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Color code: highlight, key concepts and definitions, to be added.

## Schedule

1. Measure space, measurable functions
2. Construction of Lebesgue measure on  $\mathbb{R}$ , Lebesgue integral
3. Convergence concepts, product measure
4. Independence
5. LLN

6. CLT

7. Martingale

# Chapter 1

## From Reading Group on Durrett

Reference: PTE, Folland and Tao.

### 1.1 Measure and Integration

#### 1.1.1 Measure and Probability Spaces

First of all, why should we discuss measure theory. It's for a better integration theory than the usual Riemann integral. Because we need strong conditions for arguments like:

$$\lim_n \int f_n = \int \lim_n f_n$$

to hold for Riemann integral. [TBA: Relationship with Riemann Integral](#) We'll talk about

1. Measure space, construction of measure, Lebesgue measures;
2. Integration theory
3. Convergence concepts
4. Differentiation

Let's start with an abstract measure space. Let  $\Omega$  be a space. A  $\sigma$ -algebra  $\mathcal{F}$  is a collection of subsets called *measurable sets* of  $\Omega$  such that it's closed under complements and countable unions. Also the  $\emptyset \in \mathcal{F}$ . We can assign *measure*  $\mu$  to members in  $\mathcal{F}$ . The

reason we need to restrict attention from the power set to  $\mathcal{F}$  is that there are examples that violate some properties we want a measure to have.  $\mu$  satisfies:

$$\mu(\emptyset) = 0 \quad \text{and} \quad \mu\left(\bigsqcup_i^\infty A_i\right) = \sum_i \mu(A_i) \quad \text{for} \quad (A_i) \subset \mathcal{F}$$

The triplet  $(\Omega, \mathcal{F}, \mu)$  is called a *measure space*.

A *measurable function* is  $X : (\Omega, \mathcal{F}, \mu) \rightarrow (Y, \mathcal{H})$  such that for all  $H \in \mathcal{H}$ ,  $X^{-1}(H) \in \mathcal{F}$ . Meaning that we can assign a measure to the subsets of  $\mathcal{H}$  based on the measure  $\mu$  and  $X$ . Given two measurable spaces, we can check whether a function  $X$  is measurable.

**1.1.1 Lemma.** Suppose  $\mathcal{H} = \sigma(\mathcal{G})$ , if for all  $B \in \mathcal{G}$ ,  $X^{-1}(B) \in \mathcal{F}$ , then  $X$  is measurable.

*Proof.* Let  $\mathcal{H}' = \{B \subset S \mid X^{-1}(B) \in \mathcal{F}\}$ , then we can show  $\mathcal{H}'$  is a  $\sigma$ -algebra because  $X^{-1}$  preserves complements and union. Then we have  $\mathcal{G} \subset \sigma(\mathcal{G}) = \mathcal{H} \subset \mathcal{H}'$ , meaning that all  $H \in \mathcal{H}$  satisfies the condition.  $\square$

**1.1.2 Open Sets on  $\mathbb{R}$ .** Every open set  $U \subset \mathbb{R}$  is an at most countable union of disjoint open intervals.

### Distribution and Density

*Distribution* is the probability induced by a random variable  $X$ , such that  $\mu(A) = P(X \in A)$ . *Density* is defined via some methods(To be added). It's used to change measure, for example,

$$\int g(X) dP = \int g(x)f(x) dx$$

the latter being the Lebesgue integral over  $\mathbb{R}$  to be defined later.

### Product Measure Space

Let  $(Y_\alpha, \mathcal{H}_\alpha) : \alpha \in \mathcal{A}$  be a collection of measurable spaces, then we can define a *product  $\sigma$ -algebra*  $\otimes_\alpha \mathcal{H}_\alpha = \sigma\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{H}_\alpha, \alpha \in \mathcal{A}\}$  on the product space  $\prod_\alpha \Omega_\alpha$ ,

**1.1.3 Proposition.** Let  $\Omega, \mathcal{F}$  and  $(Y_\alpha, \mathcal{H}_\alpha)$  be measurable spaces, let  $(Y, \mathcal{H}) = (\prod Y_\alpha, \otimes_\alpha \mathcal{H}_\alpha)$ .  $f : \Omega \rightarrow Y$  is  $(\mathcal{F}, \mathcal{H})$ -measurable if and only if  $f_\alpha = \pi_\alpha f$  is  $\mathcal{F}, \mathcal{H}_\alpha$ -measurable for all  $\alpha$ .

