Sep 24, 2024 (Due: 08:00 Oct 8, 2024)

1. Assume that b, δb, x, δx satisfy

$$Ax = b, \qquad A(x + \delta x) = b + \delta b,$$

where

$$A = \begin{bmatrix} 610 & 987 \\ 987 & 1597 \end{bmatrix}.$$

Construct examples such that
(1) $\|\delta b\|_\infty / \|b\|_\infty$ is very small while $\|\delta x\|_\infty / \|x\|_\infty$ is very large;
(2) $\|\delta b\|_\infty / \|b\|_\infty$ is very large while $\|\delta x\|_\infty / \|x\|_\infty$ is very small.

2. Let $Z \in \mathbb{C}^{n\times n}$ and

$$A = \begin{bmatrix} I_n & Z \\ 0 & I_n \end{bmatrix}.$$

Find $\kappa_{\mathsf{F}}(A) = \|A\|_{\mathsf{F}} \|A^{-1}\|_{\mathsf{F}}$.

3. It can be shown that Gaussian elimination without pivoting is numerically stable for solving strictly diagonally dominant linear systems, in the sense that the growth factor is bounded. Give a concrete upper bound on the growth factor.

In the exercise, let us stick to the definition of growth factor on the textbook:

$$\rho = \max_{i,j,k} \frac{|a_{i,j}^{(k)}|}{\|A\|_\infty}.$$

4. It can be shown that Gaussian elimination with partial pivoting is numerically stable for solving nonsingular tridiagonal linear systems, in the sense that the growth factor is bounded. Give a concrete upper bound on the growth factor.

5. Solve the following linear systems, using Gaussian elimination with and without pivoting:

$$\begin{bmatrix} 8 & 1 & & & & & \\ 6 & 8 & 1 & & & & \\ & 6 & 8 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 6 & 8 & 1 & \\ & & & & 6 & 8 & 1 \\ & & & & & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{98} \\ x_{99} \\ x_{100} \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 15 \\ \vdots \\ 15 \\ 15 \\ 14 \end{bmatrix}$$

and

$$\begin{bmatrix} 6 & 1 & & & & & \\ 8 & 6 & 1 & & & & \\ & 8 & 6 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 8 & 6 & 1 & \\ & & & & 8 & 6 & 1 \\ & & & & & 8 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{98} \\ x_{99} \\ x_{100} \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \\ 15 \\ \vdots \\ 15 \\ 15 \\ 14 \end{bmatrix}.$$

Compare the computed solutions with the exact ones. How can you say about the accuracy?

6. (optional) Provide a rounding error analysis for solving a triangular linear system with multiple right-hand-sides.

7. (optional) Provide a rounding error analysis for solving a symmetric positive-definite linear system through Cholesky factorization. In addition to the standard Wilkinson's rounding model, you may assume that the square root of a positive real number can be computed accurately to the machine precision, i.e.,

$$\mathrm{fl}(\sqrt{\alpha}) = \sqrt{\alpha}(1+\theta), \qquad |\theta| \le \boldsymbol{u}.$$

8. (H) Implement the cyclic reduction algorithm for solving tridiagonal linear systems. Test your implementation with the examples in Exercise 5.

(optional) How to solve banded linear systems using cyclic reduction?

9. (H) Implement the sparse matrix–vector multiplication $y = A^\top x$, where A is stored in the compressed sparse column (CSC) format.

(optional) You may play with some sparse matrices from the SuiteSparse Matrix Collection or Matrix Market.