# Time consistent equilibria in dynamic models with recursive payoffs and behavioral discounting*

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**Abstract**

We prove existence of time consistent equilibria in a class of dynamic models with recursive payoffs and generalized discounting involving both behavioral and normative applications. Our generalized Bellman equation method identifies and separates both: recursive and strategic aspects of the equilibrium problem and allows to determine the sufficient assumptions on preferences and stochastic transition to establish existence. In particular we show existence of minimal state space stationary Markov equilibrium (a time-consistent equilibrium) in a deterministic model of consumption-saving with beta-delta discounting and its generalized versions involving non-additive payoffs, general form certainty equivalents, as well as stochastic semi-hyperbolic and hyperbolic discounting models (over possibly unbounded state and unbounded above reward space). We also provide an equilibrium approximation method for a hyperbolic discounting model.

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# 1 Introduction

Since the seminal work of Ramsey (1928) and Samuelson (1937), the question of how agents discount future utility streams has been a central focus of a large body of economic research. While Koopmans (1960) axiomatized preferences resulting in dynamically consistent choice, Strotz (1956) proposed an alternative theory of dynamically inconsistent preferences. This latter work began separate and long line of research in behavioral economics that studies the observable implications of dynamic inconsistencies on the structure of intertemporal choice. With the subsequent important papers of Laibson (1997) and Harris and Laibson (2001), models where agents possess dynamically inconsistent preferences have become key in behavioral studies involving impulses, temptations as well as self-control. This work has appeared in many fields including mathematical psychology, political science, philosophy, decision theory, game theory, and especially economics.

The motivation for much of this work on dynamically inconsistent choice is found in a large empirical and experimental literature[1] that has documented the importance of "preference reversals", when agents are comparing current vs. future utilities. These issues arise in the context of many of the canonical models in economics including work studying consumption-savings, dynastic choice with altruistic or paternalistic preferences, dynamic collective household choice, distributive justice and social choice, public policy design, models of social discounting in environmental cost-benefit analysis, theories of endogenous preference formation and reference points including habit-formation, addiction, focus-weighted choice and salience, among others.

This recent work has also led to a large body of new theoretical work that seeks to both (i) provide further axiomatic foundations to time inconsistent preferences,[2] and (ii) provide needed tools for constructing equilibrium theories of coherent dynamic choice in various settings where agents have changing intertemporal tastes. The literature providing

[1] Empirical motivation for the importance of present-bias and dynamic inconsistency in choice can be found e.g. in Angeletos et al. (2001), Ameriks et al. (2007), Mcclure et al. (2007), and Cohen et al. (2020).

[2] For a recent selection of axiomatic work see e.g. Wakai (2008), Montiel Olea and Strzalecki (2014), Galperti and Strulovici (2017), Chambers and Echenique (2018), Drugeon and Ha-Huy (2021), and most recently Chakraborty (2021). In this latter paper, the author axiomatizes a general form of present-bias preferences.

a coherent equilibrium and/or optimal solutions in the presence of dynamic inconsistent preferences include the early work of Strotz (1956), Phelps and Pollak (1968), Pollak (1968) and Peleg and Yaari (1973), as well as the subsequent work over the last two decades including, in addition to Laibson (1997) and Harris and Laibson (2001), the papers of Krusell and Smith (2003), Krusell et al. (2010), Harris and Laibson (2013), Chatterjee and Eyigungor (2016), Balbus et al. (2015b, 2018), Cao and Werning (2018), Jensen (2021), Jaśkiewicz and Nowak (2021), and Bäuerle et al. (2021). The focus has been placed on obtaining sufficient conditions for the existence, characterization, and computation of optimal *time consistent* plans, i.e. planned dynamic choices that are actually followed by the agents in future periods and are not re-optimized. The literature[3] studied both: (i) short memory time consistent decision rules (e.g., dynamic choices that are Markov or semi-Markov equilibria) and (ii) long-memory solutions (e.g., subgame perfect equilibria as in Bernheim et al. (2015) or Balbus and Woźny (2016)).

One important limitation of all this existing theoretical work has been its focus on the case of *quasi-hyperbolic* discounting. Although quasi-hyperbolic discounting is an important case, it is also a somewhat special one from a theoretical perspective (as well as an empirical/experimental one). In particular, quasi-hyperbolic agents possess a very simple pattern of "1 period forward misalignment/bias" in intertemporal preferences. From a theoretical perspective, the interest of studying more general biases in dynamic preferences stems from the early work of Loewenstein and Prelec (1992, 1993), and Rubinstein (2003). More recently, Chakraborty (2021) has provided an axiomatization of present bias that involves a weakening of the stationary axiom of Koopmans (1960) or Halevy (2015), and showed this leads to a general formulation of present bias and behavioral discounting. A separate line of research consider applying revealed preferences theory of identifying violations of time consistent choice in dynamic models with generalized discounting.[4] Unfortunately, little is known about the nature of optimal/equilibrium predictions in these cases, where agents exhibit generalized / behavioral discounting.

The interest in going beyond the case of quasi-hyperbolic discounting is also motivated by empirical and experimental work. Here we only mention a small sampling of this literature. For example, in Benhabib et al. (2010), the authors experimentally study the structure of present-bias, and find strong support for present-bias, but little support of quasi-hyperbolic discounting. Chan (2017) estimates a hyperbolic discounting model,

[3] For surveys of this body of theoretical work, see the earlier papers of Fishburn and Rubinstein (1982), Frederick et al. (2002), and Noor (2009), as well the recent surveys of Ericson and Laibson (2019) and Cohen et al. (2020).

[4] For example, see Chakraborty et al. (2017) and Echenique et al. (2020).

where differences in discount factors play a key role in explaining how workers make labor supply decisions in the context of participation in welfare programs. Similarly, he finds most agents make choices exhibiting more general forms of present-bias than just quasi-hyperbolic discounting. In another recent study by Dalton et al. (2020), the authors study the role of discounting and myopia in the purchase of Medicare D drug insurance contracts, and find strong support for general forms of behavioral discounting. Similarly, Kuchler and Pagel (2021) find strong support for general forms of present-bias in the context of credit card paydowns. Present-bias also emerges naturally in dynamic collective choice problems.[5] Using experimental methods, Jackson and Yariv (2014) study a simple model of collective choice in a lab, and find that almost all subjects acting as social planners for other decision-makers exhibit some form of time-inconsistency, with the form of time-inconsistency varying across subjects; in some cases, the subjects exhibited present-bias, who in others, future-bias. In Iverson and Karp (2021), the authors study a Markov perfect equilibria in a dynamic collective model (where the decentralized economy determines aggregate savings and a planner determines climate policy) of climate with carbon taxes, where generalized behavioral discounting is exactly the one defined and studied in our paper. For a particular class of preferences and technologies (log-linear), they are able to solve the model in closed-form, and characterize the commitment devices as well as determine optimal carbon taxes.[6] In general, of course, such closed-form/parametric solutions are not possible. Finally, Mahajan et al. (2020) and Heidhues and Strack (2021) discuss methodological issues related to the identification of present-bias and behavioral discounting in econometric models. Summing up, in all of this work, the forms of present-bias that drive time-inconsistent choices appear to be consistent with more complicated forms of present bias than simple quasi-hyperbolic discounting.

An important final aspect, that bears mentioning, is work that studies the role that uncertainty plays in characterizing the nature of observed behavioral choices. There is a number of recent papers showing that preferences over time as well as over uncertain (or risky/stochastic) outcomes are intertwined. For a discussion of these issues, see the work of Loewenstein and Prelec (1992), Saito (2009), Andreoni and Sprenger (2012), Ioannou and Sadeh (2016), and Chakraborty et al. (2020), among many others. Indeed, as Halevy (2008) and Baucells and Heukamp (2012) argue: delaying a prize in time has the same effect as increasing uncertainty of getting this prize. Interestingly, as we demonstrate in

[5]See Jackson and Yariv (2015), Lizzeri and Yariv (2017) or Ebert et al. (2020) for arguments as to why time-inconsistency shows up at the social preferences level.

[6]The rare cases considering more general behavioral discounting has either focused on models that admit closed-form solutions (e.g., Young (2007)), or emphasize numerical approaches to the computation of time consistent equilibrium (e.g., Maliar and Maliar (2016) or Jensen (2021)).

the paper, uncertainty plays a critical role, even when constructing results on the existence of time consistent solutions.

Taking many of these considerations and motivations into account, we study dynamic choice models with general forms of behavioral or normative discounting rules. It is well known such models generate dynamically inconsistent preferences. The central aim of this paper is to prove existence of *stationary* time consistent equilibrium (e.g., minimal state space Markovian equilibrium) in such class of models.

The existing literature, that is most closely related, involves papers on the existence of time consistent or stationary Markov perfect equilibrium for quasi-hyperbolic decision makers under deterministic state transition as in Bernheim et al. (2015), Cao and Werning (2018), Richter (2020) and Jensen (2021), as well as stochastic state transitions, as in the work of Harris and Laibson (2001), Balbus et al. (2015b, 2018), Balbus et al. (2020b) Chatterjee and Eyigungor (2016), and Jaśkiewicz and Nowak (2021). Our contribution is to provide a unified methodological setup for equilibrium existence verification in all of these cases. Our results can be hence treated as a prerequisite of any empirical or numerical analysis of implications of various forms of discounting on allocation of scarce economic or environmental resources over current and future generations.

**Overview of the results** Before we proceed to the formalities of the paper, we begin by previewing our main results. Consider a discrete time, infinite horizon, stochastic consumption-saving model, where the sequence of time separable lifetime preferences over sequences of consumption $(c_\tau)_{\tau=t}^{\infty}$ is given any date $t$ by:

$$u(c_t) + \mathbb{E}_t \sum_{\tau=1}^{\infty} \delta_\tau u(c_{t+\tau}). \tag{1}$$

We shall refer to these preferences as $(\delta_t)$-*behavioral discounting preferences.* Notice, at any time period $t$, the consumer uses the sequence of discount factors:

$$\delta_0, \delta_1, \delta_2, \delta_3, \ldots$$

to value current and continuation utility streams (where, for convenience, we normalize $\delta_0 = 1$). A few additional remarks on these preferences are in order. First, many cases in the literature of behavioral discounting fit into this general setting. To mention a few, we have: (i) exponential discounting when $\delta_t = \delta^t$, (ii) quasi-hyperbolic discounting when $\delta_t = \beta\delta^t$ for $t \geq 1$, and (iii) hyperbolic discounting when $\delta_t = \frac{1}{1+t}$. Second, these preferences are generally time-inconsistent. That is, the discount rate between utilities in any two time periods $\tau + 1$ and $\tau$ is given by:

$$\frac{\delta_{t+1} u(c_{\tau+1})}{\delta_t u(c_\tau)},$$

for each $t \in \{0, \ldots, \tau\}$. We say the intertemporal preferences between the consecutive periods are *misaligned* whenever for some $t$:

$$\delta_t^2 \neq \delta_{t-1}\delta_{t+1}.$$

For the case of exponential discounting, preferences are aligned. For the case of quasi-hyperbolic discounting, preferences are misaligned and exhibit "1 period forward misalignment". For the case of hyperbolic discounting, these preferences also misaligned, but for *any* $t$. As a result, the preferences in (i) are time-consistent, and in both cases (ii) and (iii), time-inconsistent.

Let us consider a stochastic dynamic optimization problem, where the dynamics on the state variable (e.g. assets, production or capital levels) $s_t$ induced by sequences of current (consumption) choices is governed by a Markov transition $s_{t+1} \sim q(s_t|s_t - c_t)$, where $s_t - c_t$ denotes investment. For a feasible and measurable consumption policy $g$ mapping current state to current consumption level we can compute expected value of preferences from tomorrow onwards:

$$J(g)(s_t) = \mathbb{E}_{s_t}\left(\sum_{\tau=1}^{\infty} \delta_\tau u(g(s_{t+\tau}))\right),$$

where $\mathbb{E}_{s_t}$ is the conditional expectation operator with respect to date $t$ information. We say a measurable consumption policy $g^*$ is a *Stationary*[7] *Markov Perfect Equilibrium (SMPE)* or a *Time Consistent Equilibrium (TCE)* in a consumption-savings model with $(\delta_t)$-behavioral discounting if for any $s \in S$ we have:

$$g^*(s) \in \arg\max_{c\in[0,s]} \left\{u(c) + J(g^*)(s-c)\right\}.$$

For the moment, assume state space $S \subset \mathbb{R}$ is bounded, and the temporal return function $u : S \mapsto \mathbb{R}$ is continuous, increasing and strictly concave. Moreover, assume $q$ is stochastically increasing and stochastically continuous.[8]

[7]The question of nonstationary MPE is interesting. For the quasi-hyperbolic case, for repeated games, see Chade et al. (2008), and for dynamic games, see Balbus and Woźny (2016) and Balbus et al. (2021) for a discussion.

[8]Stochastically continuous means the transition $q$ satisfies the Feller property. For a definition of

The first main result of the paper concerns TCE in the special case of behavioral discounting model where preferences are quasi-hyperbolic with $\delta \in (0, 1)$ and $\beta \in (0, 1]$.

**Proposition 1.** *There exists a TCE in $\beta - \delta$ quasi-hyperbolic discounting model with deterministic state transition $q$.*

Notice, for the case of quasi-hyperbolic discounting consumption-savings model, we do *not require* stochastic state transitions. Given that, Proposition 1 generalizes the existing results substantially. We leave detailed literature comments until we present our main result.

Our second main result concerns the case of behavioral discounting where the sequence discount factors for the agent is given by $\delta_t \leq \delta < 1$. Here, we allow preferences for consecutive generations of selves to be misaligned in a very general ways relative to the case of the quasi-hyperbolic discounting model. For this case, we need some uncertainty in the state transition process to obtain TCE existence.[9] Our second main result can be stated as follows:

**Proposition 2.** *There exists a TCE in the $(\delta_t)$-behavioral discounting model with preferences given by (1) whenever $q$ is nonatomic.*

In fact, the existence and characterization results in this paper are *more general* than both Propositions 1 and 2, as will be made clear in the sequel, but these two propositions basically capture the central results of the paper. We make a few remarks initially about these two sets of results.

First, in all cases of TCE, we provide a *characterization* of equilibrium policies. Namely, for any TCE with consumption $g^*$, the associated equilibrium investment is *monotone* and *right-continuous* on $S$. This implies, as we show in a sequel, in models with present-bias preferences, we break all indifferences of the "current-self" in favor of the earlier selves who prefer a higher level of investment.[10]

Second, in the general version of these two propositions (Theorem 1 and Theorem 2), we can allow for both $S$ and $u$ to be *unbounded above*. In the examples, following the main results, we compute the appropriate bounds assuring continuation utilities $J$ are well defined.

stochastically increasing, see Topkis (1998), section 3.10.

[9]Without such uncertainty, counterexamples to the existence of TCE can be constructed. See discussion in example 4.

[10]This turns out to be a very important fact. For example, Caplin and Leahy (2006) show that TCE must resolve preference indifferences in this manner for positive and normative reasons.

Third, we are also able to relax the assumption of $(\delta_t)$-behavioral discounting preferences by allowing for *time-variant* preferences represented by *non-additive* aggregators with general forms of *certainty equivalents* including risk-sensitive preferences or quasi-linear means.[11]

Finally, although not following directly from Proposition 2, in the paper, we also present an existence result regarding the hyperbolic discounting model. Specifically, when characterizing TCE in the $(\delta_t)$-behavioral discounting model, we introduce the notion of a "semi-hyperbolic" model, i.e. a model where agents, have "finite" bias/misalignment. We show in what sense the TCE in the behavioral discounting model can be generated as limits of TCE in "semi-hyperbolic" models. Importantly, the hyperbolic discounting model is a special case of a behavioral discounting model where our approximation tools work. In the view of possible equilibrium indeterminacy results,[12] we think that our approximation technique (or "upper semi-continuity" of the equilibrium set) offers some stability result relative to a class of time consistent policies.

