Differential Information in Large Games with Strategic Complementaries*

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Abstract

We study equilibrium in large games of strategic complementarities (GSC) with a differential information and continuum of players. For our game, we define an appropriate notion of distributional Bayesian-Nash equilibrium in the sense of Mas-Colell (1984), and prove its existence. Further, we characterize the order-theoretic properties of the equilibrium set, provide monotone comparative statics results for ordered perturbations of the space of games, and provide explicit algorithms for computing extremal equilibrium. Our results make extensive use of the recent results on aggregating single crossing properties in Quah and Strulovici (2012). We complement these with new results on the existence of Bayesian-Nash equilibrium in the sense of Balder and Rustichini (1994) or Kim and Yannelis (1997) for large GSC, and provide analogous results for this notion of equilibrium. To obtain our results, we prove an new fixed point theorem on monotone operators in countably complete partially ordered sets. Applications of the results include riot games, "beauty contests" and common value auctions.

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1 Introduction and related literature

Beginning with the seminal papers of Schmeidler (1973) and Mas-Colell (1984), an important class of games that has been studied by game theorists and economists are games with a continuum of players indexed on a measure space. Since early results in this class of games, economists have generalized them a great deal, including extending the framework to games with incomplete or differential information. In such setting, incomplete or differential information can be modeled using player types as private information in the tradition of Harsanyi (1967) (often

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with the use of some strong law of large numbers as in Jovanovic and Rosenthal (1988), Morris and Shin (2001), Acemoglu and Jensen (2010)). Alternatively, Balder and Rustichini (1994) and Kim and Yannelis (1997) consider large games with differential information where only one state is drawn but observed according to a private σ-field. Each of these approaches is suitable to a specific economic problem at hand. For example, a private signal approach is appropriate to model features such as random taste or productivity draws, while the differential information is more appropriate to model settings like riot games, beauty contests, or common value auctions, where one can envision the situation under study as a game where there is a single true state that is differentially perceived by players.

The first approach involves the well-known technical complication of studying games with a continuum of random variables which are drawn independently from an identical distribution. In many cases, such draws do not satisfy the Strong Law of Large Numbers (see Feldman and Gilles (1985); Judd (1985) for an early discussion of this issue, as well as Alós-Ferrer (1998), or Khan, Rath, Sun, and Yu (2013) for some more recent developments). In such situations, even though there are several examples of distributions and indexation of agents that satisfy an appropriate notion of the LLN for continuum of random variables, these are either constructed somewhat arbitrarily with a substantial degree of freedom (see Judd (1985)), or require a certain level of abstraction of the space of players (e.g., see Green (1994); Sun (2006)).[1] In this paper, we follow a different tradition, that of Balder and Rustichini (1994) and Kim and Yannelis (1997), by considering large games with differential information. This setting also provides a natural interpretation of an important class of large Bayesian games. In adopting this approach, we are able to extend the results in this literature per existence and characterization of Bayesian Nash equilibrium a great deal.[2]

The second and equally important strand of literature in game theory (especially focusing on the existence of pure strategy equilibrium) which has found a large number of applications in economics (as well as operations research) concerns the so-called supermodular games (or, if one prefers, games with strategic complementaries, henceforth: GSC). See the seminal works of Topkis (1979), Vives (1990), and Milgrom and Roberts (1990) for examples. In a GSC, questions such as the existence of pure strategy Nash equilibrium does not hinge on convexity and upper-hemicontinuity concerns per best reply maps, but rather merely on the an appropriate notion of increasing best replies (in a well-defined set-theoretic sense) on a complete lattice of actions. In such a situation, the powerful fixed point theorems of Tarski (1955) and/or Veinott (1992) can be brought to bear on the existence question. Moreover, in parameterized versions of these games, one can obtain sufficient conditions for monotone equilibrium comparative statics.[3] Finally, as researchers often want to compute the Nash equilibrium comparative statics, in principle constructive methods for computing how extremal equilibrium vary in the deep parameters of the game. In such cases, these results can be directly linked to theoretical numerical methods for computing these equilibrium, hence one can ask questions concerning computable equilibrium comparative statics. As key limitation though per the existing literature on GSC is that for such games, research has primarily focused on games with a *finite* number of players.

In the current paper, we integrate these two diverse literatures, and consider the question of equilibrium existence and characterization in large games with differential information in the

[1]The space of players might be either endowed with a σ-field which is not countably generated, hence not Borel, like in Green (1994), or cardinality of the player set might be larger than continuum, as in Sun (2006).

[2]Compare also with Balder (2002) unifying the approach to equilibrium existence, and Yannelis (2009) results stressing the role of continuity of expected (interim) utility in the existence of equilibrium.

[3]There are some cases conditions for equilibrium comparative statics can be developed with topological approaches. See, for example, the discussion in Villas-Boas (1997).

presence of strategic complementaries. To do this, we extend the existing literature on GSC with a finite number of player and incomplete information (e.g., Athey (2001, 2002); or Reny (2011); Vives and Van Zandt (2007) more recently) into the context of a large game with a continuum of players. Unlike the above mentioned papers strategies, and one of the critical contributions of this paper, we obtain our existence results in the context strategies that are *not monotone* with respect to the signal, but are pointwise increasing with respect to strategies of other players.[4] Our approach is therefore more similar to that of Vives (1990) and Van Zandt (2010). As a result, this paper can also be interpreted as a direct extension of the approach taken in Balbus, Dziewulski, Reffett, and Woźny (2012), when studying games with complete information.

We start with the Mas-Colell's version of the our large game. When studying this setting, we first propose an appropriate notion of Nash equilibrium in the context of large game with differential information, and then, using the powerful fixed point machinery in the seminal work of Markowsky (1976), concerning isotone transformations of chain complete partially ordered sets[5], we prove existence of a distributional equilibrium under different set of assumptions than those studied in the extensive literature that has followed since Mas-Colell (1984).

We next turn to the Nash-Schmeidler equilibrium. Here, we obtain our results using a new generalization of Tarski-Kantorovich fixed point theorem which we prove in Theorem A.4 and its Corollary A.8.[6] Via our constructions, we are able to develop explicit methods for the computation if Nash-Schmeidler equilibrium, results that cannot be addressed directly by any known topological approaches in the existing literature. Specifically, we are able to prove all of our results using exclusively *constructive* methods. This latter fact also becomes of central importance when one considers the question of nature of equilibrium comparative statics in our class of games. In particular, we are able to prove a theorem of *computable monotone comparative statics* relative to ordered perturbations of the deep parameters of the space of primitives of a game.

The remainder of the paper is organized as follows. In Section 2, we prove existence of distributional equilibrium existence. In Section 3, we prove Bayesian-Nash equilibrium existence and equilibrium comparative statics. We then provide some economic applications of our results in Section 4. Definitions, auxiliary results, and proofs of all the results in the paper then are found in Section A and B.

2 Distributional Bayesian-Nash equilibria

We begin by describing the game we study as defined by Kim and Yannelis (1997). Let $(S, \mathcal{S}, \mu)$ denote a measure space, with S being a set of states of the world, and μ a complete probability measure defined on $\mathcal{S}$ being a completion of its Borel σ-field. Let Λ be a complete, separable and metrizable space of players. Endow Λ with a non-atomic σ-finite probability measure λ, defined on a Borel σ-field $\mathcal{L}$, such that $(\Lambda, \mathcal{L}, \lambda)$ constitutes a measure space.

[4] Keep in mind that as a game with a finite number of players is a degenerate case of our class of large games. Hence, our results extend the existing results per existence and computation of Nash equilibrium in games with a finite number of players using our new monotone operator theoretic methods, but for the case where equilibra are not necessarily monotone themselves. Therefore, we complement the important recent results obtained in Vives and Van Zandt (2007) and Van Zandt (2010).

[5] A partially ordered set $(X, \geq)$ is a chain (or a totally ordered set), if for any $x, y \in X$, either $x \geq y$, or $y \geq x$. Therefore, a partially order set (poset) $(X, \geq)$ is chain complete (CPO), if any chain $Y \subset X$, has a supremum and infimum in X. If the last condition holds only for countable subsets Y, we say that X is a countably chain complete poset (CCPO).

[6] See the Appendix of the paper for the statement and proof of these two new results.

Actions of the players are assumed to take place in $A \subset \mathbb{R}^n$, which is endowed with the Euclidean topology generating Borel σ-field $\mathcal{A}$ on A, and a natural coordinate-wise partial order $\geq$. As the equilibria of our game will be defined in terms of distributions over the Cartesian product of the space of players and their actions, we need to define a proper space of distributions. Along these lines, let $\Lambda \times A$ be the product space endowed with an order $\geq_p$, satisfying the following condition[7] : $(\alpha, a) \geq_p (\alpha', a') \Rightarrow a \geq a'$. Let $D(\Lambda \times A)$ denote the set of all measures on $\Lambda \times A$ defined on a product σ-field $\mathcal{L} \otimes \mathcal{A}$ such that for any $\nu \in D(\Lambda \times A)$, its marginal distribution on Λ is λ. Endow $D(\Lambda \times A)$ with weak* topology, and the corresponding Borel σ-field $\mathcal{D}$, as well as the partial order of first order stochastic dominance (henceforth, FOSD)[8], denoted $\succeq_D$. Clearly, $\succeq_D$ is consistent[9] with $\geq_p$. By $D_\Delta \subset D(\Lambda \times A)$, we denote the subset of probability measures on $\Lambda \times A$ with its induced topology and σ-field $\mathcal{D}_\Delta$ of a Borel sets of weak* topology.

We next define the feasible actions and payoffs contingent on the available information. Let correspondence $\tilde{A} : \Lambda \times S \rightrightarrows A$ denote a set of actions feasible to player $\alpha \in \Lambda$ who finds herself in state $s \in S$. By $r : \Lambda \times S \times D_\Delta \times A \to \mathbb{R}$, we denote the real-valued ex-post payoff function for a player, where $r(\alpha, s, \tau, a)$ is payoff value of player α, using action $a \in A$, in state $s \in S$, when the distribution of actions of other players is τ. By $\mathcal{S}_\alpha$, we denote a sub σ-field of $\mathcal{S}$ (denoting the private information of agent $\alpha \in \Lambda$), and $\pi_\alpha : S \to \mathbb{R}_+$ will be the prior distribution of agent $\alpha \in \Lambda$, where π_α is the Radon-Nikodym derivative such that $\int_S \pi_\alpha(s) d\mu(s) = 1$.

Since agents choose their actions contingent on observable signal, the distribution of actions will differ depending on the realized state of the world. For this reason, in order to properly define interim payoffs structure, we need to construct a collection of distributions over the space of players and actions, which will in turn depend on the actual signal. To do this, let $\tau : S \to D_\Delta$ be a function mapping space S to the set of probability distributions on $\Lambda \times A$. In order to avoid confusion, we shall denote the value of function τ in s by $\tau(\cdot|s)$.[10]

In some cases, we need to consider the equivalence classes of τ , which we denote by $[\tau] := \{f : S \to D_\Delta | f(\cdot|s) = \tau(\cdot|s), \mu\text{-a.e.}\}$, as well as the set of all equivalence classes containing only measurable functions, which is denoted by $\hat{T}$. By completeness of a measure μ, the measurability of τ implies a measurability of any $\tau' \in [\tau]$. We endow $\hat{T}$ with the pointwise order $\succeq_{\hat{T}}$ with respect to the equivalence classes. That is, we have the partial order given by $\tau' \succeq_{\hat{T}} \tau$ iff $\tau'(\cdot|s) \succeq_D \tau(\cdot|s)$, μ-a.e. In addition, let

$$\hat{T}_\Lambda := \{\tau \in \hat{T} | \tau(\{(\alpha, a) \in \Lambda \times A | a \in \tilde{A}(\alpha, s)\}|s) = 1, \mu\text{-a.e.}\}.$$

Finally, given the set CM of continuous and monotone functions defined on the graph of the correspondence $\tilde{A}$, define

$$\hat{T}_d := \left\{ \tau \in \hat{T}_\Lambda | \forall f \in CM, \int\limits_{\Lambda \times A} f(\alpha, a) \tau(d\alpha \times da|s) \in M(S) \right\},$$

[7] Clearly, if Λ is an ordered set, this condition is satisfied if $\geq_p$ is a product order. However, this condition may be satisfied, for example, if this order is defined as follows: $(\alpha, a) \geq_p (\alpha', a') \Leftrightarrow (\alpha = \alpha')$ and $(a \geq a')$. In this case, space of agents may be *unordered*.

[8] That is, we say that $\tau' \succeq_D \tau$ iff $\int f(a, \alpha) \tau'(d(a, \alpha)) \geq \int f(a, \alpha) \tau(d(a, \alpha))$ for every increasing, bounded and measurable $f : \Lambda \times A \to \mathbb{R}_+$. We endow the space of distributions D with the first order stochastic ordering $\succeq_P$, as the applications that we have in mind involves this partial ordering. In fact, a careful examination of our results shows, that that they hold for any *arbitrary* partial ordering on D, as long as it implies chain complete structure on the space of probability measures contained.

[9] See Wieczorek (1978) how to define an order on player-action space $\Lambda \times A$, that is consistent with order on $D(\Lambda \times A)$.

[10] Please note, that this is not equivalent to the regular conditional distribution, but to a distribution parametrized by s.

where $M(S)$ denotes the space of $\mathcal{S}$-measurable functions. Hence, $\hat{T}_d$ is a set of equivalence classes of functions mapping S to D_Δ with values (i.e. probability distributions) concentrated on the graph of $\tilde{A}(\cdot, s)$ for μ-a.e. $s \in S$. In addition, we require that for any continuous, monotone function f, $E(f) := \int_{\Lambda \times A} f(\alpha, a)\tau(d\alpha \times da|\cdot)$ be $\mathcal{S}$-measurable. The reason why we impose this additional condition will be explained in detail in the reminder of the paper.

We are now ready to characterize agent decision problems in the game. The sequence of the game is as follows. First, each player observes the state of the world $s \in S$, with respect to his private information $\mathcal{S}_\alpha$. Then, players calculate their interim payoffs defined by $v : \Lambda \times S \times \hat{T} \times A \to \mathbb{R}$,

$$v(\alpha, s, \tau, a) := \int_{\varepsilon_\alpha(s)} r(\alpha, s', \tau(\cdot|s'), a)\pi_\alpha(s'|\varepsilon_\alpha(s))d\mu(s'),$$

where

$$\pi_\alpha(s'|\varepsilon_\alpha(s)) := \begin{cases} \frac{\pi_\alpha(s')}{\int_{\varepsilon_\alpha(s)} \pi_\alpha(s'')d\mu(s'')} & if \quad s' \in \varepsilon_\alpha(s), \\ 0 & if \quad s' \notin \varepsilon_\alpha(s). \end{cases}$$

and $\varepsilon_\alpha(s)$ is the smallest set in $\mathcal{S}_\alpha$ under set inclusion containing s. Once the strategies are chosen, payoffs are distributed.

