Chapter 8 Approximation Theory

刘保东
山东大学，计算机科学与技术学院
Email:baodong@sdu.edu.cn

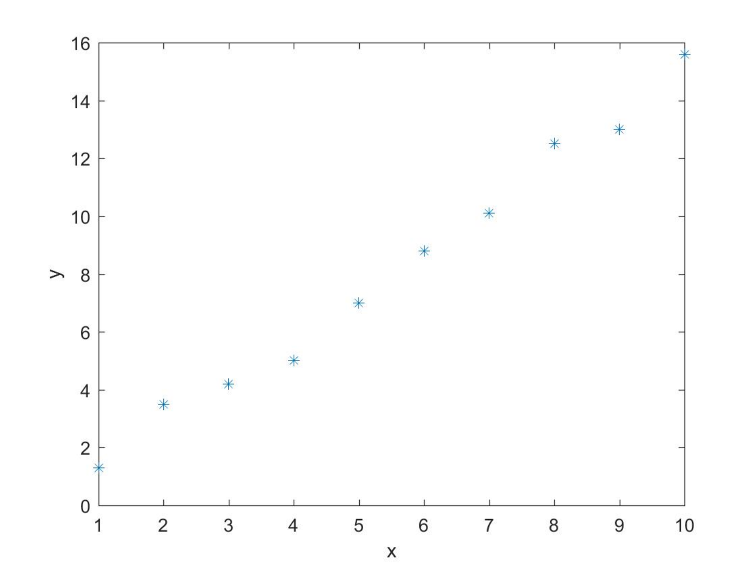
In this chapter, we will introduce the approximation theory involves **two general types of problems**.

- Approximation Problem of a Function: to find a 'simple' type of function, such as polynomial, that can be used to determine approximate values of the given functions.
- Curve Fitting Problem: to find the **"best" function** in a certain class to fit given data.

Example 1: Consider the problem of estimating the values of a function at nontabulated points, given the experimental data in Table 8.1.

x_i	y_i	x_i	y_i
1	1.3	6	8.8
2	3.5	7	10.1
3	4.2	8	12.5
4	5.0	9	13.0
5	7.0	10	15.6

Using these given data to make a graph, to view the relationship between x and y. It seems to be linear.

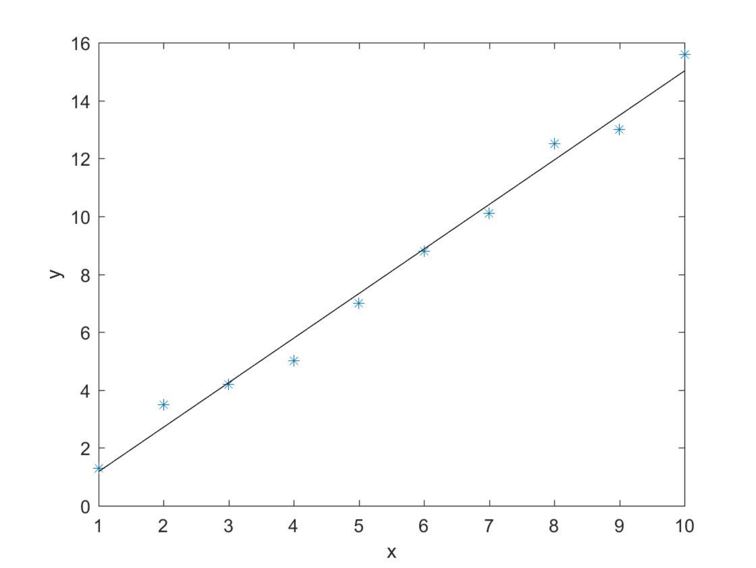


16
14
12
10
8
6
4
2
0
y
1
2
3
4
5
6
7
8
9
10
x

We fit these data with the linear polynomial, by using Matlab7.0 commands:

```
x=[1,2,3,4,5,6,7,8,9,10];
y=[1.3,3.5,4.2,5.0,7.0,8.8,10.1,12.5,13.0,15.6];
z=polyfit(x,y,1)
z=
1.5382 -0.3600
Y=1.5382*x-0.36;
plot(x,y,'*',x,Y,'r')
```

The result can be seen in the following graph.



16
14
12
10
8
6
4
2
0
y
1
2
3
4
5
6
7
8
9
10
x

Let $a_1 x_i + a_0$ denote the ith value on the approximating line and y_i be the ith given y-value.

I. Minimax Rule

- The problem of finding the equation of the best linear approximation in the absolute sense requires that values of a_0 and a_1 be found to minimize

$$E_1 = \min_{a_0, a_1} \max_{1 \leq i \leq 10} \{|y_i - (a_1 x_1 + a_0)|\}.$$

- This is commonly called a **minimax** problem and cannot be handled by elementary techniques.

II.Absolute Deviation Rule

- Another approach to determining the best linear approximation involves finding Values of a_0 and a_1 to minimize

$$E_2 = \min_{a_0, a_1} \sum_{i=1}^{10} |y_i - (a_1 x_i + a_0)|.$$

- This quantity is called the **absolute deviation**.

- To minimize this function of two variables, we need to set its partial derivatives to zero.
- That is we need to find a_0 and a_1 with

$$0 = \frac{\partial}{\partial a_0} \sum_{i=1}^{10} |y_i - (a_1 x_i + a_0)|,$$

and

$$0 = \frac{\partial}{\partial a_1} \sum_{i=1}^{10} |y_i - (a_1 x_i + a_0)|.$$

- The difficulty is that the absolute-value function is not differentiable at zero, and we may not be able to find solutions to this pair of equations.

III. Least Square Rule

- The least squares approach to this problem involves determining the best approximating line when the error involved is the sum of the squares of the differences between the y-values on the approximating line and the given y-values.
- Hence, constants a_0 and a_1 must be found that minimize the least squares error:

$$E = \min_{a_0, a_1} \sum_{i=1}^{10} [y_i - (a_1 x_i + a_0)]^2$$

直线拟合的一般形式

- The least squares method is the most convenient procedure for determining best linear approximations, but there are also important theoretical considerations that favor it.
- The **general problem of fitting the best least squares line** to a collection of data $\{(x_i, y_i)\}_{i=1}^m$ involves minimizing the total error,

$$E \equiv \min_{a_0, a_1} \sum_{i=1}^{m} [y_i - (a_1 x_i + a_0)]^2$$

- For a minimum to occur, we need

$$0 = \frac{\partial}{\partial a_0} \sum_{i=1}^{m} [y_i - (a_1 x_i + a_0))]^2 = \sum_{i=1}^{m} 2(y_i - a_1 x_i - a_0)(-1),$$

and

$$0 = \frac{\partial}{\partial a_1} \sum_{i=1}^{m} [y_i - (a_1 x_i + a_0))]^2 = \sum_{i=1}^{m} 2(y_i - a_1 x_i - a_0)(-x_i).$$