**1.1.4 Proposition (TBA).** *Product  $\sigma$ -algebra is generally smaller than the  $\sigma$ -algebra on the product space itself. On separable metric space, they coincide.*

### Probability Space and Random Variables

A measure space is a **probability space** if  $\mu = 1$ , we write it as  $(\Omega, \mathcal{F}, P)$ . A random variable/vector is a measurable function on probability space  $X : \Omega \rightarrow \mathbb{R}/\mathbb{R}^d/\mathbb{C}$  with the Borel  $\sigma$ -algebra, the  $\sigma$ -algebra generated by the open sets.

**1.1.5 Measurability.** *For countable collections of random variables  $X_n, Y_n, X+Y, XY$  and  $\sup X_n, \inf X_n, \limsup X_n, \liminf X_n$  are measurable. If  $\lim_n X_n$  exists, it's also measurable.*

*Proof.* For  $\sup_n X_n$ , let  $t \in \mathbb{R}$ ,  $\sup_n X_n \in (-\infty, t) \iff X_n < t$ , for all  $n$ , hence  $(\sup_n X_n)^{-1} = \bigcap_n X_n^{-1}(-\infty, t)$ .  $\square$

Now consider finite vector of random variables  $X^d := (X_1, \dots, X_d) : \Omega \rightarrow \mathbb{R}^d$ . Then  $X^d$  is measurable iff all  $X_i$  are measurable.

## 1.1.2 Construction of Measures

### Motivation

Now we want to show that we can actually find measure on  $\mathbb{R}$  and  $\mathbb{R}^d$  that are the most useful spaces. We start from abstract construction method inspired by the following observation of measure on  $\mathbb{R}$ .

In order to have a natural measure  $\lambda$  on  $\mathbb{R}$ , we want the following properties:

1.  $\lambda_0(a, b] = b - a$ ;
2. Measure of union of intervals should be the sum; (we can't have uncountable sum, hence restricted to countable union)
3. Apply to all intervals (we can't apply to the power set, hence restrict to the measurable sets).

Now consider the set  $\mathcal{S}_{\mathbb{R}} = \{(a, b] : a \leq b \in \mathbb{R}\}$ , it's too small, we want to extend the natural measure function  $\lambda_0$  to a larger set (we can extend to  $\mathcal{B}_{\mathbb{R}}$ , even to  $\mathcal{L}_{\mathbb{R}}$ ).

**Semi-algebra**  $\mathcal{S}$  is collection of subsets of a space  $\Omega$  that satisfies the following conditions: closed under intersection and the complements are union of finite disjoint sets in  $\mathcal{S}$ . Let  $\mu_0$  be a what I call **semi-pre-measure** to (**algebra**, **premeasure**) to (outer-measurable sets, outer measure) to (sigma-algebra, measure) where the **semi-pre-measure**  $\mu_0$  satisfies if both  $S_N = \sqcup^N S_j, S_\infty = \sqcup^\infty S_j \in \mathcal{S}$

$$\mu_0(\emptyset) = 0; \quad \mu_0(S_N) = \sum_{j=1}^N \mu_0 S_j; \quad \mu_0(S_\infty) \leq \sum_{j=1}^{\infty} \mu_0 S_j$$

We follow the steps of extension

$$(\mathcal{S}, \mu_0) \rightarrow (\mathcal{A}, \mu_1) \rightarrow (\mathcal{M}, \mu^*) \rightarrow (\sigma(\mathcal{S}), \mu)$$

that is semi-algebra and **pre-measure**  $\mu_1$  satisfies  $\mu_1 \sqcup_i^\infty A_i = \sum_i^\infty \mu_1 A_i$  as long as the union is also in  $\mathcal{A}$ . And we will show that  $\mathcal{B}_\mathbb{R} \subset \sigma(\mathcal{S})$ .

### Construction of Abstract Measure

**1.1.6**  $(\mathcal{S}, \mu_0) \rightarrow (\mathcal{A}, \mu_1)$ . We have that  $\mathcal{A} = \left\{ \sqcup_j^N S_j : S_j \in \mathcal{S} \right\}$  is an algebra. And  $\mu_1(A) = \sum_j^N \mu_0 S_j$  is a premeasure.

