Iterative Monotone Comparative Statics*

Łukasz Balbus† Wojciech Olszewski‡ Kevin Reffett§
Łukasz Woźny¶

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Abstract

We propose a novel approach to comparative statics which applies to environments where complementarities play a critical role, including environments in which the existing methods for obtaining monotone comparative statics appear inadequate. Our approach is dynamic and, methodologically, in the spirit of the celebrated "correspondence principle" introduced in Samuelson (1947). It applies even to: (a) environments with a continuum of equilibria, (b) environments in which all equilibria are unstable; and (c) chaotic environments, in which adaptive dynamic adjustment processes may not converge.

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† Faculty of Mathematics, Computer Sciences and Econometrics, University of Zielona Góra, Poland.

‡ Department of Economics, Northwestern University, USA.

§ Department of Economics, Arizona State University, USA.

¶ Corresponding author. Department of Quantitative Economics, SGH Warsaw School of Economics. Address: al. Niepodleglosci 162, 02-554 Warszawa, Poland. E-mail: lukasz.wozny@sgh.waw.pl.

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1 Introduction

Comparative statics has always been a foundational question of economic analysis. It asks how the set of optimal or equilibrium solutions of an economic model vary relative to a perturbation of the model's parameters. Such predictions are important as they contain much of the empirical content of the economic model being studied.

The purpose of this paper is to propose a new approach to comparative statics in environments where complementarities play a critical role, including environments in which the existing methods for obtaining monotone comparative statics appear inadequate. Such situations can arise in games with strategic complementarities (GSC), but also in other economic settings such as dynamic general equilibrium economies and social learning models. More specifically, our approach allows for performing comparative statics in settings where there is a continuum of equilibria, all equilibria are unstable, or chaotic settings in which a variable of interest does not converge in the long run.

Our approach to comparative statics can shortly be described as follows: We first identify sharp or tight bounds $\underline{a}(t)$ and $\overline{a}(t)$ for the location of remote iterations of a large class of adaptive learning sequences $(a^k)_{k=0}^{\infty}$ of fundamental variables (e.g., the actions of players in GSC or the prices of commodities in dynamic general equilibrium economies) starting from some initial location a^0. Having the bounds, we compare them (lower to lower and higher to higher) for different parameters t. If the lower (upper) bound for a "new" parameter exceeds the lower (upper) bound for an "old" parameter, then one can say that the choice variables increase with the parameter change (in some weak sense). We also formulate some alternative criteria by requiring that the lower bound for the new parameter exceeds the upper bound for the old parameter, or by requiring the initial a^0 be a steady state for the old parameter. In addition, one need not compare the choice variable but only some statistics thereof of interest. And as we show, such statistics happen to be comparable in the long run, even when the choice variables are not.

To better explain the nature of the paper's methodological contribution, we provide a motivating example that highlights both the limitations of the existing methods and the contributions of our new comparative statics approach.[1]

Example 1. *Our leading example is a simple game with a continuum of actions and a continuum of equilibria. The ideas are easier to explain in this setting. However, analogous arguments apply to more complicated coordination games with a finite number of actions and multiple equilibria.*

Consider the following joint venture. Players 1 and 2 choose actions a_1 *and* a_2 *from* $[0,1]$*, respectively, interpreted as their effort. The cost of taking action* a_i *for player* $i = 1, 2$ *is* ca_i*, for some* $c \in (0,1)$*. The output of the team that consists of the two players is* $2\min\{a_1, a_2\}$*. So the payoff of each player* i *is* $\min\{a_1, a_2\} - ca_i$*. This game has a continuum of equilibria: all pairs* (a_1, a_2) *such that* $a_1 = a_2$ *are equilibrium strategies.*

Suppose that players are initially playing actions a_1^0 *and* a_2^0*, and the productivity of player 1 increases, so the output becomes* $2\min\{ta_1, a_2\}$ *for some* $t > 1$*, and the payoffs are now* $\min\{ta_1, a_2\} - ca_i$ *for* $i = 1, 2$*. Intuitively, the output should increase in response to this positive productivity shock. We cannot make this conclusion, however, by comparing equilibria.*

[1] This is a slightly modified version of the team of managers game studied in Milgrom and Roberts (1990), Section 4, Example (5).

For example, if $a_1^0 = a_2^0 = a^0 \in (0,1)$ initially, the total output in this equilibrium (for $t = 1$) is equal to $2a^0$. And for $t > 1$ ***the game has a continuum of equilibria***. ***In some of them the output is higher*** *than $2a^0$, but* ***in others the output is lower*** *than $2a^0$. Indeed, any efforts a/t and $a \in [0,1]$ are equilibrium strategies. In addition,* ***all equilibria are unstable****, in the sense that for some pairs of efforts arbitrarily close to an equilibrium, the best-response dynamics does not converge to the equilibrium.*

Suppose that players adaptively learn of playing this game with the new parameter t. In discrete time, this learning happens through a sequence of action profiles $(a^k)_{k=0}^{\infty}$, starting from an action profile (not necessarily an equilibrium) $a^0 = (a_1^0, a_2^0)$. We will define the set $\mathcal{S}(a^0)$ of all "reasonable" (adaptive learning) sequences $(a^k)_{k=0}^{\infty}$, and action profiles (the bounds) $\underline{a}(t)$ and $\overline{a}(t)$ such that in the long run (i.e., for large k) a^k cannot be "much smaller" than $\underline{a}(t)$ and cannot be "much larger" than $\overline{a}(t)$ for any sequence $(a^k)_{k=0}^{\infty} \in \mathcal{S}(a^0)$. For now, suppose that $\mathcal{S}(a^0)$ contains only the sequence of best-responses to the actions of the opponents from the previous period. Then:

$$\underline{a}(t) = (\min\{a_1^0, a_2^0/t\}, \min\{a_1^0 t, a_2^0\}), \tag{1}$$

and

$$\overline{a}(t) = (\max\{a_1^0, a_2^0/t\}, \max\{a_1^0 t, a_2^0\}), \tag{2}$$

because these are the lowest and the highest elements of $\mathbb{R}^2$ such that $\underline{a}(t) \leq a^k \leq \overline{a}(t)$ in the coordinate-by-coordinate ordering for remote iterations a^k. For this game, the bounds $\underline{a}(t)$ and $\overline{a}(t)$ for the best-response sequence $(a^k)_{k=0}^{\infty}$ coincide with bounds for other reasonable adaptive sequences, but in general the bounds $\underline{a}(t)$ and $\overline{a}(t)$ are typically lower and higher, respectively, if more general adaptive sequences are allowed. Finally, in this example $br(\liminf_k a^k) = \liminf_k br(a^k)$, where br stands for the best response. We *will prove that if this (and an analogous condition for* $\limsup$*) are satisfied in a setting for all adaptive learning sequences from $\mathcal{S}(a^0)$, then both our long-run bounds $\underline{a}(t)$ and $\overline{a}(t)$ are equilibria. In general, however, $\underline{a}(t)$ and $\overline{a}(t)$ are not equilibria.*

So, suppose first that we start from an (a_1^0, a_2^0) for $t = 1$. In the process of an adaptive learning after the change of t for $t = 1$ to $t > 1$, we end up between $\underline{a}(t)$ and $\overline{a}(t)$ for $t > 1$. Since the output at $\underline{a}(1)$ is no greater than that at $\underline{a}(t)$ and the output at $\overline{a}(1)$ is no greater than that at $\overline{a}(t)$, we claim that the output should increase in response to this positive productivity shock, in some weak sense. But this sense becomes much stronger when $a_1^0 = a_2^0 = a^0$ is an equilibrium. In this case, the output at a^0 is equal to that at $\underline{a}(t)$ and is strictly smaller than that at $\overline{a}(t)$ for $t > 1$.

Note that we reached the conclusion regarding ***the output****, despite the fact that the best-response of one player (player 1) strictly decreases when t increases.*[2] *Yet the output in the long run is never lower.* ***Thus, applying our approach, we can conclude, in contrast to the previous approaches, that the output will increase in response to the increase in*** *t.*

Given the unstability of equilibria, it is plausible to expect chaotic dynamics that may not converge to any equilibrium. We emphasize that our approach also applies to this case. Since formulas (2) still apply, and $\underline{a}(t = 1) \leq \underline{a}(t > 1)$ and $\overline{a}(t = 1) \leq \overline{a}(t > 1)$ for all a^0, with strict inequality for some a^0, and so conclude that the output will weakly increase in response to the increase in t, even if we expect possible chaotic dynamics.

Our approach is iterative and methodologically in the spirit of the celebrated "correspondence principle" first presented in the work of Samuelson (1947) and then extended most notably in a series of papers by Echenique

[2] That is, the example violates an assumption that is typically made in the existing results on monotone comparative statics.

(e.g., Echenique (2002, 2004); see also McLennan (2015)). However, it improves on the existing literature by: (i) weakening the conditions for comparative statics, as it applies to environments with continuum of equilibria, which all can be unstable, and to chaotic environments, in which the initially observed outcome may not be an equilibrium and adaptive learning sequences may not converge; (ii) enlarging the set of possible adaptive learning processes, which leads to more cautious comparative statics.

The main aim of our paper is to introduce a method of conducting comparative statics, and show that it is useful in applications. We provide results that confirm that our method delivers comparative statics that is intuitively expected, even in setting where the previously available tools cannot be applied. When doing so, our propositions extend several existing results on comparative statics for parameterized monotone correspondences in sigma-complete lattices. In particular, we provide conditions under which comparative statics is conclusive for so called "mixed shocks," i.e., shocks affecting some fundamental variables positively and other fundamental variables negatively.

Related Literature For economic models in which the analysis reduces to solving to an optimization problem, there is a large set of comparative statics tools. They involve, among others, the implicit function theorem. These tools typically require strong regularity conditions on the optimization problem (e.g., the smoothness of objectives and constraints, the interiority of all optimal solutions, etc.), and comparative statics predictions are often only local in nature.[3] Alternatively, lattice programming provides a set of tools for obtaining global monotone comparative statics of optimal solutions to parameter change (see Topkis (1998) among others).

[3] There are methods for globalizing the implicit function theorem. See the celebrated work of Gale and Nikaido (1965), as well as more recent contributions (and references therewithin) of Blot (1991), Phillips (2012), and Cristea (2017).

Performing comparative statics analysis on equilibrium problems is more complicated. Especially in economic models with multiple equilibria, fixed-point comparative statics typically involves the tools of transversality and degree theory, and others from differential topology. These tools typically provide only weak local equilibrium comparative statics results, and even for these, they require stronger regularity conditions on the primitives than in the context of implicit function based comparative statics of optimization problems.[4] Alternatively, there is an extensive literature on fixed-point comparative statics for parameterized monotone operators and correspondences that transform chain-complete partially ordered sets.[5] It is especially interesting about these tools that the equilibrium comparative statics are often computable. But a general limitation of these existing order-theoretic approaches is that they typically provide limited comparative statics information in the presence of multiple equilibria. That is, the comparative statics results pertain typically to only extremal equilibria (i.e., least/minimal or greatest/maximal) and the constructive nature of the comparative statics result does not hold for iterations from any initial point.[6]

A well-known approach to studying the equilibrium comparative statics of any equilibria is embodied in the so-called "correspondence principle,"

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[4] There is an extensive literature on these approaches to equilibria comparative statics for "regular economies" based upon versions of Thom's transversality theory and Sard's theorem.

For surveys of work on regular economies, see Mas-Colell (1985, 1996), Nagata (2004), and McLennan (2018).

[5] Fixed-point comparative statics results for strong set order monotone (or "ascending") correspondences in complete lattices can be found in Topkis (1998), Chapter 2, Section 5. See also Theorem 2 in Chapter 10 in Veinott (1992) and Sabarwal (2025a) Theorem 3. For comparative statics results on extremal fixed points, see Milgrom and Roberts (1990) Theorem 6 and Milgrom and Shannon (1994) Theorem 13.

[6] See, for example, the computable comparative statics results for Nash equilibrium in interim Bayesian supermodular games in Van Zandt (2010). Also, see the discussion in Balbus et al. (2022). There, iterations need to start from least (resp., greatest) elements of the domain.

which was suggested originally in the seminal work of Samuelson (1947).[7] Here, one seeks to identify regularity conditions of optimization problems or equilibrium problems for unambiguous equilibrium comparative statics by refining away unstable equilibria, and then restricting attention to regular (or smooth) equilibria. This approach is inherently dynamic, and can be applied when equilibria are locally unique and amenable to applications of the implicit function theorem. Echenique (2002) has extended these ideas substantially, and has been able to prove stronger versions of the correspondence principle for GSC on lattices A when there is a convex set of parameters T. For example, Echenique (2002) showed that in GSC's, a continuous equilibrium selector $t \to a^*(t)$ is increasing if and only if it selects stable equilibria.[8]

Our paper shares with the correspondence principle the idea that the identification of monotone comparative statics is critically tied to a dynamic approach. That is, we view an equilibrium as the stationary point of a dynamical system, in which a new equilibrium emerges from an old equilibrium after a change in a parameter value via some dynamic adjustment process. For example, if an equilibrium at the original set of parameters is locally stable, then one can develop sufficient conditions on the behavior of this dynamical system that guarantee that starting from the equilibrium for the old parameter, the dynamical system will actually converge to the new equilibrium for small changes of the parameter.

But this leaves open many interesting questions. Aside from the obvious question of relaxing the needed topological structure and conditions

[7] See also McLennan (2015) for an interesting recent discussion of the correspondence principle, citations of the extensive literature and implications for equilibrium comparative statics.

