Time Consistent Policies and Quasi-Hyperbolic Discounting

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Summary

Since the seminal paper of Strotz (1956), the study of dynamic choice for agents with dynamically inconsistent preferences has been the focal point of much work in behavioral economics. Experimental and empirical literatures both point to the importance of such models that allow for various forms of present-bias. This includes research in such diverse areas as consumption/savings models, dynastic choice with altruism, normative models of distributive justice with intergenerational conflict, social discounting and environmental cost-benefit models, dynamic collective household models, among others.

These models often lead to time-inconsistency of the optimal solution. That is: optimal plans, under such preferences over time, are time-inconsistent and decision maker has no incentive to follow the optimal plan in the future. Out of many interesting problems economists have studied, the question of design and computation of optimal among time consistent plans (so the one that will be actually followed) has received a great attention in economic literature. This also includes important from behavioral but also numerical perspective short memory decision rules, like Markov or semi-Markov ones.

In this chapter, we present results on the existence, uniqueness, and characterization of Time Consistent Policies (TCPs) as Station-

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ary Markov Perfect Equilibria (SMPE) in a class of consumption-savings problems with quasi-hyperbolic discounting, as well as some of their extensions. In doing so, we develop a powerful *generalized Bellman equation operator* approach that facilitates the construction of TCPs. We also give sufficient conditions for the validity of the first order approach to characterizing equilibrium policies via *generalized Euler equation*, discuss the existence of monotone comparative statics in interesting deep parameters of the decision environment. We then present generalizations of our results allowing to cover unbounded returns, general certainty equivalents and multi-dimensional states. We conclude by presenting a general self-generation method characterizing non-stationary Markov perfect equilibrium policies.
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1 Introduction

The relationship between discounting and dynamic choice has been a topic of great interest in economics since the early paper of Ramsey (1928) almost a century ago. In his work, Ramsey proposed to model intertemporal preferences of a decision-maker as the weighted sum of future utilities. Subsequently, Samuelson (1937) sharpened this model of intertemporal preferences by proposing a model with exponential discounting, and this model soon became a standard approach to specifying intertemporal preferences in dynamic economies. Then, with the subsequent work of Koopmans (1960) on the axiomatic approach to recursive utility (with exponential discounting as a special case), the foundations for dynamically consistent preferences was made foundational.[1]

But the seminal paper of Strotz (1956) challenged this orthodoxy. Strotz proposed a model of dynamic choice for decisionmakers with dynamically *inconsistent* preferences. Since his pioneering suggestion, there has been a

[1]See also Halevy (2015) for a nice recent discussion of these issues.

large and ever growing literature in dynamic behavioral economics which has sought to understand dynamic decisions with dynamically inconsistent preferences. This work has appeared in such diverse fields in economics as macroeconomics, financial economics, political economy, environmental economics, health economics, and public finance. The need for such studies has been motivated by a large and growing empirical and experimental literature that has shown how important preference reversals is in our understanding of dynamic choice of decision-makers that are forced to compare current vs. future utilities.[2]

There has been at least two strands of work on the theoretical side of this literature. One strand is found in a series of papers which axiomatize the structure of preferences that exhibit various forms of dynamic inconsistency related to discounting. This includes the work of Montiel Olea and Strzalecki (2014), Chambers and Echenique (2018), and Chakraborty (2021) on quasi-hyperbolic discounting and related forms of present bias; Galperti and Strulovici (2017) on dynastic preferences with altruism and their relationship with quasi-hyperbolic discounting; Jackson and Yariv (2015) and Lizzeri and Yariv (2017) concerning the study of dynamic collective choice and time inconsistency; temptation preferences axiomatized in Gul and Pesendorfer (2001, 2004, 2005); and other related work on self-control presented in Noor (2011) and Dekel and Lipman (2012), for example. This line of work has also looked at the axiomatization of "naive" vs. "sophisticated" decisions for time inconsistent consumers (e.g., the papers of Ahn et al. (2019), and Ahn et al. (2020)) in the context of quasi-hyperbolic preferences.

To complement this work on axiomatic foundations, there has also been a voluminous literature which seeks to characterize the structure of time consistent choice in models where agents have preferences that are dynamically

[2]For work discussing the empirical motivation for the importance of present-bias and dynamic inconsistency in choice, see Angeletos et al. (2001), Ameriks et al. (2007), Halevy (2015), and Cohen et al. (2020).

inconsistent. Starting with the early work of Phelps and Pollak (1968), Pollak (1968) and Peleg and Yaari (1973) where the so-called ”game theoretic” interpretation of time consistent choice was proposed, this work has sought to identify sufficient conditions for the existence of Stationary Markov equilibria in models of quasi-hyperbolic discounting (e.g., see the work of Laibson (1997), Harris and Laibson (2001), Krusell and Smith (2003), Krusell et al. (2010), Nowak (2010), Harris and Laibson (2013), Chatterjee and Eyigungor (2016), Balbus et al. (2015c, 2018), Cao and Werning (2018), Jaśkiewicz and Nowak (2021), and Bäuerle et al. (2021), among others), as well as work the studies the long-memory subgame perfect time consistent solutions as in the papers of Bernheim et al. (2015) or Balbus and Woźny (2016). More recently, these results have been extended to more general behavioral discounting cases as in the papers of Balbus et al. (2021b), Balbus et al. (2021a), Jensen (2021), and Richter (2020).

In this literature, the question of defining what a “time-consistent solution” actually is turns out to be a very important question. Since the work of Phelps and Pollak (1968) and Peleg and Yaari (1973), as well as much of the literature that has followed, researchers have viewed a “time consistent solution” for an individual consumer as a subgame perfect equilibrium in a dynamic intrapersonal game between a collection of that consumer's “selves”. That is, it envisions the consumer as playing a dynamic game between one's current self and each of her future “selves” or “generations”, with the appropriate equilibrium concept defining a time consistent solution as a subgame-perfect Nash equilibrium (SPNE, henceforth) in this dynamic intrapersonal game.

It is important to note that this definition differs from the solution concept proposed by Strotz (1956), and further studied by Pollak (1968) and Kydland and Prescott (1977), where a decision theoretic viewpoint was taken, and the idea of *optimal* time-consistent policies was proposed where one views the time consistent solution as being "optimal" relative to the set of all possible time consistent solutions.[3] In some cases, the two approaches can be linked, but the *equivalence* of the two notions of time consistent choice is not true. As Caplin and Leahy (2006), Balbus et al. (2015c), and Balbus et al. (2021b) discuss in some detail, although the optimal time-consistent plan is a SPNE of the some game, the converse is false as from the vantage point a decision maker relative to how future ties are being broken in favour of a current self (vs. future selves). It also bears mentioning, the set of SPNE may also be very large, and most importantly, not necessarily possess the element with the *greatest value*. Hence, an *optimal* SPNE (i.e., a SPNE that corresponds to some optimal time-consistent policy) may simply not exist.

This chapter studies the question of time consistent solutions from the game theoretic approach in dynamic models where agents have present bias. The remainder of the chapter is layout as follows. In the next section, we describe a prototype of the quasi-hyperbolic discounting model, and provide a detailed discussion of its structure starting with the finite horizon case. Here, our examples compare naive vs. sophisticated solution, as well as describe what is commonly referred to as a "generalized Euler equation" in the existing literature (e.g., see Harris and Laibson (2001) and Krusell and Smith (2003)). In section 3, we then consider the infinite horizon case for a version of the quasi-hyperbolic model following a (stochastic) game-theoretic interpretation of time consistent plans (TCPs), and define a time consistent solution to be stationary Markov perfect equilibrium (SMPE) in the resulting dynamic intrapersonal game. Very importantly, in this section we introduce the new idea of a "Generalized Bellman equation" for the quasi-hyperbolic consumer, and this functional equation plays a critical role in our construction of the SMPE. We also give sufficient conditions for the existence, characterization, and computation of time consistent equilibrium

[3] See Caplin and Leahy (2006) for a nice discussion of the idea of optimal time consistent policies.

as SMPE, and as well as discuss when it is unique. We provide some examples of applications of the results, and also provide a set of monotone comparative statics results relative to the set of time consistent plans, as well as give sufficient conditions for the existence of a generalized Euler equation governing the model's SMPE for the infinite horizon case. In the next section of the paper, we consider some extensions of the model including unbounded state spaces and time inconsistent recursive utility, as well as propose a class of generalized quasi-hyperbolic discounting models. In this section, we also discuss the idea of generalized certainty equivalents, and discuss how to extended the results to models with multidimensional states. Finally, in last section of the chapter, we discuss how self-generation approaches can be used to construct more general notions of time consistent plans as subgame perfect equilibria.

2 The quasi-hyperbolic discounting model

2.1 A prototype model

We begin by describing a canonical version of the model of consumption and savings problem for a consumer with quasi-hyperbolic preferences. This model will motivate much of our discussion in this paper. Consider a discrete time, T-period, consumption-saving model, where the sequence of lifetime preferences over sequences of consumption $(c_{t+\tau})_{\tau=0}^{T}$ is given, at any date t, by:

$$u(c_t) + \mathbb{E}_t \sum_{\tau=1}^{T} \beta \delta^{\tau} u(c_{t+\tau}). \tag{1}$$

Here $\delta \in (0, 1)$ is called a long term discount factor while $\beta \in (0, 1]$ is an additional short-term impatience parameter, $\mathbb{E}_t$ denotes the mathematical expectation taken relative to date t information, and T for the moment is finite. Preferences given by (1) are the same for any t and, hence, are time invariant. For given t these preferences are non-stationary whenever $\beta < 1$.

Indeed, the ratio of discounted utilities in any two time periods $t+\tau+1$ and $t+\tau$ (for $\tau > 0$) is given by:

$$\frac{\beta\delta^{\tau+1}u(c_{t+\tau+1})}{\beta\delta^{\tau}u(c_{t+\tau})},$$

and hence is equal to $\frac{\delta u(c_{t+\tau+1})}{u(c_{t+\tau})}$ for $\tau > 0$ as compared to $\frac{\beta\delta u(c_{t+1})}{u(c_t)}$ for $\tau = 0$. These two properties, i.e. time-invariance and non-stationarity, imply time-inconsistency of sequences of preferences given by (1). Indeed, a decision maker planning choices over consumptions $\tau > 0$ period ahead would have an incentive to change its choices this period actually occurs. See Halevy (2015) proposition 4 for a formal argument. For $\beta < 1$, preferences exhibit present bias: i.e., a decision maker who planned a choice of $(c_{t+\tau})_{\tau=0}^{T}$ at time t will he an incentive to increase his consumption at the cost of reducing investment when period $t' = t+\tau$ actually occurs (as $\beta\delta < \delta$). In what follows, we say the consumer thinks of herself as a sequence of successive "selves", each self associated with particular time period t.

