Intergenerational interactions in human capital accumulation☆

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Abstract

We analyze an economy populated by a sequence of generations who decide over their consumption and investment in human capital of their immediate descendants. Our objective is to identify the impact of strategic interactions between generations on the human capital accumulation path and provide an economic application of a constructive approach to computing Markov perfect equilibria (MPE) in economies with strategic interactions by Balbus et al. (2008). We derive sufficient conditions for MPE existence and uniqueness. Then we prove analytically and show numerically that human capital accumulation path is lower in the "strategic" case than in the "non-strategic", optimal but time-inconsistent case.

Keywords: human capital, intergenerational interactions, Markov perfect equilibrium, stochastic transition, constructive approach
JEL codes: C73, I20, J22

☆Financial support from the Foundation for Polish Science (Professor Tomasz Szapiro's subsidy "Mistrz") is gratefully acknowledged. We are grateful to Rabah Amir, Timothy Kehoe, Kevin Reffett, and the participants of the 23rd Annual Congress of the European Economic Association (EEA-ESEM 2008) in Milan as well as the "Macromodels 2007" conference in Warsaw for helpful comments and suggestions. All errors are our responsibility.

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1. Introduction

Human capital is nowadays widely acknowledged to be one of the most important factors determining the differences in wealth across nations as well as their growth potential. The variable is thus present in a wide range of micro- and macroeconomic theories, including those taking an explicitly intergenerational planning perspective. In such theories, various forms of altruism (cf. Arrondel and Masson, 2006; Bertola et al., 2006; Abel and Warshawsky, 1987) are proposed to deal with the empirically grounded intergenerational correlations and linkages in wealth, human capital, social status, and occupation choice. In particular, strategic interactions across generations should be especially apparent in relation to schooling: on the one hand, a substantial fraction of investment in accumulating human capital of an individual is made by her parents, while on the other hand, the parents cannot fully anticipate what use will be eventually made of these personal assets (Becker and Tomes, 1986; Loury, 1981; Galor and Tsiddon, 1997; Orazem and Tesfatsion, 1997; Lochner, 2008).[2]

If one assumes that within each generation, people derive their utility from – among other things – the utility of their children, then there logically follows an infinite-horizon planning problem: the parents care for children who care for grandchildren who care for great-grandchildren, etc. A markedly different situation might however be encountered if the parents care for their children's consumption directly. It is then crucial if there is a way for all consecutive generations to credibly commit to their future choices. If not, we are led to the frameworks where the optimization problem becomes strategic. The impact of such strategic interactions is not clear a priori. On the one hand their presence and resulting lack of commitment may lower each generations' investment. But on the other as noted by Bernheim and Ray (1987) the higher investment today is needed to obtain the same result in terms of tommorows' consumption and utility.

The contribution of the current paper to the literature is twofold. First,

[2]The classic works within the human capital accumulation literature, such as Mincer (1958) or Ben-Porath (1967), focus primarily on the other component of investment in education which is individuals' own purposeful educational spending motivated by the expected increases in their future earnings. The Ben-Porath's model specification is however already flexible enough to allow for intergenerational transmission of human capital as well.

we identify the impact of strategic interactions between consecutive generations on the time path of human capital accumulation in an economy populated by a sequence of generations allowed to decide over their consumption levels as well as over the levels of investment in human capital of their immediate descendants. We are able to obtain clear-cut results here by computing the Markov perfect equilibrium (MPE) at the aggregated level and benchmarking the time-consistent MPE result against the optimal but time-inconsistent policy which neglects strategic interactions across generations. We then use these results to draw general conclusions about the dynamics of human capital accumulation in both types of equilibria.[3]

Secondly, given the technical and computational obstacles faced by the literature to which the current paper belongs (discussion of which is presented in the next section), this paper should also be viewed as an rigorous application and illustration of pros and cons of the novel constructive technique for computing the MPE, proposed by Balbus et al. (2008).

As far as the economic subject of this paper is concerned, our results include an analytical proof that, other things equal, (equilibrium) human capital accumulation policy is unambiguously (pointwise) smaller in the strategic model than (optimal) policy in a dynastic model. To provide this result with a quantitative edge, we also run a series of numerical exercises quantifying how large the differences between the optimal human capital accumulation decisions could be whether strategic interactions are present or not.[4] Hence this paper puts the estimations, obtained in the models neglecting such intergenerational interactions, under question.

The remainder of the article is structured as follows. Section 2 discusses the related literature, both from the substantive, and the methodological–technical angle. In Section 3 we lay out our basic model with strategic interactions and present the principal theoretical results. In Section 4 we compare this model with a benchmark model where no strategic interactions

[3]In the Appendix, we also compare these two setups to a model similarly frequently used in the literature, i.e. the one of joy-of-giving altruism (used by, among numerous others, Artige et al. (2004); Bruhin and Winkelmann (2009); Abel and Warshawsky (1987)).

[4]Additionally, we also show numerically that the joy-of-giving altruism model differs markedly from the strategic and the dynastic model, insofar the implied optimal decisions cannot be unambiguously compared against each other: for most parameter values, joy-of-giving altruism implies more human capital investment than the strategic model, but for a range of specific parametric choices, this relationship is reversed.

are allowed. Section 5 provides an illustrative numerical example for our calculations of the preceding chapters. Section 6 discusses the role of strategic interactions in shaping human capital investment decisions. Section 7 concludes. Definitions and proofs of theorems, as well as the discussion of a model with joy-of-giving altruism, have been relegated to the Appendix.

2. Related literature

The topic of intergenerational commitment and strategic interactions has been widely studied in the economic literature, both of normative and of positive nature. The former group of articles includes, among others, works by Dasgupta (1974b), Dasgupta (1974a) and Lane and Mitra (1981). According to Dasgupta (1974b), the Nash equilibrium is a concept corresponding to the universalizability criterion of distributive justice discussed by Rawls, while Lane and Mitra (1981) study Pareto (in)efficiency of a Nash equilibrium in a class of games of intergenerational altruism. The "positive" literature on strategic interactions between generations includes papers by Leininger (1986), Bernheim and Ray (1987), Bernheim and Ray (1989), Amir (1996c) and Nowak (2006). In deterministic as well as stochastic settings, these authors prove existence of a (Markov, Lipschitz continuous) perfect equilibrium in this class of games. Finally, the literature on hyperbolic discounting offers one more motivation for studying economic settings where consecutive generations (or current and future selves) play strategically in consumption decisions (see Phelps and Pollak (1968), Peleg and Yaari (1973) or more recently Laibson (1997), Bernheim et al. (1999) and Krusell and Smith (2003)). Based on these three considerations – altruistic preferences, hyperbolic discounting, and distributive justice – in our paper we let each generations' utilities be defined over their own and the successive generation's consumption, leading to strategic interactions and necessitating an application of the (Markov perfect) Nash equilibrium concept rather than just an optimal planning solution.

The commitment problem in intergenerational setups is also closely related to the issue of time (in)consistency of optimal plans which has been studied in detail by economists ever since the work of Kydland and Prescott (1977). Although Kydland and Prescott's pathbreaking contributions focused primarily on strategic interactions between the private economy and the government while the current paper deals with strategic interactions between private agents only, the conceptual and numerical problems are the same for both approaches.

From the human capital theory perspective, the investigations of the current article are based on the presumption that dynamic paths of human capital accumulation might markedly differ whether there are strategic interactions across generations involved or not (or equivalently, whether there is full or only partial commitment to future generations' choices). Basic economic intuition tells us that if strategic aspects come into play, or if commitment is only partial, the willingness to invest in future generations' human capital should be lower. The neglect of intergenerational interactions, habitually done in the literature, should thus lead to a (potentially large) overestimation of the strength of the postulated intergenerational human capital transmission mechanisms. This paper inspects under which conditions this can be a serious shortcoming of the non-strategic approaches and presents one way to alleviate it.

From the technical perspective, the point of departure of the current article is the following. The original generation (i.e. the parents) would like to choose their consumption level and the level of investment in human capital of their children optimally which requires considering the possible options the children will face in the subsequent period – when they will themselves become independent utility maximizers. The parents would therefore like to embed their children's optimization problems in their own and thus become "leaders" of such an intergenerational strategic game. Unfortunately, this procedure cannot be carried out directly: since the children's optimization problem embeds the optimization problem of their own children, and so forth *ad infinitum*, we end up with an infinite series of embedded strategic games. The problem with applying usual fixed-point arguments here is that the strategic component of the embedded games creates a "vicious circle" of strategy space which has obstructed the development of economic theories in this vein for many years (see e.g. Strotz (1955) and Phelps and Pollak (1968)). The (Markov perfect) equilibrium existence results for a deterministic incarnation of the game has been obtained by Bernheim and Ray (1983) and Leininger (1986), and for the stochastic setting – thanks to Amir (1996a,c) and Nowak (2006). These crucial technical developments are however based on topological arguments, existential rather than constructive in nature, and thus without additional results regarding uniqueness of the analyzed equilibrium, their usefulness in applied work is uncertain.

In this regard, we should also mention the methods for showing equilibrium existence in the class of dynamic games proposed by Kydland and

Prescott (1980) and Abreu et al. (1990) – the latter frequently abbreviated as APS. In this line of research, existence results come almost for free, but unfortunately almost no equilibrium characterization is available, not to mention uniqueness of the analyzed equilibria or computational possibilities[5].

