Strategic Dynamic Programming Methods for Studying Short Memory Equilibria in a Class of Stochastic Games with Uncountable Number of States*

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Abstract

We study a class of infinite horizon stochastic games with uncountable number of states. We first characterize the set of all (nonstationary) short-term (Markovian) equilibrium values by developing a new Abreu, Pearce, and Stacchetti (1990) type procedure operating in function spaces. This (among others) proofs Markov Perfect Nash Equilibrium (MPNE) existence. Moreover, we present techniques of MPNE value set approximation by a sequence of sets of discretized functions iterated on our approximated APS-type operator. This method is new and has some advantages as compared to Judd, Yeltekin, and Conklin (2003), Sleet and Yeltekin (2003) or Feng, Miao, Peralta-Alva, and Santos (2009). We show applications of our approach to altruistic growth economies and dynamic games with strategic complementarities.

keywords: stochastic games, bequest games, supermodular games, short memory (Markov) equilibria, constructive methods, computation, approximation

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1 Introduction and related literature

The existence of equilibrium in the class of discounted, infinite horizon stochastic games is an important question since the work of Shapley (1953) and includes many seminal contributions

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(see e.g. Mertens and Parthasarathy (1987) or Nowak and Raghavan (1992)). Recently however, economists focused on equilibrium existence in the class of short-memory strategies, especially over uncountable number of states. The importance of short-memory equilibria lays in (i) its simplicity, (ii) potential computational tractability, (iii) possibility of developing methods for studying monotone comparative statics or dynamics, and as a result (iv) applicability to many economic problems at hand, among others. This literature has many recent contributions, including work of Duggan (2012), who proves equilibrium existence in a class of noisy-stochastic games, of Barelli and Duggan (2014), who focus on semi-Markov equilibria, of Balbus, Jaśkiewicz, and Nowak (2014); He and Sun (2014); Nowak (2010) with few general stationary equilibrium existence results, or of Levy and McLennan (2014), who show non-existence of stationary equilibrium in a class of continuous stochastic games, among others.

Further, in recent times stochastic games have become a fundamental tool for studying dynamic economic models, where agents possess some form of limited commitment including works in (i) dynamic political economy (Lagunoff (2009)), (ii) equilibrium models of stochastic growth without commitment (Amir (1996a), Balbus, Reffett, and Woźny (2013a)), (iii) models of savings and asset prices with hyperbolic discounting (Harris and Laibson (2001), Balbus, Reffett, and Woźny (2014b)), (iv) international lending and sovereign debt (Atkeson (1991)), (v) optimal Ramsey taxation (Phelan and Stacchetti (2001)), (vi) dynamic negotiations with status quo (Duggan and Kalandrakis (2012)), or (vii) dynamic oligopoly models (Cabral and Riordan (1994)) for example. The applications of repeated, dynamic or stochastic games toolkit to analyze these phenomena results, among others, from the richness of behavior supported by a subgame perfect or sequential equilibrium (see celebrated folk theorem and its analytical tractability using recursive characterization of public equilibria of pathbreaking Abreu, Pearce, and Stacchetti (1990) contribution).

Additionally, in the literature pertaining to economic applications of stochastic games, the central concerns have been broader than the mere question of weakening conditions for the existence of subgame perfect or Markovian equilibrium. Rather, researchers have become more concerned with characterizing the properties of computational implementations, so they can study the quantitative (as well as qualitative) properties of particular subclasses of perfect equilibrium. Unfortunately, for uncountable number of states, there are only few papers and results that offer a set of rigours tools to compute, approximate or characterize the equilibrium strategies. See Balbus, Reffett, and Woźny (2013a,b, 2014a) for some recent contributions.

The aim of this paper is to address the question of short-memory equilibrium existence, characterization and computation using constructive monotone method, where our notion of monotonicity is defined using set inclusion order over spaces of value functions. Specifically, we study existence and approximation, relative to the set of MPNE for two important classes of stochastic games, namely: (i) stochastic altruistic growth economies (as analyzed by Amir (1996b) or Nowak (2006)) and (ii) supermodular games with strategic (within period) complementarities and positive externalities (as analyzed by Amir (2005), Curtat (1996) or Nowak

(2007)), although applications in other classes of games are possible (see Balbus, Reffett, and Woźny, 2012, for an early example applied to OLG economies).

We first prove existence of a *Markovian* NE via strategic dynamic programming methods similar to that proposed in the seminal work of Mertens and Parthasarathy (1987) and Abreu, Pearce, and Stacchetti (1990) (henceforth, MP/APS). We refer to this as a "indirect" method, as they focus exclusively on equilibrium values (rather than, characterizing the set of strategies that implement those equilibrium values). To mention, our method differs from those of the traditional MP/APS literature in at least two directions. Perhaps most importantly, we study the existence of short memory or Markovian equilibria, as opposed to broad classes of sequential or subgame perfect equilibria.[1] Additionally, our strategic dynamic programming method works directly in function spaces (as opposed to spaces of correspondences), by which we can avoid many of the difficult technical problems associated with measurability and numerical implementations using set-approximation techniques.[2]

Next, we propose a procedure for MPNE (value) set approximation. This differs from the approach taken by Judd, Yeltekin, and Conklin (2003) and Sleet and Yeltekin (2003) or Feng, Miao, Peralta-Alva, and Santos (2009) as our theoretical numerical methods: (i) operate directly in function spaces (ii) allow to analyze equilibria that are time/state dependent only (but are not continuation dependent), and moreover (iii) equilibria we study, are defined on a minimal state space, which greatly simplifies the approximation of the set of policies that implement particular values in the equilibrium value set.

The rest of the paper is organized as follows. We start in section 2 by presenting and discussing our method. Then in section 3 we present applications to a class of altruistic growth economy. Next, our results on Markov equilibrium existence and its value set approximation for a class of stochastic supermodular games, can be found in section 4. Finally, in the concluding section, we discuss relation of both direct and indirect methods (developed in this paper) for equilibrium value set approximation. Section 6 presents some auxiliary results.

2 Our method

The approach we take in this paper to prove existence of MPNE and approximate its (value) set, is the strategic dynamic programming approach based on the seminal work of Mertens and Parthasarathy (1987). See also Abreu, Pearce, and Stacchetti (1986, 1990) per similar

[1] It bears mentioning, we focus on short-memory Markovian equilibrium because this class of equilibrium has been the focus of a great deal of applied work. We should also mention a very interesting papers by Cole and Kocherlakota (2001), and Doraszelski and Escobar (2012) that also pursue a similar of trying to develop MP/APS type procedure in function spaces for Markovian equilibrium (i.e., methods where continuation structures are parameterized by functions) but for finite/countable number of states.

[2] In our accompanied papers (see Balbus, Reffett, and Woźny, 2012, 2014a,b), we propose an alternative *direct* method for *stationary* Markov equilibrium existence and computation. We view results of both, direct and indirect, methods as complementary and discuss them in the concluding section.

method adapted for repeated games[3]. In the original strategic dynamic programming approach, dynamic incentive constraints are handled using correspondence-based arguments. For each state $s \in S$, we shall envision agents playing a one-shot stage game with the continuation structure parameterized by a measurable correspondence of continuation values, say $v' \in \mathscr{V}$ where $\mathscr{V}$ is the space of nonempty, bounded, upper semicontinuous correspondences (for example). Imposing incentive constraints on deviations of the stage game under some continuation promised utility v', a natural operator B, that is monotone under set inclusion can be defined that transforms $\mathscr{V}$. By iterating on a "value correspondence" operator from a "greatest" element of the collection $\mathscr{V}$, the operator is shown to map down the "greatest set" $\mathscr{V}$, and then by appealing to standard "self-generation" arguments, it can be shown a decreasing subchain of subsets can be constructed, whose pointwise limit in the Hausdorff topology is the greatest fixed point of B. Just as in MP/APS for stochastic or repeated games, this fixed point turns out to be the set of sustainable values in the game, with a subgame perfect / sequential equilibrium being any measurable selection from the recursive strategy function supporting this limiting correspondence of values.

