MICROECONOMICS 1A & 1B

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Mathematical Appendix for Economics *

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Masters M1 MAEF, M1 IMMAEF & QEM1 – DU MMEF

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1 Notations

- $\mathbb{R}^n := \{x = (x^1, \ldots, x^h, \ldots, x^n) : x^h \in \mathbb{R},\ \forall\, h = 1, ..., n\}$
- $x \in \mathbb{R}^n$ and $\overline{x} \in \mathbb{R}^n$,

$$x \geq \overline{x} \Longleftrightarrow x^h \geq \overline{x}^h,\ \forall\, h = 1, ..., n$$

$$x > \overline{x} \Longleftrightarrow x \geq \overline{x} \text{ and } x \neq \overline{x}$$

$$x \gg \overline{x} \Longleftrightarrow x^h > \overline{x}^h,\ \forall\, h = 1, ..., n$$

- $x \in \mathbb{R}^n$ and $\overline{x} \in \mathbb{R}^n$, $x \cdot \overline{x}$ denotes the scalar product of x and $\overline{x}$.
- A is a matrix with m rows and n columns and B is a matrix with n rows and l columns, AB denotes the matrix product of A and B.
- H is a $n \times n$ matrix, $\mathrm{tr}(H)$ denotes the trace of H and $\det(H)$ denotes the determinant of H.
- $x \in \mathbb{R}^n$ is treated as a row matrix.
- x^T denotes the transpose of $x \in \mathbb{R}^n$, x^T is treated as a column matrix.
- f is a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$,

  f is **weakly increasing (or non-decreasing)** on X if for all x and $\overline{x}$ in X,

$$x \leq \overline{x} \Longrightarrow f(x) \leq f(\overline{x})$$

  f is **increasing** on X if for all x and $\overline{x}$ in X,

$$x \ll \overline{x} \Longrightarrow f(x) < f(\overline{x})$$

  f is **strictly increasing** on X if for all x and $\overline{x}$ in X,

$$x < \overline{x} \Longrightarrow f(x) < f(\overline{x})$$

$$f \text{ strictly increasing on } X \Longrightarrow f \text{ increasing on } X$$

$$f \text{ strictly increasing on } X \Longrightarrow f \text{ weakly increasing (or non-decreasing) on } X$$

- $X \subseteq \mathbb{R}^n$ is an open set, f is a function from X to $\mathbb{R}$ and $x \in X$,

$$\nabla f(x) := \left(\frac{\partial f}{\partial x^1}(x), \ldots, \frac{\partial f}{\partial x^h}(x), \ldots, \frac{\partial f}{\partial x^n}(x) \right)$$

denotes the **gradient** of f at x, and

$$\mathrm{H}f(x) := \begin{bmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1}(x) & \ldots & \frac{\partial^2 f}{\partial x^h \partial x^1}(x) & \ldots & \frac{\partial^2 f}{\partial x^n \partial x^1}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x^1 \partial x^h}(x) & \ldots & \frac{\partial^2 f}{\partial x^h \partial x^h}(x) & \ldots & \frac{\partial^2 f}{\partial x^n \partial x^h}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x^1 \partial x^n}(x) & \ldots & \frac{\partial^2 f}{\partial x^h \partial x^n}(x) & \ldots & \frac{\partial^2 f}{\partial x^n \partial x^n}(x) \end{bmatrix}_{n \times n}$$

denotes the **Hessian matrix** of f at x.

- $X \subseteq \mathbb{R}^n$ is an open set, $g := (g_1, \ldots, g_j, \ldots, g_m)$ is a mapping from X to $\mathbb{R}^m$ and $x \in X$,

$$\mathrm{J}g(x) := \begin{bmatrix} \frac{\partial g_1}{\partial x^1}(x) & \ldots & \frac{\partial g_1}{\partial x^h}(x) & \ldots & \frac{\partial g_1}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_j}{\partial x^1}(x) & \ldots & \frac{\partial g_j}{\partial x^h}(x) & \ldots & \frac{\partial g_j}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x^1}(x) & \ldots & \frac{\partial g_m}{\partial x^h}(x) & \ldots & \frac{\partial g_m}{\partial x^n}(x) \end{bmatrix}_{m \times n} = \begin{bmatrix} \nabla g_1(x) \\ \vdots \\ \nabla g_j(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix}_{m \times n}$$

denotes the **Jacobian matrix** of g at x.

1.1 Continuity

f is a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$.

Definition 1 (Continuous function) *f is continuous at $\overline{x} \in X$ if*

$$\lim_{x \to \overline{x}} f(x) = f(\overline{x})$$

f is continuous on X if f is continuous at every point $\overline{x} \in X$.