There is one more line of theoretical contributions closely related to our paper. Klein, Krusell, Quadrini and Ríos-Rull in a series of papers on time-consistent taxation propose an intuitive numerical technique for equilibrium computation by value function iteration. Specifically, Klein and Ríos-Rull (2003) and Klein et al. (2005) analyze[6] the Markov perfect equilibrium in a growth model without (tax policy) commitment using techniques essentially based on numerical iteration of the value function under a linear-quadratic approximation. There are, however, two problems with applications of this approach to the our case. Firstly, no controlled accuracy or error bounds are provided for these approximations. Secondly and more importantly, their method is based on differentiability of the policy function and connected strict concavity and twice differentiability of the (infinite horizon) value function which, perhaps apart from a few cases of specific functional forms representing preferences and technology, is very problematic to be shown (see e.g. the assumptions in Santos (1994), Montrucchio (1998) necessary for policy function differentiability). And although recently Klein et al. (2008) managed to solve the first mentioned problem by proposing a characterization of the time-consistent policy in terms of first order conditions (the so-called Generalized Euler Equation), the second argument, to our best knowledge, remains unsolved. Hence, as for our human capital bequest economy, there are no results available yet on the uniqueness or differentiability of the Markov perfect equilibrium (see Kohlberg (1976) and Amir (1996c) for discussion), we cannot apply the methods proposed by Klein, Krusell, Quadrini and Ríos-Rull for a constructive study.

[5]Consider for example a study of a Markovian equilibrium set for a distorted competitive economy a la Coleman (1991). If you use APS, what you can conclude existence of values that can be supported by a measurable selection from the Markovian equilibrium correspondence. If you use the direct approach a la Mirman et al. (2008), you can provide e.g. existence of a unique smooth Markov equilibrium or a complete lattice of locally Lipschitz Markovian equilibria.

[6]Klein et al. (2005) analyze a two country model and hence not only need to solve a taxation commitment problem but also find a within-period Nash equilibrium of a two country game.

Given the drawbacks of all discussed methods, the only suitable technical framework for the study of our human capital bequest economy with strategic interactions is – to our best knowledge – the one offered by Balbus et al. (2008). The reason is that these authors not only obtain the equilibrium uniqueness result (within an appropriate set of Lipschitz continuous policies) but also put forward a constructive numerical algorithm for computing the Markov perfect equilibrium in games of intergenerational altruism, based on iterating the best response map. The algorithm guarantees uniform convergence, thanks to which we are able to solve the technical problem of computing error bounds. The technique due to Balbus et al. (2008) comes, however, at a cost as well. Specifically it is restrictive in terms of requiring a specific form of stochastic transition of the state variable – here, the human capital stock. Two main features of this transition are the following: (i) it is defined in terms of distributions over the next period state space parameterized by the a current period investment and current state and (ii) it "separates" decision from distributions by requiring a certain functional form of the mixing functions. Such stochastic transition has already been widely used by Amir (1997) in optimal growth theory; by Amir, Nowak and coauthors[7] in the directly related context of dynamic games; as well as (at somehow more general level) by Magill and Quinzii (2009) in general equilibrium framework.

On the one hand our assumption on the shape of the stochastic transition function is quite general but critical for the results on uniqueness and construction of the equilibrium. On the other hand, it also has two main drawbacks. First, it requires a certain level of "mixing" (see assumption 2 for the details), and specifically cannot be reduced to the deterministic case. The second drawback is that there are no known ways yet to prove existence of appropriate price systems decentralizing firms' allocations in a general equilibrium context under stochastic technologies expressed by such probability distributions.

Recently Magill and Quinzii (2009) have proposed, however, a way to decentralize an optimal allocation in a (two-period) economy with technology being a probability distribution (over a finite number of states) rather than an Arrow-Debreu "state of nature" production function. Hence in our case, by generalizing Magill and Quinzii approach one can obtain a counterpart of the first welfare theorem and show decentralization of both optimal, nonstrate-

[7]See Amir (2002), Nowak (2007) and references therein.

gic human capital allocation as well as (accompanied with a generalized to a stochastic setting result of Lane and Leininger (1986)) of an (Markov perfect) equilibrium allocation in the strategic case. Although such characterization can be obtained, there are not known ways to show existence of a appropriate (recursive, integrable) Arrow-securities prices in the infinite horizon economy with a uncountable number of states and stochastic production technology. So these are not strategic interactions that constitute the problem for decentralization in our setup but rather the stochastic technology formulation and uncountable number of states. Finally, let is mention that this should not be consider as a serious drawback of our results, since by the mentioned counterpart of a first welfare theorem, both of our analyzed allocation can actually appear on the real markets.

The crucial contributions of this paper are therefore purely theoretical. The lack of immediate empirical applications of our theory comes from the fact that the model developed herein, though based on sound microeconomic foundations, is admittedly simplified. We are therefore convinced that it would be a stark exaggeration to calibrate it in its current form in order to draw quantitative implications aimed at discriminating between competing theories of human capital accumulation based on empirical evidence. Another reason for this limitation are mentioned problems of a general equilibrium decentralization of both the strategic and the dynastic optimization frameworks. Being aware of these theoretical and technical difficulties, our model should nevertheless be considered as an important first step: it is the first model of human capital accumulation which integrates and rigorously calculates fully-specified strategic interactions between consecutive generations into an otherwise standard framework.

3. The model

3.1. Setup

Our model economy is populated by an infinite sequence of generations whose sizes are equal and normalized to unity. Each generation $t = 0, 1, 2, \ldots$ is characterized by the common utility function U, taking values $U(c_t, c_{t+1})$, where c_t is the total consumption of generation t. We assume U to be time-separable[8] and take the form: $U(c_t, c_{t+1}) = u(c_t) + v(c_{t+1})$. The consumption

[8]We analyze the case of time-separable utility functions only because the monotone methods used in Theorem 2 rely on this assumption heavily and because this assumption

set is $Y = [0, \bar{Y}]$ where $\bar{Y} \in \mathbb{R}_+$. The unique consumption good is produced using technology f which requires two kinds of inputs: (i) time devoted to work $\hat{l}_t$, and (ii) human capital h_t. The set $H = [0, \bar{H}]$, where $\bar{H} \in \mathbb{R}_+$, represents all possible levels of human capital. We neglect all physical capital accumulation in our basic model. Human capital, on the other hand, is accumulated using technology $\tilde{g}$ taking as inputs: (i) the current level of human capital h_t, and (ii) time devoted to human capital accumulation $1-\hat{l}_t$.

Technically, our assumptions on the considered economy are the following:

Assumption 1. *Let:*

- $u, v : Y \to \mathbb{R}$ *be increasing, continuously differentiable, and satisfying* $\lim_{c\to 0} u'(c) = \lim_{c\to 0} v'(c) = \infty$*;* $(\forall c \in Y, c > 0)\;\; u'(c) < \infty$ *and* $(\forall c \in Y, c > 0)\;\; v'(c) < \infty$*. Moreover, let* u *and* v *be strictly concave and such that* $u(0) = v(0) = 0$*,*

- $f : H \times [0,1] \to Y$ *be strictly concave with respect to the second argument, twice continuously differentiable with finite partial derivatives, and satisfying* $(\forall \hat{l} \in [0,1])\;\; f(0, \hat{l}) = 0$*,* $(\forall h \in H)\;\; \lim_{\hat{l}\to 0} f_2'(h, \hat{l}) = \infty$*. Furthermore, assume that* $(\forall h \in (0, \bar{H}])\, f(h, \cdot)$ *and* $(\forall \hat{l} \in (0,1])\, f(\cdot, \hat{l})$ *are strictly increasing functions.*

Within each generation, the household chooses its consumption level c_t to maximize utility U, that is:

$$\max_{c_t} u(c_t) + v(c_{t+1}). \tag{1}$$

The neglect of physical capital accumulation requires assuming full depreciation as well. All output is thus immediately consumed: $c_t = f(h_t, \hat{l}_t)$, where $\hat{l}_t \in [0,1]$.

Human capital, on the other hand, is accumulated according to the equation: $h_{t+1} = \tilde{g}(h_t, 1 - \hat{l}_t)$, where $\tilde{g} : H \times [0,1]$ is a continuous, strictly positive function. Substituting the relations specified above into (1) and ignoring time subscripts we obtain the following household maximization problem:

$$\max_{\hat{l}\in[0,1]} u(f(h, \hat{l})) + v(f(\tilde{g}(h, 1-\hat{l}), \tilde{l})). \tag{2}$$

has been extensively used in literature. The case of non-time separable utility functions could also be analyzed nonetheless. This would require the use of results on mixed monotone operators. See Guo et al. (2004) and the applications in Balbus et al. (2008).

The problem (2) features two endogenously determined variables which are taken as given by the original generation: their own human capital level $h \in H$ and the labor choice of the next generation $\tilde{l} \in [0, 1]$.

We propose two alternative economic interpretations for our modeling approach summarized by the maximization problem (2):

- each household lives for one period and derives utility from its own consumption, $u(c_t)$, and the consumption of its immediate successor, $v(c_{t+1})$;
- each household lives for two periods but chooses the fraction of time devoted to the production of consumption goods and the fraction of time devoted to the accumulation of human capital of the subsequent generation in the first period only. Its consumption in the second period is chosen by the next generation, and thus is only indirectly influenced by the level of human capital left to the next generation.

3.2. The concept of Markov perfect equilibrium

The primary objective of this paper is to analyze closed-loop Markov perfect equilibria (MPE) of the economy specified above. To this end, we must now introduce some new notation. Namely, by $l' \in L$, where $L = \{l : (0, \overline{H}] \to [0, 1], l \in \mathscr{C}\}$, we will denote the Markov strategy of the next generation. Moreover, we shall let $\mathbf{0} \in L$ denote the constant zero function, and let $\mathbf{1} \in L$ denote a constant function whose values are always equal to 1. We shall also introduce the correspondence $D : L \times H \to [0, 1]$ defined by

$$D(l', h) = \arg\max_{\hat{l} \in [0,1]} u(f(h, \hat{l})) + v(f(\tilde{g}(h, 1 - \hat{l}), l'(\tilde{g}(h, 1 - \hat{l})))). \tag{3}$$

The best response of the current generation for next generation's strategy $l' \in L$ is therefore a selection $l(\cdot)$ from $D(l'|\cdot)$.

We adopt the following definition of MPE:

Definition 1. *A Markov perfect equilibrium (MPE) of the economy is a selection*[9] $l^* : (0, \bar{H}] \to [0, 1]$ *from* $D(l^*|\cdot)$.

