Preference relation, choice rule and utility function

• Individual decision making

– under Certainty

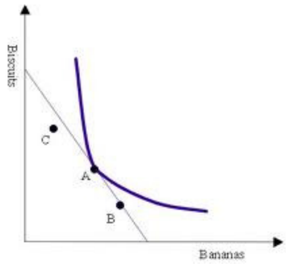
- Revelead preference and **ordinal** utility theory

Choice

Preference relation

Utility function

U(The Truth) > U(The Matrix)



If u() is a utility function, then **any strictly increasing transformation** g∘u() is a utility function representing the same preferences

• Individual decision making

– under Certainty

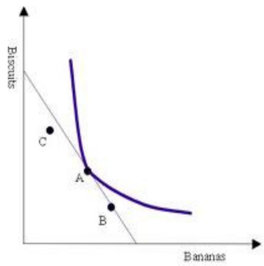
- Revealed preference and utility theory

Choice

Choice function

Utility

U(The Truth) > U(The Matrix)



- **Individual decision making**
  - **under Certainty**
    - Choice functions

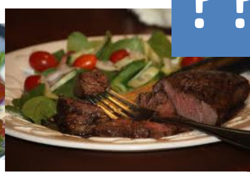
Weak axiom of revealed preference (WARP)

NOT ALLOWED

You go to a restaurant in while you are on vacation in Tuscany and you are given the following menu:

- bistecca
- pollo

???



The cook anounces that he can also serve

- trippa alla fiorentina

Preference relations

- Mathematically – binary relations in the set of decision alternatives:
  - X – decision alternatives
  - X^2 – all pairs of decision alternatives
  - $R \subset X^2$ – binary relation in X, selected subset of ordered pairs of elements of X
  - if x is in relation R with y, then we write xRy or $(x,y) \in R$

5

- Examples of relations:
  - „Being a parent of" is a binary relation on a set of human beings
  - „Being a hat" is a binary relation on a set of objects
  - *„x+y=z"* is 3-ary relation on the set of numbers
  - *„x* is better than *y* more than *x'* is better than *y'* " is a 4-ary relation on the set of alternatives.

Logical preliminary

p	q	~p	~q	p ⇒ q	⇔ ~p ∨ q	⇔ ~q ⇒ ~p
0	0	1	1	1	1	1
1	0	0	1	0	0	0
0	1	1	0	1	1	1
1	1	0	0	1	1	1

p	q	~p	~q	p ∨ q	~(p ∨ q)	⇔ ~q ∧ ~p
0	0	1	1	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	0
1	1	0	0	1	0	0

Binary relations – basic properties

- complete: xRy or yRx
- reflexive: xRx (∀x)
- irreflexive: not xRx (∀x)
- transitive: if xRy and yRz, then xRz
- symmetric: if xRy, then yRx
- asymmetric: if xRy, then not yRx
- antisymmetric: if xRy and yRx, then x=y
- negatively transitive: if not xRy and not yRz, then not xRz
  - equivalent to: xRz implies xRy or yRz
- acyclic: if x_1Rx_2, x_2Rx_3, … , $x_{n-1}Rx_n$ imply x1≠xn

Exercise – check the properties of the following relations

- R_1: (among people), to have the same colour of the eyes
- R_2: (among people), to know each other
- R_3: (in the family), to be an ancestor of
- R_4: (among real numbers), not to have the same value
- R_5: (among words in English), to be a synonym
- R_6: (among countries), to be at least as good in a rank-table of summer olympics

	R_1	R_2	R_3	R_4	R_5	R_6
complete						
reflexive						
irreflexive						
transitive						
symmetric						
asymmetric						
antisymmetric						
negatively transitive						

2 questionaires

P (for all distinct x and y in X):

How do you compare x and y? Tick one and only one of the following three options:

- I prefer x to y (this answer is denoted as $x \succ y$ or xPy).
- I prefer y to x (this answer is denoted by $y \succ x$, or yPx).
- Neither of the first two. I am indifferent (this answer is denoted by $x \sim y$ or xIy).