Exercise 2

1. f is continuous at $\overline{x} \in X$ if and only if for every open ball J of center $f(\overline{x})$ there exists an open ball B of center $\overline{x}$ such that $f(B \cap X) \subseteq J$.

2. f is continuous at $\overline{x} \in X$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - \overline{x}\| < \delta$ and $x \in X \Longrightarrow |f(x) - f(\overline{x})| < \varepsilon$.

Proposition 3 (Sequentially continuous function) *f is continuous at $\overline{x} \in X$ if and only if f is sequentially continuous at $\overline{x}$, that is, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \to \overline{x}$, we have that*

$$f(x_n) \to f(\overline{x})$$

1.2 Differentiability

$X \subseteq \mathbb{R}^n$ is an **open** set, f is a function from X to $\mathbb{R}$.

Definition 4 (Differentiable function) *f is differentiable at $\overline{x} \in X$ if*

1. *all the partial derivatives of f at $\overline{x}$ exist,*

2. *there exists a function $E_{\overline{x}}$ defined in some open ball $B(0, \varepsilon) \subseteq \mathbb{R}^n$ such that for every $u \in B(0, \varepsilon)$,*

$$f(\overline{x} + u) = f(\overline{x}) + \nabla f(\overline{x}) \cdot u + \|u\| E_{\overline{x}}(u)$$

$$\text{where } \lim_{u \to 0} E_{\overline{x}}(u) = 0$$

f is differentiable on X if f is differentiable at every point $\overline{x} \in X$.

Exercise 5 If f is differentiable at $\overline{x}$, then f is continuous at $\overline{x}$.

Definition 6 (Directional derivative) *Let $v \in \mathbb{R}^n$, $v \neq 0$. The directional derivative $D_v f(\overline{x})$ of f at $\overline{x} \in X$ in the direction v is defined as*

$$\lim_{t \to 0^+} \frac{f(\overline{x} + tv) - f(\overline{x})}{t}$$

if this limit exists and it is finite.

Proposition 7 (Differentiable function/Directional derivative) *If f is differentiable at $\overline{x} \in X$, then for every $v \in \mathbb{R}^n$ with $v \neq 0$,*

$$D_v f(\overline{x}) = \nabla f(\overline{x}) \cdot v$$

1.3 Compactness

X is a subset of $\mathbb{R}^n$.

Proposition 8 (Compact set/Subsequences) *X is compact if and only if for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of the sequence $(x_n)_{n \in \mathbb{N}}$ such that $(x_{n_k})_{k \in \mathbb{N}}$ converges to some point $\overline{x} \in X$.*[1]

Proposition 9 (Compact set) *X is compact if and only if it is closed and bounded.*

Definition 10 (Closed set) *X is closed if its complement $\mathcal{C}(X) := \mathbb{R}^n \setminus X$ is open.*

Proposition 11 (Sequentially closed) *X is closed if and only if it is sequentially closed, that is, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \to \overline{x}$, we have*

$$\overline{x} \in X$$

Definition 12 (Bounded set) *X is bounded if it is included in some ball, that is, there exists $\varepsilon > 0$ such that for all $x \in X$, $\|x\| < \varepsilon$.*

[1]Let $(x_n)_{n \in \mathbb{N}}$ be a sequence and $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers. The composed sequence $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of the sequence $(x_n)_{n \in \mathbb{N}}$.

2 Extreme Value Theorem

Theorem 13 (Extreme Value Theorem/Weierstrass Theorem) *Let f be a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$. If X is a non-empty compact set and f is continuous on X, then*

- $\exists\, x^* \in X$ *such that* $f(x^*) \geq f(x)$ *for all* $x \in X$*, and*
- $\exists\, x^{**} \in X$ *such that* $f(x^{**}) \leq f(x)$ *for al* $x \in X$*.*

3 Karush–Kuhn–Tucker Conditions

In this section, we focus on necessary and sufficient conditions in terms of first–order conditions for solving a maximization problem with inequality constraints.

In this section, we assume that

- $C \subseteq \mathbb{R}^n$ is **convex and open**,
- the following functions f and g_j with $j = 1, ..., m$ are **differentiable** on C.

$$f : x \in C \subseteq \mathbb{R}^n \longrightarrow f(x) \in \mathbb{R} \text{ and}$$
$$g_j : x \in C \subseteq \mathbb{R}^n \longrightarrow g_j(x) \in \mathbb{R},\ \forall\ j = 1, ..., m$$

Maximization problem

$$\begin{array}{ll} \max & f(x) \\ x \in C & \\ \text{subject to} & g_j(x) \geq 0,\ \forall\ j = 1, ..., m \end{array} \tag{1}$$

where f is the *objective* function, and g_j with $j = 1, ..., m$ are the *constraint* functions.