[9] We are leaving $l^*(0)$ undefined here, since under Assumptions 1 and 2, as we shall show later, it is not single-valued. The economic justification is the following: having no human capital one produces, consumes and invests nothing, but since there is a no disutility of work, any level of l could be optimal.

The MPE can be interpreted either as a subgame perfect Nash equilibrium of an sequential intergenerational game or as a time-consistent policy which is equally well suited for any generation. Since the time horizon of the economy is infinite, we concentrate on stationary Markov policies, i.e. such that in each period, the same function of the state variable h is applied.[10]

3.3. Introducing stochastic transition

Unfortunately, as discussed by Leininger (1986) and others, the standard way of obtaining results on the existence and uniqueness of MPE in similar setups – as fixed points of some self maps – is obstructed by the so-called "vicious circle" of strategy space. The problem occurs when trying to construct appropriate sets of admissible strategies/policies. Even very strong assumptions made on the strategy/policy of the subsequent generation cannot guarantee that the best response to that strategy would belong the the same strategy/policy space.

The crucial step required to solve this problem is to break the deterministic links between subsequent generations (see Amir, 1996c; Nowak, 2003). In our case, this would correspond to assuming that the transition (human capital accumulation function) $\tilde{g}$ be stochastic. Hence, we shall let $G(\cdot; h, 1-l)$ be the distribution of human capital in the subsequent period, parametrized by the current human capital level h and the time investment in education, $1-l$.

The introduction of stochastic factors in human capital accumulation is thus motivated primarily by technical reasons. Such factors have sound economic motivation, though. Indeed, (i) heredity involves randomness: the unobservable skill levels are not inherited from one's parents deterministically; (ii) human capital is not homogenous: it is technology-specific and thus up-front investment in it might (but might not) be ineffective (Chari and Hopenhayn, 1991), depending on the future pattern of technological progress; (iii) the motivation of children to learn is endogenous (Orazem and Tesfatsion, 1997). All these factors taken together make it clear that treating investment in education as a lottery where future payoffs depend on stochastic factors is quite reasonable.[11]

[10] If the horizon of the economy were finite, we could solve for non-stationary policies by backward induction.

[11] It should be noted that we rule out all systematic human capital externalities from non-relatives here (Ben-Porath, 1967; Rangazas, 2000) and assume that children's human

The following assumption on the stochastic transition follows Amir (1996c) and Nowak (2006).

Assumption 2 (Technology). *The distribution G satisfies the following conditions:*

- $\forall h \in H, \quad G(0|h,0) = 1$,
- $\forall h \in H, l \in [0,1)$,

  $$G(\cdot|h, 1-l) = (1 - g(h, 1-l))\delta_0(\cdot) + g(h, 1-l)\lambda(\cdot|h),$$

  where

- $g : H \times [0,1] \to [0,1]$ *is strictly concave with respect to the second argument, twice continuously differentiable, satisfies the condition:* $(\forall l \in [0,1]), g(0, 1-l) > 0$,
- $(\forall l \in [0,1))\, g(\cdot, 1-l)$ *and* $(\forall h \in (0, \bar{H}])\, g(h, \cdot)$ *are strictly increasing functions,*
- $(\forall h \in H) \lim_{l \to 1} g_2'(h, 1-l) = \infty$ *and* $(\forall h \in H, l < 1), 0 < g_2'(h, 1-l) < \infty$,
- $\lambda(\cdot|h)$ *is a family of Borel transition probabilities on* $(0, \bar{H}]$ *that is stochastically decreasing and continous with* h,
- δ_0 *is a probability measure concentrated at zero.*

The crucial implications of this specification are as follows: with probability $1-g(h, 1-l)$, the next generation's human capital will be zero, indicating that the investment in it has been completely ineffective. The economic interpretation of this assumption can be twofold. First, it may capture human capital-dependent mortality: the next generation's zero human capital is then a synonym for not surviving until adult age. Such a setup is in agreement with evidence: indeed, children of better educated parents face a generally lower risk of dying young. Second, this may also relate to the argument that skills are often technology-specific and that technology might change

capital is created from parental human capital, education effort, and stochastic factors only.

fast enough to make all previously acquired skills obsolete. With probability $g(h, 1-l)$, i.e. conditional on survival and non-obsolescence of skills, human capital is however drawn from a distribution λ which does not depend l. This relates to the stochastic heredity assumption, coupled with the random motivation of children to learn.

Assuming that the next generation follows a Markov strategy $l' \in L$, the maximization problem (2) augmented by the stochastic transition takes the form:

$$\max_{\hat{l}\in[0,1]} u(f(h,\hat{l})) + \int_H v(f(y,l'(y)))G(dy;h,1-\hat{l}). \tag{4}$$

Under Assumptions 1 and 2, the maximand of (4) (for a given $h \in (0,\bar{H}]$) is strictly concave and differentiable with respect to $\hat{l}$ on $(0,1)$. Furthermore, the unique optimal labor supply level l^* solves $\zeta(l^*,h,l') = 0$ whenever interior, where ζ is defined as:

$$\zeta(l,h,l') := u'(f(h,l))f_2'(h,l) - g_2'(h,1-l)\int_H v(f(y,l'(y)))\lambda(dy|h). \tag{5}$$

A MPE of the economy with stochastic transition is then a function l which solves $\zeta(l(h),h,l) = 0$ for all $h \in (0,\bar{H}]$.

3.4. Characteristics of the closed-loop MPE

Let us now comment on the possibilities of showing existence of a MPE in the given class of functions. In the paper most closely related to this one, Balbus et al. (2008) have constructed an operator whose fixed points are MPE of an economy with intergenerational altruism (see also Bernheim and Ray (1987)). The operator is defined implicitly on the set of Lipschitz continuous functions belonging to L by an appropriate first order condition. The authors find that it suffices to show continuity of such an operator, and existence of a MPE follows by the Brouwer fixed point theorem. In our particular case, however, their method fails due to the non-uniqueness of the maximizer in equation (5) for $h=0$. Specifically, for any $l' \in L$ the optimal $l^*(0) = [0,1]$. Notice also that $(\forall h \in H, h > 0)$, $l^*(h) = 1$ is the best response to $l' = \mathbf{0}$. Hence, we cannot apply those results directly.

However before showing existence we may present some of equilibrium basic properties. They will be helpful in our further analysis.

Theorem 1 (Characteristics of MPE). *Suppose that a MPE exists. Then:*

- *the set of Markov perfect equilibria of the economy has no ordered (in a pointwise order) elements in L.*
- *If $f''_{12}(\cdot,\cdot) \leq 0$ and $g''_{12}(\cdot,\cdot) \geq 0$, then l^* is strictly decreasing on $(0,\bar{H})$ wherever interior.*

Proof. See Balbus et al. (2008) for the first assertion and the Appendix for the second.

The first assertion results from the fact that an appropriate operator defined derived from the first order conditions, whose fixed points are MPE of the economy, is decreasing. The second assertion follows from established theorems on strict monotone comparative statics (Edlin and Shannon (1998); Amir (1996b)) of optimal solutions to maximization problems featuring a submodular function on a lattice. Please observe that the reverse to the second assertion need not hold. Generally, even if $f''_{12}(\cdot,\cdot) \geq 0$ and $g''_{12}(\cdot,\cdot) \leq 0$, the optimal labor supply policy l^* need not increase with h due to the strictly decreasing marginal utility.

The methods used to show uniqueness of the MPE in the setup of Balbus et al. (2008) can be used in our current model as well, disregarding the fact that in the current case, for any $l' \in L$ the optimal $l^*(0) = [0,1]$. To state this, we shall rearrange the first order condition of maximization in (4) for $h \in (0,\bar{H}]$ as:

$$\xi_h(\hat{l}) := \frac{u'(f(h,\hat{l}))f'_2(h,\hat{l})}{g'_2(h,1-\hat{l})} = \int_H v(f(y,l(y)))\lambda(dy|h). \tag{6}$$

The function $\xi_h(0,1] \to \mathbb{R}_+$, with $\xi_h(1) = 0$, introduced just above, captures the marginal utility of consumption coupled with marginal labor productivities in both sectors. Balbus et al. (2008) have showed that function ξ is continuously differentiable, strictly decreasing, and invertible with continuously differentiable inverse.

Let us also define an operator B on $P = \{\bar{l} : (0,\bar{H}] \to [0,\infty)\}$ such that for any $h \in (0,\bar{H}]$, B satisfies:

$$B\bar{l}(h) = \int_H v(f(y,\xi_h^{-1}(\bar{l}(y))))\lambda(dy|h). \tag{7}$$

The operator B is going to be central to the reasoning in the remainder of the paper: it will be used both in the proofs of our theoretical results and in their numerical implementation. Its importance stems from the fact that by definition, the fixed point of B satisfies (6).[12]

The next theorem gives the conditions under which B has a unique fixed point in P. This finding is equivalent to showing under which conditions the MPE of the considered economy, l^* exists and is unique. By E_x^f we denote the partial elasticity of a function f with respect to x: $E_x^f = \frac{\partial f(x)}{\partial x} \frac{x}{f(x)}$.

Theorem 2 (Existence and uniqueness). *Let Assumptions 1 and 2 be satisfied. Assume in addition that there exists an $r \in (0,1)$ such that for all $h \in H$ the following holds:*

$$(\forall x > 0) \quad r \geq \left[-E^v_{f(h,\xi_h^{-1}(x))} E^{f,2}_{\xi_h^{-1}(x)} E_x^{\xi_h^{-1}(x)} \right]. \tag{8}$$

Then there exists a unique MPE $l^* \in L$ of the economy under study.

Proof. See the proof of Theorem 3 in Balbus et al. (2008).

Theorem 2 provides the sufficient conditions for the existence and uniqueness of a fixed point of a MPE of the considered economy. Moreover, using further results by Balbus et al. (2008), one can straightforwardly compute it using a Picard iterative procedure.