In any date t, self t enters the period with a stock $s_t \in S$ (where $S = [0, \bar{s}]$) that is divided between the current consumption $c_t \in [0, s_t])$ and current savings/investment denoted by $i_t := s_t - c_t$, where savings/investment is placed into a (possibly) stochastic return technology. The "stock" s_t is interpreted as an productive asset or a capital. The current investment parameterizes transition probability over tomorrow's stock and is given by the first order Markov process $Q(\cdot|i_t)$.

To illustrate the model's structure, as well as the complications dynamic inconsistency creates in modeling dynamic choice, for the remainder of this section, we discuss some simple special cases of the model. Here, we shall assume the capital transition structure is deterministic and given by fixed mapping $k_{t+1} = F(k_t) - c_t$ for some increasing and continuous production function $F : \mathbb{R}_+ \to \mathbb{R}_+$. Introducing $s_t := F(k_t)$ as a state variable, we have the transition on this state be given by $Q(A|i) = \mathbf{1}_A(F(i))$, where $\mathbf{1}_A()$ is a indicator function of a Borel set $A \subseteq S$. Also, to keep things illustrative for the moment, we consider the special case of deterministic linear technology $F(k)$, and associate state s_t with returned capital $(1+r)k_t$. Later in the chapter, we shall refer to as this model as the model with "deterministic state transitions".[4]

To begin our discussion, consider a simple example of the model with $T = 3$. The preferences of the date 1 self 1 are defined then over (c_1, c_2, c_3), and given by

$$u(c_1) + \beta\delta(u(c_2) + \delta u(c_3)),$$

while the preferences of date 2 self are defined over (c_2, c_3) and given by:

$$u(c_2) + \beta\delta u(c_3)$$

with $u(c_3)$ being the preferences for the date 3 self. In what follows, we assume the utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave and twice continuously differentiable. We also impose the Inada condition on u' so that consumption choices are interior.

Since the model encompasses multiple selves, its solution depends on the beliefs that the current self formulates about all of his future selves in any given period. The literature consider two standard cases of such a belief structure. The first is the "naive" solution, in which the current self (mistakenly) believes that her future selves will continue the proposed plan. The second belief structure is the "sophisticated" solution, in which each current self correctly forecasts changes to the proposed plan by the future selves under complete markets. For $T = 3$, we first characterize the naive solution, and then the sophisticated solution, and discuss how they can differ. We take a simple model in which self 1 is endowed with a state (income or

[4] In further sections, we consider more general models with stochastic production process with $s_{t+1} = F(z_t, i_t)$ where z_t is a random shock with some distribution π_t. For a Borel set A, we can rewrite this transition process as: $Q(A|i_t) = \int_S \mathbf{1}_A(F(z, i_t))\pi_t(dz)$. Here F can be multiplicative with $F(z_t, i_t) = z_t g(i_t)$ or additive $F(z_t, i_t) = g(i_t) + z_t$, for some continuous and increasing g.

production) s_1 and divides in between c_1 and investment i_1. Such investment increased by the interest rate is returned to self 2. Consumption is nonnegative but otherwise assets can be freely moved between the periods.[5]

The naive solution to quasi-hyperbolic model To solve for the naive solution, we start with date $t = 1$ self. He solves for (c_1, c_2, c_3):

$$\max_{(c_i \geq 0)_{i=1}^3} u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3)$$
$$\text{s.t.} c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} \leq s_1,$$

Observe that only c_1 is actually realized from the optimal plan (c_1, c_2, c_3). So, the continuation plan (c_2, c_3) is then updated by date $t = 2$ self by solving

$$\max_{(c_i \geq 0)_{i=2}^3} u(c_2) + \beta\delta u(c_3)$$
$$\text{s.t.} c_2 + \frac{c_3}{(1+r)} \leq (1+r)(s_1 - c_1) := s_2.$$

Finally, since period $t = 3$ self makes no choice (i.e. $c_3 = s_3$) the actual path is then c_1 chosen by date 1 self and (c_2, c_3) chosen by self 2.

The sophisticated solution to quasi-hyperbolic model Strotz (1956) proposed a sophisticated solution to the model with changing preferences, i.e. a solution that will "actually be followed". To find it, we solve the problem by backward induction starting from date $t = 2$:

$$\max_{c_2 \geq 0, c_3 \geq 0} u(c_2) + \beta\delta u(c_3)$$
$$\text{s.t.} c_2 + \frac{c_3}{(1+r)} \leq (1+r)k_2 := (1+r)(s_1 - c_1).$$

Here, we assume self 2 is endowed with $(1 + r)k_2 = s_2$. The problem is strictly concave and hence the solution unique. We obtain the first order

[5] Although not covered in this survey, credit/borrowing constraint are important to understand dynamics of solutions to models without commitment. This is especially important for models with commitment assets. See e.g. Laibson (1997) and Woźny (2016).

condition of the optimal interior allocation:

$$\frac{u'(c_2)}{\beta\delta u'(c_3)} = 1 + r.$$

or

$$\frac{u'(c_2)}{\beta\delta u'((1+r)(k_2(1+r) - c_2)} = 1 + r.$$

Denote the optimal choice $c_2^*(k_2)$ and $c_3^*(k_2)$. We call them reaction curves of the second period self as they depend on the assets / capital obtained from the first period self. To simplify, denote by $k_3^* := k_2(1+r) - c_2^*(k_2)$, Clearly: $c_2^*(k_2) = k_2(1+r) - k_3^*(k_2)$ and $c_3^*(k_2) = (1+r)k_3^*(k_2)$. Using the implicit function theorem, under our conditions we can conclude the interior optimal reactions c_2^*, c_3^* and k_3^* are continuously differentiable functions in the asset/capital.

We now move to consider period 1 self problem:

$$\max_{c_1 \geq 0, k_2 \geq 0} u(c_1) + \beta\delta u(c_2^*(k_2)) + \beta\delta^2 u(c_3^*(k_2))$$
$$\text{s.t.} c_1 + k_2 \leq s_1.$$

That it, the first period self correctly forecasts self 2 (and 3) reaction to the left asset/capital k_2. The problem is continuous hence the solution exists. Substituting, we obtain:

$$\max_{k_2 \leq s_1} u(s_1 - k_2) + \beta\delta u(k_2(1+r) - k_3^*(k_2)) + \beta\delta^2 u((1+r)k_3^*(k_2)).$$

As the reaction function k_3^* is differentiable, we obtain the first order condition for optimal interior k_2:

$$-u'(c_1^*) + \beta\delta u'(c_2^*(k_2))[1 + r - \frac{\partial k_3^*(k_2)}{\partial k_2}] + \beta\delta^2(1+r)u'(c_3^*(k_2))\frac{\partial k_3^*(k_2)}{\partial k_2} = 0. \quad (2)$$

Recall, for self 2, we had $u'(c_2^*(k_2)) = \beta\delta(1+r)u'(c_3^*(k_2^*))$ for any k_2. Using this:

$$-u'(c_1^*) + \beta\delta u'(c_2^*(k_2))[1 + r - \frac{\partial k_3^*(k_2)}{\partial k_2}] + \delta u'(c_2^*(k_2))\frac{\partial k_3^*(k_2)}{\partial k_2} = 0,$$

and hence we have:

$$\frac{u'(c_1^*)}{\beta\delta u'(c_2^*(k_2))} = [1 + r - \frac{\partial k_3^*(k_2)}{\partial k_2}] + \frac{1}{\beta}\frac{\partial k_3^*(k_2)}{\partial k_2}$$

or:

$$\frac{u'(c_1^*)}{\beta\delta u'(c_2^*(k_2))} = 1 + r + [\frac{1}{\beta} - 1]\frac{\partial k_3^*(k_2)}{\partial k_2}.$$

Krusell et al. (2002) called this equation a "generalized Euler equation", and for our $T = 3$ model, its existence (basically by assumption). Indeed, it has the standard interpretation of the marginal rate of substitution being equal to the ratio of prices, but with the critical addition of the *corrective* factor $[\frac{1}{\beta} - 1]\frac{\partial k_3^*(k_2)}{\partial k_2}$. Notice, this term disappears when $\beta = 1$.

If $\beta \neq 1$, then this equation implies, among other things, that in order to characterize the optimal investment k_2, the first period self must *use* the (investment) reaction curve of the second period self in defining her optimal solution. This is a critical feature of this model, and is the case as from the perspective of the first period self some part of its investment will be misused for the excessive consumption in period 2. The fact that this *corrective* factor disappears for $\beta = 1$ results exactly from the principle of optimality. Indeed, whenever the preferences are *consistent*, by the principle of optimality (and envelope theorem in our example), the reaction curve of the future self is still optimal from the perspective of the current self. The term $\frac{\partial k_3^*(k_2)}{\partial k_2}$ can be computed using the implicit function theorem:

$$\frac{\partial k_3^*}{\partial k_2}(k_2) = \frac{(1 + r)\beta\delta u''(c_2^*(k_2))}{u''(c_2^*(k_2)) + (1 + r)\beta\delta u''(c_3^*(k_2))}.$$

For future reference per the next section of the paper, we also mention additionally that the problem of the first period self can be simplified by the introduction of the continuation value. This is the value from using second period self choices c_2^* and c_3^* but evaluated from the perspective of the first period self:

$$V_2(k_2) := u(c_2^*(k_2)) + \delta u(c_2^*(k_2)) = u((1 + r)k_2 - k_3^*(k_2)) + \delta u((1 + r)k_3^*(k_2))$$

Then, the problem of the first period self can be simplified to:

$$W(k_1) := \max_{k_2 \leq (1+r)k_1} u((1+r)k_1 - k_2) + \beta\delta V_2(k_2),$$

where we have also denoted the value to the fist period self problem by $W(k_1)$. V_2 is differentiable as c_2^* and c_3^* are and hence the first order condition for the optimal choice of k_2^* is simply given by:

$$u'((1+r)k_1 - k_2^*) = \beta\delta V_2'(k_2^*).$$

As we have under our conditions the "envelope":

$$V_2'(k_2) = (1+r)u'(c_2^*(k_2)) - \frac{\partial k_3^*}{\partial k_2}(k_2)[u'(c_2^*(k_2)) - (1+r)\delta u'(c_3^*(k_2))],$$

we obtain the same first order condition as in (2).

It turns out, unfortunately, that this solution approach suggested above of composing the reaction curves and constructing a first order characterization of consistent plans for the sophisticated consumer who lives for $T = 3$ periods cannot be easily extended to more periods. To see this is the case, consider her problem for the $T = 4$ model. Repeating the above reasoning, can we show c_3^*, c_4^* and k_4^* are unique and continuously differentiable as before. The objective of the second period self is continuous and differentiable. It may happens, however, that now the argmax for the second period self k_2^* is *not* unique (e.g., strict concavity in the now can fail). Moreover, any selection from the argmax certainly need not be continuously differentiable, and might actually be discontinuous. Hence, the problem of the period 1 self may not be well-defined (i.e., might not possess an optimal solution for some initial capital levels). Notice also, the generalized Euler equation representation of sequential solutions no longer holds.

