Time consistent equilibria in dynamic models with recursive payoffs and behavioral discounting*

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Abstract

We prove existence of time consistent equilibria in a wide class of dynamic models with recursive payoffs and generalized discounting involving both behavioral and normative applications. Our generalized Bellman equation method identifies and separates both: recursive and strategic aspects of the equilibrium problem and allows to precisely determine the sufficient assumptions on preferences and stochastic transition to establish existence. In particular we show existence of minimal state space stationary Markov equilibrium (a time-consistent solution) in a deterministic model of consumption-saving with beta-delta discounting and its generalized versions involving magnitude effects, non-additive payoffs, semi-hyperbolic or hyperbolic discounting (over possibly unbounded state and unbounded above reward space). We also provide an equilibrium approximation method for a hyperbolic discounting model.

Keywords: Behavioral discounting; Time consistency; Markov equilibrium; Existence; Approximation; Generalized Bellman equation; Hyperbolic discounting; Semi-hyperbolic discounting; Quasi-hyperbolic discounting

JEL classification: C61, C73

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1 Introduction

Since the seminal work of Ramsey (1928) and Samuelson (1937), the question of how agents in dynamic choice models discount future utility streams has been a central focus of large body of research. Ramsey (1928) suggested intertemporal utilities be modeled as weighted sum of future utilities, while Samuelson (1937) proposed exponential discounting. With the work of Koopmans (1960), however, on the axiomatic foundations of dynamically consistent choice, it become clear how profoundly these two situations differ. At this point in time, many researchers adopted a dynamically consistent approach, and exponential discounting as a standard to modeling preferences in various dynamic problems.

Strotz (1956) however proposed a theory of dynamically inconsistent choice, and with his paper started a new and separate line of research studying its implications in intertemporal economic models. With the important work of Laibson (1997), models with dynamically inconsistent preferences have become workhorse tools in behavioral economics that challenge the rational foundations of dynamic choice. Motivation for studying such models with dynamically inconsistent preferences is found in a large empirical and experimental literature where numerous papers have documented the importance of preference reversals when modeling how agents compare current vs. future utilities. These empirical results over the last two decades have led to a subsequent resurgence of theoretical work that seeks (i) to provide further axiomatic foundations to time inconsistent choice,[1] as well as (ii) tools constructing theories of coherent dynamic choice in various settings, where agents have changing intertemporal tastes. Work on self-control, the role of impulse and temptation, and time consistency in dynamic choice has appeared in many fields such as mathematical psychology, political science, philosophy, decision theory, game theory, as well as economics. It included studies of consumption-savings, dynastic choice with altruistic or paternalistic preferences, dynamic collective household choice, distributive justice and dynamic social choice, public policy design, models of social discounting in environmental cost-benefit analysis, theories of endogenous preference formation and reference points including theories of habit-formation, addiction, focus-weighted choice and salience, and dynamic random utility. And although much of this has focused on its positive aspects, recent work has also begun to addresses welfare issues, including how to design the optimal policy, how to assess paternalistic policies that seek to "improve" agents welfare in the presence of dynamically inconsistent choice, as well as the welfare

[1]For a recent selection of axiomatic work see e.g. Wakai (2008), Montiel Olea and Strzalecki (2014), Galperti and Strulovici (2017), Chambers and Echenique (2018), Drugeon and Ha-Huy (2018), among many others. This work includes also related work on self-control and time-inconsistent choice of Dekel and Lipman (2012), Ahn et al. (2019), and Ahn et al. (2020), Noor and Takeoka (2020b,a) among others.

implications of commitment devices that can induce self-control among consumers.[2]

Modeling coherent choice in the presence of dynamic inconsistent preferences has a long history in economics, and is found in the early papers of Strotz (1956), Phelps and Pollak (1968), Pollak (1968) and Peleg and Yaari (1973), as well as the much of subsequent work over the last two decades that has followed the work of Laibson (1997) and Harris and Laibson (2001). Out of many interesting problems economists have studied, the question of design and computation of optimal among time consistent plans (i.e., planned sequential choice policies that are followed and not re-optimized) has received a great attention in economic literature. This also includes important from behavioral and numerical perspective short memory decision rules, like Markov or semi-Markov ones. Recall that establishing the existence of consistent decision rules in (pure) Markov strategies over uncountable state space is far from trivial.

One important limitation of the existing work relative to this paper is that most of it has focused exclusively on the case of *quasi-hyperbolic* discounting.[3] Although quasi-hyperbolic discounting is a very important, it is also a somewhat special case. In particular, it has a simple pattern of "1 period forward misalignment/bias" in intertemporal preferences. Recent empirical and experimental work in both economics and psychology has found strong support for more general forms of behavioral discounting in dynamic choice models (e.g., including various versions of hyperbolic discounting), however. The work in the literature considering more general behavioral discounting has either focused on models that admit closed-form solutions (e.g., Young (2007)), or emphasize numerical approaches to the computation of time consistent equilibrium, and do not consider the question of sufficient conditions for its existence (e.g., Maliar and Maliar (2016) or Jensen (2020)). Existing approaches developed for characterizing time-consistent choice in dynamic models with quasi-hyperbolic discounting do not appear to extend to the general discounting case. Therefore, from a theoretical vantage point, the need for new tools to cover such cases of generalized discounting is important and challenging.

There has also been a great deal of empirical and experimental support for various forms of dynamic inconsistencies in intertemporal preferences, and the sources of time-inconsistencies has often been tied to some form of behavioral (or non-exponential) dis-

[2]Relative to the question of welfare, there is also a large literature on the role of commitment devices in dynamic models with time inconsistent preferences. For a nice survey of this work, see Bryan et al. (2010) and Beshears et al. (2018). A small sampling includes: Laibson (1997), Harris and Laibson (2013), Gine et al. (2010), Karlan et al. (2016), Casaburi and Macchiavello (2019), Beshears et al. (2020).

[3]For recent work see Krusell and Smith (2003), Krusell et al. (2010), Harris and Laibson (2013), Chatterjee and Eyigungor (2016), Balbus et al. (2015b, 2018), and Cao and Werning (2018). We should mention, there is a parallel important literature on self-control and impulse management in so called dual-self models (Fudenberg and Levine, 2006, 2012).

counting. For example, in early work along these lines by Laibson et al. (2007), the authors explore how high short-term discount rates are needed to explain observed borrowing behavior in US data. More recently, Duflo et al. (2011) estimate a model of naivete random quasi-hyperbolic discounting for fertilizer use in Kenya where there is a positive probability placed on time consistent choice, and find time inconsistency plays an important role in the adoption decision. Chan (2017) estimates a hyperbolic model of discounting where differences in discount factors play a key role in explaining how workers make labor supply decisions in the context of participation in welfare programs. In Dalton et al. (2020), the authors study the role of discounting and myopia in the purchase of Medicare D drug insurance contracts, and find strong support of the presence of general time inconsistent behavior and behavioral discounting. Using an experimental approach, Augenblick et al. (2015) find support for time inconsistent behavior in discounting in the context of making effort choices in real tasks. Similarly, Kuchler and Pagel (2020) document general forms of present-bias and time inconsistency in the context of credit card paydowns.

This empirical and experimental work has in turn motivated a great deal of new theoretical work seeking to characterize the structure of dynamic choice models in situations with non-exponential discounting. For surveys of this body of theoretical work, see the earlier papers of Fishburn and Rubinstein (1982), Frederick et al. (2002), and Noor (2009), as well more recent of Ericson and Laibson (2019) and Cohen et al. (2020). Some important recent theoretical contributions to this literature include Harstad (2020), who has studied the interaction between various forms of hyperbolic discounting for government policymakers and dynamic investment to study the structure of optimal investment subsidies in the presence of externalities. Halec and Yared (2019) study a prototype small open economy where the government, with present-bias objectives, is setting fiscal rules under limited commitment. This present-bias emerges naturally in many dynamic collective choice problems (e.g., see Jackson and Yariv (2015) and Lizzeri and Yariv (2017)).[4] Gottleib and Zhang (2020) study the implications of time-inconsistency on the structure of dynamic incentives in a long-term contracting problems between present-bias consumers and risk-neutral firms, and show that firms can offer contracts such that as the length of a contracting problem increases, the welfare-losses associated with present bias disappear.[5] In Iverson and Karp (2020), the authors study a Markov perfect equilibria in a dynamic model of climate with carbon taxes and generalized behavioral discounting, where the decentralized economy determines aggregate savings, and a planner determines climate policy. In Beshears et al. (2020), the authors develop a model of optimal illiquidity in an

[4]See also Becker (2012), Drugeon and Wigniolle (2020) or Ebert et al. (2020) for related results.

[5]See also the related work of Ceteman et al. (2019) in the context of a continuous time model.

economy where agents are subjected to taste shocks and have present-bias preferences. They show that the socially optimum is approximately a two-tier account system which includes completely illiquid accounts and completely liquid accounts. Finally, Heidues and Strack (2019) and Mahajan et al. (2020) discuss methodological issues related to the identification of present-bias and behavioral discounting in econometric models.

One final aspect worth mentioning is the inherent uncertain nature of the future in many of these behavioral discounting models. That is, although dynamic models of choice over time can be applied to both deterministic and stochastic environments, it is the latter that is of utmost importance for empirical studies. There is a number of recent papers showing that preferences over time as well as over stochastic outcomes are intertwined. As Halevy (2008) and Baucells and Heukamp (2012) claim: delaying a prize in time has the same effect as increasing uncertainty of getting it. Uncertainty over future states plays an important role in our analysis. We will argue, whenever preferences of consecutive generations are misaligned for more than one period ahead a certain form of transition uncertainty is necessary to obtain existence of stationary time consistent equilibrium.

Taking these considerations into account, in this paper we study various general forms of behavioral or normative discounting rules that generate dynamically inconsistent preferences. The central aim of this paper is to prove existence of time consistent equilibrium (e.g., minimal state space Markovian equilibrium) in a large class of dynamic economies with *generalized* discounting that includes in its catalog many important models from the cited literature as special cases. And the above mentioned task is only a prerequisite of any empirical analysis of implications of various forms of discounting on allocation of scarce economic or environmental resources over current and future generations under intrinsic uncertainty.

Overview of the results Before we proceed to the formalities, we begin by previewing the main results of the paper. Consider a discrete time, infinite horizon, stochastic consumption-saving model, where the sequence of time separable lifetime preferences over sequences of consumption $(c_\tau)_{\tau=t}$ is given any date t by:

$$u(c_t) + \mathbb{E}_t \sum_{\tau=1}^{\infty} \delta_\tau u(c_{t+\tau}). \tag{1}$$

We shall refer to these preferences as $(\delta_t)_{t=0}^{\infty}$-*behavioral discounting preferences*. Notice, at any time period t, the consumer uses the sequence of discount factors:

$$\delta_0, \delta_1, \delta_2, \delta_3, \ldots$$

to value current and continuation utility streams (where, for convenience, we normalize $\delta_0 = 1$). A few additional remarks on these preferences are in order. First, notice these preferences embed the discounting ideas of both Ramsey (1928) and Samuelson (1937) as special cases. Second, most cases in the literature of behavioral discounting fit into this general setting. To mention a few common special cases, we have the following: (i) exponential discounting when $\delta_t = \delta^t$, (ii) quasi-hyperbolic discounting when $\delta_t = \beta\delta^t$ for $t \geq 1$, and (iii) hyperbolic discounting when $\delta_t = \frac{1}{1+t}$. Third, these preferences are generally time-inconsistent. That is, the discount rate between utilities in any two time periods $\tau + 1$ and τ is given by:

$$\frac{\delta_{t+1} u(c_{\tau+1})}{\delta_t u(c_\tau)},$$

for any $t \in \{0, \ldots, \tau\}$. We say the intertemporal preferences between the consecutive periods are *misaligned* whenever for some t:

$$\delta_t^2 \neq \delta_{t-1}\delta_{t+1}.$$

For the special case of exponential discounting, preferences are aligned. For the case of quasi-hyperbolic discounting, preferences are misaligned and exhibit "1 period forward misalignment". For the case of hyperbolic discounting, these preferences also misaligned, but for *any* t. As a result, the preferences in (i) are time-consistent, and in both cases (ii) and (iii), time-inconsistent.

Now, let us consider the structure of a stochastic dynamic optimization problem, where the dynamics on the state variable (e.g. assets or capital levels) s_t induced by sequences of current (consumption) choices is governed by a Markov transition $s_{t+1} \sim q(s_t | s_t - c_t)$, where $s_t - c_t$ denotes investment.

Considering a sequence of feasible and measurable consumption policies $(c_t^*)_t$ mapping current state to current consumption level. We can compute its expected value from period t on:

$$U_t((c_\tau^*)_{\tau=t})(s) = \mathbb{E}_s \left(u(c_t^*(s)) + \sum_{\tau=1}^{\infty} \delta_\tau u(c_{t+\tau}^*) \right).$$

where $\mathbb{E}_s$ is the conditional expectation operator with respect to date t information. We say a sequence $(c_t^*)_t$ of measurable consumption policies is a *Markov Perfect Equilibrium (MPE)* in a consumption-savings model with (δ_t)-behavioral discounting if for any $s \in S$ and t we have:

$$c_t^*(s) \in \arg\max_{c \in [0,s]} \left\{ u(c) + \delta_1 \mathbb{E}_s U_{t+1}((c_\tau^*)_{\tau=t+1})(s-c) \right\}.$$

If additionally, this MPE is *time-invariant*,[6] i.e. $c_t^* = c^*$ we refer to this as *Time Consistent Equilibrium* (TCE).

