Markov Stationary Equilibria in Stochastic Supermodular Games with Imperfect Private and Public Information*

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Abstract

We study a class of discounted, infinite horizon stochastic games with public and private signals and strategic complementarities. Using monotone operators defined on the function space of values and strategies (equipped with a product order), we prove existence of a Stationary Markov Nash Equilibrium via constructive methods. In addition, we provide monotone comparative statics results for ordered perturbations of our space of games. We present examples from industrial organization literature and discuss possible extensions of our techniques for studying principal-agent models.
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1 Introduction and related literature

Since the class of discounted infinite horizon stochastic games was first introduced by Shapley (1953), the question of existence and characterization of equilibrium has been the object of extensive study in game theory.[1] In addition, more recently, stochastic games have become a fundamental tool for studying dynamic equilibrium in economic models where there is repeated strategic interactions among agents with limited commitment. In many such economic applications, these stochastic games with limited commitment have included games with *both* public and private information. When private information is introduced into stochastic games, the structure of equilibrium becomes difficult to analyze, as it requires one to keep track of each player's beliefs concerning the private histories of all the other players. Private information can be introduced into the structure of the game in the form of private types and/or private monitoring (see e.g. Kandori, 2002). In the former case of private types, progress has been made recently by focusing on public strategies and equilibria (see Fudenberg and Yamamoto (2011)

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[1]See Raghavan, Ferguson, Parthasarathy, and Vrieez (1991) or Neyman and Sorin (2003) for an extensive survey of results, along with references.

or applications in Athey and Bagwell (2008)). In the latter case of private monitoring, authors have often assumed that private monitoring is almost perfect (Hörner and Olszewski, 2006), or that sequential equilibrium strategies are belief-free.[2]

An additional important issue in dynamic games concerns the assumption of players' infinite memory. In recent work, economists have begun to analyze situations where players do not have arbitrarily long memory of their own and/or others past moves or states. Given this assumption, the players cannot condition their future actions on arbitrarily long histories[3]. Even in this case, the characterization of a short-memory or bounded-recall equilibria is somewhat problematic, as the punishment schemes needed to sustain equilibrium are imposed in a somewhat ad hoc manner, and can depend on the particular's of the game at hand. Further, because of structure imposed on the game in the name of analytic tractability, restrictive assumptions are often placed on player's action spaces, as well as the space of private signals/distributions, not to mention public randomization devices or necessity to use mixed strategies.

In this paper, we propose a new approach to analyze games with both public and private information (types). Our motivation is to resolve the aforementioned predicaments in the context of an important class of games. We do this by introducing a simple strategy space, as well as rational expectations concerning the opponent's private information. We also allow for uncountable multidimensional state and action spaces, and we assume that players use pure Markovian stationary strategies. In particular, Markov stationary Nash equilibrium (MSNE, henceforth) imply a few important characteristics per the structure of equilibrium, including: (i) the imposition of sequential rationality, (ii) the use of *minimal* state spaces, where the introduction of sunspots or public randomization are not necessary for the existence of equilibrium, as well as (iii) a relatively direct method to compute equilibrium. To obtain our results, our work focuses on an important class of stochastic games, namely those with strategic complementarities (GSC). It is well-known that GSC have proven very useful in applications in economics and operations research in a static context,[4] but it turns out to be difficult to adapt this toolkit to the study of dynamic equilibrium[5]. One recent attempt to analyze dynamic supermodular (extensive form) game was undertaken by Balbus, Reffett, and Woźny (2011) in a context of stochastic game with *public* signals. Here, we focus on the stochastic supermodular games with both public and private shocks, and with our new results, we are able to link the lines of literatures on dynamic supermodular games with that on Bayesian supermodular games (Van Zandt, 2010; Vives, 1990).

It is worth mentioning that in the literature pertaining to economic applications of dynamic/stochastic games, the central concern has been not only on the question of weakening conditions for the existence of equilibrium or various forms of folk theorems (see e.g. Fudenberg, Levine, and Maskin, 1994), but also on characterizing the properties of computational implementations. For such questions, one needs to provide a theory to numerical implementation that is closely tied to the arguments used to verify existence, which requires both (i) sharp characterizations of the set of equilibria being computed, and (ii) constructive fixed point methods that can be tied directly to approximation schemes. Our paper proposes a framework to address

[2]For example, this occurs when one studies the case where continuation play is independent on the beliefs on the information set (see Ely, Hörner, and Olszewski, 2005).

[3]C.f., Cole and Kocherlakota (2005), Hörner and Olszewski (2009), Barlo, Carmona, and Sabourian (2009) and Mailath and Olszewski (2011).

[4]See, for example, the excellent survey of Vives (2005) for a discussion of the extensive applications of GSC in the economics literature.

[5]The only exception to this of which we are aware is (i) the example of dynamic global game presented in Chassang (2010) , and (ii) the analyss of a large industry dynamics game studied in Sleet (2001).

both types of issues.

The rest of the paper is organized as follows. Section 2 defines the game and equilibrium concept. Then in section 3 we prove our main theorem on MSNE existence and computation. Section 4 presents three examples from industrial organization literature. Section 5 states the auxiliary theorem we use in our proofs, while section 6 concludes with a discussion of related methods.

2 The class of games

Consider an n-person infinite horizon stochastic game with private and public signals in discrete time. That is, in each period $t \in \{0, 1, 2, ...\} = \mathbb{N} \cup \{\infty\}$, every player i initially observes both public signal z^t, as well as his own private signal θ_i^t. At this stage, players simultaneously undertake actions $a^t = (a_i^t, a_{-i}^t)$, which (i) yield to each player a current period payoff, as well as (ii) parameterize a stochastic transition on states that governs the distribution of public and private signals tomorrow. At the end of the period, all actions are observed to all players, payoffs are distributed, and the game moves forward to the next period.

Formally, we define the game as the tuple $\Gamma = (Z, \Theta, A, \tilde{A}, \mu, r, q, Q)$, where the elements of these primitives are described as follows:

- Z is a public shock space, and is an interval in a vector space containing 0 vector, and we endow Z with Borel sigma-field $\mathcal{Z}$,
- $\Theta = \prod_{i=1}^{n} \Theta_i$, where Θ_i is a Polish space of private shocks for player i,
- $A = \prod_{i=1}^{n} A_i$, where $A_i \subset \mathbb{R}^k$ $(k < \infty)$ is the action space for player i, where $\mathbb{R}^k$ is equipped with its Euclidean topology, and we endow A with its product order,
- $\tilde{A}(z, \theta) = \prod_{i=1}^{n} \tilde{A}_i(z, \theta_i)$ is a weakly Borel measurable[6] A_i-valued correspondence, $(z, \theta_i) \to \tilde{A}_i(z, \theta_i)$, where $\tilde{A}_i(z, \theta_i)$ denotes a set of actions available for player i when the public shock is z and his private shock is θ_i,
- $r_i : Z \times \Theta \times A \to \mathbb{R}_+$ is player i reward function, which is assumed to be uniformly bounded by $M < \infty$,
- q is a Borel measurable transition probability from $Z \times \Theta \times A$ to Z (i.e., when a public shock is z, the vector of private shocks is $(\theta_1, ..., \theta_n)$, and actions chosen are $(a_1, ..., a_n)$, then distribution on the continuation realizations of shocks in Z is given by $q(\cdot|z, \theta, a)$),
- Q is a Borel measurable transition probability from Z to Θ (i.e., when a public shock is z, then vector of private shocks is given by $Q(\cdot|z)$). Further, let $Q_{-i}(\cdot|z, \theta_i)$ be a regular[7] conditional distribution on the "other players" private shocks Θ_{-i} (i.e., when the public shock is z, and private shock of player i is θ_i). In other words it is a posterior distribution on the other private signals, whenever agent i observes his own state and public state. In similar way, we let $Q_i(\cdot|z, \theta_{-i})$ denote a regular conditional distribution on ones "own" private shocks Θ_i.

