Oct 8, 2024 (Due: 08:00 Oct 15, 2024)

1. We have mentioned in the lecture that for complex vectors x, $y \in \mathbb{C}^n$, $\|x\|_2 = \|y\|_2 > 0$ does *not* guarantee that there exists a Householder reflection H such that $y = Hx$. Please propose a sufficient and necessary condition for the existence of a Householder reflection H so that $y = Hx$. Prove your claim.

2. Describe how to avoid cancellation when constructing Householder reflections in the Householder triangularization algorithm for complex matrices.

3. Write a program to compute the QR factorization of a complex matrix $A \in \mathbb{C}^{n\times n}$ with
(1) Cholesky QR (i.e., through the Cholesky factorization of A^*A);
(2) Householder triangularization.
Visualize the (componentwise) loss of orthogonality $|Q^*Q - I_n|$ using well-conditioned and ill-conditioned examples.
(If you use MATLAB/Octave, you may find `imagesc()` helpful.)

4. Let

$$A = \begin{bmatrix} \alpha_1 & \rho_2 & \rho_3 & \cdots & \cdots & \rho_n \\ \beta_2 & \alpha_2 & 0 & \cdots & \cdots & 0 \\ \beta_3 & 0 & \alpha_3 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \alpha_{n-1} & 0 \\ \beta_n & 0 & \cdots & \cdots & 0 & \alpha_n \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Design an efficient algorithm to compute the QR factorization of A.

5. Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Show that Q can be factorized as the product of finitely many Householder reflections, and if, in addition, $\det(Q) = 1$, Q can be factorized as the product of finitely many Givens rotations.

6. (H) Use pseudocode to describe a block algorithm (either left-looking or right-looking) for Householder triangularization. Make sure all indices are correct. For simplicity, you may assume that the number of columns of the matrix is a multiple of the block size.

7. (optional) Let w_1, w_2, …, $w_k \in \mathbb{C}^n$ be unit vectors. Try to find a matrix $T \in \mathbb{C}^{k\times k}$ such that

$$(I - 2w_1w_1^*)(I - 2w_2w_2^*)\cdots(I - 2w_kw_k^*) = I - [w_1, w_2, \ldots, w_k]T[w_1, w_2, \ldots, w_k]^*.$$

8. (optional) Design an efficient algorithm to compute the QR factorization of $R + uv^\top$, where $R \in \mathbb{R}^{n\times n}$ is an upper triangular matrix and u, $v \in \mathbb{R}^n$ are column vectors.

9. (optional) Write a program to convert a (text) STL file to another one that is more user-friendly (in the sense of supporting) for 3D printing by rotating the object. You may need to learn the STL file format if you have not played with it before.