Time consistent equilibria in dynamic models with recursive payoffs and behavioral discounting*

Łukasz Balbus[†] Kevin Reffett[‡] Łukasz Woźny[§]

November 2020

Abstract

We prove existence of time consistent equilibria in a wide class of dynamic models with recursive payoffs and generalized discounting involving both behavioral and normative applications. Our generalized Bellman equation method identifies and separates both: recursive and strategic aspects of the equilibrium problem and allows to precisely determine the sufficient assumptions on preferences and stochastic transition to establish existence. In particular we show existence of minimal state space stationary Markov equilibrium (a time-consistent solution) in a deterministic model of consumption-saving with beta-delta discounting and its generalized versions involving magnitude effects, non-additive payoffs, semi-hyperbolic or hyperbolic discounting (over possibly unbounded state and unbounded above reward space). We also provide an equilibrium approximation method for a hyperbolic discounting model.

Keywords: Behavioral discounting; Time consistency; Markov equilibrium; Existence; Approximation; Generalized Bellman equation; Hyperbolic discounting; Semi-hyperbolic discounting; Quasi-hyperbolic discounting

JEL classification: C61, C73

*We want to thank Jean-Pierre Drugeon, Tai-Wei Hu, Martin Kaae Jensen, Jawwad Noor, Christopher Phelan, Ed Prescott, David Rahman, Manuel Santos, and Jan Werner for valuable discussion during the writing of this paper, as well as participants of TUS VI Conference in Paris (2019) and Econ Theory Seminar at the University of Minnesota (2020) for interesting comments on the earlier drafts of this paper. The project was financed by the National Science Center: NCN grant number UMO-2019/35/B/HS4/00346.

[†]Faculty of Mathematics, Computer Sciences and Econometrics, University of Zielona Gora, Poland.

[‡]Department of Economics, Arizona State University, USA.

[§]Department of Quantitative Economics, Warsaw School of Economics, Warsaw, Poland. Address: al. Niepodległości 162, 02-554 Warszawa, Poland. E-mail: lukasz.wozny@sgh.waw.pl.

1 Introduction

Since the seminal work of Ramsey (1928) and Samuelson (1937), the question of how agents in dynamic choice models discount future utility streams has been a central focus of large body of research in the social sciences. Ramsey (1928) suggested intertemporal utilities be modeled as weighted sum of future utilities, while Samuelson (1937) proposed exponential discounting. With the work of Koopmans (1960), however, on the axiomatic foundations of dynamically consistent choice, it become clear how profoundly these two situations differ. At this point in time, many researchers adopted a dynamically consistent approach, and exponential discounting as the standard approach to modeling dynamic preferences in various economic problems.

Strotz (1956) however proposed a theory of dynamically inconsistent choice, and with his paper started a new and separate line of research studying the implications of dynamically inconsistent preferences in intertemporal economic models. With the important work of Laibson (1997), models with dynamically inconsistent preferences have become workhorse tools in behavioral economic models that challenge the rational foundations of dynamic choice. Motivation for studying such models with dynamically inconsistent preferences is found in a large empirical and experimental literature where numerous papers have documented the importance of preference reversals on dynamic choice when modeling how agents compare current vs. future utilities. These empirical results over the last two decades have led to a subsequent resurgence of theoretical work that seeks (i) to provide further axiomatic foundations to time inconsistent choice,[1] as well as (ii) tools constructing theories of coherent dynamic choice in various settings, where agents have changing intertemporal tastes. Work on self-control, the role of impulse and temptation, and time consistency in dynamic choice has appeared in many fields such as mathematical psychology, political science, philosophy, decision theory, game theory, as well as economics, and included studies of consumption-savings, dynastic choice with altruistic or paternalistic preferences, dynamic collective household choice, distributive justice and dynamic social choice, public policy design, models of social discounting in environmental cost-benefit analysis, theories of endogenous preference formation and reference points including theories of habit-formation, addiction, focus-weighted choice and salience, and dynamic random utility. And although much of this work on time inconsistent choice has focused on its positive aspects, recent work has also begun to addresses welfare issues, including how to design the optimal policy, how to assess paternalistic policies that seek to "improve" agents welfare in the presence of dynamically inconsistent choice, as well as the welfare implications of commitment devices that can induce self-control among consumers.[2]

[1]For a recent selection of axiomatic work see e.g. Wakai (2008), Montiel Olea and Strzalecki (2014), Galperti and Strulovici (2017), Chambers and Echenique (2018), Drugeon and Ha-Huy (2018), among many others. This work includes also papers on temptation preferences of Gul and Pesendorfer (2001, 2004, 2005), as well as related work on self-control and time-inconsistent choice of Noor (2011), Dekel and Lipman (2012), Ahn et al. (2019), and Ahn et al. (2020), Noor and Takeoka (2020b,a) among others.

[2]Relative to the question of welfare, there is also a large literature on the role of commitment devices in dynamic models with time inconsistent preferences. For a nice survey of this work, see Bryan et al. (2010) and

This paper contributes to many different strands in the literature of behavioral, dynamic economic models. Optimal plans under such preferences over time are often time-inconsistent and (perhaps surprisingly) a decision maker has no incentive to follow the optimal plan in the future. More formally, decisions are time inconsistent if plans (e.g. consumption) chosen in period t for the following $\tau, \ldots, T$ periods, say $(c^t_\tau)^T_{\tau=t}$, change with the planning date t. Our focus is on developing new tools for studying the existence and characterization of time-consistent choice in dynamic resource allocation problems where decisionmakers have preferences that exhibit general forms of behavioral discounting.[3] Modeling coherent choice in the presence of dynamic inconsistent preferences has a long history in economics, and is found in the early papers of Strotz (1956), Phelps and Pollak (1968), Pollak (1968) and Peleg and Yaari (1973), as well as the much of subsequent work over the last two decades that has followed the work of Laibson (1997) and Harris and Laibson (2001). Out of many interesting problems economists have studied, the question of design and computation of optimal among time consistent plans (i.e., planned sequential choice policies that are followed and not re-optimized) has received a great attention in economic literature. This also includes important from behavioral and numerical perspective short memory decision rules, like Markov or semi-Markov ones.

One important limitation of the existing work relative to this paper is that most of it has focused exclusively on the case of *quasi-hyperbolic* discounting (e.g., for recent work see Krusell and Smith (2003), Krusell et al. (2010), Harris and Laibson (2013), Chatterjee and Eyigungor (2016), Balbus et al. (2015b, 2018), and Cao and Werning (2018)).[4] Although quasi-hyberbolic discounting is a very important, it is also a somewhat special case. In particular, it has a simple pattern of "1 period forward misalignment/bias" in intertemporal preferences. Recent empirical and experimental work in both economics and psychology has found strong support for more general forms of behavioral discounting in dynamic choice models (e.g., including various versions of hyperbolic discounting), however. The work in the literature considering more general behavioral discounting has either focused on special cases where the models admit closed-form solutions (e.g., Young (2007)), or emphasize numerical approaches to the computation of time consistent equilibrium, and do not consider the question of sufficient conditions for its existence (e.g., Maliar and Maliar (2016) or Jensen (2020)).[5] Extending the set of tools developed for characterizing time-consistent choice in dynamic models with quasi-hyberbolic discounting to models with generalized behavioral discounting is not a trivial matter, as the

Beshears et al. (2018). A small sampling includes the papers of Laibson (1997), Harris and Laibson (2013), Gine et al. (2010), Karlan et al. (2016), Casaburi and Macchiavello (2019), Beshears et al. (2020).

[3]By "behavioral discounting" we mean all forms of discounting generated by changing tastes over utility streams.

[4]We should mention, there is a parallel important literature on self-control and impulse management in so called on dual-self models. For example, in economics, see the papers of Fudenberg and Levine (2006, 2012). The tools developed in this paper can be applied to most of these models, as they can be mapped into our language of "behavioral discounting". See Jensen (2020) for a discussion of this mapping.

[5]Per the latter numerical approach, our paper complements this work nicely as we provide sufficient conditions for existence of time consistent equilibrium with *monotone* investment/savings strategies in general models of behavioral discounting. Monotonicity of investment/savings in the general behavioral discounting model greatly simplifies the theory of approximation and computation in the numerical approach.

existing approaches to time consistent choice taken for the quasi-hyberbolic do not appear to extend to the generalized discounting case. Therefore, from a theoretical vantage point, the need for new tools to cover such cases of generalized discounting case is important and challenging.

It is important to mention that there has also been a great deal of empirical and experimental support for various general forms of dynamic inconsistencies in intertemporal preferences that can be tied to some form of behavioral (or non-exponential) discounting. For example, in early work, Laibson et al. (2007) show how high short-term discount rates are needed to explain observed borrowing behavior in US data. More recently, Duflo et al. (2011) estimate a model of naivete random quasi-hyberbolic discounting for fertilizer use in Kenya where there is a positive probability placed on time consistent choice, and find time inconsistency plays an important role in the adoption decision. Chan (2017) estimates a hyperbolic model of discounting where differences in discount factors play a key role in explaining how workers make labor supply decisions in the context of participation in welfare programs. He finds most agents appear to make time inconsistent choices exhibiting general forms of present-bias. In Dalton et al. (2020), the authors study the role of discounting and myopia in the purchase of Medicare D drug insurance contracts, and find strong support of the presence of general time inconsistent behavior and behavioral discounting. Using an experimental approach, Augenblick et al. (2015) find support for time inconsistent behavior in discounting in the context of making effort choices in real tasks. In the context of credit card paydowns, Kuchler and Pagel (2020) find strong support for general forms of present-bias and time inconsistency.[6]

This empirical work has in turn motivated a great deal of new theoretical work seeking to characterize the structure of dynamic choice models in situations with non-exponential discounting of future utility streams. For surveys of this body of theoretical work, see the earlier papers of Fishburn and Rubinstein (1982), Frederick et al. (2002), and Noor (2009), as well the recent surveys of Ericson and Laibson (2019) and Cohen et al. (2020).[7] Some important very recent theoretical contributions to this literature include Harstad (2020), who has studied the interaction between various forms of hyperbolic discounting for government policymakers and dynamic investment to study the structure of optimal investment subsidies in the presence of externalities. He shows that larger subsidies are optimal in the presence of technologies that exhibit a particular form of dynamic strategic complementarities. Halec and Yared (2019) study a prototype small open economy where the government is setting fiscal rules under limited commitment, but where the government has present-bias objectives. This present-bias emerges naturally in any dynamic collective choice problem (e.g, see Jackson and Yariv (2015) and Lizzeri and Yariv (2017))[8]. They show in this case, optimal incentives are bang-bang, with optimal rules en-

[6]For additional discussions of empirical motivation for the importance of present-bias and dynamic inconsistency in choice, see Angeletos et al. (2001), Ameriks et al. (2007) and Cohen et al. (2020).

[7]A separate line of research consider revealed preferences theory of dynamic discounting models see e.g. Echenique et al. (2016) or Dziewulski (2018).

[8]Some behavioral discounting models have implications that coincide with those coming from normative models (see Jackson and Yariv (2015); Ebert et al. (2020), among others, for arguments as to why time-inconsistency shows up at the social preferences level). See also Becker (2012) and Drugeon and Wigniolle (2020) for related

forced by maximally enforced debt limits. Gottleib and Zhang (2020) study the implications of time-inconsistency on the structure of dynamic incentives in a long-term contracting problems between present-bias consumers and risk-neutral firms, and show that firms can offer contracts such that as the length of a contracting problem increases, the welfare-losses associated with present bias disappear. They also explore the role of commitment in supporting this result, all this work done in the setting of a repeated-game. See also the related work of Ceteman et al. (2019) for studying similar questions in the context of a continuous time model. In Iverson and Karp (2020), the authors study a Markov perfect equilibria in a dynamic model of climate with carbon taxes and generalized behavioral discounting, where the decentralized economy determines aggregate savings, and a planner determines climate policy. For a particular class of preferences and technologies (log-linear), they are able to solve the model in closed-form, and characterize the nature of commitment devices and the structure of optimal carbon taxes in their model economy. In Beshears et al. (2020), the authors develop a model of optimal illiquidity in an economy where agents are subjected to taste shocks and have present-bias preferences. They show that the socially optimum is a approximately a two-tier account system which includes completely illiquid accounts and completely liquid accounts. Finally, Heidues and Strack (2019) and Mahajan et al. (2020) discuss methodological issues related to the identification of present-bias and behavioral discounting in econometric models.

One final aspect worth mentioning is the inherent uncertain nature of the future in many of these behavioral discounting models. That is, although dynamic models of choice over time can be applied to both deterministic and stochastic environments, it is the latter that is of utmost importance for empirical studies. There is a number of recent papers showing that preferences over time as well as over uncertain (or risky/stochastic) outcomes are intertwined (see Loewenstein and Prelec (1992), Saito (2009), Andreoni and Sprenger (2012), Ioannou and Sadeh (2016) among others). As Halevy (2008) and Baucells and Heukamp (2012) argue: delaying a prize in time has the same effect as increasing uncertainty of getting this prize. More specifically, uncertainty over future states plays an important role in our analysis. As we will argue, whenever preferences of consecutive generations are misaligned for more than two periods ahead a certain form of transition uncertainty is necessary to obtain existence of stationary time consistent Markov equilibria.

Taking many of these considerations into account, in this paper we study various general forms of behavioral or normative discounting rules that generate dynamically inconsistent preferences in dynamic stochastic decision problems, and study the structure and existence of time-consistent decision rules. In this paper, we focusing primarily on Markovian equilibria in a minimal state space. The task of defining, finding, characterizing, and computing (developing appropriate numerical procedures) is far from trivial. And the above mentioned tasks are only prerequisites of any empirical analysis of implications of various forms of discounting on allocation of scarce economic or environmental resources over current and future generations under

results.

intrinsic uncertainty (see Dasgupta (2008), Arrow et al. (2014) and Gerlagh and Liski (2018)). The central aim of this paper is to prove existence of time consistent equilibrium (e.g., minimal state space Markovian equilibrium) in a large class of dynamic economies with *generalized* discounting that includes in its catalog many important models from the cited literature as special cases. Such results are needed to explore both the behavioral and normative applications of these preferences in dynamic equilibrium models.

