Summary of
The Cournotian foundations of Walrasian equilibrium theory:
an exposition of recent theory

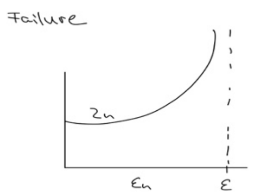
Overview:
The question the chapter addresses and summarizes is whether the Cournotian Nash Equilibrium approaches a Walrasian equilibrium as the number of players/traders growth to infinity/continuum.

In a non-cooperative Cournotian setting different players trade with each other by setting the quantity they would like to buy (bid) or offer (sell). Non-cooperativie means that the players take the quantities of other players as given and thus are not able to directly influence them. If there are only a few players in a Cournot Nash Equilibrium (CNE) might turn out that the players are not price takers but are able to influence them. Consequentially the CNE might not to be a walrasian equilibrium (WE) and thus not pareto efficient. The question is if the CNE approaches the WE as the number of players tend to infinity or a continuum of players.

There are generally 3 conditions under which the CNE approaches WE at the limit.

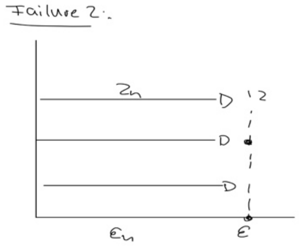
1) If zn is Cournot equilibrium (CE) for en than zn should be bounded uniformly on n. Or equivalently zn has a converging subsequence. (Thus escaping to infinity is impossible)

The mathematical interpretation is that the graph is closed (closed graph)



2) If zn converges to z and zn is a CE for en then z is a WE for the continuum economy e.

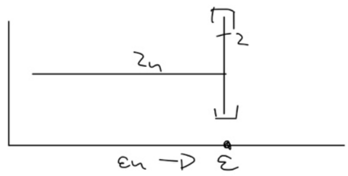
Mathimatical interpretation: Condition 1 and 2 together yield upper hemicontinuity (meaning 1) closed graph and 2) the image of a function is compact



3) If z is a WE of e then zn converges to z for a sequence zn of CE for en.

Mathimatical interpretation: yields lower hemicontinuity:

Failure 3:



Set-up:

- $l \geq 1$ # of commodities
- L = commodity space
- $\mathbb{R}^l_+ = \{v \in \mathbb{R}^l : v \geq 0\}$ = consumption set
- $P = \{\succsim_h, w_h\}_{h=1}^{m}$ = preference-endowment set

h: type of agent

(i) is agent

I: set of agents

j: is particular good

$\mathcal{E}: I \to P$ is an exchange economy with continuous agents

- $\mathcal{E}_n$: exchange economy with n = # of agents
- $x: I \to \mathbb{R}^l : \sum_{i \in I} x_i \leq 0$ is Net trade function
- m is a commodity which has no utility but only fasilitates trade between all the other goods.
- $(m_i, y_i) \in \mathbb{R}^{2L}_+$. vector with $2L$ elements
- $m: I \to \mathbb{R}^l_+$ vector with l elements
- $y: I \to \mathbb{R}^l_+$ vector with L elements
- $w \gg 0$. All agents have positive endowment.

Exchange Economy

Each trader (i) decides on the quantity to bid (buy) and the quantity to sell (offer). bids are fasilitated through commodity m. That means in order to bid for good (j) trader (i) offers m_i^j-amount of $m \geq 0$. Similarly if (i) wants to sell good (j) he offers y_i^j-quantity ≥ 0 in exchange for m_i^j.

- Thus the trader recieves y_i^j-quantities in exchange for $\left[(\underset{m}{\text{exchange price}}) \cdot m_i^j\right] = \left(\frac{z^{l+j}}{z^j}\right) \cdot m_i^j$
- Trader (j) recieves m_i^j-quantity in exchange for $\left[(\text{exchange price}_m) \cdot y_i^j\right] = \left(\frac{z^j}{z^{l+j}}\right) \cdot y_i^j$

The exchange prices are determined by the aggregate demand and supply:

$$\left.\begin{aligned} z^j &= \sum_j m_i^j \\ z^{l+j} &= \sum_i y_i^j \end{aligned}\right\} \text{clearing price is } \left(\frac{z^j}{z^{l+j}}\right)$$

which means how much of good m^j a trader recieves for 1-good of y^j.

Supply and Recieving:

The net trade of a player (i) is $x_i[m, y)$ = The difference between the quantity recieved of a good minus the quantity supplied of a good.

Supply: $\max\{0, -x_i^j\}$

The net trade is the difference btw the both.

