A Qualitative Theory of Large Games with Strategic Complementarities*

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Abstract

We study the existence and computation of equilibrium in large games with strategic complementarities. Importantly, our class of games allows to analyze economic problems without any aggregative structure. Using monotone operators (in stochastic dominance orders) defined on the space of distributions, we first prove existence of the greatest and least distributional Nash equilibrium under different set of assumptions than those in the existing literature. In addition, we provide results on computable monotone distributional equilibrium comparative statics relative to ordered perturbations of the parameters of our games that were previously only available for games with aggregative structure. We conclude by discussing the question of equilibrium uniqueness, as well as presenting applications of our results to models of social distance, large stopping games, keeping up with the Joneses but also to a general class of linear non-atomic games.
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1 Introduction

Beginning with the seminal work of Schmeidler (1973) and Mas-Colell (1984), the study of equilibrium in games with a continuum of players has been the focus of a great deal

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of work in economics.[1] In a separate, yet related set of recent papers, researchers have turned their attention to the question of the existence of equilibrium comparative statics in large games with strategic complementarities (LGSC) between players actions or ”types”. For example, for studies of existence and equilibrium comparative statics in large games of strategic complementarities, see Guesnerie and Jara-Moroni (2011), or Acemoglu and Jensen (2010, 2013). In this latter strand of work, the focus has been primarily on nonatomic *aggregative* games, in which each individual player plays against an aggregate (or, perhaps, a collection of aggregates[2]) that are constructed from the distribution of strategies that summarize the actions of all the other players in the game. In such a game questions concerning the equilibrium existence or computation of equilibrium, as well as questions pertaining to equilibrium comparative statics, are often greatly simplified.[3]

Although many important economic problems pertaining to the existence of equilibrium comparative statics can be analyzed within the context of a large aggregative game, many other relevant cannot. As we shall make clear in this paper, there are many important classes of large games where the number of agents is a continuum, and the interaction between the players inherently depends on the *entire distribution* of players actions or personal characteristics. Such games cannot be mapped into the standard existing toolkit.[4] In such games, none of the existing results in the literature provide answers either for existence, computation, or equilibrium comparative statics of equilibria in such large games.[5] Finally, the importance of studying large (nonaggregative) games with “traits” or diverse

[1]For examples of recent work, see the series of papers by Khan, Rath, Sun, and Yu (2013), Khan, Rath, Yu, and Zhang (2013), and the references therewithin.

[2]As we shall point out though later in the paper, the results in Acemoglu and Jensen (2010, 2013) are also very particular, and are difficult to apply in classes of aggregate games where aggregators are not one-dimensional. They also do not apply to large nonaggregative games.

[3]Actually, in many aggregative games with strategic complementarities, the existence and equilibrium comparative statics questions can be obtained using standard tools from the theory of game of strategic complementarities with *finite* number of players (e.g., as in the seminal work of Topkis (1979), Vives (1990), and Milgrom and Roberts (1990)). Such tools fail in more general classes of large games with strategic complementarities (LGSC). See Balbus, Dziewulski, Reffett, and Woźny (2013) for some discussion.

[4]Actually, even in aggregative games examples can arise where agent’s action may be determined by an aggregate of actions/characteristics of other agents in the direct (local) neighbourhood, where there is a continuum of such neighbourhoods as the number of agents is infinite. In such an aggregative game, even if the agents respond to an aggregate summarizing the behavioral characteristics of those playing the game in the direct neighbourhoods, as the number of such neighbourhoods may be large, the resulting game equilibrium depends on a continuum of aggregates. Hence, it is not clear how simplifying the game into an aggregative one is useful.

[5]Bear in mind, all of our results apply to large aggregative games, as such games are particular cases that fall into our class as well. Further, as will be clear by the simple example we present in the next section of the paper, none of the existence or equilibrium distributional comparative statics results in the existing literature (including those on Acemoglu and Jensen (2010, 2013), Guesnerie and Jara-Moroni (2011), and Yang and Qi (2013)) apply in very simple large *non*-aggregative games.

personal characteristics has been recently been made clear in a series of important papers including Khan, Rath, Yu, and Zhang (2013) and Khan, Rath, Sun, and Yu (2013). In these papers, the authors stress the number of traits as being an important not only to verify conditions of equilibrium existence, but also to use large games in studying actual economic problems.

In this paper, we ask a number of new important questions. First, can the methods used to verify the existence of equilibrium for GSC be extended to large games with a continuum of players without imposing any aggregative structure? If this is possible, can one develop conditions for equilibrium comparative statics results, and what will be the nature of such equilibrium comparative statics? Finally, if such equilibrium comparative statics results can be identified, can we develop a theory of computable equilibrium comparative statics for these games? In this paper, although we provide an affirmative answer to these questions, what is particularly interesting is that the methods we use to analyze LGSC turns out to be very different than those available for GSC with a finite number of players.

More specifically, we first provide sufficient conditions for distributional equilibrium to exist in LGSC, as well as provide a sharp order-theoretic characterizations of the set of equilibria. This characterization then turns out to play a central role in developing results on both equilibrium comparative statics, as well as the basis for some new results on successive approximations for computing extremal equilibria. Interestingly, in developing our results, we are able to relax many of the key continuity conditions on payoffs per existence of nonempty best reply maps found in the existing literature using more standard topological arguments (e.g., continuity conditions in weak topologies) by appealing to order-continuity conditions.

Equally as important, under our sufficient conditions for existence, we are also able to provide a powerful set of *constructive methods* computing extremal equilibrium as sigma-order limits of simple successive approximation schemes from the least (resp, the greatest) elements of spaces of candidate distributional equilibrium objects. We know of no similar results in the existing literature on the computation of extremal equilibria in LGSC. This latter result proves central in identifying sufficient conditions for the existence of *computable equilibrium comparative statics* relative to ordered perturbations of the parameters of a game. Finally, we present new sufficient conditions for existence of symmetric equilibrium in LGSC, as well as conditions where equilibrium is unique, both of which prove useful in applications.

An important point this paper makes is that although the tools used for a study of equilibrium for a standard GSC versus a LGSC are similar in a broad methodological sense, the detailed results are quite different in large games. In a LGSC, one can only provide conditions under which the set of distributional equilibria has the greatest and the least element, but the set of pure strategy equilibrium does *not* in general form a complete lattice[6]. More specifically, what we are able to show is if the best responses maps for each player are *functions*, then the set of distributional equilibria is a chain complete partially ordered set[7]. Such key differences in the structure of the set of equilibrium between standard GSC versus LGSC arise because of the particular infinite dimensional structure of large games.

Additionally, in LGSC, measurability issues characteristic for infinite-dimensional spaces mentioned above appear, and create significant impediments to proving even *existence,* let alone the presence of equilibrium monotone comparative statics. One interesting idea to the measurability problem was recently proposed in Yang and Qi (2013), who restrict their attention to monotone equilibria in types/characteristics in a class of semi-anonymous large supermodular games, when the types are chained.[8] As proposed in their paper, the results known for small GSC can be extended to LGSC for this restricted class of supermodular games assuming one focuses on equilibria in which players with higher types/characteristics use higher actions. As compared to our results, we do not impose *any* such restrictions on equilibria, and we first study the question of measurable "non monotone equilibia". In fact, our examples show that it need not be the case. We then turn our attention to the case of monotone equilibia. An immediate implication of our generalization is that techniques applied and results obtained in our and Yang and Qi (2013) paper differ along numerous important dimensions.

The remainder of the paper is organized as follows. In the next section, we begin with a motivating example of Akerlof's model of social distance. This simple model provides an example of an economic model where our results apply, but are not covered by any of the results in the existing literature. In section 3, we then provide our main results on the existence of distributional equilibrium, as well as our results on (computable) equilibrium comparative statics. We also discuss the relationship between our results, and those in the existing literature. The next section of the paper discusses conditions for the uniqueness of distributional equilibria. In section 5, we not only show how our

[6]A complete lattice is a partially ordered set X, which any subset has supremum and infimum in X.

[7]A partially ordered set (henceforth, a poset) X is chain complete (henceforth, a CPO), if for arbitrary chain $C \subset X$, $\sup(C)$ and $\inf(C)$ are each in X. If this completeness condition holds only for countable chains, we say that X is countably chain complete (henceforth, a CCPO).

[8]As we discuss in a related paper (see Balbus, Reffett, and Woźny, 2014c), the results of Yang and Qi (2013) require stronger conditions on the types/characteristics space than imposed in their paper. See the counterexample to a key lemma in the paper in section 2 of Balbus, Reffett, and Woźny (2014c).

results apply to the example of Akerlof's social distance model, but also provide three additional examples (namely, linear non-atomic games, large optimal stopping games or large keeping up with the Joneses games). Finally, in section 8, we define all the requisite mathematical terminology used in the paper, discuss all the auxiliary fixed point results we use to prove the main theorems, as well as include all our proofs.

2 Motivating example

To motivate the nature of the results in the paper, we begin by considering distributional equilibrium in a version of the *social distance* model described originally in Akerlof (1997). The model studies how agents interact to determine the distribution of social rank/status over a large number of heterogeneous individuals with individual-specific identities. Suppose there is a continuum of agents distributed over a compact interval $X \subset \mathbb{R}$, where a *location* of an individual agent is denoted by $x \in X$. Further, let $Y \subset \mathbb{R}$ be the set of all possible relative positions in society (e.g., social status/ranks), where Y is also compact and concave. In addition, each individual is also characterized by an *identity,* also an element $y \in Y$, which determines the social status/rank to which a given agent actually aspires. We shall refer to the location/identity pair (x, y) as the *characteristic* of a given agent.

Suppose the distribution of characteristics across the population is determined by a probability measure λ defined over the Borel-algebra of $X \times Y$. As a primitive of the game, we assume that every agent knows her own characteristic (x, y), as well as the distribution of characteristics across the population. In this game, we study how agents determine their optimal individual choice of social status $a \in Y$, given their location $x \in X$, and given their identity $y \in Y$.

In the model, a typical individual's payoff is influenced by two things. First, every agent aims to attain a status/rank that is a close in proximity to her true identity y. Therefore, when constructing payoffs, whenever the actual social status of the person differs from her identity y, the person will suffer a penalty per not matching to her true identity. Moreover, the further away their actual status is relative to their true identity, the more disutility the agent receives.

Second, the social status an individual attains in equilibrium will be influenced by their interactions with other agents in the game. In particular, we assume when agents interact in the game (e.g, when they match, they both suffer a disutility whenever their social status levels differ, and again this disutility increases the larger is the realized distance between their two status levels. This incorporates into the payoff structure of the game the notion that peer pressure (or peer effects) or conformism, which influences how individual agents will interact in the equilibrium of the game.

To construct payoffs, let u, $v : \mathbb{R} \to \mathbb{R}$ be a pair of continuous and decreasing functions. In addition, assume v is concave. Consider an agent characterized by a characteristic pair (x, y), who chooses a social status $a \in Y$. In particular, whenever an agent meets another individual with a social status $a' \in Y$, her utility is given by:

$$u(|a - y|) + v(|a - a'|).$$

As both functions u and v are continuous and decreasing, the objective of every player is to choose an action as close to their identity y and the other player's identity a'. Moreover, given concavity of the payoffs in v, the further away the agent is from the social status of the other agent, the steeper are the changes in the disutility.

In order to make our notation compact, denote by $\Lambda := X \times Y$, with typical elements $\alpha = (x, y)$. Suppose that the frequency of interactions of the agent with other individuals is governed by a probability measure μ_α, defined over the product Borel-algebra of Λ. Therefore, for any set $U = U_x \times Y_y$, where $U_x \subset X$ and $U_y \subset Y$, value $\mu_\alpha(U)$ is the probability of encountering an agent with a characteristic $(x', y') \in U_x \times U_y$. We require the measure μ_α to depend on $\alpha = (x, y)$, as both the location x of agents, as well as their aspirations y, are allowed to determine the structure of their social interactions with other members of the population in equilibrium. For example, it might be the case that the greater the distance between the locations of the two agents, the less frequent will be their interaction. Along similar lines, it might also be that given the distance between their chosen and true identity, agents with more similar aspirations are more likely to meet. Moreover, assume for the moment that μ_α is weakly continuous with respect to α.

It is crucial in this model that agents observe the decisions of others as they choose their own social status. We summarize the behavior of the agents by a probability measure τ defined over the Borel-algebra of $\Lambda \times Y$ with the marginal distribution λ on Λ, where τ is a probability distribution of player characteristics/social rank pairs (α, a). Hence, for any Borel sets $U \subset \Lambda$, $A \subset Y$, the mapping $\tau(U \times A)$ denotes the measure of agents with $\alpha \in \Lambda$ and social rank $a \in A$.

Given the notation, we define the decision problem faced by a typical agent in the game. The objective of a player is to choose her social status $a \in Y$ to maximize her ex-ante payoff given by

$$r(\alpha,\tau,a) := u(|a-y|) + \int_{\Lambda}\int_{Y} v(|a-a'|)\tau(da'|\alpha')\mu_{\alpha}(d\alpha'),$$

where $\alpha = (x,y)$ and $\tau(\cdot|\alpha')$ is the distribution of actions of the other players in the population conditional on the $a' \in Y$. Therefore, the payoff of an agent is an expected sum of the utilities that she receives from individual interaction with other agents. According to the definition of the payoff, the social status of an individual cannot be contingent on the social statuses of other agents, but is chosen ex-ante before any interaction with other players occurs.[9]

The equilibrium of the game is then a probability measure τ^* over $\Lambda \times Y$, with the marginal distribution over Λ equal to λ, such that

$$\tau^*\left(\{(\alpha,a) \in \Lambda \times Y \mid r(\alpha,\tau^*,a) \geq r(\alpha,\tau^*,a'),\ \text{for all } a' \in Y\}\right) = 1.$$

Loosely speaking, we require that in equilibrium almost every agent is playing a best response to the equilibrium distribution τ^*, measured with respect to the same probability distribution τ^*.