[8] See also Echenique (2002, 2004) for the precise formulations of various versions of this result. Notice, in the presence of multiple equilibria, the existence of continuous equilibrium selector is an added complication in applying Echenique (2002) results. But he is able to weaken the continuity requirements in some cases.

required to study the stability of local equilibrium comparative statics via correspondence principle based arguments, what do we do when all the equilibria are unstable? What if there is a continuum of equilibria (i.e., equilibria are indeterminate and not locally isolated)? Or what if the setting is chaotic, and observed outcomes do not converge to any equilibrium? Finally, when can a conclusive comparative statics regarding some statistics or aggregates be performed, even though the comparative statics regarding the fundamental variables is ambiguous?

We answer these questions in the reminder of the paper. In the next section, we define mathematical terminology. In section 3, we introduce our main concepts. Sections 4 and 5 contain examples and our results. The results that explain the relation of our approach to the existing results are in Section 4, and the results on mixed shocks to fundamental variables and comparative statics of their statistics or aggregates are in Section 5. In Section 6, we present applications. We include the proofs into the main text, delegating only the proofs of more technical results to Appendix.

2 Preliminaries

We start with introducing some basic definitions. A partially ordered set (or *poset*) is set A equipped with a partially order $\geq$. For $a', a \in A$, we say a' is *strictly higher* than a, and write $a' > a$, whenever $a' \geq a$ and $a' \neq a$. A poset $(A, \geq)$ is a *lattice* if for any $a, a' \in A$ there exists the *join* $a \vee a' := \sup\{a, a'\} \in A$ and there exists the *meet* $a \wedge a' := \inf\{a, a'\} \in A$. A lattice A is *complete (resp. sigma-complete)* if there exists $\bigvee B := \sup B \in A$ and $\bigwedge B := \inf B \in A$, for any (resp. countable) $B \subseteq A$. A subset $B \subset A$ is a *sublattice* of A if B is a lattice in the order induced from A, i.e. the join $a \vee a'$ and the meet $a \wedge a'$ as defined in $(A, \geq)$ belong to B for all $a, a' \in B$.

Let $(A, \geq)$ and $(B, \geq)$ be posets. A mapping $f : A \to B$ is order-preserving (or increasing) on A if $a' \geq a$ implies $f(a') \geq f(a)$ for a, a' in A. Let $F : A \rightrightarrows B$ be a nonempty-valued correspondence. Assume that for each $a \in A$ set $F(a)$ has the greatest and the least elements in A, and these elements belong to $F(a)$. Denote them with $\overline{F}(a) := \sup F(a)$ and $\underline{F}(a) := \inf F(a)$. We say F is *weakly increasing*[9] whenever $a' > a$ implies that $\overline{F}(a') \geq \overline{F}(a)$ and $\underline{F}(a') \geq \underline{F}(a)$. We say F is *strongly increasing* whenever $a' > a$ implies that $\underline{F}(a') \geq \overline{F}(a)$.

A sequence $(a^k)_{k=0}^{\infty}$ of elements of A is *increasing* if $a^{k+1} \geq a^k$ for each k. It is *strictly increasing* if $a^{k+1} > a^k$ for each k. *Decreasing* and *strictly decreasing* sequences are defined in the obvious dual manner. A *monotone sequence* then is either increasing or decreasing. We say that a increasing (resp., decreasing) sequence $(a^k)_{k=0}^{\infty}$ *converges* to $a \in A$ whenever $\bigvee_{k \geq 0} a^k = a$ (resp., $\bigwedge_{k \geq 0} a^k = a$). That is, when a is the supremum (resp., infimum) of the increasing (resp., decreasing) sequence.

Suppose A and B are sigma-complete lattices. The mapping $f : A \to B$ is *upward order continuous* (resp., *downward order continuous*) if for any increasing (resp., decreasing) sequence $(a^k)_{k=0}^{\infty}$ with $a^k \in A$, we have: $f\left(\bigvee_{k \geq 0} a^k\right) = \bigvee_{k \geq 0} f(a^k)$ $\left(\text{respectively } f\left(\bigwedge_{k \geq 0} a^k\right) = \bigwedge_{k \geq 0} f(a^k)\right)$. The mapping f is then *order continuous* if it is both upward and downward order continuous. Notice, if f is upward or downward order continuous, it is order-preserving or increasing mapping on A.[10]

3 Setting and concepts

We study an economy in which some variables or actions of the economy agents, which we call fundamental, are described by an element a of a

[9] Observe that weak monotonicity of a correspondence is a weaker notion than (Veinott-) strong set-order monotonicity. In particular, F need not be sublattice valued.

[10] If a mapping is upward (resp., downward) order continuous, it is also by definition sup (resp, inf) preserving. So our definitions here coincide with standard definitions of order continuity (e.g., Dugundji and Granas (1982), p 15).

lattice A, interpreted as the current state. For example, one can think of a as the vector of prices in an economy or the actions of players in a game. Thinking about lattice A having a vector structure is convenient while at first sight on our concepts, but other lattices are allowed and will be used in some applications of Section 6. Given the current state of fundamental variables or actions (which we will for simplicity call variables or actions), a mapping f (or a correspondence F) is a reaction function. For any a, $f(a)$ or $F(a)$ describes the way in which a would change from the current state in one period if *every agent would optimally respond to the current state* a. If $a = f(a)$ (or a belongs to $F(a)$), then the economy is in equilibrium.

3.1 Adaptive dynamic adjustment

The idea of our approach to comparative statics can be described as follows: suppose we initially observe a vector of variables or actions a^0 (or more generally a state, which is assumed to be an element of a lattice), from a set of all possible vectors A. This a^0 need not even be a Walrasian or Nash equilibrium, or an equilibrium in any other sense if we expect chaotic behavior in the studied setting. Then an exogenous parameter t of the setting changes, and this initiates an adaptive dynamic adjustment process in the setting with the new parameter. If the initially observed outcome is not an equilibrium, then adaptive learning applies also to the original setting. Assume that this dynamic process must have the form of some sequence $(a^k)_{k=0}^{\infty}$ from a class of sequences $\mathcal{S}(a^0)$. We will focus on the following, large class of sequences $\mathcal{S}(a^0)$. However, we emphasize that analysts are allowed to choose their preferred class of sequences, and perform a similar analysis.

For a correspondence $F : A \rightrightarrows A$, a sequence $(a^k)_{k=0}^{\infty}$ starting from a^0

is an *adaptive sequence* if

$$\exists_{\gamma\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ \underline{F}(\inf\{a^k,...,a^{k-\gamma+1}\}) \leq a^{k+1} \leq \overline{F}(\sup\{a^k,...,a^{k-\gamma+1}\}),$$

where it is assumed $a^{k-\gamma+1}=a^0$ for $k<\gamma-1$. Denote by $\mathcal{S}(a^0)$ the set of all adaptive sequences starting from a^0.

The idea is to be quite agnostic about the specific form of adjustments. We basically impose one postulate, namely, that agents respond to some finite history of variables or actions observed in previous periods. The parameter γ measures how far agents look into the past, and $\inf\{a^k,...,a^{k-\gamma+1}\}$ and $\sup\{a^k,...,a^{k-\gamma+1}\}$ are the extreme statistics of the variables or actions they observed in the past they look into. The correspondence $F: A \rightrightarrows A$ represents possible responses if *every agent would optimally respond to the current state a*. I

Echenique (2002) in his analysis of GSC used *convergent* sequences $(a^k)_{k=0}^{\infty}\in\mathcal{S}(a^0)$ for conducting comparative statics within equilibrium setting. However, as we demonstrate the comparative statics can be conducted for a larger set of environments then GSC, and even when there is no convergent adaptive sequence.[11]

3.2 Bounds of adaptive sequences

Suppose A is a sigma-complete lattice. At first sight, studying all sequences $(a^k)_{k=0}^{\infty}\in\mathcal{S}(a^0)$ seems intractable. However, we will show next that it reduces to studying their lower and upper bounds $\underline{a}$ and $\overline{a}$. To define these bounds, we first define by induction sequence $\underline{a}^{k,\gamma}$ (and simultaneously, sequence $\overline{a}^{k,\gamma}$). Let $\underline{a}^{0,\gamma}=\overline{a}^{0,\gamma}=a^0$ for all $\gamma\in\mathbb{N}$, and let $\underline{a}^{k+1,\gamma}=\underline{F}(\inf\{\underline{a}^{k,\gamma},...,\underline{a}^{k-\gamma+1,\gamma}\})$ and $\overline{a}^{k+1,\gamma}=\overline{F}(\sup\{\overline{a}^{k,\gamma},...,\overline{a}^{k-\gamma+1,\gamma}\})$. It is assumed that $\underline{a}^{k-l,\gamma}=\underline{a}^{0,\gamma}$ and $\overline{a}^{k-l,\gamma}=\overline{a}^{0,\gamma}$ for $l>k$. Next, for

[11] Despite using the same sets of sequences, the overlap between the Echenique's analysis and our paper is quite small.

any given γ, define $\liminf_k \underline{a}^{k,\gamma} = \bigvee_k \bigwedge_{l \geq k} \underline{a}^{l,\gamma}$ and $\limsup_k \overline{a}^{k,\gamma} = \bigwedge_k \bigvee_{l \geq k} \overline{a}^{l,\gamma}$. Observe that $\liminf_k \underline{a}^{k,\gamma}$ and $\limsup_k \overline{a}^{k,\gamma}$ exist by sigma-completeness of the lattice.

Then, for each adaptive sequence $(a^k)_{k=0}^{\infty}$, there exists a γ such that

$$\underline{a}^{k,\gamma} \leq a^k \leq \overline{a}^{k,\gamma}.$$

This implies that

$$\liminf \underline{a}^{k,\gamma} \leq \liminf a^k \leq \limsup a^k \leq \limsup \overline{a}^{k,\gamma}.$$

In addition, the sequence $\left(\liminf_k \underline{a}^{k,\gamma}\right)_{\gamma=0}^{\infty}$ is decreasing and the sequence $\left(\limsup_k \overline{a}^{k,\gamma}\right)_{\gamma=0}^{\infty}$ is increasing. Let $\underline{a} = \lim_{\gamma\to\infty} \liminf_k \underline{a}^{k,\gamma}$ and $\overline{a} = \lim_{\gamma\to\infty} \liminf_k \overline{a}^{k,\gamma}$.

By the monotonicity of sequences $\left(\liminf_k \underline{a}^{k,\gamma}\right)_{\gamma=0}^{\infty}$ and $\left(\limsup_k \overline{a}^{k,\gamma}\right)_{\gamma=0}^{\infty}$, we obtain: $\underline{a} \leq \liminf a^k \leq \limsup a^k \leq \overline{a}$ for any adaptive sequence $(a^k)_{k=0}^{\infty}$. Formally, this does not mean that $\underline{a} \leq a^k \leq \overline{a}$ for remote elements of any adaptive sequence $(a^k)_{k=0}^{\infty}$. However, this captures the intuition that for remote elements of any adaptive sequence, a^k is not much lower than $\underline{a}$, and not much greater than $\overline{a}$.

3.3 Monotone statistics

In applications, we are often interested in comparing some statistic $\varphi(a) \in \mathbb{R}$, where a in the long-run outcome for the original and new setting. More precisely, we are interested in comparing $\varphi(a^k)$ for large enough values of k. In our motivating example, φ was the output. Intuitively, if $\varphi(a^{new,k})$ in the new setting is at worst only slightly smaller than $\varphi(a^{old,k})$ for any large value of k in the original setting for any sequences $(a^{old,k})_{k=0}^{\infty}, (a^{new,k})_{k=0}^{\infty} \in \mathcal{S}(a^0)$, and this "slightly" is closer and closer to zero for larger k, then we can say that in the long-run the statistic φ weakly increases in response to the parameter change. In the realm of monotone mappings f or correspondences F, which we now restrict attention to, it makes sense to limit attention also to monotone statistics φ.

We are now ready to define monotone comparative statics in the most general case. To formalize the dependence on a parameter, we introduce a set of parameters T, and assume that correspondence $F : A \times T \rightrightarrows A$. By $\underline{a}^{k+1,\gamma}(t)$ and $\overline{a}^{k+1,\gamma}(t)$ we denote elements of the sequences constructed in Section 3.2 for $F(\cdot, t)$. We analogously define $\underline{a}(t)$ and $\overline{a}(t)$. Let mapping $\varphi : A \times T \to \mathbb{R}$ denote a statistic, or an aggregate. We now study only correspondences F and mapping φ that are monotone in a for any given t.

Definition 1. *A statistic $\varphi : A \times T \to \mathbb{R}$ weakly increases with a parameter change from t to t' if $\varphi(\underline{a}(t'), t') \geq \varphi(\underline{a}(t), t)$ as well as $\varphi(\overline{a}(t'), t') \geq \varphi(\overline{a}(t), t)$. We say such a statistic increases strongly with a parameter change if $\varphi(\underline{a}(t'), t') \geq \varphi(\overline{a}(t), t)$.*

We can define in a dual manner when a statistic decreases. We will now motivate Definition 1. Suppose that we initially observe a^0 given a parameter $t = t$ or t'. Let for a moment $\underline{a} := \underline{a}(t)$ and $\overline{a} := \overline{a}(t)$ and similarly $\underline{a}^{k,\gamma} := \underline{a}^{k,\gamma}(t)$ and $\overline{a}^{k,\gamma} := \overline{a}^{k,\gamma}(t)$. Recall that $\underline{a} \leq \liminf_k \underline{a}^{k,\gamma}$ for any γ, and that $\liminf_k \underline{a}^{k,\gamma}$ is the limit of the increasing sequence $\left(\inf_{l \geq k} \underline{a}^{l,\gamma}\right)_{l=0}^{\infty}$. Therefore, as we will consider order continuous comparative statics, for any $\varepsilon > 0$ and for any large enough value of k, it must be that $\varphi(\underline{a}) - \varepsilon < \varphi\left(\inf_{l \geq k} \underline{a}^{l,\gamma}\right)$, which in turn implies that $\varphi(\underline{a}) - \varepsilon < \varphi(\underline{a}^{l,\gamma})$ for $l \geq k$. Similarly, for any $\varepsilon > 0$ and for any large enough value of k, it must be that $\varphi(\overline{a}^{l,\gamma}) < \varphi(\overline{a}) + \varepsilon$ for $l \geq k$. In particular, $\varphi(\underline{a}) - \varepsilon < \varphi(a^k) < \varphi(\overline{a}) + \varepsilon$ for remote elements of any adaptive sequence $(a^k)_{k=0}^{\infty}$.