R (for all $x, y \in X$, not necessarily distinct):

Is x at least as preferred as y? Tick one and only one of the following two options:

- Yes
- No

2 questionaires

We exclude right away:

- A lack of ability to compare (I have no opinion, they are incomparable)
- A dependence on other factors (depends on what my parents think)
- Intensity of preferences (I somewhat prefer x, I love x and hate y)

Rational preference relation

- P is a (rational) **strict preference relation** in X, if it is:
  - **asymmetric**
  - **negatively transitive**
  - acyclic
  - transitive
  - ...
- Q is a (rational) **weak preference relation** in X, if it is:
  - **complete**
  - **transitive**
  - acyclic
  - reflexive
  - ...
- Completeness implies reflexivity (be sure that you understand)
- Asymmetry + negative transitivity implies transitivity (prove)
- Etc.

12

Relationship between strict and weak preferences

Let R be a weak preference relation (transitive, complete)

- R generates strict preference relation – P:
  - xPy, iff xRy and not yRx
- R generates indifference relation – I:
  - xIy, iff xRy and yRx

Let P be a strict preference relation (asymmetric and negatively transitive)

- P generates weak preference relation – R:
  - xRy, iff not yPx
- P generates indifference relation – I:
  - xIy, iff not xPy and not yPx

13

An example [X={a,b,c,d}]

- R={(a,a), (a,b), (a,c), (a,d), (b,a), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)}

generates:

- P={(a,c), (a,d), (b,c), (b,d), (c,d)}
- I={(a,a), (a,b), (b,a), (b,b), (c,c), (d,d)}

And the other way around:

- P={(a,c), (a,d), (b,c), (b,d), (c,d)}

generates:

- R={(a,a), (a,b), (a,c), (a,d), (b,a), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)}
- I= ={(a,a), (a,b), (b,a), (b,b), (c,c), (d,d)}

- Observe that $R=P\cup I$

14

Exercise

- X={a,b,c,d}
- P={(a,d), (c,d), (a,b), (c,b)}

- Find R and I

P vs R ($xPy \Leftrightarrow xRy \wedge \sim yRx$)

- R is complete iff P is asymmetric

- R is transitive iff P is negatively transitive

Properties of I

- I is an equivalence relation iff it is:
  - **reflexive**
  - **transitive**
  - **symmetric**
- Can we start with I as a primitive?
  - reflexive
  - symmetric
  - transitive
- No – we wouldn't be able to order the abstraction classes

17

Proof of the properties of I from the properties of R ($xIy \Leftrightarrow xRy \wedge yRx$)

- reflexive (xIx)
  - obvious – using reflexivity of R we get xRx
- transitive ($xIy \wedge yIz \Rightarrow xIz$)
  - predecessor means that $xRy \wedge yRx \wedge yRz \wedge zRy$
  - using transitivity we get $xRz \wedge zRx$, QED
- symmetric ($xIy \Rightarrow yIx$)
  - predecessor means that $xRy \wedge yRx$, QED

18

Another definition of rational preferences

- Is it enough to use a relation P that is:
  - asymmetric
  - acyclic (not necessarily negatively transitive)
- No – let's see an example

P from the previous slide – an example

- Mr X got ill and for years to come will have to take pills twice a day in an interval of exactly 12 hours. He can choose the time however.
- All the decision alternatives are represented by a circle with a circumference 12 (a clock). Let's denote the alternatives by the length of an arc from a given point (midnight/noon).
- Mr X has very peculiar preferences – he prefers y to x, if $y=x+\pi$, otherwise he doesn't care
- Thus yPx, if y lies on the circle π units farther (clockwise) than x

Exercise

- What properties does P have?
  - Asymmetry (YES)
  - negative transitivity (NO)
  - Transitivity (NO)
  - Acyclicity (YES)

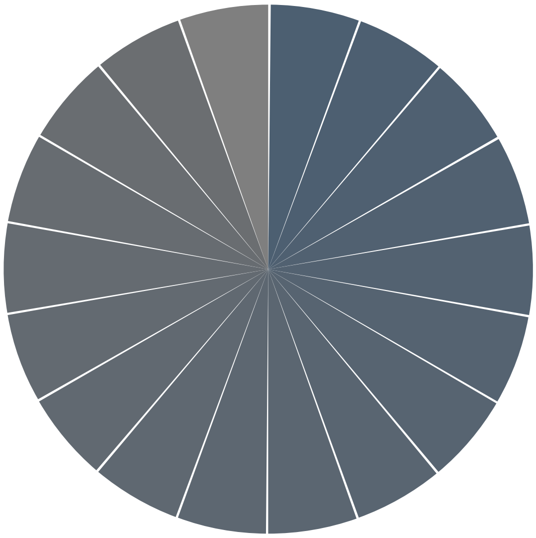
- P generates „weird" preferences:
  - Not transitive
    - $1+2\pi$ better than $1+\pi$,
    - $1+\pi$ better than 1,
    - $1+2\pi$ equally good as 1
  - Not negatively transitive
    - 1 equally good as $1+\pi/2$,
    - $1+\pi/2$ equally good as $1+\pi$,
    - 1 worse than $1+\pi$

