Lipschitz Recursive Equilibrium with a Minimal State Space and Heterogeneous Agents

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Abstract

This paper analyzes the Lucas tree model with heterogeneous agents and one asset. We show the existence of a minimal state space Lipschitz continuous recursive equilibrium using Montrucchio (1987) results. The recursive equilibrium implements a sequential equilibrium through an explicit functional equation derived from the Bellman Equation. Our method also allows to prove existence of a recursive equilibrium in a general class of deterministic or stochastic models with several assets provided there exists a Lipschitz selection on the demand correspondence. We provide examples showing applicability of our results.

Keywords: Lucas Tree Model, Recursive Equilibrium, Minimal State Space, Lipschitz Demand, Lipschitz Continuity, Heterogeneous Agents, Incomplete Markets *JEL Classification:* D50, D52

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1. Introduction

Since the work of Lucas and Prescott (1971) and Prescott and Mehra (1980), recursive equilibrium has been a key focal point of both applied and theoretical

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work in characterizing sequential equilibrium for dynamic general equilibrium models in such fields as macroeconomics, international trade, growth theory, industrial organization, financial economies, and monetary theory. Specifically, in general dynamic models with infinitely lived agents economists have focused on so-called minimal state space recursive equilibrium, i.e. a pair of stationary transition and policy functions that relate the endogenous variables in any two consecutive periods, defined on the natural state space. Apart from its simplicity, (minimal state space) recursive equilibrium is also widely used in applied or computational works, as powerful recursive methods provide algorithms to compute it efficiently. Results regarding equilibrium existence are necessary prerequisites for a theoretical and computational analysis, however.

Unfortunately, there are well known examples where recursive equilibria (in specific function spaces) in dynamic economies are non existent (see Santos (2002) for economies with taxes, Kubler and Schmedders (2002) for economies with incomplete asset markets or Krebs (2004) for economies with large borrowing limits). Some recent attempts that address the question of minimal state space recursive equilibrium existence and its approximation, include contributions of Datta, Mirman, and Reffett (2002) and Datta, Reffett, and Woźny (2018) for models with homogeneous agents, who propose a monotone maps method applied on the equilibrium version of the household first order conditions and prove equilibrium existence along with its comparative statics, using versions of Tarski fixed point theorem. Unfortunately, there are no known results on how to extend these techniques to models with heterogeneous agents and multiple assets. Next, Brumm, Kryczka, and Kubler (2017) apply some powerful results from stochastic games literature and by adding sufficient shocks prove existence of a recursive equilibrium using operators defined on households first order conditions and applying Kakutani-Fan-Gliksberg fixed point theorem on the operator defined on the Walrasian auctioneer problem. The underlying topology is weak-star and the obtained recursive equilibrium a measurable map on the state space. The measure theoretical results together with recent contributions in stochastic games allow to prove minimal state space recursive equilibrium existence without sunspots or public coordination devices.

More specifically, one of the canonical equilibrium models analyzed in the literature that significantly influenced the fields of financial economics, macroeconomics, monetary theory, optimal taxation and econometrics, was developed by Lucas Jr (1978). However, despite the model's wide application, typical assumptions involve a representative agent. In fact, presence of infinitely lived *heterogeneous* agents can be the key to explain several peculiarities of market frictions from the perspective of models with rational expectations. Apart from mentioned Brumm, Kryczka, and Kubler (2017) contribution, there are only few known results concerning recursive equilibrium existence in the Lucas three model with heterogeneous agents. These include Raad (2016), who show the existence of a possibly non-continuous recursive equilibrium with a minimal state space, however, the model assumes that agents have exogenous beliefs on portfolio transitions.[1] Kubler and Schmedders (2002) present an example of an infinite-horizon economy with Markovian fundamentals, where the recursive competitive equilibrium (defined on a state space of equilibrium asset holdings and exogenous shocks) does not exists. In their example, there must exist two different nodes of a tree such that along the equilibrium path the value of the equilibrium asset holdings is the same but such that there exist more than one equilibrium for both of the continuation economies. Although, they claim that a slight perturbation in individual endowments will restore the existence of a weakly recursive equilibrium, we detail the set of conditions that rules out Kubler and Schmedders (2002) example from the model analyzed in our paper.[2]

[1] Agents make mistakes directly or indirectly on prices by inaccurate anticipation of transition portfolios and an equilibria with rational expectations and perfect foresight can *not* be implemented in this environment. Therefore, we cannot apply Raad's result in this paper. In fact, he shows that an equilibrium allocation for an economy with agents making large enough errors on price expectations cannot be a Radner equilibrium, assuming quite general conditions on the primitives. The author also presents an example elucidating this fact even if agents make errors only on the portfolio transitions.

[2] See remark 4.19.

In this paper, we take a different approach to show the existence of a minimal state space recursive equilibrium. By minimal state space, we mean the previous period asset allocation and current state of nature.[3] We proceed basically in five steps. First, we consider a class of transition and policy functions that are Lipschitz continuous. This allows us to obtain a sup norm compact set of candidate equilibrium functions. Second, the adopted framework and the recursive demand is constructed through a selector of the Bellman correspondence which is defined *without* using the first order conditions. This is a new approach and allow to compute equilibria with occasionally binding constraints[4]. Following Montrucchio (1987) results, we assume strong conditions on the primitives to ensure a Lipschitz condition of the demand is satisfied.[5] In order to do so, we restrict our attention to models with single asset.[6] Third, another problem faced in this paper is the expansion of the implied Lipschitz constants. Here we assume conditions on the primitives that assure our operator maps back to the space of Lipschitz functions with the same constant. We define upper and lower bounds of the domains so that the effective Lipschitz constants are well behaved (i.e. non-expanding). Forth, the fixed point operator is defined using the optimization problem (defined on the candidate space of Lipschitz continuous functions) of the Walrasian auctioneer. As a result, apart from proving existence, we also establish that the constructed equilibrium is in fact Lipschitz continuous. Fifth, we use a constructive argument to explain how the sequential equilibrium can be implemented recursively by showing the consecutive relations among the endogenous variables explicitly.

Working with Lipschitz continuous functions and a sup norm, although re-

[3]It is minimal because an asset redistribution naturally influences the equilibrium prices. This is also evident in models with risk aversion heterogeneity, for instance. See also discussion in Kubler and Schmedders (2002) on weakly recursive equilibria.

[4]We present a specific example, where equilibrium policies are boundary for a subset of a state space.

[5]Every continuously differentiable function over a compact interval is Lipschitz continuous. Montrucchio (1987) theorem provides, however, the Lipschitz constant of the argmax.

[6]As we are not aware of Lipschitz selection theorems for argmax correspondences.

strictive per assumptions, allows us to avoid typical convergence problems associated with working with the set of feasible measurable functions endowed with the weak-star topology. In fact, concerning the set of measurable functions defined over uncountable domain, the Mazur lemma states that a weak-star cluster point of any subset is a pointwise cluster point of its convex hull. However, a weak-star cluster point of a typical subset may not be a pointwise cluster point of it. Importantly, this problem is present even when working in the space of randomized policies. One way to overcome this problem is to introduce some convexification devices, either via sunspots (see Duffie, Geanakoplos, Mas-Colell, and McLennan (1994)) or external noise (see Brumm, Kryczka, and Kubler (2017)) in stochastic models. Our results work for deterministic and stochastic economies and hence complement Brumm, Kryczka, and Kubler (2017) contribution. Moreover, and perhaps more importantly, working with Lipschitz continuous functions allows us to obtain a tractable and approximate space of equilibrium candidates. Although we cannot verify whether our fixed point operator is a contraction, working with Lipschitz equilibrium functions is still an important numerical advantage of our approach[7], as it is easier to characterize numerically Lipschitz function as opposed to a function that is only known to be measurable. As we do not use consumers' first order conditions, such sequential equilibrium can be computed using the dynamic programming approach and thus does not embody cumulative errors in the long run as noted by Kubler and Schmedders (2005).

Including this introduction, the paper is organized into five sections. Section 2 establishes the model. In Section 3, we define the recursive and sequential equilibrium concepts and show how they are related. Section 4 shows the existence result. We provide explicit conditions on the primitives that guarantee Lipschitz continuity of the demand correspondence on a suitable set of prices bounded away from zero and infinity. The conclusions are addressed in Section 5.

[7] See e.g. Hinderer (2005) for error bounds in approximation of Lipschitz value functions. See also Santos (2000) relating error bounds of the value and policy functions.

2. The model

2.1. Definitions

Suppose that there exists a finite set of types denoted by $\mathcal{I} = \{1, \cdots, I\}$ and such that each type $i \in \mathcal{I}$ has a continuum of agents trading in a competitive environment. Time is indexed by t in the set $\mathbb{N} = \{1, 2, \cdots\}$ for current periods and $r \in \mathbb{N} \cup \{0\}$ for future periods. In this model, the uncertainty is exogenous, in the sense of being independent of agents' actions. Each agent knows the whole set of possible exogenous variables[8] and trades contingent claims. Let $Z \subset [0, 1]^N$ for some $N \in \mathbb{N}$ be a set containing all states of nature[9] and let $\mathscr{Z}$ denote its Borel sigma-algebra. Denote by $(Z_\tau, \mathscr{Z}_\tau)$ a copy of $(Z, \mathscr{Z})$ for all $\tau \in \mathbb{N}$. Exogenous uncertainty is described by the streams $z^\tau = (z_1, \cdots, z_\tau) \in Z_1 \times \cdots \times Z_\tau = Z^\tau$ for all $\tau \in \mathbb{N}$, that is, the set of nodes of the event tree is given by $\bigcup_{\tau \in \mathbb{N}} Z^\tau$.

There is one consumption good[10] and one long lived real asset[11] with dividends characterized by a bounded, measurable function $\hat{d} : Z \to \mathbb{R}_{++}$ given in units of the consumption good. The number $\hat{d}(z)$ represents the amount of good paid by one unit of the asset in the state of nature z. By $\Theta^i \subset \mathbb{R}_+$ denote a convex set where asset choices are defined and by $C^i \subset \mathbb{R}_+$ the convex set where agent i's consumption is chosen. Moreover, write[12] $X^i = C^i \times \Theta^i$ for all

[8] Also called states of nature or exogenous shocks.

[9] Importantly, every Lipschitz continuous function is measurable, hence domains that we use in our construction allow us to work with uncountable state space.

[10] The results can be generalized for more consumption goods. The computation of Lipschitz constants used in our construction becomes cumbersome and does not bring additional economic intuition, however. For this reason we specify our main results assuming single consumption good. See Remark 6.10 in appendix for more details.

[11] We use Montrucchio (1987) conditions on the consumers maximization problem to assure existence of a Lipschitz demand. In case of more than one assets we would necessarily obtain an argmax correspondence as for some prices a typical consumer may be indifferent between some asset portfolios. As we are not aware of results characterizing Lipschitz selections from the argmax correspondences, in this paper we analyze the case of a single asset and leave the case of more assets for further research.

[12] We consider consumption sets as *subsets* of $\mathbb{R}_+$ as upper and lower bounds of the domains

$i \in \mathcal{I}$. Define the symbol without upper index as the Cartesian product (if it is not otherwise defined). For instance, write $C = \prod_{i\in\mathcal{I}} C^i$. Define analogously the symbol without upper index for functions.

Define the set of prices as $Q = \{q \in \mathbb{R}^2_{++} : q = (q_c, q_a) = (1, p)\}$. We assume that assets are given in net supply one.[13] Therefore, write

$$\overline{\Theta} = \Big\{\bar{\theta} \in \Theta : \sum_{i\in\mathcal{I}} \bar{\theta}^i = 1\Big\}.$$

Let $S = \overline{\Theta} \times Z$ be the space of state variables with a typical element denoted by $s = (\bar{\theta}, z)$ and endowed with the product topology. Write $\mathscr{S}$ as the Borel subsets of S and $(S_\tau, \mathscr{S}_\tau)$ a copy of $(S, \mathscr{S})$ for all $\tau \in \mathbb{N}$. Denote the set of all continuous functions[14] $\hat{q} : S \to Q$ by $\widehat{Q}$ with $\hat{q} = (1, \hat{p})$ and the set of all continuous functions $\hat{p} : S \to \mathbb{R}_{++}$ by $\widehat{P}$. Moreover, consider $\widehat{C}$ as the space of all continuous functions $\hat{c} : S \to C$ representing the transition of optimal consumption choices and $\widehat{\Theta}$ as the space of all continuous functions $\hat{\theta} : S \to \Theta$ representing the transition of asset distribution.[15] Finally, write $X = C \times \Theta$ and $\widehat{X} = \widehat{C} \times \widehat{\Theta}$.

Notation 2.1. Each Cartesian product of topological spaces is endowed with the product topology and any set of bounded continuous functions is endowed with the topology induced by the sup norm. The norm $||\cdot||$ in $\mathbb{R}^L$ considered here is the max norm, that is, $||y|| = \max\{|y_1|, \cdots, |y_n|\}$. Write n_y and N_y for inferior and superior boundaries of a variable y or a function $\hat{y}$ and $M_{\hat{y}}$ as the Lipschitz constant of a function $\hat{y}$. For each $y, y' \in \mathbb{R}^L$ write $y \le y'$ when $y_l \le y'_l$ for all $l \le L$ and $yy' = \sum_{l\le L} y_l y'_l$. When $y \in \mathbb{R}^L$ and $y' \in \mathbb{R}$

will play an important role in the construction of non-expanding Lipschitz constants in the proof of the existence theorem.

[13]Since we are only interested in symmetric equilibria, we assume that each agent of type i chooses the same portfolio $\bar{\theta}^i$ and, consequently, this portfolio can be viewed as the mean asset choice of agents belonging to type i.

[14]Note that we are using the "hat" symbol to denote the space of functions from S to the specified set.

[15]In the equilibrium transition $\hat{\theta}(S) \subset \bar{\Theta}$.

then write $y \leq y'$ when $y_l \leq y'$ for all $l \leq L$. For each $y, y' \in \mathbb{R}^L$ define $\max\{y, y'\} = y'' \in \mathbb{R}^L$ with $y''_l = \max\{y_l, y'_l\}$ for all $l \leq L$ and for $y' \in \mathbb{R}$ define $\max\{y, y'\} = \max\{y, (y', y', \cdots, y')\}$. For a function $\hat{y} : Y \to \mathbb{R}^L$ and $\bar{y} \in \mathbb{R}^L$, then $\hat{y} \leq \bar{y}$ stands for $\hat{y}(y) \leq \bar{y}$ for all $y \in Y$. Write $\operatorname{Int} X^i$ the interior of the set X^i relative to $\mathbb{R}^2_+$. The reverse binary relations are defined analogously.

2.2. Agents' features

In every period, preferences are represented by an instantaneous utility given by an α-concave[16] (Montrucchio, 1987) real valued function $\hat{u}^i : C^i \to \mathbb{R}$ that is strictly increasing for all $i \in \mathcal{I}$. Since $\hat{u}^i$ is concave then it has a positive directional derivative and by $\partial \hat{u}^i(c^i)(\mathring{c}^i)$ we denote the positive directional derivative of $\hat{u}^i$ evaluated at the point c^i in the direction of $\mathring{c}^i$. Sometimes we use $\partial \hat{u}^i(c^i)$ to denote $\partial \hat{u}^i(c^i)(1)$.

Each agent i has a measurable endowment $\hat{e}^i : Z \to \mathbb{R}_+$ of good and a discount factor β^i for each $i \in \mathcal{I}$.

Suppose that the spaces $\text{Prob}(Z)$ and $\text{Prob}(Z^r)$ are endowed with the weak topology and the Borel sigma-algebra for each $r \in \mathbb{N}$. Agents' subjective beliefs[17] at every fixed date r are characterized by the continuous map $\hat{\mu}^i_r : Z \to \text{Prob}(Z^r)$ for $r \in \mathbb{N}$, anticipating future exogenous states of nature given the realization of the current state of nature z. We suppose that these beliefs are predictive, i.e. for a rectangle $A_1 \times \cdots \times A_r$ the measure $\hat{\mu}^i_r$ satisfies:

$$\hat{\mu}^i_r(z)(A_1 \times \cdots \times A_r) = \int_{A_1} \cdots \int_{A_r} \hat{\lambda}^i(z_{r-1}, dz_r) \cdots \hat{\lambda}^i(z, dz_1). \tag{1}$$

where $\hat{\lambda}^i : Z \to \text{Prob}(Z)$ is a continuous probability transition rule for each $i \in \mathcal{I}$.

We follow the approach of contingent choices as given in Radner (1972). Because agents do not perfectly anticipate the future states of nature, which

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[16]As we assume a single consumption good α-concavity is equivalent on a compact domain to a (uniform) strict concavity.