Overview of the results Before we proceed to the formalities, we begin by previewing the main results of the paper. Consider a discrete time, infinite horizon, stochastic consumption-saving model, where the sequence of time separable lifetime preferences over sequences of consumption $(c_\tau)_{\tau=t}$ is given any date t by:

$$u(c_t) + \mathbb{E}_t \sum_{\tau=1}^{\infty} \delta_\tau u(c_{t+\tau}). \tag{1}$$

We shall refer to these preferences as $(\delta_t)_{t=0}^{\infty}$-*behavioral discounting preferences.*[9] Notice, at any time period t, the consumer uses the sequence of discount factors:[10]

$$\delta_0, \delta_1, \delta_2, \delta_3, \ldots$$

to value current and continuation utility streams (where, for convenience, we normalize $\delta_0 = 1$). A few additional remarks on these preferences are in order. First, notice these preferences embed the discounting ideas of both Ramsey (1928) and Samuelson (1937) as special cases. Second, most cases in the literature of behavioral discounting fit into this general setting. To mention a few common special cases, we have the following: (i) exponential discounting when $\delta_t = \delta^t$, (ii) quasi-hyperbolic discounting when $\delta_t = \beta\delta^t$ for $t \geq 1$, and (iii) hyperbolic discounting when $\delta_t = \frac{1}{1+t}$. Third, these preferences are generally time-inconsistent. That is, the discount rate between utilities in any two time periods $\tau + 1$ and τ is given by:

$$\frac{\delta_{t+1} u(c_{\tau+1})}{\delta_t u(c_\tau)},$$

for any $t \in \{0, \ldots, \tau\}$. We say the intertemporal preferences between the consecutive periods are *misaligned* whenever for some t:

$$\delta_t^2 \neq \delta_{t-1}\delta_{t+1}.$$

For the special case of exponential discounting, preferences are aligned. For the case of quasi-hyperbolic discounting, preferences are misaligned and exhibit "1 period forward misalignment". For the case of hyperbolic discounting, these preferences also misaligned, but for *any* t. As a

[9]More compactly, one might refer to these intertemporal preferences with *Ramsey discounting*, given their discounting is just a weighted-sum of future utilities. But as our motivation is mainly empirical (or behavioral), we use to refer to them as "behavioral discounting" in the paper.

[10]And has a very general form of "changing tastes" as, for example, in Hammond (1976).

result, the preferences in (i) are time-consistent, and in both cases (ii) and (iii), time-inconsistent.

Now, let us consider the structure of a stochastic dynamic optimization problem, where the dynamics on the state variable (e.g. assets or capital levels) s_t induced by sequences of current (consumption) choices is governed by a Markov transition $s_{t+1} \sim q(s_t|s_t - c_t)$, where $s_t - c_t$ denotes investment.

Considering a sequence of feasible and measurable consumption policies $(c_t^*)_t$ mapping current state to current consumption level, we can compute its expected value from period t on:

$$U_t((c_\tau^*)_{\tau=t})(s) = u(c_t^*(s)) + \mathbb{E}_s \sum_{\tau=1}^{\infty} \delta_\tau u(c_{t+\tau}^*).$$

where $\mathbb{E}_s$ is the conditional expectation operator with respect to date t information. We say a sequence $(c_t^*)_t$ of measurable consumption policies is a *Markov Perfect Equilibrium (MPE)* in a consumption-savings model with (δ_t)-behavioral discounting if for any $s \in S$ and t we have:

$$c_t^*(s) \in \arg\max_{c \in [0,s]} \left\{ u(c) + \delta_1 \mathbb{E}_s U_{t+1}((c_\tau^*)_{\tau=t+1})(s-c) \right\}.$$

If additionally, this MPE is *time-invariant*,[11] i.e. $c_t^* = c^*$ we refer to this as a *Stationary Markov Perfect Equilibrium (SMPE)*, or in this paper, simply a *Time Consistent Equilibrium*.

For the moment, assume states space $S \subset \mathbb{R}$ is bounded, and the temporal return function $u : S \mapsto \mathbb{R}$ is continuous, increasing and strictly concave. Moreover, assume q is stochastically increasing and stochastically continuous.[12]

The first main result of the paper concerns Time Consistent Equilibrium in the special case of behavioral discounting model where preferences are quasi-hyberbolic with $\delta \in (0, 1)$ and $\beta \in (0, 1]$.

Proposition 1. *There exists a Time Consistent Equilibrium in $\beta - \delta$ quasi-hyperbolic discounting model with deterministic state transition q.*

Notice, for the case of quasi-hyberbolic discounting consumption-savings models, we do *not require* stochastic state transitions. Given our weak sufficient conditions for this result, Proposition 1 generalizes substantially the existing literature.[13]

Our second main set of results concerns the case of general behavioral discounting with each $\delta_t \leq \delta < 1$. Here, we allow preferences for consecutive "generations" of selves to be misaligned

[11] The question of MPE time-consistent solutions, and more generally *sequential time consistent solutions*, is very interesting. We shall discuss MPE time consistent solutions later in the paper. For the quasi-hyperbolic case, for repeated games, see Chade et al. (2008), and for dynamic games, see Balbus and Woźny (2016) for a discussion of how one might extend the analysis to sequential time consistent solutions.

[12] The definitions of stochastically continuous and stochastically increasing we apply are standard. Stochastically continuous means the transition q satisfies the Feller property. For a standard definition of stochastically increasing, see Topkis (1998), section 3.10.

[13] For example, our main existence result for the quasi-hyberbolic case generalizes the results in Harris and Laibson (2001), Krusell and Smith (2003), Krusell et al. (2010), Bernheim et al. (2015), and Cao and Werning (2018).

in much more general ways than in the quasi-hyberbolic discounting model. For this case, we need some uncertainty in the state transition process to obtain existence of Time Consistent Equilibrium.[14] The second main result can be stated as follows:

Proposition 2. *There exists Time Consistent Equilibrium in the stochastic (δ_t)-behavioral discounting model with preferences given by (1) whenever q is nonatomic.*

In fact, our existence and characterization results in this paper are *more general* than both Propositions 1 and 2.

First, in all cases of Time Consistent Equilibrium, we will provide a sharper characterizations of equilibrium policies. Namely, for any Time Consistent Equilibrium with consumption c^*, the associated equilibrium decision rule for investment i^* is *monotone* (and right-continuous) in S. Additionally, per characterization of Time Consistent Equilibrium, in models with present-bias preferences (i.e. $\beta < 1$), we are able to break all indifference between the "current-self" in favour of the earlier selves who prefer a higher level of investment. As Caplin and Leahy (2006) show in their work related to Strotz (1956), "optimal" Time Consistent Equilibrium must resolve such indifferences in this manner for both positive and normative reasons. This is critical aspect of our existence construction, and is new relative to existing work on Time Consistent Solutions for quasi-hyperbolic models.

Second, we can allow for both S and u to be unbounded above. Relative to Proposition 2, we are also able to substantially relax the assumption of (δ_t)-behavioral discounting preferences, in particular by allowing for *non-stationary* preferences (i.e., time-dependent) represented by *non-additive* aggregators.

Finally, in characterizing time consistent equilibria in the (δ_t)-behavioral discounting model, we will introduce the notion of a "semi-hyberbolic" model, i.e. a model where agents, in a precise mathematical sense, have "finite" bias/misalignment. We will show in what sense the time consistent equilibrium in the behavioral discounting model can be generated as limits of time consistent solutions to "semi-hyberbolic" models. In such situations, our approximation results will provide a new conceptual foundation for understanding Time Consistent Equilibrium in the (δ_t)-behavioral discounting model. Importantly, the hyperbolic discounting model will be a special case of a behavioral discounting model where our approximation tools work.

Also, an important technical aspect of our approach is, we introduce a new functional equation method that robustly links recursive utility models with strategic aspects of limited commitment. Our approach substantially extends and integrates separate ideas developed in a series of contributions by Balbus et al. (2015b, 2018), Balbus et al. (2020) and Balbus (2020), among others. In doing so, we provide the first attempt of which we are aware to analyze existence of minimal state Markovian equilibrium in dynamic economies with general recursive payoffs and time-inconsistent preferences. Our results can be hence of independent interest for equilibrium

[14]Without such uncertainty in the state transition, counterexamples to the existence of time consistent equilibrium can be constructed.

existence in dynamic/stochastic games with *recursive payoffs* and *general discounting* (see Obara and Park (2017) for a recent contribution).

In the remainder of the paper, we discuss in more detail Propositions 1 and 2, as well as their generalizations. Namely, in section 2, we provide some intuition into how we approach the existence problem. In particular, we start with the motivating example of quasi-hyberbolic discounting, and use it to suggest a more general functional equation approach to other discounting problems. The key ingredient of this argument is the development of what we refer to as a "generalized Bellman operator". This then allows us to link the structure of our solution approach (taken to the quasi-hyberbolic case) to the more general class of models with (time-separable) behavioral discounting. From there, we are able to generalize the approach further to a recursive (non-separable) representations of the Time Consistent Equilibrium problem. We then use this general recursive representation to give sufficient conditions under which we can prove an existence result for this class of models. Then, in section 3, we continue exploration of the structure of our recursive approach, and analyze the case of semi-hyberbolic discounting. Although this case is interesting in itself, it also serves as a powerful tool for verifying existence of Time Consistent Solutions for the important case of hyperbolic discounting. In particular, we show precisely that one can view the hyperbolic discounting case as the limiting case of a sequence of semi-hyperbolic discounting problems, which we show in section 4. In this section, we also show how to build a powerful approach to approximating other generalized behavioral discounting models. In section 5, we show how our results can be extended to even more general models with behavioral features e.g. magnitude effects, backward looking discounting or short-lived players. We also provide many examples of special cases in the literature that fit into our setting.

2 A preliminary existence result

In this section, we consider the case of time separable quasi-hyberbolic discounting model, the most studied case in the literature. We not only study this case because of its importance, but because it provides the necessary intuition as to how we approach more general cases of behavioral discounting. That is, we use this example to build a new language for how to approach the more general case of (non-additive) discounting. We then prove existence of time consistent equilibria in this more general case, but as a corollary, we consider how the result can be specialized to the quasi-hyberbolic model.

2.1 A motivating example: quasi-hyperbolic discounting

Consider the standard, infinite horizon, stochastic consumption-savings model with quasi-hyberbolic preferences. In this model, at each period t, there is one "generation" who enters the decision problem inheriting a capital/asset stock $s_t \in S$, where $S = \mathbb{R}_+$ or $S = [0, \bar{S}] \subset \mathbb{R}_+$.[15] Generation t selects a consumption level $c_t \in [0, s_t]$, with the remaining resources $i_t = s_t - c_t$ allocated as an investment for next generation $t+1$. In general, the capital stock at $t+1$ is random, and drawn from the distribution $q(\cdot|i_t)$. The temporal utility for each generation is $u(c_t)$, where $u : S \to \mathbb{R}$ is continuous and strictly increasing function.

Then, for any stock-consumption history $(s_t, c_t)_{t=1}^{\infty}$, we denote:

$$J(c^t)(s_t) := \mathbb{E}_{s_t}\left(u(c_t) + \beta\delta \sum_{\tau=t} u(c_{\tau+1})\delta^{\tau-t}\right),$$

as generation t lifetime preferences, where $1 \geq \beta > 0$ and $1 > \delta \geq 0$, and expectations operator $\mathbb{E}_{s_t}$ is taken with respect to the realization of random variables $(s_\tau)_{\tau=t+1}$ with s_τ drawn each period from a transition distribution q. Here, as typically $c^t = (c_\tau)_{\tau=t}^{\infty}$. This objective is well-defined by the Ionescu-Tulcea theorem. Denoting by:

$$U^*(c^{t+1})(s_{t+1}) = \mathbb{E}_{s_{t+1}}\left(\sum_{\tau=t} u(c_{\tau+1})\delta^{\tau-t}\right),$$

we can rewrite this objective more conveniently as:

$$J(c^t)(s_t) = \mathbb{E}_{s_t}\left(u(c_t) + \beta\delta U^*(c^{t+1})(s_{t+1})\right). \tag{2}$$

Let $c_t^* : S \to S$ be a measurable and feasible policy, and interpret it as a Markov policy generating a history $(s_t, c_t^*(s_t))_{t=1}^{\infty}$. Suppose then the generation t deviates from c_t^* by choosing $c \in [0, s_t]$. Then, we can define a payoff:

$$P(c, (c^*)^{t+1})(s_t) := u(c) + \beta\delta \int_S U^*((c^*)^{t+1})(s_{t+1})q(ds_{t+1}|s_t - c).$$

We then have the following definition.

Definition 1. *A sequence (c_t^*) of measurable policies is a Markov Perfect Equilibrium (MPE) if for any $s \in S$ and t:*

$$c_t^*(s) \in \arg\max_{c \in [0,s]} P(c, (c^*)^{t+1})(s).$$

If additionally, the MPE is time invariant, then we refer to it as a Stationary Markov Perfect Equilibrium (SMPE) or Time Consistent Equilibrium.

Let c^* be a Time Consistent Equilibrium. It is clear for the quasi-hyperbolic discounting model, as the decisionmaker has time separable preferences, finding c^* requires *decomposing*

[15] Here, we interpret the dynamic choice model "dynastically", i.e., the infinite-horizon decisions are chosen by a collection of generations under limited commitment. Alternatively, those "generations" could represent "selves" in a model of a single agent with changing tastes as in Phelps and Pollak (1968), Peleg and Yaari (1973), or Hammond (1976). In all cases, optimal policy is modeled as an equilibrium in a dynastic game with a countable number of players.

this optimization problem into *two* functional equations and solving then. The first functional equation involves finding the *recursive* part of preferences, i.e. future value U^* computed for a given candidate policy c^*. The second functional equation then assures strategic *consistency* between the consumption policy c^* and U^* (i.e., the current choice of c^* must be a best response to the considered value U^*). These two equations describe the structure of minimal state space Markovian equilibrium self-generation for a candidate equilibrium policy c^*.

More formally, for any $s \in S$ we have:

$$U^*(c^*)(s) = u(c^*(s)) + \delta \int_S U^*(c^*)(s')q(ds'|s - c^*(s)), \tag{3}$$
$$c^*(s) \in \arg\max_{c\in[0,s]} u(c) + \beta\delta \int_S U^*(c^*)(s')q(ds'|s - c),$$

where for notational simplicity, we shall write $U^*(c^*)$ instead of $U^*((c^*)^t)$ whenever the context is clear. As observed by Balbus et al. (2018), these two functional equations can be summarized by a single *generalized* Bellman equation:

$$U^*(c^*)(s) = \frac{1}{\beta} \max_{c\in[0,s]} \left(u(c) + \beta\delta\mathbb{E}_{s-c}U^*(c^*)\right) - \frac{1-\beta}{\beta} u(c^*(s)), \tag{4}$$

where $\mathbb{E}_{s-c}$ is a short-hand notation for the conditional expectation operator in the current state $s \in S$ with respect to transition $q(\cdot|s-c)$. Here, in (4), one can think of the last element of this expression $\frac{1-\beta}{\beta}u(c^*(s))$ is the *quasi-hyberbolic dynamic inconsistency adjustment factor.* That is, this additional term depending on β appearing on the right-hand side of the maximand in (4) quasi-hyberbolic model is "added" to a standard Bellman to incorporate the fact agents have *changing* preferences over time. So, for the case of $\beta = 1$ (the case of dynamically consistent preferences with exponential discounting), this dynamic inconsistency adjustment factor reduces to 0, and the generalized Bellman operation reduces simply to the standard (time consistent) Bellman equation.[16]

It turns out this formulation of Time Consistent Equilibria in the time-separable quasi-hyberbolic case in the pair of equations in (3) represented by a single generalized Bellman in (4) can be extended in a number a directions for more general forms of behavioral discounting. For example, one can consider both (i) more general ways of evaluating certainty equivalents of future utility streams (see e.g. Kreps and Porteus, 1978)) and (ii) allow for a nonlinear aggregation of current utilities and their associated certainty equivalent (see e.g. Epstein and Zin, 1989).[17] To see how this generalization works, consider now a general *time aggregator*

[16]It is important to note that the so-called "generalized Euler equation" approach to solving time inconsistent problems is just the "first order" decomposition of the same idea we have in our generalized Bellman equation. See, for example, Harris and Laibson (2001), section 3, equation (8) for first-order analog of our generalized Bellman equation.