[6]That is, lower inverse image of an open set is open.

[7]Existence of regular conditional probability shall follow from standard conditions (e.g., see Ash (1972)).

Players know the history of both public shocks, as well as their own private shocks, and their own past actions. Let $h_i^t := (z^1, \theta_i^1, a_i^1, z^2, \theta_i^2, a_i^2, \ldots, z^t, \theta_i^t, a_i^t)$ denote this history for agent i up to period $t \in \mathbb{N} \cup \{\infty\}$, with $h_i^t \in H_i^t$ the set of all possible histories. A *strategy* for player i is a mapping $\sigma_i := (\sigma_i^1, \sigma_i^2, \ldots)$ such that for all t , $\sigma_t^i : H_i^t \to A_i$ and $\sigma_i^t(h_i^t) \in \tilde{A}_i(z^t, \theta_i^t)$ is feasible. A strategy is *Markov* if σ_i is a strategy that depends on current signals/shocks only (i.e., $\sigma_i^t(h_i^t) = \sigma_i^t(z^t, \theta_i^t)$ and σ_i^t is $Z \times \Theta_i$ Borel measurable function). A Markov strategy is *stationary* if $\sigma_i^1 = \sigma_i^2 = ... = \sigma_i$ for some measurable function σ_i. We denote by $\sigma := (\sigma_1, \ldots, \sigma_n)$ a *profile* of Markov stationary strategies.

Suppose player i knows realization of the public shocks, as well as her private shocks, but does not know a realization of private shocks of other players. If the initial public shock is z, and her initial private signal is θ_i, then player believes the initial distribution on the others' private shocks is just $Q_{-i}(\cdot|z, \theta_i)$, and the evolution of the private shocks θ_{-i}^t is a Markov chain with a distribution at any step t given by $Q_{-i}(\cdot|z^t, \theta_i^t)$.

Remark 2.1 *By our assumptions, if the current state is (z, θ), Markov stationary strategy profile is σ, then the distribution of next state (z', θ') is given by measure:*

$$\tilde{Q}(Z_0 \times T|z, \theta) := \int_{Z_0} Q(T|z')q(dz'|z, \theta, \sigma(z, \theta)),$$

where Z_0 is a measurable subset of Z and T is a measurable subset of Θ. Notice, player i does not know the realization of θ_{-i}, but knows the realization (z, θ_i), and believes that current realization on θ_{-i} is given by $Q_{-i}(\cdot|z, \theta_i)$. Because of this, he believes that the distribution on (z', θ') is given by $\tilde{Q}_i(Z_0 \times T|z, \theta_i) :=$

$$= \int_{\Theta_{-i}} \int_{Z_0} Q(T|z')q(dz'|z, \theta_i, \theta_{-i}, \sigma_i(z, \theta_i), \sigma_{-i}(z, \theta_{-i}))Q_{-i}(d\theta_{-i}|z, \theta_i).$$

Thus, for arbitrary Markov stationary strategy profile σ, the evolution of public and private state (z^t, θ_i^t) for agent i is a Markov decision process with transition probability $\tilde{Q}_i$.

The last remark requires a discussion of the structure of players' beliefs and their formation in equilibrium. Specifically, dynamic game with public states and private (information) types can potentially possess many sequential equilibria, as players can condition their action and beliefs on *arbitrary* histories. In such a case, the beliefs of a given player relative to the type and/or actions of others would matter a great deal in the construction of such sequential equilibrium. This is also true, in particular, for games with no private types (as, for example, analyzed in Abreu, Pearce, and Stacchetti (1990); APS, henceforth). However, in APS, the authors concentrate on public strategies; therefore, each player's belief about how his rivals moves is irrelevant in their approach.

Similarly, in this paper, we focus on Markov stationary strategies, and assume players' use Markovian private beliefs as well (see also Cole and Kocherlakota, 2001). When constructing Markov stationary strategies, the players condition their beliefs on the current state, as well as current private types *only.* Such a belief structure is rational in our setup, as knowing current state and own type is sufficient for forecasting the continuation structure of the game (assuming other players are using Markovian strategies and Markovian-private beliefs). Finally, what guarantees the rationality of such beliefs is our assumption that each period, the distribution on private types depends *only* on current states.

Let $h^t = (z^1, \theta^1, a^1, z^2, \theta^2, a^2, \ldots, z^t, \theta^t, a^t)$ be a history of the game up to step t, and H^t the set of all such histories. For every player, given initial public and private states, the transition among public and private states, the profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$, and a belief that others private shocks are changing according to $Q(\cdot|z^t)$, we can generate a sequence of probability measures on histories H^t $(t < \infty)$. Then, according to Iounescu-Tulcea theorem (see Bertsekas and Shreve (1978)), we know there exists a measure, say $P_i^{z,\theta_i,\sigma}$ on H^∞, and a corresponding expected value operator, say $\mathbb{E}_i^{z,\theta_i,\sigma}$, such that the objective for player i is to maximize lifetime payoffs given by:

$$\gamma_i(\sigma)(z, \theta_i) = (1 - \beta)\mathbb{E}_i^{z,\theta_i,\sigma} \left(\sum_{t=1}^{\infty} \beta^{t-1} r_i(z^t, \theta^t, \sigma_i^t(z^t, \theta^t), \sigma_{-i}^t(z^t, \theta^t)) \right),$$

A *Nash equilibrium* in our game is therefore a profile σ^* from which no unilateral deviation is profitable. That is, if σ_i is any arbitrary strategy, then σ^* is a Nash equilibrium if for every player i:

$$\gamma_i(\sigma^*)(z, \theta_i) \geq \gamma_i(\sigma_i, \sigma_{-i}^*)(z, \theta_i) \quad \forall_{(z,\theta_i) \in Z \times \Theta_i}.$$

Any Nash equilibrium that is stationary in Markov strategies is then called MSNE.

3 Main results

We this section we build our results on the existence, computation and equilibrium comparative statics of MSNE in the deep parameters of the game.

Suppose player i knows (z, θ_i) in some period, and believes that the distribution of private shocks for the other agents is $Q_{-i}(\cdot|z, \theta_i)$. If σ_{-i} is a Markov stationary strategy for the other players in the game, and her own action is a_i, then her current expected reward is given simply by

$$R_i(z, \theta_i, a_i, \sigma_{-i}) := \int_{\Theta_{-i}} r_i(z, \theta_i, \theta_{-i}, a_i, \sigma_{-i}(z, \theta_{-i}))Q_{-i}(d\theta_{-i}|z, \theta_i).$$

In line with remark 2.1, the expected value from some integrable continuation value $v_i : Z \times \Theta_i \to \mathbb{R}_+$ is computed as $E_i(z, \theta_i, a_i, \sigma_{-i}, v_i) :=$

$$= \int_{\Theta_{-i}} \int_Z \int_{\Theta_i} v_i(z', \theta_i')Q(d\theta'|z')q(dz'|z, \theta_i, \theta_{-i}, a_i, \sigma_{-i}(z, \theta_{-i}))Q_{-i}(d\theta_{-i}|z, \theta_i).$$

Define the following function space for candidate equilibrium values:

$$\mathcal{V}_i := \{v_i : Z \times \Theta_i \to [0, M] : v_i(0, \theta_i) \equiv 0, v_i \text{ is bounded and Borel measurable}\},$$

which is the set of all possible admissible continuation values in the game. Also, define a set of Markov stationary strategies to be:

$$\Sigma_i := \{\sigma : Z \times \Theta_i \to A_i : \sigma(z, \theta_i) \in \tilde{A}_i(z, \theta_i), \sigma_i \text{ is bounded and Borel measurable}\}.$$

Then, denote by $\mathcal{V} := \prod_{i=1}^{n} \mathcal{V}_i$ the product space of values, and $\Sigma := \prod_{i=1}^{n} \Sigma_i$ the product space of Markov stationary strategies, and by $\mathcal{V} \times \Sigma$ the product space of values and Markov strategies. Endow $\mathcal{V} \times \Sigma$ with its (product) pointwise partial order (i.e., $(v^1, \sigma^1) \leq (v^2, \sigma^2)$ whenever both $v_i^1(z,\theta_i) \leq v_i^2(z,\theta_i)$ and $\sigma_i^1(z,\theta_i) \leq \sigma_i^2(z,\theta_i)$ $\forall_{i=1,\dots,n}\forall_{(z,\theta_i)\in Z\times\Theta_i}$), as well as with its corresponding product topology.