Finally, we should mention that an important technical aspect of our approach is that we introduce a new functional equation method that links the existence of TCE to recursive utility models with strategic aspects under limited commitment. Our approach extends and integrates separate ideas developed in a series of contributions by Balbus et al. (2015b, 2018), Balbus et al. (2020a) and Balbus (2020), among others.

In the remainder of the paper, we discuss in more detail Propositions 1 and 2, as well as their generalizations. Namely, in section 2, we consider the quasi-hyperbolic discounting model. We show existence of TCE by proving an extended version of Proposition 1. The key ingredient of our argument is the development of what we refer to as a "generalized Bellman operator" defined on time-inconsistency aggregation mapping (TIAM). Then, in section 3, we provide the general statement and the existence result extending Proposition 2. The sufficient assumption and our method of proof, although similar to techniques used for the quasi hyperbolic discounting model at some technical level, are different with respect to stochastic state transition and the function spaces used to study TCE. In particular, in the second set of results we are not requiring the use of generalized Bellman operator. In both sections, however, our models involve recursive, time-varying payoffs and, again, a general forms of certainty equivalence operators. In section 4, we then develop and analyze the semi-hyperbolic discounting model and show

[11]Our results can be hence of independent interest for equilibrium existence in dynamic/stochastic games with *recursive payoffs* and *general discounting* (see Obara and Park (2017) for a recent contribution).

[12]See Krusell and Smith (2003) and its discussion in Cao and Werning (2018).

that one can view the hyperbolic discounting model as the limiting case of a sequence of semi-hyperbolic discounting problems. Finally, in section 5, we show how our results can be extended to even more general models with behavioral features e.g. backward looking discounting, short-lived players or magnitude effects.

## 2 Quasi-hyperbolic discounting and deterministic transitions

Consider an infinite horizon, stochastic consumption-savings model with quasi-hyperbolic preferences. At each period $t$, there is one generation,[13] who enters the decision problem inheriting a capital/asset stock $s_t \in S$, where $S = \mathbb{R}_+$ or $S = [0, \bar{S}] \subset \mathbb{R}_+$. Generation $t$ selects a consumption level $c_t \in [0, s_t]$, with the remaining resources $i_t = s_t - c_t$ allocated as an investment for next generation $t+1$. In general, the capital stock at $t+1$ is random, and drawn from the distribution $q(\cdot|i_t)$. The temporal utility for each generation is $u(c_t)$, where $u : S \to \mathbb{R}$ is continuous and strictly increasing function.

Then, for any stock-consumption history $(s_t, c_t)_{t=1}^{\infty}$, generation $t$ lifetime preferences are given by:

$$\mathbb{E}_{s_t} \left( u(c_t) + \beta\delta \sum_{\tau=t} u(c_{\tau+1}) \delta^{\tau-t} \right),$$

where $1 \geq \beta > 0$ and $1 > \delta \geq 0$, and expectation operator $\mathbb{E}_{s_t}$ is taken with respect to the realization of random variables $(s_\tau)_{\tau=t+1}^{\infty}$.

In what follows, we concentrate on stationary Markovian consumption policies, here denoted by functions $g : S \to S$, such that $g$ is measurable and feasible, i.e. $g(s) \in [0, s]$. Suppose then each of the following generations uses $g$ but generation $t$ deviates by choosing $c \in [0, s_t]$. Then, we can define a payoff:

$$u(c) + \beta\delta \int_S J(g)(s_{t+1}) q(ds_{t+1}|s_t - c),$$

where

$$J(g)(s_{t+1}) := \mathbb{E}_{s_{t+1}} \left( \sum_{\tau=t} u(g(s_{\tau+1})) \delta^{\tau-t} \right).$$

We now have the following definition.

**Definition 1.** *A measurable policy $g^* : S \to S$ is a stationary Markov Perfect equilibrium*

[13] Here, we interpret the dynamic choice model "dynastically", i.e., the infinite-horizon decisions are chosen by a collection of generations under limited commitment. Alternatively, those "generations" could represent "selves" in a model of a single agent with changing tastes as in Phelps and Pollak (1968), Peleg and Yaari (1973), or Hammond (1976).

*(SMPE), or a Time Consistent Equilibrium (TCE), if for any $s \in S$:*

$$g^*(s) \in \arg\max_{c\in[0,s]} u(c) + \beta\delta \int_S J(g^*)(s')q(ds'|s-c).$$

Since in the quasi-hyperbolic discounting model preferences from tomorrow on are stationary, finding TCE requires finding a pair $(g, U)$ solving, at any states $s \in S$, a system of functional equations:

$$U(s) = u(g(s)) + \delta \int_S U(s')q(ds'|s-g(s)), \tag{2}$$

$$g(s) \in \arg\max_{c\in[0,s]} u(c) + \beta\delta \int_S U(s')q(ds'|s-c). \tag{3}$$

Equation (2) involves finding the *recursive* part of preferences, i.e. future value $U$ computed for a given candidate policy $g$. In fact, for any feasible $g$ and under our conditions, function $J(g)$ equals $U$ solving (2). Functional equation (3) then assures strategic *consistency* between the consumption policy $g$ and $U$. It is straightforward to show that if $(g, U)$ solve system (2)-(3), they also solve a *generalized* Bellman equation:

$$U(s) = \frac{1}{\beta} \max_{c\in[0,s]} \left( u(c) + \beta\delta \int_S U(s')q(ds'|s-c) \right) - \frac{1-\beta}{\beta} u(g(s)). \tag{4}$$

Similarly, if $(g, U)$ solve (4), with $g$ being a measurable argmax selection from the maximization problem in brackets, then $(g, U)$ solve (2)-(3) and hence $g$ is a TCE.

Equation (4) has an very intuitive interpretation. One can think of the last element of this expression $\frac{1-\beta}{\beta}u(g(s))$ as the quasi-hyperbolic *dynamic inconsistency adjustment factor*. That is, this additional term depending on $\beta$ appearing on the right-hand side of the maximand in (4) is "added" to a standard Bellman equation to incorporate the fact agents have preferences *changing* over time. For the case of $\beta = 1$ (the case of dynamically consistent preferences with exponential discounting), this dynamic inconsistency adjustment factor equals 0, and the generalized Bellman operation reduces to the standard (time consistent) Bellman equation.[14]

This formulation of TCE in the time-separable quasi-hyperbolic case can be extended in a number a directions. For example, one can consider both (i) more general ways of evaluating certainty equivalents of future utility streams and (ii) allow for a nonlinear

[14]Note that the so-called "generalized Euler equation" approach to solving time inconsistent problems is the "first order" decomposition of the same idea. See, for example, Harris and Laibson (2001), section 3, equation (8) for first-order analog of our generalized Bellman equation.

aggregation of current utilities and the future certainty equivalents. To see that, for $i \in S$ by $\mathbf{E}_i(f)$ denote a certainty equivalent of an integrable function $f$ with respect to measure $q(\cdot|i)$. Then, consider dynamic preferences given by the two recursive aggregators $W_1$ and $W_2$, each mapping $S \times \mathbb{R} \to \mathbb{R}$, with the former evaluating the current preferences while the latter evaluating preferences from tomorrow onward. Then, the two functional equations (2)-(3) take a following form:

$$U(s) = W_2(g(s), \mathbf{E}_{s-g(s)}(U)) \text{ and } g(s) \in \arg\max_{c\in[0,s]} W_1(c, \mathbf{E}_{s-c}(U)). \tag{5}$$

These two equations in (5) can be mapped into a *single* one, of a form similar to (4), characterized by an *time-inconsistency aggregation mapping* (or TIAM, for short) $V : S \times S \times \mathbb{R} \to \mathbb{R}$:

$$U(s) = V(g(s), g(s), \mathbf{E}_{s-g(s)}(U)) = \max_{c\in[0,s]} V(c, g(s), \mathbf{E}_{s-c}(U)), \tag{6}$$

where the first argument of $V$ is a current consumption, the second element of $V$ is a *dynamic inconsistency adjustment factor* that corrects intertemporal preferences for the evolving structure of time-inconsistency, and the third argument is a "recursive" utility term from the next period onward. Our existence theorem will be based on this general formulation of the dynamic inconsistency problem in (5), i.e. will prove existence of value $U$ and a policy $g$ solving functional equation (6).

We present assumptions and our results at this abstract (formulation) level, as when doing so, we can obtain our results for a class of $\beta-\delta$ models with more general aggregators and certainty equivalents at once. See Example 2 and Example 3 a.o. It is clear, however, that the generalized Bellman equation for the standard, time separable quasi hyperbolic discounting model, with aggregators $W_1(x,z) = u(x) + \beta\delta z$ $W_2(x,z) = u(x) + \delta z$, is a special case of (6) with TIAM $V(x,y,z) := \frac{1}{\beta}(u(x) + \beta\delta z) - \frac{1-\beta}{\beta}u(y)$. Moreover, under expected utility $\mathbf{E}_i(U) = \int_S U(s)q(ds|i)$.

**Assumption 1** (Aggregator). *$V : S \times S \times [\vartheta, \infty) \mapsto [\vartheta, \infty)$, with $\vartheta \in \mathbb{R}$, is continuous and $(x,y,z) \mapsto V(x,y,z)$ is increasing in $(x,-y,z)$. Moreover:*

*(i) The function $z \to V(x,y,z)$ is a contraction mapping with a constant $\delta \in (0,1)$;*

*(ii) The function $s \to \zeta(s) := V(s - i_1, \phi(s), \psi(i_1)) - V(s - i_1 + (i_1 - i_2), \phi(s), \psi(i_2))$ has Strict Single Crossing Property (SSCP) for any $s \geq i_1 > i_2$ and Borel functions $\phi$ and $\psi$;*[15]

[15]Function $\zeta$ satisfies SSCP, whenever $\zeta(s_1) \geq 0$ implies $\zeta(s_2) > 0$ for any $s_2 > s_1$. Under monotonicity

*(iii) There is a sequence* $(\xi_k)_{k\in\mathbb{N}}$ *of elements of* $S$ *with* $0 < \xi_1 < \xi_2 < \ldots$, *and a sequence* $(\eta_k)_{k\in\mathbb{N}}$ *in* $\mathbb{R}_+$ *such that* $\vartheta < \eta_1 < \eta_2 < \ldots$ *with* $\eta_k \to \infty$ *where*

$$V(\xi_k, 0, \eta_{k+1}) \leq \eta_k \quad \textit{for all } k$$

*and* $r := \sup_{k\in\mathbb{N}} \frac{\eta_{k+1}}{\eta_k} \in (0, 1/\delta)$.

**Assumption 2** (Transition)**.** *The transition probability* $q(\cdot|i)$ *satisfies:*

*(i)* $i \mapsto q(\cdot|i)$ *is stochastically increasing, satisfies a Feller property, and* $q([0, \xi_{k+1}]|s) = 1$ *for all* $s \in [0, \xi_k]$;

*(ii) For any* $s \in S$, *the set of all* $i$ *such that* $q(\{s\}|i) > 0$ *is countable.*

We now define a class of Certainty Equivalent Operators (COP for short). Formally, let $\mathbf{E}_i(f)$ return a certainty equivalent of a Borel mapping $f$ with respect to measure $q(\cdot|i)$. $\mathbf{E}_i(f)$ is COP if it satisfies two conditions: (i) for any $i \in S$, $\mathbf{E}_i(\cdot)$ preserves constants i.e. for any constant $\alpha \geq \vartheta$, $\mathbf{E}_i(\alpha) = \alpha$ and (ii) for any $i \in S$, $\mathbf{E}_i(\cdot)$ is nondecreasing, i.e. for any Borel functions $f_1 : S \mapsto [\vartheta, \infty)$, $f_2 : S \mapsto [\vartheta, \infty)$ such that $f_1(s) \leq f_2(s)$ for $q(\cdot|i)$-almost $s \in S$, it holds $\mathbf{E}_i(f_1) \leq \mathbf{E}_i(f_2)$. To proceed we need the following definition.

**Definition 2.** *The COP obeys* ***Fatou-Serfozo property*** *if the following condition holds. Let* $\lim_{n\to\infty} i_n \to i$ *in* $S$, *and let* $(f_n)$ *be a sequence of* $q(\cdot|i_n)$*-essentially bounded functions, each mapping* $S \mapsto [\vartheta, \infty)$. *Then,*

*(i) The following inequalities hold*

$$\limsup_{n\to\infty} \mathbf{E}_{i_n}(f_n) \leq \mathbf{E}_i\left(\overline{\lim} f_n\right) \quad \textit{and} \quad \liminf_{n\to\infty} \mathbf{E}_{i_n}(f_n) \geq \mathbf{E}_i\left(\underline{\lim} f_n\right),$$

*where*

$$\overline{\lim} f_n(s) := \sup\left\{\limsup_{n\to\infty} f_n(s) : s_n \to s\right\} \quad \textit{and} \quad \underline{\lim} f_n(s) := \inf\left\{\liminf_{n\to\infty} f_n(s) : s_n \to s\right\};$$

*(ii) Assume* $f_n \to f$ *continuously* $q(\cdot|i)$*-almost everywhere, that is for* $q(\cdot|i)$*-almost all* $s \in S$, $f_n(s_n) \to f(s)$ *whenever* $s_n \to s$. *Then* $\lim_{n\to\infty} \mathbf{E}_{i_n}(f_n) = \mathbf{E}_i(f)$.

**Assumption 3** (Certainty Equivalent Operator)**.** *COP satisfies:*

assumptions it suffices to verify the SSCP condition for $\psi$ such that $\psi(i_1) > \psi(i_2)$. Indeed, in the opposite case, i.e. $\psi(i_2) \geq \psi(i_1)$ function $\zeta$ is negative so SSCP is satisfied trivially.

*(i) For any $i \in S$, $\mathbf{E}_i$ obeys **constant subadditivity**, i.e. for any Borel $f : S \mapsto [\vartheta, \infty)$ and $\alpha \geq \min(\vartheta, 0)$ it holds $\mathbf{E}_i(f + \alpha) \leq \mathbf{E}_i(f) + \alpha$;*

*(ii) $\mathbf{E}$ obeys the **Fatou-Serfozo** property;*

*(iii) If $f$ is a bounded and continuous from the right function, then the function $\mathbf{E}_i(f)$ is continuous from the right in $i$.*

Assumption 1 (i) is standard and together with 3 (i) assures existence of the (recursive) continuation utility $U$ for any $g$. Assumption 1 (iii) and 2 (i) assure we can use the local contractions argument for the case of unbounded states and/or unbounded above rewards. If the states space $S$ is bounded or rewards are (uniformly) bounded then these are automatically satisfied. Assumption 1 (ii) assures that (each) best response policy selection is monotone increasing on $S$. Regarding assumption 2 (ii). Observe, this assumption is satisfied for a purely deterministic transition structure and as well their convex combinations. Moreover, we allow all sets we consider (i.e. $\{i \in S : q(\{s\} | i) > 0\}$) be empty. This is the case, for example, when $q$ is non-atomic. These are the two cases mostly considered in the paper.