- These equations simplify to the **normal equations**:

$$\begin{cases} a_0 \cdot m + a_1 \sum_{i=1}^{m} x_i & = \sum_{i=1}^{m} y_i \\ a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 & = \sum_{i=1}^{m} x_i y_i . \end{cases}$$

To solve the equations, we get the solution

$$a_0 = \frac{\sum_{i=1}^{m} x_i^2 \sum_{i=1}^{m} y_i - \sum_{i=1}^{m} x_i y_i \sum_{i=1}^{m} x_i}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}$$

and

$$a_1 = \frac{m \sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}$$

一般最小二乘多项式拟合形式

The General Form of Discrete Least Square Rule

- The general problem of approximating a set of data:

$$\{(x_i, y_i) | i = 1, 2, \cdots, m\},$$

  with an algebraic polynomial

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  of degree $n < m - 1$.

- The General Form of Discrete Least Square Rule:

$$\begin{aligned} \min_{a_0, a_1, \cdots, a_n} E &= \sum_{i=1}^{m} (y_i - P_n(x_i))^2 \\ &= \sum_{i=1}^{m} (y_i - \sum_{k=0}^{n} a_k x_i^k)^2. \end{aligned}$$

- To find the suitable parameters $a_0, a_1, \cdots, a_n$, such that E gets to be minimized.
- Let

$$0 = \frac{\partial E}{\partial a_j} = -2\sum_{i=1}^{m} y_i x_i^j + 2\sum_{k=0}^{n} a_k \sum_{i=1}^{m} x_i^{j+k}.$$

for each $j = 0, 1, \cdots, n$.

- This gives $n+1$ **normal equations** in the $n+1$ unknown parameters $a_j, j = 0, 1, , \cdots, n$.

$$\sum_{k=0}^{n} a_k \sum_{i=1}^{m} x_i^{j+k} = \sum_{i=1}^{m} y_i x_i^j,$$

for each $j = 0, 1, \cdots, n$.

- Let

$$\mathbf{R} = \begin{bmatrix} x_1^n & x_1^{n-1} & \cdots & x_1 & 1 \\ x_2^n & x_2^{n-1} & \ddots & x_2 & 1 \\ x_3^n & x_3^{n-1} & \ddots & x_3 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_m^n & x_m^{n-1} & \cdots & x_m & 1 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{bmatrix}$$

- Then the above normal equations can be written as

$$\mathbf{R}^T\mathbf{R}\mathbf{a} = \mathbf{R}^T\mathbf{y}.$$

- **Note that:** These normal equations have a unique solution provided that the x_i are distinct.

(1) 幂函数: $y = \alpha x^{\beta}$ 可化为

$$\ln y = \ln \alpha + \beta \ln x.$$

(2) 指数曲线: $y = \alpha e^{\beta x}$ 可化为

$$\ln y = \ln \alpha + \beta x.$$

(3) 对数曲线: $y = \ln bx$ 可化为

$$e^{y} = bx.$$

(4) 双曲线(单支): $y = \frac{a}{x} + b$ 可化为

$$y = a\frac{1}{x} + b.$$

8.2 Orthogonal Polynomials and Least Square Approximation–正交多项式及其最小二乘逼近

- Suppose $f \in C[a,b]$ and $P_n(x)$ is a polynomial of degree at most n with form:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k.$$

- To determine a least squares approximating polynomial $P_n(x)$, define

$$\begin{aligned} E &\equiv E(a_0, a_1, \cdots, a_n) = \int_a^b [f(x) - P_n(x)]^2 \mathrm{d}x \\ &= \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k \right)^2 \mathrm{d}x. \end{aligned}$$

- Finding real coefficients $a_0, a_1, \cdots, a_n$ so that

$$\min_{a_0,a_1,\cdots,a_n} E(a_0,a_1,\cdots,a_n) = \int_a^b [f(x)-P_n(x)]^2 \mathrm{d}x$$

$$= \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k\right)^2 \mathrm{d}x.$$

$$= \int_a^b [f(x)]^2 \mathrm{d}x - 2\sum_{k=0}^{n} a_k \int_a^b x^k f(x)\mathrm{d}x + \int_a^b \left(\sum_{k=0}^{n} a_k x^k\right)^2 \mathrm{d}x,$$

- Let

$$\frac{\partial E}{\partial a_j} = 0, \quad j = 0, 1, \cdots, n.$$

- we have normal equations for $a_0, a_1, \cdots, a_n$:

$$\frac{\partial E}{\partial a_j} = -2\int_a^b x^j f(x)\mathrm{d}x + 2\sum_{k=0}^{n} a_k \int_a^b x^{j+k}\mathrm{d}x, \; j = 0, 1, \cdots, n.$$

- To find $P_n(x)$, the $(n+1)$ linear **normal equations**

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} \mathrm{d}x = \int_a^b x^j f(x) \mathrm{d}x, \quad j = 0, 1, \cdots, n.$$

- Rewrite it in linear system of equations

$$\begin{aligned}
a_0 \int_a^b 1\,dx + a_1 \int_a^b x dx + \cdots + a_n \int_a^b x^n\,dx &= \int_a^b f(x)\,dx \\
a_0 \int_a^b x dx + a_1 \int_a^b x^2\,dx + \cdots + a_n \int_a^b x^{n+1}\,dx &= \int_a^b xf(x)\,dx \\
&\vdots \\
a_0 \int_a^b x^n\,dx + a_1 \int_a^b x^{n+1}\,dx + \cdots + a_n \int_a^b x^{2n}\,dx &= \int_a^b x^n f(x)\,dx
\end{aligned}$$

- **Note that:** The normal equations always have a unique solution provided $f \in C[a, b]$.

- The coefficients in the linear system are of the form

$$\int_a^b x^{j+k}\,dx = \frac{b^{j+k+1} - a^{j+k+1}}{j+k+1},$$

for each $j, k = 0, 1, 2, \cdots, n$ and in the right side are of the form

$$\int_a^b x^j f(x)\,dx, \texttt{for}\ \ j = 0, 1, 2, \cdots, n.$$

- The matrix in the linear system is known as a **Hilbert matrix**.

Remarks:

1 The linear system does not have an easily computed numerical solution.

2 The calculations that were performed in obtaining the best nth-degree polynomial, $P_n(x)$, do not lessen the amount of work required to obtain $P_{n+1}(x)$, the polynomial of next higher degree.

- To consider the computationally efficiency, a different technique of least squares approximations will now be considered.
- To facilitate the discussion, we need some new concepts.

