Pareto optima and equilibria when preferences are incompletely known (G. Carlier and R. Dana, 2013)

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Big picture of article contents

- Exchange economy, finite number of Agents
- Incomplete preferences
- Goal: efficient and equilibrium allocations satisfying how it intuitively look like, ie
  1. Efficient allocations for the incomplete preferences coincide with the set of efficient allocations that result for some choice of utilities in the sets of each agent
  2. Equilibria for the incomplete preferences coincide with the set of equilibria that result for some choice of utilities in the sets of each agent
- Very general mathematical formulation

Incomplete preferences modeling

In short, incomplete preferences are modelled by representing them in terms of multiple utility functions, instead of a single one.

- For each consumer $i \in I$ his preferences are given by class of utility functions $\mathcal{U}_i$ consisting of multiple utility functions $u_i \in \mathcal{U}_i$
- Equivalently, instead of class of utility functions, we can think of utility correspondence (one mulit-valued function)
- Corresponding preference relation by the unanimity rule $X \succ_{\mathcal{U}_i} Y \iff \forall u_i \in \mathcal{U}_i : u_i(X) > u_i(Y)$
- Possible weaker criteria, omitted in presentation

Model setting

As a primitive, existence of two (possibly infinitly dimensional) real vector spaces E, F is assumed, along with mapping $(P, X) \in F \times E \to P \cdot X \in \mathbb{R}$.

- Mapping $\cdot(P, X)$ is separating duality mapping ie. it is bilinear and for $P \in F$ (resp. $X \in E$) if for all $X \in E$ we have $P \cdot X = 0$ then $P = 0$.
- Topology on F (resp. E): minimal locally convex and Hausdorff topology, such that for given $X \in E$ we have that $P \in F \to P \cdot X$ is continuous
- With such topology, F is a topological dual to E, ie. it can be thought of as a space of continuous linear functionals from E into $\mathbb{R}$
- We interpret E as a space of goods and F as a space of prices

Assumptions: 1

A1. For every $i \in I$ every $u_i \in \mathcal{U}_i$ is finite, concave, superdifferentiable and $\partial u_i(X)$ is compact in topology on F for every $X \in E$.

Note: For any concave function u defined on E its superdifferential at given $X \in E$ is

$$\partial u(X) = \{P \in F : \forall Y \in E : u(Y) - u(X) \leq P \cdot (Y - X)\}$$

and u is said to be superdifferentiable if for all $X \in E$ $\partial u(X) \neq \emptyset$.

Assumptions: 2

A2. For every $i \in I$ set $\mathcal{U}_i$ is convex and there is topology on $\mathcal{U}_i$ such that it is compact and mapping $u_i \in \mathcal{U}_i \to u_i(X)$ is for every X continuous in $\mathcal{U}_i$

Note: the second part of this axiom means, that $\mathcal{U}_i$ admits compact parametrization.

Assumptions: 3

A3. There exist $\Phi \in E$ such that for all $i \in I$ every $u_i \in \mathcal{U}_i$ every $P \in \partial u_i(X_i)$ one has $\Phi \cdot P > 0$ and the set

$$V_i(X_i) = \left\{ \frac{P}{\Phi P} : P \in \partial \mathcal{U}_i(X_i) \right\}$$

is compact in topology on F.

Note: this assumption might be thought of, as that there is direction in which all utility functions in all classes increase

Example: case of finite dimensional E, F

All those assumptions and topological assumptions might look scary so we show how those work in usual case of $E = F = \mathbb{R}^d$ with $\mathcal{U}_i$ being a convex set of concave functions.

- A1 is trivially satisfied,
- A2 is satisfied. First part trivially by definition of $\mathcal{U}_i$ and second part for topology of uniform convergence on compact subsets,
- A3 is satisfied if and only if there is a common vector $e \in \mathbb{R}^d$ such that all $u_i \in \mathcal{U}_i$ for all $i \in I$ are increasing in direction e.

No-trade principle and efficient allocations

Def Let X be aggregate endowment. Allocation $(X_i)_{i\in I}$ is efficient if there is no $(Y_i)_{i\in I}$ such that $Y_i \succ_{\mathcal{U}_i} X_i$.

Theorem 1. The following are equivalent

1. There exist no $(Y_i)_{i\in I}$ such that $\sum_i Y_i = 0$ and $X_i + Y_i \succ_{\mathcal{U}_i} X_i$ for all i,
2. $\bigcap_{i\in I} V_i(X_i) \neq \emptyset$,
3. there exist $P \in F$ such that for all $i \in I$ $X_i + t_i Y_i \succ_{\mathcal{U}_i} X_i$ for some $t_i > 0$ implies $P \cdot Y_i > 0$,
4. Allocation $(X_i)_{i\in I}$ is efficient,
5. There exist $(u_i)_{i\in I}$ with $u_i \in \mathcal{U}_i$ such that $(X_i)_{i\in I}$ is efficient for economy with complete preferences represented by $(u_i)_{i\in I}$.

Equilibria and welfare theorems

Def Let W_i be initial endowment of consumer i. Allocation $X^* = (X_i^*)_{i \in I}$ and prices $P^* \in F$ is an equilibrium with transfer payments if for every i, $X_i \succ_{\mathcal{U}} X_i^*$ implies $P^* X_i > P^* X_i^*$. An allocation $(X_i^*)_{i \in I}$ and prices $P^* \in F$ is an equilibrium if for every i, $P^* X_i^* = P^* W_i$ and $X_i \succ_{\mathcal{U}_i} X_i^*$ implies $P^* X_i > P^* W_i$

Equilibria and welfare theorems

Theorem 2. The following are equivalent

1. (X^*, P^*) is an equilibrium with transfers
2. $\lambda P^* \in \bigcap_{i \in I} V_i(X_i^*)$ for some $\lambda > 0$
3. There exist $(u_i)_{i \in I}$ with $u_i \in \mathcal{U}_i$ such that X^* is equilibrium with transfers for economy with complete preferences represented by $(u_i)_{i \in I}$.

Theorem 3. The following are equivalent

1. (X^*, P^*) is an equilibrium
2. $\lambda P^* \in \bigcap_{i \in I} V_i(X_i^*)$ for some $\lambda > 0$ and for every i, $P^* X_i^* = P^* W_i$
3. There exist $(u_i)_{i \in I}$ with $u_i \in \mathcal{U}_i$ such that X^* is equilibrium with transfers for economy with complete preferences represented by $(u_i)_{i \in I}$.

Equilibria and welfare theorems

Theorem 4. Following assertions hold

1. Any equilibrium allocation is efficient
2. Any efficient allocation is equilibrium allocation with transfers