Finally we comment on Assumption 3. We start by noting it is satisfied e.g. for the standard expectation

$$\mathbf{E}_i(f) = \int_S f(s) q(ds|i).$$

Noting that $q$ obeys Assumption 2, the condition 3 (ii) follows from Fatou's extension lemma.[16] The extensive literature provides other classes of COPs, e.g. taking the quasi-linear mean form:

$$\mathbf{E}_i(f) = \phi^{-1}\left(\int_S \phi(f(s)) q(ds|i)\right),$$

where $\phi : [\vartheta, \infty) \mapsto [\vartheta, \infty)$. Our lemma 15 in the Appendix of the paper assures that any quasilinear mean with continuous and strictly monotone $\phi$ obeys Fatou-Serfozo Property. Regarding constant subbadditivity, a precise characterization is provided by Theorem 12 in Marinacci and Montrucchio (2010) for the case of $\vartheta = 0$. Here, we only recall that a quasilinear mean is constant subadditive if and only if, $\phi$ is twice continuously differentiable and strictly monotone with $\phi'(x) \neq 0\,(x > 0)$, and $\phi$ is characterized by an increasing absolute risk aversion (i.e. $-\frac{\phi''(x)}{\phi'(x)}$ is increasing).

Examples satisfying our assumption 3 include the so called *entropic risk measure* (see Weil (1993)) with $\phi(s) = e^{-\gamma s}$ where $\gamma \neq 0$ as well as $\phi(s) = s^{\alpha}$ for some $\alpha > 1$ and $\vartheta \geq 0$

[16] See Lemma 3.2. in Serfozo (1982) e.g.

due to Kreps and Porteus (1978). Another example is

$$\mathbf{E}_i(f) = -\int_{\mathbb{R}} \frac{1}{\gamma} \ln \left( \int_S e^{-\gamma f(s)} q(ds|i) \right) \pi(d\gamma),$$

where $\pi$ is a Borel probability measure on an open subset of $\mathbb{R} \setminus \{0\}$ (see Mu et al., 2021).

We now define the set of candidate TCE investment functions:[17]

$$\mathcal{H} := \{h : S \mapsto S : h(s) \in [0, s] : h \text{ is increasing and right continuous}\}.$$

Under these conditions, we now have a result on the existence of TCE $g^*$, as well as provide a characterization of the corresponding investment $h^*$, where $h^*(s) := s - g^*(s)$.

**Theorem 1.** *Assume 1, 2 and 3. There exists a $g^* : S \mapsto S$ with a corresponding investment $h^* \in H$ and $U^* : S \mapsto \mathbb{R}$ such that for any $s \in S$*

$$U^*(s) = V(g^*(s), g^*(s), \mathbf{E}_{s-g^*(s)}(U^*)) = \max_{c \in [0,s]} V(c, g^*(s), \mathbf{E}_{s-c}(U^*)).$$

We now provide few examples of the quasi-hyperbolic discounting model that satisfy all of assumptions of Theorem 1 (and hence, have TCE). We start with a standard deterministic $\beta - \delta$ model. Notice, in example 1 below, we concentrate on deriving the appropriate bounds to cover utility functions unbounded above for some typical cases considered in the literature.

**Example 1.** *(Deterministic quasi-hyperbolic discounting) Consider a standard time-separable quasi-hyperbolic discounting model with $\delta \in (0, 1)$, $\beta \in (0, 1]$. Assume $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous, increasing and strictly concave. Suppose the capital transition is deterministic and given by $k_{t+1} = F(k_t) - c_t$ for some increasing and continuous production function $F : \mathbb{R}_+ \to \mathbb{R}_+$. Introducing $s_t := F(k_t)$ we have $q(A|i) = \mathbf{1}_A(F(i))$, where $\mathbf{1}_A()$ is a indicator function of a Borel set $A \subseteq S$. Assumption 2 is hence satisfied. For illustration, we now verify the conditions of assumption 1 for this special case. This problem is a special case of our model with TIAM $V(x, y, z) = \frac{1}{\beta} u(x) + \delta z - \frac{1-\beta}{\beta} u(y)$. Assumption 1 (i) is hence satisfied. The following boundary conditions suffice to assure existence of TCE for any COP satisfying Assumption 3, e.g. expected utility.*

**Example 1A. Isoelastic utility and linear technology** *Consider $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $F(k) = Rk$ for some fixed interest rate $R > 1$. See also Cao and Werning (2018) who analyze*

[17] Recall, such space was used previously in Ray (1987) or Dong (2020), more recently.

*this model. Clearly, as $u$ is strictly concave for $\sigma > 0$, the period return to savings given by $(i, s) \to u(s-i)$ has strictly increasing differences (and, hence $V$ satisfies SSCP in Assumption 1 (ii)). We now verify the bounds in 1 (iii). For $\sigma \in (0,1)$ we take $\vartheta = 0$. Let $\xi_k = \xi_0 R^{k-1}$ with arbitrary initial value $\xi_0$. We construct $\eta_k$. We assume that $\delta R^{1-\sigma} < 1$. Let*

$$\eta_k := \frac{1}{\beta}\sum_{t=1}^{\infty} u(\xi_k R^{t-1})\delta^{t-1} = \frac{1}{\beta}\frac{\xi_k^{1-\sigma}}{1-\sigma}\sum_{t=1}^{\infty}\left(\delta R^{1-\sigma}\right)^{t-1} = \frac{\xi_k^{1-\sigma}}{\beta(1-\sigma)\left(1-\delta R^{1-\sigma}\right)}.$$

*Then we have $V(\xi_k, 0, \eta_{k+1}) = \frac{1}{\beta(1-\sigma)}\xi_k^{1-\sigma} + \delta\eta_{k+1}$. By definition of $\xi_k$ and $\eta_k$ we assure*

$$\frac{1}{\beta(1-\sigma)}\xi_k^{1-\sigma} + \delta\eta_{k+1} \leq \eta_k.$$

*Clearly $\frac{\eta_{k+1}}{\eta_k} = R^{1-\sigma} \in \left(0, \frac{1}{\delta}\right)$ as required by Assumption 1 (iii).*

**Example 1B. Strictly concave utility and linear technology** *We now relax assumption that $u$ is isoelastic and require only that $u$ is strictly concave (and hence, we satisfy the SSCP in Assumption 1(ii)). Then, the required boundedness condition for Assumption 1(iii) takes the following form:*

$$\zeta(k) := \frac{1}{\beta}\sum_{t=1}^{\infty} u(kR^{t-1})\delta^{t-1} < \infty \qquad \textit{for any } k > 0.$$

*For initial $\xi_0 > 0$ we define $\xi_k = \xi_0 R^{k-1}$ and $\eta_k := \zeta(\xi_k)$. Then we have to verify the following condition*

$$\sup_{k\in\mathbb{N}} \frac{\eta_{k+1}}{\eta_k} < \frac{1}{\delta}. \tag{7}$$

*Observe that*

$$\eta_{k+1} = \frac{1}{\beta}\sum_{t=1}^{\infty} u(\xi_k R^{t})\delta^{t-1} = \frac{1}{\beta}\sum_{t=1}^{\infty}\frac{u(\xi_k R^t)}{u(\xi_k R^{t-1})}u(\xi_k R^{t-1})\delta^{t-1} \leq \omega\eta_k,$$

*where $\omega := \sup_{k>0}\frac{u\left(\xi_0 R^{k+1}\right)}{u\left(\xi_0 R^{k}\right)}$. Hence (7) is satisfied if $\omega < \frac{1}{\delta}$.*

**Example 1C. Strictly concave utility and monotone technology** *Suppose now $F$ is continuous, strictly increasing production function. Utility function is strictly concave as above. The construction of bounds in Assumption 1(iii) is almost identical as in 1B. The* *only difference is the function $\zeta$ which is generalized to*

$$\zeta(k) = \frac{1}{\beta}\sum_{t=1}^{\infty} u(F^{t-1}(k))\delta^{t-1} < \infty \quad \text{for any } k > 0,$$

*where $F^0$ is the identity function, and the constant $\omega$*

$$\omega := \sup_{k>0} \frac{u\left(F^{k+1}(\xi_0)\right)}{u\left(F^{k}(\xi_0)\right)} < \frac{1}{\delta}.$$

To the best of our knowledge, our equilibrium existence result for the quasi-hyperbolic discounting model is one of the most general in the literature. Recall, in theorem 6 of Bernheim et al. (2015), the authors prove existence of TCE in a deterministic model with CIES utility and linear technology. In Theorem 5 in Cao and Werning (2018), the authors extend this existence result to more general utility functions, but under strictly positive lower bound of the asset holding and a linear technology with small gross interest rate (i.e. $R < 1 + \frac{1-\delta}{\beta\delta}$).[18]

Theorem 1 also generalizes existing results on the stochastic versions of the quasi-hyperbolic model. For example, in Harris and Laibson (2001) the authors prove existence of the time consistent equilibrium in a smooth model with bounded intertemporal elasticity of substitution. Along that lines Balbus et al. (2018) proved equilibrium existence and uniqueness under some restrictive assumption on the stochastic transition function. Recently Balbus et al. (2020b) have also shown existence in the general model but required non-atomic transition. We should also mention the work of Chatterjee and Eyigungor (2016), who prove existence of time consistent equilibrium but in randomized strategies/lotteries. See also Jaśkiewicz and Nowak (2021) for recent progress in establishing existence of TCE in randomized strategies. Our theorem generalize all the above listed results and provide a unified methodological setup for equilibrium existence verification.

Along these lines, we continue with some additional examples. In example 2, we consider more general aggregators and show how to verify Assumption 1.

**Example 2** (Epstein-Zin Utility)**.** *Following Kreps and Porteus (1978) and Epstein and Zin (1989) we now consider a more general aggregator:*

$$W_2(x, z) = (u(x) + \delta z^{\rho})^{\frac{1}{\rho}}$$

[18]See also Jensen (2021) for a discussion of a novel approach to verify TCE using "Ego Loss minimization problem".

*for* $\rho > 1$. *Here* $u$ *is some increasing and strictly concave utility function. In case of* $\beta - \delta$ *version of this model with*

$$W_1(x,z) = (u(x) + \beta\delta z^{\rho})^{\frac{1}{\rho}}$$

*we derive TIAM:*

$$V(x,y,z) = [\frac{1}{\beta}W_1^{\rho}(x,z) - \frac{1-\beta}{\beta}u(y)]^{\frac{1}{\rho}}.$$

*It is straightforward to verify assumption 1 for this case. Indeed,* $V$ *is a contraction in* $z$ *with modulus* $\delta$ *(see Epstein and Zin (1989) A.3 or Marinacci and Montrucchio (2010)). Moreover,* $W_1^{\rho}(s-i,z)$ *has strictly increasing differences in* $(i,s)$ *and hence* $V(s-i,y,x)$ *has SSCP in* $(i,s)$. *Finally, we construct the appropriate bounds. Let* $\xi_0$ *be given. Then construct a sequence* $(\xi_k)_k$ *by taking* $\xi_{k+1}$ *such that* $q([0,\xi_{k+1}]|\xi_k) = 1$ *for given* $\xi_k$. *By assumption 2 this can be done for a deterministic or non-atomic transition. For given* $(\xi_k)$ *we need to find* $(\eta_k)$ *to satisfy* $V(\xi_k, 0, \eta_{k+1}) \leq \eta_k$ *for each* $k \in \mathbb{N}$. *Let* $\eta_k := \alpha M^k$ *and assume* $M \in \left(0, \frac{1}{\delta^{\frac{1}{\rho}}}\right)$ *and suppose* $\sup_{t\in\mathbb{N}} \frac{u(\xi_t)}{M^{t\rho}} < \infty$. *We have then*

$$\frac{u(\xi_k)}{\eta_k^{\rho}} + \delta\left(\frac{\eta_{k+1}}{\eta_k}\right)^{\rho} \leq 1. \tag{8}$$

*Substitute* $\eta_k = \alpha M^k$ *into (8) and observe* $\frac{u(\xi_k)}{\alpha^{\rho}M^{k\rho}} + \delta M^{\rho} \leq 1$. *To finish, we can take any* $\alpha$ *such that* $\alpha^{\rho} \geq \frac{\frac{u(\xi_k)}{M^{k\rho}}}{1-\delta M^{\rho}}$ *for each* $k$.

Our final example provides an application to the case of risk sensitive preferences.

**Example 3** (Risk-sensitive preferences)**.** *Consider now generalization involving the exponential certainty equivalent as defined by Weil (1993) (see also Bäuerle and Jaśkiewicz (2018) for a motivation). In such case the risk-sensitive preferences are given by*

$$u(c) - \frac{\beta\delta}{\gamma}\ln\int_S e^{-\gamma U(s')}q(ds'|s-c),$$

*where* $U(s) = u(g(s)) - \frac{\delta}{\gamma}\ln\int_S e^{-\gamma U(s')}q(ds'|s-g(s))$ *and* $\gamma > 0$. *The time aggregator is:* $W_2(x,z) := u(x) + \delta z$ *and TIAM takes the same form as in the example 1. Finally, COP given by* $-\frac{1}{\gamma}\ln\int_S e^{-\gamma f(s')}q(ds'|s-c)$ *satisfies our conditions as argued in the discussion following assumption 3. The bounds to check our assumptions hold here can be found in a similar manner to the above examples (as COP does not change the range of* $U$*).*

We now proceed with some preliminary definitions and constructions necessary to prove Theorem 1. Let $S$ be endowed with the Euclidean topology and the set $\mathcal{H}$ with the weak-star topology (i.e. the topology with the following notion of convergence $h_n \to^w h$ iff $h_n(s) \to h(s)$ whenever $h$ is continuous at $s$). By the arguments similar to Lemma 1 in Balbus et al. (2020a) $\mathcal{H}$ is weakly compact. Endow $S \times \mathcal{H}$ with its product topology. Next define the following set:

$$\mathcal{E} := \{(s,h) \in S \times \mathcal{H} : h \text{ is continuous at } s\}.$$

It is clear the evaluation function $\mathbf{e}(s,h) = h(s)$ has a continuous restriction to $\mathcal{E}$. Since $h \in \mathcal{H}$ is increasing, the section $\mathcal{E}^h := \{s \in S : (s,h) \in \mathcal{E}\}$ has a countable complement. Next, define the space $\mathbf{V}$ to be the set of real valued functions on $S \times \mathcal{H}$ such that each $f \in \mathbf{V}$:

- is bounded on any $S_k \times \mathcal{H}$, where $S_k := [0, \xi_k]$;
- is continuous from the right on $S$;
- for any $h \in \mathcal{H}$, $f$ is continuous at $(s,h)$ for all but countably many $s \in S$.[19]

Clearly $\mathbf{V}$ is a vector space. Endow it with the topology induced by the semi-norms:

$$||f||_k = \sup_{s \in S_k \times \mathcal{H}} |f(s,h)|.$$

Further define the following:

$$\mathcal{V} := \left\{ f \in \mathbf{V} : \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k} < \infty \right\} \text{ with a norm } ||f|| = \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k}.$$

Finally, $\mathcal{V}_{usc} := \{f \in \mathcal{V} : f \text{ is upper semicontinuous at any } (s,h) \in S \times \mathcal{H}\}$, and $\mathcal{U} := \{f \in \mathcal{V}_{usc} : \vartheta \leq f(s,h) \leq \eta_k, k \in \mathbb{N}, s \in S_k\}$. Lemma 2 in the Appendix shows that $\mathcal{U}$ is a closed subset of a Banach space $(\mathcal{V}, ||\cdot||)$. We are now ready to present the key steps in the proof of main theorem of this section.

