Stationary Markovian equilibrium in Altruistic Stochastic OLG models with Limited Commitment*

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Abstract

We introduce a new class of infinite horizon altruistic stochastic OLG models with capital and labor, but without commitment between overlapping generations. Under mild regularity conditions, for economies with both bounded and unbounded state spaces, continuous monotone Markov perfect Nash equilibrium (henceforth MPNE) are shown to exist, and form an antichain. For each such MPNE, we also construct corresponding stationary Markovian equilibrium invariant distributions. We then show that for many parameterizations of our economies used in applied work in macroeconomics, unique MPNE exist relative to the space of bounded measurable functions. We can then directly relate these results to those obtained by promised utility/continuation methods based upon the work of Abreu, Pearce, and Stacchetti (1990). As our results are constructive, we can provide characterizations of numerical methods for approximating MPNE, and we construct error bounds. Finally, a series of examples show potential applications and limitations of our results.

1 Introduction and related literature

Over the last two decades, there has been a renewed interest in studying dynamic general equilibrium models without commitment. The seminal

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examples of this class of models are Kydland and Prescott (1977, 1980) and Levhari and Mirman (1980). More recent work along these lines has included models of sustainable plans, altruistic growth, Ricardian equivalence, endogenous borrowing constraints, sovereign debt, monetary policy games, savings with hyperbolic discounting, and Ramsey taxation. A central issue that has emerged was the question of (i) developing appropriate version of these sorts of models in settings were one can obtain sufficiently rich characterizations of the structure of the set of subgame perfect equilibria that arise, as well as (ii) developing numerical methods for computing them. Many methodological proposals have been made.

For example, beginning with the work of Kydland and Prescott (1977, 1980) on time consistent optimal Ramsey taxation, and continuing in many recent papers (e.g, Atkeson (1991), Chari and Kehoe (1993), Sleet (1998), and Phelan and Stacchetti (2001), among others), the "promised utility" approach to constructing subgame perfect equilibrium has been proposed. In this method, one constructs a set of sustainable values for each player in an equilibrium of the game by applying strategic dynamic programming arguments.[1] In related work, other have appealed to the use of traditional dynamic programming frameworks where the (dynamic) incentive constraints introduce a recursive structure of the constraint side (in addition to the objective). In some cases, a direct approach to constructing MPNE has been implemented (e.g, see, Alj and Haurie (1983), Amir (1989, 1996b, 2002, 2005), Curtat (1996), and Nowak (2006)).[2] In other cases, where worst equilibrium are easily defined, a set of recursive methods have been proposed (e.g, Marcet and Marimon (2009) and Rustichini (1998)). Finally, appealing to Euler equations and implicit function theorem, Harris and Laibson (2001) and Klein, Krusell, and Ríos-Rull (2008) have proposed a new first order theory for MPNE via an "generalized" Euler equation approach (GEE).

Although these approaches are each promising, they are also known to suffer from some serious technical limitations. For example, when using promised utility methods, it is difficult to rigorously characterize the set of equilibrium pure strategies that sustain the set of equilibrium values.[3] When applying traditional dynamic programming methods, aside from the problem of finding suitable function spaces to solve the resulting fixed point problems, issues relating the structure and/or uniqueness of stationary Markovian equi-

[1] Its bears mentioning that in Kydland and Prescott's original work, MPNE were the focus. In this later work in promised utility methods, Markov perfection was not necessarily the focus. For a interesting survey of strategic dynamic programming methods, see the work of Pearce and Stacchetti (1997) and Sleet and Yeltekin (2003).

[2] The recursive saddlepoint methods of Marcet and Marimon (2009), as well as the generalized dynamic programming methods of Rustichini (1998), can also be viewed abstractly as within this tradition.

[3] Promised utility methods generally also need discounting.

librium arise. Further, when using the dynamic programming methods of Marcet and Marimon (2009) and Rustichini (1998), punishment schemes that sustain subgame perfect equilibrium must be imposed in an *ad hoc* manner. More troubling, for Marcet and Marimon recursive saddlepoint methods, important counterexamples exist. Finally, when applying GEE methods, the question of relating the first order theory (and the assumed smoothness of equilibrium solutions) to the equilibrium value function that solve agents dynamic programs in the game have yet to be rigorously developed.[4] So a deep understanding of how these collections of methods work has yet to be firmly established.

The central purpose of this paper is to address some of these concerns in stochastic altruistic growth models without commitment. The class of models we study are related to those in Amir (1996b) and Nowak (2006), (i.e. models of stochastic growth without commitment), but they possess a rich set of complications and frictions between generations of players/dynasties. The deterministic incarnation of our models dates back to the work of Phelps and Pollak (1968), and has been studied extensively in the recent literature (e.g., Alj and Haurie (1983), Amir (1996b), Nowak (2006) and references within). Our economy consists of a sequence of identical generations, each living one period and deriving utility from its own consumption and leisure, as well as the consumption and leisure of its successor generation. Given the lack of commitment assumed between generations, the agents in this environment face a time-consistency problem as each current generation has an incentive to deviate from a given sequence of bequests, consume a disportionate amount of current bequests, leaving little (or nothing) for subsequent generations.

A key novelty of our collection of stochastic games is that unlike all existing related work, we introduce paternalistic altruism over two objects (namely, the next generations consumption-leisure pair). This extension is nontrivial, yet important for applications, as a great deal of work in macroeconomics and dynamic public finance allow for agents preferences to be defined over consumption and leisure over their lifecycles. Although this feature does complicate matters a great deal, we are still able to obtain existence results under very general conditions. Further, we are able to given conditions under which the uniqueness results obtained in Balbus, Reffett, and Woźny (2009) can be extend to this class of models under reasonable sufficient conditions (namely, conditions involving elasticities of preferences over consumption and leisure). Therefore, we show that by introducing a stochastic transition structure into the game's state variables, and allowing

[4]For example, Klein, Krusell, and Ríos-Rull (2008) apply the implicit function theorem at the steady-state on the agents Euler equation to construct a local GEE. Unfortunately, it is not proven that on the open set near this steady state, the Euler equation is sufficient (as the equilibrium value function need not be concave). Similar issues are addressed in the first order theory of Harris and Laibson (2001).

for more general forms of intergenerational altruism than previously studied, we can still obtain a very tractable model of strategic interaction among a countable collection of generations from the viewpoint of existence and computation of MPNE.

More specifically, we prove many new interesting results about the class of models studied in this paper. We first address the question of existence of MPNE. Although existence of MPNE has been studied in versions of the model with inelastic labor supply in both the deterministic (e.g., Leininger (1986), Kohlberg (1976)) and stochastic setting (Amir (1996b), Nowak (2006), Balbus, Reffett, and Woźny (2009)), these proofs do not apply the case of elastic labor supply and more general paternalistic altruism. Next, given the importance of numerical characterizations of MPNE in the existing macroeconomics literature, we ask the question of when sufficient conditions exist under which globally stable constructive iterative procedures can be employed for constructing *unique* MPNE. We show our uniqueness results for MPNE are relative to the spaces of all Borel measurable strategies, while our existence results for MPNE take place in the space of continuous MPNE. Therefore, under our conditions for uniqueness, our global stability result for iterative methods applies relative to any initial choice of Borel measurable function that is pointwise feasible in the game. Again these conditions are general.

We next turn to the question of existence of Stationary Markov equilibrium in the game. This question has received little attention in the literature. For our games, we give conditions for existence, as well as provide conditions for uniqueness of corresponding nontrivial invariant distributions. These results hold for both the bounded or unbounded state space case. Therefore, as all our methods are constructive, we are able to provide two approximation results that allow us to construct error bounds of MPNE set, and provide a rigorous numerical approach to approximating the unique MPNE. Therefore, we are able to provide a set of sufficient conditions under which globally stable approximation schemes can be provided for our class of stochastic games. The paper concludes with a number of examples which show both the applications and limitations of our results.

These results are important for a numerous additional reasons. First, they provide a rigorous set of tools for quantitative study stochastic OLG economies with limited commitment, as we can easily tie our results to the rigorous numerical characterization (including obtaining uniform error bounds) of MPNE in this class of economies. We actually provide two such procedures. Second, the methods developed in this paper are general, and, hence, can shed some new light on other dynamic economies with time consistency issues (like Atkeson (1991); Phelan and Stacchetti (2001)). For example, the methods can be extended to models with hyperbolic discounting (like Krusell and Smith (2003); Peleg and Yaari (1973)), or more general stochastic discounted supermodular games (like Amir (2002); Curtat

(1996)); including games of multigenerational altruism with capital accumulation, fishwars, other dynamic resource extraction problems (Levhari and Mirman, 1980), among others.

From a technical perspective, we build on the approach first proposed in Balbus, Reffett, and Woźny (2009) to a similar partial commitment economies with inelastic labor supply. That is, we again build a theory of existence and computation based upon *decreasing* operators. Relative to the existence question, we extend the tools first discussed in this earlier work to a more general paternalistic altruism model. The existence result, can be hence seen as a contribution to the literature of monotone operators similar to the one proposed (in case of elastic labor supply) by Coleman (1997) and Datta, Mirman, and Reffett (2002), but only using decreasing operators (for which the existence questions per fixed points are more complicated), instead of increasing operators (where existence can be established via various versions of Tarski's theorem). Relative to uniqueness conditions, our methods in this paper are also related to those in Balbus, Reffett, and Woźny (2009), where the fixed point theorems used to show uniqueness are based on geometrical properties of monotone mappings defined in abstract cones found in the work of Guo and Lakshmikantham (1988) or Guo, Cho, and Zhu (2004). But as is clear from the paper, the existence of 2-dimensional altruism greatly complicates the characterization of sufficient conditions for global stability for the case of altruism over consumption and leisure (relative to simply consumption as in this previous work).

Finally, given the two dimensional nature of the altruism in our model, it bears mentioning the discrepancies between methods developed here or in Balbus, Reffett, and Woźny (2009), as opposed to the ones applied by Amir (1996b) or Nowak (2006). Specifically the operator used in our proof of uniqueness theorem is defined on the set of bounded functions on the states spaces and assigns for any expected utility of the next generation its best response expected utility of a current generation. This operator is hence a value function operator defined on the space of functions, whose construction is motivated by the Abreu, Pearce, and Stacchetti (1990) (APS henceforth) operator that could be defined (in this example) on the set of subsets of the value functions. This is a striking difference to the "direct methods" applied by Amir (1996b) or Nowak (2006). As a result, on the one hand, methods we develop in this paper can be seen as a generalization to a two dimensional setting of an "inverse procedure" linking choice variables with their values proposed by Coleman (2000). But on the other, and more important hand, our contribution can be seen as a consistent way of sharpening the equilibrium characterization results obtained by Kydland and Prescott (1980) or other correspondence-based approaches to the study of time consistency problems. Specifically when applying APS techniques, although existence arguments can be addressed under very general conditions, the rigorous characterization of the set of dynamic equilibrium policies (either theoretically or numerically) is typically weak. Further, it has not yet been shown how to apply APS to obtain any characterization of the long-run stochastic properties of stochastic games (i.e., equilibrium invariant distributions and/or ergodic distributions). Once again value function methods proposed in this paper should be seen as a way of circumventing the mentioned APS predicaments.

