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Color code: highlight, key concepts and definitions, to be added.

# Chapter 1

# From Reading Group on Durrett

Reference: PTE, Folland and Tao.

## 1.1 Measure and Integration

### 1.1.1 Measure and Probability Spaces

First of all, why should we discuss measure theory. It's for a better integration theory than the usual Riemann integral. Because we need strong conditions for arguments like:

$$\lim_n \int f_n = \int \lim_n f_n$$

to hold for Riemann integral. TBA: Relationship with Riemann Integral We'll talk about

1. Measure space, construction of measure, Lebesgue measures;
2. Integration theory
3. Convergence concepts
4. Differentiation

Let's start with an abstract measure space. Let $\Omega$ be a space. A $\sigma$-algebra $\mathcal{F}$ is a collection of subsets called *measurable sets* of $\Omega$ such that it's closed under complements and countable unions. Also the $\varnothing \in \mathcal{F}$. We can assign *measure* $\mu$ to members in $\mathcal{F}$. The reason we need to restrict attention from the power set to $\mathcal{F}$ is that there are examples that violate some properties we want a measure to have. $\mu$ satisfies:

$$\mu(\varnothing) = 0 \quad \text{and} \quad \mu\left(\bigsqcup_i^\infty A_i\right) = \sum_i \mu(A_i) \quad \text{for} \quad (A_i) \subset \mathcal{F}$$

The triplet $(\Omega, \mathcal{F}, \mu)$ is called a *measure space*.

A *measurable function* is $X : (\Omega, \mathcal{F}, \mu) \longrightarrow (Y, \mathcal{H})$ such that for all $H \in \mathcal{H}$, $X^{-1}(H) \in \mathcal{F}$. Meaning that we can assign a measure to the subsets of $\mathcal{H}$ based on the measure $\mu$ and $X$. Given two measurable spaces, we can check whether a function $X$ is measurable.

1.1.1 **Lemma**. *Suppose $\mathcal{H} = \sigma(\mathcal{G})$, if for all $B \in \mathcal{G}$, $X^{-1}(B) \in \mathcal{F}$, then $X$ is measurable.*

*Proof.* Let $\mathcal{H}' = \left\{B \subset S \mid X^{-1}(B) \in \mathcal{F}\right\}$, then we can show $\mathcal{H}'$ is a $\sigma$-algebra because $X^{-1}$ preserves complements and union. Then we have $\mathcal{G} \subset \sigma(\mathcal{G}) = \mathcal{H} \subset \mathcal{H}'$, meaning that all $H \in \mathcal{H}$ satisfies the condition. □

1.1.2 **Open Sets on $\mathbb{R}$.** *Every open set $U \subset \mathbb{R}$ is an at most countable union of disjoint open intervals.*

### Distribution and Density

*Distribution* is the probability induced by a random variable $X$, such that $\mu(A) = P(X \in A)$. *Density* is defined via some methods(To be added). It's used to change measure, for example,

$$\int g(X)\, \mathrm{d}P = \int g(x) f(x)\, \mathrm{d}x$$

the latter being the Lebesgue integral over $\mathbb{R}$ to be defined later.

### Product Measure Space

Let $(Y_\alpha, \mathcal{H}_\alpha) : \alpha \in \mathcal{A}$ be a collection of measurable spaces, then we can define a *product $\sigma$-algebra* $\otimes_\alpha \mathcal{H}_\alpha = \sigma\left\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{H}_\alpha, \alpha \in \mathcal{A}\right\}$ on the product space $\prod_\alpha \Omega_\alpha$,

1.1.3 **Proposition**. *Let $\Omega, \mathcal{F}$ and $(Y_\alpha, \mathcal{H}_\alpha)$ be measurable spaces, let $(Y, \mathcal{H}) = (\prod Y_\alpha, \otimes_\alpha \mathcal{H}_\alpha)$. $f : \Omega \longrightarrow Y$ is $(\mathcal{F}, \mathcal{H})$-measurable if and only if $f_\alpha = \pi_\alpha f$ is $\mathcal{F}, \mathcal{H}_\alpha$-measurable for all $\alpha$.*

**1.1.4 Proposition (TBA).** *Product $\sigma$-algebra is generally smaller than the $\sigma$-algebra on the product space itself. On separable metric space, they coincide.*

#### Probability Space and Random Variables

A measure space is a *probability space* if $\mu = 1$, we write it as $(\Omega, \mathcal{F}, P)$. A random variable/vector is a measurable function on probability space $X : \Omega \to \mathbb{R}/\mathbb{R}^d/\mathbb{C}$ with the Borel $\sigma$-algebra, the $\sigma$-algebra generated by the open sets.

