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Markov perfect equilibria in stochastic growth models with quasi-hyperbolic discounting and risk-sensitive preferences

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Abstract

We show the existence of a Markov perfect equilibrium in a non-stationary stochastic growth model with quasi-hyperbolic discounting and risk-sensitive preferences. Our proof resembles dynamic programming approach and does not involve any fixed point theorem. The risk-sensitive preferences are modelled via certainty equivalents for various utility functions including these implying risk-aversion as well as risk-seeking. Under the additional assumption of stationarity, we are able to prove the existence of a stationary Markov perfect equilibrium using the Schauder fixed point theorem.

Keywords Risk-sensitive preferences · Certainty equivalent · Quasi-hyperbolic discounting · Stochastic growth model · Markov perfect equilibrium

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1 Introduction

Beginning with the work of Strotz [50] and Phelps and Pollak [41], a long line of research in behavioural economics has explored the implications of dynamic inconsistencies in intertemporal choice. The motivation for much of this work lies in the extensive empirical and experimental literature documenting the prevalence of preference reversals, when agents compare current and future utilities. A key subclass of such models involves dynamic inconsistencies arising from non-exponential discounting, particularly quasi-hyperbolic discounting (see Harris and Laibson [23] and references cited therein). Models using quasi-hyperbolic discounting have become central to behavioural studies involving impulse and cue-driven theories of choice, temptation-driven preferences, and other models of self-control. Applications of these models have appeared in many fields, including mathematical psychology, political science, decision theory, game theory, and especially economics. Recent contributions in this line of research include applied works [17, 34], studies on the existence of Markov perfect equilibria [5, 7, 29] and papers on approximation and computation of Markov perfect equilibria such as [6, 35].

In this paper, we consider a class of stochastic growth models (see, e.g., [12, 16, 24, 49]) with uncountable state spaces, quasi-hyperbolic discounting, and risk-sensitive preferences. We allow the one-period utility and transition probability depend on the time period. It is well-known that quasi-hyperbolic discounting leads to dynamic inconsistency in preferences. We therefore envision an individual agent as a sequence of autonomous temporal *selves*, indexed by period numbers t, in which they make their decisions. Self t receives a deterministic current utility and then averages the remaining part of the utility, discounted in a quasi-hyperbolic manner. To account for risk sensitivity, we consider certainty equivalents involving the expectation operator corresponding to a probability measure induced by the strategies of the following selves and the transition probabilities. More precisely, this probability measure is constructed using the Ionescu-Tulcea theorem [40] on the space of all admissible infinite trajectories (histories or plays).

In this class of models, we prove the existence of a Markov perfect equilibrium. Our approach more closely resembles dynamic programming method and does not rely on a fixed point theorem. We provide several certainty equivalents that satisfy our assumptions, including those implying risk-aversion and risk-seeking. Under the additional assumption of stationarity, we also prove the existence of a stationary Markov perfect equilibrium using the Schauder fixed point theorem. The existence of a stationary Markov perfect equilibrium in a stochastic game with finite state and action spaces with quasi-hyperbolic discounting was stated first by Alj and Haurie [2].

This paper extends the results in [5] on the existence of Markov perfect equilibria given for a stochastic growth model, in which the agent is risk-neutral. In this way, our work links two strands of the literature in dynamic decision models: one with risk-sensitive preferences and the other with quasi-hyperbolic discounting. Here, it is worthy to mention that such an approach has been recently applied to Markov decision processes with a countable state space by Jaśkiewicz and Nowak in [31].

An early application of certainty equivalent for the exponential utility function in Markov decision processes was given by Howard and Matheson [25] and Jacobson [26]. Nowadays, this optimization criterion has found numerous applications in stochastic optimization [10, 14, 45, 52], in finance and portfolio management [20], in dynamic games [8], and in macroeconomics [3, 38]. Further comments on our results and the literature are given in Remarks 1-2.

The rest of our paper is organized as follows. In Sect. 2, we discuss and provide examples of the certainty equivalents we consider. Section 3 presents some preliminary results. Section 4 introduces the model, assumptions, and our main results on the existence of Markov perfect equilibria. It also contains comments on the invariant distributions implied by equilibrium strategies. The proofs are provided in Sect. 5.

2 Certainty equivalents

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ be the set of all positive integers.

Assume that $(\Omega, \mathcal{F}, P)$ is a probability space and $X : \Omega \to \mathbb{R}$ is a bounded, non-negative random variable. Let $\mathcal{E}(X)$ be a *certainty equivalent* of X induced by the probability measure P, i.e.,

$$\mathcal{E}(X) := \mathcal{U}^{-1}(\mathbb{E}\mathcal{U}(X)) = \mathcal{U}^{-1}\left(\int_\Omega \mathcal{U}(X)dP\right), \tag{1}$$

where $\mathcal{U} : [0, \infty) \to \mathbb{R}$ is an increasing, at least twice continuously differentiable function such that $\mathcal{U}(0) = 0$. Here, $\mathbb{E}$ is the expectation operator with respect to P. Quantity (1) is also known under the name of a *quasi-linear mean* and has a long history. Indeed, this notion has been treated already by famous researchers like Bonferroni, Kolmogorov and de Finetti, when they founded modern probability theory. For a review of the early history of certainty equivalents the reader is referred to [39].

Making use of the Taylor series expansion we obtain

$$\mathcal{U}(\mathcal{E}(X)) = \mathbb{E}\mathcal{U}(X) \approx \mathcal{U}(\mathbb{E}X) + \frac{1}{2}\mathcal{U}''(\mathbb{E}X) \text{ Var } X.$$

Note that in case of the maximiser if $\mathcal{U}$ is strictly concave, then the agent is risk-averse. If, on the other hand, $\mathcal{U}$ is strictly convex, then the agent is risk-seeking. The degree of agent's risk-aversion at level $x \geq 0$ is formally defined by the *Arrow-Pratt coefficient*, which is

$$AP(x) := -\frac{\mathcal{U}''(x)}{\mathcal{U}'(x)}. \tag{2}$$

Clearly, positive (negative) values of $AP(\cdot)$ indicate that the agent is risk-averse (risk-seeking). The reader is referred to [43] and [20] for the interpretation and a further discussion of $AP(\cdot)$. Basically, coefficient function (2) shows how risk attitude changes with the wealth level. The standard examples embrace the following classes of utility functions.

