Iterative Monotone Comparative Statics*

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February 2022

Abstract

For an increasing upper order hemi-continuous correspondence $F : A \rightrightarrows A$, where A a σ-complete lattice, we first provide tight fixed-point bounds for sufficiently large iterations $F^k(a^0)$, starting from *any* point $a^0 \in A$. We use this result for conducting iterative fixed-point comparative statics, and then apply our results to monotone games and economies. For games of strategic complementarities, we improve the correspondence principle based results of Echenique (2002) by allowing for divergent learning processes, unstable fixed points, equilibrium indeterminacies, and unordered perturbations. We also apply our results to the comparative statics of stationary equilibria in large economies and the set of recursive equilibria in macroeconomic models with indeterminacies.

Keywords: comparative statics; comparative dynamics; adaptive learning; monotone iterations; games with strategic complementarities

JEL classification: C62, C65, C72

* Łukasz Woźny acknowledges financial support by the National Science Center, Poland: NCN grant number UMO-2019/35/B/HS4/00346. Reffett acknowledges the Dean's Summer Research Award at the WP Carey School of Business at ASU for it financial support.

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1 Introduction

This paper proposes a new iterative approach to characterizing the equilibrium monotone comparative statics in economic models. Our propositions extend the existing results based upon fixed-point comparative statics for parameterized monotone correspondences in complete lattices, where the comparative statics results pertain to extremal fixed points only.[1] Our approach to identifying equilibrium comparative statics is both iterative, and viewed from the perspective of *any* initial equilibrium (or actually, *any* initial element of the domain of the correspondence). It is hence, methodologically, in the spirit of the celebrated "correspondence principle", a concept first presented in the work of Samuelson (1947), and then extended most notably in a series of papers by Echenique (e.g., Echenique (2002, 2004)), and McLennan (2015).

To understand the nature of the paper's methodological contribution, we start with a motivating example of a game that highlights both the limitations of the existing methods, as well as the contributions of our new iterative monotone comparative statics approach. The example is a modified version of the simple joint venture game first studied in Milgrom and Roberts (1990).[2]

Motivating example *Consider a game with two players, where players 1 and 2 choose actions* a_1 *and* a_2*, respectively, where* $a_i \in A_i$*,* A_i *is an interval, and* a_i *is interpreted as player* i*'s effort. The cost of taking action* a_i *for player* $i = 1, 2$ *is* ca_i*, for some* $c \in (0, 1)$*. The output of the team consisting of the two players is* $2\min\{a_1, a_2\}$*. So the payoff of each player* i *is* $\min\{a_1, a_2\} - ca_i$*. First, note that*

[1] For example, the seminal fixed-point comparative statics results for strong set order monotone (or "ascending") correspondences in complete lattices were obtained in Veinott (1992), chapter 4. See also Topkis (1998), Theorem 2.5.2. We shall define the class of parameterized monotone correspondences that our results apply to in the next section, but the class we consider covers many applications in the economics literature.

[2] See Milgrom and Roberts (1990), Section 4, Example 5.

*this game has a **continuum of equilibria**. All pairs (a_1, a_2) such that $a_1 = a_2$ are equilibrium strategies.*

*Now, suppose that players are initially playing **any** (possibly equilibrium) actions a_1^0 and a_2^0, and now the productivity of player 1 increases (so the output is $2\min\{ta_1, a_2\}$ for some $t > 1$, and the payoffs are $\min\{ta_1, a_2\} - ca_i$ for $i = 1, 2$.) Intuitively, one would think that output should increase, but we cannot make this conclusion by comparing equilibria using any of the methods in the existing literature.*

*For example, if $a_1^0 = a_2^0 = a^0 > 0$, the total output in this equilibrium (for $t = 1$) is equal to a^0. For $t > 1$, the game has a continuum of equilibria, in some of them the output is higher than a^0, but in others the output is lower than a^0. In addition, **all equilibria are unstable**.*

*Now, if we assume that learning happens through the **best-response dynamic**, starting from **an arbitrary pair** a_1^0, a_2^0, we have that $a_1^k = a_2^0/t$ and $a_2^k = ta_1^0$ for all odd k and $a_1^k = a_1^0$ and $a_2^k = a_2^0$ for all even k. So, the output is never lower than a^0 for $t > 1$, but for some values of a_1^0 and a_2^0 it happens to be strictly higher. This dynamic does not converge. However, applying less extreme dynamics, e.g., the fictitious play in which players best respond to the average of the past actions of their opponents, we converge to the equilibrium $a_1^* = (a_1^0 t + a_2^0)/2t$ and $a_2^* = (a_1^0 t + a_2^0)/2$ for $t > 1$ in which the output is $a_1^0 t + a_2^0 > a_1^0 + a_2^0$ (unless $a_1^0 = 0$).*

*Moreover, for any a_1^0, a_2^0, the **action of at least one player cannot increase** in the initial response to the increase in t, and typically one of the two players takes a strictly lower action; yet the output is **never** lower.*

The purpose of this paper is to propose an approach to equilibrium comparative statics in economic environments such as in the case of our motivating example where complementarities play a critical role, including environments in which the existing methods for obtaining monotone comparative statics appear inadequate. Such situations can arise in games with strategic complementarities, but also in other economic settings such as (dynamic) general equilibrium economies and the study of stationary equilibria in large economies where the verification of existence of equilibrium involves application of fixed-point results relying on monotone correspondences.

We begin by constructing tight lower and upper fixed-point bounds for any sequence of iterations of an increasing correspondence $F : X \rightrightarrows X$ that transforms a $\sigma-$complete lattice X starting from any initial point $x^0 \in X$. We do so by extending the recent results of Olszewski (2021b) on monotone and continuous functions.[3] With these tight fixed-point bounds, we provide a new set of results on iterative characterization of fixed-point comparative statics for parameterized monotone correspondences in $\sigma-$complete lattices. We consider monotone (or "increasing") comparative statics when the lower fixed-point *bound* under the new parameter value is no smaller than the outcome observed under the old parameter.

One critical motivation for the way we perform iterative monotone comparative statics can be found in the context of games of strategic complementarities (GSC),[4] where the best response dynamic is an extreme dynamic in which players respond to most recent actions of their opponents while disregarding the actions from the previous periods. Even under this extreme dynamic we observe in the long run only outcomes greater than the lower bound. And even if the play will converge to a Nash equilibrium, we should expect this equilibrium to be greater than the lower bound. This approach allows one to obtain new monotone comparative statics results for the case of GSC, in particular, answer the questions highlighted by the

[3] In Olszewski (2021a), he shows similar ideas can be applied without order continuity conditions, but at the expense of requiring transfinite arguments.

[4]GSC are games in which each agent's best response increases with an increase in the opponents' actions. See Vives (1990), Veinott (1992), Milgrom and Shannon (1994), and Topkis (1998) for a discussion and examples of such games.

motivating example.

To illustrate our approach, consider a game with n players, and let $BR : A \times T \rightrightarrows A$ be the parameterized best response correspondence of the game, where the space A consists of all the joint action profiles, T is a partially ordered set of parameters, each player's action space A_i is a σ-complete lattice and $A = \prod_{i=1}^{n} A_i$ is equipped with the product order. Suppose the parameter changes from t to $t' > t$. For the case, in which initial a^0 is an equilibrium of the game for the lower parameter t, we compute an equilibrium that is smaller than sufficiently large iterations of $BR^k(a^0, t')$. If this new equilibrium is greater than a^0, then we claim that the players' actions increase in response to the parameter change. We emphasize, however, that we provide conditions for monotone comparative statics even for *arbitrary* action profiles $a^0 \in A$. In particular, a^0 need not be equilibrium of a the game for t, nor a^0 need to be related (i.e., ordered) to the elements of $BR(a^0, t')$. Our comparative statics results are applicable even if the iterative adaptive learning process is *divergent*, e.g., when there are *no stable* equilibria of $BR(\cdot, t')$ or $BR(\cdot, t')$ does not have any continuous selection. Moreover, our results also hold in settings with *continuum* of equilibria, i.e., where equilibria are *not locally unique* or are *indeterminate*. We also provide conditions to state comparative statics results for so called "mixed shocks", i.e., shocks affecting some players' best responses positively and others negatively, or when BR is monotone on A but not necessarily on T. In all these cases, the set of currently available tools for conducting equilibrium comparative statics is of limited use.

Related literature Comparative statics analysis has always been a foundational tool of economic analysis. It asks how the *set* of optimal or equilibrium solutions of an economic model vary relative to a perturbation of the model's parameters. Such predictions are important as they contain much of the empirical content of the economic model being studied.

For economic models, where the equilibrium of the model is a solution to an optimization problem, there is a large set of comparative statics tools. They involve, among others, the implicit function theorem. These tools typically require strong regularity conditions on the optimization problem (e.g., the smoothness of objectives and constraints, the interiority of all optimal solutions, etc.), and comparative statics predictions are often only *local* in nature.[5] Alternatively, lattice programming provides a set of tools for obtaining *global* monotone comparative statics of optimal solutions relative to parameter change. In optimization settings, the existence of multiple optimal solutions is not an impediment to progress.

Performing comparative statics analysis on *equilibrium* problems is more complicated. Especially in economic models with multiple equilibria, fixed-point comparative statics typically involves the tools of transversality and degree theory, and other from differential topology. These tools typically provide only weak local equilibrium comparative statics results, and even for these, require stronger regularity conditions on the primitives than in the context of implicit function based comparative statics of optimization problems.[6] Alteratively, there is an extensive literature on fixed-point comparative statics for parameterized monotone operators and correspondences that transform suitably chain-complete partially ordered sets. What is especially interesting about these tools is often the equilibrium comparative static is *computable.* But a general limitation of these existing order theoretic approaches is they typically provide limited comparative statics information in the presence of multiple equilibria. That is, the comparative statics results pertain

[5] There are methods for globalizing the implicit function theorem. See the celebrated work of Gale and Nikaido (1965), as well as more recent contributions (and references therewithin) of Blot (1991), Phillips (2012), and Cristea (2017).