The game can be summarized by the tuple $\Gamma := \{(\Lambda, \mathcal{L}, \lambda), (S, \mathcal{S}, \mu), A, \tilde{A}, r, \{\pi_\alpha, \mathcal{S}_\alpha\}_{\alpha \in \Lambda}\}$. Further, its distributional Bayesian-Nash equilibrium can be define as follows:

Definition 2.1 *A distributional Bayesian-Nash equilibrium of Γ is an equivalence class $\tau^* \in \hat{T}_d$ such that μ-a.e.*

$$\tau^* \left(\left\{ (\alpha, a) | v(\alpha, s, \tau^*, a) \geq v(\alpha, s, \tau^*, a'), \forall a' \in \tilde{A}(\alpha, s) \right\} |s \right) = 1,$$

and the marginal distribution of $\tau^*(\cdot|s)$ over Λ is given by λ.

Our definition of distributional Bayesian-Nash equilibrium generalizes the concept of distributional equilibrium developed by Mas-Colell (1984) to the case of a differential information setting as well as players' characteristics (see e.g Khan, Rath, Yu, and Zhang (2013)). Observe, in the above definition, we consider an equilibrium to be an *equivalence* class of functions $\tau \in \hat{T}_d$, and not a single function; hence, given a function, the above definition need only hold in $s \in S$ μ-a.e. Eventually, we shall define equilibrium in a rather specific class of functions $\hat{T}_d$, rather than the class $\hat{T}_\Lambda$. Clearly, therefore, there might exist functions in $\hat{T}_\Lambda \backslash \hat{T}_d$ satisfying our definition of Bayesian-Nash equilibrium. However, since our existence result holds in $\hat{T}_d$, we restrict our definition solely to the space.

In order to prove existence of a distributional Bayesian-Nash equilibrium in Definition 2.1, we impose conditions summarized in the following assumption.

Assumption 2.1 *Let*

(i) $\tilde{A}$ be a complete sublattice valued, such that $\tilde{A}(\cdot, s)$ has a compact graph for all s. Moreover, let $\tilde{A}$ be weakly measurable;[11]

(ii) r is continuous and quasi-supermodular[12] *on A, satisfies single crossing properties*[13] *in (a, τ), is $\mathcal{L} \otimes \mathcal{S} \otimes \mathcal{A}$-measurable, whenever $\tau \in \hat{T}_d$;*

[11]As the action space is a subset of $\mathbb{R}^n$, complete-latticed valued implies compact-valued.
[12]See definition A.1 in the appendix.
[13]See definition A.3 in the appendix.

(iii) for λ-a.e. player, $\{r(\alpha, s, \cdot)\}_{s\in S}$ is a family of Q-summable functions on A, with functionally S-summable differences in (a, τ);[14]

(iv) $\forall\alpha$, $\mathcal{S}_\alpha$ is generated by a countable partition such that $\forall s \in S$, $\mu(\varepsilon_\alpha(s)) > 0$.

Assumptions 2.1(i),(ii),(iv) are standard. Assumption 2.1(iii), however, requires some additional comment. In this assumption, we require that the quasi-supermodularity of payoff r be preserved under summation to the interim value v (and similarly for single crossing differences).[15] This is necessary for our arguments as ordinal properties (in this case ordinal complementarity properties) are not generally preserved by aggregation. Our assumption was first proposed in Quah and Strulovici (2012), where there it was referred to as "signed-ratio monotonicity" or S-summability. There is a delicate difference, however, between the related assumption in Quah and Strulovici (2012) of S-summability, and our functional S-summability condition above. Note first, when analyzing our games with incomplete information, we shall be interested in the aggregation of ordinal difference properties for values that are estimated at *different* points in their domain. This fact changes the nature of the S-summability condition we require relative to Quah and Strulovici (2012).

To see this, for example, consider an optimization problem facing an agent who is uncertain about two possible states, say elements of the set $\{H, L\}$, where state H occurs with probability p, and state L occurs with probability $(1-p)$, respectively. Assume that the agent maximizes his expected payoff taking into account strategies of the other players, which each depend on the eventual state. Therefore, for some function $f : \{H, L\} \to D_\triangle$, the agent maximizes

$$u(H, f(H), a)p + u(L, f(L), a)(1-p),$$

where a denotes the decision variable. As $f(H)$ might not be equivalent to $f(L)$, the above function need not to have single crossing differences (even when $u(s, t, a)$ has single crossing differences in (a, t), and $\{u(s, \cdot)\}_{s\in\{H,L\}}$ is a family of functions with S-summable differences). Because of this, we need a somewhat stronger version of the S-summability condition in Quah and Strulovici (2012) to guarantee we have sufficient complementarities in the game for our methods to apply. Finally, notice if r is supermodular in a and has increasing differences in (a, τ), then our assumption (iii) obviously holds.

We now present a series of lemmas, as well as the key proposition, that will allows us to prove our main equilibrium existence result. The main tool using in our proofs of this section of the paper are based on Markowsky's fixed point theorem (see also Theorem A.2 in the Appendix), which in our context, shall be a theorem concerning the fixed points of monotone operators increasing with respect to $\succeq_{\hat{T}}$ that also transform the chain complete poset $\hat{T}_d$. Along these lines, we start by showing that $(\hat{T}_d, \succeq_{\hat{T}})$ is a chain complete poset, a property required to apply Markowsky (1976)'s theorem.

Proposition 2.1 *$(\hat{T}_d, \succeq_{\hat{T}})$ is a chain complete poset.*

Observe, this proposition greatly generalizes a well-known (and related) result in Hopenhayn and Prescott (1992) to differential information setting. It is also echnically very different result than used in Balbus, Dziewulski, Reffett, and Woźny (2012), which studies the case of our game in a setting of complete information.

[14]See Definition A.2 and A.8 in the Appendix for the characterization of the two notions.

[15]We refer the reader to Appendix A.1 for formal definitions concerning lattice theoretical properties of our game.

Next, we define an operator that shall play a central role in the proof of our existence result. To do this, first define the best reply correspondence for a player to be:

$$m(\alpha,s,\tau):=\arg\max_{a\in\tilde{A}(\alpha,s)} v(\alpha,s,\tau,a),$$

and consider the extremal selections $\overline{m}(\alpha,s,\tau):=\bigvee m(\alpha,s,\tau)$, $\underline{m}(\alpha,s,\tau):=\bigwedge m(\alpha,s,\tau)$, when they exist in $m(a,s,\tau)$. By the definition of equilibrium in Definition 2.1, for μ-a.e. s, distributional Bayesian-Nash equilibrium τ^* satisfies

$$\tau^*(\{(\alpha,a)\in\Lambda\times A|a\in m(\alpha,s,\tau^*)\}|s)=1$$

To construct this object τ^*, consider an operator that returns the greatest distributional Bayesian-Nash equilibrium selection $\overline{B}:\hat{T}_d\to\hat{T}_d$ by:

$$\overline{B}(\tau)=\left\{\tau'\in\hat{T}_d|\tau'(Gr(\overline{m}(\cdot,s,\tau))|s)=1,\mu\text{-a.e.}\right\}.$$

Notice, we can obviously define the least distributional Bayesian-Nash equilibrium selection $\underline{B}(\tau)$ in an analogous way.

To study the property of this operator, we need to show if r has increasing differences in (a,τ), and $\{r(\alpha,s,\cdot)\}_{s\in S}$ is a family functions with functionally $\mathcal{S}$-summable differences in (a,τ) with respect to a pointwise order ($\succeq_T$), v has single crossing differences in the μ-a.e. pointwise order ($\succeq_{\hat{T}}$). We state the following lemma.

Lemma 2.1 *Let $(T,\mathcal{T},\mu)$ be a non-empty, non-atomic, σ-finite measure, M be a set of $\mathcal{T}$-measurable functions $f:T\to X$, where $(X,\succeq_X)$ is a poset, endowed with σ-field $\mathcal{X}$, $u:T\times X\to\mathbb{R}$ with $u(t,x)$ a $\mathcal{T}\otimes\mathcal{X}$-measurable, single crossing function in x for all $t\in T$, and $\{u(t,\cdot)\}_{t\in T}$ be a functionally S-summable family of functions on M, with respect to the pointwise order. Then, $h:M\to\mathbb{R}$,*

$$h(f):=\int_T u(t,f(t))d\mu(t)$$

is a single crossing function with respect to μ-a.e. pointwise order.[16]

By Lemma 2.1, integration preserves $S-$summability in the μ-a.e. pointwise order. This fact, therefore, explains why in our existence constructions we do *not* work with standard pointwise partial orders.[17]

We now construct the extremal operators $\overline{B}$ and $\underline{B}$, and characterize the monotonicity properties of the pair of operators under Assumption 2.1:

Lemma 2.2 *Let Assumption 2.1 be satisfied. Then, operators $\overline{B}$ and $\underline{B}$ are well defined and $\succeq_{\hat{T}}$-isotone.*

Having these two lemmata in place, we are now able to state our main result of this section.

Theorem 2.1 (Existence) *Let Assumption 2.1 be satisfied. Then, there exists the greatest and the least distributional Bayesian-Nash distributional equilibrium of Γ in $(\hat{T}_d,\succeq_{\hat{T}})$.*

[16]Function f' dominates f in M with respect to the pointwise order, iff $\forall t\in T, f'(t)\succeq_X f(t)$. Function f' dominates function f in M with respect to μ-a.e. pointwise order, iff $f'(t)\succeq_X f(t)$ for μ-a.e. $t\in T$.

[17]Similar concerns, but in a quite different context, arise in the work of Van Zandt (2010).

Few remarks concerning this result.

First, the above theorem shows not only the existence of distributional Bayesian-Nash equilibrium, but it also characterizes the equilibrium set (i.e., there exists greatest and least elements). We should also remark that by Markowsky's theorem, both $\overline{B}$ and $\underline{B}$ have a chain complete poset of fixed points, each one of them constituting a Bayesian-Nash distributional equilibrium of the game.

Second, the sufficient conditions for existence that we impose to obtain the existence theorem *differ* from those used in Balder and Rustichini (1994) or Kim and Yannelis (1997). In particular, as compared to their work, our class of games relaxes an important payoff-continuity assumption.[18]

Finally, and very importantly, our Assumptions 2.1(ii),(iii) can be relaxed to prove existence solely of the greatest (resp., the least) distributional equilibrium of game Γ. Specifically, we can easily replace the condition (a) of quasi-supermodularity (equivalent to lattice superextremality in Li Calzi and Veinott (1992) and Veinott (1992) for real-valued functions) of payoff r in actions $a \in A$ with join- (resp., meet-) superextremality, and condition (b) concernning single crossing differences with join (resp., meet) up-crossing differences in (a, τ), and condition (c) concerning the the Q and S-summability of the payoff function with their join- (resp., meet-) counterparts, and still be able to to obtain greatest (resp., least) distributional equilibrium. This weakening of our conditions actually generalizes our results to an even broader class of large games. See Veinott (1992) and Li Calzi and Veinott (1992) for the details, as well as Section A.1 for the definitions, and corresponding results. This observation becomes useful when one is unable to show that the game in question is actually quasi-supermodular. In fact, we provide exactly such an example in Section 4.3, where we discuss the application of our results to common value auctions.

We conclude this section of the paper by consider the question of computing distributional Bayesian-Nash equilibrium. We prove two results, one pertaining to computing extremal equilibrium at fixed parameters, and another concerning the existence of computable equilibrium comparative statics (as a function of deep parameters of the game). Such a question has not be considered in any of the existing literature of which we are aware. For such computability results, we need to impose one additional condition concerning order-continuity of payoffs, which proves critical in our main results per preserving order continuity conditions (in an interval topology) to extremal selections in best reply maps. This additional assumption is the following:

Assumption 2.2 *For any monotone sequence $\{\tau_n\}$ in D_Δ, such that $\tau_n \to \tau$ in D_Δ, let $r(\alpha, s, \tau_n, a) \to r(\alpha, s, \tau, a)$.*

We remark, if r satisfies Assumption 2.1 in addition to Assumption 2.2, then as r is continuous in $a \in A$, we have $r(\alpha, s, \tau, a)$ jointly order continuous in (a, τ) for each (α, s).

Given this additional assumption, we proceed with the following corollary to our main existence result. We should mention that this result is of utmost importance for designing numerical methods aimed to compute equilibrium distributions (and proving a rigorous foundation for their use).

Corollary 2.1 *Let Assumptions 2.1 and 2.2 be satisfied, and $\bar{t}, \underline{t}$ denote the greatest and the least element of $\hat{T}_d$ respectively. Then, the greatest and least distribution Bayesian-Nash equilibrium of Γ satisfies the following successive approximation condition: $\overline{\tau}^* = \lim_{n\to\infty} \overline{B}^n(\bar{t})$, $\underline{\tau}^* = \lim_{n\to\infty} \underline{B}^n(\underline{t})$.*

[18] See also Yannelis (2009) for a discussion of payoff continuity conditions needed in previous work in the literature.

Finally, we can consider the question of computable equilibrium comparative statics. To study this question, we add similar structure to parameterized versions of our game, and discuss the existence of monotone distributional equilibrium comparative statics. Let Θ denote a partially ordered space of parameters θ. For any fixed $\theta \in \Theta$, define a parameterized version of our large game by the tuple:

$$\Gamma(\theta) := \{(\Lambda, \mathcal{L}, \lambda), (S, \mathcal{S}, \mu), A, \tilde{A}(\theta, \cdot), r(\theta, \cdot)\{\pi_\alpha, \mathcal{S}_\alpha\}_{\alpha \in \Lambda}\},$$

That is, for each θ, game $\Gamma(\theta)$ is defined as in the first part of this section. For our parameterized version of the game $\Gamma(\theta)$, we proceed with the following natural extensions of our original assumptions:

Assumption 2.3 *For any $\theta \in \Theta$, let $\Gamma(\theta)$ satisfy Assumption 2.1. Moreover, let*

(i) $\tilde{A}(\alpha, \cdot)$ be increasing in the Veinott strong set order on Θ;

(ii) r have single crossing differences in (a, θ);

(iii) family $\{r(\cdot, s, \cdot)\}_{s \in S}$ have S-summable differences in (a, θ).

The next result follows from Theorem A.8. That is, for any $\theta \in \Theta$, let $\overline{\tau}^*(\theta)$ (resp., $\underline{\tau}^*(\theta)$) be the greatest (resp., least) distributional Bayesian-Nash equilibrium in $\Gamma(\theta)$. Then we have the following monotone equilibrium comparative statics result.