Few observations about the structure of this game. First, this is a LGSC. That is, as we shall show in Section 5.1 of the paper, when the distribution of actions of players shifts with respect to the first order stochastic dominance, the optimal choices of each of the players also increase. This raises the question if the methods for a "standard" GSC can be extended to games where the number of players is a continuum. In particular, aside from existence issues, does the set of equilibria have some order theoretic structure? If so, what is the structure of equilibrium comparative statics with respect to the game's deep parameters, and can one obtain conditions where these equilibrium comparative statics are "computable" (i.e., be approximated using iterative methods)?

Per known results on this class of LGSC, it is quite easy to show that none of the existing results can be applied to this game. First of all, in our framework the agents need to observe the *entire* distribution of actions of other players in order to even write down their payoffs. In particular, this means that the externality in the game cannot be summarized by some aggregate of actions of players as in Acemoglu and Jensen (2013) (i.e., this is not an *aggregative* LGSC). Moreover, since the the set of probability distributions

[9] One interpretation the game is as follows: with probability $\mu_{\alpha}(U)$ a player α meets some player in U. Then, for such an agent in U, say α', she chooses action a' with probability $\tau(a'|\alpha')$. Given this, agent α calculates his expected payoff. In this sense, we can think of this as an interim game, but we do not analyze ex-post "matching" of agents.

is not a lattice, the externality in the game cannot be formulated as a "lattice externality" as in games discussed in, for example, Guesnerie and Jara-Moroni (2011).

Second, as the characteristic space is not chained, it is not clear how to extend any methods of Yang and Qi (2013). Further, there is no reason (per se) one needs to search for monotone equilibria (as opposed to simply measurable equilibria). Actually, it is crucial in our framework that the space of characterisics/traits is a subset of the two dimensional real space, as both the location and the true identity of an agent affect her decision. As it was stressed by Akerlof (1997) or Akerlof and Kranton (2000), both the *neighborhood* effect (the players location) and the *family background effect* (the players true identity) are the two key factors in determining social interaction and the distribution of equilibrium social distance. However, this means that the space of player characteristics is not a totally ordered set (or a chain) and so the developments of Yang and Qi (2013) (see also Balbus, Reffett, and Woźny (2014c)) cannot be applied in the above framework.

Even though none of the existing results on LGSC can be directly applied to our problem, it is possible to show that the key properties of games with strategic complementarities are preserved in the above game. In the following section we develop new tools which will allow to analyze a general class of non-atomic games with strategic complementarities. Then, we return to the above example in order to establish the main properties of this game.

3 Distributional equilibria in large games

3.1 Main results

Let Λ be a compact, perfect[10] Hausdorff set of players characteristics. Endow Λ with Borel σ-field $\mathcal{L}$, as well as a regular probability measure λ vanishing at each singleton[11]. Existence of such measure follows from Theorem 12.21 in Aliprantis and Border (2006).[12] Let $A \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) be an action set endowed with natural product order and Euclidean topology, and $\tilde{A} : \Lambda \to 2^A$ be a correspondence that assigns subsets of feasible actions to

[10]I.e., where all singletons are open sets. Or in other words that Λ has no isolated points.

[11]Observe that it does not imply that it is a nonatomic measure (see Lemma 12.18 in Aliprantis and Border (2006)).

[12]Note, we can not relax assumption that Λ is perfect. For example let Λ be a set of ordinals $[0, \omega_1]$ where ω_1 is the first uncountable number. Let $\mathcal{L}$ be an order topology and assume that λ vanishes at each singleton. Then all closed sets not containing ω_1 are countable. Hence if we take any neighbourhood of ω_1, then its complement is λ-null set. Hence any neighbourhood of ω_1 has a full measure, while $\lambda(\{\omega_1\}) = 0$. Each probability measure vanishing at singleton is not regular. Observe also that all successors are isolated points.

each player. Endow $\Lambda \times A$ with an order $\geq_p$. Assume this order satisfies an implication[13] $(\alpha', a') \geq_p (\alpha, a) \Rightarrow a' \geq a$.

By $\mathcal{A}$, we denote a family of Borel subsets on A, and by $\mathcal{M}(\Lambda \times A)$ we denote the set of *normal* probability distributions[14] on $\mathcal{L} \otimes \mathcal{A}$ with marginal distribution λ on Λ. By $\mathcal{R}$ we denote set of all regular distributions on $\mathcal{L} \otimes \mathcal{A}$. Endow $\mathcal{R}$ with partial order of *first order stochastic dominance*, and denote this partial order by[15] $\succeq_P$. Indeed, by Lemma 8.1, the space $(\mathcal{R}, \succeq_P)$ is a partially ordered set.[16]

Let

$$\mathcal{D} := \{\tau \in \mathcal{M}(\Lambda \times A) : \tau(Gr(\tilde{A})) = 1\},$$

denote the set of feasible distributions endowed with the weak topology, where $Gr(\tilde{A})$ is a graph of the correspondence $\tilde{A}$. The payoff function is denoted by $r : \Lambda \times A \times \mathcal{D} \to \mathbb{R}$.

The game is defined by the tuple $\Gamma := ((\Lambda, \mathcal{L}, \lambda), A, \tilde{A}, r)$. A *distributional equilibrium* of Γ is defined as follows[17]:

Definition 3.1 (Distributional equilibrium) *A distributional equilibrium of the game Γ is a probability measure $\tau^* \in \mathcal{D}$ such that the marginal distribution of τ^* on Λ is λ and*

$$\tau^*(\{(\alpha, a) : r(\alpha, a, \tau^*) \geq r(\alpha, a', \tau^*)\, \forall a' \in \tilde{A}(\alpha)\}) = 1.$$

Our initial interest is in the question of sufficient conditions for the existence of a distributional equilibrium for our LGSC. In order to proceed, we make the following set of assumptions:

[13] Clearly, if Λ is ordered set, this implication is satisfied if $\geq_p$ is a product order. But critically, for comparison with Yang and Qi (2013), this implication may also be satisfied if this order is defined as follows:

$$(\alpha', a') \geq_p (\alpha, a) \Leftrightarrow (\alpha = \alpha') \text{ and } (a' \geq a)$$

That is, in the case where the set of players has no nontrivial order. This allows us to study the case of distributional equilibria where pure strategies are not necessarily monotone in characteristics.

[14] Recall, a normal measure is a measure that is both inner and outer regular.

[15] That is, we say that $\tau' \succeq_P \tau$ iff $\int f(a, \alpha)\tau'(d(a, \alpha)) \geq \int f(a, \alpha)\tau(d(a, \alpha))$ for every increasing, bounded and measurable $f : \Lambda \times A \to \mathbb{R}_+$.

[16] The set of all measurable isotone functions on $\Lambda \times A$ separates the points. Take any (α_1, a_1) and (α_2, a_2). Assume a_1 is no greater then a_2. Define $f(\alpha, a) = \chi_{\{a' \in A : a \geq a_2\}}(a)$. Clearly f is isotone under assumption on $\geq_p$. Obviously it is $\mathcal{L} \otimes \mathcal{A}$ measurable, since $0 = f(\alpha_1, a_1) \neq f(\alpha_2, a_2) = 1$.

[17] Observe that $(\Lambda, \mathcal{L}, \lambda)$ is a space of agents characteristics, while Mas-Colell (1984) in his seminal paper characterizes players by their payoff functions only. We can embed Mas-Colell model into ours using the following construction: $r(\alpha, a, \tau) := \alpha(a, \tau_A)$, where $\lambda \in \Lambda$, τ_A is τ marginal on A and $\Lambda := \{\alpha \mid \Lambda \times \Delta_A \to \mathbb{R}, \alpha \text{ is continuous}\}$. Alternatively, we can interpret α as a fixed trait, e.g. α is agent's income, $a \in A(\alpha) = [0, \alpha]$ is a consumption level and payoff function is of the form: $r(\alpha, a, \tau) = u(a, \tau_A) + v(\alpha - a, \tau_A)$, where u and v are some increasing functions. See also Khan, Rath, Sun, and Yu (2013); Khan, Rath, Yu, and Zhang (2013), who analyze games with traits.

Assumption 3.1 *Assume that:*

(i) correspondence $\tilde{A}$ is complete sublattice valued[18] *and weakly $\mathcal{L}$-measurable,*

(ii) for λ-a.e. player α function r is quasi-supermodular[19] *in a, and satisfies a single crossing property*[20] *in (a, τ),*

(iii) for any $\tau \in \mathcal{D}$, $(\alpha, a) \to r(\alpha, a, \tau)$ is a Carathéodory function, i.e. is continuous on A and $\mathcal{L}$-measurable.

To proceed, we define a best response correspondence and state two lemmas concerning its structure. The best response correspondence is given by:

$$m(\alpha, \tau) := \arg \max_{a \in \tilde{A}(\alpha)} r(\alpha, a, \tau),$$

We denote its greatest element $\overline{m}(\alpha, \tau) := \bigvee m(\alpha, \tau)$, as well as its least element as $\underline{m}(\alpha, \tau) := \bigwedge m(\alpha, \tau)$.[21] We shall define the *upper distributional* operator as follows: $\forall G \in \Lambda \otimes A$, define $\overline{T} : \mathcal{R} \to \mathcal{D}$, where

$$\overline{T}(\tau)(G) := \lambda(\{\alpha : (\alpha, \overline{m}(\alpha, \tau)) \in G\}).$$

Similarly, using the least best reply $\underline{m}(\alpha, \tau)$, we can analogously define the *lower distributional* operator $\underline{T}$.

We now state our first lemma, which characterizes the monotonicity properties of the operators $\overline{T}$ and $\underline{T}$ on $(\mathcal{R}, \succeq_P)$.

[18] A complete sublattice is a sublattice $S \subset X$, such that for any subset $C \subset S$, $\vee C$ and $\wedge C$ belong to S, where sup/inf operations are taken with respect to the induced order of X. Also, since $\tilde{A}(\alpha, s) \subset A \subset \mathbb{R}^m$ is a complete sublattice, correspondence $\tilde{A}$ is compact-valued.

[19] That is, payoff r satisfies:

$$\begin{aligned} r(\alpha, a, \tau) > r(\alpha, a \wedge a', \tau) \quad &\Rightarrow \quad r(\alpha, a \vee a', \tau) > r(\alpha, a', \tau), \\ r(\alpha, a, \tau) \geq r(\alpha, a \wedge a', \tau) \quad &\Rightarrow \quad r(\alpha, a \vee a', \tau) \geq r(\alpha, a', \tau), \end{aligned}$$

for any τ, and all $a', a \in A$.

[20] That is, if for any $a' \geq a'$ and $\tau' \succeq_P \tau$ we have:

$$\begin{aligned} r(\alpha, a', \tau) > r(\alpha, a, \tau) \quad &\Rightarrow \quad r(\alpha, a', \tau') > r(\alpha, a, \tau'), \\ r(\alpha, a', \tau) \geq r(\alpha, a, \tau) \quad &\Rightarrow \quad r(\alpha, a', \tau') \geq r(\alpha, a, \tau'). \end{aligned}$$

Here notice, we do not require complementarities between (a, α) as in Yang and Qi (2013) (i.e., strategies monotone in characteristics).

[21] For the remainder of the paper, we shall use the symbols $\bigvee$ (resp., $\bigwedge$) to denote the sup (resp., inf) of an underlying set, where these operations are computed relative to the partial order imposed on the set. We shall often refer to these selections collectively as to ”extremal” selections. Extremal elements of complete sublattices are least/greatest elements.

Lemma 3.1 *Operators $\overline{T}$ and $\underline{T}$ are each well defined, map $\mathcal{R}$ into $\mathcal{D}$, and are each $\succeq_P$-increasing.*

Next, we state the order structure of the partially ordered set $(\mathcal{R}, \succeq_P)$ [22]. In particular, by Proposition 8.1, we immediately have the following lemma.

Lemma 3.2 *$(\mathcal{R}, \succeq_P)$ is a chain complete poset.*

With these two lemmas in place, we are ready to state the main theorem of the paper. Let $\overline{\delta}, \underline{\delta}$ denote the greatest and least elements of $\mathcal{D}$. Then, we have the following:

Theorem 3.1 *Under Assumption 3.1*

(i) there exist the greatest ($\overline{\tau}^*$) and the least ($\underline{\tau}^*$) distributional equilibrium of Γ,

(ii) if $\underline{T} = \overline{T}$ (i.e., the distributional best reply is a function), then the set of distributional equilibria is a chain complete poset.[23]

Additionally, if we assume that for any countable chain $\{\tau_n\} \subset \mathcal{D}$, $\tau_n \to \tau$, we have $r(\alpha, a, \tau_n, s) \to r(\alpha, a, \tau, s)$, then:

(iii) $\overline{\tau}^* = \lim\limits_{n\to\infty} \overline{T}^n(\overline{\delta})$ and $\underline{\tau}^* = \lim\limits_{n\to\infty} \underline{T}^n(\underline{\delta})$.

We next turn to the question of monotone equilibrium distributional comparative statics, and prove an important corollary to Theorem 3.1. Our concern is not only to prove the existence of such monotone equilibrium comparative statics, but to give sufficient conditions that guarantee they are *computable*. It bears mentioning at this stage that in the appendix of the paper, we prove a generalization of Markowsky's Theorem (see appendix, Theorem 8.4). This generalization plays a key role in the sequel, as it characterizes the order structure of the set of fixed points of sup (resp, inf)-preserving maps in countably chain complete partially ordered sets.