[17]These beliefs can be accurate in the case of rational expectations. But here we assume that agents always have perfect foresight with respect to price and asset transitions.

are given exogenously, rationality leads them to plan for the future at each current period contingent on all possible future trajectories of the states of nature. Therefore, we assert the definition below.

Definition 2.2. An agent i's plan is defined as the current period choice $(\boldsymbol{c}_0^i, \boldsymbol{\theta}_0^i) \in C^i \times \Theta^i$ and the streams $\{\boldsymbol{c}_r^i\}_{r\in\mathbb{N}}$ and $\{\boldsymbol{\theta}_r^i\}_{r\in\mathbb{N}}$ of measurable functions $\boldsymbol{c}_r^i : Z^r \to C^i$ and $\boldsymbol{\theta}_r^i : Z^r \to \Theta^i$ for all $r \in \mathbb{N}$ representing future plans.

In each current period, the quantity $\boldsymbol{c}_r^i(z^r)$ can be interpreted as the value planned for consumption r periods ahead if z^r is the partial history of prices actually observed during these periods. The asset plan $\{\boldsymbol{\theta}_r^j\}_{r\in\mathbb{N}}$ has an analogous interpretation.

Let $\boldsymbol{Q}$ be the set of all sequences $\{\boldsymbol{q}_r : Z^r \to Q\}_{r\geq 0}$ of measurable functions with $\boldsymbol{q}_0 \in Q$ for $r \in \mathbb{N}$. For each $i \in \mathcal{I}$ define $\boldsymbol{C}^i$ as the set of all sequences $\{\boldsymbol{c}_r^i : Z^r \to C^i\}_{r\geq 0}$ of measurable functions with $\boldsymbol{c}_0^i \in C^i$ for $r \in \mathbb{N}$. Define $\boldsymbol{\Theta}^i$ analogously for all $i \in \mathcal{I}$.

We assume that agents choose a feasible plan of consumption and savings that maximizes the expected utility, under their own beliefs, among all other feasible plans. The next definitions characterize the feasibility of a plan and how agents calculate its expected value.

Let $\hat{b}^i : \Theta^i \times Z \times Q \to C^i \times \Theta^i$ be defined as

$$\hat{b}^i(\theta_-^i, z, q) = \{(c^i, \theta^i) \in C^i \times \Theta^i : c^i + p\theta^i \leq (p + \hat{d}(z))\theta_-^i + \hat{e}^i(z)\}.$$

Let $\boldsymbol{q} \in \boldsymbol{Q}$ be a stream of contingent prices for a given $\boldsymbol{q}_0 \in Q$. For each agent $i \in \mathcal{I}$, a plan $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{C}^i \times \boldsymbol{\Theta}^i$ is feasible from $(\theta_-^i, z, \boldsymbol{q})$ if $(\boldsymbol{c}_0^i, \boldsymbol{\theta}_0^i) \in \hat{b}^i(\theta_-^i, z, \boldsymbol{q}_0)$ and for each $r \in \mathbb{N}$

$$(\boldsymbol{c}_r^i(z^r), \boldsymbol{\theta}_r^i(z^r)) \in \hat{b}^i(\boldsymbol{\theta}_{r-1}^i(z^{r-1}), z_r, \boldsymbol{q}_r(z^r)) \text{ for all } z^r \in Z^r.$$

Denote by $\boldsymbol{f}^i : \Theta^i \times Z \times \boldsymbol{Q} \to \boldsymbol{C}^i \times \boldsymbol{\Theta}^i$ a correspondence of all feasible plans for each $i \in \mathcal{I}$.

Define the agent i's expected utility $\boldsymbol{u}^i : \boldsymbol{C}^i \times Z \to \mathbb{R}$ from consuming $\boldsymbol{c}^i$

given the state $z \in Z$ by

$$\boldsymbol{u}^i(\boldsymbol{c}^i, z) = \hat{u}^i(\boldsymbol{c}_0^i) + \sum_{r\in\mathbb{N}} \int_{Z^r} (\beta^i)^r \hat{u}^i(\boldsymbol{c}_r^i(z^r))\hat{\mu}_r^i(z, dz^r).$$

Finally, define the value function $\tilde{\boldsymbol{v}}^i : \Theta^i \times Z \times \boldsymbol{Q} \to \mathbb{R}$ by:

$$\tilde{\boldsymbol{v}}^i(\theta_-^i, z, \boldsymbol{q}) = \sup\{\boldsymbol{u}^i(\boldsymbol{c}^i, z) : (\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{f}^i(\theta_-^i, z, \boldsymbol{q})\}. \tag{2}$$

The following definition characterizes agents' demand. It yields the current choice at each period given its previous and current observed variables. We assume that agents have perfect foresight, i.e, they anticipate the equilibrium stream of prices. More precisely write $\tilde{\boldsymbol{\delta}}^i : \Theta^i \times Z \times \boldsymbol{Q} \to \boldsymbol{C}^i \times \boldsymbol{\Theta}^i$ for goods and assets by:[18]

$$\tilde{\boldsymbol{\delta}}^i(\theta_-^i, z, \boldsymbol{q}) = \mathrm{argmax}\{\boldsymbol{u}^i(\boldsymbol{c}^i, z) : (\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{f}^i(\theta_-^i, z, \boldsymbol{q})\}.$$

3. Recursive and sequential equilibrium

This section defines the concepts of recursive and sequential equilibrium and establishes the relation between them. Typically, the recursive equilibrium is a function relating the variables in the sequential equilibrium between two consecutive periods.

Definition 3.1. Let $(\bar{\theta}^i)_{i\in\mathcal{I}}$ be an initial portfolio allocation and z an initial state of nature in a given period. The allocation $(\boldsymbol{c}, \boldsymbol{\theta}) \in \boldsymbol{C} \times \boldsymbol{\Theta}$ and the price $\boldsymbol{q} \in \boldsymbol{Q}$ constitute a sequential equilibrium for $\mathcal{E}$ if any any period they satisfy for all $z^r \in Z^r$:

1. optimality: $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \tilde{\boldsymbol{\delta}}^i(\bar{\theta}^i, z, \boldsymbol{q})$ for all $i \in \mathcal{I}$;
2. asset markets clearing: $\sum_{i\in\mathcal{I}} \boldsymbol{\theta}_r^i(z^r) = 1$;
3. good markets clearing: $\sum_{i\in\mathcal{I}} \boldsymbol{c}_r^i(z^r) = \hat{d}(z_r) + \sum_{i\in\mathcal{I}} \hat{e}^i(z_r)$.

[18] This correspondence can be empty, when $C^i \times \Theta^i$ is not compact.

Now, we introduce the concept of recursive equilibrium and show in the appendix that it implements the sequential equilibrium. The recursive demand is constructed using the value function. The latter is defined as the optimal value among all feasible plans, given the income and current portfolio endowments and, additionally, the transitions of the endogenous variables such as prices and asset distribution. To do so, we need to define an appropriate function spaces, where our equilibrium objects would belong to. For each $i \in \mathcal{I}$ let $\widehat{V}^i$ be the set of all uniformly bounded continuous functions $\hat{v}^i : \Theta^i \times S \to \mathbb{R}$ such that $\hat{v}^i(\,\cdot\,, s)$ is concave for each $s \in S$ and $\partial_1 \hat{v}^i$ is uniformly bounded. Assume that $\widehat{V}^i$ is endowed with the sup norm.[19] Define $\widetilde{C}^i$ as the set of all uniformly bounded continuous functions $\tilde{c}^i : \Theta^i \times S \to \mathbb{R}$ and $\widetilde{\Theta}^i$ analogously.

The definition below characterizes the demand and the indirect utilities given as transition functions.

Definition 3.2. Define the function $\hat{\delta}^i_v : \widehat{V} \times \widehat{Q} \times \widehat{\Theta} \to \widehat{V}^i$ by

$$\hat{\delta}^i_v(\hat{v}, \hat{q}, \hat{\theta})(\theta^i_-, s) = \max\left\{ \hat{u}^i(c^i) + \beta^i \int_{Z'} \hat{v}^i(\theta^i, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz') \right\} \tag{3}$$

over all $(c^i, \theta^i) \in C^i \times \Theta^i$ such that $(c^i, \theta^i) \in \hat{b}^i(\theta^i_-, z, \hat{q}(s))$ and the function $\tilde{\delta}^i_x : \widehat{V} \times \widehat{Q} \times \widehat{\Theta} \to \widetilde{C}^i \times \widetilde{\Theta}^i$ with $\tilde{\delta}^i_x = (\tilde{\delta}^i_c, \tilde{\delta}^i_\theta)$ by

$$\tilde{\delta}^i_x(\hat{v}, \hat{q}, \hat{\theta})(\theta^i_-, s) = \operatorname{argmax}\left\{ \hat{u}^i(c^i) + \beta^i \int_{Z'} \hat{v}^i(\theta^i, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz') \right\} \tag{4}$$

over all $(c^i, \theta^i) \in C^i \times \Theta^i$ such that $(c^i, \theta^i) \in \hat{b}^i(\theta^i_-, z, \hat{q}(s))$. Finally, define $\hat{\delta}^i_x : \widehat{V} \times \widehat{Q} \times \widehat{\Theta} \to \widehat{C}^i \times \widehat{\Theta}^i$ with $\hat{\delta}^i_x = (\hat{\delta}^i_c, \hat{\delta}^i_\theta)$ by

$$\hat{\delta}^i_x(\hat{v}, \hat{q}, \hat{\theta})(s) = \tilde{\delta}^i_x(\hat{v}, \hat{q}, \hat{\theta})(\bar{\theta}^i, s) \text{ for all } (\hat{v}, \hat{q}, \hat{\theta}, s) \in \widehat{V} \times \widehat{Q} \times \widehat{\Theta} \times S.$$

Remark 3.3. Notice that the policy function $\hat{\delta}^i_x$ satisfies

$$\hat{\delta}^i_x(\hat{v}, \hat{q}, \hat{\theta})(s) \in \hat{b}^i(\bar{\theta}^i, z, \hat{q}(s)) \text{ for all } (\hat{v}, \hat{q}, \hat{\theta}, s) \in \widehat{V} \times \widehat{Q} \times \widehat{\Theta} \times S. \tag{5}$$

The model with one asset allows us to write the optimal choice of consumption as a function of the price transition and the choices of current and previous

[19] Recall that $\widehat{V}$ is defined as the Cartesian product of $\widehat{V}^i$ for $i \in \mathcal{I}$.

assets. This makes clear the presentation of the model hereafter. So we have the following definition.

Definition 3.4. Consider $\check{C}^i$ the set of all functions $\check{c}^i : \Theta^i \times \Theta^i \times S \to C^i$. Define the consumption map $\check{c}^i : \widehat{Q} \to \check{C}^i$ as

$$\check{c}^i(\hat{q})(\theta^i_-, \theta^i, s) = \hat{p}(s)(\theta^i_- - \theta^i) + \hat{d}(z)\theta^i_- + \hat{e}^i(z) \text{ for all } (\theta^i_-, \theta^i, s) \in \Theta^i \times \Theta^i \times S. \quad (6)$$

Definition 3.5. The transition vector $(\hat{c}, \hat{\theta}, \hat{q}, \hat{v}) \in \widehat{C} \times \widehat{\Theta} \times \widehat{Q} \times \widehat{V}$ is a recursive equilibrium if it satisfies

1. $\hat{v}^i = \hat{\delta}^i_v(\hat{v}, \hat{q}, \hat{\theta})$ for all $i \in \mathcal{I}$;
2. $(\hat{c}^i, \hat{\theta}^i) = \hat{\delta}^i_x(\hat{v}, \hat{q}, \hat{\theta})$ for all $i \in \mathcal{I}$;
3. $\sum_{i \in \mathcal{I}} \hat{\theta}^i(s) = 1$ for all $s \in S$;
4. $\sum_{i \in \mathcal{I}} \hat{c}^i(s) = \hat{d}(z) + \sum_{i \in \mathcal{I}} \hat{e}^i(z)$ for all $s \in S$.

With our state space, this definition corresponds to the weakly recursive equilibrium as defined in Kubler and Schmedders (2002).

Example 3.6. Consider a model with one good and one asset and agents with instantaneous utility function defined by $\hat{u}^i(c^i) = \log(c^i)$ for all $c^i \in C^i$ and all $i \in \mathcal{I}$. Suppose that $Z = \{z\}$, that is, there is no exogenous uncertainty. We must impose that $C^i \subset \mathbb{R}_{++}$ and $\Theta^i \subset \mathbb{R}_{++}$ because $\hat{u}^i$ is defined only for $\mathbb{R}_{++}$. Write $\beta\bar{\theta} = \sum_{i \in \mathcal{I}} \beta^i \bar{\theta}^i$ and asset price as

$$\hat{p}(s) = (\beta\bar{\theta})\hat{d}(z)/(1 - \beta\bar{\theta}) \text{ for all } s \in S. \quad (7)$$

Lemma 6.9 in the appendix shows that the recursive equilibrium $(\hat{c}, \hat{\theta}, \hat{q}, \hat{v})$ is given for each $s \in S$ and each $i \in \mathcal{I}$ by

$$\hat{\theta}^i(s) = \beta^i \bar{\theta}^i (1 + \hat{d}(z)/\hat{p}(s)) \text{ and } \hat{c}^i(s) = (1 - \beta^i)\bar{\theta}^i(\hat{p}(s) + \hat{d}(z)). \quad (8)$$

The value function is given by

$$\hat{v}^i(\theta^i_-, s) = \hat{u}^i((1 - \beta^i)\theta^i_-)/(1 - \beta^i) + \hat{r}^i(s) \text{ for all } (\theta^i_-, s) \in \Theta^i \times S \quad (9)$$

where $\hat{r}^i : S \to \mathbb{R}$ is the fixed point of the operator $\hat{\rho}^i$ defined for each $s \in S$ by

$$\hat{\rho}^i(\tilde{r}^i)(s) = \hat{u}^i(\hat{p}(s) + \hat{d}(z)) + \beta^i \hat{u}^i\big(\beta^i(1 + \hat{d}(z)/\hat{p}(s))\big)/(1 - \beta^i) + \beta^i \tilde{r}^i(\hat{\theta}(s), z)$$

which satisfies the Blackwell's sufficient conditions[20] and hence it is a contraction. This ensures the existence of $\hat{r}^i$ satisfying the functional equation

$$\hat{r}^i(s) = \hat{u}^i(\hat{p}(s)+\hat{d}(z))+\beta^i\hat{u}^i(\beta^i(\hat{p}(s)+\hat{d}(z))/\hat{p}(s))/(1-\beta^i)+\beta^i\hat{r}^i(\hat{\theta}(s),z) \quad (10)$$

for all $s \in S$ and hence to state that $\hat{v}^i$ satisfies the Bellman equation $\hat{v}^i = \hat{\delta}^i_v(\hat{v},\hat{q},\hat{\theta})$ that is[21]

$$\hat{v}^i(\theta^i_-,s) = \max\left\{\hat{u}^i(c^i)+\beta^i\int_{Z'}\hat{v}^i(\theta^i,\hat{\theta}(s),z')\hat{\lambda}^i(z,dz')\right\} \quad (11)$$

over all $(c^i,\theta^i) \in C^i \times \Theta^i$ such that $(c^i,\theta^i) \in \hat{b}^i(\theta^i_-,z,\hat{q}(s))$. The policy functions are given for each $(\theta^i_-,s) \in \Theta^i \times S$ by[22]

$$\tilde{\theta}^i(\theta^i_-,s) = \beta^i\theta^i_-(\hat{p}(s)+\hat{d}(z))/\hat{p}(s) \text{ and } \tilde{c}^i(\theta^i_-,s) = (1-\beta^i)\theta^i_-(\hat{p}(s)+\hat{d}(z)).$$

Figures 1 and 2 show the recursive equilibrium for $\beta^1 = 1/4, \beta^2 = 3/4$ and $\hat{d}(z) = 2$. Observe that agent one[23] chooses a portfolio vanishing in the long run for any initial asset endowment (Blume and Easley, 2006).

When $\beta^i = \beta$ for all $i \in \mathcal{I}$ then the equilibrium price must be constant. Therefore, the recursive equilibrium is the corresponding steady state Lucas tree equilibrium (Lucas Jr, 1978) with homogeneous agents.[24] Explicitly,

$$\hat{p}(s) = \beta\hat{d}(z)/(1-\beta),\ \hat{\theta}^i(s) = \bar{\theta}^i \text{ and } \hat{c}^i(s) = \hat{d}(z)\bar{\theta}^i \text{ for all } s \in S.$$

The next definition provides more detail of how a recursive equilibrium implements a sequential equilibrium. Observe that each agent i has initial endowment $\theta^i_- = \bar{\theta}^i$ and optimal choices on $\overline{\Theta}$ in the equilibrium, that is, each agent chooses the mean portfolio relative to his own type.

[20]See Stokey, Lucas Jr, and Prescott (1989) for more detail.