[17]For more recent recursive preferences literature, the reader is referred to works by Le Van and Vailakis (2005), Rincon-Zapatero and Rodriguez-Palmero (2009), Martins-da Rocha and Vailakis (2010), Matkowski and Nowak (2011), Galperti and Strulovici (2017), Bich et al. (2018) and Balbus (2020).

$W(c^*(s), \mathbb{E}_{s-c^*(s)}U^*(c^*))$ that is used by the decisionmaker to evaluate current and future utilities. Then, the two functional equations linking future utility U^* and time consistent equilibrium c^* in (3) now take a following form:

$$U^*(c^*)(s) = W(c^*(s), \mathbb{E}_{s-c^*(s)}U^*(c^*)), \qquad (5)$$
$$c^*(s) \in \arg\max_{c\in[0,s]} W(c, \beta\mathbb{E}_{s-c}U^*(c^*)).$$

Similar to the quasi-hyberbolic case, in many settings (e.g., of time-inconsistent choice *without* time-separability), these two equations in (5) can be mapped into a *single* equation of a form similar to (4), where this latter single functional equation can be characterized by an *time-inconsistency aggregation mapping* $V : S \times S \times \mathbb{R} \to \mathbb{R}$ given by:

$$U^*(c^*)(s) = V(c^*(s), c^*(s), \mathbb{E}_{s-c^*(s)}U^*(c^*)) = \max_{c\in[0,s]} V(c, c^*(s), \mathbb{E}_{s-c}U^*(c^*)) \qquad (6)$$

where the first element of V is current consumption, the second element of V is a "dynamic inconsistency adjustment factor" that corrects intertemporal preferences for the evolving structure of time-inconsistency, and the third argument is a "recursive" utility term from the next period onward that is evaluated under some candidate consumption function c^*. Our existence theorem in a moment will be based on this new general formulation of the dynamic inconsistency problem in (5), and will prove existence of value U^* and a function c^* solving the single functional equation in (6).

Before we proceed, we first note that the formulation in (5) and (6) has many important examples in the literature as special cases. We discuss few of them now.

Example 1 (Time separable quasi-hyperbolic discounting). *In case of a standard, time separable quasi-hyperbolic discounting model $W(x, z) = u(x) + \delta z$, the aggregation mapping V takes the form:*

$$V(x, y, z) := \frac{1}{\beta}(u(x) + \beta\delta z) - \frac{1-\beta}{\beta}u(y).$$

Example 2 (Risk-sensitive preferences). *Consider now generalization involving the exponential certainty equivalent as defined by Weil (1993) (see also Bäuerle and Jaśkiewicz (2018) for a motivation). In such case the risk-sensitive preferences are given by*

$$u(c) - \frac{\beta\delta}{\gamma}\ln\int_S e^{-\gamma U^*(c^*)(s')}q(ds'|s-c),$$

where $U^*(c^*)(s) = u(c^*) - \frac{\delta}{\gamma}\ln\int_S e^{-\gamma U^*(c^*)(s')}q(ds'|s-c^*(s))$ and $\gamma > 0$. Then the time aggregator takes the form: $W(x, z) := u(x) + \delta z$ and the certainty equivalent for given (integrable) f is $-\frac{1}{\gamma}\ln\int_S e^{-\gamma f(s')}q(ds'|s-c)$. The aggregation mapping V takes the same form as in the example 1.

Example 3 (Kreps-Porteus Utility). *Kreps and Porteus (1978) and Epstein and Zin (1989) introduced the following CES aggregator:*

$$W(x,z) = ((u(x))^{1-\rho} + \delta z^{1-\rho})^{\frac{1}{1-\rho}}$$

for $\rho \in (0,1)$. In case of $\beta - \delta$ version of this model with $W_\beta(x,z) = ((u(x))^{1-\rho} + \beta\delta z^{1-\rho})^{\frac{1}{1-\rho}}$ we have:

$$V(x,y,z) = [\frac{1}{\beta} W_\beta^{1-\rho}(x,z) - \frac{1-\beta}{\beta}(u(y))^{1-\rho}]^{\frac{1}{1-\rho}}.$$

2.2 An existence result

We now state an initial general existence result for this class of dynamic preferences with behavioral discounting. For this result, we need the following assumptions on V and the transition probability $q(\cdot|i)$.

Assumption 1 (Aggregator). *$V : S \times S \times [\vartheta, \infty) \mapsto [\vartheta, \infty)$ is continuous, with $\vartheta \in \mathbb{R}$ and $(x,y,z) \mapsto V(x,y,z)$ is increasing in $(x,-y,z)$. Moreover:*

(i) The function $z \to V(x,y,z)$ is a contraction mapping with a constant $\delta \in (0,1)$;

(ii) The function

$$\zeta(s) = V(s - i_1, \phi(s), \psi(i_1)) - V(s - i_1 + (i_1 - i_2), \phi(s), \psi(i_2))$$

has Strict Single Crossing Property (SSCP) for any $s \geq i_1 > i_2$ and Borel functions ϕ and ψ[18];

(iii) There is a sequence ξ_k ($k \in \mathbb{N}$) of elements of S, $0 < \xi_1 < \xi_2 < \ldots$, and a sequence η_k of $\mathbb{R}_+$ such that $\vartheta < \eta_1 < \eta_2 < \ldots$ such $\eta_k \to \infty$ as $k \to \infty$ and $r := \sup_{k \in \mathbb{N}} \frac{\eta_{k+1}}{\eta_k} \in (1/\delta, \infty)$. Moreover,

$$\sup_{(x,y,z) \in [0,\xi_k]^2 \times [\vartheta, \eta_{k+1}]} |V(x,y,z)| \leq \eta_k \quad \text{for all } k,$$

or equivalently

$$\max\left(V(\xi_k, 0, \eta_{k+1}), V(0, \xi_k, \vartheta)\right) \leq \eta_k.$$

Assumption 2 (Transition). *The transition probability $q(\cdot|i)$ satisfies:*

(i) $i \mapsto q(\cdot|i)$ is stochastically increasing, satisfies a Feller property, and

$$q([0,\xi_{k+1}]|s) = 1 \quad \text{for all } s \in [0, \xi_k];$$

[18] Under our monotonicity assumptions it suffices to verify the SSCP condition for ψ such that $\psi(i_1) > \psi(i_2)$. Indeed, in the opposite case, i.e. $\psi(i_2) \geq \psi(i_1)$ function ζ is negative so SSCP is satisfied trivially.

(ii) For any $s \in S$, the set of all i such that $q(\{s\}|i) > 0$ is countable.

Assumption 1 (i) is standard. Condition (ii) assures that (each) best response equilibrium policy selection is monotone increasing on S. Assumption 1 (iii) and 2 (i) assure we can use the local contractions argument for the case of unbounded states and/or unbounded above rewards. If the states space S is bounded or rewards are (uniformly) bounded then these are automatically satisfied. Finally, we should make an important remark on assumption 2 (ii). Observe, this assumption is satisfied for a purely deterministic transition structure and as well their convex combinations. Moreover, we allow all sets we consider (i.e. $\{i \in S : q(\{s\}|i) > 0\}$) be empty. This is the case, for example, when q is non-atomic. These are the two cases mostly consider in the paper.

Now define the set of candidate time consistent equilibrium investment functions:

$$\mathcal{H} := \{h : S \mapsto S : h(s) \in [0, s] : h \text{ is increasing and right continuous}\}.$$

By the arguments similar to Lemma 1 in Balbus et al. (2020), the set $\mathcal{H}$ is weakly compact when endowed with the weak star topology (i.e. the topology with the following notion of convergence $h_n \to^w h$ iff $h_n(s) \to h(s)$ whenever h is continuous at s).

Under these conditions, we now have a very general result on the existence of Time Consistent Equilibrium c^* such that the corresponding investment $h^* \in \mathcal{H}$, where $h^*(s) := s - c^*(s)$.

Theorem 1. *Assume 1 and 2. There exists a Time Consistent Equilibrium c^* with a corresponding monotone investment $h^* \in H$. That is, if $c^* : S \mapsto S$ is the time consistent equilibrium, then there is $U^* : S \mapsto \mathbb{R}$ such that for any $s \in S$*

$$U^*(s) = \max_{c \in [0,s]} V(c, c^*(s), \mathbb{E}_{s-c}U^*) = V(c^*(s), c^*(s), \mathbb{E}_{s-c^*(s)}U^*).$$

We now proceed with some preliminary definitions, constructions, and lemmata necessary to prove this theorem. Begin by defining the following set: $\mathcal{E} := \{(s, h) \in S \times \mathcal{H} : h \text{ is continuous at } s\}$. As is usual, S is endowed with the Euclidean topology and $S \times \mathcal{H}$ is endowed with its product topology. It is well-known that the evaluation function $\mathbf{e}(s, h) = h(s)$ has a continuous restriction to $\mathcal{E}$. Since $h \in \mathcal{H}$ is increasing, the section $\mathcal{E}^h := \{s \in S : (s, h) \in \mathcal{E}\}$ has a countable complement.

Next, define the space $\mathbf{V}$ to be the set of real valued functions on $S \times \mathcal{H}$ such that each $f \in \mathbf{V}$:

- is bounded on any $S_k \times \mathcal{H}$, where $S_k := [0, \xi_k]$;
- is continuous from the right on S, and upper semicontinuous on $S \times \mathcal{H}$;
- obeys the following condition: for any $h \in \mathcal{E}$, there is a countable set $S^{f,h} \subset \mathcal{E}^h$ such that if $s \notin S^{f,h}$ then f is continuous at (s, h).

Endow the space $\mathbf{V}$ with the topology induced by the seminorms:

$$||f||_k = \sup_{s \in S_k \times \mathcal{H}} |f(s,h)|,$$

where $S_k := [0, \xi_k]$. Further define the following:

$$\mathcal{V} := \left\{ f \in \mathbf{V} : \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k} < \infty \right\},$$

and a norm on $\mathcal{V}$:

$$||f|| = \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k}.$$

Finally, define the set $\mathcal{U}$ to be:

$$\mathcal{U} := \{ f \in \mathcal{V} : ||f||_k \leq \eta_k, k \in \mathbb{N} \}.$$

We are now ready to present the key steps in the proof of main theorem of this section.

Proof of Theorem 1. Lemma 1 in the Appendix shows that $\mathcal{U}$ is a closed subset of a Banach space $(\mathcal{V}, || \cdot ||)$. We then define an operator T on $\mathcal{U}$ as follows:

$$T(f)(s,h) := \max_{i \in [0,s]} V(s - i, s - h(s), \mathbb{E}_i f(h)),$$

where $f(h) := f(\cdot, h)$. Lemma 3 shows that T is a self map on $\mathcal{U}$, while lemma 4 claims that T is a contraction mapping and thus has a unique fixed point. Denote by $f^* \in \mathcal{U}$ this unique fixed point of T in $\mathcal{U}$, and define the following mapping that characterizes the best reply correspondence for each generation:

$$BI(h)(s) = \arg\max_{i \in [0,s]} V(s - i, s - h(s), \mathbb{E}_i f^*(h)),$$

and

$$bi(h)(s) := \max BI(h)(s).$$

Lemma 5 shows that any selection of $s \mapsto BI(h)(s)$ is increasing in s. Finally, our key Lemma 7 shows that bi is continuous on compact $\mathcal{H}$. By Schauder-Tychonoff Theorem we immediately obtain the existence of a fixed point h^* of bi. Then $c^*(s) := s - h^*(s)$ is a Time Consistent Equilibrium. □

We conclude this section with an very important corollary of Theorem 1. The corollary offers a very general new existence result in a standard *deterministic* quasi-hyperbolic discounting model for the case when (i) the state space S is bounded or unbounded, and (ii) the utility function u is allowed to be unbounded above. To the best of our knowledge, this corollary with sufficient conditions for the existence of Time Consistent Equilibrium for the standard quasi-hyberbolic discounting model is the most general in the current literature (i.e., compare the result in the corollary to theorem 5 in Cao and Werning (2018) or theorem 6 in Bernheim et al. (2015)).

Corollary 1 (Deterministic quasi-hyperbolic discounting)**.** *There exists a Time Consistent Equilibrium with investment monotone in the deterministic $\beta - \delta$ model whenever u is continuous, increasing and strictly concave.*

We finish this section with an very important comment on the nature of our existence result and its interpretation.

Remark 1 (Selection from the argmax correspondence and Optimal Time Consistent Equilibrium)**.** *Our construction of Time Consistent Equilibrium in theorem 1 uses the greatest investment selection from the argmax correspondence. This selection procedure guarantees in our models with present biased preferences (i.e. $\beta < 1$), all indifference of the current self are arbitrarily resolved in favor of the earlier selves who prefer* ***higher*** *investment. In an important paper, Caplin and Leahy (2006) (following the work of Strotz (1956) argue that* optimal *time consistent solutions should resolve* ***all*** *indifference in such a manner (for not only positive reasons, but for normative interpretations of time consistent solutions). Technically, this is also critical for our existence result. To the best of our knowledge, such investment selection construction is* ***new*** *relative to the existing work on Time Consistent Solutions for quasi-hyperbolic models.*

As stressed in the remark, our construction of equilibrium in the deterministic case is novel and based on the *greatest* investment selection. Technically, whenever investment is upper semicontinuous, its associated consumption is lower semicontinuous, which assures the upper semicontinuity of the value U^*. And upper semicontinuity of U^* is critical for proving non-emptiness of the argmax correspondence. Indeed, it is not clear how the general existence for a deterministic quasi-hyperbolic discounting model can be extended using the least investment selection.[19]

3 Semi-hyperbolic discounting

We now proceed with new versions of dynamic model with time inconsistent preferences which we refer to as "semi-hyberbolic" discounting models. The semi-hyberbolic model has the flavor of quasi-hyberbolic, but allows for a more general pattern of present-bias (see Montiel Olea and Strzalecki (2014) section IV for an introduction and motivation). These models also will be useful in characterizing Time Consistent Equilibrium in more general models of behavioral discounting (e.g., the hyperbolic discounting model). To build intuition as to how to characterize

[19] Recall, the recent equilibrium existence results in certain classes of stochastic games use the *least* investment selection (see Balbus et al. (2015a, 2020) e.g.).

Time Consistent Equilibrium in this class of models, we first study the special case of $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting, a direct extension of the quasi-hyberbolic model.