Corollary 2.2 *Let Assumptions 2.1-2.3 be satisfied. Then $\overline{\tau}^*(\cdot)$ and $\underline{\tau}^*(\cdot)$ are increasing on Θ.*

Note, apart from our related paper Balbus, Dziewulski, Reffett, and Woźny (2012) on large GSC with complete information, we are not aware of any similar strong comparative statics result, with the one notable (and important) exception being Acemoglu and Jensen (2010). In Acemoglu and Jensen (2010), the authors consider aggregative games with finite number of player types, but otherwise develop similar tools. In an important sense, their approach to equilibrium comparative statics is very similar to ours, as they also impose conditions guaranteeing that the joint best response mapping has increasing selections with respect to parameter s (e.g., see Definition 3 in their paper). However, as they concentrate only on *aggregative* games, where players best respond to the average/mean action of other players, the class of games they analyze is quite different (and more restrictive) than ours. In particular, our class of games include theirs, but also allows for considerably more general classes of large games. Also, in case of a single dimensional action space A, Acemoglu and Jensen do, though, manage to show comparative statics of the extremal (aggregative) equilibria using results of Milgrom and Roberts (1994) *without* the single crossing property between player actions and aggregates. This is a very important result, and more general than ours, where we restrict attention aggregative large games. However, for multidimensional case of large aggregative games, Acemoglu and Jensen require increasing differences in the action of a player and the aggregate , which is stronger than our (ordinal) single crossing properties we use. Finally, Acemoglu and Jensen use a topological fixed point theorem (Kakutani) to show existence of an aggregate equilibrium, which makes the issues of equilibrium comparative statics and computability of equilibrium difficult (if not impossible) to address. On the contrary, we use focus exclusively order-theoretical fixed point results, where sufficient conditions to address both of these issues are very direct.

3 Bayesian Nash-Schmeidler equilibria

In the next section of the paper, we present corresponding results for an alternative description of equilibrium in a large game with differential information that is often found in the literature (namely, where equilibria are defined in terms of functions mapping players to actions as in Schmeidler (1973)). Along these lines, we begin with introducing the notation.

As in the previous section, $(S, \mathcal{S}, \mu)$ is a complete, measure space, with S being a set of states of the world, and μ a probability measure defined on $\mathcal{S}$ being a completion of its Borel σ-field. Let Λ be a compact, metrizable space of players. Endow Λ, with a non-atomic, σ-finite probability measure λ defined on a Borel σ-field $\mathcal{L}$, such that $(\Lambda, \mathcal{L}, \lambda)$ constitutes a measure space, and let $A \subset \mathbb{R}^n$ be a set of actions of players, endowed with the Euclidean topology generating Borel σ-field $\mathcal{A}$ on A, and a natural coordinate-wise order $\geq$. As we now work with a notion of equilibrium that involves joint actions of other players (as opposed to distributions), we now identify by $M(\Lambda \times S)$ the set of functions for joint-actions of players of type $\alpha \in \Lambda$ in state s to be mappings $f : \Lambda \times S \to A$ that are measurable with respect to product σ-field $\mathcal{L} \otimes \mathcal{S}$. Endow $M(\Lambda \times S)$ with its product topology and the pointwise order.

We now reconsider the components of the game, and define an alternative notion of equilibrium. As before, the correspondence of feasible actions will be $\tilde{A} : \Lambda \times S \rightrightarrows A$ which in particular assigns a set of feasible actions to player $\alpha \in \Lambda$, who finds herself in state $s \in S$. What is different in this formulation of the game, is the ex-post payoffs are given by a function $r : \Lambda \times S \times M(\Lambda \times S) \times A \to \mathbb{R}$, where $r(\alpha, s, f(\cdot, s), a)$ is the payoff value of player α, playing action a at state s, when the joint action of all other players in state s is given by $f(\cdot, s) \in M(\Lambda \times S)$. Let $\mathcal{S}_\alpha$ be a sub-σ-field of $\mathcal{S}$, denoting the private information of agent α. Moreover, let $\pi_\alpha : S \to \mathbb{R}_+$ be the prior of agent $\alpha \in \Lambda$, where π_α is the Radon-Nikodym derivative, such that $\int_S \pi_\alpha d\mu = 1$.

With this investment in notation, we are now able to describe the sequence of play in the game. First, each player observes the state of the world $s \in S$ with respect to its private information set $\mathcal{S}_\alpha$. The players then calculate their interim payoffs. If we let $\varepsilon_\alpha(s)$ be the smallest (by set inclusion) set in $\mathcal{S}_\alpha$ containing s, then the interim payoff for a player of type $\alpha \in \Lambda$ in state $s \in S$, facing joint action $f \in M(\Lambda \times S)$ is defined by function $v : \Lambda \times S \times M(\Lambda \times S) \times A \to \mathbb{R}$:

$$v(\alpha, s, f, a) := \int_{\varepsilon_\alpha(s)} r(\alpha, s', f(\cdot, s'), a) \pi_\alpha(s' | \varepsilon_\alpha(s)) d\mu(s'),$$

where $\pi_\alpha(s'|\varepsilon_\alpha(s))$ is defined as in the previous section. Finally, once the strategies are chosen, the actual state is revealed, and the payoffs of the game are distributed.

According to the above definition of the game, a feasible, pure strategy of player α is an $\mathcal{S}$-measurable selection of $\tilde{A}(\alpha, \cdot)$. Let $M(S)$ denote a set of $\mathcal{S}$-measurable functions $f : S \to \mathbb{R}$, and denote the set of all feasible strategies of player α by M_α, i.e.

$$M_\alpha := \{f \in M(S) | f(s) \in \tilde{A}(\alpha, s)\}.$$

Therefore, a joint, pure strategy of all players is an element of

$$M_\Lambda := \{f \in M(\Lambda \times S) | f(\alpha, \cdot) \in M_\alpha, \forall \alpha \in \Lambda\},$$

We now can proceed with an appropriate definition of Bayesian-Nash equilibrium for the differential information $\Gamma := \{(\Lambda, \mathcal{L}, \lambda), (S, \mathcal{S}, \mu), A, \tilde{A}, r, \{\pi_\alpha, \mathcal{S}_\alpha\}_{\alpha \in \Lambda}\}$ in the sense of Schmeidler (1973) as follows:

Definition 3.1 *A Bayesian Nash-Schmeidler equilibrium of* Γ *is a function* $f^* \in M(\Lambda \times S)$ *satisfying,*

$$\forall \alpha \in \Lambda, \forall s \in S, f^*(\alpha, s) \in \arg\max_{a \in \tilde{A}(\alpha,s)} v(\alpha, s, f^*, a).$$

We remark that our definition of Bayesian Nash-Schmeidler equilibrium in strategies is slightly different from the one stated originally in Schmeidler (1973). In his definition, Schmeidler requires that *almost every player* plays a best response strategy to the equilibrium strategy profile. In contrast, we require *every* player to be acting optimally in our notion of equilibrium in strategies for the large game (as is done, for example, in Balder and Rustichini (1994) and Kim and Yannelis (1997)).

In order to guarantee existence of a Bayesian Nash-Schmeidler equilibrium, we impose the following sufficient conditions on the primitive data of the model.

Assumption 3.1 *Assume the following:*

(i) $\tilde{A}$ *is complete sublattice valued, as well as weakly* $\mathcal{L} \otimes \mathcal{S}$*-measurable;*

(ii) function r *is continuous and quasi-supermodular on* A*, has single crossing differences in* (a, f)*, with* $r(\alpha, s, f(\cdot, s), a)$ *being a* $\mathcal{L} \otimes \mathcal{S} \otimes \mathcal{A}$*-measurable and bounded function, whenever* $f \in M(\Lambda \times A)$*;*

(iii) $\{r(\cdot, s, \cdot)\}_{s \in S}$ *is a family of* Q*-summable functions on* A*, with functionally* S*-summable differences in* (a, f)*;*

(iv) for any monotone sequence $\{f_n\}$ *in* $M(\Lambda \times S)$*, such that* $f_n \to f \in M(\Lambda \times S)$*,* $\forall \alpha \in \Lambda, s \in S, a \in A$*,* $r(\alpha, s, f_n(\cdot, s), a) \to r(\alpha, s, f(\cdot, s), a)$*;*

(v) for all $\alpha \in \Lambda$*,* $\mathcal{S}_\alpha$ *is generated by a countable partition, such that* $\forall s \in S$*,* $\mu(\varepsilon_\alpha(s)) > 0$*.*

All of the above assumptions for existence we need have been discussed previously in the paper with the exception of Assumption 3.1 (iv), which is an order continuity condition we need to obtain order continuous extremal selections. That is, unlike the previous section, in this case, this assumption will play an critical role not only relative to the question of computation and approximation of equilibria, but also to existence itself. We will remark in more detail on this issue in a moment.

Before proceeding to the main theorem of this section, we state two important lemmas. The first of these results is the following:

Lemma 3.1 *Under Assumption 3.1,* M_α *and* M_Λ *are nonempty.*

Notie, by appealing to strategic complementarities and order-theoretic constructions for our large GSC, we are able to relax several important assumptions used by different authors when obtaining our results per the nonemptiness of best replies (e.g., as compared to Balder and Rustichini (1994); Kim and Yannelis (1997)). First, we do not require the feasible action correspondence $\tilde{A}$ to be convex-valued, nor do we require any (quasi-)concavity condition of r in $a \in A$ (as we do not use any version of Kakutani/Fan-Glicksberg type theorems to obtain existence). Second, our payoffs no longer need to be continuous with respect to *joint* strategies of players (as we only require v to be order continuous on $M(\Lambda \times S)$, which is only checked only along *monotone* sequences, as opposed to weak continuity conditions that need to be checked for arbitrary nets).[19]

Before stating the main result, we shall introduce some additional notation. First, define the best-reply correspondence $BR : M(\Lambda \times S) \rightrightarrows M_\Lambda$ by:[20]

$$BR(f)(\alpha, s) := \arg \max_{a \in \tilde{A}(\alpha,s)} v(\alpha, s, f, a),$$

From the definition of Bayesian Nash-Schmeidler equilibrium in Definition 3.1, for $f^* \in M(\Lambda \times S)$ to be an equilibrium, we need $f^* \in BR(f^*)$. Along those lines, let $\overline{BR}(f) := \bigvee BR(f)$, and $\underline{BR}(f) := \bigwedge BR(f)$ denote the greatest and the least element of $BR(f)$, with respect to the pointwise order (whenever they exist). We state the following preliminary result.

Lemma 3.2 *Under Assumption 3.1, $\overline{BR} : M_\Lambda \to M_\Lambda$ (resp., $\underline{BR} : M_\Lambda \to M_\Lambda$) are well defined and increasing.*[21]

We now state the first main result of this section, which concerns the existence of equilibria in the sense of Definition 3.1. For this result, one should keep in mind that as the space of measurable functions is only a countably chain complete poset, in order to prove the new existence theorem, we must apply a generalized version of Tarski-Kantorovich Theorem (see Theorem 4.2, as well as Theorem A.3 in the Appendix Dugundji and Granas (1982)).

Theorem 3.1 (Existence) *Let Assumption 3.1 be satisfied. Then, there exists the greatest ($\overline{f}^*$) and the least ($\underline{f}^*$) Bayesian Nash-Schmeidler equilibrium. Moreover, these extremal equilibrium can be computed by successive approximation: i.e., $\lim_{n\to\infty} \overline{BR}^n(\overline{m}) = \overline{f}^*$ and $\lim_{n\to\infty} \underline{BR}^n(\underline{m}) = \underline{f}^*$, where $\overline{m}, \underline{m}$ are the greatest and the least elements of M_Λ respectively.*

Few comments on Theorem 2.1 are in order.

First, on the one hand, our existence theorem differs from the existing literature with respect to space of equilibrium objects. That is, we prove existence of Bayesian Nash-Schmeidler equilibria in *measurable* strategies, which represent a broader class of strategies than in, for example, those studied in Balder and Rustichini (1994); Kim and Yannelis (1997), who analyzed Bochner integrable strategies. More specifically, in these latter papers (and others related in the literature), the existence of Bayesian Nash-Schmeider equilibrium is based on an application of the Fan-Glicksberg fixed point theorem. In their setting, for the space of functions for equilibrium strategies, they need compactness of a joint strategy set. For this, they study a collection of bounded, Bochner integrable functions. In contrast, our argument for existence, we require the equilibrium strategy space to only be a countably chain-complete poset, which is a substantially *weaker* notion of *order completeness* (and, hence, we are able to obtain our existence results in a broader space of functions). On the other hand, our approach requires several additional assumptions as compared with Balder and Rustichini (1994); Kim and Yannelis (1997), many which are not necessary in these aforementioned papers for obtaining existence (e.g., key assumptions concerning the lattice structure of action sets, as well as quasi-supermodularity and single crossing differences of payoff functions in our game).

Second, Balder and Rustichini (1994) and Kim and Yannelis (1997) analyze large games without the assumption that the set of players is represented by a measure space (hence, without

[19]See Aliprantis and Border (2006), Chapter 8.

[20]We discuss conditions under which BR is well defined in a moment in Lemma 3.2.

[21]Recall if $M(\Lambda \times S)$ is endowed with the pointwise order (i.e. for $f', f \in M(\Lambda \times S)$, $f' \geq f$ iff $\forall \alpha \in \Lambda, s \in S, f'(\alpha, s) \geq f(\alpha, s)$), we have, for example, $\overline{BR}$ increasing if $\overline{BR}(f') \geq \overline{BR}(f)$ whenever $f' \geq f$.

the measurability assumption on a set of players). If one allows us to apply our methods applied to their alternative notion of equilibrium in the game, our results become even *stronger.* That is, in this case where the measurability requirement for equilibrium is dropped, the set of Nash equilibria is a *nonempty complete lattice under pointwise partial order* by a simple application of the Veinott (1992); Zhou (1994) version of Tarski's theorem. In such a case, we can also weaken the payoff continuity assumption in 3.1 to merely upper semicontinuity in $a \in A$, as well as drop the order continuity assumption in Assumption 3.1(iv). This remark stresses the assumption that players are represented by a measure space with σ-field, which requires additional continuity type assumption on players payoffs (the feature that is not present in small games with strategic complementaries).

