Producer Theory

Advanced Microeconomics

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1. Production sets
2. Profit maximization
3. Cost minimization
4. Aggregation
5. Efficient production

Production sets

Exogenously given technology applies over L commodities (both inputs and outputs)

Definition (production plan)

A vector $y = (y_1; :::; y_l) \in R^L$ where an output has $y_k > 0$ and an input has $y_k < 0$.

Definition (production set)

Set $Y \subseteq R^L$ of feasible production plans; generally assumed to be non-empty and closed.

Properties of production sets I

1. Y is closed (it contains its boundaries).
   Important property for the definition of production function (sup is a max).
2. $\mathbf{0} \in Y$ (shutdown)
   Uncontroversial property in the long run, not necessarily in the short run (inputs used with no outputs).
3. $y \in Y$ and $y' \leqslant y$ imply $y' \in Y$ (free disposal)
   Given a production plan if either one increases the quantity of inputs or reduces the quantity of output the new production plan is still feasible.
4. Additivity (free entry): if $y, y' \in Y$ then $y + y' \in Y$
   This implies that $ky \in Y$ for any positive integer k.

Properties of production sets II

Definition (nonincreasing returns to scale)

$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \in [0; 1]$.

Implies shutdown

Definition (nondecreasing returns to scale)

$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \geqslant 1$.

Along with shutdown, implies $\pi(p) = 0$ or $\pi(p) = +\infty$ for all p

Definition (constant returns to scale)

$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \geqslant 0$; i.e., nonincreasing *and* nondecreasing returns to scale.

Properties of production sets III

Definition (convex production set)

$y, y' \in Y$ imply $ty + (1-t)y' \in Y$ for all $t \in [0;1]$.

Vaguely "nonincreasing returns to specialization"
If $\mathbf{0} \in Y$, then convexity implies nonincreasing returns to scale

Strictly convex iff for $t \in (0,1)$, the convex combination is in the interior of Y

Characterizing Y: Transformation function I

Definition (transformation function)

Any function $F : R^L \rightarrow R$ with

1. $F(y) \leqslant 0 \Leftrightarrow y \in Y$; and
2. $F(y) = 0 \Leftrightarrow y$ is a boundary point of Y.

Can be interpreted as the amount of technical progress required to make y feasible

The set $\{y : F(y) = 0\}$ is the **production possibilities frontier** (a.k.a. **transformation frontier**)

Characterizing Y: Transformation function II

When the transformation function is differentiable, we can define the **marginal rate of transformation** of good l for good k:

Definition (marginal rate of transformation)

$$MRT_{l,k}(y) \equiv \frac{\frac{\partial F(y)}{\partial y_l}}{\frac{\partial F(y)}{\partial y_k}},$$

defined for points where $F(y) = 0$ and $\frac{\partial F(y)}{\partial y_k} \neq 0$.

Measures the extra amount of good k that can be obtained per unit reduction of good l

Equals the slope of the PPF

The single-output firm: notation

Notation will be a bit different for single-output firms:

$p \in R_+$: Price of output
$w \in R_+^{L-1}$: Prices of inputs
$q \in R_+$: Output produced
$z \in R_+^{L-1}$: Inputs used

Thus $p_{old} = (p, w)$ and $y_{old} = (q, -z)$

Characterizing Y: Production function I

Definition (production function)

For a firm with only a single output q (and inputs $-z$), defined as $f(z) \equiv \max q$ such that $(q, -z) \in Y$.

$Y = \{(q, -z) : q \leqslant f(z)\}$, assuming free disposal

Characterizing Y: Production function II

When the production function is differentiable, we can define the marginal rate of technological substitution of good l for good k:

Definition (marginal rate of technological substitution)

$$MRTS_{l,k}(z) \equiv \frac{\frac{\partial f(z)}{\partial z_l}}{\frac{\partial f(z)}{\partial z_k}},$$

defined for points where $\frac{\partial f(z)}{\partial z_k} \neq 0$.

Measures how much of input k must be used in place of one unit of input l to maintain the same level of output

The Profit Maximization Problem (PMP)

The firm's optimal production decisions are given by **supply correspondence** $y : R^L \Rightarrow R^L$

$$y(p) \equiv \arg\max_{y \in Y} p \cdot y$$

$$= \{y \in Y : p \cdot y = \pi(p)\}$$

Resulting profits are given by **profit function** $\pi(p) : R^L \rightarrow R \cup \{+\infty\}$

$$\pi(p) \equiv \max_{y \in Y} p \cdot y$$

or equivalently

$$\pi(p) \equiv max_{y:F(y) \leqslant 0} p \cdot y$$

First-order conditions: PMP I

Single-output profit maximization problem

$$\max_{z \in R_+^{L-1}} pf(z) - w \cdot z$$

where $p > 0$ is the output price and $w \in R_+^{L-1}$ are input prices.