3.1 $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting

Consider a special case of preferences in (1) where the sequence of discount factors at any date t is specified as follows:

$$1, \beta_1\beta_2\delta,\ \beta_1\beta_2^2\delta^2,\ \beta_1\beta_2^2\delta^3,\ \beta_2\beta_2^2\delta^4, \ldots$$

We shall refer to this model as the $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting. Notice, in this model, from period $t+3$ on, the discount factor becomes exponential. However, unlike in $\beta - \delta$ model, in the case of $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting, preferences are misaligned for more than just one date forward. Indeed, we have the following:

$$(\beta_1\beta_2\delta)^2 \neq \beta_1\beta_2^2\delta^2,$$

whenever $\beta_1 \neq 1$; as well as

$$(\beta_1\beta_2^2\delta^2)^2 \neq (\beta_1\beta_2\delta)(\beta_1\beta_2^2\delta^3),$$

whenever $\beta_2 \neq 1$. So although these preferences are in the spirit of $\beta - \delta$ preferences, they allow for a more general pattern of forward preference misalignment.

As before, for this model, we aim to show the existence of a Time Consistent Equilibrium c^*. For this, we seek an appropriate generalized of the "decomposition" approach to quasi-hyberbolic discounting we developed in Section 2. For $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting model, our decomposition involves three functional equations, namely:

$$\begin{aligned}
U^*(c^*)(s) &= u(c^*(s)) + \delta \int_S U(c^*)(s')q(ds'|s - c^*(s)), \\
W_1^*(c^*)(s) &= u(c^*(s)) + \beta_2\delta \int_S U(c^*)(s')q(ds'|s - c^*(s)), \\
W_2^*(c^*)(s) &= u(c^*(s)) + \beta_1\beta_2\delta \int_S W_1^*(c^*)(s')q(ds'|s - c^*(s)) \\
&= \max_{c\in[0,s]} u(c) + \beta_1\beta_2\delta \int_S W_1^*(c^*)(s')q(ds'|s - c).
\end{aligned} \tag{7}$$

We now discuss how our generalized Bellman equation approach proposed in the previous section can be extended this semi-hyperbolic discounting problem. To obtain a single functional equation linking these three functional equations, one needs again to construct corrective factors, but only now *twice*. Indeed, simplifying $U^*(c^*)(\cdot)$ with $U^*(\cdot)$ we obtain the functional equation:

$$U^*(s) = \frac{1}{\beta_1\beta_2^2} \max_{c\in[0,s]} \left\{ u(c) + \beta_1\beta_2\delta \int_S [u(c^*(s')) + \beta_2\delta \int_S U^*(s'')q(ds''|s' - c^*(s'))]q(ds'|s-c) \right\}$$
$$- [\frac{1}{\beta_1\beta_2^2} - 1]u(c^*(s)) - [\frac{1}{\beta_2} - 1]\delta \int_S u(c^*(s'))q(s'|s - c^*(s)).$$

Notice, for $\beta_2 = 1$, the second corrective factor disappears, and the problem reduces to a standard $\beta - \delta$ discounting model. Similarly, for $\beta_1 = 1$ the problem also reduces to a version of $\beta - \delta$ discounting, but not the standard quasi-hyberbolic model; rather, it would be a $\beta - \delta$ model where the additional impatience shows up between third and the second period (not the second and the first).

Next, as is clear from the above formulation, for the *deterministic* semi-hyberbolic problem, the argmax in the decisionmaker need not be necessarily well-defined in the space of investments $\mathcal{H}$.[20] We resolve this issue by considering a stochastic transitions on the state $s \in S$. Under this extra assumption, we can easily extend our existence result for the quasi-hyberbolic case to the separable semi-hyberbolic discounting. In fact, these assumptions suffice to prove existence in a more general model of discounting that we discuss in the next subsection. For this reason, we now introduce a more general version of the semi-hyberbolic model, and then state the general existence result relative to this model.

3.2 General semi-hyperbolic models

Consider a very general version of semi-hyperbolic discounting preferences that includes the $\beta_1 - \beta_2 - \delta$ semi-hyperbolic discounting model as a special case. In studying Time Consistent Equilibrium in this more general semi-hyberbolic case, we will use the existence results for this class of semi-hyberbolic models to elucidate the structure of Time Consistent Equilibrium in the (δ_t)-behavioral discounting preferences given in (1) via a limiting approximation argument.

Along these lines, first assume in that the semi-hyberbolic model is characterized by a sequence of discount factors that take the following sequential form:

$$1, \beta_1\beta_2 \ldots \beta_T, \beta_1(\beta_2 \ldots \beta_T)^2, \beta_1\beta_2^2(\beta_3 \ldots \beta_T)^3, \ldots, \beta_1\beta_2^2 \ldots \beta_{k-1}^{k-1}(\prod_{s=k}^{T} \beta_s)^k, \ldots, \prod_{\tau=1}^{T} \beta_\tau^\tau,$$

while for any $t > T$ it is:

$$\prod_{\tau=1}^{T} \beta_\tau^\tau \beta_T^{t-T},$$

Assume $\beta_T < 1$. The intuition for this formulation of the semi-hyberbolic model is that each decision maker/generation at date t is impatient up to T periods ahead and then from period T

[20] Indeed in the deterministic transition case function: $i \mapsto u(s-i) + \beta_1\beta_2\delta[u(c^*(i)) + \beta_2\delta U^*(i - c^*(i))$ may fail to be upper-semicontinuous if c^* or $s - c^*(s)$ is not usc. See also e.g. example 2 in Balbus et al. (2015a).

on the problem becomes stationary with exponential discounting at rate β_T. If additionally all $\beta_t \leq 1$ the decision maker has a *growing patience* (alike a generalized notion of present-bias).

Remark 2. *Per notation, in the previous examples, we used $\delta = \beta_T$. Now, we substitute for β_T to keep the notation concise. So, for example, we have the following special cases: for $T = 1$, we have a standard exponential discounting with β_1^t; for $T = 2$, it is a quasi-hyperbolic $\beta_1 - \beta_2$ discounting model; for $T = 3$, we have an "order two" quasi-hyperbolic $\beta_1 - \beta_2 - \beta_3$ model, etc.*

We can now again develop a functional equation representation of the consumption-savings problem for this class of semi-hyberbolic preferences. In particular, the functional equations will have the following recursive structure:

$$
\begin{aligned}
&U^*(s) = u(c^*(s)) + \beta_T \int_S U^*(s')q(ds'|s - c^*(s)),\\
&\text{and } c^*(s) \in \arg\max_{c\in[0,s]} \{u(c) + \prod_{t=1}^{T} \beta_t \int_S A_{T-1}(U^*)(s')q(ds'|s - c)\},\\
&\text{with } A_t(U^*)(s) = u(c^*(s)) + \prod_{\tau=T+1-t}^{T} \beta_\tau \int_S A_{t-1}(U^*)(s')q(ds'|s - c^*(s)),\\
&\text{where } A_0(U) := U^*.
\end{aligned}
$$

The next theorem considers our general existence for Time Consistent Equilibrium for this class of semi-hyperbolic discounting models. For this result, we will need to impose two new assumptions.

Assumption 3. *Let $u : S \to \mathbb{R}$ be continuous, increasing, strictly concave and* $\max(|u(0)|, |u(\xi_k)|) \leq (1 - \delta_T)\eta_k$.

Assumption 4. *The transition q satisfies Assumption 2. Moreover, q is nonatomic.*

With these assumptions in place, we now have the following result:

Theorem 2. *Assume 3 and 4. For any $T \geq 1$, there exists a Time Consistent Equilibrium c^* with corresponding monotone investment $h^* \in \mathcal{H}$.*

This is a central result for the case of semi-hyperbolic discounting model. Some aspects of its proof follow the lines developed for the quasi-hyperbolic discounting model. The key difference though is in the argument that concerns the continuity of best responses. That is, in the case of quasi-hyberbolic discounting, we used the space of upper semicontinuous value functions and allowed for *deterministic* transition functions. In the case of semi-hyperbolic discounting, this argument cannot proceed without the imposition of nonatomic noise relative to the state transition (see lemma 7 versus lemma 12).

To present the proof of theorem 2, we need to define certain new objects. Let $\mathbf{V}_0$ be the space of real valued functions on $S \times \mathcal{H}$ in which $f \in \mathbf{V}_0$ if and only if

- for any $k \in \mathbb{N}$, f is bounded on any $(s,h) \in S_k \times \mathcal{H}$,
- for any $h \in \mathcal{H}$ there exists a countable $S^{f,h} \subset S$ such that $f(\cdot,\cdot)$ is continuous at any (s,h) such that $s \notin S^{f,h}$.

Endow $\mathbf{V}_0$ with analogous seminorms $||\cdot||_k := \sup_{(s,h)\in S_k\times\mathcal{H}} f(s,h)$. Let $\mathcal{V}_0 \subset \mathbf{V}_0$ be the set of all functions satisfying $||f|| := \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k} < \infty$. Clearly $(\mathcal{V}_0, ||\cdot||)$ is a normed space. Define $\mathcal{U}_0 := \{f \in \mathcal{V}_0 : |f(s,h)| \leq \eta_k, \text{ for all } (s,h) \in S_k \times \mathcal{H}, k \in \mathbb{N}\}$. Let $\mathbf{V}_0^\infty$ be the countable product of $\mathbf{V}_0$ endowed with the seminorms $||f||_k^\infty := \sup_{(t,s,h)\in\mathbb{N}\times S_k\times\mathcal{H}} f_t(s,h)$. Similarly, define $\mathcal{V}_0^\infty$ and the norm on $\mathcal{V}_0^\infty$ to be $||f||^\infty := \sum_{k=1}^{\infty} \frac{||f||_k^\infty}{r^k \eta_k}$ and similarly for the set $\mathcal{U}^\infty$. In Lemma 8 we show $(\mathcal{V}_0, ||\cdot||)$ is a Banach space and $\mathcal{U}_0$ is a closed subset of $(\mathcal{V}_0, ||\cdot||)$ (hence a complete metric space). We also have the same conclusion for $(\mathcal{V}_0^\infty, ||\cdot||^\infty)$ and $\mathcal{U}_0^\infty$.

Proof. For $f \in \mathcal{U}_0$, and $t = 1, 2, \ldots, T$ we define the following operator

$$\Lambda(f)(s,h) := u(s - h(s)) + \beta_T \mathbb{E}_{h(s)}(f(h)),$$

where $\mathbb{E}_i(f(h))$ is the operator defined as in the previous section. By lemmas 8 and 9 there exists f^*, the unique fixed point of Λ. We now adapt the definition of the best response mapping as follows. Let

$$\mathcal{BI}(h)(s) := \arg\max_{i\in[0,s]} \left\{ u(s-i) + k_T \int_S A_{T-1}(s',h) q(ds'|i) \right\}.$$

where $k_t = \prod_{\tau=T+1-t}^{T} \beta_\tau$ and for any $t > 0$ we have

$$A_t(s,h) = u(s - h(s)) + k_t \int_S A_{t-1}(s',h) q(ds'|h(s)),$$

with $A_0(s,h) := f^*(s,h)$. Put

$$bi(h)(s) := \max \mathcal{BI}(h)(s).$$

Lemma 10 assures that any selection of $s \mapsto \mathcal{BI}(h)(s)$ is increasing. Next our key lemma 12 shows that the operator bi maps $\mathcal{H}$ into itself and is continuous. Hence, we find a fixed point h^* of bi. Similarly, we may choose an equilibrium as $c^*(s) = s - h^*(s)$. □

Remark 3. *Our technique allows us to generalize these existence results and also allow for more general non-additive aggregators satisfying Assumption 1. See section 5 for a more general model.*

4 Approximations, general behavioral discounting, and hyberbolic discounting

We now extend our results on Time Consistent Equilibrium to the more general class of behavioral discounting models, namely the (δ_t)-behavioral discounting. In doing so, we develop an approximation approach that allows us to relate the set of Time Consistent Equilibria in the (δ_t)-behavioral discounting model to the set of Time Consistent Equilibrium in limiting collections of semi-hyberbolic discounting models. This allows us to achieve two goals. First, we are able to extend our results in the previous sections to models with very general forms of behavioral discounting. Second, using the approximation approach, we are able to understand better the structure of (δ_t)-behavioral discounting models.

In particular, at the end of this section, we show how one can view the standard hyberbolic discounting model as a limit of a collection of semi-hyberbolic discounting models. Specifically, our approximation method allows us to construct Time Consistent Equilibrium in the very general (δ_t)-behavioral discounting with preferences as in (1) by finding an appropriate approximating sequence of semi-hyberbolic discounting models with an appropriate sequence of discount factors $(\beta_t)_{t=1}^{\infty}$. The corresponding Time Consistent Equilibrium in the limiting semi-hyberbolic case can be used to build representations of Time Consistent Equilibrium for the original problem parameterized by the discount factors $(\delta_t)_{t=1}^{\infty}$.

4.1 Limiting semi-hyperbolic discounting

We begin this section by discussing the case of limiting semi-hyperbolic discounting. A limiting semi-hyberbolic discounting model studies the T-period bias in the semi-hyberbolic discounting model as T gets arbitrarily large. For given T, denote the effective discount factors by:

$$\begin{aligned}
{}^T\delta_1 :=& \beta_1\beta_2\dots\beta_T, \\
{}^T\delta_2 :=& \beta_1(\beta_2\dots\beta_T)^2 = {}^T\delta_1 \prod_{\tau=2}^{T}\beta_\tau, \\
{}^T\delta_k :=& \beta_1\beta_2^2\dots\beta_{k-1}^{k-1}(\prod_{s=k}^{T}\beta_s)^k = {}^T\delta_{k-1}\prod_{\tau=k}^{T}\beta_\tau.
\end{aligned}$$

Hence for $k \leq T$, we have the following recursive formulation:

$$ {}^T\delta_k = {}^T\delta_{k-1}\prod_{\tau=k}^{T}\beta_\tau. \tag{8}$$

We now seek existence of Time Consistent Equilibrium in these models as $T \to \infty$, and use the result to build an approximation theory of Time Consistent Equilibrium in the (δ_t)-behavioral discounting model. Suppose that ${}^T\delta_1$ has a limit; then, any of ${}^T\delta_k$ has a limit with

$T \to \infty$. We will denote this limit by δ_k. Therefore, the recursive formula for the evolution of the successive discount factor δ_k takes the following form for any k:

$$\delta_k = \delta_{k-1} \prod_{\tau=k}^{\infty} \beta_\tau. \tag{9}$$

We then have a new result per existence of Time Consistent Equilibrium in the limiting semi-hyberbolic model relative to the (δ_t)-behavioral discounting model:

Theorem 3. *Assume 3 and 4. Consider a model with generation t preferences given by:*

$$U_t^T = u(c_t) + \mathbb{E}_t \left(\sum_{\tau=1}^{\infty} {}^T\delta_{t+\tau} u(c_{t+\tau}) \right)$$

with ${}^T\delta_t$ satisfying the above recursive formulation in (8). Then, for any T, there is a Time Consistent Equilibrium c^T, whose limit, say c^* is also a Time Consistent Equilibrium in the model with utility

$$U_t^* = u(c_t) + \mathbb{E}_t \left(\sum_{\tau=1}^{\infty} \delta_{t+\tau} u(c_{t+\tau}) \right)$$

where the sequence δ_t satisfies the recursive formulation in (9).

