A Constructive Geometrical Approach to the Uniqueness of Markov Stationary Equilibrium in Stochastic Games of Intergenerational Altruism☆

Łukasz Balbus[a], Kevin Reffett[b], Łukasz Woźny[c,*]

[a] *Institute of Mathematics, Wrocław University of Technology, 50-370 Wrocław, Poland.*
[b] *Department of Economics, Arizona State University, Tempe, AZ 85287-3806.*
[c] *Warsaw School of Economics, Theoretical and Applied Economics Department, Warsaw, Poland.*

Abstract

We provide sufficient conditions for existence and uniqueness of a monotone, Lipschitz continuous Markov stationary Nash equilibrium and implied invariant distribution in a class of intergenerational paternalistic altruism models with stochastic production. Our methods are constructive, and emphasize both order-theoretic and geometrical properties of nonlinear fixed point operators, hence can be used to build globally stable asymptotically uniformly consistent numerical schemes for approximate solutions via Picard iterations on approximate versions of our operators. Our results provide hence a new catalog of tools for rigorous analysis of Markov stationary equilibrium on minimal state spaces for overlapping generations with stochastic production, without commitment.

Keywords: stochastic games, constructive methods, intergenerational

☆The authors thank Rabah Amir, Ken Judd, David Levine, Andrzej Nowak, Ed Prescott, Manuel Santos and anonymous referees as well as the attendees at our presentation at the 13th International Symposium on Dynamic Games and Applications in Wrocław, Poland, for helpful conversations on issues relating to this paper. The usual caveats apply. Kevin Reffett gratefully acknowledges the Paris School of Economics during the Summer of 2008, as well as the Dean's Summer Grant Program at ASU in the Summer of 2010 for financial support. Łukasz Woźny gratefully acknowledge financial support from the Fulbright Program during 2006-2007 and Arizona State University, USA for hosting.

*Corresponding author. E-mail: lukasz.wozny@sgh.waw.pl. Address: Al. Niepodległości 162, 02-554 Warszawa, Poland. Fax: +48 22 849 53 12.

altruism
JEL codes: C62, C73, D91, 040

1. Introduction

Since the pioneering work of Kydland and Prescott (1977, 1980), there has been a great deal of interest in studying dynamic economies without commitment. In some cases, a lack of commitment is assumed between private agents who seek to develop enduring economic relations over time (e.g., models of strategic altruistic stochastic growth, sustainable private debt, strategic models of human capital formation, among others). In other cases, the commitment friction is between public and private agents who are trying to design equilibrium time-consistent public policy (e.g., models of Ramsey optimal taxation, sustainable sovereign/public debt, hyperbolic discounting, and monetary policy games). In the end, all of these models share one common concern, that being the concern of characterizing the set of subgame perfect equilibria. As it is well-known, in the presence of intertemporal commitment frictions, there are significant complications in even verifying the existence of subgame perfect equilibrium, let alone characterizing the set of such equilibria. Additionally, for reasons of tractability and numerical computation, researchers have often focused on pure strategy Markov stationary Nash equilibrium (MSNE) defined on "minimal" state spaces (e.g., Harris and Laibson (2001), Krusell and Smith (2003), and Klein et al. (2008)). Unfortunately, many new issues arise concerning the mathematical foundations of the suggested procedures that have been used in applications.[1]

In this paper, we propose a new set of techniques for addressing these problems within the context of an altruistic stochastic overlapping generations model of growth with strategic bequests. In our economies, we assume

[1]For example, one method that has been proposed is the so-called "Generalized Euler Equation method" (GEE). In the GEE method, researchers either (i) use the theory of distributions to develop first order expansions of the value function around each point in the state space to develop an Euler equation to characterize equilibrium, or (ii) apply the implicit function theorem to obtain local smooth characterizations of Markov equilibrium on an open set of the steady state. In both cases, the difficulty is to relate the actual optimization problem (in equilibrium) to the sufficiency of the Generalized Euler Equation on the open set of the point in the parameter space where the expansion is developed (as the value function in the equilibrium of the game is not necessarily concave on this open set).

no commitment between successor generations. An important feature of our approach is to view the problem within the context of a stochastic Markov game with a particular stochastic transition structure that has been used extensively in the literature. This structure allows us to obtain an Euler equation representation for MSNE that is both necessary and (globally) sufficient. We then use this Euler equation to show that a MSNE is any solution to a *decreasing*, continuous nonlinear operator transforming an appropriate space of candidate pure strategy equilibrium. We solve the resulting functional equation, hence existence is established. We further characterize the set of MSNE, as well as provide methods for computing MSNE accurately.

More specifically, we first provide sufficient conditions for MSNE existence in a compact subset of continuous functions, and provide a sharp characterization of the order structure of the MSNE set (i.e., they form an "antichain")[2]. As we also show that any MSNE in the game must necessarily be a fixed point of the nonlinear operator we define (as it must satisfy a necessary and sufficient equilibrium Euler equation), this result characterizes the set of MSNE in the class of continuous functions we study. For this existence result, we need very mild conditions on the primitive data of the game. Next, and perhaps most strikingly, we provide a set of sufficient conditions under which *globally stable* iterative procedures are available for computing MSNE for our class of time consistency problems. That is, we prove an uniqueness result, and show (under stronger assumptions that often are present in applied work) the result is valid relative to a very broad class of bounded measurable functions. This later uniqueness result is particularly important as the class of economies for which it holds includes parameterizations of stochastic OLG models found often in applied work. It bears mentioning that conditions for the existence of *unique* MSNE has been an open question in the literature (e.g., see the discussion in Curtat (1996) and Amir (2002)). The existence of such conditions is very important when interpreting either calibration and/or estimation results that must vary deep parameters of the economy in an effort to fit models to the data.

Although the focus of our work is motivated by interest in stochastic overlapping generations models with paternalistic altruism, we also discuss how the methods used in this paper can be adapted to other important

[2]We provide all the requisite definitions later in the paper when the results are presented.

dynamic stochastic games[3] (e.g., dynamic games with hyperbolic discounting as in Harris and Laibson (2001) and Krusell and Smith (2003), as well as settings with non-time separable preferences). For this latter extension, we develop a new ”mixed monotone” technique using methods that have no found much applications in economics till now. Therefore, in the end, our methods appear to be quite general.

Finally, as many papers that seek to study dynamic economies without commitment seek to compute MSNE (e.g., to estimate or calibrate the models to data), we provide a catalog of results characterizing the properties of simple approximation schemes (e.g., discretization methods) relative to the set of MSNE. Along these lines, we show by example how easy our methods are to implement in practice. It is useful to mention that our methods, when applicable, avoid many of the technical problems associated with both generalized Euler equation methods (henceforth, GEE), as well as promised utility methods (the latter having been used to study the set of subgame perfect equilibrium in related games).

The remainder of the paper is organized as follows: the next section discusses how our methods and result fit into the existing literature. In this section, we especially emphasize how our results complement those found in GEE literature, as well as those obtained using correspondence-based methods such as ”promised utility” methods. The third section defines the class of models we initially consider, and provides an interesting class of examples to motivate our results. As these examples assume power utility functions and Cobb-Douglas production, they contain many parameterizations of preferences and technologies that have been used in the applied literature (e.g., macroeconomics). The fourth section provides conditions under which our economies have (pure-strategy) MSNE, and we characterize the set of such MSNE using monotone methods. We also give conditions under which the set of MSNE is a singleton. In the fifth section, we provide extensions of our results based upon so-called ”mixed monotone” operators, which allow use to obtain results for the nonseparable utility case. In this section, we also describe methods that construct approximate solutions for MSNE that achieve uniform error bounds relative to a simple discretization method. The final section of the paper concludes with a discussion of applicability of our methods to other classes of stochastic games, and we include an appendix

[3]For the details we refer the reader to Balbus et al. (2010a,b).

that presents some definitions, a few abstract fixed point theorems that we use in the paper, as well as the proofs of all our results.

2. Related literature

The environment we consider has long history in economics, and is one version of a canonical set of problems studied in the time consistency and consistent plans literatures. In particular, the particular model we study dates back to the early work of Phelps and Pollak (1968) and Peleg and Yaari (1973).[4] The economy consists of a sequence of identical generations, each living one period, and deriving utility from its own consumption, as well as that of a successor generation. As agents cannot commit to plans, the "dynastic family" faces a time-consistency problem. In particular, each current generation has an incentive to deviate from a given sequence of bequests, consume a disportionate amount of current bequests, leaving little (or nothing) for subsequent generations.

Within this class of economies considered, conditions are known for the existence of semicontinuous MSNE, and have been established under quite general conditions via nonconstructive topological arguments (e.g. Leininger (1986) and Bernheim and Ray (1987))). An important step forward in characterizing the structure of equilibrium in this class of games was made in the work of Amir (1996b), where he introduces stochastic convex transitions structures into the game. He then establishes a result on the existence of MSNE. This result has been further extended in a series of recent papers by Nowak and coauthors (e.g., see (Nowak, 2006) and the citations within), where the stochastic transition structure is often assumed to be an (endogenous) mixtures of probability measures. In this body of work using the stochastic game framework, the existence results for MSNE take place in spaces of continuous functions (for both finite and infinite horizon versions of the game). To verify the existence of equilibrium in the infinite horizon version of the game, only nonconstructive topological methods have been developed[5]. Further, conditions under which MSNE is unique have yet to be

[4]Versions of our model under perfect commitment have been also studied extensively in the literature in various contexts also, beginning with the important series of papers by Laitner (1979a, 1979b, 1980, 2002), Loury (1981), and including more recent work of Alvarez (1999), Bhatt and Ogaki (2008) among others.

[5]With only few exceptions like Balbus and Nowak (2004).

established. Both of these facts have important implications for characterizing the structure of the set of MSNE, as well as allowing the possibility of providing a rigorous analysis of methods that seek to obtain approximate solutions.

Therefore, what has been missing in this literature has been methods that address two additional important questions: (i) can constructive procedures be developed for studying MSNE (which are important not only for theoretical issues such as equilibrium comparative statics, but also for characterizations of implied limiting distributions associated with pure strategies of the game, and the development of rigorous numerical methods for approximate solutions); and (ii) can results on uniqueness of MSNE be obtained (which are useful for developing globally stable iterative methods). We address both of these questions in this paper. To this end, we first provide sufficient conditions under which sharp characterizations of the set of pure-strategy MSNE can be obtained. In particular, for operator studied, we give sufficient conditions under which the set of pure-strategy MSNE forms an antichain (in standard pointwise partial order). We then develop sufficient conditions under which the set of MSNE computed by our operator is a singleton. As our methods are constructive, this result provide a rich description of a class of iterative methods for computing pure-strategy MSNE. We are able, therefore, to prove a number of interesting theorems concerning simple numerical approximation scheme that obtains asymptotic uniform error bounds for approximate solutions, which are useful in applications. In particular, our methods provide sufficient conditions for globally stable approximate solutions relative to a unique non-trivial MSNE within a class of Lipschitz continuous MSNE.

Relative to the literature on recursive methods and dynamic equilibrium, the technical innovation in our approach is integration of order-theoretic, topological, and geometrical methods into a systematic study of MSNE. This step forward is particularly important for numerical work, as we shall show. Although our techniques are related to previous work on monotone methods (e.g., see Datta and Reffett (2006) for a literature review and references), as well as the fixed point theory in ordered topological spaces as found in the work of Amann (1977), and the geometrical properties of mappings defined in abstract cones found in the work of Krasnosel'skiĭ and Zabreĭko (1984), what distinguishes the methods in this paper is our exclusive use of (iterative) fixed point theory for *decreasing* operators. To the best of our knowledge, this paper is the first application of iterative methods for decreasing operators for the study of Markov/recursive equilibrium in the literature. The fact that the operator is decreasing, of course, greatly complicates matters. For example, unlike the increasing (or "isotone") case studied in all the papers stemming from Coleman (1991), as our operators are decreasing, hence, do not possess a fixed point property relative to spaces of complete lattices and/or chain complete partially ordered sets[6]. To resolve the existence question, we integrate topological constructions into our order theoretic approach. In particular, we obtain existence via Schauder's theorem. A second complication of studying iterative methods for decreasing operators stem from the issue of 2-cycles (or so-called "fixed edges"). That is, stability (global or local) of iterations require developing geometric conditions (as opposed to simple order theoretic conditions) which have not been required in previous work (e.g. Coleman (1991, 2000), or Mirman et al. (2008)). We show that such geometric conditions are available for our case, under reasonable conditions relative to applied work, and we give explicit examples (and compute MSNE for them).

Finally, we can relate our methods to those in the existing literature on characterizing subgame perfect and/or Markov perfect equilibrium in dynamic economies without commitment. A "direct" approach to our class of problems has been taken by many authors. In this approach, appealing to more traditional dynamic programming methods, existence of stationary Markov Nash equilibrium is obtained via fixed point methods in function spaces. This approach has a long line of important contributions, including Leininger (1986), Bernheim and Ray (1987), Sundaram (1989), Curtat (1996), Amir (2002) and Nowak (2006)). Also, see the work of Marcet and Marimon (1998), and Rustichini (1998), among others, for a novel variation of this approach. A second, albeit a less direct method of equilibrium construction, is the "promised utility method", best illustrated in the seminal work of Kydland and Prescott (1980), Abreu et al. (1986, 1990, denoted by KP/APS henceforth). In this latter[7] approach, a continuation method

[6]See e.g., order theoretic fixed point theorems based upon Tarski's theorem, Markowsky's theorems, or their various extensions.

[7]Here we focus our discussion on methods of KP/APS for few reasons. Firstly, technical issues related to Marcet and Marimon's (MM henceforth) recursive saddlepoint methods remain to be worked out (e.g., Messner and Pavoni (2004)). Secondly, per MM and Rustichini's approach, it is not clear what the *ad hoc* continuation punishment in a recursive/Markov equilibrium in the game will be. As both of these questions, in principle, could

(or "promised utility" methods) based upon strategic dynamic programming arguments is developed, and authors characterize the set of equilibrium values that are sustainable in a subgame perfect or Markovian equilibrium. In this method, the set of equilibrium values induced by sequential equilibria turns out to be the maximal fixed point of a monotone operator mapping between spaces of correspondences (ordered under set inclusion). A dynamic equilibrium, then, becomes a selection from the equilibrium correspondence (along with a corresponding set of sustainable pure strategies). Applications of the APS approach are found in numerous papers including Atkeson (1991), Bernheim et al. (1999), Phelan and Stacchetti (2001), Judd et al. (2003), and Athey et al. (2005).

Although the promised utility approaches has proven very useful in some contexts, in our class of models, it suffers from a number of well-known limitations. First, for our stochastic OLG models with strategic altruism, we do not need to impose discounting, which is typically required for promised utility methods. Second, the presence of "continuous" noise in our class of dynamic games proves problematic for existing promised utility methods. In particular, this noise introduces significant complications associated with the measurability of value correspondences that represent continuation structures (as well as the possibility of constructing and characterizing measurable selections which are either equilibrium value function and/or pure strategies). Therefore, even if conditions for the existence of the greatest fixed point in spaces of measurable correspondences can be checked, minimally, one loses the constructive nature of the argument. Equally as troubling, characterizations of pure strategy equilibrium values (as well as implied pure strategies) is also difficult to obtain. This last limitation introduces significant problems in developing a rigorous theory of approximate solutions. Finally, it has not yet been shown by those that apply promised utility continuation methods how one can obtain any characterization of the long-run stochastic properties of stochastic games (i.e., equilibrium invariant distributions and/or ergodic sets)[8]. All of these issues are resolved using our methods in this paper for the class of economies studied.

