Local versions of Tarski's theorem for correspondences*

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Abstract

For a strong set order increasing (resp., strongly monotone) upper order hemicontinuous correspondence $F : A \rightrightarrows A$, where A is a complete lattice (resp., a σ-complete lattice), we provide sufficient conditions for tight fixed-point bounds for sufficiently large iterations $F^k(a^0)$, starting from any point $a^0 \in A$. Our results prove a local version of the Veinott-Zhou generalization of Tarski's theorem, as well as provide a new global version of the Tarski-Kantorovich principle for correspondences.

Keywords: monotone iterations on correspondences; Tarski's fixed-point theorem; Veinott-Zhou version of Tarski's theorem for correspondences, Tarski-Kantorovich principle for correspondences; adaptive learning.
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1 Introduction

The celebrated Tarski (1955)[1] fixed-point theorem has found numerous applications in various disciplines, especially economics. The theorem states that an increasing transformation of a complete lattice has a complete lattice of fixed points. This result has been extended to the case of monotone correspondence in the work of Veinott (1992) and Zhou (1994).[2] In the case of Tarski's original fixed point theorem, the lowest fixed point is the "limit" of the sequence of iterations starting from the lowest element of the lattice, and the highest fixed point is the "limit" of the sequence of iterations starting from the highest element of the lattice.[3] In his two recent papers, Olszewski (2021a,b) characterized the elements of the lattice which are the sharp bounds for sufficiently large iterations on an increasing function starting from *any* initial point of a domain. For example, in Olszewski (2021b), the lower bound for the sequence starting from the lowest element of this lattice is the lowest fixed point, and the upper bound for the sequence starting from the highest element of this lattice is the highest fixed point. Interestingly, the upper bound in the former case and the lower bound in the latter case must be determined as limits of the sequences obtained by iterating the limits of the sequences of *finite* iterations.

In this paper, we extend the recent result of Olszewski (2021b) on order continuous functions to the case of monotone upper order hemicontinuous correspondences. This is an important extension as fixed point mappings studied in many

[1] See also Knaster and Tarski (1928).

[2] In the Veinott-Zhou theorem, monotone means ascending in the strong set order sense. In section two of this paper, we shall refer to such correspondences as a weakly monotone. In addition, the Veinott-Zhou theorem requires that the correspondence is subcomplete and sublatticed valued.

[3] For example, see the results on constructive versions of Tarski's theorem in Cousot and Cousot (1979) and Echenique (2005). To the best of our knowledge, this paper is the first paper in the literature to study constructive and/or iterative methods to the Veinott-Zhou theorem.

economic settings are typically not single-valued. In the paper, we consider two important settings for domains for our correspondences: complete lattices and σ-complete lattices.[4] In particular, we construct fixed-point lower and upper bounds for the sequences of iterations of a weakly monotone (resp., strongly monotone) upper order hemicontinuous correspondence $F : A \rightrightarrows A$ that transforms a space A that is a complete lattice (resp., a σ-complete lattice) starting from any given initial point $a^0 \in A$.[5] In both of these important cases, we construct fixed points $\underline{a}^*$ and $\overline{a}^*$ such that sufficiently remote elements a^k of any sequence of iterations $(a^k)_{k=0}^{\infty}$ (i.e., such that $a^{k+1} \in F(a^k)$ for all k) are approximately contained between $\underline{a}^*$ and $\overline{a}^*$. As in Olszewski (2021b), the fixed points $\underline{a}^*$ and $\overline{a}^*$ are sharp or *tight*, i.e., $\underline{b} \leq \underline{a}^*$ and $\overline{a}^* \leq \overline{b}$ if fixed points $\underline{b}$ and $\overline{b}$ are such that remote *finite* iterations of F starting at a^0 are approximately located between $\underline{b}$ and $\overline{b}$.[6]

One might argue that extensions of fixed-point theorems from mappings to correspondences have worked pretty well in a variety of settings. In particular, in our setting one may consider iterations $a^{k+1} = \inf F(a^k)$, to obtain the lower fixed-point bound, and by considering iterations $a^{k+1} = \sup F(a^k)$, to obtain upper fixed-point bound. This is in fact the main idea behind the Veinott-Zhou extension of Tarski's theorem. We show, however, this idea does *not* deliver the desired extension of the main results in Olszewski (2021a,b). More specifically, it would deliver fixed-point bounds, but they are not necessarily *tight*.

[4] In economic applications, the difference between complete lattices and σ-complete lattices can be important. One such example is a fixed point problem in spaces of (Borel) measurable functions over a compact domain $A \subset \mathbb{R}^n$, where the space possesses a least and greatest element, and is endowed with a pointwise partial ordering. This space is generally only σ-complete. When this space is given an almost everywhere pointwise partial ordering, its equivalence class becomes a complete lattice (e.g., see Van Zandt (2010), lemma 5).

[5] We shall be precise in our definition of an upper order hemicontinuous correspondence in the next section of the paper.

[6] It is important to note that little is known in the existing literature even about the existence of fixed points for monotone upper order hemicontinuous correspondences that transform sigma-complete lattices. Our arguments here verify the existence, as well as provide iterative tight fixed point bounds from any initial $a^0 \in A$.

This paper is related to an important and large literature in economics that applies the Tarski-Kantorovich fixed-point principle when studying the existence of equilibria.[7] The Tarski-Kantorovich Theorem says for an order continuous transformation of a countably chain complete partially ordered set (CCPO) with least (resp., greatest) elements, the supremum (resp., infimum) of iterations from the least (resp., greatest) element of the CCPO will converge in order to the least (resp., greatest) fixed point.[8] One way of understanding the results in Olszewski (2021b) is that he shows that for order continuous functions that transform σ-complete lattice, there exists a global version of the Tarski-Kantorovich theorem, where from any element of the function's domain, elements of the fixed point set form tight bounds on sufficiently remote iterations. The second way of understanding his result is that he delivers a "local version"[9] of Tarski's theorem in the setting of an order continuous transformation of a sigma-complete lattice.