[6] There is an extensive literature on these approaches to equilibria comparative statics for "regular economies" based upon versions of Thom's transversality theory and Sard's theorem. For surveys of work on regular economies, see Mas-Colell (1985, 1996), Nagata (2004), and McLennan (2018).

typically to only *extremal* equilibria (i.e., least/minimal or greatest/maximal) and the constructive nature of the result does not hold for iterations from *any* initial point.[7]

A well-known approach to studying the equilibrium comparative statics of *any* equilibria is embodied in the so-called "Correspondence Principle", which was suggested originally in the seminal work of Samuelson (1947).[8] Here, one seeks to identify regularity conditions of optimization problems or equilibrium problems for unambiguous equilibrium comparative statics by refining away *unstable* equilibria, and then restricting attention to *regular (or smooth) equilibria*. This approach is inherently dynamic, and can be applied when equilibria are *locally unique* and amenable to applications of the implicit function theorem. Echenique (2002) has extended these ideas substantially, and is able to prove a stronger versions of the Correspondence Principle for GSC on lattices A when there is a convex set of parameters T. For example, Echenique (2002) showed that in GSC, a *continuous* equilibrium selector $t \to a^*(t)$ is increasing if and only if it selects *stable* equilibria.[9]

Our paper shares with the correspondence principle the idea that the identification of monotone comparative statics is critically tied to a *dynamic* approach. That is, one is interested in viewing an equilibrium as the stationary point of a dynamical system, in which a new equilibrium emerges from an old equilibrium after a change in a parameter value via some dynamic adjustment process. For example, if an equilibrium at the original set of parameters is *locally stable*, then

[7] See, for example, the computable comparative statics results for Nash equilibrium in Bayesian supermodular games in Van Zandt (2010). Also, see the discussion in Balbus et al. (2015a). Here, iterations need to start from least (resp., greatest) elements of the domain.

[8] See also McLennan (2015) for an interesting recent discussion of the correspondence principle, citations of the extensive literature and implications for equilibrium comparative statics.

[9] See also Echenique (2002, 2004) for the precise formulations of various versions of this result. Notice, in the presence of multiple equilibria, the existence of continuous equilibrium selectors is an added complication in applying Echenique (2002) results. But he is able to weaken the continuity requirements in some cases.

one can develop sufficient conditions on the behavior of this dynamical system that guarantees that starting from the equilibrium for the old parameter, the dynamical system will actually converge to the new equilibrium for *small* changes of the parameter. But this leaves open many interesting questions. Aside from the obvious question of relaxing the needed topological conditions required to study the stability of local equilibrium comparative statics via correspondence principle based arguments, what do we do when *all* the equilibria are *unstable*? What if there is a continuum of equilibria (i.e., equilibria are indeterminate and not locally isolated)? In Echenique (2002), he shows that in GSC, if a correspondence $BR : A \rightrightarrows A$ defined on the space of action profiles A, a complete lattice, is *strongly increasing* and upper hemi-continuous, then for *every* action profile a such that $a \leq \inf BR(a)$, a best-response sequence starting from a converges to a fixed point of BR that is *higher* than a.[10] Relative to the answers given in Echenique (2002), what can be said on comparative statics results if his requirement that the best responses to a *exceed* a is violated? This latter condition appears strong in applications. Can we perform comparative statics analysis in which the outcome for the old parameter value evolves in a process of dynamic learning to the outcome for the new parameter value? In this paper we provide answers to these questions.

The remainder of the paper is organized as follows. In the next section, we define mathematical terminology. In section 3, we generalize the results of Olszewski (2021b) to weakly and strongly increasing correspondences in lattices. In section 4, we present our two main fixed-point comparative statics results, and apply them to GSC. Section 5 contains additional applications while in the last section of the paper we make some concluding remarks. We delegated proofs of

[10] See Milgrom and Roberts (1990) and Milgrom and Shannon (1994) for some results preceding Echenique's result. See also Balbus et al. (2021) Proposition A.2. and Balbus et al. (2015b) Theorem 1 for some recent generalizations of monotone comparative statics results for dynamic games. Finally, see Heikkilä and Reffett (2006) for fixed-point results for parameterized correspondences with applications to games (and in particular, GSC).

lemmas to Appendix A.

2 Preliminaries

We start with introducing some basic definitions. A partially ordered set (or *poset*) is set A equipped with a partial order $\geq$. For $a', a \in A$, we say a' is *strictly higher* than a, and write $a' > a$, whenever $a' \geq a$ and $a' \neq a$. A poset $(A, \geq)$ is a *lattice* if for any $a, a' \in A$ there exist the *join* $a \vee a' := \sup\{a, a'\} \in A$ and the *meet* $a \wedge a' := \inf\{a, a'\} \in A$. A lattice A is *complete* if there also exist $\bigvee B := \sup B \in A$ and $\bigwedge B := \inf B \in A$ for all $B \subseteq A$. A lattice is *sigma-complete*, whenever the lattice is complete relative to any countable subset B. A subset $B \subset A$ is a *sublattice* of A if B is a lattice in the order induced from A, in particular, the infimum $a \vee a'$ and supremum $a \wedge a'$ as defined in $(A, \geq)$ belong to B for all $a, a' \in B$. A sublattice B of a lattice A is a *complete sublattice* if for any $C \subseteq B$ the supremum $\bigvee C$ and the infimum $\bigwedge C$, as defined $(A, \geq)$, exists and belong to B.

We can compare subsets of A using set relations compatible with $(A, \geq)$. Let $2^A \backslash \{\varnothing\}$ be all the nonempty subsets of A. If $(A, \geq)$ is a poset, and $B, B' \in 2^A \backslash \{\varnothing\}$, we say $B' \geq^S B$ if for all $b' \in B'$, $b \in B$, $b' \geq b$. If $(A, \geq)$ is a lattice, B and B' two nonempty subset of A, we say B' is (Veinott)-*strong set order higher* than B, denoted by $B' \geq^{SSO} B$, whenever for every $b' \in B'$ and $b \in B$, $b' \wedge b \in B$ and $b' \vee b \in B'$.

Let $F : A \rightrightarrows B$ be a nonempty-valued correspondence, where $(A, \geq)$ and $(B, \geq)$ are posets. We say F is *strongly monotone (increasing)* whenever $a' > a$ implies that $F(a') \geq^S F(a)$. Now, let $(B, \geq)$ be a lattice. We say F is *weakly monotone (increasing)* whenever $a' > a$ implies that $F(a') \geq^{SSO} F(a)$.

A sequence $(a^k)_{k=0}^{\infty}$ of elements of A is *increasing* if $a^{k+1} \geq a^k$ for each k. It is *strictly increasing* if $a^{k+1} > a^k$ for each k. *Decreasing* and *strictly decreasing* sequences can be defined in the obvious dual manner. A *monotone sequence* then is either increasing or decreasing. We say that a increasing (resp., decreasing) sequence $(a^k)_{k=0}^{\infty}$ *converges* to $a \in A$ whenever $\bigvee_{k\geq 0} a^k = a$ (resp., $\bigwedge_{k\geq 0} a^k = a$).[11] That is, when a is the supremum (resp., infimum) of the increasing (resp., decreasing) sequence. We say that F is *upper order hemicontinuous* whenever it satisfies the following condition: if any *monotone* sequence $(a^k)_{k=0}^{\infty}$ converges to a, then any *monotone* sequence $(b^k)_{k=0}^{\infty}$ such that $b^k \in F(a^k)$ for all k converges to some $b \in F(a)$.[12]

Finally, a function $f : A \mapsto B$ is order-preserving (or increasing) on A if $a \leq a'$ implies $f(a) \leq f(a')$ for a, a' in A. The function f is *upward order continuous* (resp., *downward order continuous*) if for any increasing convergent sequence (a^k) with $a^k \in A$, we have:

$$f\left(\bigvee_{k\in\mathbb{N}} a^k\right) = \bigvee_{k\in\mathbb{N}} f(a^k) \quad \left(\text{respectively } f\left(\bigwedge_{k\in\mathbb{N}} a^k\right) = \bigwedge_{k\in\mathbb{N}} f(a^k)\right).$$

The function f is then *order continuous* if it is both upward and downward order continuous. Notice, if f is upward (resp., downward) order continuous, it is order preserving (or increasing) function on A.[13]

3 Iterations on monotone correspondences

To develop our theory of iterative monotone comparative statics for monotone upper order hemicontinous correspondences in $\sigma-$complete lattices, we must first generalize the results in Olszewski (2021b) on the convergence of iterations of

[11] Notice, in definition of convergence of monotone sequences, convergence is in *order*.

[12] Notice that upper order hemicontinuity of a correspondence imposes "closure property" relative to only *monotone sequences*.

[13] If a function is upward (resp., downward) order continuous, it is also by definition sup (resp., inf) preserving. So our definitions here coincide with standard definitions of order continuity (e.g., Dugundji and Granas (1982), p. 15).

monotone functions. This generalization is important itself, but it will also become the foundation to our iterative approach to monotone comparative statics.

For any given $a^0 \in A$, we will first define a of pair fixed points (denoted by $\underline{a}^*$ and $\overline{a}^*$) of $F : A \rightrightarrows A$ that provide tight fixed-point bounds for *all* iterations of the correspondence F. To do this, we start with a basic assumption on F.

Assumption 1 *A is a sigma-complete lattice. $F : A \rightrightarrows A$ is weakly monotone and upper order hemicontinuous. Moreover, for any $a \in A$, $F(a)$ is a sub-complete sublattice of A.*

We will make Assumption 1 throughout the paper. It will be convenient to implicitly make Assumption 1 in the statements of all our results, although it will follow from the proofs that some results require only a part of Assumption 1.