The rest of the paper is organized as follows: in section 2 we present the formal model and state our assumptions, in section 3 we state our main results, and some examples of applications of our results. That last section 4 gives the proofs of all the theorems in the paper.

2 The class of Economies

We model a dynastic production economy with capital and labor but without commitment, populated by a sequence of identical generations, each living one period, each caring about its own consumption, and that of its successor generation. For simplicity, the size of each generation is assumed to be equal and normalized to unity, and there is no population growth. Apart from elastically supplying labor services $1-l$ (where l denotes leisure), any given generation divides also its (inherited) output s between current consumption c and investment $s-c$ for the next generation. The current generation receives utility for both consumption and leisure $U(c,l)$, as well as utility from its immediate successor consumption and leisure $v(c',l')$ (where (c',l') denotes next period consumption and leisure). There is a stochastic production technology summarized by stochastic transition Q that maps current savings, labor supply, and output $(s-c,1-l,s)$ into next period output s'. Time is discrete and indexed by $t=0,1,2,\ldots$. Note that such a specification can be seen as an infinite horizon stochastic game with a countable number of players.

Let K be a set of capital stock and $L:=[0,1]$ be a set of possible levels of labor that are normalized to the unit interval. We shall consider two cases for the capital stock, namely the unbounded state space case (i.e., $K:=\mathbb{R}_+$), and the bounded case (i.e., $K:=[0,S]$, where $S\in\mathbb{R}_{++}$). For any stationary, measurable policy of the next generation for consumption and leisure such that $h:=(h_1,h_2)$, with $h:K\to K\times L$, the objective function for the current generation in a MPNE is:

$$U(c,l;h,s):=u(c,l)+\int\limits_K v(h_1(y),h_2(y))Q(dy|s-c,1-l,s).$$

Before stating our assumptions, let us introduce some basic notation that we shall use throughout the paper. For $s\in K$, let the correspondence $A(s):=\{(c,l)\in K\times L, c\leq s\}$ denote the set of feasible actions for any current generation entering the period in state s, and by $\mathrm{Int}(A(s))$, denote interior of that set. Let δ_0 be a probability measure on K concentrated at point zero.

We now state a series of assumptions on preferences and transitions that we shall use for our existence results:

Assumption 1 (Preferences 1) *We assume that:*

- $u : K \times L \to \mathbb{R}_+$ *is twice continuously differentiable, strictly increasing in both arguments, supermodular and strictly concave function,*
- $v : K \times L \to \mathbb{R}_+$ *is increasing, measurable and* $\int_K v(s', 1)\lambda_k(ds'|s) < \infty$ *for all* $k = 1, \ldots, m$ *where measures* λ_k *would be specified later,*
- $v(0, l) = 0$.

A special case of these preferences, we often consider, is the case of additive separability between current consumption and leisure. This is actually the typical assumptions used in applications in macroeconomics.

Assumption 2 (Preferences 2) *given by:*

- $u(c, l) = u_1(c) + u_2(l)$, *where* u_1 *and* u_2 *are twice continuously differentiable, strictly increasing, strictly concave functions and*
- $u_1(0) = u_2(0) = 0$,
- *for* $i = 1, 2$ *the function* u_i *satisfies* $u_i'(0^+) = \infty$,
- $v : K \times L \to \mathbb{R}_+$ *is increasing, measurable and* $\int_K v(s', 1)\lambda_k(ds'|s) < \infty$ *for all* $k = 1, \ldots, m$,
- $v(0, l) = 0$.

The model of stochastic production that we adopt is a special case of that studied in a series of papers by Amir (1996a,b, 1997), Nowak and Szajowski (2003), Balbus and Nowak (2004, 2008), Nowak (2006), and Magill and Quinzii (2009). For our purposes, it proves to be convenient to assume a version of the mixing formulation for the stochastic transitions that has been studied extensively in the literature (e.g., Nowak (2006)). For the present work, we extend that specification to the case of elastic labor supply. It suffices to assume the noise mixes between two probability measures that admit densities:

Assumption 3 (Transition: general case) *In the model without absorbing state, let transition*

- *Q be given by:*

  $$Q(\cdot|s-c,1-l,s) := g(s-c,1-l)\lambda_1(\cdot|s) + (1-g(s-c,1-l))\lambda_2(\cdot|s),$$

  where the function $g : K \times L \to [0,1]$ is twice continuously differentiable, increasing in both arguments, supermodular and concave with $0 \leq g(\cdot) \leq 1$,

- *we have $(\forall c, l \in A(s))\, g(c,0) = g(0,l) = 0$,*

- *there exists a measure μ such that each measure $\lambda_k(\cdot|s)$ has a density $\rho_k(\cdot, s)$ with respect to a common measure μ, i.e. can be described as $\lambda_k(A|s) = \int\limits_A \rho_k(s', s)\mu(ds')$.*

For some of the results in the paper, we consider a slight modification of the assumption above on the stochastic transition Q for production. In particular, there are two modifications that we shall often use. The first case allows for an absorbing state, while the second imposes additive separability on the noise structure. These two different cases are given in the following two assumptions.

Assumption 4 (Transition: absorbing state) *In the model with absorbing state, let transition*

- *Q be given by:*

  $$Q(\cdot|s-c,1-l,s) := \sum_{i=1}^{m} g_i(s-c,1-l)\lambda_i(\cdot|s) + g_0(s-c,1-l)\delta_0(\cdot),$$

  where for $i = 1, \ldots, m$ functions $g_i : K \times L \to [0,1]$ are twice continuously differentiable, increasing in both arguments, supermodular, concave with $0 \leq g_i(\cdot) \leq 1$ and $\sum\limits_{i=0}^{m} g_k(\cdot) = 1$.

- *$i = 1, \ldots, m$ we have $(\forall c, l \in A(s))\, g_i(c,0) = g_i(0,l) = 0$,*

- *there exists a measure μ such that each measure $\lambda_k(\cdot|s)$ has a density $\rho_k(\cdot, s)$ with respect to a common measure μ, i.e can be described as $\lambda(A|s) = \int\limits_A \rho_k(s', s)\mu(ds')$.*

Assumption 5 (Transition: separated variables) *In the model with absorbing state, let transition*

- *Q be given by:*

  $$Q(\cdot|s-c,1-l,s) := g(s-c,1-l)\lambda(\cdot|s) + (1-g(s-c,1-l))\delta_0(\cdot),$$

  where the function $g : K \times L \to [0,1]$ is of the form $g(a,b) = g_1(a) + g_2(b)$ and each g_i is twice continuously differentiable, strictly concave and strictly increasing on K with $0 \leq g(\cdot,\cdot) \leq 1$ and $g_1(0) = g_2(0) = 0$,

- *there exists a measure μ such that $\lambda(\cdot|s)$ has a density $\rho(\cdot, s)$ with respect to measure μ, i.e. can be described as $\lambda(A|s) = \int_A \rho(s', s)\mu(ds')$,*
- *moreover the collection of the measures $\lambda(\cdot|s)$ is stochastically decreasing with s on K.*

As our assumptions on preferences are standard, we will spend some time remarking on the assumptions on transition structure and state space. In related work by Nowak (2006) or Balbus, Reffett, and Woźny (2009), the authors assume that K is a compact interval in $\mathbb{R}_+$; in Amir (1996b), the state space is taken to be unbounded $K = \mathbb{R}_+$. In particular, to show the existence of MPNE using the models of Amir (1996b), the assumption of unbounded state space is essential. This is not the case of Nowak (2006).

Additionally, following Nowak (2006), our transition Q is a convex combination of a finite number of measures λ_i (and in assumption 4,5 also δ_0) depending jointly on the state s, as well as the decision variables $s - c, l$. In particular, the functions g_i are viewed as the "weights" placed on probability measures that govern the stochastic structure of production. In what follows, we shall analyze cases with, and cases without an absorbing state. The former case is obtained by taking one of the measures (namely δ_0) to be a delta Dirac measure concentrated at point zero. The examples of transitions satisfying these assumptions (but without elastic labor supply) can be found a.o. in Nowak (2006). Also, it bears mentioning that our supermodularity assumptions on the primitives of preferences $u(c, l)$ and production g are critical for showing monotonicity of a best response operator in a model with an absorbing state.

In our approach, we can work with either bounded and unbounded state spaces. One reason we allow for both cases stems from an observation first made Balbus, Reffett, and Woźny (2009) that concerns the nondegeneracy of stationary Markov perfect equilibrium. In particular, if one assumes the following: (i) bounded state space, (ii) existence of an absorbing state 0, strict monotonicity of g, and (iii) interiority of a MPNE, this implies a positive probability of reaching an absorbing state 0 each period. This implies the Stationary Markov Equilibrium has a trivial invariant distribution. By allowing for unbounded state spaces, we can work out transitions with and without absorbing states, and obtain conditions where we can avoid this trivial outcome. We should also remark that in the existing literature, very little, if any, attention has been focused on the structure of stationary Markov equilibrium in the class of games studied. One exception, being Balbus, Reffett, and Woźny (2009) for the case of inelastic labor, and a more restrictive form of intergenerational altruism.

To understand our assumptions in the context of the existing literature, in related work on stochastic bequest economies (with inelastic labor supply), the paper of Amir (1996b) uses a different approach to characterizing the stochastic transition Q. Apart on the assumptions of the state space K that we have already discussed, the main differences between our case (following Nowak) and Amir assumptions are the following: (i) Amir assumes that transition Q, parameterized by current decisions, is (weakly) continuous, stochastically increasing and stochastically concave, while (ii) Nowak takes Q to depend on both current decisions and current state, and lets Q be given by a convex combination of a finite number of measures, where weights are given by the production process g_i.

Therefore, on the one hand, Nowak does not require stochastic monotonicity and stochastic concavity of Q, while on the other hand, Amir do not require the particular (convex combination) structure of Q. The critical results obtained in this paper follow from the monotonicity of a best response operator, where sufficient conditions can be given for Nowak structure of Q. It is not clear how such results can be easily generalized to the case of Amir's stochastic transition structure on Q. To see this, think of a transition given by assumption 3 when a measures λ_1 is stochastically dominating λ_2. This can generate Amir's transitions (see his example 2 and following comments); but, unfortunately as we argue later, this set of assumptions is not sufficient to show monotonicity of the best response operator we study (and, hence, not sufficient for uniqueness using methods developed in this paper).