[21]Note that $\hat{\lambda}^i(z) = \mathrm{dirac}(z)$.

[22]The value function $\hat{v}^i$ is strictly concave on θ^i_-. See Stokey, Lucas Jr, and Prescott (1989) Chapter 4 for more detail. Recall that $\hat{\theta} = (\hat{\theta}^i)_{i\in\mathcal{I}}$.

[23]Who has lower intertemporal discount rate.

[24]Despite the heterogeneity in the asset endowments $\bar{\theta}$.

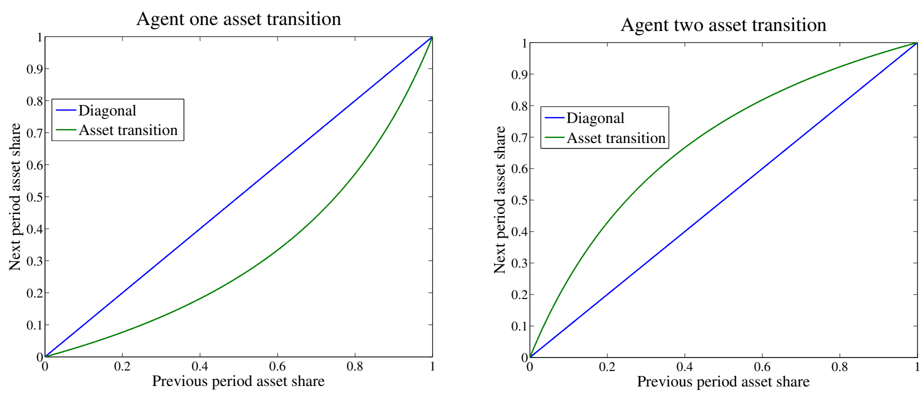

Figure 1: Graphics of asset transition $\hat{\theta}^1(\bar{\theta}^1, 1-\bar{\theta}^1, z)$ on the left and $\hat{\theta}^2(1-\bar{\theta}^2, \bar{\theta}^2, z)$ on the right.

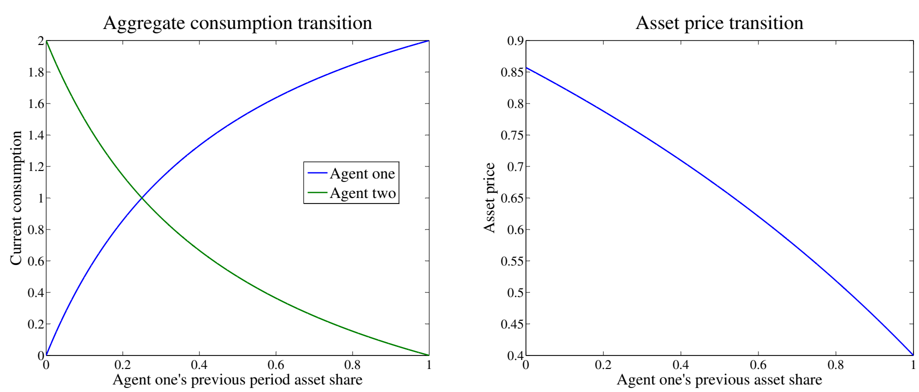

Figure 2: Graphics of consumption transitions $\hat{c}^1(\bar{\theta}^1, 1-\bar{\theta}^1, z)$ and $\hat{c}^2(\bar{\theta}^1, 1-\bar{\theta}^1, z)$ on the left and asset price transition $\hat{p}(\bar{\theta}^1, 1-\bar{\theta}^1, z)$ for $\bar{\theta}^1 \in [0, 1]$.

Definition 3.7. The transition vector $(\hat{c}, \hat{\theta}, \hat{q}) \in \widehat{C} \times \widehat{\Theta} \times \widehat{Q}$ implements the process $(\boldsymbol{c}, \boldsymbol{\theta}, \boldsymbol{q}) \in \boldsymbol{C} \times \boldsymbol{\Theta} \times \boldsymbol{Q}$ with initial condition $(\bar{\theta}, z) \in \overline{\Theta} \times Z$ if for all $z^r \in Z^r$

$$\boldsymbol{q}_0 = \hat{q}(\bar{\theta}, z), \quad \boldsymbol{\theta}_0^i = \hat{\theta}^i(\bar{\theta}, z), \quad \boldsymbol{c}_0^i = \hat{c}^i(\bar{\theta}, z)$$

and recursively for $r \in \mathbb{N}$

$$\boldsymbol{c}_r^i(z^r) = \hat{c}^i(\boldsymbol{\theta}_{r-1}(z^{r-1}), z_r) \qquad \boldsymbol{\theta}_r^i(z^r) = \hat{\theta}^i(\boldsymbol{\theta}_{r-1}(z^{r-1}), z_r) \tag{12}$$

for all $i \in \mathcal{I}$ and

$$\boldsymbol{q}_r(z^r) = \hat{q}(\boldsymbol{\theta}_{r-1}(z^{r-1}), z_r). \tag{13}$$

The next result assures that the recursive equilibrium can actually be used to construct the sequential equilibrium. We prove it in the appendix.

Theorem 3.8. *If $(\hat{c}, \hat{\theta}, \hat{q}, \hat{v}) \in \widehat{C} \times \widehat{\Theta} \times \widehat{Q} \times \widehat{V}$ is a recursive equilibrium then its implemented process $(\hat{\boldsymbol{c}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{q}}) \in \boldsymbol{C} \times \boldsymbol{\Theta} \times \boldsymbol{Q}$ with initial condition $(\bar{\theta}, z) \in \overline{\Theta} \times Z$ is a sequential equilibrium of the economy with initial state $(\bar{\theta}, z)$.*

Remark 3.9. The proof of Theorem 3.8 embodies arguments that also show the intertemporal consistency of a sequential equilibrium implemented by a recursive equilibrium. Indeed, consider a process $(\hat{\boldsymbol{c}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{q}}) \in \boldsymbol{C} \times \boldsymbol{\Theta} \times \boldsymbol{Q}$ implemented by a recursive equilibrium $(\hat{c}, \hat{\theta}, \hat{q})$ given $s_1 \in S$ at period one. Fix some period t and a realization $z^t \in Z^t$. Define the continuation $\tilde{\boldsymbol{c}}_t \in \boldsymbol{C}$ as $\tilde{\boldsymbol{c}}_{t0} = \hat{\boldsymbol{c}}_t(z^t) \in C$ and for each $r \in \mathbb{N}$ the plan $\tilde{\boldsymbol{c}}_{tr} : Z^r \to C$ as $\tilde{\boldsymbol{c}}_{tr}(z^r) = \hat{\boldsymbol{c}}_{t+r}(z^{t+r})$. Define $\tilde{\boldsymbol{q}}_t \in \boldsymbol{Q}$ and $\tilde{\boldsymbol{\theta}}_t \in \boldsymbol{\Theta}$ analogously. In the proof of Theorem 3.8 we can find that $\hat{v}^i(\hat{\boldsymbol{\theta}}_{t-1}^i(z^{t-1}), \hat{\boldsymbol{\theta}}_{t-1}(z^{t-1}), z_t) = \tilde{\boldsymbol{v}}^i(\hat{\boldsymbol{\theta}}_{t-1}^i(z^{t-1}), z_t, \tilde{\boldsymbol{q}}_t)$ and hence $(\tilde{\boldsymbol{c}}_t^i, \tilde{\boldsymbol{\theta}}_t^i) \in \tilde{\boldsymbol{\delta}}^i(\hat{\boldsymbol{\theta}}_{t-1}^i(z^{t-1}), z_t, \tilde{\boldsymbol{q}}_t)$ for all $i \in \mathcal{I}$ by equation (28). Indeed, $(\tilde{\boldsymbol{c}}_t^i, \tilde{\boldsymbol{\theta}}_t^i)$ is given according to equation[25] (12) and $(\hat{c}, \hat{\theta})$ by Item 2 of Definition 3.5. As a result, under assumptions guaranteeing the existence of equilibrium as defined in 3.5, the state space S is sufficient for characterizing the recursive equilibrium.

[25] That is, replacing r by $t + r$.

4. Existence Result

In this section, we demonstrate the existence of a recursive equilibrium with state space $S = \overline{\Theta} \times Z$.

Notation 4.1. Define $\hat{v}^i : X^i \times S \times \widehat{V} \times \widehat{\Theta} \to \mathbb{R}$ as

$$\hat{v}^i(x^i, s, \hat{v}, \hat{\theta}) = \hat{u}^i(c^i) + \beta^i \int_Z \hat{v}^i(\theta^i, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz') \tag{14}$$

for all $(i, x^i, s, \hat{v}, \hat{\theta}) \in \mathcal{I} \times X^i \times S \times \widehat{V} \times \widehat{\Theta}$.

For each $i \in \mathcal{I}$, write $\hat{w}^i : S \to C^i \times \Theta^i$ by $\hat{w}^i(s) = (\hat{e}^i(z) + \bar{\theta}^i \hat{d}(z), \bar{\theta}^i)$ for all $s \in S$.

Define the excess demand function $\hat{\xi} : \widehat{X} \times S \to \mathbb{R}^2$ with $\hat{\xi} = (\hat{\xi}_c, \hat{\xi}_a)$ as $\hat{\xi}(\hat{x}, s) = \sum_{i \in \mathcal{I}} \hat{x}^i(s) - \hat{w}^i(s)$. Write $\hat{\delta}_v = \prod \hat{\delta}^i_v$ and $\hat{\delta}_x = \prod \hat{\delta}^i_x$.

We define below the Lipschitz property. This property characterizes a boundary for the maximum slope of a function. For differentiable functions it means that the function must have bounded derivative.

Definition 4.2. Consider a function $f : Y \subset \mathbb{R}^K \to \mathbb{R}^L$.

1. We say that f is M-Lipschitz for $M \in \mathbb{R}_{++}$ if $||f(y) - f(y')|| \leq M||y - y'||$ for all $y, y' \in Y$.
2. We say that $f = (f_1, \cdots, f_L)$ is M-Lipschitz with $M \in \mathbb{R}^L_{++}$ if $f_l : Y \subset \mathbb{R}^K \to \mathbb{R}$ is M_l-Lipschitz for $l = 1, \cdots, L$.
3. We say that f is M-Lipschitz for $L = 1$ and $M \in \mathbb{R}^K_{++}$ if the k-th section $f(y_1, \cdots, \cdot, \cdots, y_K) : Y_k \subset \mathbb{R} \to \mathbb{R}$ is M_k-Lipschitz for $k = 1, \cdots, K$ and all fixed $y_\kappa \in Y_\kappa$ for $\kappa \neq k$.

Remark 4.3. Notice that a function $f : Y \subset \mathbb{R}^K \to \mathbb{R}^L \in \mathrm{Lp}(M)$ for $M \in \mathbb{R}^L_{++}$ then $f \in \mathrm{Lp}(||M||)$.

Remark 4.4. We say that $\partial \hat{u}^i \in \mathrm{Lp}(M_{\partial \hat{u}})$ when $|\partial \hat{u}^i(\dot{c}^i)(1) - \partial \hat{u}^i(\ddot{c}^i)(1)| \leq M_{\partial \hat{u}}|\dot{c}^i - \ddot{c}^i|$ for all $(\dot{c}, \ddot{c}) \in C^i \times C^i$.

We now proceed to define a set of functions that would be later useful in our construction of the fixed point operator and equilibrium bounds of relevant variables.

Notation 4.5. Consider $\widehat{F}$ the space of all continuous $\hat{f} : Y \subset \mathbb{R}^K \to \mathbb{R}^L$. Write $\mathrm{Lp}(\widehat{F}, M, n, N)$ as the set of all M-Lipschitz functions $\hat{f} \in \widehat{F}$ such that $\hat{f}(Y) \subset \prod_{l \leq L}[n_l, N_l] \subset \mathbb{R}^L$. In absence of ambiguity, we write shortly the space $\mathrm{Lp}(\widehat{F}, M, n, N)$ as $\mathrm{Lp}(M)$.

We now define a Lipschitz property of a transition probability $\hat{\lambda}$.

Definition 4.6. Consider $\widehat{F}$ as the set of all bounded continuous $\hat{f} : Z \to \mathbb{R}$. We say that a map $\hat{\lambda} : Z \to \mathrm{Prob}(Z)$ satisfies $\hat{\lambda} \in \mathrm{Lp}(M_{\hat{\lambda}})$ if and only if for each $(\dot{z}, \ddot{z}) \in Z \times Z$

$$\sup \left\{ \left| \int_Z f(z')\hat{\lambda}(\dot{z}, dz') - \int_Z f(z')\hat{\lambda}(\ddot{z}, dz') \right| : \hat{f} \in \widehat{F} \text{ and } |f| \leq 1 \right\} \leq M_{\hat{\lambda}} ||\dot{z} - \ddot{z}||.$$

The definition below used in Theorem 4.16 establishes boundaries of allocations. Despite optimal choices are bounded, under this assumption, we show that in equilibrium all allocations are interior. It is well known that those allocations also constitute an equilibrium even if the choice sets are unbounded.

Definition 4.7. Suppose that $Q \subset \{1\} \times [n_p, N_p]$, $\hat{d}(Z) \subset [n_d, N_d]$ and $\hat{e}^i(Z) \subset [n_e, N_e]$ for all $i \in \mathcal{I}$. Define $\Theta^i = [0, N_\theta]$ where $N_\theta = 1 + \gamma$ and write $C^i = [n_c, N_c]$ where $N_c = N_d + N_e + \gamma$ and $n_c = \max\{n_e - N_p - \gamma, 0\}$ for all $i \in \mathcal{I}$ and a given $\gamma > 0$ small enough. Recall that $X = C \times \Theta$ and $\widehat{X} = \widehat{C} \times \widehat{\Theta}$ with a typical element $\hat{x} \in \widehat{X}$.

Remark 4.8. Notice that[26] $\check{c}^i(\hat{q}) \in \mathrm{Lp}(M_{\check{c}\theta_-}, M_{\check{c}\theta}, M_{\check{c}s})$ where $M_{\check{c}\theta_-} = N_p + N_d$, $M_{\check{c}\theta} = N_p$ and $M_{\check{c}s} = M_{\hat{p}} N_\theta + M_{\hat{d}} N_\theta + M_{\hat{e}}$.

Definition 4.9. Given $i \in \mathcal{I}$, write $\widehat{R}^i$ for space of all continuous functions $\hat{r}^i : \Theta^i \times \Theta^i \times S \times S \to \mathbb{R}_{++}$. Define the linear map $\hat{\varphi}^i : \widehat{V} \times \widehat{Q} \to \widehat{R}^i$ for each $\hat{v} \in \widehat{V}$ and each $\hat{q} \in \widehat{Q}$ by

$$\hat{\varphi}^i(\hat{v}, \hat{q})(\theta^i_-, \theta^i, s, s') = \frac{\partial_1 \hat{v}^i(\theta^i, s')}{\partial \hat{u}^i(\check{c}^i(\hat{q})(\theta^i_-, \theta^i, s))} \text{ for all } (\theta^i_-, \theta^i, s, s') \in \Theta^i \times \Theta^i \times S \times S.$$

Moreover, define $\tilde{p}^i : \widehat{V} \times \widehat{Q} \times \widehat{\Theta} \to \widehat{P}$ for each given $(\hat{v}, \hat{q}, \hat{\theta}) \in \widehat{V} \times \widehat{Q} \times \widehat{\Theta}$ by

$$\tilde{p}^i(\hat{v}, \hat{q}, \hat{\theta})(s) = \beta^i \int_Z \hat{\varphi}^i(\hat{v}, \hat{q})(\overline{\theta}^i, \hat{\theta}^i(s), s, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz') \text{ for all } s \in S. \quad (15)$$

[26]Recall that $\check{c}^i$ is given by (6).

This definition is critical for our analysis. It follows existence of a single asset and allows us to define uniquely the next period prices via the envelope theorem.

Definition 4.10. Consider $M = (M_{\hat{q}}, M_{\hat{\varphi}})$ where $M_{\hat{\varphi}} = (M_{\hat{\varphi}\theta_-}, M_{\hat{\varphi}\theta}, M_{\hat{\varphi}s}, M_{\hat{\varphi}s'})$. Define $\widehat{V}^i_M$ as the convex set of all $\hat{v}^i \in \widehat{V}^i$ such that

1. $||\hat{v}^i|| \geq \hat{u}^i(n_e)/(1-\beta^i)$ and $||\hat{v}^i|| \leq \hat{u}^i(N_c)/(1-\beta^i)$;
2. $\hat{q} \in \mathrm{Lp}(M_{\hat{q}})$ implies $\hat{\varphi}^i(\hat{v}, \hat{q}) \in \mathrm{Lp}(M_{\hat{\varphi}})$.