We now are ready to give sufficient conditions for the existence of monotone equilibrium comparative statics for our LGSC. Consider a parameterized version of our game $\Gamma(s) := ((\Lambda, \mathcal{A}, \lambda), A(\cdot, s), \tilde{A}(\cdot, s), r(\cdot, s))$ for each $s \in S$, where the space of parameters $(S, \leq_S)$ is a partially ordered set. We make the following natural complementarity assumptions on the parameterized game $\Gamma(s)$:

[22] Our theorem generalizes the well known result that every set of probability distributions with a compact support on a metric space forms a chain complete poset (CPO) with respect to first-order stochastic dominance order to non-metrizable compact Hausdorff spaces (e.g. see Kamae, Krengel, and O'Brien, 1977).

[23] Recall, conditions for the existence of unique best replies are well-known (e.g., convex-valued action correspondences, and strictly quasi-concave payoffs).

Assumption 3.2 *Assume for* $\Gamma(s)$*:*

(i) for all $s \in S$*, the correspondence* $\tilde{A}(\cdot, s)$ *and payoff function* $r(\cdot, s)$ *satisfy Assumption 3.1,*

(ii) for all $\alpha \in \Lambda$*, the correspondence* $s \rightrightarrows \tilde{A}(\alpha, s)$ *is ascending in Veinott's strong set order*[24]*,*

(iii) for λ*-a.e. player function* r *satisfies a single crossing property in* (a, s)*.*

Under this new assumption in place, we have the following equilibrium comparative statics result. The result is essentially a corollary to Theorems 3.1 and 8.4.

Corollary 3.1 *Under Assumption 3.2 the greatest* $(\overline{\tau}^*(s))$ *and the least* $(\underline{\tau}^*(s))$ *distributional equilibrium of* $\Gamma(s)$ *exist. Moreover,* $\overline{\tau}^*, \underline{\tau}^* : S \to \mathcal{D}$*, are increasing distributional equilibrium selections.*

3.2 Remarks and discussion

We now compare our results to (i) related work in the literature concerning distributional equilibria in large games, (ii) the structure of pure strategy Nash equilibrium in GSC with a *finite* number of players, as well as (iii) literature on monotone equilibrium comparative statics for large games.

First, Theorem 3.1 establishes existence of distributional equilibria under a different set of assumptions than in Mas-Colell (1984), as well as in various related papers in the literature that followed. For example, in our case, the payoff r need *not* be weakly continuous with respect to τ (see also discussion in Rath, 1996, concerning upper semi-continuous payoffs). Instead of these sorts of conditions, we simply partially order both $\Lambda \times A$ and $\mathcal{D}$ appropriately, then add quasisupermodularity/single crossing property conditions to payoff structure for each player in their actions (as well as assume complete lattice structure of the action set), and this allows use to obtain our existence result. In this sense, our approach offers also a new method for analyzing large games with discontinuous payoffs.[25]

Second, the main theorem establishes existence of extremal distributional equilibria, which aside from generalizing some of the existence results for GSC with a finite number

[24] A correspondence $F : X \rightrightarrows Y$, for X is a poset and Y a lattice is ascending in Veinott's strong set order if for any $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$, when $x_1 \leq x_2$, $y_1 \wedge y_2 \in F(x_1)$ and $y_1 \vee y_2 \in F(x_2)$. Notice, this implies F is sublatticed-valued.

[25] For example, consider a large game of Bertrand competition, where profits may be discontinuous due to products substitutability. Observe that this cannot be modeled under assumptions used in Rath (1996).

of players, it allows us to develop new order theoretic characterizations of distributional equilibrium comparisons in deep parameters. Further, under slightly stronger continuity conditions on payoffs, it provides a method to *compute* the equilibrium comparative statics. Additionally, for the special case of the theorem where distributional best replies are *functions,* we provide an order theoretic characterization of the entire *set* of distributional equilibria. This result offers, therefore, a generalization of some important aspects of the typical existence theorem one obtains via parameterized versions of Tarski's theorem for GSC with a finite number of players (see Veinott, 1992; Zhou, 1994).

It is important to note that unlike GSC with finite set of players, for large GSC, we do *not* expect to have a complete lattice of distributional equilibria. Actually, it is only in the *best* case of *unique* distributional best replies that we can even obtain a characterization of the set of distributional equilibria as a CPO. The reason for this weaker characterization of equilibrium is very simple: although action set A is a complete lattice, the set of distributions on A is generally not.[26] As a result, instead of applying the result of Tarski, we must appeal to various generalizations of this theorem, namely, Markowsky's fixed point theorem.[27]

Third, our approach suggests direct methods for computing particular distributional equilibria (compare Topkis, 1998, Chapter 4.3) (as well as distributional equilibrium comparative statics). Under sufficient order continuity conditions on payoffs in τ, for each parameter $s \in S$, to compute extremal distributional equilibrium, one only needs to calculate the order-limit of a decreasing sequence of distributions generated by the upper distributional operator $\overline{T}$ iterating downward from the greatest element relative to the partial order $\succeq_P$ per the set of distributions $\overline{\delta}$ (resp, dually, for the lower distributional operator $\underline{T}$, iterating up from least element $\underline{\delta}$). This is true in this case as under continuity assumption of the second part of Theorem 3.1, the upper and lower distributional operators $\overline{T}$ and $\underline{T}$ are monotonically inf/sup-preserving[28]; hence, the order-limit of our iterations is attainable in a *countable* number of steps (hence, computable).

For this result on computability of extremal equilibria to be true, we need a (weak) continuity condition on payoffs in τ. Recall, in the original proof of Mas-Colell (1984) (as well as other related results in the literature), the weak continuity of payoffs with τ is

[26]For example, consider set of distributions on $A \subset \mathbb{R}^m, m \geq 2$, ordered by first order stochastic dominance ordering. In this case, it well-known that the spaces of probability measures on A is not a lattice (see Kamae, Krengel, and O'Brien, 1977); rather it is only a CPO.

[27]It is worth noting that Markowsky's Theorem *characterizes* a chain complete poset using its fixed point property relative to increasing mappings and least fixed points. In this sense, Markowsky's Theorem provides a converse result also (just as Davis, 1955, provides a converse of Tarski, 1955).

[28]See the appendix of this paper for a discussion of monotonically sup (resp., inf) preserving mappings.

critical for existence. In our case, we do *not* need such a weak continuity of payoffs for *existence*; rather, we simply need this condition for the computation of equilibrium (and for computable equilibrium comparative statics). This is a critical difference with respect to the existing literature, and is precisely where the assumption that the game is a LGSC is needed relative to the existing literature.

Fourth, we draw your attention to the restrictive assumption we need to make on action sets (e.g., the action set is required to be a complete lattice in $\mathbb{R}^m$, hence necessarily compact). This condition is required to guarantee that best response map possesses the greatest and the least element. We leave open the question of whether it is possible to generalize our results to infinite dimensional complete lattices (e.g., a class of action spaces that are complete Banach lattices) to further work (compare with Khan, 1989). It seems possible that by using order theoretic maximum theorems in Shannon (1990) and Veinott (1992), such an extension of our results to more general action spaces is possible.

Finally, the only result concerning the existence of equilibrium comparative statics for large games of which we are aware is presented in an interesting recent paper of Acemoglu and Jensen (2010). In an important sense, their approach to equilibrium comparative statics is very similar to ours, as they impose conditions guaranteeing that the joint best response mapping has increasing selections with respect to parameter s (e.g., see Definition 3 in their paper). However, there are important differences. First, as they concentrate on *aggregative* games where players best respond to the average/mean action of other players, the class of games in which they obtain equilibrium monotone comparative statics is quite different (and more restrictive) than ours. In particular, our class of games include theirs, but also allows for considerably more general classes of large games. We consider it to be an important difference. Second, in case of a single dimensional action space A, Acemoglu and Jensen manage to show comparative statics of the extremal (aggregative) equilibria using results of Milgrom and Roberts (1994) *without* the single crossing property between player actions and aggregates. This result is a very important, and more general result to the one presented even in our framework for the special case of one dimensional action spaces. However, for aggregative games with multidimensional action spaces, Acemoglu and Jensen require increasing differences in the action of a player and the aggregate which is stronger than the (ordinal) single crossing properties we use. Finally, Acemoglu and Jensen use the topological fixed point theorem of Kakutani to show existence of an aggregate equilibrium, which makes the issues of equilibrium comparative statics and computability of equilibrium difficult (if not impossible) to address. On the contrary, we use exclusively order-theoretical fixed point results, which address both of these important issues directly.

We finish this section with make six additional important remarks concerning generality of our results.

Remark 3.1 *Recall, in the original Mas-Colell (1984) paper, the game is anonymous (i.e. the reward function depends only on the marginal distribution on the action space). In other words, anonymity means that the reward function takes the form* $r(\alpha, \tau) := \tilde{r}(\alpha, \tau_A)$ *for some function* $\tilde{r}$*, where* τ_A *is a marginal distribution of* τ *on actions. Therefore, our model contains anonymous games as a special case (e.g., the case payoffs depend on a simple transformation* $\tau_A(\cdot) = \tau(\Lambda \times \cdot)$*). We also stress, however, that our analysis remains valid for non-anonymous games, as well.*

Remark 3.2 *Schmeidler (1973) studies a non-anonymous game where what is payoff relevant for each player is her own action, as well the profile of actions undertaken by each of her opponents. Therefore, for large games in this tradition, the appropriate notion of equilibrium is Nash equilibrium in* strategies *(i.e. functions mapping from the set of players to their action sets). Despite the obvious technical differences of this equilibrium concept relative to that of distributional equilibrium, it turns out our techniques advocated in this paper are applicable to this class of equilibria in strategies.*

To see how this adaptation of our techniques would proceed, let D *be a set of functions from* $\Lambda \times S$ *to* $\mathbb{R}^m$ *such that* $(\forall s \in S)\, \alpha \to f(\alpha, s)$ *is* $\mathcal{L}$*-measurable with* $f(\alpha, s) \in \tilde{A}(\alpha, s)$ *and the payoff function be denoted by* $r : \Lambda \times A \times D \times S \to \mathbb{R}$*, with a joint strategy profile being an element of* D*. Define a pointwise partial order on the strategy space of the players (note, this is different than the order used for the distributional equilibria of the previous section).*

Then, under similar assumptions as in the case of distributional equilibrium, we can prove existence and provide a stronger characterization of the set of equilibria (e.g., the existence of the greatest and the least elements of the equilibrium set). Furthermore, the fixed point theorem we apply for existence and characterization of the set of Nash equilibria in this case is different than the one used in case of distributional equilibria. This difference in fixed point construction is critical when comparing the order structure of the set of Nash equilibria. What is a critical difference in the two cases is the following: as a set of (bounded) Borel measurable functions is only a σ*-complete lattice, for the case of an uncountable number of players, the fixed point set is not a subset of a CPO (see Heikkilä and Reffett, 2006, Example 2.1.).*[29] *Therefore, to obtain even the existence in the case of Schmeidler (1973), we actually need the additional assumption of order continuity*

[29] A σ-complete lattice is a lattice which is order closed relative to *countable* subsets.

of the payoff function with strategies of all players with respect to the pointwise partial order.[30] *This is not the case for the question of existence of distributional equilibria in the previous section, where order continuity conditions are only required for approximation of extremal equilibria. Nevertheless, our order continuity assumptions are weaker than weak continuity assumed in the literature (see e.g. Khan, 1986).*[31]

Under this additional order continuity condition on payoffs, our computation and equilibrium comparative statics results (now in pointwise partial orders) are similar to those obtained for distributional equilibria (under first order stochastic dominance) in section 3.1.

Remark 3.3 *We can relate our results to the more general superextremal games with very general forms of ordinal complementarities that were introduced in Shannon (1990), LiCalzi and Veinott (1992), and Veinott (1992) with the obvious modifications of our arguments.*[32] *Notice, that if we relax Assumption 3.1 to join (resp., meet) superextremal games, a simple modification of our arguments yields existence of the greatest (resp., the least) distributional equilibrium of game* Γ. *In this case, best replies maps are join sublattices (resp., meet sublattices). This implies our upper distributional operator* $\overline{T}$ *(resp, our lower distributional operator* $\overline{T}$ *and* $\underline{T}$*) will be well-defined, and a version of the main theorem is available for upper distributional equilibria (resp., lower distributional equilibria). More specifically following LiCalzi and Veinott (1992), for the more general case, just replace quasi-supermodularity/lattice superextremal in Assumption 3.1 with join (resp., meet) superextremal payoffs* r *in individual actions* a*, and replace the assumption of a standard single crossing property with join (resp., meet) upcrossing differences. So our methods work for LGSC with weaker forms of complementarities, but at the expense of losing either least (resp., greatest) distributional equilibrium.*

Remark 3.4 *We endow the space of distributions* $\mathcal{M}(\Lambda \times A)$ *with the first order stochastic ordering* $\succeq_P$*, as the applications we have in mind involve this partial order. In fact, a*

[30] Formally, we require that payoff preserves sup/infs of the increasing/decreasing sequences in D. This is required for the extremal best response selections to be monotonically inf/sup-preserving, a feature of the best reply map we need to setup an application of Tarski-Kantorovich fixed point theorem (even for existence). See section 8 for formal definitions and proofs.

[31] As order continuity has to be verified only with respect to all sub-chains of the original set, it is straightforward to show an example of payoff functionals that are order continuous, but not weakly continuous.

[32] There is one case of the superextremal class of games that requires a more delicate argument than in the present paper: the so-called "superextremal variant" of the superextremal class (as discussed in LiCalzi and Veinott (1992), and Veinott (1992)). In these games, best replies are merely quasisublattices, and one can only guarantee the existence of appropriately isotone selections. This weak characterization of best replies greatly complicates the characterization of such selections relative to measurability, for example. We leave this one case for future research.

careful examination of the proof of our main existence theorem shows that our results can also hold for other partial orders on $\mathcal{M}(\Lambda \times A)$, as long as that partial order implies a chain complete structure of the space of probability measures.