- The logarithmic utility function $\mathcal{U}(x) = \ln(x + 1)$. The risk aversion coefficient is $AP(x) = \frac{1}{x+1}$. The certainty equivalent of a non-negative random variable X is given by

$$\mathcal{E}(X) = \exp(\mathbb{E}\ln(X + 1)) - 1. \tag{3}$$

- The exponential utility function $\mathcal{U}(x) = \operatorname{sgn}(r)(e^{rx} - 1)$ with some number $r \neq 0$ produces the constant coefficient function, i.e., $AP(x) \equiv -r$. The inverse function is either $\mathcal{U}^{-1}(y) = \frac{1}{r}\ln(1-y)$ for $y \in [0,1)$ and $r < 0$ or $\mathcal{U}^{-1}(y) = \frac{1}{r}\ln(1+y)$ for $y \in [0,\infty)$ and $r > 0$. Hence, the certainty equivalent of a random variable *X* equals

$$\mathcal{E}(X) = \frac{1}{r}\ln \mathbb{E}\exp(rX). \tag{4}$$

The exponential utility function belongs to *CARA* (Constant Absolute Risk Aversion, i.e., *AP*( ) is constant) class of utilities.

- The power utility function $\mathcal{U}(x) = \frac{1}{p}x^p$ with some positive number $p \neq 1$, gives $AP(x) = (1-p)/x$ on $[0,\infty)$. Obviously, the agent is risk-averse if $p \in (0,1)$ and risk-seeking if $p > 1$. For $p \in (0,1)$ this function is in the class of so-called *HARA* (Hyperbolic Absolute Risk Aversion, i.e., $AP(x) = \frac{1}{\gamma+\delta x}$ with $\gamma + \delta x > 0$) utilities and in this case risk-aversion decreases as wealth increases. The certainty equivalent of a bounded, non-negative random variable *X* is given by

$$\mathcal{E}(X) = (\mathbb{E}X^p)^{\frac{1}{p}}. \tag{5}$$

- The power utility function $\mathcal{U}(x) = \frac{1}{p}(x+1)^p - \frac{1}{p}$ with some negative number *p* gives $AP(x) = (1-p)/(x+1)$. This function also belongs to the *HARA* class. The certainty equivalent of *X* is as follows

$$\mathcal{E}(X) = (\mathbb{E}(X+1)^p)^{\frac{1}{p}} - 1. \tag{6}$$

- The certainty equivalent $\mathcal{E}(X)$ can also be defined for a linear, increasing function $\mathcal{U}$ and then $\mathcal{E}(X) = \mathbb{E}X$. In this case, called in the literature *risk-neutral*, $AP(x) = 0$, which means that the wealth level does not influence the risk attitude of the agent.

The certainty equivalents can be also viewed as risk measures. For instance, the negative of certainty equivalent (4) is also known as the entropic risk measure of *X*, see Section 4.1 in [20].

The certainty equivalents have been applied to various dynamic decision models. The first studies of certainty equivalent (4) in the theory of Markov decision processes (*MDPs*) date back to the pioneering works of Howard and Matheson [25] and Jacobson [26], where the certainty equivalent for the exponential function was applied. Further results for Markov control models with a general state space, discounted payoffs and a risk-sensitive agent using certainty equivalent (4) can be found in [18], where the authors establish the optimality equation and prove the existence of a Markov optimal policy. Linear-quadratic problems were considered in [26] with a finite time horizon and in [22] with the infinite time horizon. Other useful approaches to the certainty equivalents in *MDPs* with a Borel state space can be found in [11].

The importance of the certainty equivalent in macroeconomics and finance was shown in [3, 38]. Furthermore, their applications to various decision problems that use dynamic programming techniques have been well-described in a recent monograph [45]. In these applications, however, certainty equivalents are used within the framework of recursive utility theory. They are part of the so-called aggregator functions and are applied in each period of the process to evaluate its random continuation values. Some recent results along this line of research, in the context of quasi-hyperbolic discounting, can be found in [7]. The key difference between these models and ours is that we apply certainty equivalents to quasi-hyperbolically discounted random streams of one-period utilities (or payoffs) obtained along infinite trajectories of the process.

3 Preliminaries

Assume throughout this paper that $S := [0, +\infty)$, $S_+ := (0, +\infty)$ and, for each $s \in S$, $A(s) := [0, s]$. We shall use some special classes of functions to define strategies of the agents. By C we denote the set of all right-continuous mappings $c : S \to S$ such that the functions $i(s) := s - c(s)$ are non-decreasing and $0 \le c(s) \le s$ for all $s \in S$. Note that i is also right-continuous and is upper semicontinuous. Thus, $c \in C$ is lower semicontinuous. We put $I := \{i : S \to S : i(s) = s - c(s),\ c \in C\}$. Note that $i \in I$ and $c \in C$ may have at most countably many discontinuity points.

Let $u : S \to \mathbb{R}$ and $w : S \to \mathbb{R}$ be continuous functions. For $s \in S$ we set

$$A_1(s) := \arg \max_{a \in A(s)} (u(s-a) + w(a)) \quad \text{and} \quad A_2(s) := \arg \max_{b \in A(s)} (u(b) + w(s-b)).$$

The sets $A_1(s)$ and $A_2(s)$ are non-empty and compact.

Define

$$i_0(s) := \max A_1(s) \quad \text{and} \quad c_0(s) := \min A_2(s),\ s \in S.$$

By a simple adaptation of the arguments given in the proof of Theorem 6.3 in [51], we can obtain the following auxiliary result.

Lemma 1 *Assume that u and w are continuous and u is strictly concave. Then:*

(a) *The correspondence $s \to A_1(s)$ is compact-valued and strongly ascending, i.e.,*

$$(s_1 < s_2 \quad \text{and} \quad a_1 \in A_1(s_1),\ a_2 \in A_1(s_2)) \quad \text{implies that} \quad a_1 \le a_2.$$

Moreover, $i_0 \in I$.

(b) *If $i(s) \in A_1(s)$ for all $s \in S$ and s_0 is a continuity point of the function i, then $A_1(s_0)$ is a singleton.*
(c) *If s_0 is a continuity point of the function c_0, then the set $A_2(s_0)$ is a singleton.*
(d) *The only function $c \in C$ such that $c(s) \in A_2(s)$ for all $s \in S$ is $c = c_0$.*

Proof (a) Since u is strictly concave, the function $k(s, a) := u(s - a) + w(a)$ is supermodular on the set $\Delta = \{(s, a) : s \in S, 0 \le a \le s\}$. The fact that A_1 is strongly ascending follows from the proof of Lemma 3.2 in [4]. This in turn implies that i_0 is non-decreasing and from the definition of i_0, it follows that this function is right-continuous.

