Consumer demand

Lecture 3

Dual turtle

The consumer's

A dual turtle

figure

1) The maximal distance a turtle can travel in 1 day is 1 km

2) The minimal time it takes a turtle to travel 1 km is 1 day.

The equivalence between these 2 sentences relies on 2 "hidden" assumptions:

a) for (1) to imply (2), we need to assume that the turtle travels a positive distance in any period of time

[consider: the turtle ~~travels~~'s speed is 2 km/day but after half a day it must rest for half a day the max distance it can travel in 1 day is 1 km but it is able to travel this distance in only half a day]

b) for (2) to imply (1), we need to assume that the turtle cannot "jump" a positive distance in zero time.

[Consider: the turtle's speed is 1 km/day, but after a day of traveling it can "jump" 1 km. Thus it can travel 2 km in 1 day: vide a "frequent consumer" scheme" in which the number of points "jumps" after the consumer reaches a certain point level.

$M(t)$ - be the maximal distance the turtle can travel in time t and assume it is strictly increasing and continuous

if the max distance that the turtle can travel within t^* is x^* and if it covers the distance x^* in $t < t^*$, then by strict monotonicity of M the turtle can cover a distance larger than x^* in t^*, ↯

if it takes t^* for the turtle to cover the distance x^* and if it travels the distance $x > x^*$ in t^*, then by continuity of M the turtle will already be beyond the distance x^* at some $t < t^*$, a contradiction

Preferences

Def.
A monotone preference relation $\succsim$ on $X \in R_+^n$ is **homothetic** iff $x \sim y \Rightarrow \alpha x \sim \alpha y \quad \forall \alpha \geq 0$

Def. The pref. relation $\succsim$ on $X = (-\infty, +\infty) \times R_+^{n-1}$ is quasilinear wrt. commodity 1 (numeraire) iff:

(i) $x \sim y \Rightarrow (x + \alpha e_1) \sim (y + \alpha e_1)$ $\quad$ u

$e_1 = (1, 0, \dots, 0) \quad \forall \alpha \in R$ $\quad$ u

(ii) $x + \alpha e_1 \succ x \quad \forall x \quad \forall \alpha > 0$

Prop A continuous $\succsim$ on $X = R_+^n$ is homothetic iff $u(x)$ is homogeneous of degree one

$[u(\alpha x) = \alpha u(x) \quad \forall \alpha > 0]$

homogeneous function of t degree

$u(\alpha x) = \alpha^t u(x)$

$(\Rightarrow)$: $x \sim (t(x), \dots, t(x))$ so that $u(x) = t(x)$ represents $\succsim$

by homotheticity $\alpha x \sim (\alpha t(x), \dots, \alpha t(x)) \quad u(\alpha x) = \alpha t(x) = \alpha u(x)$

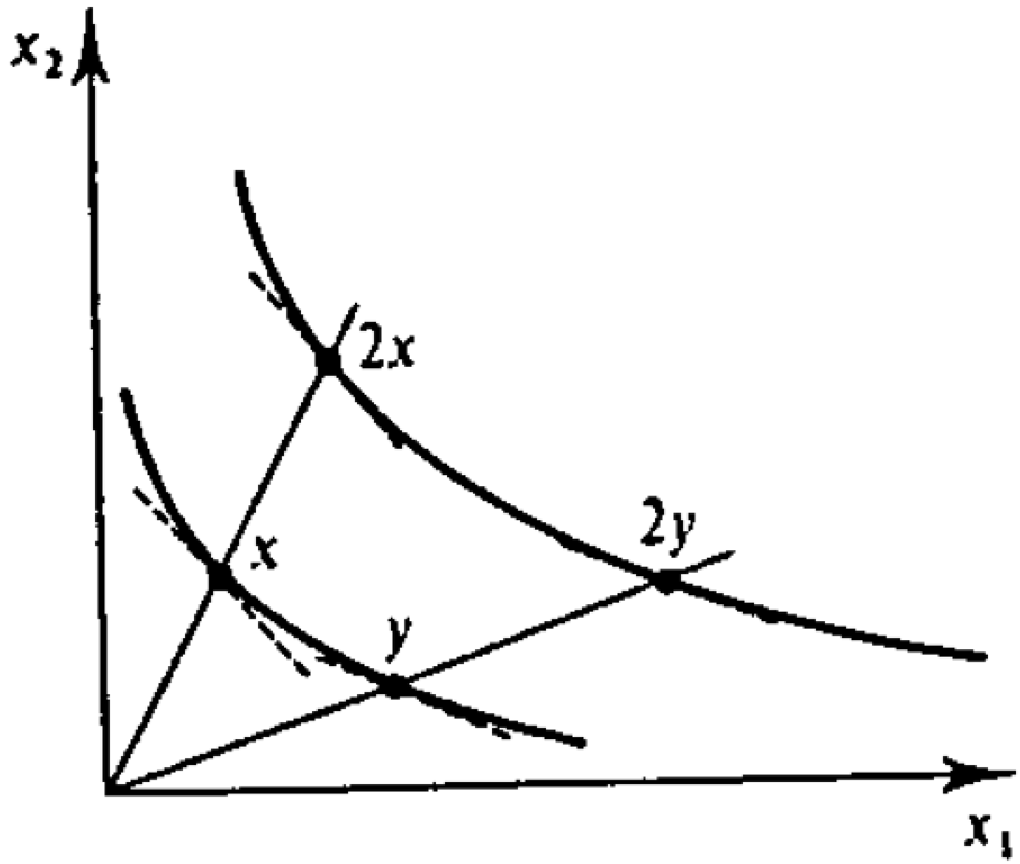
Prop. A continuous p $\succsim$ on $(-\infty, +\infty) \times R_+^{n+1}$ is quasilinear wrt. the first commodity iff

$$u(x) = x_1 + \phi(x_2, \dots, x_n)$$

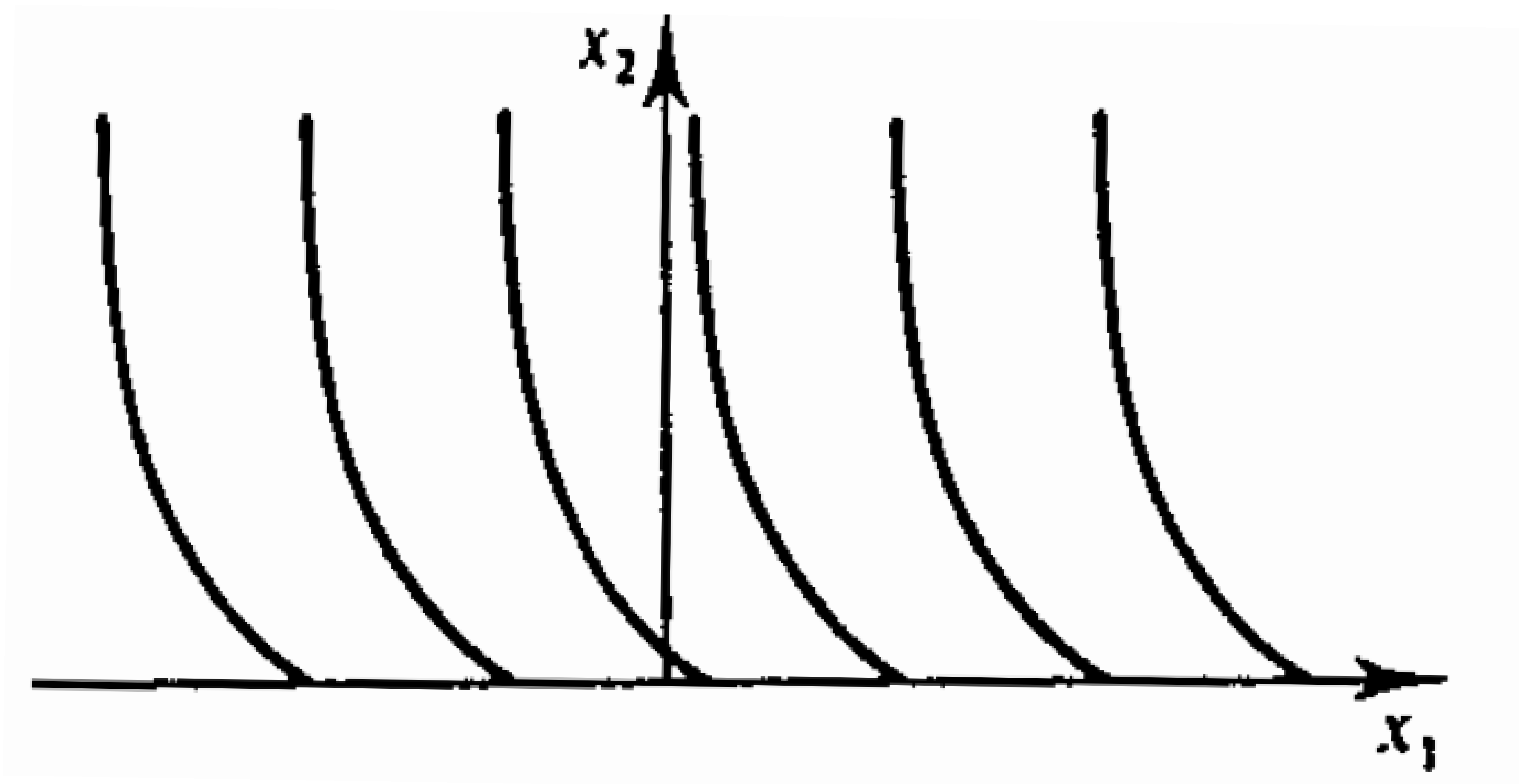
$(\Leftarrow)$ assume $u(x) = x_1 + \phi(x_2, \dots, x_n)$ we want to prove that $\succsim$ is quasilinear

$$x \sim y \Leftrightarrow x_1 + \phi(x_2, \dots, x_n) = y_1 + \phi(y_2, \dots, y_n)$$
$$\Rightarrow x_1 + \alpha + \phi(x_2, \dots, x_n) = y_1 + \alpha + \phi(y_2, \dots, y_n)$$
$$\Leftrightarrow (x + \alpha e_1) \sim (y + \alpha e_1)$$