By D we denote a set of all bounded, measurable pure strategies, i.e.:

$$D := \{h : K \to K \times L : \forall_{s \in S} h(s) \in A(s), h \text{ is bounded and measurable}\}$$

endowed with a sup norm (i.e., the topology of uniform convergence), and standard pointwise (product) order $\leq$ by:

$$(\forall \xi, \eta \in D) \quad \xi \leq \eta \quad \text{iff } (\forall s \in K)\, \xi_1(s) \leq \eta_1(s) \quad \text{and} \quad \xi_2(s) \leq \eta_2(s).$$

For $h \in D$, given continuity of the primitive data of the model (i.e., utility and stochastic production), we can define the best response map $BR(h)(s)$ as follows:

$$BR(h)(s) := \arg \max_{(c,l) \in A(s)} U(c, l; h, s).$$

Then, a *Markov perfect Nash equilibrium* (MPNE) in D is any function $h^* \in D$ such that $h^* \in BR(h^*)$.

3 Existence and approximation of MPNE

We begin our study of MPNE in this model by considering the case of stochastic production with an absorbing state. In this setting, we first prove the existence of MPNE in the set of bounded, measurable strategies D.

In particular, under assumptions 1 and 4, we can write the objective for a typical generation as:

$$U(c,l;h,s) := u(c,l) + \sum_{k=1}^{m} \int_K v(h_1(y), h_2(y))\lambda_k(dy|s)g_k(s-c, 1-l).$$

In Theorem 3.1, we state our first major existence result under the assumption of an absorbing state for stochastic production.

Theorem 3.1 (Existence of MPNE) *Under assumptions 1 and 4 there exists a MPNE. Moreover the set of MPNE in D is an anti-chain (i.e. has no ordered elements).*

This existence result in theorem 3.1 (and later, in 3.4) are obtained under very general conditions on technology and preferences. They extend the existence results previously obtained in Amir (1996b) and Nowak (2006) to the case of altruistic stochastic growth with more general paternalistic altriusm (namely, altriusm defined over multi-dimension strategies by the successor generation (in our case, consumption and leisure), as well as elastic labor supply.

As for corollaries of theorem 3.1, we can provide a further characterization of the continuity and monotonicity properties of any MPNE policy in D. We begin in corollary 1 with an additional result on the continuity properties of MPNE in the set of measurable strategies D.

Corollary 1 (Continuous MPNE) *Let assumptions 1 and 4 be satisfied. Assume additionally that*

i) there exists a μ- measurable function $\bar{\rho}$ such that $\rho_j(s', s) \leq \bar{\rho}(s')$ for each $s', s \in K$ and $j = 1, ..., m$

ii) for each function $f : K \to K$ such that $\int_K f(s')\bar{\rho}(s')\mu(ds') < \infty$ the integral $\int_K f(s')\rho(s', s)\mu(ds')$ is continous as a function of s,

iii) $\int_K v(s', 1)\bar{\rho}(s')\mu(ds') < \infty$.

Then there exists a MPNE (c^*, l^*) where $c^*(\cdot)$ and $l^*(\cdot)$ are continuous functions.

Notice, unlike the results in the case of inelastic labor supply and 1 period altruism (e.g., Amir (1996b)), our result is continuous MPNE. In particular, for elastic labor and more general intergeneration altruism, we are unable to obtain MPNE that are Lipschitzian (see discussion below). In particular, investment decisions are only continuous, but not increasing everywhere. This is also in contrast to the case of stochastic growth models with elastic labor supply and perfect commitment (where investment can be show to be increasing in the current period capital stock/output).

Finally we consider a case where MPNE is differentiable. For a function $f : K \times L \to \mathbb{R}$ by H we denote its Hessian:

$$Hes(f;c,l) = \begin{vmatrix} f^{(1,1)}(c,l) & f^{(2,1)}(c,l), \\ f^{(1,2)}(c,l) & f^{(2,2)}(c,l), \end{vmatrix}$$

and for function $u : K \times L \to \mathbb{R}$ and $g : K \times L \to \mathbb{R}$,

$$W(u,g;c,l) := u^{(1,1)}(c,l)g^{(2,2)}(s-c,1-l) + u^{(2,2)}(c,l)g^{(1,1)}(s-c,1-l)$$
$$-2u^{(1,2)}(c,l)g^{(1,2)}(s-c,1-l).$$

Corollary 2 (Differentiable MPNE) *Let assumptions 1 and 4 be satisfied with K compact. Assume additionally that*

- $u^{(1)}(0,l) = u^{(2)}(c,0) = \infty$ *for* $(c,l) \in A(s)$,
- $g^{(1)}(0,l) = g^{(2)}(c,0) = \infty$ *for* $(c,l) \in A(s)$,
- $Hes(u;c,l) \geq 0$ *and* $Hes(g;c,l) \geq 0$ *where at least one inequality is strict for all* $(c,l) \in A(s)$, *and* $W(u,g;c,l) \geq 0$,
- *For each* f, $s \to \int_K f(s')\lambda(ds'|s)$ *is differentiable.*

Then for all h $BR(h)(\cdot) \in \mathcal{C}^1(K)$ and hence MPNE is $\mathcal{C}^1$.

Amir (1996b) shows conditions when a MPNE is a bequest model (without elastic labor supply choice) is differentiable. Our result extends it to a two dimensional altruism case. The difference between our and Amir's method of showing a smooth equilibrium existence is that we require compact state space K. Also as observed by Amir (1996b) MPNE differentiability is an important result from of point of view of a Kohlberg (1976) uniqueness argument. Here we add another motivation for showing such a result. Namely differentiability of a MPNE allows one to extend a price decentralization of a MPNE allocation is our economies using methods proposed by Lane (1981) and extended by Lane and Leininger (1986). What remains to be done, is to decentralize allocations of our stochastic transition.[5]

Concerning these additional assumptions (to show differentiability of MPNE) note that $W(u,g;c,l) \geq 0$ whenever $u(c,l) = u_1(c) + u_2(l)$. The

[5]Some steps in this direction has been already taken by Magill and Quinzii (2009).

last condition in assumption for $\mathcal{C}^1$ MPNE is satisfied whenever $\lambda(A|s) := \int_A \rho(s,s')\mu(ds')$ for some finite measure μ with $\rho(\cdot,s) \in \mathcal{C}^{(1)}$ and $\int_K \sup_{s\in K} \rho^{(1)}(s,s')\mu(ds') < \infty$.

Finally, we can consider the monotonicity properties of MPNE (aside from simply the issues of investment noted above). In corollary 3, under an additional separability condition for stochastic production, we give conditions for the monotonicity properties of MPNE in D.

Corollary 3 (Monotone MPNE) *Let assumptions 1 and 5 be satisfied. Then, there exists a MPNE (c^*, l^*) with both $c^*(\cdot)$ and $l^*(\cdot)$ increasing functions.*

Combining results of theorem 3.1, corollaries 1 and 3, we obtain conditions for existence of a continuous and monotone MPNE under very general (complementarity) conditions. Notice, our result provides a *weaker* characterization of MPNE than obtained in Amir (1996b) and Nowak (2006) for models with inelastic labor supply. In particular, our characterization of a MPNE policies do not guarantee existence of an Lipschitz continuous MPNE. So by introducing two choice variables (consumption and leisure) into the game, we cannot guarantee this important additional characterization of MPNE. In Example 3.3 below, we also provide an example of our game where MPNE exist, but are neither monotone nor Lipschitz continuous. In this sense, we conclude that in stochastic altruistic growth models with elastic labor, Lipschitzian MPNE cannot be generally expected in this class of stochastic games. Further, it bears mentioning that we cannot drop Assumption 5 in corollary 3 concerning the existence of monotone Markov equilibrium.

To complete our characterization of MPNE in a baseline model, we now address the question of approximating MPNE in this game. In Theorem 3.2, we prove an important result concerning the approximation of a MPNE. To do this, we use a simple truncation/iteration argument. One can think of this as studying the structure of pointwise limits of finite iterations (and, hence, the result can be related to finite horizon truncations of our economies).

For $n \geq 1$, and given $s \in K$, we can recursively define two sequences: $\phi_{2n}(s) = BR(\phi_{2n-1})(s)$, $\phi_{2n+1}(s) = BR(\phi_{2n})(s)$ with $(\forall s \in K)\, \phi_1(s) = (0,0)$. Similarly we let $\psi_{2n}(s) = BR(\psi_{2n-1})(s)$, $\psi_{2n+1}(s) = BR(\psi_{2n})(s)$ with $(\forall s \in K)\, \psi_1(s) = (s,1)$. Observe, this can be done as under our assumptions, BR is a function (see lemma 4.3). With this notation, we present first existence of fixed edges (ϕ^d, ϕ^u) and (ψ^d, ψ^u) with a (pointwise) approximation result for a set of MPNE.

Theorem 3.2 (Approximation of MPNE set) *Let assumptions 1 and 4 be satisfied. Assume additionaly $u(c,0) = u(0,l) = 0$ for all $c \in K$ and $l \in L$. Then the following holds:*

i) there exist limits

$$(\forall s \in K) \quad \phi^d(s) = \lim_{n\to\infty} \phi_{2n-1}(s) \text{ and } \phi^u(s) = \lim_{n\to\infty} \phi_{2n}(s), \tag{1}$$

ii) as well as

$$(\forall s \in K) \quad \psi^u(s) = \lim_{n\to\infty} \psi_{2n-1}(s) \text{ and } \psi^d(s) = \lim_{n\to\infty} \psi_{2n}(s), \tag{2}$$

iii) $\phi^u = BR\left(\phi^d\right)$, $\phi^d = BR\left(\phi^u\right)$, *and* $\psi^u = BR\left(\psi^d\right)$, $\psi^d = BR\left(\psi^u\right)$,

iv) if h^* *is a MPNE then* $(\forall s \in K) \quad \phi^d(s) \le h^*(s) \le \psi^u(s)$,

v) if $\phi^d(s) = \psi^u(s)$ *for all* $s \in K$ *then there is a unique MPNE* h^**. Moreover* $h^*(s) = \phi^d(s) = \psi^u(s) = \phi^u(s) = \psi^d(s)$.

We now make an important remark on this theorem. The existence of fixed edges for iterations, as well as the limiting results in the theorem, follow directly from the monotonicity of a BR operator in the model with an absorbing state (see lemma 4.3). It turns out one way of showing these results is to adapt the methods in (Guo, Cho, and Zhu (2004), chapter 3.2) to our problem, where the computation of MNPE takes place with *decreasing* operators. Therefore, Theorem 3.2.(iv) states our first approximation result, i.e. pointwise bounds for a set of MPNE; then, Theorem 3.2.(v) provides a type of numerical stability result for iterative methods.