For the moment, assume state space $S \subset \mathbb{R}$ is bounded, and the temporal return function $u : S \mapsto \mathbb{R}$ is continuous, increasing and strictly concave. Moreover, assume q is stochastically increasing and stochastically continuous.[7]

The first main result of the paper concerns TCE in the special case of behavioral discounting model where preferences are quasi-hyperbolic with $\delta \in (0, 1)$ and $\beta \in (0, 1]$.

Proposition 1. *There exists a TCE in $\beta - \delta$ quasi-hyperbolic discounting model with deterministic state transition q.*

Notice, for the case of quasi-hyperbolic discounting consumption-savings models, we do *not require* stochastic state transitions. Given our weak sufficient conditions for this result, Proposition 1 generalizes substantially the existing literature.[8]

Our second main set of results concerns the case of general behavioral discounting with each $\delta_t \leq \delta < 1$. Here, we allow preferences for consecutive "generations" of selves to be misaligned in much more general ways than in the quasi-hyperbolic discounting model. For this case, we need some uncertainty in the state transition process to obtain existence of TCE.[9] The second main result can be stated as follows:

Proposition 2. *There exists TCE in the (δ_t)-behavioral discounting model with preferences given by (1) whenever q is nonatomic.*

In fact, the existence and characterization results in this paper are *more general* than both Propositions 1 and 2.

First, in all cases of TCE, we will provide a sharp characterizations of equilibrium policies. Namely, for any TCE with consumption c^*, the associated equilibrium decision rule for investment i^* is *monotone* (and right-continuous) in S. Additionally, per characterization of TCE, in models with present-bias preferences (i.e. $\beta < 1$), we are able to break all indifference between the "current-self" in favour of the earlier selves who prefer a higher level of investment. As Caplin and Leahy (2006) show the "optimal" TCE must resolve such preference indifferences in this manner for positive and normative reasons. This is critical aspect of our construction, and is new relative to the existing work.

[6]The question of nonstationary MPE is interesting. For the quasi-hyperbolic case, for repeated games, see Chade et al. (2008), and for dynamic games, see Balbus and Woźny (2016) for a discussion.

[7]Stochastically continuous means the transition q satisfies the Feller property. For a standard definition of stochastically increasing, see Topkis (1998), section 3.10.

[8]For example, our main existence result for the quasi-hyperbolic case generalizes these of Harris and Laibson (2001), Krusell and Smith (2003), Bernheim et al. (2015), and Cao and Werning (2018).

[9]Without such uncertaint, counterexamples to the existence of TCE can be constructed.

Second, we can allow for both S and u to be unbounded above. Relative to Proposition 2, we are also able to substantially relax the assumption of (δ_t)-behavioral discounting preferences, in particular by allowing for *time-variant* preferences represented by *non-additive* aggregators.

Finally, in characterizing TCE in the (δ_t)-behavioral discounting model, we will introduce the notion of a "semi-hyperbolic" model, i.e. a model where agents, in a precise mathematical sense, have "finite" bias/misalignment. We will show in what sense the TCE in the behavioral discounting model can be generated as limits of TCE to "semi-hyperbolic" models. In such situations, our approximation results will provide a new conceptual foundation for understanding TCE in the (δ_t)-behavioral discounting model. Importantly, the hyperbolic discounting model will be a special case of a behavioral discounting model where our approximation tools work. In the view of possible equilibrium indeterminacy results,[10] we think that our approximation (or "upper semi-continuity" of the equilibrium set) offers some stability result relative to a class of time consistent policies.

Also, an important technical aspect of our approach is, we introduce a new functional equation method that robustly links recursive utility models with strategic aspects of limited commitment. Our approach substantially extends and integrates separate ideas developed in a series of contributions by Balbus et al. (2015b, 2018), Balbus et al. (2020a) and Balbus (2020), among others. In doing so, we provide the first attempt of which we are aware to analyze existence of minimal state Markovian equilibrium in dynamic economies with general recursive payoffs and time-inconsistent preferences. Our results can be hence of independent interest for equilibrium existence in dynamic/stochastic games with *recursive payoffs* and *general discounting* (see Obara and Park (2017) for a recent contribution).

In the remainder of the paper, we discuss in more detail Propositions 1 and 2, as well as their generalizations. Namely, in section 2, we provide some intuition into how we approach the existence problem. In particular, we start with the motivating example of quasi-hyperbolic discounting, and use it to suggest a more general functional equation approach to other discounting problems. The key ingredient of this argument is the development of what we refer to as a "generalized Bellman operator". This then allows us to link the structure of our solution approach (taken to the quasi-hyperbolic case) to the more general class of models with (time-separable) behavioral discounting. From there, we are able to generalize the approach further to a recursive (non-separable) representations of the TCE problem. We then use this general recursive representation to give sufficient conditions under which we can prove an existence result for this class of models. Then, in

[10] See Krusell and Smith (2003) and its discussion in Cao and Werning (2018).

section 3, we continue exploration of the structure of our recursive approach, and analyze the case of semi-hyperbolic discounting. Although this case is interesting in itself, it also serves as a powerful tool for verifying existence of TCE for the important case of hyperbolic discounting. In particular, we show precisely that one can view the hyperbolic discounting case as the limiting case of a sequence of semi-hyperbolic discounting problems, which we show in section 4. In this section, we also show how to build a powerful approach to approximating other generalized behavioral discounting models. In section 5, we show how our results can be extended to even more general models with behavioral features e.g. magnitude effects, backward looking discounting or short-lived players. We also provide many examples of special cases in the literature that fit into our setting.

2 A preliminary existence result

We begin the paper by considering the case of time separable quasi-hyperbolic discounting model, the most studied case of behavioral discounting in the literature. It provides the necessary intuition as to how we approach more general cases of behavioral discounting, a key focal point of the paper. In particular, we use the quasi-hyperbolic example to motivate our new approach to the case of more general (non-additive) behavioral discounting. We then in this more general context prove existence of TCE, where as a corollary, we consider how the result can be specialized to the quasi-hyperbolic model.

2.1 A motivating example: quasi-hyperbolic discounting

Consider the standard, infinite horizon, stochastic consumption-savings model with quasi-hyperbolic preferences. In this model, at each period t, there is one "generation" who enters the decision problem inheriting a capital/asset stock $s_t \in S$, where $S = \mathbb{R}_+$ or $S = [0, \bar{S}] \subset \mathbb{R}_+$.[11] $c_t \in [0, s_t]$, with the remaining resources $i_t = s_t - c_t$ allocated as an investment for next generation $t+1$. In general, the capital stock at $t+1$ is random, and drawn from the distribution $q(\cdot|i_t)$. The temporal utility for each generation is $u(c_t)$, where $u : S \to \mathbb{R}$ is continuous and strictly increasing function.

[11] Here, we interpret the dynamic choice model "dynastically", i.e., the infinite-horizon decisions are chosen by a collection of generations under limited commitment. Alternatively, those "generations" could represent "selves" in a model of a single agent with changing tastes as in Phelps and Pollak (1968), Peleg and Yaari (1973), or Hammond (1976).

Then, for any stock-consumption history $(s_t, c_t)_{t=1}^{\infty}$, we denote:

$$J(c^t)(s_t) := \mathbb{E}_{s_t}\left(u(c_t) + \beta\delta \sum_{\tau=t} u(c_{\tau+1})\delta^{\tau-t}\right),$$

as generation t lifetime preferences, where $1 \geq \beta > 0$ and $1 > \delta \geq 0$, and expectations operator $\mathbb{E}_{s_t}$ is taken with respect to the realization of random variables $(s_\tau)_{\tau=t+1}$ with s_τ drawn each period from a transition distribution q. Here, as typically $c^t = (c_\tau)_{\tau=t}^{\infty}$. The objective is well-defined by the Ionescu-Tulcea theorem. Denoting by:

$$U^*(c^{t+1})(s_{t+1}) = \mathbb{E}_{s_{t+1}}\left(\sum_{\tau=t} u(c_{\tau+1})\delta^{\tau-t}\right),$$

we can rewrite this more conveniently as:

$$J(c^t)(s_t) = \mathbb{E}_{s_t}\left(u(c_t) + \beta\delta U^*(c^{t+1})(s_{t+1})\right) = u(c_t) + \beta\delta \mathbf{E}_{s_t - c_t} U^*(c^{t+1})(s_{t+1}).$$

Here $\mathbf{E}_i$ $(i \in S)$ is an expectation with respect to realization of $(s_\tau)_{\tau=t+1}^{\infty}$ with s_τ drawn each period from a transition distribution q where the initial distribution is $q(\cdot|i)$. Let $c_t^* : S \to S$ be a measurable and feasible policy, and interpret it as a Markov policy generating a history $(s_t, c_t^*(s_t))_{t=1}^{\infty}$. Suppose then the generation t deviates from c_t^* by choosing $c \in [0, s_t]$. Then, we can define a payoff:

$$P(c, (c^*)^{t+1})(s_t) := u(c) + \beta\delta \int_S U^*((c^*)^{t+1})(s_{t+1}) q(ds_{t+1}|s_t - c).$$

We then have the following definition.

Definition 1. *A sequence (c_t^*) of measurable policies is a MPE if for any $s \in S$ and t:*

$$c_t^*(s) \in arg \max_{c \in [0,s]} P(c, (c^*)^{t+1})(s).$$

If additionally, the MPE is time invariant, then we refer to it as a Stationary MPE or Time Consistent Equilibrium (TCE).

Let c^* be a TCE. It is clear for the quasi-hyperbolic discounting model, as the decision-maker has time separable preferences, finding c^* requires *decomposing* this optimization problem into *two* functional equations and solving then. The first functional equation involves finding the *recursive* part of preferences, i.e. future value U^* computed for a given candidate policy c^*. The second functional equation then assures strategic *consis-*

tency between the consumption policy c^* and U^* (i.e., the current choice of c^* must be a best response to the considered value U^*). These two equations describe the structure of minimal state space Markovian equilibrium self-generation for a candidate equilibrium policy c^*.

More formally, for any $s \in S$ we have:

$$U^*(c^*)(s) = u(c^*(s)) + \delta \int_S U^*(c^*)(s')q(ds'|s - c^*(s)), \tag{2}$$
$$c^*(s) \in \arg\max_{c\in[0,s]} u(c) + \beta\delta \int_S U^*(c^*)(s')q(ds'|s - c),$$

where for notational simplicity, we shall write $U^*(c^*)$ instead of $U^*((c^*)^t)$ whenever the context is clear. As observed by Balbus et al. (2018), these two functional equations can be summarized by a single *generalized* Bellman equation:

$$U^*(c^*)(s) = \frac{1}{\beta} \max_{c\in[0,s]} \left(u(c) + \beta\delta \mathbf{E}_{s-c} U^*(c^*)\right) - \frac{1-\beta}{\beta} u(c^*(s)). \tag{3}$$

Here, in (3), one can think of the last element of this expression $\frac{1-\beta}{\beta}u(c^*(s))$ as the *quasi-hyperbolic dynamic inconsistency adjustment factor.* That is, this additional term depending on β appearing on the right-hand side of the maximand in (3) is "added" to a standard Bellman to incorporate the fact agents have *changing* preferences over time. So, for the case of $\beta = 1$ (the case of dynamically consistent preferences with exponential discounting), this dynamic inconsistency adjustment factor reduces to 0, and the generalized Bellman operation reduces simply to the standard (time consistent) Bellman equation.[12]

It turns out this formulation of TCE in the time-separable quasi-hyperbolic case in the pair of equations in (2) represented by a single generalized Bellman in (3) can be extended in a number a directions for more general forms of behavioral discounting. For example, one can consider both (i) more general ways of evaluating certainty equivalents of future utility streams (see e.g. Kreps and Porteus, 1978)) and (ii) allow for a nonlinear aggregation of current utilities and their associated certainty equivalent (see e.g. Epstein and Zin, 1989). To see how this generalization works, consider now a general *time aggregator* $W(c^*(s), \mathbf{E}_{s-c^*(s)}U^*(c^*))$ that is used by the decisionmaker to evaluate current and future utilities. Then, the two functional equations linking future utility U^* and TCE c^*

[12]It is important to note that the so-called "generalized Euler equation" approach to solving time inconsistent problems is just the "first order" decomposition of the same idea. See, for example, Harris and Laibson (2001), section 3, equation (8) for first-order analog of our generalized Bellman equation.

in (2) now take a following form:

$$U^*(c^*)(s) = W(c^*(s), \mathbf{E}_{s-c^*(s)}U^*(c^*)), \tag{4}$$
$$c^*(s) \in \arg\max_{c\in[0,s]} W(c, \beta\mathbf{E}_{s-c}U^*(c^*)).$$

Similar to the quasi-hyperbolic case, in many settings (e.g., of time-inconsistent choice *without* time-separability), these two equations in (4) can be mapped into a *single* equation of a form similar to (3), where this latter single functional equation can be characterized by an *time-inconsistency aggregation mapping* $V : S \times S \times \mathbb{R} \to \mathbb{R}$ given by:

$$U^*(c^*)(s) = V(c^*(s), c^*(s), \mathbf{E}_{s-c^*(s)}U^*(c^*)) = \max_{c\in[0,s]} V(c, c^*(s), \mathbf{E}_{s-c}U^*(c^*)) \tag{5}$$

where the first element of V is current consumption, the second element of V is a "dynamic inconsistency adjustment factor" that corrects intertemporal preferences for the evolving structure of time-inconsistency, and the third argument is a "recursive" utility term from the next period onward that is evaluated under some candidate consumption function c^*. Our existence theorem in a moment will be based on this new general formulation of the dynamic inconsistency problem in (4), and will prove existence of value U^* and a function c^* solving the single functional equation in (5).