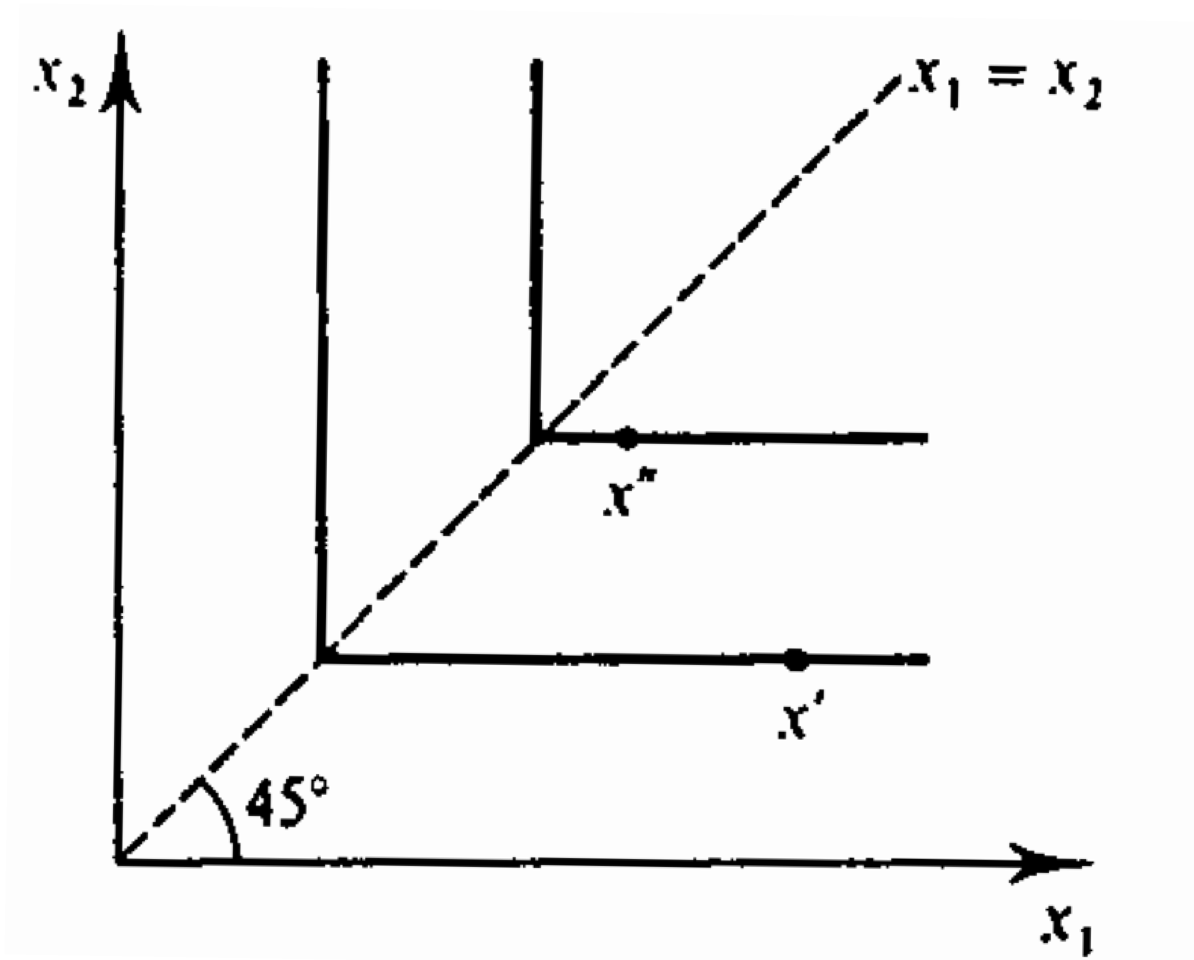
Homothetic preferences



Quasilinear preferences



Leontief preferences



$u(x_1, \cancel{y} x_2) = A \cdot x_1^{\alpha} \cdot x_2^{1-\alpha}$ Cobb-Douglas utility function

$x \sim y \quad A x_1^{\alpha} \cdot x_2^{1-\alpha} = A \cdot y_1^{\alpha} \cdot y_2^{1-\alpha}$

$\beta x \sim \beta y \quad \forall \beta \geq 0$

$$A \cdot (\beta x_1)^{\alpha} \cdot (\beta x_2)^{1-\alpha} = \cancel{A} \cdot \beta \cdot A \, x_1^{\alpha} (\cancel{x_2})^{1-\alpha}$$
$$= \beta A y_1^{\alpha} y_2^{1-\alpha}$$
$$= A (\beta y_1)^{\alpha} (\beta y_2)^{1-\alpha}$$

$$v(x_1, x_2) = \ln u(x_1, x_2)$$
$$= \ln A + \alpha \ln x_1 + (1-\alpha) \ln x_2$$

Some continuous $\succsim$ cannot be represented by a differentiable utility function

Ex. Leontief preferences:

$$x'' \succsim x' \text{ iff } \operatorname{Min}\{x_1'', x_2''\} \geq \operatorname{Min}\{x_1', x_2'\}$$

nondifferiable at $x_1 = x_2$

Lexicographic preferences

assume that $X = R_+^2$

Define $x \succsim y$ iff $x_1 > y_1$ or $(x_1 = y_1$ and $x_2 \geq y_2)$

$\succsim$ is complete, transitive, strictly monotonic

str. convex

not continuous

$x^n = (\frac{1}{n}, 0)$, $y^n = (0, 1)$

$x = \lim_{n\to\infty} x^n = (0,0)$ $y = \lim_{n\to\infty} y^n = (0,1)$

$\forall n: \quad x^n \succ y^n$

but $x \prec y$

Prop. No utility function exists that represents this preference relation

proof for every x_1, we can pick a rational number $r(x_1)$ s.t. $u(x_1,2) > r(x_1) > u(x_1,1)$

~~by~~ since $\succsim$ is lexicographic

$x_1 > x_1' \Rightarrow r(x_1) > u(x_1,1) > u(x_1',2) > r(x_1')$

Hence $r(\cdot)$ is 1:1 mapping from the set of real numbers into the set of rational numbers $\lightning$

$y_2)$

Assumption

$\succsim$ are rational, continuous, ~~st~~ locally nonsatiated str. convex preferences

⇒ we know that $\exists$ u, continuous, quasiconcave

Consumer problem (UMP)

given $p \gg 0$, $w > 0$:

$$\boxed{\max_{x \geq 0} u(x) \quad \text{s.t.} \quad p \cdot x \leq w}$$

Since $B_{p,w} = \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$ is a compact set