This paper extends the result of Olszewski (2021b) to the case of correspondences. In particular, we show the Tarski-Kantorovich principle holds globally in both: setting of the original Veinott-Zhou extension of Tarski's theorem for complete lattices, where the correspondence is additionally assumed to be upper order hemicontinuous, as well as in the setting of strongly monotone upper order hemicontinuous correspondences in σ-complete lattices, where the correspondence

[7] Some examples of work in economics applying the Tarski-Kantorovich Theorem include papers on supermodular games (e.g., Van Zandt (2010), Kunimoto and Yamashita (2020), Balbus et al. (2022)), models of production chains (Kikuchi et al., 2018), dynamic programming with unbounded returns (e.g, Kamihigashi (2014), Becker and Rincón-Zapatero (2021)) among many others), the existence of recursive equilibrium in dynamic stochastic growth models (e.g., Coleman (1991), Mirman et al. (2008), Datta et al. (2018)), computing Bewley models in macroeconomics (e.g., Li and Stachurski (2014), Açıkgöz (2018)).

[8] For example, see Jachymski et al. (2000), Theorem 1, and Dugundji and Granas (1982), p.15 for a discussion of the Tarski-Kantorovich theorem. See also Balbus et al. (2015) Theorem 1.

[9] That is, he characterizes the elements of the "local lattice" which are the sharp bounds for sufficiently large iterations on an order continuous transformation starting from any initial point of a sigma-complete lattice.

is additionally required to have the least and greatest element.

To obtain our results, we introduce a new notion of order continuity for monotone correspondences (i.e., "upper order hemicontinuity"). Upper order hemicontinuity of correspondences plays a critical role in obtaining our extensions of the result of Olszewski (2021b). Olszewski (2021a) shows that similar ideas to those in Olszewski (2021b) can be applied without order continuity conditions, but at the expense of requiring transfinite arguments. Of course, the fact that such transfinite constructions are required without order continuity is not surprising given the literature on constructive characterizations of Tarski's theorem where transfinite arguments appear indispensable (e.g., Cousot and Cousot (1979), among others). Similarly, in section 4 of this paper, we show the result for the upper order hemicontinuous correspondences can be extended to correspondences with discontinuities but its proof is more involved, requires transfinite constructions, and the extension is perhaps of less interest for economists (particularly in applications).

We believe our extensions in this paper are important because most iterations that we consider in economics (and perhaps in other areas of research) use correspondences. For example, players happen to have multiple best responses in games, including those of strategic complementarities, and consumers or producers happen to have multiple optimal bundles. Multiplicity appears when payoffs are not concave with respect to players' own actions, and consumers' or producers' choices. For example, a small reduction in an oligopolist's price may lower its current profits, but a larger reduction, which lowers the current profits by more, may make other firms exit or deter subsequent entry; or in a contest with multiple prizes whose values are convex, the increase in the expected value of prize induced by a small increase in effort may not be worth the cost of this additional effort, but a larger effort may result in a sufficient increase in prize to compensate for the effort cost.

Multi-valued best responses arise also naturally in mixed extensions of strategic form games. Mixed strategies can be represented as distributions over pure actions. When players pure actions are one-dimensional and corresponding distributions are ordered using the first-order stochastic dominance, the space of such mixed strategies is a complete lattice. Mixed extensions of games of strategic complementarities are also games of strategic complementarities (see Echenique (2003)). For such games, little is known[10] about interior mixed strategy Nash equilibria and our sharp fixed-point bound results make some progress.

2 Preliminaries

We start with introducing some basic definitions. A partially ordered set (or *poset*) is set A equipped with a partial order $\geq$. For $a', a \in A$, we say a' is *strictly higher* than a, and write $a' > a$, whenever $a' \geq a$ and $a' \neq a$. A poset $(A, \geq)$ is a *lattice* if for any $a, a' \in A$ the least upper bound of $\{a, a'\}$ (denoted by $a \vee a'$ or $\sup\{a, a'\}$) belongs to A and the greatest lower bound of $\{a, a'\}$ (denoted by $a \wedge a'$ or $\inf\{a, a'\}$) belongs to A. A lattice A is *complete* if there also exist $\bigvee B := \sup B \in A$ and $\bigwedge B := \inf B \in A$ for all $B \subseteq A$. A lattice A is σ-complete, whenever for any countable $B \subseteq A$, $\bigvee B$ and $\bigwedge B$ exist and belong to A. A subset $B \subseteq A$ is a *sublattice* of A if $a \vee a'$ and $a \wedge a'$, as defined in $(A, \geq)$, belong to B for all $a, a' \in B$. A sublattice B of a lattice A is a *subcomplete sublattice* if for any $C \subseteq B$ the supremum $\bigvee C$ and the infimum $\bigwedge C$, as defined in $(A, \geq)$, exists and belong to B.

We can compare subsets of A using set relations compatible with $(A, \geq)$. Let 2^A denote the set of all subsets of A. If $(A, \geq)$ is a poset, and $B, B' \in 2^A \backslash \{\varnothing\}$, we write $B' \geq^S B$ if for all $b' \in B'$, $b \in B$, $b' \geq b$. If $(A, \geq)$ is a lattice, B and

[10] With an exception of games of strict complementarities. See Echenique and Edlin (2004).

B' two nonempty subset of A, we say B' is (Veinott)-*strong set order higher* than B, denoted by $B' \geq^{SSO} B$, whenever for every $b' \in B'$ and $b \in B$, $b' \wedge b \in B$ and $b' \vee b \in B'$.