*Proof of Theorem 1.* Our proof is build on a two-step fixed point procedure, where first for a given candidate $h \in \mathcal{H}$ we construct recursive part of preferences (from tomorrow onward) using $V$ and later show existence of a fixed point of a best response mapping, again defined on $V$ but using recursive preferences constructed in step 1. We start by

[19]That is obeys the following condition: for any $h \in \mathcal{H}$, there is a countable set $S^{f,h} \subset S$ such that if $s \notin S^{f,h}$ then $f$ is continuous at $(s,h)$.

defining an operator $T$ on $\mathcal{U}$ as follows:

$$T(f)(s,h) := \max_{i\in[0,s]} V(s-i, s-h(s), \mathbf{E}_i f(h)),$$

where $f(h) := f(\cdot, h)$. For a given candidate policy $h$, and candidate value $f$ the operator $T$ returns the updated value of the continuation value using TIAM. Lemma 3 shows that $T$ is a self map on $\mathcal{U}$, while lemma 5 claims that $T$ is a contraction mapping and thus has a unique fixed point, say $f^* \in \mathcal{U}$. That is, for any candidate policy $h$ we obtain a unique value $f^*(h)$ solving the first requirement of the generalized Bellman equation defined on TIAM (see (6)), namely:

$$f^*(s,h) = \max_{i\in[0,s]} V(s-i, s-h(s), \mathbf{E}_i f^*(h)).$$

Next, since we consider time-inconsistent decision problems for which principle of optimality is not satysfied, we need to additionally assure that the candidate policy $h$ is consistent with the argmax of the right hand side of the generalized Bellman equation. Define the following mapping that characterizes the best reply correspondence

$$BI(h)(s) = \arg\max_{i\in[0,s]} V(s-i, s-h(s), \mathbf{E}_i f^*(h)),$$

Since $f^* \in \mathcal{U}$ we know in particular that $f^*(\cdot, h)$ is upper-semicontinuous on $S$ for any $h \in \mathcal{H}$. This assures that $BI(h)$ is well defined. Now, we take the greatest investment selection from $BI(h)$ namely:

$$bi(h)(s) := \max BI(h)(s).$$

Lemma 1 shows that any selection of $s \mapsto BI(h)(s)$ is increasing in $s$. Hence $bi(h)$ is and moreover it is right continuous. This is a key step in assuring that $bi$ is a continuous map (see our key Lemma 7) on a compact $\mathcal{H}$. Then immediately, by Schauder-Tychonoff Theorem, we obtain the existence of a fixed point $h^*$ of $bi$, i.e.

$$f^*(s,h^*) = V(s-h^*(s), s-h^*(s), \mathbf{E}_i f^*(h^*)) = \max_{i\in[0,s]} V(s-i, s-h^*(s), \mathbf{E}_i f^*(h^*)).$$

Letting $g^*(s) := s - h^*(s)$, we obtain a TCE $g^*$ with $U^* := f^*(h^*)$ a corresponding value of using policy $g^*$ from tomorrow onwards. □

We now remark on a selection from the argmax correspondence used in the proof of theorem 1 and optimal TCE. Our construction of TCE in theorem 1 uses the greatest investment selection from the argmax correspondence. This selection procedure guarantees in models with present biased preferences (i.e. $\beta < 1$), all indifference of the current self are arbitrarily resolved in favor of the earlier selves who prefer *higher* investment. In an important paper, Caplin and Leahy (2006) argue that *optimal* time consistent solutions should resolve *all* indifference in such a manner (for not only positive reasons, but for normative interpretations of time consistent solutions). Technically, this is also critical for our existence result. Such investment selection rules were also used in Bernheim et al. (2015), Cao and Werning (2018), and most recently in Jensen (2021).

As stressed, whenever investment is upper semicontinuous, its associated consumption is lower semicontinuous, which assures the upper semicontinuity of the continuation $f^*(\cdot, h)$ for any $h \in \mathcal{H}$. Such upper semicontinuity is critical for proving non-emptiness of the argmax correspondence. Indeed, it is not clear how the general existence for a deterministic quasi-hyperbolic discounting model with $\beta < 1$ can be extended using the *least* investment selection.[20] Clearly, for the future biased preferences (e.g. quasi-hyperbolic discounting with $\beta > 1$) our method requires the least investment selection which again corresponds with the argument of Caplin and Leahy (2006). See also Jensen (2021) for a similar finding.

We finish this section by considering a generalization of a quasi-hyperbolic discounting model with more than one period ahead misaligned preferences and showing why the general construction in theorem 1 fails in such case when one allows for a deterministic transition.

**Example 4** ($\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting)**.** *Consider a special case of preferences in (1) where the sequence of discount factors at any date $t$ is specified as follows:*

$$1, \beta_1\beta_2\delta,\ \beta_1\beta_2^2\delta^2,\ \beta_1\beta_2^2\delta^3,\ \beta_2\beta_2^2\delta^4, \ldots$$

*We shall refer to this model as the $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting. Notice, in this model, from period $t+3$ on, the discount factor becomes exponential. However, unlike in $\beta - \delta$ model, in the case of $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting, preferences are misaligned for more than just one date forward. Indeed, we have the following:*

$$(\beta_1\beta_2\delta)^2 \neq \beta_1\beta_2^2\delta^2,$$

[20] Recent contributions on equilibrium existence in related classes of stochastic games use the *least* investment selection (see Balbus et al. (2015a, 2020a) e.g.).

*whenever $\beta_1 \neq 1$; as well as*

$$(\beta_1\beta_2^2\delta^2)^2 \neq (\beta_1\beta_2\delta)(\beta_1\beta_2^2\delta^3),$$

*whenever $\beta_2 \neq 1$.*

*The appropriate "decomposition" approach, similar to the one we developed in Section 2, involves three functional equations, namely:*

$$\begin{aligned}
U_3(s) &= u(g(s)) + \delta \int_S U_3(s')q(ds'|s-g(s)),\\
U_2(s) &= u(g(s)) + \beta_2\delta \int_S U_3(s')q(ds'|s-g(s)),\\
U_1(s) &= u(g(s)) + \beta_1\beta_2\delta \int_S U_2(s')q(ds'|s-g(s))\\
&= \max_{c\in[0,s]} u(c) + \beta_1\beta_2\delta \int_S U_2(s')q(ds'|s-c).
\end{aligned} \tag{9}$$

*The generalized Bellman equation approach can be easily extended to cover this semi-hyperbolic discounting problem: Indeed, $(U_1, U_2, U_3)$ and $g$ solve system of equations (9) if and only if $U_3$ and $g$ (a selection from the argmax correspondence of the right hand side) solve the following functional equation defined on TIAM:*

$$\begin{aligned}
U_3(s) = \frac{1}{\beta_1\beta_2^2} \max_{c\in[0,s]} &\left\{ u(c) + \beta_1\beta_2\delta \int_S [u(g(s')) + \beta_2\delta \int_S U_3(s'')q(ds''|s'-g(s'))]q(ds'|s-c) \right\}\\
&- [\frac{1}{\beta_1\beta_2^2} - 1]u(g(s)) - [\frac{1}{\beta_2} - 1]\delta \int_S u(g(s'))q(s'|s-g(s)).
\end{aligned}$$

*Notice, TIAM above involve two corrective factors. For $\beta_2 = 1$, the second corrective factor disappears, and the problem reduces to $\beta - \delta$ discounting model. Similarly, for $\beta_1 = 1$ the problem reduces to a version of quasi-hyperbolic model, where the additional impatience shows up between third and the second period. As is clear from the above formulation, however, in the deterministic version of the semi-hyperbolic problem, the argmax on the right hand side need not be necessarily well-defined (i.e. the argmax may be empty) in the space of investments $\mathcal{H}$. Indeed, in the deterministic transition case, the objective function: $i \mapsto u(s-i) + \beta_1\beta_2\delta[u(g(i)) + \beta_2\delta U_3(i - g(i))$ may fail to be upper-semicontinuous, unless $U_3$ is and both $g$ and $s \mapsto s - g(s)$ are usc. For this reason our general existence approach based on the fixed point of TIAM may fail.*[21]

We resolve the problem raised by the above example by considering a non-atomic state

[21] See also (counter)example 2 in Balbus et al. (2015a).

transitions on $S$. Under this specified assumption, we can extend our existence result in a number of important directions. In fact, this assumption suffices to prove existence in a *much* more general model that we discuss in the next section.

# 3 Behavioral discounting and stochastic transitions

In this section, we extend our methods and results to more abstract formulations of recursive, time-inconsistent preferences. This includes models of consecutive generations to have general forms of preferences misalignment. Following the reasoning developed for a general quasi-hyperbolic discounting model in section 2, assume the existence of a sequence of abstract recursive aggregators $W_t : S \times \mathbb{R} \to \mathbb{R}$:

$$W_t(c, \mathbf{E}_{s-c}(U_{t+1})).$$

Observe, we allow the recursive aggregators to be *time-variant*. Here, $c$ again denotes consumption of the current self, and $\mathbf{E}_{s-c}(U_{t+1})$ is the certainty equivalent of the continuation utility given by $U_{t+1}$. Then, $g^*$ is a TCE if and only if

$$g^*(s) \in \arg\max_{c\in[0,s]} W_1(c, \mathbf{E}_{s-c}(U_2^*)),$$

where the sequence of recursive utilities $(U_t^*)_t$ solves for any $t$:

$$U_t^*(s) := W_t(g^*(s), \mathbf{E}_{s-g^*(s)}(U_{t+1}^*)).$$

We now state our second existence theorem.

**Theorem 2.** *Suppose*

*(i) each $W_t$ is a continuous, monotone aggregator such that $z \mapsto W_t(x, z)$ is a contraction mapping with modulus $\delta < 1$;*

*(ii) $W_1(s - i_1, \psi(i_1)) - W_1(s - i_2, \psi(i_2))$ has a SSCP for $s \geq i_1 > i_2$ and any Borel function $\psi$;*

*(iii) There is a sequence $(\xi_k)_k$ of elements of $S$, $0 < \xi_1 < \xi_2 < \ldots$, and a sequence $\eta_k$ of $\mathbb{R}_+$ such that $\vartheta < \eta_1 < \eta_2 < \ldots$ with $\eta_k \to \infty$, for which each of $W_t : S \times [\vartheta, \infty) \mapsto [\vartheta, \infty)$ obeys*

$$W_t(\xi_k, \eta_{k+1}) \leq \eta_k,$$

*Moreover,* $r := \sup_{k\in\mathbb{N}} \frac{\eta_{k+1}}{\eta_k} \in (0, 1/\delta)$.

*Then, under Assumption 2 (i) with non-atomic* $q$ *and Assumption 3, there exists a TCE* $g^*$ *with a corresponding monotone investment* $h^* \in \mathcal{H}$.

This is our central existence result for the case of general, behavioral discounting model. Some aspects of its proof follow the lines developed for the quasi-hyperbolic discounting model, but there are critical differences between the two constructions. The key difference is in our arguments concerning the continuity of value functions and the function spaces we work in. That is, in the case of quasi-hyperbolic discounting, we could work in the space of upper semicontinuous value functions. Therefore we could allow for *deterministic* transition functions. In the case of general behavioral discounting, our arguments needed to guarantee the existence of non-empty best reply maps cannot proceed without the imposition of nonatomic noise in the state transitions (see example 4). As a result, the arguments used in the proof of this existence theorem can be constructed in a more direct manner, and do not need to involve our TIAM mapping.[22]

To present the proof of theorem 2, we need to define certain new objects. Let $\mathbf{V}_0$ be the space of real valued functions on $\mathcal{N} \times S \times \mathcal{H}$ in which $f \in \mathbf{V}_0$ if and only if

- for any $t, k \in \mathbb{N}$, $f$ is bounded on any $(t, s, h) \in \mathbb{N} \times S_k \times \mathcal{H}$,
- for any $h \in \mathcal{H}$ and $t \in \mathbb{N}$, the function $f(t, \cdot, h)$ is continuous as $s$ for all but countably many $s \in S$.

Clearly $\mathbf{V}_0$ is a vector space. Endow it with a family of semi-norms:

$$||f||_k := \sup_{t\in\mathbb{N}, k\in\mathbb{N}, s\in S_k} |f(t, s, h)|.$$

Next define:

$$\mathcal{V}_0 := \left\{ f \in \mathbf{V}_0 : \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k} < \infty \right\} \text{ with a norm } ||f|| := \sum_{k=1}^{\infty} \frac{||f||_k^\infty}{r^k \eta_k}.$$

Define

$$\mathcal{U}_0^\infty := \left\{ f \in \mathcal{V}_0 : \vartheta \le f_t(s, h) \le \eta_k, (k, t) \in \mathbb{N}^2, (s, h) \in S_k \times \mathcal{H} \right\}.$$

In Lemma 8 we show $(\mathcal{V}_0, ||\cdot||)$ is a Banach space and $\mathcal{U}_0$ is a closed subset of $(\mathcal{V}_0, ||\cdot||)$ (hence a complete metric space).

[22]We thank both Referees for suggesting the possibility of a more direct approach relative to our TIAM approach in the case of stochastic transition.

*Proof of Theorem 2.* For $f \in \mathcal{V}_0$ let

$$\mathbb{T}(f)(t,s,h) = W_t(s - h(s), \mathbf{E}_{h(s)}(f_{t+1}(h))),$$

where $f_{t+1}(h)(\cdot) := f(t,s,h)$. Lemma 9 shows that $\mathbb{T}$ is a contraction mapping on $\mathcal{U}_0$ and has a unique fixed point: $f^*(t,s,h)$. Define

$$BI(h)(t,s) = \arg\max_{i \in [0,s]} W_1(s - i, \mathbf{E}_i(f_2^*(h))),$$

and $bi(h)(s) := \max BI(h)(s)$. Similarly as before (i.e. in Theorem 1), lemma 12 shows that the operator $bi$ maps $\mathcal{H}$ into itself and it is a continuous operator. This suffices to prove existence of a fixed point on convex and compact space $\mathcal{H}$. □

# 4 Approximations and hyperbolic discounting

In this section, we develop a general form of "semi-hyperbolic" discounting models in the spirit of example 4 that has the flavor of the quasi-hyperbolic model, but allows for more general pattern of present-bias.[23] We start with a description of a *finite-bias* semi-hyperbolic model, and we then study its limit version in subsection 4.2. After doing this, in section 4.3, we extend the results on TCE to the class of $(\delta_t)$-behavioral discounting. In doing so, we develop an approximation approach that allows us to relate the set of TCE in the $(\delta_t)$-behavioral discounting model to the set of TCE in limiting collections of semi-hyperbolic discounting models. In particular, in section 4.4, we show how TCE choice in the standard hyperbolic discounting model can be viewed as a limiting equilibrium behavior of a collection of semi-hyperbolic discounting models. More generally, approximation method we describe allows one to construct TCE in models with preferences as in (1) by finding an *approximating sequence* of semi-hyperbolic discounting models with an appropriate sequence of discount factors $(\beta_t)_{t=1}$. The corresponding TCE in the limiting semi-hyperbolic case can be used to build representations of TCE for the original problem parameterized by the discount factors $(\delta_t)_{t=1}$.

For the sake of exposition, and to keep things simple in this section, we assume a standard time-separable aggregators with $u : S \to \mathbb{R}$ a continuous, increasing and strictly concave utility function. This assures point (ii) of assumptions in Theorem 2 is satisfied. Next, we assume the transition $q$ is nonatomic and satisfies Assumption 2(i), and we also

[23]See also Montiel Olea and Strzalecki (2014) section IV for an introduction related models and their motivation.

assume that COP is given by a standard expected utility (and hence Assumption 3 is satisfied as well).