**1.1.5 Measurability.** *For countable collections of random variables $X_n$, $Y_n$, $X+Y$, $XY$ and $\sup X_n$, $\inf X_n$, $\limsup X_n$, $\liminf X_n$ are measurable. If $\lim_n X_n$ exists, it's also measurable.*

*Proof.* For $\sup_n X_n$, let $t \in \mathbb{R}$, $\sup_n X_n \in (-\infty, t) \iff X_n < t$, for all $t$, hence $(\sup_n X_n)^{-1} = \bigcap_n X_n^{-1}(-\infty, t)$. □

Now consider finite vector of random variables $X^d := (X_1, \ldots, X_d) : \Omega \to \mathbb{R}^d$. Then $X^d$ is measurable iff all $X_i$ are measurable.

### 1.1.2 Construction of Measures

#### Motivation

Now we want to show that we can actually find measure on $\mathbb{R}$ and $\mathbb{R}^d$ that are the most useful spaces. We start from abstract construction method inspired by the following observation of measure on $\mathbb{R}$.

In order to have a natural measure $\lambda$ on $\mathbb{R}$, we want the following properties:

1. $\lambda_0(a, b] = b - a$;
2. Measure of union of intervals should be the sum;(we can't have uncountable sum, hence restricted to countable union)
3. Apply to all intervals(we can't apply to the power set, hence restrict to the measurable sets).

Now consider the set $\mathcal{S}_{\mathbb{R}} = \{(a, b] : a \leq b \in \mathbb{R}\}$, it's too small, we want to extend the natural measure funtion $\lambda_0$ to a larger set(we can extend to $\mathcal{B}_{\mathbb{R}}$, even to $\mathcal{L}_{\mathbb{R}}$).

*Semi-algebra* $\mathcal{S}$ is collection of subsets of a space $\Omega$ that satisfies the following conditions: closed under intersection and the complements are union of finite disjoint sets in $\mathcal{S}$. Let $\mu_0$ be a what I call *semi-pre-measure* to (*algebra*, *premeasure*) to (outer-measurable sets, outer measure) to (sigma-algebra, measure) where the *semi-pre-measure* $\mu_0$ satisfies if both $S_N = \bigsqcup^N S_j, S_\infty = \bigsqcup^\infty S_j \in \mathcal{S}$

$$\mu_0(\emptyset) = 0; \quad \mu_0(S_N) = \sum^N \mu_0 S_j; \quad \mu_0(S_\infty) \leq \sum^\infty \mu_0 S_j$$

We follow the steps of extension

$$(\mathcal{S}, \mu_0) \longrightarrow (\mathcal{A}, \mu_1) \longrightarrow (\mathcal{M}, \mu^*) \longrightarrow (\sigma(\mathcal{S}), \mu)$$

that is semi-algebra and *pre-measure* $\mu_1$ satisfies $\mu_1 \bigsqcup_i^\infty A_i = \sum_i^\infty \mu_1 A_i$ as long as the union is also in $\mathcal{A}$. And we will show that $\mathcal{B}_\mathbb{R} \subset \sigma(\mathcal{S})$.