To obtain a further characterization of the set of MPNE, we need more assumptions on preferences (namely separability of utility with respect to consumption and leisure), as well as an Inada type assumptions on function g. We should mention, these assumptions are often used in the applied macroeconomics literature:

Theorem 3.3 (Uniqueness of a MPNE) *Let assumption 2 and 4 with* $m = 1$ *be satisfied i.e.*

$$Q(\cdot|s-c, 1-l, s) := g(s-c, 1-l)\lambda(\cdot|s) + (1 - g(s-c, 1-l))\delta_0(\cdot). \tag{3}$$

Assume additionally that g *is on the form* $g(c,l) = g_1(c) + g_2(l)$*, where both* g_i *are increasing, concave and twice continuously differentiable. Moreover,* $g_1'(0) = g_2'(0) = \infty$*. Finally assume that there exists a number* $\tau \in (0,1)$ *such that* $\forall (c,l) \in Int(A(s))$ *(with* $s > 0$*) we have:*

$$-\frac{\frac{v^{(1)}(c,l)}{v(c,l)}}{\frac{u_1''(c)}{u_1'(c)} + \frac{g_1''(s-c)}{g_1'(s-c)}} - \frac{\frac{v^{(2)}(c,l)}{v(c,l)}}{\frac{u_2''(l)}{u_2'(l)} + \frac{g_2''(1-l)}{g_2'(1-l)}} \le \tau. \tag{4}$$

i) Then there exists a unique MPNE h^* in D. Moreover, let $h_0 \in D$ be an arbitrary starting point in the sequence of iterations $\varphi_{n+1} = BR(\varphi_n)$ with $\varphi_1 = h_0$. Let us define $p_n(s) = \int\limits_K v(\varphi_n^1(s'), \varphi_n^2(s'))\lambda(ds'|s)$ then

$$\lim_{n\to\infty} ||p_n - p^*|| = 0, \text{ and } ||p_n - p^*|| \leq M(1 - \tau^{r^n}), \tag{5}$$

where M, τ are constants dependent on a choice of h_0.

ii) $\varphi_n \to h^*$ pointwise,

iii) If additionally K is bounded, λ has Strong Feller Property[6] *and starting point is either $(0,0)$ or $(s,1)$, then*

$$\lim_{n\to\infty} ||\varphi_n - h^*|| = 0. \tag{6}$$

Remark 1 *Note that the statement iii) in Theorem 3.3 is satisfied whenever $\varphi_2 \geq \varphi_1$ or $\varphi_2 \leq \varphi_1$ i.e. whenever first ineration is comparable with the starting point.*

A number of important things about Theorem 3.3. First, the theorem give conditions under which there exist a unique MPNE in D. These conditions, although restrictive, are actually often met in applications. We shall provide examples in a moment. Second, combining this with corollaries 1 and 3, we obtain existence of a unique, continuous and monotone MPNE. Our method of proving the uniqueness result is bases on the uniqueness of a fixed point of a particular operator defined an a (normal and solid) cone. The result is obtained by showing that under condition (4) particular operator, corresponding to best response map, is decreasing and e-convex (in the terminology of Guo and Lakshmikantham (1988)). Hence, by applying theorem 3.2.5 in Guo, Cho, and Zhu (2004), one has a unique fixed point and convergence and approximation results in Theorem 3.2.(v) follow.

Third, also observe that although in Theorem 3.2, we obtain approximation results for pointwise limits, in our context, we now get uniform convergence results. The reason for this follows again from condition (4), which guarantees that our cone in not only normal but also regular (see Guo, Cho, and Zhu (2004) for discussion). Fourth, although it is not obvious to verify whether the operator used in the proof of theorem in a contraction, by a converse to the contraction mapping theorem (see e.g. Leader (1982)), one obtains this link indirectly. This argument can be made explicit using exactly the same argument in Balbus, Reffett, and Woźny (2009) adapted to our setting, and it can give additional computation procedures and uniform error bounds for a (step function) approximation relative to unique MPNE.

[6] It means $s \to \int\limits_K f(s')\lambda(ds'|s)$ is continuous whenever f is bounded and measurable.

Finally, let us mention that the operator used in the proof of this theorem is defined on the set of bounded measurable functions on K, and assigns for any expected utility of the next generation its best response expected utility of a current generation (see proof of theorem 3.3 for the details). Hence, interesting, this operator is a operator defined on the space of *value functions*, and whose construction can be equivalently motivated by the correspondence-based strategic dynamic programming methods of Kydland and Prescott (1980) and Abreu, Pearce, and Stacchetti (1990), but only adapted to stochastic OLG models with discounting. That is, it could be defined (in this example) as a selection in strategic dynamic programming approach defined in spaces of (measurable) correspondences of continuation value functions. In this sense, we have proven that if we restrict our attention to strategic dynamic programming methods that select measurable continuation structures (ala Sleet (1998)) for our environment, this mapping would produce iterations that are described in Theorem 3.2.

That is, in some cases, strategic dynamic programming methods (at least local to a greatest fixed point) can possess geometric structure. Additionall, the way we calculate strategies associated with a particular value function is based on the (generalized) inverse procedure proposed by Coleman (2000). This indicates how all these methods can be unified in the context of our stochastic OLG model without commitment under additional conditions.

Finally observe that under conditions of theorem 3.3, we immediately obtain the following corollary to this theorem.

Corollary 4 (Finite horizon approximation) *Consider a finite horizon version of our economy, i.e. for a finite T let $(\forall t < T)$ preferences be given by $u_1(c_t)+u_2(l_t)+g(s_t-c_t,1-l_t)\int_K v(c_{t+1}(y),l_{t+1}(y))\lambda(dy|s_t)$ and for the last generation by $u_1(c_T)+u_2(l_T)$. Let assumptions of theorem 3.3 be satisfied and denote by $h_T := (h_T^1, h_T^2)$ the unique perfect equilibrium strategy of the first generation in T horizon game. Let $p_T^*(s) = \int\limits_K v(h_T^1(s'),h_T^2(s'))\lambda(ds'|s)$. Then*

$$\lim_{T\to\infty} ||p_T^* - p^*|| = 0,$$

where $p^*(s) = \int\limits_K v(h^*(s'))\lambda(ds'|s)$, and $h^*(s)$ is the unique MPNE from theorem 3.3. If λ has a Strong Feller Property and K is bounded by Theorem 3.3 we obtain

$$\lim_{T\to\infty} ||h^T - h^*|| = 0.$$

We should mention, additive separability in consumption and leisure is a typical assumption in applied lifecycle models. Further, in Balbus, Reffett, and Woźny (2009), the authors show that their uniqueness condition (with inelastic labor supply), similar to our condition (4), can be expressed in terms of elasticities of u', g' and v. In particular, by multiplying the numerator and the denominator in inequality (4) by c, we obtain the corresponding "elasticities" interpretation for our uniqueness theorem. Hence, constructing examples of where our theorems apply are quite simple.

Example 3.1 *Let $u_1(c) = c^{\alpha_1}$, $u_2(l) = l^{\alpha_2}$ and $v(c,l) = c^{\beta_1} l^{\beta_2}$. We find parameters α_i,β_i such that this model satisfies condition 4:*

$$-\frac{\frac{v^{(1)}(c,l)}{v(c,l)}}{\frac{u_1''(c)}{u_1'(c)} + \frac{g_1''(s-c)}{g_1'(s-c)}} - \frac{\frac{v^{(2)}(c,l)}{v(c,l)}}{\frac{u_2''(c)}{u_2'(c)} + \frac{g_2''(1-l)}{g_2'(1-l)}} =$$

$$= \frac{\beta_1}{1-\alpha_1 - \frac{g_1''(s-c)}{g_1'(s-c)}} + \frac{\beta_2}{1-\alpha_2 - \frac{g_2''(1-l)}{g_2'(1-l)}} \leq$$

$$\leq \frac{\beta_1}{1-\alpha_1} + \frac{\beta_2}{1-\alpha_2}.$$

Hence condition of theorem 3.3 is satisfied if $\frac{\beta_1}{1-\alpha_1} + \frac{\beta_2}{1-\alpha_2} < 1$ and g is arbitrary function satisfying assumption 4 and conditions of theorem 3.3.

Assumption on existence of an absorbing state may be restrictive, especially when the state space K is bounded (see discussion in Balbus, Reffett, and Woźny (2009)). For this reason, we now state our MPNE existence result for a model without absorbing point. Under assumptions 1 and 3, for a given strategy $h \in D$, the objective becomes now:

$$U(c,l;h,s) := u(c,l) + \beta(h,s) g(s-c, 1-l) + \gamma(h,s),$$

with

$$\gamma(h,s) := \int_K v(h_1(y), h_2(y)) \lambda_2(dy|s) \quad \text{and}$$

$$\beta(h,s) := \int_K v(h_1(y), h_2(y)) \lambda_1(dy|s) - \int_K v(h_1(y), h_2(y)) \lambda_2(dy|s).$$

We state the following theorem.

Theorem 3.4 (Existence of a continuous MPNE) *Under assumptions 1 and 3 there exists a MPNE. If in addition*

i) there exists a μ- measurable function $\bar{\rho}$ such that $\rho_j(s', s) \leq \bar{\rho}(s')$ for each $s', s \in K$ and $j = 1, ..., m$

ii) for each function $f : K \to K$ such that $\int_K f(s')\bar{\rho}(s')\mu(ds') < \infty$ the integral $\int_K f(s')\rho(s', s)\mu(ds')$ is continous as a function of s, *iii)* $\int\limits_K v(s',1)\bar{\rho}(s')\mu(ds') < \infty$.

Then $MPNE = (c^*, l^*)$*, where* $c^*(\cdot)$ *and* $l^*(\cdot)$ *are continuous functions.*

To better understand our results, we now present two additional examples showing application for results 3.1 and 3.2. For the moment, assume the bounded state space case (i.e., $K = [0, S]$, where $S \in \mathbb{R}_+$).