Let $F : A \rightrightarrows B$ be a nonempty-valued correspondence, where $(A, \geq)$ and $(B, \geq)$ are posets. We say F is *strongly monotone (increasing)* whenever $a' > a$ implies that $F(a') \geq^S F(a)$. Now, let $(B, \geq)$ be a lattice. We say F is *weakly monotone (increasing)* whenever $a' > a$ implies that $F(a') \geq^{SSO} F(a)$.

A sequence $(a^k)_{k=0}^{\infty}$ of elements of A is *increasing* if $a^{k+1} \geq a^k$ for each k. It is *strictly increasing* if $a^{k+1} > a^k$ for each k. *Decreasing* and *strictly decreasing* sequences can be defined in the obvious dual manner. A *monotone sequence* then is either increasing or decreasing. We say that an increasing (resp., decreasing) sequence $(a^k)_{k=0}^{\infty}$ *converges* to $a \in A$ whenever $\bigvee_{k \geq 0} a^k = a$ (resp., $\bigwedge_{k \geq 0} a^k = a$).[11] That is, when a is the supremum (resp., infimum) of the increasing (resp., decreasing) sequence.

We say that a correspondence F is *upper order hemicontinuous* whenever it satisfies the following condition: if any *monotone* sequence $(a^k)_{k=0}^{\infty}$ converges to a, then any *monotone* sequence $(b^k)_{k=0}^{\infty}$ such that $b^k \in F(a^k)$ for all k converges to some $b \in F(a)$.[12] Finally, a function $f : A \mapsto B$ is order-preserving (or increasing) on A if $a \leq a'$ implies $f(a) \leq f(a')$ for a, a' in A. A function f is *upward order continuous* (resp., *downward order continuous*) if for any increasing convergent sequence $(a^k)_{k=0}^{\infty}$ with $a^k \in A$, we have:

$$f\left(\bigvee_{k \in \mathbb{N}} a^k\right) = \bigvee_{k \in \mathbb{N}} f(a^k) \quad \left(\text{resp. } f\left(\bigwedge_{k \in \mathbb{N}} a^k\right) = \bigwedge_{k \in \mathbb{N}} f(a^k)\right).$$

The function f is *order continuous* if it is both upward and downward order continuous. Notice, if f is upward (resp., downward) order continuous, it is order

[11] Notice that in definition of convergence of monotone sequences, convergence is in *order*, i.e., one can define an order topology in which the convergence takes place.

[12] Notice that upper order hemicontinuity of a correspondence imposes "closure property" relative to only *monotone sequences*.

preserving (or increasing) function on A.[13]

3 Iterations on monotone upper order hemicontinuous correspondences

In this section, we will generalize the results in Olszewski (2021b) on the convergence of iterations of monotone (order) continuous functions to correspondences. We will state and prove our result under the following two alternative sets of assumptions:

Assumption 1 *A is a complete lattice. $F : A \rightrightarrows A$ is weakly monotone and upper order hemicontinuous. Moreover, for any $a \in A$, $F(a)$ is a subcomplete sublattice of A.*

Assumption 2 *A is a σ-complete lattice. $F : A \rightrightarrows A$ is strongly monotone and upper order hemicontinuous. Moreover, for any $a \in A$, the supremum and the infimum of $F(a)$ belongs to $F(a)$.*

Few comments are in order. First, upper order hemicontinuity turns out to be a natural condition that is easy to check in many economic applications. For example, in games of strategic complements (GSCs) where payoff functions are jointly continuous in action profiles which are elements of a complete lattice, the resulting best reply mappings for each player are upper order hemicontinuous as a consequence of well-known maximum theorems (e.g., Berge's theorem). Second, per Assumption 2, for economic situations that involve uncertainty (e.g., interim formulations of Bayesian supermodular games, stochastic supermodular games,

[13] If a function is upward (resp., downward) order continuous, it is also by definition sup (resp., inf) preserving. So our definitions here coincide with standard definitions of order continuity (e.g., Dugundji and Granas (1982), p. 15).

etc.), well-known measurable maximum theorems can be applied to show best replies have least and greatest elements that are measurable.[14] This latter fact provides situations in economics where our new results can be applied to settings where the domains of fixed-point mappings are σ-complete lattices.

For any given $a^0 \in A$, we will first define a of pair fixed points (denoted by $\underline{a}^*$ and $\overline{a}^*$) of $F : A \rightrightarrows A$ that provide tight fixed-point bounds for *all* iterations of the correspondence F.

Define the functions: $\underline{F} : A \to A$ and $\overline{F} : A \to A$, where

$$\underline{F}(a) := \bigwedge F(a) \quad \text{and} \quad \overline{F}(a) := \bigvee F(a).$$

Under Assumption 1, as well as Assumption 2, $\overline{F}$ and $\underline{F}$ are both well-defined selections[15] of F. We now present a number of lemmas. Our lemmas hold true under Assumption 1 as well as under Assumption 2. We will therefore not explicitly make these assumptions in the statements of the lemmas.

Lemma 1 *$\overline{F}$ (resp., $\underline{F}$) is downward order continuous (resp., upward order continuous).*

Let $\underline{a}^1 = \inf F(a^0) = \bigwedge F(a^0)$ and $\overline{a}^1 = \sup F(a^0) = \bigvee F(a^0)$ be the infimum and the supremum of $F(a^0)$; by induction, for $k = 1, 2, \ldots$ let $\underline{a}^{k+1}$ and $\overline{a}^{k+1}$ be the infimum of $F(\underline{a}^k)$ and supremum of $F(\overline{a}^k)$, i.e.

$$\underline{a}^{k+1} = \bigwedge F(\underline{a}^k) \quad \text{and} \quad \overline{a}^{k+1} = \bigvee F(\overline{a}^k).$$

[14] This latter fact follows from the fact that in such settings, by Topkis' theorem best replies/optimal solutions are sublatticed-valued, so via the Castaing representation of the correspondence, one an show the least and greatest elements of the best replies will be measurable. (See Castaing and Valadier (1977), Chapter 5, Van Zandt (2010), Theorem 10, or Hopenhayn and Prescott (1992), Proposition 2). This latter fact is particularly relevant when checking in economic applications our assumptions needed to studying iterative methods in the σ-complete lattice case, where we assume the fixed point correspondence under consideration must possess least and greatest elements. See for example the results that require Assumption 2 in section 3.