utility is continuous

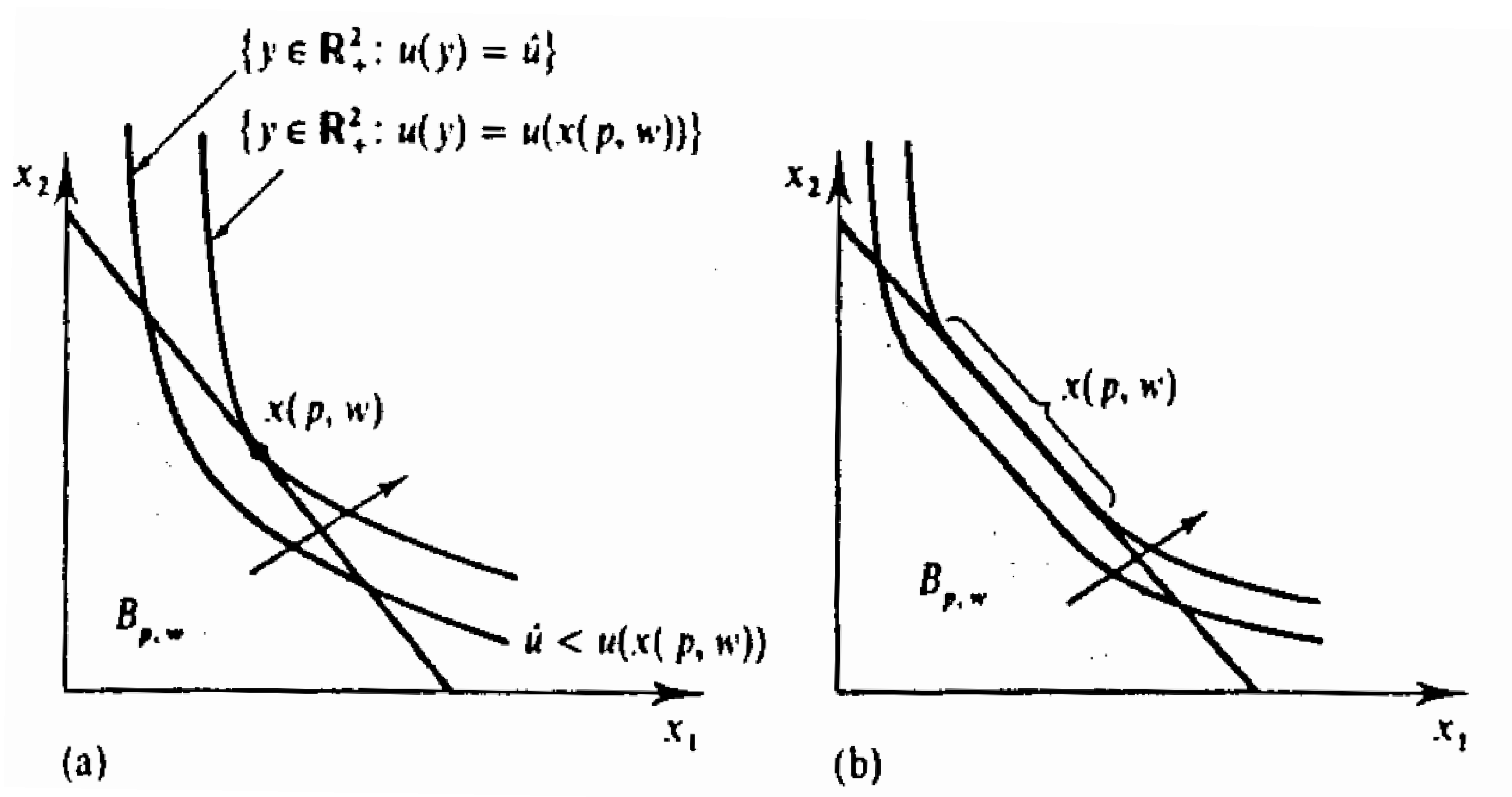
by Weierstrass thm. $\exists$ solution

$x(p,w)$ - Walrasian demand correspondence

$$x(p,w) = \underset{\substack{x \geq 0 \\ p \cdot x \leq w}}{\arg\max}\, u(x)$$

if $x(p,w)$ is single-valued then we call it Walrasian demand function

The utility maximization problem (UMP)



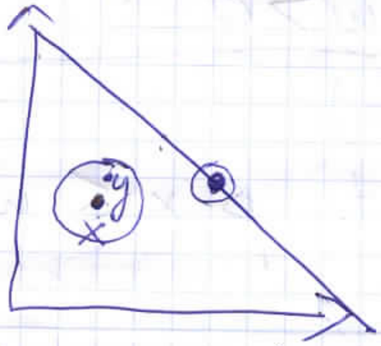
① $x(p,w)$ is <u>homogeneous of degree zero in (p,w)</u>

$$x(\alpha p, \alpha w) = x(p,w) \quad \forall p,w \quad \forall \alpha > 0$$

② <u>Walras law</u>

$$p \cdot x = w \quad \forall x \in x(p,w)$$

follows from local nonsatiation



③ (i) if $\succsim$ is convex (u is quasiconcave)

then $x(p,w)$ is a convex set

(ii) if $\succsim$ is str. convex (u is str. quasiconcave)

then $x(p,w)$ is single-valued

$u(\cdot)$ is continuously differentiable

if $x^* \in x(p,w)$ is a solution to the UMP

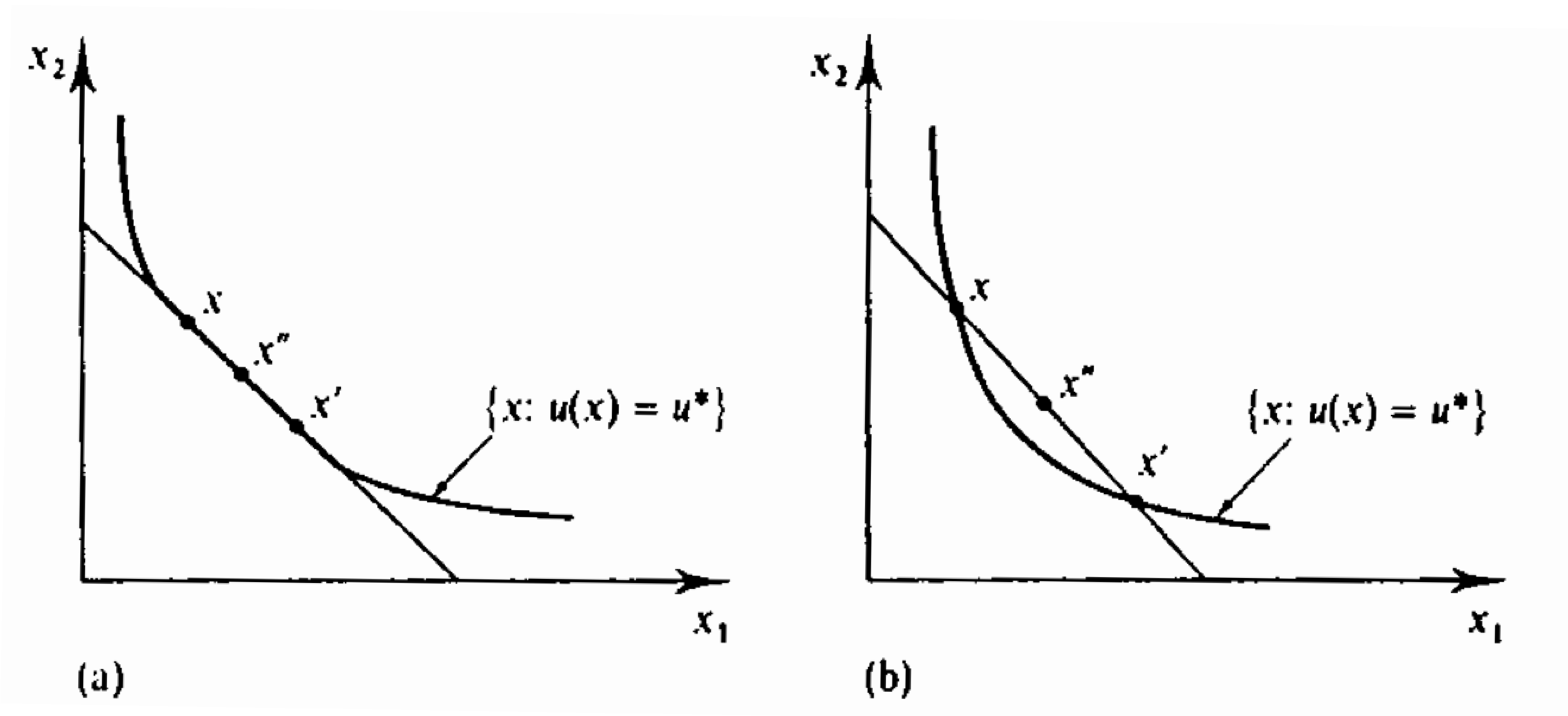
then $\exists \lambda \geq 0$ s.t.:

$\forall i = 1, \ldots, n$ $\quad \dfrac{\partial u(x^*)}{\partial x_i} \leq \lambda p_i$ with equality if $x_i^* > 0$