Definition 8.1

- The set of functions $\{\phi_0, \phi_1, \cdots, \phi_n\}$ is said to be **linearly independent** on $[a, b]$ if, whenever

$$c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0,$$

for all $x \in [a, b]$, we have $c_0 = c_1 = \cdots = c_n = 0$.
- Otherwise the set of functions is said to be **linearly dependent**.

Theorem 8.2

If $\phi_j(x)$ is a polynomial of degree j, for each $j = 0, 1, \cdots, n$, then $\{\phi_0, \phi_1, \cdots, \phi_n\}$ is linearly independent on any interval $[a, b]$.

Proof:

- Suppose $c_0, c_1, \cdots, c_n$ are real numbers for which

$$P(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0,$$

  for all $x \in [a, b]$.
- The polynomial $P(x)$ vanishes on $[a, b]$, so it must be the zero polynomial, and the coefficients of all the powers of x are zero.
- particular, the coefficient of x^n is zero.

- Since $c_n\phi_n(x)$ is the only term in $P(x)$ that contains x_n, we must have $c_n = 0$ and

$$P(x) = \sum_{j=0}^{n-1} c_j\phi_j(x).$$

- With same idea above, since the only term that contains a power of x^{n-1} is $c_{n-1}\phi_{n-1}(x)$, so this term must also be zero and

$$P(x) = \sum_{j=0}^{n-2} c_j\phi_j(x).$$

- With a similar manner, the remaining constants $c_{n-2}, c_{n-3}, \cdots, c_0$ are all zero, which implies that $\{\phi_0, \phi_1, \cdots, \phi_n\}$ is linearly independent.$\blacksquare\blacksquare$

- **Notation:** Let Π_n be the **set of all polynomials of degree at most** n.
- **Theorem 8.3:** If $\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$ is a collection of linearly independent polynomials in Π_n, then any polynomial in Π_n can be written uniquely as a linear combination of $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$. $\blacksquare$

Definition 8.4

An integrable function w is called a **weight function** on the interval I, if $w(x) \geq 0$, for all $x \in I$, but $w(x) \neq 0$ on any subinterval of I.

Suppose $\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$ is a set of linearly independent functions on $[a, b]$, $w(x)$ is a weight function for $[a, b]$, and, for $f \in C[a, b]$, a linear combination

$$P(x) = \sum_{k=0}^{n} a_k \phi_k(x).$$

is sought to minimize the error

$$\begin{aligned} & E(a_0, a_1, \cdots, a_n) \\ & = \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x) \right]^2 dx. \end{aligned} \tag{1}$$

- This problem reduces to the situation considered at the beginning of this section in the special case when $w(x) \equiv 1$ and $\phi_k(x) = x^k$, for each $k = 0, 1, \cdots, n$.
- The **normal equations** associated with this problem are derived from the fact that for each $j = 0, 1, \cdots, n$,

$$0 = \frac{\partial E}{\partial a_j} = 2 \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x) \right] \phi_j(x) dx.$$

- The system of normal equations can be written

$$\sum_{k=0}^{n} a_k \int_a^b w(x) \phi_k(x) \phi_j(x) dx = \int_a^b w(x) f(x) \phi_j(x) dx$$

for each $j = 0, 1, \cdots, n$.

We rewrite it as linear system form:

$$
\begin{aligned}
&a_0 \int_a^b w(x)\phi_0(x)\phi_0(x)\,dx + a_1 \int_a^b w(x)\phi_0(x)\phi_1(x)\,dx + \dots \\
&+a_n \int_a^b w(x)\phi_0(x)\phi_n(x)\,dx = \int_a^b w(x)f(x)\phi_0(x)\,dx \\
&a_0 \int_a^b w(x)\phi_1(x)\phi_0(x)\,dx + a_1 \int_a^b w(x)\phi_1(x)\phi_1(x)\,dx + \dots \\
&+a_n \int_a^b w(x)\phi_1(x)\phi_n(x)\,dx = \int_a^b w(x)f(x)\phi_n(x)\,dx \\
&\dots\dots \\
&a_0 \int_a^b w(x)\phi_n(x)\phi_0(x)\,dx + a_1 \int_a^b w(x)\phi_n(x)\phi_1(x)\,dx + \dots \\
&+a_n \int_a^b w(x)\phi_n(x)\phi_n(x)\,dx = \int_a^b w(x)f(x)\phi_n(x)\,dx
\end{aligned}
$$

If the functions $\phi_0, \phi_1, \cdots, \phi_n$ can be chosen so that

$$\int_a^b w(x)\phi_k(x)\phi_j(x)\,dx = \begin{cases} 0, & \text{when } j \neq k; \\ \alpha_j > 0, & \text{when } j = k. \end{cases} \tag{2}$$

then the normal equations reduce to

$$\int_a^b w(x)f(x)\phi_j(x)\,dx = a_j \int_a^b w(x)[\phi_j(x)]^2\,dx = a_j\alpha_j$$

for each $j = 0, 1, \cdots, n$, and easily solved to give

$$a_j = \frac{1}{\alpha_j}\int_a^b w(x)f(x)\phi_j(x)\,dx$$

Definition 8.5

$\phi_0, \phi_1, \cdots, \phi_n$ is said to be an **orthogonal set of functions** for the interval $[a, b]$ with respect to the weight function w if

$$\int_a^b w(x)\phi_j(x)\phi_k(x)\,dx = \begin{cases} 0, & \text{when } j \neq k; \\ \alpha_k > 0, & \text{when } j = k. \end{cases}$$

If, in addition, $\alpha_k = 1$ for each $k = 0, 1, 2, \cdots, n$, the set is said to be **orthonormal.**

Theorem 8.6

If $\phi_0, \phi_1, \cdots, \phi_n$ is an orthogonal set of functions on an interval $[a, b]$ with respect to the weight function w, then the least squares approximation to f on $[a, b]$ with respect to w is

$$P(x) = \sum_{k=0}^{n} a_k \phi_k(x).$$

where for each $k = 0, 1, 2, \cdots, n$,

$$a_k = \frac{\int_a^b w(x)\phi_k(x) f(x)\,dx}{\int_a^b w(x)[\phi_k(x)]^2\,dx} = \frac{1}{\alpha_k}\int_a^b w(x)\phi_k(x) f(x)\,dx.$$

Theorem 8.7 (Gram-Schmidt Orthogonalize Process)

The set of polynomial functions $\{\phi_0, \phi_1, \cdots, \phi_n\}$ defined in the following way is orthogonal on $[a, b]$ with respect to the weight function w.

$$\phi_0(x) = 1, \phi_1(x) = x - B_1, \texttt{for each } x \texttt{ in } [a, b],$$

where

$$B_1 = \frac{\int_a^b xw(x)[\phi_0(x)]^2\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$

and when $k \geq 2$,

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x), \texttt{for each } x \texttt{ in } [a, b],$$

where

$$B_k = \frac{\int_a^b xw(x)[\phi_{k-1}(x)]^2\,dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2\,dx}$$

Theorem 8.7 is the known Gram-Schmidt process, it gives a method that how to construct orthogonal polynomials on $[a, b]$ with respect to a weight function w.