**Construction of Abstract Measure**

**1.1.6** $(\mathcal{S}, \mu_0) \longrightarrow (\mathcal{A}, \mu_1)$. *We have that* $\mathcal{A} = \left\{\bigsqcup_j^N S_j : S_j \in \mathcal{S}\right\}$ *is an algebra. And* $\mu_1(A) = \sum_j^N \mu_0 S_j$ *is a premeasure.*

*Proof.* $\mu_1(\emptyset) = 0$. Suppose $A = \bigsqcup^\infty A_i \in \mathcal{A}$, then there exists $T_l \in \mathcal{S} : l = 1, \ldots N_T$, such that $A = \bigsqcup_l T_l$, also for each $A_j = \bigsqcup_k^{N_j} S_{j,k}$. □

Let an *outer measure* be a function over the power set such that:

$$\mu^*(\emptyset) = 0 \quad \text{and} \quad \mu^*\left(\bigsqcup^\infty A_j\right) \leq \sum_j \mu^* A_j \quad \text{and} \quad \mu^*(A) \leq \mu^*(B) \quad \text{for} \quad A \subset B$$

Also let $\mathcal{M}$ be the *outer-measurable sets*, that is the collections of $A$ such that

$$\mu^*(E) = \mu^*(E \bigcap A) + \mu^*(E \bigcap A^c) \quad \text{for all} \quad E \subset X$$

**1.1.7** $(\mathcal{A}, \mu_1) \longrightarrow (\mathcal{M}, \mu^*)$ **Caratheodory's Theorem applied to premeasures.** *We can find the outer measure induced by* $\mu_1$*:*

$$\mu_1^*(B) = \inf\left\{\sum_j^\infty \mu_1 A_j : B \subset \bigcup_j A_j\right\}$$

*we show $\mu_1^*$ is indeed an outer measure, Caratheodory states that for outer measures, $\mathcal{M}$ of $\mu_1^*$ is a $\sigma$-algebra and $\mu^*|_{\mathcal{M}}$ is complete.*

*Proof.*

1. Monotone property is easy, for countable subadditivity, we use $\varepsilon$-room method, find a cover $C_{jk}$ of $A_j$ such that :
$$\mu_1^*(A_j) > \sum_k \mu_1^* C_{jk} - \varepsilon$$
2. To show that $\mathcal{M}$ is a $\sigma$-algebra we only need to show that for $E_j \in \mathcal{M}$ and any $B \subset \Omega$, we have:
$$\mu^*(B) \geq \mu^*(B \bigcap E) + \mu^*(B \bigcap E)$$
which can be seen by considering finite sum, taking out one $E_j$ at a time and then take limit.

3. Show that the restriction is complete.

□

**1.1.8** $(\mathcal{M}, \mu^*) \longrightarrow (\sigma(\mathcal{S}), \mu)$. *We can restrict $\mu^*$ to define $\mu = \mu^*|_{\sigma(\mathcal{S})}$, and the resulting measure is complete.*

*Proof.* We show that $\sigma(\mathcal{S}) \subset \mathcal{M}$ and the restriction is complete. □

### 1.1.3 Application: Borel and Lebesgue-Stietjes Measure on $\mathbb{R}$

We verify that for any nondecreasing right-continuous function $F : \mathbb{R} \longrightarrow \mathbb{R}$, the semi-algebra $\mathcal{S}_{\mathbb{R}}$ together with the semi-pre-measure defined by

$$\lambda_0(a, b] = F(b) - F(a)$$

satisfies the conditions set in the previous section. $\mathcal{A}_{\mathbb{R}}$ be the algebra generated by $\mathcal{S}_{\mathbb{R}}$ in turn will generate $\mathcal{B}_{\mathbb{R}}$. We can extend it to a something larger, a complete measure whose domain contains $\mathcal{B}_{\mathbb{R}}$. It will be called the *Lebesgue-Stietjes measure* and *Lebesgue measurable sets* $\mathcal{M}_\lambda$. The restriction on $\mathcal{B}$ is called *Borel measure*. It's just the completion.

**1.1.9 Regularity of $\mathcal{M}_\lambda$.** *Lebesgue measurable sets are of simple form if you allow for a small error.*

1. $\lambda(A) = \inf\{\lambda U : A \subset U, U \text{ is open}\} = \sup\{\lambda K : K \subset A, K \text{ is compact}\}$
2. $A = G_\delta \setminus N_1 = F_\sigma \bigcup N_2$ *where* $N$ *are null sets.*
3. *Littlewood's First Principle: Suppose* $\lambda(A) < \infty$ *then for all* $\varepsilon > 0$*, there exists* $B$ *that is a finite union of open intervals such that* $A \triangle B < \varepsilon$.

