Comparing Markov distributional equilibrium dynamics in large games with complementarities and no aggregate risk*

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Abstract

We study a class of discounted infinite horizon stochastic games with strategic complementarities with a continuum of players. In order to analyse transition of private signals to aggregate distributions, we develop a dynamic law of large numbers, that implies a.o. no aggregate uncertainty. We define a suitable equilibrium concept, namely: Markov Stationary Equilibrium and prove its existence under a new set of assumptions, via constructive methods. Our construction allows to overcome some problems in characterizing beliefs and dynamic complementarities in this class of games. In addition, we provide constructive monotone comparative dynamics results for ordered perturbations of the space of games extending the known results from steady states or invariant measures to dynamic equilibria. For this end, we

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use our new fixed point comparative statics theorem, suitable for comparing equilibrium objects in spaces of distributions or measurable functions. Finally, we provide approximation of our large game by its small counterparts.
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1 Introduction

In this paper we consider a class of dynamic games with private types. Types determine players endowment, preferences or information and evolve stochastically according to a joint probability distribution parameterized by players' actions and types. Such games has been widely applied recently is such diverse fields as: economic growth with matching (Cole et al., 1992), models of aspirations and income distributions (Genicot and Ray, 2017), industry dynamics with investment, exit and entry decisions (Weintraub et al., 2008), models of network formation (Mele, 2017; Xu, 2018), dynamic models with identities, social dissonance and culture (Akerlof and Kranton, 2000; Bisin et al., 2011), OLG models of social norms and enforcement of laws (Acemoglu and Jackson, 2017), dynamic economies with public or private sunspots (Angeletos and Lian, 2016) or Bewley-Huggett-Ayiagari type macromodels[1] of wealth distributions and inequalities (Ma et al., 2020; Miao, 2006) to name just a few.

Such models are of particular interest with many players, allowing to model, compute or estimate dynamics of population distributions. As the number of players grows large, they become intractable, however. This is so for few reasons. Firstly, the state space with many types and players becomes large. Secondly, it becomes hard to model beliefs and their updating on both on and off the equilibrium paths. Thirdly, they become hard to compute or estimate, including studying questions involving comparative statics or dynamics. And finally, when the number of players is large, it becomes hard to argue that

[1] In Acemoglu and Jensen (2015, 2018), the relationships between large dynamic economies and large anonymous games is discussed. It is important to note that many large dynamic economies can be viewed as large aggregative games (e.g. Bewley-Huggett-Aiyagari models).

individuals indeed keep track of all the equilibrium interactions and their dynamics. Unfortunately, these problems emerge even if attention is restricted to Markovian strategies and special classes of games, like those of strategic complements (that are generally very tractable from both theoretical and applied perspectives)[2].

Out of few ideas, presented in the literature, to simplify dynamic interaction in such environments, we discuss two dominant. Some researchers advocated a class of aggregative games, where players' interactions are limited via aggregates summarizing population distributions and allow to consider beliefs concerning evolution of these aggregates only. Others have additionally simplified the equilibrium notion to consider so called oblivious strategies and concentrated analysis to stationary equilibria or stochastic steady states only. Here, we also refer a reader to an inspiring work of Kalai and Shmaya (2018), who provide some behavioral foundations for studying large but finite number of players repeated games. This important literature on equilibrium existence, computation, and comparative statics of aggregative or mean-field games is large and growing (some recent papers include contributions of Acemoglu and Jensen (2015); Adlakha et al. (2015); Doncel et al. (2016); Kwok (2019); Lacker (2018); Light and Weintraub (2019); Nutz (2018); Weintraub et al. (2008) to name just a few).

In this paper, we propose a different approach and resolve many of the mentioned theoretical and numerical problems in a new methodological framework. Firstly, we consider a large dynamic anonymous game with a measure space of players[3]. It happens that environment of a large dynamic game allows us to make agent individually negligible and limit interactions sufficiently to get tractable equilibrium conditions and characterization without revoking to a class of aggregative games. In fact, as we will show in examples our distributional approach allows to model many problems where the knowledge of the whole distribution is inevocable (e.g. games with dynamics of identities and social dissonance, dynamic quantile games, models of aspiration formation and inequality dynamics, games with private sunspots with learning and alike).

[2] Refer to Topkis (1978); Van Zandt (2010); Veinott (1992); Vives (1990) for some early contributions.

[3] Following a tradition of Bergin and Bernhardt (1992); Jovanovic and Rosenthal (1988); Karatzas et al. (1994) among others.

Secondly, we reconsider many of these questions in the context of an important subclass of games, namely those with strategic complements. Again, as we will show in the paper such class of dynamic games becomes particularly tractable exactly, when large and distributional, provided our new conditions on dynamic complementarities and belies are satisfied.[4] In particular, we identify new sufficient conditions for the existence of extensive-form large stochastic supermodular games in Markov strategies, where our games remain stochastically supermodular at both finite and infinite horizons. In doing so, we are able to develop a new set of order theoretic tools for characterizing the order structure of the set of (Markovian) distributional equilibrium. This allows us to extend the results of the theory of stochastic supermodular games to a very general setting.

Thirdly, in order to analyze transition of private signals to aggregate distributions, and handle dynamics of players' beliefs we build upon the recent results (He et al., 2017; Podczeck, 2010; Sun, 2006) and develop a dynamic exact law of large numbers (D-ELLN), that implies a.o. no aggregate uncertainty. This allows us to achieve three goals: 1. simplify analysis for allowing for independent draws of types for a continuum of players; 2. simplify dynamics of the aggregative beliefs to deterministic ones and hence 3. any individual agent do not need to update his beliefs on (product) of other players types. We think that our results per this question are of the independent interest, and can be applied in many dynamic settings where there exists idiosyncratic risk at a micro level that evolves dynamically but generates no aggregate risk.

Next, provided the D-ELLN, our environment becomes Markovian and under appropriate model of dynamic beliefs formation, we can use the powerful techniques of recursive methods to define our new equilibrium concept, namely Markov Stationary (Distributional) equilibrium. MSE involves an equilibrium action-state distribution and a law of motion of aggregate distributions, and hence resembles recursive competitive equilibrium concept known from macro literature. Our equilibrium concept is inherently dynamic, and hence we are able to characterize and compare equilibrium *transition* paths.

[4] In the paper, we discuss how the use of a measure space of agents avoids some critical problems with extensive-form stochastic supermodular games that can arise in situation where there are is a finite number of players with private information. We refer the reader to Echenique (2004a); Vives (2009) and a recent discussion of these issues in Mensch (2020).

Further, and of critical importance we are able to provide new comparative statics and comparative dynamics results. When doing so we are able to extend and complement the recent comparative statics results (Acemoglu and Jensen, 2015; Light and Weintraub, 2019) from a class of mean field games and stationary equilibria to distributional games and equilibrium dynamics.

Moreover, given the recent interest in the economic literature on oblivious equilibria, mean-field games equilibrium, as well as stationary equilibrium, we are able to develop a rigourous set of new results concerning the structure of "idealized limits" of our class of stochastic games, where the set of equilibria in finitely many player versions of the game is compared with its counterpart in the game with a measure space of players. This allows us to make precise the sense in which our large stochastic supermodular game with a measure space of players is related to similar games with a finite number of players. This latter result is particularly useful in applications, as in some settings, large dynamic economies are used as a tool to characterize the structural properties of finite player models.

Summarizing, the contribution of our paper is hence the following: 1. we provide a new, dynamic version of the exact law of large numbers. 2. We next define a concept of Markov stationary equilibrium, that involves a equilibrium action-state distribution and a law of motion of aggregate distributions. 3. We prove existence of such equilibrium under a new set of assumptions via constructive methods. 4. We also provide a new monotone comparative statics and dynamics results as well as 5. provide a new behavioral foundation of our equilibrium concept studying approximation of our game and equilibrium via its small counterparts.

We continue this discussion in more details by studying our motivating example in section 2. It introduces the class of games we investigate in this paper and shows the main problems we solve in the paper. The detailed literature discussion placing our contribution in the context of some recent studies is postponed till section 8 on related literature. The rest of the paper is organized as follows. In section 3 we state the main measure-theoretic definitions, new fixed point comparative statics results, as well as provide our version of dynamic law of large numbers necessary in the analysis of our game. Section 4 presents our model and states the main result on equilibrium existence. Our monotone comparative dynamics result is presented in section 5. Next, in section 6, we present our approximation, i.e. an idealized limit result relative to $\epsilon-$equilibrium of the large but finite anonymous stochastic game. Section 7 includes a discussion of applications. We finish the paper with the Appendix with proofs.

2 Motivating example

We begin our discussion with a simple example that captures the main features of the class of games we investigate in this paper. Suppose there is a continuum of players distributed over the interval $\Lambda = [0, 1]$. In each time period ($n = 1, \dots, \infty$), a player is endowed with capital (wealth) $t \in T = [0, 1]$. The capital can either be used for consumption c or investment a in production of future capital. Thus, we have $t = c + a$.

Future capital is determined by a stochastic technology q. Whenever a units of capital is invested, then the cumulative probability of attaining capital t' in the following period is $q(t'|a)$. We assume that $q(\cdot|a)$ is first order stochastically increasing in a.[5]

In our model, capital and consumption determine individual *status*, i.e., the player's relative position among others. Specifically, in each period every agent interacts with all the remaining players in the population. Whenever a player with capital t and consumption $c \in [0, t] \subset A := [0, 1]$ encounters an individual of status t' consuming c' the former receives the utility of $m(t-t')+w(c-c')$, where both m and w are continuous, increasing, and concave. Thus, the player benefits from meeting agents of lower capital/consumption, which may come from the feeling of superiority. Conversely, interacting with someone of a higher status/consumption makes them feel inferior.

Given a distribution μ over capital and investments (t', a') in the whole population, the payoff in a single period of an individual endowed with capital t investing a is

$$\int_{A\times T} \Big[m(t - t') + w(t - a - t' + a')\Big]\mu(da' \times dt').$$

Therefore, there is a non-trivial trade-off between current status that can be obtained via higher consumption today, and future status that follows from higher capital tomorrow

[5] In particular, we may allow for the transition to be deterministic, i.e., $q(t'|a) = 1$ if $f(a) \le t'$, and $q(t'|a) = 0$ otherwise, where f is some monotone production function.

financed through higher investments today. For this particular reason, we are interested in studying the dynamics of such an interaction.

In order to specify payoffs of the individual in the infinite horizon game, suppose that all other players play a symmetric strategy $\sigma : T \to A$ that maps the current capital t to the level of investments $\sigma(t)$. Moreover, let τ_n denote the distribution over capital levels in period n. In particular, this allows to evaluate the distribution of capital and investment levels in period n by $\mu_n(\cdot) = \tau_n\big(id_T(\cdot), \sigma(\cdot)\big)^{-1}$.

Given a strategy σ of other players, a sequence of capital distributions $\{\tau_n\}$, and the corresponding distributions $\{\mu_n\}$ of capital and investments, the payoff of a player endowed with an initial capital t_0 is

$$\max_{\{a_t\}} \left\{ (1-\beta) E_{t_0,\{\tau_t\}} \left[\sum_{n=0}^{\infty} \beta^n \int_{A\times T} \Big[m(t-t') + w(t-a-t'+a') \Big] \mu_n(da' \times dt') \right] \right\},$$

where $\beta \in (0,1)$ is a discount factor and the expectation $E_{t_0,\{\tau_t\}}$ is taken with respect to the realization of sequences $\{t_n\}$. We are interested in investigating equilibrium behaviour in such games, where the distributions of types and actions are determined by strategies of individuals and the stochastic transition, while individuals form beliefs consistent with the law of motion of private capitals and strategy of all players.

Notice that, the above problem is sequential. Although private capital draws could be random, given the sequence $\{\mu_n\}$ player's problem is a standard decision problem (that is Markov on the state space involving μ). However, this requires that players know the true evolution of capital distributions in the population. We take a different approach by allowing players to have a macro belief, i.e., a function that specifies the transition between (capital-investment) distributions: $\mu_{n+1} = \Phi(\mu_n)$. Together with the initial distribution μ_0, this allows to conjecture $\{\mu_n\}$.

The game presented above fits into the existing literature of stochastic games with the large number of agents. Nevertheless, it possesses features that we find particularly worth investigating, and which can not be addressed using the existing results.

First of all, the game is set up in a way which requires for the players to know the *whole distribution* of capital (types) and investments (actions) in the population in each period. In order to evaluate the payoff of a player in each period n, it is not sufficient to substitute the distributions τ_n or μ_n with their statistic or aggregates. In addition, knowledge of the whole distribution is necessary to determine players' beliefs about the distribution of capital in the following period.

Second of all, we are interested in determining dynamics of the above game and how the distributions of capital (types) and investments (actions) evolves across periods; rather than steady states and/or invariant distributions. To answer such questions one has to be able to compute (or at least approximate) equilibria of the dynamic game. For this reason, we introduce an equilibrium concept called *Markov Stationary Equilibrium*, that consists of a distributional equilibrium μ^* and a self-confirming macro belief Φ^*. Per the latter. The perceived (deterministic) dynamics of types distribution is required to be consistent with the actual one (governed by transition q and initial distribution of capital τ_0). We use a version of the dynamic (and exact) law of large numbers, that allows us to associate probabilities q with empirical population distributions on T. Although we can evaluate invariant distributions over types and actions in this games, it is not our main focus.

Third of all, ours is a game with strategic complementarities. Indeed, as capital and investments of other players grow and the distribution μ_n shifts with respect to the first order stochastic sense, each player finds it optimal to increase their own investments as well. Conversely, it is optimal to reduce investment as actions of other players decrease. Moreover, strategic complementarities are also present across periods. Higher (anticipated) distribution of types tomorrow incentivises each player to invest today to increase chances of obtaining higher capital (status) in the next period. Whether such dynamic complementarities are present or not, depends critically on two reinforcing conditions: increasing differences between private type (status) and anticipated population distribution the next period; and agents forming monotone beliefs, i.e. expecting higher population distribution tomorrow whenever faced with higher distribution in the current period. The order-theoretic structure of the game is particularly helpful in providing algorithms that allow to approximate paths of distributions generated by equilibrium dynamics.

From the point of view of econometric applications, it is important to determine comparative dynamics in such models. That is, analyze how changes in parameters of the game (e.g. discount factor, preference or technology parameters or the initial distribution of capital levels τ_0) affect the paths of equilibrium distributions $\{\mu_n^*\}$ (as implied by equilibrium μ^* and Φ^*). Importantly, since μ_n^* is defined over the space of types *and* actions, one needs to provide an equilibrium comparative statics result for appropriate spaces of multidimensional distributions.

Finally, for economic applications, large games serve often as an approximation of interactions among a large but finite number of individuals. Therefore, it is crucial to determine whether there is a sensible notion according to which equilibria of a game with a continuum of players would allow to approximate equilibria of large but finite interactions. This would advocate our Markov Stationary Equilibirum as an idealized limit equilibrium and a behavioral foundation of players's actions in large but finite populations.

In the remainder of this paper we address the above questions by exploiting the particular structure of the motivating example. Additionally, in section 8 we discuss that the existing methods are not applicable for studying such environments.

3 Preliminaries

In this section we introduce some mathematical notions in measure and lattice theory that will be employed in our main analysis.

3.1 On Fubini extensions and the law of large numbers

We begin by defining the notion of *super-atomless* probability space.[6] Let $(\Lambda, \mathcal{L}, \lambda)$ be a probability space. For any $E \in \mathcal{L}$ such that $\lambda(E) > 0$, let $\mathcal{L}^E := \{E \cap E' : E' \in \mathcal{L}\}$ and λ^E be the re-scaled measure from the restriction of λ to $\mathcal{L}^E$. Let $\mathcal{L}^E_\lambda$ be a set of equivalence classes of sets in $\mathcal{L}^E$ such that $\lambda^E(E_1 \triangle E_2) = 0$, for $E_1, E_2 \in \mathcal{L}^E$.[7] We endow the space with metric $d^E : \mathcal{L}^E_\lambda \times \mathcal{L}^E_\lambda \to \mathbb{R}$, given by $d^E(E_1, E_2) := \lambda^E(E_1 \triangle E_2)$.

Definition 1 (Super-atomless space)**.** A probability space $(\Lambda, \mathcal{L}, \lambda)$ is super-atomless if

[6] The following definition is by Podczeck (2009, 2010), which we find to be the most convenient for our purposes. However, equivalent definitions are provided in Hoover and Keisler (1984), who call such spaces $\aleph_1$-*atomless*, and Keisler and Sun (2009), who dubbed such spaces *rich*.

[7] We denote $E_1 \triangle E_2 := (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$.

for any $E \in \mathcal{L}$ with $\lambda(E) > 0$, $(\mathcal{L}_{\lambda}^{E}, d^{E})$ is non-separable.

Classical examples of super-atomless probability spaces include: $\{0,1\}^I$ with its usual measure when I is an uncountable set; the product measure $[0,1]^I$, where each factor is endowed with Lebesgue measure and I is uncountable;[8] subsets of these spaces with full outer measure when endowed with the subspace measure, or an atomless Loeb probability space. Furthermore, any atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure (see Podczeck, 2009).

Given a probability space $(\Lambda, \mathcal{L}, \lambda)$, a collection of random variables $(X_\alpha)_{\alpha\in\Lambda}$ is *essentially pairwise independent*, if for $(\lambda \otimes \lambda)$-almost every $(\alpha, \alpha') \in \Lambda \times \Lambda$, random variables X_α and $X_{\alpha'}$ are independent. For any set Ω and $E \subseteq (\Lambda \times \Omega)$, we denote its sections by $E_\alpha := \big\{\omega \in \Omega : (\alpha, \omega) \in E\big\}$ and $E_\omega := \big\{\alpha \in \Lambda : (\alpha, \omega) \in E\big\}$, for any $\alpha \in \Lambda$ and $\omega \in \Omega$. Similarly, for any function f defined over $\lambda \times \Omega$, let f_α and f_ω denote the section of f for a fixed α, ω, respectively. Consider the following definition.