Therefore, if, for example, $\varphi(\underline{a}(t'), t') \geq \varphi(\overline{a}(t), t)$, then $\varphi(\overline{a}^{k,\gamma}(t), t)$ cannot exceed $\varphi(\underline{a}^{k,\gamma}(t'), t')$ by more than ε for any large value of k. So, “at infinity,” $\varphi(a^k(t), t) \leq \varphi(a^k(t'), t')$ for any adaptive learning sequences

$(a^k(t))_{k=0}^{\infty}$ and $(a^k(t'))_{k=0}^{\infty}$. Similarly, if $\varphi(\overline{a}(t'), t') \leq \varphi(\underline{a}(t), t)$, then $\varphi(a^k(t'), t')) \leq \varphi(a^k(t), t)$ at infinity for any adaptive sequences $(a^k(t'))_{k=0}^{\infty}$ and $(a^k(t))_{k=0}^{\infty}$. Similar in nature, but weaker in terms of results, interpretation can be provided for weak increase with a parameter change.

Note that our approach to comparative statics is conservative. Indeed, $b \leq \varphi(\underline{a})$ implies that $b - \varepsilon < \varphi(a^{k,\gamma})$ for large enough values of k. However, the converse is in general false. This converse implication would be true if we were comparing the elements of the lattice A instead a statistic thereof. More precisely, suppose that $b < \underline{a}^{k,\gamma}$ for large enough values of k. Then $b \leq \inf_{k \geq K} \underline{a}^{k,\gamma}$ for large values of k. So $b \leq \liminf_k \underline{a}^{k,\gamma}$ for all γ, and this implies that $b \leq \underline{a}$.

3.4 Equilibrium analysis

So far we did not required equilibrium comparative statics. One may wish to postulate, however, that adaptive dynamic should result in the long run in equilibrium. Or one may simply be interested only in equilibrium analysis.[12] Under some seemingly natural conditions, the bounds $\underline{a}$ and $\overline{a}$ are in fact fixed points of F. Thus, if these conditions are satisfied, our bounds are appropriate tools for equilibrium comparative statics. We start with the case where F is a single valued mapping and then show how to extend it to correspondences.

Theorem 1. *Suppose $f : A \to A$ is an increasing mapping. If $f(\liminf_k b^k) = \liminf_k f(b^k)$ for any sequence $(b^k)_{k=0}^{\infty}$, then $\underline{a}$ is a fixed point. Similarly, if $f(\limsup_k b^k) = \limsup_k f(b^k)$ for any sequence $(b^k)_{k=0}^{\infty}$, then $\overline{a}$ is a fixed point.*

[12] We refer the reader to our related papers Balbus et al. (2025) and Olszewski (2021a,b) for results on tight fixed-point (equilibrium) bounds of best-response iterations on monotone correspondences and mappings.

We will prove the first part of the theorem. The proof of the second part is analogous.

Proof. We will prove the theorem for $\underline{a}$. The proof for $\overline{a}$ is analogous. Note first that for any γ, we have $\bigwedge_{k\geq n-\gamma} \underline{a}^{k,\gamma} \leq \inf\{\underline{a}^{j-1,\gamma}, \underline{a}^{j-2,\gamma}, ..., \underline{a}^{j-\gamma,\gamma}\}$ for all $j \geq n \geq \gamma$. Indeed, all elements under the infimum on the right-hand side also appear under the infimum on the left-hand side. By an analogous argument, $\inf\{\underline{a}^{k-1,\gamma}, \underline{a}^{k-2,\gamma}, ..., \underline{a}^{k-\gamma,\gamma}\} \leq \underline{a}^{k-\gamma,\gamma}$. Thus, letting $\underline{b}^{k,\gamma} := \inf\{\underline{a}^{k-1,\gamma}, \underline{a}^{k-2,\gamma}, \ldots, \underline{a}^{k-\gamma,\gamma}\}$,

$$\bigwedge_{k\geq n} \underline{b}^{k,\gamma} = \bigwedge_{k\geq n-\gamma} \underline{a}^{k,\gamma}. \tag{3}$$

for all $n \geq \gamma$. By the definition of $\underline{a}^{k,\gamma}$,

$$\liminf_{k\to\infty} \underline{a}^{k,\gamma} = \liminf_{k\to\infty} f(\underline{b}^{k,\gamma});$$

by assumption,

$$\liminf_{k\to\infty} f(\underline{b}^{k,\gamma}) = f(\liminf_{k\to\infty} \underline{b}^{k,\gamma});$$

and by (3),

$$f(\liminf_{k\to\infty} \underline{b}^{k,\gamma}) = f(\liminf_{k\to\infty} \underline{a}^{k,\gamma}).$$

Therefore $\liminf_{k\to\infty} \underline{a}^{k,\gamma}$ is a fixed point, and so is $\underline{a}$ by the continuity of f. $\square$

Theorem 1 generalizes to weakly increasing correspondences $F : A \rightrightarrows A$ such that $\overline{F}(\limsup_k a^k) = \limsup_k \overline{F}(a^k)$ and $\underline{F}(\liminf_k a^k) = \liminf_k \underline{F}(a^k)$ by applying its present version for mappings to functions $f = \overline{F}$ and $f = \underline{F}$.

Therefore, in the settings that satisfy the continuity conditions from Theorem 1, equilibrium comparative statics can be performed in the following way:

Definition 2. *Suppose a^0 is an equilibrium given some $t \in T$. A statistic $\varphi : A \times T \to \mathbb{R}$ increases in equilibrium with a parameter change from*

t to t' if $\varphi(\underline{a}, t') \geq \varphi(a^0, t)$, and it decreases with a parameter change if $\varphi(\overline{a}, t') \leq \varphi(a^0, t)$, where $\underline{a} = \underline{a}(t')$ and $\overline{a} = \overline{a}(t')$ and $\underline{a}(t) = \overline{a}(t) = a^0$.

Continuity conditions in Theorem 1 imply downward and upward order continuity but are stronger. We conclude with an example that shows that $\underline{a}$ and $\overline{a}$ need not be fixed points in the general case, that is, for all order continuous mappings f.

Example 2. *Let $A = \{(-1/n, 1/m) : n, m = 1, 2, ... \text{ and } n \leq m\} \cup \{(-1/n, 0) : n = 1, 2, ...\} \cup \{(-1/n, -1) : n = 1, 2, ...\} \cup \{(0, 1)\} \cup \{(0, 0)\} \cup \{(0, -1)\}$ be the lattice equipped with the ordering inherited from $\mathbb{R}^2$. Let $f : A \to A$ be the mapping defined by letting $f(-1/n, 1/m) = (-1/(n+1), 1/(m+1))$, $f(-1/n, 0) = f(-1/n, -1) = (-1/(n+1), -1)$ and $f(0, 0) = (0, -1)$, $f(0, -1) = (0, -1)$, $f(0, 1) = (0, 1)$. Then, A is a complete lattice, and f is an order-continuous mapping. Suppose that $a^0 = (-1, 1)$ then $\underline{a} = (0, 0)$ but $f(\underline{a}) = (0, -1)$.*

4 Iterative monotone comparative statics

4.1 The main results

In this section, we provide our iterative monotone comparative statics theorem. Our result (as well as our method) applies to a larger class of settings then the earlier theorems of monotone comparative statics, as well as our result improves on some existing theorems.

Let $\underline{a}(t)$ and $\overline{a}(t)$ be the bounds of adaptive learning sequences constructed in Section 3.2 starting from a^0 and iterating on $F(\cdot, t)$ for given $t \in T$.

Theorem 2. *Let A be a sigma-complete lattice and T be a poset. Endow $A \times T$ with the product order. Let $F : A \times T \rightrightarrows A$ be such that $F(a, t)$ has* *the greatest and the least elements for each a, t and suppose F is weakly increasing on $A \times T$. If $t < t'$, then:*

(i) $\underline{a}(t) \leq \underline{a}(t')$ *and* $\overline{a}(t) \leq \overline{a}(t')$;

(ii) if a^0 *is a fixed point of* $F(\cdot, t)$ *and for any* $a \in A$, $F(a, \cdot)$ *is strongly increasing, then* $a^0 \leq \underline{a}(t')$;

(iii) in point (ii), if $\underline{F}(\cdot, t')$ *is, in addition, upward order continuous, then* $\underline{a}(t')$ *is the least fixed point of* $F(\cdot, t')$ *on* $\{a \in A : a \geq a^0\}$.

Point (i) assures that the bounds of iterations starting from any initial a^0 are ordered with respect to t. Weak monotonicity of F suffices to assure this weak monotone comparative statics result. Point (ii) provides a strong monotone comparative statics if iterations start from a fixed point a^0. Strong monotonicity in point (ii) is essential; for example, the result does not hold true if $\underline{F}(a^0, t') < a^0$ is a fixed point of F.[13] Finally, point (iii) shows that under additional order continuity of $\underline{F}(\cdot, t')$ the lower bound of iterations $\underline{a}(t')$ is in fact a fixed point of $F(\cdot, t')$. Recall that all of this is in the setting of a sigma-complete lattice.

Proof. We start by proving (i). By the monotonicity of F, $\inf F(a^0, t) \leq \inf F(a^0, t')$. So, $\underline{a}^{1,\gamma}(t)$ is no greater than $\underline{a}^{1,\gamma}(t')$. Hence $\inf\{a^0, \underline{a}^{1,\gamma}(t)\} \leq \inf\{a^0, \underline{a}^{1,\gamma}(t')\}$, and $\underline{a}^{2,\gamma}(t) \leq \underline{a}^{2,\gamma}(t')$. By induction, $\underline{a}^{k,\gamma}$ obtained by iterating a^0 on $F(\cdot, t)$ is no greater than $\underline{a}^{k,\gamma}$ obtained by iterating a^0 on $F(\cdot, t')$ for any k. Hence, $\liminf_k \underline{a}^{k,\gamma}(t) \leq \liminf_k \underline{a}^{k,\gamma}(t')$ and hence $\underline{a}(t) \leq \underline{a}(t')$. We have proven the theorem for $\underline{a}$. The proof for $\overline{a}$ is analogous.

Proof of (ii). Now suppose a^0 is a fixed point of $F(\cdot, t)$. We will show, for all γ, that $a^0 \leq \underline{a}^{1,\gamma}(t')$ and the sequence $(\underline{a}^{k,\gamma}(t'))_{k=0}^{\infty}$ is increasing. Indeed,

[13] It is important to keep in mind that Theorem 2 (i)-(ii) itself does not guarantee existence of fixed points of $F(\cdot, t)$. If A is assumed to be additionally complete, then under conditions of Theorem 2 the set of fixed points of $F(\cdot, t)$ is nonempty for each t by Tarski's theorem.

$a^0 \leq \overline{F}(a^0, t) \leq \underline{F}(a^0, t') = \underline{a}^{1,\gamma}(t')$ by strong monotonicity. If $(\underline{a}^{k,\gamma}(t'))_{k=0}^{n}$ is increasing, then

$$\underline{a}^{n+1,\gamma}(t') = \underline{F}(\inf\{\underline{a}^{n-\gamma+1,\gamma}(t'), ..., \underline{a}^{n,\gamma}(t')\}, t') \geq$$
$$\geq \underline{F}(\inf\{\underline{a}^{n-\gamma,\gamma}(t'), ..., \underline{a}^{n-1,\gamma}(t')\}, t') = \underline{a}^{n,\gamma}(t'),$$

so $(\underline{a}^{k,\gamma}(t'))_{k=0}^{n+1}$ is increasing. By induction, we obtain $(\underline{a}^{k,\gamma}(t'))_{k=0}^{\infty}$ is increasing. Hence

$$a^0 \leq \liminf_k \underline{a}^{k,\gamma}(t') = \lim_k \underline{a}^{k,\gamma}(t')$$

for all γ. So, $a^0 \leq \underline{a}(t')$. Proof of (iii). For each γ the increasing sequence $(\underline{a}^{k,\gamma}(t'))_{k=0}^{\infty}$ converges to, say, $\underline{a}^{\gamma}(t')$. Now consider a sequence $(\underline{a}^{n\gamma+1,\gamma}(t'))_{n=1}^{\infty}$. Observe that:

$$\underline{a}^{n\gamma+1,\gamma}(t') = \underline{F}(\underline{a}^{(n-1)\gamma+1,\gamma}(t'), t').$$

as well as that the sequences $(\underline{a}^{n\gamma+1,\gamma}(t'))_{n=1}^{\infty}$ have the same initial value (regardless of γ). Hence $(\underline{a}^{n\gamma+1,\gamma}(t'))_{n=1}^{\infty}$ is a sequence independent of γ, and in particular $\underline{a}^{n\gamma+1,\gamma}(t') = \underline{a}^{n+1,1}(t')$. Taking the supremum we get that $\underline{a}^{\gamma}(t') = \underline{a}^{1}(t') = \underline{a}(t')$ for all γ. From the order upward continuity we get that $\underline{a}(t') = \underline{a}^{1}(t')$ is a fixed point of $\underline{F}(\cdot, t')$.

