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A generalization of Hungarian method and Hall's theorem with applications in wireless sensor networks

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ABSTRACT

In this paper, we consider various problems concerning quasi-matchings and semi-matchings in bipartite graphs, which generalize the classical problem of determining a perfect matching in bipartite graphs. We prove a generalization of Hall's marriage theorem, and present an algorithm that solves the problem of determining a lexicographically minimum g-quasi-matching (that is a set F of edges in a bipartite graph such that in one set of the bipartition every vertex v has at least $g(v)$ incident edges from F, where g is a so-called need mapping, while on the other side of the bipartition the distribution of degrees with respect to F is lexicographically minimum). We obtain that finding a lexicographically minimum quasi-matching is equivalent to minimizing any strictly convex function on the degrees of the A-side of a quasi-matching and use this fact to prove a more general statement: the optima of any component-based strictly convex cost function on any subset of L_1-sphere in $\mathbb{N}^n$ are precisely the lexicographically minimal elements of this subset. We also present an application in designing optimal CDMA-based wireless sensor networks.

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1. Introduction

Problems related to matchings and factors belong to the classical and intensively studied problems in graph theory. They are among the central concepts in textbooks on graph theory, and are the main focus of the seminal monograph of Lovász and Plummer [1] from over 20 years ago, which is still one of the most comprehensive surveys on the topic. Since the paper of Hall [2] containing a characterization of perfect matchings in bipartite graphs, many generalizations and variations of matchings and factors in (bipartite) graphs, like 2-matchings, weighted matchings and f-factors [1], have been considered. At least as much interest has been given to algorithmic issues related to matchings, where a similarly influential role is played by the famous max-flow min-cut theorem of Ford and Fulkerson [3], cf. [1]. The research in the area is still vivid, which is in part due to its applicability. Notably applications often require special properties and yield different variants of existing concepts, which were not previously covered by the theory. In this paper, we introduce and study the so-called f, g-quasi-matching as a natural generalization of matchings in bipartite graphs. Note that, as matchings in graphs are well studied concepts, we do not introduce the basic notation, but rather adopt it from Diestel [4].

A motivation for this paper and for the introduction of quasi-matchings arose from the problem of efficient (real-time) routing in wireless sensor networks [5,6], although one can imagine several other applications. For instance, a special case of the problem of finding optimal quasi-matchings was considered in [7,8] with motivation arising from certain task scheduling. In the event when a task needs to be distributed among more than one machine, we arrive at the problems dealt

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in the present paper. In the language of graph theory, we are looking for a subset F (called a g-quasi-matching) of edges in a bipartite graph $G = A + B$ such that each vertex from B has a prescribed (minimal) number of incident edges from F, and under this condition we are aiming to optimize the degrees of vertices in A with respect to F. The optimization condition is to obtain lexicographically minimum (and hence also max-degree minimum) distribution of F-degrees in A. As it turns out, g-quasi-matchings present a generalization of semi-matchings, as defined by Harvey et al. in [7] (i.e. semi-mathings are g-quasi-matchings in which the need function g is defined by $g(v) = 1$ for all $v \in B$). In particular, the main results from Sections 3 and 4 generalize a result and an algorithm from [7]. In addition, we consider the case when also vertices from A have prescribed upper bounds on F-degrees, and call this an f, g-quasi-matching. This yields the decision problem of whether such a subset $F \in E(G)$ that forms an f, g-quasi-matching exists or not.

In the next section, we fix the notation and present our main motivation from wireless sensor networks, and along the way translate the corresponding problems to graph theory. Section 3 contains a characterization of lexicographically minimum g-quasi-matchings that connects our concepts to the results of [7]. In particular, this connection enables us to prove a more general statement: the extrema of a strictly convex cost function, defined on a subset of L_1-sphere in $\mathbb{Z}^n$, are precisely the lexicographically minimal elements of this set. The main characterization of this section also paves a path for the algorithm in Section 4. There the Hungarian method is extended to the problem of finding a lexicographically minimum quasi-matching in a bipartite graph, which results in an efficient algorithm for this problem. This algorithm is presented as an off-line algorithm, although it can be interpreted as an on-line algorithm when only additions of vertices or the increase of the prescribed lower bounds occur. It is extended in Section 5 to the case when the prescribed lower bound decreases (or the vertex is deleted). In Section 6, we consider the decision version of the most general situation, where all vertices have prescribed bounds on their F-degrees, that is, the question of existence of a f, g-quasi-matchings in a bipartite graph G. Given a function f that prescribes upper bounds for F-degrees of vertices in A and a function g that prescribes lower bounds for F-degrees of vertices in B, we prove a characterization of bipartite graphs $G = A + B$ that admit an f, g-quasi-matching. This result is a generalization of the Hall's marriage theorem.

2. Preliminaries and notation

When modeling CDMA-based wireless sensor networks with graphs [5,6], the following routing problem was encountered (naturally, it can appear in any communication network with similar features). The topology of the network is given by the nodes (in our case sensor units) that are able to communicate among each other with respect to physical limitations and their mutual distance. There is a special vertex, the sink, represented by a fixed station with relatively large computational capabilities. In our model, we assume that nodes are also fixed, and they can also communicate with the sink, depending on the mentioned limitations. This yields the initial rooted graph, in which we wish to pass information from nodes to the root. While nearby nodes communicate directly with the sink, other (remote) nodes can pass information to the sink by using other nodes as communication devices. For the purpose of energy saving and latency, the number of hops from a given node to the station must be as small as possible. The overall aim is to design a routing protocol, by which each node in the network transfers information to the sink as quickly as possible. (The goal of minimizing the latency of communication has, according to [5], priority over the second goal of energy saving. Although one can imagine other types of problems (perhaps also in some types of sensor networks), where the priority is reversed or somehow mixed, we will not consider them in this paper.) 降低延迟优先与节能

Translating our problem to graphs, we wish to find a spanning tree (or, in a greater generality, spanning subgraph) in a given rooted graph using only edges that connect two different distance-levels with respect to the root. There are many such trees obtainable by an ordinary BFS-algorithm, yet they may have vertices with relatively large degree, which can cause both communication delay and large energy consumption of these nodes. Since the lifetime of the network depends on its weakest nodes, such situations need to be avoided. See [9] for more on wireless sensor networks and their routing protocols. We remark that finding a spanning tree with the smallest maximum degree in a non-rooted graphs is a rather well studied problem (see [10] and the references therein), yet it does not have much connection with the problem on rooted graphs.

Our situation can be translated to the following optimization problem. *Given a rooted graph, find a spanning tree with maximum degree as small as possible, such that each vertex has a neighbor closer to the root than itself*. This condition stipulates that the edges connecting vertices at the same distance to the root are irrelevant, and we may assume that the whole graph is bipartite. Another more general problem follows from the requirement that more than one path from a node to the sink is needed, either to provide robustness against possible node failures or to avoid communication delay due to collisions at more frequent nodes. Hence alternative paths need to be determined in advance. Then the problem is to *find a spanning subgraph with maximum degree as small as possible in which each vertex has k neighbors in the neighboring distance level that is closer to the root*. More generally, if we have a traffic estimation at the nodes, then the number of neighbors in the lower level can be assigned to each vertex individually.