Example 3.2 *In this example, Assumptions 1 and 4 are satisfied*[7]*. Let* $u(c,l) = \sqrt{s}\sqrt[4]{l}$ *and* $v(c,l) = \sqrt{cl}$*, and*

$$Q(\cdot|s-c, 1-l, s) = \sqrt{s-c}\sqrt[4]{1-l}\lambda(\cdot) + \left(1 - \sqrt{s-c}\sqrt[4]{1-l}\right)\delta_0(\cdot),$$

where, λ *is a uniform distribution on* $[0,1]$*. Then, we have*

$$U(c,l;h,s) = \sqrt{c}\sqrt[4]{l} + \xi(h)\sqrt{s-c}\sqrt[4]{1-l},$$

with $\xi(h) := \int\limits_K \sqrt{(h_1(s'))(h_2(s'))}\lambda(ds')$*. The best response map* BR *is a well defined function, and can be described:*

$$BR(h)(s) := \left(\frac{s}{1+\xi^4(h)}, \frac{1}{1+\xi^4(h)}\right)$$

By Theorem 3.2, we conclude that each perfect equilibrium has the lower bound ϕ^d *and upper bound* ψ^u*, where*

$$\phi^d(s) := \lim_{n\to\infty} \phi_{2n-1}$$

and

$$\psi^u(s) := \lim_{n\to\infty} \psi_{2n-1},$$

where, the above sequences are in the form $\phi_1(s) = (0,0)$*, and, for* $n > 1$

$$\phi_{n+1}(s) = \left(\frac{s}{1+\xi^4(\phi_n)}, \frac{1}{1+\xi^4(\phi_n)}\right).$$

At the same time $\psi_1(s) = (s,1)$ *and for* $n > 1$

$$\psi_{n+1}(s) = \left(\frac{s}{1+\xi^4(\psi_n)}, \frac{1}{1+\xi^4(\psi_n)}\right).$$

We can compute $\xi(\phi_n)$ *by the following recursive formula:* $\xi(\phi_1) = 0$ *; for* $n > 1$ *, we have*

$$\xi(\phi_{n+1}) = \frac{\int\limits_K \sqrt{s}ds}{1+\xi^4(\phi_n)} = \frac{\frac{2}{3}}{1+\xi^4(\phi_n)}.$$

[7] Apart from strict concavity of a utility for $l = 0$ and $c = 0$. But this is irrelevant for results of this example as we can easily extend BR to be a function for $s = 0$ since la our supply is constant is this case.

The same recursive formula is satisfied for $\xi(\psi_n)$. Only initial value is different, i.e, $\xi(\psi_1) = \frac{2}{3}$. Note, the function $f(x) = \frac{\frac{2}{3}}{1+x^4}$ is decreasing, hence $\xi(\phi_n)$ and $\xi(\psi_n)$ have at most two cumulation points. Both of them must be fixed point of

$$f(f(x)) = \frac{\frac{2}{3}}{1 + \left(\frac{\frac{2}{3}}{1+x^4}\right)^4}.$$

Since $f(f(x))$ has exactly one fixed point $\xi^* \approx 0,5932$, hence this is also unique fixed point of f. Hence, $\xi(\phi_n) \to \xi^*$ and $\xi(\psi_n) \to \xi^*$. Further, the functions ϕ^d and ψ^u from Theorem 3.2 are equal and

$$\phi^d(s) = \psi^u(s) = \left(\frac{s}{1+(\xi^*)^4}, \frac{1}{1+(\xi^*)^4}\right) \approx \left(\frac{s}{1.12}, \frac{1}{1.12}\right) \approx (0.89\, s, 0.89).$$

Then by Theorem 3.2, this strategy above is a unique MPNE. Observe, however, the conditions of Theorem3.3 are not satisfied by this example. On the other hand, we show an application of approximation from Theorem 3.2.

In the next example, we show that strict concavity assumptions on u and g (on the interior of their domain) are necessary for BR to be a function. This is important when developing constructive procedures for characterizing MPNE.

Example 3.3 *Let $u(c,l) = \sqrt{cl}$, $v(c,l) = 3\sqrt{cl}$. Transition probability is of the form*

$$Q(\cdot|s-c, 1-l, s) = \sqrt{(s-c)(1-l)}\lambda(\cdot) + \left(1 - \sqrt{(s-c)(1-l)}\right)\delta_0(\cdot),$$

where λ is a uniform distribution on $[0,1]$. Note, neither u neither g is strictly concave in the interior of any $A(s)$, since both functions are linear on the diagonals $d(s) := \{(c,l) : c = ls\}$. We have

$$U(c,l;h,s) = \sqrt{cl} + \xi(h)\sqrt{(s-c)(1-l)},$$

with $\xi(h) := 3\int_K \sqrt{(h_1(s'))(h_2(s'))}\lambda(ds')$. Then, the best response map $BR : D \to 2^D$ is a multifunction described by:

$$BR(h)(s) = \begin{cases} \{(1,s)\} & \text{if} \quad \xi(h) < 1, \\ \{(sl,l) : l \in [0,1]\} & \text{if} \quad \xi(h) = 1, \\ \{(0,0)\} & \text{if} \quad \xi(h) > 1. \end{cases}$$

Hence, the maximal best response (pointwise order) is

$$\overline{BR}(h)(s) = \begin{cases} (1,s) & \text{if} \quad \xi(h) \le 1, \\ (0,0) & \text{if} \quad \xi(h) > 1. \end{cases}$$

while, the minimal best response is

$$\underline{BR}(h)(s) = \begin{cases} (1,s) & \text{if} \quad \xi(h) < 1, \\ (0,0) & \text{if} \quad \xi(h) \geq 1. \end{cases}$$

Each strategy of the form $h^*(s) = (s\, l^*(s), l^*(s))$ such that $\xi(h^*) = 1$ is a MPNE. This means $l^* : K \to [0,1]$ is arbitrary Borel-measurable function satisfying:

$$\xi(h) = 3 \int_S \sqrt{s'} l^*(s') ds' = 1,$$

or, equivalently,

$$\int_S \sqrt{s'} l^*(s') ds' = \frac{1}{3}.$$

Hence, there are many examples of perfect equilibria, for example: $h^1(s) = \left(\frac{s}{2}, \frac{1}{2}\right)$, $h^2(s) = \left(\frac{5}{6}s^2, \frac{5}{6}s\right)$ and $h^3(s) = \left(\frac{5}{6}s\sqrt{s(1-s)}, \frac{5}{6}\sqrt{s(1-s)}\right)$. Note that h^3 is neither increasing with respect to s, neither Lipshitz continuous. Finally, note $\phi^d(s) = (0,0)$ and $\psi^u(s) = (s,1)$, hence, in this case, our approximation becomes trivial.

We can finally consider the question of existence of Stationary Markov equilibrium. Recall, Theorems 3.1, 3.3 and 3.4 guarantee existence of MPNE in the models with and without absorbing state. To end this section, we study the stochastic process induced by MPNE in our game, and find conditions for existence of an associated nontrivial invariant distributions existence (and, hence, we can verify the existence of nontrivial Stationary Markov Equilibria)

Theorem 3.5 (Existence of a SME) *Assume 1 and 3. If additionally $c^*(\cdot)$ and $l^*(\cdot)$ are continuous functions, λ_2 does not depend on s (i.e. $\lambda_2(\cdot|s) \equiv \lambda_2(\cdot)$) with a dense in itself support[8] and $\sup_{s\in K} g(s,1) < 1$, then there exists a unique invariant distribution.*

To see the role of assumptions in theorem 3.5 follow the example:

Example 3.4 *Let $\lambda_1(\cdot|s) = \lambda_2(\cdot|s)$ and it is a Dirac delta in $s+1$. Then $Q(\{s+1\}|s) = 1$. Note that the assumption of theorem 3.5, i.e. $\lambda_2(\cdot|s)$ does not depend on s is not satisfied. It is easy to notice that the invariant distribution does not exist.*

[8] A set A is said to be dense in itself iff for all $a \in A$ $cl(A \setminus \{a\}) = cl(A)$ where $cl(A)$ is a closure of A. In other words does not contains isolated points.

4 Proofs

In this concluding section of the paper, we present all the proofs of the results in the paper. We begin with the proofs for the model with an absorbing state.

4.1 Proofs in the model with absorbing state

In this section we assume 1 and 4. We start by extending to the set of strategies D to a set of randomized policies: $\mathcal{D}$, i.e. if $\bar{h} \in \mathcal{D}$ then $\bar{h}$ is a transition probability from K to $K \times L$ such that $\bar{h}\left(A(s)|s\right) = 1$. And similarly define $\mathcal{BR} : \mathcal{D} \to \mathcal{D}$ the following way

$$\mathcal{BR}\left(\bar{h}\right)(s) := \arg \max_{(c,l)\in A(s)} U(c, l; \bar{h}, s),$$

where

$$U(c, l; \bar{h}, s) := u(c, l) + \sum_{k=1}^{m} \int_K \int_{K\times L} v(c', l')\bar{h}(dc', dl'|s)\lambda_k(dy|s)g_k(s-c, 1-l).$$

Following Nowak (2006) or Balbus and Nowak (2008) we endow $\mathcal{D}$ with the weak-star topology. By a Caratheodory function $w : C \to R$ on $C := K \times A$ with $A := K \times L$ we mean a function such that $w(s, \cdot)$ is continuous on $A(s)$ for each $s \in K$, $w(\cdot, a)$ is Borel measurable for each $a \in A(s)$ and $s \to \max_{a\in A(s)} |w(s, a)|$ is μ-integrable over K. Since all the sets $A(s)$ are compact, $\mathcal{D}$ is compact and metrizable when endowed with the weak-star topology. For the details we refer the reader to Balder (1980) or Chapter IV in Warga (1972). Here, we only mention that a sequence $\bar{h}_n$ converges to $\bar{h}$ if and only if for every Caratheodory function we have:

$$\int\limits_K \int\limits_{A(s)} w(s, a)\bar{h}_n(da|s)\mu(ds) \to \int\limits_K \int\limits_{A(s)} w(s, a)\bar{h}(da|s)\mu(ds).$$

Observe that $\mathcal{D}$ could be treated as the set of equivalence class of correlated strategies equal μ a.e. Note that each function $h \in D$ can be treated as a member of $\mathcal{D}$ which has property $\bar{h}(s) \in \{h(s)\}$ μ a.e.

Now define an auxiliary function

$$F(c, l, \xi, s) := u(c, l) + \sum_{k=1}^{m} \xi_k g_k(s - c, 1 - l),$$

with $\xi := (\xi_1, ..., \xi_m) \in \mathbb{R}^m_{++}$. We formulate a lemma.