Now we only need to show that $\underline{a}^1(t')$ is the least fixed point of $F(\cdot, t')$ on the set $\{a \in A : a \geq a^0\}$. Consider an arbitrary $b \in A$ such that $b \geq a_0$ and that $b \in F(b, t')$. Then for every n we obtain that $\underline{a}^{n,1}(t') \leq b$. For $n = 0$ the thesis is clear. Suppose that $\underline{a}^{k,1} \leq b$ for every $k \leq n$. Then we obtain

$$\underline{a}^{n+1,1}(t') = \underline{F}(\underline{a}^{n,1}(t'), t') \leq \underline{F}(b, t') \leq b$$

since $b \in F(b, t')$. As a result $\underline{a}^{n+1,1}(t') \leq b$ for every n. Hence $\underline{a}^1(t') \leq b$, and so $\underline{a}(t') \leq b$.

□

Focusing on GSC allows us to compare our results to those based on the correspondence principle in Echenique (2002). Echenique's Theorem

3, as he introduces it, provides information about the limits of adaptive behavior after an increase in a parameter. Under his assumption that $a^0 \leq \inf F(a^0, t)$ made throughout his Theorem 3, the smallest equilibrium which is the limit of a convergent adaptive learning sequence $(a^k)_{k=0}^{\infty}$ coincides with our lower bound $\underline{a}(t)$. So, the two papers offer the same comparative statics result in this case. But our result extends Echenique's result in few dimensions: (a) our correspondence F is assumed to be only weakly (not necessarily strongly) increasing; (b) the adaptive behavior may start from an action profile a^0 that is not ordered with its image[14] under F; (c) the initial action profile a^0 need not be an equilibrium. Further, the adaptive behavior may or may not be convergent.

Observe that Theorem 2 points (i) and (ii) do not require any continuity of a correspondence F. This is in a stark difference with respect to monotone comparative statics results based upon the application of Tarski-Kantorovich fixed point theorem on sigma-complete posets (see, e.g., Balbus et al. (2022) Proposition A.2.). So relaxing the demand for a theory of equilibrium comparative statics to comparative statics of iterative bounds, allows us to obtain a new comparative statics result for discontinuous correspondences. Moreover, observe that Theorem 2 applies to lattices A that are only sigma-complete. This is a generalization with respect to comparative statics results of Veinott (1992) (Theorem 2 in Chapter 10)[15] that require A to be a complete lattice, correspondence F to be strong set order monotone and pertain to comparative statics of extremal fixed points only. Again, Theorem 2 is particularly useful in environments in which comparative statics of iterative bounds is more adequate than comparative statics of extremal equilibria (see examples in Section 6).

Finally, Theorem 2, point (iii) shows that our approach delivers similar

[14] We find point (b) important, because the Echenique assumption refers to a property of the initial equilibrium (or the starting point), not to a feature of the setting.

[15] See also Topkis (1998) Theorem 2.5.2.

monotone comparative statics to that from Sabarwal (2025a) who studies order nearest fixed points for monotone correspondences. See his Theorem 3.2(a). The difference between his and our result lays in assumptions: he considers correspondences on complete lattices without imposing order continuity while we consider correspondences on sigma-complete lattices and (in point (iii)) we impose upper order continuity.[16] Moreover, when A is a complete lattice our methods can be seen as providing constructive counterparts of Sabarwal (2025a) result, as his order nearest comparative statics are existential in nature while our are based on accumulation points of adaptive learning sequences.

4.2 Samuelson's Correspondence Principle

Samuelson (1947) argues that for stable equilibria of a smooth mapping f, one obtains local comparative statics by referring to the Implicit Function Theorem. This requires studying interior equilibria of Euclidean spaces to which the Implicit Function Theorem applies. Then, the Implicit Function Theorem guarantees the uniqueness of a close equilibrium for any value of the parameter t' close to its original value t. In turn, stability allowed Samuelson to obtain unambiguous comparative statics in a class of one-dimensional models by comparing an equilibrium for the original parameter t and the unique close equilibrium of a close parameter t'. As noticed later by several other authors (Arrow and Hahn, 1971; Echenique, 2000), the comparative statics may or may not be unambiguous in multi-dimensional models. We will observe in this section that the approach to comparative statics suggested by Samuelson coincides with our approach in the settings explored by Samuelson, no matter whether in one-dimensional or multi-dimensional setting.

[16] In fact, in points (ii) and (iii), we also impose strong monotonicity of F in t. This is inevitable if one wants to obtain bounds for *any* adaptive learning sequences.

We will interpret stability as the convergence of all adaptive learning sequences to the unique equilibrium for any given value of parameter t.

Theorem 3. *Let A be a sublattice of an Euclidean space that contains an equilibrium a^* in its interior. If all adaptive learning sequences $(a^k)_{k=0}^{\infty} \in \mathcal{S}(a^0)$ starting from some a^0 converge to a^*, then $\underline{a} = \overline{a} = a^*$.*

The theorem implies that, according to our approach, comparing equilibrium a^0 for a parameter t and the equilibrium a^* after the parameter changes to t' is equivalent to comparing the original equilibrium a^0 and $\underline{a} = \overline{a}$ for adaptive learning sequences starting at a^0. We relegate the proof of Theorem 3 to Appendix.

We now illustrate our approach to comparative statics in a classic example of GSC (i.e. Bertrand competition with heterogenous products), where Samuelson (1947) principle applies.

Example 3. *Two firms compete in prices. The reaction curve of firm $i = 1, 2$ to the price of firm $j \neq i$ is given by $p_i = a + bp_j$, where $1/b > b$.*[17] *This last inequality means that the reaction curve of firm 1 is steeper than the reaction curve of firm 2 in the system of coordinates with p_1 on the horizontal axis and p_2 on the vertical axis.*

For $\gamma = 1$, our dynamic sequences are best-response sequences. And it is well-known that the best-response sequence $(p^{k,1})_{k=0}^{\infty}$ converges to (p_1^*, p_2^*), where $p_1^* = p_2^* = \frac{a}{1-b}$ are the unique Nash equilibrium prices. Let now $\gamma = 2$. We will show that any sequence $(p^{k,2})_{k=0}^{\infty}$ from our class of sequences $\mathcal{S}(p^0)$ also converges to (p_1^*, p_2^*). The argument for an arbitrary $\gamma \geq 2$ is analogous.

If p^0 lies between the two reaction curves, then the sequence $(p^k)_{k=0}^{\infty}$ generated by the best-response dynamic is monotonic. It is decreasing when

[17] For simplicity, we restrict attention to symmetric firms, but this is inessential for the conclusions.

$p_i^0 > p_i^*$ for $i = 1, 2$, and increasing when $p_i^0 < p_i^*$ for $i = 1, 2$. Consider the former case; the arguments in the latter one are analogous. We show by induction that

$$\overline{p}^{k,2} = p^{k-1} \text{ and } \underline{p}^{k,2} = p^k \text{ for } k \geq 2. \tag{4}$$

Indeed, since $\overline{p}^{1,2} = \underline{p}^{1,2} = p^1$ and $p_i^1 \leq p_i^0$ for $i = 1, 2$, we have that $\underline{p}_i^{2,2} = br(p_j^1) = p_i^2$ and $\overline{p}_i^{2,2} = br(p_j^0) = p_i^1$. So, (4) holds for $k = 2$. Similarly, since $p_i^k \leq p_i^{k-1}$ for $i = 1, 2$, we obtain, by (4) for k, that $\underline{p}_i^{k+1,2} = br(p_j^k) = p_i^{k+1}$ and $\overline{p}_i^{k+1,2} = br(p_j^{k-1}) = p_i^k$. Obviously, (4) implies that any sequence $(p^{k,2})_{k=0}^{\infty}$ from $\mathcal{S}(p^0)$ converges to (p_1^*, p_2^*).

If p^0 does not lie between the two reaction curves, then $\underline{p}^{2,2}$ and $\underline{p}^{3,2}$ do, and $\underline{p}^{2,2} \geq \underline{p}^{3,2}$ or $\underline{p}^{2,2} \leq \underline{p}^{3,2}$ (which inequality holds depends on the position of a^0). So, the previous argument applies. The case of the upper bound is analogous.

5 Aggregate comparative statics

In this section, we provide results on monotone "aggregate" comparative statics that concern the comparative statics of some statistics or aggregates φ of fundamental variables. These results differ from most results in the existing literature, which typically concern fundamental variables themselves, like our Theorem 2. However, in some situations statistics may unambiguously increase or decrease with a change in an exogenous parameter, although the fundamental variables may not be monotone. One such situation was described in the Example 1, where the statistic φ was the aggregate (team) output and F was the joint best-response mapping. We will return to that example later in this section.

As before, assume that A is a sigma-complete lattice and T a poset, and endow $A \times T$ with the product ordering. We will assume throughout the section that an aggregate $\varphi : A \times T \to \mathbb{R}$ is order continuous on A and monotone on $A \times T$. The mapping or correspondence $F : A \times T \to A$ is monotone on A, but in contrast to the previous section it is not necessarily monotone on T. For given t and t', let $(\underline{a}^{k,\gamma}(t))_{k=0}^{\infty}$, and $(\underline{a}^{k,\gamma}(t'))_{k=0}^{\infty}$ be the sequences from Section 3, such that $a^{0,\gamma}(t) = a^{0,\gamma}(t') = a^0 \in A$; and further, $\underline{a}^{k+1,\gamma}(t) = \underline{F}(\inf\{\underline{a}^{k,\gamma}(t), ..., \underline{a}^{k-\gamma+1,\gamma}(t)\}, t)$ and $\underline{a}^{k+1,\gamma}(t') = \underline{F}(\inf\{\underline{a}^{k,\gamma}(t'), ..., \underline{a}^{k-\gamma+1,\gamma}(t')\}, t')$.

Theorem 4. *Let $t' > t$. Suppose $\varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t'), t') \geq \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t'), t')$ for any (sufficiently large) n and γ, and the following condition is satisfied:*

$$\text{if } \varphi(a, t) \leq \varphi(b, t') \text{ and } \varphi(a', t) \leq \varphi(b', t') \text{ for some } a, b, a', b' \text{ then}$$

$$\varphi\left(\underline{F}(a \wedge a', t), t\right) \leq \varphi\left(\underline{F}(b \wedge b', t'), t'\right). \tag{5}$$

Then we have $\varphi(\underline{a}(t), t) \leq \varphi(\underline{a}(t'), t')$.

Note that the hypotheses of Theorem 4, as well as the hypotheses of Theorem 5 below, concern only specific values of t and t' as well as specific elements of the adaptive sequences $(\underline{a}^k(t))_{k=0}^{\infty}$ and $(\underline{a}^k(t'))_{k=0}^{\infty}$. Condition $\varphi(\bigwedge_{k\geq n} \underline{a}^k(t'), t') \geq \bigwedge_{k\geq n} \varphi(\underline{a}^k(t'), t')$ is strong. However, as the example in Remark 1 below the proof shows, the theorem need not hold true when this condition is violated, even though condition (5) is satisfied. Moreover, verifying this condition requires computing the sequence of iterations $(\underline{a}^k(t'))_{k=0}^{\infty}$. We stated the condition to pin down what we need for the proof. However, for many statistics φ even the stronger condition that $\varphi(\bigwedge B, t) = \bigwedge\{\varphi(b, t) : b \in B\}$ (for any $t \in T$ and any countable $B \subset A$) is satisfied, and this stronger condition does not require computing any sequence. A similar comment applies to Theorem 5. The stronger condition is satisfied in our leading example.

The range of cases in which F is not monotone on T but Theorem 4 (or Theorem 5) applies is limited. However, our theorems apply to the motivating example or, for example, when φ is a median and the vectors of an Euclidean space are ordered by comparing the smallest coordinate with the smallest coordinate, etc. In addition, our theorems seem to pin down the conditions needed for monotone comparative statics when F is not monotone on T.

Finally, it at first seems surprising that the condition (5) refers to only two pairs (a, b) and (a', b'), because γ can be any positive integer in the definition of general adaptive learning sequences. So, one would expect the present condition to refer to any finite set of pairs. It is possible to require less due to the important property of general adaptive learning sequences in Lemma 1 below. Before turning to more technical details, however, it might be helpful to illustrate Theorems 4 in the setting from our motivating example.

Motivating example, continued Recall that in the motivating example the best-reply mapping was $f(a_1, a_2, t) = (\frac{a_2}{t}, ta_1)$, and the aggregate was $\varphi(a_1, a_2, t) = 2\min\{ta_1, a_2\}$. Thus $\varphi(f(a, t), t) = 2\min\{a_2, ta_1\} = \varphi(a, t)$. We also have that $\varphi(\bigwedge B, t) = \bigwedge\{\varphi(b, t) : b \in B\}$ for any $B \subset A$, because both sides of the equality are equal to the output when each player provides the lowest effort across all her efforts in B. We will show that condition (5) from Theorem 4 is satisfied. First, observe that:

$$\varphi(f(a \wedge a', t'), t') = \varphi(a \wedge a', t') =$$

$$= \varphi((\min\{a_1, a_1'\}, \min\{a_2, a_2'\}), t') = 2\min\{t'a_1, t'a_1', a_2, a_2'\}.$$

Now, $\varphi(a, t) \le \varphi(b, t')$ and $\varphi(a', t) \le \varphi(b', t')$, that is $\min\{ta_1, a_2\} \le \min\{t'b_1, b_2\}$ and $\min\{ta_1', a_2'\} \le \min\{t'b_1', b_2'\}$ imply that $\min\{ta_1, ta_1', a_2, a_2'\} \le \min\{t'b_1, t'b_1', b_2, b_2'\}$, which means that $\varphi(f(a \wedge a', t), t) \le \varphi(f(b \wedge b', t'), t')$. As a result Theorem 4 can be applied to obtain the iterative comparative statics result from the motivating example.