Now, define the functions:

$$\underline{F}(a) := \bigwedge F(a) \quad \text{and} \quad \overline{F}(a) := \bigvee F(a).$$

Under Assumption 1, $\overline{F}(a)$ and $\underline{F}(a)$ are both well-defined.

Lemma 1 *The functions $\overline{F}(a)$ and $\underline{F}(a)$ are both well defined selections of F.*[14] *Moreover, $\overline{F}$ (resp., $\underline{F}$) is downward order continuous (resp., upward order continuous).*

Let $\underline{a}^1 = \inf F(a^0) = \bigwedge F(a^0)$ and $\overline{a}^1 = \sup F(a^0) = \bigvee F(a^0)$ be the infimum and the supremum of $F(a^0)$; by induction, for $k = 1, 2, \ldots$ let $\underline{a}^{k+1}$ and $\overline{a}^{k+1}$ be the infimum of $F(\underline{a}^k)$ and supremum of $F(\overline{a}^k)$, i.e.

$$\underline{a}^{k+1} = \bigwedge F(\underline{a}^k) \quad \text{and} \quad \overline{a}^{k+1} = \bigvee F(\overline{a}^k).$$

[14] Selection of a correspondence $F : A \rightrightarrows B$ is any function $f : A \to B$ such that $f(a) \in F(a)$ for any $a \in A$.

It will be convenient to define $\underline{a}^0$ and $\overline{a}^0$ as a^0. Let $\underline{a}^\omega = \liminf_k \underline{a}^k$ and $\overline{a}^\omega = \limsup_k \overline{a}^k$. That is,

$$\underline{a}^\omega = \lim_k \bigwedge_{l \geq k} \underline{a}^l \quad \text{and} \quad \overline{a}^\omega = \lim_k \bigvee_{l \geq k} \overline{a}^l.$$

Lemma 2 *There exists $a \in F(\underline{a}^\omega)$ such that $a \leq \underline{a}^\omega$ Similarly, there exists $a \in F(\overline{a}^\omega)$ such that $a \geq \overline{a}^\omega$.*

If $\underline{a}^\omega$ is a fixed point of F, then define $\underline{a}^* = \underline{a}^\omega$; similarly, if $\overline{a}^\omega$ is a fixed point of F, then define $\overline{a}^* = \overline{a}^\omega$. Otherwise, let $\underline{a}^{\omega+1}$ be the supremum of values of $F(\underline{a}^\omega)$ that are smaller than $\underline{a}^\omega$, and let $\overline{a}^{\omega+1}$ be the infimum of values of $F(\overline{a}^\omega)$ that are greater than $\overline{a}^\omega$. That is, more formally:

$$\underline{a}^{\omega+1} = \bigvee F(\underline{a}^\omega) \cap I(\underline{a}^\omega) \quad \text{and} \quad \overline{a}^{\omega+1} = \bigwedge F(\overline{a}^\omega) \cap J(\overline{a}^\omega),$$

with $I(a) := \{a' \in A : a' \leq a\}$ and $J(a) := \{a' \in A : a' \geq a\}$. By Lemma 2, $F(\underline{a}^\omega) \cap I(\underline{a}^\omega) \neq \emptyset$, and the same is true for $F(\overline{a}^\omega) \cap J(\overline{a}^\omega)$. Hence, by Assumption 1, both $\underline{a}^{\omega+1}$ and $\overline{a}^{\omega+1}$ are well defined elements of $F(\underline{a}^\omega)$ and respectively of $F(\overline{a}^\omega)$. The following lemma follows directly from the definition $\underline{a}^{\omega+1}$ of and $\overline{a}^{\omega+1}$.

Lemma 3 *The following conditions hold:*

(i) If $\underline{a}^\omega$ is a fixed point of F, then $\underline{a}^{\omega+1} = \underline{a}^\omega$. If $\overline{a}^\omega$ is a fixed point of F, then $\overline{a}^{\omega+1} = \overline{a}^\omega$.

(ii) If $\underline{a}^\omega$ is not a fixed point of F, then $\underline{a}^{\omega+1} < \underline{a}^\omega$. If $\overline{a}^\omega$ is not a fixed point of F, then $\overline{a}^{\omega+1} > \overline{a}^\omega$.

We can now continue our iterations starting from $\underline{a}^\omega$ and $\overline{a}^\omega$. For any k we define the following sequences $(\underline{a}^{\omega+k})_{k=1}^\infty$ and $(\overline{a}^{\omega+k})_{k=1}^\infty$ recursively as follows

$$\underline{a}^{\omega+k+1} = \bigvee F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \quad \text{and} \quad \overline{a}^{\omega+k+1} = \bigwedge F(\overline{a}^{\omega+k}) \cap I(\overline{a}^{\omega+k}).$$

Indeed, it is a transfinite complement of the sequences $\underline{a}^k$ and respectively $\overline{a}^k$. This immediately yields the following result:

Lemma 4 *The sequences* $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$ *and* $\left(\overline{a}^{\omega+k}\right)_{k=0}^{\infty}$ *are both well-defined. Moreover, if any* $\underline{a}^{\omega+k_0}$ *(resp.,* $\overline{a}^{\omega+k_0}$*) is a fixed point of* F*, then the sequence* $\left(\underline{a}^{\omega+k}\right)_{k=k_0}^{\infty}$ *(resp.,* $\left(\overline{a}^{\omega+k}\right)_{k=k_0}^{\infty}$*) is constant.*

Remark. Note if $\underline{a}^{\omega}$ or $\overline{a}^{\omega}$ are not fixed points (and similarly $\underline{a}^{\omega+k}$ and $\overline{a}^{\omega+k}$ are not fixed points), and the correspondence F is *strongly monotone*, then $\underline{a}^{\omega+1}$ can be alternatively defined as any point of $F(\underline{a}^{\omega})$ that is *smaller* than $\underline{a}^{\omega}$, and $\overline{a}^{\omega+1}$ can be alternatively defined as any point of $F(\overline{a}^{\omega})$ that is *greater* than $\overline{a}^{\omega}$. This follows because by strong monotonicity, we have $\sup F(\underline{a}^{\omega+1}) \leq \inf F(\underline{a}^{\omega})$ for $\underline{a}^{\omega+1}$ as we defined it; similarly, we have $\inf F(\overline{a}^{\omega+1}) \geq \sup F(\overline{a}^{\omega})$ for $\overline{a}^{\omega+1}$ as we defined it. Therefore no a strictly between $\inf F(\underline{a}^{\omega})$ and $\underline{a}^{\omega}$ (or no a strictly between $\overline{a}^{\omega}$ and $\sup F(\overline{a}^{\omega})$) can be a fixed point. It is perhaps even computationally more efficient to define $\underline{a}^{\omega+1}$ as the infimum of $F(\underline{a}^{\omega})$, and to define $\overline{a}^{\omega+1}$ as the supremum of $F(\overline{a}^{\omega})$. However, this alternative definition is *equivalent* to our definition only when the correspondence F is *strongly increasing.*

Lemma 5 *We have: (i) the sequence* $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ *is decreasing, and its limit* $\underline{a}^*$ *is a fixed point of* F*; (ii) the sequence* $(\overline{a}^{\omega+k})_{k=0}^{\infty}$ *is increasing and its limit* $\overline{a}^*$ *is a fixed point of* F*.*

We can now state and prove the following key result:

Proposition 1 *Suppose that* $\underline{b}$ *and* $\overline{b}$ *are fixed points of* F *for which there exist an increasing sequence* $(\underline{b}^k)_{k=1}^{\infty}$ *and a decreasing sequence* $(\overline{b}^k)_{k=1}^{\infty}$ *such that* $\lim_k \underline{b}^k \geq \underline{b}$*,* $\lim_k \overline{b}^k \leq \overline{b}$*. If, for any sequence* $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$*, we have that* $\underline{b}^k \leq a^k \leq \overline{b}^k$ *for all* k*, then* $\underline{b} \leq \underline{a}^*$ *and* $\overline{a}^* \leq \overline{b}$*.*

Note that all sequences $(a^k)_{k=0}^{\infty}$ such that $a^{k+1} \in F(a^k)$ have the same first element a^0. Note further that $\left(\bigwedge_{l\geq k} \underline{a}^l\right)_{k=0}^{\infty}$ and $\left(\bigvee_{l\geq k} \overline{a}^l\right)_{k=0}^{\infty}$ are sequences $(\underline{b}^k)_{k=1}^{\infty}$ and $(\overline{b}^k)_{k=1}^{\infty}$ with the required property for $\underline{b} = \underline{a}^*$ and $\overline{b} = \overline{a}^*$.