In this paper we propose a new procedure for constructing all measurable (possibly nonstationary) Markov Nash equilibria for a class of infinite horizon stochastic games. Our approach[4] is novel, as we operate directly in *function spaces*[5], i.e. a set of bounded measurable functions on S valued in $\mathbb{R}$. See also Datta, Mirman, and Reffett (2011), where similar methods for analyzing recursive equilibria in a class of dynamic economies were originally proposed.

To see this, consider an auxiliary strategic form game G_v^s parameterized by a continuation v and state $s \in S$. By $T(v)(s)$ denote a set of all measurable Nash equilibrium (strategies or payoffs) of G_v^s. Then we define an operator $B : \mathcal{V} \to \mathcal{V}$, where $\mathcal{V} := 2^V$ for some compact subset V of function space of bounded, measurable functions. We let:

$$B(W) = \cup_{w \in W} \{v \in V : \forall s \in S\, v(s) = T(w)(s)\}.$$

Operator B is increasing[6] on $\mathcal{V}$, when endowed with a set inclusion order. By $V^* \subset V$ we denote a set of MPNE (value) functions. In the next sections we derive conditions under which the following result holds: $\cap_t B^t(\mathcal{V}) = V^* \neq \emptyset$ (t-th composition). We also provide methods

[3]Such strategic dynamic programming methods are related to the work of Kydland and Prescott (1980) (and now referred to as KP), but are distinct. In the KP approach, one develops a recursive dual method for computing equilibrium, where Lagrange multipliers become state variables. In this method, the initial condition for the dual variables has to be made consistent with the initial conditions of the incentive constrained dynamic program. To solve this last step, Kydland-Prescott propose a correspondence based method that bears a striking resemblance to MP/APS method.

[4]It bears mentioning that for dynamic games with more restrictive shocks spaces (e.g., discrete or countable), MP/APS procedure has been used extensively in economics in recent years: e.g. by Kydland and Prescott (1980), Atkeson (1991), Pearce and Stacchetti (1997), Phelan and Stacchetti (2001) for policy games, or Feng, Miao, Peralta-Alva, and Santos (2009) (FMPA henceforth) for dynamic competitive equilibrium in recursive economies.

[5]For example, see Phelan and Stacchetti (2001, p. 1500-1501), who discusses such possibility in function spaces. In this paper, we show such a procedure is analytically tractable, and discuss it computational advantages in addition.

[6]In our paper we use increasing / decreasing terms in their week meaning.

of V^* approximation in the Hausdorff distance. Before proceeding we discuss some important properties of our methods and discuss its relation to the literature.

First, let us address the differences between the traditional correspondence based MP/APS procedure and our's. Let V^* be the correspondence of sequential equilibria satisfying $B_{APS}(V^*_{APS}) = V^*_{APS}$. From the definition of MP/APS operator we know the following:

$$(\forall s \in S)(\forall \text{ number } v \in V^*_{APS}(s))(\exists \text{ measurable function } v' \in V^*_{APS} \text{ s.t. } v = T(v')(s)).$$

Specifically, observe that continuation function v' can depend on v and s hence we shall denote it by $v'_{v,s}$. Now consider our operator B and its fixed point $V^* \subset V$. We have the following property:

$$(\forall \text{ function } v \in V^*)(\exists \text{ measurable function } v' \in V^* \text{ s.t. } (\forall s \in S)\, v(s) = T(v')(s)).$$

In our method continuation v' depends on v only, hence we can denote it by v'_v.

Observe that in both methods, the profile of equilibrium decision rules: $NE(v', s)$ is generalized Markov, as it is enough to know state s and continuation function v' to make an optimal choice. In our technique, however, the dependence of v on the current state is clear: $s \to NE(s, v'_v)$. So we can verify, if the generalized Markov policy is continuous, monotone etc in s easily. In the MS/APS approach one has the following: $s \to NE(s, v'_{v,s})$, so even if NE is continuous in both variables, (generally) there is no way to control continuity of $s \to v'_{v,s}$. The best example of such discontinuous continuation selection is perhaps, the time-consistency model (see Caplin and Leahy, 2006). These problems are also the main motivation for developing a computational technique that uses specific properties of (the profile) of the equilibrium decision rules NE with respect to s (important especially, when the state space is uncountable).

On a related matter, the set of Markov Nash Equilibrium values is generally a subset of the MP/APS (subgame perfect /sequential) equilibrium (value) set. One should keep in mind, for repeated games, the two sets need not be very different[7].

This framework used to prove existence of the MPNE can be also used to define a piecewise constant approximation of the equilibrium value set V^*. Our theoretical numerical method is directly linked to the proof of equilibrium existence, hence heavily relies on the our theoretical result, i.e. it uses the fact that (i) our method operates directly in function spaces, (ii) allows to analyze equilibria that are time/state dependent only and (iii) studies equilibria defined on a minimal state space. All of this greatly simplifies the approximation of the set of policies that implement particular values in the equilibrium value set. The details are presented for the two examples we study in the next sections. In the first example we define an operator T on the space of equilibrium strategies, in the second example on the equilibrium value set.

[7]For example, as Hörner and Olszewski (2009) show, the folk theorem can hold for repeated games with imperfect monitoring and finite memory strategies. Further, Barlo, Carmona, and Sabourian (2009) show a similar results for repeated games with perfect monitoring, rich action spaces and Markov equilibria.

3 A class of altruistic growth economies

Our first model is a stochastic overlapping generations production economy with capital, but without commitment between successor generations. Time is discrete and indexed by $t = \{0, 1, 2, \ldots\}$. The economy is populated by a sequence of identical short-lived agents (or "generations"), each living one period. Given our indexation of time, we shall study a sequential infinite horizon stochastic game with a countable number of players whose "names" are indexed by time t. For simplicity, the size of each generation is assumed to be equal and normalized to unity, and there is no population growth. Any given generation divides its (inherited) output $s \in S \subset \mathbb{R}$ between current consumption c and investment in a stochastic production technology $z = s - c$, with the proceeds of this investment being left to the next generation as a bequest. As is often typical for OLG models in macroeconomic applications each new generation receives utility u from its own current consumption, as well as v from that of its immediate successor generation. As is standard in the class of models we study, there is a stochastic production technology summarized by stochastic transition Q, mapping current investment $s - c$ into the next period output, denoted by s'. This environment can be represented by a stochastic, bequest game as analyzed by Amir (1996b), Nowak (2006) and Balbus, Reffett, and Woźny (2012). With the assumptions specified above preferences of a typical generation, when the next one uses Borel measurable policy $h : S \to S$ are given by:

$$P(c, h)(s) := u(c) + \int_S v(h(s'))Q(ds'|s - c).$$

Let H be a set of bounded measurable functions f on S, such that $(\forall s \in S)\, f(s) \leq s$. A sequence of functions $\{h_t\}$, where $h_t \in H$ is a Markov Perfect Nash Equilibrium if:

$$\forall t \forall s \quad h_t(s) \in BR(h_{t+1})(s) := \arg\max_{c \in [0,s]} \{P(c, h_{t+1})(s)\}.$$

We now construct a set V^* of all MPNE in the subset of increasing and Lipschits continuous functions with modulus 1 on S denoted by $CM \subset H$, endowed with the uniform topology on compact subsets. The convergence in this topology is denoted as $\xrightarrow{u}$. We define V^* as:

$$V^* := \{h \in CM : \exists \{h_t\}_{t=1}^{\infty} \text{ s.t. } h_t = BR(h_{t+1}), h = BR(h_1)\}.$$

Assumption 1 *Let:*

- $u : S \to \mathbb{R}$ *be continuous, strictly concave and increasing,*
- $v : S \to \mathbb{R}$ *be continuous and increasing,*
- *for all* $h \in CM$ $z \to \int_S v(h(s'))Q(ds'|z)$ *be continuous, concave and increasing,*

- *for all z, $Q(\cdot|z)$ be a nonatomic measure and let support of Q be independent of z.*

Consider an operator B defined on $\mathcal{V} := 2^{CM}$, $B(\emptyset) = \emptyset$ and for $W \neq \emptyset, W \in \mathcal{V}$:

$$B(W) = \cup_{h \in W} \{h' \in CM : (\forall s \in S)\, h'(s) \in BR(h)(s)\}.$$

Lemma 1 *Assume 1. The for each $s \in S$ the correspondence $h \to BR(h)(s)$ has a sequentially closed graph*[8].