Remark 3.5 *Our analysis can be easily extended to cover equilibria that are monotone in players' characteristics if some application requires that. Specifically, similarly to the idea of Yang and Qi (2013) and following results of Balbus, Reffett, and Woźny (2014c) we can additionally assume single crossing property of r in (a, α) and assume that $\alpha \to A(\alpha)$ is Veinott's strong set order increasing to make sure, that extremal of distributional equilibria are monotone in players' characteristics.*

Remark 3.6 *In the applications we analyze we stress necessity to analyze the equilibria in the infinite-dimensional measure spaces. There are numerous models, however, where it is typically assumed that players' interact only via some aggregate (see Acemoglu and Jensen, 2010; Guesnerie and Jara-Moroni, 2011, e.g.). The results presented in this paper are still applicable to such subclass of our games but in such case our results can be strengthen[33]. Specifically suppose that players function $r(\alpha, a, g)$ depend now on some aggregate of population-action distribution $g \in G$, for some complete lattice G. Suppose aggregate value g can be determined using some monotone externality mapping $h : \mathcal{D} \to G$. One example of such externality mapping is $h(\tau) = \left(\int_\Lambda h_i(\alpha, a)\tau(d(\alpha \times a))\right)_{i=1}^n$ for some $h_i : \Lambda \times A \to \mathbb{R}$. In such case under our assumptions we can define an operator, say Ψ, mapping the complete lattice G into itself via: $\Psi(g) = h(\overline{T}(g))$, where $\overline{T}(g)(H) = \lambda(\{\alpha : (\alpha, \overline{m}(\alpha, g)\} \in H)$ for $\overline{m}(\alpha, g) = \vee \arg\max_{a \in A(\alpha)} r(\alpha, a, g)$. In such case for monotone h we can apply the fixed point theorem of Tarski on Ψ to prove existence and analyze the greatest equilibrium aggregate $\overline{g}^*$ and implied equilibrium distribution $\overline{T}(\overline{g}^*)$. More on this example in section 4.3.*

4 Symmetric equilibria and uniqueness in LGSC

4.1 Finite number of player types and symmetric equilibria

We begin this section by considering a game with finite number of player types (but, in general, a continuum of each type). Let $\{J_i\}_{i=1}^n$ be a finite collection of disjoint sets of Λ, such that $\bigcup_i J_i = \Lambda$. Let $i = 1, \ldots, n$ denote player types, i.e. $\forall i$, $\forall \alpha, \beta \in J_i$, $r(\alpha, \cdot) = r(\beta, \cdot)$ and $\tilde{A}(\alpha) \equiv \tilde{A}(\beta)$. In other words, players are identical within the same type i with respect to their payoffs and feasible strategy sets.

[33] We thank one anonymous Referee of this journal for recommending us this application.

A distributional game with finite number of players' types Γ_F is defined as in section 3, with the restriction imposed on player characteristics specified above, and a *symmetric distributional equilibrium*[34] is a measure $\tau^* \in \mathcal{M}(\Lambda \times A)$ defined as in section 3 such that additionally we have

$$\forall i, \exists a^* \in A \text{ s.t. } \tau^* \left(\left\{ (\alpha, a) \in J_i \times \{a^*\} : a \in \arg \max_{a' \in \tilde{A}(\alpha)} r(\alpha, a', \tau^*) \right\} \right) = \lambda(J_i),$$

where the marginal distribution of τ^* on Λ is λ. Hence, in a symmetric distributional equilibrium, we expect that λ almost every player of a given type plays the same strategy.

We proceed with the following proposition.

Proposition 4.1 *Under Assumption 3.1 the greatest and the least distributional equilibrium of* Γ_F *is symmetric.*

Clearly, since the game with finite number of player types is a special case of the general LGSC, we know by the previous sections, that a distributional equilibrium exists. However, in a framework with finite number of player types, they posses some additional properties, which prove very useful in numerical applications. Eventually, the above results have a practical application, which is discussed in the following section.

4.2 Remarks on symmetric equilibria in LGSC

We remark that the definition of a symmetric game and equilibrium is different than in those discussed in Milgrom and Roberts (1990) or Amir, Jakubczyk, and Knauff (2008). In these two papers, the authors dealt with symmetric quasi-supermodular games with a finite number of players. Furthermore, unlike in the previous papers, we expect players to be symmetric only within a given type i. However, it is straightforward to show that the result holds in particular for games with some i such that $J_i = \Lambda$, and $J_{i'} = \emptyset, i' \neq i$, hence satisfying symmetry among all agents. In this case, the greatest and the least equilibria are simply Dirac measures concentrated at strategies corresponding to the equilibria.

We now note several useful properties of our LGSC with finite number of types. Given that symmetric equilibria exist, our large game can be represented by a game with finite number of players in the following way.

Step 1: Construct a game with $n+1$ players, and endow each of the first n players with payoff functions $\pi_i : \tilde{A}_i \times_{j=1}^{n} \tilde{A}_j \to \mathbb{R}$, $\pi_i(a_i, b) := r(\alpha, a_i, \tau(b))$ with $\tilde{A}_i = \tilde{A}(\alpha)$,

[34] See Mas-Colell (1984), p.203 for related definition.

$\alpha \in J_i$, and $\tau(b)(\cdot) \in \mathcal{D}$ is such that[35] $\tau(b)(\cdot|\alpha) = b_i$ for $i = 1, \dots, n$ and $\alpha \in J_i$. In other words, agent i represents players of type i, where we now restrict our attention solely to symmetric distributions (i.e. where each player of type i is assigned strategy b_i). Step 2: Let the $n + 1^{\text{st}}$ player maximize payoff $-\sum_{i=1}^{n} \|a_i - b_i\|_1$ with respect to $\{b_i\}_{i=1}^{n}$ over $\times_{i=1}^{n} \tilde{A}_i$.[36]

Notice, the construction of τ implies that in the corresponding LGSC, each of the first n players does not observe their impact on the overall distribution of actions (even though there is a finite number of player types in the finite representation of the LGSC). In fact, the $n+1^{\text{st}}$ player is introduced into the game in order to equate his own strategies with those of other players. Observe, by this construction, it is always feasible (and optimal) for player $n + 1$ to set $b_i = a_i$; hence, adjust distribution τ to actions played by the remaining n agents. In this way, any Nash equilibrium of the game defined above generates distributionτ, which corresponds to a symmetric distributional equilibrium of the large game.

By Proposition 4.1, the greatest and the least elements of the equilibrium set are symmetric. Therefore, under our assumptions, one can always determine *bounds* on the equilibrium set of a LGSC by appealing to a version of its finite player/type counterpart. This observation is especially useful when approximating equilibria using numerical methods, where the space of agents is discretized. Of course, its important to keep in mind that *not every* distributional equilibrium can be determined this way (as the game might also have equilibria which are *not* symmetric).

Another distinct advantage of the finite type approach is that it allows for the usage of tools developed for finite player games analysis. In particular, we are able to determine sufficient conditions for uniqueness of equilibrium, which shall be discussed in the following section.

4.3 Uniqueness of equilibrium

In this section, we establish sufficient conditions for uniqueness a distributional equilibrium in a class of aggregative games with finite number of player types. We proceed with the description of the game.

Let $h : \mathcal{M}(\Lambda \times A) \to G$ denote an aggregate function, mapping space of measures τ to a compact set $G \subset \mathbb{R}$. Assume that payoff of a single player is dependent only on his characteristic, strategy, and the value of the aggregate in the game G (i.e. $r : \Lambda \times A \times G \to$

[35]In order to clarify, this notation means that for $\alpha \in J_i$, the conditional distribution $\tau(b)(\cdot|\alpha)$ is a Dirac delta concentrated at b_i.

[36]Where $\| \cdot \|_1$ is a taxicab norm.

$\mathbb{R}$ are the payoffs). An *aggregative distributional equilibrium* is a measure $\tau^* \in \mathcal{M}(\Lambda \times A)$, such that

$$\tau^* \left(\left\{ (\alpha, a) \in \Lambda \times A : a \in \arg \max_{a \in \tilde{A}(\alpha)} r(\alpha, a, h(\tau^*)) \right\} \right) = 1.$$

Therefore, we reproduce the definition of equilibrium stated in section 3, with a restriction of its definition to the aggregate.[37] In order to make the example complete, we state explicitly our additional assumptions needed for the large aggregative game. As before $A \subset \mathbb{R}^m_+$ is endowed with a natural product order and Euclidean topology.

Assumption 4.1 *Assume that:*

(i) $\tilde{A}_i$ *is complete lattice valued,*[38]

(ii) $r : \Lambda \times A \times G \to \mathbb{R}$ *is continuous on* A *and* $\mathcal{L}$*-measurable. Moreover, for* λ*-a.e. player,* r *is quasi-supermodular on* A*, and satisfies a single crossing property in* $(a, g) \in A \times G$*,*

(iii) $h : \mathcal{M}(\Lambda \times A) \to G$ *is increasing.*

The above assumption is a slight modification of Assumption 4.1, necessary for a specification of a large aggregative game. We proceed first with the following corollary for our main theorem's result:

Corollary 4.1 *Under Assumption 4.1 the least and the greatest equilibrium of an aggregative large game exists.*

The following result, then, provides sufficient conditions for the *unique* aggregative distributional equilibrium.

Proposition 4.2 *Consider an aggregative game with* $n < \infty$ *player types. Let Assumption 4.1 be satisfied. If in addition, we assume that*

(i) $\tilde{A}_i$ *is convex,*

(ii) for λ*-a.e. player* r *is strictly quasi-concave and twice continuously differentiable on an open set containing* A*, as well as once continuously differentiable on an open set containing* G*, such that* $\forall k = 1, \ldots, m$, $\sum_{j=1}^m \frac{\partial^2 r}{\partial a_k \partial a_j} + \frac{\partial r^2}{\partial a_k \partial g} < 0$,

[37] Observe, that by defining h as a function mapping D to G, we can define a Schmeidler counterpart of the aggregative game.

[38] Recall, that since values of $\tilde{A}_i$ are subsets of $\mathbb{R}^m$, they must also be compact.

(iii) for any symmetric $\tau \in \mathcal{M}(\Lambda \times A)$, function[39] $\bar{h} : \times_{i=1}^{n} \tilde{A}_i \to G$, $\bar{h}(a) := h(\tau_a)$, $a \in \times_{i=1}^{n} \tilde{A}_i$, *is well defined and continuously differentiable such that* $\forall i = 1, \ldots, n$, $j = 1, \ldots, m$ $\exists M < 1$, $\frac{\partial \bar{h}}{\partial a_{i,j}} \leq M$,

then there exists a unique aggregative distributional equilibrium of the game.

Note that the above proposition ensures existence of a unique *distributional equilibrium*, apart from uniqueness of a *symmetric* distributional equilibrium. In fact, the proof first establishes uniqueness of a symmetric equilibrium, which then by Theorem 4.1 implies, that it is the only equilibrium of the game.

We should also mention, although our approach to the proof of this result bears some relationship with the proof of Theorem 2.4 in Curtat (1996) (and related arguments in Gabay and Moulin (1980)), our proof is different. In particular, it uses the results of the previous section, and it is based on the fact that a large game with a finite number of types can be represented by its finite player counterpart.[40] For this reason, it is necessary to impose a relatively strong assumption on the form of an aggregate, which in addition has to be differentiable with respect to the support of the corresponding symmetric distribution.

Fortunately, it is possible to determine a broad class of aggregates satisfying our assumption. We present an example of such a function below.

Example 4.1 *Let $m = 1$, and h map distribution of player/types to an average strategy of players, i.e. $h(\tau) = M \int_A a\tau(d\alpha \times da)$, $M \in [0, 1)$. Observe, that in case of symmetric distributions, the aggregate takes the form $h(\tau) := M \sum_{i=1}^{n} \lambda(J_i) a_i$, where a_i is a strategy of type i. In this case $\bar{h}$ is differentiable with respect to a, and $\frac{\partial \bar{h}}{\partial a_i} = M\lambda(J_i) < 1$. Hence, assumption of Proposition 4.2 is satisfied.*

5 Applications

We now discuss some applications of our results. The first application continues the discussion begun in section 2. We then also provide three additional examples of large games where our theorems can be applied.

[39] τ_a denotes a profile, such that all agents from i category choose almost surely action $a_i \in \tilde{A}(\alpha)$ $\forall$ $i = 1, ..., m$. In other words τ is a step function with a_i on all sets J_i.

[40] By *represented* we mean, that there exists a game, which equilibria coincide with at least some of the equilibria of the original game. Clearly, even in the case of large games with finite number of player types, we can only generate symmetric equilibria of the original game by its finite counterpart, and not all of them.

5.1 The motivating example - continuation

We first return to the motivating example presented in Section 2. In order to make our notation consistent, denote the measure space of player characteristics by $(\Lambda, \mathcal{L}, \lambda)$, where as in Section 2, we have $\Lambda = X \times Y$. Recall, given a distribution of player characteristics and actions $\tau \in \mathcal{D}$, the objective of an agent $\alpha = (x, y)$ is to maximise

$$r(\alpha, \tau, a) := u(|a - y|) + \int_\Lambda \int_Y v(|a - a'|) d\tau(a'|\alpha') d\mu_\alpha(\alpha'),$$

with respect to $a \in Y$. As Y is a compact subset of $\mathbb{R}$, it is a complete lattice. Moreover, the set of feasible actions is common for agents with any characteristics (hence, Assumption 3.1(i) is satisfied).

By assumption, function v is concave and decreasing. Therefore, Lemma 8.3 implies that $v(|a-a'|)$ has increasing differences in (a, a'). It is easy to show that this is sufficient for payoff function $r(\alpha, \tau, a)$ to have increasing differences in (a, τ), whenever the set of distributions $\mathcal{D}$ is ordered with respect to the first order stochastic dominance $\succeq_P$. Therefore, Assumption 3.1(ii) is also satisfied.