We now state the assumptions we shall need for our existence theorem.

Assumption 3.1 *Assume that:*

- r_i *is continuous on* A *and Borel-measurable on* $Z \times \Theta$*,*
- $\tilde{A}_i$ *is a nonempty, compact sublatticed-valued correspondence,*
- r_i *is supermodular in* a_i*, has increasing differences in* (a_i, a_{-i})*, is increasing in* a_{-i}*, and*

$$r_i(0, \theta_i, \theta_{-i}, a_i, a_{-i}) \equiv 0 \quad \forall (\theta, a) \in \Theta \times A,$$

- q *is on the form*

$$q(\cdot|z,\theta,a) = p(\cdot|z,\theta,a) + (1 - p(Z|z,\theta,a))\delta_Z(\cdot),$$

  where δ_Z *is a Dirac delta on* Z *concentrated at* 0 *i.e.* $\delta_Z(\{0\}) = 1$*,* $p(\cdot|z,\theta,a)$ *is some measure such that* $p(Z|z,\theta,a) < 1$*, and* $p(Z|0,\theta,a) \equiv 0$ $\forall_{(z,\theta,a)\in Z\times\Theta\times A}$*,*
- *for* $v_i \in \mathcal{V}_i$ *denote* $p(v_i|z,\theta,a) = \int_Z v_i(z',\theta_i')p(dz'|z,\theta,a)$*, and assume that* $p(v_i|z,\theta,a)$ *is (a) continuous, supermodular and increasing in* a*, and (b) measurable in* (z,θ)*.*

Given the assumptions on preferences and stochastic transitions q, we can write down an auxiliary game that for any continuation value $v \in \mathcal{V}$, is a game of strategic complementarities with positive externalities. When q has the specific form in our conditions above, supermodularity is preserved to each players value functions recursively at each stage of the game. Although this is a powerful technical assumption, it is satisfied in many applications (e.g., see the discussion in Chassang (2010) for a particular example of this exact structure). Additionally, as we assume positive returns (i.e., $r(0,\cdot) \equiv 0$), our assumptions above assure that the expected continuation value is supermodular in its arguments, as well as monotone in a. This structure is common in the literature. For example, a stronger version of this assumption was introduced by Amir (1996), used in a series of papers by Nowak (2003, 2006, 2007), as well as studied extensively in the context of games of strategic complementarities with public information in Balbus, Reffett, and Woźny (2011). We refer the reader to our two related papers (see 2011; 2012) for a detailed discussion of the nature of these assumptions.

As the next remark indicates, though, our current form of this assumption is significantly weaker than in the existing literature.

Remark 3.1 *Observe that we do not require that* p *is a probability measure. A typical example of* p *is:* $p(\cdot|z,\theta,a) = \sum_{j=1}^J g_j(z,\theta,a)\eta_j(\cdot|z,\theta)$*, where* $\eta_j(\cdot|z,\theta)$ *are measures on* $\mathcal{Z}$ *and* $g_j : Z \times \Theta \times A \to [0,1]$ *are functions with* $\sum_{j=1}^J g_j(\cdot) \leq 1$*. However there are many examples of* p *that cannot be expressed by a linear combination of stochastic kernels, and still satisfy our assumptions. For example on* $Z = A = [0,1]$*, consider* p *having a density* $\rho_p(z'|z,\theta,a) = \xi(\theta)\left(\sqrt{z' + L(a,\theta)} - \sqrt{z'}\right)$ *for sufficiently small function* ξ *and function* L *increasing in* a*.*

Along these lines, we first introduce the following additional notation. We denote by

$$W_i(z,\theta_i,a_i,\sigma_{-i},v_i) = (1-\beta)R_i(z,\theta_i,a_i,\sigma_{-i}) + \beta E_i(z,\theta_i,a_i,\sigma_{-i},v_i),$$

the payoff to player i facing continuation v_i, with the others using strategy σ_{-i}. Therefore, we define this player's best response operator to be:

$$\mathcal{P}_i(v_i,\sigma_{-i})(z,\theta_i) = \arg\max_{a_i\in\tilde{A}_i(z,\theta_i)} W_i(z,\theta_i,a_i,\sigma_{-i},v_i),$$

as well as her corresponding best response value function to be:

$$\mathcal{T}_i(v_i,\sigma_{-i}) = \{v_i \in \mathcal{V} : v_i(z,\theta_i) = W_i(z,\theta_i,a_i,\sigma_{-i},v_i) : a_i \in \mathcal{P}_i(v_i,\sigma_{-i})(z,\theta_i)\}.$$

By $\mathcal{P}(v,\sigma) := \prod_{i=1}^{n} \mathcal{P}_i(v_i,\sigma_{-i})$, we denote a vector of best responses for all the players, and similarly by $\mathcal{T}(v,\sigma) := \prod_{i=1}^{n} \mathcal{T}_i(v_i,\sigma_{-i})$, we denote a vector of value functions for these player induced by these best replies.

To construct our MSNE, we define a few mappings. First, consider the correspondence Φ defined on the product space $\mathcal{V}\times\Sigma$, and defined to be the mapping $\Phi(v,\sigma) := \mathcal{P}(v,\sigma)\times\mathcal{T}(v,\sigma)$. Using this correspondence, we can define new mappings using the greatest (resp., least selection) from $\Phi(v,\sigma)$, and given by:

$$\overline{\Phi}(v,\sigma) = (\mathcal{T}(v,\sigma),\overline{\mathcal{P}}(v,\sigma)) \text{ (resp., } \underline{\Phi}(v,\sigma) = (\mathcal{T}(v,\sigma),\underline{\mathcal{P}}(v,\sigma)),$$

where we have $\overline{\mathcal{P}}(v,\sigma) := \prod_{i=1}^{n} \overline{\mathcal{P}}_i(v_i,\sigma_{-i})$ and similarly for $\underline{\mathcal{P}}$. Notice, this can be done by lemma 3.3 and 3.4.

We prove a number of basic observations.

Lemma 3.1 *For all Borel measurable $\sigma_{-i} : Z\times\Theta_{-i}\to A_{-i}$, $v_i : Z\times\Theta_i\to\mathbb{R}_+$ the function $W_i(z,\theta_i,a_i,\sigma_{-i},v_i)$ is a Carathéodory function in (z,θ_i) and a_i, that is: W_i is measurable in (z,θ_i) and continuous in a_i.*

Proof of lemma 3.1.: We show the desired property of both components R_i and E_i. Then we immediately have a desired property of W_i. Continuity of R_i and E_i in a_i is clear by Lebesgue Dominating Theorem.