Assumption 4.11 will provide conditions on the primitives $\{\hat{u}^i, \hat{\lambda}^i, \hat{d}, \hat{e}^i, \beta^i\}_{i \in \mathcal{I}}$ of Lucas' model and on the boundary of the price set Q so that the demand is Lipschitz according to Proposition 4.12. The Lipschitz condition on the aggregate demand is basically a sufficient condition to assure the existence of a recursive equilibrium with a minimal state space. Moreover, in case of one asset, the strong concavity is basically a sufficient condition to assure the Lipschitz property of the demand and hence, the existence of a Lipschitz recursive equilibrium. The remaining difficulty is to construct equilibrium bounds of domains. Specifically, we need to assure our fixed point operator selfmaps spaces of Lipschitz continuous functions with the same Lipschitz constants.

Assumption 4.11. Assume that there exist vectors[27]

$$\begin{aligned} \sigma_N &= (n_c, N_c, n_d, N_d, n_p, N_p) \\ \sigma_M &= (M_{\hat{\lambda}}, M_{\partial\hat{u}}, M_{\hat{d}}, M_{\check{c}}, M_{\tilde{\theta}}, M_{\hat{\theta}}, M_{\hat{p}}, M_{\hat{\varphi}}) \end{aligned} \tag{16}$$

such that for each $i \in \mathcal{I}$

1. $M_{\hat{\varphi}\theta_-} \geq (N_p + N_d) M_{\partial u} N_c M_{\check{c}\theta_-}/n_c^2$;
2. $M_{\hat{\varphi}\theta} \geq (N_p + N_d) M_{\partial u}(N_c M_{\check{c}\theta}/n_c^2 + (M_{\check{c}\theta_-} + M_{\check{c}\theta} M_{\tilde{\theta}\theta_-})/n_c)$;
3. $M_{\hat{\varphi}s} \geq (N_p + N_d) M_{\partial u} N_c M_{\check{c}s}/n_c^2$;
4. $M_{\hat{\varphi}s'} \geq (M_{\hat{p}} + M_{\hat{d}})\partial u^i(n_c/N_c) + (N_p + N_d) M_{\partial u}(M_{\check{c}\theta} M_{\tilde{\theta}s} + M_{\check{c}s})/n_c$;
5. $\alpha M_{\tilde{\theta}\theta_-} \geq M_{\partial\hat{u}}(1 + N_d/n_p)$;

[27]Recall that the other constants are given in Definition 4.7 and Remark 4.8.

6. $\alpha n_p^2 M_{\tilde{\theta}s} \geq N_{\partial\hat{u}}(M_{\hat{p}}+\beta^i(M_{\hat{\varphi}s}+M_{\hat{\varphi}s'}M_{\hat{\theta}}+N_{\varphi}M_{\hat{\lambda}}))+M_{\partial\hat{u}}M_{\check{c}s}(N_p+\beta^i N_{\varphi})$;
7. $M_{\hat{\theta}} \geq M_{\tilde{\theta}\theta\_} + M_{\tilde{\theta}s}$

where $N_{\varphi} = (N_d + N_p)\partial u^i(n_c/N_c)$.

The following proposition assures that the demand $\hat{\delta}_\theta$ is Lipschitz using Montrucchio (1987). Moreover, it assures that Lipschitz constants are not expanding, when mapping $\hat{v}$ and $\hat{\theta}$. We postpone its prove to the appendix. Recall critical conditions in Items 4, 6 and 7. Notice that it is not necessary to ensure Lipschitz conditions on the objective function in Definition 4.10 since Montrucchio (1987) imposes Lipschitz conditions only on the derivative of the objective function.

Proposition 4.12. *Consider $\{\sigma_N, \sigma_M\}$ satisfying Assumption 4.11. Then*[28]

$$\hat{\delta}_v(\hat{v},\hat{q},\hat{\theta}) \in \widehat{V}_M \text{ and } \hat{\delta}_\theta(\hat{v},\hat{q},\hat{\theta}) \in \mathrm{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta)$$

for all $(\hat{v},\hat{q},\hat{\theta}) \in \widehat{V}_M \times \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q) \times \mathrm{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta)$.

The following assumption is used directly on the next proposition.

Assumption 4.13. Assume that there exists vectors (σ_N, σ_M) as in (16) such that for each $i \in \mathcal{I}$

1. $n_p < \beta^i n_d \partial u^i(N_c/n_c)/(1 - \beta^i \partial u^i(N_c/n_c))$;
2. $N_p > \beta^i N_d \partial u^i(n_c/N_c)/(1 - \beta^i \partial u^i(n_c/N_c))$;
3. $M_{\hat{p}} > \beta^i(M_{\hat{\varphi}\theta\_} + M_{\hat{\varphi}\theta}M_{\hat{\theta}} + M_{\hat{\varphi}s} + M_{\hat{\varphi}s'}M_{\hat{\theta}} + N_\varphi M_{\hat{\lambda}})$;
4. $\beta^i \hat{u}^i(N_c) < \hat{u}^i(n_e)$.

Conditions 1 and 2 assure a suitable low and high boundary on prices ensuring excess of demand or supply of aggregate asset choices respectively. Condition 3 ensures that $\tilde{p}$ belongs to the interior of $\widehat{Q}$. This implies that the Walrasian auctioneer has positive profits for all prices outside the equilibrium set. Condition 4 implies that optimal consumption choices are interior. (Duffie,

[28]Recall that $M_{\hat{q}} = (0, M_{\hat{p}})$ and $N_q = (1, N_p)$.

Geanakoplos, Mas-Colell, and McLennan, 1994). It is summarized in the next proposition (proved in the appendix).

Proposition 4.14. *Suppose Assumption 4.13. Then there exists* $\kappa \in (0,1)$ *such that for each* $\hat{v} \in \widehat{V}_M$, $\hat{q} \in \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q)$ *and* $\hat{\theta} \in \mathrm{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta)$

1. *if* $\hat{v} = \hat{\delta}_v(\hat{v}, \hat{q}, \hat{\theta})$ *and* $(\hat{c}, \hat{\theta}) = \hat{\delta}_x(\hat{v}, \hat{q}, \hat{\theta})$ *then*

$$\tilde{p}^i(\hat{v}, \hat{q}, \hat{\theta}) \in \mathrm{Lp}(\widehat{P}, (1-\kappa)M_{\hat{p}}, (1+\kappa)n_p, (1-\kappa)N_p) \text{ for all } i \in \mathcal{I};$$

2. $\hat{c} = \hat{\delta}_c(\hat{v}, \hat{q}, \hat{\theta})$ *implies* $\hat{c}(s) > 0$ *for all* $s \in S$.

Lemma 4.15 below (proved in the appendix) shows that it is not necessary to ensure that the value function is Lipschitz for the existence theorem. Therefore, the existence theorem is based on a construction of a certain operator defined only on portfolio and price transitions. Consider $\widehat{\mathcal{V}}$ the set of all continuous maps $\hat{\nu} : \widehat{Q} \times \widehat{\Theta} \to \widehat{V}$. Since $\widehat{V}_M$ is not a closed subset of $\widehat{V}$ under the sup norm[29] we can not apply the Blackwell's sufficient conditions in order to obtain a fixed point of a contraction.

Lemma 4.15. *Suppose Assumption 4.11. Then there exists a value function* $\hat{\nu} \in \widehat{\mathcal{V}}$ *with* $\hat{\nu}(\hat{q}, \hat{\theta}) \in \widehat{V}_M$, $\hat{\nu}(\hat{q}, \hat{\theta}) = \hat{\delta}_v(\hat{\nu}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta})$ *and* $\hat{\delta}_\theta(\hat{\nu}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta}) \in \mathrm{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta)$ *for all* $(\hat{q}, \hat{\theta}) \in \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q) \times \mathrm{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta)$.

The next theorem is a central result of our paper. For this reason we present its proof but recall that many key ingredients have been already established in the lemmas. Under Assumptions 4.11 and 4.13 it assures existence of a recursive equilibrium that is Lipschitz continuous. Observe that our results work for both stochastic and deterministic economies in contrast to Brumm, Kryczka, and Kubler (2017). In order to prove this result, we consider a class of transition prices and policy functions that are Lipschitz continuous. This allows us to obtain a sup norm compact set of candidate equilibrium functions. Second, we

[29] It is actually a Banach space under a Sobolev norm. However, we do not need this topology for the existence theorem.

define the fixed point operator using the optimization problem (defined on the candidate space of Lipschitz continuous functions) of the Walrasian auctioneer. Third we apply the fixed point of Kakutani-Fan-Gliksberg. Finally we show that the fixed point of our operator satisfies the market clearing conditions.

Theorem 4.16. *Suppose that Assumptions 4.11 and 4.13 are satisfied. Then there exists a continuous recursive equilibrium* $(\hat{c}, \hat{\theta}, \hat{q}, \hat{v})$ *with* $(\hat{c}, \hat{\theta}, \hat{q})$ *Lipschitz.*[30]

Proof of Theorem 4.16. Write

$$\widehat{Y} = \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q) \times \mathrm{Lp}(\widehat{X}, M_{\hat{x}}, n_x, N_x)$$

where $\widehat{X} = \widehat{C} \times \widehat{\Theta}$, $M_{\hat{x}} = (M_{\hat{c}}, M_{\hat{\theta}})$, $n_x = (n_c, n_\theta)$ and $N_x = (N_c, N_\theta)$. The Ascoli's Theorem (Royden, 1963) assures that $\widehat{Y}$ is compact by the compactness of S. Consider $\tilde{\lambda} \in \mathrm{Prob}(S)$ any probability measure with full support[31] and write $N_\xi = ||\hat{\xi}||$. Define the function $\hat{\delta}_q : \widehat{X} \to \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q)$ as

$$\hat{\delta}_q(\hat{x}) = \mathrm{argmax}\left\{ \int_S \hat{q}(s)\hat{\xi}(\hat{x}, s)\tilde{\lambda}(ds) : \hat{q} \in \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q) \right\}.$$

Clearly, $\hat{\delta}_q$ is convex valued and has closed graph by the Dominated Convergence Theorem and the Berge Maximum Theorem (Aliprantis and Border, 1999).

Let $\hat{\delta} : \widehat{Y} \to \widehat{Y}$ be the continuous convex valued correspondence defined by:

$$\hat{\delta}(\hat{q}, \hat{x}) = \hat{\delta}_q(\hat{x}) \times \hat{\delta}_x(\hat{\nu}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta}) \text{ for all } (\hat{q}, \hat{x}) \in \widehat{Y}.$$

where $\hat{\nu}$ is given by Lemma 4.15. The operator $\hat{\delta}$ is well defined under Assumptions 4.11 and 4.13 by applying Lemma 4.15. Moreover, $\widehat{Y}$ is a nonempty compact convex space endowed with a locally convex Hausdorff topology and $\hat{\delta}$ has closed graph by the Berge Maximum Theorem (Aliprantis and Border, 1999).

[30] We could apply a fixed point argument using Assumption 4.11 to obtain a Lipschitz value function. But for this, it is necessary to use a Sobolev norm on the space $\widehat{V}$ and boundary conditions on the value functions and the set of constants guaranteeing existence of Lipschitz RCE would be more restrictive. We refer the reader to a working paper version of this paper for details per this approach.

[31] See Aliprantis and Border (1999) for the definition of the support of a measure.

Therefore, $\hat{\delta}$ has a fixed point, say, $(\hat{c}, \hat{\theta}, \hat{q})$ by the Kakutani-Fan-Gliksberg Fixed Point Theorem (Aliprantis and Border, 1999, Theorem 17.55). Write $\hat{\upsilon} = \hat{\nu}(\hat{q}, \hat{\theta})$, $(\hat{c}, \hat{\theta}) = \hat{x} = \hat{\delta}_x(\hat{\upsilon}, \hat{q}, \hat{\theta})$ and recall that $\hat{c}^i : S \to C^i$ is the i-th coordinate of $\hat{c}$ and $\hat{\theta}^i : S \to \Theta^i$ is the i-th coordinate of $\hat{\theta}$.

To show the market clearing conditions, notice that since[32] $\hat{x} = \hat{\delta}_x(\hat{\upsilon}, \hat{q}, \hat{\theta})$ then $\hat{q}(s)\hat{x}^i(s) \leq \hat{q}(s)\hat{w}^i(s)$ and hence $\hat{q}(s)(\hat{x}^i(s) - \hat{w}^i(s)) \leq 0$ for all $s \in S$ and all $i \in \mathcal{I}$. Adding over $i \in \mathcal{I}$ these budget restrictions then

$$\hat{q}(s)\hat{\xi}(\hat{x}, s) \leq 0 \quad \text{for all } s \in S. \tag{17}$$

Since $0 \in X^i$, then applying the Concave Alternative Theorem (Aliprantis and Border, 1999, Theorem 5.70) there exist $\hat{\varsigma}^i : S \to \mathbb{R}_+$, $\hat{\tau}^i : S \to \mathbb{R}^2_+$ and $\check{\tau}^i : S \to \mathbb{R}^2_+$ with $\hat{\tau}^i = (\hat{\tau}^i_c, \hat{\tau}^i_a)$ and $\check{\tau}^i = (\check{\tau}^i_c, \check{\tau}^i_a)$ such that for each $(i, s) \in \mathcal{I} \times S$ the optimal choice $\hat{x}^i(s)$ maximizes the Lagrangian[33]

$$\hat{L}(x^i, s) = \hat{\upsilon}^i(x^i, s, \hat{\upsilon}, \hat{\theta}) + \hat{\varsigma}^i(s)\hat{q}(s)(\hat{w}^i(s) - x^i) + \hat{\tau}^i(s)(N_x - x^i) + \check{\tau}^i(s)(x^i - n_x).$$

Moreover, $\hat{\tau}^i(s)(N_x - \hat{x}^i(s)) = 0$ and $\check{\tau}^i(s)(\hat{x}^i(s) - n_x) = 0$. Thus,

$$0 = \partial_1 \hat{L}(\hat{x}^i(s), s)(\mathring{x}^i) = \partial_1 \hat{\upsilon}^i(\hat{x}^i(s), s, \hat{\upsilon}, \hat{\theta})(\mathring{x}^i) - \hat{\varsigma}^i(s)\hat{q}(s)\mathring{x}^i - \hat{\tau}^i(s)\mathring{x}^i + \check{\tau}^i(s)\mathring{x}^i$$

and hence

$$\partial_1 \hat{\upsilon}^i(\hat{x}^i(s), s, \hat{\upsilon}, \hat{\theta})(\mathring{x}^i) = \hat{\varsigma}^i(s)\hat{q}(s)\mathring{x}^i + \hat{\tau}^i(s)\mathring{x}^i - \check{\tau}^i(s)\mathring{x}^i \tag{18}$$

for all $\mathring{x}^i \in \mathbb{R}^2_+$. Choosing $\mathring{x}^i = (1, 0)$ and using that $\hat{c}^i(s) > n_c$ then by (14)

$$\hat{\varsigma}^i(s) = \partial_1 \hat{\upsilon}^i(\hat{x}^i(s), s, \hat{\upsilon}, \hat{\theta})(1, 0) - \hat{\tau}^i_c(s) \leq \partial \hat{u}^i(\hat{c}^i(s)) \text{ for all } i \in \mathcal{I}. \tag{19}$$

Furthermore, choosing $\mathring{x}^i = (0, 1)$ then Equations (14), (15), (18) and Definition 4.9 imply that[34]

$$\begin{aligned} \tilde{p}^i(\hat{\upsilon}, \hat{q}, \hat{\theta})(s) &\leq (\hat{\varsigma}^i(s))^{-1} \partial_1 \hat{\upsilon}^i(\hat{x}^i(s), s, \hat{\upsilon}, \hat{\theta})(0, 1) \\ &= \hat{p}(s) + \hat{\tau}^i_a(s)/\hat{\varsigma}^i(s) - \check{\tau}^i_a(s)/\hat{\varsigma}^i(s) \end{aligned} \tag{20}$$

[32] Recall Notation 4.1 for the definition of $\hat{w}$ and $\hat{\upsilon}$.

[33] Recall Definition 4.7.