Another definition of rational preferences

- What if we take P?
  - asymmetric
  - transitive (not necessarily negatively transitive)
  - thus acyclic

- First let's try to find an example
- Then let's think about such preferences

Asymmetric, transitive, not negatively transitive relation – intuition



Asymmetric, transitive, not negatively transitive relation – example

- $X=\{R_+\}$, $xPy \Leftrightarrow x>y+5$ (I want more, but I am insensitive to small changes)

- Properties of P:
  - asymmetric – obviously
  - transitive – obviously
  - negatively transitive?
    - 11 P 5, but
    - neither 11 P 8, nor 8 P 5

- Thus I is not transitive: 11 I 8 and 8 I 5, but not 11 I 5

24

Properties of preferences – a summary

P („better than”) – asymmetric, negatively transitive

R („at least as good as”) – transitive, complete

colours, insensitiviness to small changes

P („better than”) – asymmetric, transitive

eg. Mr X

P („better than”) – asymmetric, acyclic

Preferences once more (this time strict)

- Let X represent some set of objects
- Often in economics $X \subseteq \mathrm{R}^K$ is a space of consumption bundles
  - E.g. 3 commodities: beer, wine and whisky
  - $x = (x_1, x_2, x_3)$ (x_1 cans of beer, x_2 bottles of wine, x_3 shots of whisky
- We present the consumer pairs x and y and ask how they compare
- Answer x is better than y is written $\mathbf{x} \succ \mathbf{y}$ and is read x ***is strictly preferred to*** y
- For each pair x and y there are 4 possible answers:
  - x is better than y, but not the reverse
  - y is better than x, but not the reverse
  - neither seems better to her
  - x is better than y, and y is better than x

Assumptions on strict preferences

- We would like to exclude the fourth possibility right away

Assumption 1: *Preferences are* ***asymmetric****. There is no pair x and y from X such that $x \succ y$ and $y \succ x$*

- Possible objections:
  - What if decisions are made in different time periods?
    - change of tastes
    - addictive behavior (1 cigarette $\succ$ 0 cigarettes $\succ$ 20 cigarettes changed to 20 cigarettes $\succ$ 1 cigarette $\succ$ 0 cigarettes)
    - dual-self model
  - Dependence on framing
    - E.g. Asian disease

Assumptions on strict preferences

***Assumption 2:** Preferences are **negatively transitive**: If $x \succ y$, then for any third element z, either $x \succ z$, or $z \succ y$, or both.*

- Possible objections:
  - Suppose objects in X are bundles of cans of beer and bottles of wine $x = (x_1, x_2)$
  - No problem comapring $x = (21, 9)$ with $y = (20, 8)$
  - Suppose $z = (40, 2)$. Negative transitivity demands that either $(21, 9) \succ (40, 2)$, or $(40, 2) \succ (20, 8)$, or both.
  - The consumer may say that comparing $(40, 2)$ with either $(20, 8)$ or $(21, 9)$ is to hard.
  - Negative transitivity rules this out.

Weak preferences and indifference induced from strict preferences

- Suppose our consumer's preferences are given by the relation $\succ$.

Definition: *For x and y in X,*

- *write $x \succsim y$, read "x is* ***weakly preferred*** *to y", if it is not the case that $y \succ x$.*
- *write $x \sim y$, read "x is* ***indifferent*** *to y", if it is not the case that either $x \succ y$ or $y \succ x$.*

- Problem with noncomparability: if the consumer is unable to compare $(40, 2)$ with either $(20, 8)$ or $(21, 9)$, it doesn't mean she is indifferent between them.

