Dec 10, 2024 (Due: 08:00 Dec 17, 2024)

1. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, and $b \in \mathbb{R}^n$. Suppose that $\mathcal{V}$ is a subspace of $\mathbb{R}^n$. Show that

$$x_0 = \arg\min_{x\in\mathcal{V}} \|x - A^{-1}b\|_A$$

if and only if $b - Ax_0 \in \mathcal{V}^\perp$. (The orthogonal complement is defined using the standard inner product.)

2. Suppose you are given a matrix $A \in \mathbb{R}^{m\times n}$ with full column rank (i.e., $\operatorname{rank}(A) = n$), and a vector $b \in \mathbb{R}^m$. To solve the least squares problem $\min_x \|Ax - b\|_2$, the CG method can be applied to solve the normal equation $A^\top Ax = A^\top b$. Provide a detailed iterative scheme for this method. Make sure that the operation $v \mapsto A^\top Av$ is avoided.

3. Use the conjugate gradient method to solve the 2D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

over the unit square $[0, 1]^2$ with boundary conditions

$$u(0, y) = u(1, y) = u(x, 1) = 0, \qquad u(x, 0) = \sin(\pi x).$$

Visualize the solution and the convergence history.

(optional) Use IChol-based preconditioning to accelerate the convergence of CG. (IChol means incomplete Cholesky factorization.)

4. Implement the Lanczos algorithm for symmetric eigenvalue problems. Test it with some sparse Hermitian matrices. What do you observe for the convergence of Ritz pairs and the orthogonality of the Lanczos vectors?

5. (optional) Let $p(\lambda) = \sum_{k=0}^n a_k\lambda^k$ be a real polynomial such that

$$\max_{-1\le\lambda\le1} |p(\lambda)| \le 1.$$

Show that $|a_n| \le 2^{n-1}$.

6. (optional) Let $p(\lambda)$ be a real polynomial such that $\deg p(\lambda) \le n$ and

$$\max_{-1\le\lambda\le1} |p(\lambda)| \le 1.$$

Show that $|p(\mu)| \le |T_n(\mu)|$ for any $\mu \in \mathbb{R} \setminus [-1, 1]$.