Oct 29, 2024 (Due: 08:00 Nov 5, 2024)

1. Let $(\hat{\lambda}, \hat{x})$ be an approximate eigenpair of $A \in \mathbb{C}^{n\times n}$, and $r = A\hat{x} - \hat{x}\hat{\lambda}$ be the residual. Suppose that $\|\hat{x}\|_2 = 1$. Show that there exists $E \in \mathbb{C}^{n\times n}$ such that

$$(A + E)\hat{x} = \hat{x}\hat{\lambda}, \qquad \|E\|_2 \le \|r\|_2.$$

2. Let $A_0 \in \mathbb{C}^{n\times n}$, μ_0, μ_1, …, $\mu_m \in \mathbb{C}$. Define A_1, A_2, …, A_{m+1} by

$$A_k - \mu_k I = Q_k R_k, \qquad A_{k+1} = R_k Q_k + \mu_k I,$$

for $k \in \{0, 1, \dots, m\}$, where Q_k's are unitary matrices. Show that

$$(A_0 - \mu_0 I)(A_0 - \mu_1 I) \cdots (A_0 - \mu_m I) = (Q_0 Q_1 \cdots Q_m)(R_m \cdots R_1 R_0).$$

3. Randomly generate a 1000×1000 matrix A with positive entries. Use the power method to compute $\rho(A)$. Visualize the convergence history.

4. Let A be the 200×200 Hilbert matrix (the (i, j)-entry of A is $a_{i,j} = (i + j - 1)^{-1}$). Compute the eigenvalue of A that is closest to 1 using inverse iteration and Rayleigh quotient iteration. Visualize the convergence history and report the execution time (preferably with detailed profiling).

5. Write a program to compute all eigenvectors of an upper triangular matrix with distinct diagonal entries. Compare your result with the one produced by a math library (e.g., `eig` in MATLAB/Octave).

6. (optional) Investigate the behavior of the power method applied to the following matrices:

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda & 1 \\ 0 & -\lambda \end{bmatrix},$$

where $\lambda \in \mathbb{C}$ is a given constant.

7. (optional) Implement the naive QR algorithm and visualize the *componentwise* convergence of a small example.