Distributional equilibria in dynamic supermodular games with a measure space of players and no aggregate risk*

Łukasz Balbus† Paweł Dziewulski‡ Kevin Reffett§ Łukasz Woźny¶

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Abstract

We study a class of discounted infinite horizon stochastic games with strategic complementarities with a continuum of players. In order to analyse transition of private signals to aggregate distributions, we develop a dynamic law of large numbers, that implies a.o. no aggregate uncertainty. We define a suitable equilibrium concept, namely: Markov Stationary Equilibrium and prove its existence under a new set of assumptions, via constructive methods. Our construction allows to overcome some problems in characterizing beliefs and dynamic complementarities in a class of games recently studied by Mensch (2018b). In addition, we provide constructive monotone comparative dynamics results for ordered perturbations of the space of games (extending those of Acemoglu and Jensen (2015); Light and Weintraub (2019) from steady states or invariant measures to dynamic equilibria). For this end, we use our new fixed point comparative statics theorem, suitable for comparing equilibrium objects in spaces of distributions or measurable functions. Finally, we discuss the relation of our result to the literature on equilibria in oblivious strategies of e.g. Adlakha, Johari, and Weintraub (2015); Weintraub, Benkard, and Van Roy (2008) and recent works on large but finite dynamic games (Kalai and Shmaya, 2018) providing approximation of our large game by its small counterparts. We provide numerous examples including social

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†Faculty of Mathematics, Computer Sciences and Econometrics, University of Zielona Góra, Poland.

‡Department of Economics, University of Sussex, UK.

§Department of Economics, Arizona State University, USA.

¶Department of Quantitative Economics, Warsaw School of Economics, Warsaw, Poland. Address: al. Niepodległości 162, 02-554 Warszawa, Poland. E-mail: lukasz.wozny@sgh.waw.pl.

dissonance models, dynamic global games, keeping up with the Joneses or OLG economies with intergenerational interactions.

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1 Introduction

Beginning with the seminal work in general equilibrium theory of Aumann (1964) and game theory of Schmeidler (1973) and Mas-Colell (1984), economists have long been interested in studying the equilibrium interaction of large populations of agents[1] The initial motivation for such models in Aumann's work, for example, was to make precise the concept of price-taking behaviour in an Arrow-Debreu-McKenzie (ADM) model of Walrasian competitive equilibrium. This formulation of competitive general equilibrium addressed some important epistemic questions arising in the original formulations of ADM models, where it was assumed there were a finite number of agents, yet competitive equilibrium was modelled as if each agent believed their net demand decisions were infinitesimal relative to their influence on equilibrium prices.

In such a setting of competitive equilibrium, a new question that naturally arises concerns the relationship between the structure of competitive equilibrium for ADM models when the set of agents is finite but sufficiently large vs. the situation where the set of agents was a continuum. Indeed, this question became the focal point of an interesting line of research concerning the strategic foundations of general equilibrium theory (e.g., see the monograph of Gale (2000) for a detailed discussion of this question, as well as a survey of many important developments in this literature). But similar studies of the limiting behavior of equilibrium have arise in game theory following the work of Schmeidler and MasColell, where the question of concern has been the relationship between the set of Nash equilibrium in normal-form games where the population was large but finite vs. the continuum case. Along these lines, a key recent theoretical development has been the development of idealized limit games. In the study of idealized limit games, the convergence for finite player games to games with a measure space of players case is made mathematically precise as a closed-graph property of the equilibrium set as the number of agents becomes large. Similar questions have arisen in the literature on mean-field approximations to equilibrium in finite player games.[2]

Subsequent to this initial work on large economies and large games, the study of large

[1]It is perhaps important to note that at a technical level, one should view the social equilibrium concept of Debreu (1952) as being the outcome of a generalized Nash game, where each player/agent faces a budget constraint. See Dasgupta and Maskin (2015) for a nice discussion of this fact. In many contexts, the two notions of equilibria in models with a measure space of players/agents can be directly related.

[2]See the recent work of Qiao and Yu (2014), Qiao, Yu, and Zhang (2016), Nutz and Zhang (2019), and Kalai and Shmaya (2018) for the study of idealized limit games in complete and incomplete information games. Also, see He, Sun, and Sun (2017) and He and Sun (2018) for a discussion of related issues.

dynamic economies and large anonymous sequential games with a measure space of agents was initiated. Large dynamic economics were introduced in the work of Bewley (1986), where the concept of a stationary equilibrium was first proposed and studied. Subsequent to this work, there has been a plethora of applications of related models in many fields in economics. For example, a stationary equilibrium in a prototype Bewley models is a stochastic steady state distribution over asset-idiosyncratic states for a population of agents modeled as a measure space, where this stationary distribution has the following properties: (a) given a fixed set prices (e.g., wages, interest rates, etc.), each ex-ante identical agent solves an income fluctuation problem subjected to idiosyncratic risk while facing incomplete financial markets (so they each fluctuate over time within the stochastic steady state distribution relative to their asset and idiosyncratic states, and (b) the prices parameterizing this collection of income fluctuation problems "self-generates" in equilibrium (i.e., the stationary distribution is the fixed point of an adjoint Markov operator whose kernel is parameterized by the optimal solution from the typical agents income fluctuation problem).

As becomes clear when studying such models, the dynamic "exact law of large numbers" for a continuum of random variables plays a critical role in the formulization of the stationary equilibrium problem. For example, a dynamic exact law of large numbers is needed to guarantee that the law of motion for the state variable in the model (a measure for asset-idiosyncratic states) is not a random measure, so the model in a stationary equilibrium (as well as in principle its equilibrium dynamics) involve no aggregate risk.[3] This model, and its various other incarnations, has become a workhorse model in macroeconomics, having also been extended to settings with overlapping generations settings where agents have realistic lifecycle (e.g., Huggett (1997)). Since this initial work, the large dynamic economy approach to stationary equilibrium as been applied in other areas of economics, including the entry and exit models in the spirit of Hopenhayn (1992) in industrial organization and the theory of endogenous innovation, dynamic equilibrium search and matching models in the spirit of Diamond-Mortensen-Pissarides in labour economics. For the case of large dynamic economies with aggregate risk, models in the spirit of Krusell and Smith (1998) have been dominant in macroeconomics.

Additionally, in game theory, a parallel line of studying sequential equilibrium in large anonymous games with a measure space of players first proposed in a series of papers by Jovanovic and Rosenthal (1988) for the case of no aggregate risk, and Bergin and Bernhardt (1992) and Bergin and Bernhardt (1995)) for the case of aggregate risk. As is well-known, in many cases, there exists a very close relationship between large anonymous games and large dynamic economies, so the tools of analysis in large dynamic economies and large anonymous games are closely related.[4] Finally, there has been many recent papers studying the social interaction of agents such as equilibrium models on conformism, identity and aspirations, both in static and dynamic

[3]See Huggett (1993) and Aiyagari (1994), for example, for a discussion.

[4]In Acemoglu and Jensen (2015, 2018), the relationships between large dynamic economies and large anonymous games is discussed. It is important to note that many large dynamic economies can be viewed as large aggregative games (e.g. Bewley-Huggett-Aiyagari models).

settings. In this work, for the sake of tractability, often these models assume a measure space of agents.[5]

Recently, the interest in the study of large games has been given a new motivation, as a tool to facilitate the characterization of equilibrium interaction for economies with a finite but "sufficiently large" economies/games. In these sorts of applications, this problem is that in versions of the game with a finite number of players/agents, the structure of equilibrium interaction is difficult to characterize. This makes it very difficult to analyse the models no only theoretically, but quantitatively.[6] This then makes the large economy serve as a type of asymptotic approximation tool used to characterize the distributional equilibrium structure that economies and/or games with a large number of agents will converge to. Such applications of large games in the literature include characterizing estimators in econometric models of discrete choice games, models of endogenous network formation, and matching models, as well as theoretical models of social interactions, games with spillovers, public goods, and complementarities, mean-field games, dynamic models of industry organization, distribution of income and savings in macroeconomics with incomplete markets, and games with oblivious equilibrium.[7]

In this paper, we reconsider many of these questions in the context of an important subclass of large anonymous stochastic games, namely those with strategic complementarities. In particular, we develop a new class of large stochastic supermodular games where we can develop sufficient conditions for the existence, characterization, and comparative statics and comparative dynamics of Markovian distributional equilibrium. Our methods are constructive, and we show that our results extend the results on stationary equilibrium for many interesting large dynamic economies to the setting of Markovian distributional equilibrium (hence, we can characterize and compare equilibrium *transition* paths).

In particular, we identify new sufficient conditions for the existence of extensive-form large

[5] A few examples of this extensive literature includes Bernheim (1994) for models of conformism, Genicot and Ray (2017) for models of poverty with aspiration preferences, Akerlof and Kranton (2000) and Bisin, Moro, and Topa (2011) for models of identity and culture, and Kim, Tertilt, and Yum (2019) for models of status with "Keeping up with the Joneses" preferences.

[6] A prototype of this problem is estimating endogenous network formation games or matching models. For example, see Menzel (2015, 2016) and Mele (2017) for a discussion of the relationship between the "size" of the network, the number of agents, and the quantitative tractability of resulting model. Similar issues arise in estimating social interaction agems (e.g., Brock and Durlauf (2001) and Blume, Brock, Durlauf, and Jayaraman (2015)).

[7] Here, examples include the pioneering work on mean-field/oblivious equilibrium in Weintraub, Benkard, and Van Roy (2008), Adlakha and Johari (2013), Adlakha, Johari, and Weintraub (2015) in games, the mean-field approximation to stationary equilibrium in large dynamic macroeconomies as in the applications of Nuno and Moll (2018) and Achdou, Han, Lasry, Lions, and Moll (2019), theoretical work in econometrics that estimate strategic models as discussed in Menzel (2016) for general games, Mele (2017) and Xu (2018) for estimating models of endogenous network formation, Menzel (2016) for estimating matching models, among many other recent applications of related tools. In much of this latter work, graphons are used to formalize the continuum game (e.g., see Parise and Ozdaglar (2018) for a discussion of graphon games). Such a literature is also related to important work on estimating large social interaction models such as Brock and Durlauf (2001) and Blume, Brock, Durlauf, and Jayaraman (2015), as well as population games with public goods and spillovers in Cheung and Lahkar (2018) and Lahkar and Mukherjee (2019). Many of these applications involve situations that allow for social complementarities, and can be formulated as games of strategic complements.

stochastic supermodular games, where our games remain stochastically supermodular at both finite and infinite horizons. In doing so, we are able to develop a new set of order theoretic tools for characterizing the order structure of the set of Markovian distributional equilibrium. This allows us to extend the results of the theory of stochastic supermodular games to a very general setting. We also show that the use of a measure space of agents avoids some critical problems with extensive-form stochastic supermodular games that can arise in situation where there are a finite number of players. Given the recent interest in the economics literature on such notions of equilibrium as oblivious equilibrium, mean-field equilibrium, as well as stationary equilibrium, we are able to develop a rigourous set of new results concerning the structure of "idealized limits" of our class of stochastic games, where the set of equilibria in finite player versions of our game are compared with their counterpart in the game with a measure space of players. This allows us to make precise the sense in which our large stochastic supermodular game with a measure space of players is related to similar games with a finite number of players. This latter result is particularly useful in applications, as in some settings, large dynamic economies are used as a tool to characterize the structural properties of finite player settings.

To prove our results, we must formally address an important technical question: proving an appropriate dynamic exact law of large numbers (D-ELLN) which we can apply to our class of stochastic games. Our results per this question are of independent interest, and can be applied in many dynamic settings where there exists idiosyncratic risk at a micro level that evolves dynamically but generates no aggregate risk. We then ask how do those new sufficient conditions relate to their finite player counterparts, both for models with and without complete information. We then discuss the applications of our results to various models in the literature.

Although characterizing strategic complementarities in dynamic settings has played a critical role in many literatures in economics, it is well-known that keeping an dynamic/stochastic games extensive-form supermodular is not a trivial matter. This fact has been made clear in the context of finite player dynamic/stochastic games of complete information games in a series of papers such as Curtat (1996), Echenique (2004a), Amir (2002, 2005), and Vives (2009), among others. Additional complications for finite player dynamic/stochastic games of incomplete information has been made by Balbus, Reffett, and Woźny (2013) and Mensch (2018b). In each of these situations, one complication is characterizing of sufficient conditions for the game in equilibrium to possess requisite "single crossing in continuation distribution" of the game.[8] Such a single crossing conditions can arise even in static games such as complete information with a continuum of players (e.g., see Balbus, Dziewulski, Reffet, and Woźny (2019); Balbus, Reffett, and Woźny (2015)) as well as when studying the existence of pure strategy monotone equilibria in Bayesian games,[9] or pure strategic equilibrium in Bayesian supermodular games.[10]

[8]Actually, characterizing sufficient conditions for single crossing in distribution with respect to beliefs in static large, Bayesian games with strategic complementarities is a challenge. See, for example, Balbus, Dziewulski, Reffet, and Woźny (2015) and Liu and Pei (2017).

[9]Cite Athey, McAdams, Reny, etc

[10]Cite Van Zandt Vives, our ET paper, Liu and Pei.

Important progress on these problems has been made in the context of stochastic games with complete information in papers by Curtat (1996) and Amir (2002). [11] Unfortunately, in both of those papers, although the class of stochastic supermodular games remain extensive-form supermodular for *finite* horizons, the games are not show to be extensive-form supermodular for infinite horizons, and topological (as opposed to order-theoretic) fixed-point methods must be applied to verify the existence of Markovian equilibrium. In the recent contribution of Balbus, Reffett, and Woźny (2014), sufficient conditions for infinite horizon stochastic games to remain extensive-form supermodular are provided. These results are extended to belief-free equilibrium in infinite horizon stochastic games with imperfect information in Balbus, Reffett, and Woźny (2013). These two papers in particular identify general conditions on the primitives under which the extensive form of a stochastic game remains supermodular.

In this paper, we address this issue systematically in the context of large stochastic games without aggregate risk. As a benchmark, we consider the setting of Jovanovic and Rosenthal (1988) for the case of large stochastic supermodular game. Recall, the setting studied in Jovanovic and Rosenthal (1988) is as follows: each period, each player observes its individual state and (simultaneously with the others) choose its action. The player's payoff depends on both its state and action as well as the distribution on all players states-actions. As their game is anonymous at each stage, each player does not see the impact of its actions on the aggregate distribution. Individual states are drawn each period from the distribution parameterized by player's state, action and distribution of all players on state-action space. The equilibrium concept is (nonstationary) distributive Nash equilibria, i.e. sequences of probability distribution on action-states such that almost every player is playing optimally taking action-states distributions sequence as given. Importantly, their game concerns the economy without aggregate uncertainty. That is, although each player's state is random, the uncertainty vanishes at the aggregate level. Such result is very useful, when one tries to establish distributive equilibrium existence via dynamic programming. Specifically, when distribution of players-states is constant and known to players it is not an argument of the value function but only a parameter that each player is best responding to. To establish that uncertainty vanishes at the aggregate level, Jovanovic and Rosenthal (1988) assumed the existence of an dynamic Exact law of large numbers for a continuum of random variables (e.g., a dynamic version of the Feldman and Gilles (1985) construction of a continuum of (dependent) random variables measurable with respect to players and random variables expressing individual uncertainty), and proceeded. But similar, more recent approaches to this formalization could have alternatively been employed.[12]

In a series of important papers, Bergin and Bernhardt (1992, 1995) generalize Jovanovic and Rosenthal (1988) game to the case of aggregate uncertainty. That is, their game allows for randomness of the distribution of states at the aggregate level. Specifically Bergin and

[11] In the context of a dynamic games that are not necessarily stochastic, see also Vives (2009).

[12] For additional results in this literature, see Karatzas, Shubik, and Sudderth (1994), Yang (2017), and Yang (2019).

Bernhardt (1992) and Bergin and Bernhardt (1995) add a Markovian shock to the economy that parameterize the current distribution on players-states. Then, knowing the aggregate state each player knows the (aggregate) distribution of players-states, appealing to a continuum of individual random variables possessing these properties (e.g., in their case Feldman and Gilles (1985)), the question of existence of sequential equilibrium can be addressed. Generalization provided by Bergin and Bernhardt (1992) and Bergin and Bernhardt (1995) is important as it allows to analyze economies, where the aggregate uncertainty plays a role (e.g. models of IO where competitive firm behavior is evolving over the business cycle).

In both papers, the main objective of the work is to prove the *existence* of sequential equilibrium in some appropriate context (e.g., sequential distributional equilibrium as in Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), Bergin and Bernhardt (1995)). In each case, per the question of existence, topological fixed point results of Fan-Glicksberg. Further Bergin and Bernhardt (1995) end up being the appropriate tool of analysis generalize these early results by analyzing existence of Markovian distributive equilibrium, including continuation value as a state variable. This approach can be seen as a link between large games literature, and the one on stochastic game originated by Shapley (1953)[13]. This link can be also seen in Chakrabarti (2003) paper, who shows existence of a NE in an infinite horizon stochastic game using Schmeidler (1973) equilibrium definition but, when players' payoffs depend only on the average of other strategies. His proof is based on the Mertens and Parthasarathy (1987) or Abreu, Pearce, and Stacchetti (1990) correspondence based continuation method, but applied in spaces of average values. One critical issue with all this theoretical work is that because of the general nature of the large (stochastic) game, little more than existence questions can be studied. That is, there is very little characterization of dynamic stochastic equilibrium.

In this paper, we analyze a large anonymous game as in this literature, but we focus our attention on two important additional questions: (a) the question of the existence of Markovian stationary Nash equilibrium (that induce distribution on action and types) that is additionally (b) monotone in private type. Specifically, we study a class of discounted infinite horizon stochastic games with strategic complementarities with a continuum of players. Our game is specified for no aggregative uncertainty case, hence, we first provide a new, dynamic version of the exact law of large numbers. We next define our concept of Markov Stationary equilibrium, that involves a equilibrium action-state distribution and an law of motion of aggregate distributions. This resembles recursive competitive equilibrium concept[14]. We then prove existence of such equilibrium under a new[15] set of assumptions via constructive methods. We also provide a monotone comparative statics and dynamics results as well as provide a behavioral foundation of our equilibrium concept studying approximation of our game via its small counterparts.

[13]And recently developed by Nowak and Raghavan (1992), Amir (1996) and Balbus, Reffett, and Woźny (2014).

[14]Indeed, we report recent progress on existence of recursive competitive equilibria in dynamic economies by Datta, Reffett, and Woźny (2018).

[15]Different from those of Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), Sleet (2001) or Chakrabarti (2003).

2 Related Literature

This paper contributes to many strands of the recent literature in economics. For example, it contributes to a large literature on the existence and characterization of Nash equilibrium in games with a measure space of players in both the setting of complete information and incomplete information. Aside from the original papers of Schmeidler (1973) and Mas-Colell (1984), more recent papers in this literature include the contributions by Carmona and Podczeck (2009), Jara-Moroni (2012), Khan, Rath, Sun, and Yu (2013, 2015), Khan, Rath, Yu, and Zhang (2013), He and Sun (2018), Khan and Zhang (2018), Haifeng and Wu (2019), **?**, and Luo, Qian, and Qu (2019), among others. See also the survey by Khan and Sun (2002). Relative to this large literature, first we study Markovian distributional equilibria large anonymous games. In particular, our paper provides the first sufficient conditions for the existence of least and greatest Markovian distributional equilibria in large anonymous stochastic games with a measure space of players. Importantly, we are also able to extend the existing results on *computable* equilibrium comparative statics in large static GSC reported in Balbus, Dziewulski, Reffet, and Woźny (2015, 2019), Balbus, Reffett, and Woźny (2015), and to stationary Markovian distributional equilibrium comparative dynamics. As stationary equilibrium in the context of large dynamic economies are just special cases of Stationary Markovian distributional equilibria (namely, the stationary points of their equilibrium dynamics), our results is a direct extension of these results, only proving the existence of least and greatest equilibrium *transition paths,* as well as allowing for computable equilibrium comparisons along these equilibrium trajectories from initial conditions. The additional structure is very important when testing dynamic models with complementarities in applications (e.g., to apply the quantile estimation tools developed in Echenique and Komunjer (2009) and Echenique and Komunjer (2013)).