Definition 2 (Fubini extension)**.** The probability space $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a *Fubini extension* of the natural product of probability spaces $(\Lambda, \mathcal{L}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ if:

(i) $\mathcal{L} \boxtimes \mathcal{F}$ includes all sets from $\mathcal{L} \otimes \mathcal{F}$;

(ii) for an arbitrary set $E \in \mathcal{L} \boxtimes \mathcal{F}$ and $(\lambda \otimes P)$-almost every $(\alpha, \omega) \in \Lambda \times \Omega$, the sections E_α and E_ω are $\mathcal{F}$- and $\mathcal{L}$-measurable, respectively, while

$$(\lambda \boxtimes P)(E) \;=\; \int_\Omega \lambda(E_\omega) P(d\omega) \;=\; \int_\Lambda P(E_\alpha)\lambda(d\alpha).$$

A Fubini extension is *rich*, if there is a $(\mathcal{L}\boxtimes\mathcal{F})$-measurable function $X : \Lambda \times \Omega \to \mathbb{R}$ such that the random variables $(X_\alpha)_{\alpha\in\Lambda}$ is essentially pairwise independent and the random variable X_α has the uniform distribution over $[0,1]$, for λ-almost every $\alpha \in \Lambda$.

Existence of the rich Fubini extension is proven in Proposition 5.6 of Sun (2006), for $\Lambda = [0,1]$. However, Proposition 6.2 of the same paper shows that $\mathcal{L}$ can not be a collection of Borel subsets of Λ. Indeed, following Podczeck (2010), it is necessary and

[8] Indeed, Maharam's theorem shows that the measure algebra of every super-atomless probability spaces must correspond to the countable convex combination of such spaces. See Maharam (1942).

sufficient for the space to be super-atomless. Moreover, without loss, one may assume the random variables $(X_\alpha)_{\alpha\in\Lambda}$ to be independent, rather than pairwise-independent.

A *process* is said to be $(\mathcal{L}\boxtimes\mathcal{F})$-measurable function with values in a Polish space. For any process f and set $E\in\mathcal{L}$ such that $\lambda(E)>0$, we denote the restriction of f to $E\times\Omega$ by f^E. Naturally, $\mathcal{L}^E\boxtimes\mathcal{F}:=\big\{W\in\mathcal{L}\boxtimes\mathcal{F}: W\subseteq E\times\Omega\big\}$, and $(\lambda^E\boxtimes P)$ is a probability measure re-scaled from the restriction of $(\lambda\boxtimes P)$ to $(\mathcal{L}^E\boxtimes\mathcal{F})$. The following version of (exact) Law of Large Numbers is by Sun (2006).

Theorem 1 (Law of Large Numbers). *Suppose that f is a process from a rich Fubini extension $(\Lambda\times\Omega,\mathcal{L}\boxtimes\mathcal{F},\lambda\boxtimes P)$ to some Polish space. Then, for all $E\in\mathcal{L}$ such that $\lambda(E)>0$ and P-almost every $\omega\in\Omega$, we have $\lambda(f^E_\omega)^{-1}=(\lambda^E\boxtimes P)(f^E)^{-1}$.*[9]

3.2 Lattices, chains, and fixed points

A *partial order* $\geq_X$ over a set X is a reflexive, transitive, and antisymmetric binary relation. A partially ordered set, or a *poset*, is a pair $(X,\geq_X)$ consisting of a set X and a partial order $\geq_X$. Whenever it causes no confusion, we denote $(X,\geq_X)$ with X.

For any x, $x'\in X$, their *infimum*, or the greatest lower bound, is denoted by $x\wedge x'$, and their *supremum*, or the least upper bound, by $x\vee x'$. The poset X is a *lattice* if for any x, $x'\in X$ both $x\wedge x'$ and $x\vee x'$ belong to X. Set A is a *sublattice* of X, if $A\subseteq X$ and it is a lattice with the induced order, with $x\wedge x'$ and $x\vee x'$ defined with $\geq_X$.[10]

For any subset A of a poset X, we denote the *supremum* and *infimum* of A by $\bigvee A$ and $\bigwedge A$, respectively.[11] A lattice X is *complete*, if each both $\bigvee A$ and $\bigwedge A$ belong to X, for any $A\subseteq X$. We define a *complete sublattice* analogously.

A *chain* is a completely ordered poset. A poset X is (countably) *lower chain complete* if any (countable) chain $A\subseteq X$ has its infimum in X. The poset is (countably) *upper*

[9] Given the probability space $(\Lambda,\mathcal{L},\lambda)$ and a measurable function $f:\Lambda\to Y$, we denote measure $\lambda f^{-1}(U):=\lambda\big(\{\alpha\in A: f(\alpha)\in U\}\big)$, for any measurable subset U of Y.

[10] A basic example of a lattice is the Euclidean space $\mathbb{R}^\ell$ endowed with the natural product order $\geq$, i.e., we have $x'\geq x$ if $x'_i\geq x_i$, for all $i=1,\dots,\ell$. In this case, we have $x\wedge x'$ and $x\vee x'$ are given by $(x\wedge x')_i=\min\{x_i,x'_i\}$ and $(x\vee x')_i=\max\{x_i,x'_i\}$, for all $i=1,\dots,\ell$.

[11] This is to say that, $\bigvee A$ is the least element of X such that $\bigvee A\geq a$, for all $a\in A$. Clearly, by definition, we have $x\vee x'=\bigvee\{x,x'\}$. We define $\bigwedge A$ analogously.

chain complete if any such chain has its supremum in X.

If X and Y are posets, then a function $f : X \to Y$ is *increasing* (*decreasing*) if $x' \geq_X x$ implies $f(x') \geq_Y (\leq_Y) f(x)$. Below, we present a useful generalization of Tarski's fidex point theorem, that builds on Theorem 9 in Markowsky (1976). See the Appendix for the proof.

Theorem 2. *Let $(X, \geq)$ be a lower chain complete poset with the greatest element. The set of fixed points of an increasing function $f : X \to X$ is a nonempty lower chain complete poset. Moreover, its greatest fixed point is given by $\bigvee \{x \in X : f(x) \geq x\}$.*

Given posets X, Y, function $f : X \to Y$ is *monotone sup-preserving* if, for any increasing sequence $(x_k)_{k\in\mathbb{N}}$, we have $f\big(\bigvee(x_k)_{k\in\mathbb{N}}\big) = \bigvee \big(f(x_k)\big)_{k\in\mathbb{N}}$. It is *monotone inf-preserving* if $f\big(\bigwedge(x_k)_{k\in\mathbb{N}}\big) = \bigwedge \big(f(x_k)\big)_{k\in\mathbb{N}}$, for any decreasing sequence $(x_k)_{k\in\mathbb{N}}$.

We conclude with a useful theorem, that extends the classic fixed point comparative statics results of Veinott (1992) and Topkis (1998) to countably chain complete posets. It is based on the Tarski-Kantorovich theorem (see Dugundji and Granas, 1982, Theorem 4.2). See Theorem 8 in Balbus et al. (2015c) for a proof.

Theorem 3. *Let X be a lower countably chain complete poset with the greatest element, and Θ be a poset. For any function $f : X \times \Theta \to X$ and $\theta \in \Theta$ such that f_θ is increasing and monotone inf-preserving over X, the greatest fixed point of f_θ is given by $\bigwedge \big(f_\theta^n(\bigvee X)\big)_{n\in\mathbb{N}}$.*[12] *In addition, if f is increasing in the product order and f_θ is monotone inf-preserving, for all $\theta \in \Theta$, then the greatest fixed point is increasing over Θ.*[13]

This generalization becomes critical in proving our new comparative dynamics results on a poset of equilibrium distributions and equilibrium law of motions in section 5.

[12] By f^n we denote the nth composition of f, i.e., $f^n = f \circ f \circ \ldots \circ f$ (n times).

[13] Analogously, if X is upper countably chain complete with the least element, f_t is increasing and monotone sup-preserving, the least fixed point of f_t is given by $\bigvee \big(f_t^n(\bigwedge X)\big)_{n\in\mathbb{N}}$. If f is increasing in the product order and f_t is monotone sup-preserving, for all $t \in T$, then the fixed point is increasing.

4 Large stochastic games with complementarities

In this section we introduce our general model of large stochastic games with complementarities and discuss equilibrium existence in this framework.

Consider the following stochastic game with infinite horizon. Let $(\Lambda, \mathcal{L}, \lambda)$ be a probability space of players, which we assume to be super-atomless. In each period $n = 1, 2, \ldots$ a player is endowed with a private type $t \in T \subseteq \mathbb{R}^p$, where T is compact. Let $\mathcal{T}$ denote the Borel σ-algebra corresponding to T. Given a distribution τ of types of all other players, the player can choose an action a that belongs to some set $\tilde{A}(t, \tau) \subseteq A$, where $A \subseteq \mathbb{R}^k$ is a compact space of all conceivable actions, endowed with the Borel σ-field $\mathcal{A}$. We endow both T and A with the natural product partial order $\geq$.

Let $\mathcal{M}$ be a set of probability measures on $\mathcal{T} \otimes \mathcal{A}$ and $\mathcal{M}_T$ denote the set of probability measures on $\mathcal{T}$. We endow both spaces with the induced first order stochastic dominance order.[14] The player's payoff in a particular period is determined by a bounded function $r : T \times A \times \mathcal{M} \to \mathbb{R}$, taking values $r(t, a, \mu)$, given the private type t, the action a, and the probability measure μ over types and actions of all players.

In this paper we investigate dynamic games in which private types of players are determined stochastically in each period. The transition probability is represented by a $q : T \times A \times \mathcal{M} \to \mathcal{M}_T$, that assigns the current type t of the player, her action a, and measure μ of types/actions of all players, to a probability measure $q(\cdot|t, a, \mu)$ over $\mathcal{T}$ that determines the probability of the player's type in the following period.

4.1 Problem of the player

In order to properly define the problem of a player in our dynamic setting, it is fundamental to specify how the individual is forming beliefs about types of all players in the game, based on the current distribution of types and strategies of other players.

Assumption 1. *We assume the following.*

[14] For any two probability measures μ and ν over Y, we say that μ dominates ν in the first order stochastic sense, if $\int f(y)\mu(dy) \geq \int f(y)\nu(dy)$, for any measurable, bounded function $f : Y \to \mathbb{R}$ that increases with respect to the corresponding ordering $\geq_Y$.

(i) For all $\tau \in \mathcal{M}_T$, correspondence $\tilde{A}(\cdot, \tau)$ is measurable and compact-valued.

(ii) For all $\mu \in \mathcal{M}$, function $(a,t) \to q(\cdot|t,a,\mu)$ is Borel-measurable.

The super-atomless probability space of players and Assumption 1 guarantee that the transition of private signals implies no aggregate uncertainty. Specifically, whenever the current distribution on types and actions of all players is μ, then the measure of players with privates types in some measurable set S in the following period is

$$\phi(\mu)(S) \;:=\; \int_{T\times A} q(S|t,a,\mu)\mu(dt \times da). \tag{1}$$

Theorem 4. *Under Assumption 1, there is a sampling probability space $(\Omega, \mathcal{F}, P)$ and a rich Fubini extension $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that, for any sequence $\sigma = (\sigma_n)_{n\in\mathbb{N}}$, where $\sigma_n : T \times \mathcal{M}_T \to A$, any initial state $t \in T$, and initial distribution $\tau \in \mathcal{M}_T$, there is sequence of $(\mathcal{L} \boxtimes \mathcal{F})$-measurable functions $X_n : \Lambda \times \Omega \to T$ satisfying:*

(i) For all $n \in \mathbb{N}$, the random variables $\big((X_n)_\alpha\big)_{\alpha\in\Lambda}$ are (conditional on the history) essentially pairwise independent.[15]

(ii) For all $n \in \mathbb{N}$ and P-almost every $\omega \in \Omega$, we have

$$\tau_n \;:=\; \lambda(X_n)_\omega^{-1} \;=\; (\lambda \boxtimes P)X_n^{-1};\text{[16]}$$

as well as $\mu_n(\sigma) := \lambda\big(X_n, \sigma_n(X_n, \tau_n)\big)_\omega^{-1} = (\lambda \boxtimes P)\big(X_n, \sigma_n(X_n, \tau_n)\big)^{-1}$.

(iii) The distribution of the random variable $(X_{n+1})_\alpha$, conditional on $\big((X_j)_\alpha\big)_{j\le n}$, is given by $q\big(\cdot\,|(X_n)_\alpha, \sigma_n\big((X_n)_\alpha, \mu_n(\sigma)\big), \mu_n(\sigma)\big)$.

We postpone the proof until the Appendix. As pointed out in the introduction, this theorem may be of independent interest and can be applied to dynamic settings with micro-level idiosyncratic risk that evolves dynamically but generates no aggregate risk.

We can now define the problem of the player. Let H_∞ be a set of all histories $(t_n, a_n, \tau_n)_{n\in\mathbb{N}}$, where $a_n \in \tilde{A}(t_n, \tau_n)$. By H_n we denote the set of histories up to time n, that is $H_n := \big\{(t_j, a_j, \tau_j)_{j=1}^n : a_j \in \tilde{A}(t_j, \tau_j)\big\}$. A strategy is a sequence of functions

[15] Recall that $(X_n)_\alpha$ denotes the section of X_n, for a fixed $\alpha \in \Lambda$.

[16] To clarify our notation, recall that we denote $\lambda(X_n)_\omega^{-1} = \lambda\big(\{\alpha \in \Lambda : (X_n)_\omega(\alpha) \in U\}\big)$, for any $U \in \mathcal{T}$. We define the remaining measures analogously.

$(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n : H_{n-1} \times T \times \mathcal{M}_T \to A$ is Borel-measurable in $(t_1, t_2, \ldots, t_n) \in T^n$, and $\sigma_n(h_{n-1}, t_n, \tau_n) \in \tilde{A}(t_n, \tau_n)$, where $H_0 = \{(t_0, a_0, \tau_0)\}$, for some initial t_0, a_0, τ_0.

A strategy profile is called *Markov*, if in each period n it depends on the current state (t, τ) only. Strategy is stationary, if it is time-*invariant*. By Theorem 4, any initial private state t, public state τ, a Markov strategy σ and a Markov strategy profile σ' of other players induce the unique private measure $P_{t,\tau}^{\sigma,\sigma'}$ on histories of the game[17]. This implies that the objective of the player is to maximize

$$R\big(t, (\sigma, \sigma'), \tau\big) := \\ (1-\beta)\mathbb{E}_{t,\tau}^{\sigma,\sigma'}\left[r(t, \sigma_1(t,\tau), \mu_1(\sigma')) + \sum_{n=2}^{\infty} \beta^{n-1} r\big(t_n, \sigma_n(t_n, \tau_n), \mu_n(\sigma')\big)\right], \quad (2)$$

where $\beta \in (0,1)$ is a discount factor and $\mathbb{E}_{t,\tau}^{\sigma,\sigma'}$ is the expectation induced by $P_{t,\tau}^{\sigma,\sigma'}$ and $\mu_n(\sigma') := \tau_n(id_T, \sigma'(\cdot, \tau_n))^{-1}$.

Assumption 2. *The payoff function $r : T \times A \times \mathcal{M} \to \mathbb{R}$:*

(i) is continuous over $T \times A$, for all $\mu \in \mathcal{M}$;

(ii) is monotone sup- and inf-preserving over $\mathcal{M}$, for all $(t, a) \in T \times A$;

(iii) is increasing over T, for all $(a, \mu) \in A \times \mathcal{M}$;

(iv) is supermodular over A, for all $(t, \mu) \in T \times \mathcal{M}$;[18]

(v) has increasing differences in $\big(a, (t, \mu)\big)$, and (t, μ), for all $a \in A$.[19]

Assumption 3. *The transition kernel $q : T \times A \times \mathcal{M} \to \mathcal{M}_T$:*

(i) is continuous over $T \times A$, for all $\mu \in \mathcal{M}$;

(ii) is monotone sup- and inf-preserving over $\mathcal{M}$, for all $(t, a) \in T \times A$;

(iii) is stochastically increasing in (t, a, μ);[20]

[17] For $\lambda-$a.a. $\alpha \in \Lambda$, $(X_n^\alpha)_{n\in\mathbf{N}}$ has the same distribution, which exists by the Ionescu-Tulcea Theorem (Dynkin and Yushkevich, 1979; see also Theorem 15.26 in Aliprantis and Border, 2006).

[18] A function $f : X \to \mathbb{R}$ over a lattice X is *supermodular* if $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$.

[19] Let X, T be posets. A function $f : X \times T \to \mathbb{R}$ has *increasing differences* in (x, t) if $f(x', t') - f(x, t') \geq f(x', t) - f(x, t)$, for any $x' \geq_X x$ and $t' \geq_T t$.

[20] That is, for any (t', a', μ') dominating (t, a, μ) in the corresponding product order, distribution $q(\cdot|t', a', \mu')$ first order stochastically dominates $q(\cdot|t, a, \mu)$.