Proof. The results follows from Theorem 2. $\square$

This is another central result of our paper. It allows us to approximate *general behavioral* discounting models with preferences such as (1). The key technical contribution in Theorem 3 is based on the upper semicontinuity of the *set* of Time Consistent Equilibrium with respect to the parameter T at $T = \infty$.

4.2 Approximating general behavioral discounting models

With this result in place, we are now able to explore the relationship between limiting semi-hyberbolic models and (δ_t)-behavioral discounting models even further than in Theorem 3. That is, suppose we have a (δ_t)-behavioral discounting model where the discount factors $(\delta_t)_{t=1}^{\infty}$ are given with each $\delta_t \in (0, 1)$. We now ask if we can construct a sequence of $(\beta_t)_{t=1}^{\infty}$ collection and its corresponding sequence of behavioral semi- discounting games whose Time Consistent Equilibria can approximate Time Consistent Equilibria of the (δ_t)-behavioral discounting model. The following result answers this question.

Proposition 3. *Define*

$$\beta_t := \begin{cases} \frac{\delta_1^2}{\delta_2} & if \quad t = 1 \\ \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}} & if \quad t \geq 2. \end{cases} \tag{10}$$

then a time consistent equilibrium of the semi-hyperbolic discounting model $\beta_1 - \beta_2 - \ldots$, is a time consistent equilibrium of the behavioral discounting model with $(\delta_t)_{t=1}^{\infty}$ provided $R := \lim_{t\to\infty} \frac{\delta_{t+1}}{\delta_t} = 1$.

Proof. To see that observe:

$$\frac{\delta_{t+1}}{\delta_t} = \prod_{\tau=t+1}^{\infty} \beta_\tau$$

and hence

$$\beta_t := \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}}$$

for $t > 1$. Further we have $\lim_{t\to\infty} \frac{\delta_{t+1}}{\delta_t} = \lim_{t\to\infty} \prod_{\tau=t+1}^{\infty} \beta_\tau$, that by assumptions is equal to 1. To recover β_1 proceed as follows:

$$\begin{aligned}
\delta_1 &= \beta_1 \prod_{t=2}^{\infty} \beta_t = \beta_1 \prod_{t=2}^{\infty} \frac{\delta_t^2}{\delta_{t+1}\delta_{t-1}} = \beta_1 \lim_{T\to\infty} \prod_{t=2}^{T} \frac{\delta_{t+1}^2}{\delta_{t+2}\delta_t} \\
&= \beta_1 \lim_{T\to\infty} \frac{\left(\prod_{t=2}^{T+1} \delta_t\right)^2}{\prod_{t=1}^{T} \delta_t \prod_{t=3}^{T+2} \delta_t} = \beta_1 \frac{\delta_2}{\delta_1} \lim_{T\to\infty} \frac{\delta_{T+1}}{\delta_{T+2}} = \beta_1 \frac{\delta_2}{\delta_1}.
\end{aligned}$$

Hence $\beta_1 = \frac{\delta_1^2}{\delta_2}$. □

4.3 The hyperbolic discounting case

We now use the result in the previous section to discuss how the time consistent equilibrium in the standard hyperbolic discounting model can be approximated using time consistent equilibrium in a limiting version of a semi-hyperbolic discounting model. To see how this can be done, let for any date t, the discount factor for the (δ_t)-discounting model take a specific hyperbolic form:

$$\delta_t = \frac{1}{1+t}.$$

In this case, this implies that the discount factor between any two time periods $t+1$ and t is:

$$\frac{\frac{1}{t+2}}{\frac{1}{1+t}} = \frac{t+1}{t+2}.$$

Applying our approximating formula in (10) in Proposition 3, we get:

$$\beta_{t+1} = \frac{(t+1)(t+3)}{(t+2)^2}$$

with

$$\beta_1 = \frac{3}{4}.$$

Hence, for this simple case, a Time Consistent Equilibrium of this version of the standard hyperbolic discounting model can be expressed as a limit of Time Consistent Equilibrium of the semi-hyperbolic models.

But importantly, this same argument applies to a more general form of hyperbolic discounting (e.g., see the model studied in Loewenstein and Prelec (1992)). Specifically, let for each t

$$\delta_t = (1 + \alpha t)^{-\frac{\beta}{\alpha}}.$$

Indeed, in such case, we then have:

$$\beta_t := \left(\frac{(1 + \alpha t + \alpha)(1 + \alpha t - \alpha)}{1 + \alpha t} \right)^{\frac{\beta}{\alpha}},$$

$R = 1$ and $\beta_1 := (\frac{1+2\alpha}{1+\alpha})^{\frac{\beta}{\alpha}}$.

So our approximation machinery developed in Theorem 3 and Proposition 3 is very general and flexible. Further, we are not aware of any result in the existing literature regarding Time Consistent Equilibrium existence in hyperbolic discounting game played between consecutive generations.

5 A more general existence result with additional applications

We have shown so far that many general classes of (δ_t)-behavioral discounting models can be approximated using collections of semi-hyperbolic discounting models. The restrictive assumption in that discussion is that $R = 1$. Indeed, there is a class of behavioral discounting models that cannot be approximated in this manner. In this section, we consider these time inconsistency problems, and extend our methods (and results) to even more abstract formulations of recursive (time-inconsistent) preferences. We then provide four additional examples of where this more general existence result can be applied (where our approximating technique *cannot* necessarily be applied).

5.1 The general existence result

We first state our most general existence result. Following the reasoning developed for a general quasi-hyperbolic discounting model in section 2, assume the existence of an abstract recursive aggregator $V_t : S \times S \times \mathbb{R}$ as in the functional equation (6):

$$V_t(\tilde{c}, s - \tilde{c}, \mathbb{E}_{s-\tilde{c}} U_{t+1}(c)).$$

Here $\tilde{c}$ is the current consumption, $\mathbb{E}_{s-\tilde{c}} U_{t+1}(c)$ is the certainty equivalent of the evaluation of the next generations following policy c. Corrective terms (if necessary) can be used to account for other behavioral considerations, like magnitude effects, for example (more on this in moment).

Observe, in this case, we are studying versions of the functional equation in (6) that allow the recursive aggregator to be *nonstationary*. So, the stationary case occurs, of course, when V_t is independent of t, which will be a special case of our general existence result in a moment.

For some given c, we first look for recursive utility $(U_t^*)_t$:

$$U_t^*(c)(s) = V_t(c(s), s - c^*(s), \mathbb{E}_{s-c(s)} U_{t+1}^*(c)).$$

We then seek the solutions to:

$$c^*(s) \in \arg\max_{\tilde{c}\in[0,s]} V_1(\tilde{c}, s - c^*(s), \mathbb{E}_{s-\tilde{c}} U_2^*(c^*)).$$

We now have our most general existence theorem in the paper:

Theorem 4. *Suppose Assumption 4 holds, and for any t, the continuous aggregator $(x, y, z) \mapsto V_t$ is increasing in (x, z) for each y, and obeys Assumption 1 (i)-(iii) with a common constant $\delta \in (0, 1)$. Then, there exists a Time Consistent Equilibrium c^* with corresponding monotone investment $h^* \in \mathcal{H}$.*

Proof. Let us consider $\mathcal{V}_0^\infty$ and endow it with the natural product topology. The natural family of seminorm $||\cdot||_k$ on $\mathcal{V}_0^\infty$ is defined as follows

$$||f||_k := \sup_{(t,s,h)\in\mathbb{N}\times S_k\times\mathcal{H}} |f_t(s, h)|$$

and the norm

$$||f|| = \sum_{k=1}^{\infty} \frac{||f||_k}{r^k \eta_k}.$$

Let $\mathbb{T}(f) = (T_t(f))_{t\in\mathbb{N}}$ where $f = (f_t)_{n\in\mathbb{N}}$. For $t > 1$ let

$$T_t(f)(s, h) = V_t(s - h(s), s - h(s), \mathbb{E}_{h(s)} f_{t+1}(h)).$$

Lemma 15 shows that $\mathbb{T}$ is a contraction mapping on $\mathcal{U}^\infty$ and has a unique fixed point: f^*. Define

$$BI(h)(s) = \arg\max_{i\in[0,s]} V_1(s - i, s - h(s), \mathbb{E}_i f_2^*(h)),$$

and $bi(h)(s) := \max BI(h)(s)$. Similarly as before (i.e. as in Theorem 1 and 2), lemma 18 shows that the operator bi maps $\mathcal{H}$ into itself and it is a continuous operator. This suffices to prove existence of a fixed point on convex and compact space $\mathcal{H}$. $\square$

5.2 Applications to other behavioral discounting models

We now can provide few additional applications of the main results of the paper. Let us begin with the case of generalized quasi-geometric discounting.

Example 4 (Generalized quasi-geometric discounting)**.** *Young (2007) considers a dynamic optimization model with the following sequence of discount factors:*

$$1, \tilde{\beta}_1\delta, \tilde{\beta}_1\tilde{\beta}_2\delta^2, \tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3\delta^3, \ldots$$

Therefore, between any two consecutive dates (say $t+1$ and t), the discount rate is $\tilde{\beta}_t\delta$. Suppose we have that the limit $\lim_{t\to\infty}\tilde{\beta}_t \in (0,1]$ exists and each $\tilde{\beta}_t\delta < 1$. Then, if we seek time consistent equilibria in the resulting model, we have:

$$c^*(s) \in \arg\max_{\tilde{c}\in[0,s]} u(\tilde{c}) + \tilde{\beta}_1\delta\mathbb{E}_{s-\tilde{c}}U_2(c^*).$$

where for $t \geq 2$, we also have:

$$U_t(c^*)(s) = u(c^*(s)) + \tilde{\beta}_t\delta\mathbb{E}_{s-c^*(s)}U_{t+1}(c^*).$$

Here, we can take

$$V_t(\tilde{c}, s-\tilde{c}, \mathbb{E}_{s-\tilde{c}}U(c)) = u(\tilde{c}) + \tilde{\beta}_t\delta\mathbb{E}_{s-\tilde{c}}U(c).$$

It is straightforward to see that this aggregator satisfies our assumptions in the paper, and therefore, Time Consistent Equilibrium exists whenever transition q is nonatomic, and u increasing and strictly concave. In this case $R \neq 1$ (generally) and hence our approximation technique cannot be applied.

Example 5 (Backward discounting)**.** *Following Ray et al. (2017) we consider an individual whose current utility is derived from evaluating both present and past consumption streams. Each of these streams is discounted, the former forward in the usual way, the latter backward. Specifically, assume an individual at date t evaluates consumption according to a weighted average of his own felicity (as perceived at date t) and that of a "future self" as perceived from date $T > t$. More specifically, for a generation born in $\tau = 0$ and taking the backward looking date to be $T(\tau) := T + \tau$ for some $T > 0$, her preferences are:*

$$\mathbb{E}_0\sum_{t=0}^{T}\delta^t u(c_t)[\alpha + (1-\alpha)\delta^{T-2t}] + \delta^T\mathbb{E}_{T+1}\sum_{t=T+1}^{\infty}\delta^{t-T}u(c_t)[\alpha + (1-\alpha)\delta^{-T}].$$

where α (resp. $(1-\alpha)$) is the forward (resp. backward) looking weight. Observe that from $t \geq T$ the preferences become stationary with exponential discounting δ. So put

$$W(s_{T+1}) = \mathbb{E}_{T+1}\sum_{t=T+1}^{\infty}\delta^{t-T}u(c(s_t))[\alpha + (1-\alpha)\delta^{-T}]$$

to denote the value for this stationary part (for some candidate stationary policy c). That is, *for $t \geq T$, we can take the aggregators:*

$$V_t(\tilde{c}, s - \tilde{c}, \mathbb{E}_{s-\tilde{c}}U(c)) := u(\tilde{c})[\alpha + (1-\alpha)\delta^{-T}] + \delta \mathbb{E}_{s-\tilde{c}}U(c).$$

Observe this implies that the problem resembles a finite-bias discounting model discussed in section 3. Then for $t < T$, we need to, however, construct our preferences recursively (backwards) using aggregators V_t:

$$V_t(\tilde{c}, s - \tilde{c}, \mathbb{E}_{s-\tilde{c}}U(c)) := u(\tilde{c})[\alpha + (1-\alpha)\delta^{T-2t}] + \delta \mathbb{E}_{s-\tilde{c}}U(c)$$

with $U_T(c)(s_T) = u(c(s_T))[\alpha + (1-\alpha)\delta^{-T}] + \delta^T W(s_{T+1} - c(s_{T+1}))$.

Then, in this case, we seek Time Consistent Equilibria that are solutions of the following functional equations:

$$c^*(s) \in \arg\max_{\tilde{c} \in [0,s]} u(\tilde{c})[\alpha + (1-\alpha)\delta^T] + \delta \mathbb{E}_{s-\tilde{c}}U_1(c^*).$$

Again, with $\delta < 1$ the above aggregators (V_t) satisfy our assumptions and Time Consistent Equilibrium exists whenever transition q is nonatomic, u increasing and strictly concave.

So far, in the paper, we have focused on models where this decisionmaker is infinitely-lived. It happens, our approach is also useful when attempting to understand cases where agents are short-lived. Many important problems in economics have the latter form of short-lived agents making decisions within the context of some long-term generation planning problem with examples including dynamic sustainable resource models with public policy, economic models of the transmission of human capital and endogenous preferences across generations, models of endogenous fertility, as well as related models of sustainable dynastic choice with intergenerational altruism and paternalism. One particularly relevant case is that of bequest games. We now show how our results can be applied in these models.

Example 6 (Limited time horizon discounting and bequest games)**.** *Consider a sequence of discount factors $1, \delta_1, \delta_2, \ldots, \delta_T, 0, 0, \ldots$ for some $T \geq 1$. This, therefore, is a class of T-period paternalistic bequest games with changing discount factors. To apply our results to this model, simply take:*

$$V_t(\tilde{c}, s - \tilde{c}, \mathbb{E}_{s-\tilde{c}}U(c)) = u(\tilde{c}) + \delta_t \mathbb{E}_{s-\tilde{c}}U(c).$$

Then again, we are able to verify Time Consistent Equilibria exist with monotone increasing investments whenever transition q is nonatomic. Again, observing that $\delta_{T+1} = 0$ we immediately have that the problem resembles a finite-bias discounting model discussed in section 3.

Finally, we can also allow for a discount factor to be state or choice dependent, e.g. $\beta(s)$ or $\beta(s - c)$ to account e.g. for magnitude effects in discounting (see Epstein and Hynes (1983) or Noor (2009) for a motivation).