Our work is also closely related to the line of research in large anonymous sequential games that began in a series of papers by Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), Karatzas, Shubik, and Sudderth (1994), and Bergin and Bernhardt (1995). [16] In particular, our work focus on the case of stationary Markovian distributional equilibrium for the case of large anonymous stochastic supermodular games. Relative to the question of existence and characterization of sequential distributional equilibrium, we (a) prove the existence of (minimal state space) Markovian equilibrium, (b) provide characterizations of the set of Markovian distributional equilibrium (i.e., it is countably chain complete), and (c) provide methods for *computing* equilibrium comparative dynamics and comparative statics.

Further, as work relates to this literature on large anonymous sequential stochastic games, it is also related to an number of important literature on the existence and characterization of *dynamic* exact law of large numbers (D-ELLN) that underpin all large anonymous stochastic

[16] For recent contributions, see the work of Carmona, Delarue, and Lacker (2017), Nutz (2018), Lacker (2018), Cheung and Lahkar (2018), Yang (2019), Light and Weintraub (2019), and Lahkar and Mukherjee (2019), among others. The work of Lahkar (2013) in this literature on large Bayesian supermodular games is of particular interest to note.

games and large dynamic economies. That is, the appropriate D-ELLN question in our paper is related to the one that arises in any large anonymous games as Jovanovic and Rosenthal (1988) and Bergin and Bernhardt (1992), or large dynamic economy such as Bewley models macroeconomics and various models industrial organization where there is a continuum of players/agents. In all of these settings, agents/players draw a individual source of uncertainty such that in the aggregate does not generate risk.[17] Relative to this extensive literature, we provide a new characterization of a D-ELLN that provides a conditional *independence* of player types (relative to histories of the game), and a deterministic transition of aggregate distribution on types using rich Fubini extensions in saturated or superatomless measure spaces of players.[18] This is not a mere technical detail; rather, in our setting, given the strategic interaction between players, our equilibrium construction cannot even proceed without an appropriate D-ELLN. In this sense, our work also contributes to the large literature on ELLNs and saturated measures, but in a dynamic context. In particular, our construction builds upon the important recent contributions of Sun (2006), Keisler and Sun (2009), Podczeck (2010), He, Sun, and Sun (2017), and He and Sun (2018).

This paper also extends the class of stochastic supermodular games and games of strategic complementarities (GSC) to a dynamic setting with a measure space of players. The tools related to games with strategic complementarities have found extensive applications in economics, both for games of complete information, as well as Bayesian games with incomplete information. The class of supermodular games was first introduced in operations research by Topkis (1978) and Veinott (1992), and in economics by Vives (1990) and Milgrom and Roberts (1990). Ordinal generalizations of this class of games of strategic complementarities (GSC) are provided in Shannon (1990), Veinott (1992) and Milgrom and Shannon (1994). The results were extended to Bayesian settings by Vives (1990), Vives and Van Zandt (2007) and Van Zandt (2010). A majority of this literature has focused on normal-form GSC with a finite number of players.[19] In many recent papers including Yang and Qi (2013), Balbus, Dziewulski, Reffet, and Woźny (2019); Balbus, Reffett, and Woźny (2015), and Bilancini and Boncinelli (2016), the class of supermodular games and GSC has been extended to situations of normal-form games with complete information and a measure space of players for both the case of distributional equilibria (ala Mas-Colell) and strategic equilibria (ala Schmeidler). For similar results for distributional/strategic Bayesian Nash equilibrium in games with incomplete information with a measure space of players, see Vives (1990) and Balbus, Dziewulski, Reffet, and Woźny (2015). So our work directly relates to this extensive literature.

One central goal of this paper is to extend the theory of supermodular games to a dynamic

[17]See Acemoglu and Jensen (2015) and Acemoglu and Jensen (2018) for an extensive discussion of the literature on large dynamic economies with idiosyncratic risk and no aggregate uncertainty.

[18]In much of the early literature, researcher just relied upon the construction of Feldman and Gilles (1985), among other approaches that do not focus on saturated measures. We follow the suggestion in Sun (2006), Keisler and Sun (2009), and Podczeck (2010), and make extensive use of saturation in our D-ELLN construction.

[19]An important exception is Vives (1990) where he considers versions of games of strategic complementarities with a continuum of players.

setting with a measure space of players. The original papers studying the extension of supermodular games to a dynamic setting include the recent literature on stochastic supermodular games with a finite number of players. This literature includes work for both complete information games such is studied in Curtat (1996), Amir (2002, 2005), and Balbus, Reffett, and Woźny (2014), as well as incomplete information settings studied in Balbus, Reffett, and Woźny (2013) and Mensch (2018b). This paper directly relates to this literature in many ways. First, the tools used in the current paper heavily extend that developed by Balbus, Reffett, and Woźny (2013, 2014) for the constructive study of Markovian equilibria for the finite number of players game. Relative to the results in this literature, this paper also build its results as fixed points of monotone operators on the space of functions and distributions and also use some techniques developed for large static games.[20] In doing so, we provide new sufficient conditions for preserving dynamic complementarities between the periods to player's value functions within the context of as large stochastic supermodular game. The conditions are different than those in stochastic supermodular games with a finite number of players, and importantly avoid many of the issues related to the existence of infinite horizon extensive-form supermodular games discussed in Echenique (2004b), Amir (2002, 2005). These new conditions imply when then value function has increasing differences between private type and aggregate distribution (which includes the *belief structure* of each player on the distribution of types), we obtain increasing best replies (where "increasing" is appropriately defined in our context).[21] Very importantly, with such sufficient structure in place, our large stochastic supermodular games remain extensive-form supermodular over the *infinite horizon*. This is absolutely critical for our equilibrium comparative dynamic/statics results.[22] In this sense, our paper extends the results in Balbus, Reffett, and Woźny (2014) to the case of a measure space of players, but the results are very different also. One the one hand, in Balbus, Reffett, and Woźny (2014), the authors needed a specific form of the stochastic transition to obtain their results. In this paper, because of the presence of a measure space of players, our results hold for a more general stochastic transition structure. On the other hand, as opposed to the current paper, equilibrium strategies in Balbus, Reffett, and Woźny (2014) were not necessarily monotone in types. In the current paper also, given the distributional game specification, and the structural properties implied by our D-ELLN that describes the evolution of distributional state variables on types, we are able to avoid many of problems in characterizing dynamic complementarities in actions between periods and beliefs reported recently by Mensch (2018b). As a result, we are also able to dispense with some continuity assumptions necessary to obtain existence. Finally, as in the work of Balbus, Reffett, and Woźny (2014), our proofs are constructive, where the computation of equilibrium comparative

[20]See Balbus, Dziewulski, Reffet, and Woźny (2015, 2019); Balbus, Reffett, and Woźny (2015).

[21]Notice, this need for single-crossing in distribution between private actions and aggregate distribution is *not* required in papers such as Acemoglu and Jensen (2015, 2018), and Light and Weintraub (2019). This is only because they are considering a comparative statics of *stationary equilibrium*.

[22]Keep in mind, the extensive-form games in Curtat (1996), Amir (2002, 2005) do not remain extensive-form supermodular over an infinite-horizon; hence, they are not able to obtain any equilibrium comparative statics/dynamics results for the infinite horizon cases in their papers.

statics/dynamics are also construction via simple successive approximations. In this sense, we are able to provide the applied researchers with tools allowing to approximate the equilibrium distributions.

Additionally, a critical issue that must be addressed in either GSC with a measure space of players, or Bayesian supermodular games is the question of characterizing sufficient conditions for "single crossing in distribution". For example, in Balbus, Dziewulski, Reffet, and Woźny (2015), the authors show how to extend the notion of aggregating the single crossing property in Quah and Strulovici (2012) to the setting of single crossing differences in measure. The extension is not trivial, and it plays a related and critical role in the present paper. So this paper also contributes to an important emerging literature related to "single crossing in distribution". The early papers on this question are Karlin and Rinott (1980), as well as Athey (2001) and McAdams (2003), where the latter two papers, the authors developed sufficient conditions for single-crossing differences in distribution to resolve the question of existence of monotone in type Bayesian Nash equilibrium in a game of incomplete information. See also Vives and Van Zandt (2007) and Van Zandt (2010) for similar constructions in the context of Bayesian supermodular games. Subsequent work concerning this issue, and related questions, includes the papers of Quah and Strulovici (2012), Balbus, Dziewulski, Reffet, and Woźny (2015), Liu and Pei (2018), Mensch (2018a,b,c), and Kartik, Lee, and Rappoport (2019). Our work is closely related to all of these papers as to preserve complementarity structure on our large stochastic games over time, we require the tools to characterize single cross in distribution adapted to our large stochastic game.

Importantly, our paper also contributes to the large literature on characterizing the equilibrium comparative statics and comparative dynamics for large dynamic economies and large anonymous games. The literature on large dynamic economies is extensive, beginning with the seminal papers of Bewley (1986), Hopenhayn (1992), Huggett (1993), Aiyagari (1994) among many others.[23] In particular, our results provide a foundation for a theory of comparative monotone *dynamics* results relative to ordered perturbations of the space of games and/or large dynamic economies. Specifically, we provide sufficient conditions on payoffs and transition probabilities such that sequence of equilibrium distributions as well as the aggregate law of motion (specifying transition dynamics but also rational beliefs is our game) are monotone in type for any positive shock. Interestingly, our methods extend equilibrium comparative *statics* results of Adlakha and Johari (2013), Acemoglu and Jensen (2015, 2018), and Light and Weintraub (2019), where the authors are concerned with the comparative statics of invariant distributions

[23]See Acemoglu and Jensen (2015) and and Light and Weintraub (2019) for an excellent discussion of this literature, as well Acemoglu and Jensen (2018) for a discussion of related questions in the context of behavioral neoclassical growth models.

It also bears mentioning there is a related literature on large dynamic economies with aggregate risk. Starting with Krusell and Smith (1998), and continuing with more recent work by Miao (2006), **?**, and **?**. These papers focus on the question of existence of sequential equilibria (or generalized recursive equilibrium), and obtain very little characterization of dynamic equilibrium. The no-aggregate risk case is obviously a special case of these results.

or "stochastic steady states" relative to stationary equilibrium as introduced in Bewley (1986). Further, in many of these papers, the results only apply to equilibrium *aggregates*, where one assumes in the large dynamic economic/large anonymous game the existence of a convexifying and real valued (or totally ordered) aggregate. Our approach contains this approach as a special case (as a stationary equilibrium in a large dynamic economy/large anonymous game is a special case of a Markov distributional equilibrium), but we also relax the requirement that we study equilibrium aggregates. For example, in our setting, we can study robust stationary equilibrium (and Markovian distributional equilibrium) relative to a new class of quantile games, are able to perform multi-dimensional equilibrium comparative static/dynamics relative to a (infinite dimensional) set of equilibrium distributions.

It also bears mentioning the assumptions of Acemoglu and Jensen (2015, 2018), and Light and Weintraub (2019) are not sufficient to obtain the strong results in this paper.[24] The key central difference between our work and these papers is that when studying stationary equilibrium (or mean-field equilibrium) comparative statics, one does *not* need conditions on the game that imply *single crossing in distribution* between private actions and aggregates. This is because one is only characterizing the "steady state" structure of the sequential or Markovian equilibrium. Indeed, the results of Acemoglu and Jensen (2015) do not even work for the equilibrium comparative statics/dynamics of Markovian equilibrium even in for *representative* agent versions of large dynamic economies (e.g., Mirman, Morand, and Reffett (2008)). For results in these latter results on Markovian equilibrium, or the results in the present paper on Markovian Distributional Equilibrium, one must deal with the influence of perturbations of dynamic interactions between players and their distributional counterparts via *the value function* that is needed to recursively define each player's stage game payoffs. In additional, one must study the equilibrium structure *away* from the fixed points of the equilibrium law of motion. To do this, one needs increasing differences of payoffs between types and aggregate distributions.[25] Additionally, recall we compare distributions over $\mathbb{R}^n$, and not simply there moments. Set of such objects is not a lattice, hence necessity of our new equilibrium comparative statics tools we provide in Theorem 7 in the paper. Finally, our monotone comparative statics/dynamics results are also shown to be *computable*, as we characterize the chain of parameterized equilibria converging to the one of interest for a particular parameter. In this related work on the equilibrium comparative statics of invariant measures in large dynamic economies, because equilibrium is not verified by a monotone method, its difficult to tie the computation of equilibrium comparative statics to the results on existence. This is of utmost importance for applied economists that

[24]It is also important to note that the important paper of Acemoglu and Jensen (2018) is concerned with a more general class of environments than ours, that of *behavioral* versions of Bewley models. So our results, even for the stationary equilibrium case, do not extend to their setting.

[25]This is true even in representative agent version of large dynamic economies. For example, in Mirman, Morand, and Reffett (2008), the authors needs a single crossing condition between private savings decisions and *aggregate* capital. That meant, they needed sufficient conditions for the equilibrium value functions to be supermodular in private capital and aggregate capital. They needed no related structure in shocks (as they were not looking for monotone Markov equilibrium).

calibrate the equilibrium invariant distributions' moments, or attempt to develop econometric methods for estimating equilibrium comparative statics/dynamics in data (e.g., via the quantile methods of Echenique and Komunjer (2009, 2013)).

It also bears mentioning, there is a related literature on large dynamic economies with aggregate risk. Starting with Krusell and Smith (1998), and continuing with more recent work by Miao (2006), **?**, and **?**. These papers focus on the question of existence of sequential equilibria (or generalized recursive equilibrium), and obtain very little characterization of dynamic equilibrium. Although, obviously, the no-aggregate risk case is a special case of these results, as our approach to equilibrium comparative statics/dynamics is construction and monotone, the extension of our results to the case of aggregate risk seems more direct (than, for example, the stationary equilibrium approach in Acemoglu and Jensen (2015, 2018) or the mean-field approach in Adlakha and Johari (2013), Light and Weintraub (2019) a.o.

Finally, our paper is related to the recent work on equilibria in oblivious equilibrium, mean-field equilibrium, idealized limit games, and behavioral approaches to Markovian equilibrium (where "behavioral" here means agents do not best respond, in effect, directly to the best replies of the other players of the game, rather they simple compute some summary measure of aggregate actions, and best reply to these (e.g., they might respond to a stationary distribution induced by the other player's best replies as in the concept of a stationary equilibrium in the sense of Hopenhayn (1992)). There is a large and growing literature on this approach to Markovian equilibrium in dynamic or stochastic games using this approach including papers by Weintraub, Benkard, and Van Roy (2008), Adlakha and Johari (2013), Adlakha, Johari, and Weintraub (2015), Doncel, Gast, and Gaujal (2016), Lacker (2018), Sadler (2018), and Light and Weintraub (2019), among many others.[26] For applications of this mean-field approach to applied problems in industrial organization, see Kalouptsidi (2014), Benkard, Jeziorski, and Weintraub (2015), and Ifrach and Weintraub (2016). This work is mainly driven by computability and complexity considerations, and many of these papers build methods for games in continuous time, with finitely many states, finite actions sets, symmetric equilibrium in mixed strategies, where games externalities are characterized by distributions or aggregates on states only (so not on actions). Such equilibrium is a (stationary) distribution on players states. Payoffs in such games depend on player's own action and an average of all players states. Such mean field equilibrium implies a best response *oblivious* strategy, i.e. distribution on action sets, where each players' action is optimal taking the invariant mean field distribution as given. Importantly, we study a non-aggregative games, i.e. games where players interact via the whole equilibrium distributions, and not their moments e.g.[27]

[26]See also related work of Wiecek (2017) that analyses a case of supermodular game in continuous time, in which each player moves in a discrete but different periods of time.

[27]Our methods are more general than those of Adlakha and Johari (2013), as we study different equilibrium concept, namely Markov distributional Nash equilibrium, where players use Markovian strategies as functions of private signal and aggregate distribution (as distribution on types evolves (deterministically) over time, oblivious strategies on invariant distributions are nor appropriate for our game).

In a related context, we also extend a very interesting recent result of Kalai and Shmaya (2018) on approximation of Bayesian Nash equilibrium a game with a finite number of players via *imagined-continuum* equilibrium. An imagined-continuum is a powerful, and tractable, tool that as itself is a behavioral concept of equilibrium in a Bayesian game, where although the players are playing a game with a finite but large number of players, they view the equilibrium interaction and learning (and in particular, their belief formation) as in a game with a continuum of players. For this setting, the study of stationary Markov equilibria of the game (and the evolution of beliefs in the game) using the imagined-continuum construction, and show that the equilibrium of the imagine-continuum version of the Bayesian game converges to the actual stationary Markovian equilibrium of the actual game. Our paper provides precise foundation for such an equilibrium construction, and we show that it can be applied to versions of the Kalai-Shmaya setting for the case of large stochastic supermodular games.

Our work also relates to an important recent literature that studies "idealized limit games" as in the work of Qiao and Yu (2014) and Qiao, Yu, and Zhang (2016), and Nutz and Zhang (2019) for particular applications to models of mean-field competition in models of industrial organization, as well as mean-field game limit theorems in mean-field games discussed in Lacker (2018). This work concerns the following question: as the number of players in a finite player game gets "large", in what sense does the set of Nash equilibrium converge. The critical property of the Nash equilibrium correspondence that must be checked is its "closed graph" property, which provides a precise foundation for results on asymptotic convergence of the game (and corresponding approximation results). Some interesting examples of non-convergence of mean-field game limits are provided in Nutz, Martin, and Tan (2019). Also, see Carmona, Delarue, and Lacker (2017) for a general discussion of this mean-field/idealized limit convergence literature. These results are very important, as they have turned out to be critical, for example, in econometric applications for large games. See Menzel (2016) and Leung (2019a) for a discussion. In Mele (2017), he studies the structural properties of network formation models using graphon type approximation schemes.[28] For applications of related large game approximation methods to the econometrics of endogenous network formation, and applications of idealized limits for the asymptotic properties of estimators, see Leung (2019c, 2015), Menzel (2017), Ridder and Sheng (2017) and Leung and Moon (2019), in the modelling social interactions in networks, see Baetz (2015), Xu (2018), and Leung (2019b), and in the modelling of matching, see Menzel (2015).