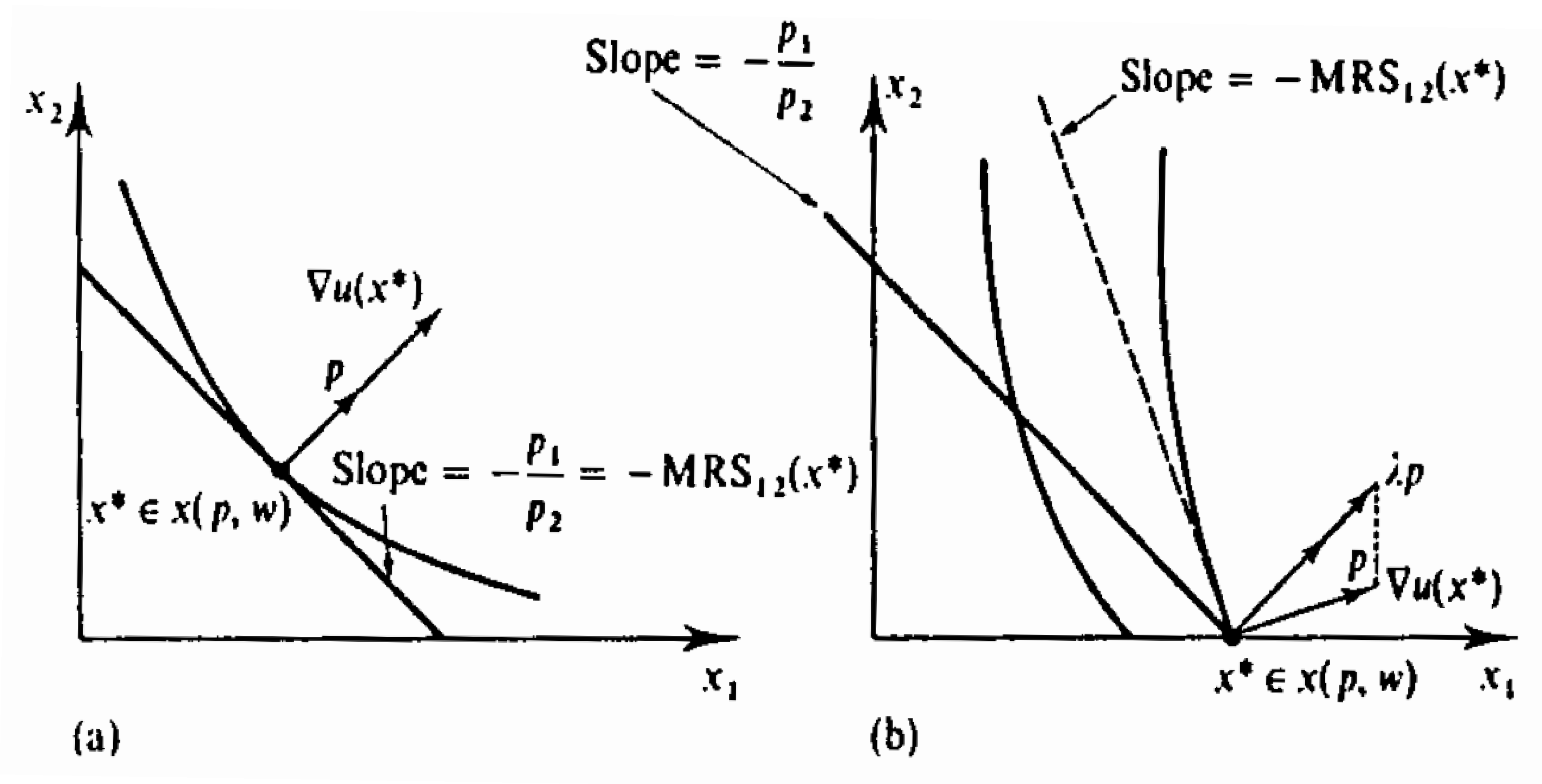
$$\nabla u(x^*) \leq \lambda \cdot p$$

$$x^* [\nabla u(x^*) - \lambda p] = 0$$

Convexity of preferences implies convexity of *x(p,w)*



Interior and boundary solution



if $\nabla u(x^*) >> 0$ ~~th~~ this is equivalent to

$$\forall i,j \quad \boxed{\frac{\partial u(x^*)/\partial x_i}{\partial u(x^*)/\partial x_j} = \frac{p_i}{p_j}}$$

let's take $x(p,w) >> 0$ to be a differentiable

$v(p,w) = u(x(p,w))$ indirect utility function

the change ~~in~~ optimal utility value resulting from Δw

$$\nabla u(x(p,w)) \cdot D_w x(p,w) \underset{\text{by KT conditions}}{=} \lambda \cdot \underbrace{p \cdot D_w x(p,w)}_{=1 \text{ by walras law}} = \lambda$$

<u>Prop</u> $v(p,w)$ is:

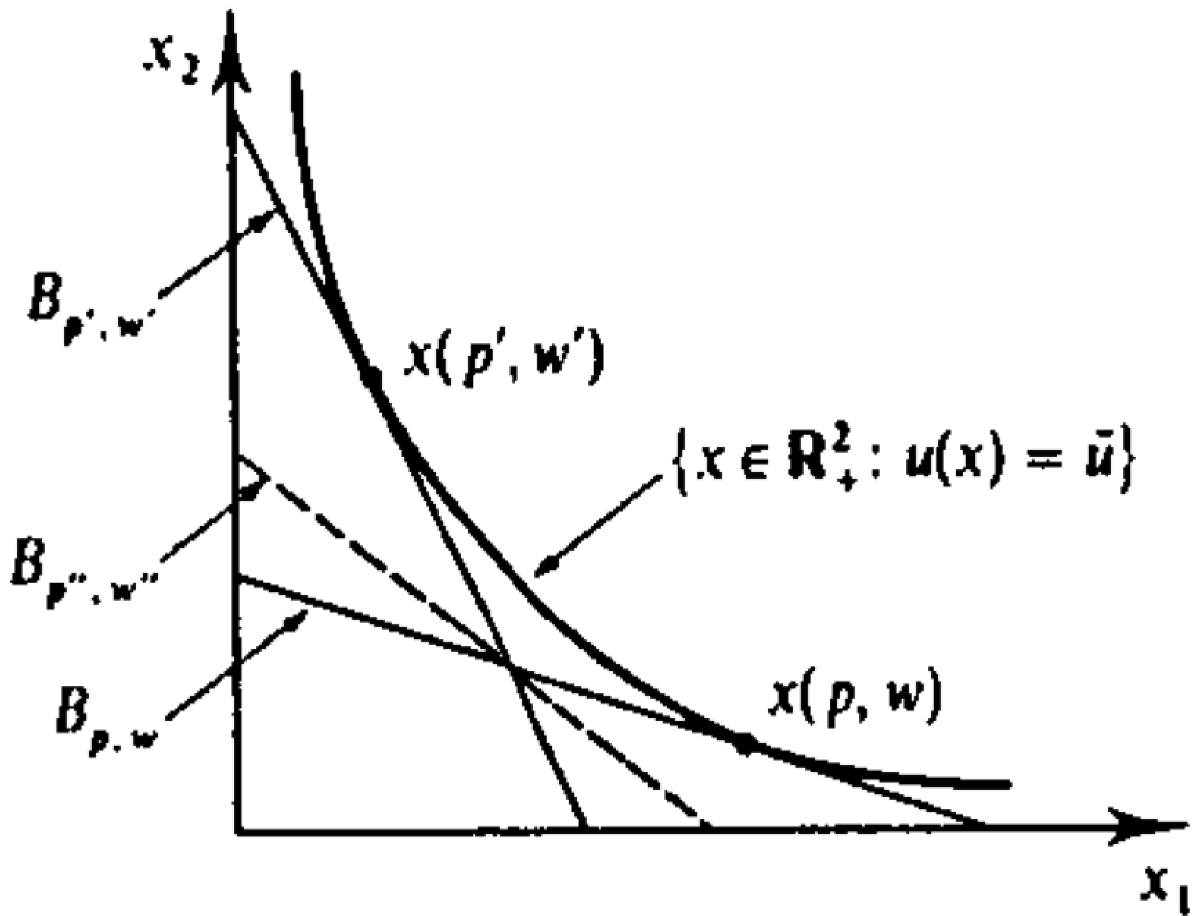
(i) homogeneous of degree zero

(ii) strictly increasing in w and nonincreasing in $p_i \ \forall i$

(iii) quasiconvex: the set $\{(p,w) : v(p,w) \leq \bar{v}\}$ is convex for any $\bar{v}$

(iv) continuous in p and w

The indirect utility function *v(p,w)* is quasiconvex



The Expenditure Minimization Problem (EMP)

for $p \gg 0$, and $u > u(0)$

$$\boxed{\begin{aligned}&\min_{x \geq 0} \; p \cdot x \\ &\text{s.t. } u(x) \geq u\end{aligned}}$$

<u>Prop.</u> $p \gg 0$ we have:

(i) if x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. The minimized expenditure level in the EMP is exactly w.

(ii) if x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. The maximized utility level in the UMP is exactly u.

