Nov 5, 2024 (Due: 08:00 Nov 12, 2024)

1. Let $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$. Suppose that $X = [x, Ax, \dots, A^{n-1}x]$ is nonsingular. Show that $X^{-1}AX$ is upper Hessenberg.

2. Let $A \in \mathbb{C}^{n\times n}$ be an *unreduced* upper Hessenberg matrix. By unreduced, we mean $A_{i+1,i} \neq 0$ for $1 \leq i \leq n-1$. Suppose that A is singular. Show that the zero eigenvalue appears at the bottom right corner after one QR sweep.

What happens if A is a singular upper Hessenberg matrix with some $A_{i+1,i} = 0$?

3. Let

$$A = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}.$$

What can you say about the convergence of
(1) the naive QR algorithm,
(2) Francis' double-shift QR algorithm?

4. Let

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Design an algorithm to compute an orthogonal matrix $Q \in \mathbb{R}^{2\times 2}$ such that

$$Q^\top AQ = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}.$$

(H) What happens if the matrix A is complex?

5. Implement the following algorithms for Hessenberg reduction:
(a) using Householder reflections;
(b) using Arnoldi process based on modified Gram–Schmidt orthogonalization.
Randomly generate a few matrices and compute the corresponding Hessenberg decomposition $A = QHQ^*$. Check the accuracy in terms of $\|Q^*AQ - H\|_{\mathsf{F}}$ and $\|Q^*Q - I\|_{\mathsf{F}}$ for your Hessenberg reduction implementations. What do you observe?

(optional) Perturb the matrix A a little bit. How do Q and H change accordingly?

6. (H) Construct an example such that the aggressive early deflation (AED) strategy does not work at all.

7. (H, optional) Write a subprogram to perform diagonal swapping of the real Schur form

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ and $A_{2,2}$ are either 1×1 or 2×2.