Finally, assuming that functions u and v are both continuous, and the function μ_α is weakly continuous with respect to α, Assumption 3.1(iii) also holds.

By Theorem 3.1, we conclude there exist the greatest and least distributional equilibria in the game. Further, as under this conditions, $r(\alpha, \tau, a)$ satisfies the requisite order continuity condition in the main theorem in τ, these extremal distributional equilibria can be approximated using iterative methods.

Finally, note that since agents care about the status of other player as well as their own identity, the extremal equilibria are trivial only in special cases. That is, in general it is not the case that in the greatest and the least equilibrium of the game the measure of agents choosing respectively the greatest and the least possible social status is 1. Therefore, the approximation methods become very useful in determining and computing distributional equilibria.

5.2 Linear non-atomic supermodular games

The game presented in the motivating example is a special case of a much larger class of games, namely *linear non-atomic supermodular games*. We now discuss how our results can be applied to this larger class of games.

As in the previous sections, assume that the measure space of player characteristics is denoted by $(\Lambda, \mathcal{L}, \lambda)$. As in section 3, let the set of all possible actions by A, and the correspondence mapping the characteristics of agents into the set of feasible strategies is denoted by $\tilde{A}$. In addition, we introduce a poset of parameters denoted by Θ. In the class of linear nonatomic supermodular games, all the agents in the population interact with each other *individually.* Therefore, the player's ex-post payoff is a sum of utilities that is determined by every separate interaction.

More specifically, suppose that the payoff from a single interaction is determined by function $u : \Lambda \times \Theta \times A \times A \to \mathbb{R}$. That is, an agent with characteristic $\alpha \in \Lambda$, given parameter $\theta \in \Theta$ and action $a \in \tilde{A}(\alpha)$ yields $u(\alpha, \theta, a, a')$ units of utility from an individual interaction with an agent playing $a' \in A$. As previously assumed, the frequency of interactions with other players will depend on the trait of a given player. So, for any $\alpha \in \Lambda$, we have μ_α a non-atomic probability measure defined over $\mathcal{L}$. Hence, for any measurable group $U \in \mathcal{L}$, $\mu_\alpha(U)$ denotes the probability that agent with characteristic α will meet an individual with a characteristic belonging to U.

Let $\tau \in \mathcal{D}$ be a distribution of characteristics and actions in the population. Given the above description of the game, the ex-post payoff of an agent with $\alpha \in \Lambda$ is defined by

$$r(\alpha, \theta, \tau, a) := \int_\Lambda \int_A u(\alpha, \theta, a, a') d\tau(a'|\alpha') d\mu_\alpha(\alpha').$$

Clearly, in order for the payoff to be well defined, we require that function $u(\alpha, \theta, a, \cdot)$ is $\mathcal{A}$-measurable for any α, θ, and a.

Assume that function u is $\mathcal{L}$-measurable, continuous and supermodular with respect to a, and has increasing differences in (a, a'), for any $\theta \in \Theta$. Moreover, let μ_α be measurable as a function of α. Given Theorem 3.1, under the above assumptions, any such linear nonatomic supermodular game will have a greatest and least distributional equilibrium (which, again, can be approximated using iterative methods).

Now, assume that for any characteristic $\alpha \in \Lambda$ and actions $a'' \geq a$, such that a'', $a \in \tilde{A}(\alpha)$, the family of functions $\{\delta_\alpha(\cdot, a')\}_{a' \in A}$, where $\delta_\alpha(\theta, a') := u(\alpha, \theta, a'', a') - u(\alpha, \theta, a, a')$, obey the signed-ratio monotonicity (see Quah and Strulovici, 2012). Then, function $r(\alpha, \theta, \tau, a)$ has single crossing differences in (a, θ). Therefore, by Corollary 3.1, the greatest and least distributional equilibrium of the game *increases* with respect to the deep parameter θ.

The key feature of the above class of games is that the payoff function is linear with respect to measure τ. Therefore, it is weakly continuous on the space of probability measures. This implies, that the additional assumption imposed before Theorem 3.1(iii) is always satisfied. Hence, no additional assumptions on the game need to be imposed in order for the result to hold.

5.3 Large stopping games

We next turn to an optimal stopping time example. Suppose that a continuum of agents are deciding how long they should each one participate in an investment project that lasts at most T periods. Each period t, each agent takes part in the investment, from which she receives a profit of $\pi(t,m)$, where m is the measure of agents participating in the project at time t. We assume can $\pi(t,m)$ may take on both positive and negative values; however, it is increasing in m. In other words, the more agents who participate in the project, the higher are the profits (or, lower are the losses) to every individual agent. Also, assume whenever the agent is not participating at the project, her payoff is equal to zero.

In the following analysis, we concentrate solely on the case where time is discrete. So, define the set of time indices by $\{1, 2, \ldots, T\}$. Suppose that the time at which the agent joins the project is determined exogenously, and that it defines the characteristic of an agent. Hence $\alpha \in \Lambda$, where $\Lambda := \{1, 2, \ldots, T\}$. The distribution of characteristics across agents is determined by some measure λ over Λ. Hence, $\lambda(\alpha)$ is the measure of individual agents that join the project at time α.

Since the time at which agents join the investment is given exogenously, they can only decide when to leave the investment. Assume that agents can leave the project only once (i..e, just like in a standard optimal stopping game). Given this, an action of a player is equivalent to a time index at which the agent decides to leave the investment. Using our notation, the set of all possible actions A is equivalent to the set of time indices Λ. [41] Moreover, the correspondence mapping agents characteristics into the set of feasible actions is defined by $\tilde{A}(\alpha) := \{\alpha, \ldots, T\}$.

Assume that a distribution of characteristics and actions for the population is given by $\tau \in \mathcal{D}$. Therefore, for any $(\alpha, a) \in \Lambda \times A$, $\tau(\{(\alpha, a)\})$ is the measure of agents joining the investment at time α, and leaving the investment at time a. Define function $F : \mathcal{D} \times \Lambda \to [0, 1]$ as

$$F(\tau, s) := \tau(\{(\alpha, a) \in \Lambda \times A \mid \alpha \leq s \leq a\}).$$

In other words, $F(\tau, s)$ is a measure of agents participating in the project at time s. Note that function $F(\tau, \cdot)$ is not a probability distribution nor a cumulative distribution. That is, first the sum of its values might not be equal to one; but second, it need not be monotone.

Given the above notation, we can define the payoff of an agent with characteristic

[41] We shall differentiate the notation of these two sets in order to avoid confusion.

$\alpha \in \Lambda$ by

$$r(\alpha, \tau, a) := \sum_{s=\alpha}^{a} \pi(s, F(\tau, s)),$$

The objective of every agent is to maximize $r(\alpha, \tau, a)$ with respect to $a \in \tilde{A}(\alpha)$.

In order to make sure that the above game satisfies the conditions stated in Assumption 3.1, we first need to show the function $F(\tau, \cdot)$ is pointwise increasing as the measure $\tau \in \mathcal{D}$ shifts upward with respect to the first order stochastic order. Along this lines, consider any τ' and τ in $\mathcal{D}$ such that $\tau' \succeq_P \tau$. By definition of the first order stochastic dominance, this implies that for any $(\alpha, a) \in \Lambda \times \Lambda$, we have

$$\tau'(\{(\alpha', a') \in \Lambda \times A \mid (\alpha', a') \leq (\alpha, a)\}) \leq \tau(\{(\alpha', a') \in \Lambda \times A \mid (\alpha', a') \leq (\alpha, a)\}).$$

Since both τ' and τ belong to $\mathcal{D}$, they have the same marginal distribution over the space of characteristics Λ. Therefore, the above condition is equivalent to

$$\tau'(\{(\alpha', a') \in \Lambda \times A \mid a' \leq a\}) \leq \tau(\{(\alpha', a') \in \Lambda \times A \mid a' \leq a\}).$$

This then implies

$$\begin{aligned} F(\tau, s) :=& \quad \tau(\{(\alpha, a) \in \Lambda \times \Lambda \mid \alpha \leq s \leq a\}) \\ \leq& \quad \tau'(\{(\alpha, a) \in \Lambda \times \Lambda \mid \alpha \leq s \leq a\}) \\ =:& \quad F(\tau', s), \end{aligned}$$

for any $s \in \Lambda$.

The above property can now be show to be sufficient for the payoff function to have increasing differences in (a, τ). Clearly, since function $\pi(s, \cdot)$ is increasing, we have

$$\begin{aligned} r(\alpha, \tau, a') - r(\alpha, \tau, a) &= \sum_{s=a}^{a'} \pi(s, F(\tau, s)) \\ &\leq \sum_{s=a}^{a'} \pi(s, F(\tau', s)) \\ &= r(\alpha, \tau', a') - r(\alpha, \tau', a), \end{aligned}$$

for any $\alpha \in \Lambda$ and $a' \geq a$ in A.

To complete the argument, as the space of characteristics and actions is finite, the payoff function is trivially continuous on $\Lambda \times A$. Therefore, the conditions specified in

Assumption 3.1 are satisfied. Moreover, once we assume that function $\pi(s, \cdot)$ is continuous for any $s \in \Lambda$, the payoff function is order continuous with respect to τ. Therefore, by Theorem 3.1, we conclude that the game has greatest and least distributional equilibrium, each can be approximated using iterative methods.

Interestingly, even though the above framework is substantially simplified and the space of characteristics and actions is finite, the externality in the above game is not a "lattice externality". Therefore, the distributional equilibria in the game cannot be studied using the methods in the existing literature (e.g., remark 3.6 or Guesnerie and Jara-Moroni (2011)). Also, note that the space of functions:

$$\{f : \Lambda \to [0, 1] \mid \exists \tau \in \mathcal{D} \text{ such that } f(s) = \tau(\{(\alpha, a) \in \Lambda \times \Lambda \mid \alpha \le s \le a\}), \ \forall s \in \Lambda\}$$

in not a lattice under the pointwise order. For example, take functions f and f' such that for some $s' \neq s$ belonging to Λ, we have $f(s) = f'(s') = 1$ and zero otherwise. Clearly, both functions belong to the above space, but it is easy to check that their pointwise join and meet do not.

Finally, we are able to determine the comparative statics of the extremal equilibria. That is, assume each period, the payoff function is parameterized by a deep parameter θ belonging to a poset Θ. Hence, each period the agent that participates in the investment receives $\pi(s, \theta, m)$, where m is the measure of players taking part in the project at the given time. Suppose that $\pi(s, \cdot, m)$ is an increasing function for any $s \in \Lambda$ and $m \in [0, 1]$. Clearly, for any α and τ, the payoff function

$$r(\alpha, \theta, \tau, a) := \sum_{s=\alpha}^{a} \pi(s, \theta, F(\tau, s)),$$

has increasing differences in (a, θ). By Corollary 3.1, this is sufficient to conclude that the extremal equilibria are increasing functions of θ.

5.4 Keeping up with the Joneses

Consider an economy with a continuum of consumers. Suppose that every agent in the economy is characterized by their initial wealth $m \geq 0$, and a number $i \in [0, 1]$. In our framework, the number i will correspond to the relative social position to which the agent refers to when choosing her consumption level. We shall define the notion formally in the remainder of the section.

Assume that the set off all possible values of wealth M is a compact subset of $\mathbb{R}_+$, where $\bar{m} := \max M$. As previously, let $\Lambda := M \times [0,1]$, and $\alpha = (m,i)$ be one of its elements. The distribution of characteristics is determined by a probability measure λ defined over the product Borel-algebra $\mathcal{L}$ of λ.

There are two markets in the economy: the *consumption good* market and the *labor* market. Every agent is endowed with m units of the initial wealth, expressed in units of the consumption good, and one unit of time that can be devoted either to labor or leisure. Given the normalized price of consumption $p = 1$ and the wage $w > 0$, the budget set is

$$B(m,w) = \{(a,n) \in \mathbb{R}_+ \times [0,1] \mid m + w \geq a + wn\},$$

where by (a,n) we denote a pair of consumption a and leisure n. Note that the set of feasible consumption levels in our framework is given by $A := [0, \bar{m}+w]$, which is compact.

Apart from the consumption and leisure, every agent takes into account the relation of her consumption to the consumption of other agents in the economy. Number $i \in [0,1]$, characterizing the agent, denotes the quantile of the distribution of consumption in the population that the agent is treating as his reference point when choosing her consumption. In other words, the higher is the agents consumption above the i'th quantile of the distribution, the better. This feature of the model incorporates the *keep up with the Joneses* effect, but in a *heterogenous* manner. That is, every agent might be characterized by a different number i the she might compare her consumption relative to a different reference quantile of the population.

Assume that the distribution of characteristics and consumption in the economy is defined by a probability measure τ over $\Lambda \times A$. Let

$$q(\tau,i) := \min\{a \in A \mid \tau(\{((m',i'),a') \in \Lambda \times A \mid a' \leq a\}) \geq i\}.$$

Hence, $q(\tau,i)$ is the i'th quantile of the distribution of consumption, given τ.

Every consumer is endowed with a pair of utility functions $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ and $v : \mathbb{R}_+ \to \mathbb{R}$. We assume that both functions are continuous, increasing, and concave. Given the distribution of consumption τ, the objective of every agent is to solve

$$\max_{(a,n)\in B(m,w)} u(a,n) + v(a - q(\tau,i)).$$

Therefore, when choosing her consumption and leisure every agent takes into account their direct effect summarized by function u, as well as the utility that comes from the relative consumption.

Note that in the above framework we model the social status differently that in the motivating example. Since agents' utility depends positively on the difference between their consumption and a certain quantile of the distribution of consumption, the social status is defined relatively to the mass of agents that consume less than a given player. We consider this to be a good approximation of the consumer choice when the social class concerns are taken into consideration. Clearly, agents want their consumption to dominate the consumption of a certain fraction of the population, rather than a certain level of consumption.

