A Constructive Study of Markov Equilibria in Stochastic Games with Strategic Complementarities*

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January 2013

Abstract

We study a class of infinite horizon, discounted stochastic games with strategic complementarities. In our class of games, we prove the existence of a Stationary Markov Nash equilibrium, as well as provide methods for constructing this least and greatest equilibrium via a successive approximation scheme. We also provide equilibrium monotone comparative statics results relative to ordered perturbations of the space of stochastic games. Under slightly stronger assumptions, we prove the stationary Markov Nash equilibrium values form a complete lattice, with least and greatest equilibrium value functions being the uniform limit of these successive approximations starting from pointwise lower and upper bounds. We also provide many examples showing applicability of our new results.

keywords: Markov equilibria, stochastic games, constructive methods
JEL codes: C62, C73

1 Introduction and related literature

Since the class of infinite horizon discounted stochastic games was introduced by Shapley (1953), and subsequently extended to more general n-player settings (e.g., Fink (1964)), the question of existence and characterization of equilibrium has been the object of extensive study in game theory[1] (see Duggan, 2012; Levy, 2012, e.g.). Moreover, recently stochastic games have become a fundamental tool for studying strategic interactions in dynamic economic models, where agents possess some form of limited commitment over time. Examples of such situations in the literature include work in such diverse fields as: (i) equilibrium models of stochastic growth without

*We thank Rabah Amir, Andy Atkeson, Ramesh Johari, Eric Maskin, Andrzej Nowak, Frank Page, Adrian Peralta-Alva, Ed Prescott, Tarun Sabarwal, Manuel Santos and Jan Werner as well as participants of 10th Society for the Advancement in Economic Theory Conference in Singapore 2010; 2011 NSF/NBER/CEME conference in Mathematical Economics and General Equilibrium Theory in Iowa; the Theoretical Economics Conference at Kansas, Paris Game Theory Seminar participants at Institut Henri Poincare, as well the associate editor and two anonymous referees of this journal for very helpful discussions during the writing of this paper. Reffett also thanks the 2010 Dean's Award in Excellence Summer grant program at ASU, as well as Centre d'Economie de la Sorbonne at Universite de Paris, I during the Summer 2012 for support of this work. Woźny acknowledges financial support from SGH Dean's Grant for Young Researchers 2010. The usual caveats apply.

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[1] For example, see Raghavan, Ferguson, Parthasarathy, and Vrieez (1991) or Neyman and Sorin (2003) for an extensive survey of results, along with references.

commitment (e.g., Majumdar and Sundaram (1991), Dutta and Sundaram (1992), Amir (1996b) or Balbus and Nowak (2004) for examples of the classic fish war problem or stochastic altruistic growth models with limited commitment between successor generations), (ii) international lending and sovereign debt (Atkeson, 1991), (iii) optimal Ramsey taxation (Phelan and Stacchetti, 2001), (iv) models of savings and asset prices with hyperbolic discounting (Harris and Laibson, 2001), (v) dynamic search with learning (e.g., see Curtat (1996) and Amir (2005)), (vi) dynamic political economy (e.g., see Lagunoff (2009), and references contained within), (vii) dynamic negotiations with status quo (see Duggan and Kalandrakis (2012)) or (viii) dynamic oligopoly models (see Cabral and Riordan (1994)), among others.

A focal point of a great deal of applied work has been on minimal state space Markov stationary Nash equilibrium (MSNE).[2] Apart from the existence and characterization of such equilibrium, more recently though, new questions concerning its computation have become a central focus of applied researchers, who seek to either calibrate or estimate particular parameters of stochastic games. When constructing approximate solutions for an equilibrium in any dynamic/stochastic game/economy, a prerequisite for the rigorous implementation of numerical methods is to have access to a sharp set of theoretical tools that both characterize the structure of elements of the set of MSNE in the economy under study, as well as identify how the equilibrium set vary in deep parameters of the game.[3] In such a situations, one must be concerned first with constructive fixed point methods that can be tied directly to numerical approximation schemes. For finite games[4], the question of existence and computation of MSNE has been essentially resolved.[5] Unfortunately, for infinite games, although the equilibrium existence question has received a great deal of attention, results that provide characterization of the MSNE set (and how it varies in deep parameters) are needed, not only to address the question of accuracy of approximation methods, but also to develop notions of qualitative and quantitative stability.[6]

The aim of this paper is to address all of these issues (i.e., existence, computation, and equilibrium comparative statics) within the context of a single unified methodological approach under a minimal set of assumptions. Our methods are *constructive* and *monotone*, where our notion of monotonicity is defined using pointwise partial order on function spaces of values or pure strategies (as opposed to, for example, set inclusion orders often used in continuation value/promised utility methods). To obtain sufficient conditions for such constructive monotone methods relative to the set of MSNE, we study an important subclass of stochastic games, namely these with strategic complementarities and positive externalities.[7]

[2]In this paper, we study both the construction of Markov stationary Nash equilibrium strategies (MSNE) and their associated values (MSNE values). The pair (MSNE, MSNE value) is what we refer to as a SMNE.

[3]For example, in a calibration exercise, one first calibrates some SMNE to the actual data, and then generates counterfactuals to the calibrated equilibrium of the model by perturbing a subset of the deep parameters of the environment. An even more difficult situation arises, when one seeks to estimate the deep structural parameters of the game (as, for instance, in the recent work in empirical industrial organization literature).

[4]By "finite game" we mean a dynamic/stochastic game with a (a) finite horizon and (b) finite state and strategy space game. By an "infinite game", we mean a game where either (c) the horizon is countable (but not finite), or (d) the action/state spaces are uncountable. We shall focus on stochastic games where both (c) and (d) are present.

[5]For example, relative to existence, see Federgruen (1978); for computation of equilibrium, see Herings and Peeters (2004); and, finally, for estimation of deep parameters, see Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008) or Pakes, Ostrovsky, and Berry (2007).

[6]There are exceptions to this remark. For example, in Balbus and Nowak (2004), a truncation argument for constructing a SMNE in symmetric games of capital accumulation is proposed. In general, though, in this literature, a unified approach to approximation and existence has not been addressed.

[7]By this we mean supermodularity of an auxiliary game. See Echenique (2004) for a notion of a supermodular extensive form game.

For these infinite horizon stochastic supermodular games, we prove a number of new results relative to Curtat (1996), Amir (2002, 2005) or Nowak (2007). First, we prove the existence of MSNE in broader spaces of (bounded, measurable) pure strategies. Second, and perhaps most importantly per applications, we develop reasonable sufficient conditions for MSNE to exist over *very general state spaces*.[8] Third, we give sufficient conditions under which the set of MSNE values forms a complete lattice of Lipschitz continuous functions. Fourth, we contribute to the literature that studies specific forms of transition kernels, and we are able to show the full power of the mixing assumption studied extensively by Nowak and coauthors in a class of stochastic games. Finally, we prove our results using minimal assumptions (i.e. we are also able to present counterexamples that violate both our assumptions and results).

Along this line, our results contribute to the recent literature on the (non-)existence of MSNE in a classes of discounted, stochastic games with absolute continuity conditions (ACC) (see Levy, 2012) or with additional noise (see Duggan, 2012). Specifically, as shall be clear in the sequel, we provide results on the existence MSNE in a stochastic game over uncountable state space, where transition between states is *not* absolutely continuous. This result complements the recent result of Levy (2012) concerning the importance of ACC; namely, that this condition is neither sufficient or necessary for the existence of MSNE. Moreover, we obtain our existence results per MSNE in pure strategies *without* introducing additional correlation or noise as is done in the work of Nowak and Raghavan (1992) or Duggan (2012).

We should also briefly mention that our method has a number advantages over correspondence-based strategic dynamic programming approach first suggested in the seminal work of Abreu, Pearce, and Stacchetti (1990), and now finding use in many papers in the literature[9]. That is, in our class of games, APS-type approaches can be used to establish sequential or non-stationary Markov equilibrium existence and approximation of the equilibrium value set for a game with a countable number of states. Our methods are able to improve known results in a number of directions. Per the question of equilibrium existence, we first note that our method works for an *uncountable* and *multi-dimensional* state space. Also, our approach allows one to approximate the equilibrium strategy itself (and not only its value). We are then able to provide conditions when the extremal SMNE can be *directly* (uniformly) approximated as the limit of sequences generated by iterations on our fixed point operator. This greatly simplifies the study of numerical procedures that can be used to implement our theoretical results in practice. Also, our approach provides a simple method to *compute* monotone equilibrium comparative statics for both values and strategies in the standard product orders, which appears sharper than equilibrium value set comparative statics in set inclusion orders as in APS type procedures. In the next section of the paper, we provide an explicit example of the power of our methods as compared to APS type methods.

Finally, unlike the existing work in the literature, our methods provide a unified approach to both finite and infinite horizon games with strategic complementarities. That is, we give conditions under which infinite horizon MSNE are simply the limits of equilibria in truncated finite horizon stochastic games. This fact is particularly important for numerical implementa-

[8] Attempts to extend existence results to stochastic games with (i) general state spaces, and (ii) discontinuous Markovian equilibrium (e.g., measurable Markov equilibrium) has been very formidable problem in the existing literature. Our work on existence, in part, contributes to this literature.

[9] It bears mentioning that for dynamic games with more restrictive shocks spaces (e.g.,discrete or countable), APS procedures has been used extensively in economics in recent years: e.g. by Phelan and Stacchetti (2001) for policy games; Feng, Miao, Peralta-Alva, and Santos (2009) for dynamic competitive equilibrium in recursive economies; or Mertens and Parthasarathy (1987) and Chakrabarti (1999) for subgame perfect/sequential equilibria in stochastic games.

tions. These can be seen, hence, as a direct generalization of Amir (1996a) results on optimal policies for discounted dynamic lattice programming models to equilibria in a general class of stochastic supermodular games.[10] As in Amir is his decision-theoretic setting, we are able to provide monotone comparative statics results on the set of MSNE equilibria for the infinite horizon stochastic game, as well as describe how equilibrium comparative statics can be computed. This is particularly important, when one seeks to construct a stable selection of the set of MSNE that are numerically (and theoretically) tractable as functions of the deep parameters of the economy/game.

The rest of the paper is organized as follows. Section 2 presents a motivating example that highlights the applicability of our results. Then section 3 states the formal definition of an infinite horizon, stochastic game. Under general conditions, in section 3.2, we propose a method for MSNE existence and computation. In section 3.4, we present related equilibrium comparative statics and equilibrium dynamics results. In section 4, we present applications of our results. Finally, section 5 discusses our results in the context of related papers on existence of MSNE in stochastic games with uncountable number of states.

2 Motivating example

We start with a simple motivating example of computing extremal MSNE in a two player stochastic game with uncountable and two-dimensional state space the highlights the essential issues raised by this paper. Each period a pair of states $(s_1, s_2) \in [0,1] \times [0,1] =: S$ is drawn and after observing it players choose one of two actions 1 or 0. The payoffs from a stage game are given in the following table:

	1	0
1	s_1, s_2	$s_1 - c, b$
0	$b, s_2 - c$	$0, 0$

This game, for example, can be thought of as a stylized partnership game in which players choose to keep putting effort into their partnership (cooperate) or to quit. That payoffs/states s_1, s_2 represent the expected returns to each player from putting effort into the partnership. Parameter c represents the losses associated with staying in the partnership, when the other player walks out; parameter b represents a potential benefit from cheating on a cooperating partner.

Assume $1 > c > b > 0$. This stage game is clearly a supermodular game with positive externalities. For such a one-shot game, we have the following pure strategy Nash equilibria depending on parameters s_1, s_2:

[10] Obtaining conditions for this extension are important. In Amir (1996a), the author provides a very general theory of supermodular growth with many state variables, where he is able to unify the theory of comparative statics (both relative to state variables and deep parameters) for both finite and infinite horizon problem. This theory can easily be extended to cases with iid stochastic shocks. An extension of this theory to stochastic supermodular games has *not* been forthcoming in the literature. That is, the (measurable) SMNE existence have only been achieved for models with a single state variable. Further, even in this case, given the application of topological fixed point theory for existence, no equilibrium comparative statics results per deep parameters have been proven.

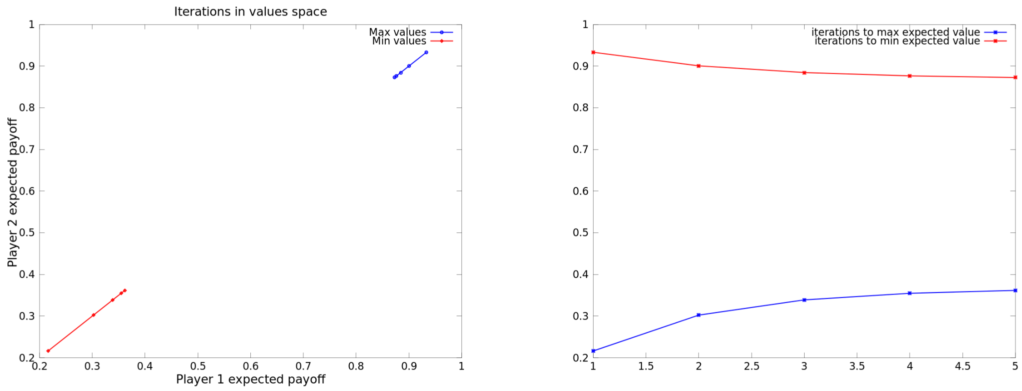

Figure 1: Convergence of iterations (expected values) from above and below to extremal MSNE expected values. Iterations in expected value space and MSNE value bounds (left panel); speed of convergence (right panel).

s_1 / s_2	$< b$	$\in (b, c)$	$> c$
$< b$	$(0, 0)$	$(0, 0)$	$(0, 1)$
$\in (b, c)$	$(0, 0)$	Two NE $(1, 1)$ and $(0, 0)$	$(1, 1)$
$> c$	$(1, 0)$	$(1, 1)$	$(1, 1)$

Let state $s = (s_1, s_2)$ be drawn from distribution $Q(\cdot|s, a)$ parameterized by current action $a = (a_1, a_2)$. Now, say the stochastic transition of state s is given by a transition kernel $Q(\cdot|s, a) = g(s, a)\lambda(\cdot|s) + (1 - g(s, a))\delta_0(\cdot)$, where $\lambda(\cdot|s)$ is a measure on S and δ_0 is a Delta Dirac concentrated at $(0, 0) \in S$. In this case, the main results of this paper show there exists the greatest and the least MSNE (see theorem 3.1), which also can be used from a numerical viewpoint to bound the set of *all* MSNE (see corollary 3.1). Moreover, our theorems show we can develop a simple successive approximation scheme to compute extremal MSNE (see theorem 3.1 and corollary 3.2).