Example 7 (Magnitude effects)**.** *Suppose the present bias discount factor β is a function of investment, i.e. $\beta : S \to [0,1]$ that is continuous and increasing. Then the aggregator takes the form:*

$$V_1(c, s-c^*(s), \mathbb{E}_{s-c}U^*(c^*)) = \max_{c\in[0,s]} (u(c) + \beta(s-c)\delta\mathbb{E}_{s-c}U_2^*(c^*))$$

where for $t > 1$:

$$U_t(c^*)(s) = V_t(c^*(s), s-c^*(s), \mathbb{E}_{s-c}U^*(c^*)) = u(c^*(s)) + \delta\mathbb{E}_{s-c^*(s)}U^*(c^*).$$

In a similar way, we can consider a case of δ being investment dependent. In such a case, one would need to impose:

$$U_t(c^*)(s) = V_t(c^*(s), s-c^*(s), \mathbb{E}_{s-c}U^*(c^*)) = u(c^*(s)) + \delta(s-c^*(s))\mathbb{E}_{s-c^*(s)}U^*(c^*).$$

It is easy to see that this specification is also a special case of the general model, and hence Time Consistent Equilibrium c^* exist in this model.

6 Concluding Remarks

In this paper, we propose a new collection of functional equation methods for proving existence of (pure strategy) Time Consistent Equilibria in a general class of dynamic models with "behavioral" discounting with recursive payoffs with a bounded or unbounded state space. Our approach allows use to link recursive utility models with the literature on the strategic aspects of stochastic games, and in particular models of dynamic choice with dynamically inconsistent preferences. Our approach includes two notable examples such as the deterministic $\beta-\delta$ quasi-hyperbolic discounting and various versions of stochastic hyperbolic discounting models. We think that the general existence methods applied in section 5 can be extended to also show existence of Time Consistent Equilibria in more general models of altruism with recursive payoffs as recently axiomatized by Galperti and Strulovici (2017). We leave it for further research.

A Appendix. Omitted lemmas and proofs

A.1 Quasi-hyperbolic discounting

We now state and prove a number of important preliminary results concerning these spaces and two important mappings defined in them. First, note the structure of the space $(\mathcal{V}, ||\cdot||)$ and its subset $\mathcal{U} \subset \mathcal{V}$.

Lemma 1. $(\mathcal{V}, ||\cdot||)$ *is a Banach space and* $\mathcal{U} \subset \mathcal{V}$ *is a closed set.*

Proof. For any $f \in \mathcal{V}$ consider $(\mathcal{V}_k, ||\cdot||_k)$, the restriction of f to $S_k \times \mathcal{H}$. Clearly $\mathcal{V}_k$ is a subset of Banach space of bounded functions on $S_k \times \mathcal{H}$, hence we only need to show $\mathcal{V}_k$ is closed. The convergence in norm $||\cdot||_k$ is equivalent to the uniform convergence on $S_k \times \mathcal{H}$. Suppose $\phi_n \rightrightarrows \phi$ as $n \to \infty$ in $||\cdot||_k$ and any of $\phi_n \in \mathcal{V}_k$. We show $\phi \in \mathcal{V}_k$. Obviously ϕ is bounded on $S_k \times \mathcal{H}$. We check further desired properties.

- *We show ϕ is right continuous on s for any fixed h.*

  Let $\epsilon > 0$ be given. Let $s_n \downarrow s^0$ and let N be such that $||\phi_N - \phi||_k < \frac{\epsilon}{2}$. We have

  $$\begin{aligned}|\phi(s_n,h) - \phi(s^0,h)| \leq & \quad |\phi(s_n,h) - \phi_N(s_n,h)| + |\phi_N(s_n,h) - \phi_N(s^0,h)| + |\phi_N(s^0,h) - \phi(s^0,h)| \\ \leq & \quad 2||\phi - \phi_N||_k + |\phi_N(s_n,h) - \phi_N(s^0,h)|.\end{aligned}$$

  Since ϕ_N is right continuous at s^0, hence taking a limit with $n \to \infty$ we have $\limsup_{n\to\infty} |\phi(s_n,h) - \phi(s^0,h)| < \epsilon$. Since ϵ is arbitrary, hence $\phi(s_n,h) \to \phi(s^0,h)$. Hence $\phi(\cdot,h)$ is right continuous.

- *We show ϕ is upper semicontinuous.* Let $(s_n,h_n) \to (s^0,h^0)$. As before $\epsilon > 0$ is given and N is such that $||\phi - \phi_N||_k < \frac{\epsilon}{2}$, Hence

  $$\begin{aligned}&\phi(s^0,h^0) - \phi(s_n,h_n) = \\ &\phi(s^0,h^0) - \phi_N(s^0,h^0) + \phi_N(s^0,h^0) - \phi_N(s_n,h_n) + \phi_N(s_n,h_n) - \phi(s_n,h_n) \geq \\ &-\epsilon + \phi_N(s^0,h^0) - \phi_N(s_n,h_n).\end{aligned}$$

  Since ϕ_N is upper semicontinuous

  $$\liminf_{n\to\infty} (\phi(s^0,h^0) - \phi(s_n,h_n)) \geq -\epsilon.$$

  Since $\epsilon > 0$, hence ϕ is upper semicontinuous.

- *We show for any $h \in \mathcal{H}$ there is a countable $\tilde{S} \subset S$ such that ϕ is continuous at any $(s,h) \in \mathcal{E}$, such that $s \notin \tilde{S}$.* Let $\tilde{S}^N \subset \mathcal{E}^h$ be a countable set such that f_N is continuous at any (s,h) with $s \notin \tilde{S}^N$. Let $\tilde{S} := \bigcup_{N=1}^{\infty} \tilde{S}^N$. Observe $\tilde{S}$ is countable and any of ϕ_N is continuous at (s,h) such that $s \notin \tilde{S}$. Since ϕ is the uniform limit of ϕ_N on any set $S_k \times \mathcal{H}$, hence ϕ is continuous at (s,h).

Consequently $\phi \in \mathcal{V}_k$ and $(\mathcal{V}_k, ||\cdot||_k)$ is Banach space. Pick any $\phi_k \in \mathcal{V}_k$ such that $\phi_{k+1}(s,h) = \phi_k(s,h)$ for any $(s,h) \in S_k \times \mathcal{H}$. Define $\phi(s,h) = \phi_k(s,h)$ whenever $s \in S_k$. Observe that $\phi(\cdot)$ is upper semicontinuous and $\phi(\cdot,h)$ is right continuous. Moreover, for any $h \in \mathcal{H}$, ϕ may be discontinuous at $(s,h) \in \mathcal{E}^h$, where s is chosen from at most countable set. Hence $\phi \in \mathcal{V}$. By Lemma 1 in Matkowski and Nowak (2011), we conclude $(\mathcal{V}, ||\cdot||$ is a Banach space. It is easy to see, $\mathcal{U}$ is a complete metric space with the metric induced by $||\cdot||$ since it is a closed subset of $\mathcal{V}$. $\square$

Lemma 2. *Let $f \in \mathcal{U}$ and suppose $h_n \to^w h$. Then if $\mu_n \to \mu$ weakly on S, then*

$$\limsup_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \leq \int_S f(s', h)\mu(ds'). \tag{11}$$

Suppose that μ is concentrated on the set of continuity points of $f(\cdot, h)$. Then

$$\lim_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') = \int_S f(s', h)\mu(ds'). \tag{12}$$

Proof. Define:

$$\overline{f}(s) = \sup\left\{\limsup_{n\to\infty} f(s_n, h_n)) : s_n \to s\right\}$$

and

$$\underline{f}(s) = \inf\left\{\liminf_{n\to\infty} f(s_n, h_n)) : s_n \to s\right\}.$$

Since f is u.s.c. hence

$$\limsup_{n\to\infty} f(s_n, h_n) \leq f(s, h)$$

whenever $s_n \to \infty$, hence $\overline{f}(s) \leq f(s, h)$. Hence and by Lemma 3.2. in Serfozo (1982) we have

$$\limsup_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \leq \int_S \overline{f}(s)\mu(ds) \leq \int_S f(s, h)\mu(ds).$$

Now suppose f is continuous at (s, h) for μ-a.a. s. Then for μ-a.a. s we have

$$\lim_{n\to\infty} f(s_n, h_n) = f(s, h)$$

whenever $s_n \to s$. Hence $f(s, h) = \underline{f}(s)$, μ-almost everywhere. Again by Lemma 3.2. in Serfozo (1982) we have

$$\liminf_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \geq \int_S \underline{f}(s)\mu(ds) = \int_S f(s, h)\mu(ds).$$

Since we have proven (11), hence

$$\int_S f(s, h)\mu(ds) \geq \limsup_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \geq \liminf_{n\to\infty} \int_S f(s', h_n)\mu_n(ds') \geq \int_S f(s, h)\mu(ds).$$

Hence (12) holds and the proof is complete. $\square$

Lemma 3. *T maps $\mathcal{U}$ into itself.*

Proof. Let $f \in \mathcal{U}$. Obviously $|T(f)(s, h)| \leq \eta_k$ for $(s, h) \in S_k \times \mathcal{H}$. Similarly as in Lemma 5 in Balbus et al. (2020) we conclude $\mathbb{E}_i f(h)$ is continuous from the right. We easily conclude $T(f)(\cdot, h)$ is right continuous. We are going to show $T(f)$ it is upper semicontinuous. Let

$(s_n, h_n) \to (s^0, h^0)$ in the corresponding topology. Pick

$$i_n \in \arg\max_{i \in [0, s_n]} V(s_n - i_n, s_n - h_n(s_n), \mathbb{E}_{i_n} f(h_n))$$

and without loss of generality suppose $i_n \to i^0$. By Assumption 2, $q(\cdot|i_n) \to q(\cdot|i^0)$ weakly. By Lemma 2 we have then

$$\limsup_{n\to\infty} \mathbb{E}_{i_n} f(h_n) \leq \mathbb{E}_{i^0} f(h^0), \tag{13}$$

We have

$$\liminf_{n\to\infty} (s_n - h_n(s_n)) \geq s^0 - \limsup_{n\to\infty} h_n(s_n) \geq s^0 - h^0(s^0), \tag{14}$$

Combining (13) and (14) we have

$$\limsup_{n\to\infty} V(s_n - i_n, s_n - h_n(s_n), \mathbb{E}_{i_n} f(h_n)) \leq T(f)(s^0, h^0). \tag{15}$$

Hence $T(f)$ is upper semicontinuous. Finally, we show $T(f)$ is continuous at any $(s,h) \in \mathcal{E}$ such that $s \notin S^{T(f),h}$, where $S^{T(f),h}$ is at most countable subset of S. We can take

$$S^{T(f),h} := \{s \in \mathcal{E}^h : q(\{s' \in S : f \text{ is continuous at } (s', h)\}|s) < 1\}$$

and clearly $S^{T(f),h}$ is countable. Now assume $(s_n, h_n) \to (s^0, h^0)$ and $s^0 \notin S^{T(f),h^0}$. Then by definition of convergence of $\mathcal{H}$ and Lemma 2 we have

$$\lim_{n\to\infty} V(s_n - i, s_n - h_n(s_n), \mathbb{E}_i f(h_n)) = V(s^0 - i, s^0 - h^0(s^0), \mathbb{E}_i f(h^0)) \tag{16}$$

for any $i \notin S^{T(f),h^0}$, in particular for s^0. Again by (13) and (14) we have

$$\begin{aligned} V(s^0 - i^0, s^0 - h(s^0), \mathbb{E}_{i^0} f(h)) &\geq \limsup_{n\to\infty} V(s_n - i_n, s_n - h_n(s_n), \mathbb{E}_{i_n} f(h_n)) \\ &\geq \liminf_{n\to\infty} V(s_n - i, s_n - h_n(s_n), \mathbb{E}_i f(h_n)) \\ &= V(s^0 - i, s^0 - h^0(s^0), \mathbb{E}_i f(h^0)). \end{aligned}$$

Since the right hand side is right continuous, hence this equality holds for any $i \in [0, s^0]$. Indeed, we can take $\tilde{i}^m \downarrow i$ as $m \to \infty$ such that $\tilde{i}^m \in S^{T(f),h^0}$, substitute i by $\tilde{i}^m$ above, and take a limit $m \to \infty$. $\square$

Lemma 4. *T is a contraction mapping on $\mathcal{U}$, and therefore has a unique fixed point in $\mathcal{U}$.*

Proof. Observe that by the standard argument

$$||T(f) - T(g)||_k \leq ||f - g||_{k+1} \quad \text{for any } k \in \mathbb{N}.$$

Hence T is 1-local contraction. By Theorem 2 in Rincon-Zapatero and Rodriguez-Palmero (2009), T is a contraction mapping on $\mathcal{U}$. By Lemma 1 and Banach Contraction Principle T has a unique fixed point. □

Lemma 5. *Let $h \in \mathcal{H}$. Then, any selection of $s \mapsto BI(h)(s)$ is increasing in s.*

Proof. Suppose that it is not the case: there are $s_1 > s_2$ and $i_1 < i_2$ such that $i_1 \in BI(h)(s_1)$ and $i_2 \in BI(h)(s_2)$. Then

$$0 \leq V(s_2 - i_2, s_2 - h(s_2), \mathbb{E}_{i_2} f^*(h)) - V(s_2 - i_2 - (i_2 - i_1), s_2 - h(s_2), \mathbb{E}_{i_1} f^*(h)).$$

But then from Assumption 1 (ii) we have

$$V(s_1 - i_2, s_1 - h(s_1), \mathbb{E}_{i_1} f^*(h)) - V(s_1 - i_2 - (i_2 - i_1), s_1 - h(s_1), \mathbb{E}_{i_2} f^*(h)) > 0$$

which contradicts $i_1 \in BI(h)(s_1)$. □

Lemma 6. *Let $h \in \mathcal{H}$. If $bi(h)$ is continuous at s, then $BI(h)(s)$ is a singleton.*

Proof. Suppose that $bi(h)$ is continuous at s and pick $y_0 \in BI(h)(s)$. By Lemma 5 we have $bi(h)(s - \delta) \leq y_0 \leq bi(s + \delta)$. Since $bi(h)$ is continuous, hence $y_0 = bi(h)(s)$, and consequently $BI(h)$ is singleton. □

Lemma 7. *The operator bi maps $\mathcal{H}$ into itself and it is a continuous operator.*