(iv) is stochastically supermodular over A, for all $(t,\mu) \in T \times \mathcal{M}$;[21]

(v) has stochastically increasing differences in $\big(a,(t,\mu)\big)$ and (t,μ), for all $a \in A$.[22]

Assumption 4. *The feasible action correspondence $\tilde{A} : T \times \mu_T \to A$:*

(i) is upper hemi-continuous;

(ii) has values that are compact lattices;

(iii) is increases with t in the sense of set inclusion;[23]

(iv) satisfies strict complementarities.[24]

Most of these assumptions are standard for a class of dynamic games with complementarities (see Curtat, 1996 or Balbus et al., 2014), with some additional monotonicity requirements for the payoff function and the transition probability. As we show in the latter part of the paper, these are indispensable to preserve strategic complementarities across periods in the extensive formulation of the game under Markovian strategies (unlike in Curtat, 1996). One example of payoff r, often assumed in applied work, that satisfies increasing differences in (t,μ) is separable and takes the form $r(t,a,\mu) = u(t,a) + v(a,\mu)$. Alternatively, our framework encompasses linear social interaction model applied recently in econometric studies by Blume et al. (2015); Kline and Tamer (2020); Kwok (2019).[25]

To exemplify the stochastic transition satisfying our assumptions, consider

$$q(\cdot|t,a,\mu) \;:=\; g(t,a,\mu)\rho(\cdot) + \big(1 - g(t,a,\mu)\big)\nu(\cdot),$$

where $g : T \times A \times \mathcal{M} \to [0,1]$ is supermodular in a, for all (t,μ); has increasing differences in $\big(a,(t,\mu)\big)$ and (t,μ), for all a; and is increasing in (a,t,μ); while ρ, ν are probability

[21] This is to say that, function $a \to \int f(t')q(dt'|t,a,\mu)$ is supermodular, for any $\mathcal{T}$-measurable, bounded, and increasing function $f : T \to \mathbb{R}$.

[22] That is, function $g(t,a,\mu) := \int f(t')q(dt'|t,a,\mu)$ has increasing differences in $\big(a,(t,\mu)\big)$ and (t,μ), for any $a \in A$ and any $\mathcal{T}$-measurable, bounded, and increasing function $f : T \to \mathbb{R}$.

[23] That is, if $t \geq t$, then $\tilde{A}(t,\tau) \subseteq \tilde{A}(t',\tau)$.

[24] $\tilde{A}$ satisfies strict complementarities iff for any $t' \geq t$, $\tau' \geq \tau$ and $a \in \tilde{A}(t,\tau')$ and $a' \in \tilde{A}(t',\tau)$ we have $a \wedge a' \in \tilde{A}(t,\tau)$ and $a \vee a' \in \tilde{A}(t',\tau')$.

[25] In these papers the payoffs take the general form of $r(t,a,\mu) = \big[\beta_1 t + \beta_2 \int_T f_1(t,t')t' \,\mu_T(dt')\big]a - \frac{1}{2}a^2 - \frac{\beta_3}{2}\big[a - \beta_4 \int_{T\times A} f_2(t')a' \,\mu(dt' \times da')\big]^2$, for some positive β_i's and linear, positive, and increasing functions f_1, f_2 that weight social interaction by measuring contextual and peer network effects respectively. Our computable monotone comparative statics/dynamics results developed in the following section can be useful for developing appropriate estimators to test equilibrium distributions in empirical models.

distributions over $\mathcal{T}$ such that ρ first order stochastically dominates ν. Such assumption was introduced by Curtat (1996), Amir (2002) and was successfully applied in some recent works of the authors. We refer the reader to the related paper Balbus et al. (2013), for a detailed discussion of the nature of these assumptions. Here we only state the following remark particularly useful is applications.

Remark. Our assumption on a stochastic transition generally implies that transition can not be deterministic. Indeed, supermodularity and increasing differences of the integrand $\int f(t')q(dt'|t,a,\mu)$ must hold for any integrable and monotone function f, which is not satisfied by general deterministic transitions. However, if $A \subseteq \mathbb{R}$ then the deterministic transition given by $q(S|t,a,\mu) = 1$ if $g(a) \in S$, and $q(S|t,a,\mu) = 0$ otherwise, for some continuous and increasing function[26] $g : A \to T$, satisfies our assumption trivially. In fact, such transition functions are often assumed in applications.

Remark. Whenever the action space A is one-dimensional and the transition function q depends only on action a, all of our results remain true even if the payoff function r is *not* increasing with respect to the type and the correspondence $\tilde{A}$ is *not* increasing in t with respect to set-inclusion (strict complementarities of $\tilde{A}$ are still crucial). This follows directly from our constructive argument presented in Section 4.3. Again, this is often the case in the applications we consider.

An important feature of our framework is that the original problem in (2) admits a recursive representation. Specifically, suppose that function $\Phi : \mathcal{M} \to \mathcal{M}$ determines the next period distribution $\Phi(\mu)$ over types and actions of all players based on the current distribution μ. In particular, given our observations from (1), the marginal of $\Phi(\mu)$ over T must be

$$\phi(\mu)(S) \;=\; \int_{T\times A} q(S|t,a,\mu)\mu(dt \times da),$$

for any measurable set S. Moreover, we restrict our attention to functions that are

[26] Note this implies that transition is independent of μ and t.

monotone inf-preserving. Formally, denote the set of all such functions by

$$\mathcal{D} := \Big\{\Phi : \mathcal{M} \to \mathcal{M} : \Phi \text{ is increasing and monotone} \\ \text{inf-preserving and } \mathrm{marg}_T(\Phi(\mu)) = \phi(\mu), \text{ for any } \mu \in \mathcal{M}\Big\}. \quad (3)$$

We denote $\mu_T := \mathrm{marg}_T(\mu)$. Endow $\mathcal{D}$ with natural componentwise order.

Using standard arguments, we show in the reminder of this section that for any initial distribution μ over types and actions of players, and functions Φ, the value corresponding to problem (2) satisfies:[27]

$$v^*(t, \mu; \Phi) = \max_{a \in \tilde{A}(t, \mu_T)} \left\{(1-\beta)r(t, a, \mu) + \beta \int_T v^*(t', \Phi(\mu); \Phi) q(dt'|t, a, \mu)\right\}. \quad (4)$$

Note, that for a given μ_0 and perceived law of motion Φ, player's problem is a Markov decision problem, with uncertainty about private signal t only. That is, sequence of aggregate distributions $(\mu_n)_{n \in \mathbb{N}}$ is deterministic. This is a consequence of the exact law of large number stated above. By standard arguments we can show that the policy correspondence is Markov with natural state space including t and μ. Definition of equilibrium requires, however, additional consistency between perceived law of motion Φ and the policy correspondence of the above Markov decision problem. Observe (more on that in a moment) that Φ also specifies beliefs players have on continuation paths of the game. Hence, when writing $v^*(t, \mu; \Phi)$ we stress that the value function and corresponding policy functions depend on beliefs (compare with construction of Markov equilibrium in Kalai and Shmaya, 2018 for a large but finite repeated game with unknown fundamentals).

4.2 Markov stationary (distributional) equilibria

We are ready to specify the notion of equilibrium in our game.

Definition 3 (Markov Stationary Equilibrium)**.** A pair $(\mu^*, \Phi^*) \in \mathcal{M} \times \mathcal{D}$ is a Markov Stationary Equilibrium (MSE) whenever:

[27] Alternatively, one may use t, τ as state variables of the value function and then compose τ and a strategy $\sigma : T \to A$ to obtain μ. In such, best response specification, strategy σ would be an additional parameter of the value function.

(i) there is a function v^* such that, for any $\mu \in \mathcal{M}$, and λ-almost every $t \in T$,

$$v^*(t, \mu; \Phi^*) = \max_{a \in \tilde{A}(t, \mu_T)} \left\{ (1-\beta) r(t, a, \mu) + \beta \int_T v^*(t', \Phi^*(\mu); \Phi^*) q(dt'|t, a, \mu) \right\};$$

(ii) there is a measurable selection σ_{μ,Φ^*} of correspondence $\Sigma_{\mu,\Phi^*} : T \rightrightarrows A$, given by

$$\Sigma_{\mu,\Phi^*}(t) := \arg \max_{a \in \tilde{A}(t, \mu_T)} \left\{ (1-\beta) r(t, a, \mu) + \beta \int_T v^*(t', \Phi^*(\mu); \Phi^*) q(dt'|t, a, \mu) \right\},$$

where $\mu^* = \mu_T^* \big(id_T, \sigma_{\mu^*,\Phi^*}\big)^{-1}$ and $\Phi^*(\mu) = \phi(\mu) \big(id_T, \sigma_{\Phi^*(\mu),\Phi^*}\big)^{-1}$ for any $\mu \in \mathcal{M}$.[28]

Our notion of equilibrium involves equilibrium distribution μ^* and Markov transition function Φ^*. It also determines the equilibrium policy $\sigma_{\mu,\Phi^*} : T \to A$ (or equivalently $\sigma^* : T \times \mathcal{M}_T \to A$). Therefore, this notion of dynamic equilibrium is not the stationary equilibrium (e.g., in the sense of Bewley). Condition (i) is a standard Bellman equation that characterizes players best reply correspondences, while (ii) involves two side consistency conditions required in MSE. The first part of conditions (ii) requires $\mu^* = \mu_T^* \big(id_T, \sigma_{\mu^*,\Phi^*}\big)^{-1}$, which is a distributional equilibrium condition for a given equilibrium law of motion. It requires that the distribution generated by the best response selection σ_{μ^*,Φ^*} to μ^* is μ^*. Observe that, although it is a static condition, it involves a *dynamic* interaction of the players with all future periods distributions (generated by the law of motion Φ^*) and summarized by the value $v^*(\cdot, \Phi^*(\mu^*); \Phi^*)$.

The second part of condition (ii) requires $\Phi^*(\mu) = \phi(\mu) \big(id_T, \sigma_{\Phi^*(\mu),\Phi^*}\big)^{-1}$, which is a dynamic condition guaranteeing that the perceived and actual laws of motions for aggregate distributions coincide. Specifically, this condition requires that the distribution on $T \times A$ *perceived* by players for the next period is the same as the distribution for the next period that is generated by the best-response selection σ.[29]

Markov transition Φ^* specifies common beliefs that players use to determine future paths of distributions. In macroeconomic literature on recursive equilibrium, such beliefs are called *rational*. These are beliefs on both "on" and "off" equilibrium paths of distributions $(\mu_n)_{n \in \mathbb{N}}$, which is guaranteed by $\Phi^*(\mu) = \phi(\mu)(id_T, \sigma_{\Phi^*(\mu),\Phi^*})^{-1}$, for any $\mu \in \mathcal{M}$.

[28] Thus $\mu^*(S) = \mu_T^*\big(\{t \in T : (t, \sigma_{\mu^*,\Phi^*}(t)) \in S\}\big)$ and $\Phi^*(\mu)(S) = \phi(\mu)\big(\{t \in T : (t, \sigma_{\Phi(\mu^*),\Phi^*}(t)) \in S\}\big)$.

[29] Since we work with no aggregate uncertainty, we do not require that Φ^* is measurable.

The appropriate conditions guaranteeing that the value function has increasing differences in both arguments (i.e., t, μ) and that the transition Φ^* is monotone, allows us to avoid problems in characterizing dynamic complementarities in actions between periods and beliefs reported by Mensch (2020). As a result, we dispense with some continuity assumptions necessary to obtain existence in that paper. This is partially due to no aggregate uncertainty assumption and the fact that any player has no influence on aggregate distribution and formation of joint beliefs (see also Kalai and Shmaya, 2018).

Theorem 5. *Under Assumptions 1–4, there exists the greatest MSE of the game.*

The above theorem requires some comment. First of all, apart from guaranteeing existence of an MSE, Theorem 5 implies that there is the greatest equilibrium that determines the upper bound for all equilibria in the space $\mathcal{M} \times \mathcal{D}$. In addition, as we show formally in the following subsection, whenever the set of maximizers corresponding to the optimization problem on the right hand-side in (4) is unique, then the set of MSE is chain complete, i.e., closed under monotone sequences of equilibria.

The next remark follows immediately from the MSE definition.

Remark. Any MSE induces a *sequential distributional equilibrium* as defined by Jovanovic and Rosenthal (1988), i.e., $(\mu_n^*)_{n\in\mathbb{N}}$, where $\mu_0^* = \mu^*$ and $\mu_n^* = \Phi^*(\mu_{n-1}^*)$

Given that a natural question arises whether there is an invariant distribution induced by MSE. Hence, the following proposition whose prove we omit.

Proposition 1 (Invariant distributions)**.** *Under assumptions 1–4, there exists the greatest invariant distribution $\bar{\nu}$ induced by the greatest MSE, i.e., $\bar{\nu} = \bar{\Phi}^*(\bar{\nu})$.*

Proposition 1 proves existence of an invariant distribution $\bar{\nu}$ and implies its approximation via iterations on $\bar{\Phi}^*$. Note that, for any MSE (μ^*, Φ^*), the pair $\big(\Phi^*(\mu^*), \Phi^*)\big)$ is also an MSE. Clearly, this implies that for any invariant distribution ν generated by Φ^*, the pair (ν, Φ^*) is also an MSE. In particular, this applies to the greatest equilibrium $(\bar{\mu}^*, \bar{\Phi}^*)$ and the corresponding greatest invariant distribution $\bar{\nu}$.[30] Notice that, all the results in

[30] However, it must be that $\bar{\nu}$ is dominated by μ^*.

the following sections: on monotone comparative statics / dynamics, approximation by small games or examples apply to a special case of stationary equilibrium $(\bar{\nu}, \bar{\Phi}^*)$ e.g.

Remark. Following an analogous argument to the one supporting Theorem 5, one can show that, under Assumptions 1 and 2, there exists the *least* MSE in $\mathcal{M} \times \mathcal{D}'$, where

$$\mathcal{D}' := \Big\{\Phi : \mathcal{M} \to \mathcal{M} : \Phi \text{ is increasing and monotone} \\ \text{sup-preserving and } \operatorname{marg}_T(\Phi(\mu)) = \phi(\mu), \text{ for any } \mu \in \mathcal{M}\Big\}.$$

This implies that the set of all MSE in $\mathcal{M} \times (\mathcal{D} \cap \mathcal{D}')$ is bounded by the greatest and the least equilibrium. In the reminder of this paper, we focus on the greatest equilibrium only. However, all the results presented below have their counterpart for the least MSE.

The proof of Theorem 5 is thoroughly discussed in the following section. An important advantage of our approach is that it is constructive and introduces an explicit algorithm that allows to approximate the *greatest* equilibrium. We need to introduce some additional notation to describe the method. For any $\mu \in \mathcal{M}$, $\Phi \in \mathcal{D}$ and function v, define

$$\Gamma(t, \mu, \Phi; v) := \arg \max_{a \in \tilde{A}(t, \mu_T)} \left\{(1-\beta) r(t, a, \mu) + \beta \int_T v(t', \Phi(\mu), \Phi) q(dt'|t, a, \mu)\right\}, \quad (5)$$

which is the set of maximizers of the player's dynamic problem. Define the greatest element of the set by $\bar{\gamma}(t, \mu, \Phi; v)$, whenever it exists.

Let $\star$ be a binary operation between $\tau \in \mathcal{M}_T$ and the set of measurable functions $h : T \to A$ returning probability measure on $T \times A$ defined as[31]:

$$\tau \star h := \tau(id_T, h)^{-1}. \quad (6)$$

Define operator $\overline{\Psi}$ that maps $\mathcal{M} \times \mathcal{D}$ to itself and takes values $\overline{\Psi}(\mu, \Phi) = (\mu', \Phi')$, where

$$\mu' := \mu_T \star \bar{\gamma}(\cdot, \mu, \Phi; v^*) \text{ and } \Phi'(\nu) := \phi(\nu) \star \bar{\gamma}\big(\cdot, \Phi(\nu), \Phi; v^*\big), \text{ for all } \nu \in \mathcal{M}, \quad (7)$$

where $v^* : T \times \mathcal{M} \times \mathcal{D} \to \mathbb{R}$ is a function solving (4).

Our construction can be summarized in the following proposition.

[31] That is, $(\tau \star h)(S) = \tau\big(\{t \in T : (t, h(t)) \in S\}\big)$, for any measurable set S.

Proposition 2 (Bounds approximation)**.** *Let $\bar{\mu}$ and $\bar{\Phi}$ be the greatest elements of $\mathcal{M}$ and $\mathcal{D}$, respectively. Under Assumptions 1–4,* $\lim_{n\to\infty} \overline{\Psi}^n(\bar{\mu}, \bar{\Phi})$ *is the greatest MSE.*

Linking the results of proposition 1 and 2 we can conclude that $\bar{\nu}$ is in fact the greatest invariant distribution from all induced by any MSE with $\Phi \in \mathcal{D}$.

4.3 Construction of equilibria

We devote this subsection to the proof of Theorem 5. We discuss the main intuition of the argument and state the auxiliary results that may be of interest themselves.

Let Assumptions 1–4 be satisfied throughout. We begin by showing that the problem of a player in (2) admits a recursive representation. In particular, for any Markov transition function $\Phi \in \mathcal{D}$, there is a unique function v satisfying equation (4).

Consider the space $\mathcal{V}$ of functions $v : T \times \mathcal{M} \times \mathcal{D} \mapsto \mathbb{R}$ such that: (i) functions v are uniformly bounded by a value $\bar{r} > 0$, (ii) $v(\cdot, \mu, \Phi)$ is increasing and continuous, for any $(\mu, \Phi) \in \mathcal{M} \times \mathcal{D}$, (iii) $v(t, \cdot, \cdot)$ is monotone inf-preserving, for any $t \in T$, (iv) v has increasing differences in $\big(t, (\mu, \Phi)\big)$. We endow $\mathcal{V}$ with natural sup-norm topology $||\cdot||_\infty$.

Lemma 1. *$\mathcal{V}$ is complete metric space.*

Given that $\mathcal{V}$ is a subset of all bounded functions, it is a subset of a Banach space. Hence, it suffices to show that the set is closed. Given that continuity, monotonicity, and increasing differences are preserved in the sup-norm convergence, the main difficulty is to show that any limit of monotone inf-preserving functions preserves this property. The proof of this claim is shown in the Appendix.

The next lemma provides an important feature of the Markov transition functions Φ.

Lemma 2. *Let $(\mu_k)_{k\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{M}$ that weakly converges to μ in $\mathcal{M}$. Let $(\Phi_k)_{k\in\mathbb{N}}$ be an decreasing sequence in $\mathcal{D}$ that pointwise weakly converges to some Φ in $\mathcal{D}$. Then $\big(\Phi_k(\mu_k)\big)_{k\in\mathbb{N}}$ weakly converges to $\Phi(\mu)$.*

Proof. Suppose that $(\mu_k)_{k\in\mathbb{N}}$ and $(\Phi_k)_{k\in\mathbb{N}}$ are both decreasing. Since each element of the set $\mathcal{D}$ increasing, it must be that $(\Phi_k\mu_k) \geq (\Phi\mu)$, for all $k \in \mathbb{N}$, and so $\lim_{k\to\infty}(\Phi_k\mu_k) \geq (\Phi\mu)$, where $\geq$ denotes the product order on $\mathcal{D}$.

