2\tilde{\psi} - 1}{\nu^2\tilde{\psi} + 1} \in [0, 1),$$

$$\beta = \frac{1-p}{m} = \frac{2}{m(\nu^2\tilde{\psi} + 1)} > 0.$$

- (b) Draw a uniform random number  $U_V$ .
- (c) Set  $\tilde{V}_{t+\Delta} = -\theta + \Psi^{-1}(U_V; p, \beta)$ , where  $\Psi$  is the exponential distribution with Dirac measure.
- (d) Given  $\tilde{V}_t$ ,  $\ln \hat{X}_t$  and the value for  $\tilde{V}_{t+\Delta}$ , compute  $\ln \hat{X}_{t+\Delta}$  from (3.8).

### 3.4 Canceling $\nu^{-1}$ out of the Broadie-Kaya discretization scheme with martingale correction

The difference of the scheme without and with martingale correction lies in the  $K_0$  term. Recall

$$K_0^* = -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1 + \frac{1}{2}K_3)\hat{V}_t, \quad \nu^2\tilde{\psi} \leq 1.5,$$

where  $A = K_2 + \frac{1}{2}K_4$ . In order to cancel out  $\nu^{-1}$  in the scheme with martingale correction, we calculate terms  $K_0^* + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta}$  all together. First, we rewrite the first part of  $K_0^*$  in terms of new parameters defined in Section 3.2 and substitute  $m = \theta + \tilde{V}_{t+\Delta} - \nu\mathring{V}_{t+\Delta}$  into it:

$$\begin{aligned} -\frac{Ab^2a}{1-2Aa} &= -\frac{1}{\nu} \frac{A\tilde{a}\tilde{b}^2}{1-2\nu A\tilde{a}} \\ &= -\frac{1}{\nu} \frac{m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}}{1 + \nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\ &= -\frac{1}{\nu} \frac{\rho[(1-\gamma)\Delta\kappa+1](\theta + \tilde{V}_{t+\Delta})}{1 + \nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\ &\quad + \frac{\frac{1}{2}(1-\gamma)\Delta\rho^2(\theta + \tilde{V}_{t+\Delta})}{1 + \nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\ &\quad + \frac{\mathring{V}_{t+\Delta}\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}}{1 + \nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}}. \end{aligned}$$

Then,

$$\begin{aligned} K_0^* + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta} &= -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1 + \frac{1}{2}K_3)\hat{V}_t + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta} \\ &= \frac{\frac{1}{2}(1-\gamma)\Delta\rho^2(\theta + \tilde{V}_{t+\Delta})}{1 + \nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\ &\quad + \frac{\mathring{V}_{t+\Delta}\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}}{1 + \nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\ &\quad + \frac{1}{2}\ln(1-2\nu\tilde{A}\tilde{a}) - \frac{1}{2}\gamma\Delta(1-\rho^2)(\theta + \tilde{V}_t) - \frac{1}{2}(1-\gamma)\Delta(\theta + \tilde{V}_{t+\Delta}) \\ &\quad + \rho[(1-\gamma)\Delta\kappa+1](\theta + \tilde{V}_{t+\Delta}) \\ &\quad + \frac{\nu\tilde{b}^{-2} - 2\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}}{1 + \nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1] - \frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}}. \end{aligned}$$

For  $K_3\hat{V}_t + K_4\hat{V}_{t+\Delta}$ , it is the same as the Broadie-Kaya discretization scheme without martingale correction.



## CHAPTER 4

### NUMERICAL EXAMPLES

We implement our modified QE scheme into GAIL and test our modified QE scheme by pricing European and American options. First we show test results for small  $\nu$  compared with the original QE scheme and Broadie and Kaya's exact sampling. Then we compare the modified QE scheme without martingale correction to the modified QE scheme with martingale correction. Our numerical results support our claim in Section 2.4 that the difference of option prices between the schemes with and without martingale correction is minor. However, the scheme with martingale correction is more computationally expensive. Then we introduce importance sampling to reduce variance and illustrate reduced runtimes and sample sizes in test results. Finally, we show how Sobol sampling with our modified scheme improves the computational efficiency in comparison to i.i.d. sampling.