Before we proceed, we first note that the formulation in (4) and (5) has many important examples in the literature as special cases. We discuss few of them now.

Example 1 (Time separable quasi-hyperbolic discounting)**.** *In case of a standard, time separable quasi-hyperbolic discounting model $W(x, z) = u(x)+\delta z$, the aggregation mapping V takes the form:*

$$V(x, y, z) := \frac{1}{\beta}(u(x) + \beta\delta z) - \frac{1-\beta}{\beta}u(y).$$

Example 2 (Risk-sensitive preferences)**.** *Consider now generalization involving the exponential certainty equivalent as defined by Weil (1993) (see also Bäuerle and Jaśkiewicz (2018) for a motivation). In such case the risk-sensitive preferences are given by*

$$u(c) - \frac{\beta\delta}{\gamma}\ln\int_S e^{-\gamma U^*(c^*)(s')}q(ds'|s-c),$$

where $U^*(c^*)(s) = u(c^*) - \frac{\delta}{\gamma}\ln\int_S e^{-\gamma U^*(c^*)(s')}q(ds'|s-c^*(s))$ and $\gamma > 0$. Then the time aggregator takes the form: $W(x, z) := u(x) + \delta z$ and the certainty equivalent for given (integrable) f is $-\frac{1}{\gamma}\ln\int_S e^{-\gamma f(s')}q(ds'|s-c)$. The aggregation mapping V takes the same *form as in the example 1.*

Example 3 (Kreps-Porteus Utility). *Kreps and Porteus (1978) and Epstein and Zin (1989) introduced the following CES aggregator:*

$$W(x,z) = ((u(x))^{1-\rho} + \delta z^{1-\rho})^{\frac{1}{1-\rho}}$$

for $\rho \in (0,1)$. In case of $\beta - \delta$ version of this model with $W_\beta(x,z) = ((u(x))^{1-\rho} + \beta\delta z^{1-\rho})^{\frac{1}{1-\rho}}$ we have:

$$V(x,y,z) = [\frac{1}{\beta}W_\beta^{1-\rho}(x,z) - \frac{1-\beta}{\beta}(u(y))^{1-\rho}]^{\frac{1}{1-\rho}}.$$

2.2 An existence result

We now state an initial general existence result for this class of dynamic preferences with behavioral discounting. For this result, we need the following assumptions on V and the transition probability $q(\cdot|i)$.

Assumption 1 (Aggregator). *$V : S \times S \times [\vartheta, \infty) \mapsto [\vartheta, \infty)$ is continuous, with $\vartheta \in \mathbb{R}$ and $(x, y, z) \mapsto V(x, y, z)$ is increasing in $(x, -y, z)$. Moreover:*

(i) The function $z \to V(x, y, z)$ is a contraction mapping with a constant $\delta \in (0, 1)$;

(ii) The function

$$\zeta(s) = V(s - i_1, \phi(s), \psi(i_1)) - V(s - i_1 + (i_1 - i_2), \phi(s), \psi(i_2))$$

has Strict Single Crossing Property (SSCP) for any $s \geq i_1 > i_2$ and Borel functions ϕ and ψ[13];

(iii) There is a sequence ξ_k ($k \in \mathbb{N}$) of elements of S, $0 < \xi_1 < \xi_2 < \ldots$, and a sequence η_k of $\mathbb{R}_+$ such that $\vartheta < \eta_1 < \eta_2 < \ldots$ such $\eta_k \to \infty$ as $k \to \infty$ and $r := \sup\limits_{k\in\mathbb{N}} \frac{\eta_{k+1}}{\eta_k} \in (1/\delta, \infty)$. Moreover,

$$\sup_{(x,y,z)\in[0,\xi_k]^2\times[\vartheta,\eta_{k+1}]} |V(x,y,z)| \leq \eta_k \quad \textit{for all } k,$$

or equivalently

$$\max\left(V(\xi_k, 0, \eta_{k+1}), V(0, \xi_k, \vartheta)\right) \leq \eta_k.$$

[13] Under our monotonicity assumptions it suffices to verify the SSCP condition for ψ such that $\psi(i_1) > \psi(i_2)$. Indeed, in the opposite case, i.e. $\psi(i_2) \geq \psi(i_1)$ function ζ is negative so SSCP is satisfied trivially.

Assumption 2 (Transition)**.** *The transition probability $q(\cdot|i)$ satisfies:*

(i) $i \mapsto q(\cdot|i)$ is stochastically increasing, satisfies a Feller property, and

$$q([0, \xi_{k+1}]|s) = 1 \quad \textit{for all } s \in [0, \xi_k];$$

(ii) For any $s \in S$, the set of all i such that $q(\{s\}\,|i) > 0$ is countable.

Assumption 1 (i) is standard. Condition (ii) assures that (each) best response equilibrium policy selection is monotone increasing on S. Assumption 1 (iii) and 2 (i) assure we can use the local contractions argument for the case of unbounded states and/or unbounded above rewards. If the states space S is bounded or rewards are (uniformly) bounded then these are automatically satisfied. Finally, we should make an important remark on assumption 2 (ii). Observe, this assumption is satisfied for a purely deterministic transition structure and as well their convex combinations. Moreover, we allow all sets we consider (i.e. $\{i \in S : q(\{s\}\,|i) > 0\}$) be empty. This is the case, for example, when q is non-atomic. These are the two cases mostly considered in the paper.

Now define the set of candidate TCE investment functions:

$$\mathcal{H} := \{h : S \mapsto S : h(s) \in [0, s] : h \text{ is increasing and right continuous}\}.$$

By the arguments similar to Lemma 1 in Balbus et al. (2020a), the set $\mathcal{H}$ is weakly compact when endowed with the weak-star topology (i.e. the topology with the following notion of convergence $h_n \to^w h$ iff $h_n(s) \to h(s)$ whenever h is continuous at s).

Under these conditions, we now have a very general result on the existence of TCE c^*, as well as providing a characterization of the monotonicity properties of the corresponding investment $h^* \in \mathcal{H}$, where $h^*(s) := s - c^*(s)$.

Theorem 1. *Assume 1 and 2. There exists a TCE c^* with a corresponding monotone investment $h^* \in H$. That is, if $c^* : S \mapsto S$ is the TCE, then there is $U^* : S \mapsto \mathbb{R}$ such that for any $s \in S$*

$$U^*(s) = \max_{c \in [0,s]} V(c, c^*(s), \mathbf{E}_{s-c}U^*) = V(c^*(s), c^*(s), \mathbf{E}_{s-c^*(s)}U^*).$$

We now proceed with some preliminary definitions, constructions, and lemmata necessary to prove this theorem. Begin by defining the following set:

$$\mathcal{E} := \{(s, h) \in S \times \mathcal{H} : h \text{ is continuous at } s\}.$$

As is usual, S is endowed with the Euclidean topology and $S \times \mathcal{H}$ is endowed with its product topology. It is well-known that the evaluation function $\mathbf{e}(s, h) = h(s)$ has a continuous restriction to $\mathcal{E}$. Since $h \in \mathcal{H}$ is increasing, the section $\mathcal{E}^h := \{s \in S : (s, h) \in \mathcal{E}\}$ has a countable complement.

Next, define the space $\mathbf{V}$ to be the set of real valued functions on $S \times \mathcal{H}$ such that each $f \in \mathbf{V}$:

- is bounded on any $S_k \times \mathcal{H}$, where $S_k := [0, \xi_k]$;
- is continuous from the right on S, and upper semicontinuous on $S \times \mathcal{H}$;
- obeys the following condition: for any $h \in \mathcal{E}$, there is a countable set $S^{f,h} \subset \mathcal{E}^h$ such that if $s \notin S^{f,h}$ then f is continuous at (s, h).

Endow the space $\mathbf{V}$ with the topology induced by the semi-norms:

$$||f||_k = \sup_{s \in S_k \times \mathcal{H}} |f(s, h)|,$$

where $S_k := [0, \xi_k]$. Further define the following:

$$\mathcal{V} := \left\{ f \in \mathbf{V} : \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k} < \infty \right\},$$

and a norm on $\mathcal{V}$:

$$||f|| = \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k}.$$

Finally, define the set $\mathcal{U}$ to be:

$$\mathcal{U} := \{ f \in \mathcal{V} : ||f||_k \leq \eta_k, k \in \mathbb{N} \}.$$

We are now ready to present the key steps in the proof of main theorem of this section.

Proof of Theorem 1. Lemma 1 in the Appendix shows that $\mathcal{U}$ is a closed subset of a Banach space $(\mathcal{V}, || \cdot ||)$. We then define an operator T on $\mathcal{U}$ as follows:

$$T(f)(s, h) := \max_{i \in [0,s]} V(s - i, s - h(s), \mathbf{E}_i f(h)),$$

where $f(h) := f(\cdot, h)$. Lemma 3 shows that T is a self map on $\mathcal{U}$, while lemma 4 claims that T is a contraction mapping and thus has a unique fixed point. Denote by $f^* \in \mathcal{U}$

this unique fixed point of T in $\mathcal{U}$, and define the following mapping that characterizes the best reply correspondence for each generation:

$$BI(h)(s) = \arg\max_{i \in [0,s]} V(s-i, s-h(s), \mathbf{E}_i f^*(h)),$$

and

$$bi(h)(s) := \max BI(h)(s).$$

Lemma 5 shows that any selection of $s \mapsto BI(h)(s)$ is increasing in s. Finally, our key Lemma 7 shows that bi is continuous on compact $\mathcal{H}$. By Schauder-Tychonoff Theorem we obtain the existence of a fixed point h^* of bi. Then $c^*(s) := s - h^*(s)$ is a TCE. $\square$

We conclude this section with an important corollary of Theorem 1. It offers a new existence result in a standard *deterministic* quasi-hyperbolic discounting model for the case when (i) the state space S is bounded or unbounded, and (ii) the utility function u is allowed to be unbounded above. To the best of our knowledge, this corollary with sufficient conditions for the existence of TCE for the standard quasi-hyperbolic discounting model is the most general in the current literature.

Corollary 1 (Deterministic quasi-hyperbolic discounting)**.** *There exists a TCE with investment monotone in the deterministic $\beta - \delta$ model whenever u is continuous, increasing and strictly concave.*

Recall, theorem 6 in Bernheim et al. (2015) proves existence of time consistent solution in a deterministic model with CIES utility and linear technology. Theorem 5 in Cao and Werning (2018) extends this existence result to other models with strictly positive lower bound of the asset holding and linear technology with small gross interest rate. These existence results can be significantly extended by allowing stochastic state transitions. For example, Harris and Laibson (2001) prove existence of the time consistent equilibrium in a smooth model with bounded intertemporal elasticity of substitution. Along that lines Balbus et al. (2018) proved equilibrium existence and uniqueness under some restrictive assumption on the stochastic transition function. Recently Balbus et al. (2020b) have also shown existence in the general model but with non-atomic transition. We should also mention the work of Chatterjee and Eyigungor (2016), who prove existence of time consistent equilibrium but in randomized strategies. Our results generalize all the above listed results and provide a unified methodological setup for equilibrium existence verification.

We finish this section with an important comment on the nature of our existence result and its interpretation.

Remark 1 (Selection from the argmax correspondence and Optimal TCE)**.** *Our construction of TCE in theorem 1 uses the greatest investment selection from the argmax correspondence. This selection procedure guarantees in our models with present biased preferences (i.e. $\beta < 1$), all indifference of the current self are arbitrarily resolved in favor of the earlier selves who prefer **higher** investment. In an important paper, Caplin and Leahy (2006) argue that* optimal *time consistent solutions should resolve **all** indifference in such a manner (for not only positive reasons, but for normative interpretations of time consistent solutions). Technically, this is also critical for our existence result. To the best of our knowledge, such investment selection construction is **new** relative to the existing work on time consistent solutions for quasi-hyperbolic models.*

As stressed in the remark, whenever investment is upper semicontinuous, its associated consumption is lower semicontinuous, which assures the upper semicontinuity of the value U^*. And upper semicontinuity of U^* is critical for proving non-emptiness of the argmax correspondence. Indeed, it is not clear how the general existence for a deterministic quasi-hyperbolic discounting model can be extended using the least investment selection.[14]

3 Semi-hyperbolic discounting

We now proceed with new versions of dynamic model with time inconsistent preferences which we refer to as "semi-hyperbolic" discounting models. The semi-hyperbolic model has the flavor of quasi-hyperbolic, but allows for a more general pattern of present-bias (see Montiel Olea and Strzalecki (2014) section IV for an introduction and motivation). These models also will be useful in characterizing TCE in more general models of behavioral discounting (e.g., the hyperbolic discounting model). To build intuition as to how to characterize TCE in this class of models, we first study the special case of $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting, a direct extension of the quasi-hyperbolic model.

3.1 $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting

Consider a special case of preferences in (1) where the sequence of discount factors at any date t is specified as follows:

$$1, \beta_1\beta_2\delta, \ \beta_1\beta_2^2\delta^2, \ \beta_1\beta_2^2\delta^3, \ \beta_2\beta_2^2\delta^4, \ldots$$

[14]Recall, the recent contribution on the question of equilibrium existence in related classes of stochastic games to those studied here use the *least* investment selection (see Balbus et al. (2015a, 2020a) e.g.).