The mathematical intuition behind Theorem 2 is the following: since the fixed point operator B is decreasing, it may have multiple, unordered fixed points. The condition in Theorem 2 asserts, however, that this operator is "e-convex" (see Guo and Lakshmikantham (1988) for details) or – in other words – it is a "local contraction" (i.e. a contraction along cone origin rays). This property is sufficient for existence of a unique fixed point. Economically, the condition (8) ("convexity" or "local contraction") could be interpreted in terms of partial elasticities: it requires that the product of elasticities of v, f and ξ_h^{-1} cannot exceed unity, i.e. that the percentage change in next-period utility v resulting from a one per-cent change in labor supply $\bar{l}$ cannot be "too high".

We leave the questions on existence and number of equilibria when condition (8) is not satisfied for further work. Instead, we shall now present our workhorse example which will be used in our subsequent numerical exercises.

[12] For a more detailed justification, see Coleman (2000) and Balbus et al. (2008).

Example 1. *Let* $U(c_1, c_2) = c_1^{\gamma_1} + \delta c_2^{\gamma_2}$*,* $f(h, l) = h^{\alpha_1} l^{\beta_1}$*. Furthermore, take any* g *satisfying Assumption 2 with* $\alpha_1, \beta_1, \gamma_1, \gamma_2 \in (0, 1)$ *and* $\delta \in (0, 1]$*. If* $1 > \beta_1(\gamma_1 + \gamma_2)$ *then there exists a unique MPE in* L*.*

PROOF. See Appendix.

4. Human capital dynamics with and without strategic interactions

In the current section, we shall compare the time-consistent Markov perfect policy l^*, discussed in the previous section, to the outcomes obtained within a similar setup which does not however allow for strategic interactions across generations.

To this end, we will focus on optimal (or full-commitment) policies. Specifically, we will consider a setup where individuals live for two periods and decide over the consumption (or labor supply) in the first period taking the consumption function of the next generation as given. In the other interpretation, individuals live for just one period, but each subsequent generation fully commits to some level of l' and reveals it to the previous generation.

In order to attain comparability of utilities across different periods, we assume that $v(\cdot) = \delta u(\cdot)$ where $\delta \in (0, 1)$ is a discount factor. Finally, to find the optimal policy (generally time inconsistent) benchmark for our Markov perfect (time consistent) policy, obtained in the previous section, we shall solve the following social planner's problem:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \delta^t u(c_t) + \sum_{t=1}^{\infty} \delta^t v(c_t) = 2 \max_{\{c_t\}} \left(\frac{u(c_0)}{2} + \sum_{t=1}^{\infty} \delta^t u(c_t) \right).$$

Observe that the similar optimization problem can be obtained when we reformulate the model such that individuals do not derive utility directly from their successors' consumption, but from their utility. Hence, generations' choices can be embedded in the first generation's optimization problem, ultimately yielding a "dynastic" model with infinite-horizon planning where each generation $t > 0$ maximizes $\sum_{\tau=t}^{\infty} \delta^{\tau - t} u(c_\tau)$.[13]

[13]Provided that the transversality condition holds: $\lim_{\tau\to\infty} \Lambda_\tau h_\tau = 0$ (where Λ is the shadow price of human capital). If the set of admissible human capital levels H is bounded, as it is in our case, this transversality condition holds for sure.

To see it formally (from $t > 0$), consider an economy populated by a sequence of generations each represented by a single household with preferences $U(c_t, V_{t+1})$ over its consumption c_t and its immediate descendants' utility V_{t+1}. Since all generations' utility functions are the same, their choices can be embedded in the first generation's optimization problem. The solution to this maximization problem corresponds to a stationary solution to an infinite-horizon dynastic model with stochastic transition in human capital levels: $\max_{\{c_\tau\}} \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(c_\tau)$, where $\delta \in (0, 1)$ is a discount factor.

For $t > 0$ the first order condition reads:

$$u'(f(h, l(h)))f_2'(h, l(h)) = \delta g_2'(h, 1 - l(h)) \int_H V(y)\lambda(dy|h), \tag{9}$$

where $V(h)$ is the Bellman's value function defined as

$$V(h) = \max_{\hat{l} \in [0,1]} \left\{ u(f(h, \hat{l})) + \delta \int_H V(y)G(dy; h, 1 - \hat{l}) \right\}. \tag{10}$$

Standard arguments of dynamic programming (see e.g. Stokey et al. (1989)) guarantee that under our assumptions the functional equation (10) has a unique solution V and that the solution corresponds to a function $l(h)$ which solves $V(h) = u(f(h, l(h))) + \delta \int_H V(y)G(dy, h, 1 - l(h))$.

The first order condition (9) guarantees that the marginal utility of consumption of the current generation, acquired thanks to an extra unit of time devoted to work, is exactly equal to the expected marginal cost in terms of utility lost by the next generation because of having marginally less human capital. Having calculated (for $t > 0$) the optimal policy $l^* : H \to [0, 1]$ in such a setup, one only needs to add the optimal decision for the first generation l_0, obtained as a solution to the maximization problem:

$$\max_{l_0} \left\{ \frac{u(f(h_0, l_0))}{2} + \delta \int_H V(y)G(dy, h_0, 1 - l_0(h_0)) \right\}.$$

Since the optimal setup rules out all strategic aspects of the decision process, the full-commitment Markov policy for the dynastic optimization economy is (generally) not a MPE of an economy with strategic interactions.[14] It

[14] A related class of models frequently encountered in the human capital accumulation literature uses the framework of joy-of-giving altruism. In such models, generations do not

turns out, however, that equilibrium policies for our basic model with strategic interactions and the optimal policy abstracting from such interactions can be directly compared:

Theorem 3 (On comparing equilibria). *Let l_{MPE} be a MPE of an economy with strategic interactions with $v(\cdot) = \delta u(\cdot)$, and l_R be the optimal stationary policy of a dynastic economy with utility u. Then $l_{MPE}(h) > l_R(h)$ for all $h \in (0, \bar{H}]$.*

Proof. See Appendix.

Theorem 3 asserts that equilibrium human capital investment is unambiguously lower in an economy with strategic interactions than in an economy using the optimal policy. The intuition behind this result is straightforward: the optimal investment policy under full commitment must exceed the equilibrium investment policy when only partial commitment between consecutive generations is possible. Indeed, under the optimal policy, the dynastic head from generation t will take into account not only the consumption of the following generation $t+1$, but of all generations from t onwards. She will therefore be willing to save more for the future than a generation t member of the strategic model: the latter person is myopic and wishes to save for her children but not for her grandchildren.

Theorem 3 provides a formal argument determining the direction of the bias incurred when a baseline model with strategic interactions is replaced with its non-strategic counterpart.[15]

derive their utility directly from their successors' consumption, but are instead interested in providing them with the means allowing for consumption. In the context of human capital accumulation it means that their utility function is $u(c_t) + v(h_{t+1})$. Hence, the decisions made by the next generation do not matter for the utility of the current generation. Unfortunately, although widely used in the literature, the "joy-of-giving" altruism utility function and hence the whole model is not directly comparable to the ones studied in this paper. Hence, we only briefly discuss the implications of joy-of-giving altruism models in the context of our argument in the Appendix.

[15]Understandably (as we will show in the Appendix) a similar clear-cut relationship does not exist between the strategic model and the model with joy-of-giving altruism. Even though each numerical example has been prepared so that direct comparisons could be possible, we find that for different parameter configurations, different results are possible. Usually it is the strategic model which puts more weight on immediate consumption and less on human capital accumulation; sometimes the result is reversed, though.

5. Numerical example

5.1. Numerical computation of the MPE

The objective of the current section is to compute numerically the equilibrium policy l^* for an economy with strategic interactions and to analyze the equilibrium dynamics of human capital accumulation given certain functional assumptions on u, v, f and G. To facilitate economic interpretation, we will concentrate on iso-elastic utility and Cobb-Douglas production functions here. We will then benchmark these numerical results against the corresponding one obtained within the non-strategic (dynastic) model discussed in the previous section.

Example 2. *Extending Example 1, let us additionally assume that* $g(h, 1-l) = \frac{1}{\bar{H}^{\alpha_2}} h^{\alpha_2}(1-l)^{\beta_2}$ *where* $\alpha_2, \beta_2 \in (0,1)$. *The function* ξ_h *is then given by:*

$$\xi_h(l) = \frac{\beta_1\gamma_1}{\beta_2} \bar{H}^{\alpha_2} h^{\alpha_1\gamma_1-\alpha_2} \frac{l^{\beta_1\gamma_1-1}}{(1-l)^{\beta_2-1}}. \tag{11}$$

Furthermore, we assume that $\beta_2 = \beta_1\gamma_1$.

The last equality assumption has been made for the sole purpose of analytical tractability: it is only when $\beta_2 = \beta_1\gamma_1$ that the ξ_h mapping is analytically invertible. Relaxing it increases the computational burden significantly but does not overturn any of our results. If $\beta_2 = \beta_1\gamma_1$, we obtain:

$$\xi_h^{-1}(\bar{l}) = \frac{\bar{l}^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}}{1+\bar{l}^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}}. \tag{12}$$

Assuming furthermore that the distribution λ is uniform on H, the MPE policy can be found as $l^*(y) = \xi_h^{-1}(\bar{l}(y))$ where $\bar{l}$ is found as the fixed point of the operator B given by

$$B\bar{l}(h) = \frac{\delta}{\bar{H}} \int_0^{\bar{H}} y^{\alpha_1\gamma_2} \left(\frac{\bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}}{1+\bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}} \right)^{\beta_1\gamma_2} dy. \tag{13}$$

As stated in Theorem 2, repeated iteration of B guarantees convergence to the MPE (see Figure 1).[16]

[16]To calculate the equilibrium policies of any of the three models numerically, we have used the discretization method discussed by Judd (1998). Matlab codes used to compute the numerical results quoted throughout the paper as well as to produce Table 1 are available from the authors upon request.