Proof of Lemma 1: First we show that $(c, h) \to P(c, h)(s)$ is continuous on $(c, h) \in [0, s] \times CM$. Let $(h_t)_{t \in \mathbb{N}} \subset CM$, $h_t(s) \to h(s)$ pointwise in S and $c_t \in BR(h_t)(s)$. Suppose $c_t \to c$. Put $z_t := s - c_t$ and $z := s - c$. Hence $v(h_t(s)) \to v(h(s))$ pointwise in s. Since $z_t \to z$ and $Q(\cdot|z)$ is nonatomic, hence by Theorem 5

$$\int_S v(h_t(s'))Q(ds'|z_t) \to \int_S v(h(s'))Q(ds'|z). \tag{1}$$

Hence the payoff function P is continuous in (h, c). As a result by Berge Maximum Theorem $BR(\cdot)(s)$ has a sequentially closed graph. ■

Lemma 2 *Let $K \subset X$. Assume K is sequentially compact. Let $\Gamma : S \times K \to \mathbb{R}$ be a correspondence. For fixed s assume that $\Gamma(s, \cdot)$ has a sequentially closed graph, that is $h_t(\cdot) \to h(\cdot)$ pointwise in S, $x_t \to x$, $x_t \in \Gamma(s, h_t)$ implies $x \in \Gamma(s, h)$. For a sequentially compact $W \subset K$ define*

$$B(W) := \bigcup_{h \in W} \{h' \in K : \forall_{s \in S}\, h'(s) \in \Gamma(s, h)\}. \tag{2}$$

Then $B(W)$ is sequentially compact.

Proof of lemma 2: Let $h'_t \in B(W)$. Then for some $h_t \in W$ and all s we have $h'_t(s) \in \Gamma(s, h_t)$. Since W is sequentially compact, we obtain existence of subsequence of h_t convergent to some $h^* \in W$. Clearly h'_t has a convergent subsequence to other $h'' \in K$. Since $\Gamma(s, \cdot)$ has a closed graph, hence $h''(s) \in \Gamma(s, h^*)$ and $h'' \in B(W)$. ■

Lemma 3 *Assume 1 and let $W \subset CM$ be a compact set in $\overset{u}{\to}$ topology. Then $B(W)$ is a compact set in $\overset{u}{\to}$ topology.*

[8] A set $G \subset X$ is said to be sequentially compact, if every converging sequence $\{f_t\} \in G$ has a limit point $f \in G$. For a topological space $(Y, \mathcal{T}_Y)$ a function $F : X \to Y$ is a *sequentially continuous* if it preserves convergence of sequences.

Proof of lemma 3: By Lemmas 1 and 2 $B(W)$ is sequentially compact, whenever W is. Observe that convergence in $\stackrel{u}{\to}$ topology on CM is equivalent to pointwise convergence. ■

Lemma 4 (Self generation) *Assume 1 and let $W \subset B(W)$. Then $W \subset V^*$*

Proof of lemma 4: Let $h \in W$. Then $h \in B(W)$, hence there exists $h = BR(h_1)$ for $h_1 \in W$ and consequently $h_t \in W$ such that $h_t = BR(h_{t+1})$. Thus $h \in V^*$. ■

With these lemmas at hand we are ready to stay our first main result of this section.

Theorem 1 (Existence) *Let assumption 1 be satisfied then B has the greatest fixed point W^*, and $\bigcap_{t=1}^{\infty} W_t = W^* = V^* \neq \emptyset$, where $W_t := B^t(\mathcal{V})$. Moreover, all W_t are sequentially compact, hence compact in $\stackrel{u}{\to}$.*

Proof of theorem 1: Observe that since CM is equicontinuous, hence by Arzela-Ascoli Theorem CM is compact with the natural sup-norm topology on all compact subsets of S. Hence it is sequentially compact. By Lemma 3 all W_t are compact sets. Moreover, by Assumption 1, $P(\cdot, h)(s)$ is concave for each $h \in CM$, $(c,s) \to P(c,h)(s)$ has doubly increasing differences[9] with $\phi(s) = s$. Hence by Theorem 2.3 in Curtat (1996) $W_t \neq \emptyset$ for all $t \in \mathbb{N}$.

Let $V^\infty := \bigcap_{t=1}^{\infty} W_t$, and W^* be the greatest fixed point of B. Then we show that $W^* \subset V^\infty$. Indeed $W^* \subset CM$, hence by monotonicity of B we have $W^* = B(W^*) \subset W_2 = B(W_1)$ and consequently $W^* \subset W_t$ follows $W^* \subset B(W_t) = W_{t+1}$ therefore $W^* \subset W_t$ for all t, hence

$$W^* \subset \bigcap_{t=1}^{\infty} W_t = V^\infty. \tag{3}$$

By monotonicity of B we have: $B(V^\infty) = B\left(\bigcap_{t=1}^{\infty} W_t\right) \subset \bigcap_{t=1}^{\infty} B(W_t) = \bigcap_{t=1}^{\infty} W_{t+1} = V^\infty$. We show the reverse inclusion. Let $h \in V^\infty$. Then there exists a sequence of Lipschitz continuous functions h_t such that $h \in BR(h_t)$ and $h_t \in W_t$. Since CM is compact we can choose a convergent subsequence. Suppose it converges to some point h_0. Since all $W_t \neq \emptyset$ hence $h_0 \in V^\infty$. By Lemma 1 $h \in BR(h_0)$. Therefore, $h \in B(V^\infty)$. Hence $V^\infty = B(V^\infty)$ and by definition $V^\infty \subset W^*$. On the other hand by (3) we have $V^\infty = W^*$.

Next we show that V^* is a fixed point. Clearly $B(V^*) \subset V^*$. We show the reverse inclusion. Let $h \in V^*$. Let $(h_t)_{t=1}^{\infty}$ be a sequence such that $(h, h_1, h_2, ...)$ is MPE. Then clearly $h = BR(h_1)$ and $h_1 \in V^*$. Hence $V^* = B(V^*)$ and we conclude that $V^* \subset W^*$. Applying again Lemma 4 we also have $W^* \subset V^*$, hence $V^* = W^* = V^\infty$. Therefore, by Lemma 14 $V^* \neq \emptyset$. ■

[9] Let $X, Y \subset \mathbb{R}$. Function $f : X \times Y \to \mathbb{R}$ has doubly increasing differences with Lipschitz continuous function $\phi : Y \to X$, if it has increasing differences and $f(\phi(y) - x, y)$ has increasing differences.

First, this theorem proves MPNE existence in the space of increasing, Lipschitz continuous functions over uncountable number of states. Second, it characterizes the set of all MPNE by an approximation scheme converging in the set inclusion order. To develop a theoretical numerical method based on this approximation scheme, however, one needs some additional assumption.