### 4.1 Semi-hyperbolic discounting

Consider a version of a semi-hyperbolic discounting model characterized by the following sequence of discount factors:

$$1, \beta_1\beta_2 \dots \beta_T, \beta_1(\beta_2 \dots \beta_T)^2, \beta_1\beta_2^2(\beta_3 \dots \beta_T)^3, \dots, \beta_1\beta_2^2 \dots \beta_{k-1}^{k-1}(\prod_{s=k}^{T} \beta_s)^k, \dots, \prod_{\tau=1}^{T} \beta_\tau^\tau,$$

while for any $t > T$ it is:

$$\prod_{\tau=1}^{T} \beta_\tau^\tau \beta_T^{t-T},$$

Assume $\beta_T < 1$. The intuition for this formulation is that each decision maker/generation at date $t$ is impatient up to $T$ periods ahead and then from period $T$ on the problem becomes stationary with exponential discounting at rate $\beta_T$. Observe that when additionally all $\beta_t \leq 1$ the decision maker has a *growing patience.*

**Remark 1.** *We have the following special cases: for $T = 1$, we have a standard exponential discounting; for $T = 2$, it is a quasi-hyperbolic $\beta_1 - \beta_2$ discounting model; for $T = 3$ , we have an "order two" quasi-hyperbolic $\beta_1 - \beta_2 - \beta_3$ model, etc. Per notation, in section 2 and example 4, we used $\delta = \beta_T$. Now, we substitute for $\beta_T$ to keep the notation concise.*

The functional equation representation of the consumption-savings problem for this class of semi-hyperbolic preferences takes the following form:

$$\begin{aligned}
&U_T(s) = u(g(s)) + \beta_T \int_S U_T(s')q(ds'|s - g(s)), \\
&\text{and } g(s) \in \arg\max_{c \in [0,s]} \{u(c) + \prod_{\tau=1}^{T} \beta_\tau \int_S U_2(s')q(ds'|s - c)\}, \\
&\text{where each } U_t(s) = u(g(s)) + \prod_{\tau=t}^{T} \beta_\tau \int_S U_{t+1}(s')q(ds'|s - g(s)).
\end{aligned}$$

It is easy to observe that semi-hyperbolic model can be obtained as a special case of the model analyzed in section 3 taking the following aggregator:

$$W_t(x,z) = \begin{cases} u(x) + z\prod_{\tau=t}^{T}\beta_\tau & if \quad t < T \\ u(x) + \beta_T z & if \quad t \geq T \end{cases}.$$

Notice, any of $W_t$ has a common Lipschitz constant $\delta \in (0,1)$ which can be taken as $\delta = \max\left(\beta_T, \prod_{\tau=t}^{T-1}\beta_\tau\right)$. What remains to be verified in order to apply Theorem 2 is to check the conditions on the state space and utility bounds. Let $\xi_0$ be given. Then, construct the sequence $(\xi_k)_k$ by taking $\xi_{k+1}$ such that $q([0,\xi_{k+1}]|\xi_k) = 1$ for each given $\xi_t$. The final assumption requires there exists a number $M \in \left(0, \frac{1}{\delta}\right)$ such that $\sup_{t\in\mathbb{N}} \frac{u(\xi_t)}{M^t} < \infty$. This suffices to assure existence of a sequence $(\eta_t)_t$ such that for any $t, k \in \mathbb{N}$ we have $W_t(\xi_k, \eta_{k+1}) \leq \eta_k$, as required by assumption $(iii)$ in Theorem 2. Indeed, we can construct a sequence $\eta_t$ in a form $\eta_t = \alpha M^t$, for some $\alpha > 0$. The sequence $\eta_t$ must satisfy

$$u(\xi_k) + \eta_{k+1}\prod_{\tau=t}^{T}\beta_\tau \leq \eta_k \text{ and } u(\xi_k) + \eta_{k+1}\beta_T \leq \eta_k$$

for any $t, k \in \mathbb{N}$, or equivalently

$$\frac{u(\xi_k)}{\eta_k} + \frac{\eta_{k+1}}{\eta_k}\prod_{\tau=t}^{T}\beta_\tau \leq 1 \text{ and } \frac{u(\xi_k)}{\eta_k} + \frac{\eta_{k+1}}{\eta_k}\beta_T \leq 1.$$

Hence $\frac{u(\xi_k)}{\eta_k} + \frac{\eta_{k+1}}{\eta_k}\delta \leq 1$. Substituting $\alpha M^t$ above into $\eta_t$ we have then

$$\sup_{t\in\mathbb{N}} \frac{u(\xi_t)}{M^t} \leq \alpha(1 - \delta M).$$

To conclude, it suffices to pick any $\alpha > \frac{\sup_{t\in\mathbb{N}} \frac{u(\xi_t)}{M^t}}{1-\delta M}$. Finally, applying theorem 2, we can hence conclude there exists a TCE of the $T$-period bias semi-hyperbolic discounting model, with investment policy in $\mathcal{H}$.

## 4.2 Limiting semi-hyperbolic discounting

We begin this section by discussing the limiting case of semi-hyperbolic discounting (i.e., the limiting semi-hyperbolic discounting model studies the $T$-period bias as $T$ gets arbitrarily large). For given $T$, denote the effective discount factors by:

$$\begin{aligned}
{}^T\delta_1 :=& \beta_1\beta_2\dots\beta_T, \\
{}^T\delta_2 :=& \beta_1(\beta_2\dots\beta_T)^2 = {}^T\delta_1\prod_{\tau=2}^{T}\beta_\tau, \\
{}^T\delta_k :=& \beta_1\beta_2^2\dots\beta_{k-1}^{k-1}(\prod_{s=k}^{T}\beta_s)^k = {}^T\delta_{k-1}\prod_{\tau=k}^{T}\beta_\tau.
\end{aligned}$$

For $k \leq T$, we then have the following recursive formulation:

$$ {}^T\delta_k = {}^T\delta_{k-1}\prod_{\tau=k}^{T}\beta_\tau. \tag{10}$$

We now seek existence of TCE in the sequence of these semi-hyperbolic models as $T \to \infty$.

Suppose that ${}^T\delta_1$ has a limit. Then, any of ${}^T\delta_k$ has a limit with $T \to \infty$. We will denote this limit by $\delta_k$. Therefore, the recursive formula for the evolution of the successive discount factor $\delta_k$ takes the following form for any $k$:

$$\delta_k = \delta_{k-1}\prod_{\tau=k}^{\infty}\beta_\tau. \tag{11}$$

We then have a new result per existence of TCE in the limiting semi-hyperbolic model relative to the $(\delta_t)$-behavioral discounting model:

**Theorem 3.** *Suppose there exists a sequence $(b_t)_t$ such that:*

$$\forall t \text{ and } \forall T \text{ we have } \eta_t\, {}^T\delta_t \leq b_t \tag{12}$$

*and that series $(b_t)_t$ is convergent. Consider a model with generation $t$ preferences given by:*

$$\mathbb{E}_s\left(u(c_t) + \sum_{\tau=1}^{\infty} {}^T\delta_\tau u(c_{t+\tau})\right) \tag{13}$$

*with ${}^T\delta_t$ satisfying the above recursive formulation in (10). Then,*

*(i) for any $T$, there is a TCE $g^T$ such that $h^T \in \mathcal{H}$, with $h^T(s) := s - g^T(s)$;*

*(ii) any limit point of the sequence $(g^T)_{T=1}^{\infty}$ in the corresponding weak-star topology, say $g^*$, is also a TCE in the model with period $t$ preferences given by:*

$$\mathbb{E}_s\left(u(c_t) + \sum_{\tau=1}^{\infty} \delta_\tau u(c_{t+\tau})\right) \tag{14}$$

*where the sequence $\delta_t$ satisfies the recursive formulation in (11).*

*Proof.* The results in (i) follows from Theorem 2. We only prove (ii). Let $t$ be the current generation whose state is $s_0$. By Lemma 14 there is a probability space $(\Omega, \mathcal{F}, P)$ and Markov chain $(\xi_\tau^T)_{\tau=1}^{\infty}$ with the transition $s \mapsto q(\cdot|s - g^T(s))$ and the current state $s_0 \in S$, and another Markov chain $(\xi_\tau)_{\tau=1}^{\infty}$ with the transition $s \mapsto q(\cdot|s - g^*(s))$ and the current state $s_0$ as well, such that for any $\tau$ and $\omega$, $\xi_\tau^T(\omega) \to \xi_\tau(\omega)$ as $T \to \infty$. Because $q$ is a nonatomic transition, we may assume without loss of generality that any of $g^*$ is continuous at $\xi_\tau(\omega)$ for any $\omega \in \Omega$. Hence for any $\omega \in \Omega$:

$$\lim_{T\to\infty} g^T(\xi_\tau^T(\omega)) = g^*(\xi_\tau(\omega)). \tag{15}$$

Suppose generation $t$ deviates and selects $c \in [0, s_0]$. In the first step, assume $s_0$ is a continuity point of $g^T$. We have then $g^T(s_0) \to g^*(s_0)$ as $T \to \infty$ and

$$\begin{aligned}
\mathbb{E}_{s_0}\left(u(g^*(s_0)) + \sum_{\tau=1}^{\infty} \delta_\tau u(g^*(s_\tau))\right) &= u(g^*(s_0)) + \int_\Omega \left(\sum_{\tau=1}^{\infty} \delta_\tau u(g^*(\xi_\tau^*(\omega)))\right) P(d\omega) \\
&= \lim_{T\to\infty}\left(u(g^T(s_0)) + \int_\Omega \left(\sum_{\tau=1}^{\infty} {}^T\delta_\tau u(g^T(\xi_\tau^T(\omega)))\right) P(d\omega)\right) \qquad (16) \\
&= \lim_{T\to\infty} \mathbb{E}_{s_0}\left(u(g^T(s_0)) + \sum_{\tau=1}^{\infty} {}^T\delta_\tau u(g^T(s_\tau))\right) \\
&= \lim_{T\to\infty} \mathbb{E}_{s_0}\left(u(g^T(s_0)) + \mathbb{E}_{s_0-g^T(s_0)}\left(\sum_{\tau=1}^{\infty} {}^T\delta_\tau u(c^T(s_\tau))\right)\right) \\
&\geq \lim_{T\to\infty}\left(u(c) + \mathbb{E}_{s_0-c}\left(\sum_{\tau=1}^{\infty} {}^T\delta_\tau u(g^T(s_\tau))\right)\right) \\
&= u(c) + \mathbb{E}_{s_0-c}\left(\sum_{\tau=1}^{\infty} \delta_\tau u(g^*(s_\tau))\right) \qquad (17)
\end{aligned}$$

where (16) and (17) follows by Dominated Convergence Theorem whose application is possible as the corresponding components of the sum are bounded by $b_t$ defined in (12). Hence $h^*(s) = s - g^*(s)$ coincides with $bi(h^*)(s)$ at any continuity point of $g^*$, where $bi(\cdot)$ is adapted to objective in (14). By nonatomicity of $q$ we easily conclude that $(bi \circ bi)(h^*)(s)$ and $bi(h^*)(s)$ coincide for any $s \in S$. Hence $bi(h^*)$ is a fixed point of $bi$. As a result, $h^T \Rightarrow bi(h^*)$ as $T \to \infty$. By previous assumption, $h^T \Rightarrow h^*$, hence $bi(h^*) = h^*$. $\square$

This result allows us to *approximate* (in the weak topology) *general behavioral discounting models* with preferences such as (1). The key technical contribution in Theorem 3 is based on the upper semicontinuity of the *set* of TCE with respect to $T$ for $T$ at $\infty$. The new condition (i.e. that the series $(b_t)$ is convergent) is required so that the resulting limiting model is well-defined.

## 4.3 Approximating general behavioral discounting models

With this result in place, we are now able to explore further the relationship between limiting semi-hyperbolic models and $(\delta_t)$-behavioral discounting models. Suppose we have a $(\delta_t)$-behavioral discounting model. We now ask the following question: can we construct a sequence of $(\beta_t)_{t=1}$ such that the TCE of the corresponding semi-hyperbolic discounting model can approximate TCE of the $(\delta_t)$-behavioral discounting model? The following result answers this question.

**Proposition 3.** *Let a sequence $(\delta_t)_t$ be given. Define*

$$\beta_t := \begin{cases} \frac{\delta_1^2}{\delta_2} & \text{if} \quad t = 1 \\ \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}} & \text{if} \quad t \geq 2. \end{cases} \tag{18}$$

*then a TCE of the semi-hyperbolic $\beta_1 - \beta_2 - \ldots$ discounting model is a TCE of the behavioral discounting model with $(\delta_t)_{t=1}$ provided $R := \lim_{t\to\infty} \frac{\delta_{t+1}}{\delta_t} = 1$.*

*Proof.* To see that observe:

$$\frac{\delta_{t+1}}{\delta_t} = \prod_{\tau=t+1}^{\infty} \beta_\tau$$

and hence

$$\beta_t := \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}}$$

for $t > 1$. Further we have $\lim_{t\to\infty} \frac{\delta_{t+1}}{\delta_t} = \lim_{t\to\infty} \prod_{\tau=t+1}^{\infty} \beta_\tau$, that by assumptions is equal to 1. To recover $\beta_1$ proceed as follows:

$$\begin{aligned}\delta_1 = \beta_1 \prod_{t=2}^{\infty} \beta_t = \beta_1 \prod_{t=2}^{\infty} \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}} &= \beta_1 \lim_{T\to\infty} \prod_{t=1}^{T} \frac{\delta_{t+1}^2}{\delta_{t+2}\delta_t} \\ &= \beta_1 \lim_{T\to\infty} \frac{\left(\prod_{t=2}^{T+1} \delta_t\right)^2}{\prod_{t=1}^{T} \delta_t \prod_{t=3}^{T+2} \delta_t} = \beta_1 \frac{\delta_2}{\delta_1} \lim_{T\to\infty} \frac{\delta_{T+1}}{\delta_{T+2}} = \beta_1 \frac{\delta_2}{\delta_1}.\end{aligned}$$

Hence $\beta_1 = \frac{\delta_1^2}{\delta_2}$. $\square$

## 4.4 The hyperbolic discounting case

We can now use the results in the previous section to discuss formally how the TCE in the hyperbolic discounting model can be approximated by TCE in limiting versions of our semi-hyperbolic discounting models. To see how such approximations can be constructed, let for any date $t$, the discount factor for the $(\delta_t)$-discounting model take a specific hyperbolic form

$$\delta_t = \left(\frac{1}{1+t}\right)^{\beta},$$

for some parameter $\beta > 1$ guaranteeing convergence of the series. Notice, this implies that the discount factor between any two time periods $t+1$ and $t$ is:

$$\frac{(\frac{1}{t+2})^{\beta}}{(\frac{1}{t+1})^{\beta}} = \left(\frac{t+1}{t+2}\right)^{\beta}.$$

Then, applying our approximating formula in (18) in Proposition 3, we obtain the following expression:

$$\beta_{t+1} = \left(\frac{(t+1)(t+3)}{(t+2)^2}\right)^{\beta}$$

where $\beta_1 = \left(\frac{3}{4}\right)^{\beta}$. Hence, for this simple case, a TCE of the hyperbolic discounting model can be expressed as a limit of a TCE of semi-hyperbolic models.

Observe in addition that this same argument applies to a more general forms of hyperbolic discounting (e.g., the model studied in Loewenstein and Prelec (1992)). Specifically, just let $\delta_t = (1+\alpha t)^{-\frac{\beta}{\alpha}}$. Indeed, in such case, we have $\beta_t := \left(\frac{(1+\alpha t+\alpha)(1+\alpha t-\alpha)}{1+\alpha t}\right)^{\frac{\beta}{\alpha}}$, $\beta_1 := (\frac{1+2\alpha}{1+\alpha})^{\frac{\beta}{\alpha}}$ with $R = 1$.