(b) Consider any continuity point $s_0 > 0$ of i. Assume that $A_1(s_0)$ has more than one point. Let $a_1 = \min A_1(s_0)$. Since A_1 is strongly ascending and i is right-continuous, it follows that

$$\lim_{s \to s_0-} i(s) \leq a_1 < i_0(s_0) = \lim_{s \to s_0+} i(s) = i(s_0).$$

This contradicts the continuity of i at s_0. Thus, $A_1(s_0)$ contains one point. Trivially, $A_1(0) = \{0\}$.

(c) It is easy to see that $c_0(s) = s - i_0(s)$ for all $s \in S$. The assertion follows from (b) since s_0 ia also a continuity point of i_0.

(d) Consider $i(s) = s - c(s)$, $s \in S$. Every continuity point s_0 of c is also a continuity point of i. Then, by (b), $A_1(s_0)$ is a singleton and $c(s_0) = s_0 - i(s_0) = s_0 - i_0(s_0)$. Thus, $c(s) = s - i_0(s)$ for all continuity points of c. Since both functions c and i_0 are right-continuous, we have $c(s) = s - i_0(s)$ for all $s \in S$. This obviously implies that $c = c_0$. $\square$

Let λ be a non-atomic probability measure on S. We assume that $\int_S s^2 \lambda(ds) < \infty$. Let $L^2(S)$ be the Hilbert space of all λ-equivalence classes of functions $\phi : S \to \mathbb{R}$ with the norm

$$\|\phi\|_2 := \sqrt{\int_S \phi^2(s) \lambda(ds)}.$$

For each $\phi \in C$ we write $\tilde{\phi}$ to denote the λ-equivalence class of ϕ in $L^2(S)$. We set $\tilde{C} := \{\tilde{\phi} \in L^2(S) : \phi \in C\}$. We can define a metric on C as follows. If $\phi,\ \psi \in C$, then $\ell(\phi, \psi) := \|\tilde{\phi} - \tilde{\psi}\|_2$.

In the sequel, we use the standard notion of *weak convergence* of a sequence $(f_n)_{n \in \mathbb{N}}$ of functions on S to $\mathbb{R}$. We say that (f_n) converges weakly to f if and only if $f_n(s) \to f(s)$ as $n \to \infty$ at any continuity point $s \in S$ of f. This weak convergence is denoted by $f_n \xrightarrow{\omega} f$.

From Proposition 2 in [30], we infer the following fact.

Lemma 2 (a) *Let $f, f_n \in C$ for all $n \in \mathbb{N}$. Then $f_n \xrightarrow{\omega} f$ as $n \to \infty$ if and only if $\lim_{n \to \infty} \ell(f_n, f) = 0$.*

(b) *C is convex and homeomorphic to the compact convex set $\tilde{C} \subset L^2(S)$.*

This lemma enables us to apply the Schauder fixed point theorem (see Corollary 17.56 in [1] or [46]) in our proof of the existence of equilibrium in a stationary model described in Sect. 4.

In the sequel, we shall use the space

$$C^\infty := C \times C \times C \times \cdots$$

endowed with the product topology. From Lemma 2, it follows that C is a compact metric space. By Tychonoff's theorem (Theorem 2.61 in [1]), C^∞ is a compact metric space, too. We recall that $(c_1^m, c_2^m, ...) \to (c_1, c_2, ...)$ in C^∞ if and only if $c_t^m \xrightarrow{\omega} c_t$ for all $t \in \mathbb{N}$ as $m \to \infty$.

Let $\Pr(S)$ be the set of all probability measures on the space S. We recall that a sequence $(\nu^m)_{m\in\mathbb{N}}$ of probability measures on S *converges weakly* to some $\nu \in \Pr(S)$ ($\nu^m \Rightarrow \nu$ in short) if, for any bounded continuous function $w : S \to \mathbb{R}$, it holds that

$$\lim_{m\to\infty} \int_S w(s)\nu^m(ds) = \int_S w(s)\nu(ds).$$

Let g and g^m be bounded Borel measurable functions on S. Define the set $D_g \subset S$ as follows. A point $s \in D_g$ if and only if there exists a sequence $s^m \to s$ as $m \to \infty$ such that $g^m(s^m) \not\to g(s)$.

The following fact is a corollary to Theorem 5.5 in [13].

Lemma 3 *Assume that D_g is a countable set, $0 \notin D_g$ and $\nu^m \Rightarrow \nu$ as $m \to \infty$. If, in addition, ν has no atoms in the set S_+, then*

$$\lim_{m\to\infty} \int_S g^m(s)\nu^m(ds) = \int_S g(s)\nu(ds).$$

In the following sections, we apply certainty equivalents to a decision process with quasi-hyperbolic discounting. Then, the underlying space Ω is the space of all admissible trajectories (histories) of the decision process, P denotes the probability measure on Ω induced by the choices of the agent and stochastic transitions according to the Ionescu-Tulcea theorem [40]. The random variable X is defined along trajectories as the total utility discounted in a quasi-hyperbolic manner.

4 The stochastic growth model and main results

4.1 The model

We assume that S is the *state space*. It represents the *set of "levels" for the renewable resource* and $A(s) = [0, s]$ is the set of *admissible actions* (possible consumption levels) in state $s \in S$. In the *stochastic growth model* with *quasi-hyperbolic preferences* and the state space S we envision an individual agent as a sequence of autonomous temporal *selves*. These selves are indexed by period numbers $t \in \mathbb{N}$ in which they make their decisions. More precisely, for any given state $s_t \in S$ at the beginning of t-th period, self t chooses a consumption level $a_t \in A(s_t)$ and the remaining part $y_t := s_t - a_t$ is invested for future *selves*. Self t's satisfaction is measured by a *period utility function* $u_\tau : S \to S$ for all $\tau \geq t$. This will be specified later assuming that all selves share the same risk aversion level.

Let q_t be a transition probability from S to S. Then, the state s_{t+1} is generated by $q_t(\cdot \mid y_t)$ depending on the investment $y_t \in A(s_t)$.

A *Markov strategy* for self t is a function $c_t \in C$. We put $i_t(s) = s - c_t(s)$, $s \in S$. This is an investment strategy (or saving plan) of self t for the following selves. For any sequence $(c_\tau) \in C^\infty$ of strategies of all selves and any $t \in \mathbb{N}$ we define $c_{[t,\infty)} := (c_t, c_{t+1}, ...) \in C^\infty$. For any state $s_t \in S$ and any $c_{[t,\infty)} \in C^\infty$ the transition probabilities $q_\tau(\cdot \mid i_\tau(s))$ induced by q_τ with $\tau \geq t$ generate due to the Ionescu-Tulcea theorem (see Proposition V.1.1 in

[40]) a unique probability measure $\mathbb{P}_{s_t}^{c[t,\infty)}$ on $S^\infty = S \times S \times \cdots$ endowed with the product σ-algebra. Let $\mathbb{E}_{s_t}^{c[t,\infty)}$ denote the expectation operator corresponding to the measure $\mathbb{P}_{s_t}^{c[t,\infty)}$. Assume that for each $\tau \in \mathbb{N}$, the function $u_\tau : S \to S$ is bounded. Note that it is non-negative.