Proof: We will proof the theorem for $\underline{a}^*$; the proof for $\overline{a}^*$ is analogous. Since $\underline{b}^l \leq a^l$ for all l, $\bigwedge_{l \geq k} \underline{b}^l \leq \bigwedge_{l \geq k} \underline{a}^l$; and since the sequence $(\underline{b}^k)_{k=1}^{\infty}$ is increasing, $\underline{b}^k = \bigwedge_{l \geq k} \underline{b}^l$, therefore $\lim_k \underline{b}^k \leq \lim_k \bigwedge_{l \geq k} \underline{a}^l = \underline{a}^\omega$. Thus, $\underline{b} \leq \underline{a}^\omega$. This completes the proof if $\underline{a}^* = \underline{a}^\omega$. If not, then $\underline{a}^\omega$ is not a fixed point and $\underline{b} < \underline{a}^\omega$. Recall $\underline{b} \in F(\underline{b})$, and $\underline{a}^{\omega+1} \in F(\underline{a}^\omega)$. As a result, $\underline{b} \vee \underline{a}^{\omega+1} \in F(\underline{a}^\omega)$ because $F(\underline{b}) \leq^{SSO} F(\underline{a}^\omega)$. Since $\underline{b} < \underline{a}^\omega$, and by Lemma 5, $\underline{a}^{\omega+1} < \underline{a}^\omega$, we have that $\underline{b} \vee \underline{a}^{\omega+1} \leq \underline{a}^\omega$. This implies that $\underline{b} \vee \underline{a}^{\omega+1} \in F(\underline{a}^\omega) \cap I(\underline{a}^\omega)$. Since $\underline{a}^{\omega+1}$ is the greatest element of this set, hence $\underline{b} \vee \underline{a}^{\omega+1} \leq \underline{a}^{\omega+1}$. So $\underline{b} \leq \underline{a}^{\omega+1}$. We show $\underline{b} \leq \underline{a}^{\omega+k}$ for any k, and consequently $\underline{b} \leq \underline{a}^*$. We have proven this thesis for $k = 1$ and suppose it is the case for some k. The proof is complete if $\underline{a}^{\omega+k}$ is a fixed point, because by Lemma 4, $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$. If $\underline{a}^{\omega+k}$ is not a fixed point, $\underline{b} < \underline{a}^{\omega+k}$, and $\underline{b} \vee \underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k})$. Moreover, $\underline{b} \vee \underline{a}^{\omega+k+1} \in I(\underline{a}^{\omega+k})$, hence $\underline{b} \vee \underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. Since $\underline{a}^{\omega+k+1}$ is the greatest element of this set, $\underline{b} \vee \underline{a}^{\omega+k+1} \leq \underline{a}^{\omega+k+1}$, consequently $\underline{b} \leq \underline{a}^{\omega+k+1}$. Thus, $\underline{b} \leq \underline{a}^{\omega+k}$ for any k, and also $\underline{b} \leq \underline{a}^*$. ■

Proposition 1 captures formally the intuition that $\underline{a}^*$ and $\overline{a}^*$ are tight fixed-point bounds between which sufficiently large iterations of F are located. It is easy to check that under some additional assumptions, typically satisfied in applications, the hypothesis of Proposition 1 can be replaced with more familiar-looking conditions that for any fixed points $\underline{b}$ and $\overline{b}$,

$$\text{if } \forall_{b < \underline{b}} \, \exists_K \, \forall_{k \geq K} \; b < a^k, \text{ then } \underline{b} \leq \underline{a}^*, \tag{1}$$

and

$$\text{if } \forall_{b > \overline{b}} \, \exists_K \, \forall_{k \geq K} \; a^k < b, \text{ then } \overline{a}^* \leq \overline{b}. \tag{2}$$

This is so, for example, if we assume that: (a) $\underline{b} = \lim_n \underline{b}^n$ for some strictly increasing sequence $(\underline{b}^n)_{n=1}^{\infty}$ and $\overline{b} = \lim_n \overline{b}^n$ for some strictly decreasing sequence $(\overline{b}^n)_{n=1}^{\infty}$; or (b) for every $\underline{a}'$ such that $\underline{a}' < \underline{a}^*$ there exist $\underline{a}''$ such that $\underline{a}' < \underline{a}'' < \underline{a}^*$, and for every $\overline{a}'$ such that $\overline{a}^* < \overline{a}'$ there exists $\overline{a}''$ such that $\overline{a}^* < \overline{a}'' < \overline{a}'$.

4 Iterative monotone comparative statics and GSCs

In this section, we provide two results for iterative monotone comparative statics for parameterized monotone correspondences. We consider their implications in an important class of models, namely for comparative statics in GSC. The first result applies to GSCs in which all actions increase in response to a parameter change. Our second result concerns GSCs in which not all actions need to increase, but some summary statistic of the actions increases in response to a parameter change. As an example of the latter case, see the game in our motivating example which has aggregative structure but not all actions increase in the parameter. Focusing on GSC allows us to compare our results to those based on the correspondence principle in Echenique (2002). But then, in the next section of the paper, we provide some additional economic applications outside the realm of GSCs.

Proposition 2 *Let A be a σ-complete lattice and T be a poset. Endow $A \times T$ with the product order. Assume $F : A \times T \rightrightarrows A$ is weakly monotone on $A \times T$, and for each t, it satisfies Assumption 1 as a correspondence on A. For any $a^0 \in A$ and $t \in T$, let $\underline{a}^{\omega}(a^0, t) = \liminf_k \underline{a}^k$ for the sequence $(\underline{a}^k)_{k=0}^{\infty}$ constructed in Section 3 by iterating on $F(\cdot, t)$ from a^0, while $\underline{a}^*(a^0, t)$ be the lower fixed-point bound constructed in Section 3. Suppose that $t' < t''$. Then:*

(i) $\underline{a}^{\omega}(a^0, t') \leq \underline{a}^{\omega}(a^0, t'')$ *and* $\underline{a}^*(a^0, t') \leq \underline{a}^*(a^0, t'')$*;*

(ii) if a^0 is a fixed point of $F(\cdot, t')$ and for any $a \in A$, $F(a, \cdot)$ is strongly monotone, then additionally $a^0 \leq \underline{a}^{\omega}(a^0, t'')$ and $a^0 \leq \underline{a}^*(a^0, t'')$;

(iii) if a^0 is a fixed point of $F(\cdot, t')$ and A is a total order then either a^0 is a fixed point of $F(\cdot, t'')$, or $a^0 \leq \underline{a}^\omega(a^0, t'')$ and $a^0 \leq \underline{a}^*(a^0, t'')$.

The result is analogous relative to $\overline{a}^\omega(a^0, t)$ and $\overline{a}^*(a^0, t)$.

We can now suggest an interpretation of this proposition in the context of an N-player normal-form game of strategic complementarities. It is well-known that adaptive learning does not guarantee convergence to Nash equilibrium.[15] The best response dynamics are an extreme form of adaptive learning[16] in which players respond to most recent actions of their opponents, disregarding the actions from the previous periods. Even under this extreme form of adaptive dynamics, we observe only outcomes greater than $\underline{a}^\omega(a^0, t)$ in the long-run. Even if we believe that the play will converge to a Nash equilibrium, we should expect this equilibrium to be greater than $\underline{a}^*(a^0, t)$, which is the smallest equilibrium such that $\underline{a}^*(a^0, t) \leq \underline{a}^\omega(a^0, t)$.

It is also important to note for comparison that Echenique (2002) interprets adaptive learning more explicitly. He suggests studying convergent sequences $(a^k)_{k=0}^\infty$ such that a^k is between the infimum of the best responses to the infimum of $a^{k-1}, ..., a^{k-\gamma}$ and the supremum of the best response to the supremum of $a^{k-1}, ..., a^{k-\gamma}$for some $\gamma > 0$. However, under his assumption that $a^0 \leq \inf F(a^0)$, the smallest equilibrium which is the limit of such a convergent sequences $(a^k)_{k=0}^\infty$ coincides with our lower fixed-point bound $\underline{a}^*(a^0, t)$. Hence, our result indeed generalizes his results.

But also note, our result extends Echenique's result in a number of dimensions: (i) our correspondence F is assumed to be only weakly (not necessarily strongly) increasing; (ii) the adaptive dynamics may start from an action profile a^0 that is not ordered with its image under F; (iii) the initial action profile a^0 need not be

[15] More precisely, this is true for uncoupled learning (see Hart and Mas-Colell (2003)).
[16] See Milgrom and Roberts (1991).

an equilibrium. Further, the adaptive dynamics may or may not be *convergent.*

To prove Proposition 2, we will need two additional lemmas.

Lemma 6 *If $F : A \times T \rightrightarrows A$ is weakly increasing in A and strongly increasing in T, then it is strongly increasing in the product $A \times T$.*

Lemma 7 *Let $\underline{A}_1 := \{a \in A : F(a) \cap I(a) \neq \emptyset\}$ and $\overline{A}_2 := \{a \in A : F(a) \cap J(a) \neq \emptyset\}$. Then, the functions*

$$a \in \underline{A}_1 \mapsto \bigvee F(a) \cap I(a) \quad \text{and} \quad a \in \overline{A}_2 \mapsto \bigwedge F(a) \cap J(a)$$

are increasing.

Proof: Proof of (i). By the monotonicity of F, $\inf F(a^0, t') \leq \inf F(a^0, t'')$. So, $\underline{a}^1$ for t' is no greater than $\underline{a}^1$ for t''. By induction, $\underline{a}^k$ obtained by iterating a^0 on $F(\cdot, t')$ is no greater than $\underline{a}^k$ obtained by iterating a^0 on $F(\cdot, t'')$. Hence, we obtain $\underline{a}^\omega(a^0, t') \leq \underline{a}^\omega(a^0, t'')$. We will now show that $\underline{a}^{\omega+k}(a^0, t') \leq \underline{a}^{\omega+k}(a^0, t'')$ for any $k \in \mathbb{N}$. It is already proven for $k = 0$. Suppose it holds for some $k \geq 0$. By Lemma 7, we have

$$\begin{aligned}
\underline{a}^{\omega+k+1}(a^0, t') &= \bigvee F(\underline{a}^{\omega+k}(a^0, t')) \cap I(\underline{a}^{\omega+k}(a^0, t')) \\
&\leq \bigvee F(\underline{a}^{\omega+k}(a^0, t'')) \cap I(\underline{a}^{\omega+k}(a^0, t'')) \\
&= \underline{a}^{\omega+k+1}(a^0, t'').
\end{aligned}$$

Consequently, $\underline{a}^{\omega+k}(a^0, t') \leq \underline{a}^{\omega+k}(a^0, t'')$ for all k, hence $\underline{a}^*(a^0, t') \leq \underline{a}^*(a^0, t'')$.