Given the monotonicity of u, the budget constraint is always binding. Hence, in any optimal solution (a, n) we have $n = (m + w - a)/w$. This allows us to reduce the number of variables in the consumer optimization problem. Given the condition, the modified payoff of the consumer is

$$r(\alpha, \tau, a) := u(a, (m + w - a)/w) + v(a - q(\tau, i)),$$

where $\alpha = (m, i)$, while the set of feasible strategies can be represented by $\tilde{A}(\alpha) = [0, (w + m)/w]$. Hence, given τ, the consumer objective is to maximize $r(\alpha, \tau, a)$ with respect to $a \in A(\alpha)$.

In order to apply our main results to the above example, we need to show that it satisfies the conditions of Assumption 3.1. Clearly, set A is a lattice, while correspondence $\tilde{A}$ is continuous and complete lattice valued. By assumptions imposed on functions u and v, function r is continuous in a and measurable with respect to the characteristic α. Therefore, it suffices to show that is has single crossing differences in (a, τ).

First, note that the quantile function $q(\cdot, i)$ is increasing on $(\mathcal{D}, \succeq_P)$ for any i. Take any $a' \geq a$ in $\tilde{A}(\alpha)$ and $\tau' \succeq_P \tau$ in $\mathcal{D}$. Then,

$$\begin{aligned} r(\alpha, \tau, a') - r(\alpha, \tau, a) &= u(a', (m + w - a')/w) - u(a, (m + w - a)/w) \\ &\quad + v(a' - q(\tau, i)) - v(a' - q(\tau, i)) \\ &\leq u(a', (m + w - a')/w) - u(a, (m + w - a)/w) \\ &\quad + v(a' - q(\tau', i)) - v(a' - q(\tau', i)) \\ &= r(\alpha, \tau', a') - r(\alpha, \tau', c), \end{aligned}$$

where the inequality is implied by monotonicity of $q(\cdot, i)$ and Lemma 8.2. Hence, function $r(\alpha, \tau, a)$ has increasing differences in (a, τ). Therefore, the above example satisfies the conditions stated in Assumption 3.1.

6 Conclusion

In this paper, we have proven the existence of distributional equilibria in a class of large games with strategic complementarities under rather general conditions that do allow for non-aggregative structures. Moreover, we provide tools for equilibrium computation and equilibrium comparative statics again without assuming players interact via some aggregates. The techniques of the paper can be generalized, however, in the few directions that we discuss here.

First, there is a set of open questions that shall be answered in subsequent work. For example, can our equilibrium existence and characterization results be generalized to (i) games that allow for infinite-dimensional spaces of actions as in Khan, Rath, and Sun (1997), and/or (ii) large Bayesian games (see Balder and Rustichini, 1994; Kim and Yannelis, 1997). Given the order theoretical tools applied in our paper, and results for standard Bayesian supermodular games (see e.g. Van Zandt, 2010), we think that at least some of these generalizations should be possible using extensions of the methods introduced in this paper.

Second, using the fixed point results in Heikkilä and Reffett (2006) for set-valued *ascending* mappings in products of posets, it should be possible to generalize the action spaces used by all players to more general posets.

Third, using recent result for dynamic supermodular stochastic games (see Balbus, Reffett, and Woźny (2014b)), we should be able extend our results to dynamic settings such as those studied in Bergin and Bernhardt (1992); Jovanovic and Rosenthal (1988) to large anonymous dynamic games with strategic complementarities (either for subgame perfect and Markovian equilibrium). This last extension is nontrivial, as it involves developing an appropriate dynamic law of large numbers for a continuum of random variables in the context of a large GSC, and will therefore be closely related to the existence and purification of equilibrium in large games (see Pascoa, 1998).

7 Related results

Since the seminal work of Schmeidler (1973) and Mas-Colell (1984), the study of equilibrium in games with a continuum of players has been the focus of a great deal of work in economics. Despite the obvious similarities in specifications of the primitives of these two versions of a large game, the definitions of equilibria proposed by these two authors display some well-known differences (see e.g. discussion in Khan, 1989). Schmeidler studies a game where what is payoff relevant for each player is her own action, as well the profile of actions undertaken by each of her opponents. Therefore, for large games in the tradition of Schmeidler, an appropriate notion of equilibrium is an equilibrium in *strategies* (i.e. functions mapping from the set of players to their action sets). In contrast, the notion of equilibrium in the large game of Mas-Colell differs from an equilibrium strategies *à la* Schmeidler a great deal. In Mas-Colell, each players' payoff depends on her own action, we well as the *distribution* of all other players actions. This latter payoff structure leads to a notion of Nash equilibrium that is defined relative to *distributions* over both players' characteristics and actions. Given this latter specification of equilibrium, the term *anonymous* game seems readily justified, as the "names" of particular players does not matter when determining each player's best reply; hence, equilibrium payoffs depend only on the distribution of the actions of the other players.[42] Despite these critical differences, both approaches seem natural and appropriate generalizations of the notion of Nash equilibria in games with a finite number of players to situations, where one might expect the marginal influence of any particular player's action on the structure of equilibrium payoffs/aggregates to be in some sense "insignificant" (see e.g. Horst and Scheinkman, 2009).

Regardless of the notion of an equilibrium that one studies, a central question that immediately arises is existence of such equilibrium. Here, in the existing literature, the toolkit used to verify the existence of equilibrium seems to have much in common. Namely, for both notions of equilibrium in the large game, its existence of equilibrium has typically been resolved appealing to topological fixed points theorems for continuous mappings that transform compact convex topological spaces.[43] For example, specific existence of equilibrium results per both notions of equilibria of large games have been presented in various interesting papers that include the following: (i) Khan (1986, 1989); Khan, Rath, and Sun (1997), Balder (1999) where the authors allow for general action spaces for players; (ii) Rath (1996), where the issue is how to generalize existence results to the case of upper semi-continuous payoffs; (iii) Balder and Rustichini (1994) and Kim and Yannelis (1997), where the question is how to generalize the results to large games with differential information;[44] (iv) Martins da Rocha and Topuzu (2008), where the question is how to generalize the results to the case of non-ordered preferences, and (v) Khan, Rath, Yu, and Sun (2005), where authors analyze games with metrizable characteristics, including

[42]Here, let us mention that anonymity can be also modeled using Schmeidler's (1973) approach, where each player's payoff depends on his/her own action and an aggregate (e.g. average) of players' strategies.

[43]That is, applying fixed point theorems in the spirit of Schauder, Fan-Glicksberg, Himmelberg, etc. See, for example, Mas-Colell (1984) and Schmeidler (1973).

[44]Compare also with Balder (2002) unifying approach to equilibrium existence and interesting discussion in Carmona and Podczeck (2009).

equilibrium in behavior/mixed strategies with appropriate laws of large numbers. For a nice survey of some recent literature, we refer the reader to Khan and Sun (2003).[45]

In a second important strand of work in game theory (for games with a *finite* number of players), researchers have focused on games with strategic complementarities (henceforth, GSC). Examples of this work (along with numerous applications) includes the seminal work of Topkis (1979), Vives (1990), and Milgrom and Roberts (1990).[46] In a GSC, due to the underlying complementarities in the game, the existence of pure strategy Nash equilibrium does not hinge on convexity arguments and/or argument concerning the upper hemicontinuity of best replies maps, but merely on the existence of monotone increasing best replies (in a well-defined set theoretic sense), as well as the complete lattice structure of agent action spaces.[47] In this situation, one can appeal to the powerful fixed point theorems of Tarski (1955, Theorem 1) or Veinott (1992, Chapter 4, Theorem 14) for increasing or ascending transformations of complete lattices to obtain nonempty complete lattice of pure strategy equilibria.[48] In addition, continuity issues per the existence of nonempty best reply maps can be resolved by appealing to order theoretic maximum theorems based upon order semicontinuity conditions on payoffs. Also, in parameterized versions of at GSC, one can exploit complementarities between parameters and best replies to obtain sufficient conditions for monotone equilibrium comparative statics.[49] Finally, as researchers applying results from the literature seek to compute Nash equilibrium at various parameter configurations, one important advantage of studying equilibrium comparative statics in a GSC is that in principle constructive methods for computing

[45]We also should mention the results in Blonski (2005) on equilibrium distributions characterization, and the results in Rashid (1983) and Carmona and Podczeck (2011) on approximation of equilibria by equilibria in games with finite number of players, as these ideas are also related to the questions raised in this paper.

[46]In the operations research literature, Topkis (1979) presented the first important results on supermodular games. In economics, the pioneering work was presented by Vives (1985), Vives (1990) and Milgrom and Roberts (1990). These latter authors extended the theoretical results known, as well as showed how to apply these methods to important problems in economics. For additional theoretical results and extensions, see the contributions of Veinott (1992), Zhou (1994), and more recently, Heikkilä and Reffett (2006). Finally for an excellent survey of this literature with applications to economic theory see Vives (2005) or Topkis (1998).

[47]Issues having to do with purification for games with finite or countable action spaces in GSC can also be resolved easily. This fact is particularly important in games with a measure space of players, as it avoids the need to deal explicitly with issues relating to the law of large numbers for a continuum of random variables that arise when purification issues are studied.

[48]For existence results appealing to Veinott's version of Tarski, see the independent work of Zhou (1994).

[49]In some cases conditions for equilibrium comparative statics can be developed with topological approaches. We should mention, though, we are aware of no equilibrium comparative statics results in the existing literature for large games where the existence of equilibirum is verified using topological fixed point theorems based upon convexity/compactness conditions.

extremal equilibrium can be developed that are tied directly to actual fixed point operators used to construct the set of pure strategy Nash equilibrium. Hence, in a GSC, one can seek to develop sufficient conditions for the existence of *computable* equilibrium comparative statics. It is important to keep in mind that all these results for GSC in the existing literature concern games with a finite number of players.

8 Technical Appendix

8.1 Auxiliary results

Markowsky (1976) presents the following theorem (see theorem 9 in his paper).

Theorem 8.1 (Markowsky, 1976) *Let $F : X \to X$ be increasing, and X a chain complete poset. Then, the set of fixed points of F is a chain complete poset. Moreover, we have $\bigvee \{x : x \leq F(x)\}$ the greatest fixed point, and $\bigwedge \{x : x \geq F(x)\}$ the least fixed point of F.*

For the sake of completeness we present its proof.

Proof: *Step 1.* We construct transfinite sequence: $f(0) = \bigwedge X$ and for $\alpha \in [0, \xi + 1]$ we have $f(\alpha) = \bigvee \{F(f(\alpha')) : \alpha' < \alpha\}$. We claim that $f(\alpha)$ is well defined and it is a chain isotone with α. For $\alpha = 0$ the proof is straightforward. Assume the thesis is satisfied for some α_0. If $\alpha_0 + 1$ is a successor of α_0 then $\{F(f(\alpha')) : \alpha' < \alpha_0\}$ is a chain, hence, has a supremum. Moreover, $F(f(\alpha_0))$ is a bound of this set. Therefore,

$$F(f(\alpha_0)) = \bigvee \{F(f(\alpha')) : \alpha' < \alpha_0\} \cup \{F(f(\alpha_0))\} = \bigvee \{F(f(\alpha')) : \alpha' < \alpha_0 + 1\} =: f(\alpha_0 + 1).$$

This implies that $\{f(\alpha') : \alpha' < \alpha_0 + 1\}$ is a chain. If α_0 is a limiting point, and all thesis are satisfied for ordinal numbers smaller then α_0, then

$$\{F(f(\alpha')) : \alpha' < \alpha_0\} = \bigcup_{\alpha < \alpha_0} \{F(f(\alpha')) : \alpha' < \alpha\}$$

Clearly the right hand side is an ascending sequence of chains, hence it is a chain. As X is chain complete, this set possess a supremum and it is $f(\alpha_0)$. By transfinite induction $f(\alpha)$ is a well defined sequence.

Step 2. We show the existence of the greatest and the least fixed points. Since $\xi + 1$ has cardinality less then cardinality X, there exist α_0 such that $f(\alpha_0) = f(\alpha_0 + 1)$. Hence the set of ordinal numbers $\{\alpha : f(\alpha) = f(\alpha+1)\}$ has the least element α_0. Let $e_0 := f(\alpha_0)$. Then $e_0 = f(\alpha_0) = f(\alpha_0+1) = F(f(\alpha_0)) = F(e_0)$. Therefore e_0 is a fixed point. Let e be arbitrary fixed point. We show that $\forall_\alpha : f(\alpha) \leq e$. We examine $P_\alpha : f(\alpha) \leq e$ by transfinite induction. For $\alpha = 0$ it is straightforward. Assume that this hypothesis is satisfied for all $\alpha' < \alpha$. Then $F(f(\alpha')) \leq F(e) = e$. Thus $f(\alpha) \leq e$. Hence $f(\alpha) \leq e$ for all α. Observe that $e_0 = f(\alpha_0) \leq e$. Hence e_0 is the least fixed point. Similarly we show existence of the greatest fixed point.

Step 3. Let $E := \{x : x \leq F(x)\}$. We show that $e_0 = \bigvee E$. First of all, observe that F maps E into itself. If $x \in E$ then $F(x) \geq x$ and by monotonicity $F(F(x)) \geq F(x)$ hence $F(x) \in E$. Observe, that all fixed points are elements of E. Moreover, observe that if we take arbitrary element of E such that $x_0 < F(x_0)$, then x_0 may not be the greatest element, since $F(x_0) \in E$. Hence only the greatest fixed point can be $\bigvee E$. To show it we can repeat the construction from step 1 with $f(0) = x_0$, with arbitrary $x_0 \in E$. Then the sequence $f(\alpha)$ is isotone and for some α_0 such that $f(\alpha_0) = f(\alpha_0+1)$ i.e. $f(\alpha_0)$ is a fixed point. Hence $x_0 \leq f(\alpha_0) \leq e_0$. Since $e_0 \in E$ hence $e_0 = \bigvee E$. Similarly we show that $\bigwedge\{x : x \geq F(x)\}$ is the least fixed point.

