Comparing Recursive Equilibrium in Economies with Dynamic Complementarities and Indeterminacy[1]

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Abstract

We prove the existence of a minimal state space recursive equilibrium (RE) for a broad class of infinite horizon dynamic general equilibrium models with positive externalities, dynamic complementarities, public policy, equilibrium indeterminacy and sunspots. These are new "multistep" monotone map methods, and apply to economies for which results on the existence of dynamic equilibria are unknown and existing methods do not apply. Our methods are *global,* based on monotone operators defined on Euler equations, and do not appeal to the theory of smooth dynamical systems that are commonly applied in the literature. Rather, using partial ordering methods, we provide a qualitative theory of equilibrium comparative statics in the presence of multiple equilibrium. These comparison results are computable via successive approximations from upper and lower bounds of particular sets of functions. We provide applications of our results to an extensive literature on local indeterminacy of dynamic equilibrium.

Keywords: Recursive Equilibrium, Supermodularity, Monotone Map Methods, Externality, Indeterminacy

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1 Introduction

Since the work of Lucas and Prescott [1971] and Prescott and Mehra [1980], recursive equilibrium (RE) have been a key focal point of both applied and theoretical work in characterizing sequential equilibrium for dynamic general equilibrium models in such fields as macroeconomics, international trade, growth theory, industrial organization, financial economies, and monetary theory.[1] When dynamic economies are Pareto optimal, in the case of homogeneous agent models and under standard concavity conditions, RE is unique, and can be computed using standard dynamic programming algorithms. In this case, equilibrium comparative statics analysis is reduced to either application of local or global implicit function theorem based smooth dynamical systems or applications of dynamic lattice programming methods to the social planner's problem. In nonoptimal economies, even the existence of dynamic equilibrium becomes complicated to prove, let alone obtain equilibrium comparative statics results. Although some recent extensions of dynamic lattice programming methods have been made for nonoptimal economies (including those with heterogeneous agents) in Mirman, Morand, and Reffett [2008] and Acemoglu and Jensen [2015], there are important nonoptimal homogeneous agent economies in which these tools are difficult to apply. Further, an extensive literature on monotone map methods has stemmed from the pioneering work of Coleman [1991], [1997], [2000] and Greenwood and Huffman [1995], but these methods are also known to fail in some nonoptimal models (see, Santos [2002], section 3.2).[2]

In this paper, we propose a new method for obtaining existence as well as equilibrium comparison in a well-studied class of nonoptimal homogeneous agent economies with dynamic complementarities. Our method extends the scope of parameterized fixed point methods to models with local indeterminacy, multiple equilibrium, and discontinuous minimal state space RE. We focus on dynamic general equilibrium models studied extensively in the literature with externalities and nonconvexities in production, public or monetary policy and monopolistic competition.[3] Very importantly, our results are based on *global* methods and make no appeal to local analysis, in contrast to the literature studying dynamic models with complementarities using the methods of smooth dynamical systems to characterize sequential equilibrium near steady states; see Benhabib and Farmer [1994], among numerous others.[4]

Our method involves "two-step" monotone maps defined on partially ordered sets. The key intuition that underlies these multi-step methods is rather simple: in the first step, we construct solutions to a parameterized fixed point problem that guarantee necessary structural restrictions implied by household optimization relative to *individual state variables*. Then, using a fixed point monotone comparative

[1] See Stokey, Lucas, and Prescott [1989].

[2] See also Datta, Mirman, and Reffett [2002], Morand and Reffett [2003], and Datta, Mirman, Morand, and Reffett [2005] for extensions of monotone map methods.

[3] The "technology" we specify can be interpreted as a "reduced-form" for production in various nonoptimal economies, including models with monopolistic competition, taxes, learning, production externalities, and even some cash-in-advance models. See Benhabib and Farmer [1994], Greenwood and Huffman [1995] and Datta, Mirman, and Reffett [2002] for examples of other economies that fit this structure.

[4] For recent papers, see Beaudry and Poitier [2007], Jaimovich [2007],[2008], Wang and Wen [2008], Guo and Harrison [2010], Antoci, Galeotti, Russi [2011], d'Albis, Augeraud-Veron, and Venditti [2012], and Huang and Meng [2012], Braga, Modesto, and Seegmuller [2014].

statics result on "first step" fixed points, we define a second step monotone operator, which verifies necessary *aggregate state consistency* conditions for a RE . The second stage fixed point structure allows us to compute *state asymmetric* RE, which is critical in the class of models with local indeterminacy e.g. As the two-step procedure verifies the existence of RE via a *monotone* operator, robust equilibrium comparative statics can be delivered in some deep parameters. An important implication is that the set of state asymmetric RE could be *huge* - consistent with results on local indeterminacy of sequential equilibrium in the literature. Note that, we do *not* need monotone RE for these methods to work, we need monotone operators defined on suitable chain complete partially ordered sets. Further, we do not need continuous RE. Indeed, our methods are designed specifically to allow for and construct discontinuous RE that are consistent with solutions to the household dynamic program.

It is, perhaps, important to point out that we obtain a rich set of robust RE comparative statics/dynamics without appealing directly to the lattice programming machinery of Topkis [1978], [1998] and Veinott [1992]. Our methods can be interpreted as an iterative class of parameterized dynamic lattice programming problems built on the household program, not that of the social planner. Our work builds also on Acemoglu and Jensen [2015], where a new approach to the existence of robust equilibrium comparative statics is proposed for large dynamic models, and where they make significant progress in obtaining sufficient conditions for robust distributional equilibrium comparative statics. Although their methods are powerful for many important classes of dynamic economies (including situations where our methods do not apply), as we show in this paper, their sufficient conditions cannot be checked even in certain homogeneous agent economies. They apply dynamic lattice programming methods to the individual agent problem and obtain sufficient *partial* monotonicity of decision rules which are then exploited to deduce aggregate equilibrium comparative statics. In this sense, even though their results are more general, in some applications, they suffer from limitations similar to that of Mirman, Morand, and Reffett [2008].

The differential approach to equilibrium comparative statics goes far back to Samuelson [1941], and is best illustrated in the seminal work of Debreu [1970], [1972], who used differential topology tools to bear on the question.[5] This method has been extended to dynamic economies by Kehoe, Levine and Romer [1990], and Santos [1992], among others. Interesting application of smooth equilibrium comparative statics is found in the extensive literature on "indeterminacy" of equilibrium in models of one-sector production with externalities e.g., see papers following the approach taken in Benhabib and Farmer [1994], Boldrin and Rustichini [1994], Benhabib and Perli [1994], and Farmer and Guo [1994].[6] These papers study determinacy of sequential equilibrium dynamics around a proposed hyperbolic point (e.g., the unique positive steady state), and it is shown that if a smooth sequential equilibrium is present, the local dynamics would be consistent with a continuum of equilibrium paths leading to the steady state. An important new approach to the study of local (and global) indeterminacy is found in the Euler equation

[5] See also MasColell [1986] for a comprehensive discussion.

[6] E.g., for recent applications of these smooth dynamical systems methods, see Santos [2002], Jaimovich [2007],[2008], Wang and Wen [2008], Guo and Harrison [2010], Antoci, Galeotti, Russi [2011], Huang and Meng [2012], Nourry, Seegmuller, and Venditt [2013], Braga, Modesto, and Seegmuller [2014].

branching methods of Stockman [2010] and Raines and Stockman [2010]. Our methods are very much in the spirit of these latter papers, but we ask different questions (i.e., we are concerned with existence of RE dynamics, and characterizing RE comparative statics; not a theory of the resulting RE dynamical system). These methods also cannot be applied to our RE as we cannot prove the existence of continuous RE dynamics in the capital stock in aggregate states. But in principle, our methods seek to complement the results and methodological approach taken in these latter two papers.

The rest of the paper is laid out as follows. In Section 2, we describe the class of homogeneous agent models analyzed in this paper. In section 3, we construct the RE operator and prove existence. In section 4, we develop equilibrium comparative statics results. In section 5 we conclude with examples and discussion of our results in comparison with the literature.

2 The Framework

We study a general class of homogeneous agent dynamic general equilibrium models with externalities and complementarities with reduced-form production function that embeds numerous other non-optimal dynamic economies with complementarities including Benhabab and Farmer [1994] and Liu and Wang [2014], among others. Time is discrete and indexed by $t \in \{0, 1, 2, ...\}$. The economy has a unit mass of identical (or a representative) infinitely-lived household with separable preferences over lifetime streams of consumption and leisure, $\{c_t\}_{t=0}^{\infty}$ and $\{l_t\}_{t=0}^{\infty}$, respectively. The household's lifetime utility is,

$$\sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(l_t)\}, \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor.

The following regular assumptions are imposed on period utility functions (omitting time subscripts).

Assumption A1. *The returns from consumption and leisure,* $u : \mathbf{R}_+ \to \mathbf{R}_+$ *and* $v : [0, 1] \to \mathbf{R}_+$ *are continuous, strictly increasing, strictly concave and continuously differentiable on* $\mathbf{R}_{++}$ *with* $u(0) = 0$, $v(0) = 0$ *and Inada-type conditions are satisfied*, i.e.

$$\lim_{c \to 0} u'(c) \to \infty; \;\; \lim_{c \to \infty} u'(c) \to 0; \;\; \lim_{l \to 0} v'(l) \to \infty.$$

Households are endowed with an initial holding of capital denoted by $k_0 > 0$ and one unit of productive time in each period. Factor and goods markets are perfectly competitive. Households own factors of production and rent them to firms. A unit mass of firms (or one representative) face production technologies given by $F(k, n, K, \hat{N})$. Technology has constant returns to scale (CRS) in private factors (k, n), where k is the individual firm's decision on capital, n its decision on labor inputs, but we also allow for the social externalities that depend on the aggregate levels of capital K and labor $\hat{N}$, respectively. We assume the following conditions on the production technology.

Assumption A2. *The production function* $F : \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ *is multiplicatively separable in private returns and the social externalities:* $F(k, n, K, \hat{N}) = f(k, n)e(K, \hat{N})$*; in addition, (a)*

$f:\mathbf{R}_{+}\times\mathbf{R}_{+}\to\mathbf{R}_{+}$ is constant returns to scale, supermodular, increasing (but increasing strictly with each argument for the positive input of the other), weakly concave jointly (but strictly concave with each argument separately for the positive input of the other), and continuously differentiable in both arguments (on $\mathbf{R}_{++}\times\mathbf{R}_{++}$) with $f(0,n)=0=f(k,0)$; (b) *the marginal products of f in capital and labor satisfy Inada-type conditions:*

$$\begin{aligned}\lim_{k\to 0} f_1(k,n) &\to \infty \text{ for all } n\in(0,1],\\ \lim_{k\to\infty} f_1(k,n) &\to 0 \text{ for all } n\in[0,1],\\ \lim_{n\to 0} f_2(k,n) &\to \infty \text{ for all } k>0;\end{aligned}$$

and (c) the social externality $e:\mathbf{R}_{+}\times\mathbf{R}_{+}\to\mathbf{R}_{+}$ is increasing and locally Lipschitz continuous jointly with $e(0,\hat{N})=0=e(K,0)=0$. In addition, (d) there exists a $k_{\max}>0$,such that $F(k,1,K,1)\leq k_{\max}$ for all $k,K\geq k_{\max}$.

Here, $k_{\max}$ stands for the maximal sustainable level of capital. Define $\mathbf{K}=[0,k_{\max}]$. We postpone further remarks on assumptions A1 and A2 till section 2.1.

2.1 Household Dynamic Program

In this subsection, we describe household and firm decision-making: with-in period choices and prices are expressed as functions of aggregate capital in that period. The law of motion or the mapping between current aggregate capital to future aggregate capital depends on individual choices. Given a level of aggregate capital, K, the maximal level of output that can be generated in any period is given by $f^M(K)=f(K,1)e(K,1)$, using all available labor in the production. This is an upper bound for feasibility of household consumption (and/or investment) and allows us to define a space of socially feasible consumption functions:

$$B^f(\mathbf{K}):=\{(C:\mathbf{K}\to\mathbf{R}_{+}|0\leq C(K)\leq f^M(K),\forall K\in\mathbf{K}\} \tag{2}$$

and, endow B^f with the topology of pointwise convergence, as well as its pointwise partial order.[7] We develop a representation of the aggregate economy parameterized by $C\in B^f$.