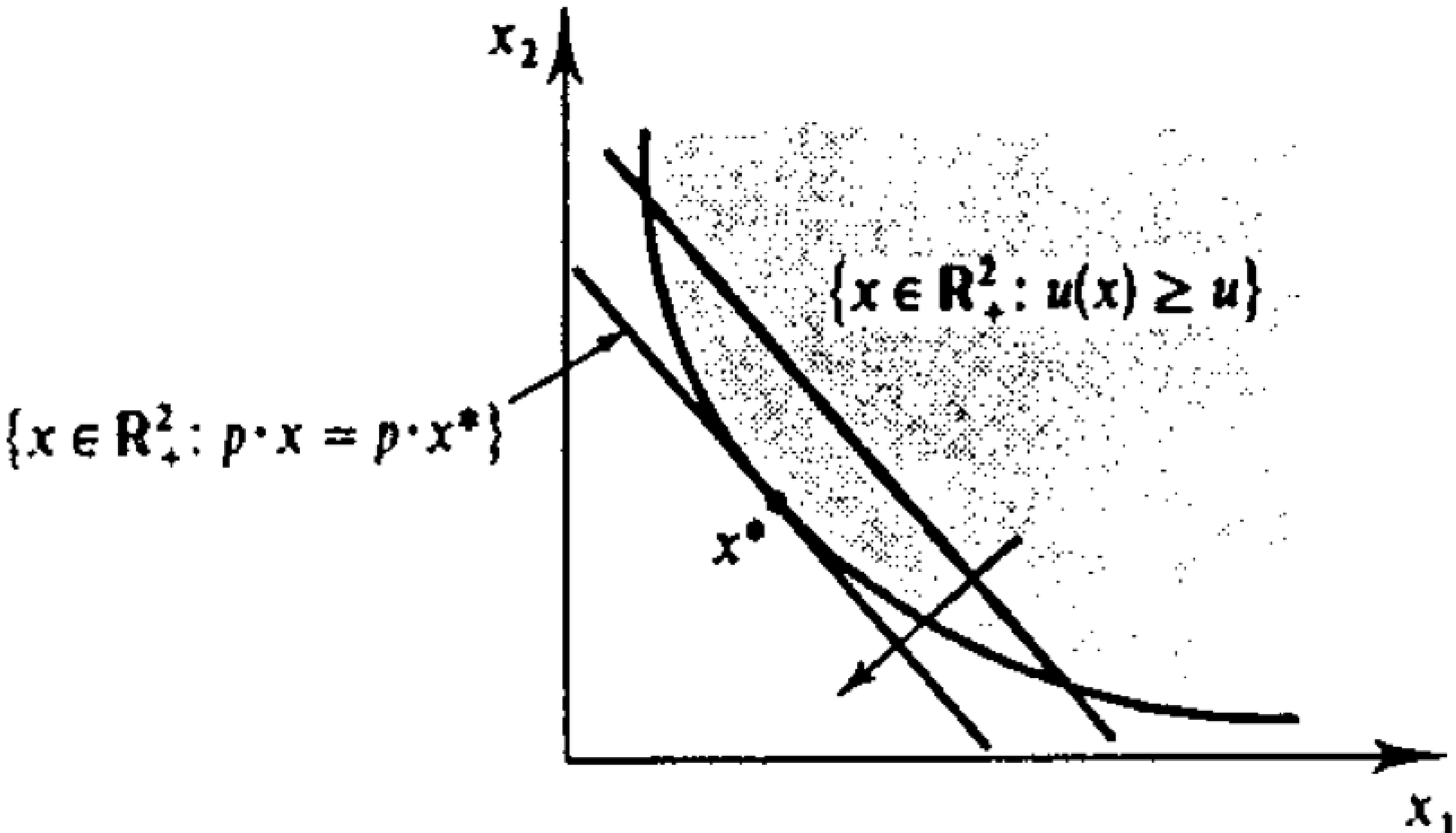
---

the solution to the EMP: $h(p,u)$

<u>Hicksian demand function</u>

optimal expenditure level $e(p,u) = p \cdot h(p,u)$

The expenditure minimization problem (EMP)



UMP	EMP
$x(p,w)$	$h(p,u)$
$v(p,w)$	$e(p,u)$

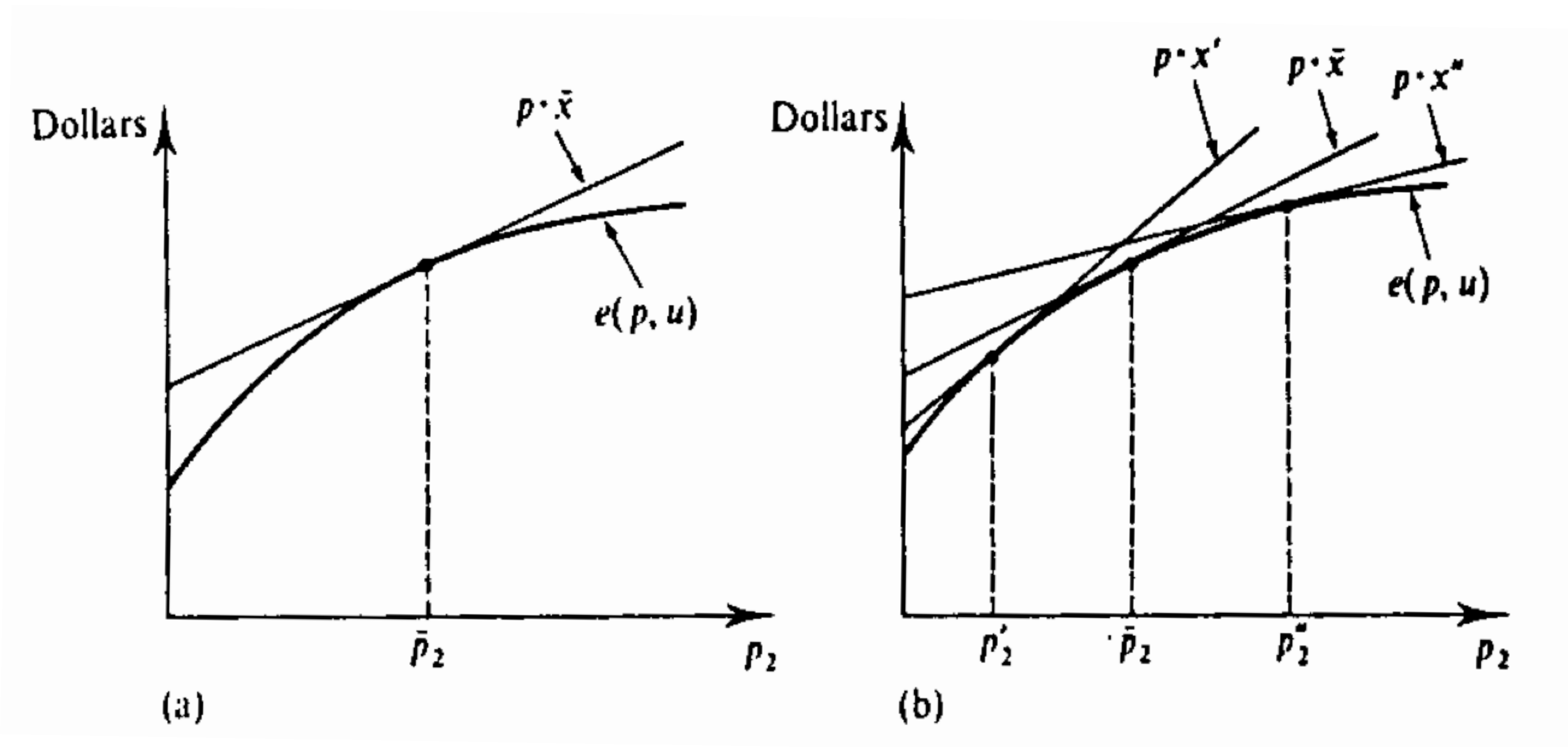
$$\boxed{\begin{aligned} h(p,u) &= x(p, e(p,u)) \\ x(p,w) &= h(p, v(p,w)) \end{aligned}}$$

Prop $e(p,u)$ is
(i) homogeneous of degree one in p
(ii) str. increasing in u and nondecreasing in p_i $\forall i$
(iii) concave in p
(iv) continuous in p and u

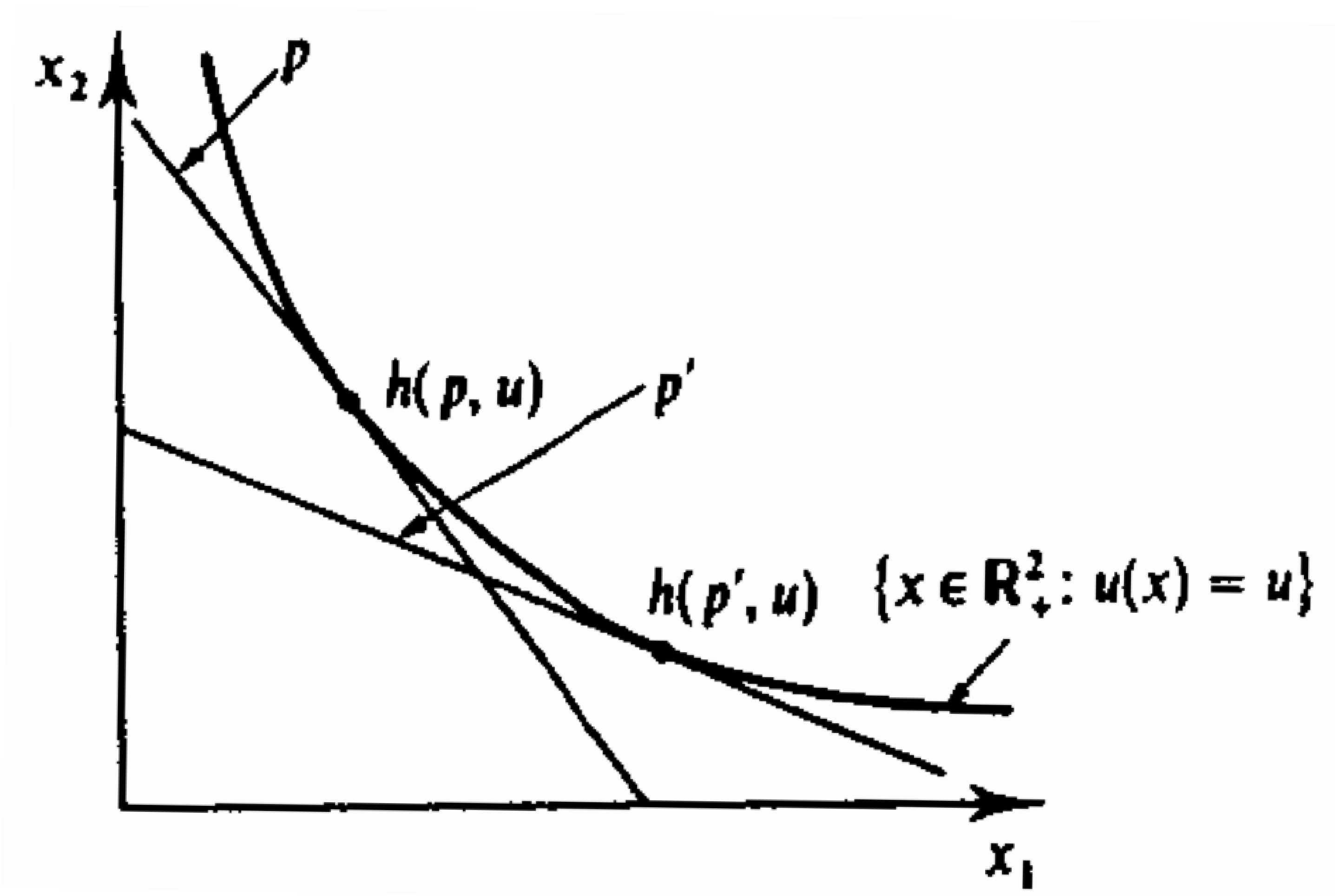
Prop for any $p \gg 0$, $h(p,u)$ has the following properties:
(i) homogeneity of degree zero in p
(ii) no excess utility: for any $x \in h(p,u)$: $u(x) = u$
(iii) if $\succsim$ is convex, then $h(p,u)$ is a convex set.