---

be studied using KP/APS approach (if the methods could be applied to our problem), we focus our discussion in this paper on KP/APS type methods.

[8]For competitive economies, progress has been made. See Peralta-Alva and Santos (2010).

3. The Model and a Motivating Example

We consider an infinite horizon dynastic production economy without commitment.[9] Time is discrete and indexed by $t = 0, 1, 2, \ldots$. The economy has one-good each period, has a single store of value which is productive, so we refer to it as capital. Households are endowed with a unit of time which they supply inelastically. A dynasty consists of a sequence of identical generations, each living one period, each caring about its successor generation. Any given generation divides its output x between current consumption c and investment $x - c$ for the successor generation. The current generation receives utility U from both its own current consumption, as well as that of its immediate successor generation. There is a stochastic production technology summarized by Q that maps current stock and current investment into next period output.

We provide a few initial formalities. Let $I(x) = [0, x]$ be the set of feasible choices of consumption for a generation with output x, $x \in I$ where the interval I is either bounded or unbounded[10], i.e. $I = [0, S]$ with $S > 0$ or $I = [0, \infty)$. By I^o denote interior of I (we use similar notation later for other sets). Let the preferences of the current generation be represented by a bounded, continuous (and, therefore, Borel measurable) utility function $U : I \times I \to \mathbb{R}_+$. The production technology $Q(\cdot|x - c, x)$ will be stochastic, and governed by probability distribution of the next generation output parameterized by current investment $x - c$ and state x. Therefore, if the successor generation follows an integrable, stationary consumption policy $h : I \to I$, the expected payoff of the generation with endowment x and consuming $c \in I(x)$ is computed as follows:

$$\mathcal{W}(c, x, h) := \int_I U(c, h(y)) Q(dy|x - c, x).$$

Assuming continuity of the problem's primitive data, by a standard application of Berge's Maximum theorem, $\arg\max_{c \in I(x)} \mathcal{W}(c, x, h)$ exists for each $x \in I$, and can be viewed as a best response of a current generation to the policy h of its successor. A pure-strategy, Markov stationary Nash equilibrium (MSNE) is a function h^* such that $h^*(x) \in \arg\max_{c \in I(x)} \mathcal{W}(c, x, h^*)$.

[9]More precisely, we are studying a dynamic stochastic production economy with both capital and labor, but with inelastic labor supply. See remarks below on the proper interpretation of our stochastic production function.

[10]To denote the latter case, we will sometimes write $S = \infty$.

Note, such an equilibrium remains an equilibrium if generations are allowed to use more general strategies (see Bertsekas and Shreve, 1978).

Before proceeding to the main theorems of the paper, we first present a simple example to motivate the nature of some of our results in the paper. To do this, we consider the special case of power utility, and Cobb-Douglas production, which has found extensive use in applied work in economics.

Example 1. *To motivate our results consider the following example. State space is $I = [0,1]$, preferences given by a time separable utility $U(c_1, c_2) = c_1^\alpha + c_2^\beta$. There is no discounting. Transition $Q(\cdot|x-c, x) = (1-(x-c)^\gamma)\delta_0(\cdot) + (x-c)^\gamma\lambda(\cdot|x)$, where δ_0 is a delta Dirac concentrated at $x = 0$, the "production" function $(x-c)^\gamma$ is of a standard Cobb-Douglas form, and λ has a cdf given by: $\frac{1-e^{-(2-x)y}}{1-e^{-(2-x)}}$. The motivation for choosing this particular form of a transition probability is given in the following sections. Let $1 > \alpha > 0, 1 > \beta > 0, 1 > \gamma > 0$.*

For this economy, by the main existence theorems of our paper (theorems 2 and 3), there exists MSNE (and its set has no ordered elements). Further, if $\alpha + \beta < 1$, there exist a unique MSNE given by a Lipschitz continuous function. Moreover, this consumption policy is a limit of the sequence of consumption policies for finite horizon version of our economy (lemma 3). By corollary 2, MSNE are continuous in the deep parameters. Finally, to any arbitrary level of accuracy (in the sup norm), we can calculate MSNE consumption policy by a picewise-constant approximation scheme using simple Picard iteration procedure (theorem 6). Example 9, then, extends these results to the case of non-separable utilities.

The results of our calculations for this example[11] *are presented in figure 1. In the left panel, we assume the capital share $\gamma = 0.33$, and vary the preference parameters α and β. In the right panel, we let the preference parameters be $\alpha = 0.6$ and $\beta = 0.3$, and vary γ. Sensitivity analysis shows the large discrepancies in consumption values and more importantly consumption function slopes.*

It bears mentioning that correspondence based methods (e.g., APS methods) do not apply to this example (e.g., we do not assume discounting). Further, GEE methods are not needed (as necessary and sufficient Euler equa-

[11]MATLAB program implementing our numerical procedure is available from authors upon request.

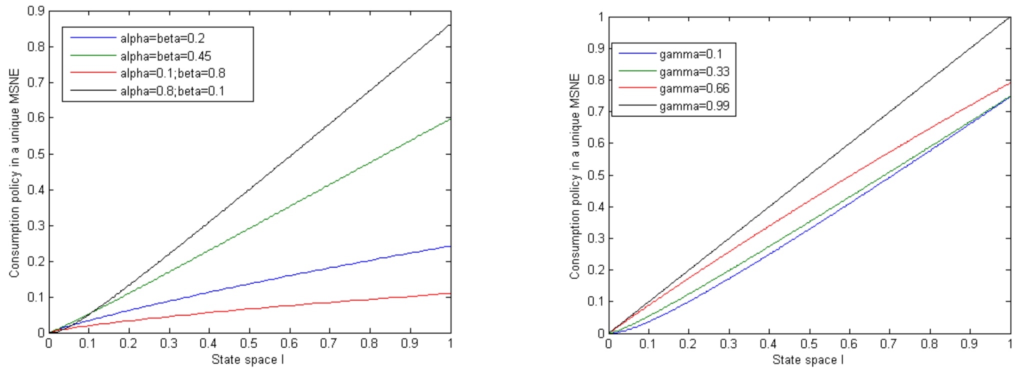

Figure 1: Consumption policy in a unique MSNE. Left panel: parameter $\gamma = 0.33$ and various α, β. Right panel: parameters $\alpha = 0.6, \beta = 0.3$ and various γ.

tions can be used to characterization best replies), nor are their application necessarily justified, as MSNE are not necessarily smooth.

4. Main results

We now present our main results for the general model. We start with a list of assumptions about preferences that we need for existence of MSNE.

Assumption 1 (Preferences). *The utility satisfy:*

- $U : I \times I \to \mathbb{R}_+$ *is of the form* $U(c_1, c_2) = u(c_1) + v(c_2)$, *where* u *and* v *are strictly increasing on* I. *Moreover* v *is bounded, continuously differentiable on*[12] I°, u *is twice continuously differentiable on* I°, *strictly concave and continuous on* I,

- $v(0) = 0$.

We use the following assumptions on transition probability Q.

Assumption 2 (Technology). *Transition probability* Q *satisfies*

- $Q(\cdot|x - c, x) = (1 - g(x - c))\delta_0(\cdot) + g(x - c)\lambda(\cdot|x)$, *where*

[12]For any set A, A° is said to be an interior of A.

- *$g : I \to [0, 1]$ is strictly increasing, concave on I and twice continuously differentiable on I°,*

- *$(\forall x \in I)$ $\lambda(\cdot|x)$ is a Borel transition probability on I, moreover $\lambda(\cdot|x)$ satisfies a Feller Property i.e. the function $x \to \int_I f(y)\lambda(dy|x)$ is continuous whenever f is continuous and bounded and for all x $\lambda(\cdot|x)$ is not Dirac delta concentrated in 0 i.e. $\lambda(\{0\}|x) < 1$ $\forall_{x\in I}$,*

- *δ_0 is a probability measure concentrated at point zero.*

Before preceding to our main existence theorems, a few remarks on these assumptions prove helpful. First, as for example in Nowak (2006), we assume Q is a convex combination (or mixture) of two distributions, with λ stochastically dominating δ_0. Hence, the stochastic structure of our game places probability $1 - g$ on the possibility that the next period capital is 0, and probability g it is drawn from λ (where these probabilities are endogenously determined). Second, notice all the nonconvexities of transition Q are given by function $1 - g$. Now, as δ_0 is a Dirac delta measure concentrated at zero, and $v(h(0)) = 0$, essentially, all such production nonconvexities are negligible. Third, a comparison between our technologies and those in Amir (1996b) (for unbounded I) can be made. On the one hand, Amir's transition Q is not dependent on the stock, but investment only. This situation in Amir may lead to a unique, but trivial, stationary Markovian equilibrium (see later in the paper, example 4), we seek to avoid. On the other hand, Amir does not require the mixing specification for stochastic production that we impose (although, he can allow it). He also does not require one of distributions to be Dirac delta.[13]

Finally, from an economic perspective, our assumptions on Q generate a large class of stochastically monotone and stochastically concave transition probabilities (see e.g. Amir (1996b)) with examples of g given by $g(x - c) = \beta(x - c)^\alpha$ or $g(x - c) = \beta(1 - e^{-(x-c)})$ where $1 > \alpha > 0$ and β is a sufficiently small positive numbers. Such functions are typical forms (Cobb-Douglas and CARA) used for technology specification in applied macroeconomic papers. Let us also mention that apart Amir and Nowak, applied papers of Horst (2005) and examples in Curtat (1996) use similar transition specification given by a convex combination of two stochastically ordered distributions.

[13]See his example 2.

Hence, the difference between ours and their approaches is that we take one of these distibutions to be Dirac delta at zero.

We now proceed with the construction of MSNE. Under assumptions 1 and 2 the value function $\mathcal{W}(c, x, h)$ is strictly concave and continuous in c on $I(x)$. Hence, for each integrable continuation strategy h for the successor generation, there is a well defined (measurable) best response operator:

$$A(h)(x) = \arg\max_{c \in I(x)} \mathcal{W}(c, x, h). \tag{1}$$

4.1. Existence

We first prove existence of MSNE within the class of Lipschitzian functions (on a minimal state space I). To do this, we first consider a subset of Lipschitiz continuous functions within the collection of continuous functions H, where H is given by the set:

$$H = \{h \in \mathscr{C}(I) | (\forall y \in I)\, 0 \leq h(y) \leq y\},$$

where $\mathscr{C}(I)$ is the set of nonnegative, continuous functions from I into I. Endow the set H with the pointwise partial order and the topology of uniform convergence on compacta of I. We will denote uniform convergence as " $\xrightarrow{u}$ " for short. Finally, note if I is an bounded interval, our topology is equivalent to sup-norm topology. Denote the zero element of H by $\underline{\theta}(x) = 0$, and the identity map $\overline{\theta}(x) = x$.

We seek MSNE within a subset of H. Let $\mathcal{C}_0$ be a set of compact subsets of $I^\circ = (0, \infty)$. For any function $\mathcal{M} : \mathcal{C}_0 \to \mathbb{R}_+$, define the collection

$$\mathcal{L}_\mathcal{M} := \{h \in H : \forall_{W \in \mathcal{C}_0} 0 \leq h(y_1) - h(y_2) \leq \mathcal{M}(W)|y_1 - y_2|, y_2 \geq y_1 \in W\}.$$

Notice, $\mathcal{L}_\mathcal{M}$ is a set of all increasing and locally Lipshitz functions with common modulus on all compact subsets of I^o. A special case of the set $\mathcal{L}_\mathcal{M}$, denoted by L_M, is the class of all increasing and Lipshitz continuous elements with common Lipschitz constant M:

$$L_M = \{h \in H | (\forall y_1 \leq y_2 \in I)\, 0 \leq h(y_2) - h(y_1) \leq M(y_2 - y_1), h(0) = 0\}.$$

Note, in the subset L_M, we have convergence $\xrightarrow{u}$ equivalent to pointwise convergence. Further, if $M_0 := \sup\{\mathcal{M}(W) : W \in \mathcal{C}_0\} < \infty$, then $\mathcal{L}(\mathcal{M}) \subseteq L_{M_0}$. Finally, if I is a closed interval (and, hence, compact), the set $\mathcal{L}_\mathcal{M}$ is equivalent to L_M for some M.

Under assumptions 1, 2, for $h \in H$, the objective function for the current generation is given by:

$$\mathcal{W}(c,x,h) = u(c) + g(x-c) \int_I v(h(y))\lambda(dy|x).$$

Further, as u and g are both differentiable on I°, we can linearize the objective $\mathcal{W}$ and define a mapping $\zeta(c,x,h)$ as follows:

$$\zeta(c,x,h) := u'(c) - g'(x-c) \int_I v(h(y))\lambda(dy|x). \tag{2}$$

We define our fixed point operator, therefore, using the Euler equation $\zeta(c,x,h) = 0$. To do this, we first study the properties of ζ in its arguments. Fix (x,h), $x \in I, x > 0$, $h \in H$. Then, given the concavity of value function $\mathcal{W}$ in c, for each (x,h), a necessary and sufficient condition for $c^*(x,h) \in I^\circ$ to be optimal is $\zeta(c^*(x,h),x,h) = 0$. Further, if $(\forall c \in I)\, \zeta(c,x,h) > 0$, then, the optimal $c^* = x$; while, if $(\forall c \in I)\, \zeta(c,x,h) < 0$, the optimal $c^* = 0$. We use the optimal policy function $c^*(x,h)$ to define a nonlinear operator A with values given pointwise by $Ah(x) := c^*(x,h)$. By the above reasoning, we have $A\underline{\theta} = \overline{\theta}$. Further, given that for all pairs (x,h), the Euler equation is necessary and sufficient for the optimal solutions $c^*(x,h)$, any MSNE must necessarily be a fixed point of our nonlinear operator.

To show that there exists some positive valued function $\mathcal{M}$ such that Ah maps $\mathcal{L}_{\mathcal{M}}$ into itself, we must make the following final assumption. Define (for a given positive, integrable f) function $p_f : I \to \mathbb{R}_+$ with: $p_f(x) := \int\limits_I f(y)\lambda(dy|x)$. Then our assumption is as follows:

Assumption 3. *Assume that:*

- $(\forall c \in I \setminus \{0\})\, u''(c) < 0$,
- *for any $x \in I \setminus \{0\}$, any*[14] *$c \in [0,x)$ and $g''(x-c) < 0$ value $\left|\frac{g'(x-c)}{g''(x-c)}\right|$ is bounded (by some positive constant M_g),*
- *collection of measures $\{\lambda(\cdot|x), x \in I\}$ is stochastically decreasing*[15] *with x,*

[14]Observe that we allow $g''(0) = -\infty$.

[15]We say that the collection of measures $\lambda(\cdot|x)$ is stochastically decreasing with x iff function $x \to \int_I f(y)\lambda(dy|x)$ is decreasing for any integrable, positive real valued and increasing f of I.