Further using the distance partition from the root, the problem naturally decomposes into finding a spanning subgraph in a bipartite graph spanned by two consecutive levels of this distance partition. The constraint on the spanning subgraph demands that, in the root-distant side of the bipartition, the degrees of vertices are prescribed: they can be 1 (derived from the original problem), have a fixed degree k (for the so-called multipath routing), or they can be determined by an arbitrary function that corresponds to estimated traffic at the nodes. Now we formally introduce the mentioned concepts.

Definition 1. Let $G = A + B$ be a bipartite graph. Given a positive integer k, a set $F \subseteq E(G)$ is a k-quasi-matching of $Y \subseteq B$, if every element of Y has at least k incident edges from F. A 1-quasi-matching of B in which every element of B has exactly 1 incident edge from F is called a semi-matching.

Semi-matchings were first introduced in Harvey et al. [7], with motivation arising from certain task scheduling. The objective was the reduction of a certain cost-function that is connected to the maximum degree of a semi-matching. As mentioned above, we consider a more general concept.

Definition 2. Let $G = A + B$ be a bipartite graph and $g: B \to \mathbb{N}$ a mapping. For a vertex $v \in B$ we call $g(v)$ the need of v, and for any $Y \subseteq B$, the need of Y is $g(Y) = \sum_{v \in Y} g(v)$. A set $F \subseteq E(G)$ is a g-quasi-matching of $Y \subseteq B$ if every element v of Y has at least $g(v)$ incident edges from F. Next, for a mapping $f: A \to \mathbb{N}$, and a vertex $u \in A$ we call $f(u)$ the capacity of u, and for any $X \subseteq A$, the capacity of X is $f(X) = \sum_{u \in X} f(u)$. A set $F \subseteq E(G)$ is an f, g-quasi-matching of $X + Y$ if every element v of Y has at least $g(v)$ incident edges from F, and every element u of X has at most $f(u)$ incident edges from F.

A g-quasi-matching of B with a constant need function, $g(v) = k$, for all $v \in B$ clearly coincides with the k-quasi-matching of B, and, in particular, when $k = 1$ this is a semi-matching. Note that f, g-quasi-matching will only be considered in Section 6, where their corresponding decision problem is characterized via a certain Hall-like condition.

Definition 3. Let $G = A + B$ be a bipartite graph and $F \subseteq E(G)$. For a vertex $v \in V(G)$, the F-degree of v, $d_F(v)$ is the degree of v in $G[F]$. The degree of F is the maximum degree in $G[F]$ of a vertex from A.

For instance, a matching of $Y \subseteq B$ is a semi-matching of Y with degree equal to 1.

Thus in one set of the bipartition, namely B, through function g the lower bounds on F-degrees of vertices are prescribed. In the other set of the bipartition, we are either aiming at the minimization of the largest degree (optimization problem), or we are facing some constraints on degrees of vertices (decision problem). We are thus interested in the following two problems.

Problem 1. Given a bipartite graph $G = A + B$ and a need function g on B, find a g-quasi-matching of B with minimum degree.

最优化问题

Problem 2. Given a bipartite graph $G = A + B$, is there an f, g-quasi-matching of $A + B$? 判定性问题

We solve the latter problem by giving a characterization that generalizes Hall's theorem in Section 6. Concerning Problem 1, we will impose even a stronger requirement, that also gives a better solution to the original problem, as we are aiming at energy preservation of the nodes. Namely, we will consider a balanced (or lexicographically minimum) distribution of degrees of vertices in A. To present it in Problem 3 we need some more notation.

Definition 4. Let $G = A + B$ be a bipartite graph, let $F \subseteq E(G)$, and $X \subseteq A$. Let $d_F(X)$ be the sequence $d_1, d_2, \ldots, d_{|X|}$ of F-degrees of vertices from X, where $d_1 \geq d_2 \geq \cdots \geq d_{|X|}$. For $Y \subseteq B$, we define $d_F(Y) = d_F(N(Y))$.

The following definition applies to all types of quasi-matchings (k-quasi-matching, g-quasi-matchings and f, g-quasi-matchings). Whenever we give a statement about a *quasi-matching* we refer to all of these.

Definition 5. Let $G = A + B$ be a bipartite graph, let F, F' be two quasi-matchings of $Y \subseteq B$. Then F is (lexicographically) greater than F', if $d_F(Y)$ is lexicographically greater than $d_{F'}(Y)$ (i.e., the first difference in the degree sequences $d_F(Y)$ and $d_{F'}(Y)$ is such that the degree from $d_F(Y)$ is greater than the degree from $d_{F'}(Y)$). A quasi-matching F of $Y \subseteq B$ that is not greater than any other quasi-matching of Y is a *minimum* quasi-matching of Y.

Clearly, a minimum quasi-matching of B has a minimum degree. It is also easy to see that in a minimum g-quasi-matching, all vertices in B have F-degree equal to their need. Thus, to solve Problem 1, we propose

Problem 3. Given a bipartite graph $G = A + B$ and a need function $g: B \to \mathbb{N}$, find a (lexicographically) minimum g-quasi-matching of B.

The application mentioned above adds some motivation to preferring the lexicographically minimum quasi-matchings to just minimum quasi-matchings: in such cases, the additional structure contributes to a uniform distribution of the communication load and energy consumption over the nodes. As we will see in Corollary 12, a lexicographically minimum matching has several other interesting properties.

Turning back to the original problem of finding a rooted spanning tree with minimum maximal degree (or balanced degrees), we see that the solution of the Problem 3 can be applied. When the degrees in the root-distant side of the bipartition are fixed, the union of optimal solutions of the subproblems on distance levels is an optimal solution to the problem on the given rooted graph.

3. Quasi-matchings

An on-line algorithm for solving Problem 3 is one of the major contributions of this paper. We start with the following easy lemma. (Recall that the *pigeonhole or Dirichlet principle* states that given a set of t objects that are placed into boxes, and there are s boxes available, then there will be a box containing at least $\lceil \frac{t}{s} \rceil$ objects.)

Lemma 6. *Let $G = A + B$ be a bipartite graph, $g: B \to \mathbb{N}$ a need function, and F a g-quasi-matching of B. Let $X \subseteq A$, with $|X| = k$, and let $Y = N(X)$ be the set of their neighbors. Let t be the number of edges with one end-vertex from Y and the other from $A - X$, and let $g(Y) = t + dk + r$, where $0 \leq r < k$ and $d \geq 0$. Then $d_F(X)$ is lexicographically greater or equal to the degree sequence with r integers $d + 1$ and $k - r$ integers d.*

Proof. Note that $d_F(X)$ is (lexicographically) the smallest only if all edges with one end-vertex from Y and the the other from $A - X$ are in F. We may thus assume without loss of generality that this is the case. Hence $\sum_{x \in X} d_F(x) = dk + r$.

If $r = 0$, then either $d_F(X)$ consists of precisely k integers d or $d_F(x)$ contains at least one integer strictly greater than d. Both distributions are lexicographically greater or equal to the distribution with k integers d.

So suppose $r > 0$ implying $k > r \geq 1$. By applying Dirichlet's principle, either X contains a vertex a with $d_F(a) > d+1 \geq 1$ (in which case $d_F(X)$ is lexicographically greater than the distribution with the largest degree $d + 1$) or there are r vertices in X with F-degree $d + 1$ and $k - r$ vertices in X with F-degree d. The claim follows. □

Recall that the original Hungarian method augments the maximum matching by flipping the matching membership of edges along alternating paths with both end-edges not in the matching. In order to adopt this approach to the more general setting of quasi-matchings, we need to tie-up such augmenting paths to particular vertices, yielding the following concepts.