*Proof.*  $\mu_1(\emptyset) = 0$ . Suppose  $A = \sqcup_i^\infty A_i \in \mathcal{A}$ , then **there exists**  $T_l \in \mathcal{S} : l = 1, \dots, N_T$ , such that  $A = \sqcup_l T_l$ , also for each  $A_j = \sqcup_k^{N_j} S_{j,k}$ .  $\square$

Let an **outer measure** be a function over the power set such that:

$$\mu^*(\emptyset) = 0 \quad \text{and} \quad \mu^*\left(\sqcup_j^\infty A_j\right) \leq \sum_j \mu^* A_j \quad \text{and} \quad \mu^*(A) \leq \mu^*(B) \quad \text{for} \quad A \subset B$$

Also let  $\mathcal{M}$  be the **outer-measurable sets**, that is the collections of  $A$  such that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all} \quad E \subset X$$

**1.1.7**  $(\mathcal{A}, \mu_1) \rightarrow (\mathcal{M}, \mu^*)$  **Caratheodory's Theorem applied to premeasures.** We can find the outer measure induced by  $\mu_1$ :

$$\mu_1^*(B) = \inf \left\{ \sum_j^\infty \mu_1 A_j : B \subset \bigcup_j A_j \right\}$$

we show  $\mu_1^*$  is indeed an outer measure, Caratheodory states that for outer measures,  $\mathcal{M}$  of  $\mu_1^*$  is a  $\sigma$ -algebra and  $\mu^*|_{\mathcal{M}}$  is complete.

*Proof.*

1. Monotone property is easy, for countable subadditivity, we use  $\varepsilon$ -room method, find a cover  $C_{jk}$  of  $A_j$  such that :

$$\mu_1^*(A_j) > \sum_k \mu_1^* C_{jk} - \varepsilon$$

2. To show that  $\mathcal{M}$  is a  $\sigma$ -algebra we only need to show that for  $E_j \in \mathcal{M}$  and any  $B \subset \Omega$ , we have:

$$\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

which can be seen by considering finite sum, taking out one  $E_j$  at a time and then take limit.

3. Show that the restriction is complete.

□

**1.1.8**  $(\mathcal{M}, \mu^*) \rightarrow (\sigma(\mathcal{S}), \mu)$ . We can restrict  $\mu^*$  to define  $\mu = \mu^*|_{\sigma(\mathcal{S})}$ , and the resulting measure is complete.

*Proof.* We show that  $\sigma(\mathcal{S}) \subset \mathcal{M}$  and the restriction is complete.

□

### 1.1.3 Application: Borel and Lebesgue-Stietjes Measure on $\mathbb{R}$

We verify that for any nondecreasing right-continuous function  $F : \mathbb{R} \rightarrow \mathbb{R}$ , the semi-algebra  $\mathcal{S}_{\mathbb{R}}$  together with the semi-pre-measure defined by

$$\lambda_0(a, b] = F(b) - F(a)$$

satisfies the conditions set in the previous section.  $\mathcal{A}_{\mathbb{R}}$  be the algebra generated by  $\mathcal{S}_{\mathbb{R}}$  in turn will generate  $\mathcal{B}_{\mathbb{R}}$ . We can extend it to a something larger, a complete measure whose domain contains  $\mathcal{B}_{\mathbb{R}}$ . It will be called the *Lebesgue-Stietjes measure* and *Lebesgue measurable sets*  $\mathcal{M}_{\lambda}$ . The restriction on  $\mathcal{B}$  is called *Borel measure*. It's just the completion.

**1.1.9 Regularity of  $\mathcal{M}_\lambda$ .** *Lebesgue measurable sets are of simple form if you allow for a small error.*

1.  $\lambda(A) = \inf \{ \lambda U : A \subset U, U \text{ is open} \} = \sup \{ \lambda K : K \subset A, K \text{ is compact} \}$
2.  $A = G_\delta \setminus N_1 = F_\sigma \cup N_2$  where  $N$  are null sets.
3. *Littlewood's First Principle: Suppose  $\lambda(A) < \infty$  then for all  $\varepsilon > 0$ , there exists  $B$  that is a finite union of open intervals such that  $\lambda(A \Delta B) < \varepsilon$ .*

*Proof.*

□

### 1.1.4 Countable Product Space and Lebesgue Measure on $\mathbb{R}^d$

*Product measurable and product measure.*

### 1.1.5 Integration

Integration and measure are the two sides of same coin: integration is the extension of measure from the space of *indicator functions*  $\mathbf{1}_A$  of measurable sets to a larger set of functions.