Let $\mathcal{U} : [0,\infty) \to \mathbb{R}$ be an increasing, at least twice continuously differentiable function such that $\mathcal{U}(0) = 0$. Observe that examples given in (3)–(6) satisfy these assumptions. The *utility* of self *t* is

$$\begin{aligned} U_t(c_{[t,\infty)})(s_t) &:= u_t(c_t(s_t)) \\ &+ \mathcal{U}^{-1}\left(\mathbb{E}_{s_t}^{c[t,\infty)}\mathcal{U}\left(\alpha\beta\sum_{\tau=t+1}^{\infty}\beta^{\tau-t-1}u_\tau(c_\tau(s_\tau))\right)\right), \end{aligned} \tag{7}$$

where $\beta \in (0,1)$ is a *long-run discount factor* and $\alpha \in (0,1]$ is a *short-run discount factor*. The idea of using utility functions involving quasi-hyperbolic discounting in the *risk-neutral case*, where the expectation operator $\mathbb{E}_{s_t}^{c[t,\infty)}$ is used, goes back to [41].

A detailed discussion of the risk-neutral case with quasi-hyperbolic discounting and some applications can be found in [5–7, 23, 29]. Note that the expression defined in (7) is non-negative and well-defined.

A *Markov Perfect Equilibrium* (*MPE*) is a sequence $c^*_{[1,\infty)} \in C^\infty$ such that for each $t \in \mathbb{N}$, every $s_t \in S$ and every $c_t \in C$, we have

$$U_t(c^*_{[t,\infty)})(s_t) \geq U_t(c_t, c^*_{[t+1,\infty)})(s_t). \tag{8}$$

A *Stationary Markov Perfect Equilibrium* (*SMPE*) is an *MPE* $c^*_{[1,\infty)} \in C^\infty$ such that $c^*_t = c^*$ for some $c^* \in C$ and for all $t \in \mathbb{N}$.

In a stationary *MPE* every self uses the same consumption strategy. One can think that every self is a short-lived player in a non-cooperative game and acts only once. The payoff function of self *t* is given by (7). Then an *MPE* is a Nash equilibrium in this non-cooperative game.

4.2 Main results

For a bounded and continuous function $w : S \to \mathbb{R}$ we set $\|w\| := \sup_{s\in S}|w(s)|$. We now formulate our main assumptions.

(U) (a) The function $\mathcal{U} : [0,\infty) \to [0,\infty)$ is either strictly concave or strictly convex. Moreover, $\mathcal{U}$ is increasing and twice continously differentiable and $\mathcal{U}(0) = 0$.
(b) For every $t \in \mathbb{N}$ the function $u_t : S \to \mathbb{R}$ is bounded, increasing, strictly concave and continuous at $s = 0$. Moreover, $u_t(0) = 0$ and $d := \sup_{t\in\mathbb{N}}\|u_t\| < \infty$.

(Q) The transition probability q_t is weakly continuous (has the Feller property) on *S*, i.e., if for each $y_0 \in S$ and $y_m \to y_0$, we have $q_t(\cdot \mid y_m) \Rightarrow q_t(\cdot \mid y_0)$ as $m \to \infty$. Moreover, for each $y \in S_+$, the probability measure $q_t(\cdot \mid y)$ is non-atomic and $q_t(\cdot \mid 0)$ has no atoms in S_+.

Assumption **(U)** imposed on u_t is rather standard. A natural example of a transition probability that satisfies Assumption **(Q)** is induced by the following recurrence equation

$$s_{t+1} = \bar{f}(y_t, \xi_t),$$

where $y_t = s_t - a_t$ is the investment in state s_t, $(\xi_t)_{t\in\mathbb{N}}$ is a sequence of i.i.d. random "shocks" having a probability distribution η. For every $z \in \mathbb{R}$ the function $\bar{f}(\cdot, z)$ is continuous and for every $y \in S$ the function $\bar{f}(y, \cdot)$ is Borel measurable. Moreover, for any Borel set D in S and investment $y \in S$ we have

$$q_t(D \mid y) := q(D \mid y) = \int_S 1_D(\bar{f}(y, z))\eta(dz), \quad t \in \mathbb{N},$$

where 1_D is the indicator function of the set D. Examples with additive or multiplicative shocks are discussed in Remark 3.2 in [4], see also [12, 24, 38, 49] and [48]. We postpone the remaining discussion concerning Assumption **(Q)** to Section 4.3. Here we only mention that Assumption **(Q)** allows for both $q(\{0\} \mid 0) = 1$ as well as $q(\{0\} \mid 0) < 1$.

We can now state our main results.

Theorem 4 *Assume that* ***(U)*** *and* ***(Q)*** *are satisfied. Then there exists an MPE* $c^*_{[1,\infty)} \in C^\infty$.

Theorem 5 *Assume that* ***(U)*** *and* ***(Q)*** *hold and the model is stationary, i.e.,* $q_t = q$ *and* $u_t = u$ *for each* $t \in \mathbb{N}$. *Then there exists an SMPE* $(c^*, c^*, \ldots)$ *with* $c^* \in C$.

Remark 1 Stationary *MPE* in risk-sensitive *MDPs* with Borel state and action spaces involving exponential function $\mathcal{U}$ and quasi-hyperbolic discounting were studied by Jaśkiewicz and Nowak in [28]. The transition probability function considered in [28] is a convex combination of finitely many non-atomic measures on the state space. First, randomized equilibria are shown to exist and then a classical purification method from the theory of statistical decision functions [19] is applied. A model similar to the one of this paper is also considered in [28] (for the exponential function $\mathcal{U}$), but under very specific assumptions implying the existence of Lipschitz continuous equilibria.

Risk-neutral *MDPs* with Borel state and action spaces and quasi-hyperbolic discounting were studied by Jaśkiewicz and Nowak in [29]. We recall that by the risk-neutrality we mean that $\mathcal{U}$ is the identity function. The main results in [29] concern randomized *MPE*. Moreover, Section 4 in [29] presents a number of examples from economics and finance. Most of them are only known to possess randomized equilibria.