In order to generate the path for aggregate capital $\{K_t\}_{t=0}^{\infty}$, we further restrict the set of consumption and investment functions to reflect that labor supply is endogenous in this economy. This restricts the space of possible investment/consumption functions we can consider. Anticipating the equilibrium conditions that govern the labor-leisure choice for a household in any RE, we posit the existence of a "contingent" aggregate labor supply, given by $N(C,K)\in[0,1]$. Here $N(C,K)$ represents a static or within-period labor-leisure choice, and is parameterized by both current aggregate state K, and consumption function C.[8]

The aggregate labor supply mapping imply restriction on the attainable level of output, hence, restrict the possible laws of motion for the aggregate state variable in any RE. In particular, when developing the

[7] In this sequel, to minimize notation, after defining a function space we delete the domain from the reference if the context is clear. For example, $B^f(\mathbf{K})$ is subsequently referred as B^f.

[8] It is shown in Lemma 3, N is decreasing in C, and increasing in K.

household's dynamic program for a candidate RE consumption function $C \in B^f$, aggregate labor supply N, households assume the law of motion on the aggregate capital stock to be:

$$K' = g(K; C, N) = f(K, N(C(K), K))e(K, N(C(K), K)) - C(K). \tag{3}$$

This law of motion generates the sequence of aggregate capital $\{K_t\}_{t=0}^{\infty}$ from $K_0 > 0$.

Next, we specify household income process as a function of aggregate capital, K. Prices of capital (r) and labor (w) are set to equal their respective marginal products,

$$\begin{aligned} r(K, \hat{N}) &= f_1(K, \hat{N})e(K, \hat{N}), \\ w(K, \hat{N}) &= f_2(K, \hat{N})e(K, \hat{N}). \end{aligned} \tag{4}$$

That is, firms are price-takers and hire capital and labor such that their individual or private marginal returns equal rental rates given aggregate (K, N). Appealing to zero profits under constant returns to scale, household income process $y(k, n, K; N(C(K), K))$ is given by:

$$y(k, n, K; N(C(K), K)) = r(K, N(C(K), K))k + w(K, N(C(K), K))n, \tag{5}$$

where for $K > 0$, the income process is real-valued. The feasible budget correspondence is,

$$\Phi(k, K; N(C(K), K)) = \{c, n, x | c + x \leq y(k, n, K; N(C(K), K)), c \geq 0, x \geq 0,\ n \in [0, 1]\},$$

where x denotes the household level of investment. Under Assumptions A1 and A2, as r, w are each continuous, the feasible correspondence Φ is a continuous correspondence, when $K > 0$.

Households use (C, N) to calculate factor prices for any $K \in \mathbf{K}_* = \mathbf{K}\backslash 0$. The household's dynamic program can be stated as follows: given $C \in B^f$ and the law of motion, $g(K; C, N) > 0$, with aggregate labor supply given by N, and the household's value function $V^*(\cdot; C, N) : \mathbf{K} \times \mathbf{K}_* \rightarrow \mathbf{R}_+$ satisfies the following parameterized Bellman equation:[9]

$$V^*(k, K; C, N) = \sup_{c,n,x \in \Phi} \{u(c) + v(1 - n) + \beta V^*(y(k, n, K; N(C(K), K)) - c, g(K; C, N); C, N)\}. \tag{6}$$

Let the optimal solutions for consumption and labor supply be given by $(c^*(k, K; C, N), n^*(k, K; C, N))$.

2.2 A Minimal State Space Recursive Equilibrium

Now, we are ready to formally define a Minimal State Space Recursive Equilibrium for this economy.

Definition 1 A *minimal state space recursive equilibrium* is a list of functions C^*, N^* for consumption and aggregate labor supply, as well as the associated value function $V^*(\cdot; C^*, N^*)$, law of motion g^*, the optimal solutions c^* and n^* and prices w, r for any $K > 0$ such that

[9] We could also allow for $u(c) = \ln(c)$. Define $\mathbf{R}^*_- = \mathbf{R} \cup -\infty$, the Bellman equation will be an upper semicontious function $V^*(\cdot; C, N) : \mathbf{K} \times \mathbf{K}_* \rightarrow \mathbf{R}^*_-$ that is continuous, when $K > 0$. Aside from that case, the range of the Bellman operator is actually $\mathbf{R}_+$.

1. $V^*(\cdot; C^*, N^*)$ satisfies (6) with $(c^*(k, K; C^*, N^*), n^*(k, K; C^*, N^*))$ being the arguments that solve the right hand side of the Bellman equation, for each k.

2. Taking prices w, r and aggregate states K, $N^*(C^*(K), K)$ as given firms maximize profit:

$$\max_{k,n \geq 0} F(k, n, K, N^*(C^*(K), K)) - r(K)k - w(K)n.$$

3. Consistency:

$$\begin{aligned} c^*(K, K; C^*, N^*) &= C^*(K), \\ n^*(K, K; C^*, N^*) &= N^*(C^*(K), K), \end{aligned}$$

with $C^*(0) = N^*(0, 0) = 0$.

4. Market clearing:

$$\begin{aligned} g^*(K; C^*, N^*) + c^*(K, K; C^*, N^*) &= F(K, N^*(C^*(K), K), K, N^*(C^*(K), K) \\ &= y(K, N^*(C^*(K), K), K; N^*(C^*(K), K)). \end{aligned}$$

For the sake of notation, list of function in the definition of the RE can be simplified and often we denote it using just: a consumption C^* and labor supply N^* with the corresponding law of motion, g^*.

We end this subsection with characterizing the first order conditions for maximization problem in (6) along any RE. Observe that by a standard argument, $V^*(\cdot; C, N)$ is strictly concave and at least once-continuously differentiable in its first argument k (e.g., Coleman [1991] and the Mirman-Zilcha Lemma). This implies under Assumptions A1 and A2, objective function on the right-hand side of (6) is strictly concave in its control variable. Therefore, we can characterize the optimal solutions in (6) by the first order conditions, which are necessary and sufficient. In particular, noting the Inada conditions, if $K > 0$, $g(K; C^*, N^*) > 0$, the optimal consumption $c^* = c^*(k, K; C^*, N^*)$ must satisfy the following functional equation,

$$u'(c^*(k, K; C^*, N^*)) - \beta u'(c^*(y_{c^*}, y_{C^*}; C^*, N^*)) f_1(\frac{y_{C^*}}{N^*(C^*(y_{C^*}), y_{C^*})}) e(y_{C^*}, N^*(C^*(y_{C^*}), y_{C^*})) = 0,$$

where $y_{c^*} = y(k, n^*(k, K; C^*, N^*), K, N^*(C^*(K), K)) - c^*$, and where $r = f_1 e$ has been substituted in the Euler equation.

Similarly, the first order condition associated with labor supply $n^*(k, K; C^*, N^*)$ in any RE is

$$\frac{v'(1 - n^*(k, K; C^*, N^*))}{u'(c^*(k, K; C^*, N^*))} - f_2(\frac{K}{N^*(C^*(K), K)}) e(K, N^*(C^*(K), K)) = 0, \tag{7}$$

and where we have used the assumption of homogeneity of production function in private returns (Assumption A2).

2.3 Necessary Properties of a Minimal State Space Recursive Equilibrium

In this subsection we highlight some necessary properties of household policies along a recursive equilibrium. These technical arguments put restrictions that constitute the basis of further construction in section 3. In particular, the household optimal solutions have continuity on individual or private capital (k) but not on aggregate capital (K) along any RE.

Lemma 2 Under A1 and A2, for any $C \in B^f$, N such that $N(C, K) \in (0, 1]$ for any $K \in \mathbf{K}_*$ *and* $g > 0$, household's optimal consumption, c^* *and labor supply,* n^* are continuous in k.

Proof. Given $C \in B^f, N$ such that $N(C, K) \in (0, 1]$ with $K \in \mathbf{K}_*$, $g(K; C, N) > 0$ household's dynamic program (6) is

$$V^*(k, K; C, N) = \max_{c,n,x \in \Phi} \{u(c) + v(1 - n) + \beta V^*(y(k, n, K; N(C(K), K)) - c, g(K; C, N); C, N)\}.$$

Under A1 and A2, by a standard argument $V^*(k, K; C, N)$ is continuous and strictly concave in k. As the feasible correspondence $\Phi(k, K; N(C(K), K))$ is nonempty, compact and convex valued, and continuous in k, each (K, C, N), and the objective on the right-hand side of the Bellman equation at $V^*(k, K; C, N)$ in (6) is continuous and strictly concave in c each (k, K, C, N), by Berge's maximum theorem, $c^*(k, K; C, N)$ and $n^*(k, K; C, N)$ are both continuous in k, each (K, C, N). ■

Therefore, by Lemma 2, although RE must *necessarily* have structural properties in *individual* state. Aside from resource feasibility, there are *no* required structural properties for a RE in aggregate state K. Our two step methods of RE construction heavily exploit this fact, and decompose our fixed point arguments relative to individual vs. aggregate state variables and isolate the discontinuities of RE to only aggregate states. This means that although multiplicities of RE might be easy to construct, obtaining sufficient conditions for RE smoothness is difficult. This casts concerns about applying smooth dynamical systems methods in characterizing multiplicity of equilibrium paths near any steady state associated with our model.

Also, Lemma 2 seems to pose serious challenge in developing rigorous applications of existing correspondence-based approaches to Generalized Markov equilibrium in the literature to compute RE in dynamic models (e.g., Kubler and Schmedders ([2003]) and Feng et. al. [2014]). For example, it is not clear how these existing generalized Markov methods can be applied to the models studied in this paper. In fact, we claim any sequential equilibrium has to satisfy a version of Lemma 2. In particular, any sequential equilibrium that is written recursively as in a Generalized Markov equilibrium on an enlarged state space involving "pseudo state variables" such as "envelopes or shadow values for capital" must deliver RE decision rules that are consistent with household value functions that are also once-continuous differentiable in individual states along a sequential equilibrium.

3 Construction of the RE

A roadmap of our approach in this RE construction is as follows: first, we solve for the equilibrium labour supply, contingent on a candidate RE consumption policy. In the second subsection, we propose suitable function spaces for a candidate RE that guarantee requisite continuity properties per Lemma 2. Next, the existence of equilibrium is proved using two-step operators We define a mapping (say, A) on a product space of functions (say, $H_1 \times H_2$) such that one subspace (H_2) is at least chain complete. The "first step" operator treat the chain complete subspace as fixed, and study fixed points of the partial map $h_1 \to A(h_1, h_2)$ in H_1. In the fourth subsection, we define "second step" operator that use as their domain H_2, and map to a subset of H_2, where we prove an RE exists.

3.1 Contingent RE Labor Supply

We construct "contingent" equilibrium representations of labor supply N, that we assume in development of the household's dynamic program in (6). It represents the "static" necessary and sufficient condition for RE labor supply. It bears mentioning this static equilibrium relationship between consumption and labor supply is generally not unique in our model, thus, in effect we have an RE labor supply correspondence say, N^*. The least and greatest selections from this correspondence, each exhibiting monotone comparative statics in $(\hat{C}, K)$ are used to parameterize "upper" and "lower" Euler equation operators.

Recall that the household decision problem defined in (6), and (7). Next, for any contingent equilibrium consumption $\hat{C}$ and aggregate labor supply $\hat{N}$, define a new mapping:

$$Z_n(n, \hat{N}, \hat{C}, K) = \frac{v'(1-n)}{u'(\hat{C})} - f_2(K, n)e(K, \hat{N}), \tag{8}$$

and a second mapping $\hat{n}^*(\hat{N}; \hat{C}, K)$ implicitly from (8):

$$Z_n(\hat{n}^*(\hat{N}; \hat{C}, K), \hat{N}, \hat{C}, K) = 0, \tag{9}$$

for $\hat{C} > 0$, all $K > 0$. Noting, the Inada conditions on v and f_2 plus the strict concavity conditions, this root is well-defined and unique. Then extend it to include the boundaries by setting $\hat{n}^*(\hat{N}; \hat{C}, K) = 1$, when $\hat{C} = 0, K > 0$ and set $\hat{n}^*(\hat{N}; \hat{C}, K) = 0$ otherwise. Lemma 3 characterizes solutions to the equation:[10]

$$\hat{n}^*(\hat{N}; \hat{C}, K) = \hat{N}.$$

[10] Under Assumption A2, when our technologies have the equilibrium wage rate w increasing in $\hat{N}$ (e.g., as in the case of Benhabib and Farmer [1994] and Liu and Wang [2014], among many others), the set of contingent RE labor supply decision will be a *correspondence*. We should note, in any *sequential equilibria* for Benhabib and Farmer models, for technologies evaluated at the so-called "indeterminacy parameters", this similar equilibrium labor supply decisions each period is also a *correspondence*. This means smoothness conditions near steady states that required to apply the Grobman-Hartman Theorem and/or stable manifold theorem are going to be problematic to check. That is, it is difficult to prove the existence of smooth equilibria, which is required to check the hypotheses needed to apply smooth dynamical systems methods to characterize the local determinacy of sequential equilibrium. This is true for both discrete and continuous time models.