if $\succsim$ is str. convex, the $h(p,u)$ is a unique element.

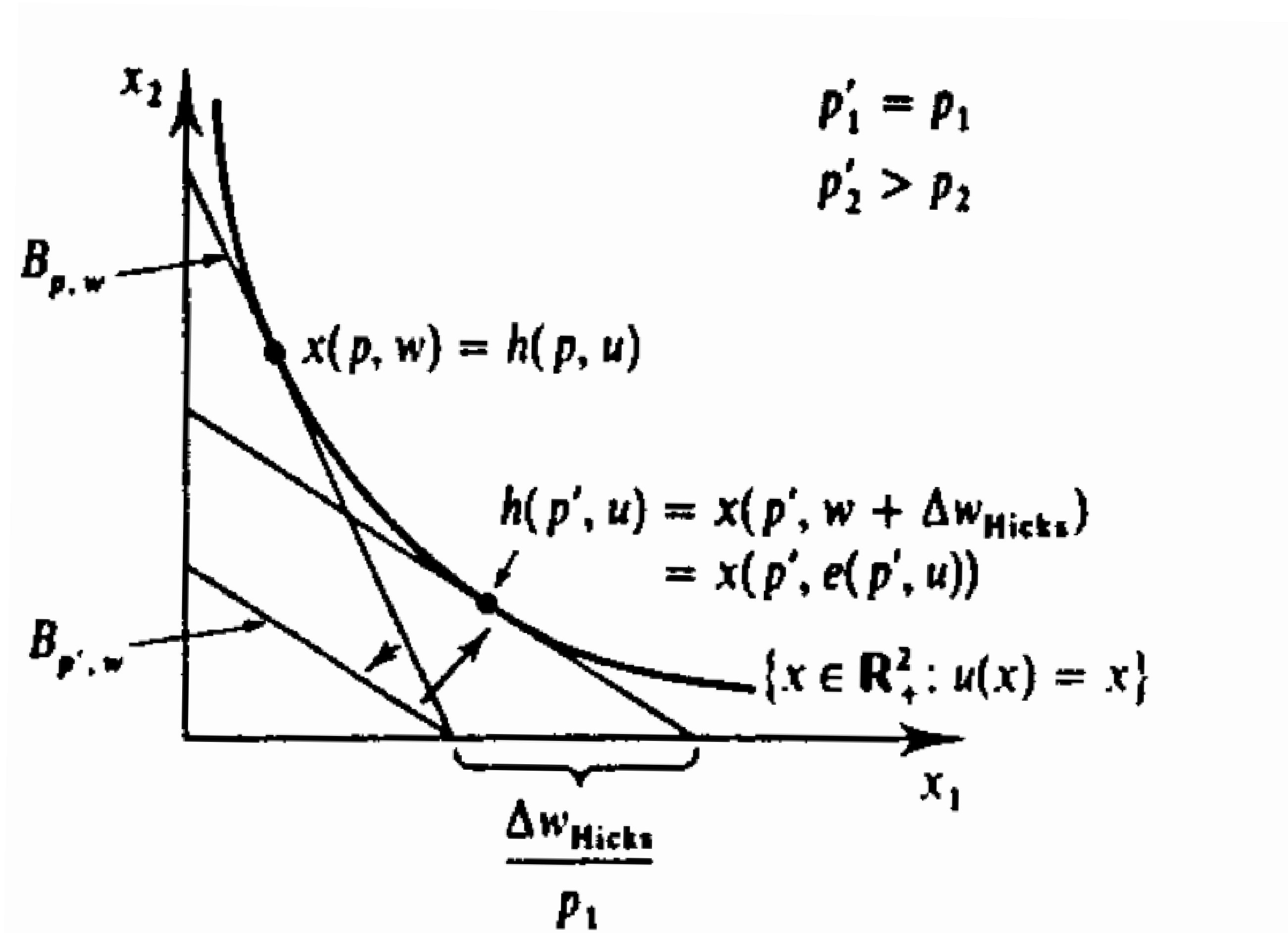
The expenditure function *e(p,w)* is concave in *p*



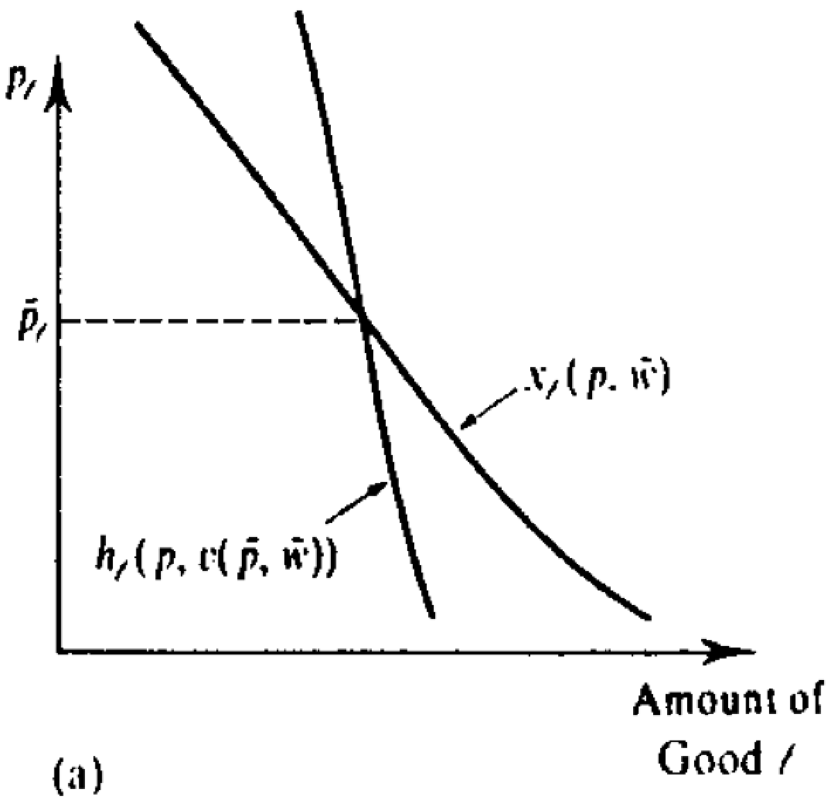
The Hicksian (or compensated) demand function

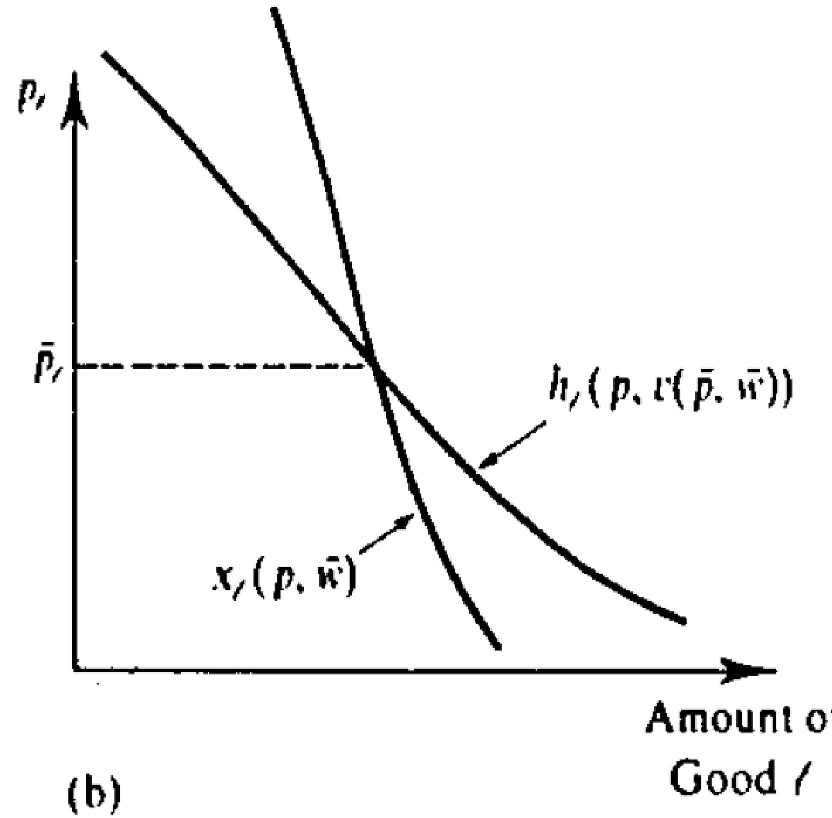


Hicksian wealth compensation



The Walrasian and Hicksian demand functions for good *i* (a) normal good (b) inferior good





