Oct 22, 2024 (Due: 08:00 Oct 29, 2024)

1. Suppose that a tall-skinny matrix $A \in \mathbb{R}^{m\times n}$ is upper bidiagonal (i.e., $a_{i,j} \neq 0$ only if $i - j \in \{0, -1\}$). Design an algorithm based on Givens rotations to solve the ridge regression problem

$$\min \|Ax - b\|_2^2 + \lambda \|x\|_2^2,$$

where λ is a given positive number.

2. Consider the constrained least squares problem

$$\min_{Cx=d} \|Ax - b\|_2.$$

In the lecture we use the QR factorization of C^* to eliminate the linear constraint $Cx = d$ and reduce the problem to a standard least squares one. Design an algorithm based on Gaussian elimination to eliminate the linear constraint.

3. The least squares problem $\min\|Ax - b\|_2$ is equivalent to an augmented linear system

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

which is Hermitian and indefinite. Similar augmented linear systems exist for the constrained least squares problem

$$\min_{Cx=d} \|Ax - b\|_2.$$

Can you figure it out?

4. Implement the Arnoldi process based on CGS/CGS2/MGS/MGS2 and visualize the loss of orthogonality. You can randomly generate a 1000×1000 matrix and generate a 30-dimensional Krylov subspace.

(H, optional) Design an algorithm to perform the Arnoldi process based on Householder orthogonalization.
Hint: use the left-looking variant.

5. Write a program to solve rank deficient least squares problems. You may assume that the rank is given. Test your program with a low-rank least squares problem. Try it with a general full-rank least squares solver and compare the results.

6. (optional) Implement the Lanczos process. Make sure you are using a short recurrence instead of using the naive Arnoldi process.

7. (optional) Implement the GTH algorithm and verify its componentwise accuracy by some examples.