The **Karush–Kuhn–Tucker conditions** associated with problem (1) are given below

$$\begin{cases} \nabla f(x) + \displaystyle\sum_{j=1}^{m} \lambda_j \nabla g_j(x) = 0 \\ \lambda_j \geq 0,\ \forall\ j = 1, ..., m \\ \lambda_j g_j(x) = 0,\ \forall\ j = 1, ..., m \\ g_j(x) \geq 0,\ \forall\ j = 1, ..., m \end{cases} \tag{2}$$

where for every $j = 1, ..., m$, $\lambda_j \in \mathbb{R}$ is called *Lagrange multiplier* associated with the inequality constraint g_j.

Definition 14 *Let $x^* \in C$, we say that the constraint j is **binding** at x^* if $g_j(x^*) = 0$. We denote*

1. $B(x^*)$ *the set of all binding constraints at* x^*, *that is*

$$B(x^*) := \{j = 1, ..., m : g_j(x^*) = 0\}$$

2. $m^* \leq m$ *the number of elements of* $B(x^*)$ *and*

3. $g^* := (g_j)_{j \in B(x^*)}$ *the following mapping*

$$g^* : x \in C \subseteq \mathbb{R}^n \longrightarrow g^*(x) = (g_j(x))_{j \in B(x^*)} \in \mathbb{R}^{m^*}$$

Theorem 15 (Karush–Kuhn–Tucker are necessary conditions) *Let* x^* *be a solution to problem (1). Assume that* ***one*** *of the following conditions is satisfied.*

1. *For all* $j = 1, ..., m$, g_j *is a* ***linear or affine*** *function.*

2. **Slater's Condition :**

   - *for all* $j = 1, ..., m$, g_j *is a* ***concave*** *function* ***or*** g_j *is a* ***quasi-concave*** *function with* $\nabla g_j(x) \neq 0$ *for all* $x \in C$, *and*
   - *there exists* $\overline{x} \in C$ *such that* $g_j(\overline{x}) > 0$ *for all* $j = 1, ..., m$.

3. **Rank Condition :** rank $\mathrm{J}g^*(x^*) = m^* \leq n$.

Then, there exists $\lambda^* = (\lambda_1^*, ..., \lambda_j^*, ..., \lambda_m^*) \in \mathbb{R}_+^m$ *such that* (x^*, λ^*) *satisfies the Karush–Kuhn–Tucker Conditions (2).*

Theorem 16 (Karush–Kuhn–Tucker are sufficient conditions) *Suppose that there exists* $\lambda^* = (\lambda_1^*, ..., \lambda_j^*, ..., \lambda_m^*) \in \mathbb{R}_+^m$ *such that* $(x^*, \lambda^*) \in C \times \mathbb{R}_+^m$ *satisfies the Karush–Kuhn–Tucker Conditions (2). Assume that*

1. f *is a* ***concave*** *function* ***or*** f *is a* ***quasi-concave*** *function with* $\nabla f(x) \neq 0$ *for all* $x \in C$, *and*

2. g_j *is a* ***quasi-concave*** *function for all* $j = 1, ..., m$.

Then, x^* *is a solution to problem (1).*

4 Concavity and quasi-concavity

In this section, we assume that C is a **convex** subset of $\mathbb{R}^n$ and f is a function from C to $\mathbb{R}$.

Concavity

Definition 17 (Concave function) *f is concave if for all $t \in [0,1]$ and for all x and $\bar{x}$ in C,*

$$f(tx + (1-t)\bar{x}) \geq tf(x) + (1-t)f(\bar{x})$$

Proposition 18 *f is concave **if and only if** the set*

$$\{(x, \alpha) \in C \times \mathbb{R} : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^{n+1}$. The set above is called hypograph of f.

Proposition 19 *C is **open** and f is **differentiable** on C. f is concave **if and only if** for all x and $\bar{x}$ in C,*

$$f(x) \leq f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x})$$

Proposition 20 *C is **open** and f is **twice continuously differentiable** on C. f is concave **if and only if** for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative semidefinite, that is, for all $x \in C$*

$$v\mathrm{H}f(x)v^T \leq 0, \ \forall\, v \in \mathbb{R}^n$$

Definition 21 (Strictly concave function) *f is strictly concave if for all $t \in]0,1[$ and for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(tx + (1-t)\bar{x}) > tf(x) + (1-t)f(\bar{x})$$

Proposition 22 *C is **open** and f is **differentiable** on C. f is strictly concave **if and only if** for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(x) < f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x})$$

Proposition 23 *C is **open** and f is **twice continuously differentiable** on C. **If** for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative definite, that is, for all $x \in C$*

$$v\mathrm{H}f(x)v^T < 0, \; \forall \, v \in \mathbb{R}^n, \; v \neq 0$$

***then** f is strictly concave.*

Quasi-concavity

Definition 24 (Quasi-concave function) *f is quasi-concave if and only if for all $\alpha \in \mathbb{R}$ the set*

$$\{x \in C : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^n$. The set above is called upper contour set of f at α.