Finally, let us also mention a special case for which the above mentioned problems are not relevant and the generalized Euler equation approach is useful, and naive and sophisticated solutions coincide. This example is unfortunately a "knife-edge" but does admit a closed-form solutions for time consistent plans.

Example 2.1 (The naive and sophisticated solutions coincide for the log utility) *Take* $u(c) = \ln(c)$. *The naive solution to the problem:*

$$\max_{\{c_i \geq 0\}_{i=1}^3} \ln(c_1) + \beta\delta \ln(c_2) + \beta\delta^2 \ln(c_3)$$
$$s.t. c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} \leq s_1,$$

gives:

$$c_1 = \frac{s_1}{1+\beta\delta+\beta\delta^2} \quad k_2 = \frac{(\beta\delta+\beta\delta^2)s_1}{1+\beta\delta+\beta\delta^2}.$$

Sophisticated solutions produces reactions:

$$c_2^* = \frac{(1+r)k_2}{1+\beta\delta} \quad c_3^* = \frac{\beta\delta(1+r)^2 k_2}{1+\beta\delta}$$

and the 1st period self preferences:

$$\ln(c_1) + \beta\delta \ln(c_2^*(k_2)) + \beta\delta^2 \ln(c_3^*(k_2)) = \ln(c_1) + \beta\delta \ln(k_2) + \beta\delta^2 \ln(k_2) + constant.$$

This constant term does not depend on k_2. *This gives the same choice:*

$$c_1^* = \frac{s_1}{1+\beta\delta+\beta\delta^2} \quad k_2^* = \frac{(\beta\delta+\beta\delta^2)s_1}{1+\beta\delta+\beta\delta^2}$$

As the first period choice of capital k_2^* *is the same, hence the choice of* c_2 *and* c_3 *also coincide for the naive and sophisticated case.*

The log utility case is a special one as the reactions are linear. The properties of the log utility case allow one to extend the generalized Euler equation approach to more periods, actually including $T = \infty$.

We show how this general class of problems can be addressed for the stochastic state transition and so called generalized Bellman equation approach.

2.2 Infinite horizon and Stationary Markov Perfect Equilibria

We now focus attention on the infinite time horizon ($T = \infty$) and investment (or consumption) policies that are Markovian and stationary. For this section, we assume the state space S is a bounded interval with $S = [0, \bar{s}]$.

Definition 2.1 (Stationary Markov policy) *The stationary Markov policy (SMP) is a Borel measurable function $h : S \mapsto S$ such that $h(s) \in [0, s]$ for any $s \in S$.*

The set of all SMP is defined as follows:

$$\mathcal{H} := \{h : S \mapsto S : h \text{ is Borel measurable, and } h(s) \in [0, s] \text{ for all } s \in S\}. \tag{3}$$

Suppose that the current period t is fixed and the consumer owns $s \in S$ amount of capital. The consumer selects today $c \in [0, s]$ for the own consumption and channels $i = s - c$ for the next period stock. This consumer predicts the future consumption is consistent with a SMP $h \in \mathcal{H}$. That is the agent consumes $h(s_\tau)$ in the period $t + \tau$ whenever the capital in this period is s_τ. The evolution of the capital stocks from $t + 1$ onward is a Markov chain $(s_\tau)_{\tau=1}^{\infty}$ whose transition probability from s_τ to $s_{\tau+1}$ takes a general form $Q(\cdot|s_\tau - h(s_\tau))$.

The distribution of s_1 (the initial distribution Markov chain) is controlled by the capital stock $s \in S$ and the current investment and is $Q(\cdot|i, s)$. To evaluate the utility from an aggregated consumption of future selves we use the continuation value function.

Definition 2.2 (Continuation Value) *The continuation value function (CV) of future selves following the stationary policy $h \in \mathcal{H}$ is defined as follows*

$$V_h(s') := \mathbb{E}\left(\sum_{\tau=1}^{\infty} u(h(s_\tau))\delta^{\tau-1}|s_1 = s'\right).$$

The expected utility of the consumer today is

$$P(c, s; h) = \mathbb{E}\left(u(c) + \beta\delta\left(\sum_{\tau=1}^{\infty} u(h(s_\tau))\delta^{\tau-1}\right)\right).$$

By definition of V_h and the property of the conditional expectation operator, noting that the distribution of s_1 is $Q(\cdot|s-c,s)$, we obtain

$$\begin{aligned} P(c,s;h) &= u(c) + \beta\delta\mathbb{E}\left(\mathbb{E}\left(\sum_{\tau=1}^{\infty} u(h(s_\tau))\delta^{\tau-1}\,|s_1\right)\right) \\ &= u(c) + \beta\delta\int_S V_h(s')Q(ds'|s-c,s). \end{aligned}$$

Next, we introduce the correspondence returning the best policies for the current self against the stationary policy of future selves.

Definition 2.3 (Best Response Correspondence) *For any $h \in \mathcal{H}$, define the Best Response correspondence (BR) as follows*

$$BR(h) := \{h \in \mathcal{H} : h(s) \in \text{arg}\max_{i\in[0,s]} P(c,s;h) \text{ for any } s \in S\}.$$

We can then define a Stationary Markov Perfect Equilibrium as follows.

Definition 2.4 (Stationary Markov Perfect Equilibrium) *A policy $h^* \in \mathcal{H}$ is a Stationary Markov Perfect Equilibrium (SMPE) if for any $s \in S$,*

$$P(h^*(s), s; h^*) \geq P(c,s;h^*).$$

In other words, SMPE is the consumption plan for every self that is optimal provided all the future selves stick to the same plan. Mathematically h^* is SMPE if and only if it is a fixed point of the best response correspondence BR. In this paper, we use terms Time Consistent Policy (TCP) and SMPE interchangeably.

2.3 Generalized Bellman equation

In the case of $\beta = 1$, we have a standard exponential discounting, and it is well known that dynamic programming techniques can then be applied to the consumers dynamic choice problem as the principle of optimality holds. Further, under standard conditions on preferences, sharper characterizations of optimal policies can also be obtained. When $\beta \in (0,1)$, the consumer preferences exhibit present-bias, and her preferences are dynamically inconsistent as in the motivating example, and a new approach to computing TCP solutions has to be proposed. In this section, we show how one can obtain a "generalized" Bellman equation that provides a recursive formulation of our non-recursive dynamic optimization problem. In doing so, we can provide a sharp characterization of how the quasi-hyperbolic discounted differs from a standard exponential discounter.

To develop a generalized Bellman equation for this problem, notice first the continuation value function in any SMP $h \in H$ must solve the following functional equation:

$$V_h(s) = u(h(s)) + \delta \int_S V_h(s')Q(ds'|s - h(s), s). \tag{4}$$

Additionally, for such h to be a SMPE we must also have:

$$W_h(s) := \max_{c\in[0,s]} u(c) + \beta\delta \int_S V_h(s')Q(ds'|s - c, s). \tag{5}$$

Combining (4) and (5) we then obtain the fundamental relation:

$$V_h(s) = \frac{1}{\beta}W_h(s) - \frac{1-\beta}{\beta}u(h(s)). \tag{6}$$

We refer to Equation (6) as our *generalized Bellman equation.*

Whenever h is a SMPE and V_h a corresponding continuation value, (h, V_h) solves the generalized Bellman equation. Similarly, whenever a pair (h, V_h) solves the generalized Bellman equation, with h being some measurable argmax selection from the maximization problem[6] then h is SMPE and V_h its corresponding continuation value.

Equation (6) is also a generalized Bellman equation as the element $\frac{1-\beta}{\beta}u(h(s))$ is the adjustment factor that must be made to the standard Bellman operator to account for changing preferences. Clearly, for $\beta = 1$, equation (6) reduces to the standard Bellman equation.

Based on equation (6), we can define an operator whose fixed points, say V^*, correspond to values for some pure strategy Markovian equilibrium

[6]That is: $W_h(s) = u(h(s)) + \beta\delta \int_S V_h(s')Q(ds'|s - h(s), s)$ for any s. .

policy. From there, we can recover the set of (pure strategy) Markovian equilibrium policy functions. Indeed, by $\mathcal{V}$ denote the set of feasible continuation value functions. Then for any $v \in \mathcal{V}$ we define

$$A(v)(s) = \max_{c\in[0,s]} \left\{ u(c) + \beta\delta \int_S v(s')Q(ds'|s-c,s) \right\},$$

and

$$B(v)(s) = \arg\max_{c\in[0,s]} \left\{ u(c) + \beta\delta \int_S v(s')Q(ds'|s-c,s) \right\}.$$

If $B(v)$ is single valued, then implementing (6) we can construct the generalized Bellman operator as follows:

$$T(v)(s) = \frac{1}{\beta}A(v)(s) - \frac{1-\beta}{\beta}u(B(v)(s)).$$

Observe that if $v^* \in \mathcal{V}$ is a fixed point of T, then it is continuation value function supported by a SMPE; that is $v^* = V_{h^*}$ where h^* is a SMPE. Moreover, having v^* we can compute $h^*(s) = B(v^*)(s)$. If $B(v)$ is not single valued the generalized Bellman equation can still be used to construct v^* by taking some selection from the argmax correspondence.

3 Existence of Stationary Markov Perfect Equilibrium

The problem of existence of Markov Perfect Equilibrium is generally an open question. Even if we assert its existence, not much is known about the characteristics or structure of the equilibrium set. An issue of especial importance is also uniqueness of Markov Perfect equilibrium. For that reason, we cannot construct an effective algorithm of construction of equilibria. For example, the successive iterations of the best response map need not be convergent, since the best response map is mostly neither monotone nor is a contraction mapping.

In this section, we provide assumptions which guarantees the existence of equilibria; then, under more restrictive assumptions, consider the question of uniqueness of equilibria (as well as convergence of successive iterations). We can search the equilibria in two alternative ways:

- **searching through policies**: find $h^* \in \mathcal{H}$, that is the best against itself. In other words, find h^* that is a fixed point of the best response map, i.e. $h^* \in BR(h^*)$.
- **searching through continuation value functions**: find $v^* \in \mathcal{V}$ that is a fixed point of generalized Bellman operator and find the policy h^* whose continuation value function is v^*. In other words, $v^* = T(v^*)$, and we reconstruct SMPE by the formula $h^* = B(v^*)$.