Third, the order imposed on $M(\Lambda \times S)$ and used in Assumption 3.1 is defined "everywhere", i.e. $f' \geq f$ iff $\forall(\alpha, s) \in \Lambda \times S, f'(\alpha, s) \geq f(\alpha, s)$. Alternatively, we may consider a case where we relax the order to $\succeq_{a.e.}$, (i.e. $f' \succeq_{a.e.} f$ iff $f'(\alpha, s) \geq f(\alpha, s)$, $\lambda \otimes \mu$-a.e). For this alternative partial order, a few important comments should be noted. First, if we let $\hat{M}(\Lambda \times S)$ denote the set of equivalence classes of functions in $M(\Lambda \times S)$ with respect to $\lambda \otimes \mu$, then $(\hat{M}(\Lambda \times S), \succeq_{a.e.})$ is a complete lattice (see Lemma 6.1 in Vives (1990)); but sup/inf in $\hat{M}(\Lambda \times S)$ are only unique up to equivalence classes. Second, the assumption concerning single crossing differences of r in (a, f) with respect to $\succeq_{a.e.}$ is significantly stronger than with respect to the "everywhere" pointwise order. Such an assumption implies, for example, if $f \simeq_{a.e.} f'$, then $BR(f) = BR(f')$. Such assumption is satisfied, for example, in a class of aggregative games. Third, the set of Nash equilibria of Γ is a nonempty, complete lattice of $\hat{M}(\Lambda \times S)$ without imposing payoff (order) continuity assumption. Therefore, equilibria are also defined via equivalence classes. Hence, with stronger assumption concerning order in which r has single crossing differences, but weaker assumptions regarding continuity, we recover a complete lattice structure of the Nash equilibrium set. Also, as argued in the remainder of the paper, analyzing games on equivalence classes of functions (rather than their elements ordered pointwise everywhere) is straightforward in many applications.[22]

Finally, similar to the literature concerning complete information quasisupermodular games with finite number of players, we can again consider a parametrized version of the game defined above, and determine how its equilibria vary with respect to parameter. Along those lines, let Θ denote a partially ordered space of parameters θ. For any fixed $\theta \in \Theta$, define game

$$\Gamma(\theta) := \{(\Lambda, \mathcal{L}, \lambda), (S, \mathcal{S}, \mu), A, \tilde{A}(\theta, \cdot), r(\theta, \cdot)\{\pi_\alpha, \mathcal{S}_\alpha\}_{\alpha \in \Lambda}\}.$$

Therefore, for each θ, game $\Gamma(\theta)$ is defined as in the first part of this section. In order to determine comparative statics of equilibria of the game, we impose the following assumptions.

Assumption 3.2 *For any $\theta \in \Theta$, let $\Gamma(\theta)$ satisfy Assumption 3.1. Moreover, let*

(i) $\tilde{A}$ be increasing in the Veinott strong set order on Θ and complete sublatticed-valued;[23]

(ii) r have single crossing differences in (a, θ);

(iii) family $\{r(\cdot, s, \cdot)\}_{s \in S}$ have S-summable differences in (a, θ).

The next result follows from Theorem A.8. For any $\theta \in \Theta$, let $\overline{f}^*(\theta)$ (resp, $\underline{f}^*(\theta)$) be the greatest (resp., least) equilibrium of $\Gamma(\theta)$ as defined as in Definition 3.1. Then, we have the following equilibrium comparative statics theorem.

[22]For a related discussion on this point, see Vives (1990), Vives and Van Zandt (2007) and Van Zandt (2010) concerning equilibria in ex-ante and interim strategies for a class of Bayesian games using equivalence classes.

[23]Sublatticed-valued is implied by ascending in Veinott's strong set order; complete sublatticed-valued is not.

Corollary 3.1 *Let Assumptions 3.1, 3.2 be satisfied. Then,* $\overline{f}^*(\cdot)$ *and* $\underline{f}^*(\cdot)$ *are increasing on* Θ.

4 Applications

We now present some applications of our results. For this purpose, we consider economic problems where we can apply our methods on large GSC: riot games, beauty contest and common value auctions. We begin with a discussion of the riot game.

4.1 Riot game

Our first example is a version of the *riot game* presented in Atkeson (2000). The game studies the aggregate behavior of a potentially angry crowd that faces riot police with the mandate of quelling collective violent actions. In this game, each of the demonstrators must decide whether to fight the police or not. If enough people join the fight, the riot police are overwhelmed by the rioters, and each rioter gets some loot $W > 0$. Otherwise, the riot police contain the riot, and each rioter gets arrested with payoff $L < 0$. Individuals who choose not to fight, leave the crowd, and get a safe payoff of 0 (regardless of whether the fight takes place or not).[24]

In our version of the game, the ability of the riot police to control the riot depends on the state of the world, say $s \in S$, and is summarized by a function $p : S \to \mathbb{R}$, which indexes the fraction of the crowd that must riot in order for the rioters (collectively) to overwhelm the police. In order to make the example more interesting, we assume that p may take values outside of the unit interval. Therefore, if $p(s) > 1$, the police always contain the riot (regardless of the number of people joining the fight). In case that $p : S \to [0, 1]$ some trivial equilibria arises, as will be discussed in the remainder of this section. Individuals are able to observe state s, as well as the ability level of the police to control the riot $p(s)$, but perceive the signal with respect to their *private* information.

4.1.1 Existence of Equilibrium

Let $(S, \mathcal{S}, \mu)$ be the measure space of the states of the world, and the private information of each individual α is represented by a countable sub σ-field $\mathcal{S}_\alpha$. Assume that each rioter, regardless of the state, chooses action $a = 1$ when willing to join the fight, and $a = 0$ otherwise. Then, for all $\alpha \in \Lambda$ and $s \in S$, $\tilde{A}(\alpha, s) = \{0, 1\}$. Moreover, let $\tau \in \hat{T}_d$ be an equivalence class of functions defined as in Section 2, mapping set S into a set of all distributions on $\Lambda \times \{0, 1\}$, denoted D_Δ, where $\tau(\{(\alpha, a)|a = 1\}|s)$ is the measure of all players joining the riot in state s.

The ex-post payoff of each individual is $r : \Lambda \times S \times D_\Delta \times A$,

$$r(\alpha, s, \tau(\cdot|s), a) := a\left[(W - L)\chi_{\{\tau(\{(\alpha,a')|a'=1\}|s) \geq p(s)\}} + L\right],$$

where χ is the indicator function. It is easy to verify that r has single crossing differences in (a, τ) for each s, since it is sufficient to show that

$$(W - L)\chi_{\{\tau(\{(\alpha,a')|a'=1\}|s) \geq p(s)\}} + L$$

increases in $\tau(\cdot|s)$, which is always true (in fact it is piecewise constant).

[24] The game is very closely related to the Diamond-Dybvig model of bank runs that have been studied in Lucas (2011) and Vives (2012). Indeed, a simple redefining of terms gives an version of Diamond-Dybvig.

To finish checking the conditions of our existence theorem, it suffices to show that $\{r(\cdot, s, \cdot)\}_{s\in S}$ has functionally S-summable differences in (a, τ). In fact, in our case, we only need to show whether the condition holds when an agent changes his strategy from $a = 0$ to $a = 1$ (as, otherwise, the condition holds trivially). Along these lines, let

$$g(\tau(\cdot|s)) := (W - L)\chi_{\{\tau(\{(\alpha,a')|a'=1\}|s)\geq p(s)\}} + L$$

Observe $g(\tau(\cdot|s)) < 0$ only if $\chi_{\{\tau(\{(\alpha,a')|a'=1\}|s)\geq p(s)\}} = 0$. Then, for any $s', s \in S$ such that $g(\tau(\cdot|s)) \leq 0$, and $g(\tau(\cdot|s')) > 0$, for any two $\tau, \tau' \in \hat{T}_d$, $\tau' \succeq_{\hat{T}} \tau$, we have

$$-\frac{g(\tau(\cdot|s))}{g(\tau(\cdot|s'))} = -\frac{L}{(W-L)} \geq -\frac{g(\tau'(\cdot|s))}{(W-L)} = -\frac{g(\tau'(\cdot|s))}{g(\tau'(\cdot|s'))}.$$

Therefore, the condition of functionally S-summable differences in (a, τ) holds, and Assumption 2.1 is satisfied.

The interim payoff is

$$v(\alpha, s, \tau, a) := \int_{\varepsilon_\alpha(s)} a\left[(W-L)\chi_{\{\tau(\{(\alpha,a)|a=1\}|s')\geq p(s')\}} + L\right]\pi_\alpha(s'|\varepsilon_\alpha(s))d\mu(s'),$$

where $\varepsilon(s)$ is defined as in Section 2. Therefore, by Theorem 2.1, there exists the greatest and the least distributional Bayesian-Nash equilibrium of the game (with respect to $\succeq_{\hat{T}}$, defined as in Section 2).

Next, we consider conditions for the existence of extremal Bayesian-Nash Schmeidler equilibrium in strategies in the sense of Section 3. Let $M(\Lambda \times S)$ be a set of $\mathcal{L} \otimes \mathcal{S}$-measurable functions $f : \Lambda \times S \to \{0, 1\}$ endowed with the pointwise order. Noting that in this setting, the fraction of rioters that join the fight is given by

$$F(s) := \int_\Lambda f(\alpha, s)d\lambda(\alpha)$$

which is increasing pointwise in f, the ex-post payoff is computed as:

$$r(\alpha, s, f, a) := a[(W - L)\chi_{\{F(s)\geq p(s)\}} + L].$$

Interestingly, the above payoff function is *not* order-continuous with respect to f (hence, Assumptions 3.1 is *not* satisfied). Therefore, we cannot directly apply Theorem 3.1 to this game to obtain existence of equilibrium in the collection $M(\Lambda \times S)$). However, if each players payoff is constant on any equivalence class of functions and equal λ-a.e., there *does* exists an equilibrium in this game defined in the *equivalence classes* of $\mathcal{L} \otimes \mathcal{S}$-measurable functions. Moreover, the set of such equilibria constitutes a complete lattice.

Therefore, aside from highlighting how to check the conditions of our theorems in an important class of examples, this example also shows the importance of disguishing partial orders in the context of Bayesian Nash-Schmeidler equilibrium even per the question of *existence.* Also, we should note in some cases, the largest and the greatest equilibrium of the game might be trivial. Observe, once $p : S \to [0, 1]$, we have for the equivalence class $\overline{\tau}$ that $\overline{\tau}(\{(\alpha, a)|a = 1\}|s) = 1$ is the greatest equilibrium, while $\underline{\tau}$, such that $\underline{\tau}(\{(\alpha, a)|a = 0\}|s) = 1$ is the least equilibrium.

4.1.2 Difficulties with Uniqueness of Equilibrium

One important question per the riot game is when does one have uniqueness of equilibrium?[25] In the original paper by Atkeson (2000), at the beginning of the game, a signal $s \in S$ is drawn from the normal distribution. Then, each player α observes a distorted value of the signal $x_\alpha = s + \zeta_\alpha$, where ζ_α is drawn form a normal distribution, identical and independent among players. In Atkeson (2000), as well as Morris and Shin (2001), equilibrium is defined by observation of the original signal x^*, at which each player is indifferent between joining or withdrawing from the riot. Moreover, the probability of drawing x^*, given state s is $p(s)$, which by the Law of Large Numbers is assumed by the authors implies that the measure of rioters in equilibrium is equal to the strength of the police.[26] Under certain assumptions imposed on distributions governing s and ζ, Morris and Shin (2001) prove uniqueness of such equilibrium.

In our framework, the question of uniqueness of equilibrium poses two main questions. First, the proof by Morris and Shin (2001) is based on an ex-ante symmetry of players whose expectations concerning s and x_α before the game are identical. In fact, knowing that players are symmetric, and the Law of Large Numbers holds for continuum of players, this enables the agents to predict the cut-off value of the observed signal. Further, in our model, the players have incomplete information about the true signal, which cannot be distinguished from other elements of the same set contained in the sub σ-algebra (rather, they receive a distorted signal). As it turns out, these two issues are crucial for uniqueness of equilibria in the presented game.

For the sake of this discussion, we shall concentrate on uniqueness in the symmetric case of the game, where every player has the same sub σ-field. Since players do not receive a distorted signal in our framework, we have to present an alternate definition of an equilibrium, to the one analyzed in Morris and Shin (2001) and Atkeson (2000). In our understanding, an equilibrium is as element $s^* \in S$, such that the expected utility of each player is equal to zero, and the measure of rioters is equal to $p(s^*)$. We shall assume that the sub σ-algebra is generated for each player by a partition of convex subsets in S, and that further s is drawn from a normal distribution. We should mention, even though the symmetric case of the game is very simple, it is easy to show that the equilibrium is *not unique*.[27]

In order to prove this, assume that s^* exists. Then, since players are symmetric, and determining their strategy with the same cut off value,

$$(W - L)\text{Prob}(s < s^* | \varepsilon(s^*)) + L = 0$$

must hold. Thus, we have

$$\text{Prob}(s < s^* | \varepsilon(s^*)) = -\frac{L}{W - L}$$

Since $-\frac{L}{W-L} \in (0, 1)$, and $\text{Prob}(s < s^* | \varepsilon(s^*))$ is continuous and increasing in s^* on $\varepsilon(s^*)$ and taking values from 0 to 1, then for every element $\varepsilon(s^*)$ in the sub σ-algebra $\mathscr{S}_\alpha$, there exists an element s^* at which interim utility of players is equal to zero, and the agents are therefore indifferent between joining the riot and withdrawing. Hence, as long, as $p(s^*) \in [0, 1]$ for each

[25] For example, when considering the version of the riot game that coincides with Diamond and Dybvig's model of bank runs, an important questtion is when is the equilibrium unique. See Lucas (2011) for discussion of the importance of this question. Unfortunately, as we shall show, the answer to this question is likely to be negative.

[26] In fact, this assumption is strong, as in general the Law of Large Numbers for continuum of identical and independent random variables does not hold. For further discussion see Alós-Ferrer (1998); Feldman and Gilles (1985); Judd (1985).

[27] In fact, the number of equilibria might even be equal to the number of elements constituting the base of the sub σ-algebra of players.

element of the sub σ-algebra, there exists an equilibrium value s^*. The above argument also shows that in general, this game has *multiple* equilibria.[28]

The same problem occurs, when analyzing equilibria defined on strategies of players, as in Sections 2 and 3. Even in the simplest cases, the game exhibits multiple equilibria, as illustrated in the following example.