## 5 Additional examples

We now provide some additional applications of our results. Throughout this section we will assume that transition $q$ satisfies Assumption 2(i) and is nonatomic. Moreover, the continuous utility $u$ is increasing and strictly concave while COP satisfies Assumption 3. All existence results in this section follow from our Theorem 2.

Let us begin with the case of generalized quasi-geometric discounting.

**Example 5** (Generalized quasi-geometric discounting). *Young (2007) considers a dynamic optimization model with the following sequence of discount factors:*

$$1, \tilde{\beta}_1\delta, \tilde{\beta}_1\tilde{\beta}_2\delta^2, \tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3\delta^3, \dots$$

*Therefore, between any two consecutive dates (say $t+1$ and $t$), the discount rate is $\tilde{\beta}_t\delta$. Suppose $\lim_{t\to\infty}\tilde{\beta}_t \in (0,1]$ exists and each $\tilde{\beta}_t\delta < 1$. Then, if we seek TCE in the resulting model, we have:*

$$g^*(s) \in \arg\max_{c\in[0,s]} u(c) + \tilde{\beta}_1\delta\mathbf{E}_{s-c}(U_2^*).$$

*where for $t \geq 2$, we also have:*

$$U_t^*(s) = u(g^*(s)) + \tilde{\beta}_t\delta\mathbf{E}_{s-g^*(s)}(U_{t+1}^*).$$

*Here, we can take*

$$W_t(c, \mathbf{E}_{s-c}(U)) = u(c) + \tilde{\beta}_t\delta\mathbf{E}_{s-c}(U).$$

*It is straightforward to see that this aggregator satisfies our assumptions, and therefore, TCE exists. Indeed, each $W_t$ is a contraction with a uniform modulus $\delta < 1$. The appropriate bounds $(\eta_k)_k$ can be computed analogously to the examples in section 3 (see page 26). In this model $R \neq 1$ (generally) and hence our approximation via sequence of semi-hyperbolic equilibria cannot be applied.*

We next consider the "backward discounting" model, introduced to the literature in a recent paper by Ray et al. (2017).

**Example 6** (Backward discounting). *Following Ray et al. (2017) we consider an individual whose current utility is derived from evaluating both present and past consumption streams. Each of these streams is discounted, the former forward in the usual way, the latter backward. Specifically, assume an individual at date $t$ evaluates consumption according to a weighted average of his own felicity (as perceived at date $t$) and that of a* *"future self" as perceived from date $T > t$. More specifically, for a generation born in $\tau = 0$ and taking the backward looking date to be $T(\tau) := T + \tau$ for some $T > 0$, her preferences are:*

$$\mathbb{E}_0 \sum_{t=0}^{T} \delta^t u(c_t)[\alpha + (1-\alpha)\delta^{T-2t}] + \delta^T \mathbb{E}_{T+1} \sum_{t=T+1}^{\infty} \delta^{t-T} u(c_t)[\alpha + (1-\alpha)\delta^{-T}].$$

*where $\alpha$ (resp. $(1-\alpha)$) is the forward (resp. backward) looking weight. Observe that from $t \geq T$ the preferences become stationary with exponential discounting $\delta$. So put*

$$U_{T+1}(s_{T+1}) = \mathbb{E}_{T+1} \sum_{t=T+1}^{\infty} \delta^{t-T} u(g(s_t))[\alpha + (1-\alpha)\delta^{-T}]$$

*to denote the value for this stationary part (for some candidate stationary policy $g$). That is, for $t \geq T$, we can take the aggregators:*

$$W_t(c, \mathbf{E}_{s-c}(U)) := u(c)[\alpha + (1-\alpha)\delta^{-T}] + \delta \mathbf{E}_{s-c}(U).$$

*Observe, this implies the problem resembles a* ***finite-bias*** *discounting model discussed in section 4. For $t < T$, we need to, however, construct our preferences recursively (backwards) using our aggregators $W_t$:*

$$W_t(c, \mathbf{E}_{s-c}(U)) := u(c)[\alpha + (1-\alpha)\delta^{T-2t}] + \delta \mathbf{E}_{s-c}(U)$$

*with $U_T(s_T) = u(g(s_T))[\alpha + (1-\alpha)\delta^{-T}] + \delta^T \mathbf{E}_{s_{T+1}-g(s_{T+1})}(U_{T+1})$.*

*Then, in this case, we seek the existence of TCE that are solutions of the following functional equation:*

$$g^*(s) \in \arg\max_{c \in [0,s]} u(c)[\alpha + (1-\alpha)\delta^T] + \delta \mathbf{E}_{s-c}(U_1^*),$$

*with $U_t^*(s) = W_t(g^*(s), E_{s-g^*(s)}(U_{t+1}^*))$. Again, with $\delta < 1$ the above aggregators ($W_t$) satisfy assumption of Theorem 2 with time-variant utility $u_t(c) := u(c)[\alpha + (1-\alpha)\delta^{T-2t}]$ whenever $t \leq T$ and $u_t(c) := u(c)[\alpha + (1-\alpha)\delta^{-T}]$ for $t \geq T$. The appropriate continuation utility bounds can be constructed similarly to the construction on page 26 for semi-hyperbolic discounting model.*

So far, in the paper, we have focused on models where this decision maker is infinitely-lived. It happens, our approach is also useful when attempting to understand cases where agents are short-lived. Important problems in economics have the latter form with examples including dynamic sustainable resource models with public policy, economic models of the transmission of human capital and endogenous preferences across generations, models of endogenous fertility, as well as related models of sustainable dynastic choice with intergenerational altruism and paternalism. One particularly relevant case is that of bequest games. We now show how our results can be applied in these models.

**Example 7** (Finitely-lived dynastic discounting and bequest games)**.** *Consider a sequence of discount factors* $1, \delta_1, \delta_2, \ldots, \delta_T, 0, 0, \ldots$ *for some* $T \geq 1$*. This, therefore, is a class of* $T$*-period paternalistic bequest games with changing discount factors. To apply our results to this model, simply take:*

$$W_t(c, \mathbf{E}_{s-c}(U)) = u(c) + \delta_t \mathbf{E}_{s-c}(U).$$

*Then again, we are able to verify TCE exist. As* $\delta_{T+1} = 0$ *we observe that the problem resembles a finite-bias discounting model discussed in section 4. Also, for this reason any sum of continuation utilities is finite and we do not need to verify bounds in condition (iii) of Theorem 2.*

Finally, we go back to the quasi-hyperbolic discounting model but now allow for a present bias discount factor to be investment dependent, to account for magnitude effects in discounting e.g. (see Epstein and Hynes (1983) or Noor (2009) for a motivation and Jaśkiewicz et al. (2014) for recent developments in optimal growth models).

**Example 8** (Magnitude effects)**.** *Suppose the present bias discount factor* $\beta$ *is a function of investment, i.e.* $\beta : S \to [0, 1]$ *that is continuous and increasing. Then the aggregator takes the form:*

$$W_1(c, \beta(s-c)\mathbf{E}_{s-c}(U_2)) = \max_{c \in [0,s]} (u(c) + \beta(s-c)\delta\mathbf{E}_{s-c}(U_2))$$

*where:*

$$U_2(s) = W_2(g(s), \mathbf{E}_{s-g(s)}(U_2)) = u(g(s)) + \delta\mathbf{E}_{s-g(s)}(U_2).$$

*Clearly both* $W_1$ *and* $W_2$ *are* $\delta$*-contractions. The appropriate continuation utility bounds can be easily constructed as on page 26.*

# 6 Concluding Remarks

This paper proposes a collection of functional equation methods for proving the existence of (pure strategy) TCE in a general class of dynamic models with "behavioral" discounting with recursive payoffs and bounded or unbounded state space. Characterizing TCE policies in such family of models, aside from the characterization we obtain in this paper, is an open question. In particular, it would be interesting to examine theoretically if one could obtain the dissaving/savings characterizations of TCE that Cao and Werning (2018) construct for the quasi-hyperbolic model for more general cases of present bias and generalized behavioral discounting. Finally, we think our general existence methods developed in section 3 can be extended to show the existence of TCE in more general models of altruism with recursive payoffs as recently axiomatized by Galperti and Strulovici (2017). We leave these questions for further research.

# A Appendix. Omitted proofs

## A.1 Quasi-hyperbolic discounting

We now state and prove a number of preliminary results.

**Lemma 1.** *Let $h \in \mathcal{H}$ and let $f \in \mathcal{U}$. Then, any selection of*

$$s \mapsto B(f,h)(s) := arg \max_{i\in[0,s]} V\left(s-i, s-h(s), \mathbf{E}_i(f(h))\right)$$

*is increasing in $s$.*

*Proof.* Suppose that it is not the case: there are $s_1 > s_2$ and $i_1 < i_2$ such that $i_1 \in B(f,h)(s_1)$ and $i_2 \in B(f,h)(s_2)$. Then

$$0 \le V(s_2 - i_2, s_2 - h(s_2), \mathbf{E}_{i_2} f(h)) - V(s_2 - i_2 - (i_2 - i_1), s_2 - h(s_2), \mathbf{E}_{i_1} f(h)).$$

But then from Assumption 1 (ii) we have

$$V(s_1 - i_2, s_1 - h(s_1), \mathbf{E}_{i_2} f(h)) - V(s_1 - i_2 - (i_2 - i_1), s_1 - h(s_1), \mathbf{E}_{i_1} f(h)) > 0$$

which contradicts $i_1 \in B(f,h)(s_1)$. □

Now we examine the structure of the space $(\mathcal{V}, ||\cdot||)$, and its subset $\mathcal{U} \subset \mathcal{V}$.

**Lemma 2.** *$(\mathcal{V}, ||\cdot||)$ is a Banach space and $\mathcal{U} \subset \mathcal{V}$ is its closed set.*

*Proof.* Let $(\mathcal{V}_k, ||\cdot||_k)$ be the set of functions from $\mathcal{V}$ restricted to $S_k \times \mathcal{H}$. Clearly it is a subspace of Banach space of bounded functions on $S_k \times \mathcal{H}$, hence we only need to show $\mathcal{V}_k$ is closed. The convergence in norm $||\cdot||_k$ is equivalent to the uniform convergence on $S_k \times \mathcal{H}$. Suppose $f_n \rightrightarrows f$ as $n \to \infty$ in $||\cdot||_k$ and any of $f_n \in \mathcal{V}_k$. We show $f \in \mathcal{V}_k$. Obviously $f$ is bounded on $S_k \times \mathcal{H}$. We check further desired properties.

- *We show $f$ is right continuous on $s$ for any fixed $h$.*

  Let $\epsilon > 0$ be given. Let $s_n \downarrow s^0$ and let $N$ be such that $||f_N - f||_k < \frac{\epsilon}{2}$. We have

$$\begin{aligned} |f(s_n,h) - f(s^0,h)| &\leq |f(s_n,h) - f_N(s_n,h)| + |f_N(s_n,h) - f_N(s^0,h)| \\ &+ |f_N(s^0,h) - f(s^0,h)| \\ &\leq 2||f - f_N||_k + |f_N(s_n,h) - f_N(s^0,h)|. \end{aligned}$$

  Since $f_N$ is right continuous at $s^0$, hence taking a limit with $n \to \infty$ we have $\limsup_{n\to\infty} |f(s_n,h) - f(s^0,h)| < \epsilon$. Since $\epsilon$ is arbitrary, hence $f(s_n,h) \to f(s^0,h)$. Hence $f(\cdot,h)$ is right continuous.

- *We show for any $h \in \mathcal{H}$ there is a countable $\tilde{S} \subset S$ such that $f$ is continuous at any $(s,h) \in S \times \mathcal{H}$, such that $s \notin \tilde{S}$.* For any $h \in \mathcal{H}$, and any $n \in \mathbb{N}$, let $\tilde{S}^n \subset S$ be a countable set such that $f_n$ is continuous at any $(s,h)$ with $s \notin \tilde{S}^n$. Let $\tilde{S} := \bigcup_{n=1}^{\infty} \tilde{S}^n$. Observe $\tilde{S}$ is countable and any of $f_n$ is continuous at $(s,h)$ whenever $s \notin \tilde{S}$. Since $f$ is the uniform limit of $f_n$ on the set $S_k \times \mathcal{H}$, hence $f$ is continuous at $(s,h)$.

Consequently $f \in \mathcal{V}_k$ and $(\mathcal{V}_k, ||\cdot||_k)$ is a Banach space. Pick any $f^k \in \mathcal{V}_k$ such that $f^{k+1}(s,h) = f^k(s,h)$ for any $(s,h) \in S_k \times \mathcal{H}$. Define $f(s,h) = f^k(s,h)$ whenever $s \in S_k$. Observe that $f(\cdot,h)$ is right continuous. Moreover, for any $h \in \mathcal{H}$, $f$ may be discontinuous at $(s,h) \in S \times \mathcal{H}$, where $s$ is chosen from at most countable set. Indeed, since $f^k \in \mathcal{V}_k$, then $f^k$ is discontinuous only in a countable set $\tilde{S}^k \subset S_k$. We can take $\tilde{S} = \bigcup_{k=1}^{\infty} \tilde{S}^k$ which is countable to conclude that $f$ is continuous at $(s,h)$ for $s \notin \tilde{S}$. Hence $f \in \mathcal{V}$. By Lemma 1 in Matkowski and Nowak (2011), we conclude $(\mathcal{V}, ||\cdot||)$ is a Banach space. We shall see, $\mathcal{U}$ is a complete metric space with the metric induced by $||\cdot||$ since it is a closed subset of $\mathcal{V}$. Assume $f_n \to f$ as $n \to \infty$ in $\mathcal{V}$, and any of $f_n$ belong to $\mathcal{U}$. We only show $f$ is upper semicontinuous. Let $(s_n,h_n) \to (s^0,h^0)$ and suppose $s_n \in S_k$ for any natural $k$. As before

$\epsilon > 0$ is given and $N$ is such that $||f - f_N||_k < \frac{\epsilon}{2}$, Hence

$$f(s^0, h^0) - f(s_n, h_n) =$$
$$f(s^0, h^0) - f_N(s^0, h^0) + f_N(s^0, h^0) - f_N(s_n, h_n) + f_N(s_n, h_n) - f(s_n, h_n) \geq$$
$$-\epsilon + f_N(s^0, h^0) - f_N(s_n, h_n).$$

Since $f_N$ is upper semicontinuous

$$\liminf_{n\to\infty}(f(s^0, h^0) - f(s_n, h_n)) \geq -\epsilon.$$

Since $\epsilon > 0$ is arbitrary, hence $f$ is upper semicontinuous. □

**Lemma 3.** *$T$ maps $\mathcal{U}$ into itself.*

*Proof.* Let $f \in \mathcal{U}$. Let $(s, h) \in S_k \times \mathcal{H}$. Then by definition of $\mathcal{U}$ we have $\vartheta \leq f(s, h) \leq \eta_k$. Hence if $i \leq s$ then by Assumption 2 we conclude $f(s', h) \leq \eta_{k+1}$ for $q(\cdot|i)$-a.a. $s' \in S$. Therefore, by definition of COP

$$\vartheta \leq \mathbf{E}_i f(h) \leq \eta_{k+1}.$$

By Assumption 1 we have

$$\vartheta \leq V(s - i, h(s), \mathbf{E}_i f(h)) \leq V(\xi_k, 0, \eta_{k+1}) \leq \eta_k.$$