3.1 Existence of SMPE. Searching through policies

In this subsection, we focus attention on the problem of existence of SMPE which is a fixed point of the correspondence BR. We make the following assumptions.

Assumption 3.1 *Assume that*

- *The utility function u is strictly concave, increasing, and $u(0) = 0$;*
- *The transition function obeys the following properties:*
  - *For $i \in S$, the transition $Q(S|i, s)$ does not depend on particular $s \in S$, call it $Q(\cdot|i)$;*
  - *For $i \in S \setminus \{0\}$, the transition $Q(\cdot|i)$ is a nonatomic measure, and $Q(\cdot|\{0\})$ is either nonatomic or $Q(\{0\}|0) = 1$;*
  - *$Q(\cdot|i)$ is stochastically increasing, that is for any increasing, Borel and bounded function $f : S \mapsto \mathbb{R}$ the function*

$$i \in S \mapsto \int_S f(s')Q(ds'|i).$$

is increasing.

Two remarks are in order.

Remark 3.1 *A typical example of u on S is the power function $u(c) = c^\alpha$, for $\alpha \in (0,1)$. Other functions satisfying the assumptions are e.g. $u(c) = \ln(1+c)$, $u(c) = 1 - e^{-c}$. For particular choice of S, for example $S = [0,1]$, we may take $u(c) = 2c - c^2$.*

Remark 3.2 *A typical example of the transition probability in economic growth theory is $s_{t+1} = G(i_t, \omega_t)$, where ω_t is a independent identically distributed shock, $G(i,\omega)$ is a continuous function in i and a Borel function in ω.*

If Q has atoms and u is unbounded below, generally the existence problem is open. We now provide an example of such a model, however, in which we can construct an analytical solution anyway.

Example 3.1 *Let $S = (0,\infty)$, $u(c) = \ln(c)$ and suppose*

$$Q(S_0|i) = \pi(\{\omega \in \Omega : \omega i^\alpha \in S_0\}),$$

$c, i \in [0,s]$ and $i + c = s$, $s > 0$. Here $\alpha \in (0,1)$ is a fixed value, and π is a log-normal distribution $\mathcal{LN}(0,\sigma^2)$. We find SMPE in the linear form: $h_k(s) = ks$ with $k > 0$. Then the capital stock $(s_t)_{t=1}^\infty$ obeys the formula:

$$s_{t+1} = \omega_t (1-k)^\alpha s_t^\alpha \quad \textit{with} \quad s_1 = s,$$

or after substitution $y_t := \ln(s_t)$, $\epsilon_t := \ln(\omega_t)$

$$y_{t+1} = \alpha \ln(1-k) + \alpha y_t + \epsilon_t. \quad \textit{with} \quad y_1 = \ln(s). \tag{7}$$

Clearly ϵ_t is an i.i.d. process with $\mathcal{N}(0,\sigma^2)$ distribution. The continuation value function is

$$v^k(s) = \mathbb{E}\left(\sum_{t=1}^\infty \ln(s_t)\delta^{t-1}\right) = \mathbb{E}\left(\sum_{t=1}^\infty y_t \delta^{t-1}\right).$$

We can compute v^k in the following way

$$v^k(s) = A_k \ln(s) + B_k \tag{8}$$

for some $A_k, B_k > 0$. We find these constants as follows:

- *find v^k as a solution of Bellman equation:*

$$v^k(s) = \ln(ks) + \delta \int_S v^k(s')Q(ds'|(1-k)s); \tag{9}$$

- *verify whether v^k obeys the* ***transversality condition****:*

$$\limsup_{t\to\infty} v^k(s_t)\delta^{t-1} \le 0$$

for almost all realizations of s_t starting from s, and if

$$\limsup_{t\to\infty} v^k(s_t)\delta^{t-1} < 0$$

then $v_t(s) = -\infty$ (see Wiszniewska-Matyszkiel (2011); Wiszniewska-Matyszkiel and Singh (2020) or Kamihigashi (2008) for details).

Substituting v^k from (8) into (9) we have

$$A_k \ln(s) + B_k = (1 + \alpha\delta A_k)\ln(s) + \ln(k) + \delta\alpha A_k \ln(1-k) + \delta B_k.$$

Hence

$$A_k = \tfrac{1}{1-\alpha\delta}, \quad \textit{and} \quad B_k = \tfrac{ln(k) + \frac{\delta\alpha \ln(1-k)}{1-\alpha\delta}}{1-\delta}.$$

As a result,

$$v^k(s) = \frac{1}{1-\alpha\delta}\ln(s) + \frac{ln(k) + \frac{\delta\alpha\ln(1-k)}{1-\alpha\delta}}{1-\delta}.$$

Now we have to verify the transversality conditions. For all t we have

$$v^k(s_t) = \frac{1}{1-\alpha\delta}\ln(s_t) + \frac{ln(k) + \frac{\delta\alpha\ln(1-k)}{1-\alpha\delta}}{1-\delta} = \frac{1}{1-\alpha\delta}y_t + B_k, \tag{10}$$

hence

$$\delta^{t-1}v^k(s_t) = \frac{\delta^{t-1}y_t}{1-\alpha\delta} + B_k\delta^{t-1}.$$

We show the expression above tends to 0 *by proving that the series $\sum_{t=1}^{\infty} y_t\delta^{t-1}$ are convergent. Let $z_t := y_t\delta^{t-1}$. By (7) we have*

$$z_{t+1} = \alpha\ln(1-k)\delta^t + \alpha\delta z_t + \delta^t\epsilon_t.$$

Hence the expectation $\mathbb{E}(z_t)$ *satisfies the following difference equations*

$$\mathbb{E}(z_{t+1}) = \alpha \ln(1-k)\delta^t + \alpha\delta\mathbb{E}(z_t).$$

Hence $\mathbb{E}(z_t)$ *satisfies*

$$\mathbb{E}(z_t) = \frac{\alpha \ln(1-k)}{\delta(1-\alpha)}\delta^t + \alpha^t\delta^t \ln(s),$$

Moreover, $Var(z_t)$ *satisfies*

$$Var(z_{t+1}) = \alpha^2\delta^2 Var(z_t) + \delta^{2t}\sigma^2$$

hence

$$Var(z_t) = \sigma^2(\alpha\delta)^{2t} + \frac{\sigma^2}{\delta^2(1-\alpha^2)}.$$

As a result, both series $\sum_{t=1}^{\infty} \mathbb{E}(z_t)$ *and* $\sum_{t=1}^{\infty} Var(z_t)$ *are convergent. By the Kolmogorov two-sequence Theorem we conclude the series* $\sum_{t=1}^{\infty} z_t$ *converges almost surely, consequently* z_t *tends to* 0 *almost surely. Hence, by definition of* z_t*, we conclude that the transversality condition holds, so* v^k *is (10) defines the continuation value function. Then, current self maximizes*

$$u(c) + \beta\delta \int_S v^k(s')Q(ds'|s-c) = \ln(s) + \frac{\beta\delta}{1-\alpha\delta}\ln(s-c) + \delta B_k$$

such that $c \in [0, s]$*. The solution is SMPE and it is*

$$h^*(s) = \frac{1}{1 + \frac{\beta\delta}{1-\alpha\delta}}s.$$

Let

$$\mathcal{I} := \{g \in \mathcal{H} : g \text{ is increasing and lower semicontinuous}\}. \tag{11}$$

We are looking for the TCP in the following class

$$\mathcal{G} := \{h \in \mathcal{H} : h(s) = s - g(s) \text{ for all } s \in S, \text{ and some } g \in \mathcal{I}\}.$$

Theorem 3.1 (Existence of SMPE) *Under Assumption 3.1 there exist a SMPE in* $\mathcal{G}$*.*

Proof: We provide a sketch of the proof. Endow $\mathcal{G}$ with the *weak topology* with the convergence $\to^*$. We only mention that this topology restricted to $\mathcal{G}$ is metrizable and the convergence $\to^*$ is equivalent to the following condition: $h_n \to^* h$ as $n \to \infty$ if and only if $h(s_n) \to h(s)$ as $n \to \infty$ whenever $s_n \to s$ as $n \to \infty$, and s is a continuity point of h. The set $\mathcal{G}$ is homeomorphic with the set of probability measures by the following transformation:

$$h \in \mathcal{G} \mapsto \eta_h,$$

where η_h is a probability measure whose cumulative distribution function is $s-h(s)$ i.e., $\eta_h([0,s)) = s-h(s)$ for $s \leq \bar{s}$ and $\eta_h([0,s)) = 1$ for $s > \bar{s}$. Hence $\mathcal{G}$ is compact. The weak topology is embedded into a topological vector space $\mathbf{G}$ of signed measures with locally bounded variations (see Jaśkiewicz and Nowak (2022) for details). For any $h \in \mathcal{G}$ we define

$$br(h)(s) := \max \arg \max_{c\in[0,s]} P(c,s;h).$$

Clearly $br(h) \in BR(h)$. We only need to show there exists a fixed point of $h \mapsto br(h)$. The operator br is well defined and maps (compact) $\mathcal{G}$ into itself (Lemma 3.2 in Balbus et al. (2015a)). Moreover, br is continuous (see proof of Theorem 1 in Balbus et al. (2015a)). Hence by the Schauder-Tychonoff fixed point theorem, br has a fixed point in $\mathcal{G}$. This fixed point is SMPE. ■

3.2 Uniqueness of SMPE. Searching through continuation values

In this subsection we apply the generalizes Bellman operator T and accept new assumptions. Comparing with the previous subsection we change assumptions on Q. We allow $Q(\cdot|i,s)$ depends on s, and relax the non-atomicity of Q but at the cost of imposing a certain "mixing" condition.

Assumption 3.2 *Assume*

- *u is strictly concave and increasing and $u(0) = 0$;*
- *transition probability Q has the following form*

$$Q(\cdot|i,s) = p(\cdot|i,s) + (1 - p(S|i,s))\delta_0(\cdot),$$

  where $\delta_0(\cdot)$ is a unit point mass (Dirac delta) concentrated in 0 and

  - *$p(\cdot|i,s)$ is a finite measure such that for any $s > 0$ and $i \in [0,s]$, $p(S|i,s) < 1$, and $p(\{0\}|0,0) = 1$;*
  - *for every bounded and Borel measurable function f such that $f(0) = 0$,*

$$i \in S \mapsto \int_S f(s')Q(ds'|i,s) \tag{12}$$

    is increasing and concave in i and continuous in (i,s).

For any $s \in S$, $v \in \mathcal{V}$ we define

$$\Pi(c,s,v) = u(c) + \beta\delta \int_S v(s')Q(ds'|s-c,s).$$

Observe that under Assumption 3.2 we have that $\Pi(\cdot,s;h)$ is strictly concave regardless on $s \in S$ and $v \in \mathcal{V}$. Since

$$B(v)(s) = \arg\max_{c\in[0,s]} \Pi(c,s;v),$$

hence $B(v)(s)$ is a singleton. Consequently the generalized Bellman operator T is well defined. We claim that T is an increasing operator.