The consumption/investment (or consumption/savings) model inspiring our present work belongs to macroeconomics (see [3] for early discussion). Assumptions made on the primitives of our model enable us to restrict attention to the class C of right-continuous strategies. This class of functions may not be enough to study other models in economics and finance, but the idea of using certainty equivalents, induced by different functions $\mathcal{U}$, together with quasi-hyperbolic discounting may lead to new valuable results.

Remark 2 In this paper we study a stochastic growth model that covers a large class of risk-sensitive preferences and quasi-hyperbolic discounting. Our approach can have a broader range of potential applications in both finance and economics.

In finance, our model can be used to the study of optimal investment portfolio choice in the presence of time-inconsistency and risk-sensitivity.[1] In such applications, an agent evaluates the discounted sequence of returns from a chosen portfolio using the utility function $\mathcal{U}$, while also exhibiting present bias, i.e., undervaluing future returns relative to current ones. In equilibrium, an agent correctly anticipates the adjustments in portfolio choices made by their future selves. Our model thus extends the potential applicability of dynamic portfolio selection models similar to those of [42, 47], or works building on the seminal contributions of Merton [37] and Samuelson [44], summarized in [33] and broadens the range of financial datasets that can be rationalized within the framework of optimal portfolio choice (see, e.g., [36] and their asset premium puzzle). A closed form solution to the Samuelson's dynamic portfolio selection problem [44] with quasi-hyperbolic discounting and the identity function $\mathcal{U}$ is given in Example 4.1 of [29]. A similar solution in the risk-sensitive case is not known. An application of risk-sensitive preferences to a dividend problem is discussed in [9].

Remark 3 The proof of Theorem 4 does not use any fixed point theorem. *MPE* are constructed for finite horizon models using an algorithm similar to that of dynamic programming. Finding an *MPE* for a finite horizon model with infinitely many states in a closed form is a difficult task. The problem becomes easier for finite state *MDPs* under risk-neutrality assumption. As an illustration one can consider Example 5.6 in [29], where *MPE* in the finite time horizon are obtained. Unfortunately, they are not convergent and exhibit some periodic properties. Selecting a convergent subsequence to an *MPE* in the infinite horizon case is theoretically possible. However, pointing out special cases, in which a convergent subsequence is easy to find is an interesting open problem. In order to get a stationary equilibrium (as in the proof of Theorem 5) a fixed point argument is needed.

Remark 4 In case of $\alpha = 1$ and in the stationary framework, (the utility function and the transition probability are independent of t) the paper [27] provides an example illustrating that the optimal policy must be Markovian. Here, we show that in our framework an *SMPE* exists. Observe that the solution concept *SMPE* is weaker than the usual optimality, i.e., an *SMPE* need not be optimal and in many cases it is a suboptimal solution, see also examples in [15].

4.3 Remarks on invariant distributions

An important issue studied in economic theory and dynamic games concerns the existence and uniqueness of an invariant (or stationary) distribution for the Markov chain induced by the transition probability and an equilibrium strategy, see [4, 12, 24, 28, 32, 38, 48] and references cited therein. A strategy $c \in C$ (or $i \in I$) is called *stochastically stable* if c and q induce a unique invariant distribution. In this subsection, based on some classical literature, we point out some cases, where an *SMPE*, say c^*, is stochastically stable.

We recall that a distribution $\mu^* \in \Pr(S)$ is called *invariant* if

$$\mu^*(B) = \int_S q(B \mid s - c^*(s))\mu^*(ds) = \int_S q(B \mid i^*(s))\mu^*(ds)$$

[1] For a special case of $\alpha = 1$, see also a casino game application in [11].

for every Borel set $B \subset S$. Here, $i^*(s) = s - c^*(s)$.

First, we want to discuss models with the transition probability having the *strong Feller (SF) property*, meaning that if $y_n \to y_0$ in S as $n \to \infty$ and $\gamma : S \to \mathbb{R}$ is a bounded Borel function, then $\int_S \gamma(s)q(ds \mid y_n) \to \int_S \gamma(s)q(ds \mid y_0)$.

Case 1: q has the *SF* property and an *SMPE c^** is continuous.

Note that the *SF* property for q holds if q has a bounded continuous density function ρ, that is,

$$q(B \mid y) = \int_B \rho(s, y)\pi(ds),$$

where $\pi \in \Pr(S)$ and $B \subset S$ is a Borel set. Additionally, it holds when q is of the form

$$q(B \mid y) = (1 - g(y))\nu_1(B) + g(y)\nu_2(B), \tag{9}$$

where $\nu_1,\ \nu_2 \in \Pr(S)$, $B \subset S$ is a Borel set and $g : S \to [0, 1]$ is a strictly concave and increasing function. An example of such function g can be: $g(y) = 1 - e^{-y}$. For some results on equilibria in models with transition functions of this type the reader is referred to [28] and references therein. Usually, it is assumed that ν_2 first-order stochastically dominates ν_1, that is, $\int_S \gamma d\nu_1 \leq \int_S \gamma d\nu_2$ for every bounded increasing function $\gamma : S \to \mathbb{R}$.

The continuity of an *SMPE* $c^* \in C$ happens only in some special situations, also in the risk-neutral case. An example for the risk-sensitive model with the exponential function $\mathcal{U}$ is shown by Propositions 2 and 3 in [28], where the transition probability q has the form as in (9) and c^* is a Lipschitz continuous function. See also [6] for assumptions implying existence of a differentiable *SMPE*.

Define

$$q^*(\cdot \mid s) := q(\cdot \mid s - c^*(s)) = q(\cdot \mid i^*(s)),\ s \in S.$$

Note that in *Case 1* the transition probability q^* has the *SF* property.

A point $y \in S$ is *accessible* for q^* if, for every $s \in S$ and every open neighbourhood $\mathcal{V}$ of y, $q^*(\mathcal{V} \mid s) > 0$. From Corollary 2.7 in [21], we conclude the following fact:

If in *Case 1* there is an accessible point $y \in S$ for q^*, then an *SMPE* c^* is stochastically stable, i.e., there exists a unique invariant distribution.

Recall, that if in addition we assume that $q(\{0\} \mid 0) = 1$, then δ_0 is the unique invariant distribution. If, however, $q(\{0\} \mid 0) < 1$, then δ_0 is not the invariant distribution. Both cases can have some natural interpretation. Condition $q(\{0\} \mid 0) = 1$ means that the resource level 0 is absorbing, i.e., it is not possible to generate any strictly positive resource level starting from 0. If, however, $q(\{0\} \mid 0) < 1$, then 0 is not an absorbing state. This condition would be satisfied if, in addition to the return from saved resources, the agent also obtained some additional income (e.g., labour income) that is received irrespective of the savings level.

