Oct 15, 2024 (Due: 08:00 Oct 22, 2024)

1. Write a program to compute the QR factorization of a general complex matrix $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ using CGS and MGS, with and without reorthogonalization. Visualize the loss of orthogonality $|Q^*Q - I_n|$ with a few examples.

2. Generate a few tall-skinny matrices with condition numbers varying from 10^0 to 10^{15}. Visualize the loss of orthogonality $\|Q^*Q - I_n\|_{\mathsf{F}}$ and the residual norm $\|A - QR\|_{\mathsf{F}}$ for Householder-QR, Cholesky-QR, CGS, MGS, etc.

3. Let $A \in \mathbb{C}^{m\times n}$. Show that $AA^\dagger$ and $I_n - A^\dagger A$, respectively, are the orthogonal projections with respect to $\mathrm{Range}(A)$ and $\mathrm{Ker}(A)$.

4. Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$. Suppose that for any $b \in \mathbb{C}^m$, $x = Xb$ is always a minimizer of the least squares problem $\min_x \|Ax - b\|_2$. Show that $AXA = A$ and $(AX)^* = AX$.

5. Find the "best" straight line that approximately passes through the data set $\{(n, \ln n) \in \mathbb{R}^2 \colon n \in \{2,3,4,5,6,7\}\}$. Visualize your result and clarify in what sense your solution is the best.

6. (optional) Generate a few least squares problems with condition numbers varying from 10^0 to 10^{15}. Choose two different kinds of right-hand sides:
(1) b is close to $\mathrm{Range}(A)$;
(2) b is far away from $\mathrm{Range}(A)$.
Compare the accuracy of the solutions produced by the following methods:
(a) solve the normal equation $A^*Ax = A^*b$ through the Cholesky factorization of A^*A;
(b) solve the augmented system

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix};$$

(c) solve the equation $Rx = Q^*b$ through Householder-QR;
(d) solve the equation $Rx = Q^*b$ through MGS.

7. (optional) Let $A \in \mathbb{R}^{m\times n}$ with full column rank. Establish the connection between the Householder-QR algorithm applied to the matrix $[0, A^\top]^\top \in \mathbb{R}^{(m+n)\times n}$ and the MGS algorithm applied to A.