Lemma 3 Say Assumptions A1 and A2 are both satisfied. Then, (a) $\hat{n}^*$ is single-valued, and continuously differentiable jointly for $(\hat{C}, K) >> 0$, $\hat{N} \in (0, 1)$. Further, (b) $\hat{n}^*(\hat{N}; \hat{C}, K)$ is increasing in $\hat{N}$ and K, and decreasing in $\hat{C}$. Finally, (c) for each $(\hat{C}, K)$, $\hat{n}^*(\hat{N}; \hat{C}, K)$ has a nonempty compact set of fixed points $N^*(\hat{C}, K) \subset [0, 1]$, with the greatest selection $\vee N^*(\hat{C}, K)$ and the least selection $\wedge N^*(\hat{C}, K)$, both continuous, increasing in K, decreasing in $\hat{C}$, and strictly positive, when $\hat{C} > 0, K > 0$.

Proof. (a) Note Z_n is strictly increasing in n, $\hat{n}^*(\hat{N}; \hat{C}, K)$ is unique for each $(\hat{C}, K)$. By the Inada conditions on v and f in n, for all $\hat{N} \in [0, 1]$, when $\hat{C} > 0$, $K > 0$, $\hat{n}^*(\hat{N}; \hat{C}, K) \in (0, 1)$. Further, when $\hat{C} > 0$, $K > 0$, $\hat{N} \in (0, 1)$, as $|\partial_n Z_n(n^*(\hat{N}; \hat{C}, K), \hat{N}, \hat{C}, K)| \neq 0$ by strict concavity of v and f, by the global implicit function theorem, root $\hat{n}^*(\hat{N}; \hat{C}, K)$ is also globally continuously differentiable (e.g., see Phillips [2012], lemma 2).

(b) The comparative statics result follows from the fact that under Assumptions A1 and A2, Z_n is strictly increasing in $(\hat{N}, \hat{C})$ and strictly decreasing in K.

(c) The set $[0, 1]$ is a complete lattice, $\hat{N} \to \hat{n}^*(\hat{N}, \hat{C}, K)$ is a increasing function on $[0, 1]$ for each $(\hat{C}, K)$, hence, by Tarski's theorem ([1955], Theorem 1), the fixed points of $\hat{n}^*(\cdot, \hat{C}, K)$ denoted by $N^*(\hat{C}, K)$ form a nonempty complete chain for all $(\hat{C}, K)$. As $\hat{n}^*$ is decreasing in K and increasing in $\hat{C}$, by Veinott's fixed point comparative statics theorem ([1992], Chapter 4, Theorem 14), the greatest fixed point $\vee N^*(\hat{C}, K)$ and least fixed point $\wedge N^*(\hat{C}, K)$, each selection well-defined, are increasing in K, and decreasing in $\hat{C}$.

Finally, the positivity of each selection, when $\hat{C} > 0$, $K > 0$,. We have $n(0; \hat{C}, K) > 0$ by the Inada conditions in Assumption A1. Finally, the continuity of the least and greatest selections $\vee N^*(\hat{C}, K)$ and $\wedge N^*(\hat{C}, K)$ follows from a modification of the transversality argument in Raines and Stockman ([2010], Propositions 4 and 5) and continuously differentiable assumption on u. ■

To slightly shorten notation, let us denote: $n^*_\vee(\hat{C}, K) := \vee N^*(\hat{C}, K)$ and $n^*_\wedge(\hat{C}, K) := \wedge N^*(\hat{C}, K)$.

We make a few remarks on Lemma 3. First, the existence of a sunspot equilibrium does not require capital externalities. By a modification of the tranversality argument in Raines and Stockman ([2010], Proposition 4 and 5), for economies satisfying Assumptions A1 and a stronger version of A2 with specific Cobb-Douglas technologies (as in Benhabib and Farmer [1994] and Liu and Wang [2014]) for the "indeterminacy parameters," there are *two* continuous selections in N^*. This implies that RE in our case is not be unique (even if each branch of our Euler equation operators we define in the next section of the paper have unique fixed points). And, sunspot equilibria exist without capital externalities driven only by labor externalities.

Under assumption A2 without the Cobb-Douglas technology specification, the Raines-Stockman results imply there is an even number of solutions for contingent labor supply N^*. The problem of studying RE with each these selections is that only the least and greatest selections are known to exhibit monotone comparative statics in $(\hat{C}, K)$; thus for other remaining selections, constructing a monotone map method (traditional or two-step methods) is more challenging.

Finally, its important to note that for a finite horizon version of our model, in the terminal period

T, when equilibrium wage rates in our economies have $w(K;\hat{N}) = f_2(K,\hat{N})e(K,\hat{N})$ are decreasing in $\hat{N}$, the solution for the "lower bound" for RE labor supply, denoted by $N_f^*(K)$, is *unique,* where $N_f^*(K)$ is the unique n solving $Z_n(n,n,f(K,n)e(K,n),K) = 0$. This is the case in models with elastic labor supply studied in Coleman [1997] and Datta, Mirman, and Reffett [2002]. In those papers, the (unique) RE can be computed as the "limit" of policy iteration type methods from (nonstationary) RE for finite horizon economies of length T. Then, the equilibrium for the terminal period economy implies a "lower bound" for RE labor supply, hence RE output, namely, $f^*(K) = f(K, N_f^*(K))e(K, N_f^*(K))$, which is used as the "upper" bound for RE consumption in the infinite horizon case, where the one period equilibrium labor supply $N_f^*(K)$ is the unique lower bound for labor supply in any RE for the infinite horizon economy. This is *not* true in this paper with assumptions A1 and A2. That is, we have *multiple equilibrium* in the terminal period for any finite horizon economy of length T. This fact requires us to be careful when constructing the maximal level of output that given contingent labor supply is possible when defining the function spaces where RE can be shown to exist. That is, it turns out RE consumption cannot exceed the level of output $f^*(K) = f(K, N_f^*(K))e(K, N_f^*(K))$, but now we have multiple candidates for $N_f^*(K)$. To see this, observe that under Assumptions A1 and A2, we can compute the set of terminal period equilibrium labor supplies $N_f^*(K)$ as the fixed point of the mapping $\hat{n}_f^*$ defined implicitly by $Z_n^f(\hat{n}_f^*(\hat{N},K),\hat{N},K) = 0$ for all $\hat{N} \in [0,1]$, $K > 0$, where

$$Z_n^f(n,\hat{N},K) = \frac{v'(1-n)}{u'(f(K,n,K,n))} - f_2(K,n)e(K,\hat{N}) \tag{10}$$

and where we set $\hat{n}_f^*(\hat{N},0) = 0$. Thus, equilibrium wages $f_2(K,n)e(K,n)$ could be rising in n, we can have multiple (but finite) number of terminal period equilibria, each continuous in K. Again, we denote the greatest and least of them as $\vee N_f^*(K)$ and $\wedge N_f^*(K)$.

We must also parameterize the space of feasible RE consumption function. In particular, we need to impose a restricted version of the upper bound for output contingent on candidate RE consumption. We use the modified production function evaluated at $n_\vee^f(K) = \vee N_f^*(K)$ and $n_\wedge^f(K) = \wedge N_f^*(K)$:

$$f_\nu^*(K) := f(K, n_\nu^f(K))e(K, n_\nu^f(K)),$$

where $\nu \in \{\vee, \wedge\}$. Observe that $f_\nu^*(K) \leq f^M(K)$ and we have a strict inequality if $K > 0$ (as $n_\wedge^*(\hat{C},K) \leq n_\vee^*(\hat{C},K) \leq 1$ for all $K \in \mathbf{K}$, with equality when $K > 0$). Then, under Assumption A2, we also have $f_\vee^*(K) \geq f_\wedge^*(K)$ for all $K \in \mathbf{K}$.

3.2 Some Useful Function Spaces

Note that, an RE is defined on the diagonal of the household's state space $\mathbf{K} \times \mathbf{K}$,

$$\mathbf{D} = \{K | (K,K) \in \mathbf{K} \times \mathbf{K}\}.$$

Thus, $\mathbf{D}$ is the space $\mathbf{K}$ embedded into $\mathbf{K}\times\mathbf{K}$. Recall the space B^f defined in equation (2). Then, our "first step" operators will always use as their domain the following space: $H_\nu \subset B^f$, for $\nu = \{\vee,\wedge\}$:

$$H_\nu(\mathbf{D}) := \quad \{h_1 : \quad \mathbf{D}\to\mathbf{R}_+|, h_1 \text{ is increasing and continuous, such that}$$
$$f^*_{h_1,\nu}(k) := f(k, n^*_\nu(h_1(k),k))e(k, n^*_\nu(h_1(k),k)) - h_1(k) \text{ is nonnegative and increasing in } k\}.$$

Endow H_ν with its pointwise partial order, and the topology of uniform convergence. Notice also, by Assumption A2, we have for any $h_1 \in H_\nu$, that $f^*_{h_1,\vee}(k) \geq f^*_{h_1,\wedge}(k)$.

The space H_ν has desirable chain completeness and compactness properties, and noted in the following Proposition.

Proposition 4 Under Assumption A2, H_ν is compact in the space of bounded, continuous functions endowed with the topology of uniform convergence (hence, chain complete under pointwise partial orders).

Proof. The compactness of H_ν follows from Coleman ([1997], Lemma 8), noting in Coleman's lemma, relative to our space H_ν, $u'(h_1(k))$ is falling in k for $h_1 \in H_\nu$. The chain completeness of H_ν follows as any compact partially ordered metric space is chain complete (e.g., Amann ([1977], Corollary 3.2)). ■

Although we often use B^f as our second step domain, we will also use the following subset of B^f to prove the existence of RE where the aggregate consumption function h is decreasing in the aggregate state K, and the implied RE investment is increasing in K:

$$B^f_m(\mathbf{D}) = \{h_2 \in B^f | h_2 \text{ is decreasing}\}.$$

Using our first and second step domains, can now define the ranges of our second step mappings when using the different domains B^f or B^f_m. In particular, we shall prove RE policies c^* exist for our economies in the function space $\mathbf{C}^*_\nu$ defined as follows:[11]

$$\mathbf{C}^*_\nu(\mathbf{D}, B^f) = \{h : \mathbf{K}\times\mathbf{K}\to\mathbf{R}_+|\ K\in\mathbf{K},\ k\to h(k,K)\in H_\nu \text{ and } K\to h(K,K)\in B^f\}. \tag{11}$$

for $\nu = \{\vee,\wedge\}$ indexes the set H_ν. Notice, RE policies in any of the spaces $\mathbf{C}^*_\nu(B)$ with $B = \{B^f,\, B^f_m\}$ is consistent with the necessary properties of any RE in Lemma 2. The following lemma is useful.

Lemma 5 Under Assumption A2 , (a) B^f and B^f_m are complete lattices. In addition, (b) $\mathbf{C}^*_\nu(B^f)$ is a complete lattice, while $\mathbf{C}^*_\nu(B^f_m)$ is subcomplete sublattice.

Proof. (a) To see B^f is a complete lattice, consider any subset $B_1 \subset B^f$. As the pointwise inf and sup operations on the elements of B preserve pointwise bounds, we have $0\leq \inf_x B_1 \leq f^M$, and $0\leq \sup_x B_1 \leq f^M$; hence, $\wedge B_1 \in B^f$ and $\vee B_1 \in B^f$. Therefore, B^f is a complete lattice. For $B_1 \subset B^f_m$, as the pointwise sup (resp, inf) operation preserves monotonicity, $\wedge B_1 \in B^f_m$ and $\vee B_1 \in B^f_m$. (b) For $B_1 \subset \mathbf{C}^*_\nu(B^f_m)$ as monotonicity in K (resp, equicontinuity at k), when $k = K$ are preserved also under

[11] Subsequent to this, we shall again omit the domain of $\mathbf{C}^*$ from the notation. So, for example, $\mathbf{C}^*(\mathbf{D}, B^f)$ will subsequently denoted by $\mathbf{C}^*(B^f)$.

arbitrary pointwise sup and inf operations on the compact set $\mathbf{D}$, $\wedge B_1 \in \mathbf{C}_\nu^*(B_m^f)$ and $\vee B_1 \in \mathbf{C}_\nu^*(B_m^f)$. Similarly, for $B_1 \subset \mathbf{C}_\nu^*(B^f)$. ■

Finally, as will be evident soon, to define our first step operator, we also need to choose a suitable upper bound for the second step iterations, i.e. $\bar{h}_2 \in B^f$ (resp., $\bar{h}_2^m \in B_m^f$) to be able to evaluate our externality mapping. For the space B^f, take $\bar{h}_2 \in B^f$ such that

$$f^M(k) \geq \bar{h}_2(k) = f_\nu^*(k) = f(k, n_\nu^f(k))e(k, n_\nu^f(k)) \tag{12}$$

with equality when $k > 0$. For the space B_m^f, as the greatest contingent labor supply $n_\vee^*(\hat{C}, K)$ is decreasing in $\hat{C}$, for any $h_2 \in B_m^f$, we have $n_\vee^*(h_2(K), K)$ is increasing in K. So, simply choose any $\bar{h}_2^m \in B_m^f$, such that $n_\vee^f(K) \leq n_\vee^*(\bar{h}_2^m(K), K) \leq 1$, with equality when $K > 0$. Then, for example, we may take

$$f^M(k) \geq \bar{h}_2^m(k) = f(k, n_\vee^*(\bar{h}_2^m(k), k))e(k, n_\vee^*(\bar{h}_2^m(k), k)) \tag{13}$$

with equality when $k > 0$.