Step 4. Finally we show that set of fixed points is a chain complete poset. Let C be some chain of fixed points. Let $c = \bigvee C$. Consider a set of upper bounds of C: $Y := \{x : x \geq c\}$. If $x \in Y$ then for all $e \in C$ we have $x \geq c \geq e$ and hence $F(x) \geq F(c) \geq F(e) = e$. Hence by definition of c we have $F(x) \geq c$, and hence $F(Y) \subset Y$. Clearly Y is chain complete poset, and $G := F|_Y$ is monotone. Hence by previous part $F|_Y$ has the least fixed point. This is the least upper bound of C in the set of fixed points of F. ■

In many cases, we will be needing a constructive version of Markowsky's theorem. The Tarski-Kantorovich fixed point theorem is one such a theorem. We now provide a generalization of the Tarski-Kantorovich result that additionally provides fixed-point comparative statics results in the spirit of Veinott's (1992) version of Tarski's theorem. See Balbus, Reffett, and Woźny (2014a) where it was first stated. For a monotone sequence $\{x_n\}_{n=0}^{\infty}$, let the top (resp, bottom) be denoted by:

$$\bigvee x_n := \sup_{n\in\mathbb{N}} x_n, \text{ and } \bigwedge x_n := \inf_{n\in\mathbb{N}} x_n.$$

By $F^n(x)$, we are referring to the n-th orbit (or iteration) of the function F from the point $x \in X$, i.e. $F^n(x) = F \circ F \circ \ldots \circ F(x)$. We now define a few key terms:

Definition 8.1 *A function $F : X \to X$ is monotonically sup-preserving (resp., monotonically inf-preserving) if for any monotone sequence $\{x_n\}_{n=0}^{\infty}$, we have: $F\left(\bigvee x_n\right) = \bigvee F(x_n)$ (resp., $F\left(\bigwedge x_n\right) = \bigwedge F(x_n)$. F is said to be monotonically sup/inf-preserving if and only if, it is both monotonically sup- and monotonically inf-preserving.*

It is worth noting that a monotonically sup (resp., inf) preserving function is necessarily increasing. The Tarski-Kantorovich Theorem (see Dugundji and Granas, 1982, Theorem 4.2) states the following:

Theorem 8.2 (Tarski-Kantorovitch) *Let X be a countably chain complete poset with the greatest $\overline{x}$ and the least element $\underline{x}$ respectively, $F : X \to X$, and Φ the fixed point set of the function F. Then:*

(i) if F is monotonically inf-preserving, $\bigwedge F^n(\overline{x})$ is the greatest fixed point of F (denoted by $\overline{\Phi}$),

(ii) if F is monotonically sup-preserving, $\bigvee F^n(\underline{x})$ is the least fixed point of F (denoted by $\underline{\Phi}$).

We now prove the following two new, but related, theorems. The first theorem pertains to the characterization of the fixed-point set of the mapping F, while the second theorem pertains to fixed-point comparative statics for parameterized versions of the mapping F.

Theorem 8.3 *Let X be a countably chain complete poset, $F : X \to X$ a monotonically sup/inf-preserving function, Φ the fixed point set of F. Then, the set of fixed points is a nonempty countably chain complete poset with*

$$\overline{\Phi} = \bigvee \{x : F(x) \geq x\}, \tag{1}$$

and

$$\underline{\Phi} = \bigwedge \{x : F(x) \leq x\}. \tag{2}$$

Proof: By Tarski-Kantorovich Theorem, F has a nonempty set of fixed points. Let e_n be an countable chain of fixed points, and $\bar{e} = \bigvee e_n$. Then,

$$F(\bar{e}) = F\left(\bigvee e_n\right) = \bigvee F(e_n) = \bigvee e_n = \bar{e}.$$

Similarly, we the show same result for $\underline{e}$=$\bigwedge e_n$. Now, we finally prove equality (1). Let x be arbitrary point such that $x \leq F(x)$. Clearly $x \leq \overline{x}$. Assume $x \leq F^n(\overline{x})$. Then,

$x \leq F(x) \leq F(F^n(\overline{x})) = F^{n+1}(\overline{x})$. Hence, $x \leq \overline{\Phi}$. Since $\overline{\Phi} \in \{x : F(x) \geq x\}$, equality (1) is proven. We prove (2) analogously. ■

Theorem 8.4 *Let X be a countably chain complete poset with the greatest and least elements, T a poset, $F : X \times T \to X$, with $F(\cdot, t)$ monotonically inf (resp. sup) preserving on X and, $F(x, \cdot)$ increasing. Then, the mapping $t \to \overline{\Phi}(t)$ (resp. $t \to \underline{\Phi}(t)$) is increasing.*

Proof: Let $t_1 \leq t_2$. From Theorem 8.3 we know that $m_i := \overline{\Phi}(t_i) = \bigvee \Gamma_i := \bigvee\{x : F(x, t_i) \leq x\}$. Note that by isotonicity of $F(x, \cdot)$ we obtain $m_1 = F(m_1, t_1) \leq F(m_1, t_2)$. Hence $m_1 \in \Gamma_2$. Since m_2 is the greatest element of Γ_2, hence $m_1 \leq m_2$. ■

In the case of chain complete poset X, we can relax continuity of F in Theorem 8.4.

8.2 Proofs of main results

Assume $(X, \leq)$ is an ordered set. Consider a compact Hausdorff topology $\mathcal{X}$ with $\mathcal{B}$ Borel σ-field on it. Let $C(X)$ be a set of continuous real valued functions on X supported on compact set. Let P be a set of isotone, measurable and bounded real valued functions. We say that ν stochastically dominates μ, and we write $\nu \succeq_P \mu$, if

$$\int_X f d\nu \geq \int_X f d\mu \text{ for all } f \in P.$$

Let $M(X)$ be a set of regular distributions on X.

Lemma 8.1 *If P separates the points, then $(M(X), \succeq_P)$ is an ordered space.*

Proof: Clearly $\succeq_P$ is reflexive and transitive. We need to show it is antisymmetric. Assume $\mu \succeq_P \nu$ and $\nu \succeq_P \mu$. Then

$$\int_X f d\mu = \int_X f d\nu \tag{3}$$

for all $f \in P$. Clearly a set $P - P$ is a Riesz subspace[50] of $C(X)$. Moreover, it contains a constant functions, hence by Stone Weierstrass Theorem $P - P$ is uniformly dense on $C(X)$, hence (3) is satisfied for all $f \in C(X)$. From Urysohn's Lemma (Aliprantis and Border (2006), Lemma 2.46) it follows that $\mu(F) = \nu(F)$ for all closed set. Since both μ

[50] See footnote 5 page 645 in Aliprantis and Border.

and ν are regular hence μ and ν are equal for all Borel sets. ∎

Having that, finally we state:

Proposition 8.1 *$(M(X), \succeq_P)$ is a chain complete poset.*

Proof: Let τ_t, $t \in Z$, be a decreasing chain. Let $T : P \to \mathbb{R}$ be defined as follows

$$T(f) = \inf_{t \in Z} \left\{ \int\limits_X f(s)\tau_t(ds) \right\},$$

and for decreasing functions $f \in -P$

$$T(f) = -\inf_{t \in Z} \left\{ \int\limits_X -f(s)\tau_t(ds) \right\} = \sup_{t \in Z} \left\{ \int\limits_X f(s)\tau_t(ds) \right\}.$$

It is easy to see T is a functional on P, which preserves addition and multiplication on positive scalar. T can be extended to the vector subspace $P - P$ in the natural way: if $f = g_1 - g_2$ for $g_i \in P$ then $T(f) = T(g_1) - T(g_2)$. It can be easily verified T is well defined. Indeed, if $f \in P - P$, then $f = g_1 - (g_1 - f)$ for $g_1 \in P$ and $f - g_1 \in P$

$$\inf_t \int_X g_1(x)\tau_t(dx) = \lim_t \int_X g_1(x)\tau_t(dx),$$

as well as

$$\inf_t \int_X g_1(x)\tau_t(dx) = \lim_t \int_X f(x) - g_1(x)\tau_t(dx).$$

Since T is isotone,

$$|T(f)| \leq ||f||_\infty, \tag{4}$$

and equality may hold since $T(1) = 1$. By the Banach Extension Theorem, there exists an extension $\hat{T}$ of T on all of $C(X)$ that satisfies (4). By The Riesz representation theorem (see Aliprantis and Border (2006), Theorem 14.12) there exists an unique regular measure $\underline{\tau}$ such that

$$\hat{T}(f) = \int\limits_X f(s)\underline{\tau}(ds),$$

for all $f \in B(X)$. Moreover, by (4) it is a probability measure. Clearly, $\underline{\tau}$ is a lower bound of the chain τ_t. We need to show it is the greatest lower bound in the class $\mathcal{M}(X)$. On the contrary, suppose there is another measure τ_0 such that τ_0 is a lower bound of τ_t but it is not dominated by $\underline{\tau}$. Then, there exists a function $\phi \in P$ such that $\int_X \phi d\tau_0 > \int_X \phi d\underline{\tau}$. As τ_0 is a lower bound of τ_t , we have

$$T(\phi) = \inf_{t\in Z}\left\{\int\limits_X \phi(s)\tau_t(ds)\right\} \geq \int_X \phi\, d\tau_0 > \int_X \phi\, d\underline{\tau} = T(\phi),$$

which is a contradiction. Hence, $\underline{\tau}$ is a greatest lower bound of τ_t. Similarly, we show the thesis for increasing sequence. ■

Proof of Lemma 3.1: We show that operator $\overline{T}$ is well defined (the same argument can be used for $\underline{T}$). Since $a \to r(\alpha, a, \tau)$ is quasi-supermodular and satisfies the single crossing property in (a, τ), under our continuity conditions on payoffs, by the standard monotonicity theorem in (Milgrom and Shannon, 1994, Theorem 4), the set of maximizers is a complete sublattice for fixed τ. Moreover, the set of maximizers is isotone in the Veinott strong set order in τ (with respect to first order stochastic dominance).

We next show that the set of maximizers has an increasing measurable selector. As r is a Carathéodory for all τ and $\tilde{A}(\alpha)$ is weakly measurable, by Theorem 18.19 in Aliprantis and Border (2006), $R_\tau(\alpha) := \max\limits_{a\in\tilde{A}(\alpha)} r(\alpha, a, \tau)$ is measurable, and the arg max correspondence $m(\alpha, \tau)$ is therefore measurable. As $m(\cdot, \tau)$ maps a measurable space into metrizable space, it is weakly measurable (see Aliprantis and Border, 2006, Theorem 18.2). Further, observe that $\overline{m}(\alpha, \tau) = (\hat{a}_1, ..., \hat{a}_m)$, where $\hat{a}_i = \max\limits_{a\in m(\alpha,\tau)} Proj_i(a)$ and $Proj_i$ denotes a projection of a vector on its i^{th} coordinate.[51] Since a projection is a continuous function in this context, by the Measurable Maximum Theorem (see Aliprantis and Border, 2006, Theorem 18.19), $\overline{m}(\alpha, \tau)$ is λ-measurable function. Therefore, by Himmelberg theorem (see lemma 18.4 in Aliprantis and Border (2006)), $\overline{T}$ is well defined and maps $\mathcal{R}$ into $\mathcal{D}$.

Step 2. As a result, for arbitrary $\alpha \in \Lambda$ we have $\overline{m}(\alpha, \tau_2) \geq \overline{m}(\alpha, \tau_1)$ whenever $\tau_2 \succeq_P \tau_1$. We now show that $\overline{T}(\tau_2) \succeq_P \overline{T}(\tau_1)$. Let $f : \lambda \times A \to \mathbb{R}$ be a increasing and measurable function. Then,

$$\begin{aligned} \int\limits_{\Lambda_0\times A_0} f(\alpha, a)\overline{T}(\tau_2)(d\alpha \times da) &= \int\limits_{\Lambda_0} f(\alpha, \overline{m}(\alpha, \tau_2))\lambda(d\alpha) \geq \\ \int\limits_{\Lambda_0} f(\alpha, \overline{m}(\alpha, \tau_1))\lambda(d\alpha) &= \int\limits_{\Lambda_0\times A_0} f(\alpha, a)\overline{T}(\tau_1)(d\alpha \times da), \end{aligned} \tag{5}$$

[51]For $a_i \in \mathbb{R}^m$ by max we denote a max of all coordinates.

where first equality follows by definition of $\overline{T}$, and (5) follows since both f and $\overline{m}(\alpha, \cdot)$ are increasing. We use a similar argument to prove monotonicity of $\underline{T}$.

Step 3. Finally we need to show that $\underline{T}$ and $\overline{T}$ maps $\mathcal{R}$ into $\mathcal{D}$. It is obvious that $\underline{T}$ and $\overline{T}$ are supported in graph of $\tilde{A}$. To finish this proof we simply need to show for all $\tau \in \mathcal{R}$ measures $\overline{T}(\tau)$ and $\underline{T}(\tau)$ are regular. Obviously marginal distribution on Λ is regular as well as on A since it is a bounded Borel measure on A which is metrizable (see Theorem 12.5 in Aliprantis and Border (2006)). Hence by Lemma 8.4 $\overline{T}(\tau)$ and $\underline{T}(\tau)$ are regular. ■

Proof of Theorem 3.1: We now show existence of the greatest equilibria (analogously, we can prove the existence of the least equilibrium).