To see how easy it is to apply our computational techniques, let us specify some parameters for the game, and compute extremal MSNE. For example, let $c = .8, b = .2$, discount factor $\beta = .9$ and assume that the function in the above specification of Q is given by

$$g(s, a) = \frac{(s_1 + s_2)}{2} \frac{(a_1 + a_2)^2}{4},$$

with λ uniformly distributed on $[0, 1] \times [0, 1]$.

In figure 1, we present the results of the computations of the greatest and the least expected values (as well as the iterations to these values): $\int_S v(s)\lambda(ds)$. Having computed the expected values in figure 1, we also construct both the greatest and the least MSNE in figure 2.

Apart from equilibrium existence and computation results, later in the paper, we prove an equilibrium monotone comparative statics theorem 3.4, which in principle could easily be computed for this game. The discussion of these results is in subsection 3.3.

This game is quite simple to compute using our techniques. Observe however, that APS type techniques in the current literature *cannot* be used to analyze this game. This is true for at least two reason. First, APS type methods fail in the case of stochastic games with *uncountable* two dimensional state spaces (as conditions that guarantee the existence of an non-empty, compact set of equilibrium values for multidimensional state space are not known).

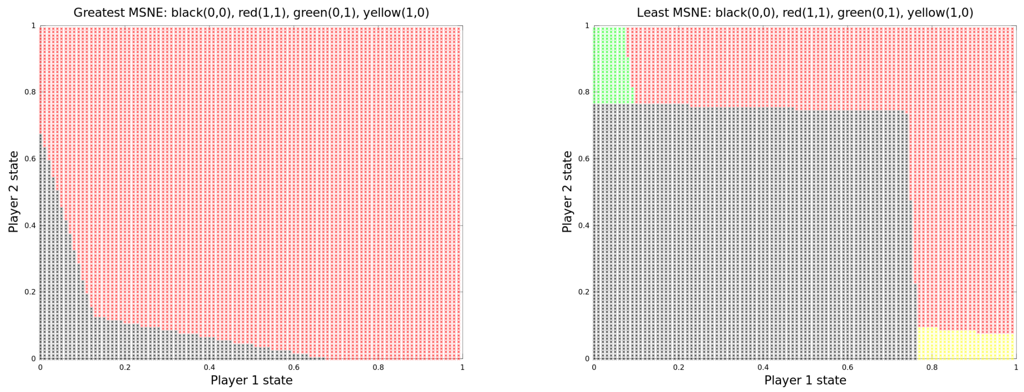

Figure 2: Greatest and least MSNE. Colored regions denote parts of the state space where a particular profile of actions is a MSNE: black: (0,0), green: (0,1), yellow: (1,0), red: (1,1).

Second, APS-type methods focus on the sequential or nonstationary Markov equilibria, and have little to say about the structure of MSNE. Finally, it bears mentioning that other topological techniques using continuity conditions also fail here as the transition probability Q need *not* be absolutely conditions (see discussion in Levy, 2012). For further discussion, see introduction or later section 5.

3 Main results

3.1 Definitions and assumptions

Consider an n-player discounted infinite horizon stochastic game in discrete time. The primitives of the class of games are given by a tuple $\left\{S, (A_i, \tilde{A}_i, \beta_i, u_i)_{i=1}^n, Q, s_0\right\}$, where $S = [0, \bar{S}] \subset \mathbb{R}^k$ is the state space, $A_i \subset \mathbb{R}^{k_i}$ player i action space with $A = \times_i A_i$, β_i is the discount factor for player i, $u_i : S \times A \to \mathbb{R}$ is the one-period payoff function, and $s_0 \in S$ the initial state of the game. For each $s \in S$, the set of feasible actions for player i is given by a correspondence $\tilde{A}_i(s)$, which is assumed to be compact Euclidean interval in $\mathbb{R}^{k_i}$ for each $s \in S$. By Q, we denote a transition function that specifies for any current state $s \in S$ and current action $a \in A$, a probability distribution over the realizations of next period states $s' \in S$.

Using this notation, we can provide a formal definition of a (Markov, stationary) strategy, payoff, and a Nash equilibrium. A *strategy* for a player i is denoted by $\Gamma_i = (\gamma_i^1, \gamma_i^2, \ldots)$, where γ_i^t specifies an action to be taken at stage t as a function of history of all states s^t, as well as actions a^t taken till stage t of the game. If a strategy depends on a partition of histories limited to the current state s_t, then the resulting strategy is referred to as *Markov*. If for all stages t, we have a Markov strategy given as $\gamma_i^t = \gamma_i$, then strategy Γ_i for player i is called a *Markov-stationary strategy*, and denoted simply by γ_i.

For a strategy profile $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$, and initial state $s_0 \in S$, the expected payoff for player i can be denoted by:

$$U_i(\Gamma, s_0) = (1 - \beta_i) \sum_{t=0}^{\infty} \beta_i^t \int u_i(s_t, a_t) dm_i^t(\Gamma, s_0),$$

where m_i^t is the stage t marginal on A_i of the unique probability distribution induced on the space of all histories for Γ, given by Ionescu–Tulcea's theorem. A strategy profile $\Gamma^* = (\Gamma_i^*, \Gamma_{-i}^*)$ is then a *Markov Stationary Nash equilibrium (MSNE)* if and only if Γ^* is feasible, and for any i, and all feasible Γ_i, we have

$$U_i(\Gamma_i^*, \Gamma_{-i}^*, s_0) \geq U_i(\Gamma_i, \Gamma_{-i}^*, s_0).$$

The aim of this section is two-fold. We first prove the existence of a MSNE. We then provide a successive approximation scheme for computing particular elements of the set of MSNE (in particular, extremal elements). An appealing aspect of methods developed in this section is that, they are essentially value iteration procedures defined in spaces of bounded measurable functions, where we incorporate the strategic restrictions imposed by the game implicitly in their definition. What is also particularly important to mention is that we can also characterize (and compute) the set of pure strategies that support particular extremal equilibrium values, and well as discuss conditions for the existence of standard monotone equilibrium comparative statics on those particular elements of the MSNE set.

Before that, we state some initial conditions on the primitives of the game that are required for the methods we discuss in this section. These conditions are stated in the following two sets of assumptions:

Assumption 1 (Preferences) *For $i = 1, \ldots, n$ let:*

- *u_i be continuous on A and measurable on S, with $0 \leq u_i(s, a) \leq \overline{u}$,*
- *$(\forall a \in A)\, u_i(0, a) = 0$,*
- *u_i be increasing in a_{-i},*
- *u_i be supermodular in a_i for each (a_{-i}, s), and has increasing differences in $(a_i; a_{-i})$,*
- *for all $s \in S$ the sets $\tilde{A}_i(s)$ be nonempty, compact intervals and $s \to \tilde{A}_i(s)$ be a measurable correspondence.*

Assumption 2 (Transition) *Let Q be given by:*

- *$Q(\cdot|s, a) = g_0(s, a)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a)\lambda_j(\cdot|s)$, where*
- *for $j = 1, ..., L$ the function $g_j : S \times A \to [0, 1]$ is continuous on A and measurable on S, increasing and supermodular in a for fixed s, and $g_j(0, a) = 0$ (clearly $\sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1$),*
- *$(\forall s \in S, j = 1, \ldots, L)\, \lambda_j(\cdot|s)$ is a Borel transition probability[11] on S,*
- *δ_0 is a probability measure concentrated at point 0.*

Although our assumptions here are related to those made in recent work by Amir (2005) and Nowak (2007), there are many important differences. Before discussing these differences explicitly in section 3.2, we first state our main result.

[11]This means among other that function $s \to \int_S v(s')\lambda_j(ds'|s)$ is measurable for any integrable v.

3.2 Existence and computation of SMNE

We first study the existence and computation of MSNE in a space of bounded measurable functions. Let $Bor(S, \mathbb{R}^n)$ be the set of Borel measurable functions from S into $\mathbb{R}^n$, and consider its following subset:

$$\mathcal{B}^n(S) := \{v \in Bor(S, \mathbb{R}^n) : \forall_i v_i(0) = 0, ||v_i|| \leq \bar{u}\}.$$

Equip the space $Bor(S, \mathbb{R}^n)$ with a pointwise partial order, and the subset $\mathcal{B}^n(S)$ with its relative partial order. Then, for a vector of continuation values $v = (v_1, v_2, \ldots, v_n) \in \mathcal{B}^n(S)$, we can consider the auxiliary one-period, n-player game G_v^s with action sets $\tilde{A}_i(s)$ and payoffs given as follows:

$$\Pi_i(v_i, s, a_i, a_{-i}) := (1 - \beta_i)u_i(s, a_i, a_{-i}) + \beta_i \int_S v_i(s')Q(ds'|s, a_i, a_{-i}).$$

We are now prepared to state the first main theorem of this section, which concerns the existence of MSNE. To provide some insight into our construction of MSNE, first note that under assumptions 1 and 2, the auxiliary game G_v^s is a supermodular game for any (v, s), hence it possesses a greatest $\overline{a}(s, v)$ and least $\underline{a}(s, v)$ (measurable[12]) pure strategy Nash equilibrium (see, Topkis (1979) and Vives (1990)) (as well as corresponding greatest $\overline{\Pi}^*(v, s)$ and least $\underline{\Pi}^*(v, s)$ equilibrium values), where the equilibrium payoffs are given as $\Pi^*(v, s) = (\Pi_1^*(v, s), \Pi_2^*(v, s), \ldots, \Pi_n^*(v, s))$. From these equilibrium payoffs, we can define a pair of extremal value operators $\overline{T}(v)(s) = \overline{\Pi}^*(v, s)$ and $\underline{T}(v)(s) = \underline{\Pi}^*(v, s)$ (as well as $T^j(v)$, which denote the j-iteration/orbit of the operator $T(v)$ from v). We can then generate recursively a sequence of lower (resp., upper) bounds for equilibrium values $\{v^j\}_{j=0}^{\infty}$ (resp., $\{w^j\}_{j=0}^{\infty}$) where $v^{j+1} = \underline{T}(v^n)$ for $j \geq 1$ from the initial guess $v^0(s) = (0, 0, \ldots, 0)$ (resp., $w^{j+1} = \overline{T}(w^j)$ from initial guess $w^0(s) = (\bar{u}, \bar{u}, \ldots, \bar{u})$) for $s > 0$ and $w^0(0) = 0$). For both lower (resp, upper) value iterations, we can then associate sequences of pure strategy Nash equilibrium strategies $\{\phi^j\}_{j=0}^{\infty}$ (resp., $\{\psi^j\}_{j=0}^{\infty}$), which are defined recursively by $\phi^j = \underline{a}(s, v^j)$ (resp., $\psi^j = \overline{a}(s, w^j)$. Existence and computation of MSNE reduces then to studying the limiting properties of this collection of (monotone) iterative processes. With this, our main existence theorem in this section is the following:

Theorem 3.1 (The successive approximation of SMNE) *Under assumptions 1 and 2 we have*

1. *(for fixed $s \in S$) $\phi^j(s)$ and $v^j(s)$ are increasing sequences and $\psi^j(s)$ and $w^j(s)$ are decreasing sequences,*

2. *for all t we have $\phi^j \leq \psi^j$ and $v^j \leq w^j$ (pointwise),*

3. *the following limits exist: $(\forall s \in S)$ $\lim\limits_{t\to\infty} \phi^j(s) = \phi^*(s)$ and $(\forall s \in S)$ $\lim\limits_{n\to\infty} \psi^j(s) = \psi^*(s)$,*

4. *the following limits exist $(\forall s \in S)$ $\lim\limits_{n\to\infty} v^j(s) = v^*(s)$ and $(\forall s \in S)$ $\lim\limits_{n\to\infty} w^j(s) = w^*(s)$,*

5. *ϕ^* and ψ^* are stationary Markov Nash equilibria in the infinite horizon stochastic game. Moreover, v^* and w^* are equilibria payoffs associated with ϕ^* and ψ^* respectively.*

[12] Lemma 6.3 shows that both extremal equilibria and their corresponding values are measurable.

Importantly, we now give pointwise bounds for successive approximations relative to any MSNE using upper and lower iterations built from the construction in Theorem 3.1. This result implies a notion of "pointwise bounds" stated as follows[13].