3.3 The First Step

We now construct our Euler equation operators. To do this, we first rewrite the equilibrium version of the household Bellman equation in (6) on the collection of functions $(h_1, h_2) \in H_\nu \times (B^f \cap [0, \bar{h}_2])$. For the rest of this section, fix the index $\nu = \{\vee, \wedge\}$.[12]

For $k, K > 0$, $h_1 > 0$, consider the following mapping:

$$Z_\nu(\hat{c}, k, K, h_1, h_2(K)) = u'(\hat{c}) - \beta u'(h_1(f_{\hat{c},\nu}^*(k)))\bar{r}(\hat{c}, h_1(f_{\hat{c},v}^*(k)), f_{h_2(K),\nu}^*(K)) \tag{14}$$

where the distorted return on capital is given by:

$$\bar{r}(\hat{c}, K, h_1(f_{\hat{c},v}^*(k)), f_{h_2(K),\nu}^*(K)) = f_1\left(\frac{f_{\hat{c},v}^*(k)}{n_\nu^*(h_1(f_{\hat{c},\nu}^*(k)), f_{h_2(K),\nu}^*(K))}\right)e(f_{h_2(K),\nu}^*(K), n_\nu^*(h_1(f_{\hat{c},\nu}^*(k)), f_{h_2(K),\nu}^*(K))).$$

Here, for $h_1 \in H_\nu$, $\bar{r}$ is increasing and continuous in $\hat{c}$, and decreasing in (h_1, h_2), noting we have defined

$$f_{\hat{c},\nu}^*(k) = f(k, n_\nu^*(\hat{c}, k))e(k, n_\nu^*(\hat{c}, k)) - \hat{c}.$$

For $K > 0$, $h_1 \in H_\nu$, $h_1 > 0$, $h_2 \in (B^f \cap [0, \bar{h}_2])$, define the mapping $\hat{c}_\nu^*$ implicitly as follows:

$$Z_\nu(\hat{c}_\nu^*(k, K, h_1, h_2(K)), k, h_1, h_2(K)) = 0. \tag{15}$$

Then, when $(h_1, h_2) \in H_\nu \times (B^f \cap [0, \bar{h}_2])$:

$$\begin{aligned} A_\nu(h_1, h_2(K))(k) &= \hat{c}_\nu^*(k, h_1, h_2(K)), \ k > 0, \ h_1 > 0, \ h_2 < \bar{h}_2 \\ &= \bar{h}_2(k) \text{ if } h_1 > 0, \ h_2 = \bar{h}_2 \text{ for any k} \\ &= 0 \text{ otherwise.} \end{aligned} \tag{16}$$

[12] Analogously, we can restrict our second step domain by $\bar{h}_2^m$.

We first study the monotonicity and order continuity properties of the operator A_ν using Proposition 15. Recall, for a mapping $f : X \to Y$, where X and Y are each countable chain complete partially ordered sets, we say f is *order continuous* if $f(\vee X') = \vee f(X')$ and $f(\wedge X')=\wedge f(X')$ for all countable chains $X' \subset X$. If X and Y are additionally Banach spaces, say f is a *compact operator* if it is (a) continuous (relative to the norm topologies on X and Y), and (b) for any bounded $X' \subset X$, $f(X') \subset X$ is relative compact. We have the following result:

LEMMA 6 Let $\bar{h}_2 \in B^f$ be given as above. Then, under Assumptions A1 and A2, (a) for any $(h_1, h_2) \in H_\nu \times \{B^f \cap [0, \bar{h}_2]\}$, $A_\nu(h_1, h_2(K)) \in H_\nu$. Further, (b)$(h_1, h_2) \to A_\nu(h_1, h_2(K))$ is order continuous on $H_\nu \times \{B^f \cap [0, \bar{h}_2]\}$.

PROOF. (a) We first prove for any $\bar{h}_2 \in B^f$, $0 \leq \bar{h}_2 \leq f^M$, with $\bar{h}_2(k) < f^M(k)$ when $k > 0$, for the order interval $[0, \bar{h}_2] \subset B^f$, $(h_1, h_2) \in H_\nu \times [0, \bar{h}_2]$, $A_\nu(h_1, h_2(K)) \in H_\nu$.

Fix $h_2 \in [0, \bar{h}_2]$. By the continuity and monotonicity properties of $h_1 \in H_\nu$, under Assumption A1 and A2, when $k > 0$, $h_1 > 0$, $Z_\nu(\hat{c}, k, K, h_1, h_2(K))$ is decreasing and continuous in $\hat{c}$, increasing and continuous in k, for each (K, h_1, h_2). Further, noting the Inada conditions on u and f and the fact that for $h_2 \in [0, \bar{h}_2]$, $\bar{h}_2(k) - h_2(k) > 0$, we therefore have (i) the existence of a unique root $\hat{c}^*_\nu(k, K, h_1, h_2(K))$ such that $Z_\nu(\hat{c}^*_\nu(k, h_1, h_2(K)), k, K, h_1, h_2(K)) = 0$, such that (ii) $\hat{c}^*_\nu$ is increasing and continuous in k for fixed (K, h_1, h_2). Therefore, noting the definition of $A(h_1; h_2(K))$ when $k = K = 0$, we have $k \to A_\nu(h_1; h_2(K))(k)$ continuous and increasing in k.

Further, if $k_1 \geq k_2$, $A_\nu(h_1; h_2(K))(k)$ is increasing in k, we have

$$u'(\hat{c}^*_\nu(k_1, K, h_1, h_2(K))) \leq u'(\hat{c}^*_\nu(k_2, K, h_1, h_2(K))).$$

This implies, by the definition of $\hat{c}^*_\nu(k, K, h_1, h_2(K))$, the second term of Z_ν must decrease when $k_1 \geq k_2$. That is, $A_\nu(h_1; h_2(K))(k) = \hat{c}^*_\nu(k, K, h_1, h_2(K))(k)$ must be such that

$$f^*_{A_\nu(h_1;h_2),\nu}(k) = f(k, n^*_\nu(A_\nu(h_1; h_2)(k), k))e(k, n^*_\nu(A_\nu(h_1; h_2)(k), k)) - A_\nu(h_1; h_2)(k)$$

is increasing in k. Noting the definition of $A_\nu(h_1, h_2(K))(k)$ elsewhere, $A_\nu(h_1; h_2(K)) \in H_\nu$ for each $h_2 \in [0, \bar{h}_2]$. As $\bar{h}_2 \in B^f$, $\bar{h}_2(k) < f^M(k)$ when $k > 0$ was arbitrary, that proves (a).

(b) Consider a function $A : X_1 \times X_2 \to Y$, where X_1, X_2 and Y nonempty and chain complete. We begin by mentioning two facts about order continuous operators. First, the operator A is order continuous jointly in $x = (x_1, x_2)$ iff it is order continuous in each argument (see Stoltenberg-Hansen et. al. [1994], Proposition 2.4). Therefore, for claim of the lemma, it suffices to check the order continuity of A_ν in each argument separately. Second, an order continuous operator is necessarily isotone (e.g., see Dugundji and Granas [1982], p. 15).

Therefore, for a $\bar{h}_2 \in B^f$, $[0, \bar{h}_2] \subset B^f$, consider $(h_1, h_2) \in H_\nu \times [0, \bar{h}_2]$. We first show $(h_1, h_2) \to A_\nu(h_1, h_2(K))$ is isotone on $H_\nu \times [0, \bar{h}_2]$. To see this, observe for $k = K > 0$, $h_1 > 0$, $h_2 \in [0, \bar{h}_2]$, as Z_ν in (14) is decreasing and continuous in $\hat{c}$, and increasing in (h_1, h_2), the operator A_ν is isotone in (h_1, h_2). Noting the definition of A elsewhere in (16), A_ν is isotone on $H_\nu \times \{B^f \cap [0, \bar{h}_2]\}$.

By Lemma 5, $H_\nu \times B^f$ is a complete lattice; hence, countably chain complete. It follows that $H_\nu \times [0, \bar{h}_2]$ is countably chain subcomplete for any $\bar{h}_2 \in B^f$ that satisfies conditions of this lemma.

We first show A_ν preserves the supremum of countable chains in H_ν, for each $h_2 \in [0, \bar{h}_2]$. When $h_2 \in [0, \bar{h}_2]$, $k = K > 0$, consider the countable chain $\{h_1^n\}$ each $h_1^n \in H_\nu$. For each $h_2 \in [0, \bar{h}_2]$, $h_1 \to A_\nu(h_1; h_2(K))$ is simply a special case of the nonlinear operator studied by Coleman [1997] (e.g., see Coleman ([1997], equation (9), lemma 5, 6, and 8). Therefore, by a result in Coleman ([1997], lemma 9), $h_1 \to A_\nu(h_1; h_2)$ is a compact operator (therefore, continuous in both the topology of uniform and the topology of pointwise convergence). Then, by Proposition 15, $h_1 \to A_\nu(h_1; h_2(K))$ is order continuous.

Next, we next show of A_ν preserves the supremum of countable chains of $\{h_2^n\}$ each $h_2^n \in [0, \bar{h}_2]$, for each $h_1 \in H_\nu$. For fixed $h_1 > 0$, $k = K > 0$, by the continuity assumptions on the derivatives of the primitives in A1 and A2, for all $k = K > 0$, we have

$$
\begin{aligned}
Z(\hat{c}_\nu^*(k, K, h_1, \vee h_2^n(K)), k, h_1, \vee h_2^n(K)) &= \vee Z(\hat{c}_\nu^*(k, K, h_1, \vee h_2^n(K)), k, K, h_1, h_2^n(K)) \\
&= Z(\vee \hat{c}_\nu^*(k, K, \vee h_1, h_2^n(K)), k, K, h_1, h_2^n(K)),
\end{aligned}
$$

where the first line follows from the fact that under assumption A2, Z is continuous pointwise in $h_2(K)$ $n_\nu^*(\hat{C}, K)$ is continuous in $\hat{C}$ for $\nu = \{\vee, \wedge\}$, and $\vee(f - h_2^n)(K) =(f - \vee\, h_2^n)(K)$, for each K; the last line follows from the fact that under Assumptions A1 and A2, as for $h_1 \in H_\nu$, Z is continuous in $\hat{x}$. Therefore, we have

$$
A_\nu(h_1; \vee h_2(K)) = \vee A(h_1; h_2(K)).
$$

The fact that A_ν preserves infimum of countable chains in $H_\nu \times [0, \bar{h}_2]$ for all $\bar{h}_2 \in B^f$, $\bar{h}_2 < f^M$ follows from a dual argument. ■

We use this lemma to show for each $h_2 \in B^f$, the operator $h_1 \to A_\nu(h_1; h_2(K))$ has a nontrivial strictly positive greatest fixed point, and this greatest fixed point is isotone in $h_2 \in [0, \bar{h}_2]$.

Lemma 7 Under Assumptions A1 and A2, for $h_2 \in [0, \bar{h}_2]$, and $\nu = \{\vee, \wedge\}$, (a) $h_1 \to A_\nu(h_1; h_2(K))$ has a greatest fixed point $h_\nu^*(h_2(K)) \in H_\nu$, with $h_\nu^*(h_2(K))(k) > 0$ when $k > 0$; (b) this fixed point can be computed by successive approximation from f_ν^* as

$$
\inf_n A_\nu^n(f_\nu^*; h_2(K)) = h_\nu^*(h_2(K)),
$$

where $\inf_n A_\nu^n(f_\nu^*; h_2(K))(k) = \lim_n A_\nu^n(f_\nu^*; h_2(K))(k)$. Finally, (c) $h_2 \to h_\nu^*$ is isotone on $[0, \bar{h}_2]$.