Step 1. Measurability of R_i. Since by Assumption 3.1 $r_i(z,\theta_i,\theta_{-i},a_i,a_{-i})$ is measurable in $Z\times\Theta\times A$, and σ_{-i} is a measurable function, $r_i(z,\theta_i,\theta_{-i},a_i,\sigma_{-i}(z,\theta_{-i}))$ is measurable in (z,θ) as well. Therefore, since $Q_i(\cdot|z,\theta_i)$ is measurable transition by Theorem 2 (point 1) in Mertens (1987) $R_i(z,\theta_i,a_i,\sigma_{-i})$ is measurable in (z,θ_i).

Step 2. Measurability of E_i. If $v_i(z',\theta_i')$ is measurable applying Mertens (1987) we obtain $z'\to\int_{\Theta_i} v_i(z',\theta_i')Q_i(d\theta_i'|z')$ is measurable. As a result if we integrate this function over z' with respect to measure $q(\cdot|z,\theta_i,\theta_{-i},a_i,\sigma_{-i}(z,\theta_{-i}))$, the new function will be measurable in (z,θ) (applying Theorem 18.19 in Aliprantis and Border (2006) or more general Theorem 2 in Mertens (1987) again). Hence we can obtain measurability of E_i in both (z,θ_i). ■

Lemma 3.2 *Under Assumptions 3.1 $E_i(z,\theta_i,a_i,\sigma_{-i},v_i)$ is supermodular in a_i increasing in σ_{-i} and has increasing differences in (a_i,σ_{-i}).*

Proof of lemma 3.2.: Observe that $E_i(z, \theta_i, a_i, \sigma_{-i}, v_i) =$

$$= \int\limits_{\Theta_{-i}} \int\limits_{Z} \int\limits_{\Theta_i} v_i(z', \theta_i')Q(d\theta'|z')q(dz'|z, \theta_i, \theta_{-i}, a_i, \sigma_{-i}(z, \theta_{-i})))Q_{-i}(d\theta_{-i}|z, \theta_i).$$

Hence the thesis follows from Assumption 3.1, as supermodularity, increasing differences and monotonicity are preserved by summation. ■

Lemma 3.3 *A correspondence $\mathcal{P}_i$ is isotone in Veinott strong set order. As a result both $(\forall i)$ $\underline{\mathcal{P}}_i$ and $\overline{\mathcal{P}}_i$ are well defined increasing operators.*

Proof of lemma 3.3.: First observe that by Lemma 3.2 W_i is supermodular in a_i. We need to show it has increasing differences in $(a_i; \sigma_{-i}, v_i)$. As increasing differences are preserved by summation, we just need to show that R_i and E_i have increasing differences separately. Observe that:

$$R_i(z, \theta_i, a_i^1, \sigma_{-i}) - R_i(z, \theta_i, a_i^2, \sigma_{-i}) =$$

$$\int\limits_{\Theta_{-i}} r_i(z, \theta_i, \theta_{-i}, a_i^1, \sigma_{-i}(z, \theta_{-i})) - r_i(z, \theta_i, \theta_{-i}, a_i^2, \sigma_{-i}(z, \theta_{-i}))Q_{-i}(d\theta_{-i}|z, \theta_i),$$

is increasing in σ_{-i} as r_i has increasing differences in (a_i, σ_{-i}). Similarly, E_i has increasing differences. To see that observe that by our assumption

$$(a_i, v_i) \to \int\limits_{\Theta_{-i}} \int\limits_{Z} \int\limits_{\Theta_i} v_i(z', \theta_i')Q(d\theta'|z')p(dz'|z, \theta_i, \theta_{-i}, a_i, \sigma_{-i}(z, \theta_{-i})))Q_{-i}(d\theta_{-i}|z, \theta_i),$$

has desired increasing differences by monotonicity of p. Thus W_i has increasing differences in $(a_i; \sigma_{-i}, v_i)$. Hence by Theorem 6.2. in Topkis (1978) $\mathcal{P}_i$ is ascending, compact and sublatticed valued correspondence from $\mathcal{V} \times \Sigma$ into itself. As a result, $\bigvee \mathcal{P}$ and $\bigwedge \mathcal{P}$ are well defined and increasing.

Now the aim is to show that $\mathcal{P}_i(v_i, \sigma_{-i})(z, \theta)$ is measurable correspondence in (z, θ). By Lemma 3.1 $W_i(z, \theta_i, a_i, \sigma_{-i}, v_i)$ is measurable in (z, θ_i) and continuous in a_i, whenever $\sigma_{-i} \in \Sigma_{-i}$ and $v_i \in \mathcal{V}_i$. Clearly $\tilde{A}_i(z, \theta_i)$ is weakly measurable correspondence. Hence by Measurable Maximum Theorem of Aliprantis and Border (2006) (Theorem 18.19) $(z, \theta_i) \to \mathcal{P}_i(\sigma_{-i}, v_i)(z, \theta_i)$ is a measurable correspondence. To show that extremal selections are measurable we use the same argument as in Balbus, Reffett, and Woźny (2011). Consider a collection of maximization problems $P_{i,j}$: $\max y_j$ such that $y \in \mathcal{P}_i(\sigma_{-i}, v_i)(z, \theta_i)$, $y = (y_1, ..., y_k) \in \mathbb{R}^k$. Observe that the every coordinate of the greatest selection of $\mathcal{P}_i(z, \theta_i)$ solves a problem $P_{i,j}$. Applying again Measurable Maximum Theorem each $y_i^{*,j}(z, \theta_i)$ is measurable and consequently $\overline{\mathcal{P}}_i(\sigma_{-i}, v_i)(z, \theta_i)$ is. Similarly we prove a measurability of $\underline{\mathcal{P}}_i$. ■

Lemma 3.4 *Operator $\mathcal{T}$ is well defined and isotone.*

Proof of lemma 3.4.: Recall that $\mathcal{T}_i(v_i, \sigma_{-i}) = \max\limits_{a_i \in \tilde{A}(z, \theta_i)} W_i(z, \theta_i, a_i, \sigma_{-i}, v_i)$. Hence the monotonicity follows directly from assumption 3.1. To show that $\mathcal{T}_i(v_i, \sigma_{-i})$ is measurable we just apply Lemma 3.1 and Measurable Maximum Theorem of Aliprantis and Border (2006) similarly as in the proof of lemma 3.3. ■

Lemma 3.5 *The operators* $\overline{\Phi}$ *and* $\underline{\Phi}$ *are isotone.*

Proof of lemma 3.5.: It follows directly from Lemmas 3.3 and 3.4. ■

Lemma 3.6 σ^* *is a MSNE equilibrium with* v^* *as a corresponding payoff iff* $(v^*, \sigma^*) \in \Phi(v^*, \sigma^*)$.

Proof of lemma 3.6.: It follows directly from principle of optimality and standard dynamic programming arguments (see Bertsekas and Shreve, 1978). Also observe that σ^* remains a MSNE if players are allowed to use more general strategies (assuming beliefs are Markov). ■

Lemma 3.7 Φ *is an u.h.c. correspondence. Moreover,* $\overline{\Phi}(v, \sigma)$ *and* $\underline{\Phi}(v, \sigma)$ *are well defined and* $\overline{\Phi}(v, \sigma), \underline{\Phi}(v, \sigma) \in \Phi(v, \sigma)$. *As a result* $\overline{\Phi}$ *is monotically inf-preserving and* $\underline{\Phi}$ *is monotically sup-preserving*[8].