Corollary 3.1 (Pointwise equilibrium bounds for SMNE) *Let assumptions 1 and 2 be satisfied and γ^* be an arbitrary stationary Markov Nash equilibrium. Then $(\forall s \in S)$ we have the equilibrium bounds $\phi^*(s) \leq \gamma^*(s) \leq \psi^*(s)$. Further, if μ^* is equilibrium payoff associated with any stationary Markov Nash equilibrium γ^*, then $(\forall s \in S)$ we have the bounds: $v^*(s) \leq \mu^*(s) \leq w^*(s)$.*

As our existence result in Theorem 3.1 is obtained under different assumptions than those found in the existing literature for stochastic supermodular games (e.g., Curtat (1996), Amir (2002, 2005) or Nowak (2007)), we feel it is useful to provide a detailed discussion of the central differences between our results and those found in the existing literature. We organize our remarks concerning these differences along a number of distinct dimensions.

First, relative to Curtat and Amir (see Amir, 2002), we do not require the payoffs or the transition probabilities to be Lipschitz continuous (which is an assumption used in both of those papers). Such conditions appear to be very strong relative to many economic applications[14]. We also do not impose any conditions on payoffs and stochastic transitions that imply "double increasing differences" in the sense of Granot and Veinott (1985) in payoff structures, or any strong concavity conditions such as strict diagonal dominance to obtain our existence result[15]. Moreover, and equally critical, we do not assume *any* increasing differences between actions and states. This last difference is also critical when comparing our results versus Amir (2005). That is, we do not require *monotone Markov equilibrium* to obtain existence (rather, we just need enough complementarity to construct *monotone operators*). So our conditions are able to distinguish between the role of monotonicity conditions needed for the existence and computation of MSNE (e.g., to obtain monotone operators in sufficiently chain complete partially ordered sets) from those conditions needed for the existence of increasing MSNE. That is not done in the existing literature. Also, as compared to Amir (2005) we do not require the class of games to have a *single* dimensional state space[16]. Finally, there are critical differences between our conditions and theirs per the specification of the stochastic transition Q. For example, in Amir (2005), to obtain existence in the infinite horizon game, he requires strong stochastic equicontinuity conditions for the distribution function Q relative to the actions a, which is critical for his existence argument. We do not need this latter assumption[17].

[13]Of course, if we sup the pointwise bounds across our compact state space, we can get an estimate of uniform bounds also. The point is our iterations are converging in a weaker sense, i.e., in the product topology.

[14]For example, such conditions rule out payoffs that are consistent with Inada type assumptions (e.g., Cobb-Douglas utility).

[15]Both of these sets of assumptions (e.g., double increasing differences and strong diagonal dominance) are required by both of these authors for existence. That is, they each need to obtain unique Lipschitz Nash equilibrium in the stage game that is *continuous* with continuation v for their eventual application of a topological fixed point theorem per existece. Similar NE uniqueness conditions are required by Nowak (2007). The difference with our setup and theirs is of utmost importance. Specifically without equilibrium uniqueness in the stage game with continuation v, authors cannot construct an upper-hemicontinuous correspondence T, a condition necessary to apply Fan-Glicksberg fixed point theorem. Similar issues arise when trying to apply Schauder's theorem (e.g. as in Curtat (1996) and Amir (2002)).

[16]Also versus Amir (2005), our feasible action correspondences $\widetilde{A}_i(s)$, and payoff/transition structures u_i and g_j are only required to be measurable with s, as opposed to upper semicontinuous as in Amir (2005).

[17]Also, as we do not require *any* form of continuity of λ_j with respect to the state s, we do not satisfy Amir's assumption T1. Further, although we both require that the stochastic transition structure Q is stochastically

Second, to obtain our results, though, we must impose a very important conditions on the stochastic transitions Q, which are *stronger* than needed for existence in the work of Curtat and Amir. In particular, we assume the transition structure induced by Q can be represented as a convex combination of $L+1$ probability measures, of which one measure is a delta Dirac concentrated at 0. As a result, with probability g_0, we set the next period state to zero; with probability g_j, the distribution is drawn from the non-degenerate distribution λ_j (where, in this latter case, this distribution does not depend on the vector of actions a, but is allowed to depend on the current state s). Also, although we assume each λ_j is stochastically ordered relative to the Dirac delta δ_0, we do not impose stochastic orders among the various measures λ_j. This "mixing" assumption for transition probabilities has been discussed extensively in the literature[18]. Surprisingly, the main strength of this assumption has not been fully used (see Nowak, 2007) until the work of Balbus, Reffett, and Woźny (2013a) in the context of paternalistic altruism economies, as well as in this present paper.[19] Clearly, the restrictive part of our assumption is that we require existence of an *absorbing state* 0 that gives the minimal value for any $v \in \mathcal{B}^n(S)$. This is required by our techniques as we need to show that operators $\overline{T}, \underline{T}$ are well defined transformations of $\mathcal{B}^n(S)$ (and hence, require $\overline{T}(v)(0) = \underline{T}(v)(0) = 0$). This assumption can be potentially very restrictive in some applications. With this issue in mind, we make a few observations on the nature of the conditions we need in our construction. First, and importantly, our assumptions can be easily generalized to allow any absorbing state $\underline{s} \in S$, such that $v(\underline{s}) = z$, where $z = \min_{s \in S} v(s)$ for any $v \in \mathcal{B}^n(S)$, and the unique Nash equilibrium value in the auxiliary game $G_v^{\underline{s}}$ has for any integrable continuation v with $v(\underline{s}) = z$. Second, $\underline{s}$ need not be minimal in S unless v is monotone. Third, we can allow for other absorbing states (and hence, the probability of reaching 0 can be reduced to *zero* (see theorem 3.5)). Finally, we are able to present applications, where such assumption is indeed satisfied. Still, clearly our absorbing state assumptions come at some cost.

Now, we discuss the sense in which our mixing assumption is "minimal" and cannot be replaced by Curtat's or Amir's more general noise specifications. To see this, consider the following example.

Example 3.1 *Follow Amir (2002) and let $Q(\cdot|s,a) = (1 - g(s,a))\lambda_1(\cdot|s) + g(s,a)\lambda_2(\cdot|s)$, where probability measure λ_2 stochastically dominates λ_1, both λ_i are atomless and $\lambda_1 \neq \lambda_2$. Let $S = [0,1]$, then:*

$$\int_S v(s')Q(ds'|s,a) = g(s,a)\left(\int_S v(s')\lambda_2(ds'|s) - \int_S v(s')\lambda_1(ds'|s)\right) + \int_S v(s')\lambda_1(ds'|s).$$

Now consider $v_t(s) = 1_{[t,1]}(s)$ and for $a_2 > a_1$ define:

$$\begin{aligned} H(t) &:= \int_S v_t(s')Q(ds'|s,a_2) - \int_S v_t(s')Q(ds'|s,a_1) = \\ &[g(s,a_2) - g(s,a_1)]\left(\int_S v_t(s')\lambda_2(ds'|s) - \int_S v_t(s')\lambda_1(ds'|s)\right). \end{aligned}$$

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supermodular with a, we do not require increasing differences with (a,s) as Amir does, nor do we require the stochastic monotonicity conditions for Q in s.

[18] For example, this mixing condition was first mentioned in Amir (1996b), and henceforth studied systematically for broad classes of stochastic games in Nowak (2003), Nowak and Szajowski (2003), Balbus and Nowak (2004) among others.

[19] In a related paper (see Balbus, Reffett, and Woźny (2013b)) we managed to generalize our mixing condition with respect to its additive / multiplicative form. Specifically in this and the current paper we can allow $Q(\cdot|s,a) = p(\cdot|s,a) + (1 - p(\cdot|s,a))\delta_0(\cdot)$ where $p(\cdot|s,a)$ is some measure such that $p(S|s,a) < 1$, and $p(S|0,a) \equiv 0$.

The family $\{v_t\}$ decreases in the product order with t. Further, by our assumptions, there exists t_0 such that $H(t_0) > 0$ and $H(0) = 0$. Hence, H is not decreasing in t, which implies the resulting function

$$(a, v) \to \int_S v(s')Q(ds'|s, a)$$

fails to have increasing differences. This implies that even though $a \to \int_S v(s')Q(ds'|s, a)$ is monotone, the auxiliary game G_v^s does not have monotone equilibrium comparative statics in v, a critical property we need in the construction in theorem 3.1 and corollary 3.1.

Third, the proof of existence in Amir (2005) is based on a Schauder fixed point theorem. For this, he needs stochastic equicontinuity conditions on the noise to get weak* continuity of a best response operator, which is transforming a nonempty, compact and convex set of monotone upper semicontinuous strategies defined on the real line into itself. In contrast, we just construct a sequence of functions whose limit is a (fixed-point) value leading to MSNE.[20] So, our method is completely constructive, and we do not need to require continuity of a best-response operator (nor compactness or convexity of a particular strategy space). So, with our changes of assumptions, our results in the end are substantially stronger than those in Amir (2005).[21]

Fourth, our limiting arguments are based on the topology of *pointwise* convergence[22]. As our function spaces are pointwise partial ordered, this allows us to state equivalently all our results using fixed point theorems for order continuous tranformation of a $\sigma-$complete poset (e.g., see Dugundji and Granas (1982), Theorem 4.1-4.2), where continuity and convergence is always characterized in terms of *order topologies*. It is precisely here where our mixing assumption on the noise has its bite, as this leads to a form of monotonicity that is preserved via the value function operator to the infinite horizon (as well as allowing us to work in spaces of measurable functions under pointwise partial orders). For example, in Curtat (1996), one can only manage to show monotonicity of an operator T in gradient partial orders (i.e., in ∂v, where ∂ denote a vector of superdifferentials of v that are defined a.e.). For such superdifferentials to be guaranteed to be well-behaved, one needs *concavity* of the equilibrium value function.[23] Under our mixing assumptions, we are not limited to such cases.

Fifth, it is useful to keep in mind that our methods apply to both finite horizon and infinite horizon games. In that sense, we unify the treatment of the question of existence, computation (and in a moment, equilibrium comparative statics) of stochastic games with strategic complementarities of any discrete time horizon[24]. This unification is not possible in any of the papers in the existing literature, as the results for finite horizon games are very different than for infinite horizon games.

[20] Compare with Nowak and Szajowski (2003) lemma 5, where a related limiting Nash equilibrium result for two player game is obtained.

[21] In particular, aside from verifying the existence of MSNE, we provide methods for constructing them, and in a moment (see section 3.4), we will show how we can obtain equilibrium monotone comparative statics for the infinite horizon game.

[22] Which for pointwise partial orders on spaces of values and strategies coincides with *order convergence* in the interval topology. See Aliprantis and Border (2003) lemma 7.16 as applied on our context.

[23] For a discussion of the relationship between concavity and the existence of superdifferentials, see Rockafellar and Wets (1998) (e.g., pp 303-304).

[24] Of course, for finite horizon games, our existence and comparative statics results can be obtained under weaker conditions; but for making arguments that stationary MSNE are limits of finite horizon games, our assumptions here are critical.

3.3 Uniform error bounds for Lipschitz continuous SMNE

We now turn to error bounds for approximate solutions. To the best of our knowledge, this is a question that has not been addressed in the current literature. We initially give two motivations for the importance of our results in this section. First, notice that the limits of iterations computing in theorem 3.1 and corollary 3.1 are only relative to *pointwise convergence*. With slightly stronger assumptions, we can obtain for those limits *uniform convergence* to least and greatest MSNE. Second, to obtain uniform error bounds, we need make some stronger assumptions on the primitives which will allow us to address the question of Lipschitz continuity of equilibrium strategies. The assumptions we will use are common in applications (e.g., compare the assumptions we impose to those in Curtat (1996) and Amir (2002)).

In this section, we assume that the state space S is endowed with a *taxi-norm* $||\cdot||_1$[25]. The spaces A_i and A are endowed with a natural sup-norm. Each function $f : S \to A$ is said to be *M-Lipschitz continuous* if and only if, for all $i = 1,\dots,n$ $||f_i(x) - f_i(y)|| \le M||x-y||_1$. Note, if f_i is differentiable, then M-Lipschitz continuity is equivalent to that each partial derivative being bounded above by M. To obtain corresponding uniform convergence and uniform approximation results, we need some additional structure, that is discussed in the following assumption.

Assumption 3 *For all i, j:*

- *u_i, g_j are twice continuously differentiable on an open set containing $S \times A$,*[26]
- *u_i is increasing in (s, a_{-i}) and satisfies cardinal complementarity*[27] *in a_i and (a_{-i}, s),*
- *u_i satisfy a strict dominant diagonal condition in a_i, a_{-i} for fixed $s \in S, s > 0$, i.e. if we denote $a_i \in \mathbb{R}^{k_i}$ as $a_i := (a_i^1, ..., a_i^{k_i})$, then*

$$\forall_{i=1,\dots,n}\forall_{j=1,\dots,k_i} \sum_{\alpha=1}^{n}\sum_{\beta=1}^{k_\alpha} \frac{\partial^2 u_i}{\partial a_i^j \partial a_\alpha^\beta} < 0,$$

- *g_j is increasing in (s, a) and has cardinal complementarity in a_i and (s, a_{-i}),*
- *g_j satisfy a strict dominant diagonal condition in a_i, a_{-i} for fixed $s \in S, s > 0$,*
- *λ_j has a Feller property,*
- *for each increasing, Lipschitz and bounded by $\overline{u}$ function f, the function $\eta_j^f(s) := \int_S f(s')\lambda_j(ds'|s)$ is increasing and Lipschitz-continuous with a constant $\bar{\eta}$,*[28]
- *$\tilde{A}_i(s) := [0, \tilde{a}_i(s)]$ and each function $s \to \tilde{a}_i(s)$ is Lipschitz continuous and isotone function*[29].

[25] Taxi-norm of vector $x = (x_1, \dots, x_k)$ is defined as $||x||_1 = \sum_{i=1}^{k} |x_i|$.

[26] Note that this implies that u_i and g_j are bounded Lipschitz continuous functions on compact $S \times A$.