For *simple functions*  $f_s = \sum_j^N \mathbf{1}_{A_j}$ , we can define  $\mu f_s = \sum_j^N \mu(A_j)$

Then for *nonnegative measurable* functions  $f$ , we can define

$$\mu f = \sup \{ \mu f_s : f_s \leq f \text{ pointwise} \}$$

If  $f = f^+ - f^-$  and the integrals of two nonnegative parts are not  $\infty$ , then we say  $f$  is *integrable* or in  $L^1$  and  $\mu f = \mu f^+ - \mu f^-$ .

**1.1.10 Littlewood's Second Principle.** *Measurable and integrable functions can be approximated by simple well-behaved functions.*

**1.1.11 Properties of Integral.** 1. *Monotone  $\mu f \leq \mu g$  if  $f \leq g$  pointwise.*

2. *Linearity for  $a > 0$ ,  $\mu(af) = a\mu f$ ,  $\mu(f + g) = \mu f + \mu g$ .*

3. *Monoton Convergence: if  $0 \leq f_n \uparrow f$  a.e., then  $\mu f_n \uparrow \mu f$ .*



*Proof.* Folland proves by choosing simple function and a scaling factor  $\alpha$ . Tao's proof is essentially the same. PTE proves via Fatou's lemma.

Suppose  $f_n \uparrow f$  a.e., if  $\mu f$  exists, we know  $\mu f_n$  must converge because it's a nondecreasing bounded sequence of numbers. Also  $\mu f_n \leq \mu f$  by monotonicity.

To show that  $\mu f \leq \lim_n \mu f_n$ , where  $\mu f = \sup \{\mu g : g \text{ is simple, } g \leq f\}$ , we only need to show that for all simple  $g \leq f$ , we have  $\mu g \leq \mu f_n$ . We can use  $\varepsilon$ -room method, let  $E_n = \{\omega : f_n \geq (1 - \varepsilon)g\}$ ,  $E_n \uparrow \Omega$ , and we have  $\int f_n \geq \int_{E_n} (1 - \varepsilon)g$ , taking  $\lim$  (**we have yet to show the monotone convergence for simple functions**), we have  $\lim \int f_n \geq (1 - \varepsilon) \int g$  for all  $\varepsilon > 0$ .  $\square$

*Remark.* These three properties fully characterise the integral. That is given a functional that satisfies the three properties we can find a measure for which the function is the integral.

#### 1.1.12 Inequalities and Controls. Useful inequalities and controls, suppose that $f \in L^1$ ;

1. *Jensen's inequality:* suppose we have a convex function  $\Phi(x)$ , then  $\mu(\Phi(f)) \geq \Phi(\mu f)$
2. *Holder's inequality.* For  $p, q : \frac{1}{p} + \frac{1}{q} = 1$ , we have  $\|fg\|_1 \leq \|f\|_p \|g\|_q$
3. *Minkowski's inequality:*  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .
4. *Chebyshev's Inequality:*
5. *For nonnegative measurable functions,  $\mu f < \infty \implies f < \infty$  a.e.,  $\mu f = 0$  implies  $f = 0$  a.e.*
6. *Moment and tail behaviour, if  $\mu$  is finite,*

$$\mu f^r < \infty \implies \sum x^{r-1} \mu(f > x) < \infty$$

*Proof.* We prove the above inequalities.

1. First we show that if  $f \in L^1$  then  $\Phi(f) \in L^1$  as well.
2. First we show that if  $f, g \in L^1$  then  $fg \in L^1$ .

$\square$

#### 1.1.13 Fubini-Tonelli.

## **1.1.6 Convergence Concepts**

**1.1.14 Monotone, Fatou and Dominated Convergence.**

**1.1.15 Uniform Integrability.**

## **1.1.7 Differentiation**

## **1.2 Independence**

### **1.2.1 Independence**

### **1.2.2 Law of Large Numbers**

### **1.2.3 Central Limit Theorem**