#### 4.1 Tests for small $\nu$

We test our scheme by pricing an European call option. The parameters are setup as in Table 4.1. We set the relative error tolerance (relTol) as 0.5% in GAIL. GAIL will return prices that satisfy our error tolerance [8]. The parameter  $K$  denotes strike price. In Table 4.2, the exact price means the price calculated by exact

Table 4.1. Parameters for numerical tests

$T$	$\Delta$	$S_0$	$K$	$V_0$	$\theta$	$\kappa$	$\rho$	$r$	relTol
5	0.2	100	90	0.04	0.09	1	-0.3	0	0.5%

sampling<sup>3</sup>. This scheme is bias-free yet computationally expensive, so we use it as a

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<sup>3</sup>The exact sampling was introduced by M. Broadie and Ö. Kaya [2] for the Heston stochastic volatility model. They applied the numerical inversion of a cumulative distribution using the characteristic function. The estimator of an asset price

benchmark. The exact sampling returns error messages for small values of  $\nu$ , so we use “-” to indicate that an error occurred. The MATLAB codes for the exact sampling are provided in Kienitz [10]. The reason why the difference between the exact price and the QE price is larger than 1% is that the sampling size we used for computing the exact price is limited by the computers’ memory. The result will be more accurate if we are able to increase its sample size. From the test result we can see that our

Table 4.2. Test of European call options with  $\nu$  close or equal to zero

$\nu$	0	1e-6	1e-5	1e-4	1e-3	1e-2	0.05	0.1	0.5
Exact price	-	-	-	-	-	-	-	-	27.782
QE	0.000	35.797e+22	14.193e+3	81.095	32.380	29.191	28.915	28.811	27.493
Modified QE	28.859	28.874	28.870	28.926	28.901	28.923	28.849	28.783	27.542
Diff	28.859	35.797e+22	14.164e+3	52.169	3.479	0.268	0.065	0.028	0.049

modified QE scheme works for all test values of  $\nu$ . For given parameters, we consider  $\nu = 0.01$  as the break point for the original QE scheme. When  $\nu$  is smaller than 0.01, the relative difference between the original QE scheme and the modified QE scheme is larger than two times of the 0.5% relative error tolerance. When  $\nu = 0.01$ , the difference is barely within the twice of the relative error tolerance. The original QE scheme gives satisfactory results for  $\nu$  larger than 0.01.

#### 4.2 Compare the modified QE scheme to the modified QE scheme with martingale correction

Now we price European call options and compare test results of our modified schemes with and without martingale correction. The parameters are setup as in Table 4.3. We test for different strike prices and results shown in Table 4.4. For each case, we repeat calculation for five times and take the average of price, time and sample size. The scheme with martingale correction takes a third more runtime

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generate using the sample stock price and variance from the exact distribution is unbiased.

than that without martingale correction while the difference of average prices are

Table 4.3. Parameters for numerical tests

$T$	$\Delta$	$S_0$	$V_0$	$\theta$	$\nu$	$\kappa$	$\rho$	$r$	relTol
5	0.2	60	0.5	0.16	0.4	1	-0.3	0	1%

Table 4.4. Compare modified QE with modified QE\_m

Option	Price (relTol=0.01)			Time (s)			Sample Size		
	K=20	K=60	K=100	K=20	K=60	K=100	K=20	K=60	K=100
Modified QE	42.782	23.505	14.208	8.33	17.70	29.42	1.08e+6	2.28e+6	3.75e+6
Modified QE_m	42.727	23.469	14.213	13.08	23.83	39.73	1.21e+6	2.22e+6	3.65e+6
Diff	0.055	0.036	0.005	-	-	-	-	-	-

within 0.2%. Because the martingale correction is more time consuming, we apply the modified QE scheme without martingale correction for the rest of numerical tests.