Proposition 1. *The MPE policy l^* is monotone. It is everywhere decreasing iff $\alpha_1\gamma_1 < \alpha_2$, everywhere increasing iff $\alpha_1\gamma_1 > \alpha_2$, and constant iff $\alpha_1\gamma_1 = \alpha_2$.*

Proof. See Appendix.

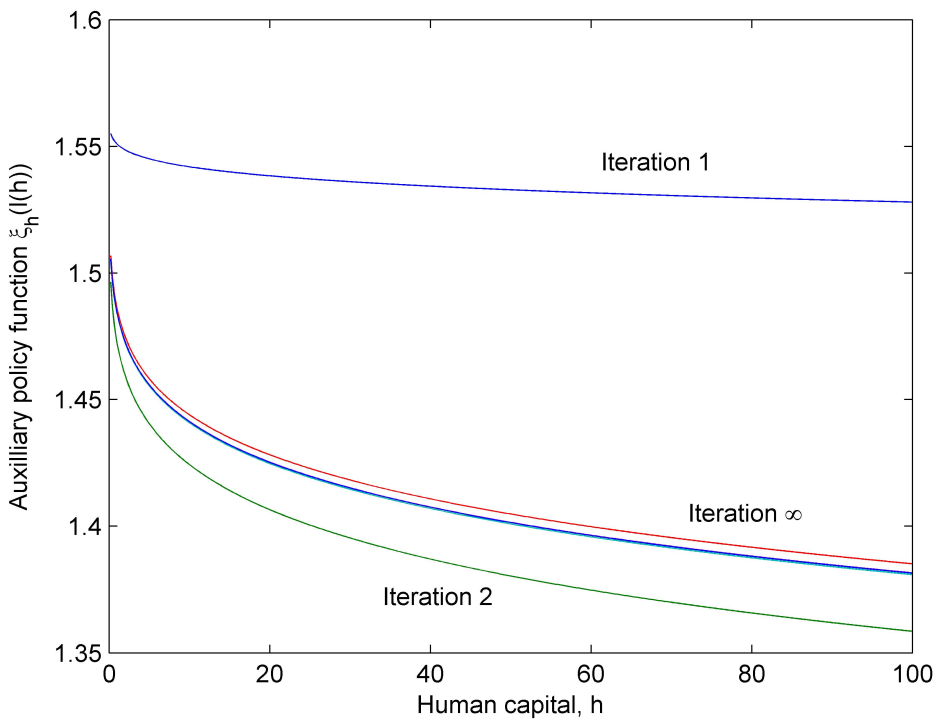

Figure 1: Convergence to the fixed point of operator B. The fixed point is the auxilliary policy function $\bar{l}(h) = \xi_h(l(h))$. Assumed parameter values: $\alpha_1 = .3; \beta_1 = .7; \alpha_2 = .3; \gamma_1 = .6; \gamma_2 = .5; \beta_2 = \beta_1\gamma_1 = .42; \bar{H} = 100; \delta = .9$.

Having specified the three cases in which the optimal labor supply policy is increasing, decreasing, or constant in the human capital endowment, let us discuss the empirical plausibility of each of the cases. The results are somewhat reassuring here. Namely, the case where $\alpha_2 > \alpha_1\gamma_1$, guaranteed to hold e.g. if $\alpha_1 \approx \alpha_2$ (i.e. if the shares of human capital in production of the consumption good and of human capital, respectively, are approximately equal), turns out to be significantly more plausible empirically than any of the other cases.[17] This case, implying that labor supply decreases (and human capital accumulation increases) with the stock of human capital, is thus going to be our benchmark case.

5.2. Dynamics

The dynamic properties of the economy are as follows. If all generations play the MPE strategy, then in the limit as $t \to \infty$, average human capital tends to $\bar{h}$ solving the implicit equation:

$$\bar{h} = 2^{\frac{1}{\alpha_2 - 1}} \bar{H} (1 - l(\bar{h}))^{\frac{\beta_2}{1-\alpha_2}}. \tag{14}$$

This result has been confirmed numerically.[18]

The distribution of human capital will also evolve over time as consecutive generations will invest different fractions of time to work and education. By definition, however, the distribution of human capital over H will have a constant density $\frac{1}{\bar{H}} g(\bar{h}, 1 - l(\bar{h})) = \frac{1}{\bar{H}^{\alpha_2+1}} \bar{h}^{\alpha_2} (1 - l(\bar{h}))^{\beta_2}$ and a probability mass $1 - g(\bar{h}, 1 - l(\bar{h})) = 1 - \frac{1}{\bar{H}^{\alpha_2}} \bar{h}^{\alpha_2} (1 - l(\bar{h}))^{\beta_2}$ concentrated at zero.

5.3. Role of the transition distribution λ

The MPE policy $l^*(h)$ depends on the underlying transition distribution λ but this impact turns out to be rather modest. As a robustness check of our earlier numerical results, we have substituted the uniform distribution λ with two alternatives:

- a triangular distribution with density

$$\varphi(h) = \begin{cases} \frac{4}{\bar{H}^2} h, & h \in (0, \frac{\bar{H}}{2}), \\ \frac{4}{\bar{H}} - \frac{4}{\bar{H}^2} h, & h \in (\frac{\bar{H}}{2}, \bar{H}); \end{cases} \tag{15}$$

- a one-point distribution[19] with all probability mass concentrated in $\bar{H}/2$: $P(h = \bar{H}/2) = 1$.

[17] Becker and Tomes (1986), Lochner (2008), among numerous others, discuss the empirical evidence that the educational effort and children's school attainments are unambiguously positively related to the parental human capital level.

[18] The results are available from the authors upon request.

[19] Note that even when λ is one-point, there remains a probability that the next generation's human capital will be zero. Hence, the assumptions and interpretations of the economy with strategic interactions studied in Section 3 are still satisfied.

As we have confirmed numerically,[20] the greatest labor supply is obtained when the distribution is uniform, and the least labor is supplied when the probability mass is concentrated at the mean human capital level. The policy for the triangular distribution falls in between these two extreme cases (uniform and one-point). The interpretation of this result is straightforward: the more risk remains that human capital of the successive generation would be low despite substantial investment, the less willing the decision maker would be to invest in human capital. Since individuals are risk-averse in this model, additional risk lowers education effort and increases labor supply which guarantees a certain payoff.

6. Numerical assessment of the role of strategic interactions

Let us now compare the equilibrium dynamics obtained in the numerical example presented above to the ones generated by the optimal-policy, dynastic model of Section 3.

Example 3. *Let* $u(c) = c^{\gamma}$, $f(h, l) = h^{\alpha_3} l^{\beta_3}$, $g(h, 1-l) = \frac{1}{\bar{H}^{\alpha_4}} h^{\alpha_4}(1-l)^{\beta_4}$. *Let the decision maker born at* t *maximize* $u(c_t) + \delta u(c_{t+1})$. *From* (9), *we obtain the first order condition for the optimal policy function* $l(h)$. *It is given as an implicit solution to the equation:*

$$\frac{l^{1-\beta_3\gamma}}{(1-l)^{1-\beta_4}} = \frac{\bar{H}^{\alpha_4}}{\delta I} h^{\alpha_3\gamma-\alpha_4}, \tag{16}$$

where $I \equiv \int_H V(y)\lambda(dy|h)$ *is a predetermined constant.*

Using the implicit function theorem, it can again be easily shown that $l(h)$ is everywhere decreasing whenever $\alpha_4 > \alpha_3\gamma$ and everywhere increasing whenever $\alpha_4 < \alpha_3\gamma$. In the special case where $\alpha_3\gamma = \alpha_4$, (16) implies that $l(h)$ is constant, independent of h. This finding paralells Proposition 1 precisely: there are absolutely no qualitative differences in the optimal policy behavior between the strategic and the non-strategic model. Quantitative differences are substantial, though, as we shall see shortly.

[20] These results are available from the authors upon request.

Moreover, just like in the strategic case, the first order condition (16) can be solved for $l^*(h)$ explicitly in the special case $\beta_3\gamma = \beta_4$. In such case,

$$l^*(h) = \frac{\left(\frac{\bar{H}^{\alpha_4}}{\delta I}\right)^{\frac{1}{1-\beta_4}} h^{\frac{\alpha_3\gamma-\alpha_4}{1-\beta_4}}}{1+\left(\frac{\bar{H}^{\alpha_4}}{\delta I}\right)^{\frac{1}{1-\beta_4}} h^{\frac{\alpha_3\gamma-\alpha_4}{1-\beta_4}}}. \tag{17}$$

What remains to be derived is the constant $I = \int_H V(y)\lambda(dy|h)$. It can be found as an implicit solution of the following equation:

$$I = \frac{\int_H y^{\alpha_3\gamma} l^*(y)^{\beta_1\gamma}\lambda(dy|h)}{1-\delta\int_H \left(\frac{y}{\bar{H}}\right)^{\alpha_4}(1-l^*(y))^{\beta_4}\lambda(dy|h)}, \tag{18}$$

with l^* defined as in (17) and thus containing I. The approximate solution to this equation can be easily computed numerically. Please note that knowing I, we can also obtain an explicit formula for the value function:

$$\begin{aligned} V(h) &= h^{\alpha_3\gamma} l^*(h)^{\beta_3\gamma} + \qquad\qquad (19)\\ &+ \left(\frac{\delta\int_H y^{\alpha_3\gamma} l^*(y)^{\beta_1\gamma}\lambda(dy|h)}{1-\delta\int_H \left(\frac{y}{\bar{H}}\right)^{\alpha_4}(1-l^*(y))^{\beta_4}\lambda(dy|h)}\right)\left(\frac{h}{\bar{H}}\right)^{\alpha_4}(1-l^*(h))^{\beta_4}. \end{aligned}$$

The direct computation of I would not have been possible if not for the introduction of stochastic transition in human capital levels. Thanks to that step, the infinite series expansion of $V(h)$ can be computed as a simple geometric series which has a closed-form sum. It also enables us to use the law of iterated expectations to convert an n-tuple integral into a product of n simple integrals.

