How to approximate functions & differential operators with particles?

$$\underbrace{f,\ \frac{\partial f}{\partial x},\ \frac{\partial^2 f}{\partial x^2}}_{F} \quad \int F(x')\,\zeta_\varepsilon(x-x')\,dx' = F_\varepsilon$$

Quadrature

Deterministic

Stochastic

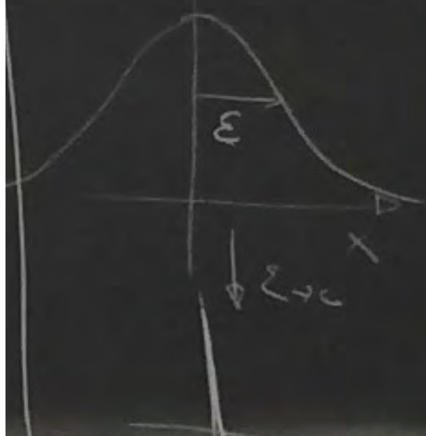
$$F_\varepsilon^h = \sum F(x_p)\,\mathrm{vol}_p\,\zeta_\varepsilon(x-x_p)$$

Monte Carlo approx

The Dirac δ-function

d dimension of function space

$$\delta(x) = \lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^d} \mathcal{J}\left(x/\varepsilon\right)$$



$$\mathcal{J}_\varepsilon(x) = \frac{1}{\sqrt{2\pi}\,\varepsilon} e^{-|x|^2/2\varepsilon^2}$$ for 1-D

& $$\mathcal{J}(x) = \frac{1}{\sqrt{2\pi}} e^{-|x|^2/2}$$

Another example of a $\mathcal{J}$ function

$$\delta(x) = \begin{cases} \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{1}{x^2+\varepsilon^2} \\ \lim_{\varepsilon \to 0} \varepsilon |x|^{\varepsilon - 1} \end{cases}$$

Properties of δ-function

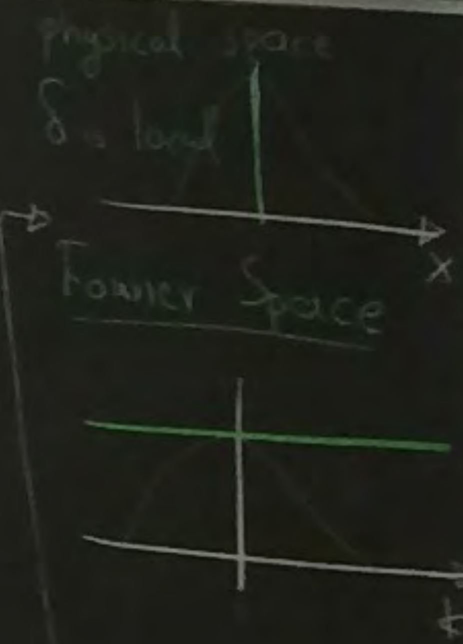
$$\int f(x)\,\delta(x)\,dx = f(0)$$

$$\downarrow$$

$$\boxed{\int \delta(x)\,dx = 1}$$

$$\int f(x)\,\delta(x-\alpha)\,dx = f(\alpha)$$

$$(\text{FT of } \delta)\ \hat{f}(k) = \int e^{-ikx}\,\delta(x)\,dx = 1$$



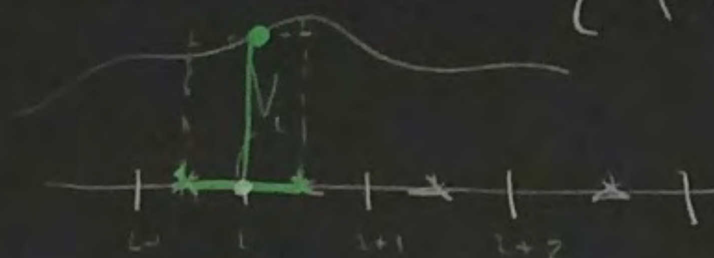
Point Particle Approximations

$$\boxed{u(x) = \int_V u(y)\,\delta(x-y)\,dy.}$$

decompose V into volumes V_k

$$= \sum_k \int_{V_k} u(y)\,\delta(x-y)\,dy$$

In 1D $\quad V_k = \left\{ (k_i - \tfrac{1}{2}) h < x_i < (k_i + \tfrac{1}{2}) h, \; i = 1 \dots, N \right\}$