Corollary 8.8

For any $n > 0$, the set of polynomial functions $\{\phi_0, \phi_1, \cdots, \phi_n\}$ given in Theorem 8.7 is linearly independent on $[a, b]$ and

$$\int_a^b w(x)\phi_n(x)Q_k(x)dx = 0,$$

for any polynomial $Q_k(x)$ of degree $k < n$.

Proof:

- Since $\phi_n(x)$ is a polynomial of degree n, Theorem 8.2 implies that $\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$ is a linearly independent set.
- Let $Q_k(x)$ be a polynomial of degree k. By Theorem 8.3 there exist numbers $c_0, c_1, \cdots, c_k$ such that

$$Q_k(x) = \sum_{j=0}^{k} c_j \phi_j(x).$$

Proof:

Thus,

$$\int_a^b w(x)Q_k(x)\phi_n(x)dx = \sum_{j=0}^{k} c_j \int_a^b w(x)\phi_j(x)\phi_n(x)dx$$
$$= \sum_{j=0}^{k} c_j \cdot 0 = 0,$$

Since $\phi_n(x)$ is orthogonal to $\phi_j(x)$ for each $j = 0, 1, \cdots, k$.■■■

Example:

The set of **Legendre Polynomial** on [-1,1] with respect to weight function $w(x) = 1$.

Using the method given in theorem 8.7, we can easily give the set of Legendre Polynomial:

$$
\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= x^2 - \frac{1}{3}, \\
P_3(x) &= x^3 - \frac{3}{5}x \\
&\vdots
\end{aligned}
$$

Note that: the Legendre Polynomials were ever mentioned in section 4.7, where their roots were used as the nodes in Gaussian Quadrature.

8.3 Chebyshev Polynomials and Economization(压缩) of Power Series

- In this section, we will introduce the set of Chebyshev Polynomials $\{T_n(x) = \cos[n \arccos x]\}$ in [-1,1] for each $n > 0$.
- First we show that $T_n(x)$ is a polynomial in x.
- We note that by definition

$$T_0(x) = \cos 0 = 1, \text{ and}$$

$$T_1(x) = \cos[\arccos x] = x.$$

- When $n > 1$, we introduce the substitution $\theta = \arccos x$ to change this equation to

$$T_n(\theta(x)) = T_n(\theta) = \cos(n\theta), \texttt{where} \quad \theta \in [0, \pi].$$

- A recurrence relation is derived by noting that

$$T_{n+1}(\theta)) = cos[(n+1)\theta] = \cos(n\theta)\cos\theta - \sin(n\theta) sin\theta$$

and

$$T_{n-1}(\theta) = cos[(n-1)\theta] = \cos(n\theta)\cos\theta + sin(n\theta) sin\theta.$$

- Adding these equations gives

$$T_{n+1}(\theta) = 2\cos(n\theta) cos\theta - T_{n-1}(\theta).$$

- Note that
$$\cos\theta = \cos(\arccos x) = x,$$
and
$$\cos(n\theta) = \cos(n\arccos x) = T_n(x),$$
- so we have for each $n \geq 1$,
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$
- Since $T_0(x) = 1$, $T_1(x) = x$, the recurrence relation implies that $T_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1}, when $n \geq 1$.

The Chebyshev polynomials are

$$
\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_2(x) &= 2xT_1(x) - T_0(x) = 2x^2 - 1, \\
T_3(x) &= 2xT2(x) - T1(x) = 4x^3 - 3x, \\
T_4(x) &= 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1. \\
&\vdots
\end{aligned}
$$

- Second, we show the orthogonality of the Chebyshev polynomials with respect to the weight function $w(x) = (1-x^2)^{-1/2}$.
- Considering

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{\cos(n \arccos x)\cos(m \arccos x)}{\sqrt{1-x^2}}\,dx$$

- Reintroducing the substitution $\theta = \arccos x$ gives

$$d\theta = -\frac{1}{\sqrt{1-x^2}}\,dx$$

and

$$\begin{aligned}\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx &= -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta)\,d\theta \\ &= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta\end{aligned}$$

Suppose $n \neq m$. Since

$$\cos(n\theta)\cos(m\theta) = \frac{1}{2}[\cos(n+m)\theta + \cos(n-m)\theta],$$

we have

$$\begin{aligned}
&\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx \\
&= \frac{1}{2}\int_0^{\pi} \cos(n+m)\theta\, d\theta + \frac{1}{2}\int_0^{\pi} \cos(n-m)\theta\, d\theta \\
&= \left[\frac{1}{2(n+m)}\sin(n+m)\theta + \frac{1}{2(n-m)}\sin(n-m)\theta\right]_0^{\pi} \\
&= 0
\end{aligned}$$

By a similar technique, it can be shown that when $n = m$,

$$\int_{-1}^{1} \frac{[T_n(x)]^2}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}, \texttt{for each } \; n \geq 1$$

Remarks:

The Chebyshev polynomials are used to minimize approximation error. We will see how they are used to solve two problems of this type:

1. An optimal placing of interpolating points to minimize the error in Lagrange interpolation;

2. A means of reducing the degree of an approximating polynomial with minimal loss of accuracy.

Theorem 8.9

- The Chebyshev polynomial $T_n(x)$ of degree $n \geq 1$ has n simple zeros in [-1, 1] at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \texttt{for each} \quad k = 1, 2, \cdots, n.$$

- Moreover, $T_n(x)$ assumes its absolute extrema(极值) at

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right) \texttt{ with } \quad T_n(\bar{x}'_k) = (-1)^k,$$

for each $k = 0, 1, \cdots, n$.

Proof:

If we let

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \texttt{for each } k = 1, 2, \cdots, n.$$

then

$$\begin{aligned} Tn(\bar{x}_k) &= \cos(n \arccos \bar{x}_k) \\ &= \cos\left(n \arccos\left(\cos\left(\frac{2k-1}{2n}\pi\right)\right)\right) \\ &= \cos\left(\frac{2k-1}{2}\pi\right) \\ &= 0, \end{aligned}$$

and each $\bar{x}_k$ is a distinct zero of T_n.

Since $T_n(x)$ is a polynomial of degree n, all zeros of $T_n(x)$ must be of this form.