The rest of the paper is organized as follows. In section 3 we state the main measure-theoretic definitions, new fixed point comparative statics results, as well as provide our version of dynamic law of large numbers necessary in the analysis of our game. Section 4 presents our model and states the main results. existence, equilibrium bounds and comparative dynamics. In section 5, we present our approximation results, and in particular study the relationship between Markovian distributional equilibria vs Markov perfect equilibrium in a large but finite game. Here, we prove an important idealized limit result relative to $\epsilon-$equilibrium that related Markov Perfect

[28] For an interesting discussion of graphon games, and their literature, see Parise and Ozdaglar (2018).

equilibrium in large anonymous stochastic games for the case of a finite (but large) number of players vs. a measure space of players. The last section of the paper includes a discussion of applications, where we develop a dynamic social distance model with cultural/identity complementarities, a dynamic model of quantile aspirations (that includes as a special case keeping up with the Joneses models), a prototype class of models with dynamic coordinations failures and learning (that includes as special cases models of dynamic bank runs, riot games, investment games, beauty contests/sentiment games, etc.), as well as overlapping generations games with social interactions.

3 Preliminaries

3.1 Measure algebras

Given a probability space $(\Lambda, \mathcal{L}, \lambda)$, by $\mathcal{N}_\lambda$ denote a family of $\lambda-$ null sets.

Definition 1. *A probability space $(\Lambda, \mathcal{L}, \lambda)$ is said to be saturated if it is atomless and for any two Polish spaces X and Y, Borel distribution τ on $X \times Y$, and any measurable function $f : \Lambda \to X$ satisfying $\lambda f^{-1} = \tau_X$, where τ_X is marginal of τ on X, there exists random variable with $g : \Lambda \to Y$ such that $\lambda(f, g)^{-1} = \tau$.*

We now present few equivalent definitions of saturated spaces, which can be found in the literature. A *measure algebra* $(\mathcal{O}, \cup^\bullet, \cap^\bullet)$ is said to be a quotient Boolean algebra containing equivalence class for equivalence relation $\sim$ on $\mathcal{L}$ defined as follows: $A \sim B$ iff $A \triangle B \in \mathcal{N}_\lambda$[29]. By $A^\bullet$, $A \in \mathcal{L}$, we denote an equivalence class containing A. $\cup^\bullet$ is defined as follows: $a_1 \cup^\bullet a_2 = (A \cup B)^\bullet$ where $A \in a_1$ and $B \in a_2$. In a similar way we define $\cap^\bullet$, as well as other operations $\setminus^\bullet$, $\triangle^\bullet$ etc. On $\mathcal{O}$ define an ordering $\subset^\bullet$ as follows: $a \subset^\bullet b$ iff $A \setminus B \in \mathcal{N}_\lambda$ for all $A \in a$ and $B \in b$. We define *principal ideal* generated by $a \in \mathcal{O}$ as follows $\mathcal{O}_a := \{b \in \mathcal{O} : b \subset^\bullet a\}$. We can also define $\mathcal{O}_E := \mathcal{O}_{E^\bullet}$. For a measure algebra $(\mathcal{O}, \cup^\bullet, \cap^\bullet)$ drop operations from the notation and and write $\mathcal{O}$ for short.

Definition 2. *A Maharam type*[30] *of $\mathcal{O}$ is the least cardinal of any family $\mathcal{U} \subset \mathcal{O}$ such that the least order closed algebra*[31] *containing $\mathcal{U}$ is $\mathcal{O}$ itself.*

Similarly, for any $a \in \mathcal{O}$, viewing $\mathcal{O}_a$ as a Boolean algebra in its own right, we define the Maharam type of $\mathcal{O}_a$. The Maharam type of $(\Lambda, \mathcal{L}, \lambda)$ is the Maharam type of $\mathcal{O}$. The Maharam type of λ is the Maharam type of $(\Lambda, \mathcal{L}, \lambda)$. The measure algebra $\mathcal{O}$ is homogenous if the Maharam type of each principal ideal except trivial (i.e. $\mathcal{N}_\lambda = \emptyset^\bullet$) is the same and is the Maharam type of $\mathcal{O}$.

[29] operation $\triangle$ means symmetric difference of two sets. That is $A \triangle B = (A \setminus B) \cup (B \setminus A)$

[30] See chapter 33 in Fremlin (2002) or original Maharam (1942) paper for details.

[31] A s et $\mathcal{U}' \subset \mathcal{O}$ is order closed if any directed subset of $\mathcal{U}'$ (with ordering $\subset^\bullet$) has supremum which belongs to $\mathcal{U}'$.

Observe that probability measure λ (or probability measure space $(\Lambda, \mathcal{L}, \lambda)$) is an atomless probability measure if and only if for each nontrivial element a of its measure algebra $\mathcal{O}$, Maharam type of its principle ideal $\mathcal{O}_a$ is infinite. A probability measure λ (or probability space $(\Lambda, \mathcal{L}, \lambda)$) is said to be *super-atomless* if and only if for each nontrivial $a \in \mathcal{O}$, the Maharam type of its principle ideal $\mathcal{O}_a$ is uncountable. λ is said to by $\kappa-$ *super-atomless* if

$$\kappa = \min\{\kappa' : \kappa' \text{ is the Maharam type of any nontrivial principle ideal}\}.$$

By Podczeck (2010) we immediately have the following theorem:

Theorem 1. *The following conditions are equivalent:*

- $(\Lambda, \mathcal{L}, \lambda)$ *with measure algebra* $\mathcal{O}$ *is super-atomless,*
- *for any* $a \in \mathcal{O}$*, the Maharam type* $\mathcal{O}_a$ *is uncountable, (is* κ *- super-atomless for at least* $\kappa = \aleph_1$*),*
- *for any not null set* $E \in \mathcal{L}$*,* $L^1_E(\lambda)$ *is not separable.*[32]

Hoover and Keisler (1984) state another formulation:

Definition 3. *Let* $(\Lambda, \mathcal{L}, \lambda)$ *be probability space and let* $\mathcal{C} \subset \mathcal{L}$ *be sub-*σ*-field. A probability space* $(\Lambda, \mathcal{L}, \lambda)$ *is atomless over* $\mathcal{C}$ *if for all* $D \in \mathcal{L} \setminus \mathcal{N}_\lambda$*, there exists* $D_0 \subset D$ *and* $L_0 \in \mathcal{C} \setminus \mathcal{N}_\lambda$ *such that*

$$0 < \lambda(D_0|\mathcal{C})(\alpha) < \lambda(D|\mathcal{C})(\alpha) \tag{1}$$

for λ*-a.a.* $\alpha \in L_0$*. For cardinality* κ*, probability space is* κ*-atomless if it is atomless over all* σ*-field generated by collection of sets with cardinality strictly less then* κ*. If* D *is a set such that for no* $L_0 \in \mathcal{L}$ *and* $D_0 \subset D$ *with* $D_0 \in \mathcal{L}$ *equation (1) is satisfied, then* D *is said to be an atom over* $\mathcal{C}$*.*

Clearly atomless space is atomless over trivial $\sigma-$ field. If probability measure is $\aleph_1$-atomless, then is is atomless over all countably generated σ-fields.

Remark 1. *Observe that, if* D *is an atom over* $\mathcal{C}$*, then the principal ideal* $\mathcal{O}_D$ *is generated by family* $\mathcal{C}_D := \{(C \cap D)^\bullet : C \in \mathcal{C}\}$*. To see that, take an arbitrary* $D_0 \subset D$ *such that* $D_0 \in \mathcal{L}$*. Then* $\lambda(D_0|\mathcal{C}) = \lambda(D|\mathcal{C})\mathbf{1}_{C_0}$ *for some* $C_0 \in \mathcal{C} \setminus N_\lambda$ *and for* $\lambda-$*a.a.* $\alpha \in \Lambda$*. As a result for all* $C \in \mathcal{C}$ *we have* $\lambda(D_0 \cap C) = \lambda(D \cap C \cap C_0)$*. If* $C := C_0$*, then we obtain* $\lambda(D_0 \cap C_0) = \lambda(D \cap C_0)$

[32] The equivalence class of all $\lambda-$ integrable functions equal λ-a.e. vanishing outside E.

and if we put $C := C_0^c$, *we obtain* $\lambda(D_0 \setminus C_0) = 0$. *Therefore,*

$$\begin{aligned}\lambda(D_0 \triangle (C_0 \cap D)) &= \lambda(D_0 \setminus (C_0 \cap D)) + \lambda((C_0 \cap D) \setminus D_0)\\ &\leq \lambda(D_0 \setminus C_0) + \lambda(D_0 \setminus D) + \lambda(C_0 \cap D) - \lambda(C_0 \cap D \cap D_0)\\ &= \lambda(C_0 \cap D) - \lambda(C_0 \cap D_0) = 0.\end{aligned}$$

Hence $D_0 \in (D \cap C_0)^\bullet$. *As a result* $\mathcal{C}_D$ *generates* $\mathcal{O}_D$. *Hence a probability space* $(\Lambda, \mathcal{L}, \lambda)$ *is superatomless if and only if it is* $\aleph_1$*-atomless.*

Classical examples of superatomless probability space include: $\{0,1\}^I$ with its usual measure when I is an uncountable set; the product measure $[0,1]^I$, where each factor is endowed with Lebesgue measure when I is uncountable[33]; subsets of these spaces with full outer measure when endowed with the subspace measure, or an atomless Loeb probability space. Further, any atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure (see Podczeck (2009)).

Combining Remark 1 and Corollary 4.5 in Hoover and Keisler (1984) we have

Theorem 2. *The following conditions are equivalent:*

(i) $(\Lambda, \mathcal{L}, \lambda)$ *is super-atomless,*

(ii) $(\Lambda, \mathcal{L}, \lambda)$ *is* $\aleph_1$*-atomless,*

(iii) $(\Lambda, \mathcal{L}, \lambda)$ *is saturated.*

3.2 Fubini extension

Let $(\Lambda, \mathcal{L}, \lambda)$ be a probability space. A collection of random variables $(X_\alpha)_{\alpha \in \Lambda}$ is *essentially pairwise independent*, if for $\lambda \otimes \lambda$- a.a. $(\alpha, \alpha') \in \Lambda \times \Lambda$ X_α and $X_{\alpha'}$ are independent random variables.

For a set $E \subset \Lambda \times \Omega$ by E^ω and E_α we denote as a corresponding sections of E. For a given function $f : \Lambda \times \Omega$ by f_α we denote the value of f, if $\alpha \in \Lambda$ is fixed. Similarly f_ω is a value of f, if $\omega \in \Omega$ is fixed.

Definition 4. *A probability space* $(\Lambda \times \Omega, \mathcal{W}, w)$ *is a* Fubini extension *of the natural product space if*

- $\mathcal{W}$ *includes all sets from* $\mathcal{L} \otimes \mathcal{F}$,

[33] Indeed, Maharam's theorem shows that the measure algebra of every superatomless probability spaces must correspond to the countable convex combination of such spaces.

- *For a* $E \in \mathcal{W}$, $\lambda \otimes P$*-a.a.* $(\alpha, \omega) \in \Lambda \times \Omega$, *any two sets* E_α *and* E^ω *are (respectively)* $\mathcal{F}$*- and* $\mathcal{L}$*-measurable and the following holds:*

$$w(E) = \int_\Omega \lambda(E^\omega) P(d\omega) = \int_\Lambda P(E_\alpha)\lambda(d\alpha).$$

A Fubini extension of $\Lambda \times \Omega$ *is said to be rich, if there exists a* $\mathcal{W}$*-measurable real valued function* S *such that a collection* $(S_\alpha)_{\alpha\in\Lambda}$ *is essentially pairwise independent random variables such that* λ*-a.a. of them has* $\mathcal{U}(0,1)$*.*

Existence of the rich Fubini extension was proved in Proposition 5.6. in Sun (2006), with $\Lambda = [0, 1]$. It is, however, shown in Proposition 6.2. in Sun (2006) that $\mathcal{L}$ can not be a collection of Borel subsets of Λ. Indeed, it is necessary and sufficient that the space of agents is superatomless. Necessity follows from Theorem 3 in Podczeck (2010), while sufficiency from Theorem 2, where it is also shown that one can take the process to be independent, and not just pairwise independent[34]. Theorem 3 shows the following generalization of these results:

Theorem 3. *If* $(\Lambda, \mathcal{L}, \lambda)$ *is super-atomless probability space, then there exists other super-atomless probability space* $(\Omega, \mathcal{F}, P)$ *such that standard product measure space on* $\Lambda \times \Omega$ *has a rich Fubini extension.*

A Fubini extension is typically denoted by $(\Lambda\times\Omega, \mathcal{L}\boxtimes\mathcal{F}, \lambda\boxtimes P)$. A *process* is said to be $\mathcal{L}\boxtimes\mathcal{F}$-measurable function with values in some Polish space. For all sets $E \in \mathcal{L} \setminus \mathcal{N}_\lambda$ and processes f, let f^E denote a restriction of f to $E \times \Omega$. Similarly for a probability space $(\Lambda, \mathcal{L}, \lambda)$ we denote $(E\times\Omega, \mathcal{L}^E\boxtimes\mathcal{F}, \lambda^E\boxtimes P)$ as follows $\mathcal{L}^E := \{E\cap E' : E' \in \mathcal{L}\}$, $\mathcal{L}^E\boxtimes\mathcal{F} := \{W \in \mathcal{L}\boxtimes\mathcal{F} : W \subset E\times\Omega\}$, and λ^E and $(\lambda^E \boxtimes P)$ are probability measures rescaled respectively from the restrictions of λ to $\mathcal{L}^E$ and $(\lambda^E \boxtimes P)$ to $\mathcal{L}^E \boxtimes \mathcal{F}$. We now introduce the following version of (strong) Law of Large Numbers.

Theorem 4 (Law of Large Numbers, Theorem 2.8. in Sun (2006))**.** *Let* f *be a process from a Fubini extension* $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *to some Polish space. Then following conditions are equivalent:*

- $(f_\alpha)_{\alpha\in\Lambda}$ *are essentially pairwise random variables,*
- *Then for all* $E \in \mathcal{L} \setminus \mathcal{N}_\lambda$*:*

$$\lambda f^E_\omega = (\lambda^E \boxtimes P)(f^E)^{-1}$$

  for P*-a.a.* $\omega \in \Omega$*.*

[34] See also Greinecker and Podczeck (2015).

3.3 Preliminaries on lattice theory

Let $(X, \leq)$ be partially ordered set (poset for short). For a subset $E \subset X$ we accept the following notation: $\bigvee A$ is a supremum of A, i.e. the least upper bound of A and $\bigwedge A$ is infimum of A, i.e. the greatest lower bound of A. We have the following definitions:

- A poset X is said to be *lattice* if each two-element subset has supremum and infimum in X. A subset $A \subset X$ is *lattice*, if it is a lattice with the induced order.
- A poset X is *complete lattice*, if each subset has supremum and infimum in X. A subset $A \subset X$ is said to be *complete lattice*, if it is a complete lattice with the induced order.
- A poset X is (resp. countably) *chain complete* if each (resp. countable) chain $A \subset X$ has supremum and infimum in X. A subset $A \subset X$ is said to be *chain complete*, if it is chain complete with the induced order.

If X is a poset and Y is another poset then a function $f : X \to Y$ is said to be *increasing (or isotone)* (*decreasing (or antitone)*) if following implication holds: $x_1 < x_2$ in X then $f(x_1) \leq (\geq) f(x_2)$. If inequality between $f(x_1)$ and $f(x_2)$ is strict then we say f is *strictly decreasing (or strictly antitone)* (resp. *strictly decreasing (antitone)*). We now state a generalization of standard Tarski fixed point theorem:

Theorem 5 (Markowsky, Theorem 9 in Markowsky (1976))**.** *Let $(X, \leq)$ be chain complete poset and $f : X \to X$ be isotone. Then set of fixed points is nonempty chain complete poset.*

We introduce the following terms:

Definition 5. *A function f from poset X to another poset Y is mon-sup preserving if for any increasing sequence $(x_n)_{n\in\mathbb{N}}$ it holds that $f\left(\bigvee(x_n)_{n\in\mathbb{N}}\right) = \bigvee(f(x_n))_{n\in\mathbb{N}}$. A function is mon-inf preserving if for any decreasing sequence $(x_n)_{n\in\mathbb{N}}$ it holds that $f\left(\bigwedge(x_n)_{n\in\mathbb{N}}\right) = \bigwedge(f(x_n))_{n\in\mathbb{N}}$. If f is mon-sup preserving and mon-inf preserving, then we say it is mon-sup-inf preserving.*

Finally we have

Theorem 6 (Tarski and Kantorovich, Theorem 4.2 in Dugundji and Granas (1982))**.** *Let X be a countably chain complete poset with the greatest $\overline{x} := \bigvee X$ and the least $\underline{x} := \bigwedge X$ elements. Assume $f : X \to X$ is isotone. Then*

- *if f is mon-sup preserving then the least fixed point is $\bigvee(f^n(\underline{x}))_{n\in\mathbb{N}}$,*
- *if f is mon-inf preserving then the greatest fixed point is $\bigwedge(f^n(\overline{x}))_{n\in\mathbb{N}}$,*

where f^n means $n-th$ composition of f (i.e. $f^n = \underbrace{f \circ f \circ ... \circ f}_{n \text{ times.}}$)

We finish this section with an important comparative statics theorem developed by Balbus, Reffett, and Woźny (2015). It extends the classic comparative statics theorems of Veinott (1992) or Topkis (1998) (theorem 2.5.2) from complete lattices to countably chain complete posets. Such extension is necessary when trying to conduce comparative statics on the poset of bounded measurable functions or space of probability distributions over $\mathbb{R}^n$ e.g. See Balbus, Reffett, and Woźny (2015) for a proof and further motivation.

Theorem 7 (Balbus, Reffett, and Woźny (2015), Theorem 8)**.** *Let X be a countably chain complete poset with the greatest and least elements and T a poset. Endow $X \times T$ with a product order. By $\underline{FP}(t)$ and $\overline{FP}(t)$ denote the least and the greatest fixed points of $f(\cdot, t)$ (if they exists). If $f : X \times T \to X$ is increasing, and mon-sup-inf preserving on X then $t \to \overline{FP}(t)$ and $t \to \underline{FP}(t)$ are increasing.*

4 Large stochastic supermodular games

We begin by defining the class of large stochastic supermodular games we consider initially in the paper. Let $(\Lambda, \mathcal{L}, \lambda)$ be a saturated probability space of agents. Let $(T, \mathcal{T})$ be space of private types of agents, where compact $T \subset \mathbb{R}^p$ and $\mathcal{T}$ a family of its Borel sets. Let $A \subset \mathbb{R}^k$ be compact action space for all agents, with $\mathcal{A}$ is a family of its Borel sets. Let $\mathcal{M}$ be a set of probability measures on $\mathcal{T} \otimes \mathcal{A}$ and $\mathcal{M}_T$ denote the set of probability measures on $\mathcal{T}$. Suppose that, if agent is of type $t \in T$, other agents types are $\tau \in \mathcal{M}_T$, then the set of feasible actions is $\tilde{A}(t, \tau) \subset A$. Let $\mathcal{A}$ be a set of Borel subsets of A. Let bounded $r : T \times A \times \mathcal{M} \to \mathbb{R}$ be an agent's payoff function, and $q : T \times A \times \mathcal{M} \to \mathcal{M}_T$ be a transition probability between private types of each agent.