### 4.3 Importance sampling

For out of the money options, we need much larger sample sizes to meet the user-defined error tolerance. In order to increase accuracy and reduce sample size, we apply importance sampling—a variance reduction technique.

As defined in (2.1), the asset price process  $X_t$  is driven by a Brownian motion. Or more precisely, our modified simulation scheme (3.7) and (3.8) is driven by a Gaussian random variable. So we can shift the mean of the Gaussian random variable to sample the asset price process more at where it gives positive payoffs [5].

For  $\mathbf{t} = (t_1, \dots, t_d)'$ ,  $t_i = \frac{T}{d}i$ ,  $i = 1, \dots, d$ , we have  $\mathbf{W} = (W_1, \dots, W_d) \sim \mathcal{N}(\mathbf{0}, \Sigma)$ , where  $\Sigma_{i,j} = \min(t_i, t_j)_{i,j=1}^d$ . For any  $c \in \mathbb{R}$ ,

$$\mathbf{Y} = (Y_1, \dots, Y_d) \sim \mathcal{N}(c\mathbf{t}, \Sigma),$$

and so the likelihood ratio is

$$\frac{f_{\mathbf{W}}(\mathbf{y})}{f_{\mathbf{Y}}(\mathbf{y})} = \frac{e^{-\frac{1}{2}(\mathbf{y}'\Sigma^{-1}\mathbf{y})}}{e^{-\frac{1}{2}((\mathbf{y}-c\mathbf{t}')\Sigma^{-1}(\mathbf{y}-c\mathbf{t}))}} = e^{-c\mathbf{y}'\Sigma^{-1}\mathbf{t} + \frac{1}{2}c^2\mathbf{t}'\Sigma^{-1}\mathbf{t}},$$

where  $f$  is the probability density function. The importance sampling estimator of the option price is then written as

$$\frac{1}{n} \sum_{k=1}^n \text{payoff}(\mathbf{Y}_k) e^{-c\mathbf{Y}_k'\Sigma^{-1}\mathbf{t} + \frac{1}{2}c^2\mathbf{t}'\Sigma^{-1}\mathbf{t}}.$$

Let  $\mathbf{q}$  denotes a vector that satisfies  $\Sigma\mathbf{q} = \mathbf{t}$ . Then  $\mathbf{q} = (0, \dots, 0, 1)'$  and the above equation can be simplifies as

$$\frac{1}{n} \sum_{k=1}^n \text{payoff}(\mathbf{Y}_k) e^{-c\mathbf{Y}_k'\mathbf{q} + \frac{1}{2}c^2\mathbf{t}'\mathbf{q}} = \frac{1}{n} \sum_{k=1}^n \text{payoff}(\mathbf{Y}_k) e^{-cY_{k,d} + \frac{1}{2}c^2T}.$$

We implement importance sampling in GAIL. Given parameters in Table 4.3, we set the value of mean shifts for tests of importance sampling as in Table 4.5. The numerical results in the following subsections will show the difference in prices, runtimes and sample sizes.

Table 4.5. Mean shifts for importance sampling

	European Call (EC)	European Put (EP)	American Put (AP)
IID Sampling	2	-0.8	-0.8
Sobol Sequences	1	-1	-0.6

**4.3.1 Simulations with IID sampling.** First we test our modified QE scheme with i.i.d. sampling. Since American call option is never optimal to exercise early, it is just like European call option. So, we omit the test for it. The test results for European call/put option and American put option are shown in Table 4.6. The ECQE (EPQE) means European call (put) option calculated by the original QE scheme. The EC/EP means European call/ put option. The AP is short for American