Definition 7. Let $G = A + B$ be a bipartite graph and $F \subseteq E(G)$ a set of edges. A *(forward) F-alternating path* from a vertex $a \in A$ to a vertex $a' \in A$ in G is a path P, starting with an edge from F, and then consecutively alternating edges not from F and edges from F (so the last edge on P is not in F). A path P from a vertex $a \in A$ to a vertex $a' \in A$ in G is a *backward F-alternating path* if the reversed path on the same edges from a' to a is a (forward) F-alternating path. An *F-augmenting path* P in G is a path from a vertex $b \in B$ to a vertex $a \in A$, such that $P - b$ is an F-alternating path from a' to a, and the edge $a'b$ is not in F.

Note that by performing F-exchange $F' = F \oplus E(P)$ of edges in an F-alternating path P from $a \in A$ to $a' \in A$, the degree of a decreases by one ($d_{F'}(a) = d_F(a) - 1$), the degree of a' increases by one ($d_{F'}(a') = d_F(a) + 1$), and all other quasi-matching-degrees remain as in F. The difference of F-degrees at the ends of F-alternating paths is the crucial property allowing us to improve quasi-matchings.

Definition 8. Let $G = A + B$ be a bipartite graph, F a quasi-matching of $Y \subseteq B$ and P an F-alternating path from $a \in A$ to $a' \in A$. The *decline* of P is $dc(P) = d_F(a) - d_F(a')$.

As it turns out, the optimality of a quasi-matching is easily analyzed if all the vertices of A are reached from some vertex of A by F-alternating paths. When this is not the case, we need to pay special attention to the edges linking the components with respect to such connectivity relation. We formalize this concept as follows.

Definition 9. Let $G = A + B$ be a bipartite graph, $F \subseteq E(G)$, and $a \in A$. The *a-section* of G is a maximal subgraph $G_a = X_a + Y_a \subseteq G$, such that there is an F-alternating path $P_{a'}$ from a to every $a' \in X_a$ and $Y_a = N_F(X_a)$ is the set of F-neighbors of X_a. Furthermore, F_a is the set of edges in F incident with X_a.

The following characterization of minimum g-quasi-matchings is in later sections essential for the generalized Hungarian method designed to find them.

Theorem 10. *Let $G = A+B$ be a bipartite graph, $g: B \to \mathbb{N}$ a need function and F a g-quasi-matching of B. Then F is a minimum g-quasi-matching of B if and only if any F-alternating path in G has decline at most* 1.

Proof. Suppose there is an F-alternating path P in G whose decline is at least two. By performing an F-exchange of edges on P, we get a g-quasi-matching F', such that F is lexicographically greater than F', a contradiction.

The converse is by induction on $g(B) = \sum_{y \in B} g(y)$. Assume that all F-alternating paths in G have decline at most 1. Let $a \in A$ be a vertex with the largest F-degree in G, and let $H = X + Y$ be the a-section in G. Note that for any $a' \in X$,

$$d_F(a) - 1 \leq d_F(a') \leq d_F(a).$$

Also note that by definition of the a-section (maximality), any edge connecting a vertex from Y to a vertex from $A - X$ is in F. Let t be the number of edges connecting a vertex from Y to a vertex from $A - X$. Then by letting $|X| = k$ and $d = d_F(a)$, we easily infer that $g(Y) = t + k(d - 1) + r$, where r is the number of vertices in X with F-degree equal to d. By Lemma 6, the distribution $d_F(X)$ coincides with the lexicographically minimum degree distribution of a g-quasi-matching. Hence, if $X = A$ (and so $t = 0$), the proof is complete.

Thus, suppose that $X \neq A$. Let $Y' = N(A - X)$, and note that $Y \cup Y' = B$, while $Y \cap Y'$ may be nonempty. Let F' be the restriction of F to the edges with one endvertex in $A - X$, and set $F'' = F - F'$ (i.e. F'' contains edges from F that have one endvertex in X). We set a need mapping g' of Y' with $g'(v) = g(v) - d_{F''}(v)$ for any $v \in Y'$. Now, any F'-alternating path in $(X - A) + Y'$ has decline at most one because F' is just the restriction of F. As $g'(Y') < g(B)$, we infer by induction hypothesis that F' is a (lexicographically) minimum g'-quasi-matching of Y'.

Let Q be a minimum g-quasi-matching. Hence $d_Q(A)$ is not greater than $d_F(A)$. In addition, we infer by Lemma 6 that the distribution $d_Q(X)$ is at least $d_F(X)$, that is, there are at least r vertices from X whose Q-degree is d. Let $p, p \geq r$ be the number of vertices in X whose Q-degree is d. Denote by Q'' the set of edges from Q that have one endvertex in X, and let $Q' = Q - Q''$. Now we introduce a need mapping g'' on Y' by setting $g''(v) = g(v) - d_{Q''}(v)$ for any $v \in Y'$. Note that $g''(Y') = g'(Y') - (p - r)$, and so

$$\sum_{u \in A-X} d_{Q'}(u) = \sum_{u \in A-X} d_{F'}(u) - (p - r). \tag{1}$$

Note also that $g'(u) \geq g''(u)$ for any $u \in Y'$. Since Q' is clearly a minimum g''-quasi-matching of $(A - X) + Y'$, we infer (again by induction hypothesis) that it has no alternating paths with decline more than 1.

We gradually increase the g''-quasi-matching Q' of $(A-X)+Y'$ to a g'-quasi-matching by using the following procedure that consists of $p - r$ steps. We denote by Q_i the quasi-matching in the i-th step of the procedure (and set $Q_0 = Q'$). In each step, we obtain Q_i from Q_{i-1} by taking a vertex $u \in Y'$ with $g''(u) < g'(u)$, for which $d_{Q_{i-1}}(u) < g'(u)$. Let P be an augmenting path from u to a vertex a_i of smallest possible Q_{i-1}-degree in $A - X$. Then we set $Q_i = Q_{i-1} \oplus E(P)$. Note that all vertices from $A - X$ on P have degree $d_{Q_i}(a_i)$ because $P - u$ is a forward Q_{i-1}-alternating path, having decline exactly 1 (unless a_i is already a neighbor of u). From this we quickly infer that there are no Q_i-alternating paths with decline more than 1, provided there were no such Q_{i-1}-alternating paths. In the last step, we get a g'-quasi-matching Q_{p-r} which thus has no alternating paths with decline more than 1. By induction hypothesis, Q_{p-r} is a minimum g'-quasi-matching of $(A-X)+Y'$, hence its degree distribution in $A - X$ coincides with $d_{F'}(A - X)$.