Lemma 4.1 *The functions*

$$C_0(l, \xi, s) := \arg \max_{c\in[0,s]} F(c, l, \xi, s),$$

and

$$L_0(c,\xi,s) := \arg\max_{l\in[0,1]} F(c,l,\xi,s),$$

are well defined. Moreover, $c \to L_0(c,\xi,s)$ *and* $l \to C_0(l,\xi,s)$ *are increasing and continuous.*

Proof *Step 1:* Fix $l \in L$, $\xi \in \mathbb{R}^m_+$ and $s \in K$. Note that $C_0(l,\xi,0) = 0$ and $L_0(c,\xi,0) = 1$. Assume that $s > 0$.
Let $C_0(l) := C_0(l,\xi,s)$ We show that C_0 is well defined. By Assumption 1 and 4 we obtain that the function F is strictly concave and hence $C_0(l)$ is well defined on $l \in [0,1]$. Also $F(c,1;\xi,s) := u(c,1)$ and hence $C_0(1) = s$

We show that $C_0(\cdot)$ is an increasing function. Note that $A(s)$ is a complete lattice. Define[9] $\eta(c,l) = U(c,l;h,s)$.

$$\eta^{(1,2)}(c,l) = u^{(1,2)}(c,l) + \sum_{k=1}^{m} \xi_k g_k^{(1,2)}(s-c,1-l) \geq 0.$$

Hence η is supermodular and on $[0,s]$ and by Topkis (1978) and Berge maximum theorem (see Aliprantis and Border (1994), theorem 17.31) C_0 is increasing and continuous in l.

Step 2: Fix $c \in I(s)$. Let

$$L_0(c) = \arg\max_{l\in L} F(c,l,\xi,s).$$

We show that L_0 is well defined function. By Assumptions 1 and 4 we have $F(s,l,\xi,s) := u(s,l)$ and hence $L_0(s) = 1$. By Assumption 1 and 4 we know that $\zeta(c,l) := F(c,l;\xi,s)$ is strictly concave. Hence $L_0(c)$ is well defined in this case. Moreover by Assumptions 1 and 4 we have

$$\zeta^{(1,2)}(c) := u^{(1,2)}(c,l) + \sum_{k=1}^{m} \xi_k g^{(1,2)}(s-c,1-l) \geq 0.$$

Hence ζ is supermodular on a complete lattice $[0,s]$ and by Topkis (1978) and of Berge maximum theorem $L_0(\cdot)$ is increasing and continuous in c. ■

Lemma 4.2 *The functions* $\xi \to C_0(l,\xi,s)$ *and* $\xi \to L_0(c,\xi,s)$ *are decreasing in product order sense.*

Proof Note that by Assumption 1 and 4 we know that both functions $c \to F(c,l,\xi,s)$ and $l \to F(c,l,\xi,s)$ are strictly concave. Let $\xi^1 \leq \xi^2$ in product

[9] For the function $f(x_1,...,x_n)$ we denote $f^{(i,j)} := \frac{\partial^2 f}{\partial x_i \partial x_j}$. Similarly we denote first order derivatives.

order sense. Then by Assumption 4 we have

$$\begin{aligned} F^{(1)}\left(c,l,\xi^1,s\right) &= u^{(1)}(c,l) - \sum_{k=1}^{m} \xi_k^1 g_k^{(1)}(s-c,1-l) \\ &\geq u^{(1)}(c,l) - \sum_{k=1}^{m} \xi_k^2 g_k^{(1)}(s-c,1-l) \\ &= F^{(1)}\left(c,l,\xi^2,s\right). \end{aligned} \tag{7}$$

Hence we easily conclude that $C_0(l,\xi^1,s) \geq C_0(l,\xi^2,s)$ (see Topkis (1978) and Berge maximum theorem (see Aliprantis and Border (1994), theorem 17.31). Similarly we prove that $\xi \to L_0(c,\xi,s)$ is decreasing. ■

Lemma 4.3 *For each $h \in D$ function $BR(h)$ is well defined and decreasing.*

Proof By assumptions 1 and 4 the function $(c,l) \to U(c,l;h,s)$ is strictly concave. Hence there is exactly one solution to the maximizing problem of $(c,l) \to U(c,l;h,s)$. Hence $BR(h)$ is well defined function $BR : D \to D$.

We show monotonicity of BR. Fix $s \in K$. Note that $U(c,l;h,s) = F(c,l;\xi(h),s)$ where k-th coordinate of the vector $\xi(h)$ is

$$\xi_k(h) = \int_K v(h_1(s'),h_2(s'))\lambda_k(ds'|s).$$

Note by Assumption 4 analogously like in (7) we obtain that $\xi \to F^{(i)}(c,l;\xi,s)$ is decreasing for $i=1,2$. Hence F is a supermodular and continuous function with (c,l) on lattice $A(s)$ and has decreasing differences with $(c,l;h)$. By Topkis (1978) the (unique) selection $BR(h)$ is decreasing for any s. ■

Now we state an auxiliary lemma:

Lemma 4.4 *BR is a pointwise continuous function i.e. if $h_n \to h$ pointwise, then $BR(h_n)(\cdot) \to BR(h)(\cdot)$ pointwise as well.*

Proof Since $(c,l) \to U(c,l;h,s)$ is strictly concave and continuous on $A(s)$ it is sufficient to show that $h \to U(c,l;h,s)$ is continuous. Note that

$$U(c,l;h,s) = F(c,l,\xi(h,s),s).$$

Clearly F is continuous in ξ. It is sufficient to show that $\xi(\cdot,s)$ is continuous in the pointwise topology. If $h_n \to h$ pointwise we obtain $\xi(h_n,s) \to \xi(h,s)$ by Assumption 1,4 and Lebesgue Dominance Theorem. Hence $U(c,l;h,s)$ is continuous in h as superposition of continuous functions. Using Berge maximum theorem the convergence holds pointwise. ■

Proof of Theorem 3.1:
First we show that $\mathcal{BR}$ is a well defined function mapping $\mathcal{D}$ to $\mathcal{D}$. Let $\bar{h} \in \mathcal{D}$. Note that

$$U(c, l; \bar{h}, s) = F(c, l; \xi(\bar{h}, s), s),$$

where

$$\begin{aligned}
\xi_k(\bar{h}, s) &:= \int\limits_K \int\limits_{A(s)} v(c', l')\bar{h}(dc', dl'|s')\lambda_k(ds'|s) \\
&= \int\limits_K \int\limits_{A(s)} v(c', l')\rho_k(s', s)\bar{h}(dc', dl'|s')\mu(ds').
\end{aligned}$$

Note that by definition of F, assumption 1 and hence strict concavity of $(c, l) \to U(c, l; h, s)$ on $A(s)$ we immediately obtain that there is unique optimal solution of maximization problem of $U(c, l; \bar{h}, s)$. Hence we have shown that $\mathcal{BR} : \mathcal{D} \to \mathcal{D}$. Moreover by strict concavity, the image of $\mathcal{BR}$ is contained in D i.e. $\mathcal{BR}(\mathcal{D}) \subset D$. Now we show that $\mathcal{BR}$ is continuous in the weak-star topology. Let $\bar{h}_n \to \bar{h}$ in the weak star topology. Note that if $a = (c, l)$ then for each $s \in K$ the function

$$w_k(s', a) := v(a)\rho_k(s', s)$$

is a Caratheodory function. Hence

$$\xi_k(\bar{h}_n, s) \to \xi_k(\bar{h}, s) \quad \text{as } n \to \infty,$$

and $U(c, l; \bar{h}_n, s) \to U(c, l; \bar{h}, s)$. Since for each ξ and s the function $(c, l) \to U(c, l; \bar{h}, s)$ is strictly concave, hence by Berge maximum theorem optimal solution of $U(c, l; \bar{h}_n, s)$ must converge to the optimal solution of $U(c, l; \bar{h}, s)$. Hence there $\mathcal{BR}(\bar{h}_n) \to \mathcal{BR}(\bar{h})$ pointwise. Since $\mathcal{BR}(\bar{h}_n)$ and $\mathcal{BR}(\bar{h})$ are pure strategies, hence this convergence also holds in the weak star topology. Hence $\mathcal{BR}$ is continuous. Since $\mathcal{D}$ is compact in this topology, hence by Schauder-Tikhonov theorem we conclude that there exists fixed point $h^* = \mathcal{BR}(h^*)$ μ a.e. Let $h^o(s) := \mathcal{BR}(h^*)(s)$ pointwise. Since $\mathcal{BR} : \mathcal{D} \to D$, hence h^o must be a stationary strategy. Since $h^o = h^*$ μ a.e. by definition of the function $\xi(h, s)$ we conclude that $\xi(h^*, s) = \xi(h^o, s)$ for each $s \in K$ and hence for each (c, l) we have $U(c, l; h^o, s) = U(c, l; h^*, s)$. Hence $h^o = h^*$ for each $s \in K$ and $BR(h^o) = h^o(s) = \mathcal{BR}(h^*)(s)$ for each $s \in K$.

Finally the antichain structure of MPNE set results directly form the fact that BR is decreasing. ■

Proof of Corollary 1:
We show that a stationary MPNE $(c^*(s), l^*(s))$ is a continuous function. It is sufficient to show that BR maps D into the set of bounded, continuous

functions on K. Let $h \in D$ be a continuous function. Let $s_n \to s_0$ as n tends to ∞. By condition (iii) of this corollary we have

$$\int_K v(h_1(s'), h_2(s'))\bar{\rho}_j(s')\mu(ds') \quad \leq \quad \int_K v(s', 1)\bar{\rho}_j(s')\mu(ds') < \infty. \tag{8}$$

Hence and by (ii) we immediately obtain

$$\int_K v(h_1(s'), h_2(s'))\lambda_j(ds'|s_n) \to \int_K v(h_1(s'), h_2(s'))\lambda_j(ds'|s_0). \tag{9}$$

Let $(c_n, l_n) := BR(h)(s_n)$ and (c_0, l_0) be an arbitrary cummulation point of (c_n, l_n).

$$U(c_n, l_n; h, s_n) \geq U(c, l; h, s_n)$$

for all $(c, l) \in A(s)$ and $n \in N$. Hence and by (9) we immediately obtain

$$U(c_0, l_0; h, s) \geq U(c, l; h, s_0),$$

for all $(c, l) \in A(s)$. Hence $(c_0, l_0) := BR(h)(s_0)$. Since BR maps D into set of continous functions, hence $(c^*(\cdot), l^*(\cdot))$ must be a continuous function. ■

Proof of Corollary 2:
We obtain FOC:

$$u^{(1)}(c, l) - \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g_k^{(1)}(s - c, 1 - l) \quad = \quad 0 \tag{10}$$

$$u^{(2)}(c, l) - \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g_k^{(2)}(s - c, 1 - l) \quad = \quad 0 \tag{11}$$

We need to show that Jacobian of the function $G(c, l) = [G_1(c, l)\, G_2(c, l)]'$ say $J(G)(c, l)$ with

$$G_1(c, l) := u^{(1)}(c, l) - \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g_k^{(1)}(s - c, 1 - l) \tag{12}$$

$$G_2(c, l) := u^{(2)}(c, l) - \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g_k^{(2)}(s - c, 1 - l) \tag{13}$$

is not zero. Note that

$$G_i^{(j)}(c, l) = u^{(i,j)}(c, l) + \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g_k^{(i,j)}(s - c, 1 - l).$$

is

$$
\begin{aligned}
J(G)(c,l) &:= \prod_{i=1}^{2}\Big(u^{(i,i)}(c,l)+\gamma(s)g^{(i,i)}(s-c,1-l)\Big) \\
&- \Big(u^{(1,2)}(c,l)+\gamma(s)g^{(1,2)}(s-c,1-l)\Big) \\
&= u^{(1,1)}(c,l)u^{(2,2)}(c,l)-\Big(u^{(1,2)}(c,l)\Big)^2 \\
&+ \gamma^2(s)\left(g^{(1,1)}(s-c,1-l)g^{(2,2)}(s-c,1-l)-\Big(g^{(1,2)}(s-c,1-l)\Big)^2\right) \\
&+ \gamma(s)\Big(u^{(1,1)}(c,l)g^{(2,2)}(s-c,1-l) \\
&+ u^{(2,2)}(c,l)g^{(1,1)}(s-c,1-l) \\
&- 2u^{(1,2)}(c,l)g^{(1,2)}(s-c,1-l)\Big) \\
&= Hes(u;c,l)+\gamma^2(s)Hes(g;s-c,1-l) \\
&+ \gamma(s)W(u,g;c,l)>0
\end{aligned}
$$

with $\gamma(s)=\int\limits_K v(h_1(s'),h_2(s'))\lambda(ds'|s)$.