Claim 3.1 *T is an increasing operator on $\mathcal{V}$.*

Proof: Clearly $A(v)$ is increasing. We show that $B(v)$ is decreasing. For proving the last assertion, we apply the standard Topkis Theorem (see Theorem 6.1 in Topkis (1978)). In this purpose we show, for any s, $(c,v) \in [0,s] \times \mathcal{V} \mapsto \Pi(c,s;v)$ has decreasing difference. For $s \in S$, let $c_1 < c_2 \leq s$ and define

$$\kappa(\cdot) := p(\cdot|s-c_1,s) - p(\cdot|s-c_2,s).$$

Clearly κ is a measure. Moreover, for any $v \in \mathcal{V}$

$$\int_S v(s')Q(ds'|s - c_2, s) - \int_S v(s')Q(ds'|s - c_1, s) = -\int_S v(s')\kappa(ds').$$

Hence

$$\Pi(c_2, s, v) - \Pi(c_1, s, v) = u(c_2) - u(c_1) - \beta\delta \int_S v(s')\kappa(ds')$$

is decreasing in v. As a result, for any s, $(c, v) \in [0, s] \times \mathcal{V} \mapsto \Pi(c, s; v)$ has decreasing difference. Hence by the standard Topkis (1978) Theorem, $B(v)$ is decreasing in v. Hence T increases with v for any $\beta \in [0, 1]$ as A is increasing with v. ■

Now we define

$$\mathcal{L} := \{h \in \mathcal{H} : \text{both } h(s) \text{ and } s - h(s) \text{ are increasing (in } s)\}.$$

By Theorems 1 and 2 in Balbus et al. (2018) we have the following results.

Theorem 3.2 (Uniqueness and attracting of SMPE) *Under Assumption 3.2 there exists a unique continuation value v^*, and corresponding unique SMPE. Moreover, for any initial $v_0 \in \mathcal{V}$ the sequence of successive iterations $v_{t+1} = T(v_t)$ for $t \geq 0$ tends uniformly to v^*. That is*

$$\lim_{t \to \infty} ||v_t - v^*||_\infty = 0. \tag{13}$$

Let h^* be the SMPE. Then h^* is increasing and if $Q(\cdot|i, s)$ does not depend on s, then $h^* \in \mathcal{L}$.

Proof: We sketch the proof. Obviously $\mathcal{V}$ is a Banach space, and it is routine to verify that $T : \mathcal{V} \mapsto \mathcal{V}$. Observe that CV corresponding to a MPE is a fixed point of the well defined operator T. By Claim 3.1, T is an increasing operator. On the other hand, for any nonnegative constant k,

$$T(v + k)(s) = T(v)(s) + \delta k.$$

Indeed, $B(v+k)(s) = B(v)(s)$ and $A(v+k)(s) = A(v)(s) + \beta\delta k$. By Blackwell Theorem (see Stokey et al. (1989)) we conclude T is a δ-contraction mapping. As a result, the fixed point v^* of T is uniquely determined and satisfies (13). Moreover, this is a unique continuation value function for a SMPE. Since $h^* = B(v^*)$, and B is well defined, hence SMPE is uniquely determined. Note that the function $(c,s) \mapsto \Pi(c,s,v^*)$ has increasing differences. Hence by Topkis (1978) h^* is increasing. If $Q(\cdot|i,s)$ does not depend on s then $(i,s) \mapsto \Pi(s-i,s,v^*)$ have increasing differences. Then, $h^* \in \mathcal{L}$. ■

We provide the following transition functions that obey Assumption 3.2.

Example 3.2 *Typical example is a linear combination between some probability measure and a unit point mass of absorbing state. For example*

$$Q(\cdot|i,s) = \sum_{l=1}^{L} g_l(i)\lambda_l(\cdot|s) + \left(1 - \sum_{l=1}^{L} g_l(i)\right)\delta_0(\cdot).$$

Here any of $g_l : S \mapsto [0,1]$ is increasing and concave such that $g_l(0) = 0$ and $\sum\limits_{l=1}^{L} g_l(i) \leq 1$ for all $i \in S$. We can find this kind of probability distribution in Rogerson (1985), Amir (1996), Szajowski (2006), Balbus and Nowak (2008) and Balbus et al. (2013), Balbus et al. (2015b). If we relax this assumption, only we can get is the existence of equilibria.

3.3 Monotone comparative statics

With Assumption 3.2 in place, we can now prove our main result on monotone comparative statics for extremal equilibrium policies. We define the operators A_θ, B_θ and T_θ as the natural adaptations to A, B and T. First of all, for $s \in S$, $c \in [0,s]$, $v \in \mathcal{V}$ and $\theta \in \Theta$ define

$$\Pi(c,s;v,\theta) := u(c) + \beta\delta \int_S v(s')Q(ds'|s-c,s,\theta).$$

Furthermore,

$$A_\theta(v)(s) = \max_{c\in[0,s]} \Pi(c,s;v,\theta),$$

$$B_\theta(v)(s) = \arg\max_{c\in[0,s]} \Pi(c, s; v, \theta),$$

and

$$T_\theta(v)(s) = \frac{1}{1-\beta} A_\theta(v)(s) - \frac{1-\beta}{\beta} u(B_\theta(v)).$$

We now specify how the change of parameter affects primitives of the model.

Assumption 3.3 *Let us assume:*

- *u does not depend on θ and obeys assumption 3.2;*
- *for any $s, i \in S$ and $\theta \in \Theta$ let $Q(\cdot|i, s, \theta) = p(\cdot|i, s, \theta) + (1 - p(S|i, s, \theta))\delta_0(\cdot)$, where for each θ $p(\cdot|i, s, \theta)$ obeys Assumption 3.2;*
- *for each $v \in \mathcal{V}$ we have $(i, \theta) \to \int_S v(s')p(ds'|i, s, \theta)$ has decreasing differences with (i, θ) and $\theta \to \int_S v(s')p(ds'|i, s, \theta)$ is decreasing on Θ.*

By Assumption 3.3 for any θ, the model obeys Assumption 3.2. As a result, there exists a unique SMPE h^*_θ and its continuation value v^*_θ. Moreover, for any $s \in S$,

$$v^*_\theta(s) = T_\theta(v^*)(s) \quad \text{and} \quad h^*_\theta(s) \in B_\theta(v^*)(s).$$

We show more, both functions increase in θ because of the following claim.

Claim 3.2 *Assume 3.2. Then, the function $B_\theta(v)$ decreases in θ and $T_\theta(v)$ decreases in $\theta \in \Theta$ and increases in $v \in \mathcal{V}$.*

Proof: By the third bullet of Assumption 3.3 it follows that for any $v \in \mathcal{V}$

$$\theta \in \Theta \mapsto \int_S v(s')p(ds'|i, s, \theta)$$

is decreasing, hence $A_\theta(v)$ is decreasing in θ. Obviously is increasing in v. For any $s \in S$ and for have for $c_1 < c_2 \le s$

$$\Pi(c_2, s; v, \theta) - \Pi(c_1, s; v, \theta) := -\int_S v(s')\kappa_\theta(ds')$$

where

$$\kappa_\theta(\cdot) := p(\cdot|s - c_1, s, \theta) - p(\cdot|s - c_2, s, \theta).$$

Obviously, κ_θ is a measure hence is increasing in v. Moreover, by the third bullet in Assumption 3.3, is increasing in θ. Hence Π has decreasing differences in (s, v) and increasing differences in (s, θ). By the standard Topkis (1978) Theorem, $B_\theta(v)(s)$ is decreasing in v and increasing in θ. As a result, $T_\theta(v)$ is decreasing in θ and increasing in v. ■

Theorem 3.3 (Monotone comparative statics) *Let Assumption 3.3 be satisfied. Then, the equilibrium policy $\theta \to h^*_\theta$ is increasing, and the continuation value $\theta \to v^*_\theta$ is decreasing.*

Proof: For any $\theta \in \Theta$, let $v_\theta : S \mapsto \mathbb{R}$ be a Borel function. Suppose that $\theta \mapsto v_\theta(s)$ is decreasing for any $s \in S$. By Theorem Claim 3.2, it follows that $T_\theta(v_\theta)$ is decreasing. Hence using the standard induction, we can show that the n-fold decomposition $T^n_\theta(\mathbf{0})$ decreases in θ. Indeed, for n it is clear. Suppose it is true for n and $T^n_\theta(\mathbf{0})$ decreases in θ. Then, by Claim 3.2 and induction hypothesis,

$$T^{n+1}_\theta(\mathbf{0}) = T_\theta(T^n_\theta(\mathbf{0}))$$

it is decreasing in θ. Consequently, $T^n_\theta(\mathbf{0})$ is decreasing in θ for any n. By Theorem 3.2, $v^*_\theta = \lim_{n\to\infty} T^n(\mathbf{0})$ decreases in θ as well. Hence, applying Claim 3.2 again we conclude $h^*_\theta = B_\theta(v^\theta)$ is increasing. ■

Similar approach can be used to obtain more specific results on comparative statics with respect to parameter β. Consider now a pair of modified operators:

$$\hat{A}(v)(s) = \max_{c\in[0,s]} \left\{ u(c) + \delta \int_S v(s')Q(ds'|s - c, s) \right\},$$
$$\hat{B}(v)(s) = \arg\max_{c\in[0,s]} \left\{ u(c) + \delta \int_S v(s')Q(ds'|s - c, s) \right\}.$$

Observe that:

$$\hat{A}(\beta v) \equiv A(v)$$
$$\hat{B}(\beta v) \equiv B(v).$$

Claim that $\hat{v}^*$ is a fixed point of operator $\hat{T}$ if and only if $v^* = \frac{\hat{v}^*}{\beta}$ is a fixed point of T, where:

$$\hat{T}(v) = \hat{A}(v)(s) - (1-\beta)u(\hat{B}(v)(s))$$

Now consider a parameterized fixed point problem: $\hat{T}_\beta(v) = \hat{A}v(s) - (1 - \beta)u(\hat{B}(v)(s))$. Observe that for each v operator $\hat{T}_\beta(v)$ is increasing in β (recall $\hat{A}$ and $\hat{B}$ do not depend on β). Under assumption 3.2 $\hat{T}_\beta$ is a monotone contraction. Hence the unique fixed point is increasing in β. This is summarized in the next claim.