To show the converse inequality, let N be an arbitrary integer. For each $k \geq N$, we have $\Phi_k(\mu_k) \leq \Phi_k(\mu_N)$. Taking the limit on both sides of the inequality as $k \to \infty$, we obtain $\lim_{k\to\infty} \Phi_k(\mu_k) \leq \Phi(\mu_N)$. By the inf-preserving property of Φ, we have $\lim_{N\to\infty} \Phi(\mu_N) = \Phi(\mu) \geq \lim_{k\to\infty} \Phi_k(\mu_k)$. This completes the proof. □

Define an operator $B : \mathcal{V} \to \mathcal{V}$ as

$$(Bv)(t, \mu, \Phi) \;:=\; \max_{a\in\tilde{A}(t,\mu_T)} \left\{(1-\beta)r(t, a, \mu) + \beta \int_T v\big(t', \Phi(\mu), \Phi\big)q(dt'|t, a, \mu)\right\}. \tag{8}$$

Some basic properties of the operator B are listed below.

Lemma 3. *For any $v \in \mathcal{V}$, function (Bv) is continuous and increasing in t, jointly monotone inf-preserving in (μ, Φ), and has increasing differences in $\big(t, (\mu, \Phi)\big)$.*

To keep our notation compact, denote the function within the brackets in (8) by

$$F(t, a, \mu; v, \Phi) \;:=\; (1-\beta)r(t, a, \mu) + \beta \int_T v\big(t', \Phi(\mu), \Phi\big)q(dt'|t, a, \mu).$$

Given Assumption 2–4, it is straightforward to show it is increasing in t, jointly continuous in (t, a) and has increasing differences in $\big(a, (t, \mu, \Phi)\big)$ and $\big(t, (\mu, \Phi)\big)$. We claim that it is also monotone inf-preserving in (μ, Φ). Clearly, suppose that $(\mu_n, \Phi_n)_{k\in\mathbb{N}}$ is a decreasing sequence that converges to (μ, Φ). By Lemma 2, we have $\Phi_n(\mu_n) \to \Phi(\mu)$. By Assumption 2 and the choice of the set $\mathcal{V}$, both $r(t, a, \mu_k) \to r(t, a, \mu)$ and $v\big(t, \Phi_k(\mu_k), \mu_k\big) \to v\big(t, \Phi(\mu), \mu\big)$. Moreover, we have that $\int_T v\big(t', \Phi_k(\mu_k), \mu_k\big)q(dt'|t, a, \mu_k) \to \int_T v\big(t', \Phi_k(\mu_k), \mu_k\big)q(dt'|t, a, \mu_k$ which follows from Lemma 10 in the Appendix.

We are ready to prove Lemma 3.

Proof of Lemma 3. Continuity of (Bv) follows from Lemma 11 in the Appendix. Monotonicity of (Bv) in t is implied by monotonicity of F and the fact that $\tilde{A}$ increases in t in the sense of set inclusion. To show that it is monotically inf-preserving in (μ, Φ), take any decreasing sequence $(\mu_k, \Phi_k)_{k\in\mathbb{N}}$ that converges to some (μ, Φ). We know that $F(t, a_k, \mu_k; v, \Phi_k) \to F(t, a, \mu; v, \Phi)$ whenever $a_k \to a$. By Lemma 11, this suffices for $(Bv)(t, \mu_k, \Phi_k) \to (Bv)(t, \mu, \Phi)$. Finally, the fact that (Bv) has increasing differences in $\big(t, (\mu, \Phi)\big)$ can be shown as in the proof of Lemma 1 in Hopenhayn and Prescott (1992). □

The following proposition follows immediately from the discussion above.

Proposition 3. *Operator $B : \mathcal{V} \to \mathcal{V}$ has a unique fixed point. As a consequence, there is exactly one solution to the equation (4) in $\mathcal{V}$.*

Indeed, Lemma 3 guarantees that B is well-defined operator that maps a complete metric space into itself. Since it is also a contraction, it has a unique fixed point v^*. Finally, showing that the value coincides with the value of the original problem (2) can be done using standard arguments. See, e.g., Theorem 9.2 in Stokey et al. (1989).

We now proceed with the second half of the argument, in which we prove existence of the greatest MSE. First, recall the definition of the correspondence Γ from (5), with its greatest selection $\bar{\gamma}$. Consider the following lemma.

Lemma 4. *For any $v \in \mathcal{V}$, function $\bar{\gamma} : T \times \mathcal{M} \times \mathcal{D} \to A$, taking values $\bar{\gamma}(t, \mu, \Phi; v)$, is well-defined, measurable in t, increasing in (t, μ, Φ), and monotone inf-preserving.*

Proof. Take any $v \in \mathcal{V}$. Clearly, we have $\Gamma(t, \mu; v, \Phi) = \arg\max_{a \in \tilde{A}(t,\mu_T)} F(t, a, \mu; v, \Phi)$. It is straightforward to verify that F is supermodular and continuous in a. Since set $\tilde{A}(t, \mu_T)$ is a complete sublattice of A, by Corollary 4.1 in Topkis (1978), set $\Gamma(t, \mu; v, \Phi)$ is a complete sublattice of A. Therefore, it admits both its greatest and least element.

We postpone the proof of measurability of $\bar{\gamma}$ until the Appendix. Monotonicity follows from increasing differences of F and the Topkis monotone comparative statics theorem.

To show that $\bar{\gamma}$ is monotone inf-preserving, let $(\mu_k, \Phi_k)_{k \in \mathbb{N}}$ be decreasing sequence converging to (μ, Φ). By the previous argument, sequence $\left(\overline{\gamma}(t, \mu_k, \Phi_k; v)\right)_{k \in \mathbb{N}}$ is decreasing. Suppose it converges to some γ, and thus $\gamma(t, \mu_k, \Phi_k; v) \geq \gamma$, for all $k \in \mathbb{N}$. Since F continuous and monotone inf-preserving, Lemma 11 guarantees that $\gamma \in \Gamma(t, \mu; v, \Phi)$. Thus, it must be $\gamma \leq \overline{\gamma}(t, \mu, \Phi; v)$, and so $\gamma \leq \overline{\gamma}(t, \mu, \Phi; v) \leq \overline{\gamma}(t, \mu_k, \Phi_k; v)$. $\square$

Next, recall the definition of operator $\star$ from (6). It possesses the following properties.

Lemma 5. *Take any measures $\tau, \tau' \in \mathcal{M}_T$ such that τ' first order stochastically dominates τ, and increasing functions $h, h' : T \times A \to A$ such that h' dominates h pointwise. Then, measure $(\tau' \star h')$ first order stochastically dominates $(\tau \star h)$.*

The proof of the above property is straightforward. Hence, we omit the argument.

Lemma 6. *Let $(\tau_k)_{k\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{M}_T$ that converges to some τ. Let $(h_k)_{k\in\mathbb{N}}$, where functions $h_k : T \times A \to A$ are increasing and monotone inf-preserving, be a pointwise decreasing sequence that converges to some h. Then $(\tau_k \star h_k) \to (\tau \star h)$ weakly.*

Proof. Indeed, for any measurable, continuous, and bounded function $f : T \times A \to \mathbb{R}$,

$$\begin{aligned}\lim_{k\to\infty} \int_{T\times A} f(t,a)(\tau_k \star h_k)(dt \times da) &= \lim_{k\to\infty} \int_T f\big(t, h_k(t)\big)\tau_k(dt)\\ &= \int_T f\big(t, h(t)\big)\tau(dt) = \int_{T\times A} f(t,a)(\tau \star h)(dt \times da)\end{aligned}$$

where the second equality follows from Lemma 10. □

In order to prove Theorem 5, take the unique function v^* that solves the equation (4). Define operator $\bar{\Psi}$ as in (7). Given monotonicity of $\bar{\gamma}$ and Lemma 5, we conclude that it is increasing. Moreover, by Lemmas 4 and 6, it is also inf-preserving.

Lemma 7. *The set $\mathcal{D}$ is a lower chain complete poset.*

Proof. Let $(\Phi_j)_{j\in J}$ be a chain of elements in $\mathcal{D}$. Let $\Phi := \bigwedge_{j\in J} \Phi_j$. It suffices to show that that Φ is monotone inf-preserving. Let $(\mu_k)_{k\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{M}$ that converges to μ. For any k, j, and increasing, measurable function $f : T \times A \to \mathbb{R}$,

$$\int_{T\times A} f(t,a)(\Phi\mu)(dt \times da) \le \int_{T\times A} f(t,a)(\Phi\mu_k)(dt \times da) \le \int_{T\times A} f(t,a)(\Phi_j\mu_k)(dt \times da).$$

As $k \to \infty$, we obtain

$$\begin{aligned}\int_{T\times A} f(t,a)(\Phi\mu)(dt \times da) &\le \liminf_{k\to\infty} \int_{T\times A} f(t,a)(\Phi\mu_k)(dt \times da)\\ &\le \limsup_{k\to\infty} \int_{T\times A} f(t,a)(\Phi\mu_k)(dt \times da) = \int_{T\times A} f(t,a)(\Phi_j\mu)(dt \times da).\end{aligned}$$

We conclude the by taking the infimum with respect to over j on the right hand-side. □

We proceed with the proof of Theorem 5.

Proof of Theorem 5. We show that there exists a fixed point of $\overline{\Psi}$ defined as in (7). First we verify the monotonicity of both coordinates of $\overline{\Psi}$. By Lemma 4 the $\overline{\gamma}(t, \mu, \Phi; v^*)$ is jointly increasing in (t, μ, Φ). Hence by Lemma 6 the expression μ' defined in (7) is jointly increasing in (μ, Φ). By the same argument Φ' is increasing as a function of $\nu \in \mathcal{M}$ and is increasing in Φ. By Lemma 6 and Lemma 4 we easily conclude that $\overline{\Psi}$ is a monotone inf-preserving selfmap on $\mathcal{M} \times \mathcal{D}$. We can hence apply Theorem 2 on a lower chain complete poset $\mathcal{M} \times \mathcal{D}$ to conclude existence of the greatest MSE. $\square$

5 Monotone comparative dynamics

In this section we discuss monotone comparative dynamics of the game discussed in Section 4. We parameterize primitives of our game with θ in a poset Θ and seek conditions under which MSE is monotone in the parameter. Observe that one such parameter can be the initial distribution on types. Given our definition of equilibrium, this means that a selection $\theta \to \mu^*(\theta)$ increases, but also the equilibrium law of motion $\theta \to \Phi^*(\theta)$ does so. Hence, the use of the term monotone comparative *dynamics* rather than monotone comparative statics. We first define a positive shock.

Assumption 5 (Positive shock)**.** *Let Θ be a poset. We assume the following.*

(i) Payoff function $r : T \times A \times \mathcal{M} \times \Theta \to \mathbb{R}$ has increasing differences in (a, θ), for all $(t, \mu) \in T \times \mathcal{M}$, and (t, θ), for all $(a, \mu) \in A \times \mathcal{M}$.

(ii) Transition kernel $q : T \times A \times \mathcal{M} \times \Theta \to \mathcal{M}_T$ is increasing over Θ and has increasing differences in (a, θ), for all $(t, \mu) \in T \times \mathcal{M}$, and (t, θ), for all $(a, \mu) \in A \times \mathcal{M}$.

(iii) Feasible action correspondence $\tilde{A} : T \times \mathcal{M} \times \Theta \to A$ has strict complementarities in (t, θ) for any $\mu \in \mathcal{M}$.

Theorem 6 (Monotone Comparative Dynamics)**.** *Suppose that the parametrized mappings $r(\cdot, \theta)$, $q(\cdot; \theta)$, and $\tilde{A}(\cdot; \theta)$ satisfy Assumptions 1–4, for all $\theta \in \Theta$. Under Assumption 5, the greatest equilibrium $\left(\bar{\mu}^*(\theta), \bar{\Phi}^*(\theta)\right)$ of the parametrized game increases on Θ.*

Proof. Let $\overline{\Psi}^{\theta}$ be the counterpart of the operator $\overline{\Psi}$ in the parametrized game with $\theta \in \Theta$. Similarly we denote ϕ^{θ} and $\overline{\gamma}^{\theta}$. Clearly, under assumptions that $q(\cdot|t, a, \mu; \theta)$ is increasing in θ. It suffices to show that $\theta \to \overline{\gamma}^\theta$ is increasing. Observe that, under our assumptions, the objective $(1-\beta)r(t,a,\mu,\theta)+\beta\int_T v^*(t',\phi(\mu),\theta)q(dt'|t,a,\mu,\theta)$ has increasing differences in (a,θ) and $v^*(t,\mu,\theta)$ has increasing differences in (t,θ), for any $\mu \in \mathcal{M}$. By Theorem 6.2. in Topkis (1978), we conclude that $\overline{\gamma}$ is increasing in θ. See also Hopenhayn and Prescott (1992). By Assumption 5 and definition we conclude that $\theta \to \phi^\theta$ is increasing. The same property is inherited by $\overline{\Psi}^\theta$ from its definition and Lemma 5. Moreover, similarly as in the proof of Theorem 5, we conclude that $\overline{\Psi}^\theta(\cdot)$ is an increasing operator, for any fixed θ. To finish this proof we only need to apply Theorem 3, recalling that a poset of distributions and poset of uniformly bounded functions are chain complete. $\square$

An immediate corollary to the above result is that under Assumptions 1–4, the greatest equilibrium increases in the initial distribution of types τ_0. Indeed, if we let $\theta = \tau_0$ and $\Theta = \mathcal{M}_T$ is ordered in the first order stochastic sense, then Assumption 5 is satisfied.

Our monotone comparative dynamics result significantly improves upon the related results in Adlakha and Johari (2013), Acemoglu and Jensen (2010, 2015), Light and Weintraub (2019). The above papers discuss comparative statics on invariant distributions and/or steady states of equilibrium aggregates. In contrast, we provide conditions under which our dynamic notion of an equilibrium is increasing with respect to the parameter. This is of utmost importance. In particular, the assumptions of Acemoglu and Jensen (2015) or Light and Weintraub (2019) are not sufficient to obtain such a strong results. Indeed, firstly conditions for comparing invariant distributions are much weaker than those of the whole dynamic equilibrium. To see that, observe the equilibrium law of motion / belief varies with parameter via $\theta \to \Phi^*(\theta)$, but also that perturbations of equilibrium distribution $\mu^*(\theta)$ propagate to future period distributions via the value $\mu^*(\theta) \to v(\cdot, \Phi^*(\mu^*(\theta))(\theta))$. Secondly, we compare whole distributions defined over multidimensional space $\mathbb{R}^n$; rather than their moments or statistics. Since space of such objects are not lattices, it is crucial to employ a new comparative statics tool introduced in Theorem 3. See also the discussion in Section 3 of Light and Weintraub (2019).

6 Approximating games with finitely many players

From the point of view of economic applications, a game of interest consists of a finite, although potentially large, number of players. In this section, we argue that our large games framework serves as an approximation of a large, but finite number of players game.

We consider a counterpart of the large game discussed previously, in which the number of players is N, where N is a large but finite number. Whenever this causes no confusion, we shall denote the set of players and its cardinality by N. We now formally define an N-player, dynamic Bayesian game, where the sequence of priors from which players types are drawn in each period n is given by τ_n. Here, we hence impose an important behavioural assumption: players believe the law of large numbers holds (in N player games) and they do not keep track of dynamics of beliefs over other players private types (see also Kalai and Shmaya (2018) imagined-continuum equilibrium concept).

As in the previous sections, the distribution τ_n is evaluated according to the formula: $\tau_1 = \tau$ where τ is an initial distribution and

$$\tau_{n+1}(Z) \;=\; \phi(\tau_n \star \sigma_n)(Z), \tag{9}$$

for $n \geq 1$. This sequence of priors is a common knowledge.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\tilde{T}_n : \Omega \to T^N$ be a random variable determining the types of players in period n. We denote $\tilde{T}_n = (\tilde{T}_n^1, \ldots, \tilde{T}_n^N)$, where $\tilde{T}_n^l$ is the random variable determining the type of agent l, drawn i.i.d. from the theoretical distribution τ_n. For any vector of types $\tilde{t} := (\tilde{t}^1, \ldots, \tilde{t}^N) \in T^N$ of players, i.e., vector of realizations of the random variable $\tilde{T}_n$, we construct the *empirical distribution* as follows

$$\hat{\tau}_n^N(\tilde{t})(Z) \;=\; \frac{\#\big\{l \in \{1, 2, \ldots, N\} : \tilde{t}^l \in Z\big\}}{N}.$$

As we aim to compare equilibrium profiles of games with different number of players, we concentrate on symmetric equilibria. For this reason, suppose that all but jth player apply a sequence of (now fixed) Markov policies $(\sigma_n)_{n\in\mathbb{N}}$. That is, any player $l \neq j$, after observing $\tilde{t}^l \in T$ and knowing the theoretical distribution τ_n at time n, chooses the action $\sigma_n(\tilde{t}^l) \in \tilde{A}(\tilde{t}^l, \tau_n)$, where function $\sigma_n : T \to A$ is a Borel measurable. Player j at time n

selects action S^j_n. For players other than j denote $S^l_n := \sigma_n(\tilde T^l_n)$ and consider a profile of random variables $S_n := (S^1_n, \ldots, S^N_n)$.