To show the second part, first note that

$$T_n'(x) = \frac{d}{dx}[\cos(n \arccos x)] = \frac{n \sin(n \arccos x)}{\sqrt{1-x^2}},$$

and that, when $k = 1, 2, \cdots, n-1$.

$$\begin{aligned} T_n'(\bar{x}_k') &= \frac{n \sin\left(n \arccos\left(\cos\left(\frac{k\pi}{n}\right)\right)\right)}{\sqrt{1 - \left[\cos\left(\frac{k\pi}{n}\right)\right]^2}} \\ &= \frac{n \sin(k\pi)}{\sin\left(\frac{k\pi}{n}\right)} = 0 \end{aligned}$$

- Since $T_n(x)$ is a polynomial of degree n, its derivative $T_n'(x)$ is a polynomial of degree $(n-1)$, and all the zeros of $T_n'(x)$ occur at these $n-1$ points.
- The only other possibilities for extrema of $T_n(x)$ occur at the endpoints of the interval [-1, 1]; that is, at $\bar{x}_n' = 1$ and at $\bar{x}_0' = -1$.
- Since for any $k = 0, 1, \cdots, n$, we have

$$\begin{aligned} T_n(\bar{x}_k') &= \cos\left(n \arccos\left(\cos\left(\frac{k\pi}{n}\right)\right)\right) \\ &= \cos(k\pi) = (-1)^k, \end{aligned}$$

  a maximum occurs at each even value of k and a minimum at each odd value.■■■

The Monic(首项系数为1) Chebyshev Polynomial

- The monic polynomials are the ones with leading coefficient 1
- The monic Chebyshev polynomials $\tilde{T}_n(x)$ are derived from the Chebyshev polynomial $T_n(x)$ by dividing by the leading coefficient 2^{n-1}.
- Hence,

$$\tilde{T}_0(x) = 1 \quad \text{and} \quad \tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x),$$

for each $n \geq 1$

.

The recurrence relationship satisfied by the Chebyshev polynomials implies that

$$\begin{aligned}
\tilde{T}_0(x) &= 1, \\
\tilde{T}_1(x) &= \frac{1}{2^0} T_1(x) = x, \\
\tilde{T}_2(x) &= x\,\tilde{T}_1(x) - \frac{1}{2}\,\tilde{T}_0(x) \\
\tilde{T}_{n+1}(x) &= x\,\tilde{T}_n(x) - \frac{1}{4}\,\tilde{T}_{n-1}(x), n \geq 2
\end{aligned}$$

Properties of $\tilde{T}_n(x)$:

1. The zeros of $\tilde{T}_n(x)$ occur at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \texttt{for each} \quad k = 1, 2, \cdots, n.$$

2. The extreme values of $\tilde{T}_n(x)$, for $n \geq 1$, occur at

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right), \texttt{with} \quad \tilde{T}_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}} \tag{3}$$

for each $k = 0, 1, 2, \cdots, n$.

Let $\tilde{\prod}_n$ denote the **set of all monic polynomials of degree** n.

The relation expressed in Eq. (3) leads to an important minimization property that distinguishes $\tilde{T}_n(x)$ from the other members of $\tilde{\prod}_n$.

Theorem 8.10

The polynomials of the form $\tilde{T}_n(x)$, when $n \geq 1$, have the property that

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\tilde{T}_n(x)| \leq \max_{x\in[-1,1]} |P_n(x)|,$$

for all $P_n(x) \in \tilde{\prod}_n$.
Moreover, equality can occur only if $P_n \equiv \tilde{T}_n$.

Proof:

Suppose that $P_n(x) \in \tilde{\prod}_n$ and

$$\max_{x\in[-1,1]} |P_n(x)| \leq \frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\tilde{T}_n(x)|.$$

Let $Q = \tilde{T}_n - P_n$. Since $\tilde{T}_n(x)$ and $P_n(x)$ are both monic polynomials of degree $n, Q(x)$ is a polynomial of degree at most $(n-1)$. Moreover, at the extreme points of $\tilde{T}_n(x)$,

$$Q(\bar{x}_k') = \tilde{T}_n(\bar{x}_k') - P_n(\bar{x}_k') = \frac{(-1)^k}{2^{n-1}} - P_n(\bar{x}_k').$$

Since

$$|P_n(\bar{x}'_k)| \le \frac{1}{2^{n-1}}, \texttt{for each } \; k = 0, 1, \cdots, n$$

we have

$$Q(\bar{x}'_k) \le 0, \texttt{when } k \texttt{ is odd}$$

and

$$Q(\bar{x}'_k) \ge 0, \texttt{when } \; k \; \texttt{ is even.}$$

Since Q is continuous, the Intermediate Value Theorem implies that $Q(x)$ has at least one zero between $\bar{x}'_j$ and $\bar{x}'_{j+1}$, for each $j = 0, 1, \cdots, n-1$. Thus Q has at least n zeros in the interval [-1,1]. But the degree of $Q(x)$ is less than n, so $Q \equiv 0$, this implies that $P_n \equiv \tilde{T}_n$. ■■■

Application I. Error Estimation for Lagrange Interpolation

Suppose that $x_0, x_1, x_2, \cdots, x_n$ are distinct points in the interval [-1,1], and $P(x)$ is the Lagrange interpolating polynomial of degree n, if $f \in C^{n+1}[-1,1]$, then, for each $x \in [-1,1]$, a number $\xi(x)$ exists in (-1,1) with

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

Generally there is no control over $\xi(x)$, so to minimize the error by shrewed placement of nodes $x_0, x_1, \cdots, x_n$, we find $x_0, x_1, \cdots, x_n$ to minimize the quantity

$$|(x - x_0)(x - x_1) \cdots (x - x_n)|$$

throughout the interval [-1,1].

Since $(x - x_0)(x - x_1) \cdots (x - x_n)$ is a monic polynomial of degree $n + 1$, we have just seen that the minimum is obtained when

$$(x - x_0)(x - x_1) \cdots (x - x_n) = \tilde{T}_{n+1}(x).$$

The maximum value of $|(x-x_0)(x-x_1)\cdots(x-x_n)|$ is smallest when x_k is chosen to be the $(k+1)$st zeros of $\tilde{T}_{n+1}$, for each $k=0,1,\cdots,n$; that is, when x_k is

$$\bar{x}_{k+1}=\cos\left(\frac{2k+1}{2(n+1)}\pi\right).$$

Since $\max_{x\in[-1,1]}|\tilde{T}_{n+1}(x)|=2^{-n}$, this also implies that

$$\begin{aligned}\frac{1}{2^n} &= \max_{x\in[-1,1]}|(x-\bar{x}_1)(x-\bar{x}_2)\cdots(x-\bar{x}_{n+1})| \\ &\leq \max_{x\in[-1,1]}|(x-x_0)(x-x_1)\cdots(x-x_n)|,\end{aligned}$$

for any choice of $x_0,x_1,\cdots,x_n$ in the interval [-1,1].