Example 4.1 *Consider an example of a riot game where the set of signals is equivalent to interval $S = [0, 1]$, which elements are distributed uniformly. A measure of $\frac{1}{2}$ of players is endowed with a trivial sub σ-algebra $\mathcal{S}_1 = \{\emptyset, S\}$, while the rest of players have information sets $\mathcal{S}_2 = \{\emptyset, S, [0, \frac{1}{2}), [\frac{1}{2}, 1]\}$. The strength of the police is determined by an affine function $p(s) = 3s - 1$. Eventually, let $-\frac{L}{W-L} = \frac{1}{2}$.*

The game has at least two equilibria. The least one, such that

$$\underline{\tau}^*(\{(\alpha, a)|\alpha \in \Lambda, a = 1\}|s) = \begin{cases} \frac{1}{2} & for \quad s \in [0, \frac{1}{2}), \\ 0 & for \quad s \in [\frac{1}{2}, 1], \end{cases}$$

and the greatest one

$$\overline{\tau}^*(\{(\alpha, a)|\alpha \in \Lambda, a = 1\}|s) = \begin{cases} 1 & for \quad s \in [0, \frac{1}{2}), \\ \frac{1}{2} & for \quad s \in [\frac{1}{2}, 1]. \end{cases}$$

4.2 Beauty contest

Next, consider a version of the *beauty contest* game (e.g., see Acemoglu and Jensen (2010), for example). Suppose that the true value of a firm is unknown. However, the players (who constitute a stock market) do receive a common signal which has to be interpreted with respect to their private information in order to value the asset of interest. Given signal s, each player α makes a public prediction about the true value by announcing $a \in \tilde{A}(\alpha, s) \subset \mathbb{R}$, where $\tilde{A}$ is well defined and convex-valued. Every agent is both interested in being close to his personal understanding of the signal, as well as predictions of other players τ being an $\mathcal{S}$-measurable function mapping the set of signals S into a set of distributions on $\Lambda \times A$ denoted by D_Δ. As in Section 2, let $\hat{T}_d$ be a set of equivalence classes of such functions with the $\succeq_{\hat{T}}$ ordering defined as in Section 2. The interim payoff of player α is then

$$v(\alpha, s, \tau, a) = \int_{\varepsilon_\alpha(s)} \left[h(\alpha, |a - H(\alpha, s')|) + g(\alpha, |a - G(\alpha, \tau(\cdot|s'))|)\right] \pi_\alpha(s'|\varepsilon_\alpha(s'))d\mu(s'),$$

where $h, g : \Lambda \times \mathbb{R}_+ \to \mathbb{R}_+$ are concave and decreasing on $\mathbb{R}_+$, $H : \Lambda \times S \to \mathbb{R}$ is $\mathcal{L} \otimes \mathcal{S}$-measurable, $G : \Lambda \times D_\Delta \to \mathbb{R}^n$ is $\mathcal{L} \otimes \mathcal{D}_\Delta$-measurable and increasing on D_Δ, $\varepsilon_\alpha(s)$ and $\pi_\alpha(s'|\varepsilon_\alpha(s))$ are both defined as in Section 2.

To show that under the above assumptions, the interim payoff v satisfies has single crossing differences in (a, τ), we prove the following lemma.

Lemma 4.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing, concave function. Then, for $X \subset \mathbb{R}$ and convex, $S \subset \mathbb{R}$, the function $g : X \times S \to \mathbb{R}$ with $g(x, s) := f(|x - s|)$ has increasing differences in (x, s).*

[28] Actually,we have that the game may have an infinite, however, countable, number of equilibria. Also, obviously, when $\mathcal{S}_\alpha = \{\emptyset, S\}$, there exists a unique equilibrium; however, apart from trivial cases, unique equilibria are very unlikely.

Under the above lemma, the ex-post payoff function $r : \Lambda \times S \times D_\Delta \times A$ given by:

$$r(\alpha, s, d, a) := -\left[h(\alpha, |a - H(\alpha, s)|) + g(\alpha, |a - G(\alpha, d)|)\right]$$

has increasing differences in (a, d). Hence, it has single crossing differences, and the family of functions $\{r(\cdot, s, \cdot)\}_{s \in S}$ is functionally S-summable in (a, τ). Therefore, Assumption 2.1 is satisfied, and by Theorem 2.1 the set of distributional Bayesian-Nash equilibria contains the greatest and the least element.[29] Additionally, equilibrium in the game need not be defined on a space of distributions. That is, once G is defined on $M(\Lambda \times S)$ and order-continuous, the game can be defined as a generalized large game in the sense of a Bayesian-Nash Schmeidler equilibrium in strategies as in Section 3, with the greatest and the least equilibrium.

Finally, even though in our example, the strategies of players are restricted to a subset of $\mathbb{R}$, the game can be extended to multi-dimensional strategy spaces. Let $\|\cdot\|$ denote a taxicab metric defined on $\mathbb{R}^n$. As shown in Topkis (1998), $f : X \times T \to \mathbb{R}$, $f(x, t) := -\|x - t\|$ is supermodular on X, and has increasing differences in (x, s) on $X \times S$ (see Example 2.6.2(g) in Topkis (1998)).[30] Hence, for any $s \in \mathbb{R}$, $H : \Lambda \times D_\Delta \to \mathbb{R}^n$ and $G : \Lambda \times D_\Delta \to \mathbb{R}^n$, increasing on D_Δ,

$$r(\alpha, s, d, a) := -\left(\|a - H(\alpha, s)\| + \|a - G(\alpha, d)\|\right),$$

is supermodular on A, and has increasing differences in (a, d). As a result, the assumptions stated in Section 2 for the existence of extremal Bayesian-Nash equilibrium in strategies are now satisfied.

4.3 Common value auctions

Assume that a measure space of agents attends a sealed, common value, multiple-unit, discriminatory auction. There is a measure $G \in \mathbb{R}_+$ of homogeneous objects which are auctioned, but each player may buy at most one unit of the good. The value of each object is $s \in S \subset \mathbb{R}$. Each player is able to perceive it only with respect to his private knowledge (e.g. as in the case of an auction for government bonds, which have commonly known face value, but each player is willing to estimate their future expected return, with this private estimate unknown).

Let $(S, \mathcal{S}, \mu)$ be a measure space of values of the good. Since the auction is discriminatory, each player pays the price equal to his bid. In this case, let $r(\alpha, s, a)$ denote payoff of player α, when the value of the good is s, and his winning bid is $a \in \tilde{A}(\alpha, s)$. If we assume r is strictly decreasing in a, every player with a loosing bid receives payoff equal to zero, A is the set of all possible bids, and $\tilde{A}(\alpha, s) \in A$ is a compact subset in $\mathbb{R}_+$, then the interim payoff of each player is

$$v(\alpha, s, \tau, a) := \int_{\varepsilon_\alpha(s)} r(\alpha, s', a)\chi_{\{\tau(\{(\alpha, a')|a' \geq a\}|s') \leq G\}}\pi_\alpha(s'|\varepsilon_\alpha(s))d\mu(s'),$$

where monotone $\tau : S \to D_\Delta$, $\tau \in \hat{T}_d$ with $\tau(\cdot|s)$ an equivalence class of functions defined as in Section 2, χ is the indicator function and family $\varepsilon_\alpha(s) := [\underline{z}_\alpha(s), \overline{z}_\alpha(s)]$ for some increasing functions $\underline{z}_\alpha$ and $\underline{z}_\alpha$.[31]

[29] Note that our approach generalizes the example presented in Acemoglu and Jensen (2010), where G is an aggregate, defined by $\int_A a\tau(s)(da)$, while in our case, G might be any measurable, increasing function mapping D_Δ to $\mathbb{R}_+$.

[30] Unfortunately, this is not true for any metric. For example, Euclidean metric is not supermodular on its domain, as shown in Topkis (1998), Example 2.6.2(h).

[31] Note that $r(\alpha, s', a)\chi_{\{\tau(\{(\alpha, a')|a' \geq a\}|s') \leq G\}}$ is not continuous on A, but only upper semi-continuous. However, our assumptions are sufficient to show that v has join up-crossing differences in (a, s) (see Definition A.7 in the Appendix) for any family $\{\tau(\cdot|s)\}_{s \in S}$ that is first order stochastically increasing in s. As a result we show that the

In the literature concerning quasi-supermodular specifications of auctions with finite number of agents, quasi-supermodularity of the interim payoff function of a given player is obtained typically through assumptions concerning the log-supermodularity of the density function of types of players. Importantly, notice this is *not* the case in this example. In fact, it is straightforward to verify that $r(\alpha, s, a)\chi_{\{\tau(\{(\alpha,a')|a'\geq a\}|s)\leq G\}}$ does *not* have single crossing differences in $(a, \tau(\cdot|s))$ (as strict inequalities that must be checked for the standard single crossing differences property are *not* preserved in $\tau(\cdot|s)$ with respect to $\succeq_D$-increases); but preservation of *weak* inequalities is preserved. This latter implication corresponds to the payoff $r(\alpha, s, a)\chi_{\{\tau(\{(\alpha,a')|a'\geq a\}|s)\leq G\}}$ having *join* (*but not meet*) up-crossing differences in $(a, \tau(\cdot|s))$ (e.g., see the Definition A.7 in the Appendix). Moreover, class of functions $\{r(\alpha, s, \cdot)\chi_{\{\tau(\{(\alpha,a')|a'\geq\cdot\}|s)\leq G\}}\}_{s\in S}$ has only functionally *join* S-summable differences in (a, τ).

Given these concerns, we explain the result under weaker conditions that we can obtain in the following remark.

Remark 4.1 *Let $\forall(\alpha, s) \in \Lambda \times S$, $r : \Lambda \times S \times A \to \mathbb{R}$ be decreasing on A, then*

$$r(\alpha, s, a)\chi_{\{\tau(\{(\alpha,a')|a'\geq a\}|s)\leq M\}}$$

has join up-crossing differences in $(a, \tau(s))$, and $\{r(\cdot, s, \cdot)\chi_{\{\tau(s)(\{(\alpha,a')|a'\geq\cdot\})\leq M\}}\}_{s\in S}$ is a class of functions with functionally join S-summable differences in (a, τ). By Theorem A.1 and Lemma 2.2, there exists a well defined, isotone operator $\overline{B}$, defined as in Section 2, which greatest fixed point (say $\overline{\tau}^*$) constitutes the greatest distributional equilibrium (in monotone strategies) of the game. Since operator $\underline{B}$ might not be well defined, nor isotone, it cannot be determined whether the least equilibrium exists.

Another issue that needs to be addressed is the computation of greatest distributional Bayesian-Nash equilibrium (in monotone strategies) $\overline{\tau}^*$. Observe, as our operator $\overline{B}$ is inf-preserving, by Theorem 2.1, we have $\lim_{n\to\infty} \overline{B}^n(\bar{t}) = \overline{\tau}^*$, where $\bar{t}$ is the greatest element of $\hat{T}_d$. Hence, the operator approximates the distribution using a conceptually simple monotone iterative procedure. This means that our method not only determines the existence of the greatest equilibrium, but also present tools for its direct computation. No similar result is available using purely topological methods.

Finally, one can modify the above game without affecting supermodular properties of the auction in question. For example, if we assume that the final price paid by winning agents is determined by some increasing aggregate $H : D_\Delta \to A$ (e.g. the average price of all winning bids), then the payoffs are given by

$$r(\alpha, s, H(\tau(\cdot|s)))\chi_{\{\tau(\{(\alpha,a')|a'\geq a\}|s)\leq M\}},$$

which again preserve the join up-crossing differences in (a, τ). This implies that all the previous results would hold.

A Definitions and auxiliary results

In this section, we present several definitions used throughout the work.

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greatest best response is increasing in s. Since $S \subset \mathbb{R}$, this greatest best response is also measurable. Then best response $\overline{B}$ maps (first order) stochastically increasing τ's into itself. It is easy to see that the set of stochastically distributions is a chain complete poset. Indeed if f is increasing and bounded then $\int_S f(s')\tau(ds'|s)$ is increasing and bounded. As a result both sup and inf over arbitrary chains preserve this property. Hence we can repeat the reasoning and results from main body of this paper.

A.1 Aggregation of quasi-supermodular functions

We first discuss the aggregation of the single crossing property in our class of models. This section extends some ideas in Quah and Strulovici (2012). Let $(X, \geq)$ be a lattice.

Definition A.1 *We say that $g : X \to \mathbb{R}$ is a (real-valued) quasi-supermodular function on X if for any two $x', x \in X$*

$$g(x) - g(x' \wedge x) \geq (>)0 \Rightarrow g(x \vee x') - g(x') \geq (>)0.$$

We now define Q-summable functions.

Definition A.2 (Q-summable functions) *Let $h, g : X \to \mathbb{R}$ be quasi-supermodular. We shall refer to h and g as to Q-summable functions on X, denoting $h \sim_{Q(X)} g$, whenever for any two unordered $x, y \in X$,*

(i) if $h(y) - h(x \wedge y) > 0$ and $g(y) - g(x \wedge y) \leq 0$,

$$-\frac{g(y) - g(x \wedge y)}{h(y) - h(x \wedge y)} \geq -\frac{g(x \vee y) - g(x)}{h(x \vee y) - h(x)};$$

(ii) if $g(y) - g(x \wedge y) > 0$ and $h(y) - h(x \wedge y) \leq 0$,

$$-\frac{h(y) - h(x \wedge y)}{g(y) - g(x \wedge y)} \geq -\frac{h(x \vee y) - h(x)}{g(x \vee y) - g(x)}.$$

Notice, the above relation is reflexive, but not transitive. We state the following results related to some results in Quah and Strulovici (2012).

Corollary A.1 *For any $\alpha \in \mathbb{R}_+$, two quasi-supermodular functions $h, g : X \to \mathbb{R}_+$ are Q-summable if and only if $\alpha h + g$ is quasi-supermodular.*

Proof F: irst, we prove that quasi-supermodularity of $\alpha h + g$ on X implies that $h \sim_{Q(X)} g$. Define $\alpha^* = -\frac{g(y)-g(x\wedge y)}{h(y)-h(x\wedge y)}$, and without loss of generality assume that $h(y) - h(x \wedge y) > 0$. Then $\alpha^*[h(y) - h(x \wedge y)] + [g(y) - g(x \wedge y)] = 0$. Because $\alpha^* h + g$ is quasi-supermodular, $\alpha^*[h(x \vee y) - h(x)] + [g(x \vee y) - g(x)] \geq 0$, and $h(x \vee y) - h(x) > 0$, since h is also quasi-supermodular. By rearranging the inequality, we obtain

$$\alpha^* = -\frac{g(y) - g(x \wedge y)}{h(y) - h(x \wedge y)} \geq -\frac{g(x \vee y) - g(x)}{h(x \vee y) - h(x)}.$$

Next we prove that if $h \sim_{\mathcal{Q}(X)} g$, then $\alpha h + g$ is quasi-supermodular on X for any $\alpha \in \mathbb{R}$. Let $\alpha[h(y) - h(x \wedge y)] + [g(y) - g(x \wedge y)] \geq (>)0$. If $h(y) - h(x \wedge y) \geq (>)0$ and $g(y) - g(x \wedge y) \geq 0$, the conclusion is obvious since h, g are quasi-supermodular on X. Now assume without loss of generality that $h(y) - h(x \wedge y) > 0$, and $g(y) - g(x \wedge y) \leq 0$, then

$$\alpha \geq -\frac{g(y) - g(x \wedge y)}{h(y) - h(x \wedge y)} \geq -\frac{g(x \vee y) - g(x)}{h(x \vee y) - h(x)},$$

and so $\alpha[h(x \vee y) - h(x)] - [g(x \vee y) - g(x)] \geq 0$, since h is quasi-supermodular on X, which completes the proof. ■

Clearly, any two supermodular functions, are at the same time Q-summable. However, the above corollary allows to determine when a sum of two quasi-supermodular functions, which are not supermodular, preserves the property. Consider the following example.

