Nov 26, 2024 (Due: 08:00 Dec 3, 2024)

1. Find an example of linear system such that the Jacobi method converges while the Gauss–Seidel method diverges. Justify your claim.

2. Find an example of positive definite linear system such that the Gauss–Seidel method converges while the Jacobi method diverges. Justify your claim.

3. Let $B \in \mathbb{C}^{n\times n}$, $g \in \mathbb{C}^n$. Suppose that $\rho(B) = 0$. Show that the iterative scheme

$$x^{(k+1)} = Bx^{(k)} + g$$

converges to the solution of $x = Bx + g$ for any initial guess $x^{(0)}$ within at most n iterations.

4. Let B, $M \in \mathbb{C}^{n\times n}$, $g \in \mathbb{C}^n$. Suppose that both M and $M - B^*MB$ are positive definite. Show that the iterative scheme

$$x^{(k+1)} = Bx^{(k)} + g$$

converges to the solution of $x = Bx + g$ for any initial guess $x^{(0)}$.

5. Numerically solve the 2D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

over the unit square $[0, 1]^2$ with boundary conditions

$$u(0, y) = u(1, y) = u(x, 1) = 0, \qquad u(x, 0) = \sin(\pi x).$$

Use the Jacobi method or the Gauss–Seidel method to solve the discretized system. Visualize the solution and the convergence history.

6. (optional) Let A be a real symmetric nonsingular matrix with positive diagonal entries. Suppose that the Gauss–Seidel method for solving $Ax = b$ converges for any initial guess. Show that A is positive definite.
(Hint: Show that $C = A - B^\top AB$ is positive definite, where $B = I - (D - L)^{-1}A$.)