Application II.

To Reduce the Degree of an Approximating Polynomial with a Minimal Loss of Accuracy.

Because the Chebyshev polynomials have a minimum maximum-absolute value that is spread uniformly on an interval, they can be used to reduce the degree of an approximation polynomial without exceeding the error tolerance.

Consider approximating an arbitrary nth-degree polynomial

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

on $[-1, 1]$ with a polynomial of degree at most $n-1$.

The object is to choose $P_{n-1}(x)$ in $\prod_{n-1}$, so that

$$\max_{x\in[-1,1]} |P_n(x) - P_{n-1}(x)|$$

is as small as possible.

We first note that $(P_n(x) - P_{n-1}(x))/a_n$ is a monic polynomial of degree n, so applying Theorem 8.10 gives

$$\max_{x\in[-1,1]} |\frac{1}{a_n}(P_n(x) - P_{n-1}(x))| \geq \frac{1}{2^{n-1}}.$$

Equality occurs precisely when

$$\frac{1}{a_n}(P_n(x) - P_{n-1}(x)) = \tilde{T}_n(x).$$

This means that we should choose

$$P_{n-1}(x) = P_n(x) - a_n \tilde{T}_n(x),$$

and with this choice we have the minimum value of

$$\begin{aligned}
&\max_{x\in[-1,1]} |(P_n(x) - P_{n-1}(x))| \\
&= |a_n| \max_{x\in[-1,1]} |\frac{1}{a_n}(P_n(x) - P_{n-1}(x))| \\
&= \frac{|a_n|}{2^{n-1}}.
\end{aligned}$$

Corollary 8.11

If $P(x)$ is the interpolating polynomial of degree at most n with nodes at the roots of $T_n(x)$, then

$$\max_{x\in[-1,1]} |f(x) - P(x)| \leq \frac{1}{2^n(n+1)!} \max_{x\in[-1,1]} |f^{(n+1)}(x)|,$$

for each $f \in C^{n+1}[-1,1]$.

Notes:

For the case of a general closed interval $[a, b]$, we can use the change of variables

$$\tilde{x} = \frac{1}{2}[(b-a)x + a + b]$$

to transform the numbers $\bar{x}_k$ in the interval $[-1, 1]$ into the corresponding number $\tilde{x}_k$ in the interval $[a, b]$.

Remarks on Polynomial Approximation

- **Advantages:** The class of algebraic polynomials has some distinct advantages for use in approximation:
- 1. there are a sufficient number of polynomials to approximate any continuous function on a closed interval to within an arbitrary tolerance;
- 2. polynomials are easily evaluated at arbitrary values;
- 3. the derivatives and integrals of polynomials exist and are easily determined.
- **Disadvantage:** Using polynomials for approximation always produces oscillation, due to the nature of polynomials with higher degree.

Rational Function:

A rational function $r(x)$ of degree N has the form

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials whose degrees sum to N.

Notes:

Every polynomial is a rational function (simply let $q(x) = 1$);

approximation by rational functions gives results that are no worse than approximation by polynomials.

- Rational functions whose numerator(分子) and denominator(分母) have the same or nearly the same degree generally produce approximation results superior to polynomial method for the same amount of computation effort. (which means that the amount of computation effort required for division is approximately the same as for multiplication.)
- Rational functions have the added advantage of permitting efficient as approximation of functions with infinite discontinuities (无穷间断点) near, but outside, the interval of approximation. Polynomial approximation is generally unacceptable in this situation.

- Suppose $r(x)$ is a rational function of degree $N = n + m$ of the form

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x_1 + \cdots + p_n x^n}{q_0 + q_1 x + \cdots + q_m x^m},$$

  that is used to approximate a function f on a closed interval I containing zero. For $r(x)$ to be defined at zero requires that $q_0 \neq 0$.
- In fact, we can assume that $q_0 = 1$, for if this is not the case we simply replace $p(x)$ by $p(x)/q_0$ and $q(x)$ by $q(x)/q_0$.
- Consequently, there are $N + 1$ parameters $q_1, q_2, \cdots, q_m$, $p_0, p_1, \cdots, p_n$ available for the approximation of $f(x)$ by $r(x)$.

The Padé approximation technique

- The Padé approximation technique is the extension of Taylor polynomial approximation to rational functions, chooses the $N+1$ parameters, so that $f^{(k)}(0) = r^{(k)}(0)$, for each $k = 0, 1, \cdots, N$, When $n = N$ and $m = 0$, the Padé approximation is just the Nth Maclaurin polynomial.
- Consider the difference

$$\begin{aligned} f(x) - r(x) &= f(x) - \frac{p(x)}{q(x)} = \frac{f(x)q(x) - p(x)}{q(x)} \\ &= \frac{f(x)\sum_{i=0}^{m} q_i x^i - \sum_{i=0}^{n} p_i x^i}{q(x)} \end{aligned}$$

-

and suppose f has the Maclaurin series expansion $f(x) = \sum_{i=0}^{\infty} a_i x^i$.

Then

$$f(x) - r(x) = \frac{\sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^{m} q_i x^i - \sum_{i=0}^{n} p_i x^i}{q(x)} \qquad (4)$$

The object is to choose the constants $q_1, q_2, \cdots, q_m$ and $p_0, p_1, \cdots, p_n$, so that

$$f^{(k)}(0) - r^{(k)}(0) = 0, \texttt{for each } \; k = 0, 1, \cdots, N.$$

This is equivalent to $f - r$ having a zero of multiplicity $N+1$ ($N+1$重根) at $x = 0$.

As a consequence, we choose $q_1, q_2, \cdots, q_m$ and $p_0, p_1, \cdots, p_n$, so that the numerator(分子) on the right side of Eq. (4),

$$\begin{aligned}(a_0 \quad + \quad & a_1 x + \cdots)(1 + q_1 x + \cdots + q_m x^m) \\ & -(p_0 + p_1 x + \cdots + p_n x^n)\end{aligned} \tag{5}$$

has no terms of degree less than or equal to N.

To simplify notation, we define

$$p_{n+1} = p_{n+2} = \cdots = p_N = 0$$

and

$$q_{m+1} = q_{m+2} = \cdots = q_N = 0.$$

We can then express the coefficient of x^k in expression (5) as

$$\left(\sum_{i=0}^{k} a_i q_{k-i}\right) - p_k,$$

So, the rational function for Padé approximation results from the solution of the $N+1$ linear equations

$$\sum_{i=0}^{k} a_i q_{k-i} = p_k, k = 0, 1, \cdots, N$$

in the $N+1$ unknowns $q_1, q_2, \cdots, q_m; p_0, p_1, \cdots, p_n$.