We are now in the position to compare the equilibrium labor supply policy function derived from the model with strategic intergenerational interactions with the alternative non-strategic scenario. To attain direct comparability of both setups, we must assure $\gamma = \gamma_1 = \gamma_2$ – in the dynastic model, the shape parameters of utility functions u and v must be equal. We shall also fix our other parameters at equal levels, $\alpha_1 = \alpha_3, \beta_1 = \beta_3, \alpha_2 = \alpha_4, \beta_2 = \beta_4$.

The results are apparent in Figure 2. Significantly more labor is supplied (and thus, less human capital is accumulated) in the case of the MPE policy in our baseline model with strategic interactions than in the optimal policy

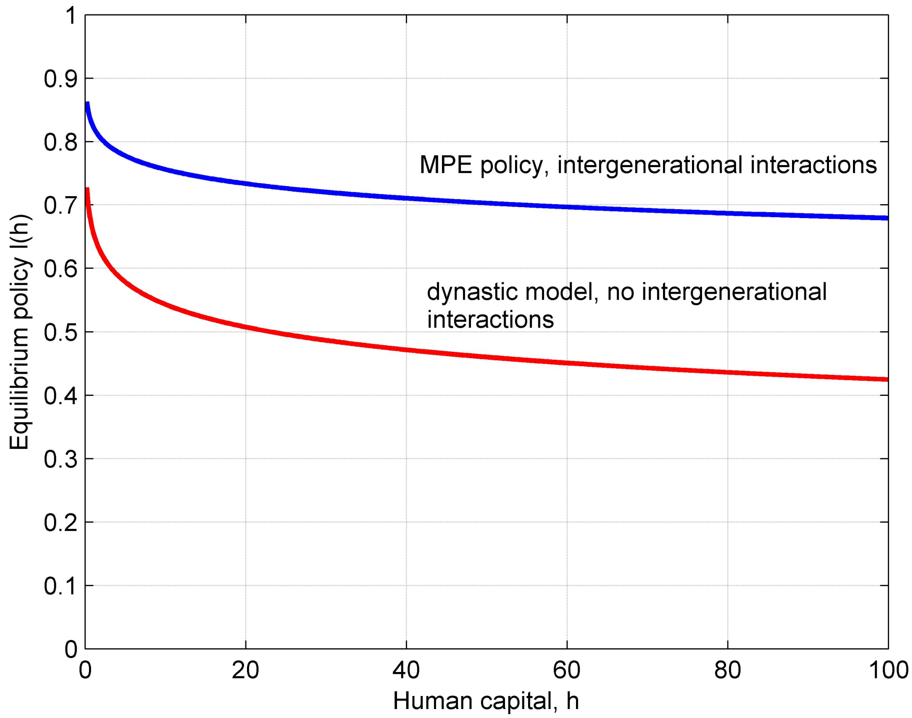

Figure 2: The difference between equilibrium policy functions $l^*(h)$ in the time-consistent policy and the optimal but time-inconsistent policy. Assumed parameter values: $\alpha_1 = .3; \beta_1 = .7; \alpha_2 = .3; \gamma = .6; \beta_2 = \beta_1\gamma_1 = .42; \bar{H} = 100; \delta = .9$.

model which does not include such interactions.[21] This directly confirms Theorem 3, providing a quantitative edge to that result.

Furthermore, even though there is a marked difference in the levels of human capital investment between the models, the shapes of the policy functions are remarkably similar. With iso-elastic utility and Cobb-Douglas production functions, and under our benchmark parametrization, labor supply functions $l^*(h)$ always decrease with h, indicating that human capital and ed-

[21]Because of its different utility function, the outcomes of the "joy-of-giving" altruism model cannot be unambiguously compared to the two alternatives discussed here. There exist certain cases (though arguably unusual) in which joy-of-giving altruism could give rise to less human capital accumulation (and more labor supply) than dynastic optimization, possibly even more than the strategic intergenerational game.

ucation effort are positively related, in line with empirical observations (e.g. Becker and Tomes (1986)).

6.1. Equilibrium investment in human capital: an interpretation

The uniform ordering of labor supply functions obtained from the models under consideration (the policy curves such as the ones depicted in Figure 2 never intersect) offers an intuitive and convincing explanation. In simple words: the more directly does child's human capital enter parent's utility function, the more willing will she be to invest in it.

The rationale is that with strategic interactions, utility acquired from second period consumption is conditional on the strategy chosen by the subsequent generation while with the optimal policy model it is certain. Bernheim and Ray (1987) identify, however, another force at work here: since in the strategic model, each generation views the investment made by their children, $(1 - l')$, as pure waste, it must invest more to obtain the same effect. The latter force turns out to have a relatively smaller impact on our results in the benchmark parametrization, but it could become dominant if β's are sufficiently small.[22]

Under dynastic optimization, utility is derived from children's utility which is a function of their human capital. In such case, the parents know exactly what would eventually be optimal for their children; because of that knowledge, they can anticipate their children's choices and solve for the social planner's first best which involves substantial human capital investment (once you care for your children's utility, you also care for your grandchildren's, great-grandchildren's, etc.). Perfect anticipation across generations is not possible in our baseline model with intergenerational interactions, though. In such a model, utility is derived from children's consumption which is decided endogenously by them in a process of utility maximization which takes into account also the grandchildren's consumption, for which the original generation does not care. This gives one more intermediate step of embeddedness: human capital $\rightarrow$ children's utility $\rightarrow$ children's consumption. In result, the interest in investing in children's human capital is smaller under this scenario. The unambiguous ordering of the strategic and the dynastic

[22]With joy-of-giving altruism, utility is derived from child's human capital directly; consequently, investment in human capital will be the highest in such case, unless β_1 and β_2 are very low, indicating that current production as well as human capital accumulation react to changes in labor supply with a very small elasticity.

model, proved formally in Theorem 3, leads to the conclusion that strategic interactions across generations are an important source of underinvestment in human capital as compared to the intergenerational first best.

6.2. Sensitivity analysis

In order to obtain a rough approximation of the magnitude of difference between equilibrium policies in the two considered models, we have carried out a numerical sensitivity analysis exercise: we have manipulated the parameters of the models under study and compared the resultant equilibrium policy functions $l^*(h)$. For each parameter configuration, we calculated two measures of distance between the functions. Since by Theorem 3, we know that $l_{MPE} > l_O$ (where MPE stands for the Markov perfect equilibrium of our baseline strategic model and O stands for "optimal", i.e. the model featuring dynastic optimization), our proposed distance measures have been defined as follows:

1. The area between l_{MPE} and l_o: $D_1 = \int_H (l_{MPE}(h) - l_O(h))dh > 0$.
2. The minimum distance between l_{MPE} and l_O: $D_2 = \inf_{h \in H} |l_{MPE}(h) - l_O(h)| > 0$.

One crucial finding which facilitates the subsequent analysis and justifies the above definitions is that the policy functions never intersect.

For simplicity of computations, we have maintained the assumption $\beta_2 = \beta_1\gamma_1$; for comparability of our results, we have also retained the condition $\gamma_1 = \gamma_2$. This limits the scope of this sensitivity analysis exercise markedly, but our intention was not to search through the whole parameter space anyway. Even under these restrictions, we find both important departures from the baseline parametrization illustrated in Figure 2 and potentially large distances between the two policy functions.

First of all, our numerical exercise confirms that equilibrium policy functions l^* from different models indeed never intersect ($D_2 > 0$). Furthermore, the numerical results on the ordering of policy functions obtained from the strategic model and from the optimal policy ($l_{MPE} > l_O$) are obviously consistent with implications of Theorem 3. The distance between these two policy functions can vary considerably, though: under some parametrizations (such as the baseline parametrization), it is large, while under others, in particular those involving radically low δ's, it may even be close to zero.

The results of our sensitivity analysis exercise have been summarized in Table 1. The baseline parametrization is: $\alpha_1 = 0.3; \beta_1 = 0.7; \alpha_2 = 0.3; \gamma =$

Table 1: Sensitivity analysis results.

Case	D_1	D_2
Close to Baseline		
Baseline	23.7462	0.1353
$\beta_1 = 0.5$	25.9257	0.1884
$\alpha_1 = 0.6$	24.2728	0.2336
$\alpha_1 = \alpha_2 = 0.6$	13.0828	0.0215
$\alpha_2 = 0.6$	13.2903	0.0043
$\beta_1 = 0.6; \gamma = 0.8$	22.3790	0.1617
$l_{MPE} \approx l_0$: low δ		
$\alpha_1 = \alpha_2 = 0.6; \delta = 0.6$	4.0759	0.0044
$\alpha_1 = \alpha_2 = 0.6; \delta = 0.3$	0.4628	0.0004
$\delta = 0.6$	7.7896	0.0296
$\beta_1 = 0.6.\gamma = 0.8; \delta = 0.6$	6.4581	0.0361
$\delta = 0.3$	0.9392	0.0026
$\beta_1 = 0.6; \gamma = 0.8; \delta = 0.3$	0.5958	0.0027

Source: own computations.

$0.6; \beta_2 = \beta_1\gamma_1 = 0.42; \bar{H} = 100; \delta = 0.9$, just like in the previous section. Unless indicated otherwise, these parameter choices are maintained throughout the table.

7. Conclusion

The purpose of the current paper has been to accomplish the two principal tasks: (i) to show how a Markov perfect equilibrium (MPE) policy function can be computed in a model with fully-specified intergenerational interactions in human capital accumulation, within an otherwise standard discrete-time framework; (ii) to compare the outcomes of the strategic model with a benchmark model which neglects intergenerational interactions. To this end, we have proven analytically that when compared to a model with dynastic optimization, our strategic model predicts unambiguously lower equilibrium investment in human capital accumulation.

