Boundary-Value Problems for Ordinary Differential Equations

- Consider a two-point boundary-value problems with the form of second-order differential equation

$$y'' = f(t, y, y'), \quad a \le t \le b \tag{1}$$

with the boundary conditions:

$$y(a) = \alpha, \ \ y(b) = \beta \tag{2}$$

Existence and Uniqueness

Theorem 1

- Suppose the function f in the boundary-value problem (1)、(2) is continuous on the set

  $$D = \{(x, y, y') | a \leq x \leq b, -\infty < y < \infty, -\infty < y' < \infty\},$$

  and that the partial derivatives f_y and $f_{y'}$ are also continuous on D.
- If
  - (i)$f_y(x, y, y') > 0$, for all $(x, y, y') \in D$
  - (ii) a constant M exists, with $|f_{y'}(x, y, y')| \leq M$, for all $(x, y, y') \in D$
- then the boundary-value problem has a unique solution.

Simple case: Linear boundary-value problems

- The linear boundary-value problem

$$\begin{cases} y'' = p(x)y' + q(x)y + r(x), \quad a \leq x \leq b; \\ y(a) = \alpha, y(b) = \beta. \end{cases} \tag{3}$$

- If $p(x)$, $q(x)$, and $r(x)$ are continuous on [a, b], and $q(x) > 0$ on $[a, b]$.
- Then the boundary-value problem (3) has a unique solution.

Solution:

To approximate the unique solution to this linear problem,

- First consider the initial-value problems

$$y'' = p(x)y' + q(x)y + r(x), \;\; a \leq x \leq b \tag{4}$$

  with $y(a) = \alpha$, and $y'(a) = 0$.

- Then consider the next initial value problem:

$$y'' = p(x)y' + q(x)y, \;\; a \leq x \leq b, \tag{5}$$

  with $y(a) = 0$, and $y'(a) = 1$.

- Both problems have a unique solution.

- Let $y_1(x)$ denote the solution to (4), and let $y_2(x)$ denote the solution to (5).
- Assume that $y_2(b) \neq 0$. Define

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x). \tag{6}$$

- Then $y(x)$ is the solution to the linear boundary problem (4).
- To see this, first note that

$$y'(x) = y_1'(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2'(x).$$

and

$$y''(x) = y_1''(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2''(x).$$

Substituting for $y_1''(x)$ and $y_2''(x)$ in this equation gives

$$\begin{aligned} y''(x) &= p(x)y_1' + q(x)y_1 + r(x) + \frac{\beta - y_1(b)}{y_2(b)}[p(x)y_2' + q(x)y_2] \\ &= p(x)\left[y_1' + \frac{\beta - y_1(b)}{y_2(b)}y_2'\right] + q(x)\left[y_1 + \frac{\beta - y_1(b)}{y_2(b)}y_2\right] + r(x) \\ &= p(x)y'(x) + q(x)y(x) + r(x). \end{aligned}$$

Moreover,

$$y(a) = y_1(a) + \frac{\beta - y_1(b)}{y_2(b)}y_2(a) = \alpha + \frac{\beta - y_1(b)}{y_2(b)} \cdot 0 = \alpha$$

and

$$y(b) = y_1(b) + \frac{\beta - y_1(b)}{y_2(b)}y_2(b) = y_1(b) + \beta - y_1(b) = \beta$$

Algorithm: Linear Shooting Method

- Uses the fourth-order Runge-Kutta technique to find the approximations to $y_1(x)$ and $y_2(x)$
- Other techniques for approximating the solutions to initial-value problems can be substituted into Step 4
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To approximate the solution of the boundary-value problem

$$-y'' + p(x)y' + q(x)y + r(x) = 0, a \le x \le b,$$

with $y(a) = \alpha$ and $y(b) = \beta$.

INPUT endpoints a, b; boundary conditions α, β; number of subintervals N.

OUTPUT approximations $w_{1,i}$ to $y(x_i)$; $w_{2,i}$ to $y'(x_i)$ for each $i = 0, 1, \cdots, N$.

Step 1 Set $h = (b - a)/N$;

$$\begin{aligned} u_{1,0} &= \alpha; \\ u_{2,0} &= 0; \\ v_{1,0} &= 0; \\ v_{2,0} &= 1. \end{aligned}$$

Step 2 For $i = 0, 1, \cdots, N-1$ do Steps 3 and 4. (The Runge-Kutta method for systems is used in Steps 3 and 4.)