We shall refer to this model as the $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting. Notice, in this model, from period $t+3$ on, the discount factor becomes exponential. However, unlike in $\beta - \delta$ model, in the case of $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting, preferences are misaligned for more than just one date forward. Indeed, we have the following:

$$(\beta_1\beta_2\delta)^2 \neq \beta_1\beta_2^2\delta^2,$$

whenever $\beta_1 \neq 1$; as well as

$$(\beta_1\beta_2^2\delta^2)^2 \neq (\beta_1\beta_2\delta)(\beta_1\beta_2^2\delta^3),$$

whenever $\beta_2 \neq 1$. So although these preferences are in the spirit of $\beta - \delta$ preferences, they allow for a more general pattern of forward preference misalignment.

As before, for this model, we aim to show the existence of a TCE c^*. For this, we seek an appropriate generalized of the "decomposition" approach to quasi-hyperbolic discounting we developed in Section 2. For $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting model, our decomposition involves three functional equations, namely:

$$\begin{aligned} U^*(c^*)(s) &= u(c^*(s)) + \delta \int_S U(c^*)(s')q(ds'|s - c^*(s)), \\ W_1^*(c^*)(s) &= u(c^*(s)) + \beta_2\delta \int_S U(c^*)(s')q(ds'|s - c^*(s)), \\ W_2^*(c^*)(s) &= u(c^*(s)) + \beta_1\beta_2\delta \int_S W_1^*(c^*)(s')q(ds'|s - c^*(s)) \\ &= \max_{c\in[0,s]} u(c) + \beta_1\beta_2\delta \int_S W_1^*(c^*)(s')q(ds'|s - c). \end{aligned} \tag{6}$$

We now discuss how our generalized Bellman equation approach proposed in the previous section can be extended this semi-hyperbolic discounting problem. To obtain a single functional equation linking these three functional equations, one needs again to construct corrective factors, but only now *twice*. Indeed, simplifying $U^*(c^*)(\cdot)$ with $U^*(\cdot)$ we obtain the functional equation:

$$\begin{aligned} U^*(s) = \frac{1}{\beta_1\beta_2^2} \max_{c\in[0,s]} &\left\{ u(c) + \beta_1\beta_2\delta \int_S [u(c^*(s')) + \beta_2\delta \int_S U^*(s'')q(ds''|s' - c^*(s'))]q(ds'|s - c) \right\} \\ &- [\frac{1}{\beta_1\beta_2^2} - 1]u(c^*(s)) - [\frac{1}{\beta_2} - 1]\delta \int_S u(c^*(s'))q(s'|s - c^*(s)). \end{aligned}$$

Notice, for $\beta_2 = 1$, the second corrective factor disappears, and the problem reduces to a standard $\beta-\delta$ discounting model. Similarly, for $\beta_1 = 1$ the problem reduces to a version of quasi-hyperbolic model, where the additional impatience shows up between third and the second period (not the second and the first).

Next, as is clear from the above formulation, for the *deterministic* semi-hyperbolic problem, the argmax in the decisionmaker need not be necessarily well-defined in the space of investments $\mathcal{H}$.[15] We resolve this issue by considering a stochastic transitions on S. Under this extra assumption, we can extend our existence result for the quasi-hyperbolic case to the separable semi-hyperbolic discounting. In fact, these assumptions suffice to prove existence in a more general model that we discuss in the next subsection.

3.2 General semi-hyperbolic models

Consider a very general version of semi-hyperbolic discounting preferences that includes the $\beta_1-\beta_2-\delta$ semi-hyperbolic discounting model as a special case. In studying TCE in this more general semi-hyperbolic case, we will use the existence results for this class of semi-hyperbolic models to elucidate the structure of TCE in the (δ_t)-behavioral discounting preferences given in (1) via a limiting approximation argument.

Along these lines, first assume in that the semi-hyperbolic model is characterized by a sequence of discount factors that take the following sequential form:

$$1, \beta_1\beta_2\ldots\beta_T, \beta_1(\beta_2\ldots\beta_T)^2, \beta_1\beta_2^2(\beta_3\ldots\beta_T)^3, \ldots, \beta_1\beta_2^2\ldots\beta_{k-1}^{k-1}(\prod_{s=k}^{T}\beta_s)^k, \ldots, \prod_{\tau=1}^{T}\beta_\tau^\tau,$$

while for any $t > T$ it is:

$$\prod_{\tau=1}^{T}\beta_\tau^\tau\beta_T^{t-T},$$

Assume $\beta_T < 1$. The intuition for this formulation of the semi-hyperbolic model is that each decision maker/generation at date t is impatient up to T periods ahead and then from period T on the problem becomes stationary with exponential discounting at rate β_T. Observe that when additionally all $\beta_t \le 1$ the decision maker has a *growing patience*.

Remark 2. *Per notation, in the previous examples, we used $\delta = \beta_T$. Now, we substitute for β_T to keep the notation concise. So, for example, we have the following special cases:*

[15] Indeed, in the deterministic transition case, the objective function: $i \mapsto u(s-i) + \beta_1\beta_2\delta[u(c^*(i)) + \beta_2\delta U^*(i-c^*(i))$ may fail to be upper-semicontinuous, unless U^* is and both c^* and $s \mapsto s - c^*(s)$ are usc. In such a case the argmax may be empty and the general existence approach based on the fixed point equation may fail. See also example 2 in Balbus et al. (2015a).

for $T = 1$, we have a standard exponential discounting with β_1^t; for $T = 2$, it is a quasi-hyperbolic $\beta_1 - \beta_2$ discounting model; for $T = 3$, we have an "order two" quasi-hyperbolic $\beta_1 - \beta_2 - \beta_3$ model, etc.

We can now again develop a functional equation representation of the consumption-savings problem for this class of semi-hyperbolic preferences. In particular, the functional equations will have the following recursive structure:

$$
\begin{aligned}
&U^*(s) = u(c^*(s)) + \beta_T \int_S U^*(s')q(ds'|s - c^*(s)),\\
&\text{and } c^*(s) \in \arg\max_{c\in[0,s]}\{u(c) + \prod_{\tau=1}^{T}\beta_\tau \int_S A_{T-1}(U^*)(s')q(ds'|s-c)\},\\
&\text{with } A_t(U^*)(s) = u(c^*(s)) + \prod_{\tau=T+1-t}^{T}\beta_\tau \int_S A_{t-1}(U^*)(s')q(ds'|s-c^*(s)),\\
&\text{where } A_0(U) := U^*.
\end{aligned}
$$

The next theorem considers our general existence for TCE for this class of models. For this result, we will need to impose two new assumptions.

Assumption 3. *Let $u : S \to \mathbb{R}$ be continuous, increasing, strictly concave and* $\max(|u(0)|, |u(\xi_k)|) \leq (1 - \delta_T)\eta_k$.

Assumption 4. *The transition q satisfies Assumption 2. Moreover, q is nonatomic.*

With these assumptions in place, we now have the following result:

Theorem 2. *Assume 3 and 4. For any $T \geq 1$, there exists a TCE c^* with corresponding monotone investment $h^* \in \mathcal{H}$.*

This is a central result for the case of semi-hyperbolic discounting model. Some aspects of its proof follow the lines developed for the quasi-hyperbolic discounting model. The key difference though is in the argument that concerns the continuity of best responses. That is, in the case of quasi-hyperbolic discounting, we used the space of upper semi-continuous value functions and allowed for *deterministic* transition functions. In the case of semi-hyperbolic discounting, this argument cannot proceed without the imposition of nonatomic noise relative to the state transition. To the best of our knowledge it is the most general existence result in such environment in the literature.

To present the proof of theorem 2, we need to define certain new objects. Let $\mathbf{V}_0$ be the space of real valued functions on $S \times \mathcal{H}$ in which $f \in \mathbf{V}_0$ if and only if

- for any $k \in \mathbb{N}$, f is bounded on any $(s,h) \in S_k \times \mathcal{H}$,
- for any $h \in \mathcal{H}$ there exists a countable $S^{f,h} \subset S$ such that $f(\cdot,\cdot)$ is continuous at any (s,h) such that $s \notin S^{f,h}$.

Endow $\mathbf{V}_0$ with analogous semi-norms $||\cdot||_k := \sup\limits_{(s,h)\in S_k\times\mathcal{H}} f(s,h)$. Let $\mathcal{V}_0 \subset \mathbf{V}_0$ be the set of all functions satisfying $||f|| := \sum\limits_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k} < \infty$. Clearly $(\mathcal{V}_0, ||\cdot||)$ is a normed space. Define $\mathcal{U}_0 := \{f \in \mathcal{V}_0 : |f(s,h)| \leq \eta_k, \text{ for all } (s,h) \in S_k \times \mathcal{H}, k \in \mathbb{N}\}$. Let $\mathbf{V}_0^\infty$ be the countable product of $\mathbf{V}_0$ endowed with the semi-norms $||f||_k^\infty := \sup\limits_{(t,s,h)\in\mathbb{N}\times S_k\times\mathcal{H}} f_t(s,h)$. Similarly, define $\mathcal{V}_0^\infty$ and the norm on $\mathcal{V}_0^\infty$ to be $||f||^\infty := \sum\limits_{k=1}^{\infty} \frac{||f||_k^\infty}{r^k \eta_k}$ and similarly for the set $\mathcal{U}^\infty$. In Lemma 8 we show $(\mathcal{V}_0, ||\cdot||)$ is a Banach space and $\mathcal{U}_0$ is a closed subset of $(\mathcal{V}_0, ||\cdot||)$ (hence a complete metric space). We also have the same conclusion for $(\mathcal{V}_0^\infty, ||\cdot||^\infty)$ and $\mathcal{U}_0^\infty$.

Proof. For $f \in \mathcal{U}_0$, and $t = 1, 2, \ldots, T$ we define the following operator

$$\Lambda(f)(s,h) := u(s - h(s)) + \beta_T \mathbf{E}_{h(s)}(f(h)),$$

where $\mathbf{E}_i(f(h))$ is the operator defined as in the previous section. By lemmas 8 and 9 there exists f^*, the unique fixed point of Λ. We now adapt the definition of the best response mapping as follows. Let

$$\mathcal{BI}(h)(s) := \arg\max_{i\in[0,s]} \left\{ u(s-i) + k_T \int_S A_{T-1}(s',h) q(ds'|i) \right\}.$$

where $k_t = \prod_{\tau=T+1-t}^{T} \beta_\tau$ and for any $t > 0$ we have

$$A_t(s,h) = u(s-h(s)) + k_t \int_S A_{t-1}(s',h) q(ds'|h(s)),$$

with $A_0(s,h) := f^*(s,h)$. Put

$$bi(h)(s) := \max \mathcal{BI}(h)(s).$$

Lemma 10 assures that any selection of $s \mapsto \mathcal{BI}(h)(s)$ is increasing. Next our key lemma 12 shows that the operator bi maps $\mathcal{H}$ into itself and is continuous. Hence, we find a fixed point h^* of bi. Similarly, we may choose an equilibrium as $c^*(s) = s - h^*(s)$. $\square$

Remark 3. *Our technique allows for more general non-additive aggregators satisfying Assumption 1. See section 5 for a more general model.*

4 Approximations, general behavioral discounting, and hyperbolic discounting

We now extend our results on TCE to the more general class of (δ_t)-behavioral discounting. In doing so, we develop an approximation approach that allows us to relate the set of TCE in the (δ_t)-behavioral discounting model to the set of TCE in limiting collections of semi-hyperbolic discounting models. This allows us to achieve two goals. First, we are able to extend our results in the previous sections to models with very general forms of behavioral discounting. Second, using the approximation approach, we are able to understand better the structure of (δ_t)-behavioral discounting models.

In particular, at the end of this section, we show how one can view the standard hyperbolic discounting model as a limit of a collection of semi-hyperbolic discounting models. Specifically, our approximation method allows us to construct TCE in models with preferences as in (1) by finding an appropriate approximating sequence of semi-hyperbolic discounting models with an appropriate sequence of discount factors $(\beta_t)_{t=1}^{\infty}$. The corresponding TCE in the limiting semi-hyperbolic case can be used to build representations of TCE for the original problem parameterized by the discount factors $(\delta_t)_{t=1}^{\infty}$.