From (1) we find that $d_{Q'}(A-X)$ is the smallest possible (noting that it can be obtained from $d_{F'}(A-X)$ by taking off $p-r$ units from vertex degrees in $A-X$) if there are exactly $p-r$ vertices in $A-X$ with Q'-degree $d-1$ and whose F'-degree is d (in all other cases, the number of vertices with F'-degree equal to d is less than the sum of $p - r$ and the number of vertices with Q'-degree equal to d, which would in turn imply that $d_F(A)$ is strictly smaller than $d_Q(A)$). Now, this implies that in other vertices of $A - X$ the distributions of $d_{Q'}$ and $d_{F'}$ are the same. Combined with distributions of degrees in X we derive that $d_F(A) = d_Q(A)$, and so F is a minimum g-quasi-matching as well. □

The 1-quasi-matchings alias semi-matchings were studied in [7]. There, the quality of semi-matchings was measured by assigning an increasing cost to the vertex degrees. In order to connect our results to theirs, we adopt the following definition.

Definition 11. Let $G = A + B$ be a bipartite graph, F a semi-matching of B, and $f : \mathbb{R}^+ \to \mathbb{R}$ a strictly (weakly) convex function. Then the function cost_f, defined as $\sum_{i=1}^{|A|} f(d_F(a_i))$ is called a strict (weak) cost function for f.

In [7], the strictly convex function $\ell(n) = \frac{1}{2}n(n+1)$ is emphasized. It is interesting in task scheduling, as it measures total latency of uniform tasks on a single machine. It is also proved that a semi-matching F has minimum $\mathrm{cost}_\ell(F)$ if and only if any F-alternating path in $G = A + B$ has decline at most 1. By Theorem 10, F-alternating paths in G have such a property if and only if F is (lexicographically) minimum semi-matching of B. The special case of Theorem 10 where the need function is constant 1 combined with the results from [7] leads to the following equivalent characteristics of the (lexicographically) minimum semi-matching.

Corollary 12. *Let $G = A + B$ be a bipartite graph, F a semi-matching of B, and $f : \mathbb{R}^+ \to \mathbb{R}$ a strictly convex function. Then the following are equivalent:*

(i) *F is a (lexicographically) minimum semi-matching of B.*
(ii) *Any F-alternating path in G has decline at most 1.*
(iii) *F has minimum $\mathrm{cost}_\ell(F)$ for $\ell(n) = \frac{1}{2}n(n+1)$.*
(iv) *F has minimum $\mathrm{cost}_f(F)$.*
(v) *L_p-norm, $1 \leq p < \infty$, of the vector $X = (d_F(a_1), \ldots, d_F(a_{|A|}))$ is minimal.*
(vi) *The variance of the vector $X = (d_F(a_1), \ldots, d_F(a_{|A|}))$ is minimal.*

Proof. The equivalence (i) $\iff$ (ii) follows from the Theorem 10. Furthermore, (ii) is equivalent to (iii) [7, Theorem 3.1] and (iv) [7, Theorem 3.5]. Finally, (iii) is equivalent to (v) [7, Theorem 3.9] and (vi) [7, Theorem 3.10]. □

Every property of Corollary 12 implies that F has a minimum $\mathrm{cost}_f(F)$ for every weakly convex function f [7, Theorem 3.5], and that L_∞-norm of the vector $X = (d_F(a_1), \ldots, d_F(a_{|A|}))$ is minimal [7, Theorem 3.12]. In both cases, the converse is not true.

Note that Theorem 10 is a key step in proving the equivalence (i) $\Leftrightarrow$ (iv) which is a statement about extrema of a strictly convex function on a subset of a multi-dimensional integer lattice $\mathbb{Z}^n$, whose elements correspond to the degrees of vertices from A with respect to semi-matchings. Although it seems that there must be a direct proof of this fact, we were unable to find one. This equivalence can be used to establish the following result.

Proposition 13. *Let $Z \subset \mathbb{Z}^n$ be a set of positive n-element partitions of an integer r. Furthermore, let $f : \mathbb{R} \to \mathbb{R}$ be any strictly convex function, and $\mathrm{cost}_f(z) = \sum_{i=1}^n f(z_i)$. Then $z' \in Z$ is a minimum of cost_f if and only if z' is the smallest element of Z with respect to the lexicographic ordering, where for comparison in this ordering, the coordinates of n-tuples in Z are sorted in non-increasing order, $z_1 \geq z_2 \geq \cdots \geq z_n$.*

Proof. Given a set Z we construct a bipartite graph $G = A + B$ as follows: $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_r\}$. For $z \in Z$, we order the elements as $z_1 \geq z_2 \geq \cdots \geq z_n$ and construct a semi-matching F_z, such that for $s_0(z) = 0$ and $s_i(z) = \sum_{j=1}^i z_j$, $i = 1, \ldots, n$, a_i is in F_z connected to all the vertices of $\{b_{s_{i-1}(z)+1}, \ldots, b_{s_i(z)}\}$. Let $E(G) = \bigcup_{z \in Z} F_z$.

We claim that G has the following properties: (a) for each $z \in Z$, there is a semi-matching F_z of G whose degree sequence is equal to z, and (b) the degree sequence of a minimum semi-matching F' of G is equal to is a lexicographically minimum element z' of Z.

First note that (a) is obvious. It is also clear that all lexicographically minimal elements of Z have the same value of cost_f. To prove (b), assume that there is a semi-matching F of G which is lexicographically smaller than F' that corresponds to z'. Then, F' is not a minimal matching and by Theorem 10, there is an F'-alternating path in G whose decline is more than 1. Let P be a shortest such path with decline more than 1 and suppose that it starts with an edge $a_i b_j \in F'$, for some $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, r\}$. But as z' is lexicographically minimum, b_j is in G not adjacent with any a_k, where $k > i$. As $d_{F'}(a_k) \geq d_{F'}(a_i)$ for $k \leq i$, we get into a contradiction with P being a shortest such path. We derive that such a path cannot exist, hence (b) is established.

Suppose that z' is a lexicographically minimal element of Z and $z \in Z$ is arbitrary. Then by (b), $F_{z'}$ is a lexicographically minimum semi-matching of G and by (a), $\mathrm{cost}_f(z) = \mathrm{cost}_f(F_z) \geq \mathrm{cost}_f(F_{z'}) = \mathrm{cost}_f(z')$, where the inequality follows from $F_{z'}$ being a lexicographically minimum semi-matching and (i) $\Rightarrow$ (iv) in Corollary 12.

Suppose that z' is the element of Z with smallest value of cost_f and $z \in Z$ is any other element. Then by (a), $\mathrm{cost}_f(F_z) = \mathrm{cost}_f(z) \geq \mathrm{cost}_f(z') = \mathrm{cost}_f(F_{z'})$. Thus, $F_{z'}$ is a semi-matching of G with smallest value of cost_f among all matchings of the form F_z. If its degree sequence is not equal to that of a minimum semi-matching of G, then it is lexicographically greater, a contradiction to (b). Then (a) implies that z' is lexicographically minimum element of Z. □

Corollary 14. *Let $G = A + B$ be a bipartite graph and let F be a (lexicographically) minimum semi-matching of B. Then there exists a maximum matching $M \subseteq F$ in G.*

Proof. Follows directly from Corollary 12 and Theorem 3.7 from [7]. □

The converse of Corollary 14 does not hold (see [7]).

4. Generalized Hungarian method

In this section, we solve Problem 3 with an algorithm of complexity $O(g(B)|E(G)|)$. We use the fact that quasi-matchings are a generalization of matchings: if we restrict ourselves to quasi-matchings with degree one, our method is a generalization of the Hungarian method of augmenting paths for finding maximum matchings in bipartite graphs.