Padé Rational Approximation

To obtain the rational approximation

$$r(x) = \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^{n} p_i x^i}{\sum_{i=0}^{m} q_i x^i}.$$

for a given function $f(x)$:

INPUT nonnegative integers m and n.

OUTPUT coefficients $q_0, q_1, \cdots, q_m$ and $p_0, p_1, \cdots, p_n$.

Step 1 Set $N = m + n$.

Step 2 For $i = 0, 1, \cdots, N$ set $a_i = \frac{f^{(i)}(0)}{i!}$. (The coefficients of the Maclaurin polynomial are $a_0, \cdots, a_N$. which could be input instead of calculated.)

Step 3 Set $q_0 = 1; p_0 = a_0$.

Step 4 For $i = 1, 2, \cdots, N$ do Steps 5-10. (Set up a linear system with matrix B.)

Step 5 For $j = 1, 2, \cdots, i-1$ if $j \leq n$ then set $b_{i,j} = 0$.

Step 6 If $i < n$ then set $b_{i,i} = 1$.

Step 15 For $j = i+1, i+2, ..., N$ do Steps 16-18. (Perform elimination.)

Step 16 Set $x_m = b_{j,i}/b_{i,i}$.

Step 17 For $k = i+1, i+2, \cdots, N+1$ set $b_{j,k} = b_{j,k} - x_m \cdot b_{i,k}$.

Step 18 Set $b_{j,i} = 0$.

Step 19 If $b_{N,N} = 0$ then OUTPUT ("The system is singular"); STOP.

Step 20 If $m > 0$ then set $q_m = b_{N,N+1}/b_{N,N}$. (Start backward substitution.)

Step 21 For $I = N-1, N-2, \cdots, n+1$ set $q_{i-n} = \frac{b_{i,N+1} - \sum_{j=i+1}^{N} b_{i,j}\, q_{j-n}}{b_{i,i}}$.

Step 22 For $i = n, n-1, ..., 1$ set $p_i = b_{i,N+1} - \sum_{j=n+1}^{N} b_{i,j}\, q_{j-n}$.

Step 23 OUTPUT($q_0, q_1, \cdots, q_m, p_0, p_1, \cdots, p_n$); STOP. (The procedure was successful.)

Chebyshev Polynomial Approximation:

Suppose we want to approximate the function f by an Nth-degree rational function r written in the form .

$$r(x) = \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)},$$

where $N = n + m$ and $q_0 = 1$.

Writing $f(x)$ in a series involving Chebyshev polynomials as

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

gives

$$f(x) - r(x) = \sum_{k=0}^{\infty} a_k T_k(x) - \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}$$

or

$$f(x) - r(x) = \frac{\sum_{k=0}^{\infty} a_k T_k(x) \sum_{k=0}^{m} q_k T_k(x) - \sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)} \tag{6}$$

The coefficients $q_1, q_2, \cdots, q_m$ and $p_0, p_1, \cdots, p_n$ are chosen so that the numerator on the right-hand side of this equation has zero coefficients for $T_k(x)$ when $k = 0, 1, \cdots, N$.

This implies that the series

$$\begin{aligned}(a_0 T_0(x) \quad + \quad & a_1 T_1(x) + \cdots)(T_0(x) + q_1 T_1(x) + \cdots + q_m T_m(x)) \\ & -(p_0 T_0(x) + p_1 T_1(x) + \cdots + p_n T_n(x))\end{aligned}$$

as no terms of degree less than or equal to N.

Two problems arise with the Chebyshev procedure that make it more difficult to implement than the Fadé method. One occurs because the product of the polynomial $q(x)$ and the series for $f(x)$ involves products of Chebyshev polynomials. This problem is resolved by making use of the relationship

$$T_i(x)\, T_j(x) = \frac{1}{2}[T_{i+j}(x) + T_{|i-j|}(x)].$$

The other problem is more difficult to resolve and involves the computation of Chebyshev series for $f(x)$. In theory, this can be down if

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

then

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx$$

and

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1-x^2}} dx, k \geq 1.$$

The use of series of sine and cosine functions to represent arbitrary functions had its beginnings in the 1750s with the study of the motion of a vibrating string(弦振动). This problem was considered by Jean d' Alembert and then taken up by the foremost mathematician of the time, Leonhard Euler. But it was Daniel Bernoulli who first advocated the use of the infinite sums of sine and cosines as a solution to the problem, sums that we now know as Fourier series. In the early part of the 19th century, Jean Baptiste Joseph Fourier used these series to study the flow of heat and developed quite a complete theory of the subject.

The first observation in the development of **Fourier series** is that, for each positive integer n, the set of functions $\{\phi_0, \phi_1, \cdots, \phi_{2n-1}\}$, where

$$\begin{aligned}\phi_0(x) &= 1/2 \\ \phi_k(x) &= \cos kx, \texttt{for each } \; k = 1, 2, \cdots, n\end{aligned}$$

and

$$\phi_{n+k}(x) = \sin kx, \texttt{for each } \; k = 1, 2, \cdots, n-1,$$

is an orthogonal set on $[-\pi, \pi]$ with respect to $w(x) \equiv 1$.

This orthogonality follows from the fact that, for every integer j, the integrals of $\sin jx$ and $\cos jx$ over $[-\pi, \pi]$ are 0, and we can rewrite products of sine and cosine functions as sums by using the **three trigonometric identities**

$$\begin{aligned}
\sin x \sin y &= \frac{1}{2}[\cos(x-y) - \cos(x+y)], \\
\cos x \cos y &= \frac{1}{2}[\cos(x-y) + \cos(x+y)], \\
\sin x \cos y &= \frac{1}{2}[\sin(x-y) + \sin(x+y)].
\end{aligned}$$

Let $\mathfrak{T}_n$ denote the set of all linear combinations of the functions $\{\phi_0, \phi_1, \cdots, \phi_{2n-1}\}$. This set is called the **set of trigonometric polynomials** of degree less than or equal to n (Notes: Some sources also include an additional function in the set, $\phi_{2n}(x) = \sin nx$.)

For a function $f \in C[-\pi, \pi]$, we want to find the continuous least squares approximation by functions in $\mathfrak{T}_n$ in the form

$$S_n(x) = \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx).$$

Since the set of functions $\{\phi_0, \phi_1, \cdots, \phi_{2n-1}\}$ is orthogonal on $[-\pi, \pi]$ with respect to $w(x) \equiv 1$, it follows from Theorem 8.6, that the appropriate selection of coefficients is

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx, \texttt{for each} \quad k = 0, 1, \cdots, n,$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx, \texttt{for each} \quad k = 1, 2, \cdots, n-1.$$

The limit of $S_n(x)$ when $n \to \infty$ is called the **Fourier series** of f. Fourier series are used to describe the solution of various ordinary and partial-differential equations that occur in physical situations.