Dependencies between rational preferences

Proposition: *If $\succ$ is asymmetric and negatively transitive, then:*

- *weak preference $\succsim$ is* ***complete*** *and* ***transitive***
- *indifference $\sim$ is* ***reflexive, symmetric*** *and* ***transitive***
- *Additionally, if $w \sim x, x \succ y$, and $y \sim z$, then $w \succ y$ and $x \succ z$.*

The first two were proved previously. The third may be proved at home.

Needed for later purposes

Additionally, we will need the following:

Proposition: *If $\succ$ is asymmetric and negatively transitive, then $\succ$ is irreflexive, transitive and acyclic.*
Proof.

- Irreflexive by asymmetry
- Transitivity:
  - Suppose that $x \succ y$ and $y \succ z$
  - By negative transitivity and $x \succ y$, either $x \succ z$ or $z \succ y$
  - Since $y \succ z$, asymmetry forbids $z \succ y$. Hence $x \succ z$
- Acyclicity:
  - If $x_1 \succ x_2$, $x_2 \succ x_3$, ..., $x_{n-1} \succ x_n$, then transitivity implies $x_1 \succ x_n$
  - Asymmetry (or irreflexivity) implies $x_1 \neq x_n$

Quod Erat Demonstrandum (QED)

Choice functions – a formal definition

- Notation:

X	set of decision alternatives
$\mathcal{B} \subset 2^X, \varnothing \notin \mathcal{B}$	available menus (non-empty subsets of X)
$C : \mathcal{B} \to \mathcal{B}$	choice function, working for every menu

- (Technical) properties:

$C(B) \neq \varnothing$	always a choice
$C(B) \subset B$	out of a menu

- If C(B) contains a single element ➔ this is the choice
- If more elements ➔ these are possible choices (not simultaneously, the decision maker picks one in the way which is not described here)

An exercise

- Let X={a,b,c}, $\mathcal{B}=2^X$
- Write down the following choice functions:
  - C_1: always a (if possible), if not – it doesn't matter
  - C_2: always the first one in the alphabetical order
  - C_3: whatever but not the last one in the alphabetical order (unless there is just one alternative available)
  - C_4: second first alphabetically (unless there is just one alternative)
  - C_5: disregard c (if technically it is possible), and if you do disregard c, also disregard b (if technically possible)

27

The solution

B	$C_1(B)$	$C_2(B)$	$C_3(B)$	$C_4(B)$	$C_5(B)$
{a}					
{b}					
{c}					
{a,b}					
{a,c}					
{b,c}					
{a,b,c}					

Choice rule induced by preference relation

- How do we relate preference relation with choice behavior?

Definition: *Given a preference relation $\succ$ on a set of objects X and a nonempty subset A of X, the* ***set of acceptable alternatives*** *from A according to $\succ$ is defined to be:*

$$c(A; \succ) = \{x \in A : \text{There is no } y \in A \text{ such that } y \succ x\}$$

Several things to note:

- $c(A; \succ)$ by definition subset of A
- $c(A; \succ)$ may contain more than one element (anything will do)

Properties of such choice rule

- In some cases, $c(A; \succ)$ may conatin no elements at all
  - $X = [0, \infty)$ with $x \in X$ representing x dollars
  - $A \subseteq X$, $A = \{1, 2, 3, ...\}$
  - Always prefers more money to less $x \succ y$ whenever $x > y$
  - Then $c(A; \succ)$ will be empty
  - The same when $A = [0, 10)$ and money is infinitely divisible
- In the examples above, $c(A; \succ)$ is empty because A is too large or not nice - it may be that $c(A; \succ)$ is empty because $\succ$ is badly behaved
  - suppose $X = \{x, y, z, w\}$, and $x \succ y$, $y \succ z$, and $z \succ x$. Then $c(\{x, y, z\}; \succ) = \emptyset$

WARP

- **Weak Axiom of Revealed Preference**: if x and y are both in A and B and if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$ (and $y \in c(A)$).
- It may be decomposed into two properties:
  - **Sen's property α:** If $x \in B \subseteq A$ and $x \in c(A)$, then $x \in c(B)$.
  - If the world champion in some game is a Pakistani, then he must also be the champion of Pakistan.
  - **Sen's property β:** If $x, y \in c(A)$, $A \subseteq B$ and $y \in c(B)$, then $x \in c(B)$.
  - If the world champion in some game is a Pakistani, then all champions (in this game) of Pakistan are also world champions.
- Observe that WARP concerns A and B such that $x, y \in A \cap B$.
  - Property α specializes to the case $A \subseteq B$
  - Property β specializes to the case $B \subseteq A$

Rational preferences induce rational choice rule

Proposition: *Suppose that $\succ$ is asymmetric and negatively transitive. Then:*

(a) *For every finite set A, $c(A;\succ)$ is nonempty*

(b) *$c(A;\succ)$ satisfies WARP*

Proof.