We now state few basic assumptions:

Assumption 1. *Assume:*

- *for all $\tau \in \mathcal{M}_T$ $\tilde{A}(\cdot, \tau)$ is measurable and compact valued correspondence,*
- *for all $\mu \in \mathcal{M}$, $(a, t) \to q(\cdot|t, a, \mu)$ is Borel measurable.*

Theorem 8. *Assume 1. Then there exists sampling probability space $(\Omega, \mathcal{F}, P)$ and a rich Fubini extension of $\Lambda \times \Omega$ say $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that for any strategy $\sigma = (\sigma_n)_{n \in \mathbb{N}}$, where $\sigma_n : T \to A$, any initial state t and initial distribution $\tau \in \mathcal{M}_T$, there is sequence of $\mathcal{L} \boxtimes \mathcal{F}$-measurable functions $X_n : \Lambda \times \Omega \to T$ such that*

(i) For all $n \in \mathbb{N}$, random vectors $(X_n^\alpha)_{\alpha \in \Lambda}$ are $\lambda \otimes \lambda$ a.s. pairwise conditionally independent,

(ii) For all $n \in \mathbb{N}$, $\lambda -$ a.a. $\alpha \in \Lambda$ and $P -$ a.a. $\omega \in \Omega$ we have

$$P(X_n^\omega)^{-1} = (\lambda \boxtimes P)(X_n)^{-1}.$$

As a result

$$\lambda(K_n^\omega)^{-1} = P(K_n^\alpha)^{-1} = (\lambda \boxtimes P)(K_n)^{-1} := \mu_n,$$

where $K_n = (X_n, \sigma_n(X_n))$.

(iii) Conditional distribution of X_{n+1}^α *under* $(X_j^\alpha)_{j\leq n}$ *is* $q(\cdot|X_n^\alpha, \sigma_n(X_n^\alpha), \mu_n)$.

(iv) For $\lambda - a.a.$ $\alpha \in \Lambda$ *a sequence* $(X_n^\alpha, Y_n^\alpha)_{n\in\mathbb{N}}$ *has a distribution* $P_{t,\mu}^\sigma$.

Proof. By Theorem 3 there exists a probability space $(\Omega, \mathcal{F}, P)$ and the rich Fubini extension of a natural product space on $\Lambda \times \Omega$. We denote it as $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Consequently, we can find $S : \Lambda \times \Omega \to [0, 1]$ such that S is a process and a family $(S^\alpha)_{\alpha\in\Lambda}$ is essentially pairwise independent with the uniform distribution on $[0, 1]$. Define $(S_n)_{n\in\mathbb{N}}$ as a set of independent copies of S. Now we are going to construct a sequence $(X_n)_{n=1}^\infty$ satisfying the thesis (i)-(iii). Let $(I, \mathcal{I}, \iota)$ be the standard interval (i.e. $I = [0, 1]$ with $\mathcal{I}$ Borel sets and ι - Lebesgue measure). Furthermore, for any $\mu \in \mathcal{M}$ we can find a- $\mathcal{I} \otimes \mathcal{T} \otimes \mathcal{A}$ function $G_\mu : I \times T \times A \mapsto T$ such that

$$\iota(G_\mu^{t,a})^{-1}(Z) = \iota\left(\{l \in I : G_\mu(l, t, a) \in Z\}\right) = q(Z|t, a, \mu)$$

for any $Z \in \mathcal{T}$[35].

For an initial distribution $\tau_1 \in \mathcal{M}_T$, we can find T-valued $I \otimes \mathcal{T}$ Borel measurable function $\tilde{G}$ such that $\tau_1 = \iota\tilde{G}^{-1}$ [36]. Put

$$X_1 := \tilde{G}(S_1). \tag{2}$$

Having the initial random variable X_1, for $n \geq 1$ let us define the following process

$$X_{n+1} = G_{\mu_n}(S_{n+1}, K_n) \tag{3}$$

where

$$\tau_n := (\lambda \boxtimes P)X_n^{-1} \quad \text{and} \quad \mu_n := (\lambda \boxtimes P)K_n^{-1}$$

and $K_n = (X_n, \sigma^*(X_n, \tau_n))$. As usual, put $K_n^\alpha(\omega) := K_n(\alpha, \omega)$ for $(\alpha, \omega) \in \Lambda \times \Omega$.
Let $\mathcal{S}_n := \sigma\{S_k : k \leq n\}$. Combining (2) and (3) we conclude that X_n is $\mathcal{S}_n$ measurable. Hence for $\lambda-$ a.a. $\alpha \in \Lambda$, X_n^α and S_{n+1}^α are independent random variables.
Now we show that (i)-(ii) are satisfied. We do it by induction with respect to n. We have the thesis (i) for $n = 1$ by essential independence of S_1 and (2). Moreover, by Theorem 4 for $P-$ a.a. $\omega \in \Omega$ the sampling distribution of X_1 $(\lambda(X_1^\omega)^{-1})$ satisfies

$$\lambda(X_1^\omega)^{-1} = (\lambda \boxtimes P)X_1^{-1} = \tau.$$

[35] See Lemma A5 in Sun (2006) for example.
[36] See again Lemma A5 in Sun (2006).

Again by Theorem 4 for $P-$ a.a. $\omega \in \Omega$ we have also

$$\lambda(K_1^{\omega})^{-1} = (\lambda \boxtimes P)K_1^{-1} := \mu_1.$$

Hence (ii) is satisfied for $n = 1$. Suppose that for some $n \in \mathbb{N}$ both (i) and (ii) are true. Observe that $(S_{n+1}^{\alpha}, X_n^{\alpha})_{\alpha\in\Lambda}$ is a family $\lambda \otimes \lambda$ -a.s. pairwise conditionally independent. It follows from induction hypothesis for X_n^{α} and from the previous observation the random variables X_n^{α} and S_{n+1}^{α} are independent λ-a.s. Hence and by (3) the family $(X_{n+1}^{\alpha})_{\alpha\in\Lambda}$ is $\lambda \otimes \lambda$ -a.s. pairwise conditionally independent. Hence the property (i) is satisfied for $n + 1$. The application of Theorem 4 provide us additionally (ii) for $n+1$ and finally we conclude that both properties (i) and (ii) are satisfied for all $n \in \mathbb{N}$.
Next we show that (iii) is satisfied. Let $\mathcal{S}_n^{\alpha} := \sigma(\{S_k^{\alpha} : k \leq n\})$ and $\Sigma_n^{\alpha} := \sigma(\{X_k^{\alpha} : k \leq n\})$. By (2) and (3) and definition of Σ_n^{α} we conclude that

$$\sigma(X_n^{\alpha}) \subset \Sigma_n^{\alpha} \subset \mathcal{S}_n^{\alpha}.$$

Hence the conditional distribution of X_{n+1}^{α} with respect to Σ_n satisfies

$$\begin{aligned} P(X_{n+1}^{\alpha} \in Z|\Sigma_n^{\alpha}) = E(P(X_{n+1}^{\alpha} \in Z|\mathcal{S}_n^{\alpha})|\Sigma_n^{\alpha}) &= E(P(G_{\mu_n}(S_{n+1}^{\alpha}, K_n^{\alpha}) \in Z|\mathcal{S}_n^{\alpha})|\Sigma_n^{\alpha}) \\ &= E(q(Z|K_n^{\alpha}, \mu_n)|\Sigma_n^{\alpha}) = q(Z|X_n^{\alpha}, \sigma^*(X_n^{\alpha}, \tau_n), \mu_n) \end{aligned}$$

for λ-a.a. $\alpha \in \Lambda$ and all $Z \in \mathcal{T}$. The last equality follows from the independence between S_{n+1}^{α} and X_n^{α}. Hence (iii) is satisfied.
Consequently for a.a. $\lambda-$a.a. $\alpha \in \Lambda$, $(X_n^{\alpha})_{n\in\mathbf{N}}$ has the same distribution which exists by the Ionescu-Tulcea Theorem (Dynkin and Yushkevich, 1979)[37]. Hence (iv) is satisfied. $\square$

Theorem 8 is our version of the dynamic law of of large numbers. We develop it to study transition of private signals and conclude it implies no aggregate uncertainty. Specifically, if the current distribution on type and action is μ_n, then by theorem 8 the next period distribution on types is

$$\tau_{n+1}(\cdot) = \int\limits_{T\times A} q(\cdot|t, a, \mu_n)\mu_n(dt \times da).$$

4.0.1 Agent Decision Problems

Let H^{∞} be a set of all histories $((t_n, a_n, \tau_n)_{n\in\mathbb{N}}$, where $a_n \in \tilde{A}(t_n, \tau_n))$. Let H^N be a set of histories up to N, that is $H^N := \{(t_n, a_n, \tau_n)_{n=1}^N, a_n \in \tilde{A}(t_n, \tau_n)\}$. The strategy of an agent is a sequence of Borel measurable functions $(\sigma_n)_{n\in\mathbb{N}}$ such that for $n \in \mathbb{N}$ σ_{n+1} maps $H^n \times T \times \mathcal{M}_T$ into A such that $\sigma_n(h, t, \tau) \in \tilde{A}(t, \tau)$, and σ_1 maps $T \times \mathcal{M}_T$ into A with $\sigma_1(t, \tau_0) \in \tilde{A}(t, \tau_0)$.

[37] Or Kolmogorov Existence Theorem (Theorem 15.26 in Aliprantis and Border (2006)).

Then, by Theorem 8, any initial private state t, public state τ and strategy profile σ induce the unique private measure P_t^σ on histories of the game. Then objective for a player is then to maximize:

$$R(\sigma, t, \tau) := (1-\beta)E^\sigma_{t,\tau}\left(\sum_{n=1}^{\infty}\beta^{n-1}r(t_n, a_n, \mu_n)\right),$$

where $\beta \in (0,1)$ is a discount factor. A strategy profile is called *Markov*, if at each period it depends on the current state (t, τ) only. Strategy is stationary, if it is time-*invariant*.

Assume that for each (t, τ) feasible action set $\tilde{A}(t, \tau)$ is a subcomplete sublattice. Next:

Assumption 2. *We assume 1 and moreover assume:*

(i) *r is supermodular in a and has increasing differences in $(a; t, \mu)$ and $(t; \mu)$,*

(ii) *q is stochastically supermodular in a and has stochastically increasing differences in $(a; t, \mu)$ and $(t; \mu)$,*

(iii) *r is increasing in t and q is stochastically increasing in (a, t, μ),*

(iv) *r, q are continuous in (t, a) and mon-sup-inf preserving in μ,*

(v) *$\tilde{A}(\cdot)$ is ascending in the set inclusion order*[38] *and ascending in the Veinott's strong set order,*[39]

(vi) *$\tilde{A}$ is uhc and compact valued correspondence.*

Most of these assumptions are standard for a class of games with complementarities, where we also require monotonicity of equilibrium strategies with private types (and marginal distributions μ_T). Interestingly, in our game increasing differences between t, μ are necessary to preserve supermodularity between the periods (via the value function) and obtain strategic complementarities in their extensive form. Two special cases of preferences that satisfy our assumptions include separable payoffs with $r(t, a, \mu) = u(t, a) + v(a, \mu)$ where u and v have increasing differences in a, t and a, μ; and $r(t, a, \mu) = u(t, a) + v(t, \mu)$, with increasing differences between a, t and t, μ. Clearly, large versions of recent specifications of *linear* social interaction models used in the econometric[40] work (see e.g. Blume, Brock, Durlauf, and Jayaraman (2015); Kline and Tamer (2020); Kwok (2019)) also fit into our setup with payoffs:

$$r(t, a, \mu) = (\beta_1 t + \beta_2 \int_T f_1(t, t')t'\mu_T(dt'))a - \frac{1}{2}a^2 - \frac{\beta_3}{2}[a - \beta_4 \int_{T\times A} f_2(t')a'\mu(dt' \times da')]^2,$$

[38] That is, $\tilde{A}$ is *ascending in the set inclusion order* if $t_1 \leq t_2$ and $\tau_1 \leq \tau_2$ then $\tilde{A}(t_1, \tau_1) \subseteq \tilde{A}(t_2, \tau_2)$.

[39] That is, $\tilde{A}$ is *ascending in Veinott's strong set order* if for any $(t, \tau) \leq (t', \tau')$, $a \in \tilde{A}(t, \tau)$ and $a' \in \tilde{A}(t', \tau') \implies a \wedge a' \in \tilde{A}(t, \tau)$ and $a \vee a' \in \tilde{A}(t', \tau')$.

[40] Here we mention, that our computable monotone comparative statics / dynamics results developed in theorem 10 can be useful for developing appropriate estimators allowing to test equilibrium distributions in empirical models.

where β_i are positive parameters and f_1, f_2 are linear, increasing (and positive on their domains) social interactions weighting functions measuring contextual- and peer- network effects.

Finally, to exemplify the stochastic transition satisfying our assumptions we recall a class of distributions:

$$q(\cdot|t, a, \mu) = g(t, a, \mu)\lambda_2(\cdot) + (1 - g(t, a, \mu))\lambda_1(\cdot)$$

where $g : A \times T \times \mathcal{M} \to [0, 1]$ is supermodular in a, and has increasing differences in $(a; t, \mu)$ and $(t; \mu)$ and is increasing in (a, t, μ); while $\lambda_2(\cdot)$ (distributions on T) stochastically dominates $\lambda_1(\cdot)$. Such assumption was introduced by Curtat (1996) and Amir (2002), and was successfully applied in some recent works of the authors (in stronger forms). We refer the reader to our related paper (see Balbus, Reffett, and Woźny (2013) for a detailed discussion of the nature of these assumptions.

If $\mu \in \mathcal{M}$ is a current distribution, then the next state has a distribution

$$\phi(\mu)(\cdot) := \int_{T \times A} q(\cdot|t, a, \mu)\mu(dt \times da).$$

We have the following lemma

Lemma 1. *Assume 2. For each $n \in \mathbb{N}$, let $\mu_n : T \mapsto \mathbb{R}$ be an increasing function and $\mu_n \in \mathcal{M}$. Suppose that, v_n is an increasing sequence, and μ_n is a stochastically increasing (decreasing) sequence. Further assume $\mu_n \to^w \tau$, and $v_n \to v$ pointwise. Then $\int_T v_n(t')\phi(\mu_n)(dt') = \int_T v(t')\phi(\mu)(dt')$.*

Proof. Observe that,

$$\int_T v_n(t')\phi(\mu_n)(dt') = \int_T \int_{T \times A} v_n(t')q(dt'|t, a, \mu_n)\mu_n(dt \times da). \tag{4}$$

By Assumption 2 $\int_{T \times A} v_n(t')q(dt'|t, a, \mu_n)$ is increasing (decreasing) function in (t, a), and is increasing (decreasing) in n. Hence by Lemma 12 the proof is complete. □

Let

$$\mathcal{D} := \{\Phi : \mathcal{M} \to \mathcal{M} : \Phi \text{ is increasing and monotonically sup-inf preserving}$$

$$\text{and } \mathrm{marg}_T(\Phi(\mu)) = \phi(\mu) \text{ for any } \mu \in \mathcal{M}\}.$$

We will denote $\mu_T := \mathrm{marg}_T(\mu)$. Endow $\mathcal{M}$ with a first stochastic dominance order and $\mathcal{D}$ with natural componentwise order.

Let $(\Theta, \leq_\Theta)$ be a partially ordered set. For each $v : T \times \Theta \to \mathbb{R}$ such that for each $\theta \in \Theta$ the function $v(\cdot, \theta)$ is Borel measurable, we define

$$H_v(t, a, \mu, \theta) := \int_T v(t', \theta)q(dt'|t, a, \mu).$$

Lemma 2. *Assume 2. If $v : T \times \Theta \to \mathbb{R}$ is increasing with t, then*

(i) $H_v(\cdot)$ is increasing in t, a, μ;

(ii) if additionally v is continuous in t and mon-sup-inf preserving in θ, then H_v is continuous in (t, a) and jointly mon-sup-inf preserving in (τ, θ);

Proof. Proof of (i) is immediate by Assumption 2. We prove (ii) only. For each $(t_k, a_k, \mu_k, \theta_k) \in T \times A \times \mathcal{M} \times \Theta$, all components increase (decrease) in k, and t, a, μ, θ are corresponding suprema (infima). Then, by Assumption 2 and Lemma 12, the proof is complete. □

Lemma 3. *Assume 2. Let $v : T \times \Theta \to \mathbb{R}$ be increasing on T and has increasing differences. Then $H_v(t, a, \mu, \theta)$ has increasing differences in $(a; t, \mu, \theta)$ and in $(t; \mu, \theta)$.*

Proof. Suppose v has increasing differences in $T \times \Theta$. We show that H_v has increasing differences in $(a; t, \mu, \theta)$. Suppose $\theta_2 \geq_\Theta \theta_1$. Then

$$H_v(t, a, \mu, \theta_2) - H_v(t, a, \mu, \theta_1) = \int_T \big(v(t', \theta_2) - v(t', \theta_1)\big)\, q(dt'|t, a, \mu), \tag{5}$$

Observe that $v(t, \theta_2) - v(t, \theta_1)$ is increasing function in t and by Assumption 2 q stochastically increases in a. In analogous way we can show that H_v has increasing differences in $(t; \theta)$. Increasing differences in $(a; t, \mu)$ and $(t; \mu)$ follow directly from Assumption 2. □

4.1 Definition of Markovian Distributional Equilibrium

Suppose that the current distribution on $T \times A$ is μ, and the next distribution is $\Phi(\mu)$. Then by standard arguments the Bellman equation associated with player's maximization problem is[41]:

$$v^*(t, \mu; \Phi) = \max_{a \in \tilde{A}(t, \mu_T)} \left((1 - \beta) r(t, a, \mu) + \beta \int_T v^*(t', \Phi(\mu); \Phi) q(dt'|t, a, \mu) \right). \tag{6}$$

Note, that for a given μ_0 and perceived law of motion Φ, players's problem is a simple Markov decision problem, with uncertainty about private signal t only. That is, sequence of aggregate distributions $\{\mu_t\}$ is deterministic. This is a consequence of the exact law of large numbers result we have stated the previous section. By standard arguments we can show that the policy correspondence is Markov with natural state space including t and μ. Definition of equilibrium requires, however, additional consistency between perceived law of motion Φ and the policy correspondence of the above Markov decision problem. Observe (more on that in a moment) that Φ also specifies beliefs players have on continuation paths of the game. Hence, when writing $v^*(t, \mu; \Phi)$ we stress that the value function and corresponding policy functions depend on beliefs

[41] Alternatively, one may use t, τ as state variables of the value function and then compose a τ and a strategy $\sigma : T \to A$ to obtain μ. In such, best response specification, strategy σ would be an additional parameter of the value function.

(compare with construction of Markov equilibrium in Kalai and Shmaya (2018) for a large but finite Bayesian game).