Hence $\vartheta \leq T(f)(s, h) \leq \eta_k$ for $(s, h) \in S_k \times \mathcal{H}$. We are going to show $T(f)$ it is upper semicontinuous. Let $(s_n, h_n) \to (s^0, h^0)$ in the corresponding topology. Pick

$$i_n \in \arg\max_{i\in[0,s_n]} V(s_n - i, s_n - h_n(s_n), \mathbf{E}_i f(h_n)) \tag{19}$$

and without loss of generality suppose $i_n \to i^0$. By Assumption 2, $q(\cdot|i_n) \to q(\cdot|i^0)$ weakly. Since $f$ is upper semicontinuous, the sequence $f(h_n)$ satisfies

$$\overline{\lim} f(h_n)(s') = \sup\left\{\limsup_{n\to\infty} f(s_n, h_n) : s_n \to s'\right\} \leq f(s', h^0) = f(h^0).$$

By Assumption 3 we have then

$$\limsup_{n\to\infty} \mathbf{E}_{i_n} f(h_n) \leq \mathbf{E}_{i^0}(\overline{\lim} f(h_n)) \leq \mathbf{E}_{i^0} f(h^0). \tag{20}$$

Observe

$$\liminf_{n\to\infty}(s_n - h_n(s_n)) \geq s^0 - \limsup_{n\to\infty} h_n(s_n) \geq s^0 - h^0(s^0), \tag{21}$$

Combining (20), (21) and Assumption 1 we have

$$\limsup_{n\to\infty} V(s_n - i_n, s_n - h_n(s_n), \mathbf{E}_{i_n} f(h_n)) \leq T(f)(s^0, h^0). \tag{22}$$

Hence $T(f)$ is upper semicontinuous. We show that for any $h \in \mathcal{H}$, $T(f)$ is continuous at any $(s, h) \in \mathcal{E}$ for all but countably many $s \in S$. We construct a countable subset of $S$ such that a continuity of $f(\cdot, h)$ fails. We can take

$$\tilde{S} := \{s \in S : q(\{s' \in S : f \text{ is continuous at } (s', h)\}|s) < 1\} \cup \left(S \setminus \mathcal{E}^{h_0}\right).$$

and clearly $\tilde{S}$ is countable. Now assume $(s_n, h_n) \to (s^0, h^0)$ and $s^0 \notin \tilde{S}$. Pick $i_n$ as in (19) and assume $i_n \to i^0$. Pick arbitrary $i \in [0, s^0) \setminus \tilde{S}$. Since $s^0 \in \mathcal{E}^{h^0}$, $h_n(s_n) \to h^0(s^0)$. By definition of $\tilde{S}$ it follows that $f(s', h_n) \to f(s', h^0)$ for $q(\cdot|i)$ almost all $s' \in S$. Then the sequence defined as $f_n(s') := f(s', h_n)$ tends to $f(s', h^0)$ continuously $q(\cdot|i)$ almost everywhere. Hence by Assumption 3 and noting the continuity of $V$ we have

$$\lim_{n\to\infty} V(s_n - i, s_n - h_n(s_n), \mathbf{E}_i f(h_n)) = V(s^0 - i, s^0 - h^0(s^0), \mathbf{E}_i f(h^0)). \tag{23}$$

Again by Assumption 3, upper semicontinuity of $f$ and consequently (20), we have then

$$\begin{aligned} V(s^0 - i^0, s^0 - h(s^0), \mathbf{E}_{i^0} f(h)) &\geq \limsup_{n\to\infty} V(s_n - i_n, s_n - h_n(s_n), \mathbf{E}_{i_n} f(h_n)) \\ &\geq \liminf_{n\to\infty} V(s_n - i, s_n - h_n(s_n), \mathbf{E}_i f(h_n)) \\ &= V(s^0 - i, s^0 - h^0(s^0), \mathbf{E}_i f(h^0)). \end{aligned} \tag{24}$$

By Assumption 3, the right hand in (24) side is right continuous. Since $S \setminus \tilde{S}$ is dense in $S$, hence this equality holds for any $i \in [0, s^0)$. Indeed, we can take $\tilde{i}^m \downarrow i$ as $m \to \infty$ such that $\tilde{i}^m \notin \tilde{S}$, substitute $i$ by $\tilde{i}^m$ above, and take a limit $m \to \infty$. To finish this proof, we only prove this inequality at $i = s^0$. Now take $\tilde{i}^m \uparrow s^0$ as $m \to \infty$ and suppose any $\tilde{i}^m \notin \tilde{S}$. Since $f(h)(\cdot)$ is continuous at $s'$ for $q(\cdot|s^0)$-almost all $s' \in S$, hence the constant sequence $f(s, h)$ converges to itself $q(\cdot|i)$ continuously in $s \in S$. As a result, $\mathbf{E}_{\tilde{i}^m} f(h^0) \to \mathbf{E}_{s^0} f(h^0)$ as $m \to \infty$. Consequently this inequality holds for $i = s^0$ as well. Combining it with (24) we have

$$T(f)(s^0,h^0) = V(s^0 - i^0, s^0 - h^0(s^0), \mathbf{E}_{i^0} f(h^0)) = \lim_{n\to\infty} T(f)(s_n,h_n).$$

Finally we show $T(f)(s,h)$ is continuous from the right in $s$. Now assume $s_n \downarrow s^0$ as $n \to \infty$ and pick $i_n$ as in (19) and $i^0$ as the limit of $i_n$. Then $h(s_n) \to h(s^0)$ as $n \to \infty$. By Lemma 1 $i_n \downarrow i^0$, and hence by Assumption 3, $\mathbf{E}_{i_n}(f(h)) \to \mathbf{E}_{i^0}(f(h))$ as $n \to \infty$. Let $i \in [0, s^0]$ be arbitrary. By Assumption 1 concerning the continuity of $V$ we have

$$\lim_{n\to\infty} T(f)(s_n,h) = \lim_{n\to\infty} V(s_n - i_n, s_n - h(s_n), \mathbf{E}_{i_n} f(h)) \geq V(s^0 - i, s^0 - h(s^0), \mathbf{E}_i f(h)). \tag{25}$$

Taking the supremum in (25) over $i \in [0, s^0]$ we have

$$\lim_{n\to\infty} T(f)(s_n,h) = T(f)(s^0,h).$$

This assures the right continuity of $T(f)(s,h)$ at $s^0 \in S$. □

**Lemma 4.** *Let $k \in \mathbb{N}$ and $i \in S_k$. Let $f, g$ be Borel and bounded on $S_{k+1}$. Then:*

$$|\mathbf{E}_i(f) - \mathbf{E}_i(g)| \leq ||f-g||_{k+1}. \tag{26}$$

*Proof.* Indeed, by Assumption 2, $q(S_{k+1}|i) = 1$ and hence for $q(\cdot|i)-$ all $s \in S$ it holds $f(s) \leq g(s) + ||f-g||_{k+1}$. As a result, under assumption Assumption 3, using monotonicity and the constant subadditivity of $\mathbf{E}_i$ we have

$$\mathbf{E}_i(f) \leq \mathbf{E}_i\left(g + ||f-g||_{k+1}\right) \leq \mathbf{E}_i(g) + ||f-g||_{k+1}.$$

Substituting $f$ by $g$ and vice-versa we have (26). □

**Lemma 5.** *$T$ is a contraction mapping on $\mathcal{U}$, and therefore has a unique fixed point in $\mathcal{U}$ .*

*Proof.* Let $k \in \mathbb{N}$ and $s \in S_k$ and let $f, g \in \mathcal{U}$. Then $h(s) \in S_k$. Pick $\tilde{i}_1 \in B(g,h)(s)$, and $\tilde{i}_2 \in B(f,h)(s)$. Both belong to $S_k$. Then

$$\begin{aligned} T(f)(s,h) - T(g)(s,h) &\leq V(s - \tilde{i}_1, s - h(s), \mathbf{E}_{\tilde{i}_1(s)}(f)) - V(s - \tilde{i}_1, s - h(s), \mathbf{E}_{\tilde{i}_1}(g)) \\ &\leq \delta \left|\mathbf{E}_{\tilde{i}_1}(f) - \mathbf{E}_{\tilde{i}_1}(g)\right|, \end{aligned}$$

where the last inequality follows from Assumption 1. Furthermore, by above and Lemma

4 we have

$$T(f)(s,h) - T(g)(s,h) \leq ||f-g||_{k+1}. \tag{27}$$

On the other hand we have

$$\begin{aligned} T(f)(s,h) - T(g)(s,h) &\geq V(s-\tilde{i}_2, s-h(s), \mathbf{E}_{\tilde{i}_2(s)}(f)) - V(s-\tilde{i}_2, s-h(s), \mathbf{E}_{\tilde{i}_2}(g)) \\ &\geq -\delta \left|\mathbf{E}_{\tilde{i}_2}(f) - \mathbf{E}_{\tilde{i}_2}(g)\right|, \end{aligned}$$

where the last inequality follows from Assumption 1. Furthermore, by above and Lemma 4 we have

$$T(f)(s,h) - T(g)(s,h) \geq -||f-g||_{k+1} \tag{28}$$

Combining (27) and (28) we have

$$|T(f)(s,h) - T(g)(s,h)| \leq ||f-g||_{k+1}$$

Taking the supremum over $s \in S_k$ and $h \in \mathcal{H}$ we have

$$||T(f) - T(g)||_k \leq ||f-g||_{k+1}.$$

Hence $T$ is 1-local contraction. By Theorem 2 in Rincon-Zapatero and Rodriguez-Palmero (2009), $T$ is a contraction mapping on $\mathcal{U}$. By Lemma 2 and Banach Contraction Principle $T$ has a unique fixed point. □

**Lemma 6.** *Let $h \in \mathcal{H}$. If $bi(h)$ is continuous at $s$, then $BI(h)(s)$ is a singleton.*

*Proof.* Suppose that $bi(h)$ is continuous at $s$ and pick $y_0 \in BI(h)(s)$. By Lemma 1, noting that $BI(h) = B(f^*, h)$ we have $bi(h)(s-\delta) \leq y_0 \leq bi(h)(s+\delta)$. Since $bi(h)$ is continuous, hence $y_0 = bi(h)(s)$, and consequently $BI(h)$ is singleton. □

**Lemma 7.** *The operator $bi$ maps $\mathcal{H}$ into itself and is continuous.*

*Proof.* By Lemma 1 it follows that $bi(h)(\cdot)$ is increasing. We show it is right continuous. Let $s_n \downarrow s^0$. We show $i_n := bi(h)(s_n) \to bi(h)(s^0)$. By Lemma 1, $i_n \downarrow i^0$ for some $i^0 \in [0, s^0]$. Since $h$ is right continuous $h(s_n) \downarrow h(s^0)$ as $n \to \infty$. Put

$$\Pi(s,i) := V(s-i, s-h(s), \mathbf{E}_i(f^*(h))).$$

Since $h$ and $i \mapsto \mathbf{E}_i(f^*(h))$ are both right continuous, hence we have

$$\Pi(s^0, i^0) = \lim_{n\to\infty} \Pi(s_n, i_n) \geq \Pi(s^0, i)$$

for all $i \in [0, s^0]$. Hence $i^0 \in BI(h)(s^0)$. Take another $i \in BI(h)(s^0)$. Again by Lemma 1 it follows $i \leq bi(h)(s_n) = i_n$ for any $n \in \mathbb{N}$. Taking a limit $n \to \infty$, $i \leq i^0$. Consequently $i_0 = bi(h)(s^0)$. We now show the continuity of $bi$ on $\mathcal{H}$. Suppose $h_n \to^w h^0$ in $\mathcal{H}$ such that $s^0$ is a continuity point of $bi(h^0)(\cdot)$. By Lemma 6 it follows that $BI(h^0)(s^0)$ is a singleton in this case. Hence we are going to show $i_n := bi(h_n)(s^0) \to i^0$ for some $i^0 \in BI(h^0)(s^0)$. Let

$$S^{f^*,h^0} := \left\{ s \in S : q\left(\{s \in S : f^* \text{ is continuous at } (s, h^0)\}\right) < 1 \right\} \cup (S \setminus \mathcal{E}^{h^0})$$

By Assumption 2 the complement of $S^{f^*,h^0}$ is at most countable. First, let us focus attention to $s^0 \notin S^{f^*,h^0}$. By definition of $S^{f^*,h^0}$ it follows that $f(\cdot, h_n)$ tends to $f(\cdot, h^0)$ continuously $q(\cdot|i)$ almost everywhere for any $i \notin S^{f^*,h^0}$. Hence by Assumption 3, for any such $i$ we have

$$\mathbf{E}_i f^*(h_n) \to \mathbf{E}_i f^*(h^0)$$

as $n \to \infty$. Moreover, $h_n(s^0) \to h(s^0)$ and since $i_n \to i^0$, then by Assumption 3

$$\lim_{n\to\infty} \mathbf{E}_{i_n} f^*(h_n)) = \mathbf{E}_{i^0} f^*(h^0)).$$

Hence for any $i \notin S^{f^*,h^0}$:

$$\begin{aligned}
V(s^0 - i^0, s^0 - h^0(s^0), \mathbf{E}_{i^0} f^*(h^0)) &= \lim_{n\to\infty} V(s^0 - i_n, s^0 - h_n(s^0), \mathbf{E}_{i_n} f^*(h_n)) \\
&\geq \lim_{n\to\infty} V(s^0 - i, s^0 - h_n(s^0), \mathbf{E}_i f^*(h_n))) \\
&= V(s^0 - i, s^0 - h^0(s^0), \mathbf{E}_i f^*(h^0))). \qquad (29)
\end{aligned}$$

Then combining the inequality above with (29) we have $i^0 \in BI(h^0)(s^0)$, consequently $i^0 = bi(h^0)(s^0)$. Hence we have proven, $bi(h_n)(s^0) \to bi(h)(s^0)$ as $n \to \infty$ whenever $s^0 \notin S^{f^*,h}$ and $s^0$ is a continuity point of $bi(h)$. To finish the proof, we need to show that this convergence is true inside $S^{f^*,h}$ as well. If $s^0 \in S^{f^*,h}$ is a continuity point of $bi(h^0)$, we may find $\delta_1 > 0$ and $\delta_2 > 0$ such that $bi(h^0)$ is both continuous at $s^0 - \delta_1$, $s^0 + \delta_2$ but $s^0 - \delta_1 \notin S^{f^*,h}$ in $s^0 + \delta_2 \notin S^{f^*,h}$ . By Assumption 2, $\delta_1$ and $\delta_2$ can be sufficiently small. Then, by the previous part of the proof

$$\begin{aligned}
bi(s^0 - \delta_1) &= \lim_{n\to\infty} bi(h_n)(s^0 - \delta_1) \leq \liminf_{n\to\infty} bi(h_n)(s^0) \\
&\leq \limsup_{n\to\infty} bi(h_n)(s^0) \leq \lim_{n\to\infty} bi(h_n)(s^0 + \delta_2) = bi(h^0)(s^0 + \delta_2).
\end{aligned}$$

Taking a limit $\delta_1 \to 0$ and $\delta_2 \to 0$ we have $bi(h_n)(s^0) \to bi(h^0)(s^0)$ as $n \to \infty$. $\square$

## A.2 Behavioral discounting

**Lemma 8.** *$(\mathcal{V}_0, ||\cdot||)$ is a Banach space, and $\mathcal{U}_0$ is a closed subset of $\mathcal{V}_0$.*