Part I: **$\mathbf{c(A;\succ)}$ is nonempty**:

- We need to show that the set $\{x \in A : \forall y \in A, y \not\succ x\}$ is nonempty
- Suppose it was empty - then for each $x \succ A$ there exists a $y \in A$ such that $y \succ x$.
- Pick $x_1 \in A$ (A is nonempty), and let x_2 be x_1's "y".
- Let x_3 be x_2's "y", and so on. In other words, take $x_1, x_2, x_3 ... \in A$, such that $...x_n \succ x_{n-1} \succ ... \succ x_2 \succ x_1$
- Since A is finite, there must exist some m and n such that $x_m = x_n$ and $m > n$.
- But this would be a cycle. Contradiction.
- So $c(A;\succ)$ is nonempty. **End of part I**.

Rational preferences induce rational choice rule

***Part II:* $\mathbf{c(A; \succ)}$ satisfies WARP:**

- Suppose x and y are in $A \cap B$, $x \in c(A, \succ)$ and $y \in c(B, \succ)$
- Since $x \in c(A, \succ)$ and $y \in A$, we have that $y \not\succ x$.
- Since $y \in c(B, \succ)$, we have that for all $z \in B$, $z \not\succ y$.
- By negative transitivity of $\succ$, for all $z \in B$ it follows that $z \not\succ x$
- This implies $x \in c(B, \succ)$.
- Similarly for $y \in c(A, \succ)$. **End of part II.**

QED

Choice rules as a primitive

- Let us now reverse the process: We observe choice and want to deduce preferences.

Definition: A choice function *on X is a function c whose domain is the set of all nonempty subsets of X, whose range is the set of all subsets of X, and that satisfies $c(A) \subseteq A$, for all $A \in X$*

- ***Assumption:*** *The choice function c is nonempty valued: $c(A) \neq \emptyset$, for all A*
- ***Assumption:*** *The choice function c satisfies* ***Weak Axiom of Revealed Preference****: If $x, y \in A \cap B$ and if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$ and $y \in c(A)$.*

Rational choice rule induces rational preferences

Proposition: *If a choice function c is nonempty valued and satisfies property α and property β (and hence WARP), then there exists a preference relation $\succ$ such that c is $c(\cdot, \succ)$*

Rational choice rule induces rational preferences

Proof.

- Define $\succ$ as follows:

$$x \succ y \Longleftarrow x \neq y \text{ and } c(\{x, y\}) = \{x\}$$

- This relation is obviously **asymmetric**.

Part I: **$\succ$ is negatively transitive**

- Suppose that $x \not\succ y$ and $y \not\succ z$, but $x \succ z$.
- $x \succ z$ implies that $\{x\} = c(\{x, z\})$, thus $z \notin c(\{x, y, z\})$ by property α
- Since $z \in c(\{y, z\})$, this implies $y \notin c(\{x, y, z\})$ again by property α
- Since $y \in c(\{x, y\})$, implies $x \notin c(\{x, y, z\})$ again by...
- Which is not possible since c is nonempty valued. Contradiction
- Hence $\succ$ is negatively transitive. **End of part I.**

Rational choice rule induces rational preferences

***Part II:* $\mathbf{c(A, \succ) = c(A)}$ for all sets A**

- Fix a set A
  (a) If $x \in c(A)$, then for all $z \in A$, $z \not\succ x$. For if $z \succ x$, then $c(\{x, z\}) = \{z\}$, contradicting property α. Thus $x \in c(A, \succ)$
  (b) If $x \notin c(A)$, then let z be chosen arbitrarily from $c(A)$. We claim that $c(\{z, x\}) = \{z\}$ - otherwise property β would be violated. Thus $z \succ x$ and $x \notin c(A, \succ)$.
  - Combining (a) and (b), $c(A, \succ) = c(A)$ for all A. **End of part II.**

QED

Utility representation

Definition: *Function $u : X \to \mathrm{R}$ represents rational preference relation $\succ$ if for all $x, y \in X$ the following holds*

$$x \succ y \iff u(x) > u(y)$$

- The representation is always well defined since $>$ on R satisfies negative transitivity and asymmetry.