[34] If $\hat{\varsigma}^i(s) = 0$ and $\hat{\xi}_a(\hat{x}, s) \leq 0$ then we have a contradiction with the fact that $\partial_1 \hat{\upsilon}^i > 0$.

for all $i \in \mathcal{I}$. Define $\tilde{p} : S \to \mathbb{R}_+$ by $\tilde{p}(s) = \max\{\tilde{p}^i(\hat{v}, \hat{q}, \hat{\theta})(s) : i \in \mathcal{I}\}$ for all $s \in S$. Then $\tilde{p} \in \mathrm{Lp}((1-\kappa)M_{\hat{p}})$ by Lemma 6.3 since $\mathcal{I}$ is finite. Given $s \in S$, consider ι such that $\tilde{p}^\iota(\hat{v}, \hat{q}, \hat{\theta})(s) = \tilde{p}(s)$. Suppose that $\hat{\xi}_a(\hat{x}, s) \leq 0$. Then $\hat{\theta}^i(s) < N_\theta$ and hence $\hat{\tau}_a^i(s) = 0$ for all $i \in \mathcal{I}$. Therefore, choosing $i = \iota$ in (20) we get,

$$\tilde{p}(s)\hat{\xi}_a(\hat{x}, s) \geq \hat{p}(s)\hat{\xi}_a(\hat{x}, s) - \check{\tau}_a^\iota(s)\hat{\xi}_a(\hat{x}, s)/\varsigma^\iota(s) \geq \hat{p}(s)\hat{\xi}_a(\hat{x}, s).$$

Suppose that $\hat{\xi}_a(\hat{x}, s) > 0$. Then there exists $i \in \mathcal{I}$ such that $\hat{\theta}^i(s) > \bar{\theta}^i \geq 0$ and $\hat{c}^i(s) < N_c$ by (6). Therefore, $\check{\tau}_a^i(s) = 0$, $\hat{\tau}_c^i(s) = 0$ and

$$\hat{\varsigma}^i(s) = \partial_1 \hat{v}^i(\hat{x}^i(s), s, \hat{v}, \hat{\theta})(1, 0) + \check{\tau}_c^i(s) \geq \partial \hat{u}^i(\hat{c}^i(s)) > 0.$$

Thus, $\tilde{p}^i(\hat{v}, \hat{q}, \hat{\theta})(s) \geq (\hat{\varsigma}^i(s))^{-1}\partial_1 \hat{v}^i(\hat{x}^i(s), s, \hat{v}, \hat{\theta})(0, 1)$ and hence by (18)

$$\tilde{p}(s)\hat{\xi}_a(\hat{x}, s) \geq \tilde{p}^i(\hat{v}, \hat{q}, \hat{\theta})(s)\hat{\xi}_a(\hat{x}, s) \geq (\hat{p}(s) + \hat{\tau}_a^i(s)/\varsigma^i(s))\hat{\xi}_a(\hat{x}, s) \geq \hat{p}(s)\hat{\xi}_a(\hat{x}, s).$$

Since $s \in S$ was given arbitrarily then for $\tilde{q} = (1, \tilde{p})$

$$\tilde{q}(s)\hat{\xi}(\hat{x}, s) \geq \hat{q}(s)\hat{\xi}(\hat{x}, s) \text{ for all } s \in S. \tag{21}$$

Notice that by definition, $\hat{\xi}(\hat{x}, \cdot) \in \mathrm{Lp}(M_{\hat{\xi}})$ for some $M_{\hat{\xi}} \in \mathbb{R}_+$. Consider

$$\zeta = \min\{\kappa n_p / N_\xi, \kappa M_{\hat{p}} / M_{\hat{\xi}}\} \tag{22}$$

Define $\check{p} : S \to \mathbb{R}_{++}$ by $\check{p}(s) = \tilde{p}(s) + \zeta\hat{\xi}_a(\hat{x}, s)$ for all $s \in S$ and $\check{q} = (1, \check{p})$. Then $\check{q} \in \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q)$ by Item 1 of Proposition 4.14 since Items 1 and 2 given in Assumption 4.13 assure the condition $n_p \leq \check{p} \leq N_p$ and Item 3 given in Assumption 4.13 assures the condition $\check{p} \in \mathrm{Lp}(M_p)$. Suppose that there exists $s \in S$ with $\hat{\xi}_a(\hat{x}, s) \neq 0$. Since $\hat{\xi}$ is continuous and $\tilde{\lambda}$ has full support, then by (21)

$$\begin{aligned}
\int_S \check{q}(s)\hat{\xi}(\hat{x}, s)\tilde{\lambda}(ds) &= \int_S \tilde{q}(s)\hat{\xi}(\hat{x}, s)\tilde{\lambda}(ds) + \int_S \zeta\hat{\xi}_a^2(\hat{x}, s)\tilde{\lambda}(ds) \\
&\geq \int_S \hat{q}(s)\hat{\xi}(\hat{x}, s)\tilde{\lambda}(ds) + \int_S \zeta\hat{\xi}_a^2(\hat{x}, s)\tilde{\lambda}(ds) \\
&> \int_S \hat{q}(s)\hat{\xi}(\hat{x}, s)\tilde{\lambda}(ds).
\end{aligned}$$

This is a contradiction since $\check{q} \in \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q)$ and $\hat{q} \in \hat{\delta}_q(\hat{x})$. Thus $\hat{\xi}_a(\hat{x}, s) = 0$ for all $s \in S$. This implies that $\hat{x}^i(s) \in \operatorname{Int} X^i$ for all $s \in S$. Therefore, all inequalities given in (17) must bind since the objective function is strictly increasing on the consumption and asset variables. This implies that $\hat{\xi}(\hat{x}, \cdot) = 0$ since $\hat{q} > 0$.

$\square$

We require demanding conditions on the recursive equilibrium (i.e. it is given by Lipschitz continuous functions on a minimal state space) hence the conditions on the primitives are demanding. In what follows, however, we show a specific example, where all assumptions are satisfied including a parametrization on the state space. Specifically, example 4.17 below elucidates how to use Assumptions 4.11 and 4.13 to ensure the existence of a recursive equilibrium with a minimal state space. It typically has the same premisses of Example 3.6 but with positive endowments, implying that asset demand can not be constant in the portfolio transition.

Example 4.17. Fix $\epsilon > 0$. Write $\overline{\Theta}_\epsilon = \epsilon\overline{\Theta}$ and $S_\epsilon = \overline{\Theta}_\epsilon \times Z$. Define $\hat{\theta}_\epsilon : \overline{\Theta} \to \overline{\Theta}_\epsilon$ by $\hat{\theta}_\epsilon(\bar{\theta}) = \epsilon\bar{\theta}$ for all $\bar{\theta} \in \overline{\Theta}$, $\hat{s}_\epsilon : S \to S_\epsilon$ by $\hat{s}_\epsilon(\bar{\theta}, z) = (\epsilon\bar{\theta}, z)$ for all $s \in S$ and $\hat{s} : S_\epsilon \to S$ by $\hat{s}(\bar{\theta}', z) = (\epsilon^{-1}\bar{\theta}', z)$ for all $s' \in S_\epsilon$. Then $\hat{\theta}_\epsilon \in \mathrm{Lp}(\epsilon)$, $\hat{s}_\epsilon \in \mathrm{Lp}(\epsilon)$ and $\hat{s} \in \mathrm{Lp}(\epsilon^{-1})$. Suppose that the functions $\hat{v}, \hat{\delta}_v, \tilde{\delta}_x, \hat{q}, \hat{\theta}, \hat{\nu}, \tilde{p}$ and $\hat{\varphi}$ now are defined over S_ϵ instead of S. Rename them as $\check{v}, \check{\delta}_v, \check{\delta}_x, \check{q}, \check{\theta}, \check{\nu}, \check{p}$ and $\check{\varphi}$ respectively. Define

$$\tilde{\delta}'_x(\hat{q}, \hat{\theta})(\theta^i_-, s) = \check{\delta}_x(\check{\nu}(\hat{q} \circ \hat{s}, \hat{\theta}_\epsilon \circ \hat{\theta} \circ \hat{s}), \hat{q} \circ \hat{s}, \hat{\theta}_\epsilon \circ \hat{\theta} \circ \hat{s})(\theta^i_-, \hat{s}_\epsilon(s))$$

and $\tilde{p}^i(\hat{v}, \hat{q}, \hat{\theta})(s) = \check{p}^i(\check{v}^i, \hat{q} \circ \hat{s}, \hat{\theta}_\epsilon \circ \hat{\theta} \circ \hat{s})(\hat{s}_\epsilon(s))$ for all $(\theta^i_-, s, \hat{q}, \hat{\theta}) \in \Theta^i \times S \times \widehat{Q} \times \widehat{\Theta}$. Suppose we find constants satisfying Assumption 4.11 for $M_{\check{p}} = \epsilon^{-1} M_{\hat{p}}$ and such that $M_{\hat{\theta}} \leq M_{\tilde{\theta}\theta_-} + \epsilon M_{\tilde{\theta}s}$. Then $\hat{\delta}'_x(\hat{q}, \hat{\theta}) \in M_{\hat{\theta}}$ if $\hat{q} \in \mathrm{Lp}(1, M_{\hat{p}})$ and $\hat{\theta} \in \mathrm{Lp}(M_{\hat{\theta}})$. Therefore, it is possible to apply Theorem 4.16 to obtain an equilibrium $(\hat{c}, \hat{\theta}, \hat{q})$. Write $\check{q} = \hat{q} \circ \hat{s}$ and $\check{\theta} = \hat{\theta}_\epsilon \circ \hat{\theta} \circ \hat{s}$. Since $M_{\hat{\theta}} = M_{\check{\theta}}$ we can decrease the price Lipschitz constant by choosing $\epsilon > 1$ and keeping the Lipschitz constant of asset transition.[35] Moreover, defining

$$\hat{v}^i(\theta^i_-, s) = \check{\delta}^i_v(\check{\nu}(\check{q}, \check{\theta}), \check{q}, \check{\theta})(\theta^i_-, \hat{s}_\epsilon(s)) \text{ for each } (i, \theta^i_-, s) \in \mathcal{I} \times \Theta^i \times S,$$

then it is easy to see that $\hat{v}^i(\theta^i_-, s) = \hat{\delta}^i_v(\hat{v}, \hat{q}, \hat{\theta})(\theta^i_-, s)$ for each $(\theta^i_-, s) \in \Theta^i \times S$ by the uniqueness of a contraction fixed point. Therefore, $\tilde{\delta}'_x(\hat{q}, \hat{\theta}) = \tilde{\delta}_x(\hat{v}, \hat{q}, \hat{\theta})$ and hence $(\hat{c}, \hat{\theta}, \hat{q}, \hat{v})$ is a recursive equilibrium.

Consider a model with one good and one asset and agents with instantaneous utility function defined by $\hat{u}^i(c^i) = \log(c^i)$ for all $c^i \in C^i$ and all $i \in \mathcal{I}$. Suppose now that there exists exogenous uncertainty and positive endowments. We found the following constants satisfying Assumptions 4.11 and 4.13. Therefore, there exists a Lipschitz recursive equilibrium for environments where the parameters are over a certain open neighborhood[36] of σ_N and σ_M. Consider $(\beta, \epsilon, N_\varphi) = (0.85, 6, 51.634484)$

$$\sigma_N = (10000, 10104.05, 2, 2.05, 31.454350, 49.052760)$$

and

$$\sigma_M = (2 \times 10^{-4}, 1 \times 10^{-8}, 5 \times 10^{-4}, M_{\check{c}}, M_{\tilde{\theta}}, 3, 0.0037679, M_{\hat{\varphi}})$$

where

$$M_{\check{c}} = (51.102760, 49.052760, 0.10435743)$$
$$M_{\tilde{\theta}} = (1.0874554, 0.31081400)$$
$$M_\varphi = (2.6386647 \times 10^{-9}, 7.8702647 \times 10^{-9}, 5.3884421 \times 10^{-12}, 0.0044027698).$$

Example 4.18. Consider the following numerical example[37]. Exogenous uncertainty is given by two states $Z = \{z_1, z_2\}$ and the transition probability is constant and uniform, that is, $\lambda(z) = (0.5.0.5)$ for all $z \in Z$. Preferences are

[35] The increasing in $M_{\tilde{\theta}s}$ is compensated by the decreasing in the price Lipschitz constant.

[36] We can also consider an open neighborhood of the utility function under a Sobolev norm involving the function and its first and second order derivatives.

[37] A Matlab code checking, whether our assumptions are satisfied is available upon request from the authors

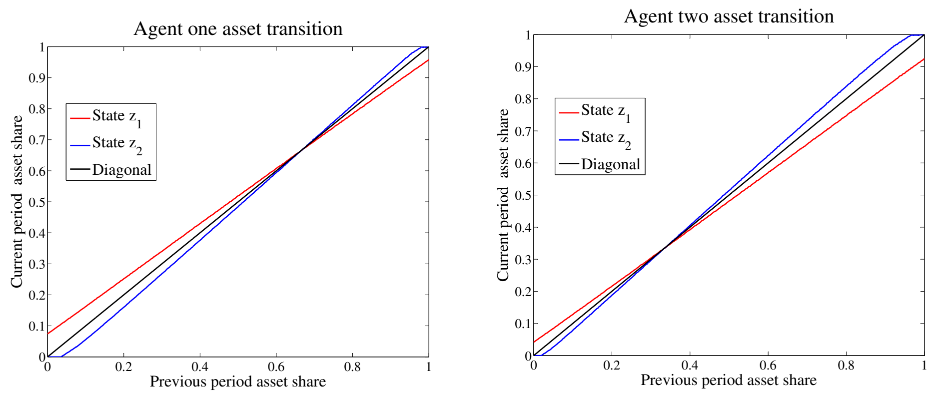

Figure 3: Graphics of $\bar{\theta}^1 \mapsto \hat{\theta}^1(\bar{\theta}^1, 1-\bar{\theta}^1, z_k)$ and $\bar{\theta}^2 \mapsto \hat{\theta}^2(1-\bar{\theta}^2, \bar{\theta}^2, z_k)$ for $k = 1, 2$.

defined by the utility function $u^i(c^i) = (c^i)^{1/2i}$ and endowments are given by $e^1(z_1) = 1$, $e^1(z_2) = 1$, $e^2(z_1) = 1$ and $e^2(z_2) = 2$. That is, agents have heterogeneity on risk aversion and aggregate wealth uncertainty. Agent one has initial asset endowment $\bar{\theta}^1 = 0.1$ and Agent two has initial asset endowment $\bar{\theta}^2 = 0.9$. Dividends are given by $\hat{d}(z_1) = 1$ and $\hat{d}(z_2) = 2$.

Figure 3 shows agents' asset transition $(\hat{\theta}^1, \hat{\theta}^2)$. Notice that $\hat{\theta}^i$ has corner solutions for $i = 1, 2$.

Figure 4 shows agents' consumption dynamics over a Monte Carlo random sampling. Considering this environment as a model of an open economy in which each agent represents a country, we clearly see formation of income cycles without considering any idiosyncratic cyclical shock.[38] For instance, country one decreases aggregate income and hence consumption and investment choices on the first periods since Equation (6) evaluated on the optimal asset choice $\boldsymbol{\theta}^i$ implies that

$$\boldsymbol{c}_t^i(z^t) + \hat{p}(\boldsymbol{\theta}_{t-1}^i(z^{t-1}), z_t)(\boldsymbol{\theta}^i(z^t) - \boldsymbol{\theta}_{t-1}^i(z^{t-1})) = \hat{d}(z_t)\boldsymbol{\theta}^i(z^{t-1}) + \hat{e}^i(z_t)$$

for all $z^t \in Z^t$ and all $t \in \mathbb{N}$.

[38] Notice that uncertainty is governed by shocks i.i.d.

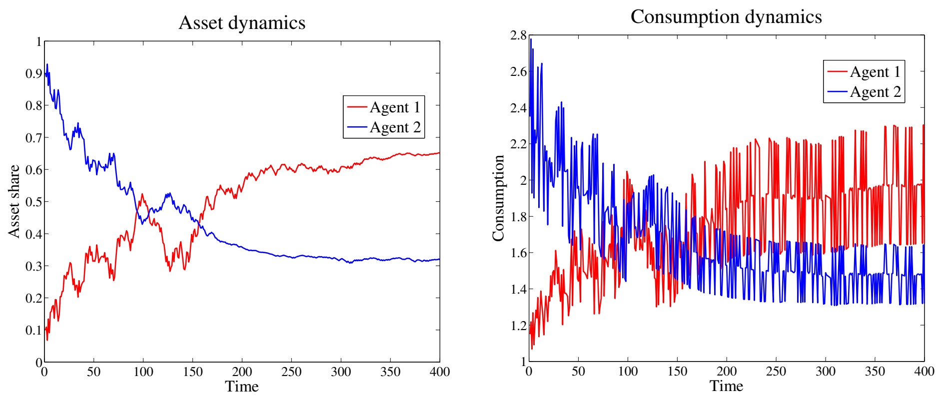

Figure 4: Graphics of $\boldsymbol{\theta}^i_t(z^t)$ and $\boldsymbol{c}^i_t(z^t)$ for $i \in \{1, 2\}$ and $t \leq 400$.