Recieving: $\max\{0, x_i^j\}$

Budget constrain:

① $\sum_j m_j^i \leq \sum_j \left(\frac{z^j}{z^{i+1}}\right) y_i^j$ = The total amount a trader (i) bids cannot exeed the quantity m he recieved before.

② $y_i^j \leq w_j^i \quad \forall j$: A trader (i) cannot offer more than he has of good (j).

Cournot-Game:

· Players might be able to influence the price.

Definition of a CNE:

A feasible allocation $(m, y): I \to \mathbb{R}^{2L}$ is CNE if $\forall i \in I \; \nexists \, (m', y') : (m', y')$ is feasible for i and

$x_i[m', y'] + w_i \succ_i x_i[m, y] + w_i$ and $m_{i'} = m_{i'}, \; y_{i'} = y_{i'}$ where $i' \neq i$.

Clearing price:

The price now might be influenced by the players decisions.

$\bar{y} = \sum_{i' \neq i} y_{i'}$

Are the aggregate amount without the quantities of player (i).

$\bar{m} = \sum_{i' \neq i} m_{i'}$

Clearing Price $= \left[\frac{\bar{m}^j}{\bar{y}^j - x_i^j}\right]$

- if $x_i^j > 0$ hence he bids then the price is increased
- if $x_i^j \leq 0$ hence he offers then the price is decreased

Quantity (i) recieves or supplies:

$$\boxed{\left[\frac{\bar{m}^j}{\bar{y}^j - x_i^j}\right] \cdot x_i^j}$$

This leads to Proposition 1.

Proposition 1:

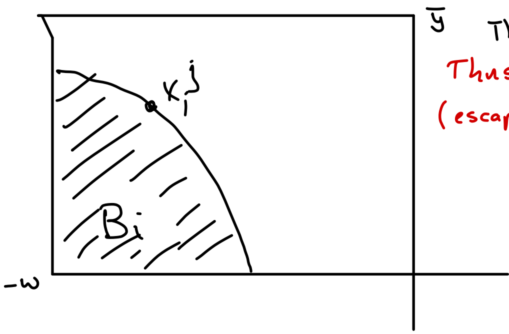
$\exists\, k > 0$ (only depending on $\mathcal{P}$): if (m, y) is a CNE for $\varepsilon : I\text{-}DP$ then $|x_i^j[m, y]| \leq k \quad \forall i, j$

Thus the gains of the net trade for traders is limited.

This holds because x_i^j is bounded by $-w_i \leq x_i \leq \bar{y}$. ① Trader (i) can't offer more than his endowment ② Trader (i) can't recieve more than the aggregate quantity supplied by all others.

(This is similar to the feasibility condition in WE)

↳ This also implies $g(x) = \sum_j \left(\frac{\bar{m}^j}{\bar{y}^j - x_i^j}\right) x_i^j \leq 0$ which means the aggregate trade across markets cannot exceed the quantities existing adjusted by price. (This is similar to the budget constrain in WE)



The budget of each (i) is bounded above and below.

Thus we can already rule out the failure of ①. (escape to infinity)

CNE → WE at the limit:

• Now we look at what happens as the number of traders $n \to \infty$. or in other words as $\varepsilon_n: I_n \to P \longrightarrow \varepsilon: I \in [0,1] \to P$. [Within the continuum $I \in [0,1]$ there are no atoms!]

① The defined exchange economy above can just be changed by using $\int$ instead of Σ and using $I \in [0,1]$

② The Budget constrain (BC) in CNE now becomes:

$$B_i(m,y) = \left\{ x_i \in \mathbb{R}^L : -w_i \leq x_i, \sum_j \frac{2^j}{2^{L+j}} \cdot x_i^j \leq 0 \right\}$$

(i) x_i is only bounded by endowments and supply.

(ii) A individual trader (i) now has no influence on the price: $\left[\sum_j \frac{2^j}{2^{L+j}}\right]$

This leads to **Definition 2:** [Assumed that all markets are open

Given $\varepsilon: [0,1] \to P$ and a feasible net trade function $x: [0,1] \to \mathbb{R}^L$ than x is walrasian if

$$\forall j \ \exists p^j \geq 0: \forall i \in [0,1] \text{ and } x_i + w_i \text{ is } \succsim_i \text{ maximal on } \left\{ v + w_i : \sum_j p^j v^j \leq 0, \ v^j = 0 \text{ for } j \right\}$$

Now we can see that CNE at the limit shows the same properties as defined in Definition 2. Consequentially we can say that a converging CNE (z_n) at the limit (z) is a WE. (This rules out the failure of Nr. ②.]

So far: We have established upper hemicontinuity!

It remains to show that every possible WE can be reached with a converging CNE (lower-hemicontinuity)
↳ I refere to chapter 7. Mas-Colell for this.