Proof of lemma 3.7.: *Step 1.* By Lemmas 3.3 and 3.4 we immediately conclude that $\overline{\Phi}(v, \sigma)$ and $\underline{\Phi}(v, \sigma)$ are well defined and $\overline{\Phi}(v, \sigma), \underline{\Phi}(v, \sigma) \in \Phi(v, \sigma)$. We need to show u.h.c. property of Φ. We need to show $(v^t, \sigma^t) \to (v, \sigma)$, $(\tilde{v}^t, \tilde{\sigma}^t) \in \Phi(v^t, \sigma^t)$, $(\tilde{v}^t, \tilde{\sigma}^t) \to (\tilde{v}, \tilde{\sigma})$ follows $(\tilde{v}, \tilde{\sigma}) \in \Phi(v, \sigma)$. Since

$$\tilde{v}_i^t(z, \theta_i) \quad = \quad W_i(z, \theta_i, \theta_{-i}, \tilde{\sigma}_i^t, \sigma_{-i}^t, \tilde{v}_i^t) = \max_{a_i \in \tilde{A}_i(z, \theta_i)} W_i(z, \theta_i, \theta_{-i}, a_i, \sigma_{-i}^t, v_i^t).$$

By Assumption 3.1 $\tilde{A}$ is nonempty compact valued correspondence as it does not depend on (σ, v). Moreover, W_i is continuous in (a_i, σ_{-i}, v_i). By Berge Maximum Theorem this implies that $\mathcal{P}_i$ is u.h.c. correspondence and

$$(\sigma_{-i}, v_i) \mapsto \max_{a_i \in \tilde{A}_i(z, \theta_i)} W_i(z, \theta_i, \theta_{-i}, a_i, \sigma_{-i}, v_i)$$

is continuous. Thus $\tilde{\sigma}_i^t \in \mathcal{P}_i(\sigma_{-i}^t, v_i^t)$, $(\sigma^t, v^t) \to (\sigma, v)$ and $(\tilde{\sigma}^t, \tilde{v}^t) \to (\tilde{\sigma}, \tilde{v})$ implies that $\tilde{\sigma}_i \in \mathcal{P}_i(\sigma_{-i}, v_i)$. Moreover, $\mathcal{T}_i$ is continuous, hence $\tilde{v}_i = \mathcal{T}_i(\sigma_{-i}, v_i)$. This implies that $(\tilde{v}, \tilde{\sigma}) \in \Phi(v, \sigma)$.

Step 2. Finally we show the last assertion. We show that $\overline{\Phi}$ is monotically inf-preserving. Let (σ^n, v^n) be decreasing sequence and $(v, \sigma) = \bigwedge_{n \in \mathbb{N}} (\sigma^n, v^n)$. As this sequence is monotone $(v^n, \sigma^n) \to (v, \sigma)$ pointwise. By previous step $\overline{\Phi}(v^n, \sigma^n) \in \Phi(v^n, \sigma^n)$. By Lemma 3.5 $\overline{\Phi}(v^n, \sigma^n)$ is decreasing sequence, hence pointwise convergent to some ϕ_0. As by previous step Φ is u.h.c., hence $\phi_0 \in \Phi(v, \sigma)$, and obviously $\phi_0 \leq \overline{\Phi}(v, \sigma)$. On the other hand observe that $\overline{\Phi}(v, \sigma) \leq \overline{\Phi}(v^n, \sigma^n)$. Taking a limit we obtain $\overline{\Phi}(v, \sigma) \leq \phi^0$. Hence $\phi_0 = \overline{\Phi}(v, \sigma)$ and $\overline{\Phi}$ is monotically inf-preserving. Similarly we show that $\underline{\Phi}$ is monotonically sup-preserving. ■

Finally, let $\phi^{t+1} = \overline{\Phi}(\phi^t)$ for $t \geq 1$ with $\phi_0(z, \theta) \equiv (\bigvee \Sigma, \bigvee \mathcal{V})$. Let $\psi^{t+1} = \underline{\Phi}(\psi^t)$ for $t \geq 1$ with $\psi_0(z, \theta) \equiv (\bigwedge \Sigma, \bigwedge \mathcal{V})$. Having that we are ready to prove the main results of the paper.

Theorem 3.1 *Let assumption 1 be satisfied. Then:*

(i) there exist limits $\phi^* = \lim_{t \to \infty} \phi^t$ *and* $\psi^* = \lim_{t \to \infty} \psi^t$,

[8] A function $F : X \to X$ is monotically sup (resp. inf)-preserving if for any increasing (resp. decreasing) sequence x_n, we have $F(\bigvee x_n) = \bigvee F(x_n)$ (resp. $F(\bigwedge x_n) = \bigwedge F(x_n)$).

(ii) ϕ_1^* *is a MSNE with* ϕ_2^* *as a corresponding payoff vector. Similarly* ψ_1^* *is a MSNE with* ψ_2^* *as a corresponding payoff vector,*

(iii) Let f^* *be a MSNE and* v^* *its corresponding payoff vector. Then* $\phi_1^* \geq f^* \geq \psi_1^*$ *and* $\phi_2^* \geq v^* \geq \psi_2^*$.

Proof of theorem 3.1.: Proof of (i). We show that ϕ^t is a monotone sequence. Clearly $\phi^2 \leq \phi^1$. Assume that $\phi^t \leq \phi^{t-1}$ for some $t > 1$. By Lemma 3.5 we then have $\phi^{t+1} \leq \overline{\Phi}(\phi^t) \leq \overline{\Phi}(\phi^{t-1}) = \phi^t$. Hence ϕ^t is antitone. Similarly we show ψ^t is isotone. As a result both of these sequences have a limit.

Proof of (ii). As $\phi^{t+1} \in \Phi(\phi^t)$, $\phi^t \to \phi^*$ by previous step, hence and by Lemma 3.7, $\phi^* \in \Phi(\phi^*)$. Similarly $\psi^* \in \Phi(\psi^*)$. By Lemma 3.6 ϕ_1^* and ψ_1^* are Nash equilibria with corresponding payoffs ϕ_2^*, and ψ_2^*.

Proof of (iii). Let f^* be an arbitrary Nash equilibrium with a corresponding payoff v^*. Then by Lemma 3.6 $(f^*, v^*) \in \Phi(f^*, v^*) \subset [\underline{\Phi}(f^*, v^*), \overline{\Phi}(f^*, v^*)]$. The last inclusion follows from Lemma 3.7. Clearly $\psi^1 \leq (f^*, v^*) \leq \phi^1$. Assume for some $t \in \mathbb{N}$:

$$\psi^t \leq (f^*, v^*) \leq \phi^t. \tag{1}$$

By definition of $\underline{\Phi}$ and $\overline{\Phi}$ and Lemma 3.5 we have

$$\psi^{t+1} = \underline{\Phi}(\psi^t) \leq \underline{\Phi}(f^*, v^*) \leq (f^*, v^*) \leq \overline{\Phi}(f^*, v^*) \leq \overline{\Phi}(\phi^t) = \phi^{t+1}.$$

Hence the inequality in (1) follows for all t. Taking a limit in (1) by step (ii) we receive $\psi_1^* \leq f^* \leq \phi_1^*$ and $\psi_2^* \leq v^* \leq \phi_2^*$. ■

Theorem 3.1 states a number of things. We start this discussion from our existence result $(ii)-(iii)$ and then move to comment on our approximation result in (i).

First, the result establishes existence of MSNE for our infinite horizon game with both public and private shocks; but it does more. It also provides bounds for constructing every MSNE. Moreover, both of these bounds are actual MSNE. We can also obtain corresponding bounds for equilibrium values.