Case 2: For the *SMPE* $c^* \in C$ obtained in Theorem 5, there exists $\epsilon > 0$ such that $\epsilon \le q(\{0\} \mid 0)$ and $q(\{0\} \mid s - c^*(s)) \ge \epsilon$ for all $s \in S_+$.

It often happens that an *SMPE* c^* is discontinuous and then, even if Assumption **(Q)** is satisfied, the transition function $q^*(\cdot \mid s) = q(\cdot \mid s - c^*(s))$ does not have the Feller property. Then in *Case 2*, it follows that, for all $s \in S$ and for each Borel set $B \subset S$, we have

$$q^*(B \mid s) \ge \epsilon\delta_0(B). \tag{10}$$

This is a special case of the Doeblin condition (see e.g., [12, 49]) and for the given c^* it can be verified. For example, (10) holds if q is of the form as in (9), $\nu_1 = \delta_0$ and $0 \le g \le 1 - \epsilon$.

From Corollary 3.3 in [12], we infer the fact:

In *Case 2* the *SMPE*c^* is stochastically stable, i.e., there exists a unique invariant distribution μ^*.

As discussed above, if we also have $q(\{0\} \mid 0) = 1$, then δ_0 is the unique invariant distribution. Clearly, $\mu^* \ne \delta_0$ when $q(\{0\} \mid 0) < 1$.

Additional results on the existence of a unique invariant distribution in various dynamic economic (game) models are obtained under a stochastic monotonicity assumption on the transition function q. Most of them deal with compact state space. For a survey, the reader is referred to [4, 24, 32, 48].

5 Proofs of Theorems 4 and 5

Throughout this section we impose Assumptions **(U)** and **(Q)** whenever they are needed.

Fix $t \in \mathbb{N}$ and assume that $k \ge t + 1$. Let $a_\tau \in S$ for every $\tau \in \mathbb{N}$. Define

$$V_{[t+1,k]}(a_{t+1}, \dots, a_k) := \mathcal{U}\left(\alpha \sum_{\tau=t+1}^{k} \beta^{\tau-t} u_\tau(a_\tau)\right)$$

and

$$V_{[t+1,\infty)}(a_{t+1}, a_{t+2}, \dots) := \mathcal{U}\left(\alpha \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u_\tau(a_\tau)\right).$$

By Assumption **(U)** we have that

$$\begin{aligned} 0 \le& \alpha \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u_\tau(a_\tau) - \alpha \sum_{\tau=t+1}^{k} \beta^{\tau-t} u_\tau(a_\tau) \\ =& \alpha \sum_{\tau=k+1}^{\infty} \beta^{\tau-t} u_\tau(a_\tau) \le \frac{d\alpha\beta^{k+1-t}}{1-\beta} =: D_k. \end{aligned} \tag{11}$$

Clearly, $D_k \leq D := \frac{d\alpha}{1-\beta}$ for all $k \geq t+1$ and $t \in \mathbb{N}$ and $\lim_{k\to\infty} D_k = 0$.

Lemma 6 *For any $t \in \mathbb{N}$ the function $V_{[t+1,k]}$ is continuous and*

$$\lim_{k\to\infty} \sup_{(a_{t+1},\ldots)\in S^\infty} \left(V_{[t+1,\infty)}(a_{t+1}, a_{t+2}, \ldots) - V_{[t+1,k]}(a_{t+1}, \ldots, a_k)\right) = 0.$$

Proof The continuity of $V_{[t+1,k]}$ is obvious. Observe that, if $\mathcal{U}$ is strictly concave, then $\mathcal{U}$ is subadditive. Hence, $0 \leq \mathcal{U}(y) - \mathcal{U}(x) \leq \mathcal{U}(y-x)$ for $y > x \geq 0$. This fact together with (11) yields that

$$0 \leq \sup_{(a_{t+1},\ldots)\in S^\infty} (V_{[t+1,\infty)}(a_{t+1}, a_{t+2}, \ldots) - V_{[t+1,k]}(a_{t+1}, \ldots a_k)) \leq \mathcal{U}(D_k).$$

Assume now that $\mathcal{U}$ is strictly convex. The derivative $\mathcal{U}'$ is under our assumptions positive and increasing. Then by the Lagrange mean value theorem and (11) we obtain

$$\begin{aligned} 0 \leq & \sup_{(a_{t+1},\ldots)\in S^\infty} (V_{[t+1,\infty)}(a_{t+1}, a_{t+2}, \ldots) - V_{[t+1,k]}(a_{t+1}, \ldots a_k)) \\ \leq & \mathcal{U}'(D) \sup_{(a_{t+1},\ldots)\in S^\infty} \left(\sum_{\tau=k+1}^{\infty} \alpha\beta^{\tau-t} u_\tau(a_\tau) \right) \leq D_k\, \mathcal{U}'(D). \end{aligned}$$

Hence, the conclusion follows. □

Let S_c denote the set of continuity points of $c \in C$. Note that $0 \in S_c$ and $S \setminus S_c$ is a denumerable set.

Lemma 7 *Let $c^m \xrightarrow{\omega} c$ in C. If $s^m \in S$ for every $m \in \mathbb{N}$ and $s^m \to s_0 \in S_c$ as $m \to \infty$, then* $\lim_{m\to\infty} c^m(s^m) = c(s_0)$.

Proof Let $s_0 \in S_c \cap S_+$. Choose any $s', s'' \in S_c$ such that $s' < s_0 < s''$. The functions $i^m(s) = s - c^m(s)$ and $i(s) = s - c(s)$ are non-decreasing and $i^m \xrightarrow{\omega} i$ as $m \to \infty$. Then for sufficiently large values of *m* we have

$$i^m(s') \leq i^m(s^m) \leq i^m(s'').$$

Therefore,

$$i(s') \leq \liminf_{m\to\infty} i^m(s^m) \leq \limsup_{m\to\infty} i^m(s^m) \leq i(s'').$$

Since S_c is dense in *S*, we can take $s' \uparrow s_0$ and $s'' \downarrow s_0$. Then, it follows that $i(s_0) = \lim_{m\to\infty} i^m(s^m)$ and consequently $c(s_0) = \lim_{m\to\infty} c^m(s^m)$. □

In the following lemma we consider any sequence $(c_1^m, c_2^m, \ldots)$ converging to $(c_1, c_2, \ldots)$ as $m \to \infty$, i.e., $c_\tau^m \xrightarrow{\omega} c_\tau$ in C for each $\tau \in \mathbb{N}$ as $m \to \infty$. The set of all continuity points of c_τ is denoted by S_{c_τ} and we note that $0 \in S_{c_\tau}$ We assume that, for each τ and $s'_\tau \in S_{c_\tau}$, (s^m_τ) is an arbitrary sequence in S converging to s'_τ as $m \to \infty$.