4.1 Limiting semi-hyperbolic discounting

We begin this section by discussing the case of limiting semi-hyperbolic discounting. A limiting semi-hyperbolic discounting model studies the T-period bias as T gets arbitrarily large. For given T, denote the effective discount factors by:

$$\begin{aligned}
{}^T\delta_1 :=& \beta_1\beta_2\dots\beta_T, \\
{}^T\delta_2 :=& \beta_1(\beta_2\dots\beta_T)^2 = {}^T\delta_1\prod_{\tau=2}^{T}\beta_\tau, \\
{}^T\delta_k :=& \beta_1\beta_2^2\dots\beta_{k-1}^{k-1}(\prod_{s=k}^{T}\beta_s)^k = {}^T\delta_{k-1}\prod_{\tau=k}^{T}\beta_\tau.
\end{aligned}$$

Hence for $k \leq T$, we have the following recursive formulation:

$$ {}^T\delta_k = {}^T\delta_{k-1} \prod_{\tau=k}^{T} \beta_\tau. \tag{7} $$

We now seek existence of TCE in these models as $T \to \infty$, and use the result to build an approximation theory of TCE in the (δ_t)-behavioral discounting model. Suppose that ${}^T\delta_1$ has a limit; then, any of ${}^T\delta_k$ has a limit with $T \to \infty$. We will denote this limit by δ_k. Therefore, the recursive formula for the evolution of the successive discount factor δ_k takes the following form for any k:

$$ \delta_k = \delta_{k-1} \prod_{\tau=k}^{\infty} \beta_\tau. \tag{8} $$

We then have a new result per existence of TCE in the limiting semi-hyperbolic model relative to the (δ_t)-behavioral discounting model:

Theorem 3. *Assume 3-4 and suppose there exists a sequence (b_t) such that:*

$$ \forall t \textit{ and } \forall T \textit{ we have } \eta_t\, {}^T\delta_t \leq b_t \tag{9} $$

and that series (b_t) is convergent. Consider a model with generation t preferences given by:

$$ U_t^T(s) = \mathbb{E}_s \left(u(c_t) + \sum_{\tau=1}^{\infty} {}^T\delta_\tau u(c_{t+\tau}) \right) \tag{10} $$

with ${}^T\delta_t$ satisfying the above recursive formulation in (7). Then,

(i) for any T, there is a TCE c^T such that $h^T \in \mathcal{H}$, with $h^T(s) := s - c^T(s)$;

(ii) any limit point of the sequence $(c^T)_{T=1}^{\infty}$ in the corresponding weak-star topology, say c^*, is also a TCE in the model with utility

$$ U_t^*(s) = \mathbb{E}_s \left(u(c_t) + \sum_{\tau=1}^{\infty} \delta_\tau u(c_{t+\tau}) \right) \tag{11} $$

where the sequence δ_t satisfies the recursive formulation in (8).

Proof. The results in (i) follows from Theorem 2. We only prove (ii). Let t be the current generation whose state is s_0. By Lemma 20 there is a probability space $(\Omega, \mathcal{F}, P)$ and

Markov chain $(\xi_\tau^T)_{\tau=1}^{\infty}$ with the transition $s \mapsto q(\cdot|s - c^T(s))$ and the current state $s_0 \in S$, and another Markov chain $(\xi_\tau)_{\tau=1}^{\infty}$ with the transition $s \mapsto q(\cdot|s - c^*(s))$ and the current state s_0 as well, such that for any τ and ω, $\xi_\tau^T(\omega) \to \xi_\tau(\omega)$ as $T \to \infty$. By Assumption 4 we may assume without loss of generality any of c^* is continuous at $\xi_\tau(\omega)$ for any $\omega \in \Omega$. Hence for any $\omega \in \Omega$:

$$\lim_{T\to\infty} c^T(\xi_\tau^T(\omega)) = c^*(\xi_\tau(\omega)). \tag{12}$$

Suppose generation t deviates and selects $c \in [0, s_0]$. In the first step, assume s_0 is a continuity point of c^T. We have then $c^T(s_0) \to c^*(s_0)$ as $T \to \infty$ and

$$\begin{aligned}
\mathbb{E}_{s_0}\left(u(c^*(s_0)) + \sum_{\tau=1}^{\infty} \delta_\tau u(c^*(s_\tau))\right) &= u(c^*(s_0)) + \int_\Omega \left(\sum_{\tau=1}^{\infty} \delta_\tau u(c^*(\xi_\tau^*(\omega)))\right) P(d\omega) \\
&= \lim_{T\to\infty}\left(u(c^T(s_0)) + \int_\Omega \left(\sum_{\tau=1}^{\infty}{}^T \delta_\tau u(c^T(\xi_\tau^T(\omega)))\right) P(d\omega)\right) \quad (13)\\
&= \lim_{T\to\infty} \mathbb{E}_{s_0}\left(u(c^T(s_0)) + \sum_{\tau=1}^{\infty}{}^T \delta_\tau u(c^T(s_\tau))\right) \\
&= \lim_{T\to\infty} \mathbb{E}_{s_0}\left(u(c^T(s_0)) + \mathbf{E}_{s_0 - c^T(s_0)}\left(\sum_{\tau=1}^{\infty}{}^T \delta_\tau u(c^T(s_\tau))\right)\right) \\
&\geq \lim_{T\to\infty}\left(u(c) + \mathbf{E}_{s_0 - c}\left(\sum_{\tau=1}^{\infty}{}^T \delta_\tau u(c^T(s_\tau))\right)\right) \\
&= u(c) + \mathbf{E}_{s_0 - c}\left(\sum_{\tau=1}^{\infty} \delta_\tau u(c^*(s_\tau))\right) \quad (14)
\end{aligned}$$

where (13) and (14) follows by Dominated Convergence Theorem whose application is possible as the corresponding components of the sum are bounded by b_t defined in (9). Hence $h^*(s) = s - c^*(s)$ coincides with $bi(h^*)(s)$ at any continuity point of c^*, where $bi(\cdot)$ is adapted to objective in (11). By Assumption 4 we easily conclude that $(bi \circ bi)(h^*)(s)$ and $bi(h^*)(s)$ coincide for any $s \in S$. Hence $bi(h^*)$ is a fixed point of bi. As a result, $h^T \Rightarrow bi(h^*)$ as $T \to \infty$. By previous assumption, $h^T \Rightarrow h^*$, hence $bi(h^*) = h^*$. $\square$

This is another central result of our paper. It allows us to approximate (in the weak topology) *general behavioral* discounting models with preferences such as (1). The key technical contribution in Theorem 3 is based on the upper semicontinuity of the *set* of TCE with respect to T at $T = \infty$. The new condition, i.e. that the series (b_t) is convergent, is required so that the limiting model is well defined.

4.2 Approximating general behavioral discounting models

With this result in place, we are now able to explore the relationship between limiting semi-hyperbolic models and (δ_t)-behavioral discounting models even further than in Theorem 3. That is, suppose we have a (δ_t)-behavioral discounting model where the discount factors $(\delta_t)_{t=1}^{\infty}$ are given with each $\delta_t \in (0,1)$. We now ask if we can construct a sequence of $(\beta_t)_{t=1}^{\infty}$ collection and its corresponding sequence of behavioral semi- discounting games whose TCE can approximate TCE of the (δ_t)-behavioral discounting model. The following result answers this question.

Proposition 3. *Define*

$$\beta_t := \begin{cases} \frac{\delta_1^2}{\delta_2} & if \quad t=1 \\ \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}} & if \quad t \geq 2. \end{cases} \tag{15}$$

then a TCE of the semi-hyperbolic discounting model $\beta_1 - \beta_2 - \ldots,$ is a TCE of the behavioral discounting model with $(\delta_t)_{t=1}^{\infty}$ provided $R := \lim_{t\to\infty} \frac{\delta_{t+1}}{\delta_t} = 1$.

Proof. To see that observe:

$$\frac{\delta_{t+1}}{\delta_t} = \prod_{\tau=t+1}^{\infty} \beta_\tau$$

and hence

$$\beta_t := \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}}$$

for $t > 1$. Further we have $\lim_{t\to\infty} \frac{\delta_{t+1}}{\delta_t} = \lim_{t\to\infty} \prod_{\tau=t+1}^{\infty} \beta_\tau$, that by assumptions is equal to 1. To recover β_1 proceed as follows:

$$\begin{aligned} \delta_1 &= \beta_1 \prod_{t=2}^{\infty} \beta_t = \beta_1 \prod_{t=2}^{\infty} \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}} = \beta_1 \lim_{T\to\infty} \prod_{t=2}^{T} \frac{\delta_{t+1}^2}{\delta_{t+2}\delta_t} \\ &= \beta_1 \lim_{T\to\infty} \frac{\left(\prod_{t=2}^{T+1} \delta_t\right)^2}{\prod_{t=1}^{T} \delta_t \prod_{t=3}^{T+2} \delta_t} = \beta_1 \frac{\delta_2}{\delta_1} \lim_{T\to\infty} \frac{\delta_{T+1}}{\delta_{T+2}} = \beta_1 \frac{\delta_2}{\delta_1}. \end{aligned}$$

Hence $\beta_1 = \frac{\delta_1^2}{\delta_2}$. ∎

4.3 The hyperbolic discounting case

We now use the result in the previous section to discuss how the TCE in the standard hyperbolic discounting model can be approximated using TCE in a limiting version of a semi-hyperbolic discounting model. To see how this can be done, let for any date t, the discount factor for the (δ_t)-discounting model take a specific hyperbolic form

$$\delta_t = \left(\frac{1}{1+t}\right)^{\beta},$$

for some parameter $\beta > 1$ guaranteeing convergence of the series. In this case, this implies that the discount factor between any two time periods $t+1$ and t is:

$$\frac{(\frac{1}{t+2})^{\beta}}{(\frac{1}{1+t})^{\beta}} = \left(\frac{t+1}{t+2}\right)^{\beta}.$$

Applying our approximating formula in (15) in Proposition 3, we get:

$$\beta_{t+1} = \left(\frac{(t+1)(t+3)}{(t+2)^2}\right)^{\beta}$$

with $\beta_1 = \left(\frac{3}{4}\right)^{\beta}$. Hence, for this simple case, a TCE of this version of the standard hyperbolic discounting model can be expressed as a limit of TCE of the semi-hyperbolic models.

This same argument applies to a more general form of hyperbolic discounting (e.g., see the model studied in Loewenstein and Prelec (1992)). Specifically, let $\delta_t = (1+\alpha t)^{-\frac{\beta}{\alpha}}$. Indeed, in such case, we then have $\beta_t := \left(\frac{(1+\alpha t+\alpha)(1+\alpha t-\alpha)}{1+\alpha t}\right)^{\frac{\beta}{\alpha}}$, $\beta_1 := (\frac{1+2\alpha}{1+\alpha})^{\frac{\beta}{\alpha}}$ with $R = 1$.

5 A more general existence result with additional applications

We have shown so far that many general classes of (δ_t)-behavioral discounting models can be approximated using collections of semi-hyperbolic discounting models. The restrictive assumption in that discussion is that $R = 1$. Indeed, there is a class of behavioral discounting models that cannot be approximated in this manner. In this section, we consider these time inconsistency problems, and extend our methods (and results) to even more abstract formulations of recursive (time-inconsistent) preferences. We then provide four additional examples of where this more general existence result can be applied (where our approximating technique *cannot* necessarily be applied).

5.1 The general existence result

We first state our most general existence result. Following the reasoning developed for a general quasi-hyperbolic discounting model in section 2, assume the existence of an abstract recursive aggregator $V_t : S \times S \times \mathbb{R}$ as in the functional equation (5):

$$V_t(\tilde{c}, s - \tilde{c}, \mathbf{E}_{s-\tilde{c}} U_{t+1}(c)).$$

Here $\tilde{c}$ is the current consumption, $\mathbf{E}_{s-\tilde{c}} U_{t+1}(c)$ is the certainty equivalent of the evaluation of the next generations following policy c. Corrective terms (if necessary) can be used to account for other behavioral considerations, like magnitude effects, for example (more on this in moment). Observe, in this case, we are studying versions of the functional equation in (5) that allow the recursive aggregator to be *time-variant*.

For some given c, we first look for recursive utility $(U_t^*)_t$:

$$U_t^*(c)(s) = V_t(c(s), s - c^*(s), \mathbf{E}_{s-c(s)} U_{t+1}^*(c)).$$

We then seek the solutions to:

$$c^*(s) \in \arg\max_{\tilde{c} \in [0,s]} V_1(\tilde{c}, s - c^*(s), \mathbf{E}_{s-\tilde{c}} U_2^*(c^*)).$$

We now have our most general existence theorem in the paper:

Theorem 4. *Suppose Assumption 4 holds, and for any t, the continuous aggregator $(x, y, z) \mapsto V_t$ is increasing in (x, z) for each y, and obeys Assumption 1 (i)-(iii) with a common constant $\delta \in (0, 1)$. Then, there exists a TCE c^* with corresponding monotone investment $h^* \in \mathcal{H}$.*

Proof. Let us consider $\mathcal{V}_0^\infty$ and endow it with the natural product topology. The natural family of seminorm $||\cdot||_k$ on $\mathcal{V}_0^\infty$ is defined as follows

$$||f||_k := \sup_{(t,s,h) \in \mathbb{N} \times S_k \times \mathcal{H}} |f_t(s, h)|$$

and the norm

$$||f|| = \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k}.$$

Let $\mathbb{T}(f) = (T_t(f))_{t\in\mathbb{N}}$ where $f = (f_t)_{n\in\mathbb{N}}$. For $t > 1$ let

$$T_t(f)(s,h) = V_t(s - h(s), s - h(s), \mathbb{E}_{h(s)} f_{t+1}(h)).$$

Lemma 15 shows that $\mathbb{T}$ is a contraction mapping on $\mathcal{U}^\infty$ and has a unique fixed point: f^*. Define

$$BI(h)(s) = \arg\max_{i\in[0,s]} V_1(s - i, s - h(s), \mathbb{E}_i f_2^*(h)),$$

and $bi(h)(s) := \max BI(h)(s)$. Similarly as before (i.e. as in Theorem 1 and 2), lemma 18 shows that the operator bi maps $\mathcal{H}$ into itself and it is a continuous operator. This suffices to prove existence of a fixed point on convex and compact space $\mathcal{H}$. □

5.2 Applications to other behavioral discounting models

We now can provide few additional applications of the main results of the paper. Let us begin with the case of generalized quasi-geometric discounting.