Assumption 2 *Assume 1 and that S is a bounded interval in $\mathbb{R}$.*

Having that we now define our numerical method in details. Let $\mathbb{R}^S$ be a Tikchonov product with its Tikchonov topology. Then each real valued function $f : S \to \mathbb{R}$ can be identified with an element of $\mathbb{R}^S$. Let $\pi_s : \mathbb{R}^S \to \mathbb{R}$ be a canonical projection on coordinate $s \in S$ (i.e. $\pi_s(f) = f(s)$). Clearly $\pi_s(\cdot)$ is a continuous function, hence maps each sequentially compact[10] subset of $\mathbb{R}^S$ to a compact set on $\mathbb{R}$. By $d_H(A, B)$ we define a Hausdorff distance between subsets $A, B \subset \mathbb{R}$ associated with a standard Euclidean space. Moreover, we have

Lemma 5 *Let Assumption 2 be satysfied and $(V_t)_{t\in\mathbb{N}}$ be a descending family of sequentially compact subsets of $\mathbb{R}^S$. Let $V = \bigcap_{t=1}^{\infty} V_t$. Then $\pi_s(V) = \bigcap_{t=1}^{\infty} \pi_s(V_t)$ $(\forall s \in S)$.*

Proof of lemma 5: It is obvious that $\pi_s(V) \subset \bigcap_{t=1}^{\infty} \pi_s(V_t)$. We now show the converse inclusion. Let $x \in \bigcap_{t=1}^{\infty} \pi_s(V_t)$. Then there is a sequence of functions $f_t \in V_t$, such that $x = f_t(s)$ for all t. Since each V_t is sequentially compact, f_t has a convergent subsequence. Thus without loss of generality assume $f_t \to f$ pointwise and $f \in V$. Obviously $x = f(s)$. Therefore $x = \pi_s(f) \in \pi_s(V)$. ■

Lemma 6 *Assume 2. Let $(W_t)_{t\in\mathbb{N}}$ be a sequence as in Theorem 1. Let $W_t(s) := \pi_s(W_t)$ for any $s \in S$. In a similar way we define V^*. Then*

$$(\forall s)\, V^*(s) = \bigcap_{t=1}^{\infty} W_t(s) = \lim_{t\to\infty} W_t(s), \tag{4}$$

where the last limit is in a Hausdorff topology on compact subsets of $\mathbb{R}$.

Proof of lemma 6: By Theorem 1 we have $V^* = \bigcap_{t=1}^{\infty} W_t$ and $\{W_t\}$ is a descending sequence. Then noting that each W_t is sequentially compact, from Lemma 5

$$V^*(s) = \pi_s(V^*) = \pi_s\left(\bigcap_{t=1}^{\infty} W_t\right) = \bigcap_{t=1}^{\infty} \pi_s(W_t) = \bigcap_{t=1}^{\infty} W_t(s) = \lim_{t\to\infty} W_t(s),$$

hence (4) is satisfied. The last equality above is clear since all W_t is $\xrightarrow{u}$ compact, which is equivalent with the sup-norm topology. ■

[10] That is each sequence contains pointwise convergent subsequence. Note that compactness and sequential compactness are equivalent if and only if S is countable.

We consider an approximation of $V^*(s)$ related to Beer (1980). Since S is interval in $\mathbb{R}$ we can define a piecewise constant multifunction in the following way: if $S = [\xi, \eta]$, then we divide $[\xi, \eta]$ into 2^j equal subintervals with equal length. Let $\mathcal{C}_j$ be a block partition set. For each block partition $C \in \mathcal{C}_j$ we define θ_C as follows

$$\theta_C(s) = \begin{cases} \bigcup_{s' \in C} V^*(s') & \text{if } s \in C \\ \emptyset & \text{otherwise,} \end{cases} \tag{5}$$

where s' is some fixed selection from C. Define $\hat{V}_j(s) = \bigcup_{C \in \mathcal{C}_j} \theta_C(s)$. Similarly we define any other correspondence from S to $\mathbb{R}$.

Theorem 2 (Computation) *Under Assumption 2 then we have* $\forall s \in S$*:*

(i) $V^*(s) \subset \ldots \subset \hat{V}_{j+1}(s) \subset \hat{V}_j(s) \subset \ldots \subset \hat{V}_1(s)$,

(ii) $\lim_{j \to \infty} d_H(\hat{V}_j(s), V^*(s)) = 0$,

(iii) let $\hat{W}_{t,j}(\cdot)$ *be a* $j-th$ *piecewise approximation for* $W_t(\cdot)$*. Then*

$$(\forall s \in S) \qquad \lim_{t,j \to \infty} d_H(\hat{W}_{t,j}(s), V^*(s)) = 0. \tag{6}$$

Proof of theorem 2: Proof of (i) is obvious. Proof of (ii): We use Theorem 1 in Beer (1980) to show that $\hat{V}^j(s) \to V^*(s)$ in the Hausdorff distance. We need to show that $V^*(\cdot)$ is u.s.c. and compact valued correspondence. By the Closed Graph Theorem (e.g. Theorem 17.11 in Aliprantis and Border (2005)) we only need to show that V^* has a closed graph.

Let $s_n \to s$, $x_n \to x$ (as $n \to \infty$). Then for all $n \in \mathbb{N}$ there is a sequence $(h_t^n)_{t \in \mathbb{N}} \subset CM$ such that $x_n = h_1^n(s_n)$ and for all $t \in \mathbb{N}$ $h_t^n(s') \in BR(h_{t+1}^n)(s')$. Since CM is compact in the $\overset{u}{\to}$ topology, we may assume for all $t \in \mathbb{N}$, $h_t^n \overset{u}{\to} h_t^*$ as $n \to \infty$. By Lemma 1 $h_t^*(s') \in BR(h_{t+1}^*)(s')$ for all t. Moreover, $x = \lim_n x_n = \lim_n h_1^n(s_n) = h_1^*(s)$, hence $x \in V^*(s)$.

Proof of (iii) By Theorem 1, point (i) and Beer (1980) we have:

$$V^* = \bigcap_{t=1}^{\infty} W_t = \bigcap_{t=1}^{\infty} \bigcap_{j=1}^{\infty} \hat{W}_t^j. \tag{7}$$

Hence by Lemma 6 we have:

$$V^*(s) = \bigcap_{t=1}^{\infty} \bigcap_{j=1}^{\infty} \hat{W}_t^j(s). \tag{8}$$

Applying Theorem 1 in Beer (1980) we obtain that each $\hat{W}_t^j(s)$ is a compact set. Moreover, there is a sequence $J_t \to \infty$ such that:

$$W_{t+1} \subset \hat{W}_{t+1}^{J_t} \subset W_t,$$

hence $V^* = \bigcap_{t=1}^{\infty} \hat{W}_{t+1}^{J_t}$. By point (i) we have for any $s \in S$, $V^*(s) = \bigcap_{t=1}^{\infty} \hat{W}_{t+1}^{J_t}(s)$. Hence we have $V^*(s) = \lim_{t\to\infty} \hat{W}_{t+1}^{J_t}(s)$ in the Hausdorff metric sense. As a result, for an arbitrary open set G containing $V^*(s)$ we have $\hat{W}_t^{J_t}(s) \subset G$. Let t_0 be a number such that this inclusion is satisfied for all $t > t_0$. Since $j \to \hat{W}_t^j(s)$ is a descending family of sets, hence $\hat{W}_t^j(s) \subset G$, whenever $t > t_0$, and $j > J_{t_0}$. ∎

4 A class of stochastic supermodular games

We consider a n-player, discounted, infinite horizon, stochastic game in discrete time. The primitives of the class of games are given by the tuple $\left\{S, (A_i, \tilde{A}_i, \beta_i, u_i)_{i=1}^n, Q, s_0\right\}$, where $S = [0, \bar{S}] \subset \mathbb{R}$ is the state space, $A_i \subset \mathbb{R}^{k_i}$ player i action space with $A = \times_i A_i$, β_i is the discount factor for player i, $u_i : S \times A \to \mathbb{R}$ is the one-period payoff function, and $s_0 \in S$ the initial state of the game. For each $s \in S$, the set of feasible actions for player i is given by $\tilde{A}_i(s)$, which is assumed to be compact Euclidean interval in $\mathbb{R}^{k_i}$. By Q, we denote a transition function that specifies for any current state $s \in S$ and current action $a \in A$, a probability distribution over the realizations of next period states $s' \in S$.