We believe that finding a constructive algorithm for computing MPE policies in models of intergenerational altruism is a significant step forward in modeling strategic linkages across generations. In this paper, we have shown that this novel tool, developed by Balbus et al. (2008), can be generalized to capture intergenerational linkages in human capital accumulation. We have shown under which conditions the MPE policy exists and is unique, we have proven its monotonicity, and also presented a workhorse example for which most calculations could be done analytically, and for which the numerical convergence of our iterative procedure to the MPE is quick and easy.

We have also presented the conditions under which the MPE labor supply policy is increasing or decreasing. These conditions are the same for the strategic and non-strategic model.

What remains to be done is, first and foremost, a generalization of the constructive algorithm for computing MPE policies into higher dimensions. This is enforced by the fact that most economic models featuring intergenerational altruism are set up with multiple choice and state variables. Another issue which ought to be dealt with is the existence of prices in a general equilibrium decentralization for both strategic and dynastic optimization models. We feel that these two steps are necessary in order to bring models with strategic interactions in human capital accumulation to the level of sophistication which is now common with models lacking such strategic interactions.

Appendix A. Definitions and proofs

Definition 2. *Let E be a real Banach space and $P \subseteq E$ be a nonempty, closed, convex set. Then:*

- *P is called a cone if it satisfies two conditions: (i) $x \in P, \epsilon > 0 \Rightarrow \epsilon x \in P$ and (ii) $x \in P, -x \in P \Rightarrow x = \theta$, where θ is a zero element of P,*
- *suppose P is a cone in E and $P^\circ \neq \emptyset$, where P° denotes the set of interior points of P, we say that P is a solid cone,*
- *every cone P in E defines an order relation $\leq$ in E as follows:*

$$x \leq y \text{ if } y - x \in P,$$

- *a cone P is said to be normal if there exists a constant $N > 0$ such that:*

$$(\forall x, y \in P) \quad \underline{\theta} \leq x \leq y \Rightarrow \|x\| \leq N\|y\|.$$

Theorem 4 (Guo et al. (2004)). *Let P be a normal solid cone in a real Banach space with partial ordering $\leq$ and $B : P \to P$ be a decreasing operator (i.e. if $l_1 < l_2 \in P$ then $Bl_2 \leq Bl_1$) satisfying:*

$$(\exists r, 0 < r < 1)(\forall l \in P^\circ), (\forall t, 0 < t < 1) \quad t^r B(tl) \leq Bl, \tag{A.1}$$

then B has a unique fixed point in P° and the following holds:

$$(\forall l_0 \in P^\circ) \quad \lim_{n\to\infty} \|l_n - l^*\| \to 0, \tag{A.2}$$

where $(\forall n \geq 1)\, l_n = B(l_{n-1})$.

Proof. of Theorem 1. The first statement is a corollary of Theorem 1 in Balbus et al. (2008).

The second statement of the theorem follows from the observation that for the given assumptions, the objective function in (4) has strictly increasing marginal returns. An application of the theorem due to Amir (1996b) and Edlin and Shannon (1998) on strict comparative statics completes the proof. □

Proof. of Example 1.

Observe that is this case elasticities of the utilities u and v as well as f are constant. Hence we may apply the Guo et al. (2004) theorem (see Theorem 4 in the Appendix) directly to the (decreasing) operator B which can be calculated explicitly for the given functions. $\square$

Proof. of Theorem 3.

Consider two families of functions parametrized by $h \in (0, \bar{H}]$, denoted as S_h,$Z_h : [0,1] \to \mathbb{R}_+$, such that for a given $h \in (0, \bar{H}]$,

$$S_h(l) = u(f(h,l)) + \delta g(h, 1-l) \int_H u(f(y, l_{MPE}(y)))\lambda(dy|h)$$

and

$$Z_h(l) = u(f(h,l)) + \delta g(h, 1-l) \int_H V(y)\lambda(dy|h),$$

where V is the value function corresponding to the Bellman equation (10).

We would like to show that for any given h, $S'_h(l) > Z'_h(l)$ in their whole domain. To this end, first note that

$$\begin{aligned} u(f(h, l_{MPE}(h))) \leq \max_{l\in[0,1]} u(f(h,l)) < \\ < \max_{l\in[0,1]}\{u(f(h,l)) + \delta g(h,1-l)\int_H V(y)\lambda(dy|h)\} = V(h). \end{aligned} \quad (A.3)$$

From the above reasoning, it immediately follows that

$$\int_H u(f(y, l_{MPE}(y)))\lambda(dy|h) < \int_H V(y)\lambda(dy|h) \quad (A.4)$$

and hence:

$$\begin{aligned} S'_h(l) = u'(f(h,l))f'_2(h,l) - \delta g'_2(h,1-l)\int_H u(f(y, l_{MPE}(y)))\lambda(dy|h) > \\ u'(f(h,l))f'_2(h,l) - \delta g'_2(h,1-l)\int_H V(y)\lambda(dy|h) = Z'_h(l), \end{aligned} \quad (A.5)$$

which completes the first part of the proof.

Now let us impose another function $T : \{1,2\} \times [0,1] \to \mathbb{R}_+$ on top of that, such that $T(1,l) = Z(l)$ and $T(2,l) = S(l)$. From inequality (A.5) we have that $T_2'(2,l) > T_2'(1,l)$, and thus T has increasing marginal returns with $i = 1,2$. For $i = 1,2$, the function $T(i,\cdot)$ defined on the lattice $[0,1]$ is thus supermodular. Hence, by the theorem due to Amir (1996b) and Edlin and Shannon (1998), we obtain that $(\forall h \in (0,\overline{H}])\, l_{MPE}(h) = \arg\max_{l\in[0,1]} T(2,l) > \arg\max_{l\in[0,1]} T(1,l) = l_R(h)$. $\square$

Proof. of Proposition 1.

In equilibrium, $\bar{l}(h) = \xi_h(l(h))$ can by defined as the right-hand side of (13).

We will now differentiate $l(h) = \xi_h^{-1}(\bar{l}(h))$ with respect to h. Observe that it is justified since ξ_h^{-1} is differentiable while from equations (12) and (13) we also have that functions η_z (where, for given $z \in [0,\infty)$, $\eta_z(h) := \xi_h^{-1}(z)$) and $\bar{l}$ are differentiable with respect to h on $(0,\bar{H})$. It is obtained that:

$$\begin{aligned}
\frac{dl(h)}{dh} &= \frac{\partial \xi_h^{-1}(\bar{l}(h))}{\partial \bar{l}(h)}\frac{\partial \bar{l}(h)}{\partial h} + \frac{\partial \xi_h^{-1}(\bar{l}(h))}{\partial h} = \\
&= \frac{1}{(1+\Xi(h))^2}\left(\frac{l(h)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_1}-1}}{1-\beta_2}\right)(\alpha_1\gamma_1-\alpha_2)\times \qquad \text{(A.6)}\\
&\times \left(1 - \frac{\frac{\beta_1\gamma_2}{1-\beta_2}\frac{\delta}{\bar{H}}\int_0^{\bar{H}} y^{\alpha_1\gamma_2}\left(\frac{\Xi(y)}{1+\Xi(y)}\right)^{\beta_1\gamma_2}\frac{1}{1+\Xi(y)}dy}{\frac{\delta}{\bar{H}}\int_0^{\bar{H}} y^{\alpha_1\gamma_2}\left(\frac{\Xi(y)}{1+\Xi(y)}\right)^{\beta_1\gamma_2}dy}\right),
\end{aligned}$$

with $\Xi(y) \equiv \bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}$. Since $\beta_1\gamma_1 = \beta_2$, and by assumption, $1 > \beta_1(\gamma_1+\gamma_2)$, it follows that $\frac{\beta_1\gamma_2}{1-\beta_2} < 1$ and thus the ratio of two integrals in the last parenthesis is smaller than one, we find the expression in the last parenthesis to be positive. In conclusion, $\frac{dl(h)}{dh} > 0$ and thus $l(h)$ is increasing in its domain iff $\alpha_1\gamma_1 > \alpha_2$, $\frac{dl(h)}{dh} < 0$ and thus $l(h)$ is decreasing in its domain iff $\alpha_1\gamma_1 < \alpha_2$, and $l(h)$ is constant iff $\alpha_1\gamma_1 = \alpha_2$. $\square$

Appendix B. The model of joy-of-giving altruism

The model with joy-of-giving altruism (and, to guarantee direct comparability, with a stochastic transition in human capital levels) can be generally specified as:

$$\max_{\hat{l}\in[0,1]} u(f(h,\hat{l})) + \int_H v(y)G(dy;h,1-\hat{l}). \qquad \text{(B.1)}$$

The crucial difference between this model and the workhorse model of the current paper consists in the fact that here, parents' utility depends directly on their children's human capital and not on their consumption ($v(h_{t+1})$ instead of $v(c_{t+1})$).

Concentrating on Markovian policies, the first order condition for optimal labor supply $l(h)$ is given by:

$$u'(f(h,l(h)))f_2'(h,l(h)) = g_2'(h,1-l(h)) \int_H v(y)\lambda(dy|h), \tag{B.2}$$

guaranteeing that the marginal utility of consumption acquired thanks to an extra unit of time devoted to work is exactly equal to the expected marginal cost in terms of lost human capital of the next generation.