(ii). Now suppose a^0 is a fixed point of $F(\cdot, t')$. We show $a^0 \leq \underline{a}^k(a^0, t'')$ for any k. For $k = 0$, we defined $\underline{a}^k(a^0, t'')$ as a^0. Suppose it is true for some $k \geq 0$. By Lemma 6, F is strongly monotone. Since $a^0 \in F(a^0, t')$ and $\underline{a}^{k+1}(a^0, t'') \in F(\underline{a}^k(a^0, t''), t'')$, we obtain that $a^0 \leq \underline{a}^{k+1}(a^0, t'')$. Consequently $a^0 \leq \underline{a}^k(a^0, t'')$ holds for any k. So $a^0 \leq \underline{a}^\omega(a^0, t'')$. We are now going to prove that $a^0 \leq$

$\underline{a}^{\omega+k}(a^0, t'')$ for any k. For $k = 0$ it is already proven. Suppose it is true for some $k \geq 0$. Then, since $a^0 \in F(a^0, t')$ and $\underline{a}^{\omega+k+1}(a^0, t'') \in F(\underline{a}^{\omega+k}(a^0, t''), t'')$, Lemma 6 implies that $a^0 \leq \underline{a}^{\omega+k+1}(a^0, t'')$. Hence the proof of (ii) is complete.

(iii). Since A is total ordered, hence either $\underline{F}(a^0, t'') \geq a^0$ or $\underline{F}(a^0, t'') \leq a^0$. In the first case, the sequence $\underline{a}^k(a^0, t'')$ is increasing in k, and by Lemma 1, $\underline{a}^\omega(a^0, t'')$ is a fixed point of $\underline{F}(\cdot, t'')$. As a result $(\underline{a}^{\omega+k}(a^0, t''))_{k=1}^\infty$ is a constant sequence equal $\underline{a}^*(a^0, t'')$. Hence $\underline{a}^*(a^0, t'') \geq a^0$. Suppose that $\underline{F}(a^0, t'') \leq a^0$. Since $\underline{F}(a^0, t'') \in F(a^0, t'')$, $a^0 \in F(a^0, t')$ and $F(a^0, t') \leq^{SSO} F(a^0, t'')$, we obtain that $a^0 \in F(a^0, t'')$. ■

Remark 1 *We now show that the strong monotonicity of $F(a, \cdot)$ cannot be relaxed in (ii) of the Proposition 2 even if $F(a, t_1) \neq F(a, t_2)$ for any distinct t_1, t_2 in T and $a \in A$. Let $T = \{0, 1\}$ and $A = [0, 1]$ with natural orders. Define*

$$F(a,0) := \begin{cases} \{0\} & if \quad a \leq \frac{1}{2} \\ [0, \frac{1}{2}] & if \quad a = \frac{1}{2} \\ \{\frac{1}{2}\} & if \quad a > \frac{1}{2}, \end{cases} \quad and \quad F(a,1) := \begin{cases} \{\frac{1}{4}\} & if \quad a \leq \frac{1}{2} \\ [\frac{1}{4}, 1] & if \quad a = \frac{1}{2} \\ \{1\} & if \quad a > \frac{1}{2}. \end{cases}$$

Obviously, this correspondence is not strongly monotone in t, but the other assumptions of Proposition 2 are satisfied. In particular, $a^0 = \frac{1}{2}$ is a fixed point of $F(\cdot, 0)$, but $\underline{a}^k(a^0, 1) = \underline{a}^{\omega+k}(a^0, 1) = \frac{1}{4}$ for any $k \in \mathbb{N}$ which violates (ii) in Proposition 2.

Observe that $a^0 = \frac{1}{2}$ is a common fixed point of $F(\cdot, 0)$ and $F(\cdot, 1)$. We can construct another, rather trivial, example in which a^0 is a fixed point of $F(\cdot, 0)$, but not $F(\cdot, 1)$. Let $A = [0, 1] \times [0, 1]$ with the product order, and $T = \{0, 1\}$ with the standard order. Let $F(a, 0) = [0, 1] \times [0, 1]$, and let $F(a, 1) = [\frac{1}{4}, 1] \times [\frac{1}{4}, 1]$ for all $a \in A$. Clearly, $a^0 = (0, 1)$ is a fixed point of $F(a, 0)$, but not $F(a, 1)$. In addition, $\underline{a}^k(a^0, 1) = \underline{a}^{\omega+k}(a^0, 1) = (\frac{1}{4}, \frac{1}{4})$ which is incomparable with a^0.

Proposition 2 has obvious implications for characterizing monotone comparative statics of any equilibrium (or fixed point) of the correspondence $F(\cdot, t)$. But we can actually provide another result on fixed-point (or equilibrium) comparative statics that might arise in setting where one seeks comparisons of "aggregates" (e.g., as in the motivating example 1). So we finish this section with a result on monotone "aggregate" comparative statics that concerns the comparative statics of some statistics or aggregates of interests ϕ that summarize the equilibrium behavior in the game or economy. For the motivating example, such statistic is aggregate (team) output. For this result, we need our correspondence $F(\cdot, t)$ to be strongly monotone.

Proposition 3 *Let A be a σ-complete lattice and T a poset. Endow $A \times T$ with the product order. Consider $F : A \times T \rightrightarrows A$. Suppose that, for each t, $F(\cdot, t) : A \rightrightarrows A$ is strongly monotone and satisfies Assumption 1. Let $\phi : A \times T \to \mathbb{R}$ be some statistic that is increasing on $A \times T$, that is also continuous on A (for each t). Suppose further the following condition is satisfied:*

$$\begin{aligned} &\textit{if } \phi(a', t') \leq \phi(a'', t'') \textit{ for some } a', a'', t' \textit{ and } t'' > t' \\ &\textit{then } \phi(\sup F(a', t'), t') \leq \phi(\inf F(a'', t''), t''). \end{aligned} \tag{3}$$

Then:

$$\phi(a^0, t') \leq \phi(\underline{a}^{\omega}(a^0, t''), t'') \textit{ and } \phi(a^0, t') \leq \phi(\underline{a}^{*}(a^0, t''), t'')$$

for any a^0 which is a fixed point of $F(\cdot, t')$.

Proof: By the monotonicity of ϕ, we have that $\phi(a^0, t') \leq \phi(a^0, t'')$. Hence by condition (3),

$$\phi(\overline{a}^1(a^0, t'), t') \leq \phi(\underline{a}^1(a^0, t''), t'').$$

Suppose that for some integer k we have that

$$\phi(\overline{a}^k(a^0, t'), t') \leq \phi(\underline{a}^k(a^0, t''), t''). \tag{4}$$

Again by condition (3) and the definitions of $\underline{a}$ and $\overline{a}$ we have

$$\phi(\overline{a}^{k+1}(a^0, t'), t') \leq \phi(\underline{a}^{k+1}(a^0, t''), t'').$$

Hence inequality (4) holds generally for all k. By the continuity of ϕ, we obtain that

$$\phi(\overline{a}^{\omega}(a^0, t'), t') \leq \phi(\underline{a}^{\omega}(a^0, t''), t''). \tag{5}$$

Since $a^0 \in F(a^0, t')$, we have that $a^0 \leq \overline{a}^1(a^0, t')$. Thus,

$$\phi(a^0, t') \leq \phi(\overline{a}^1(a^0, t'), t'),$$

by the monotonicity of ϕ, and

$$\phi(\overline{a}^1(a^0, t'), t') \leq \phi(\overline{a}^2(a^0, t'), t')$$

by the monotonicity of F on A and the monotonicity of ϕ. By induction, we show that the sequence $\phi(\overline{a}^k(a^0, t'), t')$ increases in k. Consequently,

$$\phi(a^0, t') \leq \lim_{k \to \infty} \phi(\overline{a}^k(a^0, t'), t') = \phi(\overline{a}^{\omega}(a^0, t'), t').$$

This together with (5) yields $\phi(a^0, t') \leq \phi(\underline{a}^{\omega}(a^0, t''), t'')$. By analogous arguments, we obtain that $\phi(a^0, t') \leq \phi(\underline{a}^*(a^0, t''), t'')$. ■

One might ask if the strong monotonicity in condition (3) of Proposition 3 can be relaxed to weak monotonicity (i.e., strong set order monotonicity) defined as follows:

$$\begin{aligned} &\text{if for some } a', a'', t', t'' \text{ with } t' < t'' \text{ with } \phi(a', t') \leq \phi(a'', t'') \\ &\qquad \text{then } \{\phi(\tilde{a}, t') : \tilde{a} \in F(a', t')\} \leq^{SSO} \{\phi(\tilde{a}, t'') : \tilde{a} \in F(a'', t'')\}. \end{aligned} \tag{6}$$

Then answer is no as the following example proves:

Example 1 *Let* $A = [0,1] \times [0,1]$ *and* $T = \{0,1\}$ *with usual orders,* $\phi(a_1, a_2, 0) = \phi(a_1, a_2, 1) = a_1 + a_2$*. Put* $F(a_1, a_2, 0) = [0,1] \times [0,1]$ *and* $F(a_1, a_2, 1) = \left[\frac{1}{4}, 1\right] \times \left[\frac{1}{4}, 1\right]$ *for any* $a \in A$*. Then* $a^0 = (1, 0)$ *is a fixed point of* $F(a, 0)$*;* $\underline{a}^k(a^0, 1) = \left(\frac{1}{4}, \frac{1}{4}\right)$ *for any* $k \in \mathbb{N}$ *as well as any* $k = \omega + k'$ *with* $k' \in \mathbb{N}$*. The same holds for* $\underline{a}^*(a^0, t)$*. But* $\phi(a^0, 0) = 1$ *and* $\phi(\underline{a}^*(a^0, 1), 1) = \frac{1}{2}$*, which violates the hypothesis of Proposition 3. Condition (3) is violated, because*

$$\phi\left(\frac{1}{4}, \frac{1}{4}, 1\right) = \frac{1}{2} < 2 = \phi(1, 1, 0).$$

The condition (6) holds, however. Indeed, if $t' = 0$ *then for any* $a' \in A$

$$\{\phi(\tilde{a}, t') : \tilde{a} \in F(a', t')\} = [0, 2].$$

if $t'' = 1$ *then for any* $a'' \in A$

$$\{\phi(\tilde{a}, t'') : \tilde{a} \in F(a'', t'')\} = \left[\frac{1}{2}, 2\right].$$

As a result,

$$\{\phi(\tilde{a}, t') : \tilde{a} \in F(a', t')\} \leq^{SSO} \{\phi(\tilde{a}, t'') : \tilde{a} \in F(a'', t'')\}.$$

5 Additional applications and extensions

We now provide few additional economic applications of our main propositions. In the first example, we discuss the case of equilibrium comparative statics in aggregative games with "mixed shocks". The second example shows how to apply iterative monotone comparative statics to comparing stationary equilibria of dynamic economies where the stationary equilibrium is defined over one dimensional state variable (e.g., output as in stochastic growth with iid shocks or wealth in Bewley models). In our final application, we state a proposition on comparing (potentially indeterminate) recursive equilibria in dynamic economies.