Proof. (a) Existence of greatest fixed point of $h_1 \to A_\nu(h_1; h_2(K))$: For each $h_2 \in [0, \bar{h}_2]$, as $A_\nu(\cdot; h_2(K))$ is an isotone transformation of H_ν, and H_ν is a nonempty complete lattice, hence by Tarski's theorem ([1955], Theorem 1), the set of fixed points of $h_1 \to A_\nu(h_1; h_2(K))$ is a nonempty complete lattice.

(b) Computation of greatest fixed point $h_1 \to A_\nu(h_1; h_2(K))$: For given $h_2 \in [0, \bar{h}_2]$, as $h_1 \to A_\nu(h_1; h_2(K))$ is order continuous on H_ν, consider the iterations $A^n(f_\nu^*; h_2(K))$. Then, $\{A_\nu^n(f_\nu^*; h_2(K))\}_{n=0}^\infty$

is a decreasing chain. As pointwise and uniform convergence coincide in H_ν, and pointwise convergence implies order convergence in H_ν by Proposition 15, we have for each k:

$$\begin{aligned}\lim_{n\to\infty} A_\nu^n(f_\nu^*; h_2(K))(k) &= \inf_n A_\nu^n(f_\nu^*; h_2(K))(k) \\ &= h_\nu^*(h_2(K))(k) \\ &= \vee\Psi_{A_\nu}(h_2(K))(k),\end{aligned}$$

where $\Psi_{A_\nu}(h_2(K)) \subset H_\nu$ for each $h_2 \in [0, \bar{h}_2]$ is the fixed point set of $h_1 \to A_\nu(h_1; h_2(K))$, with a trivial least fixed point $\wedge\Psi_{A_\nu} = 0$. That $h_{1,\nu}^*(h_2(K))(k)$ is strictly positive, when $k > 0$ follows from a modification of a standard argument involving the Inada conditions and iterations along RE paths (e.g., Coleman ([1997], lemma 11 and Theorem 12)).

(c) Isotonicity of h_ν^* follows from Veinott's fixed point comparative statics result (i.e., Topkis [1998], Theorem 2.5.2), noting $h_2 \to A_\nu(h_1; h_2(K))$ is increasing on $[0, \bar{h}_2]$. ■

3.4 The Second Step - Existence of RE

Using Lemma 7, we can now define our RE operator A_ν^* based on the greatest fixed point of our first step operator. For $\bar{h}_2 \in B^f$ ($\bar{h}_2 \in B_m^f$) as specified above, let the "second step" operator be defined as:

$$\begin{aligned}A_\nu^*(h_2)(k) &= h_\nu^*(h_2(k))(k) \text{ for } h_2 \in [0, \bar{h}_2] \\ &= 0 \text{ else.}\end{aligned} \tag{17}$$

where the restriction of A_ν^* to the space $h_2 \in [0, \bar{h}_2] \subset B_m^f$ is denoted by $A_{m,\nu}^*(h_2)$. Then, a RE is any fixed point h^* of A_ν^* such that, when $k > 0$, $h^*(k) > 0$, and $g^* > 0$. Further, any such RE of that is a fixed point on B_m^f will additionally have RE investment g^* monotone. Recall, the RE investment is given by

$$g_\nu^*(k) = f^*(k, n_\nu^*(C^*(k), k))e(k, n_\nu^*(C^*(k), k)) - C^*(k),$$

for any $C^* \in B^f$. Similarly, define $g_{m,\nu}^*$ for any $C^* \in B_m^f$. Recall C^* is a candidate RE consumption function, while n_ν^* are the fixed points defined in lemma 3.

Our first main theorem of the paper which concerns the existence of RE using the operator A_ν^* in equation (17) in each of the spaces $[0, \bar{h}_2]$ (resp, $[0, \bar{h}_2^m]$) where the upper bounds $\bar{h}_2 \in B^f$ (resp., $\bar{h}_2^m \in B_m^f$) are given by equations (12) (resp., 13). Recall, although for each $h_2 \in [0, \bar{h}_2]$ and any K we have: $h_\nu^*(h_2(K)) \in H_\nu$ in general: $K \to h_\nu^*(h_2(K))(K) \notin H_\nu$.

Theorem 8 Under Assumptions A1, A2, for $h_2 \in [0, \bar{h}_2] \subset B^f$ we have for $\nu = \{\vee, \wedge\}$:

(a) $A_\nu^* : [0, \bar{h}_2] \to [0, \bar{h}_2]$ has a nonempty complete lattice of fixed points $\Psi_{A_\nu^*} \subset [0, \bar{h}_2] \subset B^f$, each fixed point is a RE,

(b) operator A_ν^* is order continuous on $[0, \bar{h}_2] \subset B^f$;

(c) the least $C^*_{\nu,L}$ and the greatest $C^*_{\nu,G}$ RE can be computed as follows, for each k:

$$\vee(A^*_\nu)^n(0)(k) = C^*_{\nu,L}(k) \leq C^*_{\nu,G}(k) = \wedge(A^*_\nu)^n(\bar{h}_2)(k,k)$$
$$n^*_\nu(C^*_{\nu,L}(k),k) \geq n^*_\nu(C^*_{\nu,G}(k),k);$$

(d) for $h_2 \in [0,\bar{h}^m_2] \subset B^f_m$, for the mapping $A^*_{m,\nu} : [0,\bar{h}^m_2] \to [0,\bar{h}^m_2]$ claims (a)-(d) hold for its nonempty complete lattice of RE $\Psi_{A^*_{m,\nu}} \subset [0,\bar{h}^m_2] \subset B^f_m$.

Moreover, RE policies $c^*(\cdot,\cdot,C^*_\nu) \in \mathbf{C}^*(B^f)$ (or $\mathbf{C}^*(B^f_m)$ if we consider subspace B^f_m the respectively).

Proof. We prove parts (a)-(c) for RE in $[0,\bar{h}_2]$. The proof of (d) for RE in $[0,\bar{h}^m_2]$ relative to the claims in parts (a)-(c) for $A^*_{m,v}$ follows from a similar construction.

(a) First, observe by the Inada conditions on u and f, and the definition of the range of the first step fixed point, by construction, we have $0 \leq A^*_\nu(0) \leq A^*_\nu(\bar{h}_2) \leq \bar{h}_2$ with strict equality with $k = K > 0$, where $\bar{h}_2$ is defined in equation (13). Then, by Lemma 7(c), as A_ν is isotone, and by Lemma 5 B^f is a nonempty complete lattice, the fixed point set $\Psi_{A^*_\nu} \subset [0,\bar{h}_2]$ is a nonempty complete lattice by Tarski's Theorem.

(b) From Lemma 6, A_ν is order continuous on $H_\nu \times [0,\bar{h}_2]$. We first show this implies greatest fixed point of the partial map $h_1 \to A_\nu(h_1;h_2(K))$, is order continuous in $h_2 \in [0,\bar{h}_2]$. To see this, consider the iterations from an initial point $h^0_1 := \vee H_\nu$, with the iterations given by $\{A^n_\nu(h^0_1;h_2(K))\}^\infty_{n=0}$. As order continuity is closed under composition, and evaluation maps and projections are order continuous in chain complete partially ordered sets, we conclude by the Tarski-Kantorovich theorem (Dugundji and Granas ([1982], Theorem 4.2)):

$$\begin{aligned} A^*_\nu(h_2)(k) &= \wedge A^n_\nu(h^0_1;h_2(k))(k) = A^n_\nu(\wedge h^n_1;h_2(k))(k) \\ &= \lim_{n\to\infty} A^n_\nu(h^0_1;h_2(k))(k) = h^*_\nu(h_2(k))(k) = \vee\Psi_{A_\nu}(h_2(k))(k), \end{aligned}$$

where the convergence to $\vee\Psi_{A_\nu}(h_2(k))$ is uniform, and $\vee\Psi_{A_\nu}(h_2(k))$ is order continuous in h_2, each k.

(c) By construction, we have $0 \leq A^*_\nu(0) \leq A^*_\nu(\bar{h}_2) \leq \bar{h}_2$ with strict equality with $k = K > 0$. Then, as by part (b), $A^*_\nu(h_2)$, $h_2 \in [0,\bar{h}_2]$ is order continuous, therefore, by the Tarski-Kantorovich theorem, we have

$$\vee(A^*_\nu)^n(0) = C^*_{\nu,L} = \wedge\Psi_{A^*_\nu} \leq \vee\Psi_{A^*_\nu} = C^*_{v,G} = \wedge(A^*_\nu)^n(\bar{h}_2).$$

■

First and foremost, there are no known results on the existence of *either* sequential or recursive equilibrium under these conditions.[13] In particular, no results are known on the existence of *smooth* sequential equilibrium. It bears mentioning that, in discrete time models without proving *smooth* sequential or recursive equilibria, at least locally near the steady-state, one *cannot* apply smooth dynamical systems

[13] Note that Feng et al [2014] do not apply to this economy.

methods (as is typically done in the literature to characterize local indeterminacy of equilibria).[14] Further, as the "Santos's economy" ([2002], section 3.2) is embedded in our class of economies, we already know in our case that continuous (let alone *smooth*) RE do not exist.

Second, Coleman [1997] or Datta, Mirman, and Reffett [2002] do not handle assumption A2. In particular, these papers effectively do not consider the case of labor externalities, rather the case of elastic labor supply with income taxes and/or capital externalities under very strong restrictions. Further, Mirman, Morand, and Reffett [2008] only consider inelastic labor supply and no labor externality.

Third, equilibrium responses for labor supply is in general a *correspondence* under assumption A2. This plays a key role in our analysis. In particular, our methods for verifying RE involve *Euler equation branching methods* (see Raines and Stockman [2010] and Stockman [2010]. That is, we construct a least and greatest selections of equilibrium labor supply in each period (contingent on consumption and the capital stock), and then parameterize "upper" and "lower" Euler equation operators. In general, though, these Euler equation branches are not ordered.

Fourth, if we allow for $e(K, 0) > 0$ when $K > 0$, and $e(0, \hat{N}) > 0$ when $\hat{N} > 0$, our arguments still apply. The simplest case is $e(K, \hat{N}) = (1 - \tau(K))$, for $\tau(K) \in [0, 1)$ is discussed in section 5.

Next, we should mention, in Theorem 8, we characterize the comparative statics of any RE in individual and aggregate state variables for a *fixed* economy under Assumptions A1 and A2. That is, we prove all RE are have consumption and investment monotone in individual states (and continuous), as required by Lemma 2. Aside from these structural properties of RE, relative to aggregate states, there exist RE that have (a) monotone investment decision rules jointly in individual and aggregate states, with consumption decreasing in aggregate states, but both decision rules discontinuous in general in aggregate states; and have both investment and consumption simply bounded. So RE in our economy are state asymmetric.

4 Equilibrium Comparative Statics

We now consider RE comparative statics on the space of deep parameters of the economy relative to the set of RE equilibrium in Theorem 8. The first question is related to "capital deepening" with respect to discount rate. We do the comparative statics for RE using the operator A_{ν}^{*} in B^{f} and mention as a corollary the similar comparative statics in B_{m}^{f}.

[14] The Grobman and Hartmann theorem is the key result that is usually applied in this literature in justifying local methods to study determinacy of equilibrium via topological conjugacy arguments. In discrete time, this theorem requires sequential and/or recursive equilibrium dynamics near the steady state be *smooth* (e.g., to verify the steady state is hyperbolic and topological conjugacy). No such results on existence of smooth equilibrium dynamics in models with labor externalities (or large capital externalities) are known.

Further, given the results in Santos [1991] for Pareto optimal economies, one would assume such conditions would be *very strong*, requiring global strong concavity conditions for household dynamic programs along equilibrium paths. Hence, a global argument is needed to study indeterminancy.