Using this notation, a formal definition of a (Markov, stationary) strategy, payoff, and a Nash equilibrium can now be stated as follows. A *strategy* for a player i is denoted by $\Gamma_i = (\gamma_i^1, \gamma_i^2, \ldots)$, where γ_i^t specifies an action to be taken at stage t as a function of history of all states s^t, as well as actions a^t taken as of stage t of the game. If a strategy depends on a partition of histories limited to the current state s_t, then the resulting strategy is referred to as *Markov*. If for all stages t, we have a Markov strategy given as $\gamma_i^t = \gamma_i$, then strategy Γ_i for player i is called a *Markov-stationary strategy*, and denoted simply by γ_i. For a strategy profile $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$, and initial state $s_0 \in S$, the expected payoff for player i can be denoted by:

$$U_i(\Gamma, s_0) = (1 - \beta_i) \sum_{t=0}^{\infty} \beta_i^t \int u_i(s_t, a_t) dm_i^t(\Gamma, s_0),$$

where m_i^t is the stage t marginal on A_i of the unique probability distribution (given by Ionescu–Tulcea's theorem) induced on the space of all histories for Γ. A strategy profile $\Gamma^* = (\Gamma_i^*, \Gamma_{-i}^*)$ is a *Nash equilibrium* if and only if Γ^* is feasible, and for any i, and all feasible Γ_i, we have

$$U_i(\Gamma_i^*, \Gamma_{-i}^*, s_0) \geq U_i(\Gamma_i, \Gamma_{-i}^*, s_0).$$

For our arguments in this section, we shall require the following assumptions.

Assumption 3 *We let:*

- u_i *be continuous on* $S \times A$, *and let* u_i *be bounded by* 0 *and* $\bar{u}$,
- u_i *be supermodular in* a_i *(for any* s, a_{-i}*), and has increasing differences in* $(a_i; a_{-i}, s)$, *and be increasing in* (s, a_{-i}), *(for each* a_i*),*
- *for all* $s \in S$ *the sets* $\tilde{A}_i(s)$ *be compact intervals and multifunction* $\tilde{A}_i(\cdot)$ *be upper hemicontinuous and ascending under both (i) set inclusion i.e. if* $s_1 \le s_2$ *then* $\tilde{A}_i(s_1) \subseteq \tilde{A}_i(s_2)$, *and (ii) Veinott's strong set order* $\le_v$ *(i.e.,* $\tilde{A}_i(s_1) \le_v \tilde{A}_i(s_2)$ *if for all* $a_{1i} \in \tilde{A}_i(s_1), a_{2i} \in \tilde{A}_i(s_2)$, $a_{1i} \wedge a_{2i} \in \tilde{A}_i(s_1)$ *and* $a_{1i} \vee a_{2i} \in \tilde{A}_i(s_2)$*),*
- $Q(ds'|s, a)$ *be stochastically supermodular in* a_i *(for any* s, a_{-i}*), has stochastically increasing differences in* $(a_i; a_{-i}, s)$, *and be stochastically increasing with* a, s,
- $Q(ds'|s, a)$ *be a nonatomic measure for all* $s \in S$,
- $Q(ds'|\cdot, \cdot)$ *has a Feller property*[11] *on* $S \times A$,
- *support of* Q *is independent of* (s, a).

Balbus, Reffett, and Woźny (2014a) show, how this game under imposed assumptions, can be applied to study price competition with durable goods, time consistent policy or symmetric equilibria in public goods games, among others.

We now describe formally our method. Let V be the space of bounded, increasing, measurable value functions on S with values in $\mathbb{R}^n$:

$$V := \{v : S \to \mathbb{R}^n, \text{ such that } v \text{ is increasing and } ||v||_\infty \le \bar{u}\}.$$

Endow V with the weak topology. See Majumdar and Sundaram (1991) or Dutta and Sundaram (1992) for a proof that is a compact and metrizable topology on V. Define an auxiliary (or, super) one period n-player game $G_v^s = (\{1, \dots, n\}, \{\tilde{A}_i(s), \Pi_i\}_{i=1}^n)$, where payoffs depend on a weighted average of (i) current within period payoffs, and (ii) a vector of expected continuation values $v \in V$, with weights given by a discount factor:

$$\Pi_i(v_i, s, a) := (1 - \beta_i) u_i(s, a) + \beta_i \int_S v_i(s') Q(ds'|s, a),$$

where $v = (v_1, v_2, \dots, v_n)$, and the state $s \in S$. As $v \in V$ is increasing function, under our assumptions, G_v^s is a supermodular game. Therefore, G_v^s has a nonempty complete lattice of pure strategy Nash equilibria (e.g., see Topkis (1979) or Veinott (1992)). By $NE(v, s)$ denote the set of Nash equilibria of game G_v^s restricted to increasing functions on S.

[11] That is $\int_S f(s') Q(ds'|\cdot, \cdot)$ is continuous, whenever f is continuous and bounded.

By $\mathcal{V}$ denote the set of all subsets of V partially ordered by set inclusion. Having that, for any subset of functions $W \in \mathcal{V}$, define an operator B to be:

$$B(W) = \bigcup_{v \in W} \{w \in V : (\forall s \in S)\, w(s) = \Pi(v, s, a^*(s)),\ \text{for some } a^* \text{ s.t. } a^* \in NE(v, s)\}\,.$$

We denote by $V^* \in \mathcal{V}$ the set of equilibrium values corresponding to all monotone, Markovian equilibria of our stochastic game. It is immediate that B is both (i) increasing under set inclusion, and (ii) transforms the space $\mathcal{V}$ into itself. This fact implies, among other things, that $B : \mathcal{V} \to \mathcal{V}$ has the greatest fixed point by Tarski's theorem (as $\mathcal{V}$ ordered under set inclusion is a complete lattice). Now, as opposed to traditional MP/APS, where sequential/subgame perfect equilibrium are the focal point of the analysis, it turns out the greatest fixed point of our operator B will generate the set of *all* (monotone, possibly nonstationary) Markovian equilibria values V^* in the function space $\mathcal{V}$.

Finally, we offer iterative methods to compute the set of MPNE values in the game. Consider a sequence of subsets of (n-tuple) equilibrium value functions (on S) $\{W_t\}_{t=1}^{\infty}$ generated by iterations on our operator B from the initial subset $W_1 = V$ (with $W_{t+1} = B(W_t)$). By construction, $W_1 = V$ is mapped down (under set inclusion by B) and a subchain of decreasing subsets $\{W_t\}$ converge to V^*, the greatest fixed point of B. To formally prove such convergence in the case of uncountable state space we need to restrict to a class of increasing functions. On this set let us introduce a relation $v \approx w$ iff $v(s) \neq w(s)$ on at most countable set. Clearly it is an equivalence relation.

Lemma 7 *Assume 3. In each equivalence class there is exactly one u.s.c. function.*

Proof of lemma 7: First we show that each class possess at least one u.s.c. function. Let v be arbitrary increasing function. Then, there is at most countably many discountinuity points $S_0 := \{s_1, s_2, \ldots, ...\}$. We define $\tilde{v}(s) := v(s)$ if $s \in S \setminus S_0$ and $\tilde{v}(s_i) = \limsup_{s \to s_i} v(s)$ if $s_i \in S_0$. Clearly $\tilde{v}$ is u.s.c. function and differ with v on at most countable set.