Let $\tilde t$ be a realization of $\tilde T_n$, s^j a realization of S^j_n, and $s^l = \sigma_n(\tilde t^l)$ some realization for $l \neq j$. The empirical distribution on types-actions is given by:

$$\begin{aligned}\hat\mu^N_n(\tilde t, s^j)(D) &:= \frac{\#\{l \in \{1,2,\ldots,N\} : (\tilde t^l, s^l) \in D\}}{N} \\ &= \frac{1}{N}\sum_{l\neq j} \mathbf{1}_D\big(\tilde t^l, \sigma_n(t^l)\big) + \frac{1}{N}\mathbf{1}_D(\tilde t^j, s^j).\end{aligned}$$

Let us formulate a key lemma that asserts that empirical distribution of types-actions converges weakly to the theoretical one as the number of players increase. Thus, the type and action of each individual player becomes negligible.

Lemma 8. *For any $n \in \mathbb{N}$, let $\tilde T^{-j}_n = (\tilde T^l_n)_{l\neq j}$ be a collection of T-valued random types for $l \neq j$, drawn i.i.d. from τ_n. Let $S^l_n := \sigma_n(\tilde T^l_n)$, for all $l \neq j$. For any N, let (ξ^N, η^N) be an alternative random vector of type and policy for j, such that $(\xi^N, \eta^N) \in Gr\big(\tilde A_n(\cdot, \tau_n)\big)$ almost surely. Then $\hat\mu^N_n\big((\tilde T_{-j}, \xi^N), \eta^N\big)$ converges weakly to $(\tau_n \star \sigma_n)$, $\mathbb{P}$-almost surely.*

We now proceed to define formally histories of the game and players payoffs. Let $\mathcal{F}_n$ be the sigma-algebra generated by the sequence of random variables of types (histories) $\tilde T_k$, for $k \leq n$. The selection S^j_n for j is called *admissible* if

$$\mathbb{P}\big(S^j_n \in \tilde A_n(\tilde T^j_n, \tau_n) \,|\, \mathcal{F}_n\big) \;=\; 1, \;\mathbb{P}\text{-almost surely.}$$

The selection for players other than j is *admissible* by definition of (σ_n). For any l, assume that $\tilde T^l_n$ is a Markov chain controlled by all players, and the transition probability satisfies

$$\mathbb{P}\Big(\tilde T^l_{n+1} \in Z \big| \mathcal{F}_n\Big) \;=\; q\big(Z | \tilde T^l_n, S^l_n, \hat\mu^N_n(\tilde T_n, S^j_n)\big), \quad \text{for any } Z \in \mathcal{T}, \;\mathbb{P}\text{-almost surely.}$$

Moreover, the random variables $\tilde T^1_{n+1}, \ldots, \tilde T^N_{n+1}$ are $\mathcal{F}_n$-conditionally independent.[32] The history is generated by types and actions of all players. The set of histories up to time n is $H_n \subseteq \prod_{k=1}^n \big(\mathrm{Gr}(\tilde A_k)\big)^N$, with a generic element $h_n = (\tilde t^1_k, s^1_k, \ldots, \tilde t^N_k, s^N_k)^n_{k=1}$ and $s^l_k = \sigma_k(\tilde t^l_k)$, for all $l \neq j$. Moreover, for any k, $\tilde t^j_{k+1}$ is in the support of $q\big(\cdot\, | \tilde t^j_k, s^j_k, \hat\mu_n(\tilde t_k, s^j_k)\big)$.

[32] That is, $\tilde T_{n+1}$ has $\mathcal{F}_n$-conditional distribution $q^P\big(\cdot\,|\tilde T_n, S_n, \hat\mu^N_n(\tilde T_n, S^j_n)\big) := \bigotimes_{j=1}^N q\big(\cdot\,|\tilde T^j_n, \tilde S^j_n, \hat\mu^N_n(\tilde T_n, S^j_n)\big)$, that is $\mathbb{P}\big(\prod_{j=1}^N\{\tilde T^j_{n+1} \in Z_j\}|\mathcal{F}_n\big) = \prod_{j=1}^N q\big(Z_j|\tilde T^j_n, S^j_n, \hat\mu^N_n(\tilde T_n, S^j_n)\big)$, for any $Z_1, \ldots, Z_N$, all belonging to $\mathcal{T}$, and all $\omega \in \tilde\Omega$ (or modifying $\tilde\Omega$ on a null set if necessary).

Any initial type $\tilde{t}_1^j$ of player j, her (behavioural) policy π, policy of other players (σ_n), initial distribution for all types τ_1, and the transitions between types induce a unique private probability measure on histories and its expectation $\mathbb{E}_{\tilde{t}_1^j}^{\sigma,\pi}$. See Ionescu-Tulcea Theorem in Dynkin and Yushkevich (1979). If player j unilaterally deviates from the from Markov policy (σ_n) to $\pi = (\pi_n)$, the strategy profile is $\big((\sigma_n)_{-j}, (\pi_n)\big)$, since (σ_n) is symmetric for all players $l \neq j$. Given the initial private state $\tilde{t}_1^j = t$, player j payoff is

$$\begin{aligned}\mathcal{R}^N(\sigma^{-j}, \pi)(t) &:= (1-\beta)\mathbb{E}_t^{\sigma,\pi}\left[\sum_{n=1}^{\infty} r_n^N(\tilde{t}_n^j, s_n^j)\beta^{n-1}\right] \\ &= (1-\beta)\mathbb{E}\left[\sum_{k=1}^{\infty} r_n^N(\tilde{T}_n^j, S_n^j)\beta^{n-1}\middle|\tilde{T}_1^j = t\right],\end{aligned}$$

where r_n^N is a reward function defined as follows

$$r_n^N(t, a) := \int_{T^{N-1}} r\Big(t, a, \hat{\mu}_n^N\big((t, \tilde{t}^{-j}), a\big)\Big)\tau_n^{N-1}(d\tilde{t}^{-j}), \tag{10}$$

where $\tau_n^{N-1} = \underbrace{\tau_n \otimes \tau_n \ldots \otimes \tau_n}_{N-1 \text{ times.}}$ Similarly, let

$$q_n^N(\cdot|t, a) := \int_{T^{N-1}} q\big(\,\cdot\,|t, a, \hat{\mu}_n^N((t, \tilde{t}^{-j}), a)\big)\tau_n^{N-1}(d\tilde{t}^{-j}). \tag{11}$$

Given that the evolution of τ_n is specified in (9), policy for all players $(\sigma_n)_{n\in\mathbb{N}}$ the problem for player j is a Markov decision process with the value function

$$\tilde{v}_1^N(t) := \sup_{\pi\in\Sigma} \mathcal{R}(\sigma^{-j}, \pi)(t),$$

where Σ is the set of all feasible policies, i.e., Borel measurable functions $\pi := (\pi_n)_n^\infty$ such that $\pi_n : H_n \times T \times \mathcal{M}_T \mapsto \mathcal{M}_A$ and $\pi_k\big(\tilde{A}_n(t_n^j, \tau_n)|h_n, t_n^j\big) = 1$ for all n, $t_n^j \in T$, $h_n \in H_n$, and all π_n are Borel measurable function.[33]

Definition 4 (Approximation)**.** A profile $\hat{\sigma} = (\hat{\sigma}_n)$ is said to be an ϵ-*equilibrium* for an initial distribution τ_1, if there is some $N_0 \in \mathbb{N}$ such that for any $N > N_0$, any player $j = 1, 2, \ldots, N$, and any type $t \in T$, and any $\pi \in \Sigma$, we have

$$\epsilon + \mathcal{R}^N(\hat{\sigma})(t) \geq \mathcal{R}^N\big((\hat{\sigma})^{-j}, \pi\big)(t).$$

[33] By $\mathcal{M}_A$ we denote the set of probability distributions over A.

A symmetric action profile $\hat{\sigma}$ is an ϵ-equilibrium if it constitutes an ϵ-equilibrium for a sufficiently large N. Clearly, both ϵ and N_0 depend on the initial distribution τ_1.

Assumption 6. *We assume the following:*

(i) function r is continuous,

(ii) for any continuous and real valued function $f : T \to \mathbb{R}$, the function $(t, a, \mu) \to \int_T f(t')q(t'|t, a, \mu)$ is continuous,

(iii) for any τ, the correspondence $t \to \tilde{A}(t, \tau)$ is continuous.

Observe that, in our specification of the game with finitely many players the agents do not control the *theoretical* distribution τ_n, but only empirical one $\hat{\mu}$. For some MSE (μ^*, Φ^*), consider an associated equilibrium policy function σ^* and its associated value function v^*. For $\mu_1^* := \mu^*$ consider an associated sequential equilibrium (μ_n^*) defined recursively by iterations $\mu_{n+1}^* = \Phi^*(\mu_n^*)$. Similarly, consider associated distributions on types (τ_n^*), policies (σ_n^*), and values $v_n(t)$, where $\tau_n^* = marg_T(\mu_n^*)$, $\sigma_n^*(t) := \sigma^*(t, \mu_n^*) = \sigma_{\mu_n, \Phi^*}(t)$ and $v_n(t) := v^*(t, \mu_n^*; \Phi^*)$. We state the main theorem of this section.

Theorem 7. *Under Assumption 6, for any MSE (μ^*, Φ^*) and any $\epsilon > 0$, the sequence of implied policy functions (σ_n^*) is an ϵ-equilibrium for $\tau_1 = \tau^*$.*

Weintraub et al. (2008, 2011) and Adlakha et al. (2015) show the asymptotic Markov property of both oblivious (OE) and mean-field equilibrium (MFE). Specifically, they show that using OE and MFE in a small game on the invariant distribution becomes "approximately optimal" as the number of players tends to infinity. Such approximation requires uniqueness (and continuity) of the best reply, and moreover, as the authors work with unbounded states spaces and unbounded payoffs, they result holds under the additional *light tail condition*. A different result is provided by Kalai and Shmaya (2018), who show that any imagined-continuum equilibrium of a finite number of players (repeated) game equilibrium is an epsilon equilibrium of that small game. Moreover, the ϵ can be arbitrarily small as N tends to infinity. The authors require that the *aggregating function* (whose outcomes players assume to be given) that maps distributions to public aggregates, is Lipschitz continuous. Our results complement these earlier contributions by allowing for dynamic Markovian environment, analysis is not restricted to invariant distributions, unique continuous argmaxes nor aggregative games.[34]

7 Applications and examples

In this section we discuss several economic application of our main results.

7.1 Motivating example revisited

We return to the motivating example introduced in Section 2. Recall that, in each period the type of a player is identified with her level of capital $t \in T = [0,1]$ and actions (investments) $a \in A = [0,1]$ is chosen from the set $\tilde{A}(t,\tau) = [0,t]$. Given the distribution μ of types and actions of all players the payoff in a single period is

$$r(t,a,\tau,\theta) \ := \ \int_{A\times T} \Big[\theta_1 m(t-t') + \theta_2 w(t-a-t'+a')\Big]\mu(da\times dt),$$

where we introduce a positive parameter $\theta = (\theta_1, \theta_2)$ with respect to the initial example.

Given an investment a, the cumulative probability distribution of capital level t' in the next period is $q(t'|a)$. Thus, conditional on a macro belief Φ, the Bellman equation characterizing the player's value function in the infinite horizon game is given by

$$v(t,\tau;\Phi) = \max_{a\in\tilde{A}(t,\tau)} \left\{(1-\beta)r(t,a,\tau,\theta) + \beta\int v\big(t',\Phi(\tau);\Phi\big)q(dt'|a)\right\}.$$

It is straightforward to verify that our game satisfies Assumptions 1–4. Correspondence $\tilde{A}$ is measurable, continuous, compact valued, and increasing (both in the sense of set inclusion and strong set order). Given that functions m and w are continuous, increasing, and concave, function r is continuous over $T \times A$, increasing over T, and has increasing differences in $\big(a,(t,\mu)\big)$ and (t,μ). The function is also (trivially) supermodular in a and continuous in μ. As long as the distribution q is continuous in a,[35] the requirements for existence of the greatest Markov Sequential Equilibrium are satisfied.

[34] We also refer the reader to recent results of approximation of large static games by Carmona and Podczeck (2012, 2020) and related results in Qiao and Yu (2014); Qiao et al. (2016).

[35] For example, for a deterministic transition kernel q, given as in footnote 5, this is satisfied as long as the production function f is monotone and continuous.

Following Theorem 5, there exists the greatest equilibrium pair (μ^*, Φ^*) of distribution over types and action and the correct macro beliefs. As it was pointed out in the previous section, this pair generates the whole equilibrium path of distributions $\{\mu_n^*\}$, where $\mu_{n+1} = \Phi^*(\mu_n)$ and $\mu_1^* = \mu^*$, which allows us to investigate the actual dynamics of the model. Moreover, the sequence converges to an invariant distribution. Hence, our toolkit makes it possible to study the steady state of the model as well.

Apart from existence and approximation of equilibria, Theorem 6 allows us to say more about its comparative statics. In particular, the equilibrium (μ^*, Φ^*) and the corresponding sequence $\{\mu_n^*\}$ increase as the initial distribution of types increases in the first order stochastic sense. That is, along the equilibrium path, players invest (stochastically) more and have (stochatically) higher capital levels.

We may also analyze how the equilibrium changes with respect to the parameters θ_1, θ_2. One can easily check that, by concavity of functions m, w, function r has increasing differences in (a, θ_1) and (t, θ_1). Moreover, it has increasing differences in $(a, -\theta_2)$ and $(t, -\theta_2)$. Given that the correspondence $\tilde{A}$ and transition kernel q are independent of θ, this suffices for the equilibrium and its path to be increasing in θ_1 and decreasing in θ_2. In other words, as the weight of the capital/wealth-driven status is higher, the individuals in the population invest (stochastically) more. In contrast, as consumption-driven status becomes more important, investments decrease.

Finally, all of the above results would hold if we allowed for a more elaborate transition kernel $q(\cdot|t, a, \mu)$, that would depend not only on the action of an individual, but also on her type and the distribution of investments and capital levels in the population. However, this would require for Assumption 3 to be satisfied.

7.2 Dynamics of social distance

We next consider a dynamic model of *social distance*, described originally in Akerlof (1997). The model is related to multiple strands of the social economics literature, including models of identity and economic choice as in Akerlof and Kranton (2000), or models with endogenous social reference points, including Bernheim (1994), Brock and

Durlauf (2001), Bisin et al. (2011) and Blume et al. (2015). The model below is a dynamic extension of the static model formalized in Balbus et al. (2019).

Consider a measure space of agents. Let $T = [0,1]$ be the set of all possible social positions in the population. Each period $n = 1, \ldots, \infty$ an individual is characterized by an *identity* $t \in T$ (type), which determines the social position to which the agent aspires.

In every period an agent has to choose her own social position (action) $a \in A := [0,1]$. The set of social position feasible to agent endowed with identity t is $\tilde{A}(t,\tau) := \big[\underline{a}(t), \overline{a}(t)\big]$, where $\underline{a}$, $\overline{a} : T \to A$ are increasing functions that satisfy $\underline{a}(t) \leq t \leq \overline{a}(t)$, for all $t \in T$.

When choosing social position, there is a trade-off between *idealizm* and *conformizm*. On the one hand, the individual wants the social status a to be as close as possible to the identity t as possible. That is, given some continuous, decreasing, and concave function $m : [0,1] \to \mathbb{R}$, the agent wants to maximize $m\big(|a - t|\big)$. This represents idealizm.

On the other hand, the player experiences discomfort when interacting with agents that have different social position than herself. Whenever an agent of social position a encounters an agent of social position a', she receives utility $w\big(|a - a'|\big)$, for some continuous, decreasing, and concave function $w : [0,1] \to \mathbb{R}$. This summarizes conformizm.

Suppose that $\nu(t';t)$ is a cumulative probability distribution determining the likelihood of an agent with identity t meeting someone with identity t'. We assume that it is continuous and first order stochastically increasing in t. It captures the idea that similar minds think alike and players with similar identity are more likely to meet. Payoff of an agent of identity t and social position a, given the distribution of types/actions μ, is

$$r(t,a,\mu) \ := \ m(|a-t|) + \int_T \int_A w(|a-a'|) d\mu(a'|t') d\nu(t';t),$$

where $\mu(\cdot|t')$ is the distribution of actions of other players in the population conditional on t'. Therefore, payoff of an agent in a single period is the sum of her idealistic utility and expected conformistic payoff from interactions with other agents. In particular, our specification implies that the social position can not be contingent on the social statuses of other agents. It has to be chosen ex-ante before any interaction occurs.

Finally, following the rule that "you become who you pretend to be", we assume that the social position in a current period has a direct impact on the identity in the following period. Formally, the transition is governed by cumulative probability distribution $q(t'|a)$, that determines the likelihood of the agent acquiring identity t' in the next period, following her choice of a at the current date. Specifically, we assume that function $a \to q(\cdot|a)$ is continuous and first order stochastically increasing in a.

It is straightforward to verify that the above game admits the greatest Markov Stationary Equilibrium. Indeed, function r satisfies conditions (i), (ii) and (iv), (v) from Assumption 2. Moreover, since the transition kernel q depends only on a, it satisfies Assumption 3. Finally, as long as functions $\underline{a}, \overline{a}$ are continuous, in addition to the previously stated assumptions, correspondence $\tilde{A}(t, \tau) = \left[\underline{a}(t), \overline{a}(t)\right]$ is continuous, compact-valued, and increasing in the strong set order. Clearly, Assumption 1 holds as well.

It is crucial in the above example that the transition function q depends only on the action a. This guarantees existence of the greatest equilibrium despite the fact that the payoff function r is not increasing in t and the feasible action correspondence $\tilde{A}$ is not increasing in t with respect to set-inclusion. See the remark in Section 4.2.

Apart from equilibrium existence, we can easily determine some comparative statics in the model. First of all, it is clear that as the initial distribution of identities of players shifts in the first order stochastic sense, the equilibrium pair (μ^*, Φ^*) increases as well. Moreover, this also increases the whole equilibrium path $\{\mu^*_n\}$, where $\mu^*_1 = \mu^*$ and $\mu^*_{n+1} = \Phi^*(\mu^*_n)$.