Claim 3.3 *MCS in impatience parameter Under assumption 3.2 equilibrium consumption $h^*_\beta = \hat{B}(\hat{v}^*_\beta) = B(v^*_\beta)$ is decreasing and, analogously, equilibrium investment is increasing in β.*

3.4 Generalized Euler equations

Beginning with the work of Harris and Laibson (2001), many researchers have applied the so-called “generalized Euler equation” approach to solving dynamic/stochastic games. We now provide sufficient conditions for the existence of a unique *differentiable* pure strategy Markovian equilibrium, and state the version of the generalized Euler equation that characterizes it.

Suppose that $Q(\cdot|i,s)$ does not depend on s and we denote it by $Q(\cdot|i)$. Similarly we denote $p(\cdot|i)$

Assumption 3.4 *Assume 3.2. Additionally suppose $p(\cdot|i)$ has a density satisfying:*

$$p(S_0|i) = \int_S q(s'|i)ds',$$

for any Borel set $S_0 \subset S$. Assume additionally the following conditions:

- *For Lebesgue-a.e., $q(s'|i)$ is differentiable at any $i > 0$;*
- *Let us denote the derivative*

$$q'(s'|i) := \frac{d}{di} q(s'|i);$$

Assume $s' \in S \mapsto q'(s'|i)$ is differentiable for all $i > 0$.

Put

$$D(s) = \frac{d}{di} \int_S v^*(s') Q(ds'|i) \Big|_{i=s-h^*(s)}. \tag{14}$$

Theorem 3.4 (Generalized Euler Equations) *Assume 3.4. Then, h^* is differentiable and the following equation is satisfied for any $s > 0$:*

$$\begin{aligned} \frac{u'(h^*(s))}{\beta\delta} = D(s) \quad &= \quad \left(\frac{1}{\beta} - 1\right) \int_S u'(h^*(s)))(h^*)'(s') q'(s'|s - h^*(s)) ds' \\ &\quad - \frac{1}{\beta} \int_S u'(h^*(s')) q'(s'|s - h^*(s)) ds'. \end{aligned} \tag{15}$$

Proof: Now we derive the *Generalized Euler Equation.* Observe that h^* satisfies the First Order Conditions

$$u'(h^*(s)) - \beta\delta D(s) = 0. \tag{16}$$

Using the Fundamental Theorem of the Integral Calculus for Riemann-Stieltjes integrals (see Hewitt and Stromberg (1965) Theorem 18.19 or Amir (1997), Theorem 3.2) we have:

$$\frac{d}{di} \int_S v^*(s') Q(ds'|i) \Big|_{i=s-h^*(s)} = - \int_S (v^*)'(s') q'(s'|s - h^*(s)) ds'. \tag{17}$$

Then differentiating v^* we have

$$(v^*)'(s) = u'(h^*(s)))(h^*)'(s) + \delta D(s)(1 - (h^*)'(s)).$$

Hence

$$\begin{aligned}\int_S (v^*)'(s)q'(s|i)ds &= \int_S u'(h^*(s))(h^*)'(s)q'(s|i)ds \\ &+\delta\int_S D(s)(1-(h^*)'(s))q'(s|i)ds.\end{aligned}$$

Substituting $i = s - h^*(s)$ above, and applying (17) we obtain

$$\begin{aligned}-D(s) &= \int_S u'(h^*(s'))(h^*)'(s')q'(s'|s-h^*(s))ds' \\ &+\delta\int_S D(s')(1-(h^*)'(s'))q'(s'|s-h^*(s))ds'.\end{aligned}$$

Hence and by (16)

$$\begin{aligned}-D(s) &= \int_S u'(h^*(s')))(h^*)'(s')q'(s'|s-h^*(s))ds' \\ &+\frac{1}{\beta}\int_S u'(h^*(s'))(1-(h^*)'(s'))q'(s'|s-h^*(s))ds',\end{aligned}$$

or equivalently (15). ■

Clearly, replacing $(h^*)'(s) = 1-(g^*)'(s)$, were g^* denote equilibrium investment one obtains the generalized Euler equation:

$$\begin{aligned}\frac{u'(h^*(s))}{\beta\delta} &= \left(1-\frac{1}{\beta}\right)\int_S u'(h^*(s)))(g^*)'(s')q'(s'|g^*(s))ds' \\ &-\int_S u'(h^*(s'))q'(s'|g^*(s))ds'.\end{aligned}$$

4 Extensions

In the previous section, we have focused attention on S being a bounded interval. Using the results from Balbus et al. (2020), we can conclude that the thesis of Theorem 3.1 is satisfied for “weightly-bounded” utility functions. Then, applying the main result in Balbus et al. (2018), we deduce that the thesis of Theorem 3.2 is also valid in case of “bounded by sequence” utility functions. The definition of “weightly-bounded” function is formalized as follows.

Definition 4.1 (w-bounded function) *Let $w : S \mapsto (0, \infty)$. A function v is w-bounded if there exists $M > 0$ such that for any $s \in S$, $|v(s)| \leq M\, w(s)$.*

The space of w-bounded functions is a Banach space with the natural norm

$$||v||_w := \sup_{s \in S} \left| \frac{v(s)}{w(s)} \right|.$$

Another approach is a "boundedness by a sequence". Let $(S_j)_{j=1}^{\infty}$ be a sequence of Borel subsets of S such that any of S_j has non-empty interior and $S_1 \subset S_2 \subset \dots$, and let $\mathbf{m} := (m_j)_{j=1}^{\infty}$ be a strictly monotone sequence of positive numbers such that

$$r := \sup_{j \in \mathbb{N}} \frac{m_{j+1}}{m_j} < \infty.$$

We provide the formal definition of "bounded by a sequence".

Definition 4.2 (m-bounded) *The function v is $\mathbf{m}$-bounded if for any $j \in \mathbb{N}$, $||v||_j := \sup_{s \in S_j} |v(s)| \leq m_j$.*

We focus attention on the set of Borel measurable functions. The seminorms $||v||_j$ define a locally convex space of functions bounded on all S_j. The set of $\mathbf{m}$-bounded functions is embedded into a larger space

$$\mathbf{V} := \left\{ v : S \mapsto \mathbb{R}, v \text{ is Borel}, ||v||_j < \infty, \text{ for all } j, \text{ and } \sum_{j=1}^{\infty} \frac{||v||_j}{m_j} \zeta^{j-1} < \infty \right\}$$

for some $\zeta \in (0, 1)$. Clearly $\mathbf{V}$ is a vector space with the following norm

$$||v|| := \sum_{j=1}^{\infty} \frac{||v||_j}{m_j} \zeta^{j-1}.$$

The set of generic elements of such functions is:

$$\mathcal{V}^{\mathbf{m}} := \{ v \in \mathbf{V} : v(0) = 0, ||v||_j \leq m_j \text{ for all } j \in \mathbb{N} \}$$

Clearly $\mathcal{V}^{\mathbf{m}} \subset \mathbf{V}$. Due to Matkowski and Nowak (2011) we have

Proposition 4.1 *The tuple* $(\mathbf{V}, ||\cdot||)$ *defines a Banach space and* $\mathcal{V}^{\mathbf{m}}$ *is a closed subset of* $\mathbf{V}$.

Following Rincon-Zapatero and Rodriguez-Palmero (2003) we provide the following definition.

Definition 4.3 (1-local contraction) *The operator* $\Phi : \mathcal{V} \mapsto \mathcal{V}$ *is a 1-local contraction if there is a constant* $\xi \in (0,1)$ *such that*

$$||\Phi(v_1) - \Phi(v_2)||_j \leq \xi ||v_1 - v_2||_{j+1}.$$

Due to Rincon-Zapatero and Rodriguez-Palmero (2003, 2009)[7] we have

Proposition 4.2 *Let* $\Phi : \mathcal{V}^{\mathbf{m}} \mapsto \mathcal{V}^{\mathbf{m}}$ *be a 1-local contraction with a constant* $\xi \in (0,1)$ *and suppose* $\xi r < \zeta$. *Then* Φ *is a contraction mapping with respect to the metric induced by the norm* $||\cdot||$ *and its modulus is* $\frac{\xi r}{\zeta}$.

We provide set of assumptions for both approaches, "weightly-bounded", and "bounded by sequence".

4.1 Weighted bounded functions

We start with the assumption in case of weighted bounded felicity function.

Assumption 4.1 (w-bounded approach) *Suppose that Assumption 3.1 holds. Moreover, there exists a Borel function* $w : S \mapsto [1, \infty)$ *such that:*

- *The felicity function satisfies* $||u||_w \leq \bar{u}$;
- *The transition function satisfies*

$$\bar{Q} := \frac{\sup\limits_{i \in S} \int_S w(s')Q(ds|i)}{w(s')} < \infty$$

and

$$\delta \bar{Q} < 1.$$

[7]See also Matkowski and Nowak (2011).

Let Δ_n be the set of all non-negative Borel measures on $[0,n]$ such that any of $\eta \in \Delta_n$ satisfies $\eta([0,n]) \leq n$. Let $\Delta := \prod_{n=1}^{\infty} \Delta_n$. Endow Δ_n with the standard weak topology, and Δ with the product topology. Let $\mathcal{H}$ and $\mathcal{G}$ be as in (3) and respectively (11). Endow $\mathcal{G}$ with adapted weak topology i.e. $h_n(s) \to^w h(s)$ as $n \to \infty$ whenever h is continuous at s. There is an isomophism between $\mathcal{I}$ and a subset of product of measures on Δ. We provide the construction by Balbus et al. (2020). Any $h \in \mathcal{I}$ induces a unique element $(\eta_n)_{n=1}^{\infty} \in \Delta$ such that

$$h \in \mathcal{G} \mapsto \eta_n([0,s]) = s - h(s), \text{for } s \in [0,n].$$

In other words, $s - h(s)$ is a cumulative distribution function of $\eta_n(\cdot)$ on $[0,n]$.

Let

$$M_0 := \frac{\bar{u}}{1 - \delta \bar{Q}}$$

and let s_t be a Markov chain generated by a policy $h \in \mathcal{I}$, i.e. with the transition probability $Q(\cdot|h(s))$. Then for any t,

$$\mathbb{E}_s(u(h(s_t))) \leq \mathbb{E}_s\left(u(s_t)\right) \leq \bar{u}\mathbb{E}_s\left(w(s_t)\right).$$

We have then

$$\mathbb{E}_s \delta^t w(s_{t+1}) = \mathbb{E}_s\left(\delta^t \int_S w(s')Q(ds'|s_t - h(s_t))\right) \leq \delta\bar{Q}.\mathbb{E}_s\delta^{t-1}w(s_t).$$

Hence V_h is well defined w-bounded function, and

$$V_h(s) \leq \mathbb{E}_s\left(\sum_{t=1}^{\infty} w(s_t)\delta^{t-1}\right) \leq \frac{\bar{u}}{1-\delta\bar{Q}} w(s) = M_0 w(s).$$

Then, we may construct SMPE h^* as a fixed point of $br(h)$, which is continuous and maps $\mathcal{I}$ into itself (Lemma 13 in Balbus et al. (2020)). Hence the thesis of Theorem 3.1 are satisfied.