Example 4 (Generalized quasi-geometric discounting)**.** *Young (2007) considers a dynamic optimization model with the following sequence of discount factors:*

$$1, \tilde{\beta}_1\delta, \tilde{\beta}_1\tilde{\beta}_2\delta^2, \tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3\delta^3, \ldots$$

Therefore, between any two consecutive dates (say $t+1$ and t), the discount rate is $\tilde{\beta}_t\delta$. Suppose we have that the limit $\lim_{t\to\infty}\tilde{\beta}_t \in (0,1]$ exists and each $\tilde{\beta}_t\delta < 1$. Then, if we seek TCE in the resulting model, we have:

$$c^*(s) \in \arg\max_{\tilde{c}\in[0,s]} u(\tilde{c}) + \tilde{\beta}_1\delta\mathbf{E}_{s-\tilde{c}}U_2(c^*).$$

where for $t \geq 2$, we also have:

$$U_t(c^*)(s) = u(c^*(s)) + \tilde{\beta}_t\delta\mathbb{E}_{s-c^*(s)}U_{t+1}(c^*).$$

Here, we can take

$$V_t(\tilde{c}, s - \tilde{c}, \mathbf{E}_{s-\tilde{c}}U(c)) = u(\tilde{c}) + \tilde{\beta}_t\delta\mathbf{E}_{s-\tilde{c}}U(c).$$

It is straightforward to see that this aggregator satisfies our assumptions, and therefore, TCE exists whenever transition q is nonatomic, u increasing and strictly concave. In this case $R \neq 1$ (generally) and hence our approximation technique cannot be applied.

Example 5 (Backward discounting)**.** *Following Ray et al. (2017) we consider an individual whose current utility is derived from evaluating both present and past consumption streams. Each of these streams is discounted, the former forward in the usual way, the latter backward. Specifically, assume an individual at date t evaluates consumption according to a weighted average of his own felicity (as perceived at date t) and that of a "future self" as perceived from date $T > t$. More specifically, for a generation born in $\tau = 0$ and taking the backward looking date to be $T(\tau) := T + \tau$ for some $T > 0$, her preferences are:*

$$\mathbb{E}_0 \sum_{t=0}^{T} \delta^t u(c_t)[\alpha + (1-\alpha)\delta^{T-2t}] + \delta^T \mathbb{E}_{T+1} \sum_{t=T+1}^{\infty} \delta^{t-T} u(c_t)[\alpha + (1-\alpha)\delta^{-T}].$$

where α (resp. $(1-\alpha)$) is the forward (resp. backward) looking weight. Observe that from $t \geq T$ the preferences become stationary with exponential discounting δ. So put

$$W(s_{T+1}) = \mathbb{E}_{T+1} \sum_{t=T+1}^{\infty} \delta^{t-T} u(c(s_t))[\alpha + (1-\alpha)\delta^{-T}]$$

to denote the value for this stationary part (for some candidate stationary policy c). That is, for $t \geq T$, we can take the aggregators:

$$V_t(\tilde{c}, s - \tilde{c}, \mathbf{E}_{s-\tilde{c}} U(c)) := u(\tilde{c})[\alpha + (1-\alpha)\delta^{-T}] + \delta \mathbf{E}_{s-\tilde{c}} U(c).$$

Observe this implies that the problem resembles a finite-bias discounting model discussed in section 3. Then for $t < T$, we need to, however, construct our preferences recursively (backwards) using aggregators V_t:

$$V_t(\tilde{c}, s - \tilde{c}, \mathbf{E}_{s-\tilde{c}} U(c)) := u(\tilde{c})[\alpha + (1-\alpha)\delta^{T-2t}] + \delta \mathbf{E}_{s-\tilde{c}} U(c)$$

with $U_T(c)(s_T) = u(c(s_T))[\alpha + (1-\alpha)\delta^{-T}] + \delta^T W(s_{T+1} - c(s_{T+1}))$.

Then, in this case, we seek TCE that are solutions of the following functional equations:

$$c^*(s) \in \arg\max_{\tilde{c} \in [0,s]} u(\tilde{c})[\alpha + (1-\alpha)\delta^T] + \delta \mathbf{E}_{s-\tilde{c}} U_1(c^*).$$

Again, with $\delta < 1$ the above aggregators (V_t) satisfy our assumptions and TCE exists whenever transition q is nonatomic, u increasing and strictly concave.

So far, in the paper, we have focused on models where this decision maker is infinitelylived. It happens, our approach is also useful when attempting to understand cases where agents are short-lived. Many important problems in economics have the latter form with examples including dynamic sustainable resource models with public policy, economic models of the transmission of human capital and endogenous preferences across generations, models of endogenous fertility, as well as related models of sustainable dynastic choice with intergenerational altruism and paternalism. One particularly relevant case is that of bequest games. We now show how our results can be applied in these models.

Example 6 (Limited time horizon discounting and bequest games)**.** *Consider a sequence of discount factors $1, \delta_1, \delta_2, \ldots, \delta_T, 0, 0, \ldots$ for some $T \geq 1$. This, therefore, is a class of T-period paternalistic bequest games with changing discount factors. To apply our results to this model, simply take:*

$$V_t(\tilde{c}, s - \tilde{c}, \mathbf{E}_{s-\tilde{c}} U(c)) = u(\tilde{c}) + \delta_t \mathbf{E}_{s-\tilde{c}} U(c).$$

Then again, we are able to verify TCE exist with monotone increasing investments whenever transition q is nonatomic. Again, observing that $\delta_{T+1} = 0$ we observe that the problem resembles a finite-bias discounting model discussed in section 3.

Finally, we can also allow for a discount factor to be state or choice dependent, e.g. $\beta(s)$ or $\beta(s-c)$ to account e.g. for magnitude effects in discounting (see Epstein and Hynes (1983) or Noor (2009) for a motivation).

Example 7 (Magnitude effects)**.** *Suppose the present bias discount factor β is a function of investment, i.e. $\beta : S \to [0,1]$ that is continuous and increasing. Then the aggregator takes the form:*

$$V_1(c, s - c^*(s), \mathbf{E}_{s-c} U^*(c^*)) = \max_{c \in [0,s]} (u(c) + \beta(s-c)\delta \mathbf{E}_{s-c} U_2^*(c^*))$$

where for $t > 1$:

$$U_t(c^*)(s) = V_t(c^*(s), s - c^*(s), \mathbf{E}_{s-c} U^*(c^*)) = u(c^*(s)) + \delta \mathbf{E}_{s-c^*(s)} U^*(c^*).$$

In a similar way, we can consider a case of δ being investment dependent. In such a case, one would need to impose:

$$U_t(c^*)(s) = V_t(c^*(s), s - c^*(s), \mathbf{E}_{s-c} U^*(c^*)) = u(c^*(s)) + \delta(s - c^*(s)) \mathbf{E}_{s-c^*(s)} U^*(c^*).$$

It is easy to see that this specification is also a special case of the general model, and hence

TCE c^* exist in this model.

6 Concluding Remarks

In this paper, we propose a new collection of functional equation methods for proving existence of (pure strategy) TCE in a general class of dynamic models with "behavioral" discounting with recursive payoffs and bounded or unbounded state space. Our approach allows use to link recursive utility models with the literature on the strategic aspects of stochastic games, and in particular models of dynamic choice with dynamically inconsistent preferences. We think that the general existence methods applied in section 5 can be extended to also show existence of TCE in more general models of altruism with recursive payoffs as recently axiomatized by Galperti and Strulovici (2017). We leave this question for further research.

A Appendix. Omitted lemmas and proofs

A.1 Quasi-hyperbolic discounting

We now state and prove a number of important preliminary results concerning these spaces and two important mappings defined in them. First, note the structure of the space $(\mathcal{V}, ||\cdot||)$ and its subset $\mathcal{U} \subset \mathcal{V}$.

Lemma 1. *$(\mathcal{V}, ||\cdot||)$ is a Banach space and $\mathcal{U} \subset \mathcal{V}$ is a closed set.*

Proof. For any $f \in \mathcal{V}$ consider $(\mathcal{V}_k, ||\cdot||_k)$, the restriction of f to $S_k \times \mathcal{H}$. Clearly $\mathcal{V}_k$ is a subset of Banach space of bounded functions on $S_k \times \mathcal{H}$, hence we only need to show $\mathcal{V}_k$ is closed. The convergence in norm $||\cdot||_k$ is equivalent to the uniform convergence on $S_k \times \mathcal{H}$. Suppose $\phi_n \rightrightarrows \phi$ as $n \to \infty$ in $||\cdot||_k$ and any of $\phi_n \in \mathcal{V}_k$. We show $\phi \in \mathcal{V}_k$. Obviously ϕ is bounded on $S_k \times \mathcal{H}$. We check further desired properties.

- *We show ϕ is right continuous on s for any fixed h.*

  Let $\epsilon > 0$ be given. Let $s_n \downarrow s^0$ and let N be such that $||\phi_N - \phi||_k < \frac{\epsilon}{2}$. We have

$$\begin{aligned} |\phi(s_n, h) - \phi(s^0, h)| &\leq |\phi(s_n, h) - \phi_N(s_n, h)| + |\phi_N(s_n, h) - \phi_N(s^0, h)| + |\phi_N(s^0, h) - \phi(s^0, h)| \\ &\leq 2||\phi - \phi_N||_k + |\phi_N(s_n, h) - \phi_N(s^0, h)|. \end{aligned}$$

Since ϕ_N is right continuous at s^0, hence taking a limit with $n \to \infty$ we have $\limsup_{n\to\infty} |\phi(s_n, h) - \phi(s^0, h)| < \epsilon$. Since ϵ is arbitrary, hence $\phi(s_n, h) \to \phi(s^0, h)$. Hence $\phi(\cdot, h)$ is right continuous.

- *We show ϕ is upper semicontinuous.* Let $(s_n, h_n) \to (s^0, h^0)$. As before $\epsilon > 0$ is given and N is such that $||\phi - \phi_N||_k < \frac{\epsilon}{2}$, Hence

$$
\begin{aligned}
&\phi(s^0, h^0) - \phi(s_n, h_n) = \\
&\phi(s^0, h^0) - \phi_N(s^0, h^0) + \phi_N(s^0, h^0) - \phi_N(s_n, h_n) + \phi_N(s_n, h_n) - \phi(s_n, h_n) \geq \\
&-\epsilon + \phi_N(s^0, h^0) - \phi_N(s_n, h_n).
\end{aligned}
$$

  Since ϕ_N is upper semicontinuous

$$\liminf_{n\to\infty} (\phi(s^0, h^0) - \phi(s_n, h_n)) \geq -\epsilon.$$

  Since $\epsilon > 0$, hence ϕ is upper semicontinuous.

- *We show for any $h \in \mathcal{H}$ there is a countable $\tilde{S} \subset S$ such that ϕ is continuous at any $(s, h) \in \mathcal{E}$, such that $s \notin \tilde{S}$.* Let $\tilde{S}^N \subset \mathcal{E}^h$ be a countable set such that f_N is continuous at any (s, h) with $s \notin \tilde{S}^N$. Let $\tilde{S} := \bigcup_{N=1}^{\infty} \tilde{S}^N$. Observe $\tilde{S}$ is countable and any of ϕ_N is continuous at (s, h) such that $s \notin \tilde{S}$. Since ϕ is the uniform limit of ϕ_N on any set $S_k \times \mathcal{H}$, hence ϕ is continuous at (s, h).

Consequently $\phi \in \mathcal{V}_k$ and $(\mathcal{V}_k, || \cdot ||_k)$ is Banach space. Pick any $\phi_k \in \mathcal{V}_k$ such that $\phi_{k+1}(s, h) = \phi_k(s, h)$ for any $(s, h) \in S_k \times \mathcal{H}$. Define $\phi(s, h) = \phi_k(s, h)$ whenever $s \in S_k$. Observe that $\phi(\cdot)$ is upper semicontinuous and $\phi(\cdot, h)$ is right continuous. Moreover, for any $h \in \mathcal{H}$, ϕ may be discontinuous at $(s, h) \in \mathcal{E}^h$, where s is chosen from at most countable set. Hence $\phi \in \mathcal{V}$. By Lemma 1 in Matkowski and Nowak (2011), we conclude $(\mathcal{V}, || \cdot ||$ is a Banach space. It is easy to see, $\mathcal{U}$ is a complete metric space with the metric induced by $|| \cdot ||$ since it is a closed subset of $\mathcal{V}$. □

Lemma 2. *Let $f \in \mathcal{U}$ and suppose $h_n \to^w h$. Then if $\mu_n \to \mu$ weakly on S, then*

$$\limsup_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \leq \int_S f(s', h)\mu(ds'). \tag{16}$$

Suppose that μ is concentrated on the set of continuity points of $f(\cdot, h)$. Then

$$\lim_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') = \int_S f(s', h)\mu(ds'). \tag{17}$$

Proof. Define:

$$\overline{f}(s) = \sup\left\{\limsup_{n\to\infty} f(s_n, h_n)) : s_n \to s\right\}$$

and

$$\underline{f}(s) = \inf\left\{\liminf_{n\to\infty} f(s_n, h_n)) : s_n \to s\right\}.$$

Since f is u.s.c. hence

$$\limsup_{n\to\infty} f(s_n, h_n) \leq f(s, h)$$

whenever $s_n \to \infty$, hence $\overline{f}(s) \leq f(s, h)$. Hence and by Lemma 3.2. in Serfozo (1982) we have

$$\limsup_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \leq \int_S \overline{f}(s)\mu(ds) \leq \int_S f(s, h)\mu(ds).$$

Now suppose f is continuous at (s, h) for μ-a.a. s. Then for μ-a.a. s we have

$$\lim_{n\to\infty} f(s_n, h_n) = f(s, h)$$

whenever $s_n \to s$. Hence $f(s, h) = \underline{f}(s)$, μ-almost everywhere. Again by Lemma 3.2. in Serfozo (1982) we have

$$\liminf_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \geq \int_S \underline{f}(s)\mu(ds) = \int_S f(s, h)\mu(ds).$$

Since we have proven (16), hence

$$\int_S f(s, h)\mu(ds) \geq \limsup_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \geq \liminf_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \geq \int_S f(s, h)\mu(ds).$$

Hence (17) holds and the proof is complete. □

Lemma 3. *T maps $\mathcal{U}$ into itself.*

Proof. Let $f \in \mathcal{U}$. Obviously $|T(f)(s, h)| \leq \eta_k$ for $(s, h) \in S_k \times \mathcal{H}$. Similarly as in Lemma 5 in Balbus et al. (2020a) we conclude $\mathbf{E}_i f(h)$ is continuous from the right. We easily conclude $T(f)(\cdot, h)$ is right continuous. We are going to show $T(f)$ it is upper semicontinuous. Let $(s_n, h_n) \to (s^0, h^0)$ in the corresponding topology. Pick

$$i_n \in \arg\max_{i\in[0,s_n]} V(s_n - i_n, s_n - h_n(s_n), \mathbb{E}_{i_n} f(h_n))$$

and without loss of generality suppose $i_n \to i^0$. By Assumption 2, $q(\cdot|i_n) \to q(\cdot|i^0)$ weakly.