Theorem 9 - *Capital deepening in discount rates. Under Assumptions A1 and A2, for* $\bar{h}_2 \in B^f$ given in expression (12), *we have (a) for the least fixed point* $C^*_{\nu,L}(\beta) \in [0, \bar{h}_2]$ (resp., greatest fixed point $C^*_{\nu,G}(\beta) \in [0, \bar{h}_2]$) *for* $\beta_1 \geq \beta_2$, $C^*_{\nu,L}(\beta_1) \leq C^*_{\nu,L}(\beta_2)$ (resp, $C^*_{\nu,G}(\beta_1) \leq C^*_{\nu,G}(\beta_2)$) with RE investment $g^*_{\nu,G}(\beta_1) \geq g^*_{\nu,G}(\beta_2)$ (resp., $g^*_{\nu,L}(\beta_1) \geq g^*_{\nu,L}(\beta_2)$), and the associated labor supply $N^*_{\nu,L}(\beta_2)(k) := n^*_\nu(h^*_{\nu,L}(\beta_2)(k), k) \geq n^*_\nu(h^*_{\nu,L}(\beta_1)(k), k) =: N^*_{\nu,L}(\beta_1)(k)$ (resp., $N^*_{\nu,G}(\beta_2) \geq N^*_{\nu,G}(\beta_1)$). Also, (b) the RE comparative statics can be computed by the successive approximations as follows:

$$\vee(A_\nu)^n(0; \beta_1) = C^*_{\nu,L}(\beta_1) \leq C^*_{\nu,L}(\beta_2) = \vee(A_\nu)^n(0; \beta_2)$$
$$\wedge(A_\nu)^n(\bar{h}_2; \beta_1) = C^*_{\nu,G}(\beta_1) \leq C^*_{\nu,G}(\beta_2) = \wedge(A_\nu)^n(\bar{h}_2; \beta_2)$$

Finally (c): the claims in (a) and (b) hold for the least and greatest fixed points of $A^*_{m,v}$ on B^f_m.

Proof. Noting its dependence on the parameter β, we will do the case of RE in $C^*_\nu(\beta) \in [0, \bar{h}_2]$. The exact same argument works for $h_2 \in [0, \bar{h}^m_2]$.

Noting the definition of $A_\nu(h_1, h_2(K); \beta)$, it is decreasing in β. As by definition, $A^*_\nu(h_2; \beta)$ is the greatest fixed point of the partial map $h_1 \to A_\nu(h_1; h_2(K); \beta)$ by Veinott's fixed point comparative statics theorem, $A^*_\nu(h_2; \beta)$ is decreasing in β.

By Theorem 8(c), the mapping $h_2 \to A^*_\nu(h_2; \beta)$ is order continuous in h_2. Then, by the Tarski-Kantorovich theorem, we have

$$\begin{aligned} \vee(A^*_\nu)^n(0; \beta_1) &= C^*_{\nu,L}(\beta_1) \\ &\leq C^*_{\nu,L}(\beta_2) \\ &= \vee(A^*_\nu)^n(0; \beta_2), \end{aligned}$$

Further, as $n^*_\nu(c, k)$ is decreasing and continuous in c, we have for RE labor supply:

$$n^*_\nu(C^*_{\nu,L}(\beta_1)(k), k) \geq n^*_\nu(C^*_{\nu,L}(\beta_2)(k), k).$$

By a dual argument, we could proceed for the greatest fixed point. Noting the definition of RE investment associated with least and greatest RE consumption $C^*_{\nu,L}(\beta)$ and $C^*_{\nu,G}(\beta)$, we have RE investment $g^*_{\nu,G}(\beta_1) \geq g^*_{\nu,G}(\beta_2)$ (resp., $g^*_{\nu,L}(\beta_1) \geq g^*_{\nu,L}(\beta_2)$), which completes the proof. ■

Notice, in Theorem 9, we compare RE labor supply for different discount rate β, as well as investment and consumption. Mirman, Morand, and Reffett [2008] and Acemoglu and Jensen [2015] provide similar monotone comparison result for dynamic economies with inelastic labor supply, small capital externalities, and no labor externality. There is no obvious way to extend their results in models with elastic labor supply, labor externality, and especially, large capital externalities.

5 Applications and Discussion

In this section, we relate our contribution vis-a-vis some classical results in the literature. In particular, we apply our methods to Romer [1986], Benhabib and Farmer [1994] economies and we conclude with a detailed discussion on Santos [2002].

5.1 Romer [1986]

In the spirit of Romer [1986] economy, we introduce inelastic labor supply and no leisure-labor choice in assumptions A1 and A2. That is, consider the following special case of our general assumptions:

Assumption Romer[1986] *Modify Assumption A1 with* $v(l) = 0$ *for all* $l \in [0,1]$ *and Assumption A2 with* $\tilde{e}(K) := e(K,1)$ *rising in* K.

If $f_1(K,1)\tilde{e}(K)$ is falling in K, then there exists an unique RE by Coleman [1991, 2000] or Mirman, Morand, and Reffett [2008]. On the other hand, if $f_1(K,1)\tilde{e}(K)$ is rising in K, Theorem 8 implies existence of the least and the greatest RE. Further, Theorem 9 provides RE comparative static results. We interpret $f_1(K,1)\tilde{e}(K)$ is decreasing in K as the case of "small" externality and $f_1(K,1)\tilde{e}(K)$ increasing in K as the case of "large" externality.[15] For example, consider $f(k,1) = k^{\alpha}$ and $\tilde{e}(K) = K^{a}$ for $a, \alpha \geq 0$; with $a + \alpha < 1$ implying small externality and $a + \alpha > 1$ implying large externality. Notice that, in case of inelastic labor supply, we have a single Euler equation operator, with no Euler equation branching.[16]

5.2 Benhabib-Farmer [1994]

The economies studied in Benhabib and Farmer [1994] provide another important application of our results, as well as the (case of symmetric RE in) dynamic models with heterogeneous firms and credit constraints found in recent paper by Liu and Wang [2014], or some heterogeneous agent economies with adverse selection (e.g., Benhabib, Dong, Wang [2014]). To obtain Benhabib-Farmer [1994] style economies, we modify our assumptions as follows:

Assumption Benhabib-Farmer*[1994]* $u(c)$ *and* $v(l)$ *are each power utility in consumption and leisure (or,* $u(c) = \ln c$*) in Assumption A1, and technology is Cobb-Douglas -* $F(k,n,K,\hat{N}) = k^a n^b K^c \hat{N}^d$ *with* $f(k,n) = k^a n^b$, $e(K,\hat{N}) = K^c \hat{N}^d$, *and* $a,b,c,d > 0$, *such that* $a + b = 1$, $a + c > 1$, $b + d > 1$ *in Assumption A2.*

By Theorem 8, a complete lattice of RE exists in both $[0, \bar{h}_2]$ and $[0, \bar{h}_2^m]$, while by Theorem 9, we can compare the least and the greatest RE in each of these function spaces in the discount rate. As Benhabib and Farmer [1994] show the equivalence of the "*laissez faire*" versions of their models to monopolistic competition, our results apply to these decentralizations also. Liu and Wang [2014] have recently produced a very interesting dynamic economy, where credit constraints on heterogeneous firms generate a set of sequential equilibrium conditions in a symmetric equilibrium that are observationally equivalent to the model of Benhabib and Farmer [1994] without appealing to increasing returns directly. Thus, our results are applicable to that class of models as well.

[15] We should mention, the case of indeterminacy occuring in economies with inelastic labor such that the equilibrium return on capital could be increasing in K is mentioned in Boldrin and Rustichini [1994].

[16] Note that in this case, we need to take a bound f^M in the definition of the second step space B^f to be slightly higher: i.e., let $\hat{N} > 1$, so $f(K,\hat{N})\tilde{e}(K) > f(K,1)\tilde{e}(K)$. As the first step operator uses the upper bound for output to be $F(K,1,K,1)$, the first step greatest fixed point $h^*(h_2(K))(K) < f(K,\hat{N})\tilde{e}(K)$, hence if we take $\bar{h}_2 = f(K,1)\tilde{e}(K) < f(K,\hat{N})\tilde{e}(K)$ in the second step, $A^*(\bar{h}_2) < \bar{h}_2$.

Finally, we can develop a global theory of equilibrium comparisons of recursive sunspot equilibria for the models with multiple equilibria such as those in Benhabab and Farmer [1994] by extending the approach of Spear [1991]. Since our construction is *global,* the resulting theory of stationary sunspot equilibrium is *global.* It is not clear how to obtain such results using *local* approaches in standard methods available in the literature. For example, Spear's results (Spear [1991]) for the existence of *continuous* stationary sunspots in models with positive externalities such as Romer [1986] are not easy to extend. The complication is clear from Lemma 3. As contingent labor supply is a correspondence, it does not generally admit smooth selections; rather, only *continuous* selection that depend on steady-state capital; hence, the implicit function theorem cannot be applied at the steady-steady. The global approach avoids this complication. Further, as we generate RE with monotone dynamics in capital, we can study the question of existence of stationary sunspot equilibria using monotone Markov process methods.

5.3 Santos [2002] and Relationship of our Methods to the Literature

A common approach to studying the (local) structure of dynamic equilibrium in the literature is to use the methods of smooth dynamical systems to characterize the recursive or sequential equilibrium dynamics near a steady state. Unfortunately, these methods are difficult to apply if a locally smooth dynamic equilibrium does not exist. For rigor, one must first prove the existence of a sequential equilibrium; then, if the model is continuous time, and one can additionally prove the resulting sequential equilibrium is the solution of an autonomous differential equation having smooth extension of equilibrium dynamics near steady states.[17] In this case, one can use the Grobman-Hartman Theorem, and/or versions of the stable manifold theorem to study the local properties of RE. We are not aware of such results for the class of economies studied in this paper, even in continuous time.

Unfortunately, for discrete time models, the smoothness issue reduces to proving the existence of a smooth extension of the recursive (or sequential) equilibrium near the steady-state. This requires one to first prove the existence of a sequential equilibrium; then, to show one such sequential equilibrium can be smoothly extended over an open set of the steady state. For models considered in this paper, such methods seem to be unavailable. First, multiplicity of contingent equilibrium labor supply in Lemma 3 create difficulty in applying these methods even locally near steady states. That is, at best we may find continuous selections for contigent RE labor supply, not smooth selections. Further, the [2002] counterexample (even with inelastic labor supply) proves smooth local equilibria may not exist. More to the point, there is *non-existence* of smooth local recursive equilibria near steady states. So it does not seem easy to apply the methods of smooth dynamical systems to characterize even the local structure of RE, let alone its global structure as we do in Sections 3 and 4.

The prototype example in Santos [2002] (section 3.2) has been discussed extensively in the literature as it provides a striking example of where all known methods for the existence of minimal state RE fail. The example is important for our paper as it is a prototypical example of so-called *policy-induced* inde-

[17] See Hirsch, Smale, and Devaney [2004] (Chapter 7, especially Chapter 7.4).

terminacy.[18] It bears mentioning that this economy is *identical* to those studied in Coleman [1991],[2000] and Mirman, Morand, and Reffett [2008], except for allowing a *regressive* as opposed to a progressive income tax.

We conclude this paper by discussing how we specialize the economies studied in this paper to include the example studied in Santos [2002], as well as how all current methods fail, and our new methods are able to provide sharp characterizations of RE. Let's begin with a discussion of how to specialize our assumptions to cover his economy:

Assumption Santos[2002] *The household preferences are such that $v(l) = 0$ for all $l \in [0,1]$, the period utility function $u(c)$ is either satisfying A1 or $u(c) = \ln c$ for $c \in R_{++}$. Labor is supplied inelastically (normalized to 1). Production technology is given by $f(k,n)$, where k stands for capital and n for labor hired by the firm. Function f satisfies conditions in Assumption A2 with $e(K,\hat{N}) = 1$. In addition, there is a Lipschitz continuous and monotone income tax function, $\tau : \mathbf{K} \to [0,1)$.*

Coleman [1991] analyzes the case of progressive taxation in which τ is a monotone increasing function of aggregate state. The case of regressive taxation (or, monotone decreasing τ) includes the example in Santos ([2002], section 3.2), where it is claimed that a continuous RE does not exist. We should also mention that Peralta-Alva and Santos [2010] demonstrate (numerically) indeterminacy of sequential equilibria near the unstable steady state for the case of regressive taxation. Note that, under regressive taxation a sequential equilibrium does exist by an argument following Crettez and Morhaim [2012].

The household enters any given period with an individual level of capital $k \in \mathbf{R}_+$, facing an economy in aggregate state $K \in \mathbf{R}_+$where K is the per-capita capital stock and aggregate labor, $\hat{N} = 1$. Household income in state (k,K) is $r(K)k + w(K)$, where $r(K) := f_1(K,1)$ is the rental rate for capital and $w(K) := f_2(K,1)$ is the wage rate. Profits are zero by constant returns to scale. The income tax proceeds are redistributed as lump-sum transfer $J(K)$ back to households under a balance budget, $J(K) := \tau(K)(r(K)k + w(K))$. Denoting investment by x, the agent or household's budget constraint is,

$$c + x \leq \{(1-\tau(K))(r(K)k + w(K)) + J(K)\} =: y_\tau(k,K). \tag{18}$$

With this notation, we formally define *a minimal state space recursive equilibrium* as a list of functions C^*, V^*, c^*, w, r, J such that for each $K > 0$:

1. Taking prices w, r and law of motion $C \in B^f(\mathbf{K})$ as given $V^* : \mathbf{K} \times \mathbf{K} \times B^f \to \mathbf{R}$ satisfies the following Bellman equation for the household problem:

$$V^*(k,K;C) = \max_{c \in [0, y_\tau(k,K)]} \{u(c) + \beta V^*(y_\tau(k,K) - c, g(K;C);C)\}, \tag{19}$$

with $c^*(k,K;C)$ being the argument solving the right hand side of the Bellman equation.