Step 1. Observe that $\tilde{A}(\alpha)$ is compact-valued (hence, by Lemma 3.2, $\mathcal{D}$ is a chain complete poset). By Lemma 3.1, $\overline{T}$ is a monotone operator that maps $\mathcal{R}$ into $\mathcal{D} \subset \mathcal{R}$. MTherefore, by Theorem 8.1, we conclude the set of fixed points of $\overline{T}$ has the greatest element. Clearly all such fixed points belong to $\mathcal{D}$. Let τ^* be such a point. Then we have

$$\tau^*(\{(\alpha, a) : a \in m(\alpha, \tau^*)\}) \geq \tau^*(\{(\alpha, a) : a = \overline{m}(\alpha, \tau^*)\}) = 1,$$

which implies that τ^* is a distributive equilibrium.

Step 2. We show that τ^* is the greatest distributional equilibrium. Let τ_0 be some other equilibrium. Then, by definition of distributional equilibria

$$1 = \tau^0\left(\{(\alpha, a) : a \in m(\alpha, \tau^0)\}\right) \leq \tau^0\left(\{(\alpha, a) : a \leq \overline{m}(\alpha, \tau^0)\}\right).$$

Therefore, τ^0 is concentrated in the set $E_0 = \{(\alpha, a) : a \leq \overline{m}(\alpha, \tau^0)\}$. Taking an increasing function $f : \Lambda \times A \to \mathbb{R}$ we have

$$\begin{aligned}
\int_{\Lambda \times A} f(\alpha, a)\tau^0(d\alpha \times da) &= \int_{E_0} f(\alpha, a)\tau^0(d\alpha \times da) \\
&\leq \int_{\Lambda} f(\alpha, \overline{m}(\alpha, \tau^0))\tau^0(d\alpha \times da) \qquad (6) \\
&= \int_{\Lambda} f(\alpha, \overline{m}(\alpha, \tau^0))\lambda(d\alpha) \\
&= \int_{\Lambda} f(\alpha, a)\overline{T}(\tau^0)(d\alpha \times da),
\end{aligned}$$

where (6) follows by definition of E_0, and the last equation follows from definition of

$\overline{T}$. Therefore, $\tau^0 \preceq_P \overline{T}(\tau^0)$. By Lemma 3.1, $\overline{T}$ is increasing. Hence, by Theorem 8.3, $\tau^* \succeq_P \tau^0$, and we have τ^* the greatest distributional equilibrium.

Step 3. We next show that under additional continuity assumptions, $\overline{T}$ is monotonically inf-preserving. Let $\tau^n \in \mathcal{D}$ be a sequence, monotonically decreasing with infimum τ. This limit exists as by Lemma 3.2, the set of distributions in $\mathcal{D}$ is a chain complete. Then, we have

$$r(\alpha, \overline{m}(\alpha, \tau^n), \tau^n) \geq r(\alpha, a, \tau^n).$$

Since $\overline{m}(\alpha, \tau^n) \in \tilde{A}(\alpha)$ is a compact subset, the limit of this sequence (say m_0) exists because of Lemma 3.2 distributions form continuous chain complete poset, and satisfies $m_0 \in \tilde{A}(\alpha)$. By continuity of r, we have $r(\alpha, m_0, \tau) \geq r(\alpha, a, \tau)$ for all $a \in \tilde{A}(\alpha)$. Therefore, $m_0 \in m(\alpha, \tau)$, and

$$m_0 \leq \overline{m}(\alpha, \tau). \tag{7}$$

On the other hand, since $\tau \preceq_P \tau^n$ for all n, by monotonicity of $\overline{m}$, we have $\overline{m}(\alpha, \tau) \leq \overline{m}(\alpha, \tau^n)$ and

$$\overline{m}(\alpha, \tau) \leq \lim_{n\to\infty} \overline{m}(\alpha, \tau^n) = m_0. \tag{8}$$

Combining (7) and (8), we have desired equality: $\overline{m}(\alpha, \tau) = \lim_{n\to\infty} \overline{m}(\alpha, \tau^n)$, and we conclude $\overline{T}$ is monotonically inf-preserving.

Step 4. We next show that $\tau^* = \lim_{n\to\infty} \overline{T}^n(\bar{\delta})$. (Similarly, we construct the least distributional equilibrium using $\underline{T}$.) By Step 1, $\overline{T}$ is monotonically inf-preserving. By Lemma 3.1, $\overline{T}$ is increasing, and therefore we have the convergence of $\overline{T}^n(\bar{\delta}) \to \tau^*$ in $\mathcal{P}$ by Theorem 8.2.

Step 5. If $\overline{T} = \underline{T} := T$, by Theorem 8.1, the fixed point set of T is a chain complete poset. ■

Proof of Proposition 4.2: By Corollary 4.1 the greatest and the least equilibrium of the game exist. Moreover, by Proposition 4.1, they are both symmetric. Hence, it is sufficient to show that the game has a unique symmetric distributional equilibrium. We divide our proof into following steps.

Step 1. Define a finite game with $n+1$ players in the following way. $\forall i = 1, \ldots, n$, $\pi_i(a_i, b) = r(\alpha, a_i, \bar{h}(b)), \alpha \in J_i, a_i \in A, b \in \times_{i=1}^n \tilde{A}_i$. Each of the first n players maximizes r_i with respect to a_i over $\tilde{A}_i = \tilde{A}(\alpha), \alpha \in J_i$. The $n+1^{\text{st}}$ player payoff is then $-\sum_{i=1}^n \|a_i - b_i\|_1$, $b_i \in \tilde{A}_i$, which is maximized with respect to b over $\times_{i=1}^n \tilde{A}_i$. Endow A with a *taxicab* norm $\|\cdot\|_1$, and $\times_{i=1}^n \tilde{A}_i$ with a n times product of this norm.

Step 2. Consider one of the first n players, say player i. Let $x_i(b) := \arg\max_{a \in \tilde{A}_i} \pi_i(a, b)$.

Since r is strictly quasiconcave in a, continuous in b, and $\tilde{A}_i$ is compact and convex, x_i is a continuous function (see Berge Maximum Theorem). Moreover, due to assumption 4.1 concerning quasisupermodularity of r, x_i is also increasing (e.g. the monotoncity theorem in Milgrom and Shannon (1994)).

Step 3. Define $\tilde{\pi}_i(y,b) := \pi_i(M\phi(b)\mathbf{1} - y, b)$, where $\phi(b) = \sum_{i=1}^n \sum_{j=1}^m b_{ij}$, $b_i \in \tilde{A}_i$, $b_{ij} \in \mathbb{R}$ is the j^{th} coordinate of b_i, $y_i \in \tilde{A}_i$, and $\mathbf{1}$ is a unit vector in $\mathbb{R}^m$. Now we will show that $\tilde{\pi}_i(y,b)$ has increasing differences in (y,b).

Let y_j be the j^{th} coordinate of y. Observe, that $\frac{\partial \tilde{\pi}_i}{\partial y_j} = -\frac{\partial r}{\partial a_j}(\alpha, \cdot)$. Therefore

$$\begin{aligned}\frac{\partial^2 \tilde{\pi}_i}{\partial y_j \partial b_{k,z}} &= -M\frac{\partial \phi}{\partial b_{k,z}} \sum_{l=1}^m \frac{\partial^2 r}{\partial a_j \partial a_l}(\alpha, \cdot) - \frac{\partial^2 r}{\partial a_j \partial g}(\alpha, \cdot)\frac{\partial \bar{h}}{\partial b_{k,z}} \\ &\geq -M\left(\sum_{l=1}^m \frac{\partial^2 r}{\partial a_j \partial a_l}(\alpha, \cdot) + \frac{\partial^2 r}{\partial a_j \partial g}(\alpha, \cdot)\right) \geq 0,\end{aligned} \tag{9}$$

$\forall j,k,z,i$, where $\alpha \in J_i$. Since $\tilde{A}_i$ is a compact, convex subset of $\mathbb{R}^m_+$, denote it by $\tilde{A} \equiv [0, \tilde{a}_i]$, where $\tilde{a}_i \in \mathbb{R}^m_+$. The set of feasible y is therefore defined by $[M\phi(b)\mathbf{1} - \tilde{a}_i, M\phi(b)\mathbf{1}]$, which is ascending in the Veinott strong set order in b. Therefore, by Theorem 6.2 in Topkis (1978),

$$y_i^*(b) = \arg \max_{y \in [M\phi(b)\mathbf{1} - \tilde{a}_i, M\phi(b)\mathbf{1}]} \tilde{\pi}_i(y,b),$$

is ascending is the Veinott strong set order. Since $x_i(b)$ is defined by $y_i^*(b) = M\phi(b)\mathbf{1} - x_i(b)$, y_i^* is a function. Denote the j^{th} coordinate of $x_i(b)$ and $y_i^*(b)$ respectively by $x_{ij}(b)$ and $y_{ij}^*(b)$. By definition of $y_i^*(b)$, for any $b' \geq b$, $\forall j = 1, \ldots, m$,

$$0 \leq x_{ij}(b') - x_{ij}(b) \leq M(\phi(b') - \phi(b)) = M\|b' - b\|_1,$$

and $x_{ij}(b) - x_{ij}(b') \geq -M\|b' - b\|_1$. For any two unordered b' and b,

$$0 \leq x_{ij}(b') - x_{ij}(b) \leq x_{ij}(b' \vee b) - x_{ij}(b' \wedge b) \leq M\|b' - b\|_1,$$

since $\|b' - b\|_1 = \|b' \vee b - b' \wedge b\|_1$. Similarly, $x_{ij}(b) - x_{ij}(b') \geq -M\|b' - b\|_1$. Therefore, since $M < 1$, $\forall i, x_i$ is Lipschitz with modulus $M < 1$.

Step 4. Define operator $T : \times_{i=1}^n \tilde{A}_i \to \times_{i=1}^n \tilde{A}_i$ by $T(b) := \times_{i=1}^n x_i(b)$. Since $\forall i, x_i$ is Lipschitz with modulus M, T is a contraction. Moreover, $\times_{i=1}^n \tilde{A}_i$ is a closed subset of a Banach space $\mathbb{R}^{mn}$. By The Banach fixed point theorem (e.g., the contraction mapping theorem), T has a unique fixed point, denote it a^*. By definition, $(a = a^*, b = a^*)$ is the unique Nash equilibrium of the game defined in Step 1.

Step 5. Note that τ^* satisfying $\tau^*(\cdot|\alpha) = a_i^*$ for $\alpha \in J_i$ for $i = 1, ..., n$ is the unique symmetric distributional equilibrium of the large aggregative game with n types. Since the greatest and the least equilibria are symmetric, they must be equivalent. ■

Lemma 8.2 *Let $f : X \to \mathbb{R}$, where X is a convex subset of $\mathbb{R}$, be a concave function. Define $Y := \{(x, s) \in \mathbb{R}^2 \mid (x - s) \in X\}$. Then function $g : Y \to \mathbb{R}$, defined by $g(x, s) := f(x - s)$, has increasing differences in (x, s).*

Proof: Since f is concave, for any $x' \geq x$, and $s' \geq s$ such that f is well defined,

$$\frac{f(x' - s') - f(x - s')}{(x' - s') - (x - s')} \geq \frac{f(x' - s) - h(x - s)}{(x' - s) - (x - s)},$$

which implies that $f(x' - s') - f(x - s') \geq f(x' - s) - f(x - s)$. Since $g(x, s) = f(x - s)$, the proof is complete. ■

Lemma 8.3 *Let $f : X \to \mathbb{R}$, where X is a convex subset of $\mathbb{R}_+$, be a decreasing and concave function. Let $Y : \{(x, s) \in \mathbb{R}^2 \mid |x - s| \in X\}$. Then function $g : Y \to \mathbb{R}$, defined by $g(x, s) := f(|x - s|)$ has increasing differences in (x, s).*

Proof: First, we prove that for any convex set $X \in \mathbb{R}_+$, function $h : X \to \mathbb{R}$, $h(x) := f(|x|)$, is concave. Take any $x', x \in X$ and $\alpha \in [0, 1]$. Since X is convex, $\alpha x + (1 - \alpha)x' \in X$. Then

$$f(|\alpha x + (1 - \alpha)x'|) \geq f(\alpha|x| + (1 - \alpha)|x'|) \geq \alpha f(|x|) + (1 - \alpha)f(|x'|),$$

where the first inequality is implied by the triangle inequality and monotonicity of f, while the second follows from concavity of f. The rest is implied by Lemma 8.2. ■

Lemma 8.4 *Let X and Y be a Hausdorff topological spaces with sigma fields $\mathcal{B}_X$ and responsibly $\mathcal{B}_Y$. Let μ be a finite measure on $\mathcal{B}_X \otimes \mathcal{B}_Y$. Suppose marginals of μ on X and Y (say μ_1 and μ_2) are regular. Then μ is regular measure.*

Proof: By Theorem 12.4 in Aliprantis and Border (2006) we need to show that μ is tight. Define

$$\mathcal{E} := \{V \in \mathcal{B}_X \otimes \mathcal{B}_Y : \mu(V) = \sup\{\mu(K) : K \subset V, K \text{ is compact.}\}\}.$$

By standard argument (eg. proof of Theorem 12.5 in Aliprantis and Border (2006)) $\mathcal{E}$ is σ field. We need to show $\mathcal{E}$ includes sets on the form $U_1 \times U_2$, with $U_1 \in \mathcal{B}_X$ and $U_2 \in \mathcal{B}_Y$. Since marginals are tight, hence for given $\epsilon > 0$ there are compact sets K_1 and K_2 such that $K_i \subset U_i$ and $\mu_i(U_i \setminus K_i) < \epsilon/2$ $(i = 1, 2)$. Then

$$\mu((U_1 \times U_2) \setminus (K_1 \times K_2)) \leq \mu((U_1 \setminus K_1) \times Y) + \mu(X \times (Y \setminus K_2)) < \epsilon.$$

Hence $U_1 \times U_2 \in \mathcal{E}$. ∎

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