Formally, we say:

Definition 6. *A pair $(\mu^*, \Phi^*) \in \mathcal{M} \times \mathcal{D}$ is a Markov Stationary Distributional equilibrium (MSDE, henceforth) if*

(i) for any $\mu \in \mathcal{M}$, and a.e. $t \in T$,

$$v^*(t, \mu; \Phi^*) = \max_{a \in \tilde{A}(t, \mu_T)} \left((1-\beta) r(t, a, \mu) + \beta \int_T v^*(t'^*(\mu); \Phi^*) q(dt'|t, a, \mu) \right),$$

(ii) where argmax $\sigma_{\mu,\Phi} : T \rightrightarrows A$ of the RHS satisfies the following conditions:

$$\mu^*(\cdot) = \mu_T^*(id_T(\cdot), \sigma_{\mu^*,\Phi^*}(\cdot))^{-1},$$

$$\Phi^*(\mu)(\cdot) = \phi(\mu)(id_T(\cdot), \sigma_{\Phi^*(\mu),\Phi^*}(\cdot))^{-1}).$$

Please note that our notion of equilibrium involves both: equilibrium distribution μ^* and Markov transition Φ^*. So this notion of dynamic equilibrium is not simply stationary equilibrium in the sense of Bewley (Bewley (1986)). Condition (i) is a standard Bellman equation that characterizes players best reply correspondences, while Condition (ii) involves two side consistency conditions that are required to verify the existence of MSDE. The first part of conditions (ii) requires $\mu^*(\cdot) \in \mu_T^*(id_T(\cdot), \sigma_{\mu^*,\Phi^*}(\cdot))^{-1}$, which is a distributional equilibrium condition for a given (equilibrium) law of motion. This is a self-generating fixed point condition, simply saying that the distribution generated by the best response map σ (maximizing the right hand side of the Bellman equation) has to be the same distribution as the one that is generated by player's best response to this distribution. Observe, that although it is a static condition (as, for example in a stationary equilibrium ala Bewley), it involves a *dynamic* interaction of the players via the value function and law of motion of the measures Φ. The second part of condition (ii) has $\Phi^*(\mu)(\cdot) \in \phi(\mu)(id_T(\cdot), \sigma_{\Phi^*(\mu),\Phi^*}(\cdot))^{-1})$, which a dynamic condition that requires for every measure μ, the perceived and actual law of motions for aggregate distributions coincide. Specifically, this condition requires that the *perceived* distribution by players on $T \times A$ for the next period distribution is the same as the distribution for the next period that is actual one generated by some selection of the argmax correspondence of the players best reply correspondence. As a technical remark, as we work with no aggregate uncertainty, we do not require that Φ^* to be measurable.[42] So compare our definition of stationary equilibrium versus stationary distribution over actions and private-states.

Note that Markov transition Φ^* also specifies (common) beliefs that players use to determine future paths of distributions. In macroeconomic literature on recursive equilibrium, such beliefs

[42]Recall that monotonicity does not imply measurability (see example in Balbus, Reffett, and Woźny (2015)).

are called rational. These are beliefs on both: on and off the equilibrium paths (of $\{\mu_t\}$) as condition $\Phi^*(\mu)(\cdot) \in \phi(\mu)(id_T(\cdot), \sigma_{\Phi^*(\mu),\Phi^*}(\cdot))^{-1})$ is satisfied for any (initial) μ. Appropriate conditions, guaranteeing that the value function has increasing differences in both arguments (i.e. t, μ) and the transition Φ is monotone, allows us to avoid problems in characterizing dynamic complementarities in actions between periods and beliefs reported by Mensch (2018b). As a result, we are also able to dispense with some continuity assumptions necessary to obtain existence in Mensch (2018b) paper. This is partially due to no aggregate uncertainty assumption, and the fact that any player has no influence on aggregate distribution and formation of joint beliefs (see Kalai and Shmaya (2018)).

4.2 Equilibrium construction

Suppose that $\Phi \in \mathcal{D}$. Consider the space

$$\begin{aligned}
\mathcal{V} := \qquad & \{v : T \times \mathcal{M} \times \mathcal{D} \mapsto \mathbb{R} : \\
& v(\cdot, \mu, \Phi) \text{ is increasing and continuous } \text{ for each } (\mu, \Phi) \in \mathcal{M} \times \mathcal{D} \\
& v(t, \cdot, \cdot) \text{ is mon-sup-inf preserving for each } t \in T \\
& \underline{r} \leq \inf_{(t,\mu,\Phi)\in T\times\mathcal{M}\times\mathcal{D}} v(t, \mu, \Phi) \leq \sup_{(t,\mu,\Phi)\in T\times\mathcal{M}\times\mathcal{D}} v(t, \mu, \Phi) \leq \bar{r} \\
& v \text{ has increasing differences in } (t; \mu, \Phi)\}.
\end{aligned}$$

Endow $\mathcal{V}$ with natural sup-norm topology $||\cdot||_\infty$.

Lemma 4. *$\mathcal{V}$ is complete metric space.*

Proof. Observe that $\mathcal{V}$ is a subset of the set of all bounded functions on $T \times \mathcal{M} \times \mathcal{D}$, which is a Banach space. We only need to show $\mathcal{V}$ is a closed set. Let $v_n \rightrightarrows v$ (i.e. in a sup-norm topology) and $v_n \in \mathcal{V}$ for each $n \in \mathbb{N}$. We show that $v \in \mathcal{V}$. It is easy to see that v is supermodular in t and has increasing differences in $(t; \tau, \Phi)$. Moreover, v is continuous in t and increasing in t. We only need to show that v is mon-sup-inf preserving in (μ, Φ). Let μ_n, and Φ_n be increasing (decreasing) sequences such that $\mu_n \to^w \mu$ and $\Phi_n \to \Phi$ pointwise. Let $t \in T$ and let $\epsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ and $n \geq n_0$ we have

$$\begin{aligned}
|v(t, \mu_k, \Phi_k) - v(t, \mu, \Phi)| \leq |v(t, \mu_k, \Phi_k) - v_n(t, \mu_k, \Phi_k)| + |v_n(t, \mu_k, \Phi_k) - v_n(t, \mu, \Phi)| \\
+ v_n(t, \mu, \Phi) - v(t, \mu, \Phi) \\
\leq \frac{2}{3}\epsilon + |v_n(t, \mu_k, \Phi_n) - v_n(t, \mu, \Phi)|
\end{aligned} \tag{7}$$

Now fix $n \in \mathbb{N}$ satisfying (7). Then, since $v_n \in \mathcal{V}$, hence for large enough k it holds

$$|v_n(t, \mu_k, \Phi_n) - v_n(t, \mu, \Phi)| \leq \frac{\epsilon}{3}. \tag{8}$$

By (7) and (8) $|v(t, \mu_k, \Phi_k) - v(t, \mu, \Phi)| < \epsilon$ for large k. Hence v is mon-sup-inf preserving, consequently $v \in \mathcal{V}$. $\square$

Lemma 5. *Let $(\mu_k)_{k\in\mathbb{N}}$ be an increasing (decreasing) sequence of $\mathcal{M}$ and $\mu_k \uparrow (\downarrow)\mu$ weakly as $k \to \infty$. Let $(\Phi_k)_{k\in\mathbb{N}}$ be an increasing (decreasing) sequence of $\mathcal{D}$ and $\Phi_k(\tau) \uparrow (\downarrow)\Phi(\mu)$ weakly as $k \to \infty$ for each $\mu \in \mathcal{M}$. Then, $\Phi_k(\mu_k) \to \Phi(\mu)$ weakly as $k \to \infty$.*

Proof. Suppose that μ_k and Φ_k are both increasing. Obviously, for each $k \in \mathcal{N}$

$$\Phi_k(\mu_k) \leq \Phi(\mu),$$

hence $\lim_{k\to\infty} \Phi_k(\mu_k) \leq \Phi(\mu)$. We show the opposite inequality. Let k be arbitrary integer. Then for each $N > k$ we have

$$\Phi_N(\mu_k) \geq \Phi_N(\mu_N).$$

Taking a limit with $N \to \infty$ above we have $\lim_{N\to\infty} \Phi_N(\mu_N) \geq \Phi(\mu_k)$. Next we take a limit $k \to \infty$, and by sup-preserving property we have $\Phi(\mu) \geq \lim_{N\to\infty} \Phi_N(\mu_N)$. The proof is similar along the lines, if μ_k and Φ_k are both decreasing. $\square$

For each $v \in \mathcal{V}$ we define

$$F(t, a, \mu; v, \Phi) := (1 - \beta)r(t, a, \mu) + \beta \int_T v(t', \Phi(\mu))q(dt'|t, a, \mu),$$

Put

$$\Gamma(t, \mu; v, \Phi) = \arg \max_{a\in\tilde{A}(t,\mu_T)} F(t, a, \mu; v, \Phi),$$

and $\overline{\gamma}(t, \mu; v, \Phi) := \bigvee \Gamma(t, \mu; v, \Phi)$ and $\underline{\gamma}(t, \mu; v, \Phi) := \bigwedge \Gamma(t, \mu; v, \Phi)$ if both exist.

Define

$$B(v)(t, \mu, \Phi) := \max_{a\in\tilde{A}(t,\mu_T)} F(t, a, \mu; v, \Phi),$$

Lemma 6. *Assume 2. Let $v \in \mathcal{V}$. Then*

(i) $F(t, a, \mu; v, \Phi)$ has increasing differences in $(a; t, \mu, \Phi)$ and in $(t; \mu, \Phi)$;

(ii) $\overline{\gamma}(t, \tau, \Phi; v)$ and $\underline{\gamma}(t, \mu, \Phi; v)$ are well defined functions. Moreover, both are increasing in (t, μ, Φ);

(iii) $F(t, a, \mu; v, \Phi)$ is continuous in (t, a) and jointly mon-sup-inf preserving in (μ, Φ);

(iv) $B(v)(t, \mu, \Phi)$ is increasing in t and has increasing differences in $(t; \mu, \Phi)$;

(v) $B(v)(t,\mu,\Phi)$ is continuous in t, and is jointly mon-sup-inf preserving in (μ,Φ);

(vi) $\overline{\gamma}(t,\mu,\Phi;v)$ is mon-inf-preserving, and $\underline{\gamma}(t,\mu,\Phi;v)$ is mon-sup-preserving in (μ,Φ);

(vii) $\overline{\gamma}(t,\mu,\Phi;v)$ and $\underline{\gamma}(t,\mu,\Phi;v)$ are both measurable in t;

Proof. Let $v \in \mathcal{V}$. Put $\Theta = \mathcal{M}\times\mathcal{D}$ with corresponding product stochastic dominance order and of "pointwise order" in $\mathcal{D}$. Obviously each evaluation: $(\Phi,\mu) \in \mathcal{D}\times\mathcal{M} \to \Phi(\mu)$ is increasing. Proof of (i) follows directly from Assumption 2 and Lemma 3. The statement of monotonicity in (ii) is a direct consequence of (i) and Theorem 6.2 in Topkis (1978). We prove (iii). The continuity in (t,a) is an easy consequence of Assumption 2. We only prove the mon-sup-inf preserving property in (μ,Φ). Suppose that $(\mu_n,\Phi_n) \to (\mu,\Phi)$ and this convergence is increasing (decreasing) in both coordinate. Then, by Lemma 5 $\Phi_n(\mu_n) \to \Phi(\mu)$, and this is a monotone convergence. As a result, since $v \in \mathcal{V}$, hence $v(t,\Phi_n(\mu_n),\mu_n) \to v(t,\Phi(\mu),\mu)$. The remaining part of this proof follows from Lemma 12. The proof of statement (iv) follows from (i) and Lemma A1 in Hopenhayn and Prescott (1992). Now we prove (v). From (iii) and Lemma 13 we conclude that $B(v)(t,\mu,\Phi)$ is continuous in t. Let $\mu_n \to \mu$, $\Phi_n \to \Phi$ in increasing (decreasing) way, and $a_n \to a$. Then, by the same argument as in (iii) we conclude that $F(t,a,\mu_n;v,\Phi_n) \to F(t,a,\mu;v,\Phi)$. Hence, by Lemma 13 $B(v)(t,\mu_n,\Phi_n) \to B(v)(t,\mu,\Phi)$. We prove (vi). Let $(\mu_n,\Phi_n)_{n\in\mathbb{N}}$ be decreasing sequence convergent to (μ,Φ). Then by (ii) $\overline{\gamma}(t,\mu_n,\Phi_n;v)$ is decreasing in n, hence $\overline{\gamma}(t,\mu_n,\Phi_n;v)$ is convergent to some γ_0. By (iii) and Lemma 13 $\gamma_0 \in \Gamma^*(t,\mu;v,\Phi)$, hence $\gamma_0 \le \overline{\gamma}(t,\mu,\Phi;v)$. On the other hand, by (ii) $\gamma(t,\mu_n,\Phi_n;v) \ge \overline{\gamma}(t,\mu,\Phi;v)$, hence $\gamma_0 = \overline{\gamma}(t,\mu,\Phi;v)$. As a result $\overline{\gamma}$ is inf-preserving in (μ,Φ). By similar argument we can show that $\underline{\gamma}$ is sup-preserving in (μ,Φ). Finally we prove (vii). Observe that F is a Carathéodory function in (t,a) i.e. measurable in t and continuous in a. It follows from Assumptions 2, definition of $\mathcal{V}$ and Lemma 6 (ii). Hence, by Lemma 1 and Measurable Maximum Theorem (Theorem 18.19 in Aliprantis and Border (2006)) the correspondence $\Gamma(a,t;v,\Phi)$ is measurable in (t,a), hence by Lemma 18.2 in Aliprantis and Border (2006) is weakly measurable. For each $j = 1,2,\ldots,k$ the correspondence (drop all arguments but (t,a) from Γ and $\overline{\gamma}$ for short)

$$\Gamma^j(t,a) := \arg\max_{a'\in\Gamma(a,t)} \pi_j(a'),$$

where $\pi_j(a)$ is a projection of the vector a into $j-$ the coordinate. Obviously, π_j is a Carathéodory function as a function of (t,a). Again by Measurable Maximum Theorem $\tilde{\pi}_j(t) := \max_{a'\in\Gamma(a,t)} \pi_j(a)$ is measurable. Observe that by part (ii) of this lemma it follows that

$$\overline{\gamma}(t) = (\tilde{\pi}_1(t), \tilde{\pi}_2(t), \ldots, \tilde{\pi}_k(t)).$$

Since all coordinates are measurable, hence $\overline{\gamma}$ is measurable as well. Similarly we can prove that $\underline{\gamma}$ is measurable in t. □

Lemma 7. *Assume 2. B maps $\mathcal{V}$ into itself, and is a contraction mapping. As a result, B has an unique fixed point $v^* \in \mathcal{V}$ and satisfies (6).*

Proof. Let $V \in \mathcal{V}$. From Lemma 4 $\mathcal{V}$ is complete metric space. By Lemma 6 $B : \mathcal{V} \mapsto \mathcal{V}$. Clearly B is a $\beta-$contraction. As a result, by Banach Contraction Principle there is unique fixed point of B in $\mathcal{V}$. $\square$

4.3 The main results

Let $\star$ be a binary operation between $\tau \in \mathcal{M}_T$ and the set of measurable functions $h : T \to A$ returning probability measure on $T \times A$ defined as:

$$\tau \star h := \tau(id_T(\cdot), h(\cdot))^{-1}.$$

In the other words for all measurable $E_1 \subset T$, $E_2 \subset A$ $\tau \star h$ is a measure defined as follows:

$$(\tau \star h)(E_1 \times E_2) = \tau(\{t \in T : t \in E_1, h(t) \in E_2\}).$$

In case of correspondence $H : T \rightrightarrows A$: $\tau \star h := \tau(id_T(\cdot), H(\cdot))^{-1} := \cup_{h \in H} \tau \star h := \tau(id_T(\cdot), h(\cdot))^{-1}$, where the union is taken over all measurable selections $h \in H$.

We define the following operators: $\overline{\Psi}(\mu, \Phi) := (\mu', \Phi')$ where

$$\mu' = \mu_T \star \overline{\gamma}(\cdot, \mu, v^*, \Phi), \text{ and } \Phi'(\mu') = \phi(\mu') \star \overline{\gamma}(\cdot, \Phi(\mu'^*, \Phi) \quad \forall_{\mu' \in \mathcal{M}}.$$

Similarly, we can define $\underline{\Psi}$ using $\underline{\gamma}(t, \tau, \Phi, v^*)$ We will show that fixed point (μ^*, Φ^*) of Ψ exist and constitutes a MSE. Moreover, the sequence $(\mu_n^*)_{n \in \mathbb{N}}$ defined as $\mu_1^* = \mu^*$ and for $n \geq 1$ $\mu_{n+1}^* = \Phi^*(\mu_n^*)$ is a distributive equilibrium.

We need some properties of $\star$:

Lemma 8. *Assume 2. We have the following properties:*

(i) Suppose τ_1 and τ_2 are probability measures on T and $\tau_1 \leq_{st} \tau_2$. Further suppose that with $h_1 \leq h_2$ with $h_1 : T \times A$, $h_2 : T \times A$ increasing functions. Then

$$\tau_1 \star h_1 \leq_{st} \tau_2 \star h_2,$$

(ii) Let $\tau_N \in \mathcal{M}_T$ and $h_N : T \times A$ be increasing monotonically inf-preserving functions for each $N \in \mathbb{N}$. Then if $\tau_N \downarrow \tau$ and $h_N \downarrow h$ pointwise (as $N \to \infty$) then

$$\tau^N \star h^N \to_{weakly} \tau \star h.$$

Proof. The proof of (i) is straightforward. We only prove (ii). Let $\nu_N \in \mathcal{M}_T$ and $h_N : T \times A$ be increasing monotonically inf-preserving functions for each $N \in \mathbb{N}$. Suppose $\nu_N \downarrow \nu$ and $h_N \downarrow h$

pointwise (as $N \to \infty$) and f is continuous bounded and real valued function on $T \times A$. Then

$$\int_{T\times A} f(t,a)(\nu^N \star h^N)(dt \times da) = \int_T f(t, h_N(t))\nu_N(dt) \tag{9}$$

$$= \int_T f(t, h(t))\nu(dt = \int_{T\times A} f(t,a)(\nu \star h)(dt \times da) \tag{10}$$

where the second last equality follows from Lemma 12. □

Theorem 9. *Assume 2. There exists a MSE. Moreover, the set of all MSE has the greatest and the least element (in the coordinatewise partial order on $\mathcal{M} \times \mathcal{D}$).*

Proof. By Lemma 6 the $\overline{\gamma}(t, \tau, \Phi; v^*)$ is jointly increasing. Hence by Lemma 8 τ' is increasing in (τ, Φ). By the same argument Φ' is increasing in (τ, Φ). Hence $\overline{\Psi}$ is increasing. By Lemma 8 and Lemma 6 $\overline{\Psi}$ is monotonically inf-preserving. We can hence apply Markowski theorem (theorem 5) on a chain complete poset $\mathcal{M} \times \mathcal{D}$ to conclude existence of the greatest MSE. Similarly, we can define $\underline{\Psi}$ using $\underline{\gamma}(t, \tau, \Phi, v^*)$ and conclude existence of the least MSE. □

Remark 2. *Each MSE induces a Markov Stationary distributive Equilibrium (μ_t^*), where $\mu_0^* = \mu^*$ and $\mu_t^* = \Phi^*(\mu_{t-1}^*)$.*

Corollary 1 (Equilibrium bounds)**.** *Assume 2. We have:*

(i) A sequence $\overline{\Psi}^n(\overline{\mu}_0, \overline{\Phi}_0)$ is decreasing in n and hence convergent. Moreover, $\underline{\Psi}^n(\underline{\mu}_0, \underline{\Phi}_0)$ is increasing and hence convergent.

(ii) for any equilibrium (μ^*, Φ^*) we have

$$(\underline{\mu^*}, \underline{\Phi^*}) := \lim_{n\to\infty} \underline{\Psi}^n(\underline{\mu}_0, \underline{\Phi}_0) \le (\mu^*, \Phi^*) \le \lim_{n\to\infty} \overline{\Psi}^n(\overline{\mu}_0, \overline{\Phi}_0) := (\overline{\mu^*}, \overline{\Phi^*}).$$

Corollary 2 (Approximation)**.** *Assume 2. Under our assumptions $(\underline{\mu^*}, \underline{\Phi^*})$ and $(\overline{\mu^*}, \overline{\Phi^*})$ are MSE.*

Proof. Follows directly from Tarski-Kantorovich theorem 6 after observing that $\overline{\Psi}$ is monotonically inf preserving, while $\underline{\Psi}$ is monotonically sup preserving. □

We now proceed to characterize the extremal invariant distributions implies by extremal MSE.