Second, as is typical in the literature, to prove the existence of equilibrium, we construct auxiliary one shot game parameterized by continuation value. What is important in our method, though, is that instead of finding the set of Nash equilibria at every period, we parameterize the payoff function of every player by both continuation value function *and* strategy profile for the actions of the other players. Using this added structure, we then evaluate the best response of the player depending on the strategy of the other players and his continuation value. The advantage of this method is the simplicity of resulting computations as compared with the computations involved in the APS type methods of Cole and Kocherlakota (2001), for example. We comment more on the importance of this simplification in a moment.

Third, our method uses recent results on Bayesian supermodular games in its construction. That is, similar to the papers of Vives (1990), or Van Zandt (2010), MSNE are not necessarily monotone as a functions of states (private or public); rather, we just impose enough structure on the game to construct *operators* for value/strategy pairs that are monotone with continuation values and other player strategies. In doing this, we then obtain precisely a dynamic supermodular game. Then, when one seeks conditions sufficient to prove the existence of *monotone*

Markovian equilibrium (in states), we simply impose stronger complementarity assumptions in the primitives of the game between actions and states.[9]

Fourth, theorem provides a simple iterative algorithm that constructs the greatest and least equilibria in our infinite horizon game. More specifically, as compared with other methods (e.g., APS methods), we simultaneously iterate on operators defined in terms of *both* player values and Markovian strategies. We then show our iterations converge in order and product topologies to Markov equilibrium strategies (as well as their associated equilibrium values). One characterization that is missing here, though, are estimates of the rate of convergence, as well as the accuracy of our approximations to the least and greatest MSNE. To address these latter issues, we can introduce additional metric structure, and study the metric convergence question.

Fifth, our algorithm is simpler than that proposed in Balbus, Reffett, and Woźny (2011) for the case of public information, as we do not need to compute equilibria of the auxiliary game at *every* value function iteration. However, this simplification comes at a cost, as our iterations are *not* equilibria in truncated finite horizon games. In this sense, our method is similar to that discussed in Szajowski (2006), but very different than the one developed in Balbus and Nowak (2004, 2008), or Balbus, Reffett, and Woźny (2011) for games with public information.

Six, the approach used in the proof of theorem 3.1 reminiscent of the iterated elimination of dominated strategies (as discussed, for example, in Vives (1990)) but extended to dynamic games. Indeed, as observed by Chassang (2010), the simultaneous iterated elimination of dominated Markovian strategies (and corresponding values) leads to convergence in *order* to extremal MSNE. Recall, that this procedure heavily depends on the (Markovian) equilibrium and (Markov-private) beliefs concepts applied.

Finally, we make few more specific additional detailed remarks related to our results.

Remark 3.2 *If the best replies are unique, then we can strengthen our results by saying that the MSNE set is a countably chain complete poset. That is, MSNE set is closed under countable sup/infs of chains. It follows from our generalization of Tarski-Kantorovitch fixed point theorem (see proposition 5.1).*

Remark 3.3 *If the order on each $\mathcal{V}_i$ and Σ_i is changed to a.e. (where, a.e. refers to private and public signals), then we can conclude using Veinott (1992)/Zhou (1994) generalization of Tarski (1955) fixed point theorem that the MSNE set not only has the greatest and least elements, but is also a complete lattice. This follows from that fact the the set of of bounded, Borel-measurable functions is a sigma-complete lattice, when endowed with pointwise (everywhere) order, but is a complete lattice, when endowed with a.e. order (see Vives, 1990). In this paper we prefer to use pointwise (everywhere) order mainly for comparative statics results presented in theorem 3.2.*

We complete our discussion on the existence and characterization of MSNE with an important corollary:

Corollary 3.1 *MSNE exists in a class of stochastic games satisfying assumption 3.1 with perfect monitoring and no private information, i.e. where with probability one $\theta_i = \theta_j$ for all players.*

We now present our results on monotone equilibrium comparative statics result for the set of MSNE relative to order perturbations of the deep parameters of the game. To do this, consider

[9]See, for example Reny (2011) and citations therewithin for a related papers on monotone equilibria in Bayesian games. Also, see Curtat (1996) or Amir (2005) for monotone MSNE in stochastic supermodular games with public information.

a parameterized version of our game $\Gamma(\omega)$, where $\omega \in \Omega$ is a set of deep parameters of the game, and Ω is a poset. Specifically, we denote by $\tilde{A}(z,\theta;\omega)$, $r_i(z,\theta,a;\omega)$, $p(\cdot|z,\theta,a;\omega)$ the parameterized versions of our primitive data of the game. By $\omega \to \phi_1(\omega), \omega \to \psi_1(\omega)$, we denote the extremal selections from the MSNE correspondence.

We make the following assumptions on the parameterized primitive data of the games:

Assumption 3.2 *Assume that:*

- $\forall \omega \in \Omega$ *assumption* 3.1 *is satisfied,*
- *each* r_i *has increasing differences in* (a_i, ω) *and is increasing in* ω*,*
- $p(v_i|z,\theta,a,\omega)$ *has increasing differences with* (a_i,ω) *and is increasing in* ω *for each* $v_i \in \mathcal{V}_i$*,*
- $\tilde{A}_i(z,\theta)$ *does not depend on* ω*.*

With this parameterization complete, we can now state our central equilibrium monotone comparative statics theorem for our class of parameterized game:

Theorem 3.2 *Let 3.2 be satisfied. Then, the extremal MSNE* $\phi_1(\omega), \psi_1(\omega)$ *are monotone on* Ω*.*

Proof of theorem 3.2.: Consider a model parametrized by ω. Consider the least MSNE equilibrium as $\phi^*_\omega(z,\theta)$. We show that ϕ^*_ω is increasing in ω. To do it observe that ϕ^*_ω is a fixed point of operator $\underline{\Phi}(v,\sigma;\omega)$. Clearly by Lemma 3.5 $\underline{\Phi}$ is increasing in (v,σ) and by Lemma 3.7 is monotonically sup-preserving. We need to show that this operator is increasing in ω. Let $(\tilde{\sigma}_i(\omega), \tilde{v}_i(\omega)) := \underline{\Phi}_i(\sigma_{-i}, v_i)$. By definition of $\tilde{\sigma}_i$ it is a least selection of argmax correspondence of the function $a_i \to W_i(z,\theta_i,\theta_{-i},(a_i,a_{-i}),v_i;\omega)$ over $\tilde{A}_i(z,\theta_i)$. By Lemma 3.2 W_i is supermodular in a_i and $\tilde{A}_i$ is ascending correspondence in ω. Analogously we prove W_i has isotone differences in (a_i,ω). Hence by Topkis (1978) σ_i is increasing in ω. Since W_i is increasing in ω hence $\tilde{v}_i$ is increasing in ω. This implies that $\overline{\Phi}$ is isotone in ω. As by Lemma 3.7 $\underline{\Phi}$ is monotonically sup-preserving, and $\Sigma \times \mathcal{V}$ is countably chain complete poset, hence by Proposition 5.1 we obtain that ϕ^*_ω is the least element of the set $K_\omega := \{\phi \in \Sigma \times \mathcal{V} : \underline{P}(\phi;\omega) \leq \phi\}$. To show that ϕ^* increases in ω let $\omega_1 \leq \omega_2$. Then

$$\phi^*_{\omega_2} = \underline{P}(\phi^*_{\omega_2};\omega_2) \geq \underline{P}(\phi^*_{\omega_2};\omega_1).$$

Hence $\phi^*_{\omega_2}$ is some selection of K_{ω_1}, while $\phi^*_{\omega_1}$ is the least selection of this set. Therefore, $\phi^*_{\omega_1} \leq \phi^*_{\omega_2}$. Similarly we show that ψ^* increases in ω. ■

We should remark, we are not aware of any similar monotone comparative statics result for dynamic games in the existing literature. Here, our monotone equilibrium comparative statics result follows from the monotonicity of our operators and applications of our extension of Veinott (1992) parameterized fixed point theorem to countable chain complete posets (see proof of theorem 3.2).