On the other hand suppose there are two increasing and u.s.c. functions w and v that differ on countable sets. Let s_0 be arbitrary point in which $w(s_0) \neq v(s_0)$. Then:

$$v(s_0) = \limsup_{s \to s_0} v(s) = \lim_{n \to \infty} v(s_n),$$

for some sequence $s_n \to s_0$ and $s_n > s_0$. Without loss of generality we can assume $v(s_n) = w(s_n)$ as set of points in which v and w match is dense in S. Thus:

$$v(s_0) = \lim_{n \to \infty} v(s_n) = \lim_{n \to \infty} w(s_n) = w(s_0),$$

which contradics $v(s_0) \neq w(s_0)$. ■

Lemma 8 *Assume 3. Every sequence of increasing, real valued functions v^t on S and bounded by a common value has a pointwise convergent subsequence. As a result $\mathcal{V}$ is sequentially compact.*

Proof of lemma 8: Let $v^t : S \to \mathbb{R}$ be a sequence of increasing functions bounded by common value. By Lemma 7, there exist a sequence of the functions $\tilde{v}^t$ such that each of them is u.s.c. and differ from v^t on at most countable set, say D_t. Then we can choose a weakly convergent subsequence to some u.s.c. function $\tilde{v}$. Define $D := \bigcup_{t=1}^{\infty} D_t$. Clearly, it is a countable set, and for $S \setminus D$ we have $\tilde{v}^t(s) = v^t(s)$. On the other hand, the set of discontinuity points of v, say D_v, is also countable and on $s \in S \setminus (D \cup D_v)$ we have $v^t(s) = \tilde{v}^t(s) \to \tilde{v}(s)$ as $t \to \infty$. Since $D \cup D_v$ is countable, hence from a sequence v^t we can choose a subsequence t_n such that v^{t_n} is convergent on set $D_v \cup D$ as well. As a result $\lim_{n\to\infty} v^{t_n}(s)$ is this limit function. ■

Lemma 9 *Assume 3 and let $v^t \in \mathcal{V}, a^t(s) \in A(s)$ for all $t \in \mathbb{N}$ and $s \in S$. If $v^t \to v^*$ and $a^t \to a^*$ pointwise in s, then $\Pi(v^t, s, a^t) \to \Pi(v^*, s, a^*)$.*

Proof of lemma 9: By Assumption 3 u_i is continuous in a. Observe that

$$(v, a) \to \int_S v(s')Q(ds'|s, a)$$

is continuous by Theorem 5. Thus Π is continuous in (v, a). ■

Lemma 10 *Assume 3 and let v^t and v^* be increasing functions and bounded by common value. Let $v^t(\cdot) \to v^*(\cdot)$ pointwise and $a^t(s) \in NE(v^t, s)$ for all $s \in S$. Then, if $a^t \to a^*$ pointwise, then $a^*(s) \in NE(v^*, s)$*

Proof of lemma 10: By Assumption 3 we have continuity of r_i in a, hence for all $a_i \in A_i(s)$, and $s \in S$ holds

$$\Pi_i(v^t, s, a^t(s)) \geq \Pi_i(v^t, s, a^t_{-i}(s), a_i).$$

Taking a limit with $t \to \infty$ by Lemma 9 we obtain this inequality for a^* and v^*, hence $a^*(s) \in NE(v^*, s)$. ■

Lemma 11 *Assume 3, then B maps $\mathcal{V}$ into itself. Moreover, $B(W) \neq \emptyset$, whenever $W \neq \emptyset$.*

Proof of lemma 11: Assume that $W \neq \emptyset$. Let $v \in W$. Then v is increasing and hence by Assumption 3 $\Gamma(v, s)$ is supermodular game with a parameter s. Hence, and by Milgrom and Roberts (1994) there exist the greatest and the least selections and both are increasing in s. Again by Assumption 3 the equilibria payoffs are both increasing in s. Let w be one of such function. Thus $w \in B(W)$. ■

Lemma 12 *Assume 3 and let W be sequentially compact in a product topology. Then $B(W)$ is sequentially compact as well.*

Proof of lemma 12: Since $B(W) \subset \mathcal{V}$ and $\mathcal{V}$ is sequentially compact set in product topology which is Hausdorff, we just need to show $B(W)$ is sequentially closed. Let $(w_t)_{t\in\mathbb{N}} \subset B(W)$ and suppose $w_t \to w$ pointwise. Let $(v_t)_{t\in\mathbb{N}} \in W$ and $(a_t(\cdot))_{t\in\mathbf{N}}$ be a sequence such that $w_t(s) = \Pi(v_t, s, a_t(s))$. By Lemma 8 without loss of generality suppose $v_t \to v$ pointwise. Since W is sequentially compact, hence $v \in W$. Put D_w as a set of discontinuity points of w. Clearly D_w is at most countable. By Lemma 10, for each $s \in S$ there exists $a^*(s) \in NE(v, s)$ such that $w(s) = \Pi(v, s, a^*(s))$. Hence $w \in B(W)$. ■

Lemma 13 (Self generation) *Assume 3. Then, if $W \subset B(W)$, then $B(W) \subset V^*$.*

Proof of lemma 13: Let $w \in B(W)$. Then, we have $v_0(\cdot) := w(\cdot)$ where $w(s) = \Pi(v_1, s, \gamma^1(s))$ for some $v_1 \in W$, Nash equilibrium $\gamma^1(s) \in NE(v_1, s)$ and all $s \in S$.

Then, since $v_1 \in W$ by the assumption, $v_1 \in B(W)$. Consequently, for $v_t \in W \subset B(W)$ $(t \geq 1)$ we can choose $v_{t+1} \in W$ such that $v_t(\cdot) = \Pi(v_{t+1}, \cdot, \gamma^{t+1}(\cdot))$ and $\gamma^t(\cdot) \in NE(v_{t+1}, \cdot)$. Clearly, the Markovian strategy γ generates payoff vector w. We next need to show this is a Nash equilibrium in the stochastic game for $s \in S$. Suppose that only player i uses some other strategy $\tilde{\gamma}_i$. Then, for all t and $s \in D_t$, we have $v_t^i(s) = \Pi_i(v_{t+1}, s, \gamma^t(s)) \geq \Pi_i(v_{t+1}, s, \gamma_{-i}^{t+1}(s), \tilde{\gamma}_i^{t+1})$. If we take a T truncation $\gamma^{T,\infty} = \left(\left(\tilde{\gamma}_i^1, \gamma_{-i}^1\right), \ldots, \left(\tilde{\gamma}_i^T, \gamma_{-i}^T\right), \gamma^{T+1},\ \gamma^{T+2}, \ldots\right)$ [12], this strategy can not improve a payoff for player i. Indeed:

$$U_i(\gamma, s) \geq U_i(\gamma_{-i}, \gamma_i^{T,\infty}, s) \to U_i(\gamma_{-i}, \tilde{\gamma}_i, s)$$

as $T \to \infty$. This convergence has been obtained as u_i is bounded, and the residuum of the sum $U_i(\gamma_{-i}, \gamma_i^{T,\infty}, s)$ depending on $\left(\gamma^{T+1}, \gamma^{T+2}, \ldots\right)$ can be obtained as an expression bounded by $\bar{u}$, multiplied by β_i^T. Hence w is Nash equilibrium for $s \in S$. Thus $B(W) \subset V^*$ ■

We are now ready to summarize these results in the next theorem.

Theorem 3 *Assume 3. Then:*

1. *operator B has the greatest fixed point $W^* \neq \emptyset$ with $W^* = \lim_{t\to\infty} W_t = \bigcap_{t=1}^{\infty} W_t$,*
2. *we have $V^* = W^*$.*

Proof of theorem 3: We prove 1. As $\mathcal{V}$ is a complete lattice, B is increasing, by Tarski theorem, B has the greatest fixed point W^*. Moreover, as B is increasing, $\{W_t\}_{t=0}^{\infty}$ is a decreasing

[12] That is agent i uses strategy $\tilde{\gamma}$ up to period T and γ after that. The other agents use γ.

sequence (under set inclusion). Let $V^\infty := \lim_{t\to\infty} W_t = \bigcap_{t=1}^{\infty} W_t$. We need to show that $V^\infty = W^*$. Clearly, $V^\infty \subset W_t$ for all $t \in \mathbb{N}$; hence

$$B(V^\infty) = B\left(\bigcap_{t=1}^{\infty} W_t\right) \subset \bigcap_{t=1}^{\infty} B(W_t) = \bigcap_{t=1}^{\infty} W_{t+1} = V^\infty.$$

To show equality, it suffices to show $V^\infty \subset B(V^\infty)$. Let $w \in V^\infty$. Then, $w \in W_t$ for all t. By the definition of W_t and B, we obtain existence of the sequence $v^t \in W_t$ and Nash equilibria a^t such that

$$w(s) = \Pi(v^t, s, a^t(s)).$$

for all t and $s \in S$.