[27] That is, the payoffs are supermodular in a_i and have increasing differences in (a_i, s) and in (a_i, a_{-i}).

[28] This condition is satisfied if each of measures $\lambda_j(ds'|s)$ has a density $\rho_j(s'|s)$ and the function $s \to \rho_j(s'|s)$ is Lipschitz continuous uniformly in s'.

[29] Each coordinate $\tilde{a}_i^j$ is Lipschitz continuous. Notice, this implies that the feasible actions are Veinott strong set order isotone.

Define the set CM^N of N-tuples of increasing Lipschitz continuous functions with some constant M on S. Clearly, CM^N is a complete lattice when endowed with a partial order. It is also nonempty, convex and compact in the sup norm. It is also important to note that CM^N is closely related to the space where equilibrium is constructed in Curtat (1996) and Amir (2002). There are two key differences, though, between our game and those studied in these two papers. First, we allow the choice set A_i to depend on s (where, these authors assume A_i independent of s). [30] Second, under our assumptions, the auxiliary game G_v^0 has a continuum of Nash equilibria, and hence we need to close the Nash equilibrium correspondence in state 0. These technical differences are addressed in lemma 6.4 and proof of the next theorem.

Before we proceed, we provide a few additional definitions. For each twice continuously differentiable function $f : A \to \mathbb{R}$, we define the following mappings:

$$\mathcal{L}_{i,j}(f) := -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{k_\alpha} \frac{\partial^2 f}{\partial a_i^j \partial a_\alpha^\beta}, \qquad U_{i,j,l}^2 := \sup_{s\in S, a\in \tilde{A}(s)} \frac{\partial^2 u_i}{\partial a_i^j \partial s_l}(s,a),$$

$$G_{i,j}^1 := \sup_{s\in S, a\in \tilde{A}(s)} \sum_{\alpha=1}^{L} \frac{\partial g_\alpha}{\partial a_i^j}(s,a), \qquad G_{i,j,l}^2 := \sup_{s\in S, a\in \tilde{A}(s)} \sum_{\alpha=1}^{L} \frac{\partial^2 g_\alpha}{\partial a_i^j \partial s_l}(s,a),$$

$$M_0 := \max\left\{ \frac{(1-\beta_i)U_{i,j,l}^2 + \beta_i \bar{\eta} G_{i,j}^1 + \beta_i \bar{u} G_{i,j,l}^2}{-(1-\beta_i)\mathcal{L}_{i,j}(u_i)} : i = 1,\ldots,n, j = 1,\ldots,k_i, l = 1,\ldots,k \right\}.$$

With these definitions in mind, we can now prove our main result on Lipschitz continuous MSNE:

Theorem 3.2 (Lipschitz continuity) *Let assumptions 1, 2, 3 be satisfied. Assume additionally that each $\tilde{a}_i(\cdot)$ is Lipschitz continuous with a constant less then M_0. Then, stationary Markov Nash equilibria ϕ^*, ψ^* and corresponding values v^*, w^* are all Lipschitz continuous.*

Remark 1 *Note that we could relax assumption of transition probability, such that $\int_S v(s')Q(ds'|s,a)$ is smooth, supermodular and satisfy strict diagonal property for all Lipschitz continuous v.*

We can now study the uniform approximation of MSNE. Our results appeal to a version of Amann's theorem (e.g., Amann (1976), Theorem 6.1) to characterize the least and the greatest MSNE via successive approximations. Further, as a corollary of Theorem 3.1, we also obtain their associated value functions. For this argument, denote by $\mathbf{0}$ (by $\bar{\mathbf{u}}$ respectively) the n-tuple of function identically equal to 0 ($\bar{u}$ respectively) for all $s > 0$. Observe, as under assumptions 3, the auxiliary game has a unique NE value, we have $\underline{T}(v) = \overline{T}(v) := T(v)$. The result is as follows:

Corollary 3.2 (Uniform approximation of extremal SMNE) *Let assumptions 1, 2 and 3 be satisfied. Then $\lim_{j\to\infty} ||T^j\mathbf{0} - v^*|| = 0$ and $\lim_{j\to\infty} ||T^j\bar{\mathbf{u}} - w^*|| = 0$, with $\lim_{j\to\infty} ||\phi^j - \phi^*|| = 0$ and $\lim_{j\to\infty} ||\psi^j - \psi^*|| = 0$.*

Notice, the above corollary assures that the convergence in theorem 3.1 is *uniform.* We also obtain a stronger characterization of the set of MSNE in this case, namely, the set of MSNE equilibrium value functions form a complete lattice.

[30] That is, we allow for the case of generalized Markov stationary Nash equilibrium.

Theorem 3.3 (Complete lattice structure of SMNE value set) *Under assumptions 1, 2 and 3, the set of Markov stationary Nash equilibrium values v^* in CM^n is a nonempty complete lattice.*

The above result provides a further characterization of a MSNE strategies, as well as their corresponding set of equilibrium value functions. From a computational point of view, not only are the extremal values and strategies Lipschitzian (as known from previous work), they can also can be uniformly approximated by a simple algorithm. Also observe, is it not clear if the set of MSNE in $CM^{\sum_i k_i}$ is necessarily a complete lattice.

3.4 Monotone comparative dynamics

We now study the question of sufficient conditions under which our games exhibit equilibrium monotone comparative statics relative to both extremal fixed point values v^*, w^*, as well as the corresponding pure strategy extremal equilibria ϕ^*, ψ^*. We also consider the question of ordered equilibrium stochastic dynamics. With the question of equilibrium comparative statics in mind, we first parameterize our stochastic game by a set of parameters $\theta \in \Theta$, where Θ is some partially ordered set. One way to interpret θ is a vector whose elements include parameters representation ordered perturbations to any of the following primitive data of the game: (i) period payoffs u_i, (ii) the stochastic transitions g_j and λ_j, and (iii) feasibility correspondence A_i. Alternatively, we can think of elements θ as being policy parameters of the environment governing the setting of taxes or subsidies (as, for example, in a dynamic policy game with strategic complementarities). We shall discuss examples in the last section of the paper.

Along those lines, consider parameterized versions of Assumptions 1 and 2 as follows:

Assumption 4 (Parameterized preferences) *For $i = 1, \ldots, n$ let:*

- $u_i : S \times A \times \Theta \to \mathbb{R}$ *be a function and* $u_i(\cdot, s, \theta)$ *continuous on* A *for any* $s \in S, \theta \in \Theta$ *with* $u_i(\cdot) \leq \overline{u}$, *and* $u_i(\cdot, \cdot, \theta)$ *is measurable for all* θ,
- $(\forall a \in A, \theta \in \Theta)\, u_i(0, a, \theta) = 0$,
- u_i *be increasing in* (s, a_{-i}, θ),
- u_i *be supermodular in* a_i *for fixed* (a_{-i}, s, θ), *and has increasing differences in* $(a_i; a_{-i}, s, \theta)$,
- *for all* $s \in S, \theta \in \Theta$, *the sets* $\tilde{A}_i(s, \theta)$ *are nonempty, measurable (for given* θ*), compact intervals and a measurable multifunction that is both ascending in the Veinott's strong set order*[31], *and expanding under set inclusion*[32] *with* $\tilde{A}_i(0, \theta) = 0$.

Assumption 5 (Parameterized transition) *Let Q be given by:*

- $Q(\cdot|s, a, \theta) = g_0(s, a, \theta)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a, \theta)\lambda_j(\cdot|s, \theta)$, *where*

[31] That is, $\tilde{A}_i(s, \theta)$ is *ascending in Veinott's strong set order* if for any $(s, \theta) \leq (s', \theta')$, $a_i \in \tilde{A}_i(s, \theta)$ and $a'_i \in \tilde{A}_i(s', \theta') \Longrightarrow a_i \wedge a'_i \in \tilde{A}_i(s, \theta)$ and $a_i \vee a'_i \in \tilde{A}_i(s', \theta')$.

[32] That is, $\tilde{A}_i(s, \theta)$ is *expanding* if $s_1 \leq s_2$ and $\theta_1 \leq \theta_2$ then $\tilde{A}_i(s_1, \theta_1) \subseteq \tilde{A}_i(s_2, \theta_2)$.

- *for* $j = 1, ..., L$ *function* $g_j : S \times A \times \Theta \to [0,1]$ *is continuous with* a *for a given* s, θ, *measurable for given* θ, *increasing in* (s, a, θ), *supermodular in* a *for fixed* (s, θ), *and has increasing differences in* $(a; s, \theta)$ *and* $g_j(0, a, \theta) = 0$ *(clearly* $\sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1$*),*

- $(\forall s \in S, \theta \in \Theta, j = 1, \ldots, L)$ $\lambda_j(\cdot|s, \theta)$ *is a Borel transition probability on* S, *with each* $\lambda_j(\cdot|s, \theta)$ *stochastically increasing with* θ *and* s,

- δ_0 *is a probability measure concentrated at point* 0.

Notice, in both of these assumptions, we have added increasing difference assumptions between actions and states (as, for example, in Curtat (1996) and Amir (2002)).

We first introduce some notation. For a stochastic game evaluated at parameter $\theta \in \Theta$, denote the least and greatest equilibrium values, respectively, as v^*_θ and w^*_θ. Further, for each of these extremal values, denote the associated least and greatest MSNE pure strategies, respectively, as ϕ^*_θ and ψ^*_θ. Our first monotone equilibrium comparative statics theorem is then given in the next theorem:

Theorem 3.4 (Monotone equilibrium comparative statics) *Let assumptions 4 and 5 be satisfied. Then, the extremal equilibrium values* $v^*_\theta(s)$, $w^*_\theta(s)$ *are increasing on* $S \times \Theta$. *In addition, the associated extremal pure strategy stationary Markov Nash equilibrium* $\phi^*_\theta(s)$ *and* $\psi^*_\theta(s)$ *are increasing on* $S \times \Theta$.

In the literature on *infinite horizon* stochastic games with strategic complementarities, we are not aware of any analog result to the above concerning monotone equilibrium comparative statics as in Theorem 3.4. In particular, because of the non-constructive approach to the equilibrium existence problem (that is typically taken in the literature), it is difficult to obtain such a monotone comparative statics without fixed point uniqueness. Therefore, one key innovation of our approach of the previous section is that for the special case of our games where SMNE are monotone Markov processes, we are able to construct a sequence of parameterized monotone operators whose fixed points are extremal equilibrium selections. As the method is constructive, this also allows us to compute directly the relevant monotone selections from the set of MSNE.

Finally, we state results on dynamics and invariant distribution started from s_0 and governed by a MSNE and transition Q. Before this, let us mention that by our assumptions delta Dirac concentrated at 0 is an absorbing state (and hence we can have a trivial invariant distribution). As a result, we do not aim to prove a general result on the existence of an invariant distribution; rather, we will characterize a set of all invariant distributions, and discuss conditions when this limiting distribution is not a singleton. Along these lines, let θ be given, and let s^f_t denote a stochastic process induced by Q and equilibrium strategy f (i.e., $s_0 = s^f$ is an initial value and for $t > 0$), with s_{t+1} has a conditional distribution $Q(\cdot|s_t, f(s_t))$. By $\succeq$ we denote the first order stochastic dominance order on the space of probability measures.

We then have the following theorem:

Theorem 3.5 (Invariant distribution) *Let assumptions 1, 2 and 3 be satisfied.*

- *Then the sets of invariant distributions for processes* $s^{\phi^*}_t$ *and* $s^{\psi^*}_t$ *are chain complete (with both greatest and least elements) with respect to (first-) stochastic order.*

- *Let $\overline{\eta}(\phi^*)$ be the greatest invariant distribution with respect to ϕ^* and $\overline{\eta}(\psi^*)$ the greatest invariant distribution with respect to ψ^*. If the initial state of $s_t^{\phi^*}$ or $s_t^{\psi^*}$ is a Dirac delta in $\bar{S}$, then $s_t^{\phi^*}$ converges weakly*[33] *to $\overline{\eta}(\phi^*)$, and $s_t^{\psi^*}$ converges weakly to $\overline{\eta}(\psi^*)$, respectively.*

We make a few remarks. First, the above result is stronger than that obtained in a related theorem in Curtat (1996) (e.g., Theorem 5.2). That is, not only we do characterize the set of invariant distributions associated with extremal strategies (which he does not), but we also prove a weak convergence result per the greatest invariant selection. Second, it is worth mentioning if for almost all $s \in S$, we have $\sum_j g_j(s, \cdot) < 1$, we obtain a positive probability of reaching zero (an absorbing state) each period, and hence the only invariant distribution is delta Dirac at zero. Hence, to obtain a *nontrivial* invariant distribution, one has to assume $\sum_j g_j(s, \cdot) = 1$ for all s in *some subset* of a state space S with positive measure, e.g. interval $[S', \bar{S}] \subset S$ (see Kamihigashi and Stachurski (2010)).[34]

Second, Theorems 3.4 and 3.5 also imply results on monotone comparative dynamics (e.g., as defined by Huggett (2003)) with respect to the parameter vector $\theta \in \Theta$ induced by extremal MSNE: ϕ^*, ψ^*. To see this, we define the greatest invariant distribution $\overline{\eta}_\theta(\phi^*_\theta)$ induced by $Q(\cdot|s, \phi^*_\theta, \theta)$, and greatest invariant distribution $\overline{\eta}_\theta(\psi^*_\theta)$ induced by $Q(\cdot|s, \psi^*_\theta, \theta)$, and consider the following corollary:

Corollary 3.3 *Assume 4, 5. Additionally let assumptions of theorem 3.5 be satisfied for all $\theta \in \Theta$. Then $\overline{\eta}_{\theta_2}(\phi^*_{\theta_2}) \succeq \overline{\eta}_{\theta_1}(\phi^*_{\theta_1})$ as well as $\overline{\eta}_{\theta_2}(\psi^*_{\theta_2}) \succeq \overline{\eta}_{\theta_1}(\psi^*_{\theta_1})$ for any $\theta_2 \geq \theta_1$.*

The above results points to the importance of having constructive iterative methods for *both* strategies/values, as well as limiting distributions associated with extremal SMNE. Without such monotone iterations, we could not close our monotone comparative statics results. Further, in conclusion, we stress the fact that by weak continuity of operators used to establish invariant distributions, we can also obtain results that lead us to develop methods to *estimate* parameters θ using simulated moments methods (e.g., see Aguirregabiria and Mira (2007), for discussion of how this is done, and why it is important).