By implicit function theorem we obtain $BR(h)$ is in $\mathcal{C}^1$. Hence we show that BR maps D into a set of differentiable functions. ■

Proof of Corollary 3:

Note that the utility has now a form

$$
U(c,l;h,s)=u(c,l)+g_1(s-c)\int\limits_K v(h_1(s'),h_2(s'))\lambda(ds'|s)
$$

$$
+g_2(1-l)\int\limits_K v(h_1(s'),h_2(s'))\lambda(ds'|s).
$$

Fix arbitrary $s>0$, and $h\in D$. If $\int\limits_K v(h_1(s'),h_2(s'))\lambda(ds'|s)>0$, then by Assumption 1 and 5 this function above is strictly concave on $A(s)$, and hence maximization problem of $(c,l)\to U(c,l;h,s)$ has a unique solution. If $\int\limits_K v(h_1(s'),h_2(s'))\lambda(ds'|s)=0$, then only $(s,1)$ is optimal solution. Hence best response map BR is well defined function. Further from theorem 3.4 we obtain that BR has a fixed point.

Finally observe that by assumptions for each $h\in D$ where h is increasing, function $(c,l,s)\to U(c,l,;h,s)$ is supermodular in (c,l) and has increasing differences in (c,s) and (l,s), which follows that has increasing difference in $(c,l;s)$. Hence by Topkis (1978) theorem for each increasing $h\in D$ function $BR(h)(\cdot)$ in increasing on K. Hence BR maps increasing (bounded, measurable) functions into increasing (bounded, measurable) functions and a fixed point of BR, i.e. a MPNE is increasing on K. ■

Proof of Theorem 3.2:
Step 1. We prove (i). We show that ϕ_{2n-1} is increasing and ϕ_{2n} is decreasing. Clearly $\phi_1 \le \phi_3$ and $\phi_1 \le \phi_2$. By Lemma 4.3 and definition of sequence ϕ_n we obtain

$$\phi_2 = BR(\phi_1) \ge BR(\phi_3) = \phi_4.$$

Suppose that for some n hold $\phi_{2n} \ge \phi_{2(n+1)}$ and $\phi_{2n-1} \le \phi_{2n+1}$. By Lemma 4.3 and definition of sequence ϕ_n we obtain

$$\phi_{2n+1} = BR\left(\phi_{2n}\right) \le BR\left(\phi_{2n+2}\right) = \phi_{2n+3}.$$

Therefore

$$\phi_{2(n+2)} = BR\left(\phi_{2n+3}\right) \le BR\left(\phi_{2n+1}\right) = \phi_{2(n+1)}.$$

Finally we obtain that both sequences ϕ_{2n} and ϕ_{2n-1} are monotone and bounded. Hence there exist limits in (1).

Step 2. We prove (ii). We show that ψ_{2n-1} is decreasing and ψ_{2n} is increasing. Clearly $\psi_1 \ge \psi_3$ and $\psi_1 \ge \psi_2$. By Lemma 4.3 and definition of sequence ψ_n we obtain

$$\psi_2 = BR(\psi_1) \le BR(\psi_3) = \psi_4.$$

Suppose that for some n hold $\psi_{2n} \le \psi_{2(n+1)}$ and $\psi_{2n-1} \ge \psi_{2n+1}$. By Lemma 4.3 and definition of sequence ψ_n we obtain

$$\psi_{2n+1} = BR\left(\psi_{2n}\right) \ge BR\left(\psi_{2n+2}\right) = \psi_{2n+3}.$$

Therefore

$$\psi_{2(n+2)} = BR\left(\psi_{2n+3}\right) \ge BR\left(\psi_{2n+1}\right) = \psi_{2(n+1)}.$$

Finally we obtain that both sequences ψ_{2n} and ψ_{2n-1} are monotone and bounded. Hence there exist limits.

Step 3. We prove (iii). Note that $\phi_{2n+1} = BR(\phi_{2n})$. By Step 1 and by Lemma 4.4 we obtain $\phi^u = BR\left(\phi^d\right)$. By analogue reasoning we obtain the rest of results.

Step 4. We prove (iv). By definition of h^* we know that $h^* = BR\left(h^*\right)$. By definition of BR and ϕ_n and ψ_n we immediately obtain

$$\phi_1 \le BR(h^*) = h^* \le \psi_1. \tag{14}$$

Assume that for some n holds

$$\phi_{2n-1} \le h^* \le \psi_{2n-1}. \tag{15}$$

Note that

$$\phi_{2n+1} = BR\left(BR(\phi_{2n-1})\right) \quad \text{and} \quad \psi_{2n+1} = BR\left(BR(\psi_{2n-1})\right). \tag{16}$$

Observe that by lemma 4.3 function $BR \circ BR(\cdot)$ is increasing. Hence, combining (14), (15), (16) we obtain

$$\begin{aligned} \psi_{2n+1} &= BR\left(BR(\psi_{2n-1})\right) \geq BR\left(BR(h^*)\right) \\ &= h^* \\ &= BR\left(BR(h^*)\right) \geq BR\left(BR(\phi_{2n-1})\right) \\ &= \phi_{2n+3}. \end{aligned} \tag{17}$$

To finish the proof we just take a limit in (17).

Step 5. Proof of (v) is immediate from Theorem 3.1 and from (iv). ■

4.2 Proofs in the model with separated utility variables and absorbing state

By assumptions of Theorem 3.3 the objective becomes:

$$U(c, l; h, s) := u_1(c) + u_2(l) + \xi(h, s)[g_1(s - c) + g_2(1 - l)],$$

with $\xi(h, s) := \int_K v(h_1(y), h_2(y))\lambda(dy|s)$.

Proof of Theorem 3.3:

Step 1 We prove *i)* Let $\mathcal{P}$ be a set of bounded, Borel measurable functions $p : K \to R_+$with a pointwise partial order and the sup norm. Clearly $\mathcal{P}$ is a normal solid cone. Define an operator $T : \mathcal{P} \to \mathcal{P}$:

$$T(p)(s) = \int_K v(c_p(s'), l_p(s'))\lambda(ds'|s),$$

where $(c_p(s), l_p(s))$ is a measurable solution (refer to Brown and Purves (1973) theorem 2) of optimization problem of the function

$$H(c, l; p, s) := u_1(c) + u_2(l) + p(s)(g_1(s - c) + g_2(1 - l)).$$

Clearly H has decreasing differences in (c, p) and (l, p), hence by Topkis (1978) theorem (c^p, l^p) is decreasing. By assumption 2 $T(\cdot)$ is decreasing as well.

Now we show that the function $J(t) := t^\tau T(tp)$ $t \in (0, 1)$ is increasing for each $p \in Int\,(\mathcal{P})$ and τ from 4. Showing continuity of J at t=1 we obtain that $T(tp) \leq t^{-\tau}T(p)$, i.e. the e-convexity condition for T in theorem 3.2.5

of Guo, Cho, and Zhu (2004). Fix p from interior of $\mathcal{P}$ and $s \in K \setminus \{0\}$. Define $c(t) := c_{tp}$ and $l(t) := l_{tp}$.

First note that by $u_i'(0^+) = g_i'(0^+) = \infty$ $(i = 1, 2)$ for $t \in (0, 1)$ we have $(c(t), l(t)) \in Int(A(s))$.

Hence the equalities are satisfied

$$H^{(1)}(c(t), l(t); tp(s), s) = u_1'(c(t)) - tp(s)g_1'(s - c(t)) = 0$$

and

$$H^{(2)}(c(t), l(t); tp(s), s) = u_2'(l(t)) - tp(s)g_2'(1 - l(t)) = 0.$$

By implicit function theorem both $c(t)$ and $l(t)$ are differentiable and:

$$-c'(t) = \frac{p(s)g_1'(s - c(t)}{-\left(u_1''(c(t)) + tp(s)g_1''(s - c(t))\right)},$$

and

$$-l'(t) = \frac{p(s)g_2'(1 - l(t))}{-\left(u_2''(l) + tp(s)g_2''(1 - l(t))\right)}.$$

Then we have:

$$\begin{aligned}
-\frac{d}{dt}v(c(t), l(t)) &= -v^{(1)}(c(t), l(t))c'(t) - v^{(2)}(c(t), l(t))l'(t), \\
&= v^{(1)}(c(t), l(t))\frac{p(s)g_1'(s - c(t))}{-\left(u_1''(c(t)) + tp(s)g_1''(s - c(t))\right)}, \\
&+ v^{(2)}(c(t), l(t))\frac{p(s)g_2'(1 - l(t))}{-\left(u_2''(l(t)) + tp(s)g_2''(1 - l(t))\right)}, \\
&= -\frac{1}{t}\frac{\frac{v^{(1)}(c(t),l(t))}{v(c(t),l(t))}}{\frac{u_1''(c(t))}{u_1'(c(t))} + \frac{g_1''(s-c(t))}{g_1'(s-c(t))}}v(c(t), l(t)), \\
&- \frac{1}{t}\frac{\frac{v^{(2)}(c(t),l(t))}{v(c(t),l(t))}}{\frac{u_2''(l(t))}{u_2'(l(t))} + \frac{g_2''(1-l(t))}{g_2'(1-l(t))}}v(c(t), l(t)), \\
&\le \frac{\tau}{t}v(c(t), l(t)) \le \frac{\tau}{t}v(s, 1). \qquad (18)
\end{aligned}$$