Let $B = \{b_1, \ldots, b_n\}$ and $B_\ell = \{b_1, \ldots, b_\ell\}$, $\ell = 1, \ldots, n$. Define a mapping $g_i : B \to \mathbb{N}$ with $g_0(b) = 0$, for all $b \in B$, $\ell_1 = 1$, $\ell_i = \max\left\{j | g_{i-1}(b_j) \neq 0\right\}$ for $i > 1$, and

$$g_i(b) = \begin{cases} g_{i-1}(b) + 1; & b = b_{\ell_i} \text{ and } g_{i-1}(b) < g(b), \\ 1; & b = b_{\ell_i+1} \text{ and } g_{i-1}(b_{\ell_i}) = g(b_{\ell_i}), \\ g_{i-1}(b); & \text{otherwise} \end{cases}$$

for every $1 \leq i \leq g(B)$. That is, g_i differs from g_{i-1} only in one vertex, and in that vertex the value is by one larger; in the sequence of g_i's, we first "fill-up" b_1 until it reaches $g(b_1)$, then in $g_{g(b_1)+1}$ we move to b_2 and so on. Note that for simplicity we assume $g(b) > 0$ for all $b \in B$. We propose to find a minimum g-quasi-matching F of B using an iterative algorithm that gradually extends an g_i-quasi-matching F_i of B_ℓ using an F_{i-1}-augmenting path P_{i-1} from b_ℓ to $a \in A$ with smallest $d_{F_{i-1}}(a)$. By induction, we argue that F_i is a minimum g_i-quasi-matching of B_ℓ, thus the final F_i is a minimum g-quasi-matching of the corresponding $B_\ell = B$.

Lemma 15. *Let $G = A + B$ be a bipartite graph and $a \in A$. Using the notation of Algorithm 1, the following holds:*

$$d_{F_i}(a) = \begin{cases} d_{F_{i-1}}(a) + 1; & \text{if } a \text{ is the } A\text{-endvertex of } P_{i-1}, \\ d_{F_{i-1}}(a); & \text{otherwise}. \end{cases}$$

Algorithm 1 Iterative construction of a minimum g-quasi-matching of B.

Parameter $G = A + B$**:** a bipartite graph with $B = \{b_1, \dots, b_n\}$.
Output F**:** a minimum g-quasi-matching of B.
  Set $i = 0$, $\ell = 0$.
  Set $F_i = \emptyset$, $B_\ell = \emptyset$, $G_\ell = \emptyset$.
  while $\ell \le n$ **do**
    $\ell = \ell + 1$.
    Set $B_\ell = B_{\ell-1} \cup \{b_\ell\}$.
    Set $G_\ell = G[B_{\ell-1} \cup A]$.
    $c = 0$.
    while $c < g(b_\ell)$ **do**
      $i = i + 1$, $c = c + 1$.
      Set P_{i-1} to be an F_{i-1}-augmenting path in G_ℓ from b_ℓ to $a \in A$ with smallest possible degree $d_{F_{i-1}}(a)$.
      Set $F_i = F_{i-1} \oplus E(P_{i-1})$.
    end while
  end while
  return F_i.

Proof. The Lemma is obviously true for every vertex $a \in A \setminus V(P_{i-1})$. Since $F_i = F_{i-1} \oplus E(P_{i-1})$, $e \in F_{i-1} \cap E(P_{i-1})$ implies that $e \notin F_i$. Similarly, for every $e \in E(P_{i-1}) \setminus F_{i-1}$ we have $e \in F_i$. Therefore, the number of F_i-edges at an internal P_{i-1} vertex a is the same as the number of F_{i-1}-edges at a. However, if a is the A-endvertex of P_{i-1}, then its only P_{i-1} incident edge is not in F_{i-1} but is in F_i, so $d_{F_i}(a) = d_{F_{i-1}}(a) + 1$. □

Theorem 16. *Let $G = A + B$ be a bipartite graph. Using the notation of Algorithm 1, F_i is a minimum g_i-quasi-matching of B_ℓ in G_ℓ for $i = 1, \dots, n$.*

Proof. For $i = 1$, we have $\ell = 1$, $B_1 = \{b_1\}$ and $g_1(b_1) = 1$. Let a be any vertex from $N(b_1)$. Then $F_1 = P_0 = b_1 a$ is a minimum g_1-quasi-matching of B_1 in G_1.

Suppose now that F_{i-1} is a minimum g_{i-1}-quasi-matching of $B' = B_\ell$ (or $B' = B_{\ell-1}$) in $G' = G_\ell$ (or $G' = G_{\ell-1}$). We claim that $F_i = F_{i-1} \oplus E(P_{i-1})$ is a minimum g_i-quasi-matching of B_ℓ in G_ℓ. If this is not the case, then Theorem 10 yields an F_i-alternating path P in G_ℓ from $a' \in A$ to $a'' \in A$ with decline $d_{F_i}(a') - d_{F_i}(a'') \ge 2$. Note that every F_i-alternating subpath of P from a vertex $a \in A$ leads to a'' and every backward F_i-alternating subpath leads to a'.

Consider first the case for $E(P) \cap E(P_{i-1}) = \emptyset$. Then an edge e of P is in F_{i-1} if and only if it is in F_i. For the rest of the proof let a denote the endvertex of P_{i-1}. We distinguish three cases:

Case A1: $a \notin \{a', a''\}$.

Lemma 15 implies that $d_{F_i}(a') = d_{F_{i-1}}(a')$ and $d_{F_i}(a'') = d_{F_{i-1}}(a'')$. Thus, P is an F_{i-1}-alternating path from a' to a'' in G' with decline at least 2. A contradiction to Theorem 10, since F_{i-1} is a minimum g_{i-1}-quasi-matching of B' in G'.

Case A2: $a = a'$.

Lemma 15 implies $d_{F_i}(a') = d_{F_{i-1}}(a') + 1$ and $d_{F_i}(a'') = d_{F_{i-1}}(a'')$. Let v be the common vertex of the paths P and P_{i-1} closest to b_ℓ in P_{i-1}. Then $Q = b_\ell P_{i-1} v P a''$ (resp. $Q = b_\ell P a''$ for $v = b_\ell$) is an F_{i-1}-augmenting path in G_ℓ. Since P_{i-1} in G_ℓ is chosen so that $d_{F_{i-1}}(a)$ is minimum, we have

$$d_{F_{i-1}}(a) = d_{F_{i-1}}(a') \le d_{F_{i-1}}(a'')$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \le 0.$$

This contradicts the assumption $d_{F_i}(a') - d_{F_i}(a'') \ge 2$, as

$$d_{F_{i-1}}(a') + 1 - d_{F_{i-1}}(a'') \ge 2$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \ge 1.$$

Case A3: $a = a''$.

In this case, Lemma 15 implies that P is an F_{i-1}-alternating path from a' to a'' in G' with $d_{F_i}(a') = d_{F_{i-1}}(a')$ and $d_{F_i}(a'') = d_{F_{i-1}}(a'') + 1$. The inequality $d_{F_i}(a') - d_{F_i}(a'') \ge 2$ yields

$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') - 1 \ge 2$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \ge 3.$$

Hence, P is an F_{i-1}-alternating path in G' with decline at least 3. But this is again impossible by Theorem 10 and minimality of F_{i-1}.