Step 3 Set $x = a + ih$.

Step 4 Set

$$
\begin{aligned}
k_{1,1} &= hu_{2,i} \\
k_{1,2} &= h[p(x)u_{2,i} + q(x)u_{1,i} + r(x)]; \\
k_{2,1} &= h[u_{2,i} + \frac{1}{2}k_{1,2}]; \\
k_{2,2} &= h[p(x + \frac{h}{2})(u_{2,i} + \frac{1}{2}k_{1,2}) + q(x + \frac{h}{2})(u_{1,i} + \frac{1}{2}k_{1,1}) + r(x + \frac{h}{2})] \\
k_{3,1} &= h[u_{2,i} + \frac{1}{2}k_{2,2}]; \\
k_{3,2} &= h[p(x + \frac{h}{2})(u_{2,i} + \frac{1}{2}k_{2,2}) + q(x + \frac{h}{2})(u_{1,i} + \frac{1}{2}k_{2,1}) + r(x + \frac{h}{2})] \\
k_{4,1} &= h[u_{2,i} + k_{3,2}] \\
k_{4,2} &= h[p(x + h)(u_{2,i} + k_{3,2}) + q(x + h)(u_{1,i} + k_{3,1}) + r(x + h)] \\
u_{1,i+1} &= u_{1,i} + \frac{1}{6}[k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}]; \\
u_{2,i+1} &= u_{2,i} + \frac{1}{6}[k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}];
\end{aligned}
$$

Note: other techniques for approximating the solutions to initial-value problems can be substituted into Step 4

$$
\begin{aligned}
k'_{1,1} &= hv_{2,i} \\
k'_{1,2} &= h[p(x)v_{2,i} + q(x)v_{1,i}]; \\
k'_{2,1} &= h[v_{2,i} + \frac{1}{2}k'_{1,2}]; \\
k'_{2,2} &= h[p(x+\frac{h}{2})(v_{2,i} + \frac{1}{2}k'_{1,2}) + q(x+\frac{h}{2})(v_{1,i} + \frac{1}{2}k'_{1,1}))]; \\
k'_{3,1} &= h[v_{2,i} + \frac{1}{2}k'_{2,2}]; \\
k'_{3,2} &= h[p(x+\frac{h}{2})(v_{2,i} + \frac{1}{2}k'_{2,2}) + q(x+\frac{h}{2})(v_{1,i} + \frac{1}{2}k'_{2,1}))]; \\
k'_{4,1} &= h[v_{2,i} + k'_{3,2}] \\
k'_{4,2} &= h[p(x+h)(v_{2,i} + k'_{3,2}) + q(x+h)(v_{1,i} + k'_{3,1})]; \\
v_{1,i+1} &= v_{1,i} + \frac{1}{6}[k'_{1,1} + 2k'_{2,1} + 2k'_{3,1} + k'_{4,1}]; \\
v_{2,i+1} &= v_{2,i} + \frac{1}{6}[k'_{1,2} + 2k'_{2,2} + 2k'_{3,2} + k'_{4,2}];
\end{aligned}
$$

Step 5 Set $w_{1,0} = \alpha$; $w_{2,0} = \frac{\beta - u_{1,N}}{v_{1,N}}$;
OUTPUT $(a, w_{1,0}, w_{2,0})$.
Step 6 For $i = 1, \cdots, N$
set

$$\begin{aligned} W_1 &= u_{1,i} + w_{2,0} v_{1,i} \\ W_2 &= u_{2,i} + w_{2,0} v_{2,i}; \\ x &= a + ih; \end{aligned}$$

OUTPUT (x, W_1, W_2). (Output is $x_i, w_{1,i}, w_{2,i}$.)
Step 7 STOP. (The process is complete.)

Discrete Approximation

First, we select an integer $N > 0$ and divide the interval $[a, b]$ into $N + 1$ equal subintervals whose endpoints are the mesh points $x_i = a + ih$, for $i = 0, 1, \cdots, N$, where $h = (b - a)/N)$.