Proof. By Lemma 5 it follows that $bi(h)(\cdot)$ is increasing. We show it is right continuous. Let $s_n \downarrow s^0$. We show $i_n := bi(h)(s_n) \to bi(h)(s^0)$. By Lemma 5, $i_n \downarrow i^0$. Since h is right continuous $h(s_n) \downarrow h(s^0)$ as $n \to \infty$. Put

$$\Pi(s, i) := V(s - i, s - h(s), \mathbb{E}_i(f^*)).$$

Suppose $i \notin S^{f^*,h}$. Since h and $i \mapsto \mathbb{E}_i(f^*)$ are both right continuous, hence we have

$$\Pi(s^0, i^0) = \lim_{n \to \infty} \Pi(s_n, i_n) \geq \Pi(s^0, i)$$

for all $i \in [0, s^0)$. Hence $i^0 \in BI(h)(s^0)$ if $bi(h)(s^0) < s^0$. If we allow, $bi(h)(s^0) = s_0$, by Lemma 5 we have $i^0 \leq bi(h)(s^0) \leq bi(h)(s_n)$ for all n, hence taking a limit with $n \to \infty$ we have $i^0 = bi(h)(s^0)$. Now we show the continuity of bi on $\mathcal{H}$. Suppose $h_n \to^w h^0$ in $\mathcal{H}$ such that s^0 is a continuity point of $bi(h^0)(\cdot)$. By Lemma 6 it follows that $BI(h^0)(s^0)$ is a singleton in this case. Hence we are going to show $i_n := bi(h_n)(s^0) \to i^0$ for some $i^0 \in BI(h^0)(s^0)$. Define

$$Z^0 := \{i \in S : q(S^{f^*,h_0}|i) = 0\}.$$

By Assumption 2 the complement of Z^0 is at most countable. First, let us focus attention to $s^0 \notin Z^0$. By definition of S^{f^*,h^0}, for any $i \notin Z^0$ we have

$$\mathbb{E}_i f^*(h_n) \to \mathbb{E}_i f^*(h^0)$$

as $n \to \infty$. Moreover, $h_n(s^0) \to h(s^0)$ and if $i_n \to i$, then by Lemma 2

$$\lim_{n\to\infty} \mathbb{E}_{i_n} f^*(h_n)) = \mathbb{E}_i f^*(h^0)).$$

Hence

$$\begin{aligned} &\lim_{n\to\infty} V(s^0 - i_n, s^0 - h_n(s^0), \mathbb{E}_{i_n} f^*(h_n)) \\ &\geq \liminf_{n\to\infty} V(s^0 - i, s^0 - h_n(s^0), \mathbb{E}_i f^*(h_n))) \\ &\geq V(s^0 - i, s^0 - h^0(s^0), \mathbb{E}_i f^*(h^0))). \end{aligned} \tag{17}$$

Since the right hand side above we right continuous, hence the inequality (17) holds for any $i \in [0, s^0]$ since $s^0 \notin Z^0$. To finish the proof observe

$$\limsup_{n\to\infty} V(s^0 - i_n, s^0 - h_n(s^0), \mathbb{E}_{i_n} f^*(h_n)) \leq V(s^0 - i^0, s^0 - h^0(s^0), \mathbb{E}_{i^0} f^*(h^0)),$$

where the last inequality follows from (13). Then combining the inequality above with (17) we have $i^0 \in BI(h^0)(s^0)$, consequently $i^0 = bi(h^0)(s^0)$. Hence we have proven, $bi(h_n)(s^0) \to bi(h)(s^0)$ as $n \to \infty$ whenever $s^0 \in Z^0$ and s^0 is a continuity point of $bi(h)$. To finish the proof, we need to show that this convergence is true outside Z^0 as well. If $s^0 \notin Z^0$ is a continuity point of $bi(h^0)$, we may find $\delta_1 > 0$ and $\delta_2 > 0$ such that $bi(h^0)$ is both continuous at $s^0 - \delta_1$, $s^0 + \delta_2$ but $s^0 - \delta_1 \in Z^0$ in $s^0 + \delta_2 \in Z^0$. By Assumption 2, δ_1 and δ_2 can be sufficiently small. Then, by the previous part of the proof

$$\begin{aligned} bi(s^0 - \delta_1) &= \lim_{n\to\infty} bi(h_n)(s^0 - \delta_1) \\ &\leq \liminf_{n\to\infty} bi(h_n)(s^0) \\ &\leq \limsup_{n\to\infty} bi(h_n)(s^0) \\ &\leq \lim_{n\to\infty} bi(h_n)(s^0 + \delta_2) = bi(h^0)(s^0 + \delta_2). \end{aligned}$$

Taking a limit $\delta_1 \to 0$ and $\delta_2 \to 0$ we have $bi(h_n)(s^0) \to bi(h^0)(s^0)$ as $n \to \infty$. □

A.2 Semi-hyperbolic discounting

Lemma 8. *$(\mathcal{V}_0, ||\cdot||)$ is a Banach space and $\mathcal{U}_0 \subset \mathcal{V}_0$ is a closed set.*

Proof. For any $f \in \mathcal{V}_0$ consider $(\mathcal{V}_{k,0}, ||\cdot||_k)$, the restriction of f to $S_k \times \mathcal{H}$. Clearly $\mathcal{V}_{k,0}$ is a subset of Banach space of bounded functions on $S_k \times \mathcal{H}$, hence we only need to show $\mathcal{V}_{k,0}$ is closed. The convergence in norm $||\cdot||_k$ is equivalent to the uniform convergence on $S_k \times \mathcal{H}$. Suppose $\phi_n \rightrightarrows \phi$ as $n \to \infty$ in $||\cdot||_k$ and any of $\phi_n \in \mathcal{V}_{k,0}$. We show $\phi \in \mathcal{V}_{k,0}$. Obviously ϕ is bounded on $S_k \times \mathcal{H}$. Similarly as in Lemma 1 we may show that for any $h \in \mathcal{H}$ there is a countable $\tilde{S} \subset S$ such that ϕ is continuous at any $(s,h) \in \mathcal{E}$, such that $s \notin \tilde{S}$. Consequently $\phi \in \mathcal{V}_k$ and $(\mathcal{V}_{k,0}, ||\cdot||_k)$ is Banach space. Pick any $\phi_k \in \mathcal{V}_{k,0}$ such that $\phi_{k+1}(s,h) = \phi_k(s,h)$ for any $(s,h) \in S_k \times \mathcal{H}$. Define $\phi(s,h) = \phi_k(s,h)$ whenever $s \in S_k$. Observe that for any $h \in \mathcal{H}$, ϕ may be discontinuous at $(s,h) \in \mathcal{E}$, where s is chosen from at most countable set. Hence $\phi \in \mathcal{V}$. By Lemma 1 in Matkowski and Nowak (2011), we conclude $(\mathcal{V}_0, ||\cdot||)$ is a Banach space. It is easy to see, $\mathcal{U}_0$ is a complete metric space with the metric induced by $||\cdot||$ since it is a closed subset of $\mathcal{V}_0$. □

Lemma 9. *Λ maps $\mathcal{U}_0$ into itself and is a contraction mapping in $\mathcal{U}_0$.*

Proof. We omit the proof since it is similar as the proofs of Lemma 4. □

Lemma 10. *For any $h \in \mathcal{H}$, any selection of $\mathcal{BI}(h)(s)$ is nonempty valued and has the greatest and the least selection. Moreover, any selection of $\mathcal{BI}(h)(s)$ is increasing in s.*

Proof. We omit the proof since it is similar to the proof of Lemma 5. □

Lemma 11. *Let $h \in \mathcal{H}$ and suppose h is continuous at s. Then, if $bi(h)(s)$ is continuous at s, then $\mathcal{BI}(h)(s)$ is a singleton.*

Proof. Using Lemma 10 we repeat the same argument as in Lemma 6. □

Lemma 12. *The operator bi maps $\mathcal{H}$ into itself and it is a continuous operator.*

Proof. Let $h_n \to^w h^0$ as $n \to \infty$ and let s' be a continuity point of h^0. We have

$$\sup\left\{\limsup_{n\to\infty} f^*(s'_n, h_n) : s'_n \to s'\right\} = \inf\left\{\liminf_{n\to\infty} f^*(s'_n, h_n) : s'_n \to s'\right\} = f(s', h^0),$$

whenever $(s', h^0) \in \mathcal{E}$ and $s' \notin S^{f^*,h^0}$. Observe that for any $s'_n \to s'$ and $h_n \to^w h$ we have $h_n(s'_n) \to h^0(s')$ whenever $s' \notin S^{f^*,h^0}$ and it is a continuity point of h^0. By Assumption 4 it follows that this convergence above holds for all but countably many $s' \in S$. Let $i_n \to i^0$ in S. Hence by Lemma 2

$$\int_S f^*(s', h^0) q(ds'|i^0) = \lim_{n\to\infty} \int_S f^*(s', h_n) q(ds'|i_n). \tag{18}$$

We show that

$$\lim_{n\to\infty}\int_S A_t^*(s',h_n)q(ds'|i_n) = \int_S A_t^*(s',h^0)q(ds'|i^0). \tag{19}$$

The thesis for $t=0$ is in (18). If this thesis holds for some t, then by definition of $A_{t+1}^*(s',h)$ and (19) we have this thesis, and (19) holds for $t+1$. As a result, the function

$$(s,i,h)\in S\times S\times\mathcal{H}\mapsto u(s-i)+\prod_{t=1}^{T}\beta_t\int_S A_{T-1}^*(s',h)q(ds'|i)$$

is continuous. Let s^0 be a continuity point of $bi(h^0)(\cdot)$. Let $y_n = bi(h_n)(s^0)$ and suppose $y_n\to y^0$. Hence by Berge Maximum Theorem $y^0\in\mathcal{BI}(h^0)(s^0)$. By Lemma 11, $\mathcal{BI}(h^0)(s^0)$ is a singleton, hence $y^0 = bi(h)(s^0)$. But this implies $bi(h_n)\to^w bi(h_n)$. □

A.3 Limiting case

Lemma 13. $\prod_{k=1}^{\infty}\beta_k$ *exists and is nonzero if and only if* $\lim_{t\to\infty}\prod_{k=t}^{\infty}\beta_k = 1$.

Proof. Define $r := \prod_{k=1}^{\infty}\beta_k$, and suppose $r>0$. Then

$$-ln(r) = \sum_{k=1}^{\infty}-\ln(\beta_k).$$

Since $-\ln(\beta_k)>0$, hence the series above are convergent and

$$\lim_{t\to\infty}\sum_{k=t}^{\infty}-\ln(\beta_k) = 0. \tag{20}$$

Moreover,

$$\lim_{t\to\infty}\sum_{k=t}^{\infty}-\ln(\beta_k) = -\lim_{t\to\infty}\ln\left(\prod_{k=t}^{\infty}\beta_k\right) = -\ln\left(\lim_{t\to\infty}\prod_{k=t}^{\infty}\beta_k\right). \tag{21}$$

Combining (20) with (21) we have the thesis. Now let $r=0$. Then the right hand side in (20) yields ∞. Furthermore, by (21) we have $\lim_{t\to\infty}\prod_{k=t}^{\infty}\beta_k = 0$ □

A.4 General existence result

Lemma 14. $(\mathcal{V}_0^\infty, ||\cdot||^\infty)$ *is a Banach space, and* $\mathcal{U}_0^\infty$ *is a closed subset of* $\mathcal{V}_0^\infty$.

The proof is identical as proof of Lemma 8.

Lemma 15. $\mathbb{T}$ *is a contraction mapping on* $\mathcal{U}_0^\infty$ *and has a unique fixed point.*

Proof. We show $\mathbb{T}$ maps $\mathcal{U}_0^\infty$ into itself. Let $f \in \mathcal{U}_0^\infty$. Then for any $k \in \mathbb{N}$, $s' \in S_{k+1}$, $h \in \mathcal{H}$ and $t \in \mathbb{N}$ we have $|f_{t+1}(s', h)| \leq \eta_{k+1}$. By Assumption 2 for any $s \in S_k$ we have

$$|\mathbb{E}_{h(s)} f_{t+1}(h)| = \left| \int_S f_{t+1}(s', h) q(ds'|h(s)) \right| \leq \eta_{k+1},$$

hence

$$\left| V_t(s - h(s), s - h(s), \mathbb{E}_{h(s)} f_{t+1}(h)) \right| \leq \sup_{(x,y,z) \in S_k^2 \times [0, \eta_{k+1}]} |V_t(x, y, z)| \leq \eta_k$$

where the last equality is a consequence of Assumption 1. Furthermore, applying Lemma 2 we conclude

$$s \in S \mapsto V_t(s - h(s), s - h(s), \mathbb{E}_{h(s)} f_{t+1}(h))$$

is left continuous and continuous at any $s \notin S^{\mathbb{T}(f),h}$, where $S^{\mathbb{T}(f),h}$ is a countable subset of S. Hence $\mathbb{T}(f) \in \mathcal{U}_0^\infty$. Observe that by Assumption 1 and the standard argument

$$||\mathbb{T}(f) - \mathbb{T}(g)||_k \leq ||f - g||_{k+1} \quad \text{for any } k \in \mathbb{N}.$$

Hence is 1-local contraction. By Theorem 2 in Rincon-Zapatero and Rodriguez-Palmero (2009), $\mathbb{T}$ is a contraction mapping on $\mathcal{U}_0^\infty$. By Lemma 1 and Banach Contraction Principle $\mathbb{T}$ has a unique fixed point. □

Lemma 16. *Let $h \in \mathcal{H}$. Then, $BI(h)(s)$ is nonempty valued correspondence with the greatest and the least selection. Moreover, any selection of $BI(h)(s)$ is increasing in s.*

Proof. First we show $BI(h)(s)$ is indeed nonempty valued correspondence with the greatest and the least element. Let f^* be a unique fixed point of $\mathbb{T}$ and f_2^* be the coordinate needed to define BI. For any $h \in \mathcal{H}$ let $S^{*,h}$ be a countable subset of S such that f_2^* is continuous at any $(s, h) \in S \times \mathcal{H}$ such that $s \in S^*$. We show that the following function

$$(i, h) \in S \times \mathcal{H} \mapsto \mathbb{E}_i f_2^*(h)) = \int_S f_2^*(s, h) q(ds|i)$$

is continuous. Indeed, by Assumption 4, $q(\cdot|i)$ is nonatomic, hence $q(S^{*,h}|i) = 0$ for any $h \in \mathcal{H}$ and $i \in S$. Let $i_n \to i$ in S and $h_n \to^w h$ in $\mathcal{H}$. By Skorohod Representation Theorem, there is a probability space $(\Omega, \mathcal{F}, P)$ and random variables X_n whose distribution is $q(\cdot|i_n)$ and X whose distribution is $q(\cdot|i)$ such that $X_n \to X$ pointwise in Ω. Since $q(\cdot|i)$ is concentrated away of $S^{*,h}$, hence $X(\omega) \notin S^{*,h}$ for P- a.a. $\omega \in \Omega$. Hence $f_2^*(X_n(\omega), h_n) \to f_2^*(X(\omega), h)$ for P-a.a. ω. We have then

$$\mathbb{E}_{i_n} f_2^*(h_n)) = \int_S f_2^*(s, h_n) q(ds|i_n) = \int_\Omega f_2^*(X_n(\omega), h_n) P(d\omega) \to_{n \to \infty}$$
$$\int_\Omega f_2^*(X(\omega), h) P(d\omega) = \int_S f_2^*(s, h) q(ds|i) = \mathbb{E}_i f_2^*(h)).$$

Hence $BI(h)(s) \neq \emptyset$ and has the greatest and the least selection. The rest of proof is omitted, since is the same as the proof of Lemma 5. $\square$

By Lemma 16 we repeat the same argument as in Lemmas 6 and 7 we have the following lemma.

Lemma 17. *Let $h \in \mathcal{H}$ and suppose h is continuous at s. Then, if $bi(h)(s)$ is continuous at s then $BI(h)(s)$ is a singleton.*

Combining Lemmas 16 and 17 we have the following lemma whose proof is similar to that of lemma 12.

Lemma 18. *The operator bi maps $\mathcal{H}$ into itself and it is a continuous operator.*

References

Ahn, D. S., R. Iijima, Y. Le Yaouanq, and T. Sarver (2019): "Behavioural characterizations of naivete for time-inconsistent preferences," *Review of Economic Studies*, 86, 2319–2355.