Table 4.6. Test of option price with IID\_MC

Option	Price (relTol=0.01)			Time (s)			Sample Size		
	K=20	K=60	K=100	K=20	K=60	K=100	K=20	K=60	K=100
ECQE <sup>a</sup>	42.728	23.464	14.218	3.19	3.19	3.22	1.00e+6	1.00e+6	1.00e+6
ECw/oIS	42.750	23.497	14.196	8.22	18.40	30.34	1.06e+6	2.35e+6	3.85e+6
ECw/IS	-	23.493	14.210	-	4.85	4.67	-	6.27e+5	6.04e+5
EPQE <sup>a</sup>	2.782	23.498	54.255	3.18	3.18	3.19	1.00e+6	1.00e+6	1.00e+6
EPw/oIS	2.772	23.460	54.141	9.40	3.07	1.78	1.21e+6	4.03e+5	2.41e+5
EPw/IS	2.773	23.493	-	5.14	2.21	-	6.52e+5	2.94e+5	-
APw/oIS	2.771	23.476	54.176	9.95	3.90	2.44	1.13e+6	3.82e+5	2.25e+5
APw/IS	2.775	23.460	-	6.78	3.86	-	7.07e+5	3.80e+5	-

<sup>a</sup> This is calculated by the original QE scheme without guaranteed error tolerance.

put option. Since American call option is just like European call option, The IS is short for importance sampling.

After applying importance sampling to at the money and out of the money options, the average runtimes are reduced, especially for European call option. The average runtime is nearly a constant rather than significant correlated to strike prices. The average prices calculated with different schemes are very close for the same option with the same strike price. Our modified QE scheme gives approximation within user-defined error tolerance with constant runtime and as fast as the original QE scheme.

**4.3.2 Simulations with Sobol sequences.** Sobol sequences is a commonly used low-discrepancy sequences and is one of the sampling methods implemented in GAIL. For Sobol sequences, the time step size relax to  $\Delta = 0.5$ . The average runtimes are about 1/100 of those for i.i.d. sampling for the same relative error tolerance 1%. So, in Table 4.7, we change the relative error tolerance to 0.1%. However, when we try various mean shifts for American put option, there is no obvious reduction in runtime. This is might because that the sample size of the Sobol sequences is a factor of 2. We don't have a good explanation for this now.

Table 4.7. Test of option price with Sobol' points

Option	Price (relTol=0.01)			Time (s)			Sample Size		
	K=20	K=60	K=100	K=20	K=60	K=100	K=20	K=60	K=100
ECw/oIS	42.798	23.502	14.230	0.40	0.87	2.30	1.31e+05	2.62e+05	6.29e+05
ECw/IS	-	23.501	14.230	-	0.20	0.40	-	6.55e+04	1.31e+05
EPw/oIS	2.771	23.473	54.187	0.39	0.06	0.04	1.31e+05	1.64e+04	9.83e+03
EPw/IS	2.777	-	-	0.20	-	-	6.55e+04	-	-
APw/oIS	2.779	23.444	54.306	1.01	0.31	0.17	1.31e+05	3.28e+04	1.64e+04
APw/IS	2.774	-	-	1.05	-	-	1.31e+05	-	-

We apply importance sampling to at the money and out of the money call option, and out of the money put option. The average runtimes for at the money European call options are reduced four times and for out of the money European call options are reduced five times. The average runtimes of out of the money European put option are reduced to a half of that without importance sampling. Though Sobol sequences is a very efficient way of sampling without importance sampling, the reduction in runtime will be more obvious when calculating a large number of options. Therefore, the test results show that our modified QE scheme gives fast and accurate approximation using Sobol sequences in GAIL, which also satisfies user-defined error tolerance.

## CHAPTER 5

### DISCUSSION

The widely used QE scheme is a fast and accurate scheme for simulating the Heston model when  $\nu$  is not small but gives disastrous results for small  $\nu$ . We find a more accurate approximation for the time integral of variance process. Also we make the change of variables to cancel out  $\nu^{-1}$  in the discretization scheme of the asset price process. Our modified QE scheme can be applied for any possible value of  $\nu$ , which provides nice simulations for varied situations of the financial market.