4 Applications

There are many applications of our new results. For example, they can be used to study many examples such as dynamic (price or quantity) oligopolistic competition, stochastic growth models without commitment (and related problems of dynamic consistency), models with weak social interaction among, dynamic policy games, and interdependent security systems. In this section, we discuss four such applications of our results. We first use our results to prove existence of Markov equilibrium in a dynamic oligopoly model. We then apply the results to develop a class of dynamic interdependent security models (and in particular, a simple dynamic vaccination game). Next, we show how the methods can be used for analyzing the question of credible government public policies as in Stokey (1991). We conclude with a discussion of how our results can be used to generalize existing results per symmetric MSNE of symmetric stochastic games.

[33] That is their distributions converge weakly.

[34] It is also worth mentioning that much of the existing literature does not consider the question of characterizing the existence of invariant distributions.

4.1 Price competition with durable goods

We begin with a price competition problem with durable goods. Consider an economy with n firms competing on customers buying durable goods, that are heterogenous but substitutable to each other. Apart from price of a given good, and vector of competitors goods' prices, demand for any commodity depends on demand parameter s. Each period firms choose their prices, competing a la Bertrand with other's prices. Our aim is to analyze the Markov Stationary Nash Equilibrium of such economy.

Payoff of firm i, choosing price $a_i \in [0, \bar{a}]$ is

$$u_i(s, a_i, a_{-i}, \theta) = a_i D_i(a_i, a_{-i}, s) - C_i(D_i(a_i, a_{-i}, s), \theta),$$

where s is a (common) demand parameter, while θ is a cost function parameter. As within period game is Bertrand with heterogenous, but substitutable products, naturally the preference assumption 1 is satisfied if (a) demand D_i is increasing with a_{-i}, has increasing differences in (a_i, a_{-i}), and (b) the cost function C_i is increasing and convex. As $[0, \bar{a}]$ is single dimensional, u_i is a supermodular in a_i trivially.

Concerning the interpretation of the assumptions placed on Q in the context of this model: letting $s = 0$ be an absorbing state means that there is a probability that demand will vanish and companies will be driven out of the market. The other assumptions on transition probabilities are also satisfied if $Q(\cdot|s, a) = g_0(s, a)\delta_0(\cdot) + \sum_j g_j(s, a)\lambda_j(\cdot|s)$ and g_j, λ_j satisfy assumptions 2. Interpreting: high prices a today result in high probability for positive demand in the future, as the customer trades-off between exchanging the old product with the new one, and keeping the old product and waiting for lower prices tomorrow. Supermodularity in prices implies that the impact of a price increase on positive demand parameter tomorrow is higher when the others set higher prices. Indeed, when the company increases its price today, it may lead to a positive demand in the future (if the others have also high prices). But, if the others firms set low prices today, then such impact is definitely lower, as some clients may want to purchase the competitors good today instead. Such assumptions guarantee that the stochastic (extensive form) game has the supermodular structure for extremal strategies, the feature that is uncommon for general extensive form games (see Echenique (2004)). More specifically, if a strategy of a player is increased in the some period $t + \tau$, it leads to a higher value of all players and by our mixing transition assumption increase period t extremal strategies.

The results of the paper (theorem 3.1) prove existence of the greatest and the least Markov stationary Bertrand Equilibrium and allow to compute the equilibria, by a simple iterative procedure. The results extend, therefore, the results obtained in Curtat (1996) paper per this example to the non-monotone strategies, characterizing the monopolistic competition economy with substitutable durable goods and varying consumer preferences. Finally, our approximation procedure allow applied researcher to compute and estimate the stochastic properties of these models using the extremal invariant distributions (see theorem 3.5). Finally, if one adds assumptions of theorem 3.4 one obtain monotone comparative statics of the extremal equilibria and invariant distributions (see corollary 3.3), the results absent in the related work.

Note, to analyze such economy using the methods of Curtat (1996) or Amir (2002) , one needs to assume increasing differences between (a_i, s) and monotonicity in s. Such method allow hence to study the monotone equilibria only. The interpretation of such assumption means that high demand today imply high demand in the future. To justify this assumption, Curtat (1996) argues: that "high level of demand today is likely to result in a high level of demand tomorrow because one can assume that not all customers will be served today in the case of high demand."

Hence, in our paper, the SMNE existence is obtained under weaker assumption than those of Curtat (1996) or Amir (2002), i.e. where monotonicity assumption is questionable, as customer rationing is not a part of this game description. Hence, methods developed in section 3.2 are plausible.

4.2 Vaccination games on networks

In a recent series of papers, a number of authors have analyzed Nash equilibria a static versions of the "interdependent security game" (IDS game, henceforth). The game, first proposed in Heal and Kunreuther (2005a), has been studied extensively over the last decade.[35] This game cover numerous situations such as the designing of airline security systems, computer virus/malware prevention, prevention of pandemics, vaccination games, wildfire prevention and forestry management, biosecurity games, games concerning the control of invasive species, among others. From a structural viewpoint, there are many incarnations of the game, including versions of the game that exhibit strategic complementarities or substitutes as well as positive or negative externalities (e.g., Bauch and Earn (2004), Shafran (2008), and Sun, Yang, and Vericourt (2009)).

In this section, we consider a dynamic version of a vaccination game described in Bauch and Earn (2004) or Heal and Kunreuther (2005b), where the primitive data of the game are consistent with the presence of positive externalities and strategic complementarities.[36] In our vaccination game, agents each period decide (simultaneously) whether to vaccinate against a potential disease or not. To make the game more realistic, in our dynamic version, we allow two sources of infection transmission. The first is background environmental risk, while the second infection from agent with whom you are in contact. To formalize this latter source of risk, we assume the members of the society are arranged on a fixed network (i.e., the players form a collection of "neighborhoods"). Then, in the game, the transmission of infection is influenced not only by aggregate environmental factors (as is standard in this class of games), but from contact among neighbors in a given neighborhood.[37]

We first describe the network topography of the neighborhood structure in some detail. Players in the game are arrange over a fixed network of "neighborhoods" indexed by $i \in \{1, 2, \dots, n\} = N$. The network is described by an $n \times n$ symmetric matrix P consisting of entries $p_{ij} \in \{0, 1\}$ where "1" indicates the two players i and j are linked in a neighborhood, and 0 indicates they are not. By a neighborhood N_i for player i, we simply mean the set $N_i := \{j \in N | p_{ij} = 1\} \setminus \{i\}$. Let $\bar{N}_i$ denote the cardinality of the set N_i, and assume each N_i is nonempty.

[35] For a recent survey and list of references on emerging literature on IDS games, see the paper of Lazaka, Felegyhazi, and Buttyan (2012).

[36] In particular, our game is a version of what Heal and Kunreuther (2005a) refer to as a "class 1" games with partial protection. There are other versions of IDS games (e.g., see Heal and Kunreuther (2005b)) that are either submodular or "mixed-modular". For example one alternative formulation of the vaccine game has for low levels of vaccination, the game exhibiting strategic complementarities; and high levels of vaccination, the game exhibits strategic substitutability. This is a mixed-modular game.

[37] Although we look a particular class of vaccination games, the model we describe can be mapping into many other IDS games. For example, for related models on biosecurity games in agricultural economics, see Kobayashi and Melkonyan (2011), who consider strategic complementarities in models of biosecurity in agriculture. In these papers, the authors consider a model of "bioexclusion", where costly actions by particular farmers to treat or protect their own livestock creates a positive spillovers to all other farmers who are beginning their livestock to market, where the animals come in contact which each other, and infect each other. In Sun, Yang, and Vericourt (2009), the authors consider a related game of allocating an antiviral drug across countries during a pandemic. Finally, additional examples of our IDS game are found in the literature on risk management and wildfire prevention (e.g., Shafran (2008)).

Player $i's$ payoff depends on her own actions, and those of her neighbors. At the beginning of each period (e.g. year), all players observe a signal $s \in [0,1]$, that indicates the population susceptibility to a particular strain of the viral disease (e.g., the transmission of the virus/flu among players in your neighborhood, weather conditions that influence the probability of transmission, etc.)[38] We assume the effect of the virus is short-lived (and not fatal), so each player is not ill at the beginning of each period in the game.[39] Given this information, the agent takes an action to either vaccinate $a_i = 1$ or not $a_i = 0$. In the spirit of Sun, Yang, and Vericourt (2009), we assume the vaccine is not perfect, which in the game means that even if one vaccinates, it is still possible to get infected within the period by either chance (i.e., the background environmental factors) or by transmission from some other person in their immediate neighborhood.

We can now construct payoffs in the game, and give conditions under which the results in our paper can be applied to study the existence and computation of MSNE. First, when choosing to vaccinate or not, the payoffs player i obtains in the stage game are as follows:

$$u_i - sL_i - (1-s)q_i(a)L_i$$

if not vaccinated, or

$$u_i - c_i - v_i^E sL_i - (1-v_i^E s)v_i^I q_i(a)L_i$$

if vaccinated. Here, u_i denotes the initial utility/wealth of a healthy person, L_i the disutility of becoming ill, c_i is the (fixed) cost of vaccination, $v_i^E \in [0,1]$ (resp. $v_i^I \in [0,1]$) the effectiveness of vaccination against the contagion of illness from environmental risk sources (resp., from contact with direct neighbors), with $v_i^j = 0$ (resp. $v_i^j = 1$) indicating perfect immunity (resp. no protection), $q_i : A \to [0,1]$ endogenous probability of becoming infected by some other person if a particular profile of neighborhood vaccination decisions is taken. Then, given a continuation value in a MSNE, we simply assume each player maximizes her payoff in each stage of the game.

To map this environment into our class of stochastic supermodular games assume $v_i^I(1 - sv_i^E) \geq 1 - s$. Under these condition, we have complementarities in players' payoffs. That is, examining the payoffs' structure, when the endogenous probability of transmission of infection q_i is decreasing in the level of collective action a, the game under our assumptions exhibits positive externalities, and has increasing differences in the vector of actions a (as the more people that choose vaccinations, the lower the *indirect* risk of getting infected, and the more productive each individual vaccination actually is).[40] We should mention that our assumptions on q_i are quite natural if, for example, the *effectiveness* of individual immunity against environmental sources of risk is high (e.g. perfect with $v_i^E = 0$), and the effectiveness of immunity against the risk of infection from direct contact with neighbors is low.[41] We have also assumed, for simplicity,

[38]This signal is public information. Also, it bears mentioning, it is well-known in the epidemiology literature that the transmission of many viruses depends on weather events.

[39]We do this for easy of exposition. One could integrate additional state variables into the game to handle the case of correlation of state of health across time. One could also introduce mortality into the model, and/or develop overlapping generations versions of the game where the probability of infection depends on agent's age.

[40]We should note, a related condition to our first condition concerning the probabilities of transmission and the complementarities in the game is used in the flu pandemic game of Sun, Yang, and Vericourt (2009).

[41]Observe that our complementarity condition is consistent with "vaccine productivity", i.e. that vaccination reduces the total risk of being infected. To see that note that the probability of getting ill, if vaccinated, is $sv^E + (1-v^E s)v^I q(a)$; or when not vaccinated, it is $s + (1-s)q(a)$. Therefore, a condition for „vaccine productivity" is: $sv^E + (1-v^E s)v^I q(a) < s + (1-s)q(a)$ or equivalently: $sv^E - s + [(1-v^E s)v^I - (1-s)]q(a) < 0$. As $q(a) \geq 0$, our complementarity condition, therefore, has $[(1-v^E s)v^I - (1-s)] \geq 0$. Additionally, as extreme case has $q(a) = 1$ ($a = 0$), then our „vaccine productivity" condition above only requires $sv^E - s + [(1-v^E s)v^I - (1-s)] < 0$ or: $sv^E + (1-v^E s)v^I < 1$. The last condition is easily satisfied as all probabilities are less then one.

there are no links between periods in the game apart from the transition on the state variable s', which we assume is endogenous and drawn each period from the distribution Q (which, of course, is parameterized by the vaccination decisions of the players, as well as the current state s.) So, for example, in this game, an individual may be ill today, yet be healthy tomorrow (or vice versa).With the conditions under which this game exhibits complementarity structure now stated, all the results in the paper can be show to apply to this game under our conditions on Q (e.g., our main theorems to prove existence, as well as compute extremal MSNE).

Also, we can also interpret the absorbing state $\bar{S}$ in this model nicely in this game: namely, the absorbing state can be set to coincide with the highest susceptibility in the aggregate population to the virus. In particular, in this state, the entire population gets ill. Therefore, we refer to $\bar{S}$ as the "epidemic state", where all players receive their lowest payoffs $u_i - L_i$.