By Lemma 2 we have then

$$\limsup_{n\to\infty} \mathbb{E}_{i_n} f(h_n) \le \mathbb{E}_{i^0} f(h^0), \tag{18}$$

We have

$$\liminf_{n\to\infty}(s_n - h_n(s_n)) \ge s^0 - \limsup_{n\to\infty} h_n(s_n) \ge s^0 - h^0(s^0), \tag{19}$$

Combining (18) and (19) we have

$$\limsup_{n\to\infty} V(s_n - i_n, s_n - h_n(s_n), \mathbb{E}_{i_n} f(h_n)) \le T(f)(s^0, h^0). \tag{20}$$

Hence $T(f)$ is upper semicontinuous. Finally, we show $T(f)$ is continuous at any $(s,h) \in \mathcal{E}$ such that $s \notin S^{T(f),h}$, where $S^{T(f),h}$ is at most countable subset of S. We can take

$$S^{T(f),h} := \{s \in \mathcal{E}^h : q(\{s' \in S : f \text{ is continuous at } (s',h)\}|s) < 1\}$$

and clearly $S^{T(f),h}$ is countable. Now assume $(s_n, h_n) \to (s^0, h^0)$ and $s^0 \notin S^{T(f),h^0}$. Then by definition of convergence of $\mathcal{H}$ and Lemma 2 we have

$$\lim_{n\to\infty} V(s_n - i, s_n - h_n(s_n), \mathbb{E}_i f(h_n)) = V(s^0 - i, s^0 - h^0(s^0), \mathbb{E}_i f(h^0)) \tag{21}$$

for any $i \notin S^{T(f),h^0}$, in particular for s^0. Again by (18) and (19) we have

$$\begin{aligned} V(s^0 - i^0, s^0 - h(s^0), \mathbb{E}_{i^0} f(h)) &\ge \limsup_{n\to\infty} V(s_n - i_n, s_n - h_n(s_n), \mathbb{E}_{i_n} f(h_n)) \\ \ge \liminf_{n\to\infty} V(s_n - i, s_n - h_n(s_n), \mathbb{E}_i f(h_n)) &= V(s^0 - i, s^0 - h^0(s^0), \mathbb{E}_i f(h^0)). \end{aligned}$$

Since the right hand side is right continuous, hence this equality holds for any $i \in [0, s^0]$. Indeed, we can take $\tilde{i}^m \downarrow i$ as $m \to \infty$ such that $\tilde{i}^m \in S^{T(f),h^0}$, substitute i by $\tilde{i}^m$ above, and take a limit $m \to \infty$. □

Lemma 4. *T is a contraction mapping on $\mathcal{U}$, and therefore has a unique fixed point in $\mathcal{U}$.*

Proof. Observe that by the standard argument

$$||T(f) - T(g)||_k \le ||f - g||_{k+1} \quad \text{for any } k \in \mathbb{N}.$$

Hence T is 1-local contraction. By Theorem 2 in Rincon-Zapatero and Rodriguez-Palmero

(2009), T is a contraction mapping on $\mathcal{U}$. By Lemma 1 and Banach Contraction Principle T has a unique fixed point. $\square$

Lemma 5. *Let $h \in \mathcal{H}$. Then, any selection of $s \mapsto BI(h)(s)$ is increasing in s.*

Proof. Suppose that it is not the case: there are $s_1 > s_2$ and $i_1 < i_2$ such that $i_1 \in BI(h)(s_1)$ and $i_2 \in BI(h)(s_2)$. Then

$$0 \leq V(s_2 - i_2, s_2 - h(s_2), \mathbb{E}_{i_2} f^*(h)) - V(s_2 - i_2 - (i_2 - i_1), s_2 - h(s_2), \mathbb{E}_{i_1} f^*(h)).$$

But then from Assumption 1 (ii) we have

$$V(s_1 - i_2, s_1 - h(s_1), \mathbb{E}_{i_1} f^*(h)) - V(s_1 - i_2 - (i_2 - i_1), s_1 - h(s_1), \mathbb{E}_{i_2} f^*(h)) > 0$$

which contradicts $i_1 \in BI(h)(s_1)$. $\square$

Lemma 6. *Let $h \in \mathcal{H}$. If $bi(h)$ is continuous at s, then $BI(h)(s)$ is a singleton.*

Proof. Suppose that $bi(h)$ is continuous at s and pick $y_0 \in BI(h)(s)$. By Lemma 5 we have $bi(h)(s - \delta) \leq y_0 \leq bi(s + \delta)$. Since $bi(h)$ is continuous, hence $y_0 = bi(h)(s)$, and consequently $BI(h)$ is singleton. $\square$

Lemma 7. *The operator bi maps $\mathcal{H}$ into itself and it is a continuous operator.*

Proof. By Lemma 5 it follows that $bi(h)(\cdot)$ is increasing. We show it is right continuous. Let $s_n \downarrow s^0$. We show $i_n := bi(h)(s_n) \to bi(h)(s^0)$. By Lemma 5, $i_n \downarrow i^0$. Since h is right continuous $h(s_n) \downarrow h(s^0)$ as $n \to \infty$. Put

$$\Pi(s, i) := V(s - i, s - h(s), \mathbb{E}_i(f^*)).$$

Suppose $i \notin S^{f^*,h}$. Since h and $i \mapsto \mathbb{E}_i(f^*)$ are both right continuous, hence we have

$$\Pi(s^0, i^0) = \lim_{n \to \infty} \Pi(s_n, i_n) \geq \Pi(s^0, i)$$

for all $i \in [0, s^0)$. Hence $i^0 \in BI(h)(s^0)$ if $bi(h)(s^0) < s^0$. If we allow, $bi(h)(s^0) = s_0$, by Lemma 5 we have $i^0 \leq bi(h)(s^0) \leq bi(h)(s_n)$ for all n, hence taking a limit with $n \to \infty$ we have $i^0 = bi(h)(s^0)$. Now we show the continuity of bi on $\mathcal{H}$. Suppose $h_n \to^w h^0$ in $\mathcal{H}$ such that s^0 is a continuity point of $bi(h^0)(\cdot)$. By Lemma 6 it follows that $BI(h^0)(s^0)$ is a singleton in this case. Hence we are going to show $i_n := bi(h_n)(s^0) \to i^0$ for some $i^0 \in BI(h^0)(s^0)$. Define

$$Z^0 := \{i \in S : q(S^{f^*,h_0} | i) = 0\}.$$

By Assumption 2 the complement of Z^0 is at most countable. First, let us focus attention to $s^0 \notin Z^0$. By definition of S^{f^*,h^0}, for any $i \notin Z^0$ we have

$$\mathbb{E}_i f^*(h_n) \to \mathbb{E}_i f^*(h^0)$$

as $n \to \infty$. Moreover, $h_n(s^0) \to h(s^0)$ and if $i_n \to i$, then by Lemma 2

$$\lim_{n\to\infty} \mathbb{E}_{i_n} f^*(h_n)) = \mathbb{E}_i f^*(h^0)).$$

Hence:

$$\begin{aligned}
\lim_{n\to\infty} V(s^0 - i_n, s^0 - h_n(s^0), \mathbb{E}_{i_n} f^*(h_n)) &\geq \liminf_{n\to\infty} V(s^0 - i, s^0 - h_n(s^0), \mathbb{E}_i f^*(h_n))) \\
&\geq V(s^0 - i, s^0 - h^0(s^0), \mathbb{E}_i f^*(h^0))). \qquad (22)
\end{aligned}$$

Since the right hand side above we right continuous, hence the inequality (22) holds for any $i \in [0, s^0]$ since $s^0 \notin Z^0$. To finish the proof observe

$$\limsup_{n\to\infty} V(s^0 - i_n, s^0 - h_n(s^0), \mathbb{E}_{i_n} f^*(h_n)) \leq V(s^0 - i^0, s^0 - h^0(s^0), \mathbb{E}_{i^0} f^*(h^0)),$$

where the last inequality follows from (18). Then combining the inequality above with (22) we have $i^0 \in BI(h^0)(s^0)$, consequently $i^0 = bi(h^0)(s^0)$. Hence we have proven, $bi(h_n)(s^0) \to bi(h)(s^0)$ as $n \to \infty$ whenever $s^0 \in Z^0$ and s^0 is a continuity point of $bi(h)$. To finish the proof, we need to show that this convergence is true outside Z^0 as well. If $s^0 \notin Z^0$ is a continuity point of $bi(h^0)$, we may find $\delta_1 > 0$ and $\delta_2 > 0$ such that $bi(h^0)$ is both continuous at $s^0 - \delta_1$, $s^0 + \delta_2$ but $s^0 - \delta_1 \in Z^0$ in $s^0 + \delta_2 \in Z^0$. By Assumption 2, δ_1 and δ_2 can be sufficiently small. Then, by the previous part of the proof

$$\begin{aligned}
bi(s^0 - \delta_1) &= \lim_{n\to\infty} bi(h_n)(s^0 - \delta_1) \leq \liminf_{n\to\infty} bi(h_n)(s^0) \\
&\leq \limsup_{n\to\infty} bi(h_n)(s^0) \leq \lim_{n\to\infty} bi(h_n)(s^0 + \delta_2) = bi(h^0)(s^0 + \delta_2).
\end{aligned}$$

Taking a limit $\delta_1 \to 0$ and $\delta_2 \to 0$ we have $bi(h_n)(s^0) \to bi(h^0)(s^0)$ as $n \to \infty$. □

A.2 Semi-hyperbolic discounting

Lemma 8. *$(\mathcal{V}_0, ||\cdot||)$ is a Banach space and $\mathcal{U}_0 \subset \mathcal{V}_0$ is a closed set.*

Proof. For any $f \in \mathcal{V}_0$ consider $(\mathcal{V}_{k,0}, ||\cdot||_k)$, the restriction of f to $S_k \times \mathcal{H}$. Clearly $\mathcal{V}_{k,0}$ is a subset of Banach space of bounded functions on $S_k \times \mathcal{H}$, hence we only need to show

$\mathcal{V}_{k,0}$ is closed. The convergence in norm $||\cdot||_k$ is equivalent to the uniform convergence on $S_k \times \mathcal{H}$. Suppose $\phi_n \rightrightarrows \phi$ as $n \to \infty$ in $||\cdot||_k$ and any of $\phi_n \in \mathcal{V}_{k,0}$. We show $\phi \in \mathcal{V}_{k,0}$. Obviously ϕ is bounded on $S_k \times \mathcal{H}$. Similarly as in Lemma 1 we may show that for any $h \in \mathcal{H}$ there is a countable $\tilde{S} \subset S$ such that ϕ is continuous at any $(s,h) \in \mathcal{E}$, such that $s \notin \tilde{S}$. Consequently $\phi \in \mathcal{V}_k$ and $(\mathcal{V}_{k,0}, ||\cdot||_k)$ is Banach space. Pick any $\phi_k \in \mathcal{V}_{k,0}$ such that $\phi_{k+1}(s,h) = \phi_k(s,h)$ for any $(s,h) \in S_k \times \mathcal{H}$. Define $\phi(s,h) = \phi_k(s,h)$ whenever $s \in S_k$. Observe that for any $h \in \mathcal{H}$, ϕ may be discontinuous at $(s,h) \in \mathcal{E}$, where s is chosen from at most countable set. Hence $\phi \in \mathcal{V}$. By Lemma 1 in Matkowski and Nowak (2011), we conclude $(\mathcal{V}_0, ||\cdot||)$ is a Banach space. It is easy to see, $\mathcal{U}_0$ is a complete metric space with the metric induced by $||\cdot||$ since it is a closed subset of $\mathcal{V}_0$. □

Lemma 9. *Λ maps $\mathcal{U}_0$ into itself and is a contraction mapping in $\mathcal{U}_0$.*