- *for each continuous, positive, increasing and bounded function f, function p_f, is continuously differentiable on I^o.*
- *for each compact subset of I^o say W there exists a constant M_p^W such that $\left|\frac{p_f'(x)}{p_f(x)}\right| \leq M_p^W$, for any $x \in W$ and any continuous, positive, increasing and bounded function f with*[16] $p_f(x) \neq 0$. *Let* $M_p := \sup\{M_p^W : W \in \mathcal{C}^o\}$.

Some comments on this assumption are in order. First, requiring stochastic monotonicity of λ for the transition structure is stronger than assumptions used in Nowak (2006) to show existence of a monotone, continuous MSNE. However, to show existence of a Lipschitz continuous MSNE, Nowak (2006) assumes[17] that λ does not depend on x. Using our Euler equation approach, for λ constant, existence of MSNE in L_1 can be easily established. As we shall show in a moment, this case is not particularly interesting (e.g., see our example 4). Given this, our aim is somewhat different, and we show existence of a monotone, Lipschitz continuous MSNE with $\lambda(\cdot|x)$ dependant on x. For this result, we require some additional assumptions bounding derivatives of g and p_f.

To see how conditions on p_f in this assumption maybe satisfied, observe that if I is bounded, and for any $x \in I$, the measure $\lambda(\cdot|x)$ has a density, and the ratios of derivative (with respect to x) of each of these densities and densities itself are bounded by some constant $\bar{\rho}$ (i.e. $\left|\frac{\frac{\partial}{\partial x}\rho(y|x)}{\rho(y|x)}\right| \leq \bar{\rho}$ a.e.), then this assumption is indeed satisfied with $M_p < \infty$. So verifying this condition is direct.

Assumption 3 is important to characterize a Lipschitz continuous MSNE with $\lambda(\cdot|x)$ dependant on x in the existence proof based on properties of our fixed point operator A (defined implicitly within an equilibrium first-order condition for each generation), whose fixed points turn out to be MSNE. The next lemma provides some initial properties of the operator A.

Lemma 1. *Let assumptions 1, 2 and 3 be satisfied. Then:*

[16]In other words for every compact W such that $W \subset I$ there is a constant M_p^W such that $|p_f'(x)| \leq M_p^W p_f(x)$, for all $x \in W$ and any continuous, positive, increasing and bounded function f.

[17]See Amir (1996b) for a corresponding assumption that Q does not depend on x.

(i) there exists a compact set in the $\overset{u}{\rightarrow}$ *topology, say* $H_0 \subset H$*, and* $A(H_0) \subseteq H_0$*. Further, A is continuous and decreasing, i.e.* $\forall h_2, h_1 \in H_0$ *with* $h_2 \geq h_1$*,* $Ah_1 \geq Ah_2$*.*

(ii) assume additionally $M_p < \infty$*, and let* $M = 2 + M_p M_g$*. Then, for all* $h \in L_M$*,* $Ah \in L_M$*, A is continuous and decreasing.*

We now provide our first important result on existence of MSNE. It is important to notice that in this result, aside from proving the existence of Lipschitzian MSNE, we also characterize the order structure of the equilibrium set in L_M. In the theorem, Ψ_A denotes the set of fixed points of A.

Theorem 2. *For the operator A, under assumptions 1, 2 and 3, we have:*

(i) $\Psi_A \subset H_0$ *and is a non-empty anti-chain (i.e. has no ordered elements in* H_0*).*

(ii) If additionally $M_p < \infty$*, then, for a constant* $M := 2 + M_p M_g$*, the set of fixed points* Ψ_A *of A in* L_M *is a non-empty anti-chain (i.e. has no ordered elements in* L_M*).*

A few remarks on Theorem 2 are warranted. First, clearly the fixed points of A are MSNE for our stochastic game (as they correspond with equilibrium best responses for each generations parameterized programming problem). Further, as our operator is defined implicitly in an Euler equation (which is necessary and sufficient for a best response), this implies that the set of MSNE (in H_0) must necessarily form a nonempty antichain. Second, the conditions assumed in the theorem are not the same as the conditions of theorem 1 in (Nowak, 2006). For example, as opposed to Nowak, we show existence of Lipschitz continuous MSNE with $\lambda(\cdot|x)$ a function of x. Finally, exploiting the antitone structure of our operator Ah, we are able to sharpen the characterization of the set of MSNE (namely, they form an antichain). This fact is important, for example, if one considers questions of monotone comparative statics on the deep parameters of the game (as such conditions will be very difficult to obtain). Finally, under the conditions of the theorem, generally the set of MSNE in L_M is not a singleton[18]. We will consider the uniqueness question later in this section.

[18] One method for checking uniqueness of the fixed point of A is to show it is upward directed. The partially ordered set $(B, \leq)$ is upward directed if and only if for any $b_1, b_2 \in B$ there exists $b_3 \in B$ such that $b_1 \leq b_3$ and $b_2 \leq b_3$ (Cid-Araújo, 2004).

We finish this subsection with two common examples (of many) of probability measures that satisfy assumption 3.

Example 2. *Let* $I = [0, 1]$ *and* $\rho(y|x) = \frac{(2-x)e^{-(2-x)y}}{1-e^{-(2-x)}}$*. The distribution function is:*

$$F_\rho(y|x) = \frac{1 - e^{-(2-x)y}}{1 - e^{-(2-x)}}.$$

Note that $F_\rho(y|\cdot)$ *is a superposition of the increasing function on* $[0, 1]$ $t \to \frac{1-t^y}{1-t}$ *with increasing function* $e^{-(2-x)}$*; hence, distribution function* F_ρ *is increasing. This implies that* λ *is stochastically decreasing. Further, we can alternatively express the density as* $\rho(y|x) := \exp\left(-(2-x)y + \Phi(x)\right)$*, where* $\Phi(x) := \ln(2-x) - \ln\left(1 - e^{x-2}\right)$*. Note that:*

$$|\Phi'(x)| \leq \frac{1}{2-x} + \frac{e^{x-2}}{1-e^{x-2}} \leq 1 + \frac{e^{-1}}{1-e^{-1}} = \frac{1}{1-e^{-1}}.$$

Therefore, we have

$$\left|\frac{\partial \rho(y|x)}{\partial x}\right| \leq \rho(y|x)(y + \Phi'(x)),$$

$$\leq \rho(y|x)(1 + \frac{1}{1-e^{-1}}) = \rho(y|x)\frac{2-e^{-1}}{1-e^{-1}},$$

$$|p_f'(x)| \leq \int\limits_I f(y)\left|\frac{\partial \rho(y|x)}{\partial x}\right| dy \leq \frac{2-e^{-1}}{1-e^{-1}} \int\limits_I f(y)\rho(y|x)dy = \frac{2-e^{-1}}{1-e^{-1}} p_f(x).$$

Hence, the assumption 3 is satisfied.

The next example shows that for unbounded I, assumption 3 is also satisfied:

Example 3. *Let* $\tau \geq 0$*, and* $k > 0$ *be some positive constants,* $I = [0, \infty)$ *and* $\lambda(\cdot|x)$ *be Pareto distribution with density* $\rho(y|x) = \frac{k}{(x+\tau)^k y^{k+1}}$ *for* $y \geq \frac{1}{x+\tau}$ *and* 0 *otherwise. The distribution function is:*

$$F_\rho(y|x) = \begin{cases} 1 - \left(\frac{1}{(x+\tau)y}\right)^k & \text{if} \quad y \geq \frac{1}{x+\tau} \\ 0 & \text{otherwise.} \end{cases}$$

Let us take arbitrary continuous, positive, increasing and bounded function f. Then,

$$p_f(x) = \int\limits_{\frac{1}{x+\tau}}^{\infty} f(y)\rho(y|x)dy = \frac{k}{(x+\tau)^k} Z(x).$$

where $Z(x) := \int\limits_{\frac{1}{x+\tau}}^{\infty} \frac{f(y)}{y^{k+1}}dy$*. Observe that* $\frac{dp_f}{dx}$ *exists and*

$$p_f'(x) = -\frac{k^2}{(x+\tau)^{k+1}} Z(x) + \frac{k}{x+\tau} f\left(\frac{1}{x+\tau}\right).$$

Hence,

$$\begin{aligned} \left|\frac{p_f'(x)}{p_f(x)}\right| &\leq \frac{\frac{k^2}{(x+\tau)^{k+1}} Z(x) + \frac{k}{x+\tau} f\left(\frac{1}{x+\tau}\right)}{\frac{k}{(x+\tau)^k} Z(x)} \\ &= \frac{k}{x+\tau} + (x+\tau)^{k-1} \frac{f\left(\frac{1}{x+\tau}\right)}{Z(x)}. \end{aligned}$$

Since f is increasing, $f(y) \geq f(\frac{1}{x+\tau})$ *for* $y \geq \frac{1}{x+\tau}$ *and:*

$$\left|\frac{p_f'(x)}{p_f(x)}\right| \leq \frac{k}{x+\tau} + \frac{(x+\tau)^{k-1}}{\int\limits_{\frac{1}{x+\tau}}^{\infty} \frac{1}{y^{k+1}}dy} = \frac{2k}{x+\tau}.$$

Then, if we take arbitrary compact subset of I say W then we have

$$\left|\frac{p_f'(x)}{p_f(x)}\right| \leq \max_{x \in W} \frac{2k}{x+\tau} = M_p^W$$

for all $x \in W$*. Since* M_p^W *does not depend on* f*, assumption 3 is satisfied. Notice, for* $\tau > 0$*, this distribution satisfies stronger condition:* $M_p = \sup\{M_p^W : W \in \mathcal{C}_0\} = \frac{2k}{\tau} < \infty$*; hence, our results imply the Lipshitz continuity of MSNE strategy with modulus* $2 + M_g \frac{2k}{\tau}$*. If* $\tau = 0$*, we obtain locally Lipshitz continuous MSNE strategies.*

4.2. Uniqueness

We now turn to the question of MSNE uniqueness relative to a the class of bounded measurable pure strategies. Before doing this, we mention a few reasons why such a result is important. Aside from the fact that such uniqueness results have not been generally available on the literature on stochastic games for infinite horizon, perhaps most importantly, a uniqueness result implies iterative procedures based upon our fixed point operator that are globally stable. This is particularly important when studying the properties of approximation procedures for constructing numerical solutions. It is equally as important if one wants to either calibrate the MSNE in the deep parameters of the model, and/or estimate the model. As multiplicities create complications in interpreting the set of equilibrium at different elements of the parameter space, rigorous answers to such questions requires minimally some theory of equilibrium selection.

To make clear the structure of our arguments, we begin with an example showing that under certain assumption on transition λ the MSNE is unique. The fixed point construction we use in this example will be later used to show some more general conditions under which the set of MSNE is a singleton.

Example 4. *We will now show that under the assumption that: (i) u, v, g are increasing; (ii) u, g are bounded and concave,; (iii) one of u or g is strictly concave; (iv) $v(0) = 0$; (v) $\lambda(\cdot|x)$ is constant with x; there exists a unique MSNE of a bequest game under study. To see that let us introduce some new notation:*

$$\omega(c, x, p) = u(c) + g(x - c)p \text{ and } c_p^*(x) = \arg\max_{0 \le c \le x} \omega(c, x, p), \tag{3}$$

and let $\overline{p} = \int_I v(y)\lambda(dy)$. Now introduce an operator: $B : [0, \overline{p}] \to [0, \overline{p}]$:

$$B(p) = \int_I v(c_p^*(y))\lambda(dy),$$

and observe that B is well defined. B is also continuous since $c_p^*(x)$ is continuous with p for any x. This follows from strict concavity of ω with c and the Berge (1997) maximum theorem applied to maximization in (3). Hence, by the Lebesgue dominance theorem, we obtain continuity of B.

To show B is decreasing, we claim that best reply $c_p^*(x)$ is decreasing with p for any x. To see this, observe that $[0, x]$ is a complete lattice, and *the objective $\omega(\cdot, x, \cdot)$ has decreasing differences for any x, since $g(x-c)$ is decreasing with c. Applying Topkis (1978) theorem, noting the uniqueness of the best reply, we have $c_p^*(x)$ decreasing with p.*

Finally, by Brouwer's theorem, B has a fixed point. Further, the fixed point is unique (as, B is decreasing, and $[0, \overline{p}] \subset \mathbb{R}^1$). To show that a unique fixed point of B, say p^*, corresponds to a unique MSNE of the bequest game under study, observe by definition of a fixed point, we have $B(p^*) = \int_I v(c_{p^*}^*(y))\lambda(dy)$. Moreover, by a definition of $c_{p^*}^*$, and the fixed point property, we have:

$$\begin{aligned} c_{p^*}^*(x) &= \arg\max_{0 \le c \ \le x} \omega(c, x, p^*) \equiv \arg\max_{0 \le c \ \le x} u(c) + g(x-c)p^* = \\ &= \arg\max_{0 \le c \le x} u(c) + g(x-c) \int_I v(c_{p^*}^*(x))\lambda(dy). \end{aligned}$$

Hence, $c_{p^*}^*$ is the unique MSNE of the bequest game under study.

This example is important, as it shows when the measure $\lambda(\cdot|x)$ is constant with x, the continuation value in the game is constant. Hence, our decreasing operator B has a unique fixed point, and the game has a unique MSNE. This example also motivates why our assumption that Q is not only investment, but also state dependant, is important.

Of course, more generally, the measure $\lambda(\cdot|x)$ does depend on x. Therefore, the resulting continuation value in the game is a function (and, therefore, this simple method to solve the game does not apply). In the following discussion, we provide a set of assumptions where $\lambda(\cdot|x)$ depend (but is not necessarily decreasing) on x, yet we obtain a unique MSNE of the game under study. Let us note that the method used in example 4 is similar to that applied in the general case, i.e. we will define a decreasing value function operator B (with the key difference being that we check uniqueness of its fixed point using very different argument).

Let $P = \{p : I \to \mathbb{R}_+ | p \text{ is bounded and Borel measurable}\}$. Define an operator $B : P \to P$ as follows. For $p \in P$, compute:

$$B(p)(x) = \int_I v(c_p^*(y))\lambda(dy|x),$$

with

$$c_p^*(x) = \arg\max_{c \in I(x)} \left\{ u(c) + p(x)g(x-c) \right\}.$$

By assumption 1 and 2, we obtain

$$c_p^*(x) = \begin{cases} x & \text{if } u'(x) - p(x)g'(0) \geq 0, \\ c_0^p(x) & \text{if } u'(0) - p(x)g'(x) > 0 > u'(x) - p(x)g'(0), \\ 0 & \text{if } u'(0) - p(x)g'(x) \leq 0, \end{cases}$$

where $c_0^p(x)$ is the c solving equation $u'(c) - p(x)g'(x-c) = 0$. Now, consider the following functional equation:

$$p(x) = \int_I v\left(c_p^*(y)\right)\right) \lambda(dy|x). \tag{4}$$

This functional equation is easily shown to be well defined. Further, note we have a solution to the functional equation (4) $p^* \in P$ if and only if $h^*(x) = c_{p^*}^*(x)$ is a MSNE. We now prove the following theorem.