4.3 Time-consistent public policy

We now consider a time-consistent policy game as defined by Stokey (1991) and analyzed more recently by Lagunoff (2008). Consider a (stochastic) game between a large number of identical households and the government. We will study equilibria that treat each household identically. For any state $k \in S$ (capital level), households choose consumption c and investment i treating level of a government spending G as given. There are no security markets that household can share the risk for tomorrow capital level. The only way to consume tomorrow is to invest in the stochastic technology Q. The within period preferences for the households are given by $u(c)$ (i.e. household does not obtain utility from public spending G). The government raises revenue by levying flat tax $\tau \in [0,1]$ on capital income, to finance its public spending $G \geq 0$. Each period the government budget is balanced and its within period preferences are given by: $u(c) + J(G)$. The consumption good production technology is given by constant return to scale function $f(k)$ with $f(0) = 0$. The transition technology between states is given by a probability distribution $Q(\cdot|i,k)$, where i denotes household investment. The timing of the game in each period is that the government and household choose their actions simultaneously. Observe, in this example a natural absorbing state is $k = 0$.

To specify each players optimization problem, first we assume households and the government take price R as given, with profit maximization implying $R = f'(k)$. Assume that u, J, f are increasing, concave and twice continuously differentiable and Q is given by assumption 2 (with $L = 1$ to simplify notation). Each of the households then choose investment i to solve:

$$\max_{i \in [0,(1-\tau)Rk]} u((1-\tau)Rk - i) + g(i)\beta \int_S v_H(s)\lambda(s|k).$$

By a standard arguments, we see that the objective for the households is supermodular in i and has increasing differences in (i,t), where $t = 1-\tau$ (noting $-u''(\cdot) \geq 0$). Moreover, the objective is increasing in $t = 1-\tau$ by monotonicity assumptions on u.

The government is choosing t to solve:

$$\max_{t \in [0,1]} u(tRk - i) + J(Rk(1-t)) + g(i)\beta \int_S (v_H(s) + v_G(s))\lambda(s|k).$$

That is, the government maximizes the household utility as well as the additional utility that it obtains from public spending J, and its continuation v_G. Again, objective is supermodular in $1-\tau$ and has increasing differences in $(t = 1-\tau, i)$ as $-u''(\cdot) \geq 0$. Moreover observe, although the objective is not increasing in i, along any Nash equilibrium of the auxiliary game, the government's objective is increasing in v_H by the envelope theorem. To see that, by $(i^*, t^*)(v_H)$ denote an extremal NE of the auxiliary game and observe that:

$$
\begin{aligned}
&\frac{\partial}{\partial i}\left[u(t^*(v_H)Rk - i) + J(Rk(1-t^*(v_H))) + g(i)\beta\int_S (v_H(s)+v_G(s))\lambda(s|k)\right]_{i=i^*(v_H)} = \\
&= \left[-u'(t^*(v_H)Rk - i^*(v_H)) + g'(i^*(v_H))\beta\int_S v_H(s)\lambda(s|k)\right] + g'(i^*(v_H))\beta\int_S v_G(s)\lambda(s|k) = \\
&= g'(i^*(v_H))\beta\int_S v_G(s)\lambda(s|k) \geq 0.
\end{aligned}
$$

So, interestingly, although this model's general assumptions do not appear to satisfy the underlying sufficient conditions given in our paper, the same method developed in the paper can be extended easily to this case. That is, we are able to use our results to prove existence of MSNE, as well as compute the least and the greatest MSNE. Specifically, we can construct an operator on the space of values, that would be monotone (as the within period game is supermodular and the Nash equilibrium of such game is monotone in v_H, v_G).

Some additional interesting points of departure from this above basic specification can also be worked out, including: (i) elastic labor supply choice, or more importantly (ii) adding security markets, investment/insurance firms possessing Q and proving existence of prices decentralizing optimal investment decision i^*. Still observe, however, that here we are able to offer weak assumptions for existence of a stationary credible policy, as well as offer a variety of tools allowing for its constrictive study and computation.

4.4 Symmetric equilibria in symmetric stochastic games

Finally, we consider a special case of our stochastic game, namely, one where all players have identical preferences $u := u_i$ and action sets $\tilde{A} := \tilde{A}_i \subset \mathbb{R}$. With slight abuse of notation, we denote payoff of a player choosing a_i, when others choose a_{-i} in state s by $u(s, a_i, a_{-i})$. Now observe that for such a special case we can obtain results of theorem 3.1, corollary 3.1 and others from section 3.2 for *symmetric equilibria* dispensing assumption 1 of increasing differences of u in (a_i, a_{-i}) and supermodularity of g in a in assumption 2. Instead, to guarantee existence of the NE of the auxiliary game we need to add concavity of g in a, and concavity of u in a_i. Indeed, under such additional assumptions the auxiliary game G_v^s has the greatest and the least symmetric Nash equilibrium, both monotone in v by Corollary 2 of Milgrom and Roberts (1994). Hence, we can still construct two monotone operators $\overline{T}$,$\underline{T}$ and reconstruct the proofs of theorem 3.1 and corollary 3.1. Such modification is important, as it allows one to dispense with the restrictive assumption of (within period) strategic complementarities between players while allowing to obtain (between period) strategic complementarities (at least for selected extremal NE values), a necessary feature for our constructive arguments.

An immediate example of the importance of this generalization can be seen, when studying of symmetric SMNE in a stochastic version of a private provision of public good game. Let $u(c_i, Y)$ be a payoff from consumption of a private c_i and public good Y. Assume marginal utilities are decreasing, and both goods are complements. Endow consumer with income w to be distributed between c_i and private provision y_i. Let a public good be produced using technology $Y = F(\sum_i y_i, s)$, where F is increasing and concave in the first argument. Observe that the function $(y_i, y_{-i}) \to u(w - y_i, F(\sum_j y_j, s))$ does not have increasing differences, but has positive externalities due to free rider problem. Let s parameterize public good stock (i.e. a draw representing a stock from the previous period) or its productivity, while Q represent a process allowing to reduce a future probability of a zero output / productivity, by higher provisions $(y_1, \ldots, y_n)$ today. By theorem 3.1 and corollary 3.1 we can prove existence and approximate the greatest and least symmetric MSNE of such a game.

Finally, using this generalization, we can reconsider symmetric MSNE of an Bertrand competition with durable good example (see subsection 4.1), and relax increasing differences assumption of demand D_i with (p_i, p_{-i}) and supermodularity of g.

5 Related results on games with uncountable state spaces

In this paper, we have presented a constructive method for computing extremal MSNE for a class of stochastic supermodular games. We are able to exploit the complementarity structure of our class of games, along with some mild regularity conditions on stochastic transitions, to develop very sharp results for monotone iterative procedures that construct *both* values and pure strategies. Our paper contributes also, however, to the literature on existence of MSNE in stochastic games with uncountable state and action spaces[42].

Recall, that for the existence of stationary Markov equilibrium researchers used the auxiliary game approach, where to prove MSNE existence it suffices to show the existence of a (measurable) "fixed point" selection (i.e., a measurable selection $v^*(s) \in \Pi(v^*, s)$). This turns out to be a serious matter in a general case even if randomization is applied (see Levy (2012) for an extensive discussion of this fact). There have been many approaches to this problem in the literature. For example, in the class of correlated strategies involving i.i.d. public randomization, MSNE has been shown to exist by Nowak and Raghavan (1992). Recently, Duggan (2012) extended the paper by Nowak and Raghavan (1992) expanding the state space, where MSNE exist by appealing to the use of additional "noisy variables". The paper most related to ours is that of Nowak (2003), where by adding similar type of stochastic transition structure as ours, he is able to prove the existence of a measurable MSNE.

One can make significant progress on this problem if it is possible to select a measurable function from the Nash equilibrium mapping $\Pi(v, \cdot)$ in some nonempty, convex and compact function space. There are many approaches in the literature for implementing this idea. One obvious method is to assume that the auxiliary game has a *unique* equilibrium (and, hence, a unique equilibrium value $\Pi(v, \cdot) \in CM$, where CM is the set of Lipschitz continuous functions on the state space S). This is precisely the approach that has been recently taken by many authors in the literature (e.g., Curtat (1996) and Amir (2002)). Conditions necessary to apply this argument are strong and have been discussed already in section 3.

Another interesting idea for obtaining the needed structure to resolve these existence issues is to develop a procedure for selecting from correspondence $\Pi(v, \cdot)$ an upper semi-continuous, increasing function (i.e., distribution functions) on a compact interval of the *real line* and observing that a set of such functions is weakly compact. This approach was first explored by Majumdar and Sundaram (1991) and Dutta and Sundaram (1992) in the important class of dynamic games (e.g., dynamic resource extraction games). More recently, Amir (2005) has appealed exactly this sort of argument to a class of stochastic supermodular games, where both values and pure strategies for SMNE are shown to exist in spaces of increasing, upper semi-continuous functions. Of course, the most serious limitation of this purely topological approach in the literature is that to date, the authors have been forced to severely restrict the state space of the game, as well as

[42]Recall that Mertens and Parthasarathy (1987), Solan (1998) and Maitra and Sudderth (2007) prove existence of subgame perfect Nash equilibrium in a class of such games.

require a great deal of complementarity between actions and states (i.e., assumptions consistent with the existence of monotone Markov equilibrium), to keep the topological argument tractable.

Finally an APS type methods can be used to deduce the existence of sequential or nonstationary Markov equilibrium in stochastic games. For Markovian equilibria till now researchers using this technique (see Doraszelski and Escobar, 2012) needed to restrict the state space to countable. The reason comes from the fact that the set of measurable Nash equilibrium values in the auxiliary game fails to be weak-star compact, the fixed point operator does not preserve compactness and as a result existence may fail. The second difference relative to our method is that APS type procedure allows to approximate the equilibrium value set but not equilibrium strategy directly. Third, typical APS comparative statics argument allows to order (in set inclusion) the sets of equilibrium values but not the set of equilibria nor its particular elements as ours. These last two contributions greatly simplify the numerical procedures that need to be used to compute and calibrate equilibria in stochastic games.

6 Proofs

We first state three lemmata that prove useful in verifying the existence of SMNE in our game, and in addition, are also useful in characterizing monotone iterative procedures for constructing least and greatest SMNE (relative to pointwise partial orders on $\mathcal{B}^n(S)$). More specifically, these lemmas concerns the structure of Nash equilibria (and their associated corresponding equilibrium payoffs) in our auxiliary game G_v^s.

Lemma 6.1 (Monotone Nash equilibria in G_v^s) *Under assumptions 1 and 2, for every $s \in S$ and value $v \in \mathcal{B}^n(S)$, the game G_v^s has the maximal Nash equilibrium $\overline{a}(v,s)$, and minimal Nash equilibrium $\underline{a}(v,s)$. Moreover, both equilibria are increasing in v.*

Proof of lemma 6.1: Without loss of generality fix $s > 0$. Define auxiliary one shot game, say $\Delta(\tau)$, with an action space A, and payoff function for player i given as

$$H_i(a,\tau) := (1-\beta_i)u_i(s,a_i,a_{-i}) + \beta_i \sum_{j=1}^{L} \tau_{i,j} g_j(s,a_i,a_{-i}),$$

where $\tau := [\tau_{i,j}]_{i=1,\ldots,n,j=1,\ldots,L} \in \mathcal{T} := \mathbb{R}^{n\times L}$ is endowed with the natural pointwise order. As supermodularity of a function on a sublattice of a directed product of lattices implies increasing differences (see Topkis (1998) theorem 2.6.1) clearly, for each $\tau \in \mathcal{T}$, the game $\Delta(\tau)$ is supermodular, and satisfies all assumptions of Theorem 5 in Milgrom and Roberts (1990). Hence, there exists a complete lattice of Nash equilibria, with the greatest Nash equilibrium given by $\overline{NE}\Delta(\tau)$, and the least Nash equilibrium given by $\underline{NE}\Delta(\tau)$. Moreover, for arbitrary i, the payoff function $H_i(a,\tau)$ has increasing differences in a_i and τ; hence, $\Delta(\tau)$ also satisfies conditions of Theorem 6 in Milgrom and Roberts (1990). As a result, both $\overline{NE}\Delta(\tau)$ and $\underline{NE}\Delta(\tau)$ are increasing in τ.

Step 2: For each $s \in S$, the game G_v^s is a special case of $\Delta(\tau)$ where $\tau_{i,j} = \int_S v_i(s')\lambda_j(ds'|s)$. Therefore, by the previous step, least and greatest Nash equilibrium $\underline{a}(v,s)$ and $\overline{a}(v,s)$ are increasing in v, for each $s \in S$ ■

In our next lemma, we show that for each extremal Nash equilibrium (for state s and continuation v), we can associate an equilibrium payoff that preserves monotonicity in v. To do this, we first compute the values of greatest (resp., least) best responses given a continuation values v and state s as follows:

$$\overline{\Pi}_i^*(v, s) := \Pi_i(v_i, s, \overline{a}_i(v, s), \overline{a}_{-i}(v, s))$$

and similarly

$$\underline{\Pi}_i^*(v, s) := \Pi_i(v_i, s, \underline{a}_i(v, s), \underline{a}_{-i}(v, s)).$$

We now have the following lemma:

Lemma 6.2 (Monotone values in G_v^s) *Under assumptions 1 and 2 we have: $\overline{\Pi}_i^*(v, s)$ and $\underline{\Pi}_i^*(v, s)$ are monotone in v.*

Proof of lemma 6.2: Function Π_i is increasing with a_{-i} and v_i. For $v_2 \geq v_1$ by Lemma 6.1, we have $\underline{a}(v_2, s) \geq \underline{a}(v_1, s)$. Hence,

$$\begin{aligned}\underline{\Pi}_i^*(v^2, s) &= \max_{a_i \in \tilde{A}_i(s)} \Pi_i(v_i^2, s, a_i, \underline{a}_{-i}(v^2, s)) \geq \max_{a_i \in \tilde{A}_i(s)} \Pi_i(v_i^1, s, a_i, \underline{a}_{-i}(v^2, s)) \geq \\ &\geq \max_{a_i \in \tilde{A}_i(s)} \Pi_i(v_i^1, s, a_i, \underline{a}_{-i}(v^1, s)) = \underline{\Pi}_i^*(v^1, s).\end{aligned}$$

A similar argument proves the monotonicity of $\overline{\Pi}_i^*(v, s)$. ■

To show that $\overline{T}(\cdot)(s) = \overline{\Pi}(\cdot, s)$ and $\underline{T}(\cdot)(s) = \underline{\Pi}(\cdot, s)$ are well-defined transformations of $\mathcal{B}^n(S)$ we use standard measurable selection arguments.

