# Nov 12, 2024 (Due: 08:00 Nov 19, 2024)

**1.** Let $\hat{x} \in \mathbb{R}^n$ be an approximate eigenvector of a real symmetric matrix $A$ such that $\|\hat{x}\|_2 = 1$. Show that there exists a real symmetric matrix $\Delta A$ such that

$$(A + \Delta A)\hat{x} = \hat{x}\hat{\lambda}, \qquad \|\Delta A\|_2 \leq \|A\hat{x} - \hat{x}\hat{\lambda}\|_2,$$

where $\hat{\lambda} = \hat{x}^\top A \hat{x}$.

**2.** Given $x$, $y \in \mathbb{R}^n$. Describe in detail how to construct a rotation matrix $Q$ such that the columns of $[x, y]Q$ are orthogonal to each other.

**3.** Let $D \in \mathbb{R}^{n\times n}$ be diagonal with distinct eigenvalues, $z \in \mathbb{R}^n$ be a vector with no zero entries, and $\rho \in \mathbb{R} \setminus \{0\}$. Suppose that $(\lambda, u)$ is an eigenpair of $D + \rho z z^\top$. Show that
(1) $\lambda I - D$ is nonsingular;
(2) $(\lambda I - D)^{-1} z$ is an eigenvector of $D + \rho z z^\top$;
(3) $z^\top u \neq 0$.

**4.** Randomly generate a relatively small (e.g., $6 \times 6$) real symmetric matrix of the form $A = \operatorname{diag}\{d_1, d_2, \ldots, d_n\} + z z^\top$, where $d_i$'s are distinct, and $z$ does not have any zero entry. Visualize the function

$$f(\lambda) = 1 - \sum_{i=1}^{n} \frac{z_i^2}{\lambda - d_i}.$$

Highlight the eigenvalues of $A$ in the plot and make sure they match the roots of $f(\lambda)$. For simplicity, you may compute the eigenvalues of $A$ by existing functions from math libraries (e.g., `eig` from MATLAB/Octave).

**5.** Implement the Jacobi diagonalization algorithm for real symmetric matrices. Visualize the performance for a few matrices with different sizes. Visualize the convergence history for one example. You are encouraged to make observations on *componentwise* convergence.

You are also encouraged to try the "wrong" choice of Jacobi rotations for the cyclic Jacobi algorithm.

**6.** (optional) Let $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ be a real diagonal matrix. Let $\alpha_1$, …, $\alpha_n$ be real scalars that satisfy

$$d_n < \alpha_n < \cdots < d_i < \alpha_i < d_{i-1} < \alpha_{i-1} < \cdots < d_1 < \alpha_1.$$

Show that the $\alpha_i$'s are exact eigenvalues of $D + uu^\top$, where entries of the real vector $u$ are defined by

$$u_i = \left( \frac{\prod_{1\leq j\leq n}(\alpha_j - d_i)}{\prod_{1\leq j\leq n,\, j\neq i}(d_j - d_i)} \right)^{1/2}, \qquad (1 \leq i \leq n).$$

**7.** (optional) When bidiagonalizing an $m \times n$ matrix with $m > n$, there are two common options: bidiagonalization after QR factorization vs. direct bidiagonalization. Suppose that both left and right orthogonal transformations need to be accumulated. Calculate the cost in terms of number of floating-point operations for these options, and determine the crossover point.

Will the crossover point change if orthogonal transformations are not accumulated?