Choosing the step size h in this manner

Expanding y in a third Taylor polynomial about x_i evaluated at $x_{i+1} = x_i + h$ and $x_{i-1} = x_i - h$, we have, assuming that $y \in C^4[x_{i-1}, x_{i+1}]$

$$y(x_{i+1}) = y(x_i+h) = y(x_i)+hy'(x_i)+\frac{h^2}{2}y''(x_i)+\frac{h^3}{6}y'''(x_i)+\frac{h^4}{24}y^{(4)}(\xi_i^+)$$

for some $\xi_i^+ \in (x_i, x_{i+1})$, and

$$y(x_{i-1}) = y(x_i-h) = y(x_i)-hy'(x_i)+\frac{h^2}{2}y''(x_i)-\frac{h^3}{6}y'''(x_i)+\frac{h^4}{24}y^{(4)}(\xi_i^-)$$

for some $\xi_i^- \in (x_{i-1}, x_i)$.

If these equations are added, we have

$$y(x_{i+1})+y(x_{i-1}) = 2y(x_i)+h^2y''(x_i)+\frac{h^4}{24}[y^{(4)}(\xi_i^+)+y^{(4)}(\xi_i^+)],$$

and solving for $y''(x_i)$ gives

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{24}[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^+)]$$

The Intermediate Value Theorem 1.11 can be used to simplify the error term to give

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12}y^{(4)}(\xi_i)$$

for some $\xi_i \in [x_{i-1}, x_{i+1}]$.

This is called the centered-difference formula for $y''(x_i)$.

A centered-difference formula for $y'(x_i)$ is obtained in a similar manner

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6} y'''(\eta_i)$$

for some $\eta_i \in [x_{i-1}, x_{i+1}]$.

Forward difference formula is

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_i)}{h} - \frac{h}{2} y''(\zeta_i),$$

for some $\zeta_i \in [x_i, x_{i+1}]$

Backward difference formula: Forward difference formula is

$$y'(x_i) = \frac{y(x_i) - y(x_{i-1})}{h} + \frac{h}{2} y''(\mu_i),$$

for some $\mu_i \in [x_{i-1}, x_i]$.

The use of these centered-difference formulas in Eq. (3) results in the equation

$$\begin{aligned}\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} \quad = \quad & p(x_i)\frac{y(x_{i+1}) - y(x_{i-1})}{2h} + q(x_i)y(x_i) + r(x_i) \\ & -\frac{h^2}{12}[2p(x_i)y'''(\eta_i) - y^{(4)}(\xi_i)]\end{aligned}$$

A Finite-Difference method with truncation error of order $O(h^2)$ results by using this equation together with the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ to define the system of linear equations

$$\begin{cases} y_0 = \alpha, \ \ y_N = \beta; \\ \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - p(x_i)\frac{y_{i+1} - y_{i-1}}{2h} - q(x_i)y(x_i) = r(x_i), \end{cases} \tag{7}$$

for $i = 1, 2, \cdots, N-1$.

Rewritten (7)

$$\begin{aligned} y_0 &= \alpha,\ y_N = \beta; \\ (1+\frac{h}{2}p(x_i))y_{i-1} &- (2+q(x_i)h^2)y_i + (1-\frac{h}{2}p(x_i))y_{i+1} = h^2 r(x_i) \end{aligned}$$

for $i = 1, 2, \cdots, N-1$.

Write in linear equations as

$$\mathbf{Ay} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} -(2+q(x_1)h^2) & (1-\frac{h}{2}p(x_1)) & 0 & \cdots & 0 \\ (1+\frac{h}{2}p(x_2)) & -(2+q(x_2)h^2) & (1-\frac{h}{2}p(x_1)) & \ddots & \vdots \\ 0 & \ddots & & & \vdots \\ \vdots & & & \ddots & (1-\frac{h}{2}p(x_{N-2})) \\ 0 & 0 & 0 & (1+\frac{h}{2}p(x_{N-1})) & -(2+q(x_{N-1})h^2) \end{pmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{N-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} h^2 r(x_1) - (1 + \frac{h}{2} p(x_1)) y_0 \\ h^2 r(x_2) \\ \vdots \\ h^2 r(x_{N-2}) \\ h^2 r(x_{N-1}) - (1 - \frac{h}{2} p(x_{N-1})) y_N \end{bmatrix}$$