Prop. $p >> 0$, then $\forall p', p''$

$$\boxed{(p''-p')[h(p'',u)-h(p',u)] \leq 0}$$

compensated low of demand

proof $h(p,u)$ is optimal in the EMP

$$p'' \cdot h(p'',u) \leq p'' \cdot h(p',u)$$

$$p' \cdot h(p',u) \leq p' \cdot h(p'',u)$$

$$h(p'',u) \cdot (p''-p') \leq h(p',u) \cdot (p''-p')$$

QED

Prop. Shephard's lemma

$\forall p, u$ the following holds:

$$\boxed{h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i} \quad \forall i}$$

Envelope theorem

$$\phi(\alpha) = \min_x f(x,\alpha)$$
$$\text{s.t. } g(x,\alpha) = 0$$

at any $\bar{\alpha}$.

$$\nabla_\alpha \phi(\bar{\alpha}) = \nabla_\alpha f(x^*(\bar{\alpha}), \bar{\alpha}) - \lambda \nabla_\alpha g(x^*(\bar{\alpha}), \bar{\alpha})$$

$$e(p,u) = \min_{x \geq 0} p \cdot x \quad \text{s.t. } u(x) \geq u$$

$$\nabla_p e(p,u) = h(p,u)$$

$$\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u) \quad \forall i$$

Prop for all (p,w) and $u = v(p,w)$

$$\frac{\partial h_i(p,u)}{\partial p_j} = \frac{\partial x_i(p,w)}{\partial p_j} + \frac{\partial x_i(p,w)}{\partial w} \cdot x_j(p,w) \quad \forall i,j$$

$\boxed{\text{Slutsky equation}}$

proof
consider a consumer facing $(\bar{p},\bar{w})$ and attaining utility leve $\bar{u}$. Then we know that $\bar{w} = e(\bar{p},\bar{u})$

$\forall (p,u)$: $h_i(p,u) = x_i(p, e(p,u))$

Differentiate it wrt. p_j and evaluate it at $(\bar{p},\bar{u})$

$$\frac{\partial h_i(\bar{p},\bar{u})}{\partial p_j} = \frac{\partial x_i(\bar{p}, e(\bar{p},\bar{u}))}{\partial p_j} + \frac{\partial x_i(\bar{p}, e(\bar{p},\bar{u}))}{\partial w} \cdot \frac{\partial e(\bar{p},\bar{u})}{\partial p_j}$$

$\bar{w} = e(\bar{p},\bar{u})$ $\quad h(\bar{p},\bar{u})$

$h_j(\bar{p},\bar{u}) = x_j(\bar{p}, e(\bar{p},\bar{u})) = x_j(\bar{p},\bar{w})$

$$\boxed{\frac{\partial h_i(\bar{p},\bar{u})}{\partial p_j}} = \boxed{\frac{\partial x_i(\bar{p},\bar{u})}{\partial p_j} + \frac{\partial x_i(\bar{p},\bar{w})}{\partial w} x_j(\bar{p},\bar{w})}$$

$$\{S_{ij}\}_{i,j=1,\dots,n}$$

<u>Slutsky matrix</u>

<u>Prop.</u> Suppose that v is differentiable at $(\bar{p},\bar{w}) \gg 0$. Then

$$\boxed{x_i(\bar{p},\bar{w}) = -\frac{\partial v(\bar{p},\bar{w})/\partial p_i}{\partial v(\bar{p},\bar{u})/\partial w} \quad \forall i}$$

<u>Roy's identity</u>

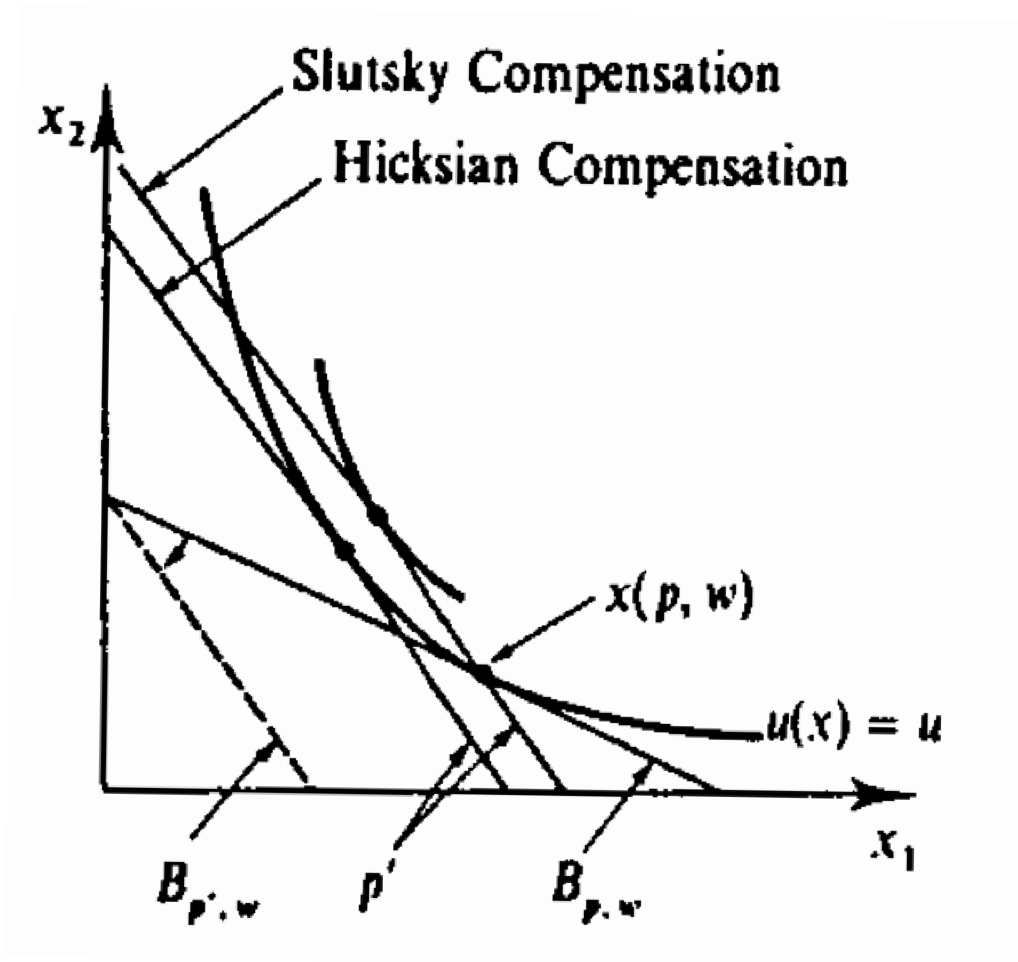
<u>Proof</u>

By envelope <u>thm.</u>

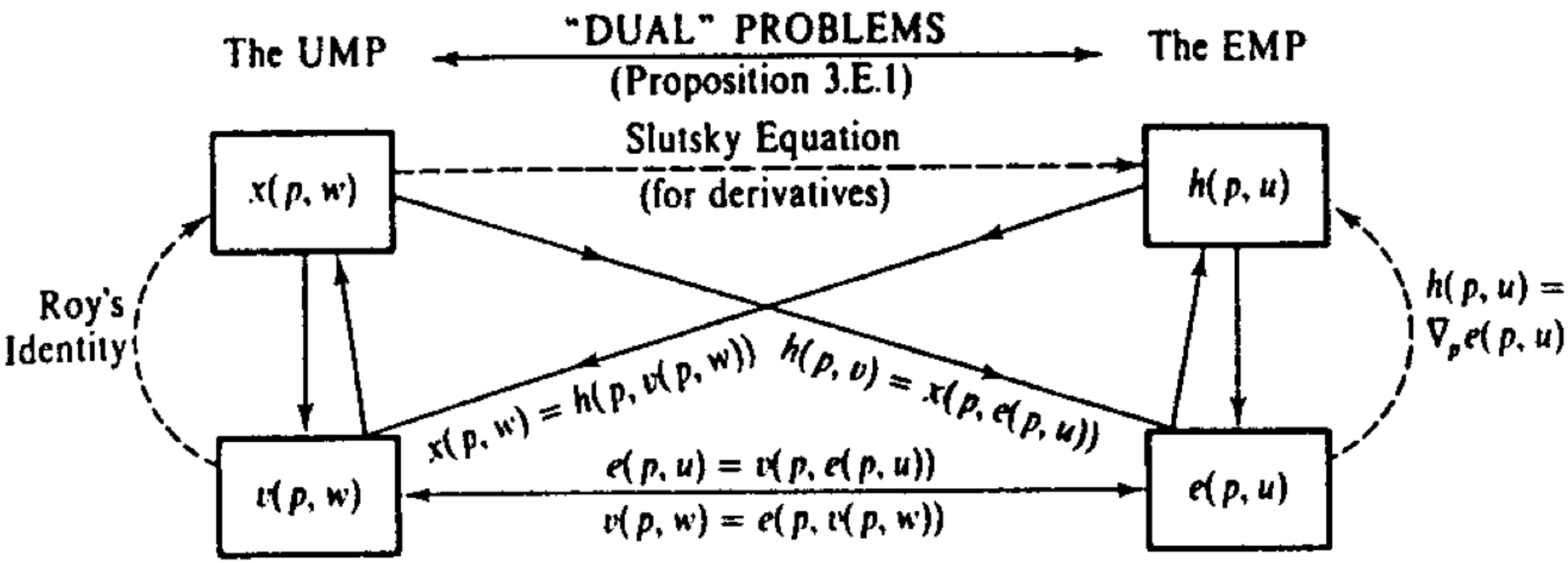
$$\max_{x \geq 0} u(x)$$
$$p \cdot x \leq w$$