4 Examples

In this section, we present three applications of our methods. In all three examples, the results of our paper can be used to verify existence of the greatest and the least Markov stationary Nash Equilibrium, but also can be applied to compute these extremal equilibria by the simple iterative procedure. Finally theorem 3.2 offers the corresponding result per monotone equilibrium comparative statics.

4.1 Dynamic price competition with private information

Consider an economy with n firms who are competing for customers buying heterogenous, but substitutable, goods. Firms have private information concerning their demand parameters $\theta_i \subset [-\epsilon_i, \epsilon_i] = \Theta_i$, and there is also a public signal $z \in Z = [0, \overline{z}] \subset \mathbb{R}^n_+$ giving each firm partial information on others' firm demand parameters. More succinctly, let the demand parameter be given by $s_i(z_i, \theta_i) = z_i + \theta_i$.

After observing z (that could, for example, reflect business cycle fluctuations), the individual parameters θ are drawn from the conditional distribution $Q(\cdot|z)$. If the other firms choose prices $a_{-i}(z, \theta_{-i})$, the interim payoff of firm i, choosing price $a_i \in [0, \bar{a}]$ is given by $u_i(z, \theta_i, a_i, a_{-i}) =$

$$\int_{\Theta_{-i}} [a_i D_i(a_i, a_{-i}(z, \theta_{-i}), z_i + \theta_i) - C_i(D_i(a_i, a_{-i}(z, \theta_{-i}), z_i + \theta_i))] Q_i(d\theta_{-i}|z, \theta_i).$$

were D_i is a demand. Normalize the profits such that if $z_i = 0$, the firm's i profit is zero (e.g. that turnover is too small to cover the costs, and the company is driven out of the market). As the within period game is Bertrand with heterogenous firms and substitutable products, the payoff assumption in 3.1 is satisfied if demand D_i is (a) increasing with others prices, and (b) has increasing differences between (a_i, a_{-i}). Also, as $[0, \bar{a}]$ is single dimensional, payoff is supermodular function of own price. Finally, assume as is standard that C_i is increasing and convex.

To interpret this model using the language of our model, let measure p on Z capture the influence on current parameters (θ, z) and prices on tomorrow's demand parameterized by vector z'. Therefore, apart from technical assumption on measurability, to apply our methods, we only require here that measure p be monotone, supermodular and continuous in prices. This latter condition can be interpreted as the demand substitution between periods (i.e., prices today imply higher probability on positive ($z \in (0, \overline{z}]$) demand parameters the next period, as consumers can wait for cheaper prices tomorrow). This effect is stronger if others set higher prices as well via the supermodularity assumption. Indeed, when the company increases its price today, it may lead to a positive demand in the future if the others have also high prices. But if the other firms set low prices today, then such impact is definitely lower, as some clients may want to purchase the competitors good today instead.

4.2 Dynamic R&D competition with positive spillovers and private costs

A second application of our results is inspired by d'Aspremont and Jacquemin (1988), who analyze a two stage game between oligopolists choosing the R&D expenditure to reduce costs in the first stage, and then in a second stage compete *a la* Cournot. The authors study the effects of R&D investment spillovers in an (subgame perfect) equilibrium, as well as its optimality. To study such a game, we analyze an infinite horizon R&D competition model, where each period, we embed the two stage game of d'Aspremont and Jacquemin (1988), which is played between n oligopolists.

Along these lines, assume that the inverse demand is given by $P(Q) = A - bQ$, where $Q = \sum_i q_i$, and the production cost functions are given by $c_i = C_i(q_i) = [\overline{z} - z_i - \theta_i - a_i - \delta \sum_{j \neq i} a_j] q_i$, where $z \in Z \subset [0, \overline{z}]^n$ is a drawn each period common cost parameter, $\theta_i \in [-\epsilon_i, \epsilon_i]$ is noise on the actual cost parameter $z_i + \theta_i$, $\delta \in [0, 1]$ is a spillover parameter, and finally a_i is a investment in a cost reduction R&D process. The cost of a_i units of R&D investment is then given by $a_i \to \gamma_i(a_i)$, which is assumed to be continuous and bounded. Apart from the within period spillovers, higher investment a_i has also intertemporal effects via p of increasing probabilities of a positive cost reduction draw tomorrow.

Every period, the profit of an oligopolist assuming the next stage a Cournot equilibrium is played is given by the function $\pi_i(z, \theta_i, a_i, a_{-i})$, where $\pi_i(z, \theta_i, a_i, a_{-i}) =$

$$
\begin{aligned}
= \quad & \frac{1}{b} \int_{\Theta_{-i}} \Bigg[\frac{A - n(\overline{z} - z_i - \theta_i - a_i - \delta \sum_{j \neq i} a_j(z, \theta_j))}{n+1} + \\
+ \quad & \frac{\sum_{j \neq i} (\overline{z} - z_j - \theta_j - a_j(z, \theta_j) - \delta \sum_{k \neq j} a_k(z, \theta_k))}{n+1} \Bigg]^2 Q_{-i}(d\theta_{-i}|z, \theta_i) - \gamma_i(a_i).
\end{aligned}
$$

Observe, for a large R&D spillovers (i.e. $\delta > .5$), the payoff is increasing in a_{-i} (e.g., the top-dog strategy effect is dominated by a spillover effect), and $\pi_i(z, \theta_i, a_i, a_{-i})$ has increasing differences in (a_i, a_{-i}) and (a_i, s). Further, the measure $p(\cdot|z, \theta, a)$ satisfies assumption 3.1 if intertemporal investment effects are self-reinforcing (i.e., if high R&D investment today has positive effects on positive cost reduction the next period, and this effect is stronger if others invest more). Finally, allowing $z = 0$ to be an absorbing state is justified, e.g. if we have $\overline{z} \geq A$, i.e. assumption ruling out production possibilities if the size of the market is too small relative to the unit production cost $\overline{z}$.

4.3 Dynamic Cournot competition with learning-by-doing and incomplete information

Finally, consider an economy where each period, n-firms compete by setting the quantity q_i of differentiated product. The goods are assumed to be behavioral complements; i.e., the consumption of one good increases purchase of the complementary products. Additionally, each firm has a individual stochastic learning-by-doing effect influencing its marginal cost function via a parameter $s_i = z_i + \theta_i$ measuring cumulative experience of the given firm.

Then, profit of a given firm is summarized by $\Pi(z, \theta_i, q_i, q_{-i}) = q_i[P_i(q_i, q_{-i}(z, \theta_{-i})) - c(z_i + \theta_i)]$. Observe that the assumptions on payoffs are satisfied if the cost c is decreasing in the learning-by-doing parameters, P_i increasing in q_{-i} (i.e., we have complementary goods), and P_i has increasing differences in (q_i, q_{-i}) (e.g., we have P_i given by the form $P_i = \gamma - q_i + \sum_{j \neq i} \delta_j q_j$).

Concerning the learning process, lets assume that joint experience vector $z \in [0, \overline{z}]^n$ is stochastic and drawn accordingly to a distribution p, while individual costs parameters $\theta \in \times_{i=1}^n [-\epsilon_i, \epsilon_i]$ which are noise in the learning effect are distributed according to Q. Finally, let $\sup P_i() < c(0 + \overline{\epsilon}_i)$. Then, the only restrictive assumption on p we require is that $q \to p(\cdot|z, \theta, q)$ is continuous, increasing and supermodular. One way of interpreting this condition (from the perspective of complementarity) is the higher output today increases the chance of a positive experience draw next period, and that effect is *stronger* if others set higher quantities, via spillovers. Under these conditions, all the main theorems of the paper can be applied.