Set up the Lagrangian and find Kuhn-Tucker conditions (assume differentiability):

$$\mathcal{L}(z, p, w, \mu) \equiv pf(z) - w \cdot z + \mu \cdot z$$

We get three (new) kinds of conditions...

First-order conditions: PMP II

1. FONCs: $p \bigtriangledown f(z^*) - w + \mu = \mathbf{0}$
2. Complementary slackness: $\mu_i \geqslant 0$ for all i
3. Non-negativity: $\mu_i \geqslant 0$ for all i
4. Original constraints: $z_i^* \geqslant 0$ for all i

First three can be summarized as: for all i,

$$p\frac{\partial f(z^*)}{\partial z_i} \leqslant w_i$$

with equality if $z_i^* > 0$
Hence, in internal equilibrium $MRTS_{l,k} = \frac{w_l}{w_k}$

First-order conditions: PMP III

If we use the alternative PMP formulation (using the transformation function) we obtain FOCs (for internal equilibrium):

$$p = \lambda \bigtriangledown F(y^*)$$

hence

$$\frac{p_l}{p_k} = MRT_{l,k}(y^*)$$

If Y is convex the necessary conditions are also sufficient

Properties of $\pi(\cdot)$ I

$\pi(\cdot)$ is **homogeneous of degree one**;
That is, if you scale all (input and output) prices up or down the same amount, you also scale profits by that amount

Proof

$$\pi(\lambda p) \equiv \max_{y \in Y} \lambda p \cdot y = \lambda \max_{y \in Y} p \cdot y = \lambda \pi(p).$$

$\square$

Properties of $\pi(\cdot)$ II

$\pi(\cdot)$ is **convex**.

Proof

Fix any p_1, p_2 and let $p_t \equiv tp_1 + (1-t)p_2$ for $t \in [0; 1]$. Then for any $y \in Y$,

$$p_t \cdot y = tp_1 \cdot y + (1-t)p_2 \cdot y$$
$$\leqslant t\pi(p_1) + (1-t)\pi(p_2).$$

Since this is true for all p_t, it holds for $\max_{y \in Y} p_t \cdot y = \pi(p_t)$:

$$\pi(p_t) \leqslant t\pi(p_1) + (1-t)\pi(p_2).$$

$\square$

Properties of $\pi(\cdot)$ III

Hotelling's lemma

$\bigtriangledown\pi(p) = y(p)$ wherever $\pi(\cdot)$ is differentiable.

Implications:

- Thus if $\pi(\cdot)$ is differentiable at p, $y(p)$ is a singleton

Properties of $y(\cdot)$ I

If Y is closed and convex, then

1. $Y = \{y \in R^L : p \cdot y \leqslant \pi(p) \text{ for all } p \gg 0\}$
2. $y(p)$ is convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if non-empty).

Properties of $y(\cdot)$ II

$y(\cdot)$ is homogeneous of degree 0
That is, a firm makes the same production choice if all (input and output) prices are scaled up or down the same amount

Proof

$$y(\lambda p) \equiv \{y \in Y : \lambda p \cdot y = \pi(\lambda p)\} = \{y \in Y : \lambda p \cdot y = \lambda \pi(p)\}$$
$$= \{y \in Y : p \cdot y = \pi(p)\} = y(p).$$

$\square$

Substitution matrix

Definition (substitution matrix)

The Jacobian of the optimal supply function,

$$Dy(p) \equiv \left[\frac{\partial y_i(p)}{\partial p_j}\right]_{i,j} \equiv \begin{bmatrix} \frac{\partial y_1(p)}{\partial p_1} & \cdots & \frac{\partial y_1(p)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n(p)}{\partial p_1} & \cdots & \frac{\partial y_n(p)}{\partial p_n} \end{bmatrix}.$$

Substitution matrix – properties

- By Hotelling's Lemma, $Dy(p) = D^2\pi(p)$, hence the substitution matrix is symmetric
- $Dy(\tilde{p})\tilde{p} = 0$ follows from homogeneity of $y(\cdot)$
- Convexity of $\pi(\cdot)$ implies positive semidefiniteness
    - Law of Supply: Supply curves must be upward sloping or for any $p, p', y \in y(p), y' \in y(p')$

$$(p - p')(y - y') \geqslant 0$$

Dividing up the problem

We separate the profit maximization problem into two parts:

1. Find a cost-minimizing way to produce a given output level q
   - Cost function
   $$c(q, w) \equiv \min_{z:f(z)\geqslant q} w \cdot z$$
   - Conditional factor demand correspondence
   $$Z^*(q, w) \equiv \arg \min_{z:f(z)\geqslant q} w \cdot z$$
   $$= \{z : f(z) \geqslant q \wedge w \cdot z = c(q, w)\}$$
2. Find an output level that maximizes difference between revenue and cost
   $$\max_{q\geqslant 0} pq - c(q, w)$$