Theorem 3. *Let conditions 1 and 2 be satisfied with I bounded. Assume that $\lim_{c \to 0^+} u'(c) = \infty$, $u''(c) < 0$ for $c > 0$, and $(\exists r, 0 < r < 1)$ such that $(\forall x \in I, x > 0)$ the following holds:*

$$(\forall c \in I^\circ(x)) \quad \frac{cv'(c)}{v(c)} \leq r \left[-\frac{cu''(c)}{u'(c)} - \frac{cg''(x-c)}{g'(x-c)} \right]. \tag{5}$$

Then, B is decreasing and has a unique fixed point p^* in P° such that

$$(\forall p_0 \in P^\circ) \quad \lim_{n \to \infty} \|p_n - p^*\| = 0, \tag{6}$$

where p_n is computed recursively as $(\forall n \geq 1)\, p_n = B(p_{n-1})$ for all p_0. Moreover, we have the following estimate of a convergence rate:

$$\|p_n - p^*\| \leq M_B(1 - \tau^{r^n}), \tag{7}$$

where $M_B > 0$ and $0 < \tau < 1$ are positive constants that depend on the choice of p_0.

Theorem 3 gives the sufficient conditions for the uniqueness of fixed points for the operator B, and this uniqueness result is robust to a very large space of functions (i.e., the space of bounded, real-valued, Borel measurable functions on I). Additionally, this unique fixed point corresponds to the unique measurable MSNE h^*. Theorem also provides error bounds, and rates of convergence of iterations on B to this unique fixed point.

If we add assumption 3, and link our results with results of theorem 2, we can also say that this unique MSNE corresponds to existence in the space[19] L_M. So we are able to prove uniqueness relative to a very large set of functions (bounded measurable functions), and existence in a very narrow set (Lipschitz continuous functions). So the result of the theorem is very strong.

The mathematical intuition behind theorem 3 is the following: since the operator B is decreasing, it may have multiple, unordered fixed points (actually, as our existence theorem shows, an nonempty antichain). The conditions in Theorem 2 assert, however, that this operator is "e-convex" (see Guo and Lakshmikantham (1988)) or in other words it is a "local contraction" (i.e. a contraction along cone origin rays). This property is a very strong infinite dimensional geometric condition, and sufficient for existence of a *unique* fixed point. From an economic perspective, the condition (5) has a very simple interpretation in terms of elasticities. That is, it requires that the sum of elasticities (in absolute values) of the primitives that generate current period returns (i.e., u' and g' with respect to c) to exceed elasticity of consumption for continuation utility v. That is, the percentage change in next-period utility v resulting from a one per-cent change in c cannot be too high[20]. Although condition (5) is restrictive (as one cannot expect MSNE uniqueness under general conditions in this class of games), it still is satisfied for general utility functions (including those often used in applications). Let use provide a simple illustration related to the initial motivating example of the paper.

Example 5. *Let I be bounded. Consider the time separable power utility function: $U(c_1, c_2) = c_1^{\alpha} + \delta c_2^{\beta}$, where $0 < \delta \leq 1$ and $0 < \alpha < 1, 0 < \beta < 1$. Observe that U satisfy assumption 1 and conditions in theorem 3, whenever stochastic production parameterized by the function g satisfies assumption 2. To see this, follow the inequalities for c given as in (5):*

$$\frac{cv'(c)}{v(c)} = \beta < 1 - \alpha \leq 1 - \alpha - \frac{cg''(x-c)}{g'(x-c)} = -\frac{cu''(c)}{u'(c)} - \frac{cg''(x-c)}{g'(x-c)},$$

where, the first inequality is satisfied by assumption, and the second follows from the strict monotonicity and concavity of g. Since, the inequality is strict, $\exists\ r$ with $0 < r < 1$, such that condition (5) holds for x in bounded I.

[19]Example 5 shows that it is indeed possible for a common functional forms of preferences and production to satisfy all these assumptions.

[20]Observe that condition (5) is equivalent to: $[\ln(v(c))]' \leq r[\ln(g'(x-c))]' - r[\ln(u'(c))]'$. We thank anonymous referee for suggesting this formulation.

So under the standard power/CRRA utility specification often used in applied macroeconomic modeling, our condition is satisfied. Further, to understand the nature of our condition, notice that by dividing inequality (5) by c, for continuous v with $v(0) = 0$, the left hand side of our inequality tends to infinity with $c \to 0$. So, essentially for our condition to hold for c close to 0, one needs the absolute risk aversion measure $\frac{-u''}{u'}$ be unbounded, and also tend to infinity with c limiting to 0. Hence, condition (5) is not satisfied for utility functions such as $u(c) = \ln(c+1)$ or CARA preferences given e.g. by $u(c) = 1 - e^{-c}$. Finally, observe that Example 5 suggests that for CRRA utilities, functional form of a "production" function g does not have to be specified, nor does $\frac{g''}{g'}$ need to be unbounded. This is important in the view of assumption 3 and theorem 2 where for MSNE in L_M we need $\frac{g'}{g''}$ to be bounded. Hence, our example 5 can satisfy assumption 3 as well.

The theorem also provides a *globally stable* successive approximation algorithm that allows us to compute the unique equilibrium, as well as provide uniform error bounds for equilibrium values approximation directly. Note, from the unique relationship between h^* and p^* (i.e., $h^*(x) = c^*_{p^*}(x)$), we are able to relate theorems on h^* with theorems that concern p^*. To see this, we simply relate iterations on the operators A, with iterations on the operator B as follows:

$$A^{n+1}h_0(x) = c^*_{B^n p_0}(x), \tag{8}$$

with $p_0(x) = \int_I v(h_0(y))\lambda(dy|x)$. By continuity of $p \to c_p(x)$ in the pointwise order, we have $A^{n+1}h_0 \to h^*$ pointwise. Since (under assumption 3) $A^{n+1}h_0 \in L_M$, pointwise convergence implies uniform. These relationships are summarized more formally in the following corollary.

Corollary 1. *Let assumption 3 and the assumptions of theorem 3 be satisfied. Then, A has a unique fixed point h^*, and*

$$(\forall h_0 \in L^\circ_M) \quad \lim_{n\to\infty} \|h_n - h^*\| = 0, \tag{9}$$

where

$$\|h_{n+1} - h^*\| \leq M_A(1 - \tau^{r^n}), \tag{10}$$

where $M_A > 0$ is a positive constant dependent on the choice of h_0.

4.3. Continuous comparative statics

In this subsection, we consider conditions under which continuous comparative statics of an MSNE equilibrium set are available. Such comparative statics are sufficient to build rigorous applications of calibration and/or estimation methods to the question of sensitivity analysis in our games.

For this, we first parameterize primitives of our economy by $\theta \in \Theta$, where Θ is a compact interval in $\mathbb{R}^m$. For each $\theta \in \Theta$, let $u(\cdot, \theta)$, $v(\cdot, \theta)$, $g(\cdot, \theta)$ be functions summarizing preferences and stochastic technologies in the previous sections of the paper (only now, we let them depend on θ), and let the probability measure we use to generate the stochastic transitions on the state be parameterized as $\lambda(\cdot|x, \theta)$. Notice initially, that when assumptions 1,2 and 3 are satisfied for any $\theta \in \Theta$, the constant $M = 2 + M_p M_g$ might depend on θ. Denote this dependence as $M(\theta)$, and observe that for compact I, we have $M(\theta) < \infty$ for all $\theta \in \Theta$. If we further assume that $\sup\limits_{\theta \in \Theta} M(\theta) \leq \bar{M}$ for some constant $\bar{M} > 0$, then we can let $LNE_{\bar{M}}(\theta)$ denote the set of MSNE belonging to $L_{\bar{M}}$ in the game with parameter θ. We often denote this correspondence as a mapping $\theta \to LNE_{\bar{M}}(\theta)$. We now have the following comparative statics theorem.

Theorem 4. *For each $\theta \in \Theta$, let assumptions 1,2 and 3 be satisfied, and I bounded. Moreover, let the mappings $(c, \theta) \in I^\circ \times \Theta \to u(c, \theta)$, $(c, \theta) \to v(c, \theta)$, $(i, \theta) \in I^\circ \times \Theta \to g(i, \theta)$ be continuous, as well as let the collection $(\theta, x) \to \lambda(\cdot|x, \theta)$ be stochastically continuous. If in addition, we assume $v(\cdot, \cdot)$ is uniformly continuous, and $\sup\limits_{\theta \in \Theta} M(\theta) \leq \bar{M}$ for some constant $\bar{M} > 0$, then the correspondence $\theta \to LNE_{\bar{M}}(\theta)$ is upper hemicontinuous (i.e. has a closed graph).*

Corollary 2. *Let assumption of theorem 4 be satisfied. If for all θ, there is a unique MSNE, then the function $\theta \to LNE_{\bar{M}}(\theta)$ is continuous.*

The theorem, as well as its corollary are very important in applications. First, obviously, the corollary gives conditions under which continuous sensitivity analysis of the equilibrium set is possible. Such a result can also be critical in developing conditions under which the simulated moments of the model converge to the actual moments of the model (e.g., Santos and Peralta-Alva (2005)).

Second, given recent work on approximating upper hemicontinuous correspondences (e.g., Beer (1980) and Feng et al. (2009)), the theorem implies that one can build step function approximation scheme to approximate the equilibrium correspondence to compute equilibrium comparative statics. For such an algorithm, it will be the case that as the "mesh" of the approximation scheme becomes finer, the approximation scheme will converge pointwise Hausdorff to the mapping $LNE_{\bar{M}}$ (uniform in the case of the corollary to the theorem). Constructing such an approximation scheme is possible because the theorem shows that the equilibrium correspondence $LNE_{\bar{M}}$ is valued in a *compact subset of a function space*, where uniform approximation schemes for arbitrary elements of this function space can easily be constructed (e.g., using various discretization schemes or piecewise linear/constant approximation schemes), as well as the fact that the theorem says the entire set of MSNE moves in an upper hemicontinuous manner. The result is important as it is difficult to imagine how one could obtain such a strong characterization of the set of MSNE using the various alternative methods in the existing literature (i.e., GEE or KP/APS).

Therefore, in the end, the theorem provides an exact analog to the correspondence based solutions methods for games (e.g., KP/APS methods) relative to the question of *computable equilibrium comparative statics.* This is in contrast to the existing methods, whose focus is on how to compute the set of Markov and/or subgame perfect equilibrium that exist at a *particular* parameter, say $\theta \in \Theta$. So, in this sense, our methods provide a new direction for correspondence-based computational methods that are an alternative to the methods in the existing literature.

Finally, returning to the corollary of the theorem, we know of no analog to this theorem in the existing literature using either GEE methods or correspondence based/promised utility methods. In particular, without our geometric approach (which require operators in function spaces to characterize the requisite geometric conditions), even if it is know that the equilibrium correspondence $LNE_{\bar{M}}$ is nonempty valued, it is not known that is a function. Such a sharp characterization of $LNE_{\bar{M}}$ is needed if one wants to have a great deal of certainty that the comparative static computed actually corresponds to the actual comparative static that arises in the equilibrium of the stochastic game.

We conclude this section with an example, of how to construct requisite bounds to characterize $LNE_{\bar{M}}$.

Example 6. *Let I be bounded. Fix constant $\theta \in (0, S]$. Continue example 2 with $\rho(y|x) = \frac{1}{\theta}\frac{\left(2-\frac{x}{\theta}\right)e^{-\left(2-\frac{x}{\theta}\right)\frac{y}{\theta}}}{1-e^{-\left(2-\frac{x}{\theta}\right)}}$ when $y \in [0, \theta]$ and 0 otherwise. Observe that the assumption 3 is satisfied for each $\theta > 0$. If additionally g satisfies* *assumption 3, we then obtain* $M_p = \frac{1}{\theta}\frac{2-e^{-1}}{1-e^{-1}}$*. Letting* $\Theta := [\epsilon, S]$*, and* $\epsilon > 0$ *we obtain:* $M(\theta) = 2 + \frac{1}{\theta}\frac{2-e^{-1}}{1-e^{-1}}M_g \leq 2 + \frac{1}{\epsilon}\frac{2-e^{-1}}{1-e^{-1}}M_g$*. Hence the conditions of theorem 4 are satisfied.*

4.4. Existence of Stationary Markov Equilibrium and Stochastic equilibrium dynamics

The results stated in theorem 2 and 3 allow us to further characterize the structure of MSNE for the economies under study. To do this, we first prove a result on the existence for Stationary Markov Equilibrium (SME). We define a Stationary Markov equilibrium to be a pair $h^* \in L_M$ (that is, a pure-strategy MSNE in L_M), and its set of associated invariant distributions on I. We prove the following result:

Theorem 5. *Let assumptions 1,2,3 be satisfied, with* $g(0) > 0$*, and* I *bounded. Assume additionally that* $||g|| < 1$*, and let* $h^* \in L_M$ *be a MSNE of the game. Then, the Markov process induced by* Q*, parameterized by* h^* *has a unique invariant distribution, and a process started from* $x_0 \in I$ *converges to this distribution.*

With results of theorem 3, we have the immediate corollary that is particularly useful in applications:

Corollary 3. *Under assumptions 1,2 and 3 with* $g(0) > 0$ *and* I *bounded, there exists a SME. If in addition other conditions of theorem 3 are satisfied, then there exists a unique SME.*

It is important to note that in our model, as $\lambda(\cdot|x)$ is stochastically decreasing, the transition probability is not a special case of that in the work of Amir (1996b). If $\lambda(\cdot|x)$ is stochastically increasing (see e.g. (Amir, 1996b)) we would easily obtain convergence to an invariant distribution (by Knaster-Tarski theorem); the problem is its uniqueness would not be guaranteed. Such a situation could be a significant complication in some applications (e.g., in calibration and/or estimation problems).

Apart from previously stated assumptions, in these results, we require that $g(0) > 0$. Many common production functions used in applied work satisfy this condition.[21] It bears mentioning, we only need this condition for our results on SME. The following example shows that under assumptions used in Nowak (2006), i.e. bounded I and $g(0) = 0$, the invariant distribution induced by a MSNE h^* is still unique, but trivial.

Example 7. *Assume 1,2,3 with I bounded but with $g(0) = 0$. Let $\{x_t : t \in N\}$ be a Markov chain generated by $Q(\cdot|x_t - h^*(x_t), x_t)$. We now show that $x_t \to 0$ almost surely and $x_t > 0$ for at most finite number of t.*

Let $p_t := Prob(x_t > 0)$. Let $E(\cdot)$ be an expected value operator induced by the chain $\{x_t : t \in N\}$. We show that $\sum\limits_{t=1}^{\infty} p_t < \infty$. Observe that $p_t = 1 - F_t(0)$, where F_t is a distribution function of x_t. Clearly, $F_t(0) := 1 - E(g(x_{t-1} - h(x_{t-1}))) := 1 - \rho_{t-1}$. By our assumptions, we have:

$$
\begin{aligned}
\rho_t &= E(g(x_t - h(x_t))) \\
&= E\left(\int_I g(x - h(x))\lambda(dx|x_{t-1})g(x_{t-1} - h(x_{t-1}))\right) \\
&\leq E\left(\int_I g(x)\lambda(dx|x_{t-1})g(x_{t-1} - h(x_{t-1}))\right) \\
&\leq \int_I g(x)\lambda(dx|0)\rho_{t-1}. \qquad (11)
\end{aligned}
$$

Since $0 < \int_I g(x)\lambda(dx|0) < 1$, we have $\sum_{t=1}^{\infty} \rho_t \leq \infty$. Since $p_t = \rho_{t-1}$, by Borel-Cantelli theorem, we obtain the occurrence $\{x_t > 0\}$ holds for the finite number of t almost surely.