Proof. We omit the proof since it is similar as the proof of Lemma 4. □

Lemma 10. *For any $h \in \mathcal{H}$, any selection of $\mathcal{BI}(h)(s)$ is nonempty valued and has the greatest and the least selection. Moreover, any selection of $\mathcal{BI}(h)(s)$ is increasing in s.*

Proof. We omit the proof since it is similar to the proof of Lemma 5. □

Lemma 11. *Let $h \in \mathcal{H}$ and suppose h is continuous at s. Then, if $bi(h)(s)$ is continuous at s, then $\mathcal{BI}(h)(s)$ is a singleton.*

Proof. Using Lemma 10 we repeat the same argument as in Lemma 6. □

Lemma 12. *The operator bi maps $\mathcal{H}$ into itself and it is a continuous operator.*

Proof. Let $h_n \to^w h^0$ as $n \to \infty$ and let s' be a continuity point of h^0. We have

$$\sup\left\{\limsup_{n\to\infty} f^*(s'_n, h_n) : s'_n \to s'\right\} = \inf\left\{\liminf_{n\to\infty} f^*(s'_n, h_n) : s'_n \to s'\right\} = f(s', h^0),$$

whenever $(s', h^0) \in \mathcal{E}$ and $s' \notin S^{f^*,h^0}$. Observe that for any $s'_n \to s'$ and $h_n \to^w h$ we have $h_n(s'_n) \to h^0(s')$ whenever $s' \notin S^{f^*,h^0}$ and it is a continuity point of h^0. By Assumption 4 it follows that this convergence above holds for all but countably many $s' \in S$. Let $i_n \to i^0$ in S. Hence by Lemma 2

$$\int_S f^*(s', h^0) q(ds'|i^0) = \lim_{n\to\infty} \int_S f^*(s', h_n) q(ds'|i_n). \tag{23}$$

We show that

$$\lim_{n\to\infty} \int_S A_t^*(s', h_n) q(ds'|i_n) = \int_S A_t^*(s', h^0) q(ds'|i^0). \tag{24}$$

The thesis for $t = 0$ is in (23). If this thesis holds for some t, then by definition of $A^*_{t+1}(s', h)$ and (24) we have this thesis, and (24) holds for $t+1$. As a result, the function

$$(s, i, h) \in S \times S \times \mathcal{H} \mapsto u(s-i) + \prod_{t=1}^{T} \beta_t \int_S A^*_{T-1}(s', h) q(ds'|i)$$

is continuous. Let s^0 be a continuity point of $bi(h^0)(\cdot)$. Let $y_n = bi(h_n)(s^0)$ and suppose $y_n \to y^0$. Hence by Berge Maximum Theorem $y^0 \in \mathcal{BI}(h^0)(s^0)$. By Lemma 11, $\mathcal{BI}(h^0)(s^0)$ is a singleton, hence $y^0 = bi(h)(s^0)$. But this implies $bi(h_n) \to^w bi(h_n)$. □

A.3 Limiting case

Lemma 13. *$\prod_{k=1}^{\infty} \beta_k$ exists and is nonzero if and only if $\lim_{t\to\infty} \prod_{k=t}^{\infty} \beta_k = 1$.*

Proof. Define $r := \prod_{k=1}^{\infty} \beta_k$, and suppose $r > 0$. Then $-ln(r) = \sum_{k=1}^{\infty} - \ln(\beta_k)$. Since $-\ln(\beta_k) > 0$, hence the series above are convergent and $\lim_{t\to\infty} \sum_{k=t}^{\infty} - \ln(\beta_k) = 0$. Moreover,

$$\lim_{t\to\infty} \sum_{k=t}^{\infty} - \ln(\beta_k) = -\lim_{t\to\infty} \ln\left(\prod_{k=t}^{\infty} \beta_k\right) = -\ln\left(\lim_{t\to\infty} \prod_{k=t}^{\infty} \beta_k\right). \tag{25}$$

Combining (A.3) with (25) we have the thesis. Now let $r = 0$. Then the right hand side in (A.3) yields ∞. Furthermore, by (25) we have $\lim_{t\to\infty} \prod_{k=t}^{\infty} \beta_k = 0$ □

A.4 General existence result

Lemma 14. *$(\mathcal{V}_0^\infty, ||\cdot||^\infty)$ is a Banach space, and $\mathcal{U}_0^\infty$ is a closed subset of $\mathcal{V}_0^\infty$.*

The proof is identical as proof of Lemma 8.

Lemma 15. *$\mathbb{T}$ is a contraction mapping on $\mathcal{U}_0^\infty$ and has a unique fixed point.*

Proof. We show $\mathbb{T}$ maps $\mathcal{U}_0^\infty$ into itself. Let $f \in \mathcal{U}_0^\infty$. Then for any $k \in \mathbb{N}$, $s' \in S_{k+1}$, $h \in \mathcal{H}$ and $t \in \mathbb{N}$ we have $|f_{t+1}(s', h)| \leq \eta_{k+1}$. By Assumption 2 for any $s \in S_k$ we have

$$|\mathbb{E}_{h(s)} f_{t+1}(h)| = \left| \int_S f_{t+1}(s', h) q(ds'|h(s)) \right| \leq \eta_{k+1},$$

hence

$$\left|V_t(s-h(s),s-h(s),\mathbb{E}_{h(s)}f_{t+1}(h))\right| \le \sup_{(x,y,z)\in S_k^2\times[0,\eta_{k+1}]} |V_t(x,y,z)| \le \eta_k$$

where the last equality is a consequence of Assumption 1. Furthermore, applying Lemma 2 we conclude

$$s\in S\mapsto V_t(s-h(s),s-h(s),\mathbb{E}_{h(s)}f_{t+1}(h))$$

is left continuous and continuous at any $s\notin S^{\mathbb{T}(f),h}$, where $S^{\mathbb{T}(f),h}$ is a countable subset of S. Hence $\mathbb{T}(f)\in\mathcal{U}_0^\infty$. Observe that by Assumption 1 and the standard argument

$$||\mathbb{T}(f)-\mathbb{T}(g)||_k \le ||f-g||_{k+1} \quad \text{for any } k\in\mathbb{N}.$$

Hence is 1-local contraction. By Theorem 2 in Rincon-Zapatero and Rodriguez-Palmero (2009), $\mathbb{T}$ is a contraction mapping on $\mathcal{U}_0^\infty$. By Lemma 1 and Banach Contraction Principle $\mathbb{T}$ has a unique fixed point. □

Lemma 16. *Let $h\in\mathcal{H}$. Then, $BI(h)(s)$ is nonempty valued correspondence with the greatest and the least selection. Moreover, any selection of $BI(h)(s)$ is increasing in s.*

Proof. First we show $BI(h)(s)$ is indeed nonempty valued correspondence with the greatest and the least element. Let f^* be a unique fixed point of $\mathbb{T}$ and f_2^* be the coordinate needed to define BI. For any $h\in\mathcal{H}$ let $S^{*,h}$ be a countable subset of S such that f_2^* is continuous at any $(s,h)\in S\times\mathcal{H}$ such that $s\in S^*$. We show that the following function

$$(i,h)\in S\times\mathcal{H}\mapsto \mathbb{E}_i f_2^*(h)) = \int_S f_2^*(s,h)q(ds|i)$$

is continuous. Indeed, by Assumption 4, $q(\cdot|i)$ is nonatomic, hence $q(S^{*,h}|i)=0$ for any $h\in\mathcal{H}$ and $i\in S$. Let $i_n\to i$ in S and $h_n\to^w h$ in $\mathcal{H}$. By Skorohod Representation Theorem, there is a probability space $(\Omega,\mathcal{F},P)$ and random variables X_n whose distribution is $q(\cdot|i_n)$ and X whose distribution is $q(\cdot|i)$ such that $X_n\to X$ pointwise in Ω. Since $q(\cdot|i)$ is concentrated away of $S^{*,h}$, hence $X(\omega)\notin S^{*,h}$ for P- a.a. $\omega\in\Omega$. Hence $f_2^*(X_n(\omega),h_n)\to f_2^*(X(\omega),h)$ for P-a.a. ω. We have then

$$\begin{aligned}\mathbb{E}_{i_n} f_2^*(h_n)) = \int_S f_2^*(s,h_n)q(ds|i_n) = \int_\Omega f_2^*(X_n(\omega),h_n)P(d\omega) \to_{n\to\infty} \\ \int_\Omega f_2^*(X(\omega),h)P(d\omega) = \int_S f_2^*(s,h)q(ds|i) = \mathbb{E}_i f_2^*(h)).\end{aligned}$$

Hence $BI(h)(s) \neq \emptyset$ and has the greatest and the least selection. The rest of proof is omitted, since is the same as the proof of Lemma 5. $\square$

By Lemma 16 we can repeat the same argument as in Lemmas 6 and 7 to obtain:

Lemma 17. *Let $h \in \mathcal{H}$ and suppose h is continuous at s. Then, if $bi(h)(s)$ is continuous at s then $BI(h)(s)$ is a singleton.*

Combining Lemmas 16 and 17 we have the following:

Lemma 18. *The operator bi maps $\mathcal{H}$ into itself and it is a continuous operator.*

A.5 Auxiliary results

Lemma 19. *Let μ_n be a sequence of measures on a Polish space Z such that $\mu_n \Rightarrow \mu$. Let $f : Z \mapsto \mathbf{R}$ be a bounded Borel measurable function and let $f_n : Z \mapsto \mathbb{R}$ be a sequence of bounded Borel measurable functions satisfying*

$$\mu(\{z \in Z : (z_n \to z \text{ as } n \to \infty \text{ in } Z) \Rightarrow (f_n(z_n) \to f(z) \text{ as } n \to \infty)\}) = 1. \tag{26}$$

Then, $\int f_n d\mu_n \to \int f d\mu$.

Proof. By Skorohod's Representation Theorem (Theorem 6.7. in Billingsley (1999)) we find a probability space $(\Omega, \mathcal{F}, P)$ and Z-valued random variables $X_n : \Omega \mapsto Z$ whose distribution is μ_n and a Z-valued random variable $X : \Omega \mapsto Z$ whose distribution is μ such that $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$. By (26) it follows that

$$\lim_{n\to\infty} f_n(X_n(\omega)) = f(X(\omega)).$$

Indeed, let

$$Z_0 := \{z \in Z : (z_n \to z \text{ as } n \to \infty \text{ in } Z) \Rightarrow (f_n(z_n) \to f(z) \text{ as } n \to \infty)\}.$$

Then,

$$P\left(\left\{\omega \in \Omega : \lim_{n\to\infty} f_n(X_n(\omega)) = f(X(\omega))\right\}\right) \geq P(\omega \in \Omega : X(\omega) \in Z_0) = \mu(Z_0) = 1.$$

Hence, and by Dominated Convergence Theorem

$$\lim_{n\to\infty} \int f_n d\mu_n = \lim_{n\to\infty} \int_\Omega f_n(X_n(\omega)) P(d\omega) = \int f d\mu.$$

□

For any T, let X_t^T be a $S-$valued Markov chain with a deterministic initial value x and a transition probability $s \in S \mapsto q(\cdot|h^T(s))$ where $h^T \in \mathcal{H}$. We denote X_t^* as a $S-$ Markov chain whose initial value x and a transition probability $s \in S \mapsto q(\cdot|h(s))$ where $h \in \mathcal{H}$. Let $Q_{s_0}^T$ be the joint distribution of $(X_t^T)_{t=1}^{\infty}$ and let Q_{s_0} be the joint distribution of $(X_t^*)_{t=1}^{\infty}$ with fixed initial distribution s_0.

Lemma 20. *For any $s_0 \in S$, $Q_{s_0}^T \Rightarrow Q_{s_0}^*$. As a result, there exists a probability space $(\Omega, \mathcal{F}, P)$ and $S-$ valued sequences $(\xi_t^T(\omega))^{\infty}$ and $(\xi_t^*(\omega))_{t=1}^{\infty}$ whose join distribution are Q^T and respectively Q^* such that $\lim_{T\to\infty} \xi_t^T(\omega) = \xi_t^*(\omega)$ for any $\omega \in \Omega$ and $t \in \mathbb{N}$.*

Proof. We show that for any integer k, $s \in S$ and any bounded and continuous $f^k : S^k \mapsto \mathbb{R}$ it holds

$$\lim_{T\to\infty} \int_{S^\infty} f(s_1, s_2, \ldots, s_k) Q_s^T(ds^\infty) = \int_{S^\infty} f(s_1, s_2, \ldots, s_k) Q_s^*(ds^\infty) \tag{27}$$

We prove this thesis by induction with respect to k. For $k = 1$ it is follows directly by Assumption 2. Suppose that (27) holds for some k. Put

$$\tilde{f}(s_1, s_2, \ldots, s_k) := \int_{S^\infty} f(s_1, s_2, \ldots, s_k, s_{k+1}) q(ds_{k+1}|h^T(s_k)).$$

Observe that by Assumption 4, any of c^T is continuous for Q_s^T and Q_s -a.a. $s^\infty \in S^\infty$. As a result, by Lemma 19, $\tilde{f}$ is a continuous function on S^k for Q_s^T and Q_s- a.a. $s^\infty \in S^\infty$. Hence substituting $\tilde{f}$ by f into (27) and applying Lemma 15.4 in Aliprantis and Border (2006) we obtain exactly (27) with $k+1$. For the second part we apply again Skorohod's Representation Theorem (Theorem 6.7. in Billingsley (1999)). □

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