Properties of $c(\cdot)$

- $c(\cdot)$ is homogeneous of degree one in w and increasing in q
- $c(\cdot)$ is concave function of w
- If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q (i.e. marginal costs are increasing in q)
- If $Z^*(\cdot)$ is single valued, then $c(\cdot)$ is differentiable with respect to w and $\bigtriangledown_w c(q, w) = Z^*(q, w)$ (Shephard's Lemma);
- If $Z^*(q, \cdot)$ is differentiable in w, then the matrix $D_w Z^*(q, w) = D^2_w c(q, w)$ is symmetric and negative semidefinite, and $D_w Z^*(q, w)w = 0$

The cost function is particularly useful when the production set is the constant returns type. In this case Hotelling's lemma inapplicable, but Shepard's lemma may be still useful.

First-order conditions: CMP

Single-output cost minimization problem

$$\min_{z \in R_+^m} w \cdot z : f(z) \geqslant q.$$

$$\mathcal{L}(z, q, w, \lambda, \mu) \equiv -w \cdot z + \lambda(f(z) - q) + \mu \cdot z$$

Applying Kuhn-Tucker here gives

$$\lambda \frac{\partial f(z^*)}{\partial z_i} \leqslant w_i$$

with equality if $z_i^* > 0$

First-order conditions: Optimal Output Problem

Optimal output problem

$$\max_{q \geqslant 0} pq - c(q, w).$$

$$\mathcal{L}(q, p, w, \mu) \equiv pq - c(q, w) + \mu q$$

Applying Kuhn-Tucker here gives

$$p \leqslant \frac{\partial c(q^*, w)}{\partial q}$$

with equality if $q^* > 0$

Comparing the problems' Kuhn-Tucker conditions

- Profit Maximization Problem: $p\frac{\partial f(z^*)}{\partial z_i} \leqslant w_i$ with equality if $z_i^* > 0$
- Cost Minimization Problem: $\lambda\frac{\partial f(z^*)}{\partial z_i} \leqslant w_i$ with equality if $z_i^* > 0$
- Optimal Output Problem: $p \leqslant \frac{\partial c(q^*,w)}{\partial q}$ with equality if $q^* > 0$

If $(q^*, z^*) > 0$, then p, λ, and $\frac{\partial c(q^*,w)}{\partial q}$ are all "the same"

Aggregate Supply I

- The absence of a budget constraint implies that individual firms' supply are not subject to wealth effects.
- Hence aggregation of production theory is simpler and requires less restrictive conditions.
- Consider J production technologies:$(Y^1, ..., Y^J)$

  Let $y^j(p,w) = \begin{pmatrix} q^j(pw,) \\ -z^j(p,w) \end{pmatrix}$ be firm $j's$ production plan.

Aggregate Supply II

- We define the following aggregate optimal production plan:

$$y(p,w) = \sum_{j=1}^{J} y^j(p,w) = \begin{pmatrix} \sum_j q^j(p,w) \\ -\sum_j z^j(p,w) \end{pmatrix}$$

- We have seen that the matrix of cross and own price effects on production plan $y^j(p,w)$: $Dy^j(p,w)$ is symmetric and positive semi-definite: the law of supply.
- Since both properties are preserved under sum then $Dy(p,w)$ is also symmetric and positive semi-definite.

Aggregate Supply II

In other words an aggregate law of supply holds.

Theorem (Existence of the Representative Producer)

In a purely competitive environment the maximum profit obtained by every firm maximizing profits separately is the same as the profit obtained if all J firms were to coordinate their choices in a joint profit maximization:

$$\pi(p, w) = \sum_{j=1}^{J} \pi^j(p, w)$$

Clearly, the intersection of aggregate supply and aggregate demand gives us a market equilibrium.

Efficiency

Definition (efficiency)

A production vector $y \in Y$ is efficient if there is no $y' \in Y$ such that $y' \geqslant y$ and $y' \neq y$.

Every efficient y must be on the boundary of Y, but there may be boundary points of Y that are not efficient

First Fundamental Theorem of Welfare Economics

Theorem

If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient

Proof

Suppose otherwise: there exists a $y' \in Y$ such that $y' \neq y$ and $y' \geqslant y$. $p \gg 0$ implies that $p \cdot y' > p \cdot y$, contradicting the assumption that y is profit maximizing.

Valid even for non-convex Y But p must be strictly positive

Second Fundamental Theorem of Welfare Economics

Theorem

Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero price vector $p \geqslant 0$.

We cannot replace by $p \gg 0$