4.2 Functions bounded by a sequence

Using the results in Balbus et al. (2018), we are going to show the thesis of Theorem 3.2 in case of $\mathbf{m}$-bounded utility functions on S. We impose the following assumptions.

Assumption 4.2 (m-bounded approach) *Assume 3.2. Moreover, suppose there exists a monotone sequence* $\mathbf{m} = (m_j)_{j=1}^{\infty}$ *such that* $m_1 > 0$ *and* $r = \sup_{j\in\mathbb{N}} \frac{m_{j+1}}{m_j} < \infty$, *and an ascending sequence of sets* $(S_j)_{j=1}^{\infty}$ *such that* $S_j \subset S$ *such that*

- *for any* j, $\sup_{s\in S_j} u(s) = (1-\beta)m_j$;
- *for any* j, $p(S_{j+1}|i,s) = p(S|i,s) = 1$ *whenever,* $s \in S_j$ *and* $i \in [0,s]$;
- *the sequence* $\mathbf{m}$ *satisfies*

$$\delta \sup_{j\in\mathbb{N}} \frac{m_{j+1}}{m_j} \leq \beta.$$

In other words, $Q(S_{j+1}|s) = 1$ whenever $s \in S_j$, that is the state visits sequentially $S_1, S_2, \ldots$ almost surely. Now we recall the definitions of $A(v)$, $B(v)$ and $T(v)$. The operator T maps $\mathcal{V}^{\mathbf{m}}$ into itself. Indeed, for $s \in S_j$

$$\begin{aligned} T(v)(s) &= u(B(v)(s)) + \delta \int_S v(s')Q(ds'|s - B(v)(s)) \\ &\leq (1-\beta)m_j + \delta m_{j+1} \leq \left(1 - \beta + \beta \sup_{j\in\mathbb{N}} \frac{m_{j+1}}{m_j}\right) m_j \leq m_j. \end{aligned}$$

Furthermore, T is a $1-$ local contraction with δ on $\mathcal{V}^{\mathbf{m}}$, hence and by Proposition 4.2 it is a contraction mapping with the metric induced by the norm on $||\cdot||$ on $\mathcal{V}^{\mathbf{m}}$. $(\mathcal{V}^{\mathbf{m}}, ||\cdot||)$ is a complete metric space, hence T has a unique fixed point in T. Hence the thesis of Theorem 3.2 are satisfied.

4.3 Recursive utility and general certainty equivalents

The current approaches rely on the assumption that the utility functions of successors defined over sequences of random levels of consumption are represented by a time-additive expected overall utility, which discounts future temporal utilities at a constant rate. This approach cannot explain some problems in economics. For example, the standard discounted utility cannot explain the *equity premium puzzle* postulated by Mehra and Prescott (1985), and the standard expectation cannot explain the Allais and Ellsberg paradoxes, see e.g. Chew (1983), Dekel (1986), Chew and Epstein (1989). To remedy these issues, some authors propose other approaches based on recursive utility and general certainty equivalents. We start with some basic introduction to these two generalizations and next provide its application to quasi-hyperbolic discounting.

4.3.1 Introduction

Let us start with the standard deterministic discounted utility function for the self t

$$V_t = \sum_{\tau=t}^{\infty} u(c_\tau)\delta^{\tau-t}$$

that obeys the Bellman equations. Koopmans (1960) provides the generalization of this approach above toward the *recursive utility function*, i.e., V_t that satisfies the equation,

$$V_t = \mathcal{A}(c_t, V_{t+1}),$$

where $\mathcal{A}$ is a function called *aggregator*. In case of the standard utility, the aggregator has an affine form $\mathcal{A}(c_t, V_{t+1}) = u(c_t) + \delta V_{t+1}$

$$V_t = u(c_t) + \delta V_{t+1}.$$

If $(c_\tau)_{\tau=t}^{\infty}$ is a sequence of lotteries adapted to some filtration $(\mathscr{F}_\tau)_{\tau=t}^{\infty}$, then V_t is a $\mathscr{F}_t$-measurable random function. Kreps and Porteus (1978) provide the following model:

$$V_t = \mathcal{A}(c_t, \mathbb{E}_t V_{t+1}),$$

where $\mathbb{E}_t$ is the expectation conditioned by $\mathscr{F}_t$. Epstein and Zin (1989) generalized the utility substituting $\mathbb{E}_t$ by a more general *certainty equivalent operator*.

Let $\mathscr{L}^1$ be the equivalence class of real valued integrable functions endowed with an order.

Definition 4.4 (Certainty Equivalent Operator (COP)) *For any t, the operator $\mathcal{M}_t$ mapping $\mathscr{L}^1$ into $\mathscr{F}_t$- measurable function is called* certainty equivalent operator *if*

- $\mathcal{M}_t(\alpha) = \alpha$ *whenever α is constant;*
- $\mathcal{M}_t(\cdot)$ *is a monotone operator on $\mathscr{L}^1$.*

The recursive utility that connects the general aggregator approach and the certainty equivalent operator was introduced by Epstein and Zin (1989) and takes the following form:

$$V_t = \mathcal{A}(c_t, \mathcal{M}_t(V_{t+1})).$$

This recursive utility is *time consistent* or *dynamically consistent.* It means that the following condition holds. For any $t \in \mathbb{N}$ let $c^t = (c_\tau)_{\tau=t}^{\infty}$ be a consumption stream, and let $\tilde{c}^t := (\tilde{c}_\tau)_{\tau=t}^{\infty}$ be another one. Then $V_t(c^t) \leq V_t(c_t, \tilde{c}^t)$ if and only if $V_{t+1}(c^{t+1}) \leq V_{t+1}(\tilde{c}^{t+1})$. In other words, the decision maker never regrets the decision made yesterday. The basic problem is the existence of recursive and uniqueness recursive utility function. The issue has the positive solution under distinct assumptions.[8]

The most common certainty equivalent operators are:

- *Entropic risk measure* postulated by Weil (1993);

$$\mathcal{M}_t(V_{t+1}) = -\frac{1}{r} \ln \left(\mathbb{E}_t \left(e^{-rV_{t+1}} \right) \right)$$

  for $r \neq 0$;

[8]For the survey on this topic, we refer the reader to Balbus (2020), Bäuerle and Jaśkiewicz (2018), Becker and Rincón-Zapatero (2021), Bich et al. (2018), Bloise and Vailakis (2018), Borovicka and Stachurski (2020), Jaśkiewicz et al. (2014), Le Van and Vailakis (2005), Marinacci and Montrucchio (2010), Martins-da Rocha and Vailakis (2010), Weil (1993) and the references therein.

- *Kreps and Porteus measure* postulated in their (1978) paper:

$$\mathcal{M}_t(V_{t+1}) = \left(\mathbb{E}_t V_{t+1}^r\right)^{\frac{1}{r}}$$

  for $r > 0$.

Both postulated measures are special cases of *quasi-linear mean*

$$\mathcal{M}_t(V_{t+1}) = \psi^{-1}\left(\ln\left(\mathbb{E}_t\left(e^{-rV_{t+1}}\right)\right)\right),$$

for some monotone and invertible function ψ (see Chew (1983) and Dekel (1986)). Another interesting operator is the so-called *max-min* operator defined by Gilboa and Schmeidler (1989), which can applied in the large body of robust control literature:

$$V_t := \min_{\theta \in \Theta} \mathbb{E}_t^\theta V_{t+1},$$

here $\mathbb{E}_t^\theta$ is the parameterized expectation, and the value θ is unknown for the self (see Balbus et al. Balbus et al. (2015b)). Some certainty equivalent operators are not provided explicitly but have application due to useful properties. For example, the certainty equivalent by Gul (1991) reflects elation and disappointment aversion.

4.3.2 Generalized quasi-hyperbolic discounting and certainty equivalents

Observe that the $\beta - \delta$ problem can be generalized along these lines as follows. Let each self t have the utility

$$W_t = \mathcal{A}_1(c_t, \mathcal{M}_t(V_{t+1})),$$

where $(V_t)_{t=1}^\infty$ is the sequence of recursive utility satisfying

$$V_t = \mathcal{A}_2(c_t, \mathcal{M}_t(V_{t+1})).$$

Here $\mathcal{A}_1$ is an aggregator connecting present consumption and the expected utility of the successor. In turn, $\mathcal{A}_2$ is an aggregator connecting a consumption of future self and the expected utility the successor. For example, our

$\beta - \delta$ problem has this form with

$$\mathcal{A}_1(c_t, V_{t+1}) = u(c_t) + \beta\delta V_{t+1}, \quad \text{and} \quad \mathcal{A}_2(c_t, V_{t+1}) = u(c_t) + \delta V_{t+1}$$

and the standard expectation. We slightly modify the model and we substitute the expectation by the risk measure:[9]

$$\mathcal{M}_t = -\ln\left(\mathbb{E}_t(e^{-V_{t+1}})\right).$$

We have that h^* is SMPE if $h^*(s) \in \arg\max_{c\in[0,s]} P(c, s; h)$ for any $s \in S$ with

$$P(c, s; h) = u(c) - \beta\delta \int_S e^{-V_h(s')} Q(ds'|s - c).$$

Here V_h solves the following Koopmans-Bellman equations

$$V_h(s) = u(h(s)) - \delta \int_S e^{-V_h(s')} Q(ds'|s - h(s)).$$

Another possible generalization involves Koopmans et al. (1964) form. Let the $\mathcal{A}_1$ be as follows:

$$\mathcal{A}_1(c_t, V_{t+1}) = \ln(1 + c_t + \beta\delta V_{t+1}),$$

and let $\mathcal{A}_2$ be as follows

$$\mathcal{A}_2(c_t, V_{t+1}) = \sqrt{c_t^2 + \delta V_{t+1}},$$

and

$$\mathcal{M}_t(V_{t+1}) = \left(\mathbf{E}_t(V_{t+1}^3)\right)^{\frac{1}{3}}.$$

The current payoff has the form

$$P(c, s; h) = \ln(1 + c + \beta\delta \left(\int_S V_h(s') Q(ds'|s - c)\right).$$

Here V_h solves the Koopmans-Bellman equations:

$$V_h(s) = \sqrt{h^2(s) + \delta \left(\int_S V_h^3(s') Q(ds'|s - h(s))\right)^{\frac{1}{3}}}.$$

The generalized Bellman equation techniques can be easily adopted to cover these extensions.

[9]In case of $\beta = 1$ the utility reduces to this studied in Bäuerle and Jaśkiewicz (2018) or in stochastic games by Asienkiewicz and Balbus (2019).