Corollary 3 (Invariant distributions)**.** *Assume 2. There exists the greatest and the least invariant distribution ($\underline{\mu}^*, \overline{\mu}^*$, respectively) implied by greatest and least MSE, i.e. $\underline{\mu}^* = \underline{\Phi}^*(\underline{\mu}^*)$ and $\overline{\mu}^* = \overline{\Phi}^*(\overline{\mu}^*)$.*

Proof. Invariant distributions are fixed points of Φ. Hence existence, follows directly from application of Markowsky (1976) fixed point theorem on a monotone map $\underline{\Phi}^*$ and observing that

$\mathcal{D}$ is a chain complete poset. Similarly, one can proceed for $\overline{\Phi^*}$. To show that the least fixed point of $\underline{\Phi^*}$ is the least invariant distribution (and the greatest fixed point of $\overline{\Phi^*}$ is the greatest invariant distributions), we use results of theorem 9. □

Our theorems prove a number of things. First they guarantees existence of MS(Distributional)E. Moreover they assure existence of the greatest and the least of such equilibria, and as a consequence gives upper and lower bounds on the whole equilibrium set. If Γ is single-valued then $\overline{\gamma}$ and $\underline{\gamma}$ coincide and theorem establishes that MSE has a particular order theoretical structure, namely its a chain complete and as a result equilibrium set is closed under monotone equilibrium sequences. Finally, and perhaps most importantly theorem guarantees that the extremal MSE can be computed iterating on operators $\underline{\Psi}$ from above and on $\overline{\Psi}$ from below.

To the best of our knowledge it is the first such strong and constructive results on existence of MSE in the literature of large games. Similar results but for games with finite number of players has been showed in Balbus, Reffett, and Woźny (2013, 2014). Although similar the methods used in profs are different. Firstly in the game with finite number of players we show existence of the MSE using operators mapping between spaces of equilibria values, and only track within period equilibrium strategies. Hence the computation part of our results concern equilibrium values. This approach can be replicated for the (large) game studied in this paper. However here we explore a different approach, and prove all results working in strategies and values directly. This allows as to characterize and compute the equilibria directly, i.e. not through values. Again we think that this approach can be used in the finite number of players version of our game. Secondly the methods concerning fixed points existence in Balbus, Reffett, and Woźny (2013, 2014) are different that the one used here. Proof of theorem our main theorem is based on the fixed point existence for monotone self-maps in chain complete posets due to Markowski). As a consequence, here we are additionally able to characterize the order structure of equilibria when Γ is single valued. From this point of view, an important aspect of our construction is that here we define an operator, whose fixed points determine equilibrium law of motion and equilibrium distribution jointly.

The application of monotone operators to the study of distributive equilibria in large games in not completely new. Sleet (2001) main existence argument is also based on the iteration of the monotone self-map in the space of distributions. However the main assumptions used in his paper to guarantee the main results are placed on the payoff function while here rather on the transition q. Specifically the main theoretical result of Sleet is stated in his lemma A4 where he proves the monotonicity of the fixed-point operator guaranteeing that the Bellman operator preserves increasing differences of the parameterized value function between states and parameter (other player continuation strategy). Having established that he solves player's optimization problem and shows that maximal selection of the argmax correspondence is increasing in the other players' continuation strategy. Finally he considers an operator on the space of increasing, usc strategies on a (single dimensional state space) and obtain an equilibrium in pure strategies as a fixed point following Knaster-Kantorovitch theorem. The final stage of his proof (operator mapping in the space of increasing, usc strategies) is similar to the arguments used by Sundaram (1989) or Amir (2005). Also, the careful reader realized that his argument will also work in the deterministic case, while our approach is suitable for stochastic only.

4.4 Monotone comparative dynamics

We finish this section with a results on monotone comparative dynamics. That is, we parameterize primitives of our game with θ in a poset Θ and seek conditions under which MPE is monotone in θ. Observe, is means not only that some selection from $\theta \rightrightarrows \mu^*(\theta)$ is monotone but also some selection from equilibrium law of motion $\theta \rightrightarrows \Phi^*(\theta)$ is. Hence we prefer to call it a monotone comparative dynamics rather than monotone comparative statics results (see e.g. Acemoglu and Jensen (2010, 2015)). We first define a positive shock.

Assumption 3 (Positive shock)**.** *On top of the previous assumptions (for each θ in a poset Θ), let:*

(i) $r(t, a, \mu; \theta)$ *has increasing differences in* (a, θ) *and in* (t, θ)

(ii) $q(t, a, \mu; \theta)$ *be increasing in* θ *and have increasing differences in* (a, θ) *and in* (t, θ)

(iii) $\tilde{A}(\cdot; \theta)$ *be strong set order ascending in* θ.

Theorem 10 (Monotone Comparative Dynamics)**.** *Under our assumptions* $\theta \to (\underline{\mu^*}(\theta), \underline{\Phi^*}(\theta))$ *and* $\theta \to (\overline{\mu^*}(\theta), \overline{\Phi^*}(\theta))$ *are increasing on* Θ.

Proof. We focus attention on $\theta \to (\overline{\mu^*}(\theta), \overline{\Phi^*}(\theta))$. Let $\overline{\Psi}^\theta$ be the counterpart of the operator Ψ in the parametrized game with $\theta \in \Theta$. Similarly we denote ϕ^θ and $\overline{\gamma}^\theta$. Clearly, under assumptions that $q(t, a, \mu; \theta)$ is increasing in θ it is sufficient to show that $\theta \to \overline{\gamma}^\theta$ is increasing. Observe that under assumptions of the theorem objective $(1 - \beta) r(t, a, \mu, \theta) + \beta \int_T v^*(t', \phi(\mu), \beta) q(dt'|t, a, \mu, \theta)$ has increasing differences in (a, θ), $v^*(t, \mu, \theta)$ has increasing differences in t, θ for any μ. By Theorem 6.2. in Topkis (1978) we easily conclude that $\overline{\gamma}$ is increasing in θ. See also Hopenhayn and Prescott (1992). By Assumption 3 and definition we conclude that $\theta \in \Theta \mapsto \phi^\theta$ is isotone. The same property is inherited by Ψ^θ from its definition and Lemma 8. Moreover, similarly as in the proof of Theorem 9 we conclude that $\overline{\Psi}^\theta(\cdot)$ is an isotone operator for any fixed θ. To finish this proof we only need to apply Theorem 7, recalling that a poset of distributions and poset of uniformly bounded functions are chain complete. □

Here we report that related results of Adlakha and Johari (2013), Acemoglu and Jensen (2015), Light and Weintraub (2019) that concern comparative statics on invariant distributions or steady states of equilibrium aggregates only, while ours characterize comparative statics on dynamic equilibrium. This is of utmost importance. Assumptions of Acemoglu and Jensen (2015) or Light and Weintraub (2019) are not sufficient to obtain such strong results. Indeed, firstly conditions for comparing invariant distributions $\mu_{int}(\theta) = \Phi^*(\mu_{int})(\theta))(\theta)$ are much weaker than those of the whole dynamic equilibrium, including influence of perturbations of dynamic interactions via the value function $\mu \to v(t', \Phi(\mu))$ as well as perturbations of the equilibrium law of motion $\theta \to \Phi^*(\theta)$. Secondly, recall that we compare distributions over $\mathbb{R}^n$, and not their moments. Set of such objects is not a lattice, hence necessity of new comparative statics tools we provide in theorem 7. See also discussion in Light and Weintraub (2019) section[43] 3.

5 MPE as approximation of a large but finite player game

It is worth to compare our results to those from mean-field equilibrium (henceforth MFE) literature. Here we mention that some authors have advocated MFE as a limit of Nash equilibria in small dynamic/stochastic games (see e.g. Doncel, Gast, and Gaujal (2016) or Lacker (2018)). For obvious reasons, this argument in limited to some equilibria only (see counterexamples in Doncel, Gast, and Gaujal (2016) among others).

Weintraub, Benkard, and Van Roy (2008) and Adlakha, Johari, and Weintraub (2015) show, however, a form of lower semicontinuity of the MFE correspondence when number of players converges to infinity, i.e. they show that for each MFE there exists a Markov perfect equilibrium of a dynamic game whose limiting (with a number of players converging to infinity) payoff corresponds to that MFE payoff. This is satisfied on invariant distribution and under, so called, light tail condition[44]. To show this results, authors need appropriate payoffs continuity assumption, as well as make sure that a sample from a sequence of measures (index by number of players n) from an MFE invariant distribution converges to this MFE.

Stronger result is provided by Kalai and Shmaya (2018), who show that any large (dynamic) game equilibrium is an epsilon MPE of a small game counterpart. Specifically, they show that every imagined equilibrium strategy is an ϵ-Nash equilibrium of its n-player game counterpart. ϵ converges to zero with $n \to \infty$. To obtain this result, authors assume that the aggregative function (mapping distributions to aggregates values) is Lipschitz continuous. As opposed to Adlakha, Johari, and Weintraub (2015), they do not require an invariant distribution, but rather show their result on the whole equilibrium path (i.e. sequences of measures over action-states). Technically, their result is based on a total variation norm and coupling of random vectors.

Following this line of research, in this section we consider the model in which the number of agents is N, where N is a large but finite natural number. The agent j_0 observes the own type X_{t,j_0} in period n and imagines that the distribution of others types is τ_t (on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$). The empirical distribution function on types is given by

$$M_{n,N}(B) = \frac{\#\{j \in \{1, 2, \dots, N : X_{n,j} \in B\}\}}{N}.$$

[43] And why a complete order $\succeq$ on the space of distributions is necessary for their results.

[44] In the appropriate $1-p$ norm. Authors prove existence of a MFE in space of distributions with such a norm.

Let (μ^*, Φ^*) be a MSE of a game with continuum players with $\sigma^* : T \times \mathcal{M}_T \to A$ an equilibrium strategy. We claim that for any $\epsilon > 0$ and large enough N, σ^* is an ϵequilibrium of a N-players game described above.

Suppose that $\sigma(t, \tau)$ is a stationary policy played by all agents. Then, the agent believes $\tau_{n+1} = \phi(\tau_n \star \sigma_{\tau_n})$ for any n, for some initial τ_1. Then the payoff is given by:

$$R(t) = \mathbb{E}\left(\sum_{n=1}^{\infty} r(X_{n,j}, \sigma(X_{n,j}, \tau_n), M^{\sigma}_{n,N})\beta^{n-1} \,|X_{1,j} = t\right),$$

where

$$M^{\sigma}_{n,N}(B) = \frac{\#\{j \in \{1, 2, \ldots, N\} : (X_{n,j}, \sigma(X_{n,j}, \tau_n)) \in B\}}{N}$$

and $(X_{n,j})_n$ is a Markov chain with a transition probability

$$\mathbb{P}(X_{n+1,j} \in B|X_{n,j}) = q(B|X_{n,j}, \sigma(X_{n,j}, \tau_n), M^{\sigma}_{n,N}).$$

For any probability measures μ_1 and μ_2 on $\mathcal{T}$ we denote the total variation distance as

$$||\mu_1 - \mu_2||_V := 2 \sup_{Z \in \mathcal{T}} |\mu_1(Z) - \mu_2(Z)|.$$

Clearly, for any measurable and bounded f we have

$$|\int_T f d\mu_1 - \int_T f d\mu_2| \le ||f||_\infty ||\mu_1 - \mu_2||_V.$$

Assumption 4. *Assume that is* $\sup_{t \in T, a \in A} ||q(\cdot|t, a, \tau') - q(\cdot|t, a, \tau)||_V \to 0$ *whenever* $||\tau' - \tau||_V \to 0$

By convention, for any set Z let us denote $\emptyset \times Z := Z$. By $\mathcal{B}(Z_1, Z_2)$ denote the set of all Borel measurable functions from Z_1 to Z_2. Let $H_1 = \emptyset$ and for $t > 1$ we denote H_t be a set of histories before n with a generic element $h_n := (s_1, a_1, \ldots, s_{n-1}, a_{n-1}) \in Gr(\tilde{A})^{n-1}$. Let $H := \bigcup_{n=1}^{\infty} H_n$. Here s_n denotes a pair of private type and distribution on agents' type.

Let

$$\Sigma_n : \{\pi_n : H_n \times T \mapsto \Delta(A) : \pi_n \in \mathcal{B}(H_n \times T, \Delta(A)), \pi_n(\mathcal{A}(t, \tau_n)|h_n, t) = 1\}$$

and $\Sigma = \prod_{n=1}^{\infty} \Sigma_n$. For any policy $\pi \in \Sigma$ we define $\pi^n : H_n \mapsto \Sigma$ as follows:

$$\pi^{(n)}(h_n) := (\pi_k)_{k=n}^{\infty}.$$

Consider the case that the agent j_0 unilaterally deviates and uses the policy $\pi \in \Sigma$ instead of σ^*. Then the payoff in step $n=1$ is

$$v_1^N(t,\pi) := \mathbb{E}\left(\sum_{n=1}^{\infty}(1-\beta)r(X_{n,j_0}, Y_{n,j_0}, M_{n,N}^Y)\beta^{n-1}\,|X_{1,j}=t\right),$$

where

$$M_{n,N}^Y(B) = \frac{\#\{j\in\{1,2,\ldots,N : (X_{n,j},Y_{n,j})\in B\}}{N}.$$

Here, $Y_{n,j} = \sigma^*(X_{n,j},\tau_n)$ whenever $j\neq j_0$ and $Y_{n,j_0} = Y_n$ where Y_n is a process of actions induced by the policy π. Then, $(X_{n,j})_n$ is a Markov chain with a transition

$$\mathbb{P}(X_{n+1,j_0}\in B|X_{n,j_0}) = q(B|X_{n,j_0},\sigma^*(X_{n,j_0},\tau_n), M_{n,N}^Y).$$

By $\to^V$ denote convergence with the total variation norm. The next lemma is immediate from Lemma 14.

Lemma 9. *For any $n\in\mathbb{N}$, $M_{n,N}^\sigma \to^V \tau_n\star\sigma_{\tau_n}$ and $M_{n,N}^Y\to^V \tau_n\star\sigma_{\tau_n}$ as $N\to\infty$.*

By Glivenko-Canteli Theorem we have that the empirical distribution uniformly converges to the theoretical distribution. A unilateral deviation does affect this fact (see Lemma 14).

By $v_n^N(t,\pi^{(n)})$ we denote the discounted expected sum from n to ∞ when j_0 uses π and all but j_0 agent use σ^*. By Bellman equation we have

$$v_n^N(t,\pi^{(n)}(h_n)) = \int_{\tilde{A}(t,\tau_n)}\left\{(1-\beta)r(t,a,M_{n,N}^Y) + \beta\int_T v_{n+1}^N(t'^{(n+1)}(h_{n+1}))q(dt'^N_{n,Y})\right\}\pi_n(da|h_n),$$

where $h_n = (s_1,a_1,\ldots,s_{n-1},a_{n-1})$.

Define

$$\mathbf{X} := \left\{(f_n)_n : f_n : T\times\Sigma :\mapsto \mathbb{R} : \forall_{n\in\mathbb{N}} f_n(t,\pi^n(h_n))\in\mathcal{B}(H_n\times T,\mathbb{R}), \sup_{n,t,\pi}|f_n(t,\pi)|\leq\bar{r}\right\}.$$

Endow $\mathbf{X}$ with the sup norm

$$||f||_\infty := \sup_{n,t,\pi}|f_n(t,\pi)|.$$

Let us define a contraction mapping $\Psi^N:\mathbf{X}\mapsto\mathbf{X}$ as follows: $\Psi^N := (\Psi_1^N,\Psi_2^N,\ldots)$ where

$$\Psi_n^N(f)(t,\pi^n(h_n)) := \int_{\tilde{A}(t,\tau_n)}\left\{r(t,a,M_{n,N}^Y) + \beta\int_T f_{n+1}(t'^{(n+1)}(h_{n+1}))q(dt'^Y_{n,N})\right\}\pi_n(da|h_n,t).$$

Observe that $(v_n^N(t,\pi))_n$ is a unique fixed point of Ψ^N in $\mathbf{X}$. Moreover, it is globally attractive. Now we have the following lemma.

Lemma 10. *There exists $\Psi^\infty : \mathbf{X} \mapsto \mathbf{X}$ such that*

$$\lim_{N\to\infty} \sup_{f\in\mathbf{X}} ||\Psi^N(f) - \Psi^\infty(f)||_\infty = 0. \tag{11}$$

Moreover, there is $\tilde{v} \in \mathbf{X}$ - a unique fixed point of Ψ^∞, such that $\lim_{N\to\infty} ||\tilde{v} - v^N||_\infty = 0$.

Proof. The existence of Ψ^∞ follows directly from Glivenko-Cantelli Theorem and Lemma 9. We show (11). Observe,

$$|\Psi_n^N(f)(t,\pi) - \Psi_n^\infty(f)(t,\pi)| \le \beta \left| \int_T f_n(t', \pi^{(n)}(h_n)) q(dt'|t, a, M_{n,N}^Y) - \int_T \int_T f_n(t', \pi^{(n)}(h_n)) q(dt'|t, a, \tau_n \star \sigma_{\tau_n}) \right|,$$

hence

$$\sup_{f\in\mathbf{X}} ||\Psi^N(f) - \Psi^\infty(f)||_\infty \le \frac{\beta}{1-\beta} \bar{r} \sup_{t\in T, a\in A} ||q(\cdot|t, a, M_{n,N}^Y) - q(\cdot|t, a, \tau_n \star \sigma_{\tau_n})||_V.$$

Applying again 9 and Assumption 4 the right hand side tends to 0 as $N \to \infty$. Hence the first part is proven.