It remains to examine the case $E(P) \cap E(P_{i-1}) \ne \emptyset$.

Case B1: $a \notin \{a', a''\}$.

Let v be the common vertex of P and P_{i-1} closest to b_ℓ in P_{i-1}. Then $Q = b_\ell P_{i-1} v P a''$ (resp. $Q = b_\ell P a''$ for $v = b_\ell$) is an F_{i-1}-augmenting path in G_ℓ. The choice of P_{i-1} implies $d_{F_{i-1}}(a) \le d_{F_{i-1}}(a'')$.

Let v' be the common vertex of P and P_{i-1} closest to a' in P. Then $R = a' P v' P_{i-1} a$ is an F_{i-1}-alternating path in G_ℓ. Since Lemma 15 implies

$$2 \le d_{F_i}(a') - d_{F_i}(a'') = d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \le d_{F_{i-1}}(a') - d_{F_{i-1}}(a),$$

R is a path with F_{i-1}-decline at least two, another contradiction to Theorem 10 and minimality of F_{i-1}.

Case B2: $a = a'$.

Let Q be the F_{i-1}-augmenting path in G_ℓ from b_ℓ to a'' as in case B1. The existence of such a path ensures that $d_{F_{i-1}}(a) - d_{F_{i-1}}(a'') \le 0$. But this is not possible, since Lemma 15 implies

$$2 \le d_{F_i}(a') - d_{F_i}(a'') = d_{F_i}(a) - d_{F_i}(a'') = d_{F_{i-1}}(a) + 1 - d_{F_{i-1}}(a'')$$

and hence $d_{F_{i-1}}(a) - d_{F_{i-1}}(a'') \ge 1$.

Case B3: $a = a''$.

Let R be the F_{i-1}-alternating path in G_ℓ from a' to a constructed as in case B1. We claim that R has decline at least three. From $d_{F_i}(a') - d_{F_i}(a'') \ge 2$ and Lemma 15, we deduce that

$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') - 1 \ge 2$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \ge 3$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a) \ge 3.$$

But this contradicts the minimality of F_{i-1}.

We conclude that F_i is a minimum g_i-quasi-matching of B_ℓ in G_ℓ. □

By setting $\ell = n$, Theorem 16 proves the correctness of Algorithm 1.

Corollary 17. *Algorithm 1 finds a minimum g-quasi-matching of B and has time-complexity $O(g(B)|E(G)|)$, where $g(B)$ is the need of B.*

Proof. As $B_n = B$, Theorem 16 establishes that B is a minimum g-quasi-matching of B. The path P_i can be found using an augmented Hungarian method: the algorithm performs a breadth-first search from the vertex b_ℓ in such way, that if the vertex whose neighbors are examined is in A, then the search proceeds along its F_{i-1} incident edges only, but from vertices of B, the search proceeds along the non-F_{i-1}-incident edges only. The search tree T produced in this manner has alternating levels of non-F_{i-1} and F_{i-1} edges, and in T there is a unique F_{i-1}-augmenting path from any vertex to B. This path starting at a vertex $a \in A$ of minimum F_{i-1}-degree is the path P_{i-1} required for Algorithm 1. The whole tree T and thus the augmenting path P_{i-1} can be constructed in $O(|E|)$ time. As there are $g(B) = \sum_{b \in B} g(b)$ iterations, the overall complexity of Algorithm 1 is $O(g(B)|E(G)|)$. □

Note that Theorem 10 can be applied to prune the tree constructed in the generalized Hungarian method in such a way that the search tree contains vertices of one F_{i-1}-degree only. If d is the minimum F_{i-1}-degree of a neighbor of b_i, then P_{i-1} need not contain any vertex of degree $d+1$. Furthermore, as soon as a vertex of F_{i-1}-degree $d-1$ is encountered, we can assume that this is the terminating vertex of P_{i-1}. These observations do not improve the theoretical complexity of the algorithm (in the worst case, for instance when G has a perfect matching, we still need to consider $O(|E(G)|)$ edges at each iteration), but they could considerably improve any practical implementation.

For semi-matchings, the complexity of our algorithm reduces to $O(|B||E(G)|)$. In an off-line setting, a very recent algorithm by Fakcharoenphol et al. [8] can solve the problem for semi-matchings in $O(m\sqrt{n}\log n)$, where m is the number of edges and n is the number of vertices in G. It is beyond the scope of this paper, but it would be a challenge to find out whether a similarly efficient off-line algorithm could be designed for general quasi-matchings. Thus we close this section with the following problem:

Problem 4. Find a more efficient algorithm for solving Problem 3. In particular, is there an algorithm with time complexity $O(|E(G)|\sqrt{g(B)}\log g(B))$?

5. On-line application of Algorithm 1

Note that each step of Algorithm 1 can be viewed as a part of an on-line procedure, where the need of a vertex, denoted b_ℓ, increases by one. In particular, this allows for immediate application of this algorithm to the on-line setting — to rearrange it for the on-line addition of a new vertex v with need $g(v)$, one only needs to perform one step of the outer *while* loop (hence the inner *while* loop which takes $O(|E(G)|)$ time is performed $g(v)$ times).

添加一个顶点：在线应用中，顶点一个接着一个的
出现，同时得到它的关联边，只修改外层while循环

However, the full on-line setting, as presented for instance in [11], also allows for removal of the vertices of B, i.e. an on-line event is not just the appearance of a new vertex, but also the disappearance of an existing vertex. In our setting, this would correspond to a wireless sensor malfunction or running out of battery, while in the task-scheduling setting (cf. again [11] for comparison), this corresponds to a task being removed from the schedule or the number of required machines for the task being decreased. In this sense, our algorithm(s) enable(s) all possible adds/removals of vertices.

Algorithm 2 describes how to augment an existing minimum quasi-matching when the need of a single vertex $b \in B$ decreases by one to obtain an optimal quasi-matching with respect to the new need function. As above, if b disappears, then this algorithm simply needs to be performed $g(b)$ times. Since $g(b)$ is bounded by $|A|$, the algorithm is clearly strongly polynomial.

Let $G = A + B$ be a bipartite graph with $B = \{b_1, \ldots, b_n\}$, and let $b \in B$, say $b = b_k$ for some k. If $g: B \to \mathbb{N}$ is a need function of B, then we denote by g_b the mapping from B to $\mathbb{N}$ with $g_b(b_i) = g(b_i)$ for $i \neq k$, and $g_b(b) = g(b) - 1$.

Algorithm 2 Obtaining a minimum g_b-quasi-matching from a minimum g-quasi-matching in $G = A + B$.

Parameter $G = A + B$**:** a bipartite graph with $B = \{b_1, \ldots, b_n\}$ and need function $g: B \to \mathbb{N}$.
Parameter F**:** a minimum g-quasi-matching of B in G.
Parameter b**:** a vertex of B.
Output F'**:** a minimum g_b-quasi-matching in G.

```
Set A_b be the set of F-neighbors of b.
Set a ∈ A_b be the vertex with largest F-degree in A_b.
if there is a backward F-alternating path P in G from a' ∈ A_b to a'' ∈ A with d_F(a'') = d_F(a') + 1 then
    set F' = F ⊕ P − a'b
else
    set F' = F − ab.
end if
return F'.
```

Theorem 18. *Let $G = A + B$ be a bipartite graph and F a minimum g-quasi-matching of B in G. Using the notation and assumptions of Algorithm 2, F' is a minimum g_b-quasi-matching of B in G.*

Proof. By Theorem 10, we need to prove that every F'-alternating path has decline at most 1 in G. Note that every F-alternating path has decline at most 1 in G, since F is minimum by assumption. There are two cases in the algorithm that we deal with separately.