[21]For example, many CES production functions. That is, for our economy assuming inelastic labor supply (hence, in equilibrium, $n = 1$), one specification for g has $\hat{g}(k, n) = G(g_1(k) + g_2(n))$, where G and g_i are strictly increasing, strictly concave, and smooth, and $g_i(0) = 0$. Then $g(0) = \hat{g}(0, 1) > 0$. For example, if g given by a standard CES production function (used commonly in the real business cycle and macroeconomics literature), we have $\hat{g}(k, n) = [k^\sigma + n^\sigma]^{\frac{1}{\sigma}}$, which has $g(k) = \hat{g}(k, 1)$ satisfying our assumption as $g(0) = \hat{g}(0, 1) = 1 > 0$. To see why the condition is needed, as we allow $g(0) = 0$, given our specification of Q, observe that the unique equilibrium invariant distribution associated with any MSNE will be concentrated at the point $x = 0$.

The above example shows the actual justification of assumption $g(0) > 0$. If we let 0 be an absorbing state then we eventually end up in a trivial invariant distribution, delta Dirac concentrated at point 0. It bears mentioning that the reasoning in above example 7 can be easily generalized to the case where I is unbounded, since by assumption 2 $\lambda(\cdot|x)$ is stochastically decreasing. Further, note that if we assume that $\lambda(\cdot|x)$ is stochastically increasing and I is unbounded, this reasoning above will *not* work. Therefore, although Amir (1996b) is not characterizing the SME of the similar bequest game, we cannot claim that under $g(0) = 0$ invariant distribution in his game would be trivial.

5. Further discussion and extensions

In what follows we (i) derive error bounds for an approximation procedure for computing MSNE of the bequest game, (ii) show uniform convergence result of equilibria in the finite horizon bequest game to the equilibria in infinite horizon game and finally (iii) we extend uniqueness result to the non-separable utility case. We begin with the question of uniform approximation schemes.

5.1. Computing MSNE

We first construct accurate approximate schemes for MSNE for the economies we consider. In particular, we discuss a simple discretization method for computing fixed points of the operator A (and, hence B via relation (8)) corresponding to a unique MSNE of our bequest economy. Following standard arguments in the literature (e.g., Fox (1973), Bertsekas (1975) and Hinderer (2005)), and exploiting the Lipschitz and uniform continuity of MSNE, we can calculate uniform error bounds for an approximation of given precision, and we can prove that a discretization procedures converge uniformly to an actual solution as its precision/mesh of the scheme gets arbitrarily large/fine. We consider the case of bounded intervals I.

Consider the following discretization scheme. Partition bounded set I into m mutually disjoint intervals $I_1, I_2, \ldots, I_m$ such that $I = \cup_{i=1}^{m} I_i$ where $x_i \in I_i$ and $P_m = \{x_1, x_2, \ldots, x_m\}$. Denote by $d_m = \max_{i=1}^{m} \sup_{x \in I_i} |x - x_i|$, i.e. the maximal grid size. Consider a function h_m, as well as an operator A_m, where h_m is a piecewise-constant approximation (i.e. a step function approximation) of h defined by:[22]

$$\begin{aligned}(\forall x \;\in\; P_m) \quad h_m(x) = h(x),\\ (\forall x \;\in\; I_i) \quad h_m(x) = h(x_i),\end{aligned}$$

and, similarly, $A_m h_m$ is a piecewise-constant approximation to Ah_m defined by:

$$\begin{aligned}(\forall x \;\in\; P_m) \quad A_m h_m(x) = Ah_m(x),\\ (\forall x \;\in\; I_i) \quad A_m h_m(x) = A_m h_m(x_i).\end{aligned}$$

So, the approximation is the following: the approximate function is set equal to the original function on the grid of the approximation, and we extend the approximation's definition to the whole compact interval I by defining the approximation to be constant in each of the subintervals I_i. This is a standard piecewise constant discretization scheme.

Now, for any $h^0 \in L_M$, let the approximation of h^0 be given by h_m^0, and define A_m^n be an n-th iteration of the approximate operator A_m from h_m^0.

Theorem 6. *Let assumption 3 and those of theorem 3 be satisfied with $M_p < \infty$, and $d_m \to 0$ as $m \to \infty$. Then, for any $h^0 \in L_M^\circ$:*

$$\lim_{n\to\infty} \lim_{m\to\infty} \|A_m^n h_m^0 - h^*\| = 0. \tag{12}$$

where $h^* \in L_M^\circ$ is the unique fixed point of operator A. Moreover, we have the following estimate for an approximation error:

$$(\forall n, m \in \mathbb{N}) \quad \|A_m^n h_m^0 - h^*\| \le (n+1)d_m M + M_A(1 - \tau^{r^n}), \tag{13}$$

where $1 > r > 0$ and $M_A > 0, 1 > \tau > 0$ are constants that are dependent on a choice of $h^0 \in L_M^\circ$.

Observe that the inequalities in (9,10) imply that operator A has properties similar to a contraction mapping. Indeed, appealing to versions of the converse to Banach's contraction mapping theorem (namely, Janos (1967)

[22] We choose a piecewise constant approximation scheme because it is arguably the simplest approximation scheme on can imagine. Obviously, similar results are available for piecewise linear, splines, and some polynomial schemes.

or more recently Hitzler and Seda (2001)), one can show that there exists a metric that induces an *equivalent* topology as the sup norm and under which the operator A is a *contraction*. Further, by the main theorem in Leader (1982), there exist direct links between iterations of the operator A in the original metric (i.e., sup metric), and the induced iterations of A under the equivalent metric under which it is now a contraction. This link proves a critical step when calculating error bounds of our approximation.

We now provide a example of how to calculate error bounds.

Example 8. *Consider parititoning I into equally-long subintervals i.e. $x_i = i\frac{I}{m}$, and $d_m = \frac{I}{m}$, $i = 0, ..., n-1$. Observe that $1 - e^x \leq -x$ for all $x \in \mathbb{R}$. Then, theorem 6 yields error bounds of approximation scheme of the form:*

$$
\begin{aligned}
||A_m^n(h_m^0) - h^*|| &\leq (n+1)d_m M + M_A(1 - \tau^{r^n}) \\
&\leq I\frac{n+1}{m}M + M_A(1 - e^{r^n \ln(\tau)}) \\
&\leq 2IM\frac{n}{m} - M_A r^n \ln(\tau) \\
&\leq \max\left(2IM, -\ln(\tau)\right)\left(\frac{n}{m} + r^n\right).
\end{aligned}
$$

5.2. Finite horizon stochastic games

We now relate MSNE in finite horizon versions of our stochastic game to those in infinite horizon case. In particular, we provide conditions when the stationary MSNE is the uniform limit of a collection of finite horizon games' equilibria. Such conditions are not known in the existing literature.

We first consider a finite horizon case of our bequest game, i.e. an economy populated by T generations, each with preferences $(\forall t < T)\, u(c_t) + g(x - c_t)\int v(c_{t+1}(y))\lambda(dy|x)$, and the terminal generation with payoff $u(c_T)$. The results on existence and uniqueness of nonstationary equilibria in this class of games are well known (e.g. see Amir (1996a)). We now consider the limiting behavior of these games. Precisely, if c_T^* is the optimal strategy of a first generation in the T-horizon bequest game, we give conditions that guarantee (i) uniqueness of MSNE in the infinite horizon game, and (ii) $\lim_{T\to\infty} c_T^* = h^*$ uniformly.

Formally we have the following lemma:

Lemma 7. *Assume 1, 2, 3 as well as conditions in theorem 3. Then h^* is the unique MSNE of the infinite horizon bequest game, and we have the nonstationary pure strategy Nash equilibrium in the finite horizon game satisfying*

$$\|c_T^* - h^*\| \to 0.$$

5.3. Extensions to Economies with Non-separable utility

Finally, we study the uniqueness of MSNE for economies where each generation has non-separable utility. For this, we appeal to the fixed point theory for so-called "mixed monotone operators", that have found little application in the economics literature. We first define a mixed monotone operator. Consider an operator $f : X \times X \to X$, with X is a partially ordered set. We say the operator f is *mixed-monotone* if *(i)* for each $y_0 \in X$, $f(x; y_0)$ is increasing in x, and *(ii)* for each $x_0 \in X$, $f(y; x_0)$ is decreasing in y. In our applications of mixed monotone operators in this section, we apply an important result on existence and uniqueness of fixed points for mixed monotone operators that is due to Guo et al. (2004), which also exploits some geometric conditions that can be checked in economic applications.

To develop our arguments for this section, we shall maintain the conditions on technology in assumption 2, but replace assumption 1 with the following new condition:

Assumption 4 (Preferences). *The utility function satisfy:*

- *$U(c)$ is given by $U(c_1, c_2) = u(c_1) + \nu(c_1)v(c_2)$ where $u, \nu, v : \mathbb{R}_+ \to \mathbb{R}_+$, are strictly increasing and continuous. Moreover u, ν are continuously differentiable and are strictly concave,*
- *$u(0) = 0$, $\nu(0) = 0$, $v(0) = 0$,*
- *$\lim_{a\to 0} u'(a) = \infty$ and $(\forall a \in I, a > 0)\, 0 < u'(a) < \infty$,*
- *$\lim_{a\to 0} g'(a) = \infty$ and $(\forall a \in I, a > 0)\, 0 < g'(a) < \infty$.*

Using lemma 1 from (Nowak, 2006), for a continuation strategy for the successor generation $h \in H$, a standard argument shows that objective function for the current generation:

$$\mathcal{W}(c, x, h) = u(c) + \nu(c)g(x - c) \int_I v(h(y))\lambda(dy|x)$$

is well-defined, strictly concave in c on $I(x)$.Therefore, at each $h \in H$, there is a unique best response for the current generation. Also, note that as u, ν and g are each continuously differentiable, we can again use the linearization $\frac{\partial \mathcal{W}}{\partial c}(c, x, h)$ to define a mapping $\zeta(c, x, h)$ as follows:

$$\zeta(c,x,h) := u'(c)+\nu'(c)g(x-c)\int_I v(h(y))\lambda(dy|x)-\nu(c)g'(x-c)\int_I v(h(y))\lambda(dy|x).$$

Then, again, using ζ, we define a new mapping $\varsigma(c, x, h_1, h_2)$:

$$\varsigma(c,x,h_1,h_2) = u'(c)+\nu'(c)g(x-c)\int_I v(h_1(y))\lambda(dy|x)-\nu(c)g'(x-c)\int_I v(h_2(y))\lambda(dy|x),$$

where $h_1, h_2 \in H$. Notice, we have decomposed the continuation structure of the game into a pair of functions (h_1, h_2). We can use this decomposition in ς to define a *mixed-monotone* operator.

To see this, first observe that ς is strictly decreasing with c and h_2 and strictly increasing with h_1. Define an operator C on $\mathscr{C}(I) \times \mathscr{C}(I)$ for $x > 0$ by: $C(h_1, \underline{\theta}) = \overline{\theta}$ (where $\underline{\theta}(y) \equiv 0$ and $\overline{\theta}(y) = y$), and if $h_2 > \underline{\theta}$ then $C(h_1, h_2)$ is an argument c which solves the equation $\varsigma(c, x, h_1, h_2) = 0$. Observe, as ς is strictly decreasing with c, the Inada conditions guarantee that C is well defined. Moreover, as ς is continuous in x and c, we have $C : \mathscr{C}(I) \times \mathscr{C}(I) \to \mathscr{C}(I)$. Finally, Inada conditions give us also that $C(h_1, h_2) \neq \underline{\theta}$. The fact that C is mixed monotone is straightforward. Observe that as opposed to the method used in section 4.2 here we define an operator not on values but on the first order condition.

We now prove the main result of this section of the paper. Instead of proving the general theorem on existence, we use power utility functions (similarly to example 5) to show conditions for MSNE uniqueness. Notice the example chooses preference that are common in applied work.

Example 9. *Let $u(c_1) = c_1^\gamma$, $\nu(c_1) = c_1^\alpha$, and $v(c_2) = c_2^\beta$, where $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$ with I bounded. If assumptions 2 and 4 on bounded I are satisfied with $\alpha + 2\beta < 1$, $\gamma \leq \alpha$, then, C has a unique fixed point on $\mathscr{C}(I)$ and condition (A.5) holds.*

6. Conclusion

This paper proposes an order-theoretic method for constructing Markov stationary Nash equilibria in games of intergenerational altruism, as well as constructing nontrivial SME. Under additional conditions, using new geometrical methods in operator theory, we obtain conditions for a unique MSNE and global stability of nontrivial SME. Our theorems are sharp. As they are also constructive, we are able to provide a rigorous relationship between approximate and actual MSNE for the economies we study. Theorems 2 and 3 present sufficient conditions for existence and uniqueness of MSNE that are often met in applied work in intergenerational models of dynamic equilibrium (e.g., overlapping generations models in macroeconomics). Using fixed points methods for mappings in abstract cones, we are able to construct a sequence of policies converging uniformly to the unique MSNE of the economy. The results can be extended to asymptotic results for approximate solutions via discretization, and we can then obtain uniform error bounds for computing unique MSNE.

We stress that the method and examples presented in the paper can be generalized and used to study the broad class of overlapping generations macroeconomies with partial commitment. Issues that remain to be studied in future work are numerous. First, one should design a decentralized OLG economy without commitment that corresponds to decentralization of the equilibria in bequest game under study, to see if a tractable dynamic general equilibrium model without commitment can be designed that would be useful for applied work in the lifecycle literature. In principle, one could compute equilibrium prices, and prove the existence of a unique decentralized stationary Markov equilibrium. Additionally, one could add labor/leisure choice into the environment. We feel that these two steps are necessary in order to bring the models with partial commitment that are common in macroeconomic applications to the level of analysis which is now common with models lacking such strategic interactions across generations.

Finally, as mentioned in the introduction, we believe that the monotone value function operator methods can be applied to other classes of economies, including time-consistency games. Specifically, in Balbus et al. (2010c) we show how our procedure to calculate a unique SMNE can be extended on multiple choice variables, allowing to include e.g. elastic labor choice in a class of OLG models with limited commitment. In Balbus et al. (2010b), we show how our transition probability and monotone operator methods can be used to compute optimal, time-consistent consumption policy of a quasi-hyperbolic consumers under uncertainty and borrowing constraints. Finally, in Balbus et al. (2010a), we prove a number of theorems allowing for a constructive study of equilibria of stochastic supermodular games including such models as Markov perfect industry dynamics (see Ericson and Pakes (1995)) or dynamic job-search (see Curtat (1996)).

Appendix A. Definitions and abstract fixed point theorems

Here we provide few definitions, as well as state a useful theorem[23] that we use in our discussion on unique MSNE.