Example 4. *Let* $u(c) = c^{\gamma_5}$, $v(h') = (h')^{\gamma_6}$, $f(h,l) = h^{\alpha_5} l^{\beta_5}$, $g(h, 1-l) = \frac{1}{\bar{H}^{\alpha_6}} h^{\alpha_6}(1-l)^{\beta_6}$. *From* (B.2), *we obtain the first order condition for the optimal policy* $l(h)$. *It is given as an implicit solution to the equation:*

$$\frac{l^{1-\beta_5\gamma_5}}{(1-l)^{1-\beta_6}} = \frac{\beta_5\gamma_5}{\delta\beta_6}(1+\gamma_6)\bar{H}^{\alpha_6-\gamma_6} h^{\alpha_5\gamma_5-\alpha_6}. \tag{B.3}$$

Using the implicit function theorem, it is straightforward to show that $l(h)$ is everywhere decreasing whenever $\alpha_6 > \alpha_5\gamma_5$ and everywhere increasing whenever $\alpha_6 < \alpha_5\gamma_5$. In the special case where $\alpha_5\gamma_5 = \alpha_6$, (B.3) implies that $l(h)$ is constant, independent of h. This finding is crucial here because it is an exact analogue to Proposition 1 and an equivalent proposition which holds for the dynastic model: whenever the MPE labor supply policy of the model with strategic interactions is decreasing/increasing, it is also decreasing/increasing in the model with "joy-of-giving" altruism.

Just like in Example 2, the above equation (B.3) can be solved for $l^*(h)$ explicitly in the special case $\beta_5\gamma_5 = \beta_6$. In such case,

$$l^*(h) = \frac{\left(\frac{\gamma_6+1}{\delta}\right)^{\frac{1}{1-\beta_6}} \bar{H}^{\frac{\alpha_6-\gamma_6}{1-\beta_6}} h^{\frac{\alpha_5\gamma_5-\alpha_6}{1-\beta_6}}}{1+\left(\frac{\gamma_6+1}{\delta}\right)^{\frac{1}{1-\beta_6}} \bar{H}^{\frac{\alpha_6-\gamma_6}{1-\beta_6}} h^{\frac{\alpha_5\gamma_5-\alpha_6}{1-\beta_6}}}. \tag{B.4}$$

For the highest available level of comparability, one has to impose $\gamma_6 = \beta_1\gamma_2$ in order to equalize the elasticities of h' in both utility functions. The functions themselves remain different, though.

Abel, A. B., Warshawsky, M., 1987. Specification of the joy of giving: Insights from altruism. Review of Economics and Statistics 70, 145–149.

Abreu, D., Pearce, D., Stacchetti, E., 1990. Toward a theory of discounted repeated games with imperfect monitoring. Econometrica 58 (5), 1041–1063.

Amir, R., 1996a. Continuous stochastic games of capital accumulation with convex transitions. Games and Economic Behavior 15, 111–131.

Amir, R., 1996b. Sensitivity analysis of multisector optimal economic dynamics. Journal of Mathematical Economics 25, 123–141.

Amir, R., 1996c. Strategic intergenerational bequests with stochastic convex production. Economic Theory 8, 367–376.

Amir, R., 1997. A new look at optimal growth under uncertainty. Journal of Economic Dynamics and Control 22 (1), 67–86.

Amir, R., 2002. Stochastic games in economics and related fields: an overview. In: Neyman, A., Sorin, S. (Eds.), Stochastic games. NATO Advanced Science Institutes Series D: Behavioural and Social Sciences. Kluwer in cooperation with NATO Scientific Affairs, Dordrecht, Boston.

Arrondel, L., Masson, A., 2006. Altruism, exchange or indirect reciprocity: What do the data on family transfers show. In: Handbook of the Economics of Giving, Altruism and Reciprocity. Elsevier B.V.

Artige, L., Camacho, C., de la Croix, D., 2004. Wealth breeds decline: Reversals of leadership and consumption habits. Journal of Economic Growth 9, 423–449.

Balbus, Ł., Reffett, K., Woźny, Ł., 2008. A constructive geometrical approach to the uniqueness of markov perfect equilibrium in stochastic games of intergenerational altruism, manuscript, Arizona State University.

Becker, G. S., Tomes, N., 1986. Human capital and the rise and fall of families. Journal of Labor Economics 4, S1–S39.

Ben-Porath, Y., 1967. The production of human capital and the life cycle of earnings. Journal of Political Economy 75, 352–365.

Bernheim, B. D., Ray, D., 1983. Altruistic growth economies. i, existence of bequest equilibria. Tech. rep., IMSSS Technical Report No. 419. Stanford University.

Bernheim, B. D., Ray, D., 1989. Markov perfect equilibria in altruistic growth economies with production uncertainty. Journal of Economic Theory 47 (1), 195–202.

Bernheim, D., Ray, D., 1987. Economic growth with intergenerational altruism. Review of Economic Studies 54, 227–242.

Bernheim, D., Ray, D., Yeltekin, S., 1999. Self-control, savings, and the low-asset trap, stanford University.

Bertola, G., Foellmi, R., Zweimueller, J., 2006. Income Distribution in Macroeconomic Models. Princeton University Press, Princeton.

Bruhin, A., Winkelmann, R., 2009. Happiness functions with preference interdependence and heterogeneity: The case of altruism within the family. Journal of Population Economics forthcoming.

Chari, V. V., Hopenhayn, H., 1991. Vintage human capital, growth, and the diffusion of new technology. Journal of Political Economy 99, 1142–1165.

Coleman, W. J., 1991. Equilibrium in a production economy with an income tax. Econometrica 59 (4), 1091–1104.

Coleman, W. J., 2000. Uniqueness of an equilibrium in infinite-horizon economies subject to taxes and externalities. Journal of Economic Theory 95, 71–78.

Dasgupta, P., 1974a. On some alternative criteria for justice between generations. Journal of Political Economy 3, 405–423.

Dasgupta, P., 1974b. On some problems arising from professor rawls' conception of distributive justice. Theory and Decision 4, 325–344.

Edlin, A. S., Shannon, C., July 1998. Strict monotonicity in comparative static. Journal of Economic Theory 81 (1), 201–219.

Galor, O., Tsiddon, D., 1997. The distribution of human capital and economic growth. Journal of Economic Growth 2, 93–124.

Guo, D., Cho, Y. J., Zhu, J., 2004. Partial ordering methods in nonlinear problems. Nova Science Publishers, Inc., New York.

Guo, D., Lakshmikantham, V., 1988. Nonlinear problems in abstract cones. Academic Press, Inc., San Diego.

Judd, K., 1998. Numerical Methods in Economics. The MIT Press, Cambridge, Massachusetts.

Klein, P., Krusell, P., Ríos-Rull, J.-V., 2008. Time-consistent public policy. Review of Economic Studies 75 (3), 789–808.

Klein, P., Ríos-Rull, J.-V., 2003. Time-consistent optimal fiscal policy. International Economic Review 44 (4), 1217–1245.

Klein, P., Vincenzo, Q., Ríos-Rull, J.-V., 2005. Optimal time-consistent taxation with international mobility of capital. Advances in Macroeconomics 5 (1), 1142–1142.

Kohlberg, E., 1976. A model of economic growth with altruism between generations. Journal of Economic Theory 13, 1–13.

Krusell, P., Smith, A., 2003. Consumption–savings decisions with quasi–geometric discounting. Econometrica 71 (1), 365–375.

Kydland, F., Prescott, E., 1980. Dynamic optimal taxation, rational expectations and optimal control. Journal of Economic Dynamics and Control 2 (1), 79–91.

Kydland, F. E., Prescott, E. C., 1977. Rules rather than discretion: The inconsistency of optimal plans. Journal of Political Economy 85, 473–491.

Laibson, D., 1997. Golden eggs and hyperbolic discounting. Quarterly Journal of Economics 112 (2), 443–77.

Lane, J., Leininger, W., 1986. On price characterization and pareto-efficiency of game equilibrium growth. Journal of Economics 46, 346–367.

Lane, J., Mitra, T., 1981. On nash equilibrium programs of capital accumulation under altruistic preferences. International Economic Review 22 (2), 309–31.

Leininger, W., 1986. The existence of perfect equilibria in model of growth with altruism between generations. Review of Economic Studies 53 (3), 349–368.

Lochner, L., 2008. Intergenerational transmission. In: Durlauf, S. N., Blume, L. E. (Eds.), The New Palgrave Dictionary of Economics. Second Edition. Palgrave Macmillan.

Loury, G. C., 1981. Intergenerational transfers and the distribution of earnings. Econometrica 49, 843–867.

Magill, M., Quinzii, M., 2009. The probability approach to general equilibrium with production. Economic Theory 39 (1), 1–41.

Mincer, J., 1958. Investment in human capital and personal income distribution. Journal of Political Economy 66, 281–302.

Mirman, L. J., Morand, O. F., Reffett, K. L., 2008. A qualitative approach to markovian equilibrium in infinite horizon economies with capital. Journal of Economic Theory 139 (1), 75–98.

Montrucchio, L., 1998. Thompson metric, contraction property and differentiability of policy functions. Journal of Economic Behavior & Organization 33 (3-4), 449–466.

Nowak, A. S., 2003. On a new class of nonzero-sum discounted stochastic games having stationary nash equilibrium points. International Journal of Game Theory 32, 121–132.

Nowak, A. S., 2006. On perfect equilibria in stochastic models of growth with intergenerational altruism. Economic Theory 28, 73–83.

Nowak, A. S., 2007. On stochastic games in economics. Mathematical Methods of Operations Research 66 (3).

Orazem, P., Tesfatsion, L., 1997. Macrodynamic implications of income-transfer policies for human capital investment and school effort. Journal of Economic Growth 2, 305–329.

Peleg, B., Yaari, M. E., 1973. On the existence of a consistent course of action when tastes are changing. Review of Economic Studies 40 (3), 391–401.

Phelps, E., Pollak, R., 1968. On second best national savings and game equilibrium growth. Review of Economic Studies 35, 195–199.

Rangazas, P. C., 2000. Schooling and economic growth: A king-rebelo experiment with human capital. Journal of Monetary Economics 46, 397–416.

Santos, M. S., 1994. Smooth dynamics and computation in models of economic growth. Journal of Economic Dynamics and Control 18 (3-4), 879–895.

Stokey, N., Lucas, R., Prescott, E., 1989. Recursive methods in economic dynamics. Harward University Press.

Strotz, R. H., 1955. Myopia and inconsistency in dynamic utility maximization. Review of Economic Studies 23 (3), 165–180.