Example A.1 *Consider two functions, $h, g : \mathbb{R}^2 \to \mathbb{R}$, defined by $h(x,y) := (x+y)^4, g(x,y) := (x+y)[(x+y)^2+1]$. It is easy to verify, that both functions are quasi-supermodular. In addition, g is supermodular, however, h is not. Still, we will show that their sum is quasi-supermodular.*

Take any two unordered elements (x',y), (x,y') in $\mathbb{R}^2$. Without loss of generality let $x' \geq x, y' \geq y$, with at least one of them being satisfied with a strict inequality. Therefore $(x',y) \vee (x,y') = (x',y'), (x',y) \wedge (x,y') = (x,y)$.

First, note that both $g(x,y') - g(x,y) > 0$ and $g(x',y) - g(x,y) > 0$. Hence we only need to verify condition (ii) of Definition A.2. Clearly, $h(x,y') - h(x,y) \leq 0$ iff $|x+y'| \leq |x+y|$. We need to verify, whether

$$-\frac{h(x,y') - h(x,y)}{g(x,y') - g(x,y)} \geq -\frac{h(x',y') - h(x',y)}{g(x',y') - g(x',y)},$$

which is equivalent to showing that $f(x) = -\frac{(x+y'^4-(x+y)^4}{(x+y'^3-(x+y)^3+(y'-y)}$ is decreasing in x. Calculating the first order derivative of f with respect to x, allows to verify, that whenever $|x+y'| \leq |x+y|$, and hence $|x'+y'| \leq |x'+y|$, the condition holds. Hence, $g+h$ is a quasi-supermodular function.

Turns out the above result holds for any finite as well as any infinite sum of Q-summable functions. We present appropriate corollaries with their proofs.

Corollary A.2 *Let $\mathscr{G} = \{g_i\}_{i=1}^m$ be a finite family of Q-summable functions $g_i : X \to \mathbb{R}$ on X. Then $\sum_{i=1}^m \alpha_i g_i$ is quasi-supermodular on X for $\alpha_i \geq 0$, $\forall i = 1, \ldots, m$.*

Proof T: ake any two functions g_i and g_j in $\mathscr{G}$, as well as any two scalars $\alpha_i, \alpha_j \in \mathbb{R}_+$. According to corollary A.1, $\alpha_i g_i + \alpha_j g_j$ is quasi-supermodular on X, since $\alpha_i g_i + \alpha_j g_j = \alpha_j[\frac{\alpha_i}{\alpha_j} g_i + g_j]$. Let $h = \alpha_i g_i + \alpha_j g_j$. Next, take any g_k from $\mathscr{G}$, and any two positive scalars α_k, and α_h. Analogously $\alpha_k g_k + \alpha_h h$ is also quasi-supermodular on X. By induction we conclude, that $\sum_{i=1}^m \alpha_i g_i$ is quasi-supermodular. ■

Next we will show, that the above property is preserved under integration.

Corollary A.3 *Let $g : X \times T \to \mathbb{R}$ be quasi-supermodular on X, for all $t \in T$, such that $\{g(\cdot,t)\}_{t \in T}$ is a family of Q-summable functions on X. Assume that for all x the function $g(x,\cdot)$ is bounded. Let $(T, \mathcal{T}, \mu)$ be a measure space, and $g(x,\cdot)$ be $\mathcal{T}$-measurable, for all $x \in X$. Then $G(x) := \int_T g(x,t) d\mu(t)$ is quasi-supermodular on X.*

Proof D: efine $h(x,y;t) := g(x,t) - g(y,t)$. Since g is $\mathcal{T}$-measurable and bounded for all fixed x, so is h. For $x', x \in X$, let us denote I_h as a bounded interval containing a union of a ranges of $h(x, x \wedge x'; \cdot)$ and $h(x \vee x', x'; \cdot)$. This implies, that there exists a sequence $\{h_n\}$ of simple functions satisfying $h_n \to h$ (pointwise), such that $h_n(x,y;t) := \sum_{i=1}^n [g(x,t_i) - g(y,t_i)] \chi_{T_i^n}(t)$, where χ is the indicator function, and $\{T_i^n : i = 1, ..., n\}$ is a partition of I_h with a mesh convergent to 0 as $n \to \infty$. Then by Corrolary A.2 h_n satisfies $h_n(x, x \wedge x'; t) \geq (>) 0 \Rightarrow h_n(x \vee x', x'; t) \geq (>) 0$.

Now assume that $\int_T h_n(x, x \wedge x'; t) d\mu(t) \geq 0$. By Corollary A.2 this implies that $\int_T h_n(x \vee x', x'; t) d\mu(t) \geq 0$, since in this case the integral is a finite sum. We claim that

$$\int_T h(x, x \wedge x'; t) d\mu(t) \geq 0 \Rightarrow \int_T h(x \vee x', x'; t) d\mu(t) \geq 0. \qquad (\star)$$

Let $h_n \to h$ pointwise. Consider two cases. The first case is the one with strict inequality. For sufficiently large n, we have $\int_T h_n(x, x \wedge x'; t) d\mu(t) > 0$, and consequently, $\int_T h_n(x \vee x', x; t) d\mu(t) \geq 0$ for such large n. This implies $\int_T h(x \vee x', x'; t) d\mu(t) \geq 0$. Alternatively, for the case $\int_T h(x, x \wedge x'; t) d\mu(t) = 0$, then for arbitrary $\epsilon \geq 0$, and sufficiently large n, we have $\epsilon + \int_T h_n(x, x \wedge x'; t) d\mu(t) > 0$. As $\epsilon + g(\cdot, t)$ is quasi-supermodular, and adding a constant does not affect Q-summability, $\epsilon + \int_T h_n(x \vee x', x; t) d\mu(t) \geq 0$. Consequently $\int_T h(x \vee x', x; t) d\mu(t) \geq -\epsilon$. Since ϵ is arbitrary, the proof of $(\star)$ is complete.

In particular,

$$\int_T h(x, x \wedge x'; t) d\mu(t) > 0 \Rightarrow \int_T h(x \vee x', x'; t) d\mu(t) \geq 0. \tag{$\star\star$}$$

It is sufficient to prove that the strict inequality is preserved above. If $\forall t \in T$, $h(x, x \wedge x', x'; t) \geq 0$, the implication is trivial. Therefore, assume that there exists a t', such that $h(x, x \wedge x', x'; t') < 0$. Take $\beta \in \mathbb{R}_+$ such that $\int_T h(x, x \wedge x'; t) d\mu(t) + \beta h(x, x \wedge x'; t') > 0$. By implication $(\star\star)$,

$$\int_T h(x \vee x', x'; t) d\mu(t) + \beta h(x \vee x', x'; t') \geq 0$$

so $\int_T h(x \vee x', x'; t) d\mu(t) > 0$. The proof is now complete. ■

Next we shall define and present several notions related to single crossing functions that we use in the paper.

Definition A.3 (Single crossing function) *We say that $f : S \to \mathbb{R}$ is a single crossing function on S if for any two $s' \geq s$, $s', s \in S$*

$$f(s) \geq (>) 0 \Rightarrow f(s') \geq (>) 0.$$

Definition A.4 (Single crossing differences) *Let $(X, \geq_X), (S, \geq_S)$ be posets. We say that $f : X \times S \to \mathbb{R}$ is a function with single crossing differences in (x, s), $x \in X, s \in S$, if for any two $x' \geq_X x$, $x', x \in X$, $g(s) = f(x', s) - f(x, s)$ is a single crossing function on S.*

The following definitions and results are taken from Quah and Strulovici (2012), who provide analogous results for finite and infinite sums of single crossing functions. We proceed with several definitions.

Definition A.5 (S-summable functions) *Let $h, g : S \to \mathbb{R}$ be single crossing functions. We shall refer to h and g as to S-summable functions on S, denoting $h \sim_{S(S)} g$, if*

(i) at any $s \in S$, such that $g(s) \leq 0$ and $h(s) > 0$,

$$-\frac{g(s)}{h(s)} \geq -\frac{g(s')}{h(s')},$$

when $s' > s$;

(ii) at any $s \in S$, such that $h(s) \leq 0$ and $g(s) > 0$,

$$-\frac{h(s)}{g(s)} \geq -\frac{h(s')}{g(s')},$$

when $s' > s$;

Corollary A.4 *For any $\alpha \in \mathbb{R}$, two single crossing functions $h, g : S \to \mathbb{R}$ are S-summable on S if and only if $\alpha h + g$ is a single crossing function on S.*

Corollary A.5 *Let $\mathscr{F} = \{f_i\}_{i=1}^m$ be a family of $f_i : S \to \mathbb{R}$ S-summable functions on S. Then $\sum_{i=1}^m \alpha_i f_i$ is a single crossing function on S for $\alpha_i \geq 0$, $\forall i = 1, \ldots, m$.*

Corollary A.6 *Let $f : S \times T \to \mathbb{R}$ be a single crossing function on S for all $t \in T$, such that $\{f(\cdot, t)\}_{t \in T}$ is a family of Ssummable functions on S. Let $(T, \mathcal{T}, \mu)$ be a measure space, and $f(s, \cdot)$ be $\mathcal{T}$-measurable, for all $s \in S$. Then, $F(s) := \int_T f(s,t) d\mu(t)$ is a single crossing function on S.*

The above results can be easily generalized to functions which are not single crossing, but only "join" or "meet" single crossing. In particular, we refer in our work to *join* up-crossing functions (e.g., motivated by the superextremal class of function discussed in Li Calzi and Veinott (1992)). In order to make the paper self complete (per Example, 4.3 in the last section of the paper), we present the following definition.

Definition A.6 *Function $f : S \to \mathbb{R}$ is join up-crossing, if for any $s' > s$, $s, s' \in S$,*

$$f(s) \geq 0 \Rightarrow f(s') \geq 0.$$

Observe, that the above definition is a slight generalization of single crossing functions, where strict inequalities are not necessarily preserved as the argument of a function increases.[32] We proceed with the following result, being a special case of Theorem 1 in Li Calzi and Veinott (1992).

Theorem A.1 *Let, $X \subset \mathbb{R}$ be a join complete lattice, and S a poset. Moreover, let $f : X \times S \to \mathbb{R}$ be continuous on X, and have join up-crossing differences in (x, s), $x \in X$, $s \in S$. Then, for any $s' \geq s$, $s', s \in S$,*

$$\bigvee \arg\max_x f(x, s') \geq \bigvee \arg\max_x f(x, s)$$

We omit the proof.

Our next concern is to guarantee, that a sum of two join up-crossing functions satisfies the property. For this reason, we construct the following order.

Definition A.7 (Join S-summable functions) *Let $h, g : S \to \mathbb{R}$ be join up-crossing functions. We shall refer to h and g as to join S-summable functions on S, denoted by $h \sim_{J(S)} g$, if*

(i) at any $s \in S$, such that $g(s), g(s') < 0$ and $h(s) \geq 0$,

$$-\frac{h(s)}{g(s)} \leq -\frac{h(s')}{g(s')},$$

when $s' > s$;

(ii) at any $s \in S$, such that $h(s), h(s') < 0$ and $g(s) \geq 0$,

$$-\frac{g(s)}{h(s)} \leq -\frac{g(s')}{h(s')},$$

when $s' > s$;

[32] Obviously we could define meet up-crossing differences in an analogous manner.

We proceed with the following lemma.

Lemma A.1 *Let $f : S \times T \to \mathbb{R}$, and $\mathscr{F} := \{f(\cdot, t)\}_{t \in T}$ be a class of join S-summable functions. Then, Corollaries A.4-A.6 hold with "single crossing functions" being replaced by "joint up-crossing functions".*

Proof W: e present only the counterpart of corollary A.4, as the other two are a straightforward modification of previous proofs.

Take $h, g \in \mathscr{F}$, and $s', s \in S$ such that $s' \geq s$. Observe that if $h(s')$ and $g(s')$ are greater or equal zero, the result is trivial. Thus, without loss of generality, it is sufficient to show that the relation holds for some $h(s) \geq 0$, and $g(s) < 0$, $g(s') < 0$.

First we prove, that if $\alpha g + h$ is join up-crossing on S, then $g \sim_{\mathcal{J}(S)} h$. Take an $\alpha^* \in \mathbb{R}$, such that $\alpha^* g(s) + h(s) = 0$. This implies that $\alpha^* g(s') + h(s') \geq 0$. By rearranging the inequality (keeping in mind that $g(s)$ is strictly negative), we obtain

$$\alpha^* = -\frac{h(s)}{g(s)} \leq -\frac{h(s')}{g(s')}.$$

To prove the opposite implication, let

$$\alpha \leq -\frac{h(s)}{g(s)} \leq -\frac{h(s')}{g(s')},$$

and $\alpha g(s) + h(s) \geq 0$. Then $\alpha g(s') + h(s') \geq 0$ ■

Remark A.1 *Observe, that by modifying Definition A.2 one can define the notion of join Q-summable functions. A slight modification of results concerning join S-summable functions is at the same time sufficient to show, that the same properties are satisfied by families of join Q-summable functions.*

Finally, we present the following definition.

Definition A.8 *Let $(M, \geq_M)$ be a poset of functions $g : \Lambda \times T \to S$, endowed with a pointwise order. Let $\mathscr{F} := \{f(\cdot, t)\}_{t \in T}$ be a family of single crossing functions, where $f : S \times T \to \mathbb{R}$. We say that $\mathscr{F}$ is a family of functionally (join) S-summable functions, if for any function $g \in M$, $\mathscr{H} := \{h(\cdot, t)\}_{t \in T}$ is (join) S-summable on M, where $h(\cdot, t) := f(g(\cdot, t), t) : \Lambda \to \mathbb{R}$.*

We proceed with the following lemma.

Corollary A.7 *Let $f : S \times T \to \mathbb{R}$, and $\mathscr{F} := \{f(\cdot, t)\}_{t \in T}$ be a class of functionally (join) S-summable functions. Then, Corollaries A.4-A.6 (respectively, Lemma A.1) hold with "S-summable" ("join S-summable") being replaced by "functionally S-summable" ("functionally join S-summable").*

The proof is straightforward, hence is omitted.

A.2 Fixed points on monotone operators on countably complete posets

Markowsky (1976) proved the following:

Theorem A.2 (Markowski's Fixed Point Theorem) *Let $f : X \to X$ be isotone function, and X a chain complete poset. Then the set of fixed points of f is a chain complete poset. Moreover $\bigvee \{x : x \leq f(x)\}$ is the greatest fixed point and $\bigwedge \{x : x \geq f(x)\}$ is the least fixed point of f.*

We now provide the following definition:

Definition A.9 *A function $f : X \to X$ is monotonically sup-preserving, if for any increasing sequence $\{x_n\}_{n=0}^{\infty}$ we have: $f(\bigvee x_n) = \bigvee f(x_n)$. We define monotonically inf-preserving functions analogously (for decreasing sequences).*

The Tarski-Kantorovich Theorem (see Theorem 4.2 in Dugundji and Granas (1982)) assures that:

Theorem A.3 *Let X be a countably chain complete poset*[33] *greatest ($\overline{x}$) and the least element ($\underline{x}$). Let $f : X \to X$ be an isotone function. Then*

(i) if f is monotonically inf-preserving then $\bigwedge f^n(\overline{x})$ is the greatest fixed point of f,

(ii) if f is monotonically sup-preserving then $\bigvee f^n(\underline{x})$ is the least fixed point of f.