Remark 4.19. Kubler and Schmedders (2002) present an example of an infinite-horizon economy with Markovian fundamentals, where the recursive competitive equilibrium does not exist. In their example there must exist two different nodes of a tree such that along the equilibrium path the value of the equilibrium asset holdings is the same but such that there exist more than one equilibrium for both of the continuation economies. The counterexample presented in section 5.2 of Kubler and Schmedders (2002) uses an economy with 2 households with state dependent CRRA preferences that are not Lipschitz at 0. Second, comparing the asset structure, they have 3 assets, some with zero dividend at particular states, and allow for short sales. All of these are ruled out by our assumption. Third, and most importantly, existence of a single asset allows us to define uniquely the next period prices via the envelope theorem (see Definition 4.9 and Equation 15). This precludes "indeterminacy" of the next period price beliefs (on the natural spate space) and hence rules out sunspot equilibria constructed in Kubler and Schmedders (2002).

5. Concluding remarks

The standard methodology used to define a recursive equilibrium with a state space containing a large set of variables is given in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994). The authors consider a state space S containing all relevant pay-off variables and a possibly empty valued correspondence $G : S \to \text{Prob}(S)$. This correspondence which embodies exogenous shocks, feasibility and *agents' first order optimality conditions*, can be interpreted as intertemporal consistency in the short run derived from some particular model. A measurable subset $S' \subset S$ is said to be self-justified if $G(s) \cap \text{Prob}(S') \neq \emptyset$ for all $s \in S'$. The set S' contains the realizations of the equilibrium variables given an initial condition on S'. Additionally, G restricted to S' yields the probability transition induced by the long-run equilibrium variables. Under regularity assumptions on G, Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) show the existence of a non-empty compact self-justified set $S' \subset S$. The Kuratowski-Ryll-Nardzewski Theorem affirms that G admits a measurable selector. Applying the Skorokhod's Theorem to this selector they find a measurable but non-necessary continuous function defined[39] on S' which relates two consecutive realizations of the equilibrium stochastic process and implements it over all periods.

Concerning a minimal state space recursive equilibrium, in related papers Kubler and Polemarchakis (2004); Spear (1985) and Hellwig (1982) point to its possible generic nonexistence, for models of overlapping generations. Despite the fact that the confirmation of this suspicion was fulfilled only with non-existence examples, Citanna and Siconolfi (2008) argue that they are actually non-robust for this class of models. Regarding the existence results, Citanna and Siconolfi (2010); Brumm and Kubler (2013), among others conclude the existence of recursive equilibrium for overlapping generations with a reduced, but not minimal, number of variables in its domain. Also Kubler and Polemarchakis (2004) shows the existence of an approximate recursive equilibrium with a minimal state space. Unfortunately, all of these results also use the first order conditions to construct the equilibrium correspondence and hence do not con-

[39] This function can depend on an extra coordinate that represents the effect of a uniform exogenous shock on the equilibrium.

firm that the implemented sequential equilibrium is arbitrarily close to an exact equilibrium (see Kubler and Schmedders (2005)). We also report important results of Citanna and Siconolfi (2010) and later Citanna and Siconolfi (2012) for economies with uncertainty and incomplete financial markets that prove a generic (in a residual set of utilities and endowments) existence of recursive equilibrium (i.e. nonconfounding simple time-homogeneous Markov equilibria) for a class of overlapping generations under assumptions of sufficient ex-ante or ex-post consumers' heterogeneity. Finally, the arguments given in Brumm and Kubler (2013) favoring the mandatory inclusion of additional variables in the state space can not be applied to the Lucas tree model analyzed in our paper because here we consider infinite lived agents and short sales is not allowed.

Acknowledgements We would like to thank Aloisio Araujo, Robert A. Becker, Geatano Bloise, Luiz H. B. Braido, Alessandro Citanna, Jose Heleno Faro, John Geanakoplos, Victor Filipe Martins-da-Rocha, Paulo Klinger Monteiro, Juan Pablo Torres-Martinez, Konrad Podczeck, Kevin Reffett, Yiannis Vailakis, participants of EWGET 2017 conference in Salamanca and EWGET 2018 conference in Paris, as well as anonymous referees for their comments that lead to improvements in this paper. We are grateful to the CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), the FAPEMIG (Fundação de Amparo à Pesquisa do Estado de Minas Gerais) project number APQ-01431-13 and the CNPQ (Conselho Nacional de Desenvolvimento Científico e Tecnológico) project number 481542-2013-2 for their support.

6. Appendix

6.1. Elementary results

Lemma 6.1. *Suppose that $X^i \subset \mathbb{R}^2_+$ is a compact convex set with $\mathbf{0} \in X^i$ and that $W^i = \mathbb{R}^2_+$. Let $\tilde{b}^i : W^i \times Q \to X^i$ be the budget correspondence defined by*

$$\tilde{b}^i(w^i, q) = \{x^i \in X^i : qx^i \leq qw^i\}.$$

Then $\tilde{b}^i$ is continuous.

Proof of Lemma 6.1 See Lemma A1 in Raad (2012).

□

The following lemmas are useful in the proof of the main result of this section. They are used to construct an operator whose fixed point is the recursive equilibrium.

Lemma 6.2. *Consider Y a metric space and $\widehat{Y}$ the space of all bounded continuous functions $\hat{y} : Y \to Y$ endowed with the sup metric. Suppose that $f : Y \times \widehat{Y} \to \mathbb{R}^L$ is bounded and continuous with $Y \times \widehat{Y}$ endowed with the product topology. Then the function $g : Y \times \widehat{Y} \to \mathbb{R}^L$ defined by $g(y, \hat{y}) = f(\hat{y}(y), \hat{y})$ is continuous.*

Proof of Lemma 6.2 Assume that[40]

$$d((y, \hat{y}), (y', \hat{y}')) = \max\{d_Y(y, y'), d_{\widehat{Y}}(\hat{y}, \hat{y}')\}.$$

Fix some $(y', \hat{y}') \in Y \times \widehat{Y}$. Given $\epsilon > 0$ take $\gamma > 0$ such that

$$d((y, \hat{y}), (y', \hat{y}')) \leq \gamma \text{ implies } ||f(y, \hat{y}) - f(y', \hat{y}')|| \leq \epsilon.$$

Using that $\hat{y}'$ is continuous then it is possible to find $\gamma' > 0$ such that[41]

$$y \in Y \text{ and } d_Y(y, y') \leq \gamma' \text{ implies } d_Y(\hat{y}'(y), \hat{y}'(y')) \leq \gamma/2.$$

Take $\gamma^- = \min\{\gamma/2, \gamma'\}$. Since $d_Y(\hat{y}(y), \hat{y}'(y')) \leq d_Y(\hat{y}(y), \hat{y}'(y)) + d_Y(\hat{y}'(y), \hat{y}'(y'))$ then

$$\begin{aligned}
d((y, \hat{y}), (y', \hat{y}')) \leq \gamma^- &\Rightarrow d_{\widehat{Y}}(\hat{y}, \hat{y}') \leq \gamma/2 \text{ and } d_Y(y, y') \leq \gamma' \\
&\Rightarrow d_Y(\hat{y}(y), \hat{y}'(y)) \leq \gamma/2 \text{ and } d_Y(\hat{y}'(y), \hat{y}'(y')) \leq \gamma/2 \\
&\Rightarrow d_Y(\hat{y}(y), \hat{y}'(y')) \leq \gamma \text{ and } d_{\widehat{Y}}(\hat{y}, \hat{y}') \leq \gamma \\
&\Rightarrow ||f(\hat{y}(y), \hat{y}) - f(\hat{y}'(y'), \hat{y}')|| \leq \epsilon \\
&\Rightarrow ||g(y, \hat{y}) - g(y', \hat{y}')|| \leq \epsilon.
\end{aligned}$$

[40] Clearly, this metric induces the product topology on $Y \times \widehat{Y}$.

[41] Observe that γ' does depend only on $(y', \hat{y}')$ which is fixed.

That is, g is continuous on the point $(y', \hat{y}') \in Y \times \widehat{Y}$. Since $(y', \hat{y}')$ was given arbitrarily, then g is continuous.

$\square$

Lemma 6.3. *Define* $\hat{m} : \mathbb{R}^L \to \mathbb{R}$ *by* $\hat{m}(y) = \max\{y_k : k \in \{1, \cdots, L\}\}$. *Then* $\hat{m} \in \mathrm{Lp}(1)$.

Proof of Lemma 6.3 Take any y_k such that $y_k = \hat{m}(y)$. Then

$$\hat{m}(y) = y_k = y_k - y'_k + y'_k \le ||y - y'|| + y'_k \le ||y - y'|| + \hat{m}(y')$$

and hence $\hat{m}(y) - \hat{m}(y') \le ||y - y'||$. By other hand, choosing y'_k such that $y'_k = \hat{m}(y')$, then

$$\hat{m}(y') = y'_k = y'_k - y_k + y_k \le ||y - y'|| + y_k \le ||y - y'|| + \hat{m}(y)$$

and thus $|\hat{m}(y) - \hat{m}(y')| \le ||y - y'||$. Therefore, $\hat{m} \in \mathrm{Lp}(1)$.

$\square$

Lemma 6.4. *Consider* $Y, Y_k \subset \mathbb{R}$ *with* $k \in \{1, 2\}$ *and* $Y' \subset \mathbb{R}^n$. *Suppose that* $f : Y_1 \times Y_2 \to Y$ *satisfies* $f \in \mathrm{Lp}(M_f)$ *and that* $g_k : Y' \to Y_k$ *satisfies* $g_k \in \mathrm{Lp}(M_{gk})$ *for* $k \in \{1, 2\}$. *Then* $h : Y' \to Y$ *defined by* $h(y) = f(g_1(y), g_2(y))$ *for all* $y \in Y'$ *satisfies* $h \in \mathrm{Lp}(M_f \max\{M_{g1}, M_{g2}\})$. *Moreover, when* $f \in \mathrm{Lp}(M_{f1}, M_{f2})$ *then* $h \in \mathrm{Lp}(M_{g1}M_{f1} + M_{g2}M_{f2})$.

Proof of Lemma 6.4

$$\begin{aligned}
|h(y) - h(y')| &= |f(g_1(y), g_2(y)) - f(g_1(y'), g_2(y'))| \\
&\le M_f \max\{|g_1(y) - g_1(y')|, |g_2(y) - g_2(y')|\} \\
&\le M_f \max\{M_{g1}, M_{g2}\}||y - y'||.
\end{aligned}$$

For the other statement, notice that

$$\begin{aligned}
|h(y) - h(y')| \le\ & |f(g_1(y), g_2(y)) - f(g_1(y'), g_2(y))| \\
&+ |f(g_1(y'), g_2(y)) - f(g_1(y'), g_2(y'))| \\
\le\ & (M_{f1}M_{g1} + M_{f2}M_{g2})||y - y'||.
\end{aligned}$$

$\square$

Lemma 6.5. *Consider $Y \subset \mathbb{R}^n$. Suppose that $f : Y \to Y$ and $g : Y \to Y$ satisfy $f \in \mathrm{Lp}(M_f)$ and $g \in \mathrm{Lp}(M_g)$. Then $f \circ g \in \mathrm{Lp}(M_f M_g)$, $f + g \in \mathrm{Lp}(M_f + M_g)$ and $fg \in \mathrm{Lp}(n(N_f M_g + N_g M_f))$.*

Proof of Lemma 6.5 Fix $y, y' \in Y$. Thus

$$||f(g(y)) - f(g(y'))|| \leq M_f ||g(y) - g(y')|| \leq M_f M_g ||y - y'||.$$

The remaining statements come directly from Lemma 6.4 for a suitable choice of f and g_k for $k \in \{1, 2\}$.

□

Lemma 6.6. *Consider $f : Y \times Z \to \mathbb{R}$ bounded continuous and $\hat{\lambda} : Z \to \mathrm{Prob}(Z)$ measurable. Assume that $f(\,\cdot\,, z) \in \mathrm{Lp}(M_f)$ for all $z \in Z$ and $\hat{\lambda} \in \mathrm{Lp}(M_{\hat{\lambda}})$. Then the function $g : Y \times Z \to \mathbb{R}$ defined by*

$$g(y, z) = \int_Z f(y, z')\hat{\lambda}(z, dz') \text{ for all } (y, z) \in Y \times Z$$

satisfies $g \in \mathrm{Lp}(M_f + N_f M_{\hat{\lambda}})$.

Proof of Lemma 6.6. Fix $(\dot{y}, \dot{z}) \in Y \times Z$ and $(\ddot{y}, \ddot{z}) \in Y \times Z$. Thus

$$\begin{aligned} |g(\dot{y}, \dot{z}) - g(\ddot{y}, \ddot{z})| \leq & \int_Z |f(\dot{y}, z') - f(\ddot{y}, z')|\hat{\lambda}(\dot{z}, dz') \\ & + N_f \left| \int_Z N_f^{-1} f(\ddot{y}, z')\hat{\lambda}(\dot{z}, dz') - \int_Z N_f^{-1} f(\ddot{y}, z')\hat{\lambda}(\ddot{z}, dz') \right| \\ & \leq (M_f + N_f M_{\hat{\lambda}})||(\dot{y}, \dot{z}) - (\ddot{y}, \ddot{z})||. \end{aligned}$$

□

Lemma 6.7. *Suppose that Y is a subset of a Hilbert Space endowed with the norm $|\cdot|$. Then for each $\ddot{y}, \dot{y} \in Y$ and $0 \leq \tau \leq 1$*

$$\tau(1 - \tau)|\ddot{y} - \dot{y}|^2 = \tau|\ddot{y}|^2 + (1 - \tau)|\dot{y}|^2 - |\tau\ddot{y} + (1 - \tau)\dot{y}|^2$$

Proof of Lemma 6.7 Consider $\langle \cdot, \cdot \rangle$ the inner product such that $|y|^2 = \langle y, y \rangle$.

Note that

$$\begin{aligned}|\tau\ddot{y}+(1-\tau)\dot{y}|^2 &= \tau^2|\ddot{y}|^2+(1-\tau)^2|\dot{y}|^2+2\tau(1-\tau)\langle\ddot{y},\dot{y}\rangle\\ &= \tau(1-\tau)(2\langle\ddot{y},\dot{y}\rangle-|\ddot{y}|^2-|\dot{y}|^2)+\tau|\ddot{y}|^2+(1-\tau)|\dot{y}|^2\\ &= -\tau(1-\tau)|\ddot{y}-\dot{y}|^2+\tau|\ddot{y}|^2+(1-\tau)|\dot{y}|^2.\end{aligned}$$

Thus,

$$\tau(1-\tau)|\ddot{y}-\dot{y}|^2=\tau|\ddot{y}|^2+(1-\tau)|\dot{y}|^2-|\tau y+(1-\tau)\dot{y}|^2.$$

□

6.2. Main results

Lemma 6.8. *Suppose*[42] *that* $||\hat{v}^i||\leq \hat{u}^i(N_c)/(1-\beta^i)$, $||\hat{v}^i||\geq \hat{u}^i(n_e)/(1-\beta^i)$ *and* $\beta^i\hat{u}^i(N_c)/<\hat{u}^i(n_e)$. *Consider* $(\tilde{c}^i,\tilde{\theta}^i)=\tilde{\delta}^i_x(\hat{v},\hat{q},\hat{\theta})$ *where* $\tilde{\delta}^i_x$ *is given by (4). Then* $\tilde{c}^i(\theta^i_-,s)>n_c$ *and*

$$\partial_1\hat{\delta}^i_v(\hat{v},\hat{q},\hat{\theta})(\theta^i_-,s)=(\hat{p}(s)+\hat{d}(z))\partial\hat{u}^i(\check{c}^i(\hat{q})(\theta^i_-,\tilde{\theta}^i(\theta^i_-,s),s)) \tag{23}$$

for all $(\theta^i_-,s)\in\Theta^i\times S$.

Proof of Lemma 6.8 Since $\hat{u}^i$ is continuous, there exists $n_c>0$ such that $\hat{u}^i(n_c)+\beta^i\hat{u}^i(N_c)/(1-\beta^i)<\hat{u}^i(n_e)/(1-\beta^i)$. In this case, any $\dot{c}^i\leq n_c$ cannot be an optimal choice because n_e is always feasible and yields higher benefit (Duffie, Geanakoplos, Mas-Colell, and McLennan, 1994). Indeed, consider $\tilde{v}^i=\hat{\delta}^i_v(\hat{v},\hat{q},\hat{\theta})$. Then $||\hat{v}^i||\leq\hat{u}^i(N_c)/(1-\beta^i)$ implies $||\tilde{v}^i||\leq\hat{u}^i(N_c)/(1-\beta^i)$ and $||\hat{v}^i||\geq\hat{u}^i(n_e)/(1-\beta^i)$ implies $||\tilde{v}^i||\geq\hat{u}^i(n_e)/(1-\beta^i)$. If $\dot{c}^i$ is optimal, then

$$\tilde{v}^i(\theta^i_-,s)\leq\hat{u}^i(\dot{c}^i)+\beta^i\hat{u}^i(N_c)/(1-\beta^i)<\hat{u}^i(n_e)/(1-\beta^i)$$

which is a contradiction. For the last claim, apply the Envelop Theorem (Milgrom and Segal, 2002) to the equation (3).