Suppose first there is no such backward F-alternating path P in G from $a' \in A_b$ to $a'' \in A$ with $d_F(a'') = d_F(a') + 1$. Then $F' = F - ab$, and note that $d_F(A) = d_{F'}(A)$ except in a where $d_{F'}(a) = d_F(a) - 1$. Hence, if there is any F'-alternating path with decline greater than 1, it ends in a. Now, no such violating path could start with a vertex from A_b, since a has the largest F-degree among these vertices. And also, no such violating path could start in any other vertex a'' of A, because that would mean there is a backward F-alternating path in G from $a \in A_b$ to $a'' \in A$ with $d_F(a'') = d_F(a') + 1$, contrary to our assumption.

Secondly, suppose there exists a backward F-alternating path in G from $a' \in A_b$ to $a'' \in A$ with $d_F(a'') = d_F(a') + 1$, and let P be a shortest such path. Then $F' = F \oplus P - a'b$, and we have $d_F(A) = d_{F'}(A)$ except in a'' where $d_{F'}(a'') = d_F(a'') - 1$. By the choice of P and the fact that there are no F-alternating paths with decline more than one, we infer that $d_F(v) = d_F(a')$ for all vertices $v \in A$ on $P \setminus \{a''\}$. Hence, for all vertices $v \in A$ on P (a'' included), we have $d_{F'}(v) = d_{F'}(a')$. For the purpose of contradiction let us suppose there is a violating F'-alternating path P' from $\hat{a}$ to $\tilde{a}$. Since F' and F differ only on P, we infer that P' must intersect P in some vertex of A. This readily implies that $d_{F'}(\hat{a}) \leq d_{F'}(a') + 1$ and $d_{F'}(\tilde{a}) \geq d_{F'}(a') - 1$. Since P' is violating, we infer that in fact $d_{F'}(\hat{a}) = d_{F'}(a') + 1$ and $d_{F'}(\tilde{a}) = d_{F'}(a') - 1$ so that the decline of P' with respect to F' is exactly 2. Now, we can easily find that there is an F-alternating path from a'' to $\tilde{a}$ in G whose decline equals 2, which is a contradiction with F being a minimum g-quasi-matching. □

From Theorem 18 and previous discussion, we infer that the augmented Hungarian method presented in this paper can be applied to the on-line problem of constructing an optimal quasi-matching of B with the set A fixed, when the vertices of B either appear or disappear one at a time. Each on-line step assures optimality of the current quasi-matching in $O(g(v)|E(G)|)$ steps. Moreover, a similar approach could be used for an on-line setting, where the vertices of A can appear or disappear. When a vertex of A of F-degree d is removed, its F-neighbors from B loose the degree with respect to a quasi-matching, which can be iteratively recovered, resulting in a patching algorithm of complexity $O(d|E(G)|)$. On the other hand, when an A-vertex of G-degree d is added, up to d vertices can be assigned to it, again resulting in a $O(d|E(G)|)$ algorithm per on-line step.

Note that when restricted to semi-matchings and just additions of vertices from B, our adaptation of Hungarian method is the same as the one from [7]. However, our proof of correctness differs in that we explicitly maintain minimality of the constructed semi-matching (in fact, even an arbitrary g-quasi-matching), after each addition (or removal) of a vertex. Furthermore, the set of possible alternating paths with decline at least two is in our approach narrowed to the vertex that is added to or removed from the graph, resulting in an efficient on-line version of the algorithm.

6. Generalized Hall's marriage theorem

In this section, we present a solution to Problem 2 by characterizing bipartite graphs $A + B$ with given $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$ that admit an f, g-quasi-matching. The result is a vast generalization of Hall's theorem.

A *network* $N = (V, A)$ is a digraph with a nonnegative capacity $c(e)$ on each edge e, and with two distinguished vertices: *source* s and *sink* t (usually, s has only outgoing, and t has only ingoing arcs). A *flow fl* assigns a value $fl(e)$ to each edge e. A flow fl is *feasible* if for each edge e, $0 \le fl(e) \le c(e)$ and the *conservation (Kirchhoff's) law* is fulfilled: for every vertex $v \in V(N) \setminus \{s, t\}$,

$$\boxed{\sum_{vx \in A(N)} fl(vx) = \sum_{xv \in A(N)} fl(xv).}$$

流入=流出

The *value* of a flow fl is $\sum_{sx \in A(N)} fl(sx)$, which is equal to $\sum_{xt \in A(N)} fl(xt)$. The famous Ford–Fulkerson (or max-flow min-cut) theorem states that the maximum value of a feasible flow in N coincides with the minimum capacity of a cut in N. (Where *cut* is the set of arcs from S to T in a S, T partition of N (i.e. $s \in S$, $t \in T$), and its *capacity* is the sum of the c-values of its edges.) More on this well-known problem and theorem can be found for instance in [1,12]. One of the several proofs of the famous Hall's marriage theorem uses the max-flow min-cut theorem, and in our generalization of this theorem, we use a similar approach.

Definition 19. Let $G = A + B$ be a bipartite graph, $f : A \to \mathbb{N}$ an availability function, and $Y \subseteq B$. For $x \in A$, let $d_Y(x) = |\{y \in Y : xy \in E(G)\}|$, that is the number of neighbors of x from Y. For $X \subset A$, let $f(X, Y) = \sum_{x \in X} \min\{f(x), d_Y(x)\}$ denote the *relative availability* of X with respect to f and Y. In particular, for $x \in X$, we write $f(\{x\}, Y)$ as $f(x, Y)$ (which is the least of $f(x)$ and $d_Y(x)$).

Intuitively, the relative availability of X with respect to f and Y presents the maximum number of edges going from X that can be used to cover Y.

Theorem 20. *Let $G = A + B$ be a bipartite graph, with $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$, a mapping $f : A \to \mathbb{N}$, and $g : B \to \mathbb{N}$. Then G has an f, g-quasi-matching of $A + B$ if and only if for every $Y \subseteq B$,*

$$f(N(Y), Y) \ge g(Y). \tag{2}$$

Proof. Suppose there is a subset $Y \subset B$ such that $\sum_{u \in N(Y)} f(u, Y) = f(N(Y), Y) < g(Y) = \sum_{v \in Y} g(v)$. Let F be an arbitrary g-quasi-matching of B in G. The vertices of Y altogether must have at least $g(Y)$ F-neighbors. As the relative availability of their neighbors $N(Y)$ is less than $g(Y)$, we derive by the pigeon-hole principle that there will be a vertex $u \in N(Y)$ such that $d_F(u) > f(u)$. Hence F is not an f, g-quasi-matching, which readily implies (since F was arbitrarily chosen) that no f, g-quasi-matching exists.