We now state our fixed point theorem on monotone operators in countably complete partially ordered sets.

Theorem A.4 *Let X be a countably chain complete poset with the greatest element ($\overline{x}$) and the least element ($\overline{x}$), $f : X \to X$ an isotone function. Then:*

•

(i) if f is monotonically inf-preserving, then

$$\overline{\Phi} = \bigvee \{x : F(x) \geq x\},$$

is the greatest fixed point of f;

(iii) if F is monotonically sup-preserving, then

$$\underline{\Phi} = \bigwedge \{x : F(x) \leq x\},$$

is the least fixed point of f.

We immediately obtain the following two important corollaries.

Corollary A.8 *Let X be a countably chain complete poset with the greatest and least elements and T a poset. If $F : X \times T \to X$ is increasing, and monotonically (sup-) inf-preserving on X, then ($t \to \underline{\Phi}(t)$) $t \to \overline{\Phi}(t)$ is isotone.*

[33] We recall the definition of a countably chain complete poset. A a partially ordered set $(Y, \geq)$ is a chain (or a totally ordered set), if for any $x, y \in X$, either $x \geq y$, or $y \geq x$. Therefore, a partially order set (poset) $(X, \geq)$ is chain complete (CPO), if any chain $Y \subset X$, has a supremum and infimum in X. If the last condition holds only for countable subsets Y, we say that X is a countably chain complete poset (CCPO).

B Proofs

Proof of proposition 2.1: Take any chain chain $T_0 \subset \hat{T}$ (w.r.t. $\succeq_{\hat{T}}$). We show that $\bigvee T_0 \in \hat{T}$. In a similar way we show that $\bigwedge T_0 \in \hat{T}$.

Let $f \in CM$. By Birkhoff (1967), Theorem 3, p. 241, there exists the least (modulo null) measurable function $\varphi(f) : S \to \mathbb{R}$ such that:[34]

$$\forall \tau \in T_0, \varphi_0(f)(s) \geq \int_{\Lambda \times A} f(\alpha, a)\tau(d\alpha \times da|s)\, \mu\, a.e..$$

We show $\varphi_0(f)$ is a linear functional for all s.[35] First we show it preserves multiplication by a positive scalar $\alpha > 0$. Then for all $\tau \in T_0$ and μ a.a. s

$$\varphi_0(\alpha f)(s) \geq \alpha \int_{\Lambda \times A} f(\alpha, a)\tau(d\alpha \times da|s)$$

Taking a supremum over $\tau \in T_0$ we have $\varphi_0(\alpha f) \geq \alpha\varphi_0(f)$. We show the reverse inequality. Observe that for all $\tau \in T_0$ and μ a.a. s

$$\alpha\alpha_0(f)(s) \geq \alpha \int_{\Lambda \times A} f(\alpha, a)\tau(d\alpha \times da|s) \tag{1}$$

$$= \int_{\Lambda \times A} \alpha f(\alpha, a)\tau(d\alpha \times da|s) \tag{2}$$

$$\tag{3}$$

Taking a supremum over $\tau \in T_0$ we have $\alpha\varphi_0(f)(s) \geq \varphi_0(\alpha f)(s)$ for μ a.a. s. As a result we have equality. To complete the proof, we simply need need to show that it preserves addition. Let f, g be arbitrary increasing functions. Then, for any $\tau \in T_0$ and μ a.a. s:

$$\begin{aligned} &\int_{\Lambda \times A} f(\alpha, a)\tau(d\alpha \times da|s) + \int_{\Lambda \times A} g(\alpha, a)\tau(d\alpha \times da|s) \\ \leq \quad &\bigvee_{\tau \in T_0} \left\{ \int_{\Lambda \times A} f(\alpha, a)\tau(d\alpha \times da|s) \right\} + \bigvee_{\tau \in T_0} \left\{ \int_{\Lambda \times A} g(\alpha, a)\tau(d\alpha \times da|s) \right\}. \end{aligned}$$

Taking supremum over the left side of a definition of φ_0, we have:

$$\varphi_0(f + g)(s) \leq \varphi_0(f)(s) + \varphi_0(g)(s).$$

To show the converse inequality, for arbitrary $\tau_1 \geq \tau_2$ and both $\tau_i \in T_0$ we have:

$$\begin{aligned} &\int_{\Lambda \times A} f(\alpha, a)\tau_1(d\alpha \times da|s) + \int_{\Lambda \times A} g(\alpha, a)\tau_2(d\alpha \times da|s) \\ \leq \quad &\int_{\Lambda \times A} f(\alpha, a)\tau_1(d\alpha \times da|s) + \int_{\Lambda \times A} g(\alpha, a)\tau_1(d\alpha \times da|s) \\ = \quad &\int\limits_{\Lambda \times A} f(\alpha, a) + g(\alpha, a)\tau_1(d\alpha \times da|s). \end{aligned}$$

[34] Observe that the supremum with this order may essentially differ from the supremum over natural pointwise order. Let Ω be a set of ordinal numbers with cardinality continuum. Consider a one-to-one function from $\xi : \Omega \to [0, 1]$. Let $E_\omega := \{\xi(\omega')|\omega' < \omega\}$. Define $\overline{\omega} := \min\{\omega \in \Omega | E_\omega \text{ is non-measurable}\}$. In other words $\varphi(f)(s) = \bigvee\{\int_{\Lambda \times A} f(\alpha, a)\tau_\omega(d\alpha \times da|s)|\omega \in \Omega\}$. For $\omega < \underline{\omega}$ consider $f_\omega(x) = \chi_{E_\omega}(x)$. Clearly f_ω is a chain with pointwise limit as indicator of non-measurable set.

[35] Note that set of increasing functions forms a cone, but not a linear space.

Taking a supremum over $\tau_1 \in T_0$ and next $\tau_2 \in T_0$ we have $\varphi_0(f)(s) + \varphi_0(g)(s) \geq \varphi_0(f+g)(s)$ μ a.e. Consequently we have equality. Hence φ_0 preserves addition and multiplication by positive value. Next we can extend a function $\varphi_0(\cdot)(s)$ to decreasing and continuous functions in a following way $\varphi_0(f) = -\varphi_0(-f)$. Clearly φ_0 preserves addition and multiplication by positive value on decreasing and continuous function as well.

Let M be a set of continuous monotone functions. Let $LM := \mathrm{lin}(M)$. Clearly,

$$LM := \{f + g | f \text{ is increasing and continuous } g \text{ is decreasing and continuous}\}.$$

We can consequently extend this functional to whole space LM which is linear in the canonical sense. Observe however that, the function $h \in LM$ have many representations $h = f + g$ $f \in CM$ and $g \in -CM$. We need to show $\varphi_0()$ is well defined and does not depend on representation of h. First we show that $\varphi_0(\mathbf{0}) = 0$ regardless on representation of $\mathbf{0}$. Let $\mathbf{0} = f + g$ with $f \in CM$ and $g \in -CM$. Then $g = -f$. Hence and by definition of $\varphi_0(f)$ for $g \in -CM$ we have

$$\varphi_0(\mathbf{0}) = \varphi_0(f) + \varphi_0(-f) = \varphi_0(f) - \varphi_0(f) = 0.$$

Now we compute $\varphi_0(h)$ for arbitrary $h \in LM$. To do it consider $h = f_1 + g_1 = f_2 + g_2$ with $f_i \in CM$ and $g_i \in -CM$. Then $\mathbf{0} = (f_1 - g_2) + (g_1 - f_2)$ and $f_1 - g_2 \in CM$ and $g_1 - f_2 \in -CM$. Since by previous step $\varphi_0(\mathbf{0}) = 0$

$$\varphi_0(f_1 - g_2) = \varphi_0(f_2 - g_1).$$

Since φ_0 preserves addition on CM we have

$$\varphi_0(f_1) - \varphi_0(g_2) = \varphi_0(f_2) - \varphi_0(g_1),$$

which means that $\varphi_0(f_1) + \varphi_0(g_1) = \varphi_0(f_2) + \varphi_0(g_2)$ and φ_0 is well defined for all $h \in LM$. To finish the proof that $\varphi_0(\cdot)(s)$ is linear we need to show that $\varphi_0(\cdot)(s)$ preserves all linear combinations of h_1 and h_2. Indeed if $h_i = f_i + g_i$ $(i = 1, 2)$ and $f_i \in CM$ $g_i \in -CM$ and for $\alpha < 0$ and $\beta < 0$ we have:

$$\begin{aligned}
\varphi_0(\alpha h_1 + \beta h_2) &= \varphi_0(\alpha f_1 + \beta f_2) + \varphi_0(\alpha g_1 + \beta g_2) && (4)\\
&= -\varphi_0(-\alpha f_1 - \beta f_2) - \varphi_0(-\alpha g_1 - \beta g_2) && (5)\\
&= -\varphi_0(-\alpha f_1) - \varphi_0(-\beta f_2) - \varphi_0(-\alpha g_1) - \varphi_0(-\beta g_2) && (6)\\
&= \alpha\varphi_0(f_1) + \beta\varphi_0(f_2) + \alpha\varphi_0(g_1) + \beta\varphi_0(g_2) && (7)\\
&= \alpha\varphi_0(h_1) + \beta\varphi_0(h_2),
\end{aligned}$$

where (4) and (5) follow by definition of φ_0, (6) is satisfied since φ_0 preserves addition on CM and $-CM$ and (7) is satisfied since φ_0 preserves multiplication by positive scalar. Hence φ_0 is linear operator on LM. Observe that as $\varphi_0(f)$ is positive on LM (i.e. $\xi(\cdot) \geq 0$), for $\xi \in LM$ follows $\varphi_0(\xi) > 0$. That is, $\xi = f + g \geq 0$ for f-increasing and g-decreasing follows from $f(s) \geq -g(s)$. Since both f and $-g$ are both increasing, we have $\varphi_0(f) \geq -\varphi_0(g)$, and $\varphi_0(\xi) \geq 0$.

Next, we show that there is positive extension of the functional on the whole space of continuous functions. As $\Lambda \times A$ is compact and metrizable., f is bounded from above and below, so it is contained between two constant functions which are clearly each in LM. Therefore, there exists a monotone sublinear function, e.g. $p(f) := \max_{a \in \tilde{A}(\alpha,s)} f(\alpha, s)$, such that $p(f)(s) \geq \varphi_0(f)(s)$

on LM. Observe that LM is a Riesz subspace of a set of continuous functions on $\Lambda \times A$. Thus, by Theorem 8.31 in Aliprantis and Border (2006), there is a positive extension of the operator φ_0 on the whole space of continuous, real-valued functions supported on $Gr(\tilde{A})$. Again, by the Riesz-Markov Theorem (see Theorem 14.12 in Aliprantis and Border (2006)), there exists a unique regular measure $\tau_0(\cdot|s)$ such that:

$$\varphi_0(f)(s) = \int_{\Lambda \times A} f(\alpha, s)\tau_0(d\alpha \times da|s).$$

Thus, it is a continuous functional with norm 1 (as τ is a probability measure). By definition, $\varphi_0(f)(\cdot)$ is measurable whenever f is increasing and measurable, hence $\tau_0 \in \hat{T}_d$ and τ_0 is the supremum of $\tau \in T_0$ in $(\hat{T}_d, \prec_p)$. It is easy to verify that whenever we have for all $\tau \in T_0$, $s \in S$, the mapping $\tau(\cdot|s)$ having support on the graph of $\tilde{A}(\cdot, s)$, so does τ_0. Moreover, since marginal distribution of τ on Λ is λ, hence marginal of τ_0 is λ as wwell. Then, to finish the proof, just take $f \equiv 1$, so that $\varphi_0(1)(s) = \int_{\Lambda\times A} \tau(d\alpha \times da|s)$. Similarly. we show that decreasing chains have infimum in $\hat{T}_d$. ■

Proof of lemma 2.1: Since u is $\mathcal{T} \otimes \mathcal{X}$-measurable, and M is the space of $\mathcal{T}$-measurable functions, $u(\cdot, f(\cdot))$ is $\mathcal{T}$-measurable. Moreover, as $\forall t \in T, u(t, \cdot)$ is a single crossing function, and $\{u(t, \cdot)\}_{t\in T}$ is a functionally S-summable family on M with respect to the pointwise order, v is a single crossing function with respect to the same ordering.

Now, we will show, that v is also a single crossing function on M with respect to μ-a.e. pointwise order. Let $f', f \in M$, such that $f'(t) \succeq_X f(t), \mu$-a.e., and let T' denote a set of points where $f'(t) < f(t)$. Clearly, $\mu(T') = 0$. Assume $0 \leq h(f)$. Therefore:

$$0 \leq h(f) = \int_T u(t, f(t))d\mu(t) = \int_{T\setminus T'} u(t, f(t))d\mu(t).$$

Since $\{u(t, \cdot)\}_{t\in T}$ is a functionally S-summable family on M with respect to the pointwise order, so is $\{u(t, \cdot)\}_{t\in T\setminus T'}$. Hence, $0 \leq \int_{T\setminus T'} u(t, f'(t))d\mu(t) = \int_T u(t, f'(t))d\mu(t) = h(f')$. ■

Proof of lemma 2.2: We prove the result for $\overline{B}$. The proof for $\underline{B}$ is analogous. First, $v(\alpha, s, \tau, \cdot)$ is continuous on A. Moreover, by Ely and Pęski (2006), Lemma 9, it is also measurable on $\mathcal{L}$, hence v is Carathéodory. By Assumption 2.1(ii),(iii), as well as Corollaries A.3, A.6 and Proposition 2.1, v has single crossing differences in (a, f), with respect to μ-a.e. pointwise order $\succeq_{\hat{T}}$.

Since $\tilde{A}(\alpha, s)$ is compact, by Berge's Maximum Theorem (see Berge (1997), p. 116), $m(\alpha, s, \tau) := \arg\max_{a\in\tilde{A}(\alpha,s)} v(\alpha, s, \tau, a)$ is nonempty. In addition, by Milgrom and Shannon (1994) or Veinott (1992) generalization of Topkis Topkis (1978) Monotonicity Theorem, m is a subcomplete sublattice of $\tilde{A}(\alpha, s)$ with the greatest and the least element, and is isotone in the Veinott's strong set order in τ. From the Measurable Maximum Theorem (see Aliprantis and Border (2006), Theorem 18.19), it follows that m is $\mathcal{L} \otimes \mathcal{S}$-measurable (and hence, weakly measurable as A is a metrizable space, which admits a measurable selection. See Aliprantis and Border (2006), Lemma 18.2.). Therefore, $\overline{m}(\alpha, s, \tau)$ exists, and is increasing on $\hat{T}_d$.