Definition 1. *Let E be a real Banach space and $P \subseteq E$ be a nonempty, closed, convex set. Then:*

- *P is called a cone if it satisfies three conditions: (i) $p \in P, \epsilon > 0 \Rightarrow \epsilon p \in P$, (ii) $p \in P, -p \in P \Rightarrow p = \theta$, where θ is a zero element of P, and (iii) $p \in P$, $q \in P \Rightarrow p + q \in P$,*
- *suppose P is a cone in E and $P^\circ \neq \emptyset$, where P° denotes the set of interior points of P, we say that P is a solid cone,*
- *a cone P is regular iff each increasing sequence which has an upper bound in order has a limit,*
- *a cone P is said to be normal if there exists a constant $N > 0$ such that:*
$$(\forall p_1, p_2 \in P) \quad \underline{\theta} \leq p_1 \leq p_2 \Rightarrow \|p_1\| \leq N\|p_2\|.$$

Theorem 8 (Guo et al. (2004)). *Let P be a normal solid cone in a real Banach space with partial ordering $\leq$ and $B : P^\circ \to P^\circ$ be a decreasing operator (i.e. if $p_1 < p_2 \in P$ then $Bp_2 \leq Bp_1$) satisfying:*

$$(\exists r, 0 < r < 1)(\forall p \in P^\circ)(\forall t, 0 < t < 1) \quad t^r B(tp) \leq Bp, \tag{A.1}$$

then B has a unique fixed point p^* in P° and:

$$(\forall p_0 \in P^\circ) \quad \lim_{n\to\infty} \|p_n - p^*\| \to 0, \tag{A.2}$$

where $(\forall n \geq 1)\, p_n = B(p_{n-1})$. Moreover we have the following estimate of convergence rate:

$$\|p_n - p^*\| \leq M_B(1 - \tau^{r^n}), \tag{A.3}$$

where $M_B > 0$ and $0 < \tau < 1$ are positive constants dependent on the choice of p_0.

[23] See theorem 3.2.5 in chapter 3.2 of Guo et al. (2004).

It is important to stress that Guo et al. (2004) establish uniqueness results under weaker conditions than we use in our work (A.1). We use a stronger version of their result, which guarantees other conclusions of theorem 8.

Definition 2. *Let P be a cone in real Banach space with partial ordering $\leq$ and operator $C : P \times P \to P$. If C is increasing with the first argument and decreasing with the second, i.e. $(\forall p', p, q', q \in P)(p' \geq p)(q' \geq q)$ we have $C(p', q) \geq C(p, q)$ and $C(p, q) \geq C(p, q')$, we say that C is a mixed monotone operator. If $\exists p^* \in P$ such that $C(p^*, p^*) = p^*$, then p^* is called a fixed point of C.*

Theorem 9. *Let P be a normal solid cone in real Banach space with partial ordering $\leq$ and $C : P^\circ \times P^\circ \to P^\circ$ be a mixed monotone operator. Assume that there exists a constant $0 < r < 1$ such that:*

$$(\forall 0 < t < 1)(\forall p, q \in P^\circ) \quad C(tp, t^{-1}q) \geq t^r C(p, q). \tag{A.4}$$

Then C has a unique fixed point $p^* \in P^\circ$ and for any $p_0, q_0 \in P^\circ$:

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = p^*, \tag{A.5}$$

where $(\forall n \geq 1)$ we define $p_n = C(p_{n-1}, q_{n-1})$ and $q_n = C(q_{n-1}, p_{n-1})$.

Appendix B. Proofs

Proof (of lemma 1). *Step 1*: We prove (i). Without loss of generality assume $S = \infty$. Let us construct $H_0 \subset H$ such that H_0 contains increasing functions and is compact in the $\xrightarrow{u}$ topology. This set will take a form $H_0 := \mathcal{L}_{\mathcal{M}}$ for some positive valued function $\mathcal{M}$. Before finding this function we show that $\mathcal{L}_{\mathcal{M}}$ is compact with endowed topology $\xrightarrow{u}$. Let $h \in \mathcal{L}_{\mathcal{M}}$ be arbitrary. Note that for all $y \in I$ $0 \leq h(y) \leq y$. Next observe that $\mathcal{L}_{\mathcal{M}}$ is equicontinuous for all selections of $\mathcal{M}$. Let us take $y_0 \in I^0$ and sufficiently smal $\epsilon \in \left(0, \frac{y_0}{2}\right)$. Define $W_0 := \left[\frac{y_0}{2}, \frac{3}{2}y_0\right]$. Clearly y_0 is interior point of W_0. Let $M_0 := \mathcal{M}(W_0)$. Observe $[y_0 - \epsilon, y_0 + \epsilon] \subset W_0$. Then if we take $\delta < \min\left(\frac{\epsilon}{M_0}, \epsilon\right)$, then $\sup_{h\in\mathcal{L}_{\mathcal{M}}} |h(y_0) - h(y')| < \epsilon$ for all $y' \in (y_0 - \delta, y_0 + \delta)$. For $y_0 = 0$ we have $\sup_{h\in\mathcal{L}_{\mathcal{M}}} |h(y') - h(0)| = \sup_{h\in\mathcal{L}_{\mathcal{M}}} h(y') \leq y' < \epsilon$ for $\delta = \epsilon$. Hence $\mathcal{L}_{\mathcal{M}}$ is equicontinuous. Clearly $\mathcal{L}_{\mathcal{M}}$ is pointwise closed, hence by the Arzela-Ascoli theorem (e.g., Kelley (1955), Theorem 17, p. 233), $\mathcal{L}_{\mathcal{M}}$ is compact in the topology $\xrightarrow{u}$.

Let $x \in I, x > 0$, and $h \in H$ be increasing. Clearly Ah is well-defined (i.e., nonempty and single-valued). We now find function $\mathcal{M}$ such that $Ah \in \mathcal{L}_{\mathcal{M}}$. First, we show that $Ah(\cdot)$ is increasing whenever h is. As (by assumption 2 and 3) the function $x \to \zeta(c, x, h)$ (ζ is defined in (2)) is increasing and $c \to \zeta(c, x, h)$ (by assumption 1, 2) is strictly decreasing, there exists x_1 and x_2 such that $0 \leq x_1 < x_2 \leq S$, and such that if $x \in (0, x_1]$ then $Ah(x) = 0$, if $x \in [x_2, S)$ then $Ah(x) = x$, and if $x \in (x_1, x_2)$ then $Ah(x)$ is a zero of the function $\zeta(\cdot, x, h)$ (i.e. the argument for which this function reaches 0). Note that, if $(0, x_1] = \emptyset$, then $x_1 = 0$ and if $(x_2, S) = \emptyset$, then $x_2 = S$. Moreover, $Ah(\cdot)$ is increasing and continuous on intervals $(0, x_1]$, (x_1, x_2) and $[x_2, S)$. Note that since $Ah(x_1) = 0$ and $Ah(x_2) = x_2$, hence for each $x \in [x_1, x_2]$ we have $Ah(x_1) \leq Ah(x) \leq Ah(x_2)$, which implies that $Ah(\cdot)$ is increasing on all I. If $(0, x_1] = \emptyset$ or $(x_2, S) = \emptyset$, then this monotonicity holds as well.

To see that $Ah(\cdot)$ is Lipschitz on all $W \in \mathcal{C}_0$ define $w(c, x, p) := u(c) + p(x)g(x - c)$ for some given continuous, decreasing, nonnegative function p. Observe the relation $\mathcal{W}(c, x, h) = w(c, x, p_{v \circ h})$, where p_f was defined in section 4.1. Denote the unique argument maximizing w with respect to c as $c^*(x)$. Also let c_0 denote a zero element of a function $\frac{\partial}{\partial c} w(c, x, p)$, if it exist and observe that:

$$c^*(x) = \begin{cases} x & \text{if } u'(x) - p(x)g'(0) > 0, \\ c_0(x) & \text{if } u'(0) - p(x)g'(x) > 0 > u'(x) - p(x)g'(0), \\ 0 & \text{if } u'(0) - p(x)g'(x) < 0. \end{cases}$$

following from concavity of u and g[24]. Since $0 \leq c_0(x) \leq x$ there exist points $x_i \in I$, $i = 1, 2$ such that $0 \leq x_1 < x_2 < S$ such that $c^*(x) = x$ for $x \in (x_2, S)$, $c^*(x) = c_0(x)$ for $x \in [x_1, x_2]$ and $c^*(x) = 0$ for $x \in [0, x_1)$. Note that $I_2 := (x_2, S)$ or $I_1 := [0, x_1)$ can be empty sets. Without loss of generality assume that both sets are nonempty. Note that on I_1 c^* is Lipschitz continuous with a constant 0 and on I_2 c^* has a Lipschitz constant 1. It is sufficient to show that c_0 is also Lipschitz continuous on all compact subsets $W \subset (x_1, x_2)$. Note that on (x_1, x_2) $c_0(x) \in (0, x)$ and $p(x) > 0$. Hence and by Implicit Function Theorem the derivative c_0' exists and:

$$c_0'(x) = \frac{p(x)g''(x - c_0(x)) + p'(x)g'(x - c_0(x))}{u''(c_0(x)) + p(x)g''(x - c_0(x))}.$$

[24]We write $u'(0) := \lim_{c \to 0^+} u'(c)$ for short.

We now put $p(x) = \int\limits_I v(h(y))\lambda(dy|x)$ for some increasing $h \in H \setminus \{0\}$ such that (x_1, x_2) is a nonempty set. Let $x \in W$ Further

$$\begin{aligned} |c_0'(x)| &= \frac{p(x)g''(x - c_0(x))}{u''(c_0(x)) + p(x)g''(x - c_0(x))} + \left|\frac{p'(x)g'(x - c_0(x))}{u''(c_0(x)) + p(x)g''(x - c_0(x))}\right| \\ &\leq 1 + \frac{-p'(x)g'(x - c_0(x))}{-u''(c_0(x)) - p(x)g''(x - c_0(x))} \leq 1 + \frac{-p'(x)g'(x - c_0(x))}{-p(x)g''(x - c_0(x))} \\ &\leq 1 + M_p^W M_g < \infty, \end{aligned}$$

where M_g and M_p^W are bounds of $\left|\frac{p'}{p}\right|$ and $\left|\frac{g'}{g''}\right|$ respectively. These constants exist by assumption 3. Hence $Ah(\cdot)$ is Lipschitz continuous on all compact subsets $W \subset (s_1, s_2)$ with modulus $1 + M_p^W M_g$. To show it is Lipschitz continuous on all compact W it is sufficient to show it is continuous. But it is easy to verify since $x \to \int\limits_I v(h(y))\lambda(dy|x)$ is continuous since h is continuous. Hence continuity of $Ah(\cdot)$ follows from continuity of u and g and Berge (1997) maximum theorem. Therefore, $c^*(x)$ is a Lipschitz continuous function with Lipschitz constant $2 + M_p^W M_g$ i.e. sum of the Lipschitz constant on subintervals. Setting $\mathcal{M}(W) := 2 + M_p^W M_g$ we conclude that $Ah \in \mathcal{L}_{\mathcal{M}}$. Obviously if we take a function $h \in L_M$ and $p(x)$ such that $(x_1, x_2) = \emptyset$, then $Ah = \underline{\theta}$ or $Ah = \overline{\theta}$ and $A(h) \in \mathcal{L}_{\mathcal{M}}$ as well. This implies that A maps $\mathcal{L}_{\mathcal{M}}$ into itself.

We next show that A is continuous (in $\overset{u}{\to}$ convergence) on $\mathcal{L}_{\mathcal{M}}$. We have $Ah_n \to Ah$ pointwise when $h_n \to h$, by continuity of functions u, g, v and Lebesgue's dominated convergence theorem. Since $\mathcal{L}_{\mathcal{M}}$ is equicontinuous, the topology of pointwise and $\overset{u}{\to}$ convergence coincide in $\mathcal{L}_{\mathcal{M}}$ hence, $Ah_n \overset{u}{\to} Ah$ uniformly on all compact subsets of I.

Finally, A is decreasing. Let $h_1 \leq h_2$. Since $h \to \zeta(c, x, h)$ is decreasing we have $\zeta(c, x, h_1) \geq \zeta(c, x, h_2)$. If $Ah_2(x) = 0$ the hypothesis holds trivially. Let $Ah_2(x) \in (0, x]$. Then $\zeta(Ah_2(x), x, h_2) \geq 0$. Let $\zeta(Ah_2(x), x, h_2) = 0$, and $\zeta(Ah_1(x), x, h_1) = 0$. Then $\zeta(Ah_2(x), x, h_1) \geq 0$. In this case $\zeta(\cdot, x, h_1)$ is decreasing and we obtain $Ah_1 \geq Ah_2$. If $\zeta(Ah_1(x), x, h_1) > 0$ then we immediately obtain $Ah_1(x) = x$, and $Ah_1 \geq Ah_2$ as well. Finally let $Ah_2(x) = x$. Then for all $c \in (0, x)$ we have $0 \leq \zeta(c, x, h_2) \leq \zeta(c, x, h_1)$ and $Ah_1(x) = x$ as well.

Step 2. We prove (ii). We repeat the reasoning from the previous step. We just need to show that there exists a compact set say $H_0 \subset H$ such that $A(H_0) \subset H_0$. This is $H_0 := L_M$ for $M := 2 + M_g M_p$. Repeating reasoning from the previous step we obtain $|c_0'(x)| \leq 1 + M_p M_g$ for $x \in (x_1, x_2)$.

Proof (of theorem 2). *Step 1.* Proof of (i). Let H_0 be a set from lemma 1. As it was mentioned before H_0 is compact in $\xrightarrow{u}$. Clearly H_0 is convex. Lemma 1 asserts that A is continuous. Hence, by Schauder's theorem (e.g., Kuratowski (1966), p. 544), $\Psi_A \subset H_0$ is nonempty. Suppose now that Ψ_A has two ordered fixed points in L_M say, $h, \hat{h} \in \Psi_A$. Then for all $x \in I$, $h(x) \leq \hat{h}(x)$. By monotonicity property of the operator A (lemma 1) we obtain $h(x) = Ah(x) \geq A\hat{h}(x) = \hat{h}(x)$. Hence $h(x) = \hat{h}(x)$. Hence each pair of the ordered fixed points is a pair of identical elements. Therefore, Ψ_A is antichained.

Step 2. Proof of (ii). Let M be a number from lemma 1. By the proof of lemma 1 set L_M is closed, relatively compact and hence compact (in topology $\xrightarrow{u}$). The convexity of L_M is obvious. Lemma 1 asserts that A is continuous. Hence, by Schauder's theorem (e.g., Kuratowski (1966), p. 544), $\Psi_A \subset L_M$ is nonempty. We repeat reasoning from previous step to obtain that Ψ_A is antichain.

Proof (of theorem 3). The strategy of the proof is to show that conditions of Guo et al. (2004) theorem (see theorem 8 in appendix) are satisfied. Clearly P is normal solid cone with natural product order and sup-norm. Observe that P° is a set of strictly positive functions.