Observe that for $c_{[1,\infty)} \in C^\infty$, we have

$$\begin{aligned}
&\mathbb{E}^{c_{[t,\infty)}}_{s_t} V_{[t+1,k]}(a_{t+1}, ..., a_k) = \mathbb{E}^{c_{[t,\infty)}}_{s_t} V_{[t+1,k]}(c_{t+1}(s_{t+1}), ..., c_k(s_k)) \\
&= \int_S \cdots \int_S \mathcal{U}\left(\alpha \sum_{\tau=t+1}^{k} \beta^{\tau-t} u_\tau(c_\tau(s_\tau))\right) \\
&\quad q_{k-1}(ds_k \mid i_{k-1}(s_{k-1})) \cdots q_t(ds_{t+1} \mid i_t(s_t)),
\end{aligned}$$

where we recall that $i_\tau(s) = s - c_\tau(s)$, $s \in S$ and $\tau \in \mathbb{N}$.

Lemma 8 *For each $k \geq t+1$ we have*

(a) $\lim_{m\to\infty} \mathbb{E}^{c^m_{[t,\infty)}}_{s^m_t} V_{[t+1,k]}(a_{t+1}, \ldots, a_k) = \mathbb{E}^{c_{[t,\infty)}}_{s'_t} V_{[t+1,k]}(a_{t+1}, \ldots, a_k)$.

(b) $\lim_{m\to\infty} \mathbb{E}^{c^m_{[t,\infty)}}_{s^m_t} V_{[t+1,\infty)}(a_{t+1}, a_{t+2}, \ldots) = \mathbb{E}^{c_{[t,\infty)}}_{s'_t} V_{[t+1,\infty)}(a_{t+1}, a_{t+2} \ldots)$.

Proof (a) Assume that $c^m_{[t,\infty)} \to c_{[t,\infty)}$ in C^∞ as $m \to \infty$. Put

$$g^m(\cdot) = V_{[t+1,k]}(c^m_{t+1}(s^m_{t+1}), ..., c^m_{k-1}(s^m_{k-1}), c^m_k(\cdot))$$

and

$$g(\cdot) = V_{[t+1,k]}(c_{t+1}(s'_{t+1}), ..., c_{k-1}(s'_{k-1}), c_k(\cdot)).$$

By Lemma 7 and Assumption **(U)** the functions g^m and g satisfy conditions in Lemma 3. Moreover, Lemma 7 and Assumption **(Q)** yield that

$$q_\tau(\cdot \mid s^m_\tau - c^m_\tau(s^m_\tau)) \Rightarrow q_\tau(\cdot \mid s'_\tau - c_\tau(s'_\tau))$$

or equivalently

$$q_\tau(\cdot \mid i^m_\tau(s^m_\tau)) \Rightarrow q_\tau(\cdot \mid i_\tau(s'_\tau)) \tag{12}$$

for any $\tau \in \mathbb{N}$. Therefore, Lemma 3 implies that

$$\begin{aligned}
&\int_S V_{[t+1,k]}(c^m_{t+1}(s^m_{t+1}), ..., c^m_{k-1}(s^m_{k-1}), c^m_k(s_k)) q_{k-1}(ds_k \mid i^m_{k-1}(s^m_{k-1})) \to \\
&\int_S V_{[t+1,k]}(c_{t+1}(s'_{t+1}), ..., c_{k-1}(s'_{k-1}), c_k(s_k)) q_{k-1}(ds_k \mid i_{k-1}(s'_{k-1}))
\end{aligned} \tag{13}$$

as $m \to \infty$. Now consider

$$g^m(\cdot) = \int_S V_{[t+1,k]}(c^m_{t+1}(s^m_{t+1}), ..., c^m_{k-1}(\cdot), c^m_k(s_k)) q_{k-1}(ds_k \mid i^m_{k-1}(\cdot))$$

and

$$g(\cdot) = \int_S V_{[t+1,k]}(c_{t+1}(s'_{t+1}), ..., c_{k-1}(\cdot), c_k(s_k))q_{k-1}(ds_k \mid i_{k-1}(\cdot)).$$

Now using (13), we observe that these functions also satisfy the assumptions of Lemma 3. This fact and (12) for $\tau = k-2$ yield that

$$\begin{aligned}&\int_S \int_S V_{[t+1,k]}(c^m_{t+1}(s^m_{t+1}), ..., c^m_{k-1}(s_{k-1}), c^m_k(s_k))\\ &q_{k-1}(ds_k \mid i^m_{k-1}(s_{k-1}))q_{k-2}(ds_{k-1} \mid i^m_{k-2}(s^m_{k-2})) \to\\ &\int_S \int_S V_{[t+1,k]}(c_{t+1}(s'_{t+1}), ..., c_{k-1}(s_{k-1}), c_k(s_k))\\ &q_{k-1}(ds_k \mid i_{k-1}(s_{k-1}))q_{k-2}(ds_{k-1} \mid i_{k-2}(s'_{k-2}))\end{aligned}$$

as $m \to \infty$. Continuing this procedure we finally get the conclusion.

(b) This part follows from point (a) and the uniform approximation shown in Lemma 6. □

Lemma 9 *For each $s_t \in S_{c_t}$ $\lim_{m\to\infty} U_t(c^m_{[t,\infty)})(s_t) = U_t(c_{[t,\infty)})(s_t)$.*

Proof The continuity of the inverse function $\mathcal{U}^{-1}$, Lemma 8(b) and Assumption **(U)** give the result. □

For convenience of notation for $c_{[t+1,\infty)} = (c_{t+1}, c_{t+2}, ...)$ and $s_{t+1} \in S$ we set

$$\begin{aligned}J_{t+1}(c_{[t+1,\infty)})(s_{t+1}) :=&\mathbb{E}^{c_{[t+1,\infty)}}_{s_{t+1}} V_{[t+1,\infty)}(c_{t+1}(s_{t+1}), \ldots)\\ =&\mathbb{E}^{c_{[t+1,\infty)}}_{s_{t+1}} \mathcal{U}\Big(\alpha \sum_{\tau=t+1}^{\infty} \beta^{\tau-t-1} u_\tau(c_\tau(s_\tau))\Big).\end{aligned}$$

Let $s_t \in S$ and $a \in A(s_t)$. Then, we define

$$\begin{aligned}&P_t(s_t, a, c_{[t+1,\infty)}) := u_t(a)\\ &+\mathcal{U}^{-1}\left(\int_S J_{t+1}(c_{[t+1,\infty)})(s_{t+1})q_t(ds_{t+1} \mid s_t - a)\right).\end{aligned} \tag{14}$$

Clearly, $P_t(s_t, c_t(s_t), c_{[t+1,\infty)}) = U_t(c_{[t,\infty)})(s_t)$ for $s_t \in S$.