In case of our motivating example the result can also be verified directly. If $a_2^0/t > a_1^0$ (which is equivalent to $a_2^0 > ta_1^0$), then $\underline{a}_1^{1,\gamma} = \overline{a}_1^{1,\gamma} = a_1^1 = a_2^0/t$ and $\underline{a}_2^{1,\gamma} = \overline{a}_2^{1,\gamma} = a_2^1 = ta_1^0$. Therefore $(\underline{a}_1^{2,\gamma}, \underline{a}_2^{2,\gamma}) = (a_1^0, ta_1^0)$ for $\gamma \geq 2$, and $(\overline{a}_1^{2,\gamma}, \overline{a}_2^{2,\gamma}) = (a_2^0/t, a_2^0)$. Finally, $(\underline{a}_1^{k,\gamma}, \underline{a}_2^{k,\gamma}) = (\underline{a}_1^{2,\gamma}, \underline{a}_2^{2,\gamma})$ and $(\overline{a}_1^{k,\gamma}, \overline{a}_2^{k,\gamma}) = (\overline{a}_1^{2,\gamma}, \overline{a}_2^{2,\gamma})$ for $k > 2$ and all $\gamma \geq 2$.

If $a_1^0 > a_2^0/t$ (which is equivalent to $ta_1^0 > a_2^0$), then again $\underline{a}_1^{1,\gamma} = \overline{a}_1^{1,\gamma} = a_1^1 = a_2^0/t$ and $\underline{a}_2^{1,\gamma} = \overline{a}_2^{1,\gamma} = a_2^1 = ta_1^0$. Therefore $(\underline{a}_1^{2,\gamma}, \underline{a}_2^{2,\gamma}) = (a_2^0/t, a_2^0)$ and $(\overline{a}_1^{2,\gamma}, \overline{a}_2^{2,\gamma}) = (a_1^0, ta_1^0)$ for $\gamma \geq 2$. And again, $(\underline{a}_1^{k,\gamma}, \underline{a}_2^{k,\gamma}) = (\underline{a}_1^{2,\gamma}, \underline{a}_2^{2,\gamma})$ and $(\overline{a}_1^{k,\gamma}, \overline{a}_2^{k,\gamma}) = (\overline{a}_1^{2,\gamma}, \overline{a}_2^{2,\gamma})$ for $k > 2$ and all for $\gamma \geq 2$.

This gives us the following results:

(a) If $a_1^0 \leq a_2^0 \leq ta_1^0$, then the output in $\underline{a}(t > 1)$ is $2a_2^0$, which is equal to the total output in $\underline{a}(t = 1)$.

(b) If $a_2^0 > ta_1^0$, then the output in $\underline{a}(t > 1)$ is $2ta_1^0$, which is greater than $2a_1^0$, the total output in $\underline{a}(t = 1)$.

(c) If $a_2^0 < a_1^0$, then the output in $\underline{a}(t > 1)$ is $2a_2^0$, which is equal to the total output in $\underline{a}(t = 1)$. $\square$

As mentioned, in the proof of Theorem 4, an important role is played by the following lemma:

Lemma 1. *Let $\gamma \geq 2$ and consider a sequence $(\underline{a}^{k,\gamma})_{k=0}^{\infty}$, where $\underline{a}^{k+1,\gamma} = f(\bigwedge\{\underline{a}^{k,\gamma}, ..., \underline{a}^{k-\gamma+1,\gamma}\})$ for some increasing mapping $f : A \to A$ and given $a^{0,\gamma} \in A$. Then:*

1. *for any $n \geq 0$, $j \in \{1, ..., \gamma-1\}$ we have $\underline{a}^{n\gamma+j,\gamma} = f(\underline{a}^{n\gamma,\gamma} \wedge \underline{a}^{n\gamma+j-1,\gamma})$,*
2. *for any $n \geq 1$ we have $\underline{a}^{n\gamma,\gamma} = f(\underline{a}^{(n-1)\gamma,\gamma} \wedge \underline{a}^{n\gamma-1,\gamma})$.*

Due to Lemma 1, we could state condition 5 in terms of pairs elements of a lattice (a and b, and a' and b') instead of any finite sequences of elements. At first sight, Lemma 1 is surprising, but as the reader can see in the proof, the two elements: $\underline{a}^{n\gamma,\gamma}$ and $\underline{a}^{n\gamma+j-1,\gamma}$, or $\underline{a}^{(n-1)\gamma,\gamma}$ and

$\underline{a}^{n\gamma-1,\gamma}$, encode everything contained in $\underline{a}^{k,\gamma}, ..., \underline{a}^{k-\gamma+1,\gamma}$ that is necessary for computing $\underline{a}^{k+1,\gamma}$. We relegate the proof of Lemma 1 to Appendix, and we will proceed to proving Theorem 4.

Proof of Theorem 4. Since φ is monotone on T we have: $\varphi(a^0, t) \leq \varphi(a^0, t')$. By condition (5), taking $a = a' = a^{0,\gamma}(t)$ and $b = b' = a^{0,\gamma}(t')$ we obtain: $\varphi(\underline{a}^{1,\gamma}(t), t) \leq \varphi(\underline{a}^{1,\gamma}(t'), t')$. Applying condition (5) again we obtain: $\varphi(\underline{a}^{2,\gamma}(t), t) \leq \varphi(\underline{a}^{2,\gamma}(t'), t')$. Continuing by induction, if $\varphi(\underline{a}^{k-j,\gamma}(t), t) \leq \varphi(\underline{a}^{k-j,\gamma}(t'), t')$ for some k and each $j \in \{0, 1, \ldots, k\}$, then $\varphi(\underline{a}^{k+1,\gamma}(t), t) \leq \varphi(\underline{a}^{k+1,\gamma}(t'), t')$. Indeed, this follows from condition (5) and Lemma 1. As a result, we obtain: $\varphi(\underline{a}^{k,\gamma}(t), t) \leq \varphi(\underline{a}^{k,\gamma}(t'), t')$ for all k. Since $\bigwedge_{k\geq n} \underline{a}^{k,\gamma} \leq \underline{a}^{k,\gamma}$ for any $k \geq n$, by monotonicity of φ on A we have: $\varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t), t) \leq \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t), t) \leq \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t'), t') \leq \varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t'), t')$, where the last inequality follows from assumption: $\varphi(\bigwedge_k \underline{a}^{k,\gamma}(t'), t') \geq \bigwedge_k \varphi(\underline{a}^{k,\gamma}(t'), t')$. Taking the limit with n and γ and using the continuity of φ on A we conclude that $\varphi(\underline{a}(t), t) \leq \varphi(\underline{a}(t'), t')$.$\square$

Note that we used the monotonicity of φ on T only to obtain that $\varphi(a^{0,\gamma}(t), t) \leq \varphi(a^{0,\gamma}(t'), t')$.

Remark 1. *Observe that the following always holds:* $\varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t'), t') \leq \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t'), t')$*, and so condition* $\varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t'), t') \geq \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t'), t')$ *in fact implies* $\varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t'), t') = \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t'), t')$*. To see that this condition is critical, let us reconsider the sequence of best-response iterations (i.e. adaptive learning sequences with* $\gamma = 1$*) in our motivating example. Specifically, consider two sequences:* $(a^k(t))_{k=0}^{\infty}$ *and* $(a^k(t'))_{k=0}^{\infty}$*, where* $a^0(t) = a^0(t') = (x, x)$ *for some number* $x \in (0, 1]$*. Then* $a^k(t) = (\frac{x}{t}, xt)$ *for any odd* k *and* $a^k(t) = a^0(t)$ *for any even* k*. Similarly,* $a^k(t') = (\frac{x}{t'}, xt')$ *for any odd* k *and* $a^k(t') = a^0(t')$ *for any even* k*. This implies that* $\underline{a}^{k,\gamma}(t) = (\frac{x}{t}, x)$ *and* $\underline{a}^{k,\gamma}(t') = (\frac{x}{t'}, x)$ *for* $k, \gamma \geq 2$ *whenever* $t' > 1$ *and* $t > 1$*.*

But let now the aggregate statistics be given by: $\varphi(a_1, a_2, t) = a_1 + a_2$. *Assume* $t' > t > 1$. *Then we have:* $\varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t'), t') = (1 + \frac{1}{t'})x < 2x = \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t'), t')$, *and so the assumption of Theorem 4 is violated. Observe that in this case the conclusion does not hold either. Indeed:* $\varphi(\underline{a}(t), t) = (1 + \frac{1}{t})x > (1 + \frac{1}{t'})x = \varphi(\underline{a}(t'), t')$, *even though* $\varphi(a^k(t), t) \leq \varphi(a^k(t'), t')$ *for any* k.

A result analogous to that of Theorem 4 holds for upper bounds if we replace $\varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t'), t') \geq \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t'), t')$ with $\varphi(\bigvee_{k\geq n} \overline{a}^{k,\gamma}(t'), t') \leq \bigvee_{k\geq n} \varphi(\overline{a}^{k,\gamma}(t'), t')$ and condition (5) with

$$\text{if } \varphi(a, t) \leq \varphi(b, t') \text{ and } \varphi(a', t) \leq \varphi(b', t') \text{ for some } a, b, a', b'$$

$$\varphi\left(\overline{F}(a \vee a', t), t\right) \leq \varphi\left(\overline{F}(b \vee b', t'), t'\right).$$

We then obtain: $\varphi(\overline{a}(t), t) \leq \varphi(\overline{a}(t'), t')$. For this analogous result and for Theorem 5, we need the following lemma, which is an analogue of Lemma 1 for $(\overline{a}^{k,\gamma}(t'))_{k=0}^{\infty}$ and its proof is omitted.

Lemma 2. *Let* $\gamma \geq 2$ *and consider the sequence* $(\overline{a}^{k,\gamma})_{k=0}^{\infty}$, *where* $\overline{a}^{k+1,\gamma} = f(\bigvee\{\overline{a}^{k,\gamma}, ..., \overline{a}^{k-\gamma+1,\gamma}\})$ *for some increasing mapping* $f : A \to A$ *and given* $a^{0,\gamma} \in A$. *Then:*

1. *for any* $n \geq 0$, $j \in \{1, ..., \gamma-1\}$ *we have* $\overline{a}^{n\gamma+j,\gamma} = f(\overline{a}^{n\gamma,\gamma} \vee \overline{a}^{n\gamma+j-1,\gamma}\})$,
2. *for any* $n \geq 1$ *we have* $\overline{a}^{n\gamma,\gamma} = f(\overline{a}^{(n-1)\gamma,\gamma} \vee \overline{a}^{n\gamma-1,\gamma})$.

Our next and last theorem concerns the iterations started from a^0, a fixed point of $F(\cdot, t)$.

Theorem 5. *Let* $t < t'$ *and suppose* $a^0 \in F(a^0, t)$; *moreover* $\varphi(\bigwedge_{k\geq n} \underline{a}^{k,\gamma}(t'), t') \geq \bigwedge_{k\geq n} \varphi(\underline{a}^{k,\gamma}(t'), t')$ *for any (sufficiently large)* n *and* γ, *and the following condition is satisfied:*

$$\textit{if } \varphi(a, t) \leq \varphi(b, t') \textit{ and } \varphi(a', t) \leq \varphi(b', t') \textit{ for some } a, b, a', b' \textit{ then}$$

$$\varphi\left(\overline{F}(a \vee a', t), t\right) \leq \varphi\left(\underline{F}(b \wedge b', t'), t'\right). \tag{6}$$

Then we have $\varphi(a^0, t) \leq \varphi(\underline{a}(t'), t')$.

Proof of Theorem 5. From monotonicity of φ on T we have: $\varphi(a^0, t) \leq \varphi(a^0, t')$. By condition (6), taking $a = a' = a^0$ and $b = b' = a^0$, we obtain: $\varphi(\overline{a}^{1,\gamma}(t), t) \leq \varphi(\underline{a}^{1,\gamma}(t'), t')$. Applying condition (6) again, this time to $a = a^0$, $a' = \overline{a}^{1,\gamma}(t)$, and $b = a^0$, $b' = \underline{a}^{1,\gamma}(t')$, we obtain: $\varphi(\overline{a}^{2,\gamma}(t), t) \leq \varphi(\underline{a}^{2,\gamma}(t'), t')$. Continuing by induction, if $\varphi(\overline{a}^{k-j,\gamma}(t), t) \leq \varphi(\underline{a}^{k-j,\gamma}(t'), t')$ for some k and each $j \in \{0, 1, \ldots, k\}$ then $\varphi(\overline{a}^{k+1,\gamma}(t), t) \leq \varphi(\underline{a}^{k+1,\gamma}(t'), t')$. To see why, observe that $\overline{a}^{k+1,\gamma}(t) = \overline{F}(\overline{a}^{k,\gamma}(t) \vee \overline{a}^{l,\gamma}(t), t)$ by Lemma 2. Similarly, $\underline{a}^{k+1,\gamma}(t') = \underline{F}(\underline{a}^{k,\gamma}(t') \wedge \underline{a}^{l,\gamma}(t'), t')$ by Lemma 1. Note that l is the same in the formula for $\overline{a}^{k+1,\gamma}(t)$ and $\underline{a}^{k+1,\gamma}(t')$. Applying condition (6), $\varphi(\overline{a}^{k+1,\gamma}(t), t) \leq \varphi(\underline{a}^{k+1,\gamma}(t'), t')$. Observe that $(\overline{a}^{k,\gamma})_{k=0}^{\infty}(t)$ is increasing for all γ. Indeed, since $\overline{a}^{0,\gamma}(t) \leq \overline{F}(a^0, t) = \overline{a}^{1,\gamma}(t)$ and $\overline{a}^{k+1,\gamma}(t) = \overline{F}(\overline{a}^{k,\gamma}(t), t)$, $(\overline{a}^{k,\gamma}(t))_{k=0}^{\infty}$ is increasing by the monotonicity of $\overline{F}$ on A. Consequently,

$$\varphi(\bigvee_{k \geq n} \overline{a}^{k,\gamma}(t), t) \leq \bigwedge_{k \geq n} \varphi(\underline{a}^{k,\gamma}(t'), t') \leq \varphi(\bigwedge_{k \geq n} \underline{a}^{k,\gamma}(t'), t'). \tag{7}$$