Proposition: *If u represents $\succ$, then for any strictly increasing function $f : \mathrm{R} \to \mathrm{R}$, the function $v(x) = f(u(x))$ represents $\succ$ as well.* ***Proof.***

$$x \succ y$$
$$u(x) > u(y)$$
$$f(u(x)) > f(u(y))$$
$$v(x) > v(y)$$

QED

Minimal element in a finite set

Lemma:

In any finite set $A \subseteq X$, there is a minimal element (similarly, there is also a maximal element).

Proof:

By induction on the size of A. If A is a singleton, then by completeness its only element is minimal. For the inductive step, let A be of cardinality $n+1$ and let $x \in A$. The set $A - \{x\}$ is of cardinality n and by the inductive assumption has a minimal element denoted by y. If $x \succsim y$, then y is minimal in A. If $y \succsim x$, then by transitivity $z \succsim x$ for all $z \in A - \{x\}$, and thus x is minimal.

Utility representation for finite sets

Claim:

If $\succsim$ is a preference relation on a finite set X, then $\succsim$ has a utility representation with values being natural numbers.

Proof:

We will construct a sequence of sets inductively. Let X_1 be the subset of elements that are minimal in X. By the above lemma, X_1 is not empty. Assume we have constructed the sets $X_1, \ldots, X_k$. If $X = X_1 \cup X_2 \cup \ldots \cup X_k$, we are done. If not, define X_{k+1} to be the set of minimal elements in $X - X_1 - X_2 - \cdots - X_k$. By the lemma $X_{k+1} \neq \emptyset$. Because X is finite, we must be done after at most $|X|$ steps. Define $U(x) = k$ if $x \in X_k$. Thus, $U(x)$ is the step number at which x is "eliminated". To verify that U represents $\succsim$, let $a \succ b$. Then $a \notin X_1 \cup X_2 \cup \cdots X_{U(b)}$ and thus $U(a) > U(b)$. If $a \sim b$, then clearly $U(a) = U(b)$.

Utility representation result I

Definition: *A preference relation* $\succ$ *on* X *is continuous if for all* $x, y \in X$*,* $x \succ y$ *implies that there is an* $\epsilon > 0$ *such that* $x' \succ y'$ *for any* x' *and* y' *such that* $d(x, x') < \epsilon$ *and* $d(y, y') < \epsilon$*.*

Proposition: *Assume that* X *is a convex subset of* R^{n}*. If* $\succ$ *is a continuous preference relation on* X*, then* $\succ$ *has a continuous utility representation.*

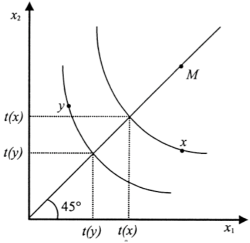
Utility representation result II

Monotonicity:

The relation $\succsim$ satisfies *monotonicity at the bundle* y if for all $x \in X$,
if $x_k \geq y_k$ for all k, then $x \succsim y$, and
if $x_k > y_k$ for all k, then $x \succ y$.

The relation $\succsim$ satisfies *monotonicity* if it satisfies monotonicity at every $y \in X$.

Proposition: *Any preference relation satisfying monotonicity and continuity can be represented by a utility function*



Proof

- Take any bundle $x \in \mathrm{R}^n_+$.
- It is at least as good as the bundle $0 = (0, \ldots, 0)$
- On the other hand $M = (\max_k\{x_k\}, \ldots, \max_k\{x_k\})$ is at least as good as x
- Both 0 and M are on the main diagonal
- By continuity there is a bundle on the main diagonal that is indifferent to x
- By monotonicity this bundle is unique, denote it by $(t(x), \ldots, t(x))$.
- Let $u(x) = t(x)$. We show that u represents the preferences:
- By transitivity, $x \succsim y \iff (t(x), \ldots, t(x)) \succsim (t(y), \ldots, t(y))$
- By monotonicity this is true if and only if $t(x) \geqslant t(y)$

QED