Lemma 6.3 (Measurable equilibria and values of G_v^s) *Under assumptions 1 and 2 we have:*

- $\overline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$ *and* $\underline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$,
- *functions* $s \to \overline{a}(v, s)$ *and* $s \to \underline{a}(v, s)$ *are measurable for any* $v \in \mathcal{B}^n(S)$.

Proof of lemma 6.3: For $v \in \mathcal{B}^n(S)$ and $s \in S$, define the function $F_v : A \times S \to \mathbb{R}$ as follows:

$$F_v(a, s) = \sum_{i=1}^n \Pi_i(v_i, s, a) - \sum_{i=1}^n \max_{z_i \in \widetilde{A}_i(s)} \Pi_i(v_i, s, z_i, a_{-i}).$$

Observe $F_v(a, s) \leq 0$. Consider the problem:

$$\max_{a \in \times_i^n \tilde{A}_i(s)} F_v(a, s).$$

By assumption 1 and 2, the objective F_v is a Carathéodory function, and the (joint) feasible correspondence $\widetilde{A}(s) = \times_i \widetilde{A}_i(s)$ is weakly-measurable. By a standard measurable maximum theorem (e.g. theorem 18.19 in Aliprantis and Border (2003)), the correspondence $N_v : S \twoheadrightarrow \times_i A_i(s)$ defined as:

$$N_v(s) := \arg \max_{a \in \widetilde{A}_i(s)} F_v(a, s),$$

is measurable with nonempty compact values. Further, observe that $N_v(s)$, by definition, is a set of all Nash equilibria for the game G_v^s. Therefore, to finish the proof of our first assertion, for some player i, consider a problem $\max_{a\in N_v(s)} \Pi_i(v_i, s, a)$. Again, by the measurable maximum theorem, the value function $\overline{\Pi}_i^*(v, s)$ is measurable. A similar argument shows each $\underline{\Pi}_i^*(v, s)$ is measurable. Therefore by theorem 4.1 in Himmelberg (1975), we have for value operators $\overline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$ and $\underline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$.

To show the second assertion of the theorem, for some player i, again consider a problem of $\max_{a\in N_v(s)} a_i^j$ for some $j \in \{1, 2, \ldots, k_i\}$. Again, appealing to the measurable maximum theorem and theorem 4.1 in Himmelberg (1975), the (maximizing) selection $\overline{a}(v, s)$ (respectively, $\underline{a}(v, s)$) is measurable with s. ■

Proof of theorem 3.1: Poof of 1. Clearly $\phi^1 \leq \phi^2$ and $v^1 \leq v^2$. Suppose $\phi^t \leq \phi^{t+1}$ and $v^t \leq v^{t+1}$. By the definition of the sequence $\{v^t\}$ and lemma 6.2 , we have $v^{t+1} \leq v^{t+2}$. Then, by Lemma 6.1, definition of $\{\phi^t\}$, and the induction hypotheses, we obtain $\phi^{t+1}(s) = \underline{a}(v^{t+1}, s) \leq \underline{a}(v^{t+2}, s) = \phi^{t+2}(s)$. Similarly, we obtain monotonicity of ψ^t and w^t.

Proof of 2: Clearly, the thesis is satisfied for $t = 1$. By induction, suppose that the thesis is satisfied for some t. Since $v^t \leq w^t$, by Lemma 6.2, we obtain

$$v^{t+1}(s) = \underline{\Pi}^*(v^t, s) \leq \underline{\Pi}^*(w^t, s) \leq \overline{\Pi}^*(w^t, s) = w^{t+1}(s).$$

Then, by Lemma 6.1, we obtain

$$\begin{aligned} \phi^{t+1}(s) &= \underline{a}(v^{t+1}, s) \\ &\leq \underline{a}(w^{t+1}, s) \quad \text{and hence} \\ &\leq \overline{a}(w^{t+1}, s) = \psi^{t+2}(s). \end{aligned}$$

Proof of 3-4: It is clear since for each $s \in S$, the sequences of values v^t, w^t and associated pure strategies ϕ^t and ψ^t are bounded. Further, by previous step, they are monotone.

Proof of 5: By definition of v^t and ϕ^t, we obtain

$$\begin{aligned} v_i^{t+1}(s) &= (1-\beta_i)u_i(s, \phi^t(s)) + \beta_i \sum_{j=1}^{L} g_j(s, \phi^t(s)) \int_S v_i^t(s')\lambda_j(ds'|s) \\ &\geq (1-\beta_i)u_i(s, a_i, \phi_{-i}^t(s)) + \beta_i \sum_{j=1}^{L} g_j(s, a_i, \phi_{-i}^t(s)) \int_S v_i^t(s')\lambda_j(ds'|s), \end{aligned}$$

for arbitrary $a_i \in \tilde{A}_i(s)$. By the continuity of u_i and g and the Lebesgue Dominance Theorem, if we take a limit $t \to \infty$, we obtain

$$\begin{aligned} v_i^*(s) &= (1-\beta_i)u_i(s, \phi^*(s)) + \beta_i \sum_{j=1}^{L} g_j(s, \phi^*(s)) \int_S v_i^*(s')\lambda_j(ds'|s) \\ &\geq (1-\beta_i)u_i(s, a_i, \phi_{-i}^*(s)) + \beta_i \sum_{j=1}^{L} g_j(s, a_i, \phi_{-i}^*(s)) \int_S v_i^*(s')\lambda_j(ds'|s), \end{aligned}$$

which, by lemma 6.3, implies that ϕ^* is a pure stationary (measurable) Nash equilibrium, and v^* is its associated (measurable) equilibrium payoff. Analogously, we have ψ^* a pure strategy (measurable) Nash equilibrium, and w^* its associated (measurable) equilibrium payoff. ■

Proof of corollary 3.1: Step 1. We prove the desired inequality for equilibria payoffs. Since $0 \leq \mu^* \leq \overline{u}$, by Lemma 6.2 and definition of v^t and w^t, we obtain

$$v_1 \leq \mu^* \leq w_1.$$

By induction, let $v_t \leq \mu^* \leq w_t$. Again, from Lemma 6.2 we have:

$$\begin{aligned} v_{t+1} &= \underline{\Pi}^*(v_t, s) \leq \underline{\Pi}^*(\mu^*, s) \\ &\leq \mu^*(s) \leq \overline{\Pi}^*(\mu^*, s) \leq \overline{\Pi}^*(w_t, s) = w_{t+1}. \end{aligned}$$

Taking a limit with t we obtain desired inequality for equilibria payoffs.

Step 2: By previous step and Lemma 6.1, we obtain:

$$\begin{aligned} \phi^*(s) &= \underline{a}(v^*, s) \leq \underline{a}(\mu^*, s) \\ &\leq \gamma^*(s) \leq \overline{a}(\mu^*, s) \leq \overline{a}(w^*, s) = \psi^*. \end{aligned}$$

■

For fixed continuation value v let:

$$M_{i,j,l} := \sup_{s \in S, a \in A(s)} \frac{\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_l}}{-\sum_{\tilde{i}=1}^{n} \sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}},$$

$$M := \max \{ M_{i,j,l} : i = 1, \ldots, n, \text{ and } j = 1, \ldots, k_i, \text{ and } l = 1, \ldots, k \}.$$

By assumption 3, the constant M is a strictly positive real number.

Lemma 6.4 *Let assumptions 1, 2, 3 be satisfied and constraint functions $\tilde{a}_i \in CM^{k_i}$. Fix $v \in \mathcal{B}_n(S)$, and assume it is Lipschitz continuous. Consider an auxiliary game G_v^s. Then, there is a unique Nash equilibrium in this game $a^*(v, s)$ and belongs to $CM^{\sum_i k_i}$.*

Proof of lemma 6.4: Let $s > 0$ and let $v \in \mathcal{B}^n(S)$ be Lipschitz continuous function. To simplify we drop v from our notation. Let $x^1(s) = \tilde{a}(s)$and $x_i^{t+1}(s) := \arg \max_{a_i \in \tilde{A}_i(s)} \Pi_i(s, a_i, x_{-i}^t(s))$for $n \geq 1$.This is well defined by strict concavity of $\Pi_i$in a_i. Clearly, $x^1$is nondecreasing and Lipschitz continuous with a constant less that M. By induction, assume that this thesis holds for $t \in \mathbb{N}$. Note that $(s, a_i) \to \Pi_i(s, a_i, x_{-i}^t(s))$has increasing differences. Indeed if we take $s_1 \leq s_2$and $y_1 \leq y_2$then $x_{-i}^t(s_1) \leq x_{-i}^t(s_2)$and

$$\begin{aligned} & \Pi_i(s_1, y_2, x_{-i}^t(s_1)) - \Pi_i(s_1, y_1, x_{-i}^t(s_1)), \\ \leq\ & \Pi_i(s_1, y_2, x_{-i}^t(s_2)) - \Pi_i(s_1, y_1, x_{-i}^t(s_2)), \\ \leq\ & \Pi_i(s_2, y_2, x_{-i}^t(s_2)) - \Pi_i(s_2, y_1, x_{-i}^t(s_2)). \end{aligned}$$

Therefore, since $\tilde{A}_i(\cdot)$ is ascending in the Veinott strong set order, by Theorem 6.1 in Topkis (1978) we obtain that $x_i^{t+1}(\cdot)$ is isotone. We show that $x_i^{t+1}(\cdot)$ is Lipschitz continuous with a constant M. To do this we check hypotheses of Theorem 2.4(ii) in Curtat (1996). Define $\varphi(s) = s_1 + \ldots + s_k$. Define $\mathbf{1}_i := (1, 1, \ldots, 1) \in \mathbb{R}^{k_i}$. We show that the function $(s, y) \to \Pi^*(s, y) :=$

$\Pi_i(s, M\varphi(s)\mathbf{1}_i - y, x^t_{-i}(s))$ has increasing differences. Note that $M\varphi(s) - \tilde{a}_i(s) \leq y \leq M\varphi(s)$. We show that the collection of the sets $Y(s) := [M\varphi(s) - \tilde{a}_i(s), M\varphi(s)]$ is ascending in the Veinott strong set order. Let $s_1 \geq s_2$ in product order Then,

$$\begin{aligned} M\varphi(s_1) - \tilde{a}^j_i(s_1) - \Big(M\varphi(s_2) - \tilde{a}^j_i(s_2)\Big) &= \\ &= M||s_1 - s_2||_1 - |\tilde{a}^j_i(s_1) - \tilde{a}^j_i(s_2)| \geq 0. \end{aligned}$$

This, therefore, implies that lower bound of $Y(s)$ is increasing with s. Clearly upper bound of $Y(s)$ is increasing as well. Hence $Y(s)$ is ascending in the Veinott's strong set order.

Note that since for all $s_l \to x^t(s)$ is monotone and continuous, hence must be differentiable almost everywhere (Royden (1968)). By M Lipschitz property of x^t we conclude that each partial derivative is bounded by M. Hence we have for all $l = 1, \ldots, k$, $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$:

$$\frac{\partial \Pi^*_i}{\partial y^j_i} = -\frac{\partial \Pi_i}{\partial a^j_i}.$$

Next we have (for fixed s_{-k}):

$$\begin{aligned} \frac{\partial^2 \Pi^*_i}{\partial y^j_i \partial s_k} &= -\frac{\partial^2 \Pi_i}{\partial a^j_i \partial s_k} - M\sum_{\alpha=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_i a^\alpha_i}\frac{\partial \varphi}{\partial s_k} - \sum_{\tilde{i}\neq i}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a^j_i \partial a^{\tilde{j}}_{\tilde{i}}}\frac{\partial x^*_{\tilde{i},\tilde{j}}}{\partial s_k}. \\ &\geq -\frac{\partial^2 \Pi_i}{\partial a^j_i \partial s_k} - M\sum_{\alpha=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_i a^\alpha_i} - \sum_{\tilde{i}\neq i}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a^j_i \partial a^{\tilde{j}}_{\tilde{i}}} M \\ &= -\frac{\partial^2 \Pi_i}{\partial a^j_i \partial s_k} - \sum_{\tilde{i}=1}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a^j_i \partial a^{\tilde{j}}_{\tilde{i}}} M \\ &= \left(-\sum_{\tilde{i}=1}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a^j_i \partial a^{\tilde{j}}_{\tilde{i}}}\right)\left(M - \frac{\frac{\partial^2 \Pi_i}{\partial a^j_i \partial s_k}}{-\sum_{\tilde{i}=1}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a^j_i \partial a^{\tilde{j}}_{\tilde{i}}}}\right) \geq 0 \end{aligned}$$

almost everywhere. Since $\frac{\partial \Pi^*_i}{\partial a^j_i}$ is continuous, by Theorem 6.2. in Topkis (1978) the solution of the optimization problem $y \to \Pi^*(v, s, y)$ (say $y^*(s, v)$) is isotone. From definition of $y^*(s, v)$ and x^{t+1} if $s_1 \leq s_2$ we have

$$0 \leq x^{t+1}_{i,j}(s_1) - x^{t+1}_{i,j}(s_2) \leq M(\varphi(s_1) - \varphi(s_2)) = M||s_1 - s_2||_1.$$

Analogously we prove appropriate inequality whenever $s_1 \geq s_2$. If s_1 and s_2 are incomparable then

$$x^{t+1}_{i,j}(s_1) - x^{t+1}_{i,j}(s_2) \leq x^{t+1}_{i,j}(s_1 \vee s_2) - x^{t+1}_{i,j}(s_1 \wedge s_2) \leq M||s_1 - s_2||_1$$

since $||s_1 - s_2||_1 = ||s_1 \vee s_2 - s_1 \wedge s_2||_1$. Similarly we prove that:

$$x^{t+1}_{i,j}(s_1) - x^{t+1}_{i,j}(s_2) \geq -M||s_1 - s_2||_1.$$

But this implies that x^{t+1} is also M-Lipschitz continuous, which implies that each x^t is M-Lipschitz continuous. Since Π_i has increasing differences in (a_i, a_{-i}) hence by Theorem 6.2 in

Topkis (1978) we know that the operator $x \to \arg \max_{y_i \in \tilde{A}_i(s)} \Pi_i(s, y_i, x_{-i})$ is increasing. Therefore $x^t(s)$ must be decreasing in t. This implies that there exists $a^* = \lim_{n\to\infty} x^t$ which is isotone and M-Lipschitz continuous. Uniqueness of Nash Equilibria follows from assumption 3 and Gabay and Moulin (1980), hence $a^* = a^*(s, v)$ for $s > 0$.