[15] A selection of a correspondence $F : A \rightrightarrows B$ is any function $f : A \to B$ such that $f(a) \in F(a)$ for any $a \in A$.

It will be convenient to define $\underline{a}^0$ and $\overline{a}^0$ as a^0. Let $\underline{a}^\omega = \liminf_k \underline{a}^k$ and $\overline{a}^\omega = \limsup_k \overline{a}^k$. That is,

$$\underline{a}^\omega = \lim_k \bigwedge_{l \geq k} \underline{a}^l \quad \text{and} \quad \overline{a}^\omega = \lim_k \bigvee_{l \geq k} \overline{a}^l.$$

Lemma 2 *There exists $a \in F(\underline{a}^\omega)$ such that $a \leq \underline{a}^\omega$ Similarly, there exists $a \in F(\overline{a}^\omega)$ such that $a \geq \overline{a}^\omega$.*

If $\underline{a}^\omega$ is a fixed point of F, then let $\underline{a}^* = \underline{a}^\omega$; similarly, if $\overline{a}^\omega$ is a fixed point of F, then define $\overline{a}^* = \overline{a}^\omega$. Otherwise, under Assumption 1, let $\underline{a}^{\omega+1}$ be the supremum of values of $F(\underline{a}^\omega)$ that are smaller than $\underline{a}^\omega$, and let $\overline{a}^{\omega+1}$ be the infimum of values of $F(\overline{a}^\omega)$ that are greater than $\overline{a}^\omega$; under Assumption 2, let $\underline{a}^{\omega+1}$ be any element of F$(\underline{a}^\omega)$ smaller than $\underline{a}^\omega$, and let $\overline{a}^{\omega+1}$ be any element of F$(\overline{a}^\omega)$ greater than $\overline{a}^\omega$. That is, more formally:

$$\underline{a}^{\omega+1} = \bigvee F(\underline{a}^\omega) \cap I(\underline{a}^\omega) \quad \text{and} \quad \overline{a}^{\omega+1} = \bigwedge F(\overline{a}^\omega) \cap J(\overline{a}^\omega),$$

with $I(a) := \{a' \in A : a' \leq a\}$ and $J(a) := \{a' \in A : a' \geq a\}$ under Assumption 1 and

$$\underline{a}^{\omega+1} \in F(\underline{a}^\omega) \text{ and } \underline{a}^{\omega+1} < \underline{a}^\omega \text{ and } \overline{a}^{\omega+1} \in F(\overline{a}^\omega),\ \overline{a}^{\omega+1} > \overline{a}^\omega$$

under Assumption 2.

By Lemma 2, $F(\underline{a}^\omega) \cap I(\underline{a}^\omega) \neq \emptyset$, and the same is true for $F(\overline{a}^\omega) \cap J(\overline{a}^\omega)$. Hence, by each of our two assumptions, both $\underline{a}^{\omega+1}$ and $\overline{a}^{\omega+1}$ are well defined elements of $F(\underline{a}^\omega)$ and respectively of $F(\overline{a}^\omega)$. The following lemma follows directly from the definition $\underline{a}^{\omega+1}$ of and $\overline{a}^{\omega+1}$.

Lemma 3 *(i) If $\underline{a}^\omega$ is a fixed point of F, then $\underline{a}^{\omega+1} = \underline{a}^\omega$. If $\overline{a}^\omega$ is a fixed point of F, then $\overline{a}^{\omega+1} = \overline{a}^\omega$.*

(ii) If $\underline{a}^\omega$ is not a fixed point of F, then $\underline{a}^{\omega+1} < \underline{a}^\omega$. If $\overline{a}^\omega$ is not a fixed point of F, then $\overline{a}^{\omega+1} > \overline{a}^\omega$.

We can now continue our iterations starting from $\underline{a}^{\omega}$ and $\overline{a}^{\omega}$. For any k we define the following sequences $(\underline{a}^{\omega+k})_{k=1}^{\infty}$ and $(\overline{a}^{\omega+k})_{k=1}^{\infty}$ recursively as follows

$$\underline{a}^{\omega+k+1} = \bigvee F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \quad \text{and} \quad \overline{a}^{\omega+k+1} = \bigwedge F(\overline{a}^{\omega+k}) \cap J(\overline{a}^{\omega+k}).$$

under Assumption 1 and under Assumption 2 as follows:

$$\underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k}) \text{ and } \underline{a}^{\omega+k+1} < \underline{a}^{\omega+k}; \ \overline{a}^{\omega+k+1} \in F(\overline{a}^{\omega+k}), \ \overline{a}^{\omega+k+1} > \overline{a}^{\omega+k},$$

unless $\underline{a}^{\omega+k}$ is a fixed point, in which case $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$ (and analogously for $\overline{a}^{\omega+k}$).