We now need to prove that $\overline{m}(\alpha, s, \tau)$ is a measurable selection of $m(\alpha, s, \tau)$. Define $\overline{m}(\alpha, s, \tau) = (\overline{m}_1, \ldots, \overline{m}_n)$. Again by the Measurable Maximum Theorem, function $\overline{m}_i(\cdot, \tau) := \max_{a_i\in m(\cdot,\tau)} a_i$ is $\mathcal{L} \otimes \mathcal{S}$-measurable for any τ and $i = 1, \ldots, n$. Hence, $\overline{m}(\cdot, \tau)$ is also $\mathcal{L} \otimes \mathcal{S}$-measurable, hence $\overline{B}(\tau)(s)$ is measurable in s as well.

Next we show that $\overline{B}$ is increasing. Fix an arbitrary, increasing function $f : \Lambda \times A \to \mathbb{R}$. For an arbitrary $\alpha \in \Lambda$, we prove that $\tau \to \overline{m}(\alpha, s, \tau)$ is isotone, whenever $\tau \in \hat{T}_d$. Let $\tau \succeq_{\hat{T}} \tau'$. Fix arbitrary $s \in S$, such that $\tau(s)$ stochastically dominates $\tau'(s)$. We have:

$$\begin{aligned}\int_{\Lambda\times A} f(\alpha, a)\overline{B}(\tau)(d\alpha \times da) &= \int_{\Lambda\times A} f(\alpha, \overline{m}(\alpha, s, \tau))\lambda(d\alpha) \\ \geq \quad \int_{\Lambda\times A} f(\alpha, \overline{m}(\alpha, s, \tau'))\lambda(d\alpha) &= \int_{\Lambda\times A} f(\alpha, a)\overline{B}(\tau')(d\alpha \times da),\end{aligned}$$

where first equality follows from the definition of $\overline{B}(\tau)$, concentrated on the graph of $\overline{m}(\alpha, s, \tau)$. The inequality follows from monotonicity of $\overline{m}$ in τ. Since the set of $s \in S$ satisfying the above equation has the full measure, $\overline{B}(\tau) \succeq_{\hat{T}} \overline{B}(\tau')$. ■

Proof of theorem 2.1: By Lemma 2.2, $\overline{B}$ is isotone. Moreover, by Proposition 2.1, $\hat{T}_d$ is a chain complete poset. Hence, by Markovsky's Theorem (see Appendix, Theorem A.2), $\overline{B}$ has a chain complete poset of fixed points, with the greatest and the least element. Denote the greatest element of the set by $\overline{\tau}^*$. Then, by definition, $\overline{\tau}^*$ constitutes distributional Bayesian-Nash equilibrium of Γ.

Next, we will prove that $\overline{\tau}^*$ is the greatest equilibrium of the game. Take any other equilibrium of the game τ'. Fix $s \in S$, such that $\tau'\left(\{(\alpha, a) | a \in m(\alpha, s, a)\} | s\right) = 1$. Clearly $\{(\alpha, a) | a \in m(\alpha, s, a)\}$ is a set of full measure. Let $f : \Lambda \times A \to \mathbb{R}$ be increasing. Then,

$$\int_{\Lambda\times A} f(\alpha, a)\tau'(d\alpha \times da|s) \leq \int_{\Lambda\times A} f(\alpha, \overline{m}(\alpha, s, \tau'))\lambda(d\alpha) \leq \int_{\Lambda\times A} f(\alpha, a)\overline{B}(\tau')(d\alpha \times da|s).$$

Therefore, $\tau' \prec_{\hat{T}} \overline{B}(\tau')$. Since $\overline{T}$ is isotone, by Markovsky's Theorem $\overline{\tau}^* \succeq_{\hat{T}} \tau'$. We prove existence of the least equilibrium analogously, using operator $\underline{B}$. ■

Proof of corollary 2.1: In order to prove the result, we shall use Tarski-Kantorovich Theorem (see Appendix, Theorem A.3). It is sufficient to show that $\overline{B}$ and $\underline{B}$ are respectively monotonically inf- and sup-preserving.

Take any decreasing sequence $\{\tau^n\}$ in $\hat{T}_d$, such that $\tau^n \to \tau$ in $\hat{T}_d$. By Assumption 2.2 and Lebesgue Dominated Convergence Theorem, $m(\alpha, s, \tau^n) \to m(\alpha, s, \tau)$, in particular, since $\{\overline{m}(\alpha, s, \tau^n)\}$ form a monotone sequence in $\tilde{A}(\alpha, s)$ (which is a complete lattice), $\overline{m}(\alpha, s, \tau^n) \to \overline{m}(\alpha, s, \tau)$. Hence, for any $s \in S$, by Lebesgue Dominated Convergence Theorem $\overline{B}_s(\tau^n) \to \overline{B}_s(\tau)$, and so $\overline{B}$ is inf-preserving. Since $\overline{B}$ is isotone, the hypothesis of Theorem A.3 are satisfied. To prove the second part of the corollary, we conduct a similar argument. ■

Proof of corollary 2.2: By Assumptions 2.1-2.3, for any $\tau \in \hat{T}_d$, $\overline{B}(\tau)$ (respectively $\underline{B}(\tau)$) is increasing in θ and monotonically inf-preserving (respectively sup-preserving) on $\hat{T}_d$. Therefore, by Theorem A.8, $\overline{\tau}^*(\cdot)$ and $\underline{\tau}^*(\cdot)$ are increasing on Θ. ■

Proof of lemma 3.1: Since $A \subset \mathbb{R}^n$, any compact subset of A is closed. Hence, by Assumption 3.1(i), $\tilde{A}$ has non-empty, closed values. Moreover, it maps measurable space into a complete metric space (hence, a Polish space). Therefore, by Kuratowski-Ryll-Nardzewski Selection Theorem (see Aliprantis and Border (2006), Theorem 18.13), $\tilde{A}(\alpha, \cdot)$ and $\tilde{A}$ include a measurable selection. ■

Proof of lemma 3.2: By Assumption 3.1(ii), r is continuous in a. By Lebesgue Dominated Convergence Theorem, so is v. Since r is $\mathcal{L} \otimes \mathcal{S}$-measurable, and $f \in M(L \times S)$, then $r \circ f$ is $\mathcal{L} \otimes \mathcal{S}$-measurable. By Lemma 9 in Ely and Pęski (2006), so is v. Therefore, v is Carathéodory in $(a, (\alpha, s))$. By Assumptions 3.1(ii),(iii), as well as Corollaries A.3, A.6, v is quasi-supermodular on A, with single crossing differences in (a, f).

Recall that A is a separable, metric space, $(S, \mathcal{S})$ is a measurable space, and $\tilde{A}$ is a well defined and weakly measurable, with compact values. Therefore, by the Measurable Maximum Theorem (see Aliprantis and Border (2006), Theorem 18.19), $\arg\max_{a \in \tilde{A}(\alpha,s)} v(\alpha, s, f, a)$ is well defined with compact values, $\mathcal{L} \otimes \mathcal{S}$-measurable, and admits a measurable selection. Hence, BR is well defined. In addition, since it maps a measurable space into a metrizable space, it is also weakly measurable.

In addition, by Milgrom and Shannon (1994) or Veinott (1992) generalization of Topkis' Monotonicity Theorem, it is a complete lattice, with the greatest and the lest element, isotone in the Veinott order in f.

Since $BR(f)$ is a complete lattice, isotone in f, both $\overline{BR}, \underline{BR}$ exist and are increasing in f (pointwise). Now, we prove that they are each measurable selections of $BR(f)$. First consider $\overline{BR}(f)$. Let $\overline{BR}(f) := (\bar{f}_1, \ldots, \bar{f}_n)$. Again by the Measurable Maximum Theorem, function $\bar{f}_i(\cdot) := \max_{a_i \in BR(f)(\cdot)} a_i$ is $\mathcal{L} \otimes \mathcal{S}$-measurable for any f and $i = 1, \ldots, n$, hence $\overline{BR}(f)$ is also $\mathcal{L} \otimes \mathcal{S}$-measurable. Analogously, we prove that $\underline{BR}(f)$ is $\mathcal{L} \otimes \mathcal{S}$-measurable. ■

Proof of theorem 3.1: Lemma 3.2 implies that both $\overline{BR}, \underline{BR} : M_\Lambda \to M_\Lambda$ are well defined. We claim that $\overline{BR}$ is monotonically inf-preserving, while $\underline{BR}$ is monotonically sup-preserving. To see this, take an decreasing sequence $\{f_n\}$, $f^n \to f$ in M_Λ. Observe that $f = \bigwedge f^n$. Since $\overline{BR}$ is increasing, $\bigwedge \overline{BR}(f_n) = \lim_{n\to\infty} \overline{BR}(f_n)$. On the other hand, $\overline{BR}(\bigwedge f_n) = \overline{BR}(f)$. It is therefore sufficient to show that $\lim_{n\to\infty} \overline{BR}(f_n) = \overline{BR}(f)$. By Assumption 3.1(iv), and Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \bigvee \left\{ \arg \max_{a \in \tilde{A}(\alpha,\cdot)} v(\alpha, \cdot, f_n, a) \right\} = \bigvee \left\{ \arg \max_{a \in \tilde{A}(\alpha,\cdot)} v(\alpha, \cdot, f, a) \right\},$$

hence $\lim_{n\to\infty} \overline{BR}(f_n) = \lim_{n\to\infty} \bigvee BR(f_n) = \bigvee BR(f) = \overline{BR}(f)$, and so $\overline{BR}(\bigwedge f^n) = \bigwedge \overline{BR}(f^n)$. Analogously, we prove that $\underline{BR}$ is monotonically sup-preserving.

As M_Λ is a countably chain complete poset, by the generalization of Knaster-Tarski Theorem (see Theorem A.3 in the Appendix), $\overline{BR}$ (resp., $\underline{BR}$) have greatest (resp, least) fixed points, in addition to equilibria, in Γ.

Denote the greatest fixed point of $\overline{BR}$ by $\overline{f}^*$ and the least point of $\underline{BR}$ by $\underline{f}^*$. For an arbitrary equilibrium f_0, by Knaster-Tarski Theorem , $\underline{f}^* = \bigwedge\{f : \underline{BR}(f) \le f\} \le f_0 \le \bigvee\{f : \overline{BR}(f) \le f\} = \overline{f}^*$, which completes the proof. ■

Proof of corollary 3.1: By Assumptions 3.1, 3.2, $\overline{BR}(f)$ (respectively $\underline{BR}(f)$) is increasing in θ and monotonically inf-preserving (respectively sup-preserving). By Theorem A.8, $\overline{f}^*(\cdot)$ and $\underline{f}^*(\cdot)$ are increasing on Θ. ■

Proof of lemma 4.1: First we shall prove, that for any convex set $X \in \mathbb{R}$, function $h : X \to \mathbb{R}$, such that $h(x) := f(|x|)$ is concave. Take any $x', x \in X$, and $\alpha \in [0, 1]$. Since X is convex,

$\alpha x + (1-\alpha)x' \in X$. Then

$$f(|\alpha x + (1-\alpha)x'|) \geq f(\alpha|x| + (1-\alpha)|x'|) \geq \alpha f(|x|) + (1-\alpha)f(|x'|),$$

where the first inequality is implied by triangle inequality, and monotonicity of f, while the second by concavity of f.

Since h is concave, for any $x' \geq x$, $x', x \in X$, and $s' \geq s$, such that h is well defined,

$$\frac{h(x'-s') - h(x-s')}{(x'-s') - (y-s')} \geq \frac{h(x'-s) - h(x-s)}{(x'-s) - (y-s)},$$

which implies that $h(x'-s') - h(x-s') \geq h(x'-s) - h(x-s)$. Since $g(x,s) = h(x-s)$, the proof is complete. ■

Proof of remark 4.1: Take $a' \geq a$, and $\tau' \succeq_{\hat{T}} \tau$. Assume that

$$r(\alpha, s, a')\chi_{\{\tau(\{(\alpha,a'')|a''\geq a'\}|s)\leq G\}} \geq r(\alpha, s, a)\chi_{\{\tau(\{(\alpha,a'')|a''\geq a\}|s)\leq G\}},$$

Since $\chi_{\{\tau(\{(\alpha,a'')|a''\geq a\}|s)\leq G\}}$ is increasing and r is decreasing in a, this takes place either when $\chi_{\{\tau(\{(\alpha,a'')|a''\geq a'\}|s)\leq G\}} = 1$ and $\chi_{\{\tau(\{(\alpha,a'')|a''\geq a\}|s)\leq G\}} = 0$, or $\chi_{\{\tau(\{(\alpha,a'')|a''\geq a'\}|s)\leq G\}} = 0$, and $\chi_{\{\tau(\{(\alpha,a'')|a''\geq a\}|s)\leq G\}} = 0$.

Consider the first case. For this, $r(\alpha, s, a') \geq 0$. As $\chi_{\{\tau(\{(\alpha,a'')|a''\geq a\}|s)\leq G\}}$ decreases in τ, $\chi_{\{\tau'(\{(\alpha,a'')|a''\geq a'\}|s)\leq G\}}$ is equal to 1 or 0. If it is equal to 1, the implication is true. If it is equal to zero, the result also holds. In the second case the result is trivial.

To prove that collection $\{r(\alpha, s, \cdot)\chi_{\{\tau(\{(\alpha,a'')|a''\geq\cdot\}|s)\leq G\}}\}_{s\in S}$ is functionally join S-summable, take any two $s', s \in S$, and any $a' \geq a$ in A, as well as $\tau' \succeq_{\hat{T}} \tau$ in $\hat{T}_d$. Without loss of generality assume that

$$r(\alpha, s, a')\chi_{\{\tau(\{(\alpha,a'')|a''\geq a'\}|s)\leq G\}} - r(\alpha, s, a)\chi_{\{\tau(\{(\alpha,a'')|a''\geq a\}|s)\leq G\}} \geq 0,$$

and

$$r(\alpha, s', a')\chi_{\{\tau(\{(\alpha,a'')|a''\geq a'\}|s')\leq G\}} - r(\alpha, s', a)\chi_{\{\tau(\{(\alpha,a'')|a''\geq a\}|s')\leq G\}} < 0,$$

Observe, that this only takes place when

$$\chi_{\{\tau(\{(\alpha,a'')|a''\geq a'\}|s')\leq G\}} = \chi_{\{\tau(\{(\alpha,a'')|a''\geq a\}|s)\leq G\}} = 1,$$

which implies that

$$\chi_{\{\tau'(\{(\alpha,a'')|a''\geq a'\}|s')\leq G\}} = \chi_{\{\tau'(\{(\alpha,a'')|a''\geq a\}|s')\leq G\}} = 1,$$

hence, the denominator is constant. Since the nominator in the definition or join S-summable relation (see Definition A.7) is positive and decreasing, the condition holds for any two arbitrary s' and s. The proof is complete. ■

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