7.3 Parenting and endogenous preferences for consumption

Our tools can be applied to dynamic games with short-lived agents, where individuals make decisions in one period only, but their actions propel dynamics for future generations. This example is inspired by the literature on paternalistic bequests, keeping up with the Joneses, and growth with social ranking and endogenous preferences (see Cole et al., 1992, Doepke and Zilibotti, 2008, 2017 and Genicot and Ray, 2017).

Consider a society populated with a measure space of single-parent single-child families. Each individual (a parent) lives for a single period and a parent-child sequence forms a dynasty. The type of a parent is determined by her lifetime income $y \in [0, 1]$ and a parameter $i \in [0, 1]$ that summarizes preferences of the individual toward consumption.

Therefore, the space of types $t = (y, i)$ is given by $T = [0, 1]^2$.

In each period, the income can be devoted to consumption c and investment (savings) s. Thus, we have $y = c + s$. Consumption yields immediate utility $u(c, g)$, where parameter g represents *propensity to consume*. Formally, we assume that function u is continuous and concave in c, and has increasing differences in (c, g). That is, the higher the parameter g, the higher is the marginal utility of consumption.

In our paternalistic setting, a parent evaluates the well-being of her child according to $w(t', \tau')$, where $t' = (y', i')$ is the the future type of the child and τ' is a distribution of types the next period. We assume that w is increasing in type. Therefore, the parent values high income and high propensity to consume of the child — since the parent cares only about her immediate descendant, she wants the child to consume as much as possible. Moreover, we assume that w has increasing differences in (t', τ'), i.e., the higher is the future distribution on types the higher is the parent's incremental benefit the child's type.

Each parent devotes (e.g., educational) effort $e \in [0, 1]$ to shape preferences of her child (i.e., raise their aspiration level). The cost of effort is given by $C(e, \mu_E)$, where μ_E denotes the distribution of efforts in the population. We assume that the cost function is continuous and increasing with e, and has decreasing differences in (e, μ_E) — the higher effort in the population, the easier it is for an individual to influence her child.

Given our description, the action of an individual is $a = (s, e)$ and the action space is $A = [0, 1]^2$. Savings s and effort e affect both the future income and preferences of the child. Let the cumulative distribution $q(t'|s, e)$ determine the probability of the future type of the child being $t' = (y', i')$, where q is stochastically increasing in both arguments and supermodular. Thus, investment s and effort e are complements. Indeed, from the parent's perspective higher effort (that skews preference of the child towards consumption) makes marginal investment/bequest more valuable. In such a scenario the higher amount of child's income will be devoted to consumption that pleases a paternalistic parent.

Finally, the marginal propensity to consume g is generated endogenously for each individual via "keeping up with the Joneses" motivation. Formally, we have $g = \theta\Gamma(t, \mu_C)$, for some positive parameter θ and an increasing function Γ, that depends both on the type of the player and the distribution of consumption levels across population. For example,

$$\Gamma(t, \mu_C) \ := \ \inf\left\{c \in [0,1] : i \leq \mu_C(c' : c' \leq c)\right\},$$

where $t = (y, i)$. That is, Γ is equal to the i'th quantile of consumption in the population.

Given our initial description, the objective of a parent of type $t = (y, i)$ is to maximize

$$u\big(y - s, \theta\Gamma(t, \mu_C)\big) + \int_{[0,1]} w\big(t', \Phi_T(\mu)\big) q(dt'|s, e) - C(e, \mu_E),$$

with respect to $(s, e) \in \tilde{A}(t, \tau) = [0, y] \times [0, 1]$. Here $\Phi_T(\mu)$ is the projected next period distribution on types in the population. Note that w is not a value function in the sense discussed Section 4, but a paternalistic evaluation of child's welfare. In particular, preferences of a parent may be misaligned with future preferences of the child.

To verify whether assumptions of our theorems are satisfied, consider an increasing Markov strategy: $\sigma : T \to A$, with σ_s and σ_e being its projections on both coordinates. Then for some measurable set B, we have $\mu_C(B) = \tau\big(\{t \in T : [y - \sigma_s(t)] \in B\}\big)$, $\mu_E(B) = \tau\big(\{t : \sigma_e(t) \in B\}\big)$, and $\Phi_T(\mu)(B) = \int_T q\big(B|\sigma_s(t), \sigma_e(t)\big)\tau(dt)$. Then, higher σ implies first order stochastic dominance increase of μ_E and $\Phi_T(\mu)$, but first order stochastic dominance decrease of μ_C.[36] Increasing differences of $u(c, g)$, $w(t', \tau')$, and $-C(e, \mu_E)$, together with assumptions on the transition q suffice to show our results.

Specifically, using an argument analogous to the one presented in Section 4.3, one can show that the above games admits the greatest MSE (μ^*, Φ^*). Moreover, the monotone comparative dynamics and equilibrium approximation can be applied. In particular, one can show that the equilibrium is decreasing with respect to the parameter θ.

It is important to point out that the above observations are true even though payoff function is not necessarily increasing with respect to the state t, nor it has increasing differences in (t, μ). In fact, whenever function Γ is specified as above, the latter assumption does not hold. In our main argument the additional assumptions are crucial to show particular properties of the value function in the infinite horizon problem. Analyzing a game with short-lived agents allows to dispense such assumptions.

[36]Indeed, for any measurable and increasing $f : [0,1] \to \mathbb{R}$ we have: $\int_C f(c)\mu'_C(dc) = \int_T f(y - \sigma'_s(t))\tau(dt) \leq \int_T f(y - \sigma_s(t))\tau(dt) = \int_C f(c)\mu_C(dc)$, where σ'_s is pointwise higher than σ_s.

7.4 Legal norms and public enforcement

Here we discuss a version of the model of social/legal norms an public enforcement as in Acemoglu and Jackson (2017). Suppose there is a continuum of agents, each endowed with at type $t \in [0,1]$. Let $L \in [0,1]$ be the social/legal norm in the society. In each period, an individual randomly interacts with other members of the population. Before any interaction takes place, the individual of type t has to choose an action $a \in [0,t]$. We say that action a is "legal" if $a \leq L$. Otherwise, we say that it is "illegal".

Whenever an agent of type t playing action a encounters an agent playing action $\tilde{a}$, the bilateral public enforcement takes place. If both action a and $\tilde{a}$ are legal, i.e., both $a, \tilde{a} \leq L$, the players are allowed to play the selected actions. If action a is illegal, while $\tilde{a}$ is legal, the latter agent forces the former to abide the law. This means, that the former agent has to change her action to L. Analogously, if $a \leq L$ but $\tilde{a} > L$, the latter agent has to change her action to L. Finally, if both a, $\tilde{a}$ are illegal, the agents play their chosen actions, since none of the agents has the moral ground to enforce the legal action.[37]

In each interaction, an individual agent cares about two things. One hand, she wants her actual action (i.e., the one after a potential enforcement) to as close to her type as possible. This yields utility $u\big(|t-a+\mathbf{1}_{a>L}\mathbf{1}_{\tilde{a}\leq L}(a-L)|\big)$, for some continuous, decreasing, and concave function u, where $\mathbf{1}_{\tilde{a}\leq L}$ is the indicator function. In addition, the agents wants her chosen action a, to be as close as possible to the (potentially enforced) action of the other player. This yields utility $v\big(|a-\tilde{a}+\mathbf{1}_{a\leq L}\mathbf{1}_{\tilde{a}>L}(a-L)|\big)$, where v is continuous, concave, and decreasing. Finally, suppose that $\theta \geq 0$ is a fine imposed for those who brake the law and are caught. Given our description, the payoff in a single period of an agent of type t choosing action a is given by

$$\begin{aligned} r(t,a,\mu) \;&:=\; \int_{[0,1]} \Big[u\big(|t-a+\mathbf{1}_{a>L}\mathbf{1}_{\tilde{a}\leq L}(a-L)|\big) \\ &\qquad\qquad + \; v\big(|a-\tilde{a}+\mathbf{1}_{a\leq L}\mathbf{1}_{\tilde{a}>L}(\tilde{a}-L)|\big) - \theta\mathbf{1}_{a>L}\mathbf{1}_{\tilde{a}\leq L}\Big]\mu_A(d\tilde{a}), \end{aligned}$$

where μ_A is the probability distributions over actions in the population. The set of

[37] It is straightforward to extend the above model in order to incorporate imperfect and/or exogenous (police) enforcement. In order to simplify notation, we discuss only the most basic form of the game.

constraints is then given by $\tilde{A}(t,\tau) := [0,t]$. Finally, suppose that the future type t' of a player is determined stochastically by her current action a, where the cumulative probability distribution $q(t'|a)$ is increasing in a in the first order stochastic sense.

One can easily check that the assumptions necessary for our results to hold are satisfied.[38] Our techniques allow to analyze dynamics of population enforcement and formation of legal norms. In particular, the above game admits the greatest MSE (as well as the corresponding equilibrium path). The monotone comparative dynamics, as well as the approximation results hold. In particular, one can show that the equilibrium decrease in the fine θ.

7.5 Dynamics for models with sunspots, coordination failures, and learning

We next consider a prototypical coordination game based on Angeletos and Lian (2016), recently applied to beauty contests, bank runs, riot games, or currency attacks.[39]

In particular, we consider a simple class of dynamic beauty contests that nicely fit our model. Consider a game where each player based on its private signal t chooses an action a. Action is potentially costly and its cost can depend on t, say via the utility function $u(t,a)$ of a player type t. Let $t \in T$, with T a poset, and assume u is increasing in t and has increasing differences in (t,a). Player t has a payoff that depends also on action taken by other players, say $\int_A g(a,a')\mu_A(da')$, where g also has increasing differences between a, a'. As is standard in global games and dynamic coordinations games with complementarities, we study symmetric monotone in type equilibria. That is, we assume each player assumes the other players in the game are using some increasing strategy $\sigma : T \to A$, so she will hence have a joint payoff given by:

$$r(t,a,\mu) := u(t,a) + \int_T g(a,\sigma(t'))\mu_T(dt').$$

[38] Note that function r is upper semi-continuous in action a, rather that continuous. However, this is sufficient for our results to hold.

[39] See Morris and Shin (2002) for an extensive discussion of this literature. See also Carmona et al. (2017) for an interesting recent application of mean-field methods to a related class of games.

Such payoff satisfies our assumptions in the previous section, so all our tools can be applied to study the existence of the greatest MSE.[40]

Alternatively, out framework can be applied to riot games, where

$$r(t,a,\mu) \;:=\; a\left[\int_S (t_1+L)\mathbf{1}_{\{R(\mu)\geq s\}}\nu(ds)-L\right]-c(a,t_2),$$

for some player type by $t=(t_1,t_2)$ and compact interval $S\subseteq\mathbb{R}$. Thus, taking the risky action $a=1$ allows the player to win t_1 if a sufficient number $R(\mu):=\mu\big(\{(t,a):a=1\}\big)$ of players takes a risky (and costly) action, or loose L otherwise. The strength s of the police is distributed according to measure ν. Whenever the cost function is decreasing in t_2 and $c(0,t_2)=0$ (for normalization), the dynamic game can be solved using our methods for a variety of transition functions q allowing to model inertia, habit formation, and dynamic social externalities. See also Morris and Yildiz (2016) applications.

7.6 Idiosyncratic risk under multidimensional production externalities and technological dynamics

Our model can be also applied to analyze dynamics and technological progress in a class of large economies of agents facing (idiosyncratic) private productivity risk under incomplete markets. Specifically, inspired by an original model of Romer (1986) and some recent works on Bewley-Huggett-Ayiagari type models with ex-ante identical but ex-post heterogeneous agents, we can consider dynamics of capital and labour population distributions under endogenous labor supply, production externalities and non-convexities of production technology.[41]

Our economy is populated by a measure of producers, each endowed with capital $t\in T:=[0,1]$, unit of time and a private technology f that transforms private inputs into finished outputs. Productivity of this technology depends also on externality summarized by the distribution of capital levels and labor decisions in the economy: $\mu_{T\times L}$. Specifically, each agent working $l\in L:=[0,1]$ is able to produce $y=f(t,l,\mu_{T\times L})$ units of a

[40] Note that, we can even dispense monotonicity of u with respect to t as long as the transition function q depends only on one-dimensional action a.

[41] See also Angeletos and Calvet (2005) for a related study.

single-dimensional consumption/investment good. Importantly, we assume that production externalities are summarized by the whole distribution of other producers decisions. Our reduced form of technology allows for nontrivial interactions with market leaders, closely related companies, or competitive fringe in both capital and labor dimensions. We assume the continuous function f is increasing with respect to all arguments and possess increasing differences in (t, l), in (t, μ) and (l, μ). In particular, the private technologies endowed to each agent need not be convex.

The output of a particular producer can be devoted to consumption c or investment i that increases the level of capital holdings in the next period; hence, $c+i = y$. When c units of the output are consumed and labor supply is l, the agent receives $U(c, l) = u(c)+v(1-l)$ units of instantaneous utility, where we assume both $u, v : \mathbb{R} \to \mathbb{R}$ are smooth, concave and strictly increasing. Whenever $i \in I := [0, 1]$ units of the good are invested, the capital in the next period is determined stochastically with respect to probability measure $q(\cdot|i)$.[42]

Following some known complementarity conditions for joint monotone controls (see Hopenhayn and Prescott, 1992 and Mirman et al., 2008) we need to assume that: $\frac{-u''}{u'} \leq \frac{f''_{12}}{f'_1 f'_2}$. This condition requires that the degree of complementarity between private capital and labour is high relative to the curvature of the utility function and thus suffices for increasing differences of payoffs with t, l. Similar restrictions on the curvature and complementarities of preferences and technologies need to be imposed with respect to pairs $t, \mu_{T\times L}$ and $l, \mu_{T\times L}$.[43] They suffice for increasing differences of payoff with $t, \mu_{T\times L}$ and for increasing differences with respect to $l, \mu_{T\times L}$.

Given Theorem 5, we conclude[44] there exists the greatest and the least MSE for this large dynamic nonmarket economy (interpreted as a large anonymous game) and the extreme equilibria can be approximated using iterative methods. Moreover our MSE

[42] Our methods allow to analyze *two sector economies*: a consumption good sector with technology f and investment good sector with stochastic technology $q(\cdot|t, i, l, \mu)$ where μ is a distribution on types T and decisions $L \times I$. In the example we consider a simple version of q depending on investment i only.

[43] In case of differentiable $f(t, l, g)$, where $g = G(\mu_{T\times L})$ is some real valued statistics summarizing production externalities, the two required conditions read as follows: $\frac{-u''}{u'} \leq \frac{f''_{13}}{f'_1 f'_3}$, $\frac{-u''}{u'} \leq \frac{f''_{23}}{f'_2 f'_3}$.

[44] Notice, in our setting, the correspondence $A(t, l, \mu_{T\times L}) = [0, f(t, l, \mu_{T\times L})] \times L$ does not have strict complementarities. To assure that the value function v^* in (4) preserves increasing differences in (t, μ) we need to use constructions of Mirman et al. (2008) (Lemmas 11, 12, and Theorems 3, 4). They show that under assumptions stated on u, v, and f the value function posses increasing differences in t and μ.

serves as a limit of interactions with large but finite number of producers. Finally, natural comparative dynamics results follow with respect to model parameters from theorem 6.

This example is important as it highlights the differences between our results and those in the existing literature on equilibrium comparative statics for large dynamic economies (see, e.g., Acemoglu and Jensen, 2015, 2018). Specifically, we consider existence of Markov stationary transitional dynamics and comparative dynamics results. For existence of stationary equilibria and their comparative statics, given single dimensional aggregates summarizing production externalities, one can apply Theorem 5 and Lemma 1 in Acemoglu and Jensen (2015).[45] In particular, our sufficient conditions on the primitives that guarantee each player's value function has increasing differences in (t, μ) are not crucial.

8 Related literature

This paper contributes to many strands of the recent literature in economics that we briefly discuss here.

Firstly, our work contributes to the line of research in large anonymous sequential games that began in a series of papers by Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992) or Karatzas et al. (1994). Relative to these papers our work focus on the case of minimal state space stationary Markovian distributional equilibrium. Further and of independendt interest, we contribute to an important literature on the existence and characterization of *dynamic* exact law of large numbers (D-ELLN) that underpin all large anonymous stochastic games or economies. In these settings, agents/players draw a individual source of uncertainty such that in the aggregate does not generate risk. Relative to this literature, we provide a new characterization of a D-ELLN that provides a conditional *independence* of player types (relative to histories of the game), and a deterministic transition of aggregate distribution on types using rich Fubini extensions in saturated or superatomless measure spaces of players. This is not a mere technical detail;

[45] We should mention, in Acemoglu and Jensen (2015), to identify positive shocks one needs to require additional structure on primitives to preserve increasing differences between individual states and the shock parameters. That requires more assumptions than noted in their Lemma 1.

rather, in our setting, given the strategic interaction between players, our equilibrium construction cannot even proceed without an appropriate D-ELLN. Our construction builds upon the important recent contributions of Sun (2006), Keisler and Sun (2009) and Podczeck (2010).

Secondly, our paper also extends the class of games of strategic complementarities (GSC) to a dynamic setting with a measure space of players. Following the inspiring work of Van Zandt (2010) in few recent papers including Balbus et al. (2015a, 2019, 2015b), and Bilancini and Boncinelli (2016), the class of supermodular games and GSC has been extended to situations of normal-form games with complete and incomplete information. Simultaneously, a number of papers studied dynamic GSC with complete and incomplete information.[46] This paper directly relates to this literature in many ways. First, the tools used in the current paper heavily extend that developed by Balbus et al. (2013, 2014) for the constructive study of Markovian equilibria for the finite number of players game. In doing so, we provide sufficient conditions for preserving dynamic complementarities between the periods to player's value functions. The conditions allow to avoid many of the issues related to notion of extensive-form supermodular games discussed in Vives (2009), Echenique (2004b), Amir (2002) e.g. These new conditions imply that the value function has increasing differences between private type and aggregate distribution. Very importantly, with these sufficient structure in place, our large stochastic supermodular games remain extensive-form supermodular over the *infinite horizon*. This is critical for our equilibrium comparative dynamic/statics results. Given the distributional game specification, and the structural properties implied by our D-ELLN that describes the evolution of distributional state variables on types, we are able to avoid many of problems in characterizing dynamic complementarities in actions between periods and beliefs reported recently by Mensch (2020). Finally, as in the work of Balbus et al. (2014), our proofs are constructive, where the computation of equilibrium comparative statics/dynamics are also construction via simple successive approximations. In this sense, we are able to provide the applied researchers with tools allowing to approximate the equilibrium distributions.