Finally,

$$\begin{aligned} ||v^N - \tilde{v}||_\infty = & \quad ||\Psi^N(v^N) - \Psi^\infty(\tilde{v})||_\infty \\ \le & \quad ||\Psi^N(v^N) - \Psi^N(\tilde{v})||_\infty + ||\Psi^N(\tilde{v}) - \Psi^\infty(\tilde{v})|| \\ \le & \quad \beta ||v^N - \tilde{v}||_\infty + \sup_{f\in\mathbf{X}} ||\Psi^N(f) - \Psi^\infty(f)||_\infty. \end{aligned}$$

Rearranging this term and using the first part of this lemma we have

$$\limsup_{N\to\infty} ||v^N - \tilde{v}||_\infty \le \frac{1}{1-\beta} \limsup_{N\to\infty} \sup_{f\in\mathbf{X}} ||\Psi^N(f) - \Psi^\infty(f)||_\infty = 0.$$

□

Theorem 11. *For any $\epsilon > 0$ there is N_0 such that for any $N > N_0$, σ^* is ϵ - equilibrium in N-player game.*

Proof. Observe that $v_1^N(t,\pi)$ corresponds the payoff in the N player game, when all but j_0 agent uses σ^* and j_0 uses π. Moreover, $\tilde{v}^\infty(t,\pi)$ corresponds the payoff in infinitely (but countably) many players' game with similar profile as before. Applying Glivenko-Cantelli Theorem and lemma 9 Nash equilibrium in the game with countably many players is in fact equivalent to the distributional equilibrium of a large game, and the stationary policy σ^* is Nash equilibrium. Hence and by Lemma 10 and Lemma 15 we immediately have the thesis. □

6 Applications and examples

6.1 Production externalities, idiosyncratic risk, and technological dynamics

We first consider the application suggested in Acemoglu and Jensen (2015) to the model of Romer (1986) but adapted to heterogeneity with idiosyncratic risk, endogenous labor supply, and non-convexities in both production on some aspect of preferences. Such a model then is a large dynamic economy version of the original model in Romer (1986) in the spirit of a Bewley model, and is closely related to versions of the Bewley model studied in Huggett (1997) and Angeletos and Calvet (2005). This example is important as it highlights the differences between our results (as applied to large dynamic economies) and those in the existing literature on "robust equilibrium comparative statics" for large dynamic economies (e.g., Acemoglu and Jensen (2015) and Acemoglu and Jensen (2018)). That is, if one only seeks *stationary equilibrium* comparative statics, we do not need conditions on primitives in the previous sections to guarantee the existence of such equilibrium comparative statics. In paritcular, we do not need to impose sufficient conditions on primitive that guarantee each players value function has increasing difference $(t, \mu_{T\times L})$ in a symmetric stationary Markovian distributional equilibrium; rather, the results on stationary equilibrium comparative statics just follow from Acemoglu and Jensen (2015) (Theorem 5, and Lemmas 1 and 2) assuming $u(c)$ is strictly concave.[45] But for our stronger comparative dynamics results (e.g., per equilibrium transition paths), we need the additional structure in this paper. To keep things simple, we assume each agent has type-specific production function, and that technology differs only by an efficiency unit shock to capital. [46]

The economy is populated by a saturated probability space of consumers $(\Lambda, \mathcal{L}, \lambda)$. We denote the set of all possible levels of capital by $T := \left[0, \bar{t}\right]$, where $\bar{t} > 0$. Endow T with the Borel algebra $\mathcal{T}$, and let $\mathcal{M}_T$ be the set of probability measures over T. Similar to Romer (1986), assume each agent possesses a private technology f that transforms private capital into finished output good, but the productivity of this technology depends also on an externality summarized by the distribution of capital levels and labor decisions in the economy $\mu_{T\times L}$. So, for example, each agent working $l \in [0,1] := L$ is able to produce $f(t, l, \mu_{T\times L})$ units of a single-dimensional consumption good. We assume the function $f : T \times [0,1] \times \mathcal{M}_{T\times L} \to \mathbb{R}_+$ is increasing with respect to all arguments and possess increasing differences (in t, l, in t, μ and l, μ). In particular, the private technologies endowed to each agent need not be convex. To bound the problem, we assume the existence of greatest possible level of output that equal to $f(\bar{t}, 1, \delta_{1,\bar{t},1})$, where $\delta_{\bar{t},1}$

[45] We should mention, in Acemoglu and Jensen (2015), to identify positive shocks as they discuss, they do need to require additional structure on primitives to preserve increasing differences/supermodularity between individual states and the positive shock. That requires more assumptions than noted in Lemmas 1 and 2, and those assumptions in many cases are closely related to the "preference-technology" complementarity assumptions we impose below in condition (PT-a) and (PT-b).

[46] Alternatively, you could make agents ex ante identical, and given them each an endowment of a unit of time, subject that unit of time to an efficiency unit shock $t \in [t_l, t_h]$ where $0 < t_l < t_b < \infty$ where under no aggregate risk, the mean of the labor shock equals 1. The interpretation of the model is that each agent as a production function $f(k, t, \mu_{T\times L})$ where now t denotes the agents technology in a given period.

is the Dirac measure concentrated at $(l,t) = (1,\bar{t})$. Then, let the action space for each player be given by $A := \left[0, f(\bar{t}, 1, \delta_{\bar{t},1})\right] \times [0,1]$, and endow this set with the Borel algebra $\mathcal{A}$. Finally, denote the set of probability measures over $\mathcal{T} \otimes \mathcal{A}$ by $\mathcal{M}$.

The output $f(t,l,\mu_t)$ for a particular player t can be either consumed or invested in order to increase the level of efficient capital holdings the next period, hence, $c+a = f(t,l,\mu_{T\times L})$. When c units of the output are consumed and labor supply is l, the agent receives $U(c,l) = u(c)+v(1-l)$ units of instantaneous utility, where we assume $u : \mathbb{R} \to \mathbb{R}$ to be continuous, concave, strictly increasing, and smooth, and v is continuous, smooth, and strictly increasing. On the other hand, whenever a units of the good are invested, the capital in the next period is determined stochastically with respect to probability measure $q(\cdot|t,a,\mu)$, where $\mu \in \mathcal{M}$ is a measure over pairs (t,a,l).

In order to simplify the analysis, we concentrate on the investment decisions of each consumer. Denote the set of feasible investment decisions of the consumer by $\left[0, f(t,l,\mu_{T\times L})\right]$. Suppose that the sum of discounted utilities from the next period till the end of time is given by $v : T \to \mathcal{M} \to \mathbb{R}$. Then, the optimization problem of a consumer of type t can be described as

Suppose for any $c, l, t, \mu_{T\times L}$, we assume the following restrictions on the joint curvature/complementarities in consumption preferences and technologies (PT): (PT-a) $\frac{-u''(c)}{u'(c)} \leq \frac{f''_{13}(t,l,\mu_{T\times L})}{f'_1(t,l,\mu_{T\times L})f'_3(t,l,\mu_{T\times L})}$ (i.e., that the degree of complementarity between private and aggregate capital is high relative to the curvature of the utility function, which suffices for increasing differences of payoffs with $t, \mu_{T\times L}$); (PT-b)$\frac{-u''(c)}{u'(c)} \leq \frac{f''_{23}(t,l,\mu_{T\times L})}{f'_2(t,l,\mu_{T\times L})f'_3(t,l,\mu_{T\times L})}$ for increasing differences of payoff with $l, \mu_{T\times L}$ and $\frac{-u''(c)}{u'(c)} \leq \frac{f''_{12}(t,l,\mu_{T\times L})}{f'_1(t,l,\mu_{T\times L})f'_2(t,l,\mu_{T\times L})}$ for increasing differences with respect to t, l. [47] Notice, in our setting, the correspondence $A(t,l,\mu_{T\times L}) = [0, f(t,l,\mu_{T\times L})]$ does not have strict cardinal complementarities in $(t, \mu_{T\times L})$ for each l; hence, we cannot preserve increasing differences to the value function $v^*(t,\mu;\Phi)$ in (6) using standard value function projection theorems. [48] But under our assumptions stated on u, v, and f (including the curvature conditions in (PT-a) and (PT-b) on u), noting the separability of preferences, we can adapt the results in Mirman, Morand, and Reffett (2008) on nonsmooth envelope theorems in dynamic lattice programming problems to verify that $\mathrm{v}^*(t,\mu_{T\times L};\Phi)$ in this case has increasing differences in $(t,\mu_{T\times L})$ for each Φ in this model.[49]

Given Theorem 9, we there concludes there exists the greatest and the least MSE for this

[47]Note, these two conditions are closely related to similar codiitions in the literature. The condition on complementarities between $t, \mu_{T\times L}$ is precisely the condition used in Mirman, Morand, and Reffett (2008) for a representative agent version of this environment, only adapted to the heterogeneous agent case. These conditions are also closely related to the sufficient condition for complementarity for joint monotone controls in a stochastic growth model with Markov shocks (e.g., see for discussion Hopenhayn and Prescott (1992), p. 1403).

[48]For example, Topkis (1998), Theorem 2.7.6 and Hopenhayn and Prescott (1992), Proposition 2 do not apply.

[49]In particular, we can directly adapt the arguments in Mirman, Morand, and Reffett (2008), Lemmas 11 and 12, and Theorems 3 and 4. Notice, we do not have monotone equilibrium controls/dynamics for aggregates in labor/leisure; rather, we only have monotone dynamics for the aggregates for mean capital. This is an interesting counterpart to the results reported for Bewley models in Huggett (1997), for example.

large dynamic economy (interpreted as a large anonymous game). If we now assume additionally function f is continuous in t and sup-inf preserving in $\mu_{T\times L}$ is sup-inf preserving, as $u(c)$ and $v(l)$ are continuous and smooth, the return function of this model $r(t, \mu_{T\times L}, a)$ $=u(f(t, 1-a_2, \mu_{T\times L}) - a_1)+v(a_2)$ is continuous in (t, a) and sup-inf preserving in $\mu_{T\times L}$. Therefore, the remaining conditions stated in Assumption 2 are satisfied, and we conclude that the extreme equilibria can be approximated using iterative methods.

Finally, if we additionally want to consider the question of monotone dynamics for labor/leisure aggregates, we can consider the case of Greenwood-Huffman-Hercowitz (GHH) no-income effect preferences for consumption and leisure. That is, let each agent have preferences each period given by $U(c, l) = u(c + v(l))$ where $u(z)$ and $v(l)$ satisfy our previous conditions, with $z = c + v(l)$. In this case, given the quasi-linear preference specification of preferences, the agents decision on labor-leisure is static, and does not depend on her dynamic choices for saving and investment. In this case, labor supply can be show to be increasing in aggregate capital also. Then, under our complementarity conditions on u and f noted in (PT-a) and (PT-b) we also get monotone dynamics for labor aggregates.

We should note, though, that in the case of the model with GHH preferences, the Inada conditions are weak (and only apply to the composite consumption "good" $z = c + v(l)$); hence, there is an new issue that arise of verifying sufficient conditions that guarantee that capital aggregates do not collapse to a mean capital level of zero. To avoid this situation, one can start the economies distribution of capital sufficiently "near" the stationary equilibrium level, and interpret this initial condition as a perturbation of stationary equilibrium, and the transitional dynamics back to stationary equilibrium aggregates as the object of interest from the vantage point of equilibrium transitional comparative dynamics.

6.2 Dynamics of social distances

We next consider a dynamic version of the *social distance* model described originally by Akerlof (1997), and recently formalized in a static setting in Balbus, Dziewulski, Reffet, and Woźny (2019). The model is important, as it is also related to many other strands of the social economics literature, including models of identity and economic choice as in Akerlof and Kranton (2000), as well as models with endogenous social reference points (for example, the conformity models of Bernheim (1994), the social interaction models of Brock and Durlauf (2001) and Blume, Brock, Durlauf, and Jayaraman (2015), social models of addiction such as **?**, as well as models with economic choice with cultural dimensions as in Bisin, Moro, and Topa (2011)). In the model we study, we characterizing the existence of least and greatest equilibrium dynamics for the distribution of social ranks/statuses over a large number of heterogeneous individuals. The model is very general, and can be reinterpreted or amended to fit many economic situations where agents preferences are endogenous, contain some important social dimensions, and the interaction in equilibrium is over a large number agents in the society.

Consider a saturated measure space of agents distributed over Λ, (say, a fixed α is a *location* of an individual agent). Let $T \subset \mathbb{R}$ be the set of all possible positions in the society (e.g., social statuses/ranks), where T is compact and convex. Each period an individual is characterized by an *aspiration* $t \in T$, which determines the social status/rank to which the agent aspires. We shall refer to the t as the *characteristic* of an agent.

We first describe the within period game. Each period, every agent knows his own characteristic t, as well as the distribution of characteristics across the population. In this game, we study how agents determine their optimal individual choice of social status $a \in [0, \gamma t] \subset T$. Value γt bounds the set of actions available for every agent, and parameter γ scales how far (in social distance) an agent can get from his true identity in one period.

The payoff of an individual is determined as follows. On the one hand, every agent has preferences of what level of status/rank they want to attain in proximity an aspired identity t. The agent, therefore, will suffer a penalty whenever his social status a does not match to his true identity t. That is status is valued, and the further away the actual status is from the true identity, the more disutility the agent receives from that status assignment. On the other hand, the individual payoff is affected by interactions with other agents in the status game. In particular, we assume that the players meet by random assignment. Whenever an individual agent meets another agent, he suffers a disutility if his social status a differs from the social status a' of the other individual. [50] In particular, the disutility for the agent increases the greater is the distance between the two levels of social status. This incorporates a form of peer pressure or conformism to the game.

To formalize this ideas, let u, $v : \mathbb{R} \to \mathbb{R}$ be a pair of continuous, decreasing functions. In addition, assume that v is concave. Consider an agent characterized by t, who chooses a social status $a \in [0, \gamma t]$. Whenever an agent randomly meets (or is randomly assigned) another individual with a social status $a' \in T$, his utility is given by:

$$u(|a - t|) + v(|a - a'|).$$

As both functions u and v are decreasing, the objective of every player is to choose an action as close as possible to their true identity t and the identity a' of the other player. Moreover, given concavity of function v, the further away is the status of the agent from the social rank of the other player, the steeper are the changes in the disutility.

Suppose further that the frequency of interactions of the agent with other individuals is governed by a probability measure ν_t, defined over the Borel-field of T. Therefore, for any set U, value $\nu_t(U)$ is the probability of encountering an agent with a characteristic $t' \in U$. We assume that the measure ν_t depends on t, as e.g. agents with similar aspirations are more likely to meet.

Let μ be a probability measure defined over the Borel-field of $T \times A$. Suppose that the

[50] Notice, this feature is very related to the idea of conformity in Bernheim (1994) and some models of aspiration preferences (e.g., Genicot and Ray (2017)).

marginal distribution of μ. Clearly, μ is a probability distribution of player characteristics and social ranks (t, a). Hence, μ denotes the measure of agents with particular characteristic t and social rank $a \in A$.

Given the notation, we define the decision problem faced by a typical agent in the game. The objective of a player is to choose his social status $a \in Y$ that maximizes his expected payoff given by

$$r(t, a, \mu) := u(|a-t|) + \int_T \int_T v(|a-a'|) d\mu(a'|t') d\nu_t(t'),$$

where $\mu(\cdot|t')$ is the distribution of actions of other players in the population conditional on $t' \in \times Y$. Therefore, the payoff of an agent is the sum of utilities that he receives from individual interactions with other agents. According to the above definition, the social status of an individual cannot be contingent on the social statuses of other agents, but has to be chosen ex-ante before any interaction occurs.

Next, given the type $t \in T$, chosen action a and distribution τ the next period distribution over private types is determined by the transition q. We assume that when agent chooses an higher action and aspires to the higher types the more likely it is to get a draw of a high type t' the next period. This is given by our specification.

Formally, since the set of feasible actions is $[0, \gamma t] \subset Y$ our assumption is satisfied. By our assumption, function v is concave and decreasing. Therefore, $v(|a-a'|)$ has increasing differences in (a, a'). It is easy to show that our conditions here are sufficient for payoff function $r(t, a, \tau)$ to have increasing differences with (a, τ).

We next show the equilibrium value of r is monotone in t. Consider an optimal strategy $\phi(t)$, where we drop ν to simply notation. We first claim that $\phi(t) - t$ is decreasing in t. To see this, put $z := a - t$ and consider the mapping:

$$\max_{z \in [-t, \gamma - 1t]} u(z) + \int_T \int_T v(|z + t - a'|) d\mu(a'|t') d\nu_t(t')$$

The objective has increasing differences in $(z, -t)$ and the constraint set is strong set order increasing in $-t$ for $\gamma \leq 1$. By Topkis (1978) theorem, we conclude that the optimal policy $z\,(= \phi(t) - t)$ is decreasing in t. To show that the function

$$\int_T \int_T v(|\phi(t) - \phi(t')|) d\mu(\phi(t')|t') d\nu_t(t')$$

is monotone in t, assume ν_t puts positive measure on $[t, \bar{T}]$. Hence, we have

$$r(t, \phi(t), \mu) = u(|\phi(t) - t|) + \int_T \int_T v(|\phi(t) - \phi(t')|) d\mu(\phi(t')|t') d\nu_t(t')$$

which under our additional assumptions is monotone t.

Knowing all this, we conclude by the main theorems of the previous section that there are greatest and least Markov Stationary Nash equilibrium, each implying equilibrium distributions over types. Importantly, our equilibrium is not stationary in the sense that the measure over types is invariant in time. On the contrary, our equilibrium distribution over identities may vary each period according to the law of motion that governs the evolution of states over time.

Finally, and importantly, as our primitives also satisfy the order-continuity assumption needed for the constructive results on computing extremal Markovian distributional equilibria. This allows us to iteratively compute the extremal equilibria by successive approximation. Moreover, notice since agents strictly care about both the status of other players as well as their own true identity, the extremal equilibria are only trivial in some very special cases. Therefore, the approximation methods may be very useful in determining and computing the distributional equilibria itself for both the least and greatest Markovian distributional equilibria.

6.3 Dynamics in quantile aspiration models

A third application of our results is to models with social interaction that fit within the "keeping up with the Jones" framework, but with a very general formation of endogenous aspiration preferences. This model related to the recent aspirations model of Genicot and Ray (2017), but without the formulation of endogenous reference points that generates a convex-concave interaction (as in, for example, prospect theory and the literature on reference-dependent gain-loss utility). We do this to ensure a global complementarities between consumption c and aspiration consumption q. What is interesting and new here is well allow but a very general formulation of aspiration reference points. In particular, they are quantile aspiration reference point preferences. .

Consider an economy consisting of a saturated probability space of consumers $(\Lambda, \mathcal{L}, \lambda)$. Suppose at any time period, each agent is characterized by his wealth $t \in [0, \bar{t}]$. Denote $T := [0, \bar{t}]$, and endow the set with the Borel algebra $\mathcal{T}$. Each period, the consumer may spend his wealth on the current consumption, or save it for the next period. Therefore, $c + a = t$, where c and a denote respectively the level of consumption and savings chosen by the consumer. In order to make our presentation more transparent, let $A := [0, \bar{t}]$, which denotes the set of all possible consumption/savings levels.

We are not ready to define the quantile aspiration preferences. For a typical agent with wealth t, when choosing her consumption, the consumer takes into account the i'th percentile of consumption in the economy (as an endogenous reference point). In particular, whenever choosing the level of consumption c, she receives an instantaneous utility $u(c, q_i)$ where q_i denotes the i'th percentile of consumption in the economy. Suppose also for convenience that $u(0, q_i) = 0$ and u has increasing differences in c, q_i and is increasing and concave with respect to the first argument. Observe that, whenever μ is the distribution of wealth/investment pairs (t, a) in the economy, then the contribution of aspirations to preferences can be computed as follows: let

$$q_i(\mu) := \inf\{c : i \leq \mu(\{(t, t - c') : c' \leq c\})\}.$$

Then, the function q_i is decreasing with respect to μ. We can then let period payoffs for this agent be: $r(t,a,\mu) := u\big(a, \theta q_i(\mu)\big)$ for some parameter $\theta > 0$ that measures the strength of negative externalities/aspiration preferences.

The optimization problem of the consumer is defined as:

$$\max_{a\in[0,t]} u\big(t-a, \theta q_i(\mu)\big) + \beta \int_{T\times A} v(t', \mu'_T) q(dt'|t,a),$$

For any distribution μ, we can compute the implied marginal distribution μ_A on A. Increasing differences between a, μ_A are implied by increasing differences assumption on u. While monotonicity of q with respect to the second argument imply increasing differences between a, t and monotonicity with t.

We finally note, we can identify a simple equilibrium comparative statics/dynamic prediction of the model. In particular, the Markovian distributional equilibrium is decreasing in θ, as for each agent, we have increasing differences between $(a, -\theta)$.

6.4 Dynamic for models with sunspots, coordination failures, and learning

We next consider a prototypical coordination game based on Angeletos and Lian (2016), recently applied to beauty contests, bank runs, riot games, or fixed exchange currency attacks (see for example Morris and Shin (2002) for an extensive discussion of this literature, as well as Carmona, Delarue, and Lacker (2017) for an interesting recent application of mean-field methods to a related class of games). In particular, we consider a simple class of dynamic beauty contests that nicely fit our class of models. Other models in the literature on dynamic coordination games and global games also can be shown to fit out setting.