5.1 Comparing equilibria in aggregative games with mixed shocks

As argued in motivating example a shock or parameter change may affect players or agents adversely, i.e., for some the shock is positive and for some negative. In particular, the initial change of the parameter may increase actions / decisions taken by some and decrease for the others. Hence, the initial change may affect the action / decision so that it is not comparable in a product order to the initial profile. In a class of aggregative games, our results are suited to provide comparative statics results for such shocks as well. Below we present a general construction and an illustrative example.

Consider a family of N-player aggregative games $(\Gamma_t)_t$ parameterized by $t \in T$. All actions a_i, $i = 1, 2, ..., N$, belong to $\mathbb{R}^k$. The aggregate is given by the sum of all actions, but the extensions to additively separable aggregates[17] studied by Acemoglu and Jensen (2013) are straightforward. As the game is aggregative, the best response correspondence of each player i can be written as $BR_i(z_i, t)$ where $z_i := \sum_{j \neq i} a_j$, i.e., a partial aggregate z_i sums actions of all but player i. Let $z = (z_i)_{i=1}^N$ be a profile of such (partial) aggregates. An *aggregate best response* is a correspondence $B : \mathbb{R}^{kN} \times T \rightrightarrows \mathbb{R}^{kN}$ where $B(z,t) = (B_i(z,t))_{i=1}^N$ and

$$B_i(z,t) := \sum_{j \neq i} BR_j(z_j, t).$$

That is, B maps a profile of partial aggregates and returns updated profiles of such partial aggregates. The next proposition assures the equivalence between Nash equilibria of Γ_t and fixed points of $B(\cdot, t)$.

Proposition 4 *$a^* = (a_i^*)_{i=1}^N$ is a Nash equilibrium of Γ_t if and only if $z^* = (z_i^*)_{i=1}^N$ is a fixed point of $B(\cdot, t)$.*

[17] Examples include aggregates given by $g(a) = H(\sum_i h_i(a_i))$, where H and each h_i are strictly increasing mappings $\mathbb{R} \to \mathbb{R}$.

Proof: Suppose a^* is a Nash equilibrium. Clearly for each i we have $a_i^* \in BR_i(\sum_{k\neq i} a_k^*, t)$. As a result,

$$z_i^* := \sum_{j\neq i} a_j^* \in \sum_{j\neq i} BR_j(\sum_{k\neq j} a_k^*, t) = \sum_{j\neq i} BR_j(z_j^*, t) =: B_i(z^*, t).$$

Since i was arbitrary, we have $z^* \in B(z^*, t)$. For the other direction, suppose z^* is a fixed point of $B(\cdot, t)$. Recover a vector a^* such that:

$$a_k^* = \frac{\sum_{i\neq k} z_i^* - (N-2)z_k^*}{N-1}.$$

As z^* is a fixed point we have

$$a_k^* \in \frac{\sum_{i\neq k}\sum_{j\neq i} BR_j(z_j^*, t) - (N-2)\sum_{i\neq k} BR_i(z_i^*, t)}{N-1}.$$

Computing the sums we obtain:

$$a_k^* \in BR_k(z_k^*, t).$$

for each k. ■

Consequently, best response (or BR) dynamic can be represented as B-dynamic, and B-dynamic has a corresponding BR-dynamic. Indeed, $z_k^t = \sum_{j\neq k} a_j^t$ and $a_k^t = \frac{\sum_{j\neq k} z_j^t - (N-2)z_k^t}{N-1}$ are both continuous hence an adaptive learning process converges in BR if and only if it converges in B.

Clearly, for $N > 2$ examples can be constructed such that, even though not every BR_i is monotone on $\mathbb{R}^{kN} \times T$, operator B is monotone on $\mathbb{R}^{kN} \times T$. Then one can apply our results from section 3 to the monotone correspondence B. The following example illustrates this point.

Example 2 *Consider a government planning a tax reform. There are three agents in the economy. The first is a high (productive) type and two others are low (productivity) types. Due to positive externalizes the actual productivity of each agent* *depends on actions (economic activity level) taken by others. The government is willing to allocate subsidy t for each of the low productivity agents as an incentive to increase their economic activity. The government is willing to accept the cost of such reform at the amount of $\frac{3}{2}t$ for some $t \in [0,1]$. Since the total subsidy is $2t$ the government needs to tax the productive agent in the amount of $\frac{1}{2}t$. The subsidy or tax is linear and hence imposed for each unit of agent's activity.*

Let $a_1, a_2, a_3 \in [0,1]^3$ denote the activity levels and define the payoffs of each agent:

$$\begin{cases} u_1(a_1,a_2,a_3,t) &= \frac{2}{3}a_1\left(a_2+a_3+1-\frac{1}{2}t\right)-a_1^2 \\ u_2(a_1,a_2,a_3,t) &= \frac{2}{3}a_2\left(a_1+a_3+t\right)-a_2^2 \\ u_3(a_1,a_2,a_3,t) &= \frac{2}{3}a_3\left(a_1+a_2+t\right)-a_3^2. \end{cases}$$

The last term in each payoff denotes the quadratic cost of activity level.

Clearly, the initial increase in t causes a mixed shock to an economy, it makes the productive agent react by taking lower activity level but at the time increases activities of 2 and 3. Clearly the game is aggregative.

The best response of each agents are as follows:

$$\begin{cases} BR_1(a_2,a_3,t) &= \frac{1}{3}\left(z_1+1-\frac{1}{2}t\right) \\ BR_2(a_1,a_3,t) &= \frac{1}{3}(z_2+t) \\ BR_3(a_1,a_2,t) &= \frac{1}{3}(z_3+t), \end{cases}$$

with $z_1 := a_2+a_3$, $z_2 := a_1+a_3$ and $z_3 = a_1+a_2$. The (partial) aggregate best response is hence

$$\begin{cases} B_1(z_2,z_3,t) &:= \frac{z_2+z_3+2t}{3} \\ B_2(z_1,z_3,t) &:= \frac{z_1+z_3+1+\frac{1}{2}t}{3} \\ B_3(z_1,z_2,t) &:= \frac{z_1+z_2+1+\frac{1}{2}t}{3}. \end{cases}$$

Observe that B is monotone in t. It summarizes the fact that at the aggregate a decrease in the first agent's activity level is compensated by an increase of others. We can apply Proposition 2 directly on B to conclude on the fixed-points bounds *of iterations from any a^0. Indeed, we verify that the fixed point of $B(\cdot, t)$ is*

$$\begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} = \begin{bmatrix} \frac{5}{4}t + \frac{1}{2} \\ \frac{7}{8}t + \frac{3}{4} \\ \frac{7}{8}t + \frac{3}{4} \end{bmatrix}.$$

And hence the Nash equilibrium (a_1^*, a_2^*, a_3^*) is

$$\begin{cases} a_1^* &= \frac{1}{4}t + \frac{1}{2} \\ a_2^* &= \frac{5}{8}t + \frac{1}{4} \\ a_3^* &= \frac{5}{8}t + \frac{1}{4}. \end{cases}$$

Via the tax reform the government managed to increase the activities of all agents.

5.2 Comparing stationary distributions in monotone economies

Our results can be applied to the comparative statics of stationary distributions of output or income associated with infinite horizon stochastic growth models with nonconvexities. The production or income available at period t is $y_t \in Y$, where $Y = [0, \bar{Y}] \subset \mathbb{R}_+$. Agent selects a consumption level $c_t \in [0, y_t]$, with the remaining resources $i_t = z_t - c_t$ allocated as an investment. The evolution of income is given by a continuous, strictly increasing production function $y_{t+1} = f(i_t, z_{t+1})$, where z_{t+1} is a random shock drawn each period from distribution π over a finite set Z. For simplicity, we assume full depreciation. The temporal utility is given by a continuous, strictly increasing and strictly concave function $u : Y \mapsto \mathbb{R}$. The agent's objective then is to maximize her expected discounted payoffs over an infinite horizon, given an initial state $y_0 \in Y$ and discount $\beta \in (0,1)$. Denote the value of this optimization problem by $v^*(y_0)$. This problem admits a recursive representation, where $v = v^*$ is the unique solution to the Bellman equation:

$$v(z) = \max_{i \in [0,y]} u(y - i) + \beta \int_Z v(f(i, z'))d\pi(z').$$

Let the policy correspondence be denoted by

$$H^*(y, \beta) = \arg\max_{i \in [0,y]} u(y - i) + \beta \int_Z v(f(i, z'))d\pi(z').$$

The objective is supermodular in i, and since u is strictly increasing and strictly concave, the objective has strictly increasing differences in $(i; y, \beta)$ (as the continuation return is linear in β). Then, by an application of the Topkis (1998) Theorem (e.g., Theorem 2.8.4), the policy correspondence $H^*(y, \beta)$ is a nonempty upper hemicontinous and jointly strongly monotone in (y, β).