Since $\mathcal{V}$ is sequentially compact, without loss of generality, assume v^t converges to v^*. Moreover, $v^* \in V^\infty$, since W_t is a descending set of sequentially compact sequences in the product topology. Fix arbitrary $s \in S$. Without loss of generality, let $a^t \to a^*$, where a^* is some point from A. By Lemma 10 a^* is a Nash equilibrium in the static game $\Gamma(v^*, s)$.

We obtain $w \in B(V^\infty)$. Hence, V^∞ is a fixed point of B, and, by definition $V^\infty \subset W^*$.

To finish the proof, we simply need to show $W^* \subset V^\infty$. Since $W^* \subset V$, $W^* = B(W^*) \subset B(V) = W_1$. By induction, we have $W^* \subset W_t$ for all t; hence, $W^* \subset V^\infty$. Therefore, $W^* = V^\infty$, which completes the proof.

We prove 2. First show that V^* is a fixed point of operator B. Clearly $B(V^*) \subset V^*$. So we just need to show the reverse inclusion. Let $v \in V^*$ and $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a profile supporting v. By assumption 3, $\gamma_{2,\infty} = (\gamma_2, \gamma_3, \ldots)$ must be a Nash equilibrium μ almost everywhere (i.e. a set of initial states S_0 which $\gamma_{2,\infty}$ is not a Markov equilibrium must satisfy $\mu(S_0) = 0$). Define a new profile $\tilde{\gamma}(s) = \gamma_{2,\infty}$ for $s \notin S_0$ and $\tilde{\gamma}(s) = \gamma$ if $s \in S_0$. Let $\tilde{v}$ be equilibrium payoff generated by $\tilde{\gamma}$. Clearly, $\tilde{v} \in V^*$ and is μ measurable and also $v(s) = \Pi(\tilde{v}, s, \gamma_1)$. Thus $v \in B(V^*)$ and hence $V^* \subset B(V^*)$. As a result $B(V^*) = V^*$.

Finally by definition (greatest fixed point) of W^* we conclude that $V^* \subset W^*$. To obtain the reverse inclusion we apply lemma 13. Indeed $W^* \subset B(W^*)$, and, therefore, $W^* \subset V^*$ and we obtain that $V^* = W^*$.

Finally, observe that by previous steps $V^* = \bigcap_{t=1}^{\infty} W_t$ and W_t is set inclusion descending sequence and by Lemma 12 all sets W_t are sequentially compact. Hence and by Lemma 14 $V^* \neq \emptyset$. ■

The above theorem establishes among others that, the stochastic game has a (possibly nonstationary) Markov Nash Equilibrium in monotone strategies on a minimal state space of current state variables. Observe that conditions to establish that fact are weaker than one imposed by Curtat (1996) or Amir (2002). Specifically we do not require smoothness of the primitives nor any diagonal dominance conditions that assure that the auxiliary game has a unique Nash equilibrium, that is moreover continuous with the continuation value. The assumption we impose here are almost the same as those from the work of Amir (2005).

Similarly as in case of bequest model we propose an approximation of $V^*(s)$ by corresponding piecewise constant correspondence $\hat{W}_t^j$. The difference is that V^* is a set of equilibria values, but the construction of approximations is similar.

Theorem 4 *Under assumption 3 for all $s \in S$:*

(i) $V^*(s) \subset ... \subset \hat{V}_{j+1}(s) \subset \hat{V}_j(s) \subset \ldots \subset \hat{V}_1(s)$,

(ii) $\lim_{j\to\infty} d_H(\hat{V}_j(s), V^*(s)) = 0$,

(iii) Let $\hat{W}_t^j(\cdot)$ be a $j-th$ piecewise approximation for $W_t(\cdot)$. Then

$$\lim_{t,j\to\infty} d_H(W_{t,j}(s), V^*(s)) = 0. \tag{9}$$

Proof of Theorem 4: Proof of (i) is obvious. Proof of (ii): we apply Theorem 1 in Beer (1980). $V^*(s)$ is a canonical projection of $V^*(s)$ in a Tikchonov topology. We need to show that $V^*(s)$ has a closed graph.

Let $s_n \to s$ $(n \to \infty)$, $x_n \in V^*(s_n)$ for all n and $x_n \to x$. By Theorem 3 and Lemma 5, $x_n \in W_t(s_n)$ for all $t \in \mathbb{N}$. Then there exists a sequence $(v_t^n)_{n\in\mathbb{N}}$, such that $x_n = v_t^n(s_n)$ with $v_t^n \in W_t$. Consequently, there exists $(v_\tau^n)_{\tau=1}^{t-1} \in \prod_{\tau=1}^{t-1} W_\tau$ such that $(\forall_{s\in S})$ $v_\tau^n(s) = \Pi(v_{\tau-1}^n, s, a_\tau^n)$ for some $a_\tau^n(s) \in NE(v_{\tau-1}^n, s)$. Without loss of generality (using Lemma 8) suppose, if we take a limit $n \to \infty$ then we obtain $v_\tau^n \to v_\tau$ pointwise in s for all $\tau \leq t$. Using Lemmas 8 and 12 we have sequential compactness of all W_τ and $v_\tau \in W_\tau$. By Lemmas 9 and 10 have $(\forall_{s\in S})$ $v_\tau(s) = \Pi(v_{\tau-1}, s, a_\tau(s))$ for some selection $a_\tau(s) \in NE(v_{\tau-1}, s)$. Applying again Lemma 10 we have then $x = \lim_n x_n = \Pi(v^{t-1}, s, a^t(s))$ for some selection $a^t(s) \in NE(v^{t-1}, s)$. Hence $x \in W_t(s)$. Since $t \in \mathbb{N}$ is arbitrary, hence and by Lemma 5 $x \in V^*(s)$.

Proof of (iii). Observe that by Theorem 3 all W_t are sequentially compact. Hence all $W_t(s)$ is compact set for all s. The remainder of this proof is similar as a part (iii) of Theorem 2. ■

5 Concluding remarks

In this paper, under mild conditions, we develop a strategic dynamic programming approach to a class of stochastic games, and provide numerical method for computing the equilibrium value set that is associated with MPNE in the game. Aside from allowing us to operate with uncountable number of states, we are also able to produce strategic dynamic programming methods that focus on *Markovian* equilibrium (as opposed to more general sequential equilibrium). The set of MPNE include both stationary and nonstationary Markovian equilibrium, but in all cases, MPNE exists on the *minimal* state spaces (i.e., compact interval of the real line).

The characterization of a sequential NE strategy obtained using standard MP/APS method in general is very weak (i.e., all one can conclude, is that there exists some measurable value function that is a selection from V^*). In our case, as we focus on the operator B, that maps in space of *functions,* we resolve this selection issue at the stage of defining the operator. Few additional comments are now in order.