$$\frac{\partial v(\bar{p},\bar{u})}{\partial p_i} = -\underbrace{\lambda}_{= \frac{\partial v(p,w)}{\partial w}} x_i(\bar{p},\bar{w})$$

Hicksian vs Slutsky wealth compensation



Duality relationships between the UMP and the EMP



Topics not covered in the lecture

- How to evaluate a welfare change due to a price change?
  - Utility value is meaningless (ordinal utility is unique up a strictly increasing transformation)
  - We should evaluate it in money terms – a money metric indirect utility function

$$e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w)), \text{ where } \bar{p} \text{ is a reference price vector}$$

$$\text{if } \bar{p} = p^0, \text{ then it is Equivalent Variation}$$

$$\text{if } \bar{p} = p^1, \text{ then it is Compensating Variation}$$

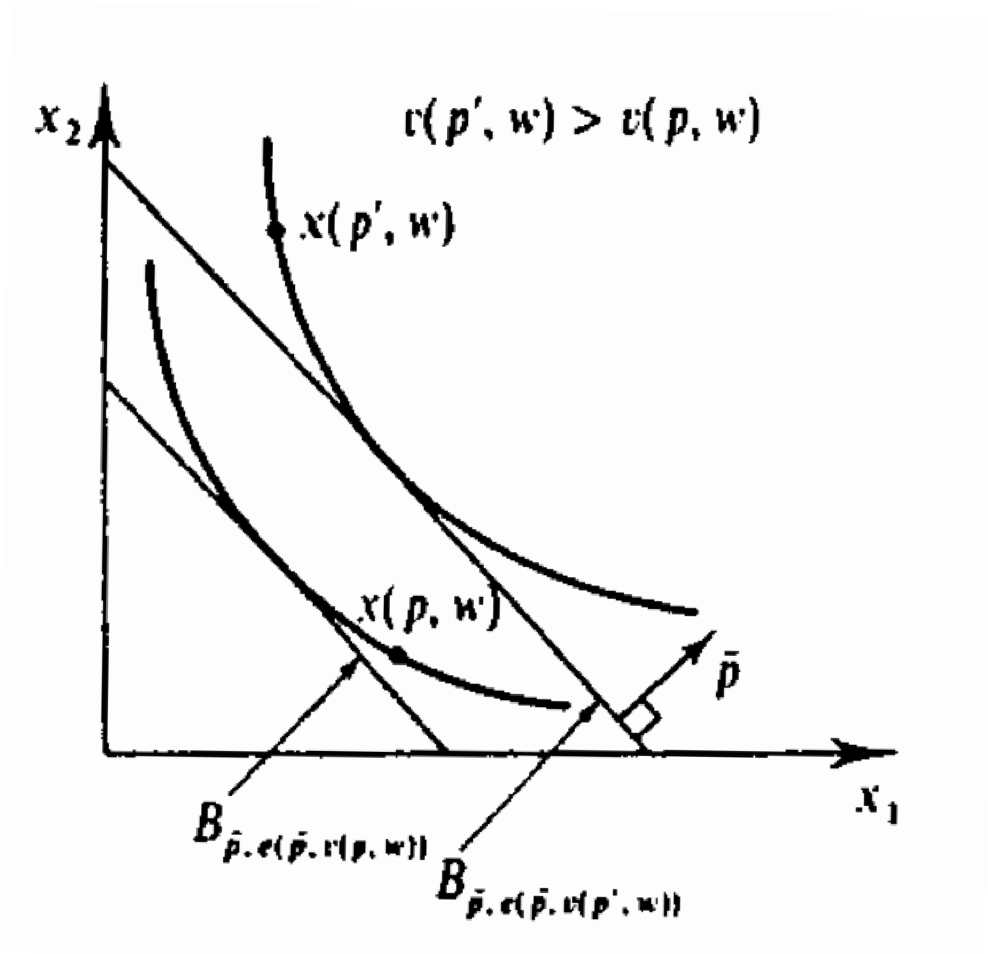
$$\text{Let's define } u^0 = v(p^0, w) \text{ and } u^1 = v(p^1, w)$$

Then:

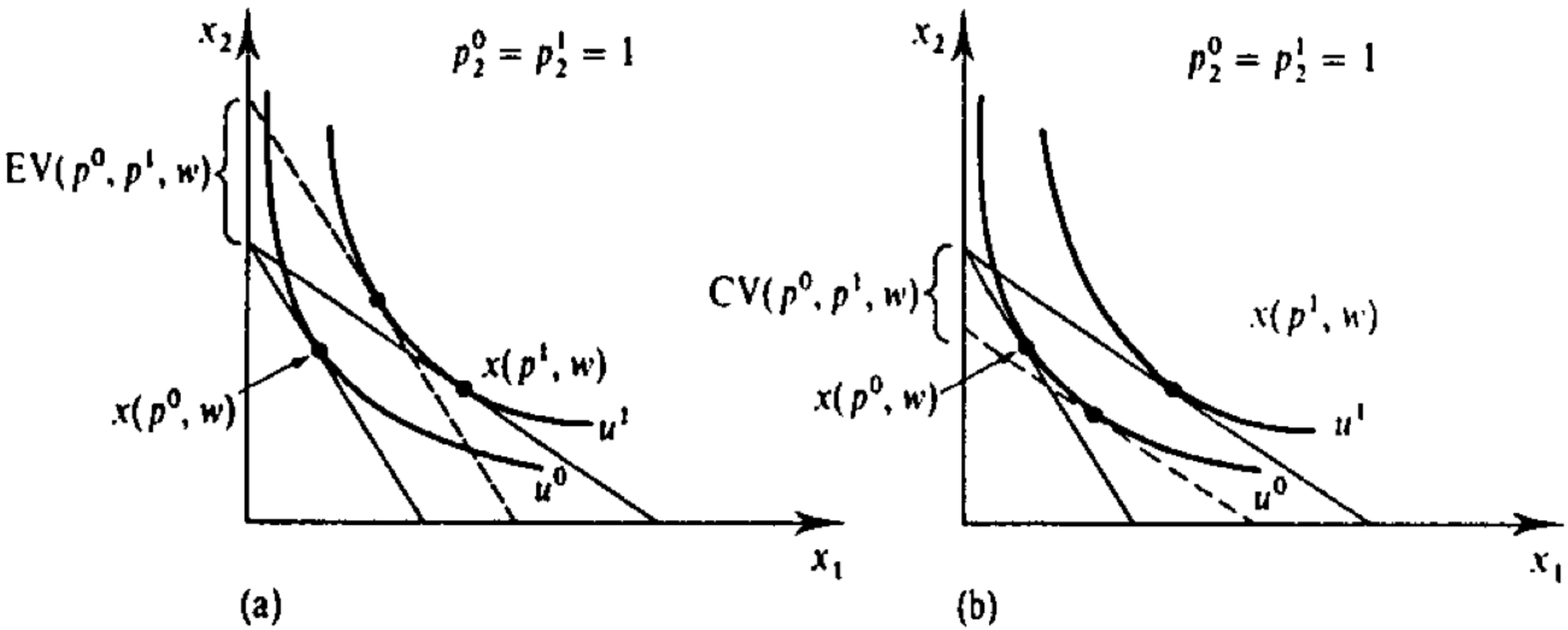
$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w$$

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0)$$

A money metric indirect utility function



The equivalent and compensating variation measures of welfare change.



Equivalent and Compensating Variation

Using Shephards lemma:

$$\frac{\partial e(p,u)}{\partial p_1} = h_1(p,u)$$

We obtain:

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \overline{p}_{-1}, u^1)\,dp_1$$

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \overline{p}_{-1}, u^0)\,dp_1$$

where $\overline{p}_{-1} = (\overline{p}_2, \overline{p}_3, \ldots, \overline{p}_n)$

The equivalent variation and the compensating variation