Let $\mathcal{M}(Y)$ denote a set of measures on Y endowed with the first-order stochastic dominance and the weak convergence of measures. Recall that $\mathcal{M}(Y)$ endowed with the first-order stochastic dominance order is a complete lattice in the case that $Y \subset \mathbb{R}_+$ (see, for example, Kamae et al. (1977)). For a measurable set $B \subset Y$ define the stochastic income transition with $Q(B|i) := \int_Z 1_B(f(i, z'))d\pi(dz')$. For any selector $h_\beta(\cdot) \in H^*(\cdot, \beta)$, define the associated adjoint Markov operator:

$$\Lambda_{h_\beta}\mu(B) = \int_Y Q(B|h_\beta(y))\mu(dy)$$

and the associated induced adjoint Markov correspondence:

$$\Lambda\mu(B) = \{\Lambda_{h_\beta}\mu(B)\}_{h_\beta \in H^*(\cdot,\beta)}.$$

As H^* is strongly monotone, then by a result in Huggett (2003) (Theorem 1), Λ is strongly monotone on $M(Y)$. It is also weakly upper hemicontinous.

We can now apply our Proposition 2 to characterize the iterative monotone comparative statics of the stationary income distributions. That is, when $\beta_1 \leq \beta_2$, for *any* initial measure $\mu_0 \in \mathcal{M}(Y)$, we have the following lower (resp., upper) fixed-point bounds for the set of stationary equilibrium elements $\mu_l^*(\mu_0; \beta_1) \leq \mu_l^*(\mu_0; \beta_2)$ (resp., $\mu_u^*(\mu_0; \beta_1) \leq \mu_u^*(\mu_0; \beta_2)$), and from any stationary equilibrium at the discount rate β_1, say μ_{β_1}, our iterations for the operator Λ_{β_2} from μ_{β_1} satisfies the following: $\mu_{\beta_1} \leq \mu_l^*(\mu_{\beta_1};\beta_2) = \mu_{\beta_2}$ where μ_{β_2} is a stationary equilibrium for the economy at β_2.

We remark that a similar reasoning can be applied to study the stationary equilibrium distribution in large dynamic economies in the spirit of Bewley or Huggett/Aiyagari models without aggregate risk. Indeed, interpreting μ as a distributions of income over Y in some large economy, we can study monotone comparative statics of stationary or invariant income distributions after the monotone exogenous shock to the policy function h in the income fluctuation problem of the shocks governed by Q for any initial income distribution μ^0.

We mention, our results extend the stationary equilibrium comparative statics for monotone economies based upon the work of Hopenhayn and Prescott (1992), Huggett (2003), and Acemoglu and Jensen (2015).

5.3 Comparing recursive equilibria in dynamic models with indeterminate equilibria

We finally show how one can apply our results to monotone map methods in the study of recursive equilibrium (RE) comparative statics in macroeconomic models.[18]

There is a continuum of identical agents born each period who live for two periods. In the first period of life, they are endowed with a unit of time which they supply inelastically to the firm at the prevailing wage $w(s)$, and they consume and save. In the second period of life, they consume their savings which are subjected to a stochastic return $r(s')$. Here s and s' denote vectors of state variables in the current and the following periods. Preferences are time separable with constant discounting at rate $\beta \in (0,1)$ and given by $u(c_1) + \beta v(c_2)$ where consumption

[18] This is a large literature in macroeconomics. See Coleman (1991), Mirman et al. (2008) and Datta et al. (2018) for examples of applications of monotone map methods.

when young (resp., old) is denoted by c_1 (resp., c_2), and $u : \mathbb{R}_+ \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$ are twice continuously differentiable, strictly increasing, strictly concave, with $\lim_{c\to 0^+} u'(c) = \infty = \lim_{c\to 0^+} v'(c)$.

The reduced-form technology is given by: $F(k, n, K, N, z) = f(k, n)e(K, N, z)$ where we assume f is a technology transforming private inputs of capital and labor (k, n), where total factor productivity depends also, via $e(K, N, z)$, on per capital aggregates of capital and labor (K, N) and a shock $z \in Z$ drawn each period from a distribution π on a finite set Z. We assume f is constant returns to scale, increasing (but increasing strictly with each argument for the positive input of the other), weakly concave jointly (but strictly concave with each argument separately for the positive input of the other) and twice continuously differentiable. Moreover $r(k, z) := f_1(k, 1)e(K, 1, z)$ is decreasing in k and increasing[19] in K for $K > 0$, $\lim_{k\to 0^+} r(k, z) = \infty$, $\lim_{k\to 0^+} r(k, z_{\max})k = 0$; $w(k, z) := f_2(k, 1)e(k, 1, z)$ is increasing in k with $\lim_{k\to 0^+} w(k, z) = 0$; both r and w are increasing in z for all k. Finally, there exists a maximal sustainable capital stock $k_{\max}$ (i.e., $\forall k \geq k_{\max}$ and $\forall z \in Z$, $F(k, 1, k, 1, z) \leq k_{\max}$), with $F(0, 1, 0, 1, z) = 0$. Many examples of technologies that satisfy these assumptions can be given (see, e.g., Datta et al. (2018)). Further, under this assumption, we can restrict attention to compact state spaces for capital $X \subset \mathbb{R}_+$.

Anticipating $n = 1 = N$ in any equilibrium with inelastic labor supply, we also require $k = K$ and denote a vector of state variables by $s = (K, z) \in S = X \times Z$. Households take the candidate aggregate saving $h \in W$, with $W = \{h : S \to \mathbb{R}_+, 0 \leq h \leq w\}$, as given. Together with π, h describes the law of motion for the

[19] Recall $f_1(k, 1)e(K, 1, z)$ under our assumptions is *mixed monotone* in (k, K), i.e., decreasing in k and increasing in K. This is a critical feature that creates the possibility for equilibrium indeterminacy in this class of models (see Santos (2002) and Datta et al. (2018) for discussion). By equilibrium indeterminacy, we mean there can exist a continuum of equilibria.

aggregate variable. Taking this, a young agent solves:

$$\max_{y\in[0,w(s)]} u(w(s)-y)+\beta\int_Z v(r(h(s),z')y)d\pi(z'),$$

Let $\hat{y}(s;h)$ be the optimal solution to this household problem. It is unique under our assumptions.

Labor and capital markets are competitive hence by profit maximization we obtain $w(K,z) = F_2(K,1,K,1,z)$ and $r(K,z) = F_1(K,1,K,1,z)$. A Recursive Equilibrium (RE) for our economy is a law of motion $h^* \in W$ and a policy function $y^* \in W$ such that we have $y^*(s) = \hat{y}(s;h^*) = h^*(s)$ for any $s \in S_{++}$ whenever $h^*(s) > 0$, and $h^*(s) = 0$ otherwise. Here $S_{++} := X_{++} \times Z$ with $X_{++} \subset \mathbb{R}_{++}$. Market clearing is implied by the formulation of the household problem.

We now consider the question of capital deepening, i.e., the comparative statics of the set of RE in the discount rate β. For this reason we introduce the nonlinear operator A_β mapping W defined implicitly in the household equilibrium Euler equation. Specifically, for $h(s) > 0$, define $A_\beta h(s)$ as the unique y solving:

$$u'(w(s)-y)-\beta\int_Z u'(f_1(h(s),1)e(y,1,z')y)f_1(y,1)e(h(s),1,z')d\pi(z')=0, \quad (7)$$

and $A_\beta h(s) = 0$ whenever $h(s) = 0$. Therefore, any function $h^*_\beta \in W$ is an RE law of motion if and only if it is a non-zero fixed point of the operator A_β. Endow W with its pointwise partial order $\leq$. Then $(W, \leq)$ is a complete lattice. Under our assumptions, A_β is order continuous (hence, monotone) self map on H. Moreover there exists $h_0 \in W$ such that $\forall h \geq h_0$, $A_\beta h > h$ on S^*.[20] Next define the set $W^m = \{h \in W, h \text{ increasing}\}$. W^m is subcomplete in W.

We now consider the operator A_β that transforms the space $(W^m \cap [h_0, w], \leq)$. For any function $h^0 \in W^m \cap [h_0, w]$, $\beta_1 \leq \beta_2$, applying the results of Proposition 2, we first have the lower and upper fixed-point bounds given by: $h^\omega_l(h^0;\beta_1) \leq$

[20]See McGovern et al. (2013) Proposition 2.

$h_l^\omega(h^0;\beta_2)$ and $h_l^*(h^0;\beta_1) \leq h_l^*(h^0;\beta_2)$ and similarly $h_u^\omega(h^0;\beta_1) \leq h_u^\omega(h^0;\beta_2)$ and $h_u^*(h^0;\beta_1) \leq h_u^*(h^0;\beta_2)$. Importantly, as operator A_β is single-valued and order continuous, taking initially *any* RE $h_{\beta_1}^*$ in $W^m \cap [h_0, w]$, we have $h_{\beta_1}^* \leq h_l^*(h_{\beta_1}^*;\beta_2)$ where $h_l^*(h_{\beta_1}^*;\beta_2)$ is an RE generated by A_{β_2} for the economy with discount rate β_2. So even in the presence of a possible of continuum of RE in the space $(W^m \cap [h_0, w], \leq)$, from any RE $h_{\beta_1}^*$ of the less "patient" economy, there is an RE for the more patient economy that majorizes the RE $h_{\beta_1}^*$.

We finish this section with a few additional remarks. First, analogously Proposition 2 can be used to conduct RE comparative statics in *infinite horizon* models with equilibrium indeterminacies (see Benhabib and Farmer (1994) and Datta et al. (2018)). Second, similar construction can be applied to characterize RE comparative statics in nonconvex nonoptimal dynamic economies where the optimal household decisions are increasing correspondences (see e.g. Mirman et al. (2008)).