The last inequality follows directly from (4). Hence if $c(t, s) := c(t)$ then the derivative of $v(c(t, s), l(t, s))$ is integrable with respect to probabilistic measure $\lambda(\cdot|s)$ since $v(s, 1)$ is integrable by assumption 2. Hence by (18) and (4) we obtain:

$$
\begin{aligned}
t\frac{d}{dt}\left(\int\limits_K v(c(t,s'),l(t,s'))\lambda(ds'|s)\right) &= t\int\limits_K \frac{d}{dt}v(c(t,s'),l(t,s'))\lambda(ds'|s) \\
&\geq -\tau\int\limits_K v(c(t,s),l(t,s))\lambda(ds'|s) \\
&= -\tau T(tp).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
J'(t) &= t^{\tau-1}\left(\tau T(tp) + t\left(\int\limits_K \frac{\partial}{\partial t}v(c(t,s'),l(t,s'))\lambda(ds'|s)\right)\right) \\
&\geq t^{\tau-1}\left(\tau T(tp) - \tau T(tp)\right) \\
&= 0.
\end{aligned}
$$

Hence $J(t)$ is increasing on $(0,1)$. Since J is continuous, J is decreasing on all $[0,1]$. Hence by Guo, Cho, and Zhu (2004) we obtain that T posses unique fixed point say p^*. Moreover, each sequence of iterations $p_{n+1} = T(p_n)$ (with p_0 arbitrary starting point) converges to p^*. Moreover, (5) and (6) hold. Hence there exists unique $h^* := (c^*,l^*)$ such that $p^*(s) = \int\limits_K v(c^*(s'),l^*(s'))\lambda(ds'|s)$, where the pair $(c^*(s),l^*(s))$ solves optimization problem of the function $(c,l)\to H(c,l;p^*,s)$. Moreover, (c^*,l^*) is a unique perfect equilibrium.

Step 2 Fix $s>0$. For $(c,l)\in A(s)$ and $\xi>0$ let $F(c,l;\xi) := u_1(c) + u_2(l) + \xi\left(g_1(s-c)+g_2(1-l)\right)$. Then H is on the form $H(c,l;p,s) = F(c,l;p(s))$. Note that from assumptions 2 and 5 F is continuous as a function of $(c,l;\xi)$. Clearly F is strictly concave on compact set $A(s)$. Hence and by Berge maximum theorem (see Aliprantis and Border (1994), theorem 17.31) we obtain $\varphi_n \to h^*$.

Step 3 Since λ has Strong Feller Property p^* must be continuous. since $p^*(s) = \int\limits_K v(c^*(s'),l^*(s'))\lambda(ds'|s)$. Note that

$$
h^*(s) = \arg\max_{(c,l)\in A(s)} H(c,l;p^*,s).
$$

and $s\to H(c,l;p,s)$ is continuous since p^* is. Hence conditions of Berge maximum theorem are satisfied and hence $h^*(\cdot)$ is continuous.

Note that by theorem 3.2 $\varphi_{2n}\to h^*$ and $\varphi_{2n-1}\to h^*$ and both sequences are monotone, since BR is decreasing operator. Since K is compact, hence both subsequences satisfy condition of Dini Theorem and we obtain uniform continuity of φ_n. ■

4.3 Proofs in the model without absorbing state

We now turn to a transition without an absorbing state (see assumption 3). By assumptions 1 and 3 the objective becomes:

$$U(c,l;h,s) := u(c,l) + \beta(h,s)g(s-c,1-l) + \gamma(h,s),$$

with $\beta(h,s) := \int_K v(h_1(y),h_2(y))\lambda_1(dy|s) - \int_K v(h_1(y),h_2(y))\lambda_2(dy|s)$, and $\gamma(h,s) := \int_K v(h_1(y),h_2(y))\lambda_2(dy|s)$.

Define $G(c,l;\beta,\gamma,s) := u(c,l) + \beta g(s-c,1-l) + \gamma$ with $\beta \in R$ and $\gamma \in \mathbb{R}_+$. We start with some preliminary lemma.

Lemma 4.5 *For each $\beta \in R$, $\gamma \in R_+$ and $s \in K$ the function $(c,l) \to G(c,l;\beta,\gamma,s)$ has a unique maximum.*

Proof Since $G(\cdot,\cdot;\beta,\gamma,s)$ is continuous on $A(s)$, hence the set of maximization problem must be nonempty. We show that optimal solution is unique. If $\beta > 0$ by Assumption 3 we obtain uniqueness of optimal solution since $(c,l) \to G(c,l;\beta,\gamma,s)$ is strictly concave. If $\beta < 0$ by Assumption 3 we obtain unique solution as well, moreover it is $(s,1)$. ■

Lemma 4.6 *Let $\beta_n \to \beta$ and $\gamma_n \to \gamma$. Let $h^n(s) = \arg\max_{(c,l)\in A(s)} G(c,l;\beta_n,\gamma_n,s)$ and $h(s) = arg\max_{(c,l)\in A(s)} G(c,l;\beta,\gamma,s)$. Then $h^n \to h$ pointwise.*

Proof Let $\beta_n \le 0$ and $\beta \le 0$. Then $h^n \equiv (s,1) = h$. If Let $\beta_n \ge 0$ and $\beta \ge 0$ then $(c,l) \to G(c,l;\beta_n,\gamma_n,s)$ and $(c,l) \to G(c,l;\beta_n,\gamma_n,s)$ are strictly concave on $A(s)$ Moreover, G is continuous with respect to (β,γ). Hence conditions of Berge maximum theorem are satisfied and desired convergence hold. ■

Proof of Theorem 3.4:
By lemma 4.5 and and assumption 1 we immediately obtain that there is unique optimal solution of maximization problem of $U(c,l;\bar{h},s)$. Hence we have shown that $\mathcal{BR} : \mathcal{D} \to \mathcal{D}$ is well defined. Moreover, the image of $\mathcal{BR}$ is contained in D i.e. $\mathcal{BR}(\mathcal{D}) \subset D$. Now we show that $\mathcal{BR}$ is continuous in the weak-star topology. Let $\bar{h}_n \to \bar{h}$ in the weak star topology. Note that if $a = (c,l)$ then for each $s \in K$ the function

$$w_k(s',a) := v(a)\rho_k(s',s)$$

is a Caratheodory function. Hence

$$\beta(\bar{h}_n,s) \to \beta(\bar{h},s) \quad \text{as } n \to \infty,$$

and

$$\gamma(\bar{h}_n, s) \to \gamma(\bar{h}, s) \quad \text{as } n \to \infty,$$

and hence $U(c, l; \bar{h}_n, s) \to U(c, l; \bar{h}, s)$. By Lemma 4.6 optimal solution of $U(c, l; \bar{h}_n, s)$ must converge to the optimal solution of $U(c, l; \bar{h}, s)$. Hence there $\mathcal{BR}(\bar{h}_n) \to \mathcal{BR}(\bar{h})$ pointwise and hence in the weak star topology. Hence $\mathcal{BR}$ is continuous. Therefore by Schauder-Tikhonov Theorem we conclude that there exists fixed point $h^* = \mathcal{BR}(h^*)$ μ a.e. Let $h^o := \mathcal{BR}(h^*)$ pointwise. Since $\mathcal{BR} : \mathcal{D} \to D$, hence h^o must be a stationary strategy. Since $h^o = h^*$ μ a.e. by definition of the functions $\beta(h, s)$ and $\gamma(h, s)$ we conclude that $\beta(h^*, s) = \beta(h^o, s)$ and $\gamma(h^*, s) = \gamma(h^o, s)$ for each $s \in K$ and hence for each (c, l) we have $U(c, l; h^o, s) = U(c, l; h^*, l)$. Hence $h^o = h^*$ for each $s \in K$ and $h^*(s) = \mathcal{BR}(h^*)(s)$ for each $s \in K$.

Finally to show continuity of a MPNE follow the reasoning in the proof of corollary 1. ■

Proof of Theorem 3.5:

Let (c^*, l^*) be given MNPE. For a transition probability $Q(\cdot|s - c^*(s), 1 - l^*(s), s)$ let us define a corresponding Markov operator $H : \mathcal{C}(K) \to \mathcal{C}(K)$ as

$$H(f)(s) := g(s - c^*(s), 1 - l^*(s)) \int_K f(s')\lambda_1(ds'|s)$$

$$+(1 - g(s - c^*(s), 1 - l^*(s))) \int_K f(s')\lambda_2(ds').$$

Observe that operator H is stable hence $Q(\cdot|s - c^*(s), 1 - l^*(s), s)$ has a Feller property. We now show that H is also quasi-compact[10]. To see that let us also define an operator L:

$$L(f)(s) := (1 - g(s - c^*(s), 1 - l^*(s))) \int_K f(s')\lambda_2(ds').$$

in $\mathcal{C}(K)$. Endow $\mathcal{C}(K)$ with the sup norm and denote a unit ball in $\mathcal{C}(K)$ by $\mathcal{B}$. Note that:

$$L(\mathcal{B}) = \left\{ (1 - g(s - c^*(s), 1 - l^*(s))) \int_K f(s')\lambda_2(ds') : f \in \mathcal{B} \right\}.$$

Note that

$$L(\mathcal{B}) = \{(1 - g(s - c^*(s), 1 - l^*(s)))\, \alpha : \alpha \in [0, 1]\}.$$

[10]Endow $\mathcal{C}(K)$ with the sup norm. An operator $H : \mathcal{C}(K) \to \mathcal{C}(K)$ is said to be quasi-compact if there exists a natural number n and a compact operator L such that $||H^n - L|| < 1$.

is the compact set. Hence L is a compact operator. Then:

$$\begin{aligned}|H(f)(s)-L(f)(s)| &= \left|g(s-c^*(s),1-l^*(s))\int\limits_K f(s')\lambda_1(ds'|s)\right| \\ &\le g(s-c^*(s),1-l^*(s))\int\limits_K \left|f(s')\right|\lambda_1(ds'|s) \\ &\le \sup_{s\in K} g(s,1)<1.\end{aligned}$$

This completes that H is quasi-compact. Finally applying theorem 3.3 from Futia (1982) we get that H is equicontinuous. We take arbitrary element $s_0 \in \text{supp}(\lambda_2)$. Since $Int(\text{supp}(\lambda_2))$ is dense in itself we obtain $U_\varepsilon := (s_0 - \varepsilon, s_0+\varepsilon)\cap Int(\text{supp}(\lambda_2)) \neq \emptyset$, for all ε. Hence $Q(U_\varepsilon|s-c^*(s),1-l^*(s),s)>0$. Hence Q satisfies uniqueness criterion 2.11 in Futia (1982) . Therefore thesis of this theorem follows directly from his theorem 2.12. ■

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