Note that $B\underline{\theta}(x) = \int_I v(y)\lambda(dy|x)$, where $\underline{\theta}(x) \equiv 0$. Let $x \in I$, $0 < t < 1$, and $p \in P^\circ$ be given. We need to show B maps P° into itself. First observe that

$$\lim_{c \to 0^+} \frac{\omega(c, x, p)}{\partial c} = \lim_{c \to 0^+} \left(u'(c) - p(x)g'(x - c)\right) = \infty.$$

Hence by definition of c_p^* we obtain that $c_p^*(y) > 0$ for $y > 0$. Hence $B(p) \in P^\circ$. As v is strictly positive on $I \setminus \{0\}$, and $\lambda(\cdot|x)$ is not Dirac delta at 0, this implies that $B(p)(x) := \int\limits_I v(c_p^*(y))\lambda(dy|x) > 0$, for all $x \in I$.

For a given $r, 1 > r > 0$, consider a function $\phi_r : [0, 1] \to \mathbb{R}_+$, $\phi_r(t) = t^r B(tp)(x)$. We will show that ϕ_r is increasing with t on $(0, 1)$. Adding continuity of ϕ_r from the left at 1, we will conclude that $\exists\, r, 0 < r < 1$ such that $\phi_r(t) \leq \phi_r(1)$; hence, $t^r B(tp) \leq Bp$ as required by Guo et al. (2004) theorem.

Note that the function $\omega(c, x, p) = u(c) + p(x)g(x - c)$ has decreasing differences in (c, p). By Topkis (1978) $p \to c_p^*(x)$ is decreasing and continuous in the topology $\xrightarrow{u}$. By the same argument $t \to c_{tp}^*$ is decreasing and continuous. By definition of B it is sufficient to show that the function $t \to t^r v(c_{tp}^*(y))$

is increasing for all $y \in I$. Clearly $c^*_{tp}(0) = 0$ for all $t \in (0, 1]$. For arbitrary $y \in I^\circ$, let us divide interval $T^0 := (0, 1]$ into two disjoints parts $T^0 = T^y_1 \cup T^y_2$, where $T^y_1 := \left\{t \in T^0 : c^*_{tp}(y) \in (0, 1)\right\}$ and $T^y_2 := \{t \in T^0 : c_{tp}(y) = y\}$. Note that T^y_1 is open and T^y_2 is closed. Since $t \to t^r v(c^*_{tp}(y))$ is continuous, we just need to show that this function is increasing in all T^y_i. It is easy to see that this function is increasing on T^y_2. Let $t \in T^y_2$, and $c(t) := c^*_{tp}(y)$. Clearly $c(t)$ solves an equation $\frac{\partial}{\partial c}\omega(c, y, tp) = 0$. Since $u''(c) < 0$ for $c > 0$, by Implicit Function Theorem we obtain that $c'(t) := \frac{d}{dt}c^*_{tp}(y)$ exists and:

$$c'(t) = \frac{p(y)g'(y - c(t))}{u''(c(t)) + tp(y)g''(y - c(t))}. \tag{B.1}$$

Further:

$$\frac{d}{dt}(t^r v(c(t)) = rt^{r-1}v(c(t)) + t^r v'(c(t))c'(t) = t^{r-1}v(c(t))\left(r + t\frac{v'(c(t))}{v(c(t))}c'(t)\right).$$

By equation (B.1):

$$\begin{aligned}
&t\frac{v'(c(t))}{v(c(t))}c'(t) = t\frac{v'(c(t))}{v(c(t))}\frac{p(y)g'(y - c(t))}{u''(c(t)) + tp(y)g''(y - c(t))} = \\
&= \frac{\frac{v'(c(t))}{v(c(t))}}{\frac{u''(c(t))}{tp(y)g'(y-c(t))} + \frac{g''(y-c(t))}{g'(y-c(t))}} = \frac{\frac{v'(c(t))}{v(c(t))}}{\frac{u''(c(t))}{u'(c(t))} + \frac{g''(y-c(t))}{g'(y-c(t))}} = \\
&= -\frac{\frac{v'(c(t))}{v(c(t))}}{-\frac{u''(c(t))}{u'(c(t))} - \frac{g''(y-c(t))}{g'(y-c(t))}} \geq -r,
\end{aligned} \tag{B.2}$$

where the last inequality follows from (5). Combining (B.1) and (B.2) we obtain that $\frac{d}{dt}(t^r v(c(t)) \geq 0$, and therefore $t^r v(c(t)) = t^r v(c^*_{tp}(y))$ is increasing on the interval T^y_2. Hence we obtain monotonicity of $\phi_r(\cdot)$ on the whole interval $[0, 1]$.

As a result we have that $t^r B(tp) \leq Bp$ for any $t, 0 < t < 1$, any $p \in P^\circ$ as in theorem 8. Therefore, we conclude that B has a unique fixed point in P°, and conditions (6), (7) hold.

Proof (of corollary 1). Fix $x \in I$ and for any nonnegative constant p consider $l(p) = c_p(x)$ i.e. consider l as a function from $\mathbb{R}$ to $\mathbb{R}$. Let us take $p \in [0, \bar{p}]$ with $\bar{p} = \int\limits_I v(y)\lambda(dy|0)$. We show that l is a Lipschitz continuous function on $[0, \bar{p}]$. Since $p \to c_p(x)$ is decreasing and continuous, hence there exist number η such that $c_p(x) = x$ for all $p \in [0, \eta]$, and $c_p(x) \in (0, x)$ for all $p \in (\eta, \bar{p}]$. Inada condition on u guarantees that $c_p(x) > 0$, whenever $p > 0$. Without loss of generality assume that $0 < \eta < S$. With fixed x we have $l'(p) := \frac{\partial}{\partial p} c_p(x) = 0$ for $p \in [0, \eta]$. Let $p \in (\eta, S)$. Then $\frac{\partial}{\partial p}\omega(l(p), x, p) = 0$. By Implicit Function Theorem we have:

$$|l'(p)| = \frac{-g'(x - l(p))}{u''(l(p)) + pg''(x - l(p))} \leq$$

$$\frac{-g'(x - l(p))}{pg''(x - l(p))} \frac{pg''(x - l(p))}{u''(l(p)) + pg''(x - l(p))} \leq \frac{M_g}{p}.$$

Hence for all $\epsilon > 0$ and $p \in [\epsilon, \bar{p}]$ we have

$$|l(p_1) - l(p_2)| \leq \frac{M_g}{\epsilon}|p_1 - p_2|. \tag{B.3}$$

Let $p_{\max}(x) := \int_I v(y)\lambda(dy|x)$. Clearly it is decreasing and continuous function. Let

$$p^1_{\max}(x) := B(p_{\max})(x) = \int_I v\left(c_{p_{\max}}(y)\right)\lambda(dy|x).$$

Since $\lim_{c \to 0^+} u'(c) = \infty$, $c_{p_{\max}}(y) > 0$ for all $y > 0$. Noting that $p^1_{\max}(\cdot)$ is continuous on compact I (as $p_{\max}$ is continuous), we conclude that $\min_{x \in I} p^1_{\max}(x) > 0$. Therefore

$$p^1_{\max}(x) \geq \bar{\epsilon} := \min_{x \in I} p^1_{\max}(x) > 0.$$

Consider a sequence p_n defined in (7). Note that $B(p_0) = p_1 \leq p_{\max} = B(\underline{\theta})$ as by theorem 3 operator B is decreasing. Suppose $p_{\max} \geq p_n \geq p^1_{\max}$ for some n. Applying B we obtain $p^1_{\max} \leq p_{n+1}$. This implies that $p^n(x) \geq p^1_{\max}(x) \geq \bar{\epsilon}$ for all n and x. By definition of l we have $l\left(B^n p_0\right) = c_{B^n p_0}(x) = A^{n+1}(h_0)(x)$. Therefore,

$$|A^{n+1}(h_0)(x) - h^*(x)| = |c_{B^n p_0}(x) - c_{p^*}(x)| \leq \frac{M_g}{\bar{\epsilon}}|B^n p_0(x) - p^*(x)| \leq M_A|1 - \tau^{r^n}|,$$

with $M_A := \frac{M_g}{\bar{\epsilon}} M_B$. The last inequality follows from definition of $l(\cdot)$ and (B.3).

Proof (of theorem 4). Let $h(\cdot|\theta_n) \in LNE_{\bar{M}}(\theta_n)$, $\theta_n \to \theta$ and $h(\cdot|\theta_n) \xrightarrow{u} h_0$. We show that $h_0 \in LNE_{\bar{M}}(\theta)$. From definition of $h(\cdot|\theta_n)$ we have

$$u(h(x|\theta_n)) + \int_I v(h(y|\theta_n),\theta_n)\lambda(dy|x,\theta_n)g(x-h(x|\theta_n)) \tag{B.4}$$

$$\geq u(c) + \int_I v(h(y|\theta_n),\theta_n)\lambda(dy|x,\theta_n)g(x-c), \tag{B.5}$$

for all $c \in [0,x]$. For fixed x we need to show that the convergence

$$J_n := \int_I v(h(y|\theta_n),\theta_n)\lambda(dy|x,\theta_n) \to \int_I v(h_0(y),\theta)\lambda(dy|x,\theta) := J \tag{B.6}$$

is satisfied. We have

$$|J_n - J| \leq \int_I |v(h(y|\theta_n),\theta_n) - v(h_0(y),\theta)|\lambda(dy|x,\theta_n) \tag{B.7}$$

$$+ \left| \int_I v(h_0(y),\theta)\lambda(dy|x,\theta_n) - \int_I v(h_0(y),\theta)\lambda(dy|x,\theta) \right|. \tag{B.8}$$

Let $\delta_n := \int_I v(h_0(y),\theta)\lambda(dy|x,\theta_n) - \int_I v(h_0(y),\theta)\lambda(dy|x,\theta)$. Since $v(h_0(y))$ is continuous, hence and by Feller property of $\lambda(\cdot|x,\theta)$ we obtain $\delta_n \to 0$. Observe that since $v(\cdot,\cdot)$ is uniformly continuous and $h(\cdot|\theta_n) \xrightarrow{u} h_0(\cdot)$, hence $v(h(\cdot|\theta_n),\theta_n) \xrightarrow{u} v(h_0(\cdot),\theta)$. Since I is bounded, hence for all $\varepsilon > 0$ there is n_ε such that for all $n > n_\varepsilon$ we have

$$||v(h(\cdot|\theta_n),\theta_n) - v(h_0(\cdot),\theta)|| < \varepsilon.$$

Therefore for $n > n_\varepsilon$ we have

$$|J_n - J| \leq \varepsilon + \delta_n.$$

Taking a limit $n \to \infty$ and next $\varepsilon \to 0$ we have $|J_n - J| \to 0$. Hence if we take a limit in (B.4) we obtain that h_0 is a Nash equilibrium in the game with θ. Moreover, $h_0 \in L_{\bar{M}}$, hence $h_0 \in LNE_{\bar{M}}(\theta)$.

Proof (of theorem 5). Let a MSNE $h^* \in L_M$ be given. For a transition probability $Q(\cdot|x-h^*(x),x)$ let us define a corresponding Markov operator $T:\mathscr{C}(I)\to\mathscr{C}(I)$ by:

$$Tf(x) = g(x-h^*(x))\int_I f(y)\lambda(dy|x) + (1-g(x-h^*(x)))f(0).$$

Observe that operator T is stable hence $Q(\cdot|x-h^*(x),x)$ has a Feller property. We now show that T is also quasi-compact[25]. To see that let us also define an operator L:

$$Lf(x) = (1-g(x-h(x)))f(0),$$

in $\mathscr{C}(I)$. Endow $\mathscr{C}(I)$ with the sup norm and denote a unit ball in $\mathscr{C}(I)$ by K. Let f be an arbitrary element from K. Note that:

$$L(K) = \left\{(1-g(x-h(x)))f(0) : f\in K\right\}.$$

Clearly:

$$L(K) = \left\{\alpha(1-g(x-h(x))) : \alpha\in[0,1]\right\},$$

is the compact set. Hence L is a compact operator. Let $f\in K$ Then:

$$\begin{aligned}|Tf(x)-Lf(x)| &= \left|g(x-h(x))\int_I f(y)\lambda(dy|x)\right| \leq \\ &\leq g(x-h(x))\int_I |f(y)|\lambda(dy|x) \leq ||g||_\infty < 1.\end{aligned}$$

Hence $||T-L||<1$ and this completes that T is quasi-compact.

Finally applying theorem 3.3 from Futia (1982) we get that T is equicontinuous. Further we observe that $Q(0|S-h^*(S),S)>0$ and $Q(0|0,0)>0$. In theorem 2.12 Futia (1982) shows that if an operator T is equicontinuous and Q satisfy the above mixing condition then the Markov process induced by Q and h^* has a unique invariant distribution μ^*. Moreover we get that from any initial $x_0\in I$, the measure on I induced by Q and h^* converges to μ^*.

[25] An operator $T:\mathscr{C}(I)\to\mathscr{C}(I)$ is said to be quasi-compact iff there exists a natural number n and a compact operator L such that $||T^n-L||<1$.

Proof (of theorem 6). For any $x \in I_i$:

$$
\begin{aligned}
&|A_m^n h_m^0(x) - h^*(x)| \\
&\leq |A(A_m^{n-1} h_m^0)(x_i) - A(A^{n-1}h^0)(x_i)| + \\
&|A^n h^0(x_i) - A^n h^0(x)| + |A^n h^0(x) - h^*(x)|.
\end{aligned} \tag{B.9}
$$

where above follows from the definition of A_m and a triangle inequality. Further, by theorem 2 A maps L_M into L_M, and by point (ii) in theorem 3 we obtain:

$$
|A^n h^0(x_i) - A^n h^0(x)| + |A^n h^0(x) - h^*(x)| \leq M d_m + M_A \left(1 - \tau^{r^n}\right). \tag{B.10}
$$

Next, observe in the sup norm:

$$
\begin{aligned}
&||A(A_m^{n-1} h_m^0) - A(A^{n-1}h^0)|| \leq \sup_{k=1,2,\ldots} ||A^k(A_m^{n-1}h_m^0) - A^k(A^{n-1}h^0)|| \\
&\leq ||A_m^{n-1}h_m^0 - A^{n-1}h^0||,
\end{aligned}
$$

where the first inequality follows from the definition of the sup operator, and the second follows by point 9 in Leader (1982) theorem. Specifically we can apply the main theorem in Leader (1982) as A is continuous, uniformly contractive and L_M compact. As a result we conclude that, there exists a number k for iterations of A giving smaller distance than the sup distance between any two starting points, e.g. $A_m^{n-1}h_m^0$ and $A^{n-1}h^0$, applying the above n times, we obtain:

$$
\begin{aligned}
&||A(A_m^{n-1}h_m^0) - A(A^{n-1}h^0)|| \leq ||A_m^{n-1}h_m^0 - A^{n-1}h^0|| \leq \\
&\leq M d_m + ||A(A_m^{n-2}h_m^0) - A(A^{n-2}h^0)|| \leq \ldots \leq \\
&(n-1)M d_m + ||h_m^0 - h^0|| = nM d_m.
\end{aligned} \tag{B.11}
$$

Combining the expressions (B.9), (B.10) and (B.11) we obtain:

$$
\begin{aligned}
||A_m^n h_m^0 - h^*|| \leq &\quad ||A(A_m^{n-1}h_m^0) - A(A^{n-1}h^0)|| + d_m \\
+||A^n h^0 - h^*|| \leq &\quad (n+1)M d_m + M_A(1 - \tau^{r^n}).
\end{aligned} \tag{B.12}
$$

The first assertion follows from (B.12) by taking limits with $m \to \infty$, and next with $n \to \infty$.