[46] See the seminal papers of Curtat (1996), Amir (2005), or more recent Mensch (2020) contribution for incomplete information settings.

Importantly, our paper also contributes to the large literature on characterizing the equilibrium comparative statics and dynamics for large dynamic economies and games. The literature on large dynamic economies is extensive and we refer the reader to Acemoglu and Jensen (2015) and Light and Weintraub (2019) for an excellent discussion and citations. In particular, our results provide a foundation for a theory of comparative monotone *dynamics* results relative to ordered perturbations of the space of games/economies. Specifically, we provide sufficient conditions on payoffs and transition probabilities such that sequence of equilibrium distributions as well as the aggregate law of motion (specifying transition dynamics but also rational beliefs is our game) are monotone in type for any positive shock. Interestingly, our methods extend equilibrium comparative *statics* results of Adlakha and Johari (2013), Acemoglu and Jensen (2015, 2018), and Light and Weintraub (2019), where the authors are concerned with the comparative statics of invariant distributions or "stochastic steady states". Further, in many of these papers, the results only apply to equilibrium *aggregates*, where one assumes in the large dynamic economic/large anonymous game the existence of a convexifying and real valued (or totally ordered) aggregate. Our approach contains this approach as a special case. For example, in our setting, we can study stationary equilibrium (and Markovian distributional equilibrium) of a new class of quantile games, and are able there to perform multi-dimensional equilibrium comparative static/dynamics relative to a (infinite dimensional) set of equilibrium distributions. It also bears mentioning the assumptions of Acemoglu and Jensen (2015, 2018), and Light and Weintraub (2019) are not sufficient to obtain results of our paper. The key central difference between our work and these papers is that when studying stationary equilibrium (or mean-field equilibrium) comparative statics, one does *not* need conditions on the game that imply *single crossing in distribution* between private actions and aggregates. This is because one is only characterizing the "steady state" structure of the sequential or Markovian equilibrium. For the results in the present paper on Markovian Distributional Equilibrium, one must deal with the influence of perturbations of dynamic interactions between players and their distributional counterparts via *the value function* that is needed to recursively define each player's stage game payoffs. In additional, one must study the equilibrium structure *away* from the fixed points of the equilibrium law of motion. To do this, one needs increasing differences of payoffs between types and aggregate distributions.[47] Additionally, recall we compare distributions over $\mathbb{R}^n$, and not simply there moments. Set of such objects is not a lattice, hence necessity of our new equilibrium comparative statics tools we provide in Theorem 3 in the paper. Finally, our monotone comparative statics/dynamics results are also shown to be *computable*, as we characterize the chain of parameterized equilibria converging to the one of interest for a particular parameter. This is of utmost importance for applied economists that calibrate the equilibrium invariant distributions' moments, or attempt to develop econometric methods for estimating equilibrium comparative statics/dynamics in data (e.g., via the quantile methods of Echenique and Komunjer (2009, 2013)).

Finally, our paper is related to the recent work on oblivious (or stationary) equilibrium and mean-field games. There is a large and growing literature including papers by Weintraub et al. (2008, 2011), Adlakha and Johari (2013), Adlakha et al. (2015), Doncel et al. (2016), Lacker (2018), and Light and Weintraub (2019), among many others.[48] This work is mainly driven by computability and complexity considerations, and many of these papers build methods for games in continuous time, with finitely many states, finite actions sets, symmetric equilibrium in mixed strategies, where games externalities are characterized by distributions or aggregates on states only (so not on actions). Equilibrium of such games is a (stationary) distribution on players states. Such mean field equilibrium implies a best response *oblivious* strategy, i.e. distribution on action sets, where each players' action is optimal taking the invariant mean field distribution as given. For some recent progress on this line of literature we refer the reader to e.g. Adlakha et al. (2015). In a related context, we also extend a very interesting recent result of Kalai and Shmaya (2018) on foundations of epsilon Bayesian Nash equilibrium of a finite

[47] Actually, characterizing sufficient conditions for single crossing in distribution with respect to beliefs in static large, Bayesian games with strategic complementarities is a challenge. See, for example, Balbus et al. (2015a) and Liu and Pei (2017).

[48] For related work on large dynamic supermodular games see the work of Wiecek (2017), who analyses a case of supermodular game in continuous time, in which each player moves in a discrete but different period of time. Moreover, Adlakha and Johari (2013), study a mean-field version on our large dynamic supermodular game with single dimmentional actions and strategic interaction via distribution on types. For such environment they show existence and of a mean-field equilibrium, i.e. an oblivious strategy and invariant distribution.

number of players game via *imagined-continuum* equilibrium.[49] An imagined-continuum is a powerful, and tractable, tool that as itself is a behavioral concept of equilibrium in a Bayesian game, where although the players are playing a game with a finite number of players, they view the equilibrium interaction and learning (and in particular, their belief formation) as in a game with a continuum of players. For this setting, we show that the equilibrium of the imagine-continuum version of the Bayesian game converges to the stationary Markovian equilibrium of the actual game. Our paper extends Kalai-Shmaya setting to non-stationary equilibria and does so without imposing assumption that players take the equilibrium aggregates as given.

A Appendix

A.1 Auxiliary results

Lemma 9. *Let* $(\Xi, \leq)$ *be a poset with its order topology. Suppose* $f_k : \Xi \to \mathbb{R}$ *is a sequence of increasing functions and monotone inf-preserving. Then if* $x_k \downarrow x$ *in* Ξ *and* $f_k \downarrow f$ *pointwise* $(k \to \infty)$*. Then* $f_k(x_k) \to f(x)$*.*

Proof. Let $n \in \mathbb{N}$. Since f_k is decreasing sequence of increasing functions and $x_k \downarrow x$ hence for $k > n$

$$f(x) \leq f_k(x_k) \leq f_k(x_n).$$

We take a limit $k \to \infty$ and we obtain

$$f(x) \leq \liminf_{k\to\infty} f_k(x_k) \leq \limsup_{k\to\infty} f_k(x_k) \leq f(x_n).$$

to finish the proof we just take a limit $k \to \infty$. $\square$

[49] A "reversed" question concern the following question: as the number of players in a finite player game gets "large", in what sense does the set of Nash equilibrium converge. Some interesting examples of non-convergence of mean-field game limits are provided in Nutz et al. (2019). Also, see Carmona et al. (2017) for a general discussion of this mean-field/idealized limit convergence literature. These results are very important, as they have turned out to be critical, for example, in econometric applications for large games. See Menzel (2016) for a discussion.

Lemma 10. *Let $(\nu_k)_{k\in\mathbb{N}}$ be a sequence of probability measures on common Polish space S and $(h_k)_{k\in\mathbb{N}}$ be a sequence of bounded measurable and bounded real valued functions on S. Suppose $\nu_k \downarrow \nu$ (i.e. in stochastic dominance order and weak topology) and $h_k \downarrow h$. Then $\lim_{k\to\infty} \int h_k d\nu_k = \int h d\nu$.*

Proof. It is a consequence of Lemma 9 with Ξ as a space of bounded measurable real valued functions on S, and $f_k(x) := \int_S x(s)\nu_k(ds)$, $x_k(s) = h_k(s)$. □

Lemma 11. *Let S_1 and S_2 be topological spaces and $f : S_1 \times S_2 \mapsto \mathbb{R}$ be jointly continuous function. Let $\Gamma : S_1 \mapsto S_2$ be a continuous and compact valued correspondence. Put $\Gamma^*(x) := arg \max\limits_{y\in\Gamma(x)} f(x, y)$. Let $x_k \to x$ in S_1, and $y_k \to y$ in S_2, and $y_k \in \Gamma^*(x_k)$. Suppose that $x_k \to x$, and $y_k \to y$ as $k \to \infty$. Then $y \in \Gamma^*(x)$.*

Proof. Let $y' \in \Gamma(x)$ be given. By continuity of Γ at x, for $k \in \mathbb{N}$ there is $y'_k \in \Gamma(x_k)$ such that $y'_k \to y'$. Observe that since $y_k \in \Gamma^*(x_k)$, hence $f(x_k, y_k) \geq f(x_k, y'_k)$ for all $k \in \mathbb{N}$. By joint continuity of f we have $f(x, y) \geq f(x, y')$. Since $y' \in \Gamma(x)$ is arbitrary, hence $y \in \Gamma^*(x)$. □

A.2 Proofs

Proof of Theorem 2. We follow the argument in Echenique (2005). Let $\bar{x}$ be the greatest element of X. Let $\mathscr{I}$ be a set of ordinal numbers with cardinality strictly greater than X. Let us define the following transfinite sequence with the initial element $x_0 = \bar{x}$ and for $i \in \mathscr{I} \setminus \{0\}$ we define:

$$x_i = \bigwedge \{f(x_j) : j < i\}.$$

We prove that x_i is a well defined decreasing sequence. Clearly $x_1 = f(x_0) \leq x_0$. Suppose that $(x_j)_{j<i}$ is well defined and decreasing for some i. Then $(f(x_j))_{j<i}$ is a decreasing sequence. Hence it has an infimum, and it is exactly x_i. Consequently x_j is well defined and decreasing on $[0, i]$. Hence by transfinite induction, the transfinite sequence $(x_i)_{i\in\mathscr{I}}$ is well defined and decreasing. Since $\mathscr{I}$ has the cardinality strictly greater than X, there exists no one to one mapping between $\mathscr{I}$ and X. Consequently we find the least element

$\bar{i} \in \mathscr{I}$ of the set $\{i \in \mathscr{I} : x_i = x_{i+1}\}$. Then $x_{\bar{i}} = x_{\bar{i}+1} = f(x_{\bar{i}})$, and $e^* := x_{\bar{i}}$ is a fixed point of f. We show, $e^* = \bigvee\{x \in X : f(x) \geq x\}$. Put $\mathcal{X} := \{x \in X : f(x) \geq x\}$ Obviously $e^* \in \mathcal{X}$. Pick another $y \in \mathcal{X}$. Obviously $y \leq x_0$ and suppose there is $i \in \mathscr{I}$ such that $y \leq x_j$ for any $j < i$. Since $y \in \mathcal{X}$, hence and by transfinite induction hypothesis we have $y \leq f(y) \leq f(x_j)$. Hence $y \leq \bigwedge\{f(x_j) : j \leq i\}$, and consequently $y \leq x_i$ for any $i \in \mathscr{I}$. In particular, $y \leq x_{\bar{i}} = e^*$. □

Proof of Theorem 4. By Proposition 5.6 of Sun (2006) and Theorem 1 in Podczeck (2010) there is a probability space $(\Omega, \mathcal{F}, P)$ and a rich Fubini extension of a natural product space on $\Lambda \times \Omega$, denoted by $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Consequently, we can find a process $\eta : \Lambda \times \Omega \to [0,1]$ such that the family $(\eta_\alpha)_{\alpha \in \Lambda}$ is essentially pairwise independent with the uniform distribution on $[0,1]$.

Define $(\eta_n)_{n \in \mathbb{N}}$ as a set of independent copies of η. We construct a sequence $(X_n)_{n=1}^{\infty}$ satisfying the thesis (i)–(iii). Let $(I, \mathcal{I}, \iota)$ be the standard interval, i.e., $I = [0,1]$, with $\mathcal{I}$ Borel sets, and the Lebesgue measure ι. Furthermore, for any $\mu \in \mathcal{M}$ there is a $(\mathcal{I} \otimes \mathcal{T} \otimes \mathcal{A})$-measurable function $G^\mu : I \times T \times A \mapsto T$ such that

$$\iota\left(G^\mu_{(t,a)}\right)^{-1}(Z) \;=\; \iota\Big(\big\{l \in I : G^\mu(l,t,a) \in Z\big\}\Big) \;=\; q(Z|t,a,\mu),$$

for any $Z \in \mathcal{T}$.[50] For any initial distribution $\tau_0 \in \mathcal{M}_T$, there exists a T-valued $(I \otimes \mathcal{T})$-measurable function $\tilde{G}$ such that $\tau_0 = \iota\tilde{G}^{-1}$.[51] Put $X_1 := \tilde{G}(\eta_1)$. Having the initial random variable X_1, define the following process $X_{n+1} = G^{\mu_n}(\eta_{n+1}, K_n)$, for $n > 1$, where $K_n := (X_n, \sigma(X_n, \tau_n))$ and

$$\tau_n \;:=\; (\lambda \boxtimes P)X_n^{-1} \quad \text{and} \quad \mu_n \;:=\; (\lambda \boxtimes P)K_n^{-1}.$$

As usual, put $(K_n)_\alpha(\omega) := K_n(\alpha, \omega)$ for $(\alpha, \omega) \in \Lambda \times \Omega$.

Let $\mathcal{S}_n := \sigma\big(\{\eta_k : k \leq n\}\big)$. By definition of X_1 and X_{n+1}, we conclude that X_n is $\mathcal{S}_n$-measurable. Hence, $(X_n)_\alpha$ and $(\eta_{n+1})_\alpha$ are independent, for λ-almost every $\alpha \in \Lambda$.

Now we show that (i)–(ii) are satisfied. We do it by induction with respect to n. For $n = 1$, the claim holds by essential independence of η_1 and X_1. Moreover, by Theorem 1,

[50] For example, see Lemma A5 in Sun (2006).

[51] Again, see Lemma A5 in Sun (2006).

for P-almost every $\omega \in \Omega$ the sampling distribution $\lambda(X_1)_\omega^{-1}$ of X_1, i.e., satisfies

$$\lambda(X_1)_\omega^{-1} = (\lambda \boxtimes P)X_1^{-1} = \tau.$$

Again by Theorem 1, for P-almost all $\omega \in \Omega$, we have

$$\lambda(K_1)_\omega^{-1} = (\lambda \boxtimes P)K_1^{-1} := \mu_1.$$

Hence, (ii) is satisfied for $n = 1$. Suppose that for some $n \geq 1$ both (i) and (ii) are true. Observe that $((\eta_{n+1})_\alpha, (X_n)_\alpha)_{\alpha\in\Lambda}$ is a family $(\lambda \otimes \lambda)$-almost everywhere pairwise conditionally independent random variables. It follows from induction hypothesis for $(X_n)_\alpha$ and from the previous observation the random variables $(X_n)_\alpha$ and $(\eta_{n+1})_\alpha$ are independent λ-almost surely. Hence, by construction of X_{n+1}, the family $((X_{n+1})_\alpha)_{\alpha\in\Lambda}$ is $(\lambda \otimes \lambda)$-almost surely pairwise conditionally independent. Hence the property (i) is satisfied for $(n + 1)$. The application of Theorem 1 we obtain (ii) for $(n + 1)$; hence, (i) and (ii) hold for all $n \geq 1$.

Next we show that (iii) is satisfied. Let $(\mathcal{S}_n)_\alpha := \sigma(\{(\eta_k)_\alpha : k \leq n\})$ and $(\Sigma_n)_\alpha := \sigma(\{(X_k)_\alpha : k \leq n\})$. By definition of X_n and $(\Sigma_n)_\alpha$ we conclude that

$$\sigma((X_n)_\alpha) \subset (\Sigma_n)_\alpha \subset (\mathcal{S}_n)_\alpha.$$

Let E be the standard expectation with respect to P. Hence the conditional distribution of $(X_{n+1})_\alpha$ with respect to $(\Sigma_n)_\alpha$ satisfies

$$\begin{aligned} P((X_{n+1})_\alpha \in Z|(\Sigma_n)_\alpha) = & \quad E(P((X_{n+1})_\alpha \in Z|(\mathcal{S}_n)_\alpha)|(\Sigma_n)_\alpha) \\ = & \quad E(P(G^{\mu_n}((\eta_{n+1})_\alpha, (K_n)_\alpha) \in Z|(\mathcal{S}_n)_\alpha)|(\Sigma_n)_\alpha) \\ = & \quad E(q(Z|(K_n)_\alpha, \mu_n)|(\Sigma_n)_\alpha) = q(Z|(X_n)_\alpha, \sigma^*((X_n)_\alpha, \tau_n), \mu_n) \end{aligned}$$

for λ-a.a. $\alpha \in \Lambda$ and all $Z \in \mathcal{T}$. The last equality follows from the independence between $(\eta_{n+1})_\alpha$ and $(X_n)_\alpha$. Hence (iii) is satisfied. $\square$

Proof of Lemma 1. Suppose $v_n \in \mathcal{V}$ for all $n \in \mathbb{N}$ and $v_n \rightrightarrows v$. Furthermore, let μ_k, and Φ_k be decreasing sequences in $\mathcal{M}$ and respectively in $\mathcal{D}$ such that $\mu_k \to \mu$ weakly and $\Phi_k \to \Phi$ pointwise as $k \to \infty$. Let $t \in T$ and let $\epsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ and $n \geq n_0$ we have

$$\begin{aligned} |v(t,\mu_k,\Phi_k) - v(t,\mu,\Phi)| &\leq |v(t,\mu_k,\Phi_k) - v_n(t,\mu_k,\Phi_k)| + |v_n(t,\mu_k,\Phi_k) - v_n(t,\mu,\Phi)| \\ &\quad + |v_n(t,\mu,\Phi) - v(t,\mu,\Phi)| \\ &\leq \frac{2}{3}\epsilon + |v_n(t,\mu_k,\Phi_k) - v_n(t,\mu,\Phi)| \end{aligned} \tag{12}$$