1. Existence of the greatest fixed point of B follows from a standard application of Tarski fixed point theorem. However, to establish MPNE existence and convergence of iterations from $\mathcal{V}$ on B one needs to show that that $\cap_t B^t(\mathcal{V}) \neq \emptyset$. For this we show that B is an order-continuous operator, i.e. maps compact sets to compact sets and apply Tarski-Kantorovich theorem (e.g., Dugundji and Granas (1982), Theorem 4.2, p. 15). For this one needs a topology on elements of V. Related papers often use weak* topology (see Chakrabarti, 1999). As pointed by Nowak and Raghavan (1992) or Nowak (2007) the set of measurable selections from the Nash equilibrium value correspoindence is not weak* compact without imposing additional strong assumptions (see Mertens and Parthasarathy, 1987). For this reason, in this paper we use uniform and weak topologies instead. Moreover, in our application of the existence theorem, as order convergence and topological convergence coincide, the lower envelope of the subchains generated by $\{W_t\}$ under set inclusion is equal to the topological limit of the sequence, which greatly simplifies computations in principle.

2. For any strategic dynamic programming argument (e.g., traditional MP/APS in spaces of correspondences or our method in function spaces), for the method to make sense, it requires the auxiliary game has a (measurable) Nash equilibrium (or decision problem has a measurable solution). In our situation we assume continuity of the primitives or in the supermodular game, that auxiliary game is a game of strategic complementarity. Of course, alternative topological conditions could be imposed but measurability of the value function has to be carefully analyzed.

3. The assumption of the full (or invariant) support of Q is critical for this "recursive type" method to work. That is, by having a Markov selection of values from V^*, we can generate supporting Markov (time or state dependent) Nash equilibrium.

4. Observe that our procedure does not imply that the (Bellman type) equation $B(V^*) = V^*$ is satisfied for a *particular* value function v^*; rather, only by a *set* of value functions V^*. Hence, generally existence of a stationary Markov Nash equilibrium cannot be deduced using these arguments. Also, our method is in contrast to the original MP/APS method, where for any $w(s)$, one finds a continuation v; hence, the construction for that method becomes *pointwise*, and for any state $s \in S$, one can select a different continuation function v. This implies the equilibrium strategies that sustain any given point in the equilibrium value set are not only state dependent, but also continuation dependent. In our method this is not the case. This difference, in principle, could have significant implications for computation.

5. Our construction is related to the Cole and Kocherlakota (2001) study of Markov-private

information equilibria by the MP/APS type procedure in function spaces. As compared to their study our treats different class of games (with general payoff functions, public signals and uncountable number of states) though. Also, recently and independently of our results Doraszelski and Escobar (2012) established an MP/APS type procedure in function spaces for Markovian equilibria in repeated games with imperfect monitoring. Again their construction differs from ours as they require a finite number of actions, countable number of states and payoff irrelevant shocks. All of these are necessary in their approach to preserve the measurability of a value function, but not is our case as measurability requirement is easily satisfied for extremal equilibria/best responses.

Similarly, approximation results presented in this paper require few comments. Recall, our method approximates the set of equilibrium values on uncountable number of states. This is in contrast with all related papers that will be discussed in the moment.

6. In comparison to Judd, Yeltekin, and Conklin (2003) and Sleet and Yeltekin (2003) method of MP/APS set approximation, our does not rely on *convexification* of the set of continuation values. Such step usually involves introducing sunspots or correlations devices into the game. On the contrary our method does not need any of this convexification or correlation[13].

7. Our discretization method is directly linked to the theorem / proof of the equilibrium (value set) existence. This is critical as we want to work with uncountable number of states. This is in contrast to the FMPA approach which is useful for finite/countable number of states only.

8. The argument presented in this paper share some properties of Chang (1998) proposal of discretization technique for equilibrium value set approximation. This has been formalized by FMPA, who use Beer (1980) result on approximation of correspondences by a step correspondence. Our approach is different as we approximate the set of functions, not a correspondence. Of course any set of function on a common domain can be represented as a correspondence, but doing such a step we loose important characterization of selections from this correspondence, however. In our approach we select functions at the moment of defining an operator or every step of its approximation. Hence once the set of interest is computed, any selection we take is approximate Markovian equilibrium value, while in the FMPA approach one needs to select measurable equilibrium value of interest once at the end of the computation, when selection of Markovian equilibrium (on the minimal state space) can be problematic (see Datta, Mirman, and Reffett (2011) for a discussion). Finally, at each point of our iteration we select not only equilibrium value but also equilibrium strategy, hence for each value we can directly compute Nash equilibrium that supports the value of interest. Such advancement is possible as we define equilibria on the minimal state space that are time/state dependent only, and as a result we can operate directly in function spaces.

9. For a stochastic supermodular game, in accompanied paper Balbus, Reffett, and Woźny

[13]Also Cronshaw (1997) proposes a Newton method for equilibrium value set approximation but cannot prove that his procedure converge to the greatest fixed point of our interest.

(2014a), show additionally that under some mixing assumptions on stochastic transitions, we are able to exploit the complementarity structure of our class of games, and develop very sharp results for iterative procedures on *both* values and pure strategies, which provides very important improvements over MP/APS type methods that we develop in the current paper. This brings few interesting points. First under assumptions of both papers (and this can be done indeed), one is able to show how to compute both the greatest and least stationary MPNE values w^*, v^*, as well as the associated extremal pure strategy equilibrium. Of course, they are also Markov NE and $w^*, v^* \in V^* = B(V^*)$.

10. As for every iteration of our operator we can keep track of the greatest and the least NE value, clearly, our iterative methods from the other paper verify that our MP/APS value set $W_t \subseteq [v_t, w_t]$. So in such case, our iterations from the other paper provide interval bounds on the iterations on our strategic dynamic programming operator. Further, as the partial orders we use in all cases are chain-complete (i.e., both pointwise and set inclusion orders), we conclude that $V^* \subseteq [v^*, w^*]$. That is, the set of value functions that are associated with MPNE belongs to an ordered interval between least and greatest SMNE. So although we cannot show that W_t (for $t \geq 2$) or V^* are ordered intervals of functions, we use our iterative methods to calculate the two bounds using direct techniques of the accompanied paper.

11. This observation leads us to a final important point linking our direct methods with MP/APS approach. Namely, in Abreu, Pearce, and Stacchetti (1990), they show that under certain assumption any value from $V^*(s)$ can be obtained in a bang-bang equilibrium for a *repeated* game, i.e. one using extremal values from $V^*(s)$. Direct and constructive methods of Balbus, Reffett, and Woźny (2014a) can be hence used to compute two of such extremal values that support equilibrium punishment schemes that actually implement MPNE. This greatly sharpens the method by which we support all MPNE in our collection of dynamic games.

Similar comparisons can be obtained for a class of stationary MPN equilibria in a bequest economy (see Balbus, Reffett, and Woźny (2013a) for recent results obtained using direct methods).

6 Appendix: auxiliary results

Endow the real line $\mathbb{R}$ with a natural Euclidian topology and Lebesgue measure. Moreover, let $(X, \mathcal{T})$ where $X = \prod_{s\in S} \mathbb{R}$, and $\mathcal{T}$ is a product topology that is generated by cylinders, i.e. $C = \prod_{s\in S} C(s)$ where $\{s : C(s) \neq \mathbb{R}\}$ is finite. For a set $G \in X$ let $\pi_s(G)$ be its canonical projection on coordinate s.

Lemma 14 *Let $X = \prod_{s\in S} K$ where K is compact set in $\mathbb{R}^m$, and $(X, \mathcal{T})$ be a product topology. Let G_t be a sequence of sequentially compact subsets of X. Assume $G_t(s) \neq \emptyset$ for all $s \in S$ and G_t is descending in the set inclusion order. Then $G^\infty := \bigcap_{t=1}^{\infty} G_t \neq \emptyset$.*

Theorem 5 *Let $f_n : \mathbb{R} \to \mathbb{R}$ be a set of increasing functions and $\mu(\cdot)$ is a nonatomic measure. If $f_n \to f$ weakly and $\mu_n \to \mu$ weakly then $\lim_{n\to\infty} \int_S f_n d\mu_n \to \int_S f d\mu$.*

Proof of theorem 5: See Theorem 4.6 in Majumdar and Sundaram (1991) or Lemma 3.7 in Balbus, Jaśkiewicz, and Nowak (2014) for example. ■

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