For the converse, let $f(N(Y), Y) \ge g(Y)$ hold for all $Y \subseteq B$. We introduce two additional vertices: a that is connected to all vertices $a_i \in A$, and b, connected to all $b_j \in B$. Construct a digraph G', by choosing a direction of all edges from G as follows: from a to each $a_i \in A$, from vertices of A to their neighbors in B, and from each b_j to b. Next, construct a network out of the digraph G', by setting flow capacities $c : E(G') \to \mathbb{N}$ as follows: $c(aa_i) = f(a_i)$, $c(a_ib_j) = 1$ (for $a_ib_j \in E(G)$), and $c(b_jb) = g(b_j)$. Note that there exists a flow of size $g(B)$ in G' if and only if there exists an f, g-quasi-matching of $A + B$. By max-flow min-cut theorem, the maximum flow value coincides with the minimum cut capacity in the network G'.

Let C be a minimum cut in the network, and let Z be the set of vertices from B for which $b_jb \in C$. Let $Y = B \setminus Z$. Since C is a cut, for every vertex $b_j \in Y$ and every neighbor a_i of b_j, we have either $a_ib_j \in C$ or $aa_i \in C$ (since C is minimum, we may assume that both does not happen). Denote by K the set of vertices a_i from $N(Y)$ such that $aa_i \in C$ and let $L = N(Y) \setminus K$. For $b_j \in Y$, let m_j denote the number of its neighbors in L (which coincides with the number of its incident edges that are from C). Note that

$$\sum_{j, b_j \in Y} m_j = \sum_{a_i \in L} d_Y(a_i) \ge f(L, Y).$$

Now,

$$\begin{aligned} |C| &= g(Z) + f(K) + \sum_{j, b_j \in Y} m_j \\ &\ge g(Z) + f(K, Y) + f(L, Y) \\ &\ge g(Z) + f(N(Y), Y) \\ &\ge g(Z) + g(Y) = g(B) \end{aligned}$$

where in the last inequality (2) is used. The result now readily follows. □

Let us state the most obvious corollaries of the theorem. First, if f is not involved, i.e. if $f(u) = d(u)$ for all $u \in A$, then $f(N(Y), Y) = \sum_{u \in N(Y)} d_Y(u) = \sum_{v \in Y} d(v)$, and (2) turns into a much simpler condition $\sum_{v \in Y} d(v) \geq g(Y)$ for every $Y \subseteq B$.

If we want that each vertex in A covers only one vertex from B, that is $f(u) = 1$ for all $u \in A$, we get $f(N(Y), Y) = \sum_{u \in N(Y)} 1 = |N(Y)|$, and the condition (2) reads $|N(Y)| \geq g(Y)$ for every $Y \subseteq B$. If, in addition, $g(v) = 1$ for all $v \in B$, we get $|N(Y)| \geq |Y|$ for all $Y \subseteq B$ which is exactly Hall's condition. On the other hand, this implies that $A + B$ has a perfect matching of vertices from B. Thus Hall's theorem is a corollary of Theorem 20.

One of the common formulations of Hall's theorem is in terms of systems of distinct representatives. Let us formulate also Theorem 20 in this sense.

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a family of sets, with $S = \cup_{i=1}^{m} A_i = \{b_1, \ldots, b_n\}$, and let there be mappings $f : \mathcal{A} \to \mathbb{N}$, and $g : S \to \mathbb{N}$. We say that the family $\mathcal{A}$ has a *(lower) system of f, g-representatives* if to every set $A_i \in \mathcal{A}$ we associate at most $f(A_i)$ representatives from S, and every vertex $b_j \in S$ is a representative of at least $g(b_j)$ sets from $\mathcal{A}$. In this terminology, Theorem 20 reads as follows.

Corollary 21. *A family of sets $\mathcal{A}$ has a lower system of f, g-representatives if and only if for every subset $Y \subseteq S$ we have*

$$\sum_{A_i \in \mathcal{A}} \min\{f(A_i), |A_i \cap Y|\} \geq \sum_{b_j \in Y} g(b_j).$$

By duality, since the interpretation of the roles of sets and vertices in Theorem 20 can be reversed, we have another corollary expressed in similar terms. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a family of sets, with $S = \cup_{j=1}^{n} B_j = \{a_1, \ldots, a_m\}$, and let there be mappings $f : S \to \mathbb{N}$, and $g : \mathcal{B} \to \mathbb{N}$. We say that the family $\mathcal{B}$ has *an upper system of f, g-representatives* if to every set $B_j \in \mathcal{B}$, we associate at least $g(B_j)$ representatives from S, and every vertex $a_i \in S$ is a representative of at most $f(a_i)$ sets from $\mathcal{B}$. In this terminology, we infer from Theorem 20:

Corollary 22. *A family of sets $\mathcal{B}$ has an upper system of f, g-representatives if and only if for every sub family $Y \subseteq \mathcal{B}$ we have*

$$\sum_{a_i \in S} \min\{f(a_i), |Y(a_i)|\} \geq \sum_{B_j \in Y} g(B_j),$$

where $Y(a_i) = \{B_j \in Y : a_i \in B_j\}$ (i.e. $|Y(a_i)|$ is the number of sets from the family Y that contain a_i).

From the above corollaries, one can easily find formulations when one or both of the mappings f, g is not involved or is constant (say, equal to 1). The resulting formulations are mostly easier and nicer than the above and could also be applicable.

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References

[1] L. Lovász, M.D. Plummer, Matching Theory, Akademiai Klado, Budapest, 1986.
[2] P. Hall, On representatives of subsets, J. London. Math. Soc. 10 (1935) 26–30.
[3] L.R. Ford, D.R. Fulkerson, Flows in Networks, Princeton Univ. Press, Princeton NJ, 1962.
[4] R. Diestel, Graph Theory, fourth ed., in: Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Heidelberg, 2010.
[5] K. Benkič, Proposed use of a CDMA technique in wireless sensor networks. in: CD proceedings of 2007 14th International Workshop on Systems, Signals and Image Processing (IWSSIP) and 6th EURASIP Conference Focused on Speech & Image Processing, Multimedia Communications and Services (EC-SPIMCS), Faculty of Electrical Engineering and Computer Science, Maribor, Slovenia, 2007, pp. 1–6.
[6] B.H. Liu, N. Bulusu, H. Pham, S. Jha, CSMAC: a novel DS-CDMA based mac protocol for wireless sensor networks, Global Telecommunications Conference Workshops, IEEE, 2004.
[7] N.J.A. Harvey, R.E. Ladner, L. Lovász, T. Tamir, Semi-matchings for bipartite graphs and load balancing, Journal of Algorithms 59 (2006) 53–78.
[8] J. Fakcharoenphol, B. Laekhanukit, D. Nanongkai, Faster algorithms for semi-matching problems, presented at 37th International Colloquium on Automata, Languages and Programming, Bordeaux, France, 2010, arXiv:1004.3363/2010.
[9] J.N. Al-Karaki, A.E. Kamal, Routing techniques in wireless sensor networks: a survey, The Hashemite University, Iowa state university, IEEE Wireless Communications, 2004.
[10] M.X. Goemans, Minimum bounded-degree spanning trees, in: Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science, 2006, pp. 273–282.
[11] Y. Azar, On-line load balancing, in: A. Fiat, G. Woeginger (Eds.), Online Algorithms: The State of the Art, in: Lecture Notes in Comput. Sci., vol. 1442, Springer-Verlag, Berlin, 1998.
[12] D.B. West, Introduction to Graph Theory, Prentice Hall, New Jersey, 2001.