Indeed, it is a transfinite complement of the sequences $\underline{a}^{k}$ and respectively $\overline{a}^{k}$. This yields the following results:

Lemma 4 *The sequences $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$ and $\left(\overline{a}^{\omega+k}\right)_{k=0}^{\infty}$ are both well-defined. Moreover, if any $\underline{a}^{\omega+k_0}$ (resp., $\overline{a}^{\omega+k_0}$) is a fixed point of F, then the sequence $\left(\underline{a}^{\omega+k}\right)_{k=k_0}^{\infty}$ (resp., $\left(\overline{a}^{\omega+k}\right)_{k=k_0}^{\infty}$) is constant.*

Lemma 5 *(i) The sequence $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ is decreasing, and its limit $\underline{a}^{*}$ is a fixed point of F; (ii) the sequence $(\overline{a}^{\omega+k})_{k=0}^{\infty}$ is increasing and its limit $\overline{a}^{*}$ is a fixed point of F.*

This competes our construction of fixed points $\underline{a}^{*}$ and $\overline{a}^{*}$. It possibly appears as a puzzling feature of the construction that $\underline{a}^{k+1}$ is defined as the infimum of $F(\underline{a}^{k})$, while $\underline{a}^{\omega+k+1}$ is defined under Assumption 1 as the supremum of $F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. (A similar question concerns $\overline{a}^{k+1}$ and $\overline{a}^{\omega+k+1}$.) For the definition of $\underline{a}^{k+1}$ we had no choice. It had to be the infimum of $F(\underline{a}^{k})$ to guarantee $\underline{a}^{*}$ is indeed a lower bound for the large iterations of F. In turn, if we defined $\underline{a}^{\omega+k+1}$ as the infimum of $F(\underline{a}^{\omega+k})$, then $\underline{a}^{*}$ would still be a fixed-point lower bound for the iterations of F, but it could not be the sharp one. This is illustrated by the following example.[16]

[16] Actually, we originally defined $\underline{a}^{\omega+k+1}$ as the infimum of $F(\underline{a}^{\omega+k})$, which led us to incorrectly conjecture that we had to assume strong monotonicity of F in order to generalize the results of Olszewski (2021a,b) to correspondences.

Example. Recall Example 1 from Olszewski (2021b) in which X is a sublattice of $\mathbb{R}^2$ equipped with the coordinate-by-coordinate ordering that consists of points: (-1,0), (0,0), (1,0), (0,1), (1,1), (2,1). Olszewski defined a mapping $f : X \to X$ such that $\underline{a}^{\omega} = \liminf f^k(0,1) = (0,0)$, but $f(0,0) = (-1,0)$.

Consider a sublattice $A = X \cup I$ of $\mathbb{R}^2$, where $I = \{(y,0) : y \in [-4,-1]\}$ equipped with the coordinate-by-coordinate ordering. Extend mapping f to a correspondence $F : A \rightrightarrows A$ by letting $F(y,0) = \{(z,0) : z \in [-4,-2]\}$ for $y \in [-3,-1]$, and $F(y,0) = (-4,0)$ for $y \in [-4,-3)$.[17] That is, $F = f$ on X, and F on I is illustrated in Figure 1, in which we identified I with the interval $[-4,-1]$.

If we defined $\underline{a}^{\omega+1}$ as $\inf F(\underline{a}^{\omega}) = (-4,0)$, then we would obtain $\underline{a}^* = (-4,0)$, and this would not be a sharp fixed point bound for the sequence $(a^k)_{k=0}^{\infty}$. This sharp fixed point bound is $\underline{a}^* = (-2,0)$, and this $\underline{a}^*$ is indeed obtained if $\underline{a}^{\omega+1}$ is defined as $\bigvee F(\underline{a}^{\omega}) \cap I(\underline{a}^{\omega}) = (-2,0)$, as we do.

A similar puzzling feature may concern the definition of $\underline{a}^{\omega+k+1}$ and $\overline{a}^{\omega+k+1}$ under Assumption 2. However, for strongly monotone F is, if $\underline{a}^{\omega+k}$ is not a fixed point of F, then $\sup F(a) \leq \inf F(\underline{a}^{\omega+k})$ for all $a \in F(\underline{a}^{\omega+k})$. So, no element of $F(\underline{a}^{\omega+k})$ can be a fixed point possibly except $\inf F(\underline{a}^{\omega+k})$. Thus, by defining $\underline{a}^{\omega+k+1}$ in the way in which we do, we can be sure that we will not "jump down" over any fixed point.

We can now state and prove the following key result.

Proposition 1 *Both under Assumption 1 or under Assumption 2, the following statements hold true: (i) An increasing sequence* $\left(\bigwedge_{l\geq k} \underline{a}^l\right)_{k=0}^{\infty}$ *converges to* $\underline{a}^{\omega} \geq \underline{a}^*$*, and for any sequence* $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$ *for all* k*, we have that* $\left(\bigwedge_{l\geq k} \underline{a}^l\right)_{k=0}^{\infty} \leq a^k$*. A decreasing sequence* $\left(\bigvee_{l\geq k} \overline{a}^l\right)_{k=0}^{\infty}$ *converges to* $\overline{a}^{\omega} \leq \overline{a}^*$*, and*

[17] Note that $F(-1,0) = \{(z,0) : z \in [-4,-2]\}$, while $f(-1,0) = (-1,0)$ in Olszewski (2021b).

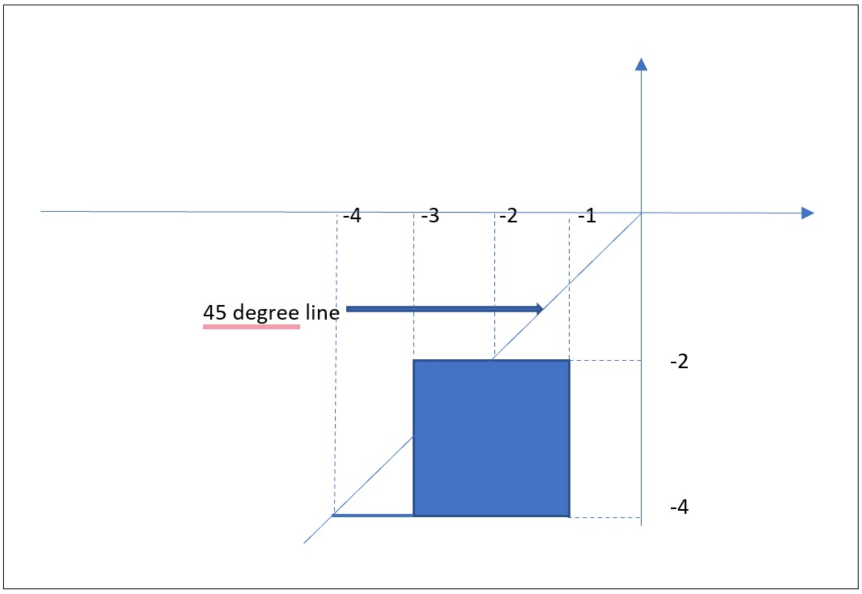

Figure 1: The graph correspondence $F \mid I$ from Example 1.