Our modified scheme is implemented in GAIL, a software library that automatically evaluates multidimensional integrals. This allows us to take advantage of multiple properties of GAIL. First, the estimated price is calculated to satisfy a user-defined tolerance. Second, GAIL has a variety of sampling methods including i.i.d., Sobol, lattice and more. In this thesis, we try the Sobol sequences and obtain fast convergence. Last but not least, GAIL has more than ten types of options implemented. So, GAIL with our modified scheme provides a powerful option pricing tool for the Heston stochastic volatility model.

There are several problems that we want to work on in the future. The importance sampling technique improves the simulation significantly. We choose proper mean shift values by hand. For the future work, we hope to choose those automatically given different parameter settings. Moreover, for multiple underlying assets, we hope to choose mean shift for each underlying asset automatically. There are more variance reduction techniques such as control variates and antithetic variates. We want to implement and test those other variance reduction techniques. At last, we hope to apply our future work to different class of options and other financial derivatives.

APPENDIX A  
PROOF FOR THE ERROR ORDER OF THE APPROXIMATION OF THE  
INTEGRAL OF  $V$



We want to show the error of our approximation for  $\int_t^{t+\Delta} V_u du$  is  $o(\nu\Delta)$ . Recall the process

$$dV_t = \kappa(\theta - V_t) dt + \nu\sqrt{V_t} dW_t, \quad V_0 \text{ given,}$$

defined in (2.2). It is not easy to prove directly from process  $V$ . So, we define a new process  $U$  and let  $U_t$  satisfy a similar stochastic differential equation:

$$dU_t = -\kappa U_t dt + \nu\sqrt{V_t} dW_t, \quad U_0 = 0. \quad (\text{A.1})$$

Let  $Y_t = V_t - U_t$ , and note that  $Y_t$  satisfies a differential equation:

$$dY_t = \kappa(\theta - Y_t) dt, \quad Y_0 = V_0.$$

Moreover,  $Y_t = V_t$  when  $\nu = 0$ . So, we apply similar deduction to  $Y_t$  as in Section 3.1:

$$\int_t^{t+\Delta} Y_u du = \Delta[\gamma Y_t + (1 - \gamma)Y_{t+\Delta}],$$

where  $\gamma = (1 - e^{\kappa\Delta} + \kappa\Delta)/\kappa\Delta(1 - e^{\kappa\Delta})$ .

Now we have the integral of  $V_t$  written in terms of  $V$  and  $U$  as follows:

$$\begin{aligned} \int_t^{t+\Delta} V_u du &= \int_t^{t+\Delta} (Y_u + U_u) du \\ &= \Delta[\gamma Y_t + (1 - \gamma)Y_{t+\Delta}] + \int_t^{t+\Delta} U_u du \\ &= \Delta[\gamma V_t + (1 - \gamma)V_{t+\Delta}] - \left\{ \Delta[\gamma U_t + (1 - \gamma)U_{t+\Delta}] - \int_t^{t+\Delta} U_u du \right\}. \end{aligned} \quad (\text{A.2})$$

Our problem is to show that the error in approximating  $\int_t^{t+\Delta} U_u du$  by  $\Delta[\gamma_1 U_t + (1 - \gamma_1)U_{t+\Delta}]$  is  $o(\nu\Delta)$ .

We can write the solution of  $U_t$  as

$$U_t = \nu e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{V_s} dW_s.$$

Now we rewrite the integral of  $U_t$  by using integration by parts twice and (A.1):