4.4 Multidimensional states

The generalized Bellman equation approach and results presented in section 3.2 extend easily to the multi-dimensional state space. To see that let the state space be $S = [0, \bar{s}] \subset \mathbb{R}^n$ and, for each $s \in S$, the action set $A(s) \subset A \subset \mathbb{R}^m$ for some A. Period utility is now $u : A \to \mathbb{R}_+$. Assumption 3.2 need to be generalized as follows:

Assumption 4.3 *Assume*

- *for each* $s \in S$ *set* $A(s)$ *is a compact and subcomplete sublattice with* $A(0) = \{0\}$ *moreover* $s \mapsto A(s)$ *is measurable,*
- u *is continuous, increasing and supermodular with* $u(0) = 0$*;*
- *transition probability* $Q(\cdot|a, x) = p(\cdot|s, a) + (1 - p(S|a, x))\delta_0(\cdot)$, *satisfies:*
  - $p(\cdot|a, s)$ *is a finite measure such that for any* $s > 0$ *and* $a \in A(s)$, $p(S|i, s) < 1$, *and* $p(\{0\}|0, 0) = 1$*;*
  - *for every bounded and Borel measurable function* f *such that* $f(0) = 0$,

$$a \mapsto \int_S f(s')Q(ds'|a, s) \tag{18}$$

    is decreasing and supermodular in a *and continuous in* (a, s).

Under assumption 4.3 the generalized Bellman equation approach can still be used to analyze SMPE but now, as the $B(v)(s)$ can be multivalued, some of its selection need to be specified. Balbus et al. (2015c) use the greatest and the least selection, i.e. $\overline{B}(v)(s)$ and $\underline{B}(v)(s)$. This allows to specify two operators $\overline{T}$ and $\underline{T}$, respectively. As $B(v)$ is multivalued equilibrium uniqueness is not guaranteed but its existence and approximation can constructed by applying the following theorem.

Theorem 4.1 *Let assumption 4.3 hold. Then, the set of equilibrium continuation values possesses the least $v^\star = \underline{T}(v^\star)$ and the greatest $w^\star = \overline{T}(w^\star)$ elements corresponding to the greatest $\overline{h}^\star$ and the least $\underline{h}^\star$ stationary, time-consistent Markov policies.*

5 Self-generation approach

In this last section inspired by the work of Abreu et al. (1990) or Mertens and Parthasarathy (1987), but adopted from the case of repeated games to dynamic games and short memory equilibria (see also Doraszelski and Escobar (2012)), we discuss a self-generation approach to constructing a large set of non-stationary equilibria in this model. This method relies on the successive approximation of the sets of functions or equilibria value. The advantage of this method is relatively weak assumption on felicity functions and the transition probability. The drawback is that this method allows to verify the existence of equilibria in a set of non-stationary strategies. In particular, after verifying the existence of the equilibria, we never know whether is stationary or not even if the model is stationary. In the next subsection we present briefly the results by Balbus and Woźny (2016). Similar method can be found in non-stationary models as well. For example see Balbus et al. (2020), or in case of bequest games, see Balbus et al. (2017).

We come back to the model from Section 3.1. For our main result here, we need the following assumption.

Assumption 5.1 *Assume 3.1 and moreover assume that for any $s_0 \in S$ the set $Z_0(s_0) := \{i \in S : Q(\{s_0\}|i) > 0\}$ is at most countable.*

This new assumption is not very restrictive, as is illustrated by the following example.

Remark 5.1 *Observe that if $Q(\cdot|i)$ is nonatomic, then any of the set $Z_0(s_0)$ is empty and if $Q(\cdot|i)$ is deterministic such that $Q(A|i) = \mathbf{1}_A(G(i))$ for some* *strictly increasing G, then any of $Z_0(s_0)$ has at most one element. We have a similar conclusion if for some $\alpha \in (0,1)$ and strictly increasing G:*

$$Q(A|i) = \alpha\tilde{Q}(A|i) + (1-\alpha)\mathbf{1}_A(G(i)),$$

whenever $\tilde{Q}(\cdot|i)$ is nonatomic.

The generic set of equilibria is again $\mathcal{G}$ endowed with the weak topology. For any $\mathcal{W} \subset \mathcal{G}$ we define

$$\mathcal{B}(\mathcal{W}) := \bigcup_{h\in\mathcal{W}} \arg\max_{c\in[0,s]} P(c,s;h).$$

In other words, $\mathcal{B}$ maps $2^{\mathcal{G}}$ into itself. Obviously $(2^{\mathcal{G}}, \subset)$ is complete lattice and $\mathcal{B}$ is increasing under set inclusion (i.e. $\mathcal{W}_1 \subset \mathcal{W}_2$ implies $\mathcal{B}(\mathcal{W}_1) \subset \mathcal{B}(\mathcal{W}_2)$). By the main theorem in Tarski (1955), there exists a nonempty complete lattice of fixed points (and in particular, the greatest fixed point under set inclusion $\mathcal{W}^*$).

Define the sequence of iterations

$$\mathcal{W}^{t+1} = \mathcal{B}(\mathcal{W}^t)$$

for $\mathcal{W}^0 = \mathcal{G}$. Now we introduce the following definition.

Definition 5.1 (Self-generating) *Let $\mathcal{W} \subset \mathcal{G}$. We say that $\mathcal{W}$ is **self-generating** if $\mathcal{W} \subset \mathcal{B}(\mathcal{W})$.*

We denote $\mathcal{E} \subset \mathcal{G}$ as the set of all equilibria $\mathcal{E}$. The following lemma shows $\mathcal{E}$ is the greatest self generating set.

Lemma 5.1 (Self-generating property) *If $\mathcal{W}$ is self generating, then $\mathcal{W} \subset \mathcal{E}$.*

Theorem 5.1 (Construction of equilibria values) *Let $\mathcal{E}$ to be the set of equilibria. Then, $\mathcal{E}$ is non-empty and it is the greatest fixed point of $\mathcal{B}$. Moreover, and*

$$\mathcal{E} = \bigcap_{t=1}^{\infty} \mathcal{W}_t.$$

Proof: Due to series of Lemmas in Balbus et al. (2020) we have that $\mathcal{B}(\mathcal{W})$ is a nonempty weakly compact set whenever $\mathcal{W}$ is. As a result, $\bigcap_{t=1}^{\infty} \mathcal{W}^t \neq \emptyset$ and is weakly compact. Hence we have $\mathcal{W}^* \subset \bigcap_{t=1}^{\infty} \mathcal{W}^t$.

On the other hand,

$$\mathcal{B}\left(\bigcap_{t=1}^{\infty} \mathcal{W}^t\right) \subset \mathcal{W}^{t+1},$$

for any t, hence taking intersection over t, we have

$$\mathcal{B}\left(\bigcap_{t=1}^{\infty} \mathcal{W}^t\right) \subset \bigcap_{t=1}^{\infty} \mathcal{W}^t$$

Furthermore,

$$\mathcal{B}\left(\bigcap_{t=1}^{\infty} \mathcal{W}^t\right) \supset \bigcap_{t=1}^{\infty} \mathcal{W}^t.$$

Indeed, $h \in \bigcap_{t=1}^{\infty} \mathcal{W}^t$, hence $h(s) \in \arg\max_{c \in [0,s]} P(c, s; h_t)$ for all s and some sequence h_t such that $h_t \in \mathcal{W}^t$. Since any of $\mathcal{W}^t$ is compact and $\mathcal{W}^t$ is descending, without loss of generality suppose $h_t \to^w \tilde{h}$ for some $\tilde{h} \in \bigcap_{t=1}^{\infty} \mathcal{W}^t$. By Lemma 7 c) in Balbus et al. (2020), $P(c, s; h)$ is continuous in h, hence $h(s) \in \arg\max_{c \in [0,s]} P(c, s; \tilde{h})$, hence $h \in \mathcal{B}(\bigcap_{t=1}^{\infty} \mathcal{W}^t)$. Consequently,

$$\mathcal{W}^* = \bigcap_{t=1}^{\infty} \mathcal{W}^t.$$

Let $\mathcal{E} \subset \mathcal{G}$ be the set of all equilibria continuation values. Observe that $h \in \mathcal{E}$ implies that there is h_t a sequence from $\mathcal{E}$ such that for any $s \in S$, $h(s) \in \arg\max_{c \in [0,s]} P(c, s; h_1)$, $h_1(s) \in \arg\max_{c \in [0,s]} P(c, s; h_2)$, $h_2(s) = \arg\max_{c \in [0,s]} P(c, s; h_2)$ and so on. Consequently $h_1 \in \mathcal{E}$, hence $h \in \mathcal{B}(\mathcal{E})$. In other words $\mathcal{E} \subset \mathcal{B}(\mathcal{E})$, hence $\mathcal{E}$ is self generating.

Obviously any fixed point of $\mathcal{W}$ is self-generating. In particular $\mathcal{W}^*$ is self-generating, hence by Lemma 5.1 we have

$$\mathcal{W}^* \subset \mathcal{E}. \tag{19}$$

But $\mathcal{E} \subset \mathcal{W}^t$ for any $t \in \mathbb{N}$. hence

$$\mathcal{E} \subset \bigcap_{t=1}^{\infty} \mathcal{W}^t = \mathcal{W}^*.$$

Therefore, by (19),

$$\mathcal{E} = \bigcap_{t=1}^{\infty} \mathcal{W}^t = \mathcal{W}^*.$$

■

6 Concluding remarks

In this chapter, we have presented a series of results on existence and characterization of TCPs in a canonical version of dynamic choice problem for a quasi-hyperbolic consumer. We have also discussed some natural generalizations of the model. We have not only studied the case of TCPs as SMPE equilibria, but have also consider the case of more general forms of (nonstationary) Markov perfect equilibria via self-generation methods. In the case of TCP as SMPE equilibria, we have also given conditions under which uniqueness of TCPs can be established, and we have also discussed when monotone comparative statics of TCP in natural parameters of the model are possible (i.e., discount rates). Finally, we have discuss when sufficient conditions are present for the existence of generalized Euler equation, and we mention the additional extensions of our methods to more general models of dynamic biases.

There is an emerging literature that studies the question of the structure of TCPs in this more general behavioral discounting case. This work includes not only models with present bias, but models with future-bias, with backward discounting including hyperbolic models, models with more general recursive aggregators that generate dynamically inconsistent preferences, models of altruistic dynastic choice than the present chapter. There are also some more general results on the existence of SMPE in *deterministic* models for the quasi-hyperbolic case (e.g., see Bernheim et al. (2015), Cao and Werning (2018) and Balbus et al. (2021b)), as well as work on long-memory solutions for TCPs in models with more general discounting features (see for example Balbus et al. (2021a) where many additional results are presented).

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