*Proof.* □

### 1.1.4 Countable Product Space and Lebesgue Measure on $\mathbb{R}^d$

*Product measurable* and *product measure*.

### 1.1.5 Integration

Integration and measure are the two sides of same coin: integration is the extension of measure from the space of *indicator functions* $\mathbf{1}_A$ of measurable sets to a larger set of functions.

For *simple functions* $f_s = \sum_j^N \mathbf{1}_{A_j}$, we can define $\mu f_s = \sum_j^N \mu(A_j)$

Then for *nonnegative measurable* functions $f$, we can define

$$\mu f = \sup\{\mu f_s : f_s \leq f \quad \text{pointwise}\}$$

If $f = f^+ - f^-$ and the integrals of two nonnegative parts are not $\infty$, then we say $f$ is *integrable* or in $L^1$ and $\mu f = \mu f^+ - \mu f^-$.

**1.1.10 Littlewood's Second Principle.** *Measurable and integrable functions can be approximated by simple well-behaved functions.*

**1.1.11 Properties of Integral.**

1. *Monotone* $\mu f \leq \mu g$ *if* $f \leq g$ *pointwise.*
2. *Linearity for* $a > 0$, $\mu(af) = a\mu f$, $\mu(f+g) = \mu f + \mu g$.
3. *Monoton Convergence: if* $0 \leq f_n \uparrow f$ *a.e., then* $\mu f_n \uparrow \mu f$.

*Proof.* Folland proves by choosing simple function and a scaling factor $\alpha$. Tao's proof is essentially the same. PTE proves via Fatou's lemma.

Suppose $f_n \uparrow f$ a.e., if $\mu f$ exists, we know $\mu f_n$ must converge because it's a nondecreasing bounded sequence of numbers. Also $\mu f_n \leq \mu f$ by monotonicity.

To show that $\mu f \leq \lim_n \mu f_n$, where $\mu f = \sup\{\mu g : g \text{ is simple}, g \leq f\}$, we only need to show that for all simple $g \leq f$, we have $\mu g \leq \mu f_n$. We can use $\varepsilon$-room method, let $E_n = \{\omega : f_n \geq (1-\varepsilon)g\}$, $E_n \uparrow \Omega$, and we have $\int f_n \geq \int_{E_n}(1-\varepsilon)g$, taking lim, we have $\lim \int f_n \geq (1-\varepsilon)\int g$ for all $\varepsilon > 0$. □

*Remark.* These three properties fully characterise the integral. That is given a functional that satisfies the three properties we can find a measure for which the function is the integral.

#### 1.1.12 Inequalities and Controls. *Useful inequalities and controls*

1. *Jensen's inequality: suppose we have a convex function $\Phi(x)$, then $\mu(\Phi(f)) \geq \Phi\mu f$*
2. *Holder's inequality. For $p, q : \frac{1}{p} + \frac{1}{q} = 1$, we have $\|fg\|_1 \leq \|f\|_p \|g\|_q$*
3. *Minkowski's inequality: $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.*
4. *Chebyshev's Inequality:*
5. *For nonnegative measurable functions, $\mu f < \infty \implies f < \infty$ a.e., $\mu f = 0$ implies $f = 0$ a.e.*
6. *Moment and tail behaviour, if $\mu$ is finite,*

$$\mu f^r < \infty \implies \sum x^{r-1}\mu(f > x) < \infty$$

#### 1.1.13 Fubini-Tonelli.

### 1.1.6 Convergence Concepts

#### 1.1.14 Monotone, Fatou and Dominated Convergence.

#### 1.1.15 Uniform Integrability.

### 1.1.7 Differentiation

## 1.2 Independence

### 1.2.1 Independence

### 1.2.2 Law of Large Numbers

### 1.2.3 Central Limit Theorem