□

[42] Benveniste and Scheinkman (1979) present a similar result.

Lemma 6.9. *Write* $\beta\bar{\theta} = \sum_{i\in\mathcal{I}} \beta^i \bar{\theta}^i$. *Under assumptions of Example 3.6, the recursive equilibrium is given for each* $s \in S$ *by* $\hat{p}(s) = \beta\bar{\theta}\hat{d}(z)/(1-\beta\bar{\theta})$,

$$\hat{\theta}^i(s) = \beta^i(\hat{p}(s)+\hat{d}(z))\bar{\theta}^i/\hat{p}(s) \text{ and } \hat{c}^i(s) = (1-\beta^i)(\hat{p}(s)+\hat{d}(z))\bar{\theta}^i.$$

Proof of Lemma 6.9 Consider $\tilde{v} = \hat{\delta}_v(\hat{v},\hat{q},\hat{\theta})$ and $(\tilde{c},\tilde{\theta}) = \tilde{\delta}_x(\hat{v},\hat{q},\hat{\theta})$. Then

$$\tilde{v}^i(\theta^i_-, s) = \max\left\{\hat{u}^i(-\hat{p}(s)\theta^i + (\hat{p}(s)+\hat{d}(z))\theta^i_-) + \beta^i\hat{v}^i(\theta^i, \hat{\theta}(s), z)\right\} \tag{24}$$

over all $\theta^i \in \Theta^i$ such that $\check{c}^i(\hat{q})(\theta^i_-, \theta^i, s) \geq 0$ where we recall that $\hat{v}^i(\theta^i_-, s) = \hat{u}^i((1-\beta^i)\theta^i_-)/(1-\beta^i) + \hat{r}^i(s)$ for all $(\theta^i_-, s) \in \Theta^i \times S$. Therefore, the first order condition[43] of Equation (24) evaluated on $\dot{\theta}^i$ is

$$(1-\beta^i)\hat{p}(s)\dot{\theta}^i = -\beta^i\hat{p}(s)\dot{\theta}^i + \beta^i(\hat{p}(s)+\hat{d}(z))\theta^i_-. \tag{25}$$

Thus $\dot{\theta}^i = \tilde{\theta}^i(\theta^i_-, s) = \beta^i\theta^i_-(1+\hat{d}(z)/\hat{p}(s))$ is the unique solution that satisfies (25) for all $(\theta^i_-, s) \in \Theta^i \times S$. Moreover, using that $\tilde{v}^i = \hat{\delta}^i_v(\hat{v},\hat{q},\hat{\theta})$ then

$$\tilde{v}^i(\theta^i_-, s) = \hat{u}^i(\tilde{c}^i(\theta^i_-, s)) + \beta^i\hat{v}^i(\tilde{\theta}^i(\theta^i_-, s), \hat{\theta}(s), z) \text{ for all } (\theta^i_-, s) \in \Theta^i \times S.$$

Since $\tilde{c}^i(\theta^i_-, s) = (1-\beta^i)\theta^i_-(\hat{p}(s)+\hat{d}(z))$ for all $(\theta^i_-, s) \in \Theta^i \times S$ then by (10)

$$\begin{aligned}
\tilde{v}^i(\theta^i_-, s) &= \hat{u}^i(\hat{p}(s)+\hat{d}(z)) + \hat{u}^i((1-\beta^i)\theta^i_-) \\
&\quad + \beta^i\hat{u}^i(\beta^i(\hat{p}(s)+\hat{d}(z))/\hat{p}(s))/(1-\beta^i) \\
&\quad + \beta^i\hat{u}^i((1-\beta^i)\theta^i_-)/(1-\beta^i) + \beta^i\hat{r}^i(\hat{\theta}(s), z) \\
&= \hat{u}^i((1-\beta^i)\theta^i_-)/(1-\beta^i) + \hat{r}^i(s) \\
&= \hat{v}^i(\theta^i_-, s)
\end{aligned}$$

for all $(\theta^i_-, s) \in \Theta^i \times S$. Therefore, $\tilde{v}^i = \hat{v}^i$.

Finally, notice that for each $s \in S$

$$\hat{d}(z)/\hat{p}(s) = (1-\beta\bar{\theta})/(\beta\bar{\theta}) \text{ and } \hat{p}(s)+\hat{d}(z) = \hat{d}(z)/(1-\beta\bar{\theta}).$$

[43] The strict concavity of $\hat{u}^i$ and $\hat{v}^i$ on the first coordinate and the INADA condition are sufficient for optimality of the solution given by the first order condition.

Thus

$$\sum_{i \in \mathcal{I}} \hat{\theta}^i(s) = (1 + \hat{d}(z)/\hat{p}(s))(\beta\bar{\theta}) = 1$$

and

$$\sum_{i \in \mathcal{I}} \hat{c}^i(s) = (\hat{p}(s) + \hat{d}(z))(1 - \beta\bar{\theta}) = \hat{d}(z).$$

$\square$

Proof of Theorem 3.8. Since the market clearing conditions come directly from the definition of the recursive equilibrium, it is sufficient to prove that $(\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{\theta}}^i) \in \tilde{\boldsymbol{\delta}}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$ for all $z \in Z$ and all $i \in \mathcal{I}$. Fix $s = (\bar{\theta}, z)$, let $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{f}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$ be a feasible plan and define

$$\boldsymbol{u}_r^i(\boldsymbol{c}^i, z) = \hat{u}^i(\boldsymbol{c}_0^i) + \sum_{\tau=1}^{r} \int_{Z^\tau} (\beta^i)^\tau \hat{u}^i(\boldsymbol{c}_\tau^i(z^\tau))\hat{\mu}_\tau^i(z, dz^\tau).$$

Consider $(\hat{c}, \hat{\theta}, \hat{q}, \hat{v}) \in \widehat{C} \times \widehat{\Theta} \times \widehat{Q} \times \widehat{V}$ satisfying

$$\hat{v} = \hat{\delta}_v(\hat{v}, \hat{q}, \hat{\theta}) \text{ and } (\hat{c}, \hat{\theta}) = \hat{\delta}_x(\hat{v}, \hat{q}, \hat{\theta}). \tag{26}$$

Then

$$\begin{aligned} \hat{v}^i(\bar{\theta}^i, s) &= \sup \left\{ \hat{u}^i(c^i) + \beta^i \int_Z \hat{v}^i(\theta^i, \hat{\theta}(s), z_1)\hat{\lambda}^i(z, dz_1) \right\} \\ &\geq \hat{u}^i(\boldsymbol{c}_0^i) + \beta^i \int_Z \hat{v}^i(\boldsymbol{\theta}_0^i, \hat{\theta}(s), z_1)\hat{\lambda}^i(z, dz_1). \end{aligned} \tag{27}$$

where the sup in the first equation is over all $(c^i, \theta^i) \in C^i \times \Theta^i$ such that $(c^i, \theta^i) \in \hat{b}^i(\bar{\theta}^i, z, \hat{q}(s))$. The above inequality comes from the fact that $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ is feasible[44] and hence $(\boldsymbol{c}_0^i, \boldsymbol{\theta}_0^i) \in \hat{b}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}}_0) = \hat{b}^i(\bar{\theta}^i, z, \hat{q}(s))$ by the price recursive relation given in Definition 3.7. Since $\hat{\boldsymbol{c}}_0 = \hat{c}(s)$ and $\hat{\boldsymbol{\theta}}_0 = \hat{\theta}(s)$ then by Definition 3.5 Item 2

$$(\hat{\boldsymbol{c}}_0^i, \hat{\boldsymbol{\theta}}_0^i) = \hat{\delta}_x^i(\hat{v}, \hat{q}, \hat{\theta})(s)$$

that is

$$\hat{v}^i(\bar{\theta}^i, s) = \hat{u}^i(\hat{\boldsymbol{c}}_0^i) + \beta^i \int_Z \hat{v}^i(\hat{\boldsymbol{\theta}}_0^i, \hat{\theta}(s), z_1)\hat{\lambda}^i(z, dz_1).$$

[44] That is, $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{f}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$.

Recall that $(\hat{\theta}(s), z_1) = (\hat{\boldsymbol{\theta}}_0, z_1)$ for each $z_1 \in Z$. Using (26) again then

$$\begin{aligned}\hat{v}^i(\boldsymbol{\theta}_0^i, \hat{\theta}(s), z_1) &= \sup\left\{\hat{u}^i(c^i) + \beta^i \int_Z \hat{v}^i(\theta^i, \hat{\theta}(\hat{\boldsymbol{\theta}}_0, z_1), z_2)\hat{\lambda}^i(z_1, dz_2)\right\}\\ &\geq \hat{u}^i(\boldsymbol{c}_1^i(z_1)) + \beta^i \int_Z \hat{v}^i(\boldsymbol{\theta}_1^i(z_1), \hat{\theta}(\hat{\boldsymbol{\theta}}_0, z_1), z_2)\hat{\lambda}^i(z_1, dz_2).\end{aligned}$$

where the sup in the first equation is over all $(c^i, \theta^i) \in \hat{b}^i\big(\boldsymbol{\theta}_0^i, z_1, \hat{q}(\hat{\boldsymbol{\theta}}_0, z_1)\big)$. The above inequality comes from the fact that $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ is feasible and hence $(\boldsymbol{c}_1^i(z_1), \boldsymbol{\theta}_1^i(z_1)) \in \hat{b}^i(\boldsymbol{\theta}_0^i, z_1, \hat{\boldsymbol{q}}_1(z_1)) = \hat{b}^i(\boldsymbol{\theta}_0^i, z_1, \hat{q}(\hat{\boldsymbol{\theta}}_0, z_1))$ for all $z_1 \in Z$. Indeed, the recursive relations in Definition 3.7 implies that $\hat{\theta}(s) = \hat{\boldsymbol{\theta}}_0$ and hence $\hat{\boldsymbol{q}}_1(z_1) = \hat{q}(\hat{\boldsymbol{\theta}}_0, z_1) = \hat{q}(\hat{\boldsymbol{\theta}}_0, z_1)$. Since $\hat{\boldsymbol{c}}_1(z_1) = \hat{c}(\hat{\boldsymbol{\theta}}_0, z_1)$ and $\hat{\boldsymbol{\theta}}_1(z_1) = \hat{\theta}(\hat{\boldsymbol{\theta}}_0, z_1)$ then replacing $(\bar{\theta}, z)$ by $(\hat{\boldsymbol{\theta}}_0, z_1)$ in Definition 3.5 Item 2

$$(\hat{\boldsymbol{c}}_1^i(z_1), \hat{\boldsymbol{\theta}}_1^i(z_1)) = \hat{\delta}_x^i(\hat{v}, \hat{q}, \hat{\theta})(\hat{\boldsymbol{\theta}}_0, z_1)$$

and hence

$$\hat{v}^i(\hat{\boldsymbol{\theta}}_0^i, \hat{\theta}(s), z_1) = \hat{u}^i(\hat{\boldsymbol{c}}_1^i(z_1)) + \beta^i \int_Z \hat{v}^i\big(\hat{\boldsymbol{\theta}}_1^i(z_1), \hat{\theta}(\hat{\boldsymbol{\theta}}_0, z_1), z_2\big)\hat{\lambda}^i(z_1, dz_2).$$

Replacing the previous inequalities[45] of $\hat{v}^i$ in (27) then

$$\begin{aligned}\hat{v}^i(\bar{\theta}^i, s) &\geq \hat{u}^i(\boldsymbol{c}_0^i) + \beta^i \int_Z \hat{u}^i(\boldsymbol{c}_1^i(z_1))\hat{\lambda}^i(z, dz_1)\\ &+ (\beta^i)^2 \int_Z \int_Z \hat{v}^i(\boldsymbol{\theta}_1^i(z_1), \hat{\theta}(\hat{\boldsymbol{\theta}}_0, z_1), z_2)\hat{\lambda}^i(z_1, dz_2)\hat{\lambda}^i(z, dz_1)\\ &= \hat{u}^i(\boldsymbol{c}_0^i) + \beta^i \int_Z \hat{u}^i(\boldsymbol{c}_1^i(z_1))\hat{\mu}_1^i(z, dz_1)\\ &+ (\beta^i)^2 \int_{Z^2} \hat{v}^i(\boldsymbol{\theta}_1^i(z_1), \hat{\theta}(\hat{\boldsymbol{\theta}}_0, z_1), z_2)\hat{\mu}_2^i(z, dz^2)\\ &= \boldsymbol{u}_1^i(\boldsymbol{c}^i, z) + (\beta^i)^2 \int_{Z^2} \hat{v}^i(\boldsymbol{\theta}_1^i(z_1), \hat{\boldsymbol{\theta}}_1(z_1), z_2)\hat{\mu}_2^i(z, dz^2).\end{aligned}$$

It follows from induction on r that

$$\hat{v}^i(\bar{\theta}^i, s) \geq \boldsymbol{u}_{r-1}^i(\boldsymbol{c}^i, z) + (\beta^i)^r \int_{Z^r} \hat{v}^i(\boldsymbol{\theta}_{r-1}^i(z^{r-1}), \hat{\boldsymbol{\theta}}_{r-1}(z^{r-1}), z_r)\hat{\mu}_r^i(z, dz^r).$$

[45] See Stokey and Lucas Chapter 9 for more detail about the composition of the stochastic kernels $\hat{\lambda}^i$.

Taking the limit as $r \to \infty$ and using that $\hat{v}^i$ is bounded then $\hat{v}^i(\bar{\theta}^i, s) \geq \boldsymbol{u}^i(\boldsymbol{c}^i, z)$ for all $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{f}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$ since $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ was chosen arbitrarily. Therefore, we conclude by (2) that $\hat{v}^i(\bar{\theta}^i, s) \geq \tilde{\boldsymbol{v}}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$.

Define recursively,[46]

$$(\hat{\boldsymbol{c}}^i_r(z^r), \hat{\boldsymbol{\theta}}^i_r(z^r)) = \hat{\delta}^i_x(\hat{v}, \hat{q}, \hat{\theta})(\hat{\boldsymbol{\theta}}^i_{r-1}(z^{r-1}), z_r) \text{ for each } r \in \mathbb{N}. \tag{28}$$

Therefore, $(\hat{\boldsymbol{c}}^i_r(z^r), \hat{\boldsymbol{\theta}}^i_r(z^r)) \in \hat{b}^i(\hat{\boldsymbol{\theta}}^i_{r-1}(z^{r-1}), z_r, \hat{q}(\hat{\boldsymbol{\theta}}_{r-1}(z^{r-1}), z_r))$ for all $r \in \mathbb{N}$ by (5) and hence $(\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{\theta}}^i) \in \boldsymbol{f}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$ since $\hat{\boldsymbol{q}}_r(z^r) = \hat{q}(\hat{\boldsymbol{\theta}}_{r-1}(z^{r-1}), z_r)$ for all $r \in \mathbb{N}$ by (13).

Replacing $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ by $(\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{\theta}}^i)$ in the previous arguments then all inequalities must bind and hence $\hat{v}^i(\bar{\theta}^i, s) = \boldsymbol{u}^i(\hat{\boldsymbol{c}}^i, z) \leq \tilde{\boldsymbol{v}}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$. Therefore, $\hat{v}^i(\bar{\theta}^i, s) = \tilde{\boldsymbol{v}}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$ and $(\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{\theta}}^i) \in \tilde{\boldsymbol{\delta}}^i(\bar{\theta}^i, z, \hat{\boldsymbol{q}})$.

□

Proof of Proposition 4.12. Assumption 4.11 assures that $\widehat{V}_M$ is invariant under the operator $\hat{\delta}_v$ defined by (3), that is, for each $i \in \mathcal{I}$

$$\hat{\delta}^i_v(\hat{v}, \hat{q}, \hat{\theta})(\theta^i_-, s) = \max\left\{\hat{u}^i(c^i) + \beta^i \int_Z \hat{v}^i(\theta^i, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz')\right\}$$

over all $(c^i, \theta^i) \in C^i \times \Theta^i$ such that $(c^i, \theta^i) \in \hat{b}^i(\theta^i_-, z, \hat{q}(s))$. Indeed, consider $(\hat{v}, \hat{q}, \hat{\theta}) \in \widehat{V}_M \times \mathrm{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q) \times \mathrm{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta)$ and write $\tilde{v}^i = \hat{\delta}^i_v(\hat{v}, \hat{q}, \hat{\theta})$. To show that[47] $\tilde{v}^i \in \widehat{V}^i_M$, first note that Condition 1 of Definition 4.10 is satisfied using the arguments given in Lemma 6.8.