Ahn, D. S., R. Iijima, and T. Sarver (2020): "Naivete about temptation and self-control: Foundations for recursive naive quasi-hyperbolic discounting," *Journal of Economic Theory*, 189.

Ameriks, J., A. Caplin, J. Leahy, and T. Tyler (2007): "Measuring self-control problems," *American Economic Review*, 97.

Andreoni, J. and C. Sprenger (2012): "Estimating time preferences from convex budgets," *American Economic Review*, 102, 3333–56.

Angeletos, G.-M., D. Laibson, A. Repetto, J. Tobacman, and S. Weinberg (2001): "The hyberbolic consumption model: calibration, simulation, and empirical evaluation," *Journal of Economic Perspectives*, 15, 47–68.

Arrow, K. J., M. L. Cropper, C. Gollier, B. Groom, G. M. Heal, R. G. Newell, W. D. Nordhaus, R. S. Pindyck, W. A. Pizer, P. R. Portney, T. Sterner, R. S. J. Tol, and M. L. Weitzman (2014): "Should governments use a declining discount rate in project analysis?" *Review of Environmental Economics and Policy*, 8, 145–163.

Augenblick, N., M. Niederle, and C. Sprenger (2015): "Working over time: dynamic inconsistency in real effort tasks," *Quarterly Journal of Economics*, 130, 1067–1115.

Balbus, Ł. (2020): "On recursive utilities with non-affine aggregator and conditional certainty equivalent," *Economic Theory*, 70, 551–577.

Balbus, Ł., A. Jaśkiewicz, and A. S. Nowak (2015a): "Stochastic bequest games," *Games and Economic Behavior*, 90, 247–256.

——— (2020): "Equilibria in altruistic economic growth models," *Dynamic Games and Applications*, 10, 1–18.

Balbus, Ł., K. Reffett, and Ł. Woźny (2015b): "Time consistent Markov policies in dynamic economies with quasi-hyperbolic consumers," *International Journal of Game Theory*, 44, 83–112.

——— (2018): "On uniqueness of time-consistent Markov policies for quasi-hyperbolic consumers under uncertainty," *Journal of Economic Theory*, 176, 293 – 310.

Balbus, Ł. and Ł. Woźny (2016): "A strategic dynamic programming method for studying short-memory equilibria of stochastic games with uncountable number of states," *Dynamic Games and Applications*, 6, 187–208.

Baucells, M. and F. H. Heukamp (2012): "Probability and time trade-off," *Management Science*, 58, 831–842.

Bäuerle, N. and A. Jaśkiewicz (2018): "Stochastic optimal growth model with risk sensitive preferences," *Journal of Economic Theory*, 173, 181 – 200.

Becker, R. A. (2012): "Optimal growth with heterogeneous agents and the twisted turnpike: An example," *International Journal of Economic Theory*, 8, 27–47.

Bernheim, B. D., D. Ray, and S. Yeltekin (2015): "Poverty and self-control," *Econometrica*, 83, 1877–1911.

Beshears, J., D. Choi, James J.and Laibson, and B. C. Madrian (2018): "Behavioral household finance," in *Handbook of Behavioral Economics - Foundations and Applications 1*, ed. by D. Bernheim, D. Laibson, and S. DellaVigna, Elsevier, vol. 1, 1st ed.

Beshears, J., J. J. Choi, C. Clayton, C. Harris, D. Laibson, and B. C. Madrian (2020): "Optimal illiquidity," NBER Working Papers 27459.

Bich, P., J.-P. Drugeon, and L. Morhaim (2018): "On temporal aggregators and dynamic programming," *Economic Theory*, 66, 787–817.

Bryan, G., D. Karlan, and S. Nelson (2010): "Commitment devices," *Annual Review of Economics*, 2, 671–698.

Cao, D. and I. Werning (2018): "Saving and dissaving with hyperbolic discounting," *Econometrica*, 86, 805–857.

Caplin, A. and J. Leahy (2006): "The recursive approach to time inconsistency," *Journal of Economic Theory*, 131, 134–156.

Casaburi, L. and R. Macchiavello (2019): "Demand and supply of infrequent payments as a commitment device: evidence from Kenya," *American Economic Review*, 109, 523–55.

Ceteman, D., F. Feng, and C. Urgen (2019): "Contracting with non-exponential discounting: moral hazard and dynamic inconsistency," MS.

Chade, H., P. Prokopovych, and L. Smith (2008): "Repeated games with present-biased preferences," *Journal of Economic Theory*, 139, 157–175.

Chambers, C. P. and F. Echenique (2018): "On multiple discount rates," *Econometrica*, 86, 1325–1346.

Chan, M. K. (2017): "Welfare dependence and self-control: an empirical analysis," *Review of Economic Studies*, 84, 1379–1423.

Chatterjee, S. and B. Eyigungor (2016): "Continuous Markov equilibria with quasi-geometric discounting," *Journal of Economic Theory*, 163, 467–494.

Cohen, J., K. M. Ericson, D. Laibson, and J. M. White (2020): "Measuring time preferences," *Journal of Economic Literature*, 58, 299–347.

Dalton, C. M., G. Gowrisankaran, and R. J. Town (2020): "Salience, myopia, and complex dynamic incentives: evidence from Medicare Part D," *Review of Economic Studies*, 87, 822–869.

Dasgupta, P. (2008): "Discounting climate change," *Journal of Risk and Uncertainty*, 37, 141–169.

Dekel, E. and B. L. Lipman (2012): "Costly self-control and random self-indulgence," *Econometrica*, 80, 1271–1302.

Drugeon, J.-P. and T. Ha-Huy (2018): "A not so myopic axiomatization of discounting," Documents de recherche 18-02, Centre d'Études des Politiques Économiques (EPEE), Université d'Evry Val d'Essonne.

Drugeon, J.-P. and B. Wigniolle (2020): "On Markovian collective choice with heterogeneous quasi-hyperbolic discounting," *Economic Theory*, DOI: 10.1007/s00199-020-01291-z.

Duflo, E., M. Kremer, and J. Robinson (2011): "Nudging farmers to use fertilizer: theory and experimental evidence from Kenya," *American Economic Review*, 101, 2350–90.

Dziewulski, P. (2018): "Revealed time preference," *Games and Economic Behavior*, 112, 67 – 77.

Ebert, S., W. Wei, and X. Y. Zhou (2020): "Weighted discounting—On group diversity, time-inconsistency, and consequences for investment," *Journal of Economic Theory*, 189, 105089.

Echenique, F., T. Imai, and K. Saito (2016): "Testable implications of models of intertemporal choice: exponential discounting and its generalizations," MS.

Epstein, L. G. and J. A. Hynes (1983): "The rate of time preference and dynamic economic analysis," *Journal of Political Economy*, 91, 611–635.

Epstein, L. G. and S. E. Zin (1989): "Substitution, risk aversion, and the remporal behavior of consumption and asset returns: a theoretical framework," *Econometrica*, 57, 937–969.

Ericson, K. M. and D. Laibson (2019): "Intertemporal choice," in *Handbook of Behavioral Economics - Foundations and Applications 2*, ed. by D. Bernheim, D. Laibson, and S. DellaVigna, Elsevier, vol. 2.

Fishburn, P. C. and A. Rubinstein (1982): "Time preference," *International Economic Review*, 23, 677–694.

Frederick, S., G. Loewenstein, and T. O'Donoghue (2002): "Time discounting and time preference: a critical review," *Journal of Economic Literature*, 40, 351–401.

Fudenberg, D. and D. K. Levine (2006): "A dual-self model of impulse control," *American Economic Review*, 96, 1449–1476.

——— (2012): "Timing and self-control," *Econometrica*, 80, 1–42.

Galperti, S. and B. Strulovici (2017): "A theory of intergenerational altruism," *Econometrica*, 85, 1175–1218.

Gerlagh, R. and M. Liski (2018): "Consistent climate policies," *Journal of the European Economic Association*, 16, 1–44.

Gine, X., D. Karlan, and J. Zinman (2010): "Put your money where your butt is: a commitment contract for smoking cessation," *American Economic Journal: Applied Economics*, 2, 213–35.

Gottleib, D. and X. Zhang (2020): "Long-term contracting with time-inconsistent agents," *Econometrica*, forthcoming.

Gul, F. and W. Pesendorfer (2001): "Temptation and self-control," *Econometrica*, 69, 1403–1435.

——— (2004): "Self-control and the theory of consumption," *Econometrica*, 72, 119–158.

——— (2005): "The Revealed Preference Theory of Changing Tastes," *Review of Economic Studies*, 72, 429–448.

Halec, M. and P. Yared (2019): "Fiscal rules and discretion under limited commitment," Tech. rep., NBER Working Paper, 25463.

Halevy, Y. (2008): "Strotz meets allais: diminishing impatience and the certainty effect," *The American Economic Review*, 98, 1145–1162.

Hammond, P. J. (1976): "Changing tastes and coherent dynamic choice," *Review of Economic Studies*, 43, 159–173.

Harris, C. and D. Laibson (2001): "Dynamic choices of hyperbolic consumers," *Economet-*

rica, 69, 935–57.

——— (2013): "Instantaneous gratification," *Quarterly Journal of Economics*, 128, 205–248.

HARSTAD, B. (2020): "Technology and time inconsistency," *Journal of Political Economy*, 128, 2653–2689.

HEIDUES, P. AND P. STRACK (2019): "Identifying present-bias from the timing of choices," MS.

IOANNOU, C. A. AND J. SADEH (2016): "Time preferences and risk aversion: tests on domain differences," *Journal of Risk and Uncertainty*, 53, 29–54.

IVERSON, T. AND L. KARP (2020): "Carbon taxes and climate commitment with non-constant time preference," *Review of Economic Studies*.

JACKSON, M. O. AND L. YARIV (2015): "Collective dynamic choice: the necessity of time inconsistency," *American Economic Journal: Microeconomics*, 7, 150–178.

JENSEN, M. K. (2020): "A dual self approach to the computation of time-consistent equilibria," MS.

KARLAN, D., M. MCCONNELL, S. MULLAINATHAN, AND J. ZINMAN (2016): "Getting to the top of mind: how reminders increase saving," *Management Science*, 62, 3393–3411.

KOOPMANS, T. C. (1960): "Stationary ordinal utility and impatience," *Econometrica*, 28, 287–309.

KREPS, D. M. AND E. L. PORTEUS (1978): "Temporal resolution of uncertainty and dynamic choice theory," *Econometrica*, 46, 185–200.

KRUSELL, P., B. KURUSCU, AND A. A. SMITH JR. (2010): "Temptation and taxation," *Econometrica*, 78, 2063–2084.

KRUSELL, P. AND A. SMITH (2003): "Consumption–savings decisions with quasi–geometric discounting," *Econometrica*, 71, 365–375.

KUCHLER, T. AND M. PAGEL (2020): "Sticking to your plan: The role of present bias for credit card paydown," *Journal of Financial Economics*.

LAIBSON, D. (1997): "Golden eggs and hyperbolic discounting," *Quarterly Journal of Economics*, 112, 443–77.

LAIBSON, D., A. REPETTO, AND J. TOBACMAN (2007): "Estimating discount functions with consumption choices over the lifecycle," NBER Working Papers 13314.

LE VAN, C. AND Y. VAILAKIS (2005): "Recursive utility and optimal growth with bounded or unbounded returns," *Journal of Economic Theory*, 123, 187–209.

LIZZERI, A. AND L. YARIV (2017): "Collective self-control," *American Economic Journal: Microeconomics*, 9, 213–44.

LOEWENSTEIN, G. AND D. PRELEC (1992): "Anomalies in intertemporal choice: evidence and an interpretation," *The Quarterly Journal of Economics*, 107, 573–597.

MAHAJAN, A., C. MICHEL, AND A. TAROZZI (2020): "Identification of time-inconsistent models: the case of insecticide nets," Tech. rep., NBER Working Paper, 27198.

MALIAR, L. AND S. MALIAR (2016): "Ruling out multiplicity of smooth equilibria in dynamic games: a hyperbolic discounting example," *Dynamic Games and Applications*, 6, 243–261.

MARTINS-DA ROCHA, V. F. AND Y. VAILAKIS (2010): "Existence and uniqueness of a fixed point for local contractions," *Econometrica*, 78, 1127–1141.

MATKOWSKI, J. AND A. NOWAK (2011): "On discounted dynamic programming with unbounded returns," *Economic Theory*, 46, 455–474.

MONTIEL OLEA, J. L. AND T. STRZALECKI (2014): "Axiomatization and measurement of quasi-hyperbolic discounting," *The Quarterly Journal of Economics*, 129, 1449–1499.

NOOR, J. (2009): "Hyperbolic discounting and the standard model: Eliciting discount functions," *Journal of Economic Theory*, 144, 2077 – 2083.

——— (2011): “Temptation and revealed preference,” *Econometrica*, 79, 601–644.

Noor, J. and N. Takeoka (2020a): “Constrained optimal discounting,” MS.

——— (2020b): “Optimal discounting,” MS.

Obara, I. and J. Park (2017): “Repeated games with general discounting,” *Journal of Economic Theory*, 172, 348 – 375.

Peleg, B. and M. E. Yaari (1973): “On the existence of a consistent course of action when tastes are changing,” *Review of Economic Studies*, 40, 391–401.

Phelps, E. and R. Pollak (1968): “On second best national savings and game equilibrium growth,” *Review of Economic Studies*, 35, 195–199.

Pollak, R. A. (1968): “Consistent planning,” *Review of Economic Studies*, 35, 201–208.

Ramsey, F. P. (1928): “A mathematical theory of saving,” *The Economic Journal*, 38, 543–559.

Ray, D., N. Vellodi, and R. Wang (2017): “Backward discounting,” MS.

Rincon-Zapatero, J. P. and C. Rodriguez-Palmero (2009): “Corrigendum to ”Existence and uniqueness of solutions to the Bellman equation in the unbounded case” Econometrica, Vol. 71, No. 5 (September, 2003), 1519-1555,” *Econometrica*, 77, 317–318.

Saito, K. (2009): “A relationship between risk and time preferences,” Discussion Papers 1477, Northwestern University, Center for Mathematical Studies in Economics and Management Science.

Samuelson, P. A. (1937): “A note on measurement of utility,” *Review of Economic Studies*, 4, 155–161.

Serfozo, R. (1982): “Convergence of Lebesgue integrals with varying measures,” *Sankhya: The Indian Journal of Statistics*, 44, 380–402.

Strotz, R. H. (1956): “Myopia and inconsistency in dynamic utility maximization,” *Review of Economic Studies*, 23, 165–180.

Topkis, D. M. (1998): *Supermodularity and complementarity*, Frontiers of economic research, Princeton University Press.

Wakai, K. (2008): “A model of utility smoothing,” *Econometrica*, 76, 137–153.

Weil, P. (1993): “Precautionary savings and the permanent income hypothesis,” *The Review of Economic Studies*, 60, 367–383.

Young, E. R. (2007): “Generalized quasi-geometric discounting,” *Economics Letters*, 96, 343–350.