Thus, by the monotonicity and continuity of φ, we obtain $\varphi(a^0, t) \leq \varphi(\limsup_n \overline{a}^{n,\gamma}(t), t) = \varphi(\lim_n \overline{a}^{n,\gamma}(t), t)$. Combining this with inequality (7), and referring again to the continuity of φ, we get: $\varphi(a^0, t) \leq \varphi(\liminf_n \underline{a}^{n,\gamma}(t'), t')$. Since this last inequality holds for all γ, we have that $\varphi(a^0, t) \leq \varphi(\underline{a}(t'), t')$, as desired.$\square$

Motivating example, continued Finally, observe that, although condition (5) in Theorem 5 is not satisfied for arbitrary a, a', b, b',[18] it is satisfied for the elements of the two sequences $(\overline{a}^{k,\gamma}(t))_{k=0}^{\infty}$ and $(\underline{a}^{k,\gamma}(t'))_{k=0}^{\infty}$ to which condition (6) is applied in the proof of Theorem 5. Indeed, let $t' \geq t$ and

[18] Taking $a = (2, 1), b = (2, 4), a' = (0, 1), b' = (2, 0)$ we have that $\varphi(a, 1) = 2 \leq 8 = \varphi(b, 2)$ and $\varphi(a', 1) = 0 \leq 0 = \varphi(b', 2)$ but $\varphi(f(a \vee a', 1), 1) = 2 > 0 = \varphi(f(b \wedge b', 2), 2)$.

consider a fixed point $a^0 = (\frac{x}{t}, x)$ (for some $x \in (0,1]$) with the aggregate output of $2x$. Since f is single valued, the sequence $(\overline{a}^{k,\gamma}(t))_{k=0}^{\infty}$ is constant. The sequence $(\underline{a}^{k,\gamma}(t'))_{k=0}^{\infty}$ starting from a^0 is given by $\underline{a}^{1,\gamma}(t') = (\frac{x}{t'}, \frac{xt'}{t})$ and $\underline{a}^{2,\gamma}(t') = (\frac{x}{t'}, x)$ which is a fixed point of $f(\cdot, t')$ for $t' > t$. The value of the aggregate is: $\varphi(a^0, t) = 2x = \varphi(a^0, t')$ and $\varphi(\overline{a}^{1,\gamma}(t), t) = 2x = \varphi(\underline{a}^{1,\gamma}(t'), t')$, and $\varphi(f(a^0 \vee \overline{a}^{1,\gamma}(t), t), t) = \varphi(\overline{a}^{2,\gamma}(t), t) = 2x = \varphi(\underline{a}^{2,\gamma}(t'), t') = \varphi(f(a^0 \wedge \underline{a}^{1,\gamma}(t'), t'), t')$. The conclusion of Theorem 5 is hence that $\varphi(a^0, t) = 2x \leq 2x = \varphi(\underline{a}(t'), t')$ as stated in the motivating example. It is straightforward to show that $\varphi(\overline{a}(t'), t') = 2x\frac{t'}{t}$.

6 Applications

6.1 Social learning on networks

Our first application is to studying social learning on networks. DeGroot's model, in which agents take weighted averages of the opinions they observe, is a commonly applied approach to studying social learning. Obviously, this very specific type of learning cannot well describe many real-life situations of interest. Cerreia-Vioglio et al. (2023) recently suggested a more general model, in which an opinion aggregator is a mapping that satisfies certain axioms. In their model of an economy of n agents, the opinion profile is represented by a vector $a \in A = [0,1]^n$, and learning is represented by an opinion aggregator $F : [0,1]^n \to [0,1]^n$ that is monotone with respect to coordinate-by-coordinate ordering on $A = [0,1]^n$.[19] We will illustrate our approach to comparative statics by applying it to the setting studied by Cerreia-Vioglio et al. (2023).

Our first result says that when agents assign higher weights to higher

[19] In addition to monotonicity, they impose two other axioms: normalization ($F(k, ..., k) = (k, ..., k)$ for all $k \in [0,1]$) and translation invariance ($F(a_1+k, ..., a_n+k) = F(a_1, ..., a_n) + (k, ..., k)$ whenever it makes sense). They all are satisfied in our application.

opinions, then all agents have higher opinions in the long-run. More specifically, suppose that S is the set of all possible weight vectors, that is, $S = \{s = (s_1, ..., s_n) \in [0, 1]^n : s_1 + ... + s_n = 1\}$, where s_i, the i-th coordinate of any s, represents the weight assigned to the i-th highest opinion. The set $T = S^n$ of weighting profiles (different agents may have different weight vectors) is partially ordered by $\preccurlyeq$, the coordinate-by-coordinate first-order stochastic dominance. Now, we consider two opinion aggregators $F : A \times T \to A$, which are defined as follows: each agent orders opinions in the input vector from the lowest to the highest, and her output opinion is the weighted average or the weighted median of the input opinions, where the agent uses her own weights.

Proposition 1. *If $t' \preccurlyeq t''$, then: $\underline{a}(t') \leq \underline{a}(t'')$ and $\overline{a}(t') \leq \overline{a}(t'')$; and if a^0 is a fixed point of $F(\cdot, t')$, then $a^0 \leq \underline{a}(t'')$.*

That is, if the agents shift their weights towards higher opinions, then the opinion of each of them increases in the long-run in response to this shift.

Note that this result holds true even though the opinions need not converge in the long run. Therefore, Proposition 1 could not have been obtained by referring the previously existing methods of performing comparative statics. In addition, it follows from the proof that F can be defined in a variety of other ways, for example, one may use the correspondence F that takes values between the weighted average and the weighted median, more precisely, $F(a) = \{b \in A : F^1(a) \wedge F^2(a) \leq b \leq F^1(a) \vee F^2(a)\}$, where F^1 is the weighted mean and F^2 is the weighted median.

Proof. We will check the key assumption of Theorem 2 that F is weakly increasing on $A \times T$. All other assumptions are obviously satisfied. Suppose that $a' \leq a''$ and $t' \preccurlyeq t''$. Then it follows directly from the definition of first-order stochastic dominance and the definition of F that $F(a', t') \leq F(a', t'')$.

To complete the proof we must show that $F(a', t'') \leq F(a'', t'')$. We will show that $(a')^{(k)} \leq (a'')^{(k)}$, where $a^{(k)}$ denotes the k-th lowest coordinate of a, and this will complete the proof. Indeed, if the $k-1$ lowest coordinates of a' are the same as the $k-1$ lowest coordinates of a'', then $(a')^{(k)}$ is the lowest of the remaining $n-k+1$ coordinates of a', and $(a'')^{(k)}$ is the lowest of the corresponding $n-k+1$ coordinates of a''. Then, $(a')^{(k)} \leq (a'')^{(k)}$, because $a' \leq a''$ in the coordinate-by-coordinate ordering. Otherwise, $(a'')^{(l)}$ for an $l < k$, is the coordinate of a'' corresponding to one of the $n-k+1$ highest coordinates of a'. This coordinate of a'' is higher than the coordinate of a' it corresponds to, and so $(a'')^{(l)} \geq (a')^{(k)}$. This completes the proof, because $(a'')^{(k)} \geq (a'')^{(l)}$ by definition. $\square$

We finally illustrate by an example that our approach allows for comparative statics even *beyond* of what is covered in our results. More specifically, one may be interested how different initial opinions affect the opinions in the long-run. So, consider a group of agents $N = \{1, 2, 3\}$ who share their opinions $a^0 \in [0, 1]^3$. Suppose that the weights assigned to the other agents are represented by the matrix

$$W = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.6 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}.$$

Note that this time agents assign weights by identity not according to the ranking of opinions, that is, the entry in column j and row i of the matrix represents the weight assigned by agent i to the opinion of agent j.

Consider the aggregation induced by the median. For example, $a^1 = F(a^0) = (0.6, 0.6, 0.6)$ for $a^0 = (0.6, 0.6, 0.6)$ and $a^1 = F(a^0) = (0.6, 0.4, 0.6)$ for $a^0 = (0.8, 0.6, 0.4)$. The median aggregator satisfies the conditions required by of Cerreia-Vioglio et al. Actually, the aggregator was used as an example in their paper.

When $a^0 = (0.6, 0.6, 0.6)$, then $\underline{a} = \overline{a} = (0.6, 0.6, 0.6)$. When $a^0 = (0.8, 0.6, 0.4)$, then $\underline{a}^{k,1} = \overline{a}^{k,1} = (0.6, 0.4, 0.6)$ for odd k and $\underline{a}^{k,1} = \overline{a}^{k,1} = (0.6, 0.6, 0.4)$ for even k. Thus, $\liminf_{k=\infty} \underline{a}^{k,1} = (0.6, 0.4, 0.4)$ and $\limsup_{k=\infty} \overline{a}^{k,1} = (0.6, 0.6, 0.6)$. For $\gamma \geq 2$, $\underline{a}^{1,\gamma} = \overline{a}^{1,\gamma} = (0.6, 0.4, 0.6)$; and for $k \geq 2$, $\underline{a}^{k,\gamma} = (0.4, 0.4, 0.4)$ and $\overline{a}^{k,\gamma} = (0.6, 0.6, 0.6)$. So, $\liminf_{k=\infty} \underline{a}^{k,\gamma} = (0.4, 0.4, 0.4)$ and $\limsup_{k=\infty} \overline{a}^{k,\gamma} = (0.6, 0.6, 0.6)$. This gives $\underline{a} = (0.4, 0.4, 0.4)$ and $\overline{a} = (0.6, 0.6, 0.6)$.

Therefore we conclude that if the initial opinions change from $(0.6, 0.6, 0.6)$ to $(0.8, 0.6, 0.4)$, then the opinions of all agents go down in the long run. Intuitively, the reason is that the opinions of agent 1 are less influential than the opinions of agent 3.

6.2 Network effects in a Bertrand competition

Our second application concerns firm specific network effects in oligopolistic markets with differentiated products. We refer the reader to Katz and Shapiro (1985) for a motivation and framework for studying networks effects in the Cournot industry.[20] In contrast to their paper, we analyze industries described by Bertrand price competition.

We start with an algebraic illustration. Consider a 2-player Bertrand competition with differentiated products. Suppose the demand of firm i facing competitor j is given by: $d_i(p_i, p_j) = z_i - 0.5p_i + \delta_i p_j$. Assume $\delta_i \in (0, 1)$. Marginal costs are equal to c_1 and c_2. Profit function of each company is given by: $\pi_i(p_i, p_j) = (z_i - 0.5p_i + \delta_i p_j)(p_i - c_i)$. The best responses of each company are given by: $br_i(p_j) = z_i + \delta_i p_j + 0.5c_i$ and the unique Nash equilibrium is (p_1^{NE}, p_2^{NE}), where: $p_i^{NE} = \frac{z_i}{1-\delta_1\delta_2} + \frac{\delta_i z_j}{1-\delta_1\delta_2} + 0.5\frac{\delta_i c_j + c_i}{1-\delta_1\delta_2}$. Computing the equilibrium outputs (q_1^{NE}, q_2^{NE}) we obtain:

$$q_i^{NE} = 0.5\frac{z_i}{1-\delta_1\delta_2} + 0.5\frac{\delta_i z_j}{1-\delta_1\delta_2} + 0.5\frac{c_i}{1-\delta_1\delta_2}[\delta_1\delta_2 - 0.5] + 0.25\frac{\delta_i c_j}{1-\delta_1\delta_2}.$$

[20] See also Amir et al. (2021) for a comparison of industry and firm-specific network effects.

For now, we set $c_i = c_j = 0$.

Following the literature that studies the industry viability in the presence of network externalities we now analyze quantity dynamic as a mapping of firm specific network effects. To do that let us assume that the market size parameters (z_i, z_j) in the demand function are increasing functions of own expected production levels (a_i, a_j). This can be a result of a demand side economies of scale driven, e.g., by the snob or bandwagon effects. We start from a simple example where $z_i := g_i(a_i)$ for some increasing mapping $g_i : \mathbb{R}_+ \to \mathbb{R}_+$. The Bertrand equilibrium production levels (for zero marginal costs) are now as follows: $q_1^{NE} = \frac{1}{2}\frac{g_1(a_1)}{1-\delta_1\delta_2} + \frac{1}{2}\frac{\delta_1 g_2(a_2)}{1-\delta_1\delta_2}$, $q_2^{NE} = \frac{1}{2}\frac{g_2(a_2)}{1-\delta_1\delta_2} + \frac{1}{2}\frac{\delta_2 g_1(a_1)}{1-\delta_1\delta_2}$. The own network effect $g_1(a_1)$ increases directly Nash equilibrium output q_1^{NE}, while a competitor's network effect $(g_2(a_2))$ increases firm 1 Nash equilibrium output only indirectly via equilibrium prices in a Bertrand competition. We will call it a *spillover effect*. We impose the rational expectation equilibrium (REE) condition requiring that in REE $a_i = q_i^{NE}$, i.e. the expected and realized production levels coincide for both firms. Network / production size dynamics starting from given $a^0 = (a_1^0, a_2^0)$ is given by iterating on function $f(a_1^k, a_2^k) := \left(\frac{1}{2}\frac{g_1(a_1^k)}{1-\delta_1\delta_2} + \frac{1}{2}\frac{\delta_1 g_2(a_2^k)}{1-\delta_1\delta_2}, \frac{1}{2}\frac{g_2(a_2^k)}{1-\delta_1\delta_2} + \frac{1}{2}\frac{\delta_2 g_1(a_1^k)}{1-\delta_1\delta_2}\right)$. The REE are fixed points of f.