Proposition 25 *f is quasi-concave **if and only if** for all $t \in [0, 1]$ and for all x and $\bar{x}$ in C,*

$$f(tx + (1 - t)\bar{x}) \geq \min\{f(x), f(\bar{x})\}$$

Proposition 26 *C is **open** and f is **differentiable** on C. f is quasi-concave **if and only if** for all x and $\bar{x}$ in C,*

$$f(x) \geq f(\bar{x}) \Longrightarrow \nabla f(\bar{x}) \cdot (x - \bar{x}) \geq 0$$

Proposition 27 *C is **open** and f is **differentiable** on C. **If** f is quasi-concave $\underline{\text{and } \nabla f(x) \neq 0 \text{ for all } x \in C}$, **then** for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(x) > f(\bar{x}) \Longrightarrow \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

Proposition 28 *C is **open** and f is **twice continuously differentiable** on C. **If** f is quasi-concave, **then** for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative semidefinite on $\mathrm{Ker}\nabla f(x)$, that is, for all $x \in C$*

$$v \in \mathbb{R}^n \text{ and } \nabla f(x) \cdot v = 0 \Longrightarrow v\mathrm{H}f(x)v^T \leq 0$$

Definition 29 (Strictly quasi-concave function) *f is strictly quasi-concave if and only if for all $t \in]0, 1[$ and for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(tx + (1 - t)\bar{x}) > \min\{f(x), f(\bar{x})\}$$

Proposition 30 *C is **open** and f is **differentiable** on C.*

1. ***If** for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(x) \geq f(\bar{x}) \Longrightarrow \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

   ***then** f is strictly quasi-concave.*

2. ***If** f is strictly quasi-concave $\underline{\text{and } \nabla f(x) \neq 0 \text{ for all } x \in C}$, **then** for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(x) \geq f(\bar{x}) \Longrightarrow \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

Proposition 31 *C is **open** and f is **twice continuously differentiable** on C. **If** for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative definite on* $\mathrm{Ker}\nabla f(x)$*, that is, for all $x \in C$*

$$v \in \mathbb{R}^n,\ v \neq 0 \text{ and } \nabla f(x) \cdot v = 0 \Longrightarrow v\mathrm{H}f(x)v^T < 0$$

***then** f is strictly quasi-concave.*

Remark 32 *We remark that*

$$\begin{array}{ccccc}
f \text{ linear or affine} & \Rightarrow & f \text{ concave} & \Leftarrow & f \text{ strictly concave} \\
 & & \Downarrow & & \Downarrow \\
 & & f \text{ quasi-concave} & \Leftarrow & f \text{ strictly quasi-concave}
\end{array}$$

We remind the definitions and some properties of negative definite/semidefinite matrices. Let H be a $n \times n$ **symmetric** matrix.

Definition 33

1. *H is negative semidefinite if $v\mathrm{H}v^T \leq 0$ for all $v \in \mathbb{R}^n$.*

2. *H is negative definite if $v\mathrm{H}v^T < 0$ for all $v \in \mathbb{R}^n$ with $v \neq 0$.*

Proposition 34

1. *H has n real eigenvalues. We denote $\lambda_1, ..., \lambda_n$ the eigenvalues of H.*

2. *H is negative semidefinite if and only $\lambda_i \leq 0$ for every $i = 1, ..., n$.*

3. *H is negative definite if and only $\lambda_i < 0$ for every $i = 1, ..., n$.*

Proposition 35

1. *If H is negative semidefinite, then* $\mathrm{tr}(H) \leq 0$ *and* $\det(H) \geq 0$ *if n is even,* $\det(H) \leq 0$ *if n is odd.*

2. *If H is negative definite, then* $\mathrm{tr}(H) < 0$ *and* $\det(H) > 0$ *if n is even,* $\det(H) < 0$ *if n is odd.*

We remark that if $n = 2$, then the conditions stated in the proposition above also are sufficient conditions, that is

1. H is negative semidefinite if and only if $\mathrm{tr}(H) \leq 0$ and $\det(H) \geq 0$.

2. H is negative definite if and only if $\mathrm{tr}(H) < 0$ and $\det(H) > 0$.

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