$$\begin{aligned}
\int_t^{t+\Delta} U_u \, du &= U_u(u-t-\Delta\gamma)|_t^{t+\Delta} - \int_t^{t+\Delta} (u-t-\Delta\gamma) \, dU_u \\
&= \Delta[\gamma U_t + (1-\gamma)U_{t+\Delta}] + \int_t^{t+\Delta} (u-t-\Delta\gamma) \kappa U_u \, du \\
&\quad - \nu \int_t^{t+\Delta} (u-t-\Delta\gamma) \sqrt{V_u} \, dW_u \\
&= \Delta[\gamma U_t + (1-\gamma)U_{t+\Delta}] + \frac{1}{2} \kappa U_u [(u-t-\Delta\gamma)^2 - \Delta^2 \gamma^2] |_t^{t+\Delta} \\
&\quad - \int_t^{t+\Delta} \frac{1}{2} [(u-t-\Delta\gamma)^2 - \Delta^2 \gamma^2] \kappa \, dU_u \\
&\quad - \nu \int_t^{t+\Delta} (u-t-\Delta\gamma) \sqrt{V_u} \, dW_u \\
&= \Delta[\gamma U_t + (1-\gamma)U_{t+\Delta}] + \kappa U_{t+\Delta} \Delta^2 \left( \frac{1}{2} - \gamma \right) \\
&\quad + \int_t^{t+\Delta} (u-t) \left[ \frac{(u-t)}{2} - \Delta\gamma \right] \kappa^2 U_u \, du \\
&\quad - \nu \int_t^{t+\Delta} h(u) \sqrt{V_u} \, dW_u, \tag{A.3}
\end{aligned}$$

where  $h(u) = \kappa(u-t)[(u-t)/2 - \Delta\gamma] + u-t-\Delta\gamma$ . Since  $U_t = O(\nu)$ , it is easy to see that the second term in (A.3) is  $O(\nu\Delta^2)$ . By Hölder's inequality, the third term of (A.3) is

$$\begin{aligned}
&\int_t^{t+\Delta} \left( \frac{(u-t)^2}{2} - (u-t)\Delta\gamma \right) \kappa^2 U_u \, du \\
&\leq \Delta\gamma \kappa^2 \sqrt{\int_t^{t+\Delta} \left( \frac{(u-t)^2}{2} - (u-t)\Delta\gamma \right)^2 \, du} \int_t^{t+\Delta} U_u^2 \, du \\
&= O(\nu\Delta^3).
\end{aligned}$$

Now we want to show that

$$I = \nu \int_t^{t+\Delta} h(u) \sqrt{V_u} \, dW_u = O(\nu\Delta^{1.5}).$$

Since  $V_t$  is a Cox-Ingersoll-Ross process, we know that its expectation given  $V_0$  is

$\mathbb{E}(V_t) = V_0 e^{-\kappa u} + \theta(1 - e^{-\kappa u})$  [4], which is  $O(1)$ . Also, since  $\max_{t \leq u \leq t+\Delta} |h(u)| = O(\Delta)$ ,

$$\begin{aligned} \mathbb{E}\left(\nu^2 \int_t^{t+\Delta} [h(u)]^2 V_u \, du\right) &= \nu^2 \int_t^{t+\Delta} [h(u)]^2 \mathbb{E}(V_u) \, du \\ &= \nu^2 \int_t^{t+\Delta} O(\Delta^2) \, du \\ &= O(\nu^2 \Delta^3) \\ &< \infty, \end{aligned}$$

we have  $I \in \mathcal{L}^2$  and  $I$  is an Itô integral. By Theorem 4.7 of Klebaner [11], Itô integral  $I$  is a continuous zero mean square integrable martingale. Thus, by the Itô isometry,

$$\begin{aligned} \text{Var}(I) &= \mathbb{E}(I^2) - [\mathbb{E}(I)]^2 \\ &= \mathbb{E}\left(\nu^2 \int_t^{t+\Delta} [h(u)]^2 V_u \, du\right) \\ &= O(\nu^2 \Delta^3). \end{aligned}$$

So,  $I$  is a normal distributed random variable with mean zero and variance of order  $O(\nu^2 \Delta^3)$ . The Itô integral  $I$  is  $O(\nu \Delta^{1.5})$ . This implies that

$$\int_t^{t+\Delta} U_u \, du = \Delta(\gamma U_t + (1 - \gamma)U_{t+\Delta}) + O(\nu \Delta^{1.5})$$

in distribution, and so by (A.2)

$$\int_t^{t+\Delta} V_u \, du = \Delta(\gamma V_t + (1 - \gamma)V_{t+\Delta}) + O(\nu \Delta^{1.5}),$$

also in distribution.

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