5 Appendix: auxiliary result

Here we state and prove the following proposition of independent interest.

Proposition 5.1 *Let X be a countably chain complete poset (i.e. if $x_n \in X$ is monotone sequence then its supremum and infimum belongs to X) with the greatest element $\overline{\theta}$ and the least element $\overline{\theta}$. Let $F : X \to X$ be an isotone function. Then:*

(i) If F is monotonically inf preserving[10] *then $\overline{\Phi} := \bigwedge F^n(\overline{\theta})$ is the greatest fixed point and if F is monotonically sup preserving then $\underline{\Phi} := \bigvee F^n(\underline{\theta})$ is the least fixed point.*

(ii) If F is monotonically inf preserving function then

$$\overline{\Phi} = \bigvee \{x : F(x) \geq x\}. \tag{2}$$

(iii) If F is monotonically sup preserving function then

$$\underline{\Phi} = \bigwedge \{x : F(x) \leq x\}. \tag{3}$$

(iv) If F is monotonically sup and inf preserving function, then its fixed point set is a countably chain complete poset.

Proof of proposition 5.1.: Proof of (i): Assume F is monotonically inf-preserving. Clearly $F(\overline{\theta}) \leq \overline{\theta}$. If for some n, $F^n(\overline{\theta}) \geq F^{n+1}(\overline{\theta})$, then $F^{n+1}(\overline{\theta}) = F(F^n(\overline{\theta})) \geq F(F^{n+1}(\overline{\theta})) = F^{n+2}(\overline{\theta})$. Hence, $F^n(\overline{\theta})$ is decreasing, and $\overline{\phi}$ is well defined. Since F is monotonically inf-preserving, we have

$$F(\overline{\Phi}) \quad = \quad F\left(\bigwedge F^n(\theta)\right) = \bigwedge F^{n+1}(\theta) = \overline{\Phi}.$$

Therefore, $\overline{\phi}$ is fixed point of F. We show it is the greatest fixed point. Let us take arbitrary fixed point $e = F(e)$. Clearly, $e \leq \overline{\theta}$, and $e = F(e) \leq F(\overline{\theta})$. If $e \leq F^n(\overline{\theta})$, then $e = F(e) \leq F^{n+1}(\overline{\theta})$. Therefore, $e \leq F^n(\overline{\theta})$ for all n, which implies $e \leq \overline{\Phi}$. Similarly, we prove that $\underline{\Phi}$ is well defined and it is the least fixed point of F. We prove analogously the case, when F is is monotonically sup-preserving.

Proof of (ii) We prove equality (2). Let x be arbitrary point such that $x \leq F(x)$. Clearly $x \leq \overline{\theta}$. Assume $x \leq F^n(\overline{\theta})$. Then, $x \leq F(x) \leq F(F^n(\overline{\theta})) = F^{n+1}(\overline{\theta})$. Hence, $x \leq \overline{\Phi}$. Since $\overline{\Phi} \in \{x : F(x) \geq x\}$, equality (2) is proven.

Proof of (iii). We prove (3) analogously.

Proof of (iv). Let e_n be an countable chain of fixed points. Let $\bar{e} = \bigvee e_n$. It exists in X as X is a countable chain complete. Then,

$$F(\bar{e}) \quad = \quad F\left(\bigvee e_n\right) = \bigvee F(e_n) = \bigvee e_n = \bar{e}.$$

Similarly, we prove the thesis for decreasing sequences. ■

6 Conclusions and related techniques

This paper proposes a new set of monotone methods for a class of discounted, infinite horizon stochastic games with both public and private signals, as well as strategic complementarities. The role of strategic complementarities in the development of our methods is critical, as they allow us to study the Markovian equilibrium in our class of games directly.[11]

[10] For definition see footnote 8.

[11] I.e. using "primal" representations of agent decision problems as opposed to dual representations as in recursive saddlepoint methods, as is typical in many of the existing approaches to similar problems in the literature.

Our analysis shares some of the properties of the belief-free equilibria studied in Ely, Hörner, and Olszewski (2005), as we assume players have (rational) Markovian beliefs that depend only on public and individual signals, and we do not need to model beliefs off the equilibrium path as in their work. Also, as Markovian equilibria are adopted here, we do not allow players to impose punishments schemes inconsistent with Markovian strategies, which is also related to work using belief-free equilibria. Further, in our model, public information amounts to signalling the distribution of private information and past moves, rather than signalling current opponents' actions.[12] Finally, our analysis is very closely related to ideas that are behind the methods proposed in Cole and Kocherlakota (2001), who develop methods for solving for (nonstationary) Markov equilibria with Markov beliefs via APS-type methods applied in function spaces.

Per extensions of the results in future work, perhaps the most critical class of models where our stochastic games approach seems most appropriate is the study of Markovian equilibrium in dynamic principal-agent problems, where we have both unobservable information or actions, which is well-known to greatly complicate the nature of dynamic equilibrium arrangements. In this literature, there are at least three other techniques used to study similar dynamic principal-agent problems, namely: (i) APS methods, (ii) recursive saddlepoint method and (iii) first order approaches. The APS approach has proven very useful for verifying existence of sequential equilibrium in broad classes of both repeated and dynamic games (see Atkeson, 1991). This approach focuses on the computation of the equilibrium *value set*, without a sharp characterization of sequential equilibrium strategies that support *any* equilibrium value in the equilibrium value set. Further, when these games have state variables (like capital stocks or shocks), additional issues arise in the presence of public and private information over *uncountable* state spaces. That is, the APS method becomes significantly more complicated as the set of *measurable* Nash equilibrium values need not be *closed* in any useful topology (e.g. weak-star topology).

Another important set of techniques for studying limited commitment problems are the so-called "recursive saddlepoint methods" as discussed originally in the seminal work of Kydland and Prescott (1980), and further developed in Messner, Pavoni, and Sleet (2012) for example. These methods have been shown to be very useful to compute equilibrium in some classes of incentive problems with private information or actions, where primal and dual optimization problems can be appropriately linked. One immediate limitation of such methods is that "punishment schemes" are typically assumed to be "exogenous", and specified in an *ad hoc* manner. Further there are subtle issues associated with the existence and computation of recursive saddlepoints themselves, which is needed to guarantee KKT multipliers are useful and placed in appropriate dual spaces.

Finally, the first order approaches developed in Ábrahám and Pavoni (2008); Mitchell and Zhang (2010) are often useful when they can be rigorously applied. In particular, when problems are *concave* in equilibrium, one can precisely link the first order conditions for optimization problem with its actual solutions, e.g. by showing that these first order conditions are not only necessary but also (locally) sufficient[13]. In this sense, the requirements needed to apply such methods are similar to recursive saddlepoint methods. Unfortunately, as in recursive saddlepoint problems, when state variables are present (as, for example, in a dynamic game), conditions on primitives that imply concavity of the value function are very difficult to obtain.

The techniques developed in this current paper have an important technical advantage over

[12]That is, in our game players' actions are observable at the end of the period, but then are forgotten the next period and only signalled by a public state.

[13]Observe that our tools actually allow to obtain conditions, where players best replies are characterized by both necessary and sufficient first order conditions (see Woźny and Growiec (2012) for the details).

all this work, as in the present method, one works directly with both equilibrium strategies and values simultaneously per the existence and computation of equilibrium question without necessity to use first order conditions, duality or importantly restricting our results to the ones available using APS-type techniques.

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