Finally $\Pi(0, a) = 0$ for all $a \in A$ hence we can define $a^*(0) := \lim_{s\to 0^+} a^*(s)$ and obtain a unique Nash equilibrium $a^*(s, v)$ that is isotone and M-Lipschitz continuous in s. ■

Proof of theorem 3.2: To simplify notation, let $L = 1$ and hence $g(s, a) := g_1(s, a)$ and $\eta(f)(s) := \eta_1^f(s)$. Let $v \in \mathcal{B}^n(S)$ be Lipschitz continuous. Under assumptions 3, $a^*(s, v)$ is a well defined (for $s > 0$) as auxiliary game satisfies conditions of Gabay and Moulin (1980). Let $\pi_i(v_i, s, a) = (1-\beta_i)u_i(s, a) + \beta_i\eta_i(v_i)g(s, a)$, and observe that $\pi_i(v_i, s, \cdot)$ has also strict diagonal property, and obviously has cardinal complementarities. Here, note that

$$\mathcal{L}_{i,j}(\pi_i(v_i, s, \cdot)) = (1-\beta_i)\mathcal{L}_{i,j}(u_i(s, \cdot)) + \beta_i\eta_i(v_i)(s)\mathcal{L}_{i,j}(g(s, \cdot)) < 0.$$

Note further that applying Royden (1968) and continuity of the left side of expression below we have

$$\begin{aligned}
\frac{\frac{\partial^2 \pi_i(v,s,\cdot)}{\partial a_i^j \partial s_l}}{-\sum_{\tilde{i}=1}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \pi_i(v,s,\cdot)}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}} &= \frac{(1-\beta_i)\frac{\partial u_i}{\partial a_i^j \partial s_l} + \beta_i \frac{\partial \eta(v)(s)}{\partial s_l}\frac{\partial g(s,a)}{\partial a_i^j} + \beta_i\eta(v)(s)\frac{\partial^2 g(s,a)}{\partial a_i^j \partial s_l}}{-(1-\beta_i)\mathcal{L}_{i,j}(u_i) - \beta_i\eta(v)(s)\mathcal{L}_{i,j}(g)} \\
&\leq \frac{(1-\beta_i)U_{i,j,l}^2 + \beta_i\bar{\eta}G_{i,j}^1 + \beta_i\bar{u}G_{i,j,l}^2}{-(1-\beta_i)\mathcal{L}_{i,j}(u_i)} \leq M_0.
\end{aligned}$$

By Lemma 6.4, we know that $a^*(\cdot, v) \in CM_0^{\sum_i k_i}$. The following argument shows that $Tv(s)$ is Lipschitz continuous.

$$\begin{aligned}
|T_iv(s_1) - T_iv(s_2)| &\leq (1-\beta_i)|u_i(s_1, a^*(s_1, v)) - u_i(s_2, a^*(s_2, v))| \\
&+ \beta_i|\eta(v_i)(s_1) - \eta(v_i)(s_2)|g(s_1, a^*(s_1, v)) \\
&+ \beta_i\eta(v_i)(s_2)|g(s_1, a^*(s_1, v)) - g(s_2, a^*(s_2, v))| \\
&\leq (U_1 + M_0U_2 + \bar{\eta} + (G_1 + G_2M_0)\bar{u})||s_1 - s_2||_1 \\
&= M_1||s_1 - s_2||_1,
\end{aligned}$$

where

$$U_1 := \sum_{l=1}^{h} \sup_{s\in S, a\in\tilde{A}(s)} \frac{\partial u_i}{\partial s_l}(s, a), \qquad U_2 := \sum_{i=1}^{m}\sum_{j=1}^{k_i} \sup_{s\in S, a\in\tilde{A}(s)} \frac{\partial u_i}{\partial a_i^j}(s, a),$$

$$G_1 := \sum_{l=1}^{h} \sup_{s\in S, a\in\tilde{A}(s)} \frac{\partial g}{\partial s_l}(s, a), \qquad G_2 := \sum_{i=1}^{m}\sum_{j=1}^{k_i} \sup_{s\in S, a\in\tilde{A}(s)} \frac{\partial g}{\partial a_i^j}(s, a),$$

and $M_1 := U_1 + M_0U_2 + \bar{\eta} + (G_1 + G_2M_0)\bar{u}$. Hence image of operator T is a subset of CM_1^n. Therefore the thesis is proven. ■

Proof of theorem 3.3: On CM^n define a function $T(v)(s) = \Pi^*(v,s)$. By a standard argument (e.g. Curtat (1996)) $T : CM^n \to CM^n$is continuous and increasing on CM^n. By Tarski (1955) theorem, it therefore has a nonempty complete lattice of fixed points, say $FP(T)$. Further, for each fixed point $v^*(\cdot) \in FP(T)$, there is a corresponding unique stationary Markov Nash equilibrium $a^*(v^*, \cdot)$. ■

Lemma 6.5 *Let X be a lattice, Y be a poset. Assume (i) $F : X \times Y \to \mathbb{R}$ and $G : X \times Y \to \mathbb{R}$ have increasing differences, (ii) that $\forall\ y \in Y$, $G(\cdot, y)$ and $\gamma : Y \to \mathbb{R}_+$ are increasing functions. Then, function H defined by $H(x,y) = F(x,y) + \gamma(y)G(x,y)$ has increasing differences.*

Proof of lemma 6.5: Under the hypotheses of the lemma, it suffices to show that $\gamma(y)G(x,y)$ has increasing differences (as increasing differences is a cardinal property and closed under addition). Let $y_1 > y_2$, $x_1 > x_2$ and $(x_i, y_i) \in X \times Y$. By the hypothesis of increasing differences of G, and monotonicity of γ and $G(\cdot, y)$, we have the following inequality

$$\gamma(y_1)(G(x_1,y_1) - G(x_2,y_1)) \geq \gamma(y_2)(G(x_1,y_2) - G(x_2,y_2)).$$

Therefore,

$$\gamma(y_1)G(x_1,y_1) + \gamma(y_2)G(x_2,y_2) \geq \gamma(y_1)G(x_2,y_1) + \gamma(y_2)G(x_1,y_2).$$

■

Proof of theorem 3.4: Step 1. Let v_θ be a function $(s,\theta) \to v_\theta(s)$ that is increasing. By assumption 5 and lemma 6.5, the payoff function $\Pi_i(v_\theta, s, a, \theta)$ has increasing differences in (a_i, θ). Further, Π_i clearly also has increasing differences in (a_i, a_{-i}). As $\widetilde{A}_i()$ is ascending in Veinott's strong set order, by Theorem 6 in Milgrom and Roberts (1990), the greatest and the least Nash equilibrium in the supermodular game $G^s_{v,\theta}$ are increasing selections. Further, by the same argument as in Lemma 6.2, as $\widetilde{A}_i()$ is also ascending under set inclusion by assumption, we obtain monotonicity of corresponding equilibria payoff.

Step 2: Note, for each θ, the parameterized stochastic game satisfies conditions of Theorem 3.1. Further, noting the initial values of the sequence of $w^t_\theta(s)$ and $v^t_\theta(s)$ (constructed in Theorem 3.1) do not depend on θ and are isotone in s, by the previous step, each iteration of both sequences of values is increasing with respect to (s,θ). Also, each of the iterations of $\phi^t_\theta(s)$ and $\psi^t_\theta(s)$ are also increasing in (s,θ). Therefore, as the pointwise partial order is closed, the limits of these sequences preserve this partial ordering, and the limits are increasing with respect to (s,θ). ■

For each equilibrium strategy f, define the operator

$$T^o_f(\eta)(A) = \int\limits_S Q(A|s, f(s))\eta(ds). \tag{1}$$

where η^* is said to be invariant with respect to f if and only if it is a fixed point of T^o_f.

Proof of theorem 3.5: By theorem 3.4 both ϕ^* and ψ^* are increasing functions. Let v be increasing function. By Assumption 2

$$\int_S v(s) T^o_{\phi^*}(\eta)(ds) = \sum_{j=1}^{L} g_j(s, \phi^*(s)) \int_S v(s')\lambda_j(ds'|s)\eta(ds).$$

Since by assumption, for each $s \in S$, the function under integral is increasing, the right-side increase pointwise whenever η is stochastically increase. Moreover, as the family of probability measures on a compact state space S ordered by $\succeq$ (first order stochastic dominance) is chain complete (as it is a compact ordered topological space, e.g., Amann (1977), lemma 3.1 or corollary 3.2). Hence, $T^o_{\phi^*}$ satisfies conditions of Markowsky (1976) theorem (Theorem 9), and we conclude that the set of invariant distributions is a chain complete with greatest and least invariant distributions (see also Amann (1977), Theorem 3.3). By a similar construction, the same is true for the operator $T^o_{\psi^*}$.

To show the second assertion, we first prove that $T^o_{\phi^*}(\cdot)(A)$ is weakly continuous (i.e. if $\eta_t \to \eta$ weakly then $T^o_{\phi^*}(\eta_t) \to T^o_{\phi^*}(\eta)$ weakly). Let $\eta_t \to \eta$ weakly, and v be continuous. By Feller property of $\lambda_j(\cdot|s)$, we have $s \to \int_S v(s')\lambda_j(s'|s)$ continuous. Therefore,

$$\int_S \int_S v(s')\lambda_j(ds'|s)\eta_t(ds) \to \int_S \int_S v(s')\lambda_j(s'|s)\eta(ds).$$

This, in turn, implies

$$T^o_{\phi^*}(\eta_t) \to T^o_{\phi^*}(\eta)$$

weakly. Let $\eta_t^{\phi^*}$ be a distribution of $s_t^{\phi^*}$ and $\eta_1^{\phi^*} = \delta_{\bar{S}}$. By the previous step, η_t is stochastically decreasing. It is, therefore, weakly convergent to some η^*. By continuity of T^o, we have $\eta^* = T^o_{\phi^*}(\eta^*)$. By definition of $\overline{\eta}(\phi^*)$, we immediately obtain $\overline{\eta}(\phi^*) \preceq \eta^*$. By the stochastic monotonicity of $T^o_{\phi^*}(\cdot)$, we can recursively obtain that $\delta_{\bar{S}} \succeq \eta_t^{\phi^*} \succeq \overline{\eta}(\phi^*)$, and hence $\eta^* \succeq \overline{\eta}(\phi^*)$. As a result, we conclude $\eta^* = \overline{\eta}(\phi^*)$. Similarly, we show convergence of the sequence of distributions $s_t^{\psi^*}$. ■

Proof of corollary 3.3: By theorem 3.5, there exists greatest fixed points for $T^{o,\theta_2}_{\phi^*}$ and $T^{o,\theta_1}_{\phi^*}$. Also, $T^{o,\theta}_{\phi^*}$ is weakly continuous. Further, $\theta \to T^{o,\theta}_{\phi^*}$ is an increasing map under first stochastic dominance on a chain complete poset of probability measures on the compact state set.

Consider a sequence of iterations from a $\delta_{\bar{S}}$ generated on $T^{o,\theta}_{\phi^*}$ (the operator defined in (1) but associated with $Q(\cdot|s,a,\theta)$). Observe, by the Tarski-Kantorovich theorem (Dugundji and Granas (1982), theorem 4.2), we have

$$\sup_t T^{t,o,\theta_2}_{\phi^*} = \overline{\eta}_{\theta_2} \text{ and } \sup_t T^{t,o,\theta_2}_{\phi^*} = \overline{\eta}_{\theta_2}$$

As for any t, we also have $T^{t,o,\theta_2}_{\phi^*} \succeq T^{t,o,\theta_1}_{\phi^*}$. Therefore, by weak continuity (and the fact that $\succeq$ is a closed order), we obtain:

$$\overline{\eta}_{\theta_2} = \sup_t T^{o,\theta_2}_{\phi^*} \succeq \sup_t T^{t,o,\theta_1}_{\phi^*} = \overline{\eta}_{\theta_1}.$$

Similarly we proceed for ψ^*. ■

Remark 2 *Since norms $||\cdot||_1$ and $||\cdot||$ are equivalent on $\mathbb{R}^h$ and for all $s \in S$ $||s||_1 \leq h||s||$, hence we have that v^*, w^*, ϕ^* and ψ^* each Lipschitz continuous when we endow S with the maximum norm.*

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