Second, for the Condition 2 of Definition 4.10 consider $\check{c}^i$ as in (6) and

$$\check{v}^i(\theta^i_-, \theta^i, s) = \hat{u}^i(\check{c}^i(\hat{q})(\theta^i_-, \theta^i, s)) + \beta^i \int_Z \hat{v}^i(\theta^i, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz') \tag{29}$$

for all $(\theta^i_-, \theta^i, s) \in \Theta^i \times \Theta^i \times S$. We claim that $\tilde{v}^i$ is concave on θ^i_-. Indeed, pick

$$\dot{\theta}^i = \operatorname{argmax}\left\{\check{v}^i(\dot{\theta}^i_-, \theta^i, s) \text{ over all } \theta^i \in \Theta^i \text{ such that } \check{c}^i(\hat{q})(\dot{\theta}^i_-, \theta^i, s) \geq 0\right\}$$

[46]This plan is measurable by the Measurable Maximum Theorem (Aliprantis and Border, 1999).

[47]The following arguments also show directly that $\tilde{v}^i \in \widehat{V}^i$.

and

$$\ddot{\theta}^i = \operatorname{argmax}\Big\{\breve{v}^i(\ddot{\theta}^i_-, \theta^i, s) \text{ over all } \theta^i \in \Theta^i \text{ such that } \breve{c}^i(\hat{q})(\ddot{\theta}^i_-, \theta^i, s) \geq 0\Big\}.$$

Then for $\dot{\tau}, \ddot{\tau} \in [0,1]$ with $\dot{\tau} + \ddot{\tau} = 1$

$$\tilde{v}^i(\dot{\tau}\dot{\theta}^i_- + \ddot{\tau}\ddot{\theta}^i_-, s) \geq \dot{\tau}\tilde{v}^i(\dot{\theta}^i_-, s) + \ddot{\tau}\tilde{v}^i(\ddot{\theta}^i_-, s)$$

because $\hat{u}^i$ is concave and

$$\breve{c}^i(\hat{q})(\dot{\tau}\dot{\theta}^i_- + \ddot{\tau}\ddot{\theta}^i_-, \dot{\tau}\dot{\theta}^i + \ddot{\tau}\ddot{\theta}^i, s) = \dot{\tau}\breve{c}^i(\hat{q})(\dot{\theta}^i_-, \dot{\theta}^i, s) + \ddot{\tau}\breve{c}^i(\hat{q})(\ddot{\theta}^i_-, \ddot{\theta}^i, s) \geq 0.$$

Moreover, $\breve{v}^i(\theta^i_-, \cdot\,, s)$ is $\alpha(\hat{p}(s))^2$-concave for each $(\theta^i_-, s) \in \Theta^i \times S$. Indeed, consider $(\dot{\tau}, \ddot{\tau}) \in [0,1]^2$ with $\dot{\tau} + \ddot{\tau} = 1$. By hypothesis, $\hat{v}^i(\,\cdot\,, s)$ is concave and $\hat{u}^i$ is α-concave and hence

$$\begin{aligned}
\hat{u}^i(\breve{c}^i(\hat{q})(\theta^i_-, (\dot{\tau}\dot{\theta}^i + \ddot{\tau}\ddot{\theta}^i), s)) &\geq \dot{\tau}\hat{u}^i(\breve{c}^i(\hat{q})(\theta^i_-, \dot{\theta}^i, s)) + \ddot{\tau}\hat{u}^i(\breve{c}^i(\hat{q})(\theta^i_-, \ddot{\theta}^i, s)) \\
&\quad + \alpha\dot{\tau}\ddot{\tau}|\breve{c}^i(\hat{q})(\theta^i_-, \dot{\theta}^i, s) - \breve{c}^i(\hat{q})(\theta^i_-, \ddot{\theta}^i, s)|^2/2 \\
&\geq \dot{\tau}\hat{u}^i(\breve{c}^i(\hat{q})(\theta^i_-, \dot{\theta}^i, s)) + \ddot{\tau}\hat{u}^i(\breve{c}^i(\hat{q})(\theta^i_-, \ddot{\theta}^i, s)) \\
&\quad + \alpha(\hat{p}(s))^2\dot{\tau}\ddot{\tau}|\dot{\theta} - \ddot{\theta}|^2/2.
\end{aligned}$$

Consider $\tilde{\theta}^i = \tilde{\delta}^i_\theta(\hat{v}, \hat{q}, \hat{\theta})$ where $\tilde{\delta}^i_\theta$ is given by (4). Then

$$\tilde{\theta}^i(\theta^i_-, s) = \operatorname{argmax}\left\{\breve{v}^i(\theta^i_-, \theta^i, s) \text{ over all } \theta^i \in \Theta^i : \breve{c}^i(\hat{q})(\theta^i_-, \theta^i, s) \geq 0\right\}.$$

To see the Lipschitz constants on the sections of $\partial_1\tilde{v}^i$, note that

$$\begin{aligned}
\partial_2\breve{v}^i(\theta^i_-, \theta^i, s) &= -\hat{p}(s)\partial\hat{u}^i\big(\breve{c}^i(\hat{q})(\theta^i_-, \theta^i, s)\big) + \beta^i \int_Z \partial_1\hat{v}^i(\theta^i, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz') \\
&= \partial\hat{u}^i(\breve{c}^i(\hat{q})(\theta^i_-, \theta^i, s))\bigg(-\hat{p}(s) + \beta^i \int_Z \frac{\partial_1\hat{v}^i(\theta^i, \hat{\theta}(s), z')}{\partial\hat{u}^i(\breve{c}^i(\hat{q})(\theta^i_-, \theta^i, s))}\hat{\lambda}^i(z, dz')\bigg) \\
&= \partial\hat{u}^i(\breve{c}^i(\hat{q})(\theta^i_-, \theta^i, s))\bigg(-\hat{p}(s) + \beta^i \int_Z \hat{\varphi}^i(\theta^i_-, \theta^i, s, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz')\bigg)
\end{aligned}$$

Since $\hat{p}$ and $\hat{d}$ do not depend on θ^i_-, then we can apply Theorem 3.1 given in (Montrucchio, 1987) pointwise on s to find the Lipschitz constant of $\tilde{\theta}^i$ on the variable θ^i_-. Indeed, by Lemmas 6.4 and 6.5

$$\partial_2\breve{v}^i(\,\cdot\,, \theta^i, s) \in \mathrm{Lp}(\hat{p}(s)M_{\partial\hat{u}}(\hat{p}(s) + \hat{d}(z))) \text{ for all } s \in S.$$

Therefore, $\tilde{\theta}^i(\,\cdot\,,s) \in \mathrm{Lp}(M_{\partial\hat{u}}(1+\hat{d}(z)/\hat{p}(s))/\alpha)$, that is,

$$\tilde{\theta}^i(\,\cdot\,,s) \in \mathrm{Lp}(M_{\partial\hat{u}}(1+N_d/n_p)/\alpha) \text{ for all } s \in S. \tag{30}$$

Moreover, $\partial_2\check{v}^i(\theta^i_-,\theta^i,\,\cdot\,) \in \mathrm{Lp}(M_{\partial\check{v}s})$ where

$$M_{\partial\check{v}s} = N_{\partial\hat{u}}(M_{\hat{p}} + \beta^i(M_{\hat{\varphi}s} + M_{\hat{\varphi}s'}M_{\hat{\theta}} + N_\varphi M_{\hat{\lambda}})) + M_{\partial\hat{u}}M_{\check{c}s}(N_p + \beta^i N_\varphi).$$

Therefore, applying Theorem 3.1 given in (Montrucchio, 1987)

$$\tilde{\theta}^i(\theta^i_-,\,\cdot\,) \in \mathrm{Lp}(M_{\tilde{\theta}s}) \text{ where } M_{\tilde{\theta}s} = M_{\partial\check{v}s}/(\alpha n_p^2). \tag{31}$$

By definition

$$\tilde{v}^i(\theta^i_-,s) = \max\left\{\check{v}^i(\theta^i_-,\theta^i,s):\ \text{over all } \theta^i \in \Theta^i \text{ such that } \check{c}^i(\hat{q})(\theta^i_-,\theta^i,s) \geq 0\right\}$$

over all $\theta^i \in \Theta^i$ such that $\check{c}^i(\hat{q})(\theta^i_-,\theta^i,s) \geq 0$. Recall that $\tilde{v}^i(\,\cdot\,,s)$ is concave for each fixed $s \in S$. Applying Lemma 6.8 we get

$$\partial_1\tilde{v}^i(\theta^i_-,s) = (\hat{p}(s)+\hat{d}(z))\partial\hat{u}^i(\check{c}^i(\hat{q})(\theta^i_-,\tilde{\theta}^i(\theta^i_-,s),s)) \text{ for all } (\theta^i_-,s) \in \Theta^i \times S. \tag{32}$$

Thus, $\hat{\varphi}^i(\tilde{v}^i,\hat{q}) \in \mathrm{Lp}(M_{\hat{\varphi}})$ by Assumption 4.11 Items 1, 2, 3, 4.

Finally, to show that

$$\hat{\delta}_\theta(\hat{v},\hat{q},\hat{\theta}) \in \mathrm{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta) \tag{33}$$

notice that $\hat{\delta}_\theta(\hat{v},\hat{q},\hat{\theta})(s) = \tilde{\theta}^i(\bar{\theta}^i,s)$ for all $s \in S$. Thus equations (30) and (31) jointly with Conditions 5, 6 and 7 of Assumption 4.11 imply (33).

□

Proof of Proposition 4.14. Consider $(\tilde{c}^i,\tilde{\theta}^i) = \tilde{\delta}^i_x(\hat{v},\hat{q},\hat{\theta})$. By Equation (23) given in Lemma 6.8

$$\partial_1\hat{v}^i(\theta^i_-,s) = (\hat{p}(s)+\hat{d}(z))\partial\hat{u}^i(\check{c}^i(\hat{q})(\theta^i_-,\tilde{\theta}^i(\theta^i_-,s),s)) \text{ for all } (\theta^i_-,s) \in \Theta^i \times S.$$

Therefore, $n_\varphi \geq (n_p+n_d)\partial u^i(N_c/n_c)$ and $N_\varphi \leq (N_p+N_d)\partial u^i(n_c/N_c)$. Recall that

$$\tilde{p}^i(\hat{v},\hat{q},\hat{\theta})(s) = \beta^i \int_Z \hat{\varphi}^i(\bar{\theta}^i,\hat{\theta}^i(s),s,\hat{\theta}(s),z')\hat{\lambda}^i(z,dz') \text{ for all } s \in S$$

and observe that Assumption 4.13 Items 1 and 2 imply that

$$n_p < \beta^i(n_p + n_d)\partial u^i(N_c/n_c) \text{ and } N_p > \beta^i(N_p + N_d)\partial u^i(n_c/N_c).$$

Therefore, $n_p < \tilde{p}(s) < N_p$ for all $s \in S$. Moreover, it is straightforward to conclude that $\tilde{p} \in \mathrm{Lp}((1-\kappa)M_{\tilde{p}})$ for some $\kappa \in (0,1)$ by Condition 3 of Assumption 4.13.

For Item 2 of Proposition 4.14, use Condition 4 of Assumption 4.13, Condition 1 of Definition 4.10 and Lemma 6.8.

□

Proof of Lemma 4.15 Clearly, $\hat{\delta}_v$ is continuous by the Berge Maximum Theorem (Aliprantis and Border, 1999), Lemma 6.1 and Lemma 6.2. Consider any $\hat{\nu}_1 \in \widehat{\mathcal{V}}$ with $\hat{\nu}_1(\widehat{Q} \times \widehat{\Theta}) \subset \widehat{V}_M$ and define recursively for $n > 1$

$$\hat{\nu}_n(\hat{q}, \hat{\theta}) = \hat{\delta}_v(\hat{\nu}_{n-1}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta}) \text{ for all } (\hat{q}, \hat{\theta}) \in \widehat{Q} \times \widehat{\Theta}.$$

Fix an arbitrary $(\hat{q}, \hat{\theta}) \in \widehat{Q} \times \widehat{\Theta}$. Then $\{\hat{\nu}_n(\hat{q}, \hat{\theta})\}_{n \in \mathbb{N}}$ is a Cauchy sequence on the sup norm (Stokey, Lucas Jr, and Prescott, 1989) converging to $\hat{\nu}(\hat{q}, \hat{\theta})$ and clearly $\hat{\nu}(\hat{q}, \hat{\theta}) = \hat{\delta}_v(\hat{\nu}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta})$ since $\hat{\delta}_v$ is continuous. Applying Lemma 6.8 we get[48] as in (32)

$$\partial_1 \hat{\nu}^i_n(\hat{q}, \hat{\theta})(\theta^i_-, s) = (\hat{p}(s) + \hat{d}(z))\partial \hat{u}^i(\check{c}^i(\hat{q})(\theta^i_-, \tilde{\delta}^i_\theta(\hat{\nu}_{n-1}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta})(\theta^i_-, s), s))$$

for all $(\theta^i_-, s) \in \Theta^i \times S$ where we recall that $\check{c}^i$ is given by (6). Moreover, $\hat{\varphi}^i(\hat{\nu}^i_n(\hat{q}, \hat{\theta}), \hat{q}) \in \mathrm{Lp}(M_{\hat{\varphi}})$ for all $n \in \mathbb{N}$ by the same arguments given in Proposition 4.12. Therefore,

$$\partial_1 \hat{\nu}^i(\hat{q}, \hat{\theta})(\theta^i_-, s) = (\hat{p}(s) + \hat{d}(z))\partial \hat{u}^i(\check{c}^i(\hat{q})(\theta^i_-, \tilde{\delta}^i_\theta(\hat{\nu}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta})(\theta^i_-, s), s))$$

because $\{\tilde{\delta}^i_\theta(\hat{\nu}_{n-1}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta})\}_{n \in \mathbb{N}}$ converges on the sup norm by the Berge Maximum Theorem which ensures the continuity of $\tilde{\delta}^i_\theta$. Thus, all arguments given in Proposition 4.12 can be replicated again to show that $\hat{\delta}_\theta(\hat{\nu}(\hat{q}, \hat{\theta}), \hat{q}, \hat{\theta}) \in$

[48] Recall that $\hat{q} = (1, \hat{p})$.

$\text{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta)$ for all $(\hat{q}, \hat{\theta}) \in \text{Lp}(\widehat{Q}, M_{\hat{q}}, n_q, N_q) \times \text{Lp}(\widehat{\Theta}, M_{\hat{\theta}}, n_\theta, N_\theta)$ and that $\hat{\nu}(\hat{q}, \hat{\theta}) \in \widehat{V}_M$.

$\square$

Remark 6.10. For J goods and $\hat{u}^i : C^i \subset \mathbb{R}^J_+ \to \mathbb{R}$ an α-concave utility function it is easy to see that all arguments above can be applied. Indeed, assume that the good one has unitary price, write $c^i_{-1} = (c^i_2, \cdots, c^i_J)$, $\hat{q}_{-1} = (\hat{q}_2, \cdots, \hat{q}_J)$, define

$$\check{c}^i_1(\hat{q})(\theta^i_-, \theta^i, c^i_{-1}, s') = \hat{p}(s)(\theta^i_- - \theta^i) - \hat{q}_{-1}(s)c^i_{-1} + \hat{d}(z)\theta^i_- + \hat{e}^i(z)$$

and

$$\check{v}^i(\theta^i_-, \theta^i, c^i_{-1}, s') = \hat{u}^i(\check{c}^i_1(\hat{q})(\theta^i_-, \theta^i, c^i_{-1}, s'), c^i_{-1}) + \beta^i \int_Z \hat{v}^i(\theta^i, \hat{\theta}(s), z')\hat{\lambda}^i(z, dz').$$

Then

$$\begin{aligned} \partial_3 \check{v}^i(\theta^i_-, \theta^i, c^i_{-1}, s')(\mathring{c}^i_{-1}) = &- \sum_{j\geq 2} \hat{q}_j(s)\mathring{c}^i_j \partial_1 \hat{u}^i(\check{c}^i_1(\hat{q})(\theta^i_-, \theta^i, c^i_{-1}, s'), c^i_{-1}) \\ &+ \sum_{j\geq 2} \mathring{c}^i_j \partial_j \hat{u}^i(\check{c}^i_1(\hat{q})(\theta^i_-, \theta^i, c^i_{-1}, s'), c^i_{-1}). \end{aligned}$$

Therefore, define

$$\check{\varphi}(\hat{v}, \hat{q})(\theta^i_-, \theta^i, c^i_{-1}, s, s') = \hat{v}^i(\theta^i, s')/\partial_1 \hat{u}^i(\check{c}^i_1(\hat{q})(\theta^i_-, \theta^i, c^i_{-1}, s), c^i_{-1})$$

for all $(\theta^i_-, \theta^i, c^i_{-1}, s, s') \in \Theta^i \times \Theta^i \times C^i_{-1} \times S \times S$.

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