Discrete Least Square Approximation in the Sense of Trigonometric Polynomials

Suppose that a collection of $2m$ paired data points $\{(x_j, y_j)\}_{j=0}^{2m-1}$ is given, with the first elements in the pairs equally partitioning a closed interval.

For convenience, we assume that the interval is $[-\pi, \pi]$, so,

$$x_j = -\pi + \left(\frac{j}{m}\right)\pi, \texttt{for each } j = 0, 1, \cdots, 2m-1. \quad (7)$$

If it is not $[-\pi, \pi]$, a simple linear transformation could be used to translate the data into this form.

The goal in the discrete case is to determine the trigonometric polynomial $S_n(x)$ in $\mathfrak{T}_n$ that will minimize

$$E(S_n) = \sum_{j=0}^{2m-1} [y_j - S_n(x_j)]^2.$$

To do this we need to choose the constants $a_0, a_1, \cdots, a_n; b_1, b_2, \cdots, b_{n-1}$ so that

$$E(S_n) = \sum_{j=0}^{2m-1} \left\{ y_j - \left(\frac{a_0}{2} + a_n \cos nx_j + \sum_{k=1}^{n-1} (a_k \cos kx_j + b_k \sin kx_j) \right) \right\}^2 \tag{8}$$

is a minimum.

The determination of the constants is simplified by the fact that the set $\{\phi_0, \phi_1, \cdots, \phi_{2n-1}\}$ is orthogonal with respect to summation over the equally spaced points $\{x_j\}_{j=0}^{2m-1}$ in $[-\pi, \pi]$. By this we mean that for each $k \neq l$,

$$\sum_{j=0}^{2m-1} \phi_k(x_j)\phi_l(x_j) = 0.$$

To show this orthogonality. we use the following lemma.

Lemma 8.12

If the integer r is not a multiple of $2m$, then

$$\sum_{j=0}^{2m-1} \cos rx_j = 0, \text{ and } \sum_{j=0}^{2m-1} \sin rx_j = 0$$

Moreover, if r is not a multiple of m, then

$$\sum_{j=0}^{2m-1} (\cos rx_j)^2 = m, \text{ and } \sum_{j=0}^{2m-1} (\sin rx_j)^2 = m.$$

Proof:

Euler's Formula states that if $i^2 = -1$, then for every real number z. we have

$$e^{iz} = \cos z + i \sin z.$$

Applying this result gives

$$\begin{aligned} \sum_{j=0}^{2m-1} \cos rx_j + i \sum_{j=0}^{2m-1} \sin rx_j &= \sum_{j=0}^{2m-1} (\cos rx_j + i \sin rx_j) \\ &= \sum_{j=0}^{2m-1} e^{irx_j} \end{aligned}$$

But

$$e^{irx_j} = e^{ir(-\pi + j\pi/m)} = e^{-ir\pi} \cdot e^{irj\pi/m},$$

so

$$\sum_{j=0}^{2m-1} \cos rx_j + i \sum_{j=0}^{2m-1} \sin rx_j = e^{-ir\pi} \sum_{j=0}^{2m-1} e^{irj\pi/m}.$$

Since $\sum_{j=0}^{2m-1} e^{irj\pi/m}$ is a geometric series with first term 1 and ratio $e^{ir\pi/m} \neq 1$, we have

$$\sum_{j=0}^{2m-1} e^{irj\pi/m} = \frac{1-(e^{ir\pi/m})^{2m}}{1-e^{ir\pi/m}} = \frac{1-e^{2ir\pi}}{1-e^{ir\pi/m}}$$

But $e^{2ir\pi} = \cos 2r\pi + i\sin 2r\pi = 1$, so $1 - e^{2ir\pi} = 0$ and

$$\sum_{j=0}^{2m-1} \cos rx_j + i\sum_{j=0}^{2m-1} \sin rx_j = e^{-ir\pi}\sum_{j=0}^{2m-1} e^{irj\pi/m} = 0$$

This implies that both

$$\sum_{j=0}^{2m-1} \cos rx_j = 0, \quad \text{and} \quad \sum_{j=0}^{2m-1} \sin rx_j = 0$$

If r is not a multiple of m, these sums imply that

$$\begin{aligned}\sum_{j=0}^{2m-1} (\cos rx_j)^2 &= \sum_{j=0}^{2m-1} \frac{1}{2}(1 + \cos 2rx_j) \\ &= \frac{1}{2}\left[\sum_{j=0}^{2m-1} 1 + \sum_{j=0}^{2m-1} \cos 2rx_j\right] \\ &= \frac{1}{2}(2m + 0) = m\end{aligned}$$

and, similarly, that

$$\sum_{j=0}^{2m-1} (\sin rx_j)^2 = m. \quad \blacksquare\blacksquare\blacksquare.$$

Now let's show the orthogonality of the set $\{\phi_0, \phi_1, \cdots, \phi_{2n-1}\}$, which means that for $k \neq l$, we have

$$\sum_{j=0}^{2m-1} \phi_k(x_j)\phi_l(x_j) = 0.$$

Consider, for example, the case

$$\sum_{j=0}^{2m-1} \phi_k(x_j)\phi_{n+l}(x_j) = \sum_{j=0}^{2m-1} (\cos kx_j)(\sin lx_j).$$

Since

$$\cos kx_j \sin lx_j = \frac{1}{2}[\sin(l+k)x_j + \sin(l-k)x_j]$$

and $(l+k)$ and $(l-k)$ are both integers that are not multiples of $2m$, by Lemma 8.12, implies that

$$\begin{aligned}
\sum_{j=0}^{2m-1} (\cos kx_j)(\sin lx_j) &= \frac{1}{2}\left[\sum_{j=0}^{2m-1} \sin(l+k)x_j + \sum_{j=0}^{2m-1} \sin(l-k)x_j\right] \\
&= \frac{1}{2}(0+0) = 0.
\end{aligned}$$

This technique is used to show that the orthogonality condition is satisfied for any pairs of the functions and is used to produce the following result.

Theorem 8.13

The constants in the summation

$$S_n(x) = \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx)$$

that minimize the least squares sum

$$E(a_0, a_1, \cdots, a_n; b_1, b_2, \cdots, b_{n-1}) = \sum_{j=0}^{2m-1} (y_j - S_n(x_j))^2$$

are

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j, k = 0, 1, \cdots, n,$$

and

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j, k = 1, 2, \cdots, n,$$