Now fix $n \in \mathbb{N}$ satisfying (12). Then, since $v_n \in \mathcal{V}$, hence for large enough k it holds

$$|v_n(t,\mu_k,\Phi_n) - v_n(t,\mu,\Phi)| \leq \frac{\epsilon}{3}. \tag{13}$$

By (12) and (13) $|v(t,\mu_k,\Phi_k) - v(t,\mu,\Phi)| < \epsilon$ for large k. Hence v is mon-sup-inf preserving, consequently $v \in \mathcal{V}$. □

Continuation of the proof to Lemma 4. We prove (vi). Finally we prove (vii). Observe that F is a Carathéodory function in (t,a) i.e. measurable in t and continuous in a. It follows from Assumptions 2, definition of $\mathcal{V}$ and Lemma 4 (ii). Hence, by Lemma 1 and Measurable Maximum Theorem (Theorem 18.19 in Aliprantis and Border (2006)) the correspondence $\Gamma(a,t;v,\Phi)$ is measurable in (t,a), hence by Lemma 18.2 in Aliprantis and Border (2006) is weakly measurable. For each $j = 1,2,\ldots,k$ the correspondence (drop all arguments but (t,a) from Γ and $\overline{\gamma}$ for short)

$$\Gamma^j(t,a) := \arg\max_{a' \in \Gamma(a,t)} \pi_j(a'),$$

where $\pi_j(a)$ is a projection of the vector a into $j-$ the coordinate. Obviously, π_j is a Carathéodory function as a function of (t,a). Again by Measurable Maximum Theorem $\tilde{\pi}_j(t) := \max_{a' \in \Gamma(a,t)} \pi_j(a)$ is measurable. Observe that by part (ii) of this lemma it follows that

$$\overline{\gamma}(t) = (\tilde{\pi}_1(t), \tilde{\pi}_2(t), \ldots, \tilde{\pi}_k(t)).$$

Since all coordinates are measurable, hence $\overline{\gamma}$ is measurable as well. Similarly we can prove that $\underline{\gamma}$ is measurable in t. □

Proofs for the approximation results

Proof of Lemma 8 Suppose that $f : T \times A \mapsto \mathbb{R}$ is a continuous (bounded) function i.e. $f \in C(T \times A)$. Obviously

$$\frac{1}{N} f(\xi^N(\omega), \eta^N(\omega)) \to 0, \text{ surely, i.e. for any } \omega \in \Omega.$$

Moreover, applying standard Kolmogorov Law of Large Numbers Theorem we have

$$\lim_{N\to\infty} \frac{1}{N-1} \sum_{l\neq j} f(\tilde{T}_l, \sigma_n(\tilde{T}_l)) = \int_T f(t, \sigma_n(t))\tau_n(dt) = \int_{T\times A} f(t,a)\tau_n \star \sigma_n(dt\times da), \quad \mathbb{P}-a.s.$$

Consequently for $\mathbb{P}$-a.a. $\omega \in \Omega$

$$\lim_{N\to\infty} \int_{T\times A} f(t,a)\hat{\mu}_n^N((\tilde{T}_{-j}, \xi^N), \eta^N) = \int_{T\times A} f(t,a)\tau_n \star \sigma_n(dt\times da). \tag{14}$$

Let $\mathbf{F}$ be a countable and dense set in $C(T \times A)$. Let $\tilde{\Omega} \subset \Omega$ be a set such that any element of $\mathbf{F}$ obeys (14). Then $\mathbb{P}(\tilde{\Omega}) = 1$. We show that (14) holds for any $f \in C(T \times A)$ whenever $\omega \in \tilde{\Omega}$. Let $\epsilon > 0$ be given. Since $\mathbf{F}$ is dense in $C(T \times A)$ we can pick $f_0 \in \mathbf{F}$ such that $||f - f_0||_\infty < \frac{\epsilon}{3}$. Then

$$\int_{T\times A} |f(t,a) - f_0(t,a)|\hat{\mu}_n^N((\tilde{T}_{-j}, \xi^N), \eta^N)(dt\times da) \leq \frac{\epsilon}{3} \tag{15}$$

and

$$\int_{T\times A} |f(t,a) - f_0(t,a)|\tau_n \star \sigma_n(dt\times da) \leq \frac{\epsilon}{3}. \tag{16}$$

Hence

$$\begin{aligned} \left| \int_{T\times A} f(t,a)\hat{\mu}_n^N((\tilde{T}_{-j}, \xi^N), \eta^N)(dt\times da) - \int_{T\times A} f(t,a)\tau_n \star \sigma_n(dt\times da) \right| \\ \leq \int_{T\times A} |f(t,a) - f_0(t,a)|\hat{\mu}_n^N((\tilde{T}_{-j}, \xi^N), \eta^N)(dt\times da) \\ + \int_{T\times A} |f(t,a) - f_0(t,a)|\tau_n \star \sigma_n(dt\times da) \\ + \left| \int_{T\times A} f_0(t,a)\hat{\mu}_n^N((\tilde{T}_{-j}, \xi^N), \eta^N)(dt\times da) - \int_{T\times A} f_0(t,a)\tau_n \star \sigma_n(dt\times da) \right| \\ \leq \frac{2}{3}\epsilon + \left| \int_{T\times A} f_0(t,a)\hat{\mu}_n^N((\tilde{T}_{-j}, \xi^N), \eta^N)(dt\times da) - \int_{T\times A} f_0(t,a)\tau_n \star \sigma_n(dt\times da) \right|, \end{aligned} \tag{17}$$

where the last inequality follows from (15) and (16). Since $\omega \in \tilde{\Omega}$, hence there exists an integer N_0 such that for any $N > N_0$

$$\left| \int_{T\times A} f_0(t,a)\hat{\mu}_n^N((\tilde{T}_{-j},\xi^N),\eta^N)(dt \times da) - \int_{T\times A} f_0(t,a)\tau_n \star \sigma_n(dt\times da) \right| < \frac{\epsilon}{3}. \quad (18)$$

Combining (17) and (18) for $N > N_0$ we have

$$\left| \int_{T\times A} f(t,a)\hat{\mu}_n^N((\tilde{T}_{-j},\xi^N),\eta^N)(dt \times da) - \int_{T\times A} f(t,a)\tau_n \star \sigma_n(dt\times da) \right| < \epsilon. \quad (19)$$

Since $\epsilon > 0$, the inequality (19) implies that the equation (14) holds for f and $\omega \in \tilde{\Omega}$. Since $f \in C(T\times A)$ is arbitrary and $\mathbb{P}(\tilde{\Omega}) = 1$, it follows that

$$\hat{\mu}_n^N((\tilde{T}_{-j},\xi^N),\eta^N) \Rightarrow \tau_n \star \sigma_n \quad \text{almost surely.}$$

□

Recall that $\tilde{v}_1^N(t) := \sup_{\pi\in\Sigma} \mathcal{R}(\sigma^{-j},\pi)(t)$. Then, Bellman equations for optimal values $\tilde{v}_n^N$ updated for any $n \in \mathbb{N}$ have the form

$$\tilde{v}_n^N(t) = \max_{a\in\tilde{A}(t,\tau_n)} \left\{ (1-\beta)r_n^N(t,a) + \beta \int_T \tilde{v}_{n+1}^N(t')q_n^N(ds'|t,a) \right\}. \quad (20)$$

Let us define $\mathbf{C}$ as a set of continuous real valued functions on T commonly bounded by $\bar{r}$. Clearly, it is a closed subset of Banach space. The metric in product space $\mathcal{C} := \mathbf{C}^\infty$ is embedded in the natural Banach space with a norm defined as follows. For $v = (v_n)_{n\in\mathbb{N}}$ we put

$$||v||^\zeta := \sum_{n=1}^\infty \frac{1}{\zeta^{n-1}} \sup_{t\in T} |v_n(t)|,$$

where $\zeta \in \left(0, \frac{1}{\beta}\right)$ is a fixed value. Clearly, $v^N \to v$ in $||\cdot||^\zeta$ as $N\to\infty$ if and only if, $v_n^N \rightrightarrows v_n$ for any $n \in \mathbb{N}$ as $N \to \infty$.

Let $v \in \mathcal{C}$ and $t \in T$. Put $B^N(v)(t) := \left(B_n^N(v)(t)\right)_{n\in\mathbb{N}}$ where

$$B_n^N(v)(t) = \max_{a\in\tilde{A}(t,\tau_n)} \left\{ (1-\beta)r_n^N(t,a) + \beta \int_T v_{n+1}(t')q_n^N(dt'|t,a) \right\}.$$

Similarly, define $\mathcal{B}^N(v)(t) := \left(\mathcal{B}_n^N(v)(t)\right)_{n\in\mathbb{N}}$ where

$$\mathcal{B}_n^N(v)(t) = (1-\beta)r_n^N(t,\sigma_n(t)) + \beta \int_T v_{n+1}(t')q_n^N(dt'|t,\sigma_n(t)).$$

For $v \in \mathcal{C}$, let $B^\infty(v)(t) := (B^\infty_n(v)(t))_{n\in\mathbb{N}}$ where

$$B^\infty_n(v) := \max_{a\in\tilde{A}} \left\{ (1-\beta) r_n(t,a) + \beta \int_T v_{n+1}(t') q_n(dt'|t,a) \right\},$$

where

$$r_n(t,a) := r(t,a,\tau_n \star \sigma_n) \qquad \text{and } q_n(\cdot|t,a) := q(\cdot|t,a,\tau_n \star \sigma_n),$$

for $(t,a) \in Gr(\tilde{A}(\cdot,\tau_n))$. Similarly define $\mathcal{B}^\infty(v)(t) := (B^\infty_n(v)(t))_{n\in\mathbb{N}}$ where

$$\mathcal{B}^\infty_n(v)(t) = (1-\beta) r_n(t,\sigma_n(t)) + \beta \int_T v_{n+1}(t') q_n(dt'|t,\sigma_n(t)).$$

Now we prove basic properties of B^N and B^∞.

Lemma 12. *Let σ be a Borel measurable function with respect to the first argument. Then*

(i) All $B^N, \mathcal{B}^N_n$, B^∞ and $\mathcal{B}^\infty_n$ map $\mathcal{C}$ into itself;

(ii) B^N, $\mathcal{B}^N_n$, B^∞ and $\mathcal{B}^N_n$ are $\beta\zeta$-contraction mappings on $\mathcal{C}$;

(iii) If $v^N \to v$ in $\mathcal{C}$ as $N \to \infty$, then $B^N(v^N) \to B^\infty(v)$, and $\mathcal{B}^N(v^N) \to \mathcal{B}^\infty(v)$ in $\mathcal{C}$ as $N \to \infty$;

(iv) let $\tilde{v}^N$ be the fixed point of B^N and let $\tilde{v}^\infty$ be the fixed point of B^∞ in $\mathbf{C}$; Then $||\tilde{v}^N - \tilde{v}^\infty||_\infty \to 0$ as $N \to \infty$;

(v) let $\check{v}^N$ be the fixed point of $\mathcal{B}^N$ and let $\check{v}^\infty$ be the fixed point of $\mathcal{B}^\infty$ in $\mathbf{C}$; Then $||\check{v}^N - \check{v}^\infty||_\infty \to 0$ as $N \to \infty$;

Proof. Proof of (i) Let $v \in \mathcal{C}$. Observe by Assumptions 6 for any n and N the following functions

$$\Pi^N_n(t,a,v) = (1-\beta) r^N_n(t,a) + \beta \int_T v_{n+1}(t') q^N_n(dt'|t,a)$$

and

$$\Pi^\infty_n(t,a,v) = (1-\beta) r_n(t,a) + \beta \int_T v_{n+1}(t') q_n(dt'|t,a),$$

are both continuous in (t,a). Since $B^N_n(v)(t) = \max_{a\in\tilde{A}(t,\tau_n)} \Pi^N_n(t,a,v)$ and $B^\infty_n(v)(t) = \max_{a\in\tilde{A}(t,\tau_n)} \Pi^\infty_n(t,a,v)$, hence by Berge Maximum Theorem, the proof of (i) complete.

Proof of (ii). It is routine to verify, for $v, w \in \mathcal{C}$

$$||B_n^N(v) - B_n^N(w)||_\infty \leq \beta ||v_{n+1} - w_{n+1}||_\infty.$$

Dividing both sides by ζ^{n-1} and summing over n we obtain

$$||B_n^N(v) - B_n^N(w)||^\zeta = \sum_{n=1}^{\infty} \frac{||B_n^N(v) - B_n^N(w)||_\infty}{\zeta^{n-1}} \leq \beta\zeta \sum_{n=1}^{\infty} ||v_n - w_n||_\infty = \beta\zeta ||v_n - w_n||_\infty.$$

Similar argument works with B^∞.

Proof of (iii). Suppose that for all $N \in \mathbb{N}$, $(t^N, a^N) \in \tilde{A}(t^N, \tau_n)$, and $v^N \to v$ in $(\mathcal{C}, ||\cdot||_\infty)$, and $(t^N, a^N) \to (t, a)$ as $N \to \infty$. We show that

$$\Pi_n^N(t^N, a^N, v^N) \to \Pi_n^\infty(t, a, v) \tag{21}$$

By Lemma 8 and Assumption 6 we have that $r_n^N(t^N, a^N) \to r_n(t, a)$ and $q_n^N(\cdot|t^N, a^N) \to^* q_n(\cdot|t, a)$ as $N \to \infty$. Hence (21) holds. Furthermore, by (i) it follows that there is t^N such that

$$\sup_{t \in T} |B_n^N(v^N)(t) - B_n^\infty(v)(t)| = ||B_n^N(v^N)(t^N) - B_n^\infty(v)(t^N)||.$$

Without loss of generality suppose $t^N \to t$ as $t^N \to t$. Combining definition of r_n and q_n in (10) and (11), next Lemma 8 and finally (21) it follows that the right hand side above tends to 0. Hence, $||B^N(v^N) - B^\infty(v^\infty)||^\zeta \to 0$ as $N \to \infty$.

Proof of (iv). Observe that

$$\begin{aligned} ||\tilde{v}^N - \tilde{v}^\infty||^\kappa = ||B^N(\tilde{v}^N) - B^\infty(\tilde{v}^\infty)||^\zeta &\leq ||B^N(\tilde{v}^N) - B^N(\tilde{v}^\infty)||^\zeta + ||B^N(\tilde{v}^\infty) - B^\infty(\tilde{v}^\infty)||^\zeta \\ &\leq \beta\zeta ||\tilde{v}^N - \tilde{v}^\infty||^\zeta + ||B^N(\tilde{v}^\infty) - B^\infty(\tilde{v}^\infty)||^\zeta, \end{aligned}$$

where the last inequality follows from (ii). Hence

$$||\tilde{v}^N - \tilde{v}^\infty||^\kappa \leq \frac{1}{1 - \beta\zeta} ||B^N(\tilde{v}^\infty) - B^\infty(\tilde{v}^\infty)||^\zeta.$$

To finish the proof we only take $N \to \infty$, since by (iii) the right hand side above tends to 0.

Proof of (v) is similar to (iv). □

Lemma 13. *Consider MDP and suppose that $(\tau_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ are implied sequences of distribution on types and policies for some MSE (μ^*, Φ^*). Then the sequences of value functions $\bar{v}$ for (μ^*, Φ^*) is a common fixed point of B^∞ and $\mathcal{B}^\infty$. As a result, $\bar{v} = \tilde{v}^\infty = \check{v}^\infty$.*

Proof. By Lemma 12 it follows that B^∞ and $\mathcal{B}^\infty$ are both contractions on $\mathcal{C}$, hence we only need to show $\bar{v}$ is the fixed point of B^∞ and $\mathcal{B}^\infty$. By definition of $\bar{v}$, v^*, μ_n and τ_n, for any $t \in T$ we have $\bar{v}_n(t) = v^*(t, \tau_n, \Phi^*)$ and

$$\begin{aligned}
\bar{v}_n(t) &= \max_{a\in\tilde{A}(t,\tau_n)} \left\{(1-\beta)r(t,a,\mu_n) + \beta\int_T v^*(t', \mu_{n+1}, \Phi^*)q(dt'|t,a,\mu_n)\right\} \\
&= \max_{a\in\tilde{A}(t,\tau_n)} \left\{(1-\beta)r(t,a,\mu_n) + \beta\int_T \bar{v}_{n+1}(t')q(dt'|t,a,\mu_n)\right\} \\
&= \max_{a\in\tilde{A}(t,\tau_n)} \left\{(1-\beta)r(t,a,\tau_n\star\sigma_n) + \beta\int_T \bar{v}_{n+1}(t')q(dt'|t,a,\tau_n\star\sigma_n)\right\} \\
&= B_n^\infty(\bar{v}_{n+1})(t).
\end{aligned}$$

Hence $\bar{v} = B^\infty(\bar{v})$ and by uniqueness of the fixed point of B^∞, $\bar{v} = \tilde{v}^\infty$. By the same argument we obtain $\bar{v} = \mathcal{B}^\infty(\bar{v})$, and $\bar{v} = \check{v}$. □

Proof of Theorem 7. Let $\epsilon > 0$. Let $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ be an associated sequential policy function for (μ^*, Φ^*). Suppose that player j unilaterally deviates from σ using π instead. By definition, for any $t \in T$ we have

$$\mathcal{R}^N((\sigma)^{-j}, \pi)(t) - \check{v}_1^N(t) \leq \tilde{v}_1^N(t_1^j) - \check{v}_1^N(t_1^j) \leq ||\tilde{v}_1^N - \check{v}_1^N||_\infty. \tag{22}$$

By Lemma 12, $\tilde{v}_1^N \rightrightarrows v_1^\infty$ and $\check{v}_1^N \rightrightarrows \check{v}_1^\infty$. Observe that since the policy is $\sigma = \sigma^*$ and the initial state is $\tau_1 = \tau^*$, hence by Lemma 13 $\check{v}_1^\infty = v^\infty$.

As a result, for large enough N $||\tilde{v}_1^N - \check{v}_1^N||_\infty < \epsilon$, hence the right hand side in (22) is less than ϵ. □

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