Dec 3, 2024 (Due: 08:00 Dec 10, 2024)

1. Implement your own GMRES solver. Test it with at least two systems of sparse linear equations (for symmetric and nonsymmetric coefficient matrices) with 1000+ unknowns and plot the residual history.

2. Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Let r_k be the residual vector at kth iterate produced by the steepest descent (SD) method when solving the linear system $Ax = b$. Show that if $r_{k+1} = 0$, then r_k is an eigenvector of A.

3. Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Let x_k be the approximate solution at kth iterate when applying the steepest descent (SD) method to the linear system $Ax = b$. Show that

$$f(x_{k+1}) \le \big(1 - \kappa^{-1}\big) f(x_k),$$

where $f(x) = x^\top Ax - 2b^\top x$ and $\kappa = \|A\|_2 \|A^{-1}\|_2$.

4. Use the steepest descent (SD) method to solve the linear system

$$\begin{bmatrix} 20 & \\ & 1 \end{bmatrix} x = 0$$

with the initial guess $x_0 = [1, 5]^\top$. Plot the intermediate approximations x_k's for $0 \le k \le 10$.

5. (H) Derive the Lanczos algorithm for computing a few largest eigenvalues of AB^{-1}, where A and B are given large sparse Hermitian positive definite matrices. Note that B^{-1} is not explicitly available.

6. (optional) Implement GMRES and FOM, with right preconditioning. Use an artificial example to test the convergence of GMRES and FOM, with and without preconditioning.

One possible way to construct an artificial example for preconditioning is as follows: Create random lower and upper bidiagonal matrices L_0 and U_0, respectively. Perturb L_0 and U_0 a little bit (with fillins) to obtain denser triangular matrices L and U. Then you can test GMRES and FOM with $A = LU$ and $M = L_0 U_0$. You are certainly free to try other examples.

7. (H, optional) Derive Sorensen's implicit restarting procedure for the Arnoldi decomposition.