The results of our paper can be directly applied to conduct comparative statics of the firm specific network effects dynamics for any pair of monotone functions (g_1, g_2). For example:

Proposition 2. *For any initial a^0 production level, the long-run bounds: $\underline{a}$ and $\overline{a}$ of the adaptive sequences are increasing in parameters $(\delta_1, \delta_2) \in (0,1)^2$. That is, higher product substitutability leads to higher long-run network effects and hence production levels. If a^0 is the REE for (δ_1, δ_2) then for any higher parameters (δ_1', δ_2') we have $a^0 \leq \underline{a}(\delta_1', \delta_2')$.*

Proposition 2 follows directly from Theorem 2 when we observe that f is monotone in a and δ_1, δ_2. It is also intuitive, the higher the parameters of product substitutability, the higher the spillover effects of the network effects and hence the higher the long-run realized production levels. If marginal costs (c_1, c_2) are non-zero we also conclude:

Proposition 3. *The long-run bounds: $\underline{a}$ and $\overline{a}$ are increasing in marginal costs c_1, c_2, provided $\delta_1\delta_2 \in (0.5, 1)$.*

Its proof is again an application of Theorem 2. It it enough to observe that the q_i^{NE} is increasing with c_j and with c_i provided $\delta_1\delta_2 \in (0.5, 1)$. The condition $\delta_1\delta_2 \in (0.5, 1)$ means that the spillover effects are high enough to assure that an increase in marginal costs, e.g. in c_1, leads to an increase of equilibrium output q_i^{NE} of both firms $i = 1, 2$. This is a sufficient condition for long-run production bounds to be increasing in marginal costs.

Both propositions in this example cannot be obtained using the previously existing methods. The result holds even when there are multiple REE, even when the adjustment process does not converge and importantly, for any initial production level. We finish this example with a numerical illustration which shows that our long-run bounds are often computable.

Numerical example Suppose $g_1(a) = g_2(a) = 2(a - (a-1)(a-0.5)a)$ for $a \in Z = [0, 1.14]$ with $\delta_1 = 0.02, \delta_2 = 0.05$ and $c_1 = c_2 = 0$. These functions guarantee multiplicity of REE, which is a common feature of oligopoly models with network effects. Indeed, in our example there are 3 stable REE: $(0, 0)$, $(0.04645, 1.0045)$ and $(1.0386, 1.0823)$. Dynamics of network externalities in this model has an intuitive interpretation. Let us start with Picard iterations ($\gamma = 1$). If both firms start with small levels of $a^0 = (a_1^0, a_2^0)$ such iterative dynamics will lead to stable REE $(0, 0)$ and the industry will not be viable. If both start with large levels of (a_1^0, a_2^0) (e.g. above 0.5) the iterative dynamics will converge to a greatest REE. There is an intermediate case as well, if the first firm starts from a small but the second from a large level, e.g. $(0.4, 0.48)$, the dynamics converges to the asymetric equilibrium $(0.04645, 1.0045)$ where one firm dominates the market while the competitor has a small market share. Industry viability depends also on the considered learning process, i.e. for the initial point $a^0 = (0.2, 0.48)$ although for $\gamma = 1$ the industry dynamics converges to $(0, 0)$, for any $\gamma \geq 2$ the lower bound is $\underline{a} = (0, 0)$ and the upper bound is $\overline{a} = (0.04645, 1.0045)$. The reason for this is a fact that $f(a^0)$ is not ordered with respect to a^0 and the sequence $(a^{k,\gamma})_{k=0}^{\infty}$ is not monotone. Again, the previously existing methods would not deliver these results.

Our results can be also applied to more general (than linear) demand and costs functions, in particular these that generate multiple Nash equilibria in the underlying Bertrand competition, provided the analyzed monotone correspondence F possesses the greatest and the least selections. Also, more general forms of network effects are allowed, e.g. *direct* network externalities of firm j on a demand of firm i, captured by including a_j as an additional argument of mapping g_i.

6.3 Parameterized interim Bayesian supermodular games

Our approach provides new results on the existence and comparative statics of pure strategy equilibria in the interim formulation of Bayesian supermodular games.[21] The primitives of the game are given by a tuple $(I, T, (\mathcal{F}_i, A_i, u_i, p_i)_{i\in I}, \Theta)$, where I is the set of players. The types are given by T_i with sigma-field $\mathcal{F}_i$ for each $i \in I$ (where we let $T = \prod_{i\in\{0\}\cup I} T_i$ with the product sigma-field). The component T_0 (with sigma-field $\mathcal{F}_0$) represents residual uncertainty not observed by the players (or the set of

[21] The class of interim Bayesian games is broader than the ex-ante Bayesian games. Moreover, the notion of interim Bayesian Nash equilibrium is stronger than the notion of an ex-ante Bayesian Nash equilibrium. See Van Zandt (2010) for details and the global games literature (e.g., Morris and Shin (2002) and Morris et al. (2016)) for applications.

possible payoffs). Type dependent belief functions for the players are given by $p_i : T_i \to M_{-i}$, where M_{-i} is the set of probability measures on T_{-i} with the product sigma-field $\mathcal{F}_{-i}$. For each set $F_{-i} \in \mathcal{F}_{-i}$, the mapping $t_t \to p_i(F_{-i} \mid t_i)$ is measurable. The action set for each $i \in I$ is A_i, which is a compact metric lattice (with its Borel sigma-field), and we denote the space of joint action profiles by A. Finally, Θ is a partially ordered set of parameters.

The players' payoffs are given by $u_i : A \times T \times \Theta \to \mathbb{R}$ with u_i bounded. We additionally assume: (a) u_i is continuous and supermodular in player's own action, (b) u_i is measurable in t, (c) u_i has increasing differences between (a_i, a_{-i}), and (d) u_i has (i) increasing differences (resp, (ii) strictly increasing differences) between (a_i, θ). Equilibria of this game were studied by Van Zandt (2010).

The set of interim Bayesian Nash equilibrium (BNE) at each parameter $\theta \in \Theta$ coincides with the set of fixed-points of the correspondence defined by the players' joint best response correspondences. That is, the BNE are fixed points of a parameterized, weakly increasing correspondence $F : S \times \Theta \rightrightarrows S$, where S denotes the set of strategies - measurable functions mapping individual types (a sigma-field) to individual actions (a complete lattice) - endowed with the pointwise partial order. S is a thus a sigma-complete lattice with greatest and least elements.[22]

Our Theorem 2, applied to the best response correspondence F, provides new results on equilibrium comparative statics with respect to θ. Under assumptions (a)-(d)(i), the best response correspondence admits least and greatest selections that are, respectively, upward order continuous and

[22] In the ex-ante formulation of our Bayesian supermodular game (e.g., see van Zandt and Vives (2007), as well as Sabarwal (2025b)), one works with equivalence classes of measurable strategies ordered almost everywhere pointwise. The set of such strategies is a complete lattice. Existence of ex-ante pure strategy BNE then follows from an application of the Veinott-Zhou version of Tarski's fixed point theorem and order-nearest equilibrium comparative statics follow from recent results in Sabarwal (2025a).

downward order continuous (see Van Zandt (2010)). Theorem 2 (i) guarantees that the bounds of adaptive learning sequences iterated on a best response correspondence with higher parameter are higher than the bounds of adaptive learning sequences iterated on a best response correspondence with lower parameter, *irrespective* of the initial profile of strategies (in particular, even when that profile is not an equilibrium profile). Moreover, under condition (d)(ii) (instead of (d)(i)) Theorem 2 (iii) ensures than, in the long run, adaptive learning sequences started from an initial interim BNE for a smaller parameter, but iterated on a best response correspondence at a higher parameter, are bounded below by an order nearest interim BNE of a game with higher parameter.[23]

6.4 Distributional dynamics and long-run income distribution in monotone economies

We first provide an algebraic illustration of our techniques applied to a distributional comparative statics in the spirit of Camacho et al. (2018) (Section 6) and then apply our techniques to obtain general monotone comparative statics of the long-run income dynamics in monotone economies.

Distributional income dynamics Consider a continuum of agents of size 1, where each individual is characterized by its wealth or income. Normalize wealth / income on $[0,1]$ with a standard order. In this algebraic example, we model a distribution of wealth / income in the population using a family of beta distributions[24] $Be(x,y)$ with shape parameters (x,y).

[23] It bears mentioning, for the interim formulation of this game, since the space of measurable functions S is not a complete lattice, none of the results in Sabarwal (2025a) apply. Relative to Van Zandt (2010), our results provide iterative monotone comparative statics starting from *any initial profile* of strategies.

[24] A density of $Be(x,y)$ is $\rho(w;x,y) = \frac{1}{\xi(x,y)} w^{x-1}(1-w)^{y-1}\mathbf{1}_{[0,1]}(w)$, where $\xi(x,y)$ is the normalized constant. It is well-known, $\xi(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, where $\Gamma(a) := \int_0^\infty w^{a-1}e^{-w}dw$ for every $a>0$.

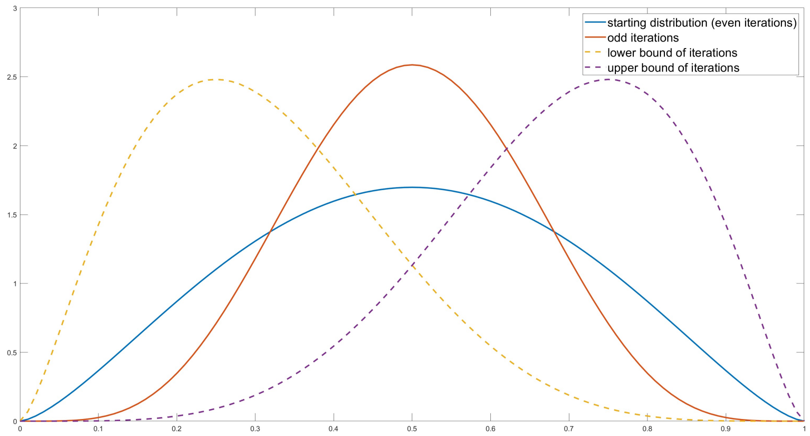
For example $Be(1,1)$ is the uniform distribution on $[0,1]$. Assume that $(x,y) \in [\varepsilon, D-\varepsilon]^2$ for some small $\epsilon > 0$, and large enough $D > \varepsilon$. Suppose that processes in the economy (a summary of earnings, innovations, materialization of risks but also governmental policies like taxes or subsidies, etc.) affect the wealth / income distribution via mapping f. In particular, let $A := \{Be(x,y) : (x,y) \in [\varepsilon, D-\varepsilon]^2\}$, and let $f : A \to A$ be defined, e.g., as follows $f(Be(x,y)) := Be(D-y, D-x)$. Endow A with the first order stochastic dominance. It is well known that the distribution $Be(x', y')$ stochastically dominates $Be(x,y)$ if $x' > x$ and $y > y'$. As a result, f is an increasing mapping. The set of fixed points of f is a set of invariant measures under this map that represent the stable income / wealth distribution. Such fixed points are given by: $\{Be(x, D-x) : x \in [\varepsilon, D-\varepsilon]\}$. This is totally ordered set with the least element $Be(\varepsilon, D-\varepsilon)$ and the greatest element $Be(D-\varepsilon, \varepsilon)$.

Now illustrate our results by studying distributional income / wealth dynamic $\{\mu^{k,\gamma}\}_{k=0}^{\infty}$ governed by f. For given x_0, let $\mu^0 := Be(x_0, x_0)$. Then for any odd k, $\mu^{k,1} = Be(D-x_0, D-x_0)$ and for even k, $\mu^{k,1} = Be(x_0, x_0)$. As a consequence, $\underline{\mu} = Be(\min(x_0, D-x_0), \max(x_0, D-x_0))$, and $\overline{\mu} = Be(\max(x_0, D-x_0), \min(x_0, D-x_0))$.

For example, take $D = 8$ and $\mu = Be(2.5, 2.5)$. We get $\mu^{k,1} = Be(5.5, 5.5)$ for odd k, and $\mu^{k,1} = Be(2.5, 2.5)$ for even k. We get $\underline{\mu} = Be(2.5, 5.5)$ and $\overline{\mu} = Be(5.5, 2.5)$. Both are fixed points of f. Figure 1 illustrates the iterations.

Comparing long-run income distributions in monotone economies Our results can be applied to the comparative statics of long-run distributions of output or income associated with more abstract infinite-horizon stochastic growth models with nonconvexities. Let the production or income available at period t be $y_t \in Y$, where $Y = [0, \bar{Y}] \subset \mathbb{R}_+$. Agent selects a consumption

Figure 1: Cycles and long-run distribution dynamics for a family of beta distributions.



level $c_t \in [0, y_t]$, with the remaining resources $i_t = z_t - c_t$ allocated as an investment. The evolution of income is given by $y_{t+1} = f(i_t, z_{t+1})$ where f is a continuous, strictly increasing production function and z_{t+1} is a random shock drawn each period from distribution π over a finite set Z. For simplicity, we assume full depreciation. The temporal utility is given by a continuous, strictly increasing and strictly concave mapping $u : Y \to \mathbb{R}$. The agent's objective then is to maximize her expected discounted payoffs over an infinite horizon, given an initial state $y_0 \in Y$ and discount $\beta \in (0, 1)$. Denote the value of this optimization problem by $v^*(y_0)$. This problem admits a recursive representation, where $v = v^*$ is the unique solution to the Bellman equation: $v(z) = \max_{i \in [0,y]} u(y - i) + \beta \int_Z v(f(i, z'))d\pi(z')$.