Approximate integrals with mid-point rule

$$\int_{V_k} g(x)\,dx = w_k\, g(x_k) = h \cdot g(x_k)$$

$$u(x) \approx u^h(x) = \sum_k w_k\, u(x_k)\, \delta(x - x_k)$$

$$\underbrace{(x_k, \gamma_k)}_{\text{Point Particle Approximation}}$$

$$= \sum_k h\, u(x_k)\, \delta(x - x_k)$$

$$\boxed{u^h(x) = \sum \gamma_k\, \delta(x - x_k)}$$

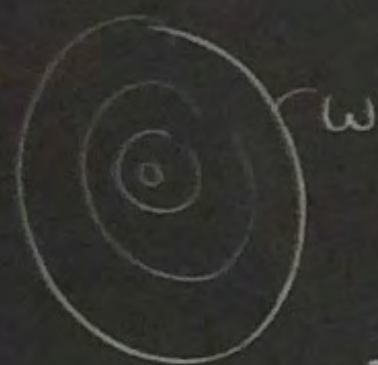
LOCATION

with $\gamma_k = h \cdot u(x_k)$

strength or weight

$$\underbrace{\int u^h(x)\,dx} = \sum \gamma_k \int \delta(x - x_k)\,dx =$$

$$= \underbrace{\sum \gamma_k}$$



$$\omega^h(x) = \sum \gamma_k \, \delta(x - x_k)$$

$$\underbrace{\Gamma = \int \omega^h(x)\,dx} = \sum \gamma_k$$

Pollutant Transport

$$c(x,t) \to \sum \overbrace{\boxed{c_k^{(t)} vol_k(t=0)}}^{\gamma_k} \delta(x - \chi_k)$$

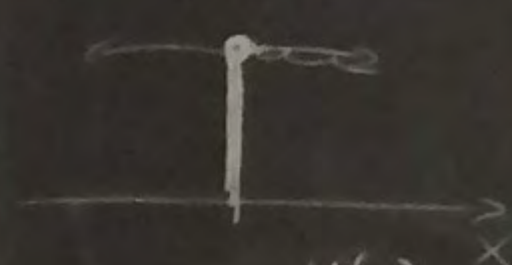
$$\boxed{\frac{\partial c}{\partial t} + u(x)\frac{\partial c}{\partial x} = 0} \to \text{FD}$$

$$\downarrow$$

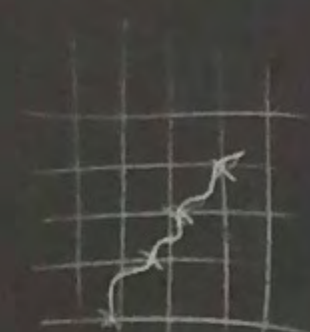
$$\frac{dc(x)}{dt} = 0 \quad \& \quad \frac{dx}{dt} = u$$

$$\downarrow$$

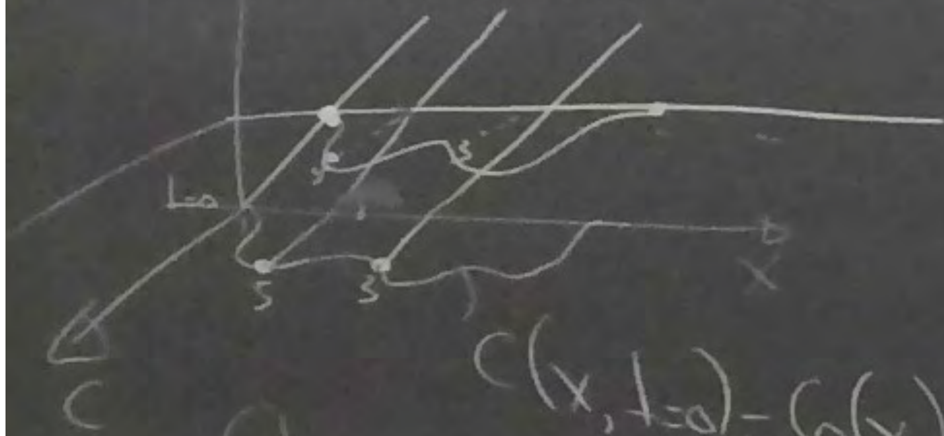
$$\frac{d\gamma_k}{dt} = 0 \qquad \frac{dx_k}{dt} = u(x_k)$$



u(x)



$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = 0 \qquad u = \text{const} = \alpha$$



$$c(x,t) = c_0(x - \alpha t)$$

$$c(x, t=0) = c_0(x)$$

Characteristics

$$u(x) = \int u(y)\, \delta(x-y)\, dy$$

$$= \int u(y) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^d} \mathcal{J}\left(\frac{|x-y|}{\varepsilon}\right) dy$$

$$= \lim_{\varepsilon \to 0} \int u(y)\, \mathcal{J}_\varepsilon(x-y)\, dy$$

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x)$$

with

$$\boxed{u_\varepsilon(x) = \int u(y)\, \mathcal{J}_\varepsilon(x-y)\, dy}$$

SMOOTH FUNCTION APPROXIMATION

filtering = smoothing

What is the error I do when smoothing is performed?