[18] The question of policy induced indeterminancy has been discussed extensively in the recent literature. The early literature includes papers by Schmidt-Grohe and Uribe [1997], Guo and Lansing [1998], and Guo and Harrison [2004], while more recently, work by Nourry, Seegmuller, and Venditti [2013] and Nishimura, Seegmuller, and Venditti [2015], among others, have addressed this issue.

2. Taking prices w, r as given firms solve the profit maximization problem:

$$\max_{k,n\geq 0} f(k,n) - r(K)k - w(K)n.$$

3. Consistency:

$$\begin{aligned} C^*(K) &= c^*(K, K; C^*), \\ C^*(0) &= 0. \end{aligned}$$

4. Government budget balance: $J(K) = \tau(K)[r(K)k + w(K)]$ and

5. Market clearing: $c^*(K, K; C^*)+ g^*(K, K; C^*) = f(K, 1) = y_\tau(K, K)$.

Next, for the sake of notation, list of function in the definition of the RE are simplified and denoted by using a consumption function C^* and the corresponding law of motion g^*. The unique optimal solution $c^*(k, K; C^*)$ in (19) has strong structural properties in the individual state k (although not in aggregate state K). The next proposition is a counterpart of lemma 2 from section 2.3.

Proposition 10 Under assumptions, (a) an RE consumption function $c^*(k, K; C^*)$ is increasing and Lipschitz in k, and (b) an RE investment function $x^*(k, K; C^*) = y_\tau(k, K) - c^*(k, K; C^*)$ is increasing and Lipschitz in k.

Proof. A standard argument shows that under our assumptions, the value function $V^*(k, K; C^*)$ is strictly concave and continuous in k, for each K, has a smooth envelope

$$V_1^*(k, K; C^*) = u'(c^*(k, K; C^*))r(K)(1 - \tau(K)),$$

and $V_1^*(k, K; C)$ is decreasing in k; hence, consumption $c^*(k, K; C^*)$ is increasing in k which proves one part of (a). To prove (b), notice that the necessary and sufficient first order characterization of the unique optimal investment $x^* = x^*(k, K; C^*)$ is,

$$u'((y_\tau - x^*)(k, K, C^*)) - \beta u'((y_\tau - x^*)(x^*, g^*(K); C^*)\ r(g^*(K))(1 - \tau(g^*(K)) = 0, \tag{20}$$

where $g^*(K) = \tilde{f}(K) - C^*(K)$. From (a), a consumption $c^*(k, K; C^*) = (y_\tau - x^*)(k, K; C^*)$ is monotone increasing in k, as when k rises, the left-hand side of (20) falls in k. As the continuation consumption $c^*(k', K'; C^*) = (y_\tau - x^*)(k', K'; C^*)$ is also monotone increasing in k, this implies $x^*(k, K; C^*)$ is increasing in k, which proves part of (b).

Finally, since $y_\tau(k, K)$ is Lipschitz for $K > 0$, $c^*(k, K; C^*)$ and $x^*(k, K; C^*)$ are both Lipschitz with a Lipschitz constant bounded by the Lipschitz constant of $y_\tau(k, K)$, namely $F_1(K, 1)$ when $K > 0$. ■

We make a few remarks.

First, according to Proposition 10, *any* RE in this economy must be *continuous* in individual state k, so any discontinuities in a RE must occur only in aggregate states K.

Second, if we approach the question of existence and characterization of RE via dynamic lattice programming methods (e.g., as discussed in Mirman, Morand, and Reffett [2008] and Acemoglu and Jensen [2015]) it is easy to see issues that arise under regressive taxation as the return on capital is not monotone in g. That is, a lattice programming argument cannot determine whether $c^*(k, K; C^*)$ is increasing or decreasing in C^*, and without further characterization of equilibrium single crossing properties; hence, existence of an RE would need to be verified, for example, by a topological argument. This complicates the question of constructing monotone equilibrium comparative statics in the deep parameters of the economy (β, τ). Further, obtaining sharp characterizations of RE that have joint monotonicity properties of RE investment in both individual and aggregate states (i.e., monotonicity of $K \to c^*(K, K; C^*)$ in any RE C^*) directly via application of dynamic lattice programming, as in Mirman, Morand, and Reffett [2008], is not possible as requisite single-crossing properties are not evident. Actually, single-crossing properties are only shown to be held in *particular subclasses* of RE.

Third, if we try to apply Coleman's monotone map method to verify existence of any RE with regressive taxation, we also run into problems. To see that, following Coleman [1991], let us take a "guess" at future consumption function $C \in H$ where

$$H(\mathbf{D}) = \{C : D \to D | 0 \leq C(K) \leq f(K),\ C \text{ is continuous and increasing with } f - C \text{ increasing in } K\}.$$

Next, we define a mapping $Z_c(\hat{c}, k, K, C)$ based on the Euler equation as follows - for $C \in H$, $0 < C(K) < \tilde{f}(K)$, $K > 0$,

$$Z_c(\hat{c}, k, K, C) = u'(\hat{c}) - \beta u'(C(y_\tau(k, K) - \hat{c}))r(\tilde{f}(K) - \hat{c})[1 - \tau(\tilde{f}(K) - \hat{c})]. \tag{21}$$

The Coleman monotone map operator is,

$$\begin{aligned} A_c(C)(K) &= \hat{c}^*(K, K, C) \text{ such that } Z_c(\hat{c}^*(k, K, C), k, K, C) = 0, \\ &= 0 \text{ if } C = 0. \end{aligned} \tag{22}$$

Under progressive taxation, everything works well: that is, A_c is single-valued, isotone and has a nontrivial fixed point, which is a recursive equilibrium.[19] Also, the fixed point can be computed by successive approximations as the limits of non-stationary recursive equilibria for finite horizon economies. Further, one can show that the fixed point is increasing in β, and decreasing in τ.[20] However, under regressive taxation, the Coleman monotone map operator A_c is *not single-valued*; rather, it is nonempty upper hemicontinuous correspondence and does not necessarily admit a continuous selection in its first argument k, for each K, let alone a Lipschitz section as in Proposition 10. Therefore, as Santos [2002] points out, the Coleman monotone map method cannot verify existence of an RE or characterize equilibrium comparative statics.

Our methods in this paper do work for the case of regressive taxation. Since we are interested in comparing RE, we specify explicitly how our new Euler equation operator depends on (β, τ), the deep

[19] Keep in mind, $C = 0$ is a trivial fixed point.

[20] Here, "decreasing in τ" is in the pointwise partial order: $\tau_1 \geq \tau_2$ if $\tau_1(K) \geq \tau_2(K)$ for all K.

parameters of the economy. Let $h_1 \in H$, $h_2 \in B^f$, for $0 < h_1 < \tilde{f}$, $k > 0$, define

$$Z(\hat{c}, k, K, h_1; h_2(K), \beta, \tau) = u'(\hat{c}) - \beta u'(h_1(\tilde{f}(k) - \hat{c}))r(\tilde{f}(k) - \hat{c})(1 - \tau(\tilde{f}(K) - h_2(K))). \tag{23}$$

Next, define the operator A as follows:

$$\begin{aligned} A(h_1, h_2, K; \beta, \tau)(k) &= \hat{c}^*(k, K, h_1, h_2; \beta, \tau) \text{ s.t. } Z(\hat{c}^*(k, K, h_1, h_2; \beta, \tau), k, K, h_1; h_2, \beta, \tau) = 0, \quad (24) \\ &= 0 \text{ otherwise.} \end{aligned}$$

Notice, the operator defined in (24) differs from Coleman's operator defined in (22) only by how it treats the "tax" in the second term of (21) versus (23). In this method, we add an additional step to the computation of the fixed point (compose τ with $h_2 \in B^f$) which allows us to study the complementarity structure of the household equilibrium Euler equation in "two-steps". In effect, the decomposition of the equilibrium fixed point problem deconstructs the single-crossing property for the household's problem into a single crossing property (in equilibrium) in two parts, one part isolating "individual" state dynamic complementarities, and a second part isolating "aggregate" state dynamic complementarities.

In terms of the example we are considering in this section: for any $h_2 \in B^f$, $h_1 \to A(h_1, h_2, K; \beta, \tau)$ is isotone. Also, the partial map $h_1 \to A(h_1, h_2, K; \beta, \tau)$ is *precisely* Coleman's "monotone map" operator embedded as a "first-step" operator in a "two-step" procedure. As H is a complete lattice under pointwise partial order, by Tarski's theorem, our first-step operator has a complete lattice of fixed points $\Psi_A \subset H$ for each (h_2, K, β, τ) with a trivial greatest fixed point at 0, and a *unique* (least) strictly "interior" fixed point $h^*(h_2, K, \beta, \tau) \in H$.[21]

Next, we use the least fixed point to define a second step operator

$$\begin{aligned} A^*(h_2; \beta, \tau)(K) &= h^*(h_2, K; \beta, \tau)(K), \;\; h_2 \in B^f, K > 0 \\ &= 0 \text{ else.} \end{aligned}$$

By Veinott's fixed point comparative statics result (e.g., Topkis ([1998], Theorem 2.5.2), $h^*(h_2, K; \beta, \tau)$ is increasing in h_2 on B^f for each (β, τ), with $h^*(h_2, K; \beta, \tau)$ also increasing in β, and decreasing in τ. By construction, $A^*(h_2; \beta, \tau) \in B^f$ and B^f is a nonempty complete lattice. Therefore, by Tarski's theorem, the fixed point set of operator A^* denoted by $\Psi_{A^*}(\beta, \tau)$ is a nonempty complete lattice and *each fixed point is an RE.*

Importantly, by construction for any $C^*(\beta, \tau) \in \Psi_{A^*}(\beta, \tau) \subset B^f$ we have that the optimal solution to the household dynamic program $(k, K) \to c^*(k, K, C^*; \beta, \tau) \in \mathbf{C}^*(\mathbf{D}, B^f(\mathbf{K}))$, where we recall that $\mathbf{C}^*(\mathbf{D}, B^f(\mathbf{K}))$ is the set of candidate RE consumption policy functions.

Further, as A^* is increasing in β and decreasing in τ, by a standard fixed point comparative statics argument, under *either* progressive or regressive taxation, the least RE, $\wedge\Psi_{A^*}(\beta, \tau)$, and the greatest RE, $\vee\Psi_{A^*}(\beta, \tau)$, are increasing in β and decreasing in τ.

Notice that we can provide sharper characterization of *some* RE (namely, some RE has stronger monotonicity properties for investment). That is, assume additionally at function $K \to \tilde{f}(K) - h_2(K)$

[21] Existence follows from Coleman [1991] and uniqueness of strictly positive fixed point for the "first step" operator follows from Coleman [2000]. These are also derived as corollaries in our paper.

is increasing. Denoting by $A^*_m(h_2;\beta,\tau)$ the restriction of the mapping $A^*(h_2;\beta,\tau)$ to B^f_m. Now $A^*_m : B^f_m \to B^f_m$ and, as B^f_m is a subcomplete lattice of B^f, the fixed point set of A^*_m, by Tarski's theorem, is nonempty complete lattice of RE consumption functions having an associated RE investment monotone increasing (but, in general, discontinuous). Finally, as $A^*_m(h_2;\beta,\tau)$ restricted B^f_m is also increasing in β, and decreasing in τ, we have the same robust equilibrium comparative statics relative to (β,τ) obtained in Coleman [1991] and Mirman, Morand, and Reffett [2008] not only for progressive taxes, but also for the case of regressive taxes. The difference between the cases is that, in general, we have multiple equilibria with regressive taxes as we show multiple *subclasses* of RE along with robust equilibrium comparative statics.

The main limitation of our approach is obtaining sufficient structure for the multi step *monotone* method to work. Specifically, if the tax structure is not monotone our methods cannot be applied. Also, the case of nonseparable preferences are complicated to handle even in models for which sequential equilibrium is known to exist e.g., Crettez, Morhaim [2012].