For a beauty contest, consider a game where each player based on its private signal t chooses an action a. Action is potentially costly and its cost can depend on t, say via the utility function for a type t player given by $u(t,a)$. We assume that $t \in T$, with T a poset, and assume u has increasing differences between (t,a). Assume additionally u is increasing in t. Player t has a payoff that depends also on action taken by other players, say $\theta \int_A g(a,a')\mu_A(da')$, where g also has increasing differences between a, a'. As is standard in global games and dynamic coordinations games with complementarities, we study symmetric monotone in type equilibria. That is, we assume each player assumes the other players in the game are using some increasing strategy $\phi : T \to A$, so she will hence have a joint payoff given by:

$$r(t,a,\mu) := u(t,a) + \int_T g(a, \phi(t'))\mu_T(dt').$$

Observe that such payoff satisfies our assumptions in the previous section, so all our tools can be applied to study the existence of least and greatest Markovian distributional equilibria.

We should mention, typical examples of u and g for dynamic beauty contests in the literature include $u(t,a) = -(a-t)^2$, $g(a,a') = -(a-a^{'})^2$, with the action space being $\tilde{A}(t) = [0,t]$, where

t is a private information concerning an asset value, while $-(a-a')^2$ measures the distance from other player's valuation decision. An alternative specification for our model could be one of an dynamic *aggregative game*, where the payoff of a typical player t is given by:

$$r(t,a,\mu) := u(t,a) + g(a, \int_T \phi(t')\mu_T(dt')).$$

We can suppose that the first period signals t are drawn around the true state of the world $s \in S$. Here τ_n serves as a coordination device (even though distribution τ_n can be very far form the true state s), and the monotone transition on the type states (e.g. $t' \sim q(\cdot|t,a)$) can allow for *learning* effects in the game (e.g., the learning potentially also depending on action taken a as discussed in Angeletos and Lian (2016)).

Another example involves riot games or bunk runs. Here, we can take the payoffs to be:

$$r(t,a,\mu) := a[\int_{-S}^{S} (W-L)1_{\{R(\mu)\geq S-s\}}\nu(ds) + L] - c(a,t),$$

where taking a risky action ($a=1$) allows the player to win $W>0$ if sufficiently many other players take a risky (and costly) action, or loose $L<0$ if this is not the case. The fraction necessary to "win" in the period is $S-s$, and distribution of s is given by $\nu(\cdot)$ with $R(\mu) = \mu(\{(a,t) : a = 1\})$. Clearly, for increasing W, this game has increasing differences in a, μ and a, t and t, μ provided cost function c has decreasing differences in a, t. Moreover, the payoff is increasing in t whenever cost function c is decreasing in t. Again, in this model, the types t' can evolve over time according to transition function $q(\cdot|t,a,\mu)$, where this mapping can possess many features such as involving inertia via t, habit formation via a, or social externalities via μ making it cheaper to take a risky action in the future. See also recent Morris and Yildiz (2016) applications.

6.5 Short lived players and MPE in overlapping generations economies

In this subsection we show how to extend our methods to a class of games with short lived players, typically applied to OLG economies with social interactions. So, consider a game with a sequence of overlapping short lived players, each active for two consecutive periods. Specifically consider a set of players' traits as $C = \{y, o\}$, where y denote a young player (say born in period t) and o denotes an old player in period t (so born in period $t-1$). Each period Λ young layers are born. Including initial old generation (old in period $n=1$), each period we have Λ young and Λ old players.

Each player living for two periods, chooses action a, a' (in the first and the second period respectively), against population distribution μ, μ' with payoff given by

$$\max_{a\in\tilde{A}(t,\tau)} r_y(t,a,\mu) + \beta \int_T \tilde{v}(t', \Phi(\mu)) q(dt', t, a, \mu),$$

where $\tilde{v}(t', \mu') = \max_{a' \in \tilde{A}(t', \tau')} r_o(t', a', \mu')$ and r_y, r_o are payoff functions of young and old players. Initial old generation solves $\max_{a' \in \tilde{A}(t, \tau)} r_o(t, a', \mu)$. Now μ is a joint probability distribution on $H \times T \times A$ (with coordinate partial order) that specifies distributions of types and actions of young and old players. A special case of young players interacting only with young population (μ_y), old with old population (μ_o) also fits our framework. Assuming payoffs r_y, r_o and transition q satisfy our assumptions we can extend all our results to this class of games with short lived players. In is worth mentioning, that in this example we introduce players characteristics and hence show how to extend our tools and analysis to a class of dynamic semi-anonymous games with traits (see also Khan, Rath, Sun, and Yu (2013); Khan, Rath, Yu, and Zhang (2013)).

Similar OLG games with social interaction have been recently analyzed by Acemoglu and Jackson (2014, 2017) in their studies on evolution of social norms and enforcement of laws. Specifically, we can consider an OLG version of Acemoglu and Jackson (2017) of a random matching with an uncountable number of players and payoffs of the form $r_i(t, a, \mu) =$

$$
\begin{aligned}
&-\int_A \left[(-t + a + I_{\{a>L\}}[\theta_2 + (1-\theta_2)I_{\{\tilde{a} \le L\}}][L - a])^2 + \theta_1(a - \tilde{a} - I_{\{\tilde{a}>L\}}[\theta_2 + (1-\theta_2)I_{\{a \le L\}}][L - \tilde{a}])^2\right. \\
&\left.+(1-\theta_2)\theta_3 I_{\{\tilde{a} \le L\}} I_{\{a>L\}}\right] \mu_A(d\tilde{a}) - \theta_4 \int_A \tilde{a}\mu_A(d\tilde{a}) - \theta_2\theta_3 I_{\{a>L\}}
\end{aligned}
$$

with types space $T = [0, 1]$, action set $\tilde{A}(t) = [0, t]$. Here, $L \in [0, 1]$ is a law, i.e. an upper bound on behavior of agents, $\theta_1 \in (0, 1)$ is a social sensitivity, $\theta_3 \ge 0$ is a fine imposed on law-breakers (conditional on public enforcement with probability $\theta_2 \in [0, 1]$), while $\theta_4 \ge 0$ captures negative externality of the average behavior across population. Term $-(-t + a)^2$ measures mismatch between player's behavior and his/her type, while term $-(a - \tilde{a})^2$ measures a mismatch between behaviour of the matched players. In case of action exceeding the law ($I_{\{a>L\}}$), with probability θ_2 there will be a public enforcement and with probability $(1 - \theta_2)$ there will be a whistleblower [51] (which is only possible if the matched agent obeys the law i.e. $I_{\{\tilde{a} \le L\}}$, both bringing action a to level L). The same concerns the situation when the matched partner is breaking the law. This can reduce a mismatch between partners by adjusting $\tilde{a}$ to L.

With such specification, choice set is a complete lattice and payoff has increasing differences between a and μ, as well as a, t. Here, the payoff function is increasing in t but only upper-semicontinuous with a. As far as dynamics is concerned, we can consider a simple example of $q(\cdot|t)$ that does not depend on a neither μ, but also interesting cases where distribution of players types depend dynamically on social externalizes μ or involves habit formation, i.e. dependence of q of a. Monotone comparative statics/dynamics can also be easily obtained (MPE increasing in $-\theta_1, \theta_2, -\theta_3$), noting that the objective has increasing differences in $a, -\theta_1$ and $a, -\theta_3$, as well as that objective has increasing differences in a, θ_2 and t, θ_2.

[51] Alike Acemoglu and Jackson (2017) here, we assume that players wistleblow whenever allowed.

7 Appendix

Lemma 11. *Let $(X, \leq)$ be a poset with its order topology. Suppose $f_N : X \to \mathbb{R}$ is a sequence of increasing functions and monotonically inf-preserving. Then if $x_N \downarrow x$ in X and $f_N \downarrow f$ pointwise ($N \to \infty$). Then $f_N(x_N) \to f(x)$.*

Proof. Since f_N is decreasing sequence of increasing functions and $x_N \downarrow x$ hence for $N > k$

$$f(x) \leq f_N(x_N) \leq f_N(x_k).$$

We take a limit $N \to \infty$ and we obtain

$$f(x) \leq \liminf_{N\to\infty} f_N(x_N) \leq \limsup_{N\to\infty} f_N(x_N) \leq f(x_k).$$

to finish the proof we just take a limit $k \to \infty$. ☐

Lemma 12. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on common Polish space S and $(h_n)_{n\in\mathbb{N}}$ be a sequence of bounded measurable and bounded real valued functions on S. Suppose $\nu_n \downarrow \nu$ (i.e. in stochastic dominance order and weak topology) and $h_n \downarrow h$. Then $\lim_{n\to\infty} \int h_n d\nu_n = \int h d\nu$.*

Proof. It is a consequence of Lemma 11 with X as a space of bounded measurable real valued functions on S, and $f_n(x) := \int_S x(s)\nu_n(ds)$, $x_n(s) = h_n(s)$. ☐

Lemma 13. *Let Θ_1 and Θ_2 be topological spaces and $f : \Theta_1 \times \Theta_2 \mapsto \mathbb{R}$ be jointly continuous function. Let $\Gamma : \Theta_1 \mapsto \Theta_2$ be a continuous and compact valued correspondence. Put $\Gamma^*(x) := arg \max_{y\in\Gamma(x)} f(x, y)$. Let $x_n \to x$ in Θ_1, and $y_n \to y$ in Θ_2, and $y_n \in \Gamma^*(x_n)$. Suppose that $x_n \to x$, and $y_n \to y$ as $n \to \infty$. Then $y \in \Gamma(x)$.*

Proof. Let $y' \in \Gamma(x)$ be given. By continuity of Γ in x, for $n \in \mathbb{N}$ there is $y'_n \in \Gamma(x_n)$ such that $y'_n \to y'$. Observe that since $y_n \in \Gamma^*(x_n)$, hence $f(x_n, y_n) \geq f(x_n, y'_n)$ for all $n \in \mathbb{N}$. By joint continuity of f we have $f(x, y) \geq f(x, y')$. Since $y' \in \Gamma(x)$ is arbitrary, hence $y \in \Gamma^*(x)$. ☐

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_1, \ldots, X_N)$ an independent sampling of $\mathbb{R}^k$ valued function with distribution μ. Let

$$\hat{F}^N(x) := \frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{X_i \leq x},$$

be an empirical distribution function and F be the real distribution function. By Glivenko-Cantelli Theorem $\hat{F}^N \rightrightarrows F$ as $N \to \infty$. Let $\hat{\mu}^N$ be an empirical distribution, i.e. generated by $\hat{F}^N$:

$$\hat{\mu}^N(Z) := \frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_Z(X_j),$$

for $Z \in \mathcal{B}(\mathbb{R}^k)$.

We prove

Lemma 14. *Let $(X_1, \ldots, X_N)$ be a sampling of $\mathbb{R}^k$ valued function with distribution μ. If $k = 1$, then it holds*

$$\lim_{N\to\infty} \sup_{Z\in\mathcal{B}(\mathbb{R})} |\mu^N(Z) - \mu(Z)| \leq ||\hat{F}^N - F||_\infty \quad \mathbb{P} - a.e.. \tag{12}$$

As a result, for any k the total variation of the signed measure $\hat{\mu}^N - \mu$ tends to 0.

Proof. Consider the single dimensional case i.e. the random variables have values in $\mathbb{R}^k$. Let $Z = (z_1, z_2]$ for some $z_1 < z_2$. Pick $\epsilon > 0$. Observe

$$\sup_{z_1<z_2} |\hat{\mu}^N((z_1, z_2]) - \mu((z_1, z_2])| \leq ||\bar{F}^N - F||_\infty.$$

Indeed,

$$\begin{aligned} |\hat{\mu}^N((z_1, z_2]) - \mu((z_1, z_2])| \leq & \quad |\hat{F}^N(z_2) - F(z_2) - (\hat{F}^N(z_1) - F(z_1)|)| \\ \leq & \quad \max\left(|\hat{F}^N(z_2) - F(z_2)|, |\hat{F}^N(z_1) - F(z_1)|\right) \leq \quad ||\hat{F}^N - F||_\infty. \end{aligned} \tag{13}$$

Let $Z = \{z\}$. Observe that

$$|\hat{\mu}^N(\{z\}) - \mu(\{z\})| = \quad \lim_{z_1\uparrow z} |\hat{F}^N(z) - \hat{F}(z) - (\hat{F}^N(z_1) - \hat{F}(z_1))| \leq ||\hat{F}^N - F||. \tag{14}$$

Now let $Z = (z_1, z_2)$. Combining (13) and (14) we have

$$|\hat{\mu}^N(Z) - \mu(Z)| = |\hat{\mu}^N((z_1, z_2]) - \hat{\mu}((z_1, z_2]) - (\hat{\mu}^N(\{z_2\}) - \mu(\{z_2\}))| \leq ||\hat{F}^N - F||_\infty. \tag{15}$$

Similarly we prove that the inequality (12) holds for any set $Z = [z_1, z_2]$. Now take any open set Z. Obviously Z can be written as $Z = \bigcup_{k=1}^\infty O_k$ where $\{O_k : k \in \mathbb{N}\}$ is a family of pairwise disjoined open intervals. Let $O_k = (z_k, z_{k+1})$. Let $Z_n = \bigcup_{k=1}^n O_k$. We show that

$$|\hat{\mu}^N(Z_n) - \mu(Z_n)| \leq ||\hat{F}^N - F||_\infty \tag{16}$$

for any n. For $n = 1$ this thesis is already proven in (15). Suppose it is satisfied with $n \in \mathbb{N}$. We show the thesis with $n + 1$. By renumbering the elements if necessary suppose $z_1 < z_2 < z_3 < \ldots < z_{n+2}$. Let O'_k be defined as follows: $O'_k = O_k$ for $k = 1, 2, \ldots, n - 1$ and $O'_n = (z_n, z_{n+2})$.

We have $Z_{n+1} = \left(\bigcup_{k=1}^{n} O'_k\right) \setminus [z_n, z_{n+1}]$ Put $Z' = \bigcup_{k=1}^{n} O'_k)$ and $Z'' = [z_n, z_{n+1}]$. By induction hypothesis

$$|\hat{\mu}^N(Z') - \mu(Z')| \leq ||\hat{F}^N - F||_\infty. \tag{17}$$

By (13), (14) and (15)

$$|\hat{\mu}^N(Z'') - \mu(Z'')| \leq ||\hat{F}^N - F||_\infty. \tag{18}$$

By (17) and (18) we have

$$|\hat{\mu}^N(Z_{n+1}) - \mu(Z_{n+1})| = |\hat{\mu}^N(Z') - \mu(Z') - (|\hat{\mu}^N(Z'')) - \mu(Z'')|)| \leq ||\hat{F}^N - F||_\infty. \tag{19}$$

Taking a limit in the inequality above we have (12) for any open set. By taking complements, we have similar results for any closed set Z. Now suppose that Z is Borel. Pick $\epsilon > 0$. By Theorem 12.5 in Aliprantis and Border (2006) any Borel measure on metrizable space is inner and outer regular. Hence we can pick open O and closed set K such that $K \subset Z \subset O$

$$\mu(O \setminus K) \leq \frac{\epsilon}{2} \tag{20}$$

$$\hat{\mu}^N(O \setminus K) \leq \frac{\epsilon}{2}. \tag{21}$$

Hence

$$\hat{\mu}^N(Z) - \mu(Z) \leq \hat{\mu}(O) - \mu(O) + \mu(O) + \epsilon/2 \leq ||F^N - F||_\infty + \mu(O) - \mu(Z) + \epsilon/2 \leq ||\hat{F}^N - F||_\infty + \epsilon$$

and

$$\hat{\mu}^N(Z) - \mu(Z) \geq \hat{\mu}(K) - \mu(K) + \mu(K) - \epsilon/2 \geq -||\hat{F}^N - F|| - (\mu(Z) - \mu(K)) - \epsilon/2 \geq -||\hat{F}^N - F||_\infty - \epsilon.$$

Finally

$$|\hat{\mu}^N(Z) - \mu(Z)| \leq ||F^N - F||_\infty + \epsilon.$$

Since ϵ is arbitrarily small, the proof is complete for the single dimensional case.

Now suppose that the sampling functions are $\mathbb{R}^k$ valued. Let $\psi : [0,1] \mapsto \mathbb{R}^k$ be a Borel measurable function induces isomorphism. That is, every Borel set Z in $\mathbb{R}^k$ set has a form $Z = \psi(Z_0)$ for some $Z_0 \subset [0,1]$. Then we may define $Y_j = \psi^{-1}(X_j)$. Let $\hat{\nu}^N$ and ν be empirical and theoretical distributions of Y_j. Then $\{X_j \in Z\} = \{Y_j \in \psi^{-1}(Z)\}$ and the empirical distribution of X_j obeys

$$\sup_{Z\in\mathcal{B}(\mathbb{R}^k)} |\hat{\mu}^N(Z) - \mu(Z)| = \sup_{Z\in\mathcal{B}(\mathbb{R}^k)} |\hat{\nu}^N(\psi^{-1}(Z)) - \nu(\psi^{-1}(Z))| \leq ||F_\nu^N - F_\nu||_\infty. \tag{22}$$

By Glivenko-Cantelli Theorem we easily conclude that the expression above tends to 0 as $N \to \infty$. $\square$

Lemma 15. *Let $N \in \mathbb{N}$ and let $D^N := D_1 \times D_2 \times \ldots \times D_N$ and $D = D_1 \times D_2 \times \ldots$. For any N consider the one shot N person game with payoff function $w_j^N : D^N \mapsto \mathbb{R}$, such that there exists a limit*

$$w_j(x_1, x_2, \ldots) = \lim_{N\to\infty} w_j^N(x_1, \ldots, x_N).$$

Suppose that $(x_1^*, x_2^*, \ldots)$ is a Nash equilibrium with countably many players, when the payoff for j is w_j and

$$\lim_{N\to\infty} \sup_{j\in\mathbb{N}, x_j\in D_j} |w_j^N(x_{-j}^*, x_j) - w_j(x_{-j}^*, x_j)| = 0. \tag{23}$$

Let $(x_1^*, x_2^*, \ldots)$ be a Nash equilibrium in the game with a payoff w. Then, for any $\epsilon > 0$ there is N_0 such that $(x_1^*, x_2^*, \ldots, x_N^*)$ is $\epsilon-$Nash equilibrium.

Proof. Pick $\epsilon > 0$. Take N_0 such that for any $N > N_0$ it holds

$$\sup_{j\in\mathbb{N}, x_j\in D_j} |w_j^N(x_{-j}^*, x_j) - w_j(x_{-j}^*, x_j)| < \frac{\epsilon}{2}.$$

This is possible by (23). Then, since $(x_1^*, x_2^*, \ldots)$ is NE with w_j and x_j is any alternative policy for agent j, then

$$\epsilon + w^N(x_1^*, \ldots, x_j^*, \ldots, x_N^*) \geq w(x_1^*, x_2^*, \ldots, x_j^*, \ldots) - \frac{\epsilon}{2}$$
$$\geq w(x_1^*, x_2^*, \ldots, x_j, \ldots) - \frac{\epsilon}{2} \geq w^N(x_1^*, x_2^*, \ldots, x_{j-1}^*, x_j, x_{j+1}^*, \ldots, x_N^*).$$

$\square$

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