Lemma 10 *The function $a \to P_t(s_t, a, c_{[t+1,\infty)})$ defined in (14) is continuous on $A(s_t)$.*

Proof We may consider $s_t > 0$. Assume that $a^m, a \in A(s_t)$ and $a = \lim_{m\to\infty} a^m$. The result follows from Lemma 3 with

$$g^m(\cdot) = g(\cdot) = J_{t+1}(c_{[t+1,\infty)})(\cdot)$$

and $\nu^m(\cdot) = q_t(\cdot \mid s_t - a^m)$ and $\nu(\cdot) = q_t(\cdot \mid s_t - a)$. □

For any $c_{[t+1,\infty)} \in C^\infty$ and $s_t \in S$ define

$$BR_t(c_{[t+1,\infty)})(s_t) := \arg \max_{a \in A(s_t)} P_t(s_t, a, c_{[t+1,\infty)})$$

and

$$br_t(c_{[t+1,\infty)})(s_t) := \min BR_t(c_{[t+1,\infty)})(s_t).$$

By Lemma 1(d), we note that $br_t(c_{[t+1,\infty)}) \in C$. The set $BR_t(c_{[t+1,\infty)})(s)$ is regarded as the set of all best responses of self *t* in state *s*, given that the following selves are going to use $c_{[t+1,\infty)} \in C^\infty$. Under our assumptions, this set is non-empty and compact (see Lemma 2(b)).

Lemma 11 *The best response mapping* $br_t : C^\infty \to C$ *is continuous.*

Proof Assume that $c^m_{[1,\infty)} \to c_{[1,\infty)}$ as $m \to \infty$. Clearly, $c^m_{[t+1,\infty)} \to c_{[t+1,\infty)}$ as $m \to \infty$. Put $\phi^m := br_t(c^m_{[t+1,\infty)})$ and $\phi := br_t(c_{[t+1,\infty)})$. Since C is a compact metric space, the sequence (ϕ^m) has an accumulation point, say $\varphi \in C$. We need to show that $\varphi = \phi$. Without loss of generality, assume that $\phi^m \xrightarrow{\omega} \varphi$ as $m \to \infty$. Let $s_t = s \in S_\varphi$, i.e., *s* is a continuity point of φ. We have

$$P_t(s, \phi^m(s), c^m_{[t+1,\infty)}) \geq P_t(s, a, c^m_{[t+1,\infty)}) \quad \text{for all} \quad a \in A(s).$$

This inequality means that

$$\begin{aligned} &u_t(\phi^m(s)) + \mathcal{U}^{-1}\left(\int_S J_{t+1}(c^m_{[t+1,\infty)})(s')q_t(ds' \mid s - \phi^m(s))\right) \\ \geq &u_t(a) + \mathcal{U}^{-1}\left(\int_S J_{t+1}(c^m_{[t+1,\infty)})(s')q_t(ds' \mid s - a)\right) \end{aligned}$$

for all $a \in A(s)$. By Lemmas 3, 7 and 9, when $m \to \infty$, it follows that

$$\begin{aligned} &u_t(\varphi(s)) + \mathcal{U}^{-1}\left(\int_S J_{t+1}(c_{[t+1,\infty)})(s')q_t(ds' \mid s - \varphi(s)))\right) \\ \geq &u_t(a) + \mathcal{U}^{-1}\left(\int_S J_{t+1}(c_{[t+1,\infty)})(s')q_t(ds' \mid s - a))\right) \end{aligned}$$

for all $a \in A(s)$. Thus, $\varphi(s) \in BR_t(c_{[t+1,\infty)})(s)$. Since $s \in S_\varphi$, the set $BR_t(c_{[t+1,\infty)})(s)$ has one point (Lemma 1(c)). Therefore, $\varphi(s) = \phi(s)$. Note that $S \setminus S_\varphi$ is a countable set and hence S_φ is dense in *S*. Both functions φ and ϕ are right-continuous. Thus, $\varphi(s) = \phi(s)$ for all $s \in S$ and the proof is complete. □

Proof of Theorem 4 Let $c^o(s) = 0$ for all $s \in S$ and c^o_∞ be a constant sequence of strategies for all selves, i.e., $c^o_\infty := (c^o, c^o, \ldots)$. Now we define a sequence $(c^n_{[1,\infty)})$ as follows: $c^n_{[1,\infty)} = (c^n_1, c^n_2, ...) \in C^\infty$ where $c^n_t := c^o$ for each $t \geq n+1$, $c^n_n := br_n(c^o_\infty)$ and

$$c^n_t := br_t((c^n_{t+1}, ..., c^n_n, c^o, c^o, ...)) \quad \text{for} \quad 1 \leq t \leq n-1.$$

Since C^∞ is a compact metric space, there exists an increasing sequence (n') of positive integers and a sequence $c^*_{[1,\infty)} = (c^*_1, c^*_2, ...) \in C^\infty$ such that $c^{n'}_t \xrightarrow{\omega} c^*_t$ as $n' \to \infty$ for all $t \in \mathbb{N}$. By Lemma 11, it follows that $c^*_t = br_t((c^*_{t+1}, c^*_{t+2}, ...))$ for all $t \in \mathbb{N}$. Thus, $c^*_{[1,\infty)}$ is an *MPE*. □

Proof of Theorem 5 We can restrict attention to constant sequences in C^∞, i.e., we assume that each self uses a strategy $c \in C$. Every such a constant sequence $(c, c, \ldots) \in C^\infty$ can be identified with its element $c \in C$. Hence, we can write *P*(*s*, *a*, *c*) for $P_t(s, a, c_{[t,\infty)})$, where $a \in A(s)$, $c \in C$. Then C becomes the domain for the best response mapping *br*. By Lemmas 2 and 11 and the Schauder fixed point theorem [1, 46], there exists $c^* \in C$ such that $c^* = br(c^*)$. Clearly, the constant sequence $(c^*, c^*, ...)$ is an *SMPE*. □

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Declarations

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