6 Concluding remarks

This paper proposes a new iterative approach the monotone comparative statics of fixed points for increasing correspondences in $\sigma-$complete lattices. We are able to show how to apply the results to GSCs, and in that context relate our results to those obtained via the correspondence principle in the work of Echenique (2002, 2004). We also provide new results on the comparative statics of aggregates in GSCs where the players are subjected to "mixed shocks". Finally, we show how to apply our result to the literature on monotone methods for dynamic economies, and how to use the tools for the iterative monotone comparative statics of the set of dynamic equilibria (e.g., set of recursive equilibria and/or stationary equilibria).

There are important limitations to our work that we are now considering, of which two directions seem particularly important. One important question concerns generalizing results on monotone comparative statics to games of strategic substitutes (GSSs).[21] For example, the correspondence principle has been applied to such games, but with limited success. Here, the complications for our approach are numerous. First, there are issues with existence of equilibria in this general class of games, where sufficient conditions often center around topological and convexities considerations, none of which are present in this paper. Second, although it is true that for decreasing operators one could consider their second orbits (which are monotone increasing), one immediate problem is obtaining sufficient conditions on the primitive data of GSS, such that this second orbit exhibit monotone comparative statics. See Roy and Sabarwal (2010) for a discussion of this complication. Another problem from the viewpoint of sharp fixed-point bounds, decreasing operators cycle and possess so-called "fixed edges", which greatly complicates developing sufficiently *tight* fixed-point bounds from which equilibrium comparative statics can be inferred.

An additional important new direction one could consider would be the relaxation of the requirement that the domains transformed in our parameterized fixe-point problems are $\sigma-$complete lattices. This assumption rules out many economic applications of our results where monotonicity is present. For example, as our application to monotone Markov processes and stochastic monotonicity in Section 5.2 makes clear, as spaces of probability measures ordered by the first-order stochastic dominance are generally not lattices, our results cannot be applied to stationary equilibria in monotone economies over multi-dimensional state spaces. So it would be important to consider our approach in more general partially order space than $\sigma-$complete lattices (e.g., directed countably chain complete partially

[21] For work on GSSs, and a discussion of their importance, see the papers of Dubey et al. (2006), Roy and Sabarwal (2010, 2012), Acemoglu and Jensen (2013) and Barthel and Hoffmann (2019) among others.

ordered sets). We leave these issues for further work.

A Appendix

Proof to Lemma 1. Since $F(a)$ is a complete sublattice of A, both $\overline{F}(a)$ and $\underline{F}(a)$ are well-defined selections of F. Furthermore, since F is weakly monotone, $\overline{F}$ and $\underline{F}$ are both increasing. Indeed, if $a' < a''$ then $\underline{F}(a') \wedge \underline{F}(a'') \in F(a')$ and $\underline{F}(a') \vee \underline{F}(a'') \in F(a'')$. As a result

$$\underline{F}(a') \leq \underline{F}(a') \wedge \underline{F}(a'').$$

Hence $\underline{F}(a') \wedge \underline{F}(a'') = \underline{F}(a')$ and consequently $\underline{F}(a') \leq \underline{F}(a'')$. Similarly we prove the monotonicity of $\overline{F}$. Now, we prove the upward continuity of $\underline{F}$. We omit a similar proof that $\overline{F}$ is downward continuous. Let $(a^k)_{k=1}^{\infty}$ be an increasing sequence in A such that $a = \bigvee_{k\in\mathbb{N}} a^k$. Let $b^k := \underline{F}(a^k)$. By the previous parts of the proof we conclude that $b^k \in F(a^k)$ for any $k \in \mathbb{N}$, and b^k is increasing. Let $b = \bigvee b^k$. By upper hemicontinuity of F we obtain $b \in F(a)$. Hence, $\underline{F}(a) \leq b$. On the other hand, $\underline{F}(a) \geq b^k$ for any k. It follows from the definition of b^k and the monotonicity of $\underline{F}(a)$. Hence $b \leq \underline{F}(a)$. Together with $\underline{F}(a) \leq b$, we have $b = \underline{F}(a)$, and hence the upward continuity. ■

Proof to Lemma 2. We will prove the claim for $\underline{a}^{\omega}$; the proof for $\overline{a}^{\omega}$ is analogous. The sequence $\left(\bigwedge_{l\geq k} \underline{a}^l\right)_{k=0}^{\infty}$ is an increasing sequence whose supremum is $\underline{a}^{\omega}$. Let $b^k = \underline{F}\left(\bigwedge_{l\geq k} \underline{a}^l\right)$. By Lemma 1, we know $\underline{F}$ is an increasing function, hence b^k is increasing as well. From Lemma 1 which states $\underline{F}$ is upward continuous, we conclude that

$$a := \bigvee_{k\in\mathbb{N}} b^k = \underline{F}(\underline{a}^{\omega}) \in F(\underline{a}^{\omega}).$$

To finish the proof, we must show that $a \leq \underline{a}^{\omega}$. Since $\bigwedge_{l\geq k} \underline{a}^l \leq \underline{a}^l$ for all $l \geq k$, we have that $b^k \leq \underline{a}^{l+1}$ for all $l \geq k$ by the monotonicity of $\underline{F}$ and the definition of $\underline{a}^{l+1}$. So, $b^k \leq \bigwedge_{l\geq k+1} \underline{a}^l \leq \underline{a}^{\omega}$, which gives that $a = \lim_k b^k \leq \underline{a}^{\omega}$. ■

Proof to Lemma 4. We will show the hypothesis for the sequence $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$; the proof for the sequence $\left(\overline{a}^{\omega+k}\right)_{k=0}^{\infty}$ is analogous. That is, we will show by induction that $\underline{a}^{\omega+k+1}$ is well-defined for any $k \geq 0$, and if $\underline{a}^{\omega+k}$ is a fixed point, then $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$.

For $k = 0$, this holds true by Lemma 3. Suppose that $\underline{a}^{\omega+k}$ is a fixed point of F for some $k > 0$. Then $\underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \neq \emptyset$, so $\underline{a}^{\omega+k+1}$ is well-defined by Assumption 1. In addition, $\underline{a}^{\omega+k}$ must be $\bigvee F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. Hence $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$ by the definition of $\underline{a}^{\omega+k+1}$.

Suppose now that $\underline{a}^{\omega+k}$ is not a fixed point of F. By induction hypothesis $\underline{a}^{\omega+k-1}$ is neither a fixed point of F, because then $\underline{a}^{\omega+k} = \underline{a}^{\omega+k-1}$ would also be a fixed point. Hence $\underline{a}^{\omega+k-1} > \underline{a}^{\omega+k}$. By Assumption 1, $F(\underline{a}^{\omega+k}) \leq^{SSO} F(\underline{a}^{\omega+k-1})$. Take any $a' \in F(\underline{a}^{\omega+k})$. Since $\underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k-1})$, it must be that $a' \wedge \underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k})$ and obviously $a' \wedge \underline{a}^{\omega+k} \in I(\underline{a}^{\omega+k})$. As a result $F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \neq \emptyset$. Thus, $\underline{a}^{\omega+k+1}$ is well-defined. ■

Proof to Lemma 5. We will prove this lemma for $\underline{a}^*$; the proof for $\overline{a}^*$ is analogous. By construction and Lemma 4, $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$ is a well-defined and decreasing sequence. Let $\underline{a}^*$ be its limit. Since $\underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k})$ for all k, by taking a limit as $k \to \infty$ and applying the upper hemicontinuity of F we have $\underline{a}^* \in F(\underline{a}^*)$.

Proof to Lemma 6. Take $a_1 \leq a_2$ in A and $t_1 \leq t_2$ in T. Let $b_1 \in F(a_1, t_1)$ and $b_2 \in F(a_2, t_2)$ be arbitrary elements. Pick any $b' \in F(a_1, t_2)$. Then $b_2 \wedge b' \in$

$F(a_1, t_2)$, because $F(a_1, t_2) \leq^{SSO} F(a_2, t_2)$. And since $F(a_1, t_1) \leq^{S} F(a_1, t_2)$, $b_1 \leq b' \wedge b_2$. Consequently $b_1 \leq b_2$.

Proof to Lemma 7. Put $\Phi(a) := \bigvee F(a) \cap I(a)$ and $\Psi(a) := \bigwedge F(a) \cap J(a)$.

Let $a_1, a_2 \in \underline{A}_1$ and $a_1 < a_2$. Then, $\Phi(a_1) \in F(a_1)$ and $\Phi(a_2) \in F(a_2)$. Moreover, $\Phi(a_1) \leq a_1$ and $\Phi(a_2) \leq a_2$. Thus, $\Phi(a_1) \vee \Phi(a_2) \in F(a_2)$ because $F(a_1)^{SSO} \leq F(a_2)$. But $\Phi(a_1) \vee \Phi(a_2) \leq a_1 \vee a_2 = a_2$. Consequently $\Phi(a_1) \vee \Phi(a_2) \in F(a_2) \cap I(a_2)$. Since $\Phi(a_2)$ is the greatest element from this set, $\Phi(a_1) \vee \Phi(a_2) \leq \Phi(a_2)$, we have that $\Phi(a_1) \leq \Phi(a_2)$.

Now, let $a_1, a_2 \in \overline{A}_2$ and $a_1 < a_2$. Then, $\Psi(a_1) \in F(a_1)$ and $\Psi(a_2) \in F(a_2)$. Thus, $\Psi(a_1) \wedge \Psi(a_2) \in F(a_1)$, because $F(a_1) \leq^{SSO} F(a_2)$. Moreover, $\Psi(a_1) \wedge \Psi(a_2) \geq a_1 \wedge a_2 = a_1$, so $\Psi(a_1) \wedge \Psi(a_2) \in J(a_1)$. Hence $\Psi(a_1) \wedge \Psi(a_2) \in F(a_1) \cap J(a_1)$. Since $\Psi(a_1)$ is the least element from this set, we have that $\Psi(a_1) \leq \Psi(a_1) \wedge \Psi(a_2)$. Consequently $\Psi(a_1) \leq \Psi(a_2)$. ■

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