We conclude with a generalization of equilibrium comparative statics results for a class of distorted economies with elastic labor supply. First note that the economy with an income tax corresponds to a "reduced form" production specification in the spirit of assumption A2. In particular, we can rewrite the income process for our economy $y(k,n,K;\hat{N})$ in equation (5) as a "reduced form" production function $f(k,n)e(K,\hat{N})$ (e.g., see Greenwood and Huffman [1995], Coleman [2000] and Datta, Mirman, and Reffett [2002]). In this case, you can define $\hat{e}(K,\hat{N}) = e(K,\hat{N})(1-\tau(K)) \in [0,1)$, where e satisfies assumption A2, and τ is either progressive or regressive. Further, noting that under constant returns to scale, in equilibrium after imposing the balanced budget rule, we can define our operator as in the main section, and verify the existence of RE in Theorem 8.

We assume proceeds of the income tax are returned as lump-sum transfers $J(K)$ to the household, where these transfers satisfy a balanced-budget taxation rule

$$J(K) = \tau(K)y(K, N^*(C^*(K),K), K; N^*(C^*(K),K)),$$

where N^* is any RE labor supply in Theorem 8. Then, when $k = K$, noting constant returns to scale and zero profits in private returns for the firms, the income process in a RE for a household is

$$\begin{aligned} y_\tau(k, N^*(C^*(k),k), k, N^*(C^*(k),k)) &= (1-\tau(k))\{r(k,N^*(C^*(k),k))k + w(k,N^*(C^*(k),k))N^*(C^*(k),k)\} + J(k) \\ &= f(k,N^*(C^*(k),k))e(k,N^*(C^*(k),k)). \end{aligned}$$

Under Assumptions A2, an RE exists and can be computed by Theorem 8. To obtain our RE comparison results, we substitute the equilibrium relationship $y_\tau(k,N^*(k),k;N^*(k)) = f(k,N^*(k))e(k,N^*(k))$, into the definition of A_ν in equations (14), (15), and (16), noting the dependence of $A_\nu(h_1,h_2(K);\tau)$ on the tax, we have the following important result:

Theorem 11 - *Policy Comparative Statics. Under Assumptions A1 and A2, for $\bar{h}_2 \in B^f$ (a) for the least RE $C^*_{\nu,L}(\tau) \in [0,\bar{h}_2]$* (resp., greatest RE $C^*_{\nu,G}(\tau) \in [0,\bar{h}_2]$ *we have for* $\tau_1(K) \geq \tau_2(K)$ for all $K \in \mathbf{K}$, *then* $C^*_{\nu,L}(\tau_1) \geq C^*_{\nu,L}(\tau_2)$ (resp, $C^*_{\nu,G}(\tau_1) \geq C^*_{\nu,G}(\tau_2)$) with RE investment $g^*_{\nu,G}(\tau_1) \leq\ g^*_{\nu,G}(\tau_2)$ (resp., $g^*_{\nu,L}(\tau_1) \leq\ g^*_{\nu,L}(\tau_2)$), and the associated RE labor supply $n^*_\nu(C^*_{\nu,L}(\tau_2)(k), k) \leq n^*_\nu(C^*_{\nu,L}(\tau_1)(k), k)$ (resp, $n^*_\nu(C^*_{\nu,G}(\tau_2)(k), k) \leq n^*_\nu(C^*_{\nu,G}(\tau_1)(k), k)$). (b) these RE comparative statics can be computed as follows:

$$\begin{aligned}
\vee(A^*_\nu)^n(0;\tau_1) &= C^*_{\nu,L}(\tau_1) \geq C^*_{\nu,L}(\tau_2) = \vee(A^*_\nu)^n(0;\tau_2) \\
\wedge(A^*_\nu)^n(\bar{h}_2;\tau_1) &= C^*_{\nu,G}(\tau_1) \geq C^*_{\nu,G}(\tau_2) = \wedge(A^*_\nu)^n(\bar{h}_2;\tau_2).
\end{aligned}$$

Finally (c) the claims in (a) and (b) hold for the least and greatest RE computed using $A^*_{m,v}$ for $h_2 \in [0, \bar{h}^m_2] \subset B^f_m$.

Proof. We prove (b) first; then (a) follows from the argument directly. By the order continuity of $h_2 \to A^*_{m,v}(h_2;\tau)$ on B^f_m by the Tarski-Kantorovich theorem, we have

$$\begin{aligned}
\vee(A^*_{m,\nu})^n(0;\tau_1) &= h^*_\nu(\tau_1) \geq h^*_\nu(\tau_2) = \vee(A^*_{m,\nu})^n(0;\tau_2) \\
\text{with } n^*_\nu(h^*_{m,\nu}(\tau_1)(k), k) &\leq n^*_\nu(h^*_{m,\nu}(\tau_2)(k), k),
\end{aligned}$$

with $g^*_{m,\nu}$ increasing in (k) for each τ, and the last line follows from the fact that $n^*_\nu(c,k)$ is continuous and decreasing in c. (a) For each $h_1 \in H_\nu$, $h_2 \in B^f_m$, using the fact that under lump-sum transfers, $y_\tau(k, N^*(C^*(k),k), k; N^*(C^*(k),k)) = f(k, N^*(C^*(k),k))e(k, N^*(C^*(k),k))$ is independent of τ in any RE, when $k > 0$, $h_1 > 0$, $h_2 < y^*_{f,v}$, by the definition $A_\nu(h_1, h_2(K);\tau)$ is increasing in τ. Then, as in the proof of previous theorem, by Veinott's fixed point comparative statics theorem has $A^*_{m,\nu}(h_2;\tau)$ is increasing in τ. ■

6 Appendix: Fixed Point Theory in Ordered Spaces

6.1 Mathematical Terminology

Posets and lattices: A *partially ordered set* (or poset) is a set P ordered with a reflexive, transitive, and antisymmetric relation. If any two elements of $C \subset P$ are comparable, C is referred to as a linearly ordered set, or chain. A *lattice* is a set L ordered with a reflexive, transitive, and antisymmetric relation $\geq$ such that any two elements x and x' in L have a least upper bound in L , denoted $x \wedge x'$, and a greatest lower bound in L, denoted $x \vee x'$. $L_1 \subset L$ is a *sublattice* of L if it contains the sup and the inf (with respect to L) of any pair of points in L_1. A lattice is *complete* if any subset L_1 of L has a least upper bound and a greatest lower bound in L. L_1 is *subcomplete* if it is complete and a sublattice. In a poset P, if every subchain in $C \subset P$ is complete, then C is referred to as a *chain complete poset* (or CPO). If every countably subchain in C is complete, then C is referred to as a *countably chain complete poset* (or CCPO). Let $[a) = \{x | x \in P, x \geq a\}$ be the *upperset* of a, and $(b] = \{x | x \in P, x \leq b\}$ the *lowerset* of b. We say P is an *ordered topological space* if $[a)$ and $(b]$ are closed in the topology on P. An order interval is defined to be $[a, b] = [a) \cap (b]$, $a \leq b$.

Isotone (or order preserving) mappings on a poset: Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be Posets. A mapping $f : X \to Y$ is *increasing* (or isotone) on X if $f(x') \geq_Y f(x)$, when $x' \geq_X x$, for $x, x' \in X$. If

$f(x') >_Y f(x)$ when $x' >_X x$, we say f is *strictly increasing.* [22] The mapping $f : X \to Y$ is *join preserving* (resp, *meet preserving)* if we have for any countable chain C, $f(\vee C) = \vee f(C)$ (resp, $f(\wedge C) = \wedge f(C)$). A mapping that is both join and meet preserving is *order continuous.*

A correspondence (or multifunction) $F : X \to 2^Y$ is *ascending* in a binary set relation $\rhd$ on 2^Y if $F(x') \rhd F(x)$, when $x' \geq_X x$. Let $\mathbf{X}$ be a poset, $\mathbf{Y}$ a lattice, and define the relation $\rhd = \geq_v$ on the range $\mathbf{L(Y)}$ of all nonempty sublattices of $\mathbf{Y}$, where for $L_1, L_2 \in \mathbf{L(X)}$ we say $L_1 \geq_v L_2$ in *Veinott's Strong Set order* if for all $x_2 \in L_2$, $x_1 \in L_1$, $x_1 \vee x_2 \in L_1$, $x_1 \wedge x_2 \in L_2$.

Fixed points. Let $F : X \to 2^X$ be a non-empty valued correspondence. $x \in X$ is a fixed point of F if $x \in F(x)$. If F is a function, a fixed point is $x \in X$ such that $x = F(x)$. For $F : X \times T \to 2^X$ denote by $\Psi_F(t) : T \to 2^X$ the fixed point correspondence of F at $t \in T$.

6.2 Some Useful Fixed Point Theorems

One critical result we use through the paper is Veinott's version of Tarski's theorem. His result is stated in the next proposition.

Proposition 12 Veinott ([1992], chapter 4, Theorem 14). Let X be a nonempty complete lattice, T a poset, $F : X \times T \to 2^X$ a nonempty, subcomplete-valued correspondence that is Veinott's strong set order ascending. Then, (i) $\Psi_F(t)$ is a nonempty complete lattice, and (ii) $\vee \Psi_F(t)$ and $\wedge \Psi_F(t)$ are isotone selections from $T \to X$.

Tarski's original theorem (Tarski ([1955]), Theorem 1) occurs as a special case of Proposition 12, where $F(x,t) = f(x)$, and $f : X \to X$ is a function. An important extension of Tarski's theorem is given by Markowsky ([1976], theorem 9) and is stated in the next proposition. The fixed point comparative statics result in the proposition per least (resp, greatest) fixed points is a corollary of a theorem proven in Heikkila and Reffett ([2006], Theorem 2.1), which in turn implies $t \to \Psi_F(t)$ is weak-induced ascending upward and downward.

Proposition 13 Let X be a CPO, T a poset, $f : X \times T \to X$ increasing in x, each $t \in T$. Then, (i) $t \to \Psi_F(t)$ is nonempty CPO, (ii) $t \to \wedge \Psi_F(t)$ (resp, $\vee\ \Psi_F(t)$) are increasing selections.

For our results, we will need constructive versions of Proposition 12 and Proposition 13. For this, we will assume for each $t \in T$, the partial map $f_t : X \to X$ is order continuous. For this case, we have the following version of Tarski-Kantorovich-Markowsky theorem. The characterization of the fixed point set in (i) is from Balbus, Reffett, and Woźny [2014]. The computability result is the classic Tarski-Kantorovich theorem (e.g., Dujundgi and Granas ([1982], Theorem 4.2)). There is a dual version for the greatest selections.

[22] To avoid using references to "isotone mapping", we will often use the more traditional terminology in economics of "increasing". In the literature on partially ordered sets, an "increasing map" often denotes something slightly different (e.g., $f(x') \geq_Y f(x)$ when $x' >_X x$ for x, $x' \in X$).

Proposition 14 Let X be a CCPO, T a poset, $f : X \times T \to X$ order continuous in x, each t , and $\exists$ a $x_L \in X$ such that $x_L \leq f(x_L, t)$. Denote by $\Psi_f(t) : T \to 2^X$ the fixed point correspondence of f at $t \in T$. Then, (i) $\Psi_F(t)$ is nonempty CCPO. Further, the iterations $\sup_n f^n(x_L, t) = \wedge\Psi_f(t)$. Finally, if in addition, X and T are each continuous domains, and f is additionally order continuous on T, then the mapping $t \to \wedge\Psi_f(t)$ is order continuous.

6.3 Order and Uniform Topologies

Consider a mapping $f : X \to Y$, where X and Y are each countable chain complete partially ordered sets. We say f is *order continuous* if $f(\vee X') = \vee f(X')$ and $f(\wedge X')=\wedge f(X')$ for all countable chains $X' \subset X$. If X and Y are additionally Banach spaces, say f is a *compact operator* if it is (a) continuous (relative to the norm topologies on X and Y), and (b) for any bounded $X' \subset X$, $f(X') \subset X$ is relatively compact. We have the following result:

Proposition 15 Say $X(S)$ a collection of functions on $S = [0, 1]$, $X(S)$ compact in the topology of uniform convergence and endowed with the pointwise partial order, $f : X(S) \to X(S)$ is isotone and compact. Then, f is order continuous on $X(S)$.

Proof. As $X(S)$ is compact in the topology of uniform convergence, $X(S)$ is compact in the topology of pointwise convergence (as pointwise and uniform convergence coincide in $X(S)$); hence, $X(S)$ is chain complete in the pointwise partial order on $X(S)$ (Amann, [1977], corollary 3.2). As $f : X(S) \to X(S)$ is isotone and continuous on $X(S)$ in the topology of uniform convergence, f is continuous in the pointwise topology. This implies f is continuous in the interval topology of $X(S)$ associated with pointwise partial orders (as interval topology in this case coincides with the uniform topology/pointwise topology in $X(S)$). Hence, f is order continuous in pointwise partial orders. ■

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