Proof (of lemma 7). Observe that $c_1^* = \overline{\theta}$ and that $c_T^* = A^{T-1}\overline{\theta}$. Since conditions in theorem 8 are satisfied (via relation (8)) we obtain $\lim_{T\to\infty} A^{T-1}\overline{\theta} = h^*$ uniformly.

PROOF (OF EXAMPLE 9). Let $h_1, h_2 \in \mathcal{C}(I)^\circ$ and $0 < t < 1$ be given. For simplicity, for a given $x \in I$ denote $C(h_1, h_2)(x) = \tilde{C}$ and $C(th_1, t^{-1}h_2)(x) = \tilde{C}_t$. Observe $\tilde{C} > \tilde{C}_t$. From the definition of $\tilde{C}$ and $\tilde{C}_t$ we obtain:

$$\gamma \tilde{C}^{\gamma-1} + \alpha \tilde{C}^{\alpha-1} g(x-\tilde{C}) \int_I h_1^\beta(y)\lambda(dy|x) - \tilde{C}^\alpha g'(x-\tilde{C}) \int_I h_2^\beta(y)\lambda(dy|x) = 0,$$

Similarly, for $\tilde{C}_t$, we have:

$$\gamma \tilde{C}_t^{\gamma-1} + \alpha \tilde{C}_t^{\alpha-1} g(x-\tilde{C}_t) \int_I t^\beta h_1^\beta(y)\lambda(dy|x) - \tilde{C}_t^\alpha g'(x-\tilde{C}_t) \int_I t^{-\beta} h_2^\beta(y)\lambda(dy|x) = 0,$$

Solving for $\int_I h_2^\beta(y)\lambda(dy|x)$ from the latter equation, and substituting the result into the former, we have:

$$\begin{aligned}&t^\beta \tfrac{\tilde{C}^\alpha g'(x-\tilde{C})}{\tilde{C}_t^\alpha g'(x-\tilde{C}_t)} \left[\gamma \tilde{C}_t^{\gamma-1} + \alpha \tilde{C}_t^{\alpha-1} g(x-\tilde{C}_t) \textstyle\int_I t^\beta h_1^\beta(y)\lambda(dy|x)\right] = \\ &\left[\gamma \tilde{C}^{\gamma-1} + \alpha \tilde{C}^{\alpha-1} g(x-\tilde{C}) \textstyle\int_I h_1^\beta(y)\lambda(dy|x)\right].\end{aligned}$$

As g is strictly concave, and $0 < \alpha < 1$, we have $\frac{\tilde{C}^\alpha g'(x-\tilde{C})}{\tilde{C}_t^\alpha g'(x-\tilde{C}_t)} > 1$; hence:

$$\begin{aligned}&t^\beta \left[\gamma \tilde{C}_t^{\gamma-1} + \alpha \tilde{C}_t^{\alpha-1} g(x-\tilde{C}_t) \textstyle\int_I t^\beta h_1^\beta(y)\lambda(dy|x)\right] < \\ &< \left[\gamma \tilde{C}^{\gamma-1} + \alpha \tilde{C}^{\alpha-1} g(x-\tilde{C}) \textstyle\int_I h_1^\beta(y)\lambda(dy|x)\right].\end{aligned} \tag{B.13}$$

We now show by contradiction that $\tilde{C}_t \geq t^{\frac{2\beta}{1-\alpha}} \tilde{C}$. Assume $\exists\, \tilde{x} \in I, \tilde{x} > 0$ such that $\tilde{C}_t(\tilde{x}) < t^{\frac{2\beta}{1-\alpha}} \tilde{C}(\tilde{x})$. It follows that $t^{2\beta} \tilde{C}_t^{\alpha-1}(\tilde{x}) > \tilde{C}^{\alpha-1}(\tilde{x})$. Moreover, as g and h_1 are nonnegative, and $g(\tilde{x} - \tilde{C}_t(\tilde{x})) > g(\tilde{x} - \tilde{C}(\tilde{x}))$, we have

$$\alpha t^\beta \tilde{C}_t^{\alpha-1}(\tilde{x}) g(\tilde{x}-\tilde{C}_t(\tilde{x})) \int_I t^\beta h_1^\beta(y)\lambda(dy|x) \geq \alpha \tilde{C}^{\alpha-1}(\tilde{x}) g(\tilde{x}-\tilde{C}(\tilde{x})) \int_I h_1^\beta(y)\lambda(dy|x). \tag{B.14}$$

Since $\gamma \leq \alpha$, we have $\frac{\beta}{1-\gamma} \leq \frac{\beta}{1-\alpha}$. With $0 < t < 1$, we obtain $t^{\frac{2\beta}{1-\alpha}} \tilde{C}(\tilde{x}) < t^{\frac{\beta}{1-\alpha}} \tilde{C}(\tilde{x}) \leq t^{\frac{\beta}{1-\gamma}} \tilde{C}(\tilde{x})$. Combining this result with the assumption , we have: $\tilde{C}_t(\tilde{x}) < t^{\frac{\beta}{1-\gamma}} \tilde{C}(\tilde{x})$. Which leads to the following:

$$t^\beta \tilde{C}_t^{\gamma-1}(\tilde{x}) > \tilde{C}^{\gamma-1}(\tilde{x}). \tag{B.15}$$

Adding inequalities (B.14) and (B.15), we obtain a contradiction (B.13) at $\tilde{x}$. Hence $\tilde{C}_t \geq t^{\frac{2\beta}{1-\alpha}} \tilde{C}$.

By assumption $2\beta + \alpha < 1$ hence $r := \frac{2\beta}{1-\alpha} < 1$; therefore, the hypotheses of condition (A.4) in a theorem 9 is satisfied. Since $\mathscr{C}(I)$ is a subset of a normal solid cone in a real Banach space, and C mixed monotone with $C(\cdot,\cdot) \neq \underline{\theta}$, we conclude from theorem 9 (see appendix) the existence of a unique fixed point of C on $\mathscr{C}(I)$.

Abreu, D., Pearce, D., Stacchetti, E., 1986. Optimal cartel equilibria with imperfect monitoring. J. Econ. Theory 39 (1), 251–269.

Abreu, D., Pearce, D., Stacchetti, E., 1990. Toward a theory of discounted repeated games with imperfect monitoring. Econometrica 58 (5), 1041–1063.

Alvarez, F., 1999. Social mobility: the barro-becker children meet the laitner-loury dynasties. Rev. Econ. Stud. 2 (1), 65–103.

Amann, H., 1977. Order structures and fixed points, sAFA 2, ATTI del 2. Seminario di Analisi Funzionale e Applicazioni. MS.

Amir, R., 1996a. Continuous stochastic games of capital accumulation with convex transitions. Games Econ. Behav. 15, 111–131.

Amir, R., 1996b. Strategic intergenerational bequests with stochastic convex production. Econ. Theory 8, 367–376.

Amir, R., 2002. Discounted stochastic games. Annals of Operations Research 114, 39–56.

Athey, S., Atkeson, A., Kehoe, P. J., 2005. The optimal degree of discretion in monetary policy. Econometrica 73 (5), 1431–1475.

Atkeson, A., 1991. International lending with moral hazard and risk of repudiation. Econometrica 59 (4), 1069–89.

Balbus, Ł., Nowak, A. S., 2004. Construction of nash equilibria in symmetric stochastic games of capital accumulation. Math. Methods Operations Res. 60, 267–277.

Balbus, Ł., Reffett, K., Woźny, Ł., 2010a. Constructive study of markov equilibria in stochastic games with strategic complementarities, Wroclaw University of Technology, Arizona State University, Warsaw School of Economics.

Balbus, Ł., Reffett, K., Woźny, Ł., 2010b. On constructive monotone methods for studying time-consistent markov policies of quasi-hyperbolic consumers under uncertainty, Wroclaw University of Technology, Arizona State University, Warsaw School of Economics.

Balbus, Ł., Reffett, K., Woźny, Ł., 2010c. Stationary markovian equilibrium in altruistic stochastic olg models with limited commitment, Wroclaw University of Technology, Arizona State University, Warsaw School of Economics.

Beer, G., 1980. The approximation of upper semicontinuous multifunction by step multifunction. Pacific Journal Of Mathematics 87 (1), 11–19.

Berge, C., 1997. Topological spaces. Including a treatment of multi-valued functions, vector spaces and convexity. Dover Publications, Inc., New York.

Bernheim, D., Ray, D., 1987. Economic growth with intergenerational altruism. Rev. Econ. Stud. 54, 227–242.

Bernheim, D., Ray, D., Yeltekin, S., 1999. Self-control, savings, and the low-asset trap, stanford University.

Bertsekas, D., 1975. Convergence of discretization procedures in dynamic programming. IEEE Transactions on Automatic Control 20, 415–419.

Bertsekas, D., Shreve, S., 1978. Stochastic Optimal Control The Discrete Time Case. Academic Press, New York.

Bhatt, V., Ogaki, M., 2008. Tough love and intergenerational altruism. Tech. Rep. #544, RCER Working paper.

Cid-Araújo, J., 2004. The uniqueness of fixed points for decreasing operators. Applied Mathematics Letters 17, 861–866.

Coleman, Wilbur John, I., 1991. Equilibrium in a production economy with an income tax. Econometrica 59 (4), 1091–1104.

Coleman, Wilbur John, I., 2000. Uniqueness of an equilibrium in infinite-horizon economies subject to taxes and externalities. J. Econ. Theory 95, 71–78.

Curtat, L., 1996. Markov equilibria of stochastic games with complementarities. Games Econ. Behav. 17, 177–199.

Datta, M., Reffett, K. L., 2006. Isotone recursive methods: the case of homogeneous agents. In: Kazuo, N., Dana, R. A., Le Van, C., Mitra, T. (Eds.), Handbook on optimal growth. Springer.

Ericson, R., Pakes, A., 1995. Markov-perfect industry dynamics: A framework for empirical work. Rev. Econ. Stud. 62 (1), 53–82.

Feng, Z., Miao, J., Peralta-Alva, A., Santos, M., 2009. Numerical simulation of nonoptimal dynamic equilibrium models. Tech. rep.

Fox, B., 1973. Discretizing dynamic programs. Journal of Optimization Theory and Applications 11 (3), 228–234.

Futia, C. A., 1982. Invariant distributions and the limiting behavior of markovian economic models. Econometrica 50 (2), 377–408.

Guo, D., Cho, Y. J., Zhu, J., 2004. Partial ordering methods in nonlinear problems. Nova Science Publishers, Inc., New York.

Guo, D., Lakshmikantham, V., 1988. Nonlinear problems in abstract cones. Academic Press, Inc., San Diego.

Harris, C., Laibson, D., 2001. Dynamic choices of hyperbolic consumers. Econometrica 69 (4), 935–57.

Hinderer, K., 2005. Lipschitz continuity of value functions in markovian decision processes. Math. Methods Operations Res. 62, 3–22.

Hitzler, P., Seda, A. K., 2001. A "converse" of the banach contraction mapping theorem. Journal of Electrical Engineering 52 (10/s), 3–6.

Horst, U., 2005. Stationary equilibria in discounted stochastic games with weakly interacting players. Games Econ. Behav. 51 (1), 83–108.

Janos, L., 1967. A converse of banach's contraction theorem. Proceedings of the American Mathematical Society 18 (2), 287–289.

Judd, K. L., Yeltekin, S., Conklin, J., 2003. Computing supergame equilibria. Econometrica 71 (4), 1239–1254.

Kelley, J., 1955. General Topology. Van Nostrand Press.

Klein, P., Krusell, P., Ríos-Rull, J., 2008. Time-consistent public policy. Rev. Econ. Stud. 75 (3), 789–808.

Krasnosel'skiĭ, M., Zabreĭko, P., 1984. Geometrical methods of nonlinear analysis. Springer-Verlag.

Krusell, P., Smith, A., 2003. Consumption–savings decisions with quasi–geometric discounting. Econometrica 71 (1), 365–375.

Kuratowski, K., 1966. Topology, vol 1. Academic Press.

Kydland, F., Prescott, E., 1977. Rules rather than discretion: the inconsistency of optimal plans. J. Polit. Economy 85 (3), 473–91.

Kydland, F., Prescott, E., 1980. Dynamic optimal taxation, rational expectations and optimal control. J. Econ. Dynam. Control 2 (1), 79–91.

Laitner, J., 1979a. Household bequest behaviour and the national distribution of wealth. Rev. Econ. Stud. 46 (3), 467–83.

Laitner, J., 1979b. Household bequests, perfect expectations, and the national distribution of wealth. Econometrica 47 (5), 1175–93.

Laitner, J., 1980. Intergenerational preference differences and optimal national saving. J. Econ. Theory 22 (1), 56–66.

Laitner, J., 2002. Wealth inequality and altruistic bequests. American Economic Review 92 (2), 270–273.

Leader, S., 1982. Uniformly contractive fixed points in compact metric spaces. Proceedings of the American Mathematical Society 86 (1), 153–158.

Leininger, W., 1986. The existence of perfect equilibria in model of growth with altruism between generations. Rev. Econ. Stud. 53 (3), 349–368.

Loury, G., 1981. Intergenerational transfers and the distribution of earnings. Econometrica 49 (4), 843–67.

Marcet, A., Marimon, R., 1998. Recursive contracts. Economics working papers, European University Institute.

Messner, M., Pavoni, N., 2004. On the recursive saddle point method. Levine's bibliography, UCLA Department of Economics.

Mirman, L. J., Morand, O. F., Reffett, K. L., 2008. A qualitative approach to markovian equilibrium in infinite horizon economies with capital. J. Econ. Theory 139 (1), 75–98.

Nowak, A. S., 2006. On perfect equilibria in stochastic models of growth with intergenerational altruism. Econ. Theory 28, 73–83.

Peleg, B., Yaari, M. E., 1973. On the existence of a consistent course of action when tastes are changing. Rev. Econ. Stud. 40 (3), 391–401.

Peralta-Alva, A., Santos, M., 2010. Problems in the numerical simulation of models with heterogeneous agents and economic distortions. Journal of the European Economic Association 8 (2-3), 617–625.

Phelan, C., Stacchetti, E., 2001. Sequantial equilibria in a ramsey tax model. Econometrica 69 (6), 1491–1518.

Phelps, E., Pollak, R., 1968. On second best national savings and game equilibrium growth. Rev. Econ. Stud. 35, 195–199.

Rustichini, A., 1998. Dynamic programming solution of incentive constrained problems. J. Econ. Theory 78 (2), 329–354.

Santos, M., Peralta-Alva, A., 2005. Accuracy of simulations for stochastic dynamic models. Econometrica 73 (6), 1939–1976.

Sundaram, R. K., 1989. Perfect equilibrium in non-randomized strategies in a class of symmetric dynamic games. J. Econ. Theory 47 (1), 153–177.

Topkis, D. M., March-April 1978. Minimazing a submodular function on a lattice. Operations Research 26 (2), 305–321.