<u>Error</u>: $\left|u(x) - u_\varepsilon(x)\right|$

$$u(x) - u_\varepsilon(x) = u(x)\cdot 1 - \int u(y)\, J_\varepsilon(x-y)\, dy.$$

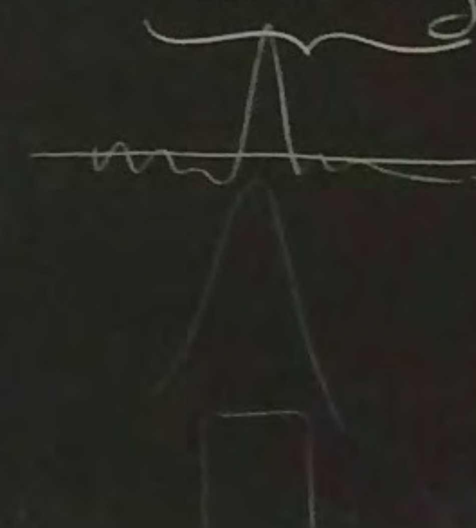
$$\left(\underline{\text{Require}} \text{ that } \underbrace{\int J_\varepsilon(x)\, dx}_{=1} = \int \delta(x)\, dx = 1 \right)$$

$$E = \underbrace{u(x)} \underbrace{\int J_\varepsilon(x-y)\, dy}_{\text{Partitions of unity}} - \int u(y)\, J_\varepsilon(x-y)\, dy$$

$$E = \int \underbrace{\left[u(x) - u(y)\right]}_{\text{Taylor series}} J_\varepsilon(x-y)\, dy.$$

Taylor series

$$\underbrace{u(x) - u(y)} = (x-y)\frac{\partial u}{\partial x} + \frac{(x-y)^2}{2}\frac{\partial^2 u}{\partial x^2} + \text{h.o.t.}$$



$$E = \varepsilon\,\frac{\partial u}{\partial x}\cancel{\int (x-y)\,J_\varepsilon(x-y)\,dy}$$

$$+\ \varepsilon^2\,\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\cancel{\int (x-y)^2\,J_\varepsilon(x-y)\,dy}$$

$$+\ \varepsilon^3\,\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\int (x-y)^3\,\underline{J_\varepsilon(x-y)\,dy}$$

$$\text{Error} \sim \mathcal{O}(\varepsilon^r)$$

$$\text{for } J_\varepsilon(x) = \frac{1}{\varepsilon^d}\,J\!\left(\frac{|x|}{\varepsilon}\right)$$

$$\int x^a J(x)\,dx = 0 \quad \text{for } 1 \le |a| \le r-1$$

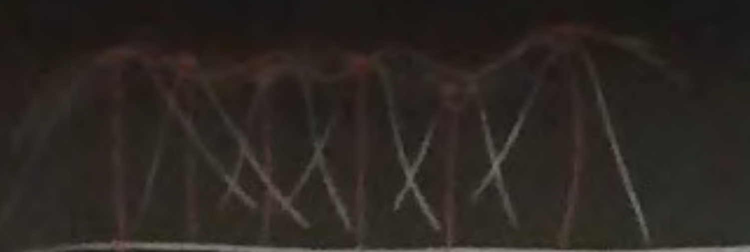
$$\int x^r J(x)\,dx < \infty$$

$$\int J(x)\,dx = 1$$

$$\int (x-y)\,\underbrace{J_\varepsilon(x-y)}\,dy = \int (x-y)\,\underbrace{\frac{1}{\varepsilon}\,J\!\left(\frac{x-y}{\varepsilon}\right)}\,dy$$

$$= -\varepsilon^2\int \frac{x-y}{\varepsilon}\,\underbrace{\frac{1}{\varepsilon}\,J\!\left(\frac{x-y}{\varepsilon}\right)}\,d\!\left(\frac{x-y}{\varepsilon}\right)$$

$$= -\varepsilon\int z\,\cancelto{0}{J(z)\,dz}$$



$$\omega = \sum \gamma_k \delta(x - x_k)$$

$$\frac{dx_k}{dt} = u_k$$

$$\frac{D\omega}{Dt} = 0 = \frac{\partial \omega}{\partial t} + u \nabla \omega = 0$$

$$\frac{dx}{dt} = u \qquad \frac{d\gamma_k}{dt} = 0$$

$$\nabla^2 u = \nabla \times \omega$$

$$f(0) = \int f(x)\,\delta(x)\,dx$$

$$1 = \int \delta(x)\,dx \rightarrow \int J(x)\,dx = 1$$

$f(x) = x$ $\quad 0 = \int x\,\delta(x)\,dx \rightarrow \int J(x)\,x\,dx = 0$ $\quad r = 1$

$f(x) = x^2$ $\quad 0 = \int x^2 \delta(x)\,dx$ $\quad \int J(x)\,x^2\,dx = 0$

$f(x) = x^n$ $\quad 0 = \int x^n \delta(x)\,dx$

Any even function is $\mathcal{O}(\varepsilon^2)$

any positive function is $\mathcal{O}(\varepsilon^2)$

$$J(\rho) = \frac{2(2-\rho^2)}{\pi(1+\rho^2)^4} \rightarrow r = 3$$

Mass $\rightarrow$ Concentration

$$\frac{\partial C}{\partial x}(x) = \sum \gamma_k \frac{\partial J_\varepsilon(x - x_k)}{\partial x}$$