Let the policy correspondence be given by $H^*(y, \beta) = \arg\max_{i \in [0,y]} u(y - i) + \beta \int_Z v^*(f(i, z'))d\pi(z')$. Since u is strictly increasing and strictly concave, the objective has strictly increasing differences in $(i; y, \beta)$. Then, by an application of the Topkis (1998) Theorem (e.g., Theorem 2.8.4), the policy correspondence[25] H^* is a nonempty and jointly strongly increasing

[25]Recall, since f is not necessarily concave in i the policy is not necessarily unique.

in (y, β).

Let $\mathcal{M}(Y)$ denote a set of measures on Y endowed with the first-order stochastic dominance and the weak convergence of measures. $\mathcal{M}(Y)$ is a complete lattice[26]. For a measurable set $B \subset Y$ define the stochastic income transition with $Q(B|i) := \int_Z 1_B(f(i, z'))d\pi(dz')$. For any selector $h_\beta(\cdot) \in H^*(\cdot, \beta)$, define the associated adjoint Markov operator: $\Lambda_{h_\beta}\mu(B) = \int_Y Q(B|h_\beta(y))\mu(dy)$ and the associated adjoint Markov correspondence: $\Lambda\mu(B) = \{\Lambda_{h_\beta}\mu(B)\}_{h_\beta \in H^*(\cdot,\beta)}$. Since H^* is strongly increasing, Λ is strongly increasing[27] on $M(Y)$.

We can now apply our Theorem 2 to characterize the iterative monotone comparative statics of the stationary income distributions.

Proposition 4. *When $\beta_1 \leq \beta_2$, for any initial measure $\mu_0 \in \mathcal{M}(Y)$, the lower (resp., upper) bounds for long-run income distribution dynamics are increasing, i.e. $\underline{\mu}(\beta_1) \leq \underline{\mu}(\beta_2)$ (resp., $\overline{\mu}(\beta_1) \leq \overline{\mu}(\beta_2)$), and from any stationary equilibrium at the discount rate β_1, say μ_{β_1}, we conclude that iterations on Λ_{β_2} from μ_{β_1} satisfy $\mu_{\beta_1} \leq \underline{\mu}(\beta_2)$.*

We remark that a similar reasoning can be applied to study the stationary equilibrium distribution in large dynamic economies in the spirit of Bewley or Huggett/Aiyagari models without aggregate risk. Indeed, interpreting μ as a distributions of income over Y in some large economy, we can study monotone comparative statics of stationary or invariant income distributions after the monotone exogenous shock to the policy function h in the income fluctuation problem of the shocks governed by Q for any initial income distribution μ^0.

In addition, our results extend the stationary equilibrium comparative statics for monotone economies based upon the work of Hopenhayn and Prescott (1992), Huggett (2003), and Acemoglu and Jensen (2015) as they

[26]See, for example, Kamae et al. (1977)).

[27]See Huggett (2003) Theorem 1.

impose either unique (and stable) stationary equilibrium distribution or, under multiplicity, concentrate on extremal stationary equilibrium distributions only.

6.5 Comparing recursive equilibria in dynamic models with indeterminate equilibria

We finally show how to apply our results to obtain monotone comparative statics of (minimal state space) recursive equilibria (RE) in macroeconomic models with multiplicities.[28]

Consider a simple stochastic OLG economy with production. There is a continuum of identical agents born each period who live for two periods. In the first period of life, they are endowed with a unit of time which they supply inelastically to the firm at the prevailing wage $w(s)$, and they consume and save. In the second period of life, they consume their savings which are subjected to a stochastic return $r(s')$. Here s and s' denote vectors of aggregate state variables in the current and the following periods. Preferences are time separable with discount rate $\beta \in (0, 1)$ and are given by $u(c_1) + \beta v(c_2)$, where consumption when young (resp., old) is denoted by c_1 (resp., c_2), and $u : \mathbb{R}_+ \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$ are smooth, strictly increasing, strictly concave, with $\lim_{c\to 0^+} u'(c) = \infty = \lim_{c\to 0^+} v'(c)$.

The reduced-form technology is given by $f(k, n)e(K, N, z)$ where f is a technology transforming private inputs of capital and labor (k, n), and the externality $e(K, N, z)$ is a total factor productivity that depends on per capital aggregates of capital and labor (K, N) and a shock $z \in Z = [z_l, z_h] \subset \mathbb{R}_{++}$ which is drawn each period from a first-order Markov process with stationary transition $\pi(z, z')$ that satisfies a Feller property. We let f satisfy typical assumptions, namely: it is constant returns to scale, in-

[28] See Coleman (1991) and Mirman et al. (2008) for motivation and other references.

creasing (but increasing strictly with each argument for the positive input of the other), weakly concave jointly (but strictly concave with each argument separately for the positive input of the other) and twice continuously differentiable. Moreover $r(k, z) := f_1(k, 1)e(K, 1, z)$ is decreasing in k and increasing in K for $K > 0$,[29] $\lim_{k\to 0^+} r(k, z) = \infty$, $\lim_{k\to 0^+} r(k, z_{\max})k = 0$; $w(k, z) := f_2(k, 1)e(k, 1, z)$ is increasing in k with $\lim_{k\to 0^+} w(k, z) = 0$; both r and w are increasing in z for all k. These assumptions will be later used to show that a RC is a fixed point of a *monotone* operator defined using a Euler equation. Finally, $f(0, 1)e(0, 1, z) = 0$ for any z and there exists a maximal sustainable capital stock[30] denoted by $k_{\max}$. This last assumptions assures that the state space for capital is a compact set $X \subset \mathbb{R}_+$. Many specific examples of technologies that satisfy these assumptions can be given (see, e.g., Datta et al. (2018)).

We now proceed to define a RE. Anticipating that $n = 1 = N$ and $k = K$ in any RE, and denoting the aggregate vector of state variables by $s = (K, z) \in S = X \times Z$, we consider the existence of RE in a class of investment functions W with pointwise partial orders. In particular, let $W = \{h : S \to \mathbb{R}_+,\ 0 \le h \le w,\ h(k, z)$ increasing in k, and measurable$\}$. Together with π, $h \in W$ describes the law of motion for the aggregate variables. Taking this, a young agent solves: $\max_{y\in[0,w(s)]} u(w(s)-y)+\beta \int_Z v(r(h(s), z')y)\pi(z, dz')$. Let $\hat{y}(s; h)$ be the optimal solution to this household problem for $h \in W$. The optimal solution is unique under our assumptions. Labor and capital markets are competitive hence by profit maximization, in an RE, $w(K, z) = f_2(K, 1)e(K, 1, z)$ and $r(K, z) = f_1(K, 1)e(K, 1, z)$. An RE of this economy in the space W is a law of motion $h^* \in W$ and policy function $y^* \in W$ such

[29] This model is known to have multiple equilibria. Recall $f_1(k, 1)e(K, 1, z)$ under our assumptions is *mixed monotone* in (k, K), i.e., decreasing in k and increasing in K. This is a critical feature that creates the possibility for equilibrium indeterminacy (i.e. a continuum of equilibria) in this class of models (see Santos (2002) and Datta et al. (2018) for discussion).

[30] $\forall k \ge k_{\max}$ and $\forall z \in Z$, $F(k, 1, k, 1, z) \le k_{\max}$.

that $y^*(s) = \hat{y}(s; h^*) = h^*(s) \in W$ for each $s \in S_{++}$ whenever $h^*(s) > 0$. Here $S_{++} := X_{++} \times Z$ with $X_{++} \subset \mathbb{R}_{++}$. Market clearing is implied by the formulation of the household problem.

We now proceed to conduct a comparative statics exercise. Specifically, consider the question of capital deepening in the discount rate β to the set of RE in this economy. Define a nonlinear operator F_β on W as follows: for $h(s) > 0$, $h \in W$, let $F_\beta h(s)$ be the unique y solving:

$$u'(w(s)-y)-\beta \int_Z v'(f_1(h(s),1)e(y,1,z')y)f_1(y,1)e(h(s),1,z')\pi(z,dz') = 0, \tag{8}$$

with $F_\beta h(s) = 0$, whenever $h(s) = 0$ in any state s. Therefore, any mapping $h^*_\beta \in W$ is an RE law of motion if and only if it is a non-zero fixed point of the operator F_β in W. It is easy to prove F_β is a monotone operator on W. Further, under our assumptions, it can be show there exists a continuous $h_0 \in W$ such that $\forall h \geq h_0 > 0$, $F_\beta h > h_0$ on S_{++}.[31] Let $A = W \cap [h_0, w]$, and endow A with its relative pointwise partial order $\leq$.

Observe that Theorem 2 does not guarantee existence of RE. To see that the set of RE is non-empty in the analyzed economy observe that $(A, \leq)$ is a sigma-complete lattice and $F_\beta : A \to A$ is an order-continuous self map, hence, there exists an RE by an application of a Tarski-Kantorovich fixed point theorem.[32]

Applying Theorem 2 to the operator F_β we obtain:

Proposition 5. *For any initial $h^0 \in A$ and $\beta_1 \leq \beta_2$, we have: (i) the lower bounds for RE satisfy $\underline{h}(\beta_1) \leq \underline{h}(\beta_2)$ and the upper bounds satisfy $\overline{h}(\beta_1) \leq \overline{h}(\beta_2)$, (ii) taking initially any RE $h^*_{\beta_1} \in A$ but iterating on F_{β_2} we also have $h^*_{\beta_1} \leq \underline{h}(\beta_2)$.*

[31] See McGovern et al. (2013) Proposition 2. In fact, $h_0 \in W$ is continuous jointly in (k, z).

[32] RE also exist (see Morand and Reffett (2007)) in subsets of W, where the elements of these subsets are additionally (i) lower or (ii) upper semicontinuous in k.

We conclude with two remarks about Proposition 5. First, in the interpretation, this proposition says that even in the presence of a possible continuum of RE in $(A, \leq)$, from any initial RE $h^*_{\beta_1} \in A$ of the less "patient" economy, the iterative process of computing the RE for the more patient economy is bounded below by $h^*_{\beta_1} \in A$.

Second, in dynamic stochastic equilibrium models with uncountable shocks the analyzed function space A (with the pointwise partial orders) is only sigma-complete. Hence, none of the results of Echenique (2002, 2004) can be applied. Yet, based on Theorem 2, we obtain a result on iterative monotone comparative statics of (monotone and measurable) RE. If the shocks are discrete, then the space $(A, \leq)$ is a complete lattice and by Tarski's theorem the set of RE on $(A, \leq)$ is a nonempty complete lattice for each β. The novelty of applying Theorem 2 in such settings is that it provides a monotone comparative statics of the long run (adaptive learning) bounds, and not only the comparative statics of extremal RE.

Applications of Theorem 2 can also be proposed in macroeconomic models with infinitely lived agents as well as with equilibrium indeterminacies (see for example Benhabib and Farmer (1994) and Datta et al. (2018)). Even in nonconvex, nonoptimal dynamic economies, where the optimal household decisions are correspondences, Theorem 2 can be applied to obtain similar iterative monotone comparative statics result as these correspondences satisfy conditions of Theorem 2 under standard assumptions.[33]

A Appendix

Proof to Theorem 3. We will show that $\underline{a} \geq a^*$; the proof that $\overline{a} \leq a^*$ is analogous. Let $a \ll a^*$, by this we mean that all coordinates of a are

[33] See Mirman et al. (2008) for examples of such economies and assumptions guaranteeing existence of RE.

smaller than the corresponding coordinates of a^*. Since $(\underline{a}^{k,\gamma})_{k=0}^{\infty}$ is an adaptive learning sequence, $\underline{a}^{k,\gamma} > a$ for large enough values of k. Therefore, $\bigwedge_{l\geq k} \underline{a}^{l,\gamma} \geq a$ for large enough values of k, what implies that $\liminf_k \underline{a}^{k,\gamma} \geq a$. This is true for all γ. Since $\underline{a}$ is the limit of the decreasing sequence $(\liminf_k \underline{a}^{k,\gamma})_{\gamma=1}^{\infty}$, we obtain that $\underline{a} \geq a$. Since this is true for all $a \ll a^*$, we have that $\underline{a} \geq \sup\{a \in A : a \ll a^*\} = a^*$, because a^* is in the interior of the Euclidian space. Finally, $\underline{a} \geq a^*$, $\overline{a} \leq a^*$ and $\underline{a} \leq \overline{a}$ deliver the result.

Proof to Lemma 1 For $n = 0$ by definition and for $n > 1$ by induction: for any $1 \leq j \leq \gamma$ we have $\underline{a}^{n\gamma+j,\gamma} = f(\bigwedge\{\underline{a}^{n\gamma+k,\gamma} : k = 0, 1, \ldots, j-1\})$. Since $\bigwedge\{\underline{a}^{n\gamma+k,\gamma} : k = 0, 1, \ldots, j-1\} \leq \bigwedge\{\underline{a}^{n\gamma+k,\gamma} : k = 0, 1, \ldots, j-2\}$, we have $\underline{a}^{n\gamma+j,\gamma} = f(\bigwedge\{\underline{a}^{n\gamma+k,\gamma} : k = 0, 1, \ldots, j-1\}) \leq f(\bigwedge\{\underline{a}^{n\gamma+k,\gamma} : k = 0, 1, \ldots, j-2\}) = \underline{a}^{n\gamma+j-1,\gamma}$ for $j \geq 2$ by the monotonicity of f. Thus the sequence $(\underline{a}^{n\gamma+j,\gamma})_{j=1}^{\gamma}$ is decreasing. As a consequence, $\underline{a}^{n\gamma+j,\gamma} = f(\underline{a}^{n\gamma+j-1,\gamma} \wedge \underline{a}^{n\gamma,\gamma})$ for any $1 \leq j \leq \gamma$. This yields the lemma.

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