*Proof.* Consider $(\mathcal{V}_{0,k}, ||\cdot||_k)$, the space of restrictions of $f$ to $\mathbb{N} \times S_k \times \mathcal{H}$. Obviously it is a subspace of a Banach space of bounded functions on $\mathbb{N} \times S_k \times \mathcal{H}$. We show it is a Banach space by proving that $\mathcal{V}_{0,k}$ is a closed set. Let $f_n$ is a sequence of $\mathcal{V}_{0,k}$ and suppose $f_n \to f$ in the norm. It is easy to see $f(t,s,h)$ is bounded on $\mathbb{N} \times S_k \times \mathcal{H}$. Since $f_n \in \mathcal{V}_{0,k}$ hence for any $(t,h)$, there is a countable $\tilde{S}_n \subset S_k$, such that $f_n(t,\cdot,h)$ is continuous at $s$ for any $s \in \mathcal{S}_k$. Then $\mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}_k$ is countable and any of $f_n(t,\cdot,h)$ is continuous at $s \notin \mathcal{S}$. Since $f$ is a uniform limit of $f_n$, hence is continuous at any $s \notin \mathcal{S}$. Now consider the sequence $f^k \in \mathcal{V}_{0,k}$ such that for any $k \in \mathbb{N}$, $f^{k+1}(t,s,h) = f^k(t,s,h)$ for any $s \in S_k$. Then we can well define $f(t,s,h) := f^k(t,s,h)$ whenever $s \in S_k$. Obviously $f$ is bounded on any $\mathbb{N} \times S_k \times \mathcal{H}$. Since $f^k \in \mathcal{V}_{0,k}$, hence for any $t \in \mathbb{N}$ and $h \in \mathcal{H}$, there is a countable set $\mathcal{S}_0^k \subset S_k$ such that $f^k(t,\cdot,h)$ is continuous at any $s \notin \mathcal{S}_0^k$. Then $f$ is continuous at any $s \in \mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}_k$. Clearly $\mathcal{S}$ is countable. By Lemma 1 in Matkowski and Nowak (2011), we conclude $(\mathcal{V}_0, ||\cdot||$ is a Banach space. Obviously $\mathcal{U}_0$ is a closed subset of $\mathcal{V}_0$. $\square$

**Lemma 9.** *$\mathbb{T}$ is a contraction mapping on $\mathcal{U}_0^\infty$ and has a unique fixed point.*

*Proof.* We show $\mathbb{T}$ maps $\mathcal{U}_0$ into itself. Let $f \in \mathcal{U}_0$. Then for any $k \in \mathbb{N}$, $s' \in S_{k+1}$, $h \in \mathcal{H}$ and $t \in \mathbb{N}$ we have $\vartheta \le f(t+1,s',h) \le \eta_{k+1}$. By Assumption 2 and 3 for any $s \in S_k$ we have

$$\vartheta \le \mathbf{E}_{h(s)} f_{t+1}(h) \le \eta_{k+1}.$$

Hence

$$\vartheta \le W_t(s-h(s), \mathbf{E}_{h(s)} f_{t+1}(h)) \le W_t(\xi_k, \eta_{k+1}) \le \eta_k.$$

Now let $\mathcal{S} \subset S$ be a countable set such that $f(t,s,h)$ is continuous at any $s \notin \mathcal{S}$. Let $s \in \mathcal{E}^h$. Observe that $S \setminus \mathcal{E}^h$ is countable. Suppose $s_n$ is a sequence tending to $s$ as $n \to \infty$. Then $h(s_n) \to h(s)$ as $n \to \infty$. Furthermore, since the transition $q$ ranges over the nonatomic measures, it follows that $q(\mathcal{S}|h(s)) = 0$. Then for $q(\cdot|h(s))$-almost all $s' \in S$, constant of functions $f(t+1,s'_n,h) \to f(t+1,s',h)$ as $n \to \infty$. In other words, the constant sequence of functions $f_n(\cdot) = f(t+1,\cdot,h)$ tends to itself continuously at $s'$. By Assumptions 3 and 2, $\mathbf{E}_{h(s_n)} f_{t+1}(h) \to \mathbf{E}_{h(s)} f_{t+1}(h)$ as $n \to \infty$. Hence

$$W_t(s_n - h(s_n), \mathbf{E}_{h(s_n)} f_{t+1}(h)) \to W_t(s-h(s), \mathbf{E}_{h(s)} f_{t+1}(h)) \quad \text{as } n \to \infty.$$

Hence $\mathbb{T}(f)(t,\cdot,h)$ is continuous at any $s \in \mathcal{E}^h$. Since $S \setminus \mathcal{E}^h$ is countable, $\mathbb{T}(f) \in \mathcal{U}_0$. Now we show a local contraction property. Pick $f, g \in \mathcal{U}_0$. By Assumption 2, for any natural $k$ and $s \in S_k$ we have $q(S_{k+1}|h(s)) = 1$. Hence by Assumption 3, and consequently the non-expansive property of $\mathbf{E}_{h(s)}(\cdot)$ in Lemma 4 we have

$$\left|\mathbf{E}_{h(s)} f_{t+1}(h) - \mathbf{E}_{h(s)} g_{t+1}(h)\right| \leq ||f - g||_{k+1}. \tag{30}$$

Since $W_t$ has a Lipschitz constant with $\delta$, we have then

$$\left|W_t(s - h(s), \mathbf{E}_{h(s)} f_{t+1}(h)) - W_t(s - h(s), \mathbf{E}_{h(s)} g_{t+1}(h))\right| \leq \left|\mathbf{E}_{h(s)} f_{t+1}(h) - \mathbf{E}_{h(s)} g_{t+1}(h)\right|.$$

Taking a limit above with $(t, s, h) \in \mathbb{N} \times S_k \times \mathcal{H}$, and combining with (30) we have

$$||\mathbb{T}(f) - \mathbb{T}(f)||_k \leq \delta ||f - g||_{k+1}.$$

Since $k$ is arbitrary, hence $\mathbb{T}$ is a 1-local contraction. By Theorem 2 in Rincon-Zapatero and Rodriguez-Palmero (2009), $\mathbb{T}$ is a contraction mapping on $\mathcal{U}_0^\infty$. By Lemma 2 and Banach Contraction Principle $\mathbb{T}$ has a unique fixed point. □

**Lemma 10.** *Let $h \in \mathcal{H}$. Then, $BI(h)$ is nonempty valued correspondence with the greatest and the least selection. Moreover, any selection of $BI(h)$ is increasing in $s$.*

*Proof.* First we show $BI(h)(s)$ is indeed nonempty valued correspondence with the greatest and the least element. Let $f^*$ be a unique fixed point of $\mathbb{T}$ and $f_2^*$ be the coordinate needed to define $BI$ (i.e. $f_2^*(s,h) := f^*(2,s,h)$. For any $h \in \mathcal{H}$ let $S^{*,h}$ be a countable subset of $S$ such that $f_2^*$ is continuous at any $(s,h) \in S \times \mathcal{H}$ such that $s \notin S^*$. Since $q(\cdot|i)$ is nonatomic for any $i \in S$, hence $q(S^{*,h}|i) = 1$. Hence the constant sequence $f_n(s) = f_2^*(s,h)$ tends to itself continuously. Therefore, by Assumption 3 it follows that $\mathbf{E}_i(f_2^*(h))$ is continuous in $i \in S$. Hence

$$i \in S \mapsto W_1\left(s - i, \mathbf{E}_i(f_2^*(h))\right)$$

is continuous. Hence $BI(h)(s) \neq \emptyset$ and has the greatest and the least element. By Lemma 1 it follows that any selection of $BI(h)$ is increasing. □

By Lemma 10 we can repeat the same argument as in Lemma 6 to obtain:

**Lemma 11.** *Let $h \in \mathcal{H}$ and suppose $h$ is continuous at $s$. Then, if $s \mapsto bi(h)(s)$ is continuous at $s$ then $BI(h)(s)$ is a singleton.*

Combining Lemmas 10 and 11 we have the following:

**Lemma 12.** *The operator bi maps $\mathcal{H}$ into itself and it is a continuous operator.*

Its proof in analogous to the proof of Lemma 7. It is enough to substitute $V$ with $W_1$ and recall from Lemma 10 that $i \in S \mapsto W_1(s-i, \mathbf{E}_i(f_2^*(h)))$ is continuous.

## A.3 Approximating general behavioral discounting models

**Lemma 13.** $\prod\limits_{k=1}^{\infty} \beta_k$ *exists and is nonzero if and only if* $\lim\limits_{t\to\infty} \prod\limits_{k=t}^{\infty} \beta_k = 1$.

*Proof.* Define $r := \prod\limits_{k=1}^{\infty} \beta_k$, and suppose $r > 0$. Then $-ln(r) = \sum\limits_{k=1}^{\infty} -\ln(\beta_k)$. Since $-\ln(\beta_k) > 0$, hence the series above are convergent and $\lim\limits_{t\to\infty} \sum\limits_{k=t}^{\infty} -\ln(\beta_k) = 0$. Moreover,

$$\lim_{t\to\infty} \sum_{k=t}^{\infty} -\ln(\beta_k) = -\lim_{t\to\infty} \ln\left(\prod_{k=t}^{\infty} \beta_k\right) = -\ln\left(\lim_{t\to\infty} \prod_{k=t}^{\infty} \beta_k\right). \tag{31}$$

Combining (A.3) with (31) we have the thesis. Now let $r = 0$. Then the right hand side in (A.3) yields $\infty$. Furthermore, by (31) we have $\lim\limits_{t\to\infty} \prod\limits_{k=t}^{\infty} \beta_k = 0$ □

For any $T$, let $X_t^T$ be a $S-$valued Markov chain with a deterministic initial value $x$ and a transition probability $s \in S \mapsto q(\cdot|h^T(s))$ where $h^T \in \mathcal{H}$. We denote $X_t^*$ as a $S-$ Markov chain whose initial value $x$ and a transition probability $s \in S \mapsto q(\cdot|h(s))$ where $h \in \mathcal{H}$. Let $Q_{s_0}^T$ be the joint distribution of $(X_t^T)_{t=1}^{\infty}$ and let $Q_{s_0}$ be the joint distribution of $(X_t^*)_{t=1}^{\infty}$ with fixed initial distribution $s_0$.

**Lemma 14.** *For any $s_0 \in S$, $Q_{s_0}^T \Rightarrow Q_{s_0}^*$. As a result, there exists a probability space $(\Omega, \mathcal{F}, P)$ and $S-$ valued sequences $(\xi_t^T(\omega))^{\infty}$ and $(\xi_t^*(\omega))_{t=1}^{\infty}$ whose join distribution are $Q^T$ and respectively $Q^*$ such that $\lim\limits_{T\to\infty} \xi_t^T(\omega) = \xi_t^*(\omega)$ for any $\omega \in \Omega$ and $t \in \mathbb{N}$.*

*Proof.* We show that for any integer $k$, $s \in S$ and any bounded and continuous $f^k : S^k \mapsto \mathbb{R}$ it holds

$$\lim_{T\to\infty} \int_{S^{\infty}} f(s_1, s_2, \ldots, s_k) Q_s^T(ds^{\infty}) = \int_{S^{\infty}} f(s_1, s_2, \ldots, s_k) Q_s^*(ds^{\infty}) \tag{32}$$

We prove this thesis by induction with respect to $k$. For $k = 1$ it is follows directly by Assumption 2. Suppose that (32) holds for some $k$. Put

$$\tilde{f}(s_1, s_2, \ldots, s_k) := \int_{S^\infty} f(s_1, s_2, \ldots, s_k, s_{k+1}) q(ds_{k+1}|h^T(s_k)).$$

Observe that by nonatomicity of $q$ any of $g^T$ is continuous for $Q_s^T$ and $Q_s$ -a.a. $s^\infty \in S^\infty$. As a result, by Lemma 15 letting $\phi$ be an identity mapping, $\tilde{f}$ is a continuous function on $S^k$ for $Q_s^T$ and $Q_s$- a.a. $s^\infty \in S^\infty$. Hence substituting $\tilde{f}$ by $f$ into (32) and applying Lemma 15.4 in Aliprantis and Border (2006) we obtain exactly (32) with $k + 1$. For the second part we apply again Skorohod's Representation Theorem (Theorem 6.7. in Billingsley (1999)). $\square$

## A.4 Auxiliary results

**Lemma 15.** *Suppose $\mu_n \to \mu$ weakly on $S$, and all $\mu_n$ are concentrated on a common bounded interval $S_0 \subset S$. Let $f_n$, be a sequence of Borel functions all commonly bounded on $S_0$. Furthermore, let $\phi$ be a strictly monotone and continuous real valued function whose domain is included in the range of any $f_n$. Then*

$$\limsup_{n\to\infty} \phi^{-1}\left(\int_S \phi(f_n(s'))\mu_n(ds')\right) \le \phi^{-1}\left(\int_S \phi(\limsup_{n\to\infty} f_n(s'))\mu(ds')\right). \tag{33}$$

*and*

$$\liminf_{n\to\infty} \phi^{-1}\left(\int_S \phi(f_n(s'))\mu_n(ds')\right) \ge \phi^{-1}\left(\int_S \phi(\liminf_{n\to\infty} f_n(s',h))\mu(ds')\right). \tag{34}$$

*Suppose that $\mu$ is concentrated on the set of the points in which $f_n \to f$ continuously. Then*

$$\lim_{n\to\infty} \phi^{-1}\left(\int_S \phi(f_n(s'))\mu_n(ds')\right) = \phi^{-1}\left(\int_S \phi(f(s'))\mu(ds')\right). \tag{35}$$

*Proof.* For instance we prove (33) and (35) and we prove this fact for decreasing $\phi$. The proof of (34) is similar, and the same in case of increasing $\phi$. Since $\mu_n \to \mu$, hence by the Skorohod's Representation Theorem (Theorem 6.7. in Billingsley (1999)) we find a probability space $(\Omega, \mathcal{F}, P)$, a sequence of random variables $X_n$ and a random variable $X$ on $\Omega$ whose distribution is $\mu_n$ and respectively $\mu$ such that $X_n(\omega) \to X(\omega)$ pointwise in

$\omega \in \Omega$. We have then

$$\begin{aligned}
\liminf_{n\to\infty} \int_S \phi(f_n(s))\mu_n(ds) &= \liminf_{n\to\infty} \int_S \phi(f_n(X_n(\omega)))P(d\omega) \\
&\geq \int_S \liminf_{n\to\infty} \phi(f_n(X_n(\omega)))P(d\omega) \qquad (36) \\
&\geq \int_S \phi(\limsup_{n\to\infty} f_n(X_n(\omega)))P(d\omega) \qquad (37) \\
&\geq \int_S \phi(\overline{\lim} f_n(X(\omega)))P(d\omega) \qquad (38) \\
&= \int_S \phi(\overline{\lim} f_n(s))\mu(ds),
\end{aligned}$$

where the inequality in (36) follows from the standard Fatous Lemma, the inequality in (37) occurs since $\phi$ is decreasing and continuous, and the inequality (38) follows from definition of $\overline{\lim} f_n$. Noting that $\phi^{-1}$ is continuous and decreasing we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \phi^{-1}\left(\int_S \phi(f_n(s))\mu_n(ds)\right) &\leq \phi^{-1}\left(\liminf_{n\to\infty} \int_S \phi(f_n(s))\mu_n(ds)\right) \\
&\leq \phi^{-1}\left(\int_S \phi(\overline{\lim} f_n(s))\mu(ds)\right).
\end{aligned}$$

Now suppose that $f_n \to f$ continuously $\mu$-almost everywhere. Then

$$f_n(X_n(\omega)) \to f(X(\omega)) \qquad (39)$$

for any $\omega \in \Omega$. Then we repeat the above lines substituting inequalities $\geq$ with $=$, and substituting $\limsup$ and $\liminf$ with $\lim$. Then the counterpart of (36) follows from the standard Dominated Convergence Theorem and the counterpart of (37) and (38) follow from the continuity of $\phi$ and respectively from (39). As a result we have (35). □

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