for any sequence $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$ *for all* k*, we have that* $a^k \leq \left(\bigvee_{l \geq k} \overline{a}^l \right)_{k=0}^{\infty}$.

(ii) Suppose that $\underline{b}$ *is fixed point of* F *for which there exist an increasing sequence* $(\underline{b}^k)_{k=1}^{\infty}$ *such that* $\lim_k \underline{b}^k \geq \underline{b}$*, and for any sequence* $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$*, we have that* $\underline{b}^k \leq a^k$ *for all* k*, then* $\underline{b} \leq \underline{a}^*$*. Suppose that* $\overline{b}$ *is fixed point of* F *for which there exist an decreasing sequence* $(\overline{b}^k)_{k=1}^{\infty}$ *such that* $\lim_k \overline{b}^k \leq \overline{b}$*, and for any sequence* $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$*, we have that* $a^k \leq \overline{b}^k$ *for all* k*, then* $\overline{b} \geq \overline{a}^*$*.*

Proof: We will prove the theorem for $\underline{a}^*$; the proof for $\overline{a}^*$ is analogous. Part (i) follows directly from the definitions and previous results. We will prove part (ii). Since $\underline{b}^l \leq a^l$ for all l, $\bigwedge_{l \geq k} \underline{b}^l \leq \bigwedge_{l \geq k} \underline{a}^l$; and since the sequence $(\underline{b}^k)_{k=1}^{\infty}$ is increasing, $\underline{b}^k = \bigwedge_{l \geq k} \underline{b}^l$, therefore $\lim_k \underline{b}^k \leq \lim_k \bigwedge_{l \geq k} \underline{a}^l = \underline{a}^{\omega}$. Thus, $\underline{b} \leq \underline{a}^{\omega}$. This completes the proof if $\underline{a}^* = \underline{a}^{\omega}$. If not, then $\underline{a}^{\omega}$ is not a fixed point and $\underline{b} < \underline{a}^{\omega}$. Recall

$\underline{b} \in F(\underline{b})$, and $\underline{a}^{\omega+1} \in F(\underline{a}^{\omega})$. Under Assumption 1, $\underline{b} \vee \underline{a}^{\omega+1} \in F(\underline{a}^{\omega})$ because $F(\underline{b}) \leq^{SSO} F(\underline{a}^{\omega})$. Since $\underline{b} < \underline{a}^{\omega}$, and by Lemma 5, $\underline{a}^{\omega+1} < \underline{a}^{\omega}$, we have that $\underline{b} \vee \underline{a}^{\omega+1} \leq \underline{a}^{\omega}$. This implies that $\underline{b} \vee \underline{a}^{\omega+1} \in F(\underline{a}^{\omega}) \cap I(\underline{a}^{\omega})$. Since $\underline{a}^{\omega+1}$ is the greatest element of this set, hence $\underline{b} \vee \underline{a}^{\omega+1} \leq \underline{a}^{\omega+1}$. So $\underline{b} \leq \underline{a}^{\omega+1}$. Under Assumption 2, $\underline{b} \leq \underline{a}^{\omega+1}$ because $\underline{b} \leq \underline{a}^{\omega}$, so the strong monotonicity of F implies $\underline{b} \leq \sup F(\underline{b}) \leq \inf F(\underline{a}^{\omega}) \leq \underline{a}^{\omega+1}$. We show $\underline{b} \leq \underline{a}^{\omega+k}$ for any k, and consequently $\underline{b} \leq \underline{a}^{*}$. We have proven this thesis for $k = 1$ and suppose it is the case for some k. The proof is complete if $\underline{a}^{\omega+k}$ is a fixed point, because by Lemma 4, $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$. If $\underline{a}^{\omega+k}$ is not a fixed point, $\underline{b} < \underline{a}^{\omega+k}$, and $\underline{b} \vee \underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k})$. Moreover, $\underline{b} \vee \underline{a}^{\omega+k+1} \in I(\underline{a}^{\omega+k})$, hence $\underline{b} \vee \underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. Since $\underline{a}^{\omega+k+1}$ was defined as the greatest element of this set under Assumption 1, $\underline{b} \vee \underline{a}^{\omega+k+1} \leq \underline{a}^{\omega+k+1}$, consequently $\underline{b} \leq \underline{a}^{\omega+k+1}$. Under Assumption 2, $\underline{b} \leq \underline{a}^{\omega+k+1}$ because $\underline{b} \leq \underline{a}^{\omega+k}$, so strong monotonicity of F implies that $\underline{b} \leq \sup F(\underline{b}) \leq \inf F(\underline{a}^{\omega+k}) \leq \underline{a}^{\omega+k+1}$. Thus, $\underline{b} \leq \underline{a}^{\omega+k}$ for any k, and also $\underline{b} \leq \underline{a}^{*}$. ∎

Proposition 1 captures formally the intuition that $\underline{a}^{*}$ and $\overline{a}^{*}$ are tight fixed-point bounds between which sufficiently large iterations of F are located.

Remark. In Proposition 1, we could alternatively require the sequence $(\underline{b}^{k})_{k=1}^{\infty}$ to be decreasing, and the sequence $(\overline{b}^{k})_{k=1}^{\infty}$ to be increasing. (Recall that we define no other convergent sequences.) Then, $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ would be such a decreasing sequence for $\underline{a}^{*}$, and $\left(\overline{a}^{\omega+k}\right)_{k=0}^{\infty}$ would be such an increasing sequence for $\overline{a}^{*}$. The hypothesis of Proposition 1 would still hold true, because $\underline{b}^{k} \leq \underline{a}^{k}$ for all k implies that

$$\underline{b} \leq \lim_{k} \underline{b}^{k} \leq \liminf_{k} \underline{a}^{k} = \underline{a}^{\omega}.$$

Then, the arguments analogous to those from the proof of Proposition 1 yield $\underline{b} \leq \underline{a}^{\omega+k}$ for all k, which implies that $\underline{b} \leq \underline{a}^{*}$. The proof that $\overline{b} \geq \overline{a}^{*}$ is analogous.

4 Iterations on discontinuous correspondences

One may wonder whether Proposition 1 can be extended to discontinuous weakly monotone correspondences, or whether the result for mappings from Olszewski (2021a) can be extended to correspondences. There are two possible extensions. First, one may ask if there exist tight fixed-point bounds for sequences of finite iterations starting from an arbitrary point of a lattice. The answer to this question is negative, even for mappings, as the following example shows.

Example. Let $A = [0, 1) \cup \{2 - 1/n : n = 1, 2, \ldots\} \cup \{2, 3\}$ with the lattice structure inherited from the reals. Let $f : A \to A$ be given by $f(a) = a$ for a from $[0, 1)$, $f(a) = 2 - 1/(n+1)$ for $a = 2 - 1/n$, and $f(a) = 3$ for $a = 2, 3$. Points $a < 1$ and $a = 3$ are the fixed points of mapping f. For $a^0 = 1$, the sequence of finite iterations $a^n = f^n(a^0) = 2 - 1/(n + 1)$ is increasing and converges to $a = 2$. Thus, $a = 3$ is the tight fixed-point upper bound for this sequence of iterations, and any $a < 1$ is a fixed-point lower bound. This implies that the tight fixed-point lower bound does not exist.

The result from Olszewski (2021a) can be extended, but the cost of relaxing our continuity conditions is that we must introduce transfinite sequences. In addition, we must restrict attention to iterating a correspondence that transforms a complete lattice A. More precisely, the following result can be obtained by minimally modifying the proof from Olszewski (2021a).

Let $\alpha > |A|$, where $|A|$ stands for the cardinality of A, be a cardinal number. For every $\underline{a}_0 = \overline{a}^0 = a^0 \in A$, and every weakly monotone correspondence $F : A \rightrightarrows A$, say that $(a_\beta)_{\beta<\alpha}$ is *a sequence of transfinite iterations* of F if:

$$a_\beta \in F(a_{\beta-1}) \text{ if } \beta \text{ has a predecessor } \beta - 1;$$

and

$$\bigvee_{\gamma<\beta}\bigwedge_{\gamma\leq\delta<\beta} a_\delta \leq a_\beta \leq \bigwedge_{\gamma<\beta}\bigvee_{\gamma\leq\delta<\beta} a^\delta \text{ if } \beta \text{ is a limit ordinal.}$$

In addition, distinguish two special sequences of transfinite iterations

$$\underline{a}^\beta =: \begin{cases} \inf F(a^{\beta-1}) \text{ if } \beta \text{ has a predecessor } \beta-1 \\ \bigvee_{\gamma<\beta}\bigwedge_{\gamma\leq\delta<\beta} a^\delta \text{ if } \beta \text{ is a limit ordinal.} \end{cases} \tag{1}$$

and

$$\overline{a}^\beta =: \begin{cases} \sup F(a^{\beta-1}) \text{ if } \beta \text{ has a predecessor } \beta-1 \\ \bigwedge_{\gamma<\beta}\bigvee_{\gamma\leq\delta<\beta} a^\delta \text{ if } \beta \text{ is a limit ordinal.} \end{cases} \tag{2}$$

Proposition 2 *Suppose that $(A,\leq)$ is a complete lattice, and $F : A \rightrightarrows A$ is a weakly monotone correspondence such that $F(a)$ has the smallest and the greatest element for all $a \in A$. Let $\alpha > |A|$ be a regular cardinal number.*[18] *Then, for any $a_0 = a^0 \in A$, there exist $\underline{\beta}, \overline{\beta} < \alpha$ such that $\underline{a}_\beta = \underline{a}_{\underline{\beta}}$ for all $\underline{\beta} \leq \beta < \alpha$, and $\overline{a}^\beta = \overline{a}^{\overline{\beta}}$ for all $\overline{\beta} \leq \beta < \alpha$. In particular, $\underline{a}_{\underline{\beta}}$ and $\overline{a}^{\overline{\beta}}$ are fixed points of F.*

Moreover, $\underline{a}_{\underline{\beta}}$ is the greatest fixed point $\underline{a}$ of F with the property that $\underline{a} \leq a_\beta$ for sufficiently large $\beta < \alpha$ and for all sequences of transfinite iterations $(a_\beta)_{\beta<\alpha}$, and $\overline{a}^{\overline{\beta}}$ and the smallest fixed point $\overline{a}$ of F with the property that $a^\beta \leq \overline{a}$ for sufficiently large $\beta < \alpha$ and for all sequences of transfinite iterations $(a_\beta)_{\beta<\alpha}$.

It is possible to obtain a somewhat stronger result than Proposition 2, which requires a somewhat more involved proof. However, since transfinite sequences are unlikely to be of interest for economists, we will not present and discuss this result in this paper.

[18] A regular cardinal number α is defined by the following property: No set of cardinality α can be represented as the union of a family of subsets such that each subset from the family has a cardinality smaller than α, and the family itself is of a cardinality smaller than α.

5 Appendix

Proof of Lemma 1. Assume 1. Since F is weakly monotone, $\overline{F}$ and $\underline{F}$ are both weakly increasing. Indeed, if $a' < a''$ then $\underline{F}(a') \wedge \underline{F}(a'') \in F(a')$. As a result,

$$\underline{F}(a') \leq \underline{F}(a') \wedge \underline{F}(a'').$$

Hence $\underline{F}(a') \wedge \underline{F}(a'') = \underline{F}(a')$ and consequently $\underline{F}(a') \leq \underline{F}(a'')$. Similarly we prove the monotonicity of $\overline{F}$. The argument under Assumption 2 is straightforward. We prove the upward continuity of $\underline{F}$. Its proof is the same under Assumption 1 or Assumption 2. Let $(a^k)_{k=1}^{\infty}$ be an increasing sequence in A such that $a = \bigvee_{k\in\mathbb{N}} a^k$. Let $b^k := \underline{F}(a^k)$. We conclude that $b^k \in F(a^k)$ for any $k \in \mathbb{N}$, and b^k is increasing. Let $b := \bigvee b^k$. Since b^k belongs to $F(a^k)$ and the sequence $(b^k)_{k=1}^{\infty}$ is increasing, b belongs to $F(a)$ by upper hemicontinuity of F. Hence, $\underline{F}\,(a) \leq b$. On the other hand, $\underline{F}(a) \geq b^k$ for any k. It follows from the definition of b^k and the monotonicity of $\underline{F}$. Hence $b \leq \underline{F}(a)$. Together with $\underline{F}\,(a) \leq b$, we have $b = \underline{F}(a)$, and hence the upward continuity. We omit a similar proof that $\overline{F}$ is downward continuous. ■

Proof of Lemma 2. We will prove the claim for $\underline{a}^{\omega}$; the proof for $\overline{a}^{\omega}$ is analogous. The sequence $\left(\bigwedge_{l\geq k} \underline{a}^l\right)_{k=0}^{\infty}$ is an increasing sequence whose supremum is $\underline{a}^{\omega}$. Let $b^k = \underline{F}\left(\bigwedge_{l\geq k} \underline{a}^l\right)$. By Lemma 1, we know $\underline{F}$ is an increasing function, hence b^k is increasing as well. In addition, since from Lemma 1, $\underline{F}$ is upward continuous, we conclude that

$$a := \bigvee_{k\in\mathbb{N}} b^k = \underline{F}(\underline{a}^{\omega}) \in F(\underline{a}^{\omega}).$$

To finish the proof, we must show that $a \leq \underline{a}^{\omega}$. Since $\bigwedge_{l\geq k} \underline{a}^l \leq \underline{a}^l$ for all $l \geq k$, we have that $b^k \leq \underline{a}^{l+1}$ for all $l \geq k$ by the monotonicity of $\underline{F}$ and the definition of

$\underline{a}^{l+1}$. So, $b^k \leq \bigwedge_{l \geq k+1} \underline{a}^l \leq \underline{a}^\omega$, which gives that $a = \lim_k b^k \leq \underline{a}^\omega$. ■

Proof of Lemma 4. We will show the hypothesis for the sequence $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$; the proof for the sequence $\left(\overline{a}^{\omega+k}\right)_{k=0}^{\infty}$ is analogous. That is, we will show by induction that $\underline{a}^{\omega+k+1}$ is well-defined for any $k \geq 0$, and if $\underline{a}^{\omega+k}$ is a fixed point, then $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$.

For $k = 0$, this holds true by Lemma 3. Suppose that $\underline{a}^{\omega+k}$ is a fixed point of F for some $k > 0$. Then $\underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \neq \emptyset$, so $\underline{a}^{\omega+k+1}$ is well-defined by Assumption 1. In addition, $\underline{a}^{\omega+k}$ must be $\bigvee F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. Hence $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$ by the definition of $\underline{a}^{\omega+k+1}$. Under Assumption 2, $\underline{a}^{\omega+k+1}$ is defined as $\underline{a}^{\omega+k}$.

Suppose now that $\underline{a}^{\omega+k}$ is not a fixed point of F. By induction hypothesis $\underline{a}^{\omega+k-1}$ is neither a fixed point of F, because then $\underline{a}^{\omega+k} = \underline{a}^{\omega+k-1}$ would also be a fixed point. Hence $\underline{a}^{\omega+k-1} > \underline{a}^{\omega+k}$. By Assumption 1, $F(\underline{a}^{\omega+k}) \leq^{SSO} F(\underline{a}^{\omega+k-1})$. Take any $a' \in F(\underline{a}^{\omega+k})$. Since $\underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k-1})$, it must be that $a' \wedge \underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k})$ and obviously $a' \wedge \underline{a}^{\omega+k} \in I(\underline{a}^{\omega+k})$. As a result $F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \neq \emptyset$. Thus, $\underline{a}^{\omega+k+1}$ is well-defined. Under Assumption 2, $\underline{a}^{\omega+k+1}$ is defined as an arbitrary element of $F(\underline{a}^{\omega+k})$ smaller than $\underline{a}^{\omega+k}$. Such an element exists by Lemma 2. So, $\underline{a}^{\omega+k+1}$ is well-defined. ■

Proof of Lemma 5. We will prove this lemma for $\underline{a}^*$; the proof for $\overline{a}^*$ is analogous. By construction and Lemma 4, $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$ is a well-defined and decreasing sequence. Let $\underline{a}^*$ be its limit. Since $\underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k})$ for all k, by taking a limit as $k \to \infty$ and applying the upper